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Abstract

The classical non-holomorphic Eisenstein series $E_p^{\mathrm{par}}(z,s)$ on the upper half-plane $\mathbb{H}$ is associated to a parabolic fixed point p of a Fuchsian subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ of the first kind. Hyperbolic and elliptic analogues of $E_p^{\mathrm{par}}(z,s)$ were also studied, namely non-holomorphic Eisenstein series which are associated to a pair of hyperbolic fixed points of Γ or a point in the upper half-plane, respectively. In particular, von Pippich derived Kronecker limit type formulas for elliptic Eisenstein series on the upper half-plane.

In the present thesis we consider hyperbolic and elliptic Eisenstein series in the n-dimensional hyperbolic upper half-space $\mathbb{H}^n$ for a discrete group Γ of orientation-preserving isometries of $\mathbb{H}^n$ which has finite hyperbolic volume. Here we realize these isometries as certain matrices with entries in the Clifford numbers. We define the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ associated to a pair (Q_1,Q_2) of hyperbolic fixed points of Γ and the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$ associated to a point $Q \in \mathbb{H}^n$. First we prove the absolute and locally uniform convergence of these series for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$. Then we derive some other basic properties of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ and $E^{\mathrm{ell}}_Q(P,s)$ like Γ-invariance, smoothness and certain differential equations that are satisfied by these Eisenstein series.

We establish the meromorphic continuations of the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ and the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$ in s to the whole complex plane. For that we employ the relations between these Eisenstein series and the so-called hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$, which is meromorphically continued to all $s \in \mathbb{C}$ by means of its spectral expansion. In this way we also establish the meromorphic continuation of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ via its spectral expansion, and further obtain the meromorphic continuation of $E^{\mathrm{ell}}_Q(P,s)$ by expressing it in terms of $K^{\mathrm{hyp}}(P,Q,s)$. Moreover, we determine the possible poles of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ and $E^{\mathrm{ell}}_Q(P,s)$.

Using the aforementioned meromorphic continuations, we investigate the behaviour of the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ and the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$ at the point $s=0$ via their Laurent expansions. We determine the first two terms in the Laurent expansions of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ and $E^{\mathrm{ell}}_Q(P,s)$ at $s=0$ for arbitrary n and Γ. Eventually, we refine the Laurent expansion of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ for $n=2$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and $n=3$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$, as well as the Laurent expansion of $E^{\mathrm{ell}}_Q(P,s)$ for $n=3$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$, and obtain Kronecker limit type formulas in these specific cases.

Zusammenfassung

Die klassische nicht-holomorphe Eisensteinreihe $E_p^{\mathrm{par}}(z,s)$ auf der oberen Halbebene $\mathbb{H}$ ist assoziiert zu einem parabolischen Fixpunkt p einer Fuchsschen Gruppe $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ erster Art. Hyperbolische und elliptische Analoga von $E_p^{\mathrm{par}}(z,s)$ wurden ebenfalls untersucht; diese sind nicht-holomorphe Eisensteinreihen, die zu einem Paar hyperbolischer Fixpunkte von Γ bzw. einem Punkt in der oberen Halbebene assoziiert sind. Insbesondere bewies von Pippich Kroneckersche Grenzformeln für elliptische Eisensteinreihen auf der oberen Halbebene.

In der vorliegenden Arbeit betrachten wir hyperbolische und elliptische Eisensteinreihen im n-dimensionalen hyperbolischen oberen Halbraum $\mathbb{H}^n$ für eine diskrete Gruppe Γ orientierungserhaltender Isometrien von $\mathbb{H}^n$, die endliches hyperbolisches Volumen besitzt. Hierbei realisieren wir diese Isometrien durch bestimmte Matrizen mit Einträgen in den Clifford-Zahlen. Wir definieren die hyperbolische Eisensteinreihe $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$, die zu einem Paar (Q_1,Q_2) hyperbolischer Fixpunkte von Γ assoziiert ist, und die elliptische Eisensteinreihe $E^{\mathrm{ell}}_Q(P,s)$, die zu einem Punkt $Q \in \mathbb{H}^n$ assoziiert ist. Zunächst beweisen wir die absolute und lokal gleichmäßige Konvergenz dieser Reihen für $s \in \mathbb{C}$ mit $\mathrm{Re}(s) > n-1$. Anschließend zeigen wir einige weitere grundlegende Eigenschaften von $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ und $E^{\mathrm{ell}}_Q(P,s)$ wie Γ-Invarianz, Glattheit und bestimmte Differentialgleichungen, welche diese Eisensteinreihen erfüllen.

Wir etablieren die meromorphen Fortsetzungen der hyperbolischen Eisensteinreihe $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ und der elliptischen Eisensteinreihe $E^{\mathrm{ell}}_Q(P,s)$ in s auf die gesamte komplexe Ebene. Dazu nutzen wir die Relationen zwischen diesen Eisensteinreihen und der sogenannten hyperbolischen Kernfunktion $K^{\mathrm{hyp}}(P,Q,s)$, die mit Hilfe ihrer Spektralentwicklung in alle $s \in \mathbb{C}$ meromorph fortgesetzt wird. Auf diese Weise etablieren wir auch die meromorphe Fortsetzung von $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ über ihre Spektralentwicklung, und erhalten außerdem die meromorphe Fortsetzung von $E^{\mathrm{ell}}_Q(P,s)$, indem wir sie in Termen von $K^{\mathrm{hyp}}(P,Q,s)$ ausdrücken. Ferner bestimmen wir die möglichen Pole von $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ und $E^{\mathrm{ell}}_Q(P,s)$.

Unter Verwendung der oben genannten meromorphen Fortsetzungen untersuchen wir das Verhalten der hyperbolischen Eisensteinreihe $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ und der elliptischen Eisensteinreihe $E^{\mathrm{ell}}_Q(P,s)$ im Punkt $s=0$ mittels ihrer Laurent-Entwicklungen. Wir bestimmen die ersten beiden Terme in den Laurent-Entwicklungen von $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ und $E^{\mathrm{ell}}_Q(P,s)$ um $s=0$ für beliebige n und Γ. Schließlich verfeinern wir die Laurent-Entwicklung von $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ für $n=2$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ und $n=3$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$, sowie die Laurent-Entwicklung von $E^{\mathrm{ell}}_Q(P,s)$ für $n=3$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$, und erhalten Kroneckersche Grenzformeln in diesen konkreten Fällen.

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Contents

Introduction

This thesis deals with the study of certain complex-valued functions on the n-dimensional hyperbolic space, called hyperbolic and elliptic Eisenstein series. These functions generalize the known parabolic, hyperbolic and elliptic Eisenstein series on the complex upper half-plane. In the introduction we give a motivation for considering these Eisenstein series, present a summary of our main results and outline the structure of this work.

Non-holomorphic Eisenstein series on the upper half-plane

Classically, in the theory of automorphic functions on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ the non-holomorphic Eisenstein series associated to the cusp ∞ of the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ is for $z = x + iy \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ given by

$$E_\infty^{\mathrm{par}}(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2, \\ \gcd(c,d)=1}} \frac{y^s}{|cz+d|^{2s}},$$

where Γ_∞ denotes the stabilizer of ∞ in Γ and $\gamma z = (az+b)(cz+d)^{-1}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. This series converges absolutely and locally uniformly for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. The function $E_\infty^{\mathrm{par}}(z,s)$ is invariant in z under the action of Γ, i.e. $E_\infty^{\mathrm{par}}(\gamma z,s) = E_\infty^{\mathrm{par}}(z,s)$ for any $\gamma \in \Gamma$, and an eigenfunction of the hyperbolic Laplace operator

$$\Delta_{\mathbb{H}} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

on $\mathbb{H}$ with eigenvalue $s(1-s)$. Moreover, it is holomorphic in s for $\mathrm{Re}(s) > 1$, and has a meromorphic continuation in s to the whole complex plane with a simple pole at $s = 1$. The famous Kronecker limit formula gives the Laurent expansion of $E_\infty^{\mathrm{par}}(z,s)$ at this pole, stating that

$$E_\infty^{\mathrm{par}}(z,s) = \frac{3}{\pi} \cdot \frac{1}{s-1} - \frac{1}{2\pi} \log\left(|\Delta(z)| \,\mathrm{Im}(z)^6\right) + \frac{6 - 72\,\zeta'(-1) - 6\,\log(4\pi)}{\pi} + \mathrm{O}(s-1), \quad (0.1)$$

where $\Delta(z) = (E_4(z)^3 - E_6(z)^2)/1728$ is the unique normalized cusp form of weight 12 for Γ and

$$E_k(z) = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2, \\ \gcd(c,d)=1}} \frac{1}{(cz+d)^k} \qquad (k = 4, 6)$$

denotes the holomorphic Eisenstein series of weight k. From the functional equation of $E_\infty^{\mathrm{par}}(z,s)$, relating its values for s and $1-s$, one derives the Laurent expansion

$$E_\infty^{\mathrm{par}}(z,s) = 1 + \log\left(|\Delta(z)|^{1/6}\,\mathrm{Im}(z)\right) \cdot s + \mathrm{O}(s^2) \quad (0.2)$$

at $s = 0$. Since $E_\infty^{\mathrm{par}}(z,s)$ is harmonic with respect to $\Delta_{\mathbb{H}}$ exactly for $s = 0$ and $s = 1$, it is natural to study its behaviour at these two points.
More generally, for a cusp (parabolic fixed point) p of a Fuchsian subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ of the

first kind there is an associated non-holomorphic Eisenstein series, which for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ is given by

$$E_p^{\mathrm{par}}(z,s) = \sum_{\gamma \in \Gamma_p \backslash \Gamma} \mathrm{Im}(\sigma_p^{-1}\gamma z)^s.$$

Here Γ_p is the stabilizer of p in Γ and $\sigma_p \in \mathrm{PSL}_2(\mathbb{R})$ is a so-called parabolic scaling matrix of p, satisfying $\sigma_p\infty = p$. We call $E_p^{\mathrm{par}}(z,s)$ the parabolic Eisenstein series associated to the cusp (parabolic fixed point) p. Again, the series converges absolutely and locally uniformly for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Also the parabolic Eisenstein series $E_p^{\mathrm{par}}(z,s)$ is invariant in z under the action of Γ and an eigenfunction of $\Delta_{\mathbb{H}}$ with eigenvalue $s(1-s)$. It is holomorphic in s for $\mathrm{Re}(s) > 1$, and admits a meromorphic continuation to all $s \in \mathbb{C}$ (see, e.g., [Kub73], [Iwa02]).

In 1979, Kudla and Millson introduced hyperbolic analogues of parabolic Eisenstein series, namely non-holomorphic Eisenstein series associated to a pair of hyperbolic fixed points of a Fuchsian subgroup of the first kind (see [KM79]). While Kudla's and Millson's functions are 1-forms, scalar-valued hyperbolic Eisenstein series twisted with modular symbols were later studied in [Ris04], and von Pippich considered scalar-valued hyperbolic Eisenstein series associated to a pair of hyperbolic fixed points of a general Fuchsian subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ of the first kind in her Diploma thesis [Pip05] (see also [JKP10]). More precisely, given a pair (h_1,h_2) of hyperbolic fixed points of Γ with stabilizer $\Gamma_{(h_1,h_2)}$, for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ the hyperbolic Eisenstein series associated to (h_1,h_2) is given by

$$E_{(h_1,h_2)}^{\mathrm{hyp}}(z,s) = \sum_{\gamma \in \Gamma_{(h_1,h_2)} \backslash \Gamma} \cosh(d_{\mathbb{H}}(\gamma z, \mathcal{L}_{(h_1,h_2)}))^{-s},$$

where $\mathcal{L}_{(h_1,h_2)}$ denotes the unique geodesic in $\mathbb{H}$ connecting h_1 and h_2 and $d_{\mathbb{H}}(\gamma z, \mathcal{L}_{(h_1,h_2)})$ is the hyperbolic distance between γz and $\mathcal{L}_{(h_1,h_2)}$ in the upper half-plane. As in the parabolic case, the series defining $E_{(h_1,h_2)}^{\mathrm{hyp}}(z,s)$ converges absolutely and locally uniformly for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. The hyperbolic Eisenstein series $E_{(h_1,h_2)}^{\mathrm{hyp}}(z,s)$ is smooth and invariant in z under the action of Γ and a holomorphic function in s for $\mathrm{Re}(s) > 1$. Though $E_{(h_1,h_2)}^{\mathrm{hyp}}(z,s)$ is not an eigenfunction of the hyperbolic Laplace operator, it satisfies the shift equation

$$\Delta_{\mathbb{H}}\, E_{(h_1,h_2)}^{\mathrm{hyp}}(z,s) = s(1-s)\, E_{(h_1,h_2)}^{\mathrm{hyp}}(z,s) + s^2\, E_{(h_1,h_2)}^{\mathrm{hyp}}(z,s+2),$$

thus, it is harmonic at the point $s = 0$. In [JKP10] the authors established the meromorphic continuation of $E_{(h_1,h_2)}^{\mathrm{hyp}}(z,s)$ to all $s \in \mathbb{C}$ and showed that it has a double zero at $s = 0$.

Elliptic analogues of the above non-holomorphic Eisenstein series were introduced by Jorgenson and Kramer in their unpublished work [JK04] (see also [JK11]). These are called elliptic Eisenstein series and associated to elliptic fixed points or arbitrary points in $\mathbb{H}$. Kramer's student von Pippich comprehensively studied elliptic Eisenstein series for an arbitrary Fuchsian subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ of the first kind in her PhD thesis [Pip10]. Given a point $w \in \mathbb{H}$ with stabilizer subgroup Γ_w, for $z \in \mathbb{H}$ with $z \neq \gamma w$ for any $\gamma \in \Gamma$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ the elliptic Eisenstein series associated to w is given by

$$E_w^{\mathrm{ell}}(z,s) = \sum_{\gamma \in \Gamma_w \backslash \Gamma} \sinh(d_{\mathbb{H}}(\gamma z, w))^{-s}.$$

The series converges absolutely and locally uniformly for $z \in \mathbb{H}$ with $z \neq \gamma w$ for any $\gamma \in \Gamma$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. The elliptic Eisenstein series $E_w^{\mathrm{ell}}(z,s)$ has a singularity in z at every Γ-translate of the point w. It is smooth and invariant in z under the action of Γ, wherever it is defined, and holomorphic in s for $\mathrm{Re}(s) > 1$. Moreover, $E_w^{\mathrm{ell}}(z,s)$ satisfies the differential shift equation

$$\Delta_{\mathbb{H}}\, E_w^{\mathrm{ell}}(z,s) = s(1-s)\, E_w^{\mathrm{ell}}(z,s) - s^2\, E_w^{\mathrm{ell}}(z,s+2),$$

which makes it plausible to study its meromorphic continuation to the harmonic point $s = 0$. This was also done by von Pippich who proved in [Pip10] (see also [Pip16]) that the elliptic Eisenstein series $E_w^{\mathrm{ell}}(z, s)$ has a meromorphic continuation to all $s \in \mathbb{C}$, and gave its Laurent expansion at $s = 0$. For the specific group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and its elliptic fixed points i and $\rho = \exp\left(\frac{2\pi i}{3}\right)$ she further determined the Kronecker limit type formulas

$$\begin{aligned} E_i^{\mathrm{ell}}(z,s) &= -\log\left(|E_6(z)|\,|\Delta(z)|^{-1/2}\right) \cdot s + \mathrm{O}(s^2), \\ E_\rho^{\mathrm{ell}}(z,s) &= -\log\left(|E_4(z)|\,|\Delta(z)|^{-1/3}\right) \cdot s + \mathrm{O}(s^2). \end{aligned}$$

In [Pip16], von Pippich also studied the relation of $E_w^{\mathrm{ell}}(z, s)$ to the automorphic Green's function.

The goal of this thesis is to generalize the above hyperbolic and elliptic Eisenstein series for higher dimensional hyperbolic spaces. We define hyperbolic and elliptic Eisenstein series in the n-dimensional hyperbolic upper half-space $\mathbb{H}^n$ for any $n \geq 2$, give some of their basic properties, establish their meromorphic continuations to the whole complex plane and investigate their behaviour at the point $s = 0$.

The n-dimensional hyperbolic space and the group $\mathrm{PSL}_2(C_{n-1})$

For $n \in \mathbb{N}$, $n \geq 2$, the upper half-space model of the n-dimensional hyperbolic space (hyperbolic n-space) is the set

$$\mathbb{H}^n = \{P = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n \mid x_{n-1} > 0\}.$$

Its boundary is $\{P = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n \mid x_{n-1} = 0\} \cong \mathbb{R}^{n-1}$ together with the point ∞ and we write $\widehat{\mathbb{R}}^{n-1} = \mathbb{R}^{n-1} \cup \{\infty\}$. In the coordinates $x_0, \ldots, x_{n-1}$ the hyperbolic line element on $\mathbb{H}^n$ is given by

$$ds^2_{\mathbb{H}^n} = \frac{dx_0^2 + \cdots + dx_{n-1}^2}{x_{n-1}^2}.$$

Then the hyperbolic distance of $P, Q \in \mathbb{H}^n$ derived from $ds^2_{\mathbb{H}^n}$ is denoted by $d_{\mathbb{H}^n}(P, Q)$ and the hyperbolic volume element on $\mathbb{H}^n$ by $\mu_{\mathbb{H}^n}(P)$. Moreover, the hyperbolic Laplace operator associated with $ds^2_{\mathbb{H}^n}$ is given by

$$\Delta_{\mathbb{H}^n} = -x_{n-1}^2 \left(\frac{\partial^2}{\partial x_0^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}\right) + (n-2)\, x_{n-1} \frac{\partial}{\partial x_{n-1}}.$$

For $n = 2$ this yields again the upper half-plane

$$\mathbb{H}^2 = \mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \cong \{z = x + iy \in \mathbb{C} \mid y > 0\},$$

while for $n = 3$ we obtain the upper half-space model

$$\mathbb{H}^3 = \{(x, y, r) \in \mathbb{R}^3 \mid r > 0\} \cong \{P = z + jr \mid z \in \mathbb{C},\, r \in \mathbb{R},\, r > 0\}$$

of hyperbolic 3-space.

The Möbius transformations of $\mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ can be realized as a certain group of 2×2-matrices with entries in the Clifford numbers. This approach was first used in 1902 by Vahlen (see [Vah02]), later rediscovered and improved by Maass (see [Maa49]) and also used by Ahlfors (see [Ahl85a], [Ahl86]). For $n \in \mathbb{N}$ we define the Clifford numbers C_n as the associative algebra over $\mathbb{R}$ generated by elements $i_1, \ldots, i_{n-1}$, satisfying the relations $i_k^2 = -1$ for $k = 1, \ldots, n-1$ and $i_k i_l = -i_l i_k$ for $k, l = 1, \ldots, n-1$ with $k \neq l$. Any $a \in C_n$ can be uniquely written in the form $a = \sum_I a_I\, I$, where $a_I \in \mathbb{R}$, the sum runs over all products $I = i_{\nu_1} \cdots i_{\nu_k}$ with $1 \leq \nu_1 < \cdots < \nu_k \leq n-1$ and the empty product $I = \emptyset$ is interpreted as 1. We equip C_n with the square norm, so for $a = \sum_I a_I\, I$

we have $|a|^2 = \sum_I a_I^2$.
We define three involutions of C_n: the map $' : a \mapsto a'$, which replaces i_k by $-i_k$ $(k = 1, \dots, n-1)$, the map $^* : a \mapsto a^*$, which reverses the order of the factors in each product $I = i_{\nu_1} \cdots i_{\nu_k}$, and the map $^- : a \mapsto \overline{a} = (a')^*$, which is the composition of the previous two.
An element $x \in C_n$ of the form $x = x_0 + x_1 i_1 + \cdots + x_{n-1} i_{n-1}$ with $x_0, \dots, x_{n-1} \in \mathbb{R}$ is called a vector. The subspace $V_n \subseteq C_n$ of all vectors can naturally be identified with $\mathbb{R}^n$ which enables us to regard $\mathbb{R}^n$ as a subspace of C_n. Any $x \in V_n \setminus \{0\}$ is multiplicatively invertible, so the non-zero vectors generate a multiplicative group Γ_n, called the Clifford group. Then any $a \in \Gamma_n$ has the multiplicative inverse $a^{-1} = \overline{a}/|a|^2$.

The set

$$\mathrm{SL}_2(C_n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \Gamma_n \cup \{0\},\ ab^*, cd^* \in V_n,\ ad^* - bc^* = 1 \right\}$$

of matrices whose entries are certain Clifford numbers is a group under matrix multiplication, and we can consider the quotient group $\mathrm{PSL}_2(C_n) = \mathrm{SL}_2(C_n)/\{\pm I\}$. In the simplest cases $n = 1, 2$ we rediscover the well-known groups $\mathrm{PSL}_2(C_1) = \mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(C_2) = \mathrm{PSL}_2(\mathbb{C})$.
Now let $n \in \mathbb{N}$ with $n \geq 2$. Using the identification of $\mathbb{R}^{n-1}$ and V_{n-1}, the group $\mathrm{PSL}_2(C_{n-1})$ acts on $\widehat{\mathbb{R}}^{n-1}$ via the formula

$$\left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\, P \right) \mapsto \gamma P = (aP + b)\,(cP + d)^{-1}, \tag{0.3}$$

where for $P = \infty$ we define $\gamma\infty := ac^{-1}$ if $c \neq 0$, and $\gamma\infty := \infty$ if $c = 0$, and in the case $cP + d = 0$ and $aP + b \neq 0$ we set $\gamma P := \infty$. In this way the right-hand side of (0.3) is always well-defined. The group $\mathrm{PSL}_2(C_{n-1})$ is isomorphic to the group of orientation-preserving Möbius transformations of $\widehat{\mathbb{R}}^{n-1}$. Further, it acts doubly transitively on $\widehat{\mathbb{R}}^{n-1}$, i.e. for any $P, Q, R, S \in \widehat{\mathbb{R}}^{n-1}$ with $P \neq Q$ and $R \neq S$ there exists $\gamma \in \mathrm{PSL}_2(C_{n-1})$ such that $\gamma P = R$ and $\gamma Q = S$.
Taking into account that $\mathrm{PSL}_2(C_{n-1}) \subseteq \mathrm{PSL}_2(C_n)$ and that $\widehat{\mathbb{R}}^{n-1}$ can be viewed as a subspace of $\widehat{\mathbb{R}}^n$, the action of $\mathrm{PSL}_2(C_{n-1})$ naturally extends to $\widehat{\mathbb{R}}^n$ via (0.3) and $\mathrm{PSL}_2(C_{n-1})$ also acts on the hyperbolic space $\mathbb{H}^n \subseteq \widehat{\mathbb{R}}^n$. The group $\mathrm{PSL}_2(C_{n-1})$ is isomorphic to the group of orientation-preserving Möbius transformations of $\mathbb{H}^n$. It acts transitively on $\mathbb{H}^n$, i.e. for any $P, Q \in \mathbb{H}^n$ there exists $\gamma \in \mathrm{PSL}_2(C_{n-1})$ such that $\gamma P = Q$. The hyperbolic line element $ds^2_{\mathbb{H}^n}$, the hyperbolic distance $d_{\mathbb{H}^n}(P, Q)$, the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ and the hyperbolic volume element $\mu_{\mathbb{H}^n}(P)$ are all invariant under the action of $\mathrm{PSL}_2(C_{n-1})$.

To consider an analogue of Fuchsian subgroups of the first kind in $\mathrm{PSL}_2(C_{n-1})$, we endow it with a topology, and a subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ is called discrete if the induced topology on Γ is discrete. This is the case if and only if Γ acts discontinuously on $\mathbb{H}^n$, i.e. for any compact subset $K \subseteq \mathbb{H}^n$ there are only finitely many $\gamma \in \Gamma$ with $\gamma(K) \cap K \neq \emptyset$. If $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ is a discrete subgroup, then for any $P \in \mathbb{H}^n$ the stabilizer subgroup $\Gamma_P = \{\gamma \in \Gamma \mid \gamma P = P\}$ is finite and the orbit $\Gamma P = \{\gamma P \mid \gamma \in \Gamma\}$ is a discrete subset of $\mathbb{H}^n$.
For a discrete subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ the set $\Gamma\backslash\mathbb{H}^n = \{\Gamma P \mid P \in \mathbb{H}^n\}$ of orbits can be identified with a fundamental domain $\mathcal{F}_\Gamma$, i.e. a non-empty, connected, open subset of $\mathbb{H}^n$ such that distinct points of $\mathcal{F}_\Gamma$ are not equivalent with respect to Γ and every orbit ΓP contains a point in the closure $\overline{\mathcal{F}_\Gamma}$. Every discrete subgroup admits a fundamental domain which is not unique, but all fundamental domains $\mathcal{F}_\Gamma$ for Γ have the same hyperbolic volume, called the volume $\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)$ of $\Gamma\backslash\mathbb{H}^n$, respectively of Γ. We call Γ cofinite if $\mathrm{vol}(\Gamma\backslash\mathbb{H}^n) < \infty$.
In the following we consider discrete and cofinite subgroups of $\mathrm{PSL}_2(C_{n-1})$ which generalize Fuchsian subgroups $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ of the first kind and for which we will define Eisenstein series in $\mathbb{H}^n$. Important examples are the well-known modular group $\mathrm{PSL}_2(\mathbb{Z}) \subseteq \mathrm{PSL}_2(\mathbb{R})$ for $n = 2$, and $\mathrm{PSL}_2(\mathbb{Z}[i]) \subseteq \mathrm{PSL}_2(\mathbb{C})$ in the case $n = 3$.

Elements of $\mathrm{PSL}_2(C_{n-1})$ are classified in terms of the number and the location of their fixed points in $\mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ as follows: A matrix $\gamma \in \mathrm{PSL}_2(C_{n-1})$, $\gamma \neq I$, is called parabolic if it has exactly one fixed point in $\widehat{\mathbb{R}}^{n-1}$ and no fixed points in $\mathbb{H}^n$, loxodromic if it has exactly two fixed points in $\widehat{\mathbb{R}}^{n-1}$ and no fixed points in $\mathbb{H}^n$, and elliptic if it has a (not necessarily unique) fixed point in $\mathbb{H}^n$. Moreover, a loxodromic element is called hyperbolic if it is conjugate in $\mathrm{PSL}_2(C_{n-1})$ to a matrix of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$. This classification of an element is invariant under conjugation in $\mathrm{PSL}_2(C_{n-1})$ and depends only on its conjugacy class.
A point $P \in \mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ is called a parabolic, hyperbolic, elliptic or loxodromic fixed point of a discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ if it is a fixed point of a parabolic, hyperbolic, elliptic or loxodromic element of Γ, respectively.

Parabolic Eisenstein series in $\mathbb{H}^n$ and spectral expansion

Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. An automorphic function with respect to Γ is a function $f : \mathbb{H}^n \to \mathbb{C}$ which satisfies $f(\gamma P) = f(P)$ for any $\gamma \in \Gamma$ and $P \in \mathbb{H}^n$. It is a well-defined function on the quotient $\Gamma \backslash \mathbb{H}^n$. Together with the addition and scalar multiplication of functions the automorphic functions with respect to Γ form a complex vector space $\mathcal{A}(\Gamma \backslash \mathbb{H}^n)$. The inner product of $f_1, f_2 \in \mathcal{A}(\Gamma \backslash \mathbb{H}^n)$ is defined by

$$\langle f_1, f_2 \rangle = \int_{\Gamma \backslash \mathbb{H}^n} f_1(P) \, \overline{f_2(P)} \; \mu_{\mathbb{H}^n}(P),$$

provided that the integral exists. The $\mu_{\mathbb{H}^n}$-measurable functions $f \in \mathcal{A}(\Gamma \backslash \mathbb{H}^n)$ with $\langle f, f \rangle < \infty$ together with the inner product $\langle \cdot, \cdot \rangle$ form a complex Hilbert space $\mathcal{L}^2(\Gamma \backslash \mathbb{H}^n)$.

An important example of an automorphic function in $\mathbb{H}^n$ is the parabolic Eisenstein series associated to a cusp of Γ. We call a parabolic fixed point η of a discrete subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ a cusp of Γ if its stabilizer subgroup $\Gamma_\eta = \{\gamma \in \Gamma \mid \gamma\eta = \eta\}$ contains a free abelian subgroup of rank $n-1$. By C_Γ we denote a complete set of Γ-inequivalent cusps of Γ and we set $c_\Gamma = |C_\Gamma|$. If Γ is also cofinite, every parabolic fixed point is a cusp and the number c_Γ is finite.
Let $\eta_j \in C_\Gamma$ ($j \in \{1, \ldots, c_\Gamma\}$) and $\sigma_{\eta_j} \in \mathrm{PSL}_2(C_{n-1})$ be an element which satisfies $\sigma_{\eta_j}\infty = \eta_j$ and a certain normalization condition. Then for $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ we define the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_j}(P, s)$ associated to the cusp $\eta_j \in C_\Gamma$ by

$$E^{\mathrm{par}}_{\eta_j}(P, s) = \sum_{\gamma \in \Gamma_{\eta_j} \backslash \Gamma} x_{n-1}(\sigma_{\eta_j}^{-1} \gamma P)^s,$$

where $x_{n-1}(\sigma_{\eta_j}^{-1}\gamma P)$ is the x_{n-1}-coordinate of $\sigma_{\eta_j}^{-1}\gamma P$. This series converges absolutely and locally uniformly for $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$. The parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_j}(P, s)$ is an automorphic function with respect to Γ and an eigenfunction of $\Delta_{\mathbb{H}^n}$ with eigenvalue $s(n-1-s)$. It admits a meromorphic continuation in s to the whole complex plane which has no poles with $\mathrm{Re}(s) = \frac{n-1}{2}$ and only finitely many poles with $\mathrm{Re}(s) > \frac{n-1}{2}$; these are located in the real interval $(\frac{n-1}{2}, n-1]$ and are simple. For all these properties we refer to [CS80].

The spectrum of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ on $\mathcal{L}^2(\Gamma \backslash \mathbb{H}^n)$ consists of discrete and continuous spectrum. We enumerate the eigenvalues of the discrete spectrum by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and write $\lambda_j = \left(\frac{n-1}{2}\right)^2 + r_j^2 = s_j(n-1-s_j)$, i.e. $s_j = \frac{n-1}{2} + ir_j$ with $r_j \geq 0$ or $r_j \in \left[-i\,\frac{n-1}{2}, 0\right)$. Further, we choose an orthonormal basis $\{\psi_j(P) \mid j \in \mathbb{N}_0\}$ of the eigenfunctions for the discrete eigenvalues, where each $\psi_j(P)$ is an eigenfunction for λ_j, and the eigenfunction associated to $\lambda_0 = 0$ is given by $\psi_0(P) = \mathrm{vol}(\Gamma \backslash \mathbb{H}^n)^{-1/2}$. The eigenvalues of the continuous spectrum are of the form $\lambda = \left(\frac{n-1}{2}\right)^2 + t^2 = s(n-1-s)$, i.e. $s = \frac{n-1}{2} + it$ with $t \in \mathbb{R}$, and the corresponding eigenfunctions are the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}\left(P, \frac{n-1}{2} + it\right)$ ($k = 1, \ldots, c_\Gamma$).

In this setting, referring to [CS80] and [Söd12], every $f \in \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ admits a spectral expansion of the form

$$f(P) = \sum_{j=0}^{\infty} a_j\, \psi_j(P) + \frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^{\infty} a_{t,\eta_k}\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt, \tag{0.4}$$

where the coefficients are given by $a_j = \langle f, \psi_j \rangle$ and

$$a_{t,\eta_k} = \int_{\Gamma\backslash\mathbb{H}^n} f(Q)\, \overline{E^{\mathrm{par}}_{\eta_k}}\Big(Q, \frac{n-1}{2} + it\Big)\, \mu_{\mathbb{H}^n}(Q),$$

and the series (0.4) converges in the $\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$-norm. If $k_0 = \lfloor \frac{n}{4} \rfloor + 1$ and $f \in C^{2k_0}(\mathbb{H}^n) \cap \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ such that $\Delta^l_{\mathbb{H}^n} f \in \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ for $l = 0, \ldots, k_0$, then the spectral expansion of f converges absolutely and uniformly on compact subsets of $\mathbb{H}^n$ and equation (0.4) holds true as a pointwise relation.

Hyperbolic and elliptic Eisenstein series in hyperbolic n-space

After these preliminaries we give an overview of our results about hyperbolic and elliptic Eisenstein series in $\mathbb{H}^n$ in the next few sections.

Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. By H_Γ we denote a complete set of Γ-inequivalent pairs of hyperbolic fixed points of Γ. For $(Q_1, Q_2) \in H_\Gamma$ there is an element $\sigma_{(Q_1,Q_2)} \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma_{(Q_1,Q_2)} 0 = Q_1$ and $\sigma_{(Q_1,Q_2)} \infty = Q_2$, called a hyperbolic scaling matrix of (Q_1, Q_2). Let $\Gamma_{(Q_1,Q_2)} = \big\{\gamma \in \Gamma \bigm| \gamma Q_1 = Q_1,\, \gamma Q_2 = Q_2\big\}$ denote the stabilizer subgroup of (Q_1, Q_2) in Γ, then its subset

$$\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} = \big\{\gamma \in \Gamma_{(Q_1,Q_2)} \bigm| \gamma = I \text{ or } \gamma \text{ is hyperbolic}\big\}$$

is an infinite cyclic group, which we call the hyperbolic stabilizer subgroup of (Q_1, Q_2) in Γ. More precisely, there exists $\mu \in \mathbb{R}$, $\mu > 1$, such that

$$\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} = \left\{ \sigma_{(Q_1,Q_2)} \begin{pmatrix} \mu^m & 0 \\ 0 & \mu^{-m} \end{pmatrix} \sigma^{-1}_{(Q_1,Q_2)} \;\middle|\; m \in \mathbb{Z} \right\} / \{\pm I\}.$$

We write $\mathcal{L}_{(Q_1,Q_2)}$ for the unique geodesic in $\mathbb{H}^n$ connecting $Q_1, Q_2 \in \widehat{\mathbb{R}}^{n-1}$. Its image under the natural projection $\pi_{\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}} : \mathbb{H}^n \to \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\mathbb{H}^n$ is a closed geodesic in $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\mathbb{H}^n$ which we denote by $L_{(Q_1,Q_2)}$, and we write $l_{(Q_1,Q_2)}$ for its hyperbolic length.
Then for $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ we define the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P, s)$ associated to the pair $(Q_1, Q_2) \in H_\Gamma$ of hyperbolic fixed points by

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P, s) = \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(Q_1,Q_2)}))^{-s}. \tag{0.5}$$

Here we remark that Eisenstein series in $\mathbb{H}^3$ associated to a hyperbolic or loxodromic element of a discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{C})$ were considered in [Iri19b], while Eisenstein series in $\mathbb{H}^n$ which are associated to an involution and also called "hyperbolic Eisenstein series" were defined in [Iri19a].
We prove in Lemma 4.1.4 that the series (0.5) converges absolutely and locally uniformly for $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$. In the proof we first assume that $Q_1 = 0$ and $Q_2 = \infty$ and we fix $P \in \mathbb{H}^n$. Choosing the representatives such that $1 \leq |\gamma P| < \mu^2$ for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma$, we get the bound

$$\Big| E^{\mathrm{hyp}}_{(0,\infty)}(P, s) \Big| \leq \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma} x_{n-1}(\gamma P)^{\sigma},$$

where $\sigma = \mathrm{Re}(s)$. Subsequently, using a result about eigenfunctions of $\mathrm{PSL}_2(C_{n-1})$-invariant integral operators, we bound the last series as

$$\sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma} x_{n-1}(\gamma P)^{\sigma} \leq \frac{1}{|\Lambda(P)|}\sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma}\int_{B_{\varepsilon(P)}(\gamma P)} x_{n-1}(Q)^{\sigma}\ \mu_{\mathbb{H}^n}(Q) \leq \frac{2^{n-1}\,\mu^{2\sigma}}{|\Lambda(P)|\,(\sigma-n+1)},$$

where $\Lambda(P)\in\mathbb{C}$ and $B_{\varepsilon(P)}(\gamma P)$ is the open hyperbolic ball with center γP and sufficiently small radius $\varepsilon(P)>0$, with $\Lambda(P)$ and $\varepsilon(P)$ depending on P. From this we obtain the absolute and locally uniform convergence of $E^{\mathrm{hyp}}_{(0,\infty)}(P,s)$ for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$. Eventually, if $K\subseteq\mathbb{H}^n$ is a compact subset, the constants $\Lambda(P)$ and $\varepsilon(P)$ can be chosen uniformly for all $P\in K$.
If $(Q_1,Q_2)\in H_\Gamma$ is an arbitrary pair of hyperbolic fixed points, then the discrete and cofinite subgroup $\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)}\subseteq\mathrm{PSL}_2(C_{n-1})$ has the hyperbolic fixed points 0 and ∞, and the hyperbolic Eisenstein series for Γ and (Q_1,Q_2) can be written in terms of the hyperbolic Eisenstein series for $\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)}$ and $(0,\infty)$. In this way, the absolute and locally uniform convergence of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ follows from the specific case $Q_1=0$, $Q_2=\infty$.

We easily see from its definition that the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ is invariant in P under the action of Γ. Moreover, from the proof of its convergence we conclude that for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ it is bounded on $\Gamma\backslash\mathbb{H}^n$ and an element of $\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$.
In Lemma 4.1.7 we show that for $P=(x_0,\ldots,x_{n-1})\in\mathbb{H}^n$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ is infinitely often continuously differentiable with respect to $x_0,\ldots,x_{n-1}$. To that aim, for fixed $\gamma\in\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\Gamma$ we write $g_\gamma(P)=\cosh(d_{\mathbb{H}^n}(\gamma P,\mathcal{L}_{(Q_1,Q_2)}))$ in terms of the coordinates $x_0,\ldots,x_{n-1}$ and see that $g_\gamma(P)^{-s}$ is infinitely often continuously differentiable with respect to them. Hence, for any multi-index $\boldsymbol{\alpha}\in\mathbb{N}_0^n$ the derivative

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}}g_\gamma(P)^{-s} \tag{0.6}$$

exists and is continuous, and we are left to prove that the respective series of partial derivatives summed over all $\gamma\in\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\Gamma$ converges absolutely and uniformly on compact subsets $K\subseteq\mathbb{H}^n$, provided that $\mathrm{Re}(s)>n-1$. This is done by proving in several steps that for any $\boldsymbol{\alpha}\in\mathbb{N}_0^n$ and $\gamma\in\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\Gamma$ the derivative (0.6) can be majorized on K by a finite sum of summands of the form $|p(s)|\cdot C(K)\cdot g_\gamma(P)^{-\sigma}$, where $\sigma=\mathrm{Re}(s)$, $p\in\mathbb{Z}[X]$ and $C(K)>0$ is a constant depending only on K. Then the desired absolute and uniform convergence on K follows from the convergence of the series (0.5), provided that $\mathrm{Re}(s)>n-1$.
Though $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ is no eigenfunction of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$, it still fulfils the shift equation

$$\Delta_{\mathbb{H}^n}\,E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)=s(n-1-s)\,E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)+s^2\,E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s+2), \tag{0.7}$$

which is proven in Lemma 4.1.8 using so-called hyperbolic coordinates.

Now let $Q\in\mathbb{H}^n$ be a point with stabilizer subgroup Γ_Q, then for $P\in\mathbb{H}^n$ with $P\neq\gamma Q$ for any $\gamma\in\Gamma$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ we define the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$ associated to Q by

$$E^{\mathrm{ell}}_Q(P,s)=\sum_{\gamma\in\Gamma_Q\backslash\Gamma}\sinh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s}. \tag{0.8}$$

In Lemma 4.2.4 we prove that the series (0.8) converges absolutely and locally uniformly for $P\in\mathbb{H}^n$ with $P\neq\gamma Q$ for any $\gamma\in\Gamma$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$. For that we first fix $P\in\mathbb{H}^n$ with $P\neq\gamma Q$ for any $\gamma\in\Gamma$ and majorize the elliptic Eisenstein series as

$$\left|E^{\mathrm{ell}}_Q(P,s)\right|\leq C_1(P)^{-\sigma}\sum_{\gamma\in\Gamma_Q\backslash\Gamma}\exp(-\sigma\,d_{\mathbb{H}^n}(\gamma P,Q)),$$

where $\sigma = \mathrm{Re}(s)$ and $C_1(P) > 0$ is a constant depending only on P. Then we show that for $r \in \mathbb{R}$, $r > 0$, the counting function $N_Q^{\mathrm{ell}}(r;P) = |\{\gamma \in \Gamma_Q\backslash\Gamma \mid d_{\mathbb{H}^n}(\gamma P, Q) < r\}|$ satisfies the estimate $N_Q^{\mathrm{ell}}(r;P) \leq C_2(P) \cdot \exp((n-1)\,r)$, where the constant $C_2(P) > 0$ depends only on P. From this we infer that

$$\sum_{\gamma\in\Gamma_Q\backslash\Gamma} \exp(-\sigma\, d_{\mathbb{H}^n}(\gamma P, Q)) = \lim_{R\to\infty} \int_0^R \exp(-\sigma r)\; dN_Q^{\mathrm{ell}}(r;P) \leq \frac{\sigma\, C_2(P)}{\sigma - n + 1},$$

which gives us the absolute and locally uniform convergence for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$. Finally, for a compact subset $K \subseteq \mathbb{H}^n$ not containing any Γ-translate of Q the constants $C_1(P)$ and $C_2(P)$ can be chosen uniformly for all $P \in K$.

The proof of its convergence implies that for $P \in \mathbb{H}^n$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ is bounded as $P \to \eta_j$, where $\eta_j \in C_\Gamma$ ($j = 1, \ldots, c_\Gamma$) is a cusp of Γ. It is directly seen from its definition that $E_Q^{\mathrm{ell}}(P,s)$ is invariant in P under the action of Γ. Following the same idea as for the hyperbolic Eisenstein series, we further find in Lemma 4.3.8 that for $P = (x_0, \ldots, x_{n-1}) \in \mathbb{H}^n$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ is infinitely often continuously differentiable with respect to $x_0, \ldots, x_{n-1}$. It also satisfies a shift equation under the hyperbolic Laplace operator, namely

$$\Delta_{\mathbb{H}^n}\, E_Q^{\mathrm{ell}}(P,s) = s(n-1-s)\, E_Q^{\mathrm{ell}}(P,s) + s(n-2-s)\, E_Q^{\mathrm{ell}}(P,s+2),$$

which we verify in Lemma 4.2.8 using so-called elliptic coordinates centered at Q.

For $P, Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ we additionally define the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ by

$$K^{\mathrm{hyp}}(P,Q,s) = \sum_{\gamma\in\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s}. \tag{0.9}$$

The series (0.9) converges absolutely and locally uniformly for $P, Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$. Immediately by definition the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ is symmetric in the variables P and Q and invariant under the action of Γ in both P and Q. Moreover, we find that $K^{\mathrm{hyp}}(P,Q,s)$, as a function in P, is bounded on $\Gamma\backslash\mathbb{H}^n$ and an element of $\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$. Also the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ is infinitely often continuously differentiable with respect to the coordinates of $P = (x_0, \ldots, x_{n-1}) \in \mathbb{H}^n$. Under the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ with respect to P we find the shift equation

$$\Delta_{\mathbb{H}^n}\, K^{\mathrm{hyp}}(P,Q,s) = s(n-1-s)\, K^{\mathrm{hyp}}(P,Q,s) + s(s+1)\, K^{\mathrm{hyp}}(P,Q,s+2). \tag{0.10}$$

We see in the next section how both hyperbolic and elliptic Eisenstein series can be expressed in terms of $K^{\mathrm{hyp}}(P,Q,s)$.

Spectral expansions and meromorphic continuations

In order to establish the meromorphic continuations of hyperbolic and elliptic Eisenstein series in s to the whole complex plane, we first compute the spectral expansions of $K^{\mathrm{hyp}}(P,Q,s)$ and $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$. This is done similarly to the case $n = 2$ (see [Pip10] and [JKP10]).

Using $K^{\mathrm{hyp}}(P,Q,s) \in C^\infty(\mathbb{H}^n) \cap \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ and the differential equation (0.10), we see in Theorem 5.1.1 that for $P, Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the hyperbolic kernel function admits the spectral expansion

$$K^{\mathrm{hyp}}(P,Q,s) = \sum_{j=0}^{\infty} a_{j,Q}(s)\, \psi_j(P) + \frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\; dt, \tag{0.11}$$

which converges absolutely and locally uniformly, and we compute the coefficients as

$$a_{j,Q}(s) = \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big)\,\overline{\psi_j}(Q),$$
$$a_{t,\eta_k,Q}(s) = \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+it}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-it}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-it\Big).$$

Afterwards, in Proposition 5.2.1 we show that for $P\in\mathbb{H}^n$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ can be written as a line integral of the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ along the closed geodesic $L_{(Q_1,Q_2)}$ as

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = \frac{2^{1-s}\,\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)^2}\int_{L_{(Q_1,Q_2)}} K^{\mathrm{hyp}}(P,Q,s)\;ds_{\mathbb{H}^n}(Q). \tag{0.12}$$

This formula generalizes the respective result of [JPS16] for $n=2$. We first compute the above integral over $K^{\mathrm{hyp}}(P,Q,s)$ for the hyperbolic fixed points $Q_1=0$, $Q_2=\infty$, where we can identify the closed geodesic $L_{(0,\infty)}$ with the subset

$$\big\{(0,\ldots,0,x_{n-1})\in\mathbb{H}^n \mid x_{n-1}\in\big[1,\exp(l_{(0,\infty)})\big)\big\}$$

of $\mathbb{H}^n$, and then deduce the identity (0.12) for general $(Q_1,Q_2)\in H_\Gamma$.
Making use of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)\in C^\infty(\mathbb{H}^n)\cap\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ and the differential equation (0.7), we show in Theorem 5.2.2 that for $P\in\mathbb{H}^n$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ the hyperbolic Eisenstein series admits the spectral expansion

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = \sum_{j=0}^{\infty} b_j(s)\,\psi_j(P) + \frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} b_{t,\eta_k}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\;dt, \tag{0.13}$$

which converges absolutely and locally uniformly. Substituting the spectral expansion of $K^{\mathrm{hyp}}(P,Q,s)$ into (0.12) gives us the coefficients

$$b_j(s) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big)\int_{L_{(Q_1,Q_2)}}\overline{\psi_j}(Q)\;ds_{\mathbb{H}^n}(Q),$$
$$b_{t,\eta_k}(s) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+it}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-it}{2}\Big)\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-it\Big)\;ds_{\mathbb{H}^n}(Q).$$

Using the spectral expansion (0.11), we prove in Theorem 6.1.1 that the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ admits a meromorphic continuation in s to the whole complex plane. In the proof we first establish the meromorphic continuation of the series in (0.11) arising from the discrete spectrum. The explicit formula for the coefficient $a_{j,Q}(s)$ gives its meromorphic continuation to the whole complex plane. Using Stirling's asymptotic formula for the gamma function and the bound

$$\sup_{P\in\mathbb{H}^n}|\psi_j(P)| = \mathrm{O}\left(r_j^{n/2}\right) \qquad (r_j\to\infty)$$

for the eigenfunction $\psi_j(P)$, for any $j\in\mathbb{N}_0$ with $r_j\geq 0$ we get

$$|a_{j,Q}(s)\,\psi_j(P)| = \mathrm{O}\left(r_j^{\mathrm{Re}(s)+\frac{n-1}{2}}\,\exp\left(-\frac{\pi\,r_j}{2}\right)\right) \quad (r_j\to\infty)$$

with an implied constant depending on s. Since by a result of Lax and Phillips (see [LP82]) there exist only finitely many $j\in\mathbb{N}_0$ with $r_j\in\big[-i\,\frac{n-1}{2},0\big)$, the infinite series over j in (0.11) converges absolutely and locally uniformly for all $s\in\mathbb{C}$ and defines a holomorphic function away from the poles of $a_{j,Q}(s)$.

Afterwards, for a cusp $\eta_k \in C_\Gamma$ $(k = 1, \ldots, c_\Gamma)$ we establish the meromorphic continuation of the integral

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt, \tag{0.14}$$

which arises from the continuous spectrum and is holomorphic in the half-plane $\mathrm{Re}(s) > \frac{n-1}{2}$. In a first step we use the residue theorem to establish its meromorphic continuation to the strip $\frac{n-1}{2} - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} + \varepsilon$ for some sufficiently small $\varepsilon > 0$. Using the residue theorem for a second time, we then obtain the meromorphic continuation of the integral (0.14) to the strip $\frac{n-1}{2} - 2 < \mathrm{Re}(s) < \frac{n-1}{2}$, which is essentially given by the integral itself, together with two additional summands involving the parabolic Eisenstein series associated to η_k. These two steps together provide the meromorphic continuation of (0.14) to the strip $\frac{n-1}{2} - 2 < \mathrm{Re}(s) \leq \frac{n-1}{2}$. Continuing the two-step process sketched above, we inductively derive the meromorphic continuation of the integral (0.14) to the strip $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) \leq \frac{n-1}{2} - 2m$ for any $m \in \mathbb{N}_0$, which gives us the meromorphic continuation of the continuous part of the spectral expansion (0.11) in s to the whole complex plane.
By analyzing the continued function we additionally see in Theorem 6.1.1 that the possible poles of the function $\Gamma(s)\,\Gamma\left(s - \frac{n-1}{2}\right)^{-1} K^{\mathrm{hyp}}(P,Q,s)$ are located at the following points:

(i) $s = \frac{n-1}{2} \pm i r_j - 2N$, where $j, N \in \mathbb{N}_0$ and $\lambda_j = s_j(n-1-s_j) = \left(\frac{n-1}{2}\right)^2 + r_j^2$ is the eigenvalue of the eigenfunction $\psi_j(P)$.

(ii) $s = n - 1 - \rho - 2N$, where $N \in \mathbb{N}_0$ and $w = \rho$ is a pole of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,w)$ for some $k \in \{1, \ldots, c_\Gamma\}$ with $\rho \in \left(\frac{n-1}{2}, n-1\right]$.

(iii) $s = \rho - 2N$, where $N \in \mathbb{N}_0$ and $w = \rho$ is a pole of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,w)$ for some $k \in \{1, \ldots, c_\Gamma\}$ with $\mathrm{Re}(\rho) < \frac{n-1}{2}$.

We also compute the residues at the poles of type (i). Further, in Corollary 6.1.3 we obtain that the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ admits a simple pole at $s = n-1$ with residue

$$\mathrm{Res}_{s=n-1}\, K^{\mathrm{hyp}}(P,Q,s) = \frac{2\,\pi^{n/2}}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma\left(\frac{n}{2}\right)}.$$

Proceeding as for the hyperbolic kernel function, we then establish the meromorphic continuation of the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ in s to the whole complex plane via its spectral expansion (0.13) in Theorem 6.2.1. We see that also the possible poles of the function $\Gamma\left(\frac{s}{2}\right)^2\,\Gamma\left(s - \frac{n-1}{2}\right)^{-1} E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ are located at the points of types (i), (ii) and (iii) above, and we compute the residues at the poles of type (i). In particular, we conclude in Corollary 6.2.3 that $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits a simple pole at $s = n-1$ with residue

$$\mathrm{Res}_{s=n-1}\, E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = \frac{2\, l_{(Q_1,Q_2)}\, \pi^{\frac{n-1}{2}}}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma\left(\frac{n-1}{2}\right)}.$$

To derive the meromorphic continuation of the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$, we follow the idea given in [Pip10] and first show in Proposition 6.3.1 that for $P, Q \in \mathbb{H}^n$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ it has a representation as an infinite sum of shifted hyperbolic kernel functions as

$$E^{\mathrm{ell}}_Q(P,s) = \frac{1}{|\Gamma_Q|} \sum_{k=0}^{\infty} \frac{\left(\frac{s}{2}\right)_k}{k!}\, K^{\mathrm{hyp}}(P,Q,s+2k), \tag{0.15}$$

where $\left(\frac{s}{2}\right)_k = \Gamma\left(\frac{s}{2} + k\right)\,\Gamma\left(\frac{s}{2}\right)^{-1}$ denotes the Pochhammer symbol. After proving that the series on the right-hand side of (0.15) converges absolutely and locally uniformly, the assertion follows by inserting the definition of $K^{\mathrm{hyp}}(P,Q,s)$ and changing the order of summation.

Using (0.15), we obtain in Theorem 6.3.2 the meromorphic continuation of $E_Q^{\mathrm{ell}}(P,s)$ in s to the whole complex plane. For $m \in \mathbb{N}_0$ we write the elliptic Eisenstein series as

$$E_Q^{\mathrm{ell}}(P,s) = \frac{1}{|\Gamma_Q|} \sum_{l=0}^{m} \frac{\left(\frac{s}{2}\right)_l}{l!} K^{\mathrm{hyp}}(P,Q,s+2l) + \frac{1}{|\Gamma_Q|} \sum_{l=m+1}^{\infty} \frac{\left(\frac{s}{2}\right)_l}{l!} K^{\mathrm{hyp}}(P,Q,s+2l), \tag{0.16}$$

and show that the infinite sum in (0.16) converges absolutely and locally uniformly on the half-plane $\mathcal{H}_m = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > n-1-2(m+1)\}$. Together with the meromorphic continuation of $K^{\mathrm{hyp}}(P,Q,s+2l)$ $(l = 0,\ldots,m)$ this gives us the meromorphic continuation of the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ to the half-plane $\mathcal{H}_m$ and, as $m \in \mathbb{N}_0$ can be chosen arbitrarily, to the whole complex plane. Moreover, we show that the possible poles of the function $\Gamma\left(s - \frac{n-1}{2}\right)^{-1} E_Q^{\mathrm{ell}}(P,s)$ are again located at the points of types (i), (ii) and (iii). Especially, we find that $E_Q^{\mathrm{ell}}(P,s)$ admits a simple pole at $s = n-1$ with residue

$$\mathrm{Res}_{s=n-1}\, E_Q^{\mathrm{ell}}(P,s) = \frac{2\,\pi^{n/2}}{|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma\left(\frac{n}{2}\right)},$$

which is proven in Corollary 6.3.4.

Kronecker limit formulas

The formula for the meromorphic continuation of the hyperbolic Eisenstein series to $s = 0$ and the knowledge of the coefficients $b_j(s)$ and $b_{t,\eta_k}(s)$ enable us to determine a Laurent expansion of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ at this point, which depends on the dimension of $\mathbb{H}^n$. In Proposition 7.1.2 we prove that for $n \equiv 0 \mod 2$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned}
E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) - \frac{2\,\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{\left\lfloor\frac{n-1}{4}\right\rfloor} \frac{(-1)^l}{l!}\, \Gamma\Big(s - \frac{n-1}{2} + l\Big) \\
\cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(P,s+2l) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2l)\, ds_{\mathbb{H}^n}(Q) \\
= \left(\frac{l_{(Q_1,Q_2)}\,\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1-n}{2}\right)}{2\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \sum_{l=1}^{2} G_{n,(Q_1,Q_2),l}(P) \right) \cdot s \\
+ \left(\frac{l_{(Q_1,Q_2)}\,\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1-n}{2}\right)\left(\gamma + \psi^{(0)}\left(\frac{1-n}{2}\right)\right)}{4\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \sum_{l=1}^{4} H_{n,(Q_1,Q_2),l}(P) \right) \cdot s^2 + \mathrm{O}(s^3),
\end{aligned} \tag{0.17}$$

where $G_{n,(Q_1,Q_2),l}(P)$ $(l = 1,2)$ and $H_{n,(Q_1,Q_2),l}(P)$ $(l = 1,2,3,4)$ are Γ-invariant functions which are given explicitly in the proof.
For $n \equiv 3 \mod 4$ we obtain a similar Laurent expansion, which is stated in Proposition 7.1.4 and, in contrast to (0.17), starts with a constant term. An almost identical formula holds in the case $n \equiv 1 \mod 4$ (see Proposition 7.1.6), where in the proof we have to take into consideration that the point $s = 0$ lies on the line $\mathrm{Re}(s) = \frac{n-1}{2} - 2 \cdot \frac{n-1}{4}$ with $\frac{n-1}{4} \in \mathbb{N}$.

Considering the case $n = 2$, in Proposition 7.2.2 we first give a Laurent expansion at $s = 0$ of the hyperbolic Eisenstein series on $\mathbb{H}$ for a general Fuchsian subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ of the first kind. After that we treat the specific case of the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Using the Laurent expansions (0.1) and (0.2) of the parabolic Eisenstein series $E^{\mathrm{par}}_\infty(z,s)$, in Theorem 7.2.4 we find a Kronecker limit formula for the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s)$ for $\mathrm{PSL}_2(\mathbb{Z})$, namely

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s) = \Bigg(\frac{1}{2} \int_{L_{(Q_1,Q_2)}} \log\big(|\Delta(z)|\, |\Delta(w)| \operatorname{Im}(z)^6 \operatorname{Im}(w)^6 \big) \; ds_{\mathbb{H}}(w) \\ + 3\, l_{(Q_1,Q_2)} \big(24\, \zeta'(-1) + \log(8\pi^2) - 1 \big) + \sum_{l=3}^{4} H_{2,(Q_1,Q_2),l}(z) \Bigg) \cdot s^2 + \mathrm{O}(s^3).$$

For $n = 3$ we derive a Laurent expansion at $s = 0$ of the hyperbolic Eisenstein series in $\mathbb{H}^3$ for a general discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{C})$ in Proposition 7.3.2. Then we consider the specific group $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$. Employing the results from chapter 8 of [EGM13] and a functional equation for the parabolic Eisenstein series $E^{\mathrm{par}}_{\infty}(P,s)$ associated to the only cusp ∞ of $\mathrm{PSL}_2(\mathbb{Z}[i])$, we find the Laurent expansions of $E^{\mathrm{par}}_{\infty}(P,s)$ at the points $s = 2$ and $s = 0$. With these expansions we prove a Kronecker limit formula for the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ for $\mathrm{PSL}_2(\mathbb{Z}[i])$ in Theorem 7.3.6. Precisely, we have

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = -\frac{3\, l_{(Q_1,Q_2)}\, \pi^3}{8\, \zeta_{\mathbb{Q}(i)}(2)} \\ + \Bigg(\frac{\pi^3}{8\, \zeta_{\mathbb{Q}(i)}(2)} \int_{L_{(Q_1,Q_2)}} \log\big(\eta_{\mathbb{Q}(i)}(P)\, \eta_{\mathbb{Q}(i)}(Q)\, r(P)\, r(Q) \big) \; ds_{\mathbb{H}^3}(Q) - \frac{l_{(Q_1,Q_2)}\, \pi\, C_{\mathbb{Q}(i)}}{2} \\ - \frac{l_{(Q_1,Q_2)}\, \pi^3}{8\, \zeta_{\mathbb{Q}(i)}(2)} \left(\log\Big(|\Delta(i)|^{1/6} \Big) + 2\,(1 - \gamma + \log(2)) + \frac{\zeta'_{\mathbb{Q}(i)}(2)}{\zeta_{\mathbb{Q}(i)}(2)} \right) \Bigg) \cdot s + \mathrm{O}(s^2), \quad (0.18)$$

where $r(P)$ is the r-coordinate of $P \in \mathbb{H}^3$, $\zeta_{\mathbb{Q}(i)}(s)$ denotes the Dedekind zeta function of the imaginary quadratic field $\mathbb{Q}(i)$, $C_{\mathbb{Q}(i)} \in \mathbb{C}$ is a constant, $\eta_{\mathbb{Q}(i)} : \mathbb{H}^3 \to \mathbb{R}$ is a function which satisfies $\eta_{\mathbb{Q}(i)}(\delta P) = ||cP + d||^2\, \eta_{\mathbb{Q}(i)}(P)$ for any $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and γ denotes the Euler-Mascheroni constant.

Moreover, we use the decomposition (0.16) of the elliptic Eisenstein series, the formula for the meromorphic continuation of the hyperbolic kernel function to $s = 0$ and the knowledge of the coefficients $a_{j,Q}(s)$ and $a_{t,\eta_k,Q}(s)$ to compute a Laurent expansion of $E^{\mathrm{ell}}_{Q}(P,s)$ at this point, which again depends on the dimension of the hyperbolic space. For $n \equiv 0 \mod 2$ we show in Proposition 8.1.2 that the elliptic Eisenstein series $E^{\mathrm{ell}}_{Q}(P,s)$ admits a Laurent expansion at $s = 0$ of the form

$$E^{\mathrm{ell}}_{Q}(P,s) - \frac{1}{|\Gamma_Q|} \sum_{l=0}^{\frac{n}{2}-1} \frac{2^{s+2l}\, \pi^{\frac{n-1}{2}} \left(\frac{s}{2}\right)_l}{l!\, \Gamma(s+2l)} \sum_{l'=l}^{\left\lfloor \frac{n-1}{4} \right\rfloor} \frac{(-1)^{l'-l}}{(l'-l)!} \Gamma\Big(s - \frac{n-1}{2} + l + l'\Big) \\ \cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2l')\, E^{\mathrm{par}}_{\eta_k}(P, s+2l') \\ = \frac{\pi^{\frac{n-1}{2}}\, \Gamma\left(\frac{1-n}{2}\right)}{|\Gamma_Q| \operatorname{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|} \sum_{m=1}^{2} G_{n,Q,m}(P) + \frac{1}{|\Gamma_Q|} \sum_{l=1}^{\frac{n}{2}-1} \sum_{m=1}^{2} G_{n,Q,l,m}(P) \\ + \Bigg(\frac{\pi^{\frac{n-1}{2}}\, \Gamma\left(\frac{1-n}{2}\right) \left(\gamma + \log(4) + \psi^{(0)}\left(\frac{1-n}{2}\right)\right)}{2\, |\Gamma_Q| \operatorname{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|} \sum_{m=1}^{5} H_{n,Q,m}(P) \\ + \sum_{l=1}^{\frac{n}{2}-1} \frac{4^{l-1}\, \pi^{\frac{n-1}{2}}\, (l-1)!\, \Gamma\left(\frac{1-n}{2} + l\right)}{l\,(2l-1)!\, |\Gamma_Q| \operatorname{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|} \sum_{l=1}^{\frac{n}{2}-1} \sum_{m=1}^{4} H_{n,Q,l,m}(P) \Bigg) \cdot s + \mathrm{O}(s^2), \quad (0.19)$$

where $G_{n,Q,m}(P)$ $(m = 1,2)$, $G_{n,Q,l,m}(P)$ $\big(l = 1, \ldots, \frac{n}{2} - 1,\ m = 1,2\big)$, $H_{n,Q,m}(P)$ $(m = 1,2,3,4,5)$ and $H_{n,Q,l,m}(P)$ $\big(l = 1, \ldots, \frac{n}{2} - 1,\ m = 1,2,3,4\big)$ are Γ-invariant functions which are given explicitly in the proof.

In Proposition 8.1.4 we obtain a similar Laurent expansion for $n \equiv 3 \mod 4$, but which starts with the s^{-1}-term. In the case $n \equiv 1 \mod 4$ there is an analogous formula (see Proposition 8.1.6), where in the proof we have to pay attention that $\frac{n-1}{4} \in \mathbb{N}$ holds true.

Making use of (0.19) with $n = 2$, in Proposition 8.2.2 we rediscover the Laurent expansion at $s = 0$ of the elliptic Eisenstein series on $\mathbb{H}$ for a general Fuchsian subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ of the first kind which was established in [Pip10].
As for the hyperbolic Eisenstein series, we then consider the case $n = 3$. In Proposition 8.3.2 we first obtain a Laurent expansion at $s = 0$ of the elliptic Eisenstein series in $\mathbb{H}^3$ for a general discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{C})$. Then we let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$ and use the Laurent expansions of the parabolic Eisenstein series $E^{\mathrm{par}}_\infty(P, s)$ at $s = 2$ and $s = 0$ to derive a Kronecker limit formula for the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P, s)$ for $\mathrm{PSL}_2(\mathbb{Z}[i])$ in Theorem 8.3.3. It is given by

$$\begin{aligned} E^{\mathrm{ell}}_Q(P,s) = &-\frac{3\,\pi^3}{4\,|\Gamma_Q|\,\zeta_{\mathbb{Q}(i)}(2)}\cdot\frac{1}{s} + \frac{\pi^3}{4\,|\Gamma_Q|\,\zeta_{\mathbb{Q}(i)}(2)}\log\left(\eta_{\mathbb{Q}(i)}(P)\,\eta_{\mathbb{Q}(i)}(Q)\,r(P)\,r(Q)\right) - \frac{\pi\,C_{\mathbb{Q}(i)}}{|\Gamma_Q|} \\ &- \frac{\pi^3}{4\,|\Gamma_Q|\,\zeta_{\mathbb{Q}(i)}(2)}\left(\log\left(|\Delta(i)|^{1/6}\right) - 2\,(1+\gamma) + \log(32) + \frac{\zeta'_{\mathbb{Q}(i)}(2)}{\zeta_{\mathbb{Q}(i)}(2)}\right) + \mathrm{O}(s), \qquad (0.20) \end{aligned}$$

where $\zeta_{\mathbb{Q}(i)}(s)$, $C_{\mathbb{Q}(i)}$, $\eta_{\mathbb{Q}(i)}$ and γ are as in (0.18).

It is possible to establish Kronecker limit formulas similar to (0.18) and (0.20) also for hyperbolic and elliptic Eisenstein series in $\mathbb{H}^3$ for $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K)$, where K is an imaginary quadratic field with ring of integers $\mathcal{O}_K$ and class number 1. However, in this thesis we do not address this task and leave it for future research.

Outline of this thesis

We quickly describe the outline of this work. In the chapters 1, 2 and 3 we fix definitions and notations that we employ during the thesis and recall known results. Subsequently, we present and prove our results about hyperbolic and elliptic Eisenstein series in n-dimensional hyperbolic space in the chapters 4, 5, 6, 7 and 8.

In chapter 1 we introduce the n-dimensional hyperbolic space. We briefly present some of the most common models for it, especially the upper half-space model $\mathbb{H}^n$, and further define hyperbolic and elliptic coordinates in $\mathbb{H}^n$.
Then we turn to the Möbius transformations of hyperbolic n-space in chapter 2. To that aim we first consider the Clifford numbers C_n and a few of their elementary properties. We introduce two certain groups $\mathrm{GL}_2(C_n)$, $\mathrm{SL}_2(C_n)$ of matrices with entries in the Clifford numbers and explain how these groups act on $\mathbb{H}^{n+1} \cup \widehat{\mathbb{R}}^n$ via Möbius transformations. Afterwards, we consider an analogue of Fuchsian subgroups $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ of the first kind, namely discrete and cofinite subgroups of $\mathrm{PSL}_2(C_{n-1})$. We establish the classification of elements of $\mathrm{PSL}_2(C_{n-1})$ as parabolic, hyperbolic, elliptic and loxodromic. Finally, we treat fixed points of parabolic, hyperbolic and elliptic elements and their respective stabilizer subgroups in $\mathrm{PSL}_2(C_{n-1})$ and a discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$.
Chapter 3 collects several topics and results needed later in this thesis. First we determine the radial eigenfunctions of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ in elliptic coordinates. Then we look at $\mathrm{PSL}_2(C_{n-1})$-invariant integral operators and eigenfunctions of these operators. After that we introduce the notion of automorphic functions in $\mathbb{H}^n$ and consider the definition and basic properties of parabolic Eisenstein series in $\mathbb{H}^n$. At the end of the chapter we establish the spectral expansion of a square-integrable automorphic function.
In chapter 4 we introduce the main objects of this thesis. We define the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P, s)$ associated to a pair (Q_1, Q_2) of hyperbolic fixed points of a discrete and

cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ and investigate its basic properties. Subsequently, we do the same for the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$ associated to a point $Q \in \mathbb{H}^n$ and a discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$. These functions generalize the known hyperbolic and elliptic Eisenstein series on the upper half-plane $\mathbb{H}$. Further, we introduce and study the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ which is closely related to hyperbolic and elliptic Eisenstein series.
Chapter 5 deals with the computation of some spectral expansions. First we determine the spectral expansion of the hyperbolic kernel function. After that we prove a representation of the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ as a certain integral of $K^{\mathrm{hyp}}(P,Q,s)$ which enables us to derive the spectral expansion of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$.
In chapter 6 we use the results from the previous chapter to establish the meromorphic continuations of hyperbolic and elliptic Eisenstein series in s to the whole complex plane and determine the possible poles. Using their spectral expansions, we obtain the meromorphic continuations of the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ and the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ to all $s \in \mathbb{C}$. Afterwards, we also derive the meromorphic continuation of the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$ to all $s \in \mathbb{C}$ by expressing it in terms of the hyperbolic kernel function.
The aim of chapter 7 is to find an analogue of the Kronecker limit formula for the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ in $\mathbb{H}^n$. For arbitrary dimension n and an arbitrary discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ we first compute the first two terms in the Laurent expansion of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ at the point $s=0$. Then we consider the specific examples $n=2$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and $n=3$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$, and derive a formula of Kronecker limit type for $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ in both of these cases.
Finally, in chapter 8 we look for an analogue of the Kronecker limit formula also for the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$. We determine the first two terms in the Laurent expansion of $E^{\mathrm{ell}}_Q(P,s)$ at the point $s=0$ for arbitrary dimension n and an arbitrary discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$. Subsequently, we recall von Pippich's results about Kronecker limit formulas for the elliptic Eisenstein series in the specific case $n=2$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, and then prove a formula of Kronecker limit type for $E^{\mathrm{ell}}_Q(P,s)$ in the case $n=3$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$.

Notations

We fix a few general notations that we will use during the course of this thesis.

- $\mathbb{N} = \{1,2,3,\dots\}$ denotes the natural numbers.
- $\mathbb{N}_0 = \{0,1,2,\dots\}$ denotes the natural numbers with zero.
- $\mathbb{Z}$ denotes the integers.
- $\mathbb{Q}$ denotes the rational numbers.
- $\mathbb{R}$ denotes the real numbers.
- $\mathbb{C}$ denotes the complex numbers.
- $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real part and the imaginary part of $z \in \mathbb{C}$, respectively.
- $\log(x)$ denotes the natural logarithm of $x \in \mathbb{R}$, $x > 0$.
- $\sqrt{z} = z^{1/2}$ denotes the principal branch of the square root of a complex number $z \in \mathbb{C}$, so that $\arg(\sqrt{z}) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- $\mathrm{Log}(z)$ denotes the principal branch of the complex logarithm of $z \in \mathbb{C} \setminus \{0\}$, so that $\mathrm{Im}(\mathrm{Log}(z)) \in (-\pi, \pi]$. Then we have $\mathrm{Log}(z) = \log(z)$ for $z \in \mathbb{R}$, $z > 0$.

1. Hyperbolic n-space

In this first chapter we introduce the n-dimensional hyperbolic space (or short: hyperbolic n-space) which is a central object in this thesis. We start with a short presentation of the most common models for the n-dimensional hyperbolic space. In the second section we define different types of coordinates for points in the upper half-space model $\mathbb{H}^n$ of the hyperbolic n-space.

1.1. The n-dimensional hyperbolic space

The n-dimensional hyperbolic space is the unique simply connected, n-dimensional, complete Riemannian manifold of constant sectional curvature -1. There are several possible realizations of this space, each of them suitable for the study of different aspects of hyperbolic space. All of these models are isometric to each other. In this section we present the three most common realizations. Throughout the section let $n \in \mathbb{N}$ with $n \geq 2$.

1.1.1. The upper half-space model

During the course of this thesis we will solely work with the upper half-space model which realizes hyperbolic n-space as the upper half of the vector space $\mathbb{R}^n$.

Definition 1.1.1. The *upper half-space model* $\mathbb{H}^n$ *of hyperbolic* n*-space* is the set

$$\mathbb{H}^n = \{P = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n \mid x_{n-1} > 0\}.$$

So the space $\mathbb{H}^n$ consists of all points with n components and real entries whose last entry is strictly positive. Its boundary $\partial\mathbb{H}^n$ is given by

$$\{P = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n \mid x_{n-1} = 0\} \cong \mathbb{R}^{n-1},$$

together with the point ∞, i.e. $\partial\mathbb{H}^n = \mathbb{R}^{n-1} \cup \{\infty\}$.

Notation 1.1.2. For the boundary $\partial\mathbb{H}^n$ we introduce the notation

$$\widehat{\mathbb{R}}^{n-1} := \mathbb{R}^{n-1} \cup \{\infty\}.$$

The usual Euclidean norm on $\mathbb{R}^n$ naturally induces a norm on $\mathbb{H}^n$. For $P = (x_0, \ldots, x_{n-1}) \in \mathbb{H}^n$ it is given by

$$|P| = \sqrt{x_0^2 + \cdots + x_{n-1}^2}.$$

In the rectangular coordinates $P = (x_0, \ldots, x_{n-1})$ the *hyperbolic line element* $ds^2_{\mathbb{H}^n}$ on $\mathbb{H}^n$ is given by

$$ds^2_{\mathbb{H}^n} = \frac{dx_0^2 + \cdots + dx_{n-1}^2}{x_{n-1}^2} = \frac{|dP|^2}{x_{n-1}^2}. \tag{1.1}$$

Then the *hyperbolic distance* $d_{\mathbb{H}^n}(P, Q)$ from $P \in \mathbb{H}^n$ to $Q \in \mathbb{H}^n$ derived from $ds^2_{\mathbb{H}^n}$ is given by

$$d_{\mathbb{H}^n}(P, Q) = \inf_{\gamma} \left(\int_{\gamma} ds^2_{\mathbb{H}^n} \right)^{1/2}, \tag{1.2}$$

where the infimum is taken over all continuous paths $\gamma : [0, 1] \to \mathbb{H}^n$ with $\gamma(0) = P$ and $\gamma(1) = Q$.

For $P = (x_0, \ldots, x_{n-1}) \in \mathbb{H}^n$ and $Q = (y_0, \ldots, y_{n-1}) \in \mathbb{H}^n$ the hyperbolic distance satisfies the identity

$$\begin{aligned}\cosh(d_{\mathbb{H}^n}(P,Q)) &= 1 + \frac{|P-Q|^2}{2\,x_{n-1}\,y_{n-1}} = 1 + \frac{(x_0-y_0)^2 + \cdots + (x_{n-1}-y_{n-1})^2}{2\,x_{n-1}\,y_{n-1}} \\ &= \frac{(x_0-y_0)^2 + \cdots + (x_{n-2}-y_{n-2})^2 + x_{n-1}^2 + y_{n-1}^2}{2\,x_{n-1}\,y_{n-1}}. \end{aligned} \tag{1.3}$$

The *hyperbolic Laplace operator* $\Delta_{\mathbb{H}^n}$, also called *Laplace-Beltrami operator*, associated with the hyperbolic line element $ds^2_{\mathbb{H}^n}$, is given by

$$\Delta_{\mathbb{H}^n} = -x_{n-1}^2 \left(\frac{\partial^2}{\partial x_0^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} \right) + (n-2)\, x_{n-1} \frac{\partial}{\partial x_{n-1}}. \tag{1.4}$$

Moreover, the *hyperbolic volume element* $\mu_{\mathbb{H}^n}(P)$ on $\mathbb{H}^n$ with respect to the rectangular coordinates $P = (x_0, \ldots, x_{n-1})$ is given by

$$\mu_{\mathbb{H}^n}(P) = \frac{dx_0 \cdots dx_{n-1}}{x_{n-1}^n}. \tag{1.5}$$

The geodesics in $\mathbb{H}^n$ are the half-circles and lines which are orthogonal to the boundary $\partial\mathbb{H}^n$.

Example 1.1.3.

(a) For $n = 2$ we write $\mathbb{H}^2 =: \mathbb{H}$ which is the well-known upper half-plane

$$\mathbb{H} = \{(x,y) \in \mathbb{R}^2 \mid y > 0\} \cong \{z = x + iy \in \mathbb{C} \mid y > 0\}.$$

We usually write $z \in \mathbb{H}$ in the form $z = x + iy$ with $x, y \in \mathbb{R}$, $y > 0$. For the norm $|z|$ of $z = x + iy \in \mathbb{H}$ we then have $|z| = \sqrt{x^2 + y^2}$. In the coordinates x, y the hyperbolic line element $ds^2_{\mathbb{H}}$ on $\mathbb{H}$ is given by

$$ds^2_{\mathbb{H}} = \frac{dx^2 + dy^2}{y^2}.$$

The hyperbolic Laplace operator $\Delta_{\mathbb{H}}$ derived from $ds^2_{\mathbb{H}}$ and the hyperbolic volume element $\mu_{\mathbb{H}}(z)$ on $\mathbb{H}$ are given by

$$\Delta_{\mathbb{H}} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \qquad \mu_{\mathbb{H}}(z) = \frac{dx\,dy}{y^2},$$

respectively. For the hyperbolic distance $d_{\mathbb{H}}(z,w)$ from $z \in \mathbb{H}$ to $w \in \mathbb{H}$ we have the formulas

$$d_{\mathbb{H}}(z,w) = \log \left(\frac{|z - \overline{w}| + |z - w|}{|z - \overline{w}| - |z - w|} \right) \tag{1.6}$$

(see e.g. [Bea12], p. 130) and

$$\cosh(d_{\mathbb{H}}(z,w)) = 1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)} = \frac{(\mathrm{Re}(z) - \mathrm{Re}(w))^2 + \mathrm{Im}(z)^2 + \mathrm{Im}(w)^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}. \tag{1.7}$$

(b) For $n = 3$ we obtain the upper half-space model

$$\begin{aligned}\mathbb{H}^3 = \{(x,y,r) \in \mathbb{R}^3 \mid r > 0\} &\cong \{P = z + jr \mid z \in \mathbb{C},\, r \in \mathbb{R},\, r > 0\} \\ &\cong \{P = x + iy + jr \mid x, y, r \in \mathbb{R},\, r > 0\}\end{aligned}$$

of hyperbolic 3-space which is a subset of the quaternions $\mathbb{R}[i,j,k]$ with the standard basis $\{1, i, j, k\}$. The norm $|P|$ of $P = z + jr = x + iy + jr \in \mathbb{H}^3$ is given by

$$|P| = \sqrt{|z|^2 + r^2} = \sqrt{x^2 + y^2 + r^2}.$$

In the coordinates x, y, r the hyperbolic line element $ds^2_{\mathbb{H}^3}$, the hyperbolic Laplace operator $\Delta_{\mathbb{H}^3}$ and the hyperbolic volume element $\mu_{\mathbb{H}^3}(P)$ are given by

$$ds^2_{\mathbb{H}^3} = \frac{dx^2 + dy^2 + dr^2}{r^2}, \qquad \Delta_{\mathbb{H}^3} = -r^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2}\right) + r\frac{\partial}{\partial r}, \qquad \mu_{\mathbb{H}^3}(P) = \frac{dx\, dy\, dr}{r^3},$$

respectively. The hyperbolic distance $d_{\mathbb{H}^3}(P, Q)$ from $P = z_1 + jr_1 \in \mathbb{H}^3$ to $Q = z_2 + jr_2 \in \mathbb{H}^3$ satisfies

$$\cosh(d_{\mathbb{H}^3}(P, Q)) = 1 + \frac{|P-Q|^2}{2\, r_1\, r_2} = \frac{|z_1 - z_2|^2 + r_1^2 + r_2^2}{2\, r_1\, r_2}.$$

1.1.2. The unit ball model

Another model for the n-dimensional hyperbolic space is the realization as the interior of the unit ball in $\mathbb{R}^n$.

Definition 1.1.4. The *unit ball model* $\mathbb{B}^n$ *of hyperbolic n-space* is the set

$$\mathbb{B}^n = \{P = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n \mid |P| < 1\}.$$

The boundary $\partial\mathbb{B}^n$ is given by

$$\partial\mathbb{B}^n = \mathbb{S}^{n-1} = \{P = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n \mid |P| = 1\},$$

which is the unit sphere in $\mathbb{R}^n$.

Definition 1.1.5. A point $P \in \partial\mathbb{B}^n$ is called a *point at infinity.*

In the rectangular coordinates $P = (x_0, \ldots, x_{n-1})$ the *hyperbolic line element* $ds^2_{\mathbb{B}^n}$ on $\mathbb{B}^n$ is given by

$$ds^2_{\mathbb{B}^n} = \frac{4\,(dx_0^2 + \cdots + dx_{n-1}^2)}{(1 - (x_0^2 + \cdots + x_{n-1}^2))^2} = \frac{4\,|dP|^2}{(1 - |P|^2)^2}.$$

Then the *hyperbolic distance* $d_{\mathbb{B}^n}(P, Q)$ from $P \in \mathbb{B}^n$ to $Q \in \mathbb{B}^n$ obtained from $ds^2_{\mathbb{B}^n}$ is given by

$$d_{\mathbb{B}^n}(P, Q) = \inf_{\gamma}\left(\int_{\gamma} ds^2_{\mathbb{B}^n}\right)^{1/2},$$

where the infimum is taken over all continuous paths $\gamma : [0, 1] \to \mathbb{B}^n$ with $\gamma(0) = P$ and $\gamma(1) = Q$.

For $P = (x_0, \ldots, x_{n-1}) \in \mathbb{B}^n$ and $Q = (y_0, \ldots, y_{n-1}) \in \mathbb{B}^n$ the hyperbolic distance can be written as

$$\begin{aligned}\cosh(d_{\mathbb{B}^n}(P, Q)) &= 1 + \frac{2\,|P-Q|^2}{(1 - |P|^2)\,(1 - |Q|^2)} \\ &= 1 + \frac{2\,\left((x_0 - y_0)^2 + \cdots + (x_{n-1} - y_{n-1})^2\right)}{\left(1 - (x_0^2 + \cdots + x_{n-1}^2)\right)\,\left(1 - (y_0^2 + \cdots + y_{n-1}^2)\right)}.\end{aligned}$$

The *hyperbolic Laplace operator* $\Delta_{\mathbb{B}^n}$ derived from the hyperbolic line element $ds^2_{\mathbb{B}^n}$ is given by

$$\begin{aligned}\Delta_{\mathbb{B}^n} &= -\,\frac{(1 - |P|^2)^2}{4}\left(\frac{\partial^2}{\partial x_0^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}\right) - (n-2)\,\frac{1 - |P|^2}{2}\left(x_0\,\frac{\partial}{\partial x_0} + \cdots + x_{n-1}\,\frac{\partial}{\partial x_{n-1}}\right) \\ &= -\,\frac{\left(1 - (x_0^2 + \cdots + x_{n-1}^2)\right)^2}{4}\left(\frac{\partial^2}{\partial x_0^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}\right) \\ &\quad - (n-2)\,\frac{1 - (x_0^2 + \cdots + x_{n-1}^2)}{2}\left(x_0\,\frac{\partial}{\partial x_0} + \cdots + x_{n-1}\,\frac{\partial}{\partial x_{n-1}}\right),\end{aligned}$$

and the *hyperbolic volume element* $\mu_{\mathbb{B}^n}(P)$ on $\mathbb{B}^n$ with respect to the rectangular coordinates $P = (x_0, \ldots, x_{n-1})$ is given by

$$\mu_{\mathbb{B}^n}(P) = \frac{2^n\, dx_0 \cdots dx_{n-1}}{(1 - |P|^2)^n} = \frac{2^n\, dx_0 \cdots dx_{n-1}}{(1 - (x_0^2 + \cdots + x_{n-1}^2))^n}.$$

1.1.3. The hyperboloid model

The hyperboloid model realizes hyperbolic n-space as the positive sheet of a two-sheeted hyperboloid in $\mathbb{R}^{n+1}$.

Definition 1.1.6. For $P = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ and $Q = (y_0, \ldots, y_n) \in \mathbb{R}^{n+1}$ the *Lorentzian inner product* $P \circ Q$ is defined by

$$P \circ Q = -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n.$$

The space $\mathbb{R}^{n+1}$ together with the Lorentzian inner product is called *Lorentzian* $(n+1)$*-space* and is denoted by $\mathbb{R}^{1,n}$.

In Lorentzian $(n+1)$-space also imaginary lengths are possible. The sphere of unit imaginary radius

$$\mathbb{F}^n = \{P \in \mathbb{R}^{n+1} \mid P \circ P = -1\}$$

is a hyperboloid with two sheets, hence it is not connected. This issue is solved by discarding one of the sheets.

Definition 1.1.7. The *hyperboloid model* $\mathbb{F}^n_+$ *of hyperbolic n-space* is defined as the positive sheet of $\mathbb{F}^n$, i.e. as the set

$$\mathbb{F}^n_+ = \{P = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid P \circ P = -1,\, x_0 > 0\}.$$

In the rectangular coordinates $P = (x_0, \ldots, x_n)$ the *hyperbolic line element* $ds^2_{\mathbb{F}^n_+}$ on $\mathbb{F}^n_+$ is given by

$$ds^2_{\mathbb{F}^n_+} = -dx_0^2 + dx_1^2 + \cdots + dx_n^2.$$

The *hyperbolic distance* $d_{\mathbb{F}^n_+}(P, Q)$ from $P \in \mathbb{F}^n_+$ to $Q \in \mathbb{F}^n_+$ derived from $ds^2_{\mathbb{F}^n_+}$ is given by

$$d_{\mathbb{F}^n_+}(P, Q) = \inf_\gamma \left(\int_\gamma ds^2_{\mathbb{F}^n_+} \right)^{1/2},$$

where the infimum is taken over all continuous paths $\gamma : [0, 1] \to \mathbb{F}^n_+$ with $\gamma(0) = P$ and $\gamma(1) = Q$.

If $P = (x_0, \ldots, x_n) \in \mathbb{F}^n_+$ and $Q = (y_0, \ldots, y_n) \in \mathbb{F}^n_+$, the hyperbolic distance satisfies the formula

$$\cosh(d_{\mathbb{F}^n_+}(P, Q)) = -P \circ Q = x_0 y_0 - x_1 y_1 - \cdots - x_n y_n$$

with the Lorentzian inner product $\circ$ as in Definition 1.1.6.

The *hyperbolic Laplace operator* $\Delta_{\mathbb{F}^n_+}$ associated with the hyperbolic line element $ds^2_{\mathbb{F}^n_+}$ is given by

$$\Delta_{\mathbb{F}^n_+} = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2},$$

while the *hyperbolic volume element* $\mu_{\mathbb{F}^n_+}(P)$ on $\mathbb{F}^n_+$ with respect to the rectangular coordinates $P = (x_0, \ldots, x_n)$ is given by

$$\mu_{\mathbb{F}^n_+}(P) = dx_1 \cdots dx_n.$$

1.2. Coordinates in $\mathbb{H}^n$

During this thesis we will mainly work in the rectangular coordinates $P = (x_0, \ldots, x_{n-1})$ of the upper half-space $\mathbb{H}^n$. However, for some computations it will be more convenient to employ some other coordinate system. We now introduce two different coordinates for points in $\mathbb{H}^n$, namely the so-called hyperbolic and elliptic coordinates, that will be useful in further chapters. Throughout the section let $n \in \mathbb{N}$ with $n \geq 2$.

1.2.1. Hyperbolic coordinates

For fixed $r > 0$ the set $\mathbb{S}^{n-1}_{\mathbb{H}^n,r} := \{P \in \mathbb{H}^n \mid |P| = r\}$ of points in $\mathbb{H}^n$ with norm r is the upper half of a Euclidean $(n-1)$-sphere with center 0 and radius r. Thus, any point $P \in \mathbb{H}^n$ is uniquely determined by its norm and $(n-1)$-many angles describing the location of P on the half-sphere $\mathbb{S}^{n-1}_{\mathbb{H}^n,r}$. For these angles we can take the usual n-dimensional polar coordinates which leads to the definition of the following coordinates.

Definition 1.2.1.

(a) For $n = 2$ and $z = (x, y) = x + iy \in \mathbb{H}$ we have

$$x = e^u \cos(\theta), \quad y = e^u \sin(\theta),$$

where the *hyperbolic coordinates* $u = u(z) \in \mathbb{R}$ and $\theta = \theta(z) \in (0, \pi)$ are given by

$$\begin{aligned} u(z) &:= \log(|z|) = \log\left(\sqrt{x^2 + y^2}\right), \\ \theta(z) &:= \measuredangle(z, \text{positive } x\text{-axis}) = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right). \end{aligned}$$

(b) For $n \geq 3$ and $P = (x_0, \ldots, x_{n-1}) \in \mathbb{H}^n$ we have

$$x_0 = e^u \cos(\theta_1) \prod_{j=2}^{n-1} \sin(\theta_j), \quad x_1 = e^u \prod_{j=1}^{n-1} \sin(\theta_j), \quad x_k = e^u \cos(\theta_k) \prod_{j=k+1}^{n-1} \sin(\theta_j)$$

for $k = 2, \ldots, n-1$, where the *hyperbolic coordinates* $u = u(P) \in \mathbb{R}$, $\theta_1 = \theta_1(P) \in [0, 2\pi)$, $\theta_k = \theta_k(P) \in [0, \pi]$ $(k = 2, \ldots, n-2)$ and $\theta_{n-1} = \theta_{n-1}(P) \in [0, \frac{\pi}{2})$ are given by

$$\begin{aligned} u(P) &:= \log(|P|) = \log\left(\sqrt{x_0^2 + \cdots + x_{n-1}^2}\right), \\ \theta_1(P) &:= \measuredangle(\text{projection of } P \text{ on the } x_0\text{-}x_1\text{-plane, positive } x_0\text{-axis}) \\ &= \begin{cases} \arccos\left(\frac{x_0}{\sqrt{x_0^2+x_1^2}}\right), & x_1 \geq 0, \\ 2\pi - \arccos\left(\frac{x_0}{\sqrt{x_0^2+x_1^2}}\right), & x_1 < 0, \end{cases} \\ \theta_k(P) &:= \measuredangle(P, \text{positive } x_k\text{-axis}) = \arccos\left(\frac{x_k}{\sqrt{x_0^2 + \cdots + x_{n-1}^2}}\right) \quad (k = 2, \ldots, n-1). \end{aligned}$$

Remark 1.2.2. As the unique geodesic in $\mathbb{H}^n$ through $P = (x_0, \ldots, x_{n-1}) \in \mathbb{H}^n$ and the "north pole" $(0, \ldots, 0, |P|)$ of the half-sphere $\mathbb{S}^{n-1}_{\mathbb{H}^n,|P|}$ is perpendicular to the x_{n-1}-axis, the hyperbolic distance from P to $(0, \ldots, 0, |P|)$ is equal to the hyperbolic distance from P to the positive x_{n-1}-axis. Using formula (1.3), for $n \geq 3$ its hyperbolic cosine can be expressed in terms of the angle $\theta_{n-1}(P)$ of the hyperbolic coordinates as

$$\cosh(d_{\mathbb{H}^n}(P, (0, \ldots, 0, |P|))) = \frac{x_0^2 + \cdots + x_{n-1}^2 + |P|^2}{2\, x_{n-1}\, |P|} = \frac{2|P|^2}{2\, x_{n-1}\, |P|} = \frac{|P|}{x_{n-1}} = \frac{1}{\cos(\theta_{n-1}(P))},$$

whereas in the case $n = 2$ we have

$$\cosh(d_{\mathbb{H}}(z, (0, |z|))) = \cosh(d_{\mathbb{H}}(z, i|z|)) = \frac{x^2 + y^2 + |z|^2}{2\, y\, |z|} = \frac{2|z|^2}{2\, y\, |z|} = \frac{|z|}{y} = \frac{1}{\sin(\theta(z))}.$$

Through these relations, for $n \geq 3$ it is possible to replace the coordinate $\theta_{n-1}(P)$ in Definition 1.2.1 (b) by $v(P) := d_{\mathbb{H}^n}(P, (0, \ldots, 0, |P|))$, while for $n = 2$ one can replace the coordinate $\theta(z)$ by $v(z) := d_{\mathbb{H}}(z, i|z|)$.

As an example we determine the hyperbolic line element $ds^2_{\mathbb{H}}$, the hyperbolic volume element $\mu_{\mathbb{H}}(z)$ and the hyperbolic Laplace operator $\Delta_{\mathbb{H}}$ in terms of the hyperbolic coordinates u, θ on the upper half-plane $\mathbb{H}$. The case $n \geq 3$ is then considered in the subsequent lemma.

Example 1.2.3. For $z = (x, y) = x + iy \in \mathbb{H}$ we have

$$x = e^u \cos(\theta), \quad y = e^u \sin(\theta)$$

with the hyperbolic coordinates

$$u(z) = \log\left(\sqrt{x^2+y^2}\right), \quad \theta(z) = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right).$$

Computing

$$\begin{aligned} dx &= \frac{\partial x}{\partial u}\, du + \frac{\partial x}{\partial \theta}\, d\theta = e^u \cos(\theta)\, du - e^u \sin(\theta)\, d\theta, \\ dy &= \frac{\partial y}{\partial u}\, du + \frac{\partial y}{\partial \theta}\, d\theta = e^u \sin(\theta)\, du + e^u \cos(\theta)\, d\theta, \end{aligned}$$

we get

$$\begin{aligned} dx^2 &= e^{2u} \cos(\theta)^2\, du^2 - 2e^{2u} \cos(\theta) \sin(\theta)\, du\, d\theta + e^{2u} \sin(\theta)^2\, d\theta^2, \\ dy^2 &= e^{2u} \sin(\theta)^2\, du^2 + 2e^{2u} \cos(\theta) \sin(\theta)\, du\, d\theta + e^{2u} \cos(\theta)^2\, d\theta^2. \end{aligned}$$

Thus, in hyperbolic coordinates the hyperbolic line element is given by

$$\begin{aligned} ds^2_{\mathbb{H}} &= \frac{dx^2+dy^2}{y^2} = \frac{e^{2u}\left(\cos(\theta)^2+\sin(\theta)^2\right) du^2 + e^{2u}\left(\sin(\theta)^2+\cos(\theta)^2\right) d\theta^2}{e^{2u} \sin(\theta)^2} \\ &= \frac{e^{2u}\, du^2 + e^{2u}\, d\theta^2}{e^{2u} \sin(\theta)^2} = \frac{du^2+d\theta^2}{\sin(\theta)^2}. \end{aligned}$$

From this we easily obtain the volume element

$$\mu_{\mathbb{H}}(z) = \sqrt{\frac{1}{\sin(\theta)^2} \cdot \frac{1}{\sin(\theta)^2}}\, du\, d\theta = \frac{du\, d\theta}{\sin(\theta)^2}.$$

Finally, we can derive the hyperbolic Laplace operator as

$$\Delta_{\mathbb{H}} = -\sin(\theta)^2 \left(\frac{\partial}{\partial u}\left(\frac{\sin(\theta)^2}{\sin(\theta)^2}\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial \theta}\left(\frac{\sin(\theta)^2}{\sin(\theta)^2}\frac{\partial}{\partial \theta}\right)\right) = -\sin(\theta)^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial \theta^2}\right).$$

Lemma 1.2.4. *Let $n \geq 3$. In terms of the hyperbolic coordinates the following assertions hold true.*

(a) The hyperbolic line element $ds^2_{\mathbb{H}^n}$ has the form

$$ds^2_{\mathbb{H}^n} = \frac{1}{\cos(\theta_{n-1})^2}\, du^2 + \sum_{k=1}^{n-2}\left(\prod_{j=k+1}^{n-2} \sin(\theta_j)^2\right) \tan(\theta_{n-1})^2\, d\theta_k^2 + \frac{1}{\cos(\theta_{n-1})^2}\, d\theta_{n-1}^2.$$

(b) The hyperbolic volume element $\mu_{\mathbb{H}^n}(P)$ has the form

$$\mu_{\mathbb{H}^n}(P) = \frac{\prod_{j=2}^{n-1} \sin(\theta_j)^{j-1}}{\cos(\theta_{n-1})^n}\, du\, d\theta_1 \cdots d\theta_{n-1}.$$

(c) The hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ has the form

$$\Delta_{\mathbb{H}^n} = -\cos(\theta_{n-1})^2 \frac{\partial^2}{\partial u^2} - \sum_{k=1}^{n-2} \frac{1}{\left(\prod_{j=k+1}^{n-2} \sin(\theta_j)^2\right) \tan(\theta_{n-1})^2} \frac{\partial^2}{\partial \theta_k^2} - \cos(\theta_{n-1})^2 \frac{\partial^2}{\partial \theta_{n-1}^2}$$
$$- \sum_{k=2}^{n-2} \frac{(k-1)\cos(\theta_{n-1})^2}{\left(\prod_{j=k+1}^{n-1} \sin(\theta_j)^2\right) \tan(\theta_k)} \frac{\partial}{\partial \theta_k} - \frac{n-2}{\tan(\theta_{n-1})} \frac{\partial}{\partial \theta_{n-1}}.$$

Proof.

(a) We compute

$$dx_0 = \frac{\partial x_0}{\partial u}\, du + \sum_{l=1}^{n-1} \frac{\partial x_0}{\partial \theta_l}\, d\theta_l = e^u \cos(\theta_1) \left(\prod_{j=2}^{n-1} \sin(\theta_j)\right) du - e^u \left(\prod_{j=1}^{n-1} \sin(\theta_j)\right) d\theta_1$$
$$+ \sum_{l=2}^{n-1} e^u \cos(\theta_1) \cos(\theta_l) \left(\prod_{\substack{j=2,\\ j\neq l}}^{n-1} \sin(\theta_j)\right) d\theta_l,$$

$$dx_1 = \frac{\partial x_1}{\partial u}\, du + \sum_{l=1}^{n-1} \frac{\partial x_1}{\partial \theta_l}\, d\theta_l = e^u \left(\prod_{j=1}^{n-1} \sin(\theta_j)\right) du + e^u \cos(\theta_1) \left(\prod_{j=2}^{n-1} \sin(\theta_j)\right) d\theta_1$$
$$+ \sum_{l=2}^{n-1} e^u \sin(\theta_1) \cos(\theta_l) \left(\prod_{\substack{j=2,\\ j\neq l}}^{n-1} \sin(\theta_j)\right) d\theta_l$$

and

$$dx_k = \frac{\partial x_k}{\partial u}\, du + \sum_{l=1}^{n-1} \frac{\partial x_k}{\partial \theta_l}\, d\theta_l = \frac{\partial x_k}{\partial u}\, du + \sum_{l=k}^{n-1} \frac{\partial x_k}{\partial \theta_l}\, d\theta_l$$
$$= e^u \cos(\theta_k) \left(\prod_{j=k+1}^{n-1} \sin(\theta_j)\right) du - e^u \left(\prod_{j=k}^{n-1} \sin(\theta_j)\right) d\theta_k$$
$$+ \sum_{l=k+1}^{n-1} e^u \cos(\theta_k) \cos(\theta_l) \left(\prod_{\substack{j=k+1,\\ j\neq l}}^{n-1} \sin(\theta_j)\right) d\theta_l$$

for $k = 2, \ldots, n-1$. Squaring $dx_0, \ldots, dx_{n-1}$, we obtain

$$dx_0^2 = e^{2u} \cos(\theta_1)^2 \left(\prod_{j=2}^{n-1} \sin(\theta_j)^2\right) du^2 + e^{2u} \left(\prod_{j=1}^{n-1} \sin(\theta_j)^2\right) d\theta_1^2$$
$$+ \sum_{l=2}^{n-1} e^{2u} \cos(\theta_1)^2 \cos(\theta_l)^2 \left(\prod_{\substack{j=2,\\ j\neq l}}^{n-1} \sin(\theta_j)^2\right) d\theta_l^2 - 2e^{2u} \cos(\theta_1) \sin(\theta_1) \left(\prod_{j=2}^{n-1} \sin(\theta_j)^2\right) du\, d\theta_1$$
$$+ \sum_{l=2}^{n-1} 2e^{2u} \cos(\theta_1)^2 \cos(\theta_l) \sin(\theta_l) \left(\prod_{\substack{j=2,\\ j\neq l}}^{n-1} \sin(\theta_j)^2\right) du\, d\theta_l$$
$$- \sum_{l=2}^{n-1} 2e^{2u} \cos(\theta_1) \sin(\theta_1) \cos(\theta_l) \sin(\theta_l) \left(\prod_{\substack{j=2,\\ j\neq l}}^{n-1} \sin(\theta_j)^2\right) d\theta_1\, d\theta_l$$
$$+ \sum_{\substack{l_1,l_2=2,\\ l_1<l_2}}^{n-1} 2e^{2u} \cos(\theta_1)^2 \cos(\theta_{l_1}) \sin(\theta_{l_1}) \cos(\theta_{l_2}) \sin(\theta_{l_2}) \left(\prod_{\substack{j=2,\\ j\neq l_1, j\neq l_2}}^{n-1} \sin(\theta_j)^2\right) d\theta_{l_1}\, d\theta_{l_2},$$

$$
\begin{aligned}
dx_1^2 &= e^{2u} \left(\prod_{j=1}^{n-1} \sin(\theta_j)^2 \right) du^2 + e^{2u}\, \cos(\theta_1)^2 \left(\prod_{j=2}^{n-1} \sin(\theta_j)^2 \right) d\theta_1^2 \\
&+ \sum_{l=2}^{n-1} e^{2u}\, \sin(\theta_1)^2\, \cos(\theta_l)^2 \left(\prod_{\substack{j=2,\\ j\neq l}}^{n-1} \sin(\theta_j)^2 \right) d\theta_l^2 + 2e^{2u}\, \cos(\theta_1)\, \sin(\theta_1) \left(\prod_{j=2}^{n-1} \sin(\theta_j)^2 \right) du\, d\theta_1 \\
&+ \sum_{l=2}^{n-1} 2e^{2u} \sin(\theta_1)^2\, \cos(\theta_l)\, \sin(\theta_l) \left(\prod_{\substack{j=2,\\ j\neq l}}^{n-1} \sin(\theta_j)^2 \right) du\, d\theta_l \\
&+ \sum_{l=2}^{n-1} 2e^{2u} \cos(\theta_1)\, \sin(\theta_1)\, \cos(\theta_l)\, \sin(\theta_l) \left(\prod_{\substack{j=2,\\ j\neq l}}^{n-1} \sin(\theta_j)^2 \right) d\theta_1\, d\theta_l \\
&+ \sum_{\substack{l_1,l_2=2,\\ l_1<l_2}}^{n-1} 2e^{2u}\, \sin(\theta_1)^2\, \cos(\theta_{l_1})\, \sin(\theta_{l_1})\, \cos(\theta_{l_2})\, \sin(\theta_{l_2}) \left(\prod_{\substack{j=2,\\ j\neq l_1,\, j\neq l_2}}^{n-1} \sin(\theta_j)^2 \right) d\theta_{l_1}\, d\theta_{l_2}
\end{aligned}
$$

and

$$
\begin{aligned}
dx_k^2 &= e^{2u}\, \cos(\theta_k)^2 \left(\prod_{j=k+1}^{n-1} \sin(\theta_j)^2 \right) du^2 + e^{2u} \left(\prod_{j=k}^{n-1} \sin(\theta_j)^2 \right) d\theta_k^2 \\
&+ \sum_{l=k+1}^{n-1} e^{2u}\, \cos(\theta_k)^2\, \cos(\theta_l)^2 \left(\prod_{\substack{j=k+1,\\ j\neq l}}^{n-1} \sin(\theta_j)^2 \right) d\theta_l^2 - 2e^{2u}\, \cos(\theta_k)\, \sin(\theta_k) \left(\prod_{j=k+1}^{n-1} \sin(\theta_j)^2 \right) du\, d\theta_k \\
&+ \sum_{l=k+1}^{n-1} 2e^{2u} \cos(\theta_k)^2\, \cos(\theta_l)\, \sin(\theta_l) \left(\prod_{\substack{j=k+1,\\ j\neq l}}^{n-1} \sin(\theta_j)^2 \right) du\, d\theta_l \\
&- \sum_{l=k+1}^{n-1} 2e^{2u} \cos(\theta_k)\, \sin(\theta_k)\, \cos(\theta_l)\, \sin(\theta_l) \left(\prod_{\substack{j=k+1,\\ j\neq l}}^{n-1} \sin(\theta_j)^2 \right) d\theta_k\, d\theta_l \\
&+ \sum_{\substack{l_1,l_2=k+1,\\ l_1<l_2}}^{n-1} 2e^{2u}\, \cos(\theta_k)^2\, \cos(\theta_{l_1})\, \sin(\theta_{l_1})\, \cos(\theta_{l_2})\, \sin(\theta_{l_2}) \left(\prod_{\substack{j=k+1,\\ j\neq l_1,\, j\neq l_2}}^{n-1} \sin(\theta_j)^2 \right) d\theta_{l_1}\, d\theta_{l_2}
\end{aligned}
$$

for $k = 2, \ldots, n-1$. Making use of the identity $\cos(\theta_j)^2 + \sin(\theta_j)^2 = 1$ ($j = 1, \ldots, n-1$) multiple times, a tedious but straightforward addition gives us

$$
dx_0^2 + dx_1^2 + \sum_{k=2}^{n-1} dx_k^2 = e^{2u}\, du^2 + \sum_{k=1}^{n-2} e^{2u} \left(\prod_{j=k+1}^{n-1} \sin(\theta_j)^2 \right) d\theta_k^2 + e^{2u}\, d\theta_{n-1}^2.
$$

We end up with the hyperbolic line element

$$
\begin{aligned}
ds_{\mathbb{H}^n}^2 &= \frac{dx_0^2 + \cdots + dx_{n-1}^2}{x_{n-1}^2} = \frac{e^{2u}\, du^2 + \sum_{k=1}^{n-2} e^{2u} \left(\prod_{j=k+1}^{n-1} \sin(\theta_j)^2 \right) d\theta_k^2 + e^{2u}\, d\theta_{n-1}^2}{e^{2u}\, \cos(\theta_{n-1})^2} \\
&= \frac{1}{\cos(\theta_{n-1})^2}\, du^2 + \sum_{k=1}^{n-2} \left(\prod_{j=k+1}^{n-2} \sin(\theta_j)^2 \right) \tan(\theta_{n-1})^2\, d\theta_k^2 + \frac{1}{\cos(\theta_{n-1})^2}\, d\theta_{n-1}^2.
\end{aligned}
$$

(b) Using part (a), we obtain for the hyperbolic volume element that

$$
\begin{aligned}
\mu_{\mathbb{H}^n}(P) &= \sqrt{\frac{1}{\cos(\theta_{n-1})^2} \cdot \prod_{k=1}^{n-2} \left(\left(\prod_{j=k+1}^{n-2} \sin(\theta_j)^2 \right) \tan(\theta_{n-1})^2 \right) \cdot \frac{1}{\cos(\theta_{n-1})^2}}\, du\, d\theta_1 \cdots d\theta_{n-1} \\
&= \frac{\sqrt{\left(\prod_{j=2}^{n-2} \sin(\theta_j)^{2(j-1)} \right) \tan(\theta_{n-1})^{2(n-2)}}}{\cos(\theta_{n-1})^2}\, du\, d\theta_1 \cdots d\theta_{n-1} \\
&= \frac{\left(\prod_{j=2}^{n-2} \sin(\theta_j)^{j-1} \right) \tan(\theta_{n-1})^{n-2}}{\cos(\theta_{n-1})^2}\, du\, d\theta_1 \cdots d\theta_{n-1} \\
&= \frac{\prod_{j=2}^{n-1} \sin(\theta_j)^{j-1}}{\cos(\theta_{n-1})^n}\, du\, d\theta_1 \cdots d\theta_{n-1}.
\end{aligned}
$$

(c) The hyperbolic Laplace operator derived from the hyperbolic line element $ds^2_{\mathbb{H}^n}$ is now given by

$$
\begin{aligned}
\Delta_{\mathbb{H}^n} =& - \frac{\cos(\theta_{n-1})^n}{\prod_{j=2}^{n-1} \sin(\theta_j)^{j-1}} \frac{\partial}{\partial u} \left(\frac{\left(\prod_{j=2}^{n-1} \sin(\theta_j)^{j-1} \right) \cos(\theta_{n-1})^2}{\cos(\theta_{n-1})^n} \frac{\partial}{\partial u} \right) \\
& - \frac{\cos(\theta_{n-1})^n}{\prod_{j=2}^{n-1} \sin(\theta_j)^{j-1}} \sum_{k=1}^{n-2} \frac{\partial}{\partial \theta_k} \left(\frac{\prod_{j=2}^{n-1} \sin(\theta_j)^{j-1}}{\cos(\theta_{n-1})^n \left(\prod_{j=k+1}^{n-2} \sin(\theta_j)^2 \right) \tan(\theta_{n-1})^2} \frac{\partial}{\partial \theta_k} \right) \\
& - \frac{\cos(\theta_{n-1})^n}{\prod_{j=2}^{n-1} \sin(\theta_j)^{j-1}} \frac{\partial}{\partial \theta_{n-1}} \left(\frac{\left(\prod_{j=2}^{n-1} \sin(\theta_j)^{j-1} \right) \cos(\theta_{n-1})^2}{\cos(\theta_{n-1})^n} \frac{\partial}{\partial \theta_{n-1}} \right) \\
=& - \cos(\theta_{n-1})^2 \frac{\partial^2}{\partial u^2} - \sum_{k=1}^{n-2} \frac{1}{\left(\prod_{j=k+1}^{n-2} \sin(\theta_j)^2 \right) \tan(\theta_{n-1})^2} \frac{\partial^2}{\partial \theta_k^2} - \cos(\theta_{n-1})^2 \frac{\partial^2}{\partial \theta_{n-1}^2} \\
& - \frac{\cos(\theta_{n-1})^n}{\prod_{j=2}^{n-1} \sin(\theta_j)^{j-1}} \sum_{k=1}^{n-2} \frac{\partial}{\partial \theta_k} \left(\frac{\prod_{j=2}^{n-1} \sin(\theta_j)^{j-1}}{\cos(\theta_{n-1})^n \left(\prod_{j=k+1}^{n-2} \sin(\theta_j)^2 \right) \tan(\theta_{n-1})^2} \right) \frac{\partial}{\partial \theta_k} \\
& - \frac{\cos(\theta_{n-1})^n}{\prod_{j=2}^{n-1} \sin(\theta_j)^{j-1}} \frac{\partial}{\partial \theta_{n-1}} \left(\frac{\prod_{j=2}^{n-1} \sin(\theta_j)^{j-1}}{\cos(\theta_{n-1})^{n-2}} \right) \frac{\partial}{\partial \theta_{n-1}}.
\end{aligned}
\tag{1.8}
$$

Computing the partial derivatives

$$
\begin{aligned}
&\frac{\partial}{\partial \theta_1} \left(\frac{\prod_{j=2}^{n-1} \sin(\theta_j)^{j-1}}{\cos(\theta_{n-1})^n \left(\prod_{j=2}^{n-2} \sin(\theta_j)^2 \right) \tan(\theta_{n-1})^2} \right) = 0, \\
&\frac{\partial}{\partial \theta_k} \left(\frac{\prod_{j=2}^{n-1} \sin(\theta_j)^{j-1}}{\cos(\theta_{n-1})^n \left(\prod_{j=k+1}^{n-2} \sin(\theta_j)^2 \right) \tan(\theta_{n-1})^2} \right) \\
&\quad = \frac{\partial}{\partial \theta_k} \left(\frac{\left(\prod_{j=2}^{k} \sin(\theta_j)^{j-1} \right) \left(\prod_{j=k+1}^{n-2} \sin(\theta_j)^{j-3} \right) \sin(\theta_{n-1})^{n-4}}{\cos(\theta_{n-1})^{n-2}} \right) \\
&\quad = \frac{\left(\prod_{j=2}^{k-1} \sin(\theta_j)^{j-1} \right) (k-1) \sin(\theta_k)^{k-2} \cos(\theta_k) \left(\prod_{j=k+1}^{n-1} \sin(\theta_j)^{j-3} \right)}{\cos(\theta_{n-1})^{n-2}}
\end{aligned}
$$

for $k = 2, \dots, n-2$, and

$$\frac{\partial}{\partial\theta_{n-1}}\left(\frac{\prod_{j=2}^{n-1}\sin(\theta_j)^{j-1}}{\cos(\theta_{n-1})^{n-2}}\right) = \frac{\partial}{\partial\theta_{n-1}}\left(\prod_{j=2}^{n-2}\sin(\theta_j)^{j-1}\,\tan(\theta_{n-1})^{n-2}\right)$$
$$= \frac{(n-2)\,\prod_{j=2}^{n-2}\sin(\theta_j)^{j-1}\,\tan(\theta_{n-1})^{n-3}}{\cos(\theta_{n-1})^2},$$

we obtain

$$-\frac{\cos(\theta_{n-1})^n}{\prod_{j=2}^{n-1}\sin(\theta_j)^{j-1}}\frac{\partial}{\partial\theta_1}\left(\frac{\prod_{j=2}^{n-1}\sin(\theta_j)^{j-1}}{\cos(\theta_{n-1})^n\left(\prod_{j=2}^{n-2}\sin(\theta_j)^2\right)\tan(\theta_{n-1})^2}\right) = 0,$$

$$-\frac{\cos(\theta_{n-1})^n}{\prod_{j=2}^{n-1}\sin(\theta_j)^{j-1}}\frac{\partial}{\partial\theta_k}\left(\frac{\prod_{j=2}^{n-1}\sin(\theta_j)^{j-1}}{\cos(\theta_{n-1})^n\left(\prod_{j=k+1}^{n-2}\sin(\theta_j)^2\right)\tan(\theta_{n-1})^2}\right)$$
$$= -\frac{\cos(\theta_{n-1})^2\,(k-1)\,\cos(\theta_k)}{\sin(\theta_k)\left(\prod_{j=k+1}^{n-1}\sin(\theta_j)^2\right)} = -\frac{(k-1)\,\cos(\theta_{n-1})^2}{\left(\prod_{j=k+1}^{n-1}\sin(\theta_j)^2\right)\tan(\theta_k)}$$

for $k = 2, \dots, n-2$, and

$$-\frac{\cos(\theta_{n-1})^n}{\prod_{j=2}^{n-1}\sin(\theta_j)^{j-1}}\frac{\partial}{\partial\theta_{n-1}}\left(\frac{\prod_{j=2}^{n-1}\sin(\theta_j)^{j-1}}{\cos(\theta_{n-1})^{n-2}}\right)$$
$$= -\frac{(n-2)\,\cos(\theta_{n-1})^{n-2}\,\tan(\theta_{n-1})^{n-3}}{\sin(\theta_{n-1})^{n-2}} = -\frac{n-2}{\tan(\theta_{n-1})}.$$

Inserting these identities into formula (1.8) yields the desired result

$$\Delta_{\mathbb{H}^n} = -\cos(\theta_{n-1})^2\frac{\partial^2}{\partial u^2} - \sum_{k=1}^{n-2}\frac{1}{\left(\prod_{j=k+1}^{n-2}\sin(\theta_j)^2\right)\tan(\theta_{n-1})^2}\frac{\partial^2}{\partial\theta_k^2} - \cos(\theta_{n-1})^2\frac{\partial^2}{\partial\theta_{n-1}^2}$$
$$-\sum_{k=2}^{n-2}\frac{(k-1)\,\cos(\theta_{n-1})^2}{\left(\prod_{j=k+1}^{n-1}\sin(\theta_j)^2\right)\tan(\theta_k)}\frac{\partial}{\partial\theta_k} - \frac{n-2}{\tan(\theta_{n-1})}\frac{\partial}{\partial\theta_{n-1}}.$$

□

1.2.2. Elliptic coordinates

Let $Q \in \mathbb{H}^n$ be fixed. Then for fixed $r > 0$ the set $\{P \in \mathbb{H}^n \mid d_{\mathbb{H}^n}(P,Q) = r\}$ of points in $\mathbb{H}^n$ with the hyperbolic distance r to Q is a hyperbolic $(n-1)$-sphere with center Q and radius r. It is also a Euclidean sphere, though with different center and radius. Thus, any point $P \in \mathbb{H}^n$ is uniquely determined by its hyperbolic distance to Q and some vector which describes the location of P on that sphere. This gives rise to define the following coordinates.

Definition 1.2.5. Let $Q \in \mathbb{H}^n$ be fixed. For $P \in \mathbb{H}^n$ we define the *elliptic coordinates* $\varrho_Q = \varrho_Q(P) \in [0,\infty)$, $\zeta_Q = \zeta_Q(P) \in \mathbb{S}^{n-1}$ *centered at* Q, where

$$\varrho_Q(P) := d_{\mathbb{H}^n}(P,Q)$$

and $\zeta_Q(P)$ denotes the unit vector at Q that is tangent to the unique geodesic in $\mathbb{H}^n$ through Q and P. By the usual identificiation of the unit tangent space at Q and the unit sphere $\mathbb{S}^{n-1}$ we write $\zeta_Q(P) \in \mathbb{S}^{n-1}$.

In the literature the coordinates from Definition 1.2.5 are often called *geodesic polar coordinates.*

Remark 1.2.6. The location of $P \in \mathbb{H}^n$ on the hyperbolic sphere of radius $d_{\mathbb{H}^n}(P,Q)$ around Q can also be uniquely described by $(n-1)$-many angles $\vartheta_{Q,1}(P), \ldots, \vartheta_{Q,n-1}(P)$. Hence, it is possible to replace the coordinate $\zeta_Q(P)$ in Definition 1.2.5 by the coordinates $\vartheta_{Q,1}(P), \ldots, \vartheta_{Q,n-1}(P)$.

Example 1.2.7. For $n = 2$ and $z = (x,y) = x + iy \in \mathbb{H}$ we can define the elliptic coordinates $\varrho = \varrho(z)$, $\vartheta = \vartheta(z)$ centered at $i \in \mathbb{H}$ as

$$\varrho(z) = d_{\mathbb{H}}(z,i), \qquad \vartheta(z) = \measuredangle(\mathcal{L}, T_z),$$

where $\mathcal{L}$ denotes the positive y-axis in $\mathbb{H}$ and T_z is the tangent at the unique geodesic in $\mathbb{H}$ through i and z at the point i. These coordinates are connected to the rectangular x-y-coordinates via the relations

$$x = \frac{\sinh(\varrho)\,\sin(\vartheta)}{\cosh(\varrho) + \sinh(\varrho)\,\cos(\vartheta)}, \qquad y = \frac{1}{\cosh(\varrho) + \sinh(\varrho)\,\cos(\vartheta)}$$

(see, e.g., [Pip05]). In the elliptic coordinates ϱ, ϑ the hyperbolic line element $ds^2_{\mathbb{H}}$, the hyperbolic volume element $\mu_{\mathbb{H}}(z)$ and the hyperbolic Laplace operator $\Delta_{\mathbb{H}}$ are of the form

$$\begin{aligned} ds^2_{\mathbb{H}} &= d\varrho^2 + \sinh(\varrho)^2\, d\vartheta^2, \\ \mu_{\mathbb{H}}(z) &= \sinh(\varrho)\, d\varrho\, d\vartheta, \\ \Delta_{\mathbb{H}} &= -\frac{\partial^2}{\partial\varrho^2} - \frac{1}{\tanh(\varrho)}\frac{\partial}{\partial\varrho} - \frac{1}{\sinh(\varrho)^2}\frac{\partial^2}{\partial\vartheta^2}. \end{aligned}$$

The hyperbolic line element, the hyperbolic volume element and the hyperbolic Laplace operator in terms of the elliptic coordinates centered at $Q \in \mathbb{H}^n$ are given in the following lemma. These formulas follow from results about polar coordinates on Riemannian manifolds, since the elliptic coordinates centered at Q are obtained from polar coordinates on the tangent space at Q. We omit the details here.

Lemma 1.2.8. *Let $Q \in \mathbb{H}^n$ be fixed. In terms of the elliptic coordinates centered at Q the following assertions hold true.*

(a) The hyperbolic line element $ds^2_{\mathbb{H}^n}$ has the form

$$ds^2_{\mathbb{H}^n} = d\varrho_Q^2 + \sinh(\varrho_Q)^2\, |d\zeta_Q|^2,$$

where $|d\zeta_Q|^2$ denotes the line element on the unit sphere $\mathbb{S}^{n-1}$.

(b) The hyperbolic volume element $\mu_{\mathbb{H}^n}(P)$ has the form

$$\mu_{\mathbb{H}^n}(P) = \sinh(\varrho_Q)^{n-1}\, d\varrho_Q\, d\nu_{n-1}(\zeta_Q),$$

where $d\nu_{n-1}(\zeta_Q)$ denotes the volume element on the unit sphere $\mathbb{S}^{n-1}$.

(c) The hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ has the form

$$\Delta_{\mathbb{H}^n} = -\frac{\partial^2}{\partial\varrho_Q^2} - (n-1)\,\frac{1}{\tanh(\varrho_Q)}\frac{\partial}{\partial\varrho_Q} - \frac{1}{\sinh(\varrho_Q)^2}\,\Delta_{\mathbb{S}^{n-1}},$$

where $\Delta_{\mathbb{S}^{n-1}}$ denotes the Laplace operator on the unit sphere $\mathbb{S}^{n-1}$.

2. Groups acting on hyperbolic n-space

After the introduction of hyperbolic n-space we now turn to the groups that act on this space via Möbius transformations; these are certain groups of Clifford matrices. In the first section we present Clifford numbers and the Clifford group and some of their basic properties. We define two groups $\mathrm{GL}_2(C_n)$, $\mathrm{SL}_2(C_n)$ of matrices whose entries are certain Clifford numbers in the second section. Afterwards, we explain how these groups act on the upper half-space $\mathbb{H}^{n+1}$ and its boundary $\widehat{\mathbb{R}}^n$ via Möbius transformations in the third section, and give important properties of this action. In the fourth section we consider discrete, cofinite subgroups $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ which generalize the classical Fuchsian subgroups $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ of the first kind. In the chapter's fifth section we establish the notion of parabolic, hyperbolic, elliptic and loxodromic elements of $\mathrm{PSL}_2(C_{n-1})$, while we treat their fixed points in $\mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ and the respective stabilizer subgroups in the sixth section. The notations used in this final section will be employed in the later chapters to define three different types of Eisenstein series in $\mathbb{H}^n$.

2.1. Clifford numbers

There are several different ways to treat the theory of Möbius transformations in $\mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$. While it is possible to express the orientation-preserving Möbius transformations in terms of the group $\mathrm{PSO}(n,1)$, we follow another approach involving matrices with certain Clifford numbers as entries. It was first used in 1902 by Vahlen in [Vah02] and later rediscovered by Maass in [Maa49]. Before we consider these matrices, we introduce Clifford numbers and some basic properties that we will require later. We omit most of the proofs here. Throughout this section we let $n \in \mathbb{N}$.

Definition 2.1.1. We define the *Clifford algebra* C_n as the associative algebra over $\mathbb{R}$ generated by $n-1$ elements $i_1, \dots, i_{n-1}$, satisfying the relations

(i) $i_k^2 = -1$ for $k = 1, \dots, n-1$,

(ii) $i_k i_l = -i_l i_k$ for $k, l = 1, \dots, n-1$ with $k \neq l$.

Sometimes it might be convenient to introduce the additional generating element $i_0 := 1$. The elements of C_n are called *Clifford numbers*.

It is immediately clear by definition that C_n is a subalgebra of C_{n+1} for any $n \in \mathbb{N}$. We note that condition (ii) in Definition 2.1.1 implies that multiplication in C_n is not commutative for $n \geq 3$.

Remark 2.1.2. Most authors define the Clifford algebra as being generated by n elements $i_1, \dots, i_n$ instead of $i_1, i_2, \dots, i_{n-1}$. However, Definition 2.1.1 is in line with the definition in [Ahl85a] and [Söd12]. Moreover, it will suit our setting better in that a certain group of matrices with entries in C_{n-1} (instead of entries in C_{n-2}) will act on the n-dimensional hyperbolic space $\mathbb{H}^n$, and that the space V_n of vectors of C_n (see Definition 2.1.8) can be identified with $\mathbb{R}^n$ (instead of $\mathbb{R}^{n+1}$).

Remark 2.1.3. An element $a \in C_n$ can be uniquely written in the form $a = \sum_I a_I\, I$, where $a_I \in \mathbb{R}$ and the summation runs over all products $I = i_{\nu_1} \cdots i_{\nu_k}$ with $1 \leq \nu_1 < \cdots < \nu_k \leq n-1$. The empty product $I = \emptyset$ is also permitted and interpreted as the real number 1. Hence, C_n is a real vector space of dimension 2^{n-1}.

Definition 2.1.4. Let $a = \sum_I a_I\, I \in C_n$. The coefficient $a_\emptyset$ of the empty product is called the *real part* of a and denoted by $\mathrm{Re}(a)$. The sum $\sum_{I \neq \emptyset} a_I\, I$ of all other terms is called the *imaginary part* of a and is denoted by $\mathrm{Im}(a)$.

Definition 2.1.5. We equip the space C_n with the *square norm*, i.e. for $a = \sum_I a_I I \in C_n$ we have $|a|^2 = \sum_I a_I^2$.

In terms of the real and the imaginary part we can write $|a|^2 = \mathrm{Re}(a)^2 + |\mathrm{Im}(a)|^2$ for the norm of $a \in C_n$.

Definition 2.1.6.

(a) The *degree* of a product $I = i_{\nu_1} \cdots i_{\nu_k}$ is the number k.

(b) The *degree* of an element $a = \sum_I a_I I \in C_n$ is the highest occurring degree of a product I with $a_I \neq 0$.

(c) An element $a = \sum_I a_I I \in C_n$ is called *homogeneous* if all products I with $a_I \neq 0$ have the same degree.

Example 2.1.7. As an example we consider the Clifford algebra in the simplest cases $n = 1, 2, 3$.

(a) For $n = 1$ the Clifford algebra C_1 can simply be identified with the real numbers $\mathbb{R}$.

(b) In the case $n = 2$ we obtain the real associative algebra generated by the element $i := i_1$ with $i^2 = -1$, so C_2 can be identified with the complex numbers $\mathbb{C}$.

(c) For $n = 3$ we can identify C_3 with the quaternion algebra $\mathcal{H}$, where the basic quaternions i, j, k are represented by i_1, i_2 and $i_1 i_2$.

Though the full Clifford algebra itself is a vector space, we use the term "vector" only for specific elements of C_n.

Definition 2.1.8. An element $x \in C_n$ of the form $x = x_0 + x_1 i_1 + \cdots + x_{n-1} i_{n-1}$, where $x_0, \ldots, x_{n-1} \in \mathbb{R}$, is called *vector*. The subspace of C_n consisting of all vectors is denoted by V_n.

It is immediately seen that V_n is an n-dimensional real vector space which can naturally be identified with $\mathbb{R}^n$ via

$$x_0 + x_1 i_1 + \cdots + x_{n-1} i_{n-1} \in V_n \quad \longleftrightarrow \quad (x_0, x_1, \ldots, x_{n-1}) \in \mathbb{R}^n.$$

In section 2.3 we will use this identification to interpret $\mathbb{R}^n$ as a subspace of C_n.

The square of a vector $x = x_0 + x_i i_1 + \cdots + x_{n-1} i_{n-1} \in V_n$ is again a vector, since

$$(x_k i_k)(x_l i_l) + (x_l i_l)(x_k i_k) = x_k x_l i_k i_l + x_k x_l i_l i_k = x_k x_l i_k i_l - x_k x_l i_k i_l = 0$$

holds true for $k, l = 1, \ldots, n-1$ with $k \neq l$. In particular, we have the formula

$$x^2 = \mathrm{Re}(x)^2 + 2\,\mathrm{Re}(x)\,\mathrm{Im}(x) - |\mathrm{Im}(x)|^2.$$

The same is true for any power x^k with $k \in \mathbb{N}$ and for any polynomial in x with real coefficients. An interesting property is that this carries over even to fractional powers. For instance, any $x \in V_n$ with $\mathrm{Im}(x) \neq 0$ has the two vector square roots

$$\sqrt{x} = \pm \left(\sqrt{\frac{|x| + \mathrm{Re}(x)}{2}} + \frac{\mathrm{Im}(x)}{|\mathrm{Im}(x)|} \sqrt{\frac{|x| - \mathrm{Re}(x)}{2}} \right) \in V_n$$

(see, e.g., formula (1.3) in [Ahl86]). Further, we note that any $x_0 \in \mathbb{R}$ with $x_0 \geq 0$ has its usual two real square roots in V_n, whereas for $n \geq 3$ any $x_0 \in \mathbb{R}$ with $x_0 < 0$ has infinitely many square roots in V_n, namely all $y \in V_n$ with $\mathrm{Re}(y) = 0$ and $|y|^2 = |\mathrm{Im}(y)|^2 = -x_0$.

Similar to complex conjugation, we have three commonly used involutions, respectively conjugations, in C_n which are introduced in the next definition.

Definition 2.1.9. We define the following functions from C_n to C_n:

(a) Let $' : C_n \to C_n$ be the map which replaces i_k by $-i_k$ ($k = 1, \dots, n-1$). It is called the *main involution* or *main conjugation*, and $a' := {}'(a)$ is called the *main conjugate* of a.

(b) Let $^* : C_n \to C_n$ be the map which reverses the order of the factors in each product $I = i_{\nu_1} \cdots i_{\nu_k}$. It is called the *reverse involution* or *reverse conjugation*, and $a^* := {}^*(a)$ is called the *reverse conjugate* of a.

(c) Let $^- : C_n \to C_n$ be the map which combines the previous two, i.e. for $a \in C_n$ we set $\overline{a} := (a')^* = (a^*)'$. It is called the *complex involution* or *complex conjugation*, and $\overline{a} := {}^-(a)$ is called the *complex conjugate* of a.

Clearly, all of these maps are involutions of C_n. The involution $'$ defines an algebra automorphism of C_n, i.e. we have $(a+b)' = a' + b'$ and $(ab)' = a'b'$ for any $a, b \in C_n$. The other two involutions * and $^-$ are anti-automorphisms of C_n, i.e. we have $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$, as well as $\overline{a+b} = \overline{a} + \overline{b}$ and $\overline{ab} = \overline{b}\,\overline{a}$ for any $a, b \in C_n$. Moreover, for any product $I = i_{\nu_1} \cdots i_{\nu_k}$ with $1 \le \nu_1 < \cdots < \nu_k \le n-1$ we simply have

$$I' = (-1)^k I, \qquad I^* = (-1)^{k(k-1)/2} I, \qquad \overline{I} = (-1)^{k(k+1)/2} I.$$

For $x \in V_n$ it is obvious that $x^* = x$, and therefore $\overline{x} = x'$. Further, a straightforward computation yields the identity $x\overline{x} = \overline{x}x = |x|^2$ for any $x \in V_n$. So any non-zero vector is multiplicatively invertible in C_n with inverse $x^{-1} = \overline{x}/|x|^2 \in V_n$. Since products of invertible elements are again invertible, the non-zero vectors in C_n generate a group with respect to multiplication.

Definition 2.1.10. The multiplicative group that consists of all products of elements of $V_n \setminus \{0\}$ is called the *Clifford group* and is denoted by Γ_n.

Example 2.1.11. For $n = 1, 2, 3$ we have $\Gamma_1 = \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$, $\Gamma_2 = \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ and $\Gamma_3 = \mathcal{H} \setminus \{0\}$ for the quaternion algebra $\mathcal{H}$. In the remaining cases, i.e. if $n \ge 4$, the Clifford group Γ_n is a proper subset of $C_n \setminus \{0\}$. This means that there are Clifford numbers that cannot be written as a product of vectors.

In [Ahl85a], Ahlfors proved the following properties for elements of the Clifford group.

Proposition 2.1.12. *For any $a, b \in \Gamma_n$ the following assertions hold true.*

(a) $a\overline{a} = \overline{a}a = |a|^2$.

(b) $|ab| = |a|\,|b|$.

By applying the main involution $'$, part (a) of the proposition also shows that $a'a^* = a^*a' = |a|^2$ for any $a \in \Gamma_n$. Moreover, it yields that, in analogy to vectors, there is a simple formula for the inverse of an element of the Clifford group.

Corollary 2.1.13. *Any $a \in \Gamma_n$ has the multiplicative inverse $a^{-1} = \overline{a}/|a|^2$.*

It is easy to see that inverting an element commutes with all three involutions from Definition 2.1.9, i.e. for any $a \in \Gamma_n$ the identities $(a')^{-1} = (a^{-1})'$, $(a^*)^{-1} = (a^{-1})^*$ and $\overline{a}^{-1} = \overline{a^{-1}}$ hold true.

Example 2.1.14. Proposition 2.1.12 does not hold true for general elements of C_n.

(a) Let $a = 1 + i_1 i_2 i_3 \in C_4$, then we have $\overline{a} = a$ and $a^2 = 2 + 2i_1 i_2 i_3 = 2a$. Consequently, we get $a\overline{a} = \overline{a}a = a^2 = 2a$, but $|a|^2 = 2$, meaning that in general $a\overline{a} \neq |a|^2$. This also shows that $a = 1 + i_1 i_2 i_3$ cannot be written as a product of vectors.

(b) If we view $a = 1 + i_1 i_2 i_3$ as an element of C_5, then $a^2 i_4 = 2i_4 + 2i_1 i_2 i_3 i_4 \neq 0$. On the other hand one can compute that $a i_4 a = 0$, so that $|a^2 i_4| \neq |a i_4 a|$ and $|a i_4 a| \neq |a i_4|\,|a|$. This proves that in general $|ab| \neq |ba|$ and $|ab| \neq |a|\,|b|$.

However, we have the following weaker results for general Clifford numbers (see, e.g., [Wat93], Theorem 1).

Proposition 2.1.15. *For any $a, b \in C_n$ the following assertions hold true.*

(a) $|a|^2 = \mathrm{Re}(a\overline{a}) = \mathrm{Re}(\overline{a}a)$.

(b) If either $a \in \Gamma_n$ or $b \in \Gamma_n$, then $|ab| = |a|\,|b|$.

(c) $\sum_I a_I b_I = \mathrm{Re}\left(a\overline{b}\right) = \mathrm{Re}(\overline{a}b)$.

(d) If $a, b \in V_n$, then $\sum_I a_I b_I = \frac{1}{2}\left(a\overline{b} + b\overline{a}\right)$.

There is also a different characterization of elements of the Clifford group via bijective and orientation-preserving isometries of $V_n \cong \mathbb{R}^n$ (see, e.g., [Ahl85a]).

Proposition 2.1.16. *If $a \in \Gamma_n$ and $x \in V_n$, then also $ax(a')^{-1} \in V_n$ and the map $V_n \to V_n$, $x \mapsto ax(a')^{-1}$, is a bijective and orientation-preserving isometry. Conversely, any $a \in C_n$ such that the map $V_n \to V_n$, $x \mapsto ax(a')^{-1}$, is a bijective and orientation-preserving isometry already satisfies $a \in \Gamma_n$. Moreover, this yields the isomorphisms $\Gamma_n/\mathbb{R}^\times \cong \mathrm{SO}(n)$ and $\Gamma_n/\mathbb{R}^+ \cong \mathrm{PSO}(n)$.*

We close this section with two elementary results on Clifford numbers that will be of frequent use in the remainder of this chapter. The first one can be found, e.g., in [Ahl86], while for the second one we give a proof.

Proposition 2.1.17. *Let $a, b \in \Gamma_n$. Then $ab^{-1} \in V_n$ if and only if $a^*b \in V_n$, and $b^{-1}a \in V_n$ if and only if $ba^* \in V_n$.*

Lemma 2.1.18. *Let $a \in C_{n+1}$. If $a \in C_n \subseteq C_{n+1}$, then we have $i_n a = a' i_n$.*

Proof. We write $a = \sum_I a_I\, I$ with $a_I \in \mathbb{R}$, and where the summation runs over all products $I = i_{\nu_1} \cdots i_{\nu_k}$ with $1 \leq \nu_1 < \cdots < \nu_k \leq n-1$. Then for any such product I we have

$$i_n\, I = i_n i_{\nu_1} \cdots i_{\nu_k} = (-1)^k\, i_{\nu_1} \cdots i_{\nu_k} i_n = (-1)^k\, I\, i_n = I'\, i_n.$$

This leads to

$$i_n a = \sum_I a_I\, i_n\, I = \sum_I a_I\, I'\, i_n = \Big(\sum_I a_I\, I\Big)' i_n = a' i_n.$$

□

2.2. Clifford matrices

In this section we introduce two groups of 2×2-matrices whose entries are Clifford numbers which are subject to certain conditions. We mainly follow Ahlfors' approach in [Ahl86], though we additionally define a slightly more general group. A different development of Clifford matrices and their relationship to Möbius transformations is given, for instance, by Waterman in [Wat93]. As the results of this section are known, we do not carry out all of the proofs. Throughout the section let again $n \in \mathbb{N}$.

Definition 2.2.1. We define the sets

$$\mathrm{GL}_2(C_n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \Gamma_n \cup \{0\},\ ab^*, cd^* \in V_n,\ ad^* - bc^* \in \mathbb{R} \setminus \{0\} \right\},$$

$$\mathrm{SL}_2(C_n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \Gamma_n \cup \{0\},\ ab^*, cd^* \in V_n,\ ad^* - bc^* = 1 \right\}$$

of matrices with entries in the Clifford numbers C_n. An element of $\mathrm{GL}_2(C_n)$ or $\mathrm{SL}_2(C_n)$ is called a *Clifford matrix.*

From the definition we immediately see that $\mathrm{SL}_2(C_n) \subseteq \mathrm{GL}_2(C_n)$.

Remark 2.2.2. As not all Clifford numbers are permitted as entries in Definition 2.2.1, more precise notations would actually be $\mathrm{GL}_2(\Gamma_n)$ and $\mathrm{SL}_2(\Gamma_n)$ instead of $\mathrm{GL}_2(C_n)$ and $\mathrm{SL}_2(C_n)$. These alternate notations are indeed used by Ahlfors in [Ahl86]. If we were really strict, we would even have to write $\mathrm{GL}_2(\Gamma_n \cup \{0\})$ and $\mathrm{SL}_2(\Gamma_n \cup \{0\})$. However, our notations agree with the ones used in [Wat93] and [Söd12].

The notations $\mathrm{GL}_2(C_n)$ and $\mathrm{SL}_2(C_n)$ already suggest that both sets form a group which is indeed the case.

Theorem 2.2.3. *The sets $\mathrm{GL}_2(C_n)$ and $\mathrm{SL}_2(C_n)$ of Clifford matrices are groups under matrix multiplication.*

Proof. For $\mathrm{SL}_2(C_n)$ see, e.g., [Ahl86]. The proof for $\mathrm{GL}_2(C_n)$ is similar. □

It is now justified to introduce the following notions.

Definition 2.2.4. We call $\mathrm{GL}_2(C_n)$ the *general linear group* and $\mathrm{SL}_2(C_n)$ the *special linear group* over the Clifford numbers C_n.

Example 2.2.5. For $n = 1, 2$ the sets $\mathrm{GL}_2(C_n)$ and $\mathrm{SL}_2(C_n)$ reduce to well-known groups of matrices.

(a) In the case $n = 1$ we have $\mathrm{GL}_2(C_1) = \mathrm{GL}_2(\mathbb{R})$ and $\mathrm{SL}_2(C_1) = \mathrm{SL}_2(\mathbb{R})$.

(b) For $n = 2$ we obtain $\mathrm{GL}_2(C_2) = \mathrm{GL}_2(\mathbb{C})$ and $\mathrm{SL}_2(C_2) = \mathrm{SL}_2(\mathbb{C})$.

We give a few further properties of Clifford matrices.

Lemma 2.2.6. *For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(C_n)$ the following assertions hold true.*

(a) a^*c, $b^*d \in V_n$.

(b) $d^*a - b^*c = ad^* - bc^*$.

Proof.

(a) If $a = 0$ or $c = 0$, it is clear that $a^*c = 0 \in V_n$. Now let $a \neq 0$ and $c \neq 0$. By the definition of $\mathrm{GL}_2(C_n)$ and Proposition 2.1.17 we have $a^{-1}b, c^{-1}d \in V_n$. Writing

$$1 = (ad^* - bc^*)^{-1}(ad^* - bc^*) = (ad^* - bc^*)^{-1}\, a \left(d^*(c^*)^{-1} - a^{-1}b\right) c^*,$$

we then obtain

$$\begin{aligned} (c^*a)^{-1} = a^{-1}(c^*)^{-1} &= (ad^* - bc^*)^{-1}\left(d^*(c^*)^{-1} - a^{-1}b\right) \\ &= (ad^* - bc^*)^{-1}\left((c^{-1}d)^* - a^{-1}b\right) = (ad^* - bc^*)^{-1}\left(c^{-1}d - a^{-1}b\right) \in V_n. \end{aligned}$$

This gives us $c^*a = \left((c^*a)^{-1}\right)^{-1} \in V_n$ and $a^*c = (c^*a)^* \in V_n$.

For $b = 0$ or $d = 0$ the condition $b^*d = 0 \in V_n$ is again obvious. If $b \neq 0$ and $d \neq 0$, then by Proposition 2.1.17 we have $b^{-1}a \in V_n$ as $ba^* = (ab^*)^* \in V_n$, and $d^{-1}c \in V_n$ as $dc^* = (cd^*)^* \in V_n$. Now we write

$$1 = (ad^* - bc^*)^{-1}(ad^* - bc^*) = (ad^* - bc^*)^{-1}\, b \left(b^{-1}a - c^*(d^*)^{-1}\right) d^*,$$

which gives us

$$\begin{aligned} (d^*b)^{-1} = b^{-1}(d^*)^{-1} &= (ad^* - bc^*)^{-1}\left(b^{-1}a - c^*(d^*)^{-1}\right) \\ &= (ad^* - bc^*)^{-1}\left(b^{-1}a - (d^{-1}c)^*\right) = (ad^* - bc^*)^{-1}\left(b^{-1}a - d^{-1}c\right) \in V_n. \end{aligned}$$

This implies $d^*b = \left((d^*b)^{-1}\right)^{-1} \in V_n$ and $b^*d = (d^*b)^* \in V_n$.

(b) If $d = 0$, then $c \neq 0$ and there is $\lambda \in \mathbb{R} \setminus \{0\}$ such that $-bc^* = ad^* - bc^* = \lambda$. This implies $b = -\lambda(c^*)^{-1}$ and

$$d^*a - b^*c = -(-\lambda(c^*)^{-1})^*c = \lambda c^{-1}c = \lambda = ad^* - bc^*.$$

Now let $d \neq 0$. From $b^*cd^* = b^*(cd^*)^* = b^*dc^* = (b^*d)^*c^* = d^*bc^*$, where we made use of cd^*, $b^*d \in V_n$, we obtain

$$\begin{aligned}(ad^* - bc^*)\,|d|^2 &= d^*\,(ad^* - bc^*)\,d' = d^*ad^*d' - d^*bc^*d' = d^*ad^*d' - b^*cd^*d' \\ &= d^*a\,|d|^2 - b^*c\,|d|^2 = (d^*a - b^*c)\,|d|^2.\end{aligned}$$

Dividing this identity by $|d|^2 \neq 0$ yields $d^*a - b^*c = ad^* - bc^*$, as asserted.

□

Definition 2.2.7. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(C_n)$ the expression $\mathrm{pdet}(\gamma) := ad^* - bc^*$ is called the *pseudo-determinant* (sometimes also called *Clifford determinant*) of γ.

By Definition 2.2.1 we have $\mathrm{pdet}(\gamma) \in \mathbb{R} \setminus \{0\}$ for any $\gamma \in \mathrm{GL}_2(C_n)$, and $\mathrm{pdet}(\gamma) = 1$ for any $\gamma \in \mathrm{SL}_2(C_n)$.

We see in the next two lemmas that the pseudo-determinant satisfies several properties of a determinant.

Lemma 2.2.8. *The pseudo-determinant is multiplicative, i.e. for any $\gamma, \delta \in \mathrm{GL}_2(C_n)$ we have*

$$\mathrm{pdet}(\gamma\delta) = \mathrm{pdet}(\gamma)\,\mathrm{pdet}(\delta).$$

Proof. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\delta = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, then

$$\gamma\delta = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

Further, we have ef^*, $gh^* \in V_n$, which implies $ef^* = (ef^*)^* = fe^*$ and $gh^* = (gh^*)^* = hg^*$. Using this together with $\mathrm{pdet}(\delta) \in \mathbb{R} \setminus \{0\}$, we compute

$$\begin{aligned}\mathrm{pdet}(\gamma\delta) &= (ae + bg)\,(cf + dh)^* - (af + bh)\,(ce + dg)^* \\ &= (ae + bg)\,(f^*c^* + h^*d^*) - (af + bh)\,(e^*c^* + g^*d^*) \\ &= a\,(ef^* - fe^*)\,c^* + a\,(eh^* - fg^*)\,d^* + b\,(gf^* - he^*)\,c^* + b\,(gh^* - hg^*)\,d^* \\ &= a\,(eh^* - fg^*)\,d^* - b\,(eh^* - fg^*)^*\,c^* = a\,\mathrm{pdet}(\delta)\,d^* - b\,\mathrm{pdet}(\delta)^*\,c^* \\ &= (ad^* - bc^*)\,\mathrm{pdet}(\delta) = \mathrm{pdet}(\gamma)\,\mathrm{pdet}(\delta).\end{aligned}$$

□

Lemma 2.2.9. *If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(C_n)$ and $\gamma^* := \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$, then also $\gamma^* \in \mathrm{GL}_2(C_n)$ and*

$$\gamma\gamma^* = \gamma^*\gamma = \begin{pmatrix} \mathrm{pdet}(\gamma) & 0 \\ 0 & \mathrm{pdet}(\gamma) \end{pmatrix}.$$

Proof. First we note that $a, b, c, d \in \Gamma_n \cup \{0\}$ implies that also $a^*, -b^*, -c^*, d^* \in \Gamma_n \cup \{0\}$. Moreover, by Lemma 2.2.6 we have

$$d^*(-b^*)^* = -d^*b = -(b^*d)^* = -b^*d \in V_n, \qquad -c^*(a^*)^* = -c^*a = -(a^*c)^* = -a^*c \in V_n$$

and $d^*(a^*)^* - (-b^*)(-c^*)^* = d^*a - b^*c = ad^* - bc^* \in \mathbb{R} \setminus \{0\}$. This shows that $\gamma^* \in \mathrm{GL}_2(C_n)$.

Making use of $ab^*, cd^*, a^*c, b^*d \in V_n$ and $d^*a - b^*c = \mathrm{pdet}(\gamma)$, a simple computation now gives us

$$\gamma\gamma^* = \begin{pmatrix} ad^* - bc^* & -ab^* + ba^* \\ cd^* - dc^* & -cb^* + da^* \end{pmatrix} = \begin{pmatrix} ad^* - bc^* & -ab^* + (ab^*)^* \\ cd^* - (cd^*)^* & (ad^* - bc^*)^* \end{pmatrix} = \begin{pmatrix} \mathrm{pdet}(\gamma) & 0 \\ 0 & \mathrm{pdet}(\gamma) \end{pmatrix}$$

and

$$\gamma^*\gamma = \begin{pmatrix} d^*a - b^*c & d^*b - b^*d \\ -c^*a + a^*c & -c^*b + a^*d \end{pmatrix} = \begin{pmatrix} d^*a - b^*c & (b^*d)^* - b^*d \\ -(a^*c)^* + a^*c & (d^*a - b^*c)^* \end{pmatrix} = \begin{pmatrix} \mathrm{pdet}(\gamma) & 0 \\ 0 & \mathrm{pdet}(\gamma) \end{pmatrix}.$$

□

Corollary 2.2.10. *For* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(C_n)$ *its inverse matrix* $\gamma^{-1} \in \mathrm{GL}_2(C_n)$ *is given by*

$$\gamma^{-1} = \frac{1}{\mathrm{pdet}(\gamma)}\gamma^* = \frac{1}{ad^* - bc^*}\begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}.$$

Similar to $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$, the special linear group $\mathrm{SL}_2(C_n)$ has a quite simple set of generators.

Proposition 2.2.11. *The group* $\mathrm{SL}_2(C_n)$ *is generated by the matrices*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad (x \in V_n). \tag{2.1}$$

Proof. At first we note that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^3.$$

Thus, the identity

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

implies that any matrix $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$ with $y \in V_n$ is a product of the matrices (2.1).
Now any element $\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$ with $y \in V_n \setminus \{0\}$ can be written as

$$\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} = \begin{pmatrix} y & y^2 - y \\ y^{-1} & 1 \end{pmatrix}\begin{pmatrix} y & 1 - y \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y^2 - y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 - y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Since both the square and the inverse of y are again vectors, this proves that any of these matrices is a product of the claimed generators.
Next we consider matrices of the form $\begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix}$ with $a \in \Gamma_n$. Then there are non-zero vectors $y_1, \ldots, y_k \in V_n \setminus \{0\}$ with $a = y_1 \cdots y_k$, so that

$$(a^*)^{-1} = ((y_1 \cdots y_k)^*)^{-1} = (y_k^* \cdots y_1^*)^{-1} = (y_k \cdots y_1)^{-1} = y_1^{-1} \cdots y_k^{-1}.$$

We obtain

$$\begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix} = \begin{pmatrix} y_1 & 0 \\ 0 & y_1^{-1} \end{pmatrix} \cdots \begin{pmatrix} y_k & 0 \\ 0 & y_k^{-1} \end{pmatrix},$$

which implies that any matrix $\begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix}$ with $a \in \Gamma_n$ is a product of the elements (2.1).

Finally, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(C_n)$ be an arbitrary matrix. If $c = 0$, then we have $a \neq 0$, $d = (a^*)^{-1}$, $a^{-1}b \in V_n$ (since $ab^* \in V_n$) and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & (a^*)^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}$$

is a product of the claimed generating elements. And if $c \neq 0$, then $b = (ad^* - 1)\,(c^*)^{-1}$, $ac^{-1} \in V_n$ (as $a^*c \in V_n$), $c^{-1}d \in V_n$ (since $cd^* \in V_n$) and $c^{-1}d = (c^{-1}d)^* = d^*(c^*)^{-1}$. Thus, we get that

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & (ad^* - 1)\,(c^*)^{-1} \\ c & d \end{pmatrix} = \begin{pmatrix} (c^*)^{-1} & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & c^{-1}d \end{pmatrix} \\ &= \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (c^*)^{-1} & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix} \end{aligned}$$

is a product of the matrices (2.1). This completes the proof. □

It is easy to see that $\{\lambda I \mid \lambda \in \mathbb{R} \setminus \{0\}\}$ is a subgroup of $\mathrm{GL}_2(C_n)$ and $\{\pm I\}$ is a subgroup of $\mathrm{SL}_2(C_n)$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ denotes the 2×2-identity matrix. Thus, we can consider the respective quotient groups which will appear as certain groups of Möbius transformations in the next section.

Definition 2.2.12. We define the quotient groups

$$\begin{aligned} \mathrm{PGL}_2(C_n) &:= \mathrm{GL}_2(C_n)/\{\lambda I \mid \lambda \in \mathbb{R} \setminus \{0\}\}, \\ \mathrm{PSL}_2(C_n) &:= \mathrm{SL}_2(C_n)/\{\pm I\}. \end{aligned}$$

Remark 2.2.13. To ease notation we will usually denote elements of $\mathrm{PGL}_2(C_n)$, respectively of $\mathrm{PSL}_2(C_n)$, by matrices in $\mathrm{GL}_2(C_n)$, respectively in $\mathrm{SL}_2(C_n)$, representing them.

2.3. The action of Clifford matrices

In [Ahl85a] and [Ahl86] Ahlfors explained how Möbius transformations of $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ can be described by the action of Clifford matrices. We essentially follow him, and subsequently see how this action extends to the hyperbolic space $\mathbb{H}^{n+1}$. Again we omit some of the proofs here.

Let $n \in \mathbb{N}$. By the natural identification of $\mathbb{R}^n$ with the space V_n of vectors in C_n, any $P \in \mathbb{R}^n$ can be written in the form

$$P = x_0 + x_1 i_1 + \cdots + x_{n-1} i_{n-1}$$

with $x_0, \ldots, x_{n-1} \in \mathbb{R}$.

Remark 2.3.1. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(C_n)$ and $P \in \mathbb{R}^n \cong V_n$ we have $c, d \in \Gamma_n \cup \{0\}$ and $cd^* \in V_n$. If $c = 0$, then $cP + d = d$ is either zero or invertible. In case that $c \neq 0$, we can write $cP + d = c\,(P + c^{-1}d)$. As $cd^* \in V_n$ is equivalent to $c^{-1}d \in V_n$, the number $P + c^{-1}d$ is in $\Gamma_n \cup \{0\}$, and so is $c\,(P + c^{-1}d)$. Hence, $cP + d$ is either zero or invertible.
Moreover, because of the condition $ad^* - bc^* \in \mathbb{R} \setminus \{0\}$ the equations $aP + b = 0$ and $cP + d = 0$ cannot be fulfilled simultaneously since otherwise

$$ad^* - bc^* = a(-cP)^* - (-aP)c^* = -aPc^* + aPc^* = 0$$

would hold.

Definition 2.3.2. For a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(C_n)$ and an element $P \in \mathbb{R}^n$ we set

$$\gamma P := (aP + b)\,(cP + d)^{-1}. \tag{2.2}$$

We have to clarify how to interpret the right-hand side of (2.2), in order to assign it a value for $P = \infty$ and to permit it to take the value ∞.

In general, for two Clifford numbers $a, b \in C_n$, where b is multiplicatively invertible, also ab^{-1} is a well-defined element of C_n. In the case $b = 0$ and $a \neq 0$ we set $ab^{-1} := \infty$. By our considerations in Remark 2.3.1, in this way the right-hand side of formula (2.2) is always well-defined, either as an element of C_n or as ∞.
Finally, for $P = \infty$ we define $\gamma\infty := ac^{-1}$ if $c \neq 0$, and $\gamma\infty := \infty$ if $c = 0$.

Definition 2.3.3. We say that the matrix $\gamma \in \mathrm{GL}_2(C_n)$ *induces* the map $\widehat{\mathbb{R}}^n \to C_n \cup \{\infty\}$ that we have defined above. The map induced by γ will be denoted by $\widetilde{\gamma}$.

Definition 2.3.4. A map $\widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ is called a *Möbius transformation* if it is a finite composition of reflections in hyperplanes and inversions in spheres.

As a matter of fact, for any $\gamma \in \mathrm{GL}_2(C_n)$ the map $\widetilde{\gamma}$ even defines a Möbius transformation, especially it maps to $V_n \cup \{\infty\}$ which is identified with $\widehat{\mathbb{R}}^n$. This is a particular consequence of the following theorem.

Theorem 2.3.5. *The group* $\mathrm{PGL}_2(C_n)$ *is isomorphic to the full group of Möbius transformations of* $\widehat{\mathbb{R}}^n$*: Each* $\gamma \in \mathrm{GL}_2(C_n)$ *induces a Möbius transformation* $\widetilde{\gamma} : \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$*, and if* $\gamma_1, \gamma_2 \in \mathrm{GL}_2(C_n)$ *induce* $\widetilde{\gamma}_1, \widetilde{\gamma}_2$*, then the product* $\gamma_1\gamma_2$ *induces the composite map* $\widetilde{\gamma}_1 \circ \widetilde{\gamma}_2 : \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$*. Conversely, every Möbius transformation* $g : \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ *is induced by the Clifford matrices* $\lambda\gamma$ $(\lambda \in \mathbb{R} \setminus \{0\})$ *for some* $\gamma \in \mathrm{GL}_2(C_n)$*.*

Proof. See, e.g., [Wat93].

□

If we restrict γ to the subgroup $\mathrm{SL}_2(C_n) \subseteq \mathrm{GL}_2(C_n)$, it is now clear that the induced maps $\widetilde{\gamma} : \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ $(\gamma \in \mathrm{PSL}_2(C_n))$ form a certain subgroup of the full group of Möbius transformations. The precise statement is formulated in the next theorem.

Theorem 2.3.6. *The group* $\mathrm{PSL}_2(C_n)$ *is isomorphic to the group of orientation-preserving Möbius transformations of* $\widehat{\mathbb{R}}^n$*: Each* $\gamma \in \mathrm{SL}_2(C_n)$ *induces an orientation-preserving Möbius transformation* $\widetilde{\gamma} : \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$*, and if* $\gamma_1, \gamma_2 \in \mathrm{SL}_2(C_n)$ *induce* $\widetilde{\gamma}_1, \widetilde{\gamma}_2$*, then the product* $\gamma_1\gamma_2$ *induces the composite map* $\widetilde{\gamma}_1 \circ \widetilde{\gamma}_2 : \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$*. Conversely, every orientation-preserving Möbius transformation* $g : \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ *is induced by the pair* $\pm\gamma$ *of Clifford matrices for some* $\gamma \in \mathrm{SL}_2(C_n)$*.*

Proof. See, e.g., [Ahl86] or [Wat93].

□

Notation 2.3.7. From now on we drop the notation $\widetilde{\gamma}$ and denote both the matrix and its induced Möbius transformation by γ.

The action of $\mathrm{GL}_2(C_n)$ and $\mathrm{SL}_2(C_n)$ extends to $\widehat{\mathbb{R}}^{n+1}$ in a natural way: By the definition of the Clifford group Γ_n we have $\Gamma_n \subseteq \Gamma_{n+1}$, which yields $\mathrm{GL}_2(C_n) \subseteq \mathrm{GL}_2(C_{n+1})$ and likewise $\mathrm{SL}_2(C_n) \subseteq \mathrm{SL}_2(C_{n+1})$. Further, $\widehat{\mathbb{R}}^n$ can naturally be considered as a subspace of $\widehat{\mathbb{R}}^{n+1}$. Therefore, the action of any matrix $\gamma \in \mathrm{GL}_2(C_n)$ extends automatically to $\widehat{\mathbb{R}}^{n+1}$ via formula (2.2), and $\widehat{\mathbb{R}}^n$ is mapped on itself. Clearly, the same is true for $\mathrm{SL}_2(C_n)$.

From this extended action one obtains the following equivalent characterization of the group $\mathrm{GL}_2(C_n)$.

Proposition 2.3.8. *The group* $\mathrm{GL}_2(C_n)$ *is equal to the set*

$$\left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in C_n,\ \gamma \text{ induces a bijection } \widehat{\mathbb{R}}^{n+1} \to \widehat{\mathbb{R}}^{n+1},\ P \mapsto (aP+b)\,(cP+d)^{-1}\right\} \tag{2.3}$$

of matrices.

Proof. See, e.g., [Wat93]. In fact, Waterman used (2.3) as definition of $\mathrm{GL}_2(C_n)$ and proved that this agrees with Definition 2.2.1.

□

Lemma 2.3.9. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(C_n)$ *and* $P, Q \in \mathbb{R}^{n+1}$ *such that* $\gamma P, \gamma Q \in \mathbb{R}^{n+1}$. *Then the following assertions hold true.*

(a) We have

$$\gamma P - \gamma Q = \mathrm{pdet}(\gamma)\,((cQ+d)^*)^{-1}\,(P-Q)\,(cP+d)^{-1}.$$

(b) We have the infinitesimal formula

$$d(\gamma P) = \mathrm{pdet}(\gamma)\,((cP+d)^*)^{-1}\,dP\,(cP+d)^{-1}.$$

(c) The linear distortion of γ *is*

$$|\gamma'(P)| = \frac{|\mathrm{pdet}(\gamma)|}{|cP+d|^2}.$$

(d) The norm of $\gamma P - \gamma Q$ *is given by*

$$|\gamma P - \gamma Q| = \frac{|\mathrm{pdet}(\gamma)|\,|P-Q|}{|cP+d|\,|cQ+d|} = |P-Q|\,\sqrt{|\gamma'(P)|}\,\sqrt{|\gamma'(Q)|}.$$

Proof.

(a) We first note that $cP+d \neq 0$ and $cQ+d \neq 0$ as by assumption $\gamma P, \gamma Q \in \mathbb{R}^{n+1}$. Because of $(\gamma Q)^* = \gamma Q$ we can write

$$\begin{aligned}\gamma P - \gamma Q = \gamma P - (\gamma Q)^* &= (aP+b)\,(cP+d)^{-1} - \left((aQ+b)\,(cQ+d)^{-1}\right)^* \\ &= (aP+b)\,(cP+d)^{-1} - (Qc^*+d^*)^{-1}\,(Qa^*+b^*) \\ &= (Qc^*+d^*)^{-1}\Big((Qc^*+d^*)\,(aP+b) - (Qa^*+b^*)\,(cP+d)\Big)\,(cP+d)^{-1}.\end{aligned}$$

Taking into account that $c^*a = (a^*c)^* = a^*c$, $d^*b = (b^*d)^* = b^*d$ and $d^*a - b^*c = ad^* - bc^* \in \mathbb{R}\setminus\{0\}$ by Lemma 2.2.6, the inner bracket equals

$$\begin{aligned}&(Qc^*+d^*)\,(aP+b) - (Qa^*+b^*)\,(cP+d) \\ &\quad = Qc^*aP + Qc^*b + d^*aP + d^*b - Qa^*cP - Qa^*d - b^*cP - b^*d \\ &\quad = (d^*a - b^*c)\,P - Q\,(a^*d - c^*b) = (d^*a-b^*c)\,P - Q\,(d^*a-b^*c)^* \\ &\quad = (ad^*-bc^*)\,(P-Q) = \mathrm{pdet}(\gamma)\,(P-Q).\end{aligned}$$

Finally, this yields

$$\gamma P - \gamma Q = \mathrm{pdet}(\gamma)\,((cQ+d)^*)^{-1}\,(P-Q)\,(cP+d)^{-1}.$$

(b) Letting $Q \to P$ in part (a), we obtain the asserted formula.

(c) By part (b) we have

$$|d(\gamma P)| = |\mathrm{pdet}(\gamma)| \cdot |((cP+d)^*)^{-1}| \cdot |dP| \cdot |(cP+d)^{-1}|,$$

hence

$$|\gamma'(P)| = \frac{|d(\gamma P)|}{|dP|} = \frac{|\mathrm{pdet}(\gamma)|}{|(cP+d)^*|\,|cP+d|} = \frac{|\mathrm{pdet}(\gamma)|}{|cP+d|^2}.$$

(d) The equality

$$|\gamma P - \gamma Q| = \frac{|\mathrm{pdet}(\gamma)|\,|P-Q|}{|cP+d|\,|cQ+d|}$$

is a direct consequence of (a). The second one follows from

$$\sqrt{|\gamma'(P)|} = \frac{\sqrt{|\mathrm{pdet}(\gamma)|}}{|cP+d|} \qquad \text{and} \qquad \sqrt{|\gamma'(Q)|} = \frac{\sqrt{|\mathrm{pdet}(\gamma)|}}{|cQ+d|}.$$

□

The next lemma enables us to conclude that the group $\mathrm{SL}_2(C_n)$ also acts on the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$.

Lemma 2.3.10. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(C_n)$ *and* $P = x_0 + x_1 i_1 + \cdots + x_n i_n \in \mathbb{R}^{n+1}$ *such that* $\gamma P \in \mathbb{R}^{n+1}$. *If we write* $\gamma P = y_0 + y_1 i_1 + \cdots + y_n i_n$, *then*

$$y_n = \frac{\mathrm{pdet}(\gamma)\, x_n}{|cP+d|^2}.$$

Proof. First we note that $\gamma P \in \mathbb{R}^{n+1}$ implies $cP + d \neq 0$. Setting $P_0 := x_0 + \cdots + x_{n-1} i_{n-1}$ so that $P = P_0 + x_n i_n$, we compute

$$\begin{aligned}
\gamma P &= (aP+b)\,(cP+d)^{-1} = \frac{(aP+b)\,(\overline{cP+d})}{|cP+d|^2} \\
&= \frac{(aP_0 + a x_n i_n + b)\,(\overline{cP_0 + c x_n i_n + d})}{|cP+d|^2} = \frac{(aP_0 + a x_n i_n + b)\,\left(\overline{P_0}\,\overline{c} - x_n i_n \overline{c} + \overline{d}\right)}{|cP+d|^2} \\
&= \frac{a|P_0|^2 \overline{c} + aP_0 \overline{d} + b\overline{P_0}\,\overline{c} + b\overline{d} + a x_n^2 \overline{c} + a x_n i_n \overline{P_0}\,\overline{c} - aP_0 x_n i_n \overline{c} + a x_n i_n \overline{d} - b x_n i_n \overline{c}}{|cP+d|^2}.
\end{aligned}$$

By Lemma 2.1.18 we have $i_n \overline{P_0} = P_0^* i_n = P_0 i_n$, $i_n \overline{d} = d^* i_n$ and $i_n \overline{c} = c^* i_n$, which leads to

$$\begin{aligned}
\gamma P &= \frac{a\,(|P_0|^2 + x_n^2)\,\overline{c} + b\overline{d} + aP_0\overline{d} + b\overline{P_0}\,\overline{c} + (ad^* - bc^*)\, x_n i_n}{|cP+d|^2} \\
&= \frac{a|P|^2\overline{c} + b\overline{d} + aP_0\overline{d} + b\overline{P_0}\,\overline{c}}{|cP+d|^2} + \frac{\mathrm{pdet}(\gamma)\, x_n}{|cP+d|^2}\, i_n.
\end{aligned}$$

□

The $(n+1)$-dimensional hyperbolic space

$$\mathbb{H}^{n+1} = \{P = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_n > 0\}$$

can naturally be identified with the subset

$$\{x_0 + x_1 i_1 + \cdots + x_n i_n \mid x_0, \ldots, x_n \in \mathbb{R},\ x_n > 0\} \subseteq V_{n+1}$$

of the Clifford numbers C_{n+1}. If $P \in \mathbb{H}^{n+1}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(C_n)$, then $\gamma P \neq \infty$, since the equation $cP + d = 0$ is satisfied if and only if $P = -c^{-1}d$, which is either ∞ or an element of $\mathbb{R}^n$ by Proposition 2.1.17. Thus, Lemma 2.3.10 shows that $\gamma P \in \mathbb{H}^{n+1}$, provided that $\mathrm{pdet}(\gamma) > 0$. So from Lemma 2.3.10 and its proof we can draw the following conclusion.

Theorem 2.3.11. *The group* $\mathrm{PSL}_2(C_n)$ *is isomorphic to the group of orientation-preserving Möbius transformations of* $\mathbb{H}^{n+1}$*, where* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(C_n)$ *acts on* $\mathbb{H}^{n+1}$ *via* (2.2)*. If* $P = P_0 + x_n i_n \in \mathbb{H}^{n+1}$ *with* $P_0 = x_0 + x_1 i_1 + \cdots + x_{n-1} i_{n-1}$*, then*

$$\gamma P = y_0 + y_1 i_1 + \cdots + y_n i_n = Q_0 + y_n i_n,$$

where

$$\begin{aligned} Q_0 &= \frac{a\overline{c}|P|^2 + b\overline{d} + aP_0\overline{d} + b\overline{P_0}\overline{c}}{|cP+d|^2} = \frac{(aP_0+b)\,(\overline{cP_0+d}) + a\overline{c}x_n^2}{|cP_0+d|^2 + |c|^2 x_n^2}, \\ y_n &= \frac{x_n}{|cP+d|^2} = \frac{x_n}{|cP_0+d|^2 + |c|^2 x_n^2}. \end{aligned}$$

Proof. The only assertions that are left to prove are the identities

$$a\overline{c}|P|^2 + b\overline{d} + aP_0\overline{d} + b\overline{P_0}\overline{c} = (aP_0+b)\,(\overline{cP_0+d}) + a\overline{c}x_n^2$$

and

$$|cP+d|^2 = |cP_0+d|^2 + |c|^2 x_n^2.$$

The first identity follows from

$$\begin{aligned} (aP_0+b)\,(\overline{cP_0+d}) + a\overline{c}x_n^2 &= (aP_0+b)\,(\overline{P_0}\overline{c} + \overline{d}) + a\overline{c}x_n^2 \\ &= a|P_0|^2\overline{c} + aP_0\overline{d} + b\overline{P_0}\overline{c} + b\overline{d} + a\overline{c}x_n^2 \\ &= a\overline{c}\,(|P_0|^2 + x_n^2) + b\overline{d} + aP_0\overline{d} + b\overline{P_0}\overline{c} \\ &= a\overline{c}|P|^2 + b\overline{d} + aP_0\overline{d} + b\overline{P_0}\overline{c}. \end{aligned}$$

For the second identity we note that $cP_0 + d \in C_n$, from which we conclude that

$$|cP+d|^2 = |cP_0 + d + cx_n i_n|^2 = |cP_0+d|^2 + |cx_n i_n|^2 = |cP_0+d|^2 + |c|^2 x_n^2.$$

□

Remark 2.3.12. As we want to work with hyperbolic n-space $\mathbb{H}^n$ instead of hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$ in this thesis, from now on we let $n \in \mathbb{N}$ with $n \geq 2$ and consider the group $\mathrm{SL}_2(C_{n-1})$ acting on $\mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$.

Example 2.3.13. We illustrate the action of elements of $\mathrm{SL}_2(C_{n-1})$ on $\mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ with a few examples.

(a) The matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda \in \mathbb{R} \setminus \{0\}$ acts on $\mathbb{H}^n \cup \mathbb{R}^{n-1}$ as a dilation $P \mapsto \lambda^2 P$ by the factor λ^2, while $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \infty = \infty$.

(b) The element $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ with $\mu \in V_{n-1} \cong \mathbb{R}^{n-1}$ acts on $\mathbb{H}^n \cup \mathbb{R}^{n-1}$ as a translation $P \mapsto P + \mu$ by μ, and we have $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \infty = \infty$.

(c) The matrix $\begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}$ with $a \in \Gamma_{n-1}$, $|a| = 1$, acts on $\mathbb{H}^n \cup \mathbb{R}^{n-1}$ as $P \mapsto aP(a')^{-1} = aPa^*$, and $\begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix} \infty = \infty$. We will see in section 2.5 that this map is a composition of rotations.

(d) The element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acts on $(\mathbb{H}^n \cup \mathbb{R}^{n-1}) \setminus \{0\}$ as inversion $P \mapsto -P^{-1} = -\overline{P}/|P|^2$, while we have $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} 0 = \infty$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \infty = 0$.

The next proposition gives us information about the transitivity of the action of $\mathrm{SL}_2(C_{n-1})$ on $\mathbb{H}^n$ and $\widehat{\mathbb{R}}^{n-1}$.

Proposition 2.3.14. *The following assertions hold true.*

(a) The group $\mathrm{SL}_2(C_{n-1})$ acts transitively on $\mathbb{H}^n$, i.e. for any $P, Q \in \mathbb{H}^n$ there exists an element $\gamma \in \mathrm{SL}_2(C_{n-1})$ such that $\gamma P = Q$.

(b) The group $\mathrm{SL}_2(C_{n-1})$ acts doubly transitively on $\widehat{\mathbb{R}}^{n-1}$, i.e. for any $P, Q, R, S \in \widehat{\mathbb{R}}^{n-1}$ with $P \neq Q$ and $R \neq S$ there exists an element $\gamma \in \mathrm{SL}_2(C_{n-1})$ such that $\gamma P = R$ and $\gamma Q = S$.

Proof.

(a) Let $P = P_0 + x_{n-1} i_{n-1}$ and $Q = Q_0 + y_{n-1} i_{n-1}$. Letting

$$\sigma_P := \begin{pmatrix} \sqrt{x_{n-1}} & P_0/\sqrt{x_{n-1}} \\ 0 & 1/\sqrt{x_{n-1}} \end{pmatrix} \in \mathrm{SL}_2(C_{n-1}), \qquad \sigma_Q := \begin{pmatrix} \sqrt{y_{n-1}} & Q_0/\sqrt{y_{n-1}} \\ 0 & 1/\sqrt{y_{n-1}} \end{pmatrix} \in \mathrm{SL}_2(C_{n-1}),$$

we have

$$\sigma_P i_{n-1} = \left(\sqrt{x_{n-1}}\, i_{n-1} + \frac{P_0}{\sqrt{x_{n-1}}} \right) \sqrt{x_{n-1}} = x_{n-1} i_{n-1} + P_0 = P,$$

and likewise $\sigma_Q i_{n-1} = Q$. Because of $\sigma_P^{-1} P = i_{n-1}$ the matrix $\gamma := \sigma_Q \sigma_P^{-1} \in \mathrm{SL}_2(C_{n-1})$ now satisfies $\gamma P = \sigma_Q \sigma_P^{-1} P = \sigma_Q i_{n-1} = Q$.

(b) We start by showing that there is a matrix $\sigma_{(P,Q)} \in \mathrm{SL}_2(C_{n-1})$ such that $\sigma_{(P,Q)} 0 = P$ and $\sigma_{(P,Q)} \infty = Q$. If $P, Q \in \mathbb{R}^{n-1}$, then we have

$$\begin{pmatrix} (P-Q)^{-1} & -(P-Q)^{-1} P \\ 1 & -Q \end{pmatrix} \in \mathrm{SL}_2(C_{n-1})$$

as $\left((P-Q)^{-1}\right)^{-1} \left(-(P-Q)^{-1} P\right) = -P \in V_n$ and $1^{-1}(-Q) = -Q \in V_n$, where we also apply Proposition 2.1.17, and it has pseudo-determinant

$$(P-Q)^{-1}(-Q)^* + (P-Q)^{-1} P = (P-Q)^{-1}(-Q+P) = 1.$$

Thus, its inverse matrix

$$\sigma_{(P,Q)} := \begin{pmatrix} (P-Q)^{-1} & -(P-Q)^{-1} P \\ 1 & -Q \end{pmatrix}^{-1} = \begin{pmatrix} -Q & P(P-Q)^{-1} \\ -1 & (P-Q)^{-1} \end{pmatrix}$$

is also in $\mathrm{SL}_2(C_{n-1})$ and satisfies the conditions $\sigma_{(P,Q)} 0 = P(P-Q)^{-1} \left((P-Q)^{-1}\right)^{-1} = P$ and $\sigma_{(P,Q)} \infty = (-Q)(-1)^{-1} = Q$.
In the case $P \in \mathbb{R}^{n-1}$ and $Q = \infty$ the element $\sigma_{(P,Q)}$ can be chosen as

$$\sigma_{(P,Q)} := \begin{pmatrix} 1 & P \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(C_{n-1})$$

since $\sigma_{(P,Q)} 0 = P$ and $\sigma_{(P,Q)} \infty = \infty = Q$, whereas for $P = \infty$ and $Q \in \mathbb{R}^{n-1}$ we can set

$$\sigma_{(P,Q)} := \begin{pmatrix} -Q & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(C_{n-1})$$

because $\sigma_{(P,Q)} 0 = \infty = P$ and $\sigma_{(P,Q)} \infty = (-Q)(-1)^{-1} = Q$. Completely analogous we find $\sigma_{(R,S)} \in \mathrm{SL}_2(C_{n-1})$ with $\sigma_{(R,S)} 0 = R$ and $\sigma_{(R,S)} \infty = S$.
Now the matrix $\gamma := \sigma_{(R,S)} \sigma_{(P,Q)}^{-1} \in \mathrm{SL}_2(C_{n-1})$ satisfies $\gamma P = \sigma_{(R,S)} \sigma_{(P,Q)}^{-1} P = \sigma_{(R,S)} 0 = R$ and $\gamma Q = \sigma_{(R,S)} \sigma_{(P,Q)}^{-1} Q = \sigma_{(R,S)} \infty = S$.

□

A further significant result is that the hyperbolic line element, the hyperbolic distance, the hyperbolic Laplace operator and the hyperbolic volume element on $\mathbb{H}^n$, introduced in subsection 1.1.1, are all invariant under the action of $\mathrm{SL}_2(C_{n-1})$.

Proposition 2.3.15. *The following assertions hold true.*

(a) The hyperbolic line element $ds^2_{\mathbb{H}^n}$ on $\mathbb{H}^n$, given in (1.1)*, is invariant under the action of $\mathrm{SL}_2(C_{n-1})$, i.e. for any $\gamma \in \mathrm{SL}_2(C_{n-1})$ and $P \in \mathbb{H}^n$ we have $ds^2_{\mathbb{H}^n}(\gamma P) = ds^2_{\mathbb{H}^n}(P)$.*

(b) For any $P, Q \in \mathbb{H}^n$ and $\gamma \in \mathrm{SL}_2(C_{n-1})$ the hyperbolic distance $d_{\mathbb{H}^n}(P,Q)$, given in (1.2)*, satisfies*

$$d_{\mathbb{H}^n}(\gamma P, \gamma Q) = d_{\mathbb{H}^n}(P,Q).$$

In other words: the elements of $\mathrm{SL}_2(C_{n-1})$ act on $\mathbb{H}^n$ as isometries with respect to the hyperbolic metric.

(c) The hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$, given in (1.4)*, is an $\mathrm{SL}_2(C_{n-1})$-invariant differential operator on $\mathbb{H}^n$, i.e. for any $f \in C^2(\mathbb{H}^n)$, $\gamma \in \mathrm{SL}_2(C_{n-1})$ and $P \in \mathbb{H}^n$ we have $\Delta_{\mathbb{H}^n}(f(\gamma P)) = (\Delta_{\mathbb{H}^n} f)(\gamma P)$.*

(d) The hyperbolic volume element $\mu_{\mathbb{H}^n}(P)$ on $\mathbb{H}^n$, given in (1.5)*, is invariant under the action of $\mathrm{SL}_2(C_{n-1})$, i.e. for any $\gamma \in \mathrm{SL}_2(C_{n-1})$ and $P \in \mathbb{H}^n$ we have $\mu_{\mathbb{H}^n}(\gamma P) = \mu_{\mathbb{H}^n}(P)$.*

Proof.

(a) Let $P = x_0 + x_1 i_1 + \cdots + x_{n-1} i_{n-1} \in \mathbb{H}^n$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(C_{n-1})$. If we write $\gamma P = y_0 + y_1 i_1 + \cdots + y_{n-1} i_{n-1}$, then by Lemma 2.3.9 (b) and Lemma 2.3.10 we have

$$\begin{aligned} ds^2_{\mathbb{H}^n}(\gamma P) &= \frac{dy_0^2 + \cdots + dy_{n-1}^2}{y_{n-1}^2} = \frac{|d(\gamma P)|^2}{y_{n-1}^2} = \frac{|dP|^2\,|cP+d|^4}{|(cP+d)^*|^2\,|cP+d|^2\,x_{n-1}^2} \\ &= \frac{|dP|^2}{x_{n-1}^2} = \frac{dx_0^2 + \cdots + dx_{n-1}^2}{x_{n-1}^2} = ds^2_{\mathbb{H}^n}(P). \end{aligned}$$

This proves that $ds^2_{\mathbb{H}^n}$ is $\mathrm{SL}_2(C_{n-1})$-invariant.

(b) The assertion already follows from the definition of $d_{\mathbb{H}^n}(P,Q)$ and the $\mathrm{SL}_2(C_{n-1})$-invariance of the hyperbolic line element $ds^2_{\mathbb{H}^n}$.
Alternatively, if $P, Q \in \mathbb{H}^n$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(C_{n-1})$, then formula (1.3), Lemma 2.3.9 (d) and Lemma 2.3.10 give us

$$\begin{aligned} \cosh(d_{\mathbb{H}^n}(\gamma P, \gamma Q)) &= 1 + \frac{|\gamma P - \gamma Q|^2}{2\,x_{n-1}(\gamma P)\,x_{n-1}(\gamma Q)} = 1 + \frac{|P-Q|^2\,|cP+d|^2\,|cQ+d|^2}{2\,|cP+d|^2\,|cQ+d|^2\,x_{n-1}(P)\,x_{n-1}(Q)} \\ &= 1 + \frac{|P-Q|^2}{2\,x_{n-1}(P)\,x_{n-1}(Q)} = \cosh(d_{\mathbb{H}^n}(P,Q)). \end{aligned}$$

Applying the inverse hyperbolic cosine to both sides of this equality, we obtain the desired identity.

(c) As $\Delta_{\mathbb{H}^n}$ is derived from the $\mathrm{SL}_2(C_{n-1})$-invariant hyperbolic line element $ds^2_{\mathbb{H}^n}$, this is a consequence of part (a).

(d) The $\mathrm{SL}_2(C_{n-1})$-invariance of $\mu_{\mathbb{H}^n}(P)$ also follows immediately from part (a).

□

2.4. Discrete and cofinite subgroups

Classically, a Fuchsian subgroup of the first kind is a discrete subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ such that a fundamental domain $\mathcal{F}_\Gamma \subseteq \mathbb{H}$ for Γ has finite hyperbolic volume. In this section we look at an analogue of Fuchsian subgroups of the first kind in the more general setting of Clifford matrices. These subgroups of $\mathrm{PSL}_2(C_{n-1})$ were also considered, e.g., in [Söd12].

Since we have $\Gamma_{n-1} \cup \{0\} \subseteq C_{n-1}$ and the Clifford algebra C_{n-1} is a vector space over $\mathbb{R}$ of dimension 2^{n-2}, we can identify $\mathrm{GL}_2(C_{n-1})$ and $\mathrm{SL}_2(C_{n-1})$ as topological spaces with subsets of $\left(\mathbb{R}^{2^{n-2}}\right)^4 = \mathbb{R}^{2^n}$.

Definition 2.4.1. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(C_{n-1})$ we define its *norm* as

$$||\gamma|| := \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}.$$

Then $\mathrm{GL}_2(C_{n-1})$ and $\mathrm{SL}_2(C_{n-1})$ are topological groups with respect to the metric $d(\gamma, \delta) = ||\gamma - \delta||$ ($\gamma, \delta \in \mathrm{GL}_2(C_{n-1})$) that is induced by this norm. The same holds true for the respective quotient spaces $\mathrm{PGL}_2(C_{n-1})$ and $\mathrm{PSL}_2(C_{n-1})$.

Definition 2.4.2. A subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ is *discrete* if the induced topology on Γ is discrete, i.e. if for any $\gamma \in \Gamma$ the set $\{\gamma\}$ is open in Γ.

A subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ is discrete if and only if the identity I is isolated from $\Gamma \setminus \{I\}$, i.e. if there is a neighbourhood of I which contains no element of $\Gamma \setminus \{I\}$. Furthermore, any discrete subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ is countable (see, e.g., [His94], Definition 1.15).

Definition 2.4.3. A subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ acts *discontinuously* on $\mathbb{H}^n$ if for any compact subset $K \subseteq \mathbb{H}^n$ the number

$$\left|\{\gamma \in \Gamma \mid \gamma(K) \cap K \neq \emptyset\}\right|$$

is finite.

Now we have the equivalent characterization of a discrete subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ that it acts discontinuously on $\mathbb{H}^n$.

Proposition 2.4.4. *A subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ is discrete if and only if it acts discontinuously on $\mathbb{H}^n$.*

Proof. See, e.g., [Rat94], Theorem 5.3.5. □

We give a few useful corollaries of this proposition.

Corollary 2.4.5. *Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete subgroup. Then for any $P \in \mathbb{H}^n$ the stabilizer subgroup $\Gamma_P = \{\gamma \in \Gamma \mid \gamma P = P\}$ is finite.*

Proof. As the set $\{P\}$ is compact in $\mathbb{H}^n$ and Γ acts discontinuously on $\mathbb{H}^n$, there exist only finitely many $\gamma \in \Gamma$ with $\gamma(\{P\}) \cap \{P\} \neq \emptyset$, i.e. with $\gamma P = P$.

□

Corollary 2.4.6. *Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete subgroup. Then for any $P \in \mathbb{H}^n$ the orbit $\Gamma P = \{\gamma P \mid \gamma \in \Gamma\}$ is a closed discrete subset of $\mathbb{H}^n$.*

Proof. The assertion follows by combining Proposition 2.4.4 with [Rat94], §5.3., Lemma 5.

□

Corollary 2.4.7. *Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete subgroup. Then for any $P, Q \in \mathbb{H}^n$ there exist at most finitely many $\gamma \in \Gamma$ with $\gamma P = Q$.*

Proof. If there is no $\gamma \in \Gamma$ with $\gamma P = Q$, we are done. Suppose there exists $\gamma_1 \in \Gamma$ with $\gamma_1 P = Q$ and let $\gamma_2 \in \Gamma$ be an arbitrary element with $\gamma_2 P = Q$. Then we have $\gamma_1 P = \gamma_2 P$, so that $\gamma_1^{-1}\gamma_2 P = P$ and $\gamma_1^{-1}\gamma_2 \in \Gamma_P$. This yields $\gamma_2 \in \gamma_1 \Gamma_P$ which is a finite set by Corollary 2.4.5. □

For a discrete subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ acting on $\mathbb{H}^n$ we can consider the set

$$\Gamma\backslash\mathbb{H}^n = \{\Gamma P \mid P \in \mathbb{H}^n\}$$

of orbits under Γ. Endowing $\Gamma\backslash\mathbb{H}^n$ with the quotient topology, i.e. the finest topology in which the natural projection $\pi_\Gamma : \mathbb{H}^n \to \Gamma\backslash\mathbb{H}^n$, $P \mapsto \Gamma P$, is continuous, it becomes a topological space.

The action of a discrete subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ on $\mathbb{H}^n$, respectively the quotient $\Gamma\backslash\mathbb{H}^n$, can be visualized by a fundamental domain.

Definition 2.4.8. Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete subgroup. A subset $\mathcal{F}_\Gamma \subseteq \mathbb{H}^n$ is called a *fundamental domain* for Γ if it satisfies the following conditions:

(i) $\mathcal{F}_\Gamma$ is a domain in $\mathbb{H}^n$, i.e. a non-empty, connected, open subset of $\mathbb{H}^n$.

(ii) If $P, Q \in \mathcal{F}_\Gamma$ with $P \neq Q$, then P and Q are not equivalent with respect to Γ, i.e. there is no $\gamma \in \Gamma$ with $\gamma P = Q$.

(iii) Every orbit ΓP of $P \in \mathbb{H}^n$ by Γ contains a point in the closure $\overline{\mathcal{F}_\Gamma}$ of $\mathcal{F}_\Gamma$ in $\mathbb{H}^n$.

Definition 2.4.9. A convex polyhedron $\mathcal{P}_\Gamma \subseteq \mathbb{H}^n$ whose interior is a fundamental domain for Γ is called *convex fundamental polyhedron* for Γ.

Any discrete subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ admits a fundamental domain and a convex fundamental polyhedron. It can be constructed as a so-called Dirichlet domain and Dirichlet polyhedron.

Definition 2.4.10. Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete subgroup and $Q \in \mathbb{H}^n$ with trivial stabilizer subgroup $\Gamma_Q = \{I\}$. The *Dirichlet domain with center* Q is defined as

$$\mathcal{D}_\Gamma(Q) := \{P \in \mathbb{H}^n \mid d_{\mathbb{H}^n}(P,Q) < d_{\mathbb{H}^n}(\gamma P, Q) \text{ for all } \gamma \in \Gamma, \gamma \neq I\}$$

if $\Gamma \neq \{I\}$, and as $\mathcal{D}_\Gamma(Q) := \mathbb{H}^n$ in case that $\Gamma = \{I\}$. Its closure $\overline{\mathcal{D}_\Gamma(Q)}$ is called *Dirichlet polyhedron with center* Q.

In the situation of the previous definition there always exists such a point $Q \in \mathbb{H}^n$ with trivial stabilizer subgroup. This is, e.g., a consequence of Theorem 6.6.12 in [Rat94].

Proposition 2.4.11. *Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete subgroup and $Q \in \mathbb{H}^n$ with $\Gamma_Q = \{I\}$. Then the Dirichlet domain $\mathcal{D}_\Gamma(Q)$ is a convex fundamental domain for Γ, and the Dirichlet polyhedron $\overline{\mathcal{D}_\Gamma(Q)}$ is a convex fundamental polyhedron for Γ.*

Proof. This follows, e.g., from [Rat94], Theorem 6.6.13 and Theorem 6.7.1. □

Remark 2.4.12. A fundamental domain $\mathcal{F}_\Gamma$ for Γ is not unique because for any $\gamma \in \Gamma$ also $\gamma\mathcal{F}_\Gamma$ is a fundamental domain. However, by the Γ-invariance of the hyperbolic volume element $\mu_{\mathbb{H}^n}(P)$ (which is an immediate consequence of its $\mathrm{SL}_2(C_{n-1})$-invariance) all fundamental domains $\mathcal{F}_\Gamma$ for Γ have the same hyperbolic volume

$$\mathrm{vol}_{\mathbb{H}^n}(\mathcal{F}_\Gamma) = \int_{\mathcal{F}_\Gamma} \mu_{\mathbb{H}^n}(P),$$

see also [Rat94], Theorem 6.7.2.

This fact justifies the following definition.

Definition 2.4.13. Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete subgroup and $\mathcal{F}_\Gamma \subseteq \mathbb{H}^n$ a fundamental domain for Γ. We call $\mathrm{vol}(\Gamma\backslash\mathbb{H}^n) := \mathrm{vol}_{\mathbb{H}^n}(\mathcal{F}_\Gamma)$ the *volume* of $\Gamma\backslash\mathbb{H}^n$, respectively of Γ.

In general this volume might be infinite, which makes it reasonable to introduce a notion for discrete subgroups with finite hyperbolic volume.

Definition 2.4.14. Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete subgroup.

(a) Γ is called *cofinite* if $\mathrm{vol}(\Gamma\backslash\mathbb{H}^n) < \infty$.

(b) Γ is called *cocompact* if it has a fundamental domain $\mathcal{F}_\Gamma$ such that the closure $\overline{\mathcal{F}_\Gamma}$ is compact in $\mathbb{H}^n$.

In the further course of this thesis we will consider discrete, cofinite subgroups of $\mathrm{PSL}_2(C_{n-1})$ and their action on $\mathbb{H}^n$.

Remark 2.4.15. Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. At some points in this thesis we might call a subset $\mathcal{F}_\Gamma \subseteq \mathbb{H}^n$ a *fundamental domain* for Γ if it is connected and contains exactly one point of every orbit ΓP of $P \in \mathbb{H}^n$ by Γ. Since in this case the natural projection $\pi_\Gamma : \mathbb{H}^n \to \Gamma\backslash\mathbb{H}^n$ induces an isomorphism $\mathcal{F}_\Gamma \cong \Gamma\backslash\mathbb{H}^n$, in a slight abuse of notation we will sometimes identify the quotient $\Gamma\backslash\mathbb{H}^n$ with the subset $\mathcal{F}_\Gamma \subseteq \mathbb{H}^n$ and points in $\Gamma\backslash\mathbb{H}^n$ with their preimages in $\mathcal{F}_\Gamma$.

Remark 2.4.16.

(a) For $n = 2$, a discrete subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1}) = \mathrm{PSL}_2(\mathbb{R})$ is called a *Fuchsian subgroup*. It is cofinite if and only if every point on the boundary $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is a limit point of an orbit Γz for some $z \in \mathbb{H}$ (see, e.g., [Miy06], Theorem 1.9.1). A Fuchsian subgroup which satisfies this condition is called *Fuchsian subgroup of the first kind*.

(b) In the case $n = 3$, a discrete subgroup of $\mathrm{PSL}_2(C_{n-1}) = \mathrm{PSL}_2(\mathbb{C})$ is called a *Kleinian group*.

Example 2.4.17. The following examples of discrete, cofinite subgroups $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ will be of particular interest in this thesis.

(a) For $n = 2$, the well-known *modular group* $\Gamma = \mathrm{PSL}_2(\mathbb{Z}) \subseteq \mathrm{PSL}_2(\mathbb{R})$ is discrete, hence it is a Fuchsian subgroup. A fundamental domain for Γ is given by the set

$$\mathcal{F}_\Gamma = \left\{ z \in \mathbb{H} \;\middle|\; |\mathrm{Re}(z)| < \frac{1}{2},\, |z| > 1 \right\},$$

which has the finite hyperbolic volume $\mathrm{vol}_{\mathbb{H}}(\mathcal{F}_\Gamma) = \pi/3$. Thus, $\mathrm{PSL}_2(\mathbb{Z})$ is cofinite and therefore a Fuchsian subgroup of the first kind.

(b) Let $n = 3$ and let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$ and discriminant d_K. Then the subgroup $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K) \subseteq \mathrm{PSL}_2(\mathbb{C})$ is discrete, thus it is a Kleinian group. The quotient $\Gamma\backslash\mathbb{H}^3$ has the finite hyperbolic volume

$$\mathrm{vol}\left(\Gamma\backslash\mathbb{H}^3\right) = \frac{|d_K|^{3/2}}{4\pi^2}\, \zeta_K(2),$$

with the Dedekind zeta function

$$\zeta_K(s) = \sum_{\substack{I \subseteq \mathcal{O}_K \text{ ideal}, \\ I \neq \{0\}}} \mathcal{N}(I)^{-s},$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $\mathcal{N}(I)$ denotes the norm of I. Consequently, the group $\mathrm{PSL}_2(\mathcal{O}_K)$ is cofinite.

In the particular case $K = \mathbb{Q}(i)$ we have $\mathcal{O}_K = \mathbb{Z}[i]$ and $d_K = d_{\mathbb{Q}(i)} = -4$, and we obtain the discrete and cofinite subgroup $\mathrm{PSL}_2(\mathbb{Z}[i]) \subseteq \mathrm{PSL}_2(\mathbb{C})$. The set

$$\mathcal{F}_\Gamma = \left\{ P = z + jr \in \mathbb{H}^3 \;\middle|\; |\mathrm{Re}(z)| < \frac{1}{2},\, 0 < \mathrm{Im}(z) < \frac{1}{2},\, |P| > 1 \right\}$$

is a fundamental domain for $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$ and $\Gamma\backslash\mathbb{H}^3$ has the hyperbolic volume

$$\mathrm{vol}_{\mathbb{H}^3}(\mathcal{F}_\Gamma) = \frac{2\,\zeta_{\mathbb{Q}(i)}(2)}{\pi^2}.$$

2.5. Parabolic, hyperbolic, elliptic and loxodromic elements

According to the usual classification of Möbius transformations based on the number and the location of their fixed points in $\mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$, we now define parabolic, hyperbolic, elliptic and loxodromic elements of $\mathrm{SL}_2(C_{n-1})$ and $\mathrm{PSL}_2(C_{n-1})$. We also give important characterizations of these elements in Remark 2.5.14 and Remark 2.5.15.

Definition 2.5.1. A point $P \in \mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ is a *fixed point* of a Clifford matrix $\gamma \in \mathrm{SL}_2(C_{n-1})$ if $\gamma P = P$.

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(C_{n-1})$, then $P \in \mathbb{H}^n \cup \mathbb{R}^{n-1}$ is a fixed point of γ if and only if it satisfies the equation $aP + b = P\,(cP + d)$, while ∞ is a fixed point of γ if and only if $c = 0$. Further, we note that 0 is a fixed point of γ if and only if $b = 0$.

In contrast to the cases $n = 2$ and $n = 3$, the fixed point equation $aP + b = P\,(cP + d)$ is not trivially solvable anymore for $n \geq 4$ as multiplication in C_{n-1} is no longer commutative. However, the Brouwer fixed point theorem implies that any $\gamma \in \mathrm{SL}_2(C_{n-1})$ has at least one fixed point in $\mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$.

Clearly, for $\gamma = \pm I$ every $P \in \mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ is a fixed point of γ. The non-identity elements of $\mathrm{SL}_2(C_{n-1})$ can be classified in terms of the number and the location of their fixed points in $\mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ as follows.

Definition 2.5.2. Let $\gamma \in \mathrm{SL}_2(C_{n-1})$ with $\gamma \neq \pm I$.

(a) γ is called *parabolic* if it has exactly one fixed point in $\widehat{\mathbb{R}}^{n-1}$ and no fixed points in $\mathbb{H}^n$.

(b) γ is called *loxodromic* if it has exactly two fixed points in $\widehat{\mathbb{R}}^{n-1}$ and no fixed points in $\mathbb{H}^n$.

(c) γ is called *elliptic* if it has a fixed point in $\mathbb{H}^n$. It is called *regular elliptic* if its fixed point in $\mathbb{H}^n$ is unique, and *non-regular elliptic* if it has more than one fixed point in $\mathbb{H}^n$.

An element of $\mathrm{PSL}_2(C_{n-1})$ is called *parabolic*, *loxodromic*, *elliptic*, *regular elliptic* or *non-regular elliptic* if its preimages in $\mathrm{SL}_2(C_{n-1})$ have this property.

Remark 2.5.3. Let $\gamma \in \mathrm{SL}_2(C_{n-1})$ with $\gamma \neq \pm I$. For any Clifford matrix $\sigma \in \mathrm{SL}_2(C_{n-1})$, the elements γ and $\sigma\gamma\sigma^{-1}$ are conjugate. If $P \in \mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ is a fixed point of γ, then

$$(\sigma\gamma\sigma^{-1})(\sigma P) = \sigma\gamma(\sigma^{-1}\sigma)P = \sigma\gamma P = \sigma P,$$

so that $\sigma P \in \mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ is a fixed point of $\sigma\gamma\sigma^{-1}$. Moreover, for $P \in \mathbb{H}^n$ we have $\sigma P \in \mathbb{H}^n$, and $P \in \widehat{\mathbb{R}}^{n-1}$ implies that $\sigma P \in \widehat{\mathbb{R}}^{n-1}$. Conversely, if $P \in \mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ is a fixed point of $\sigma\gamma\sigma^{-1}$, then

$$(\sigma\gamma\sigma^{-1})P = P \quad\Longrightarrow\quad \gamma(\sigma^{-1}P) = (\sigma^{-1}\sigma)\gamma\sigma^{-1}P = \sigma^{-1}\big((\sigma\gamma\sigma^{-1})P\big) = \sigma^{-1}P,$$

which shows that $\sigma^{-1}P \in \mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ is a fixed point of γ. Furthermore, $P \in \mathbb{H}^n$ implies that $\sigma^{-1}P \in \mathbb{H}^n$, and for $P \in \widehat{\mathbb{R}}^{n-1}$ we have $\sigma^{-1}P \in \widehat{\mathbb{R}}^{n-1}$.
This shows that the number and the location (in $\mathbb{H}^n$ or $\widehat{\mathbb{R}}^{n-1}$) of fixed points of γ does not change under conjugation and is the same for the whole conjugacy class

$$[\gamma] = \{\sigma\gamma\sigma^{-1} \mid \sigma \in \mathrm{SL}_2(C_{n-1})\}.$$

Consequently, the classification of an element $\gamma \in \mathrm{SL}_2(C_{n-1})$, $\gamma \neq \pm I$, in Definition 2.5.2 is invariant under conjugation in $\mathrm{SL}_2(C_{n-1})$ and depends only on the conjugacy class $[\gamma]$.

Definition 2.5.4. A matrix $\gamma \in \mathrm{SL}_2(C_{n-1})$ is called *normalized* if it has the form

$$\gamma = \begin{pmatrix} gc & * \\ c & cg \end{pmatrix}$$

with $c \in \Gamma_{n-1}$ and $g \in V_{n-1}$.

The upper right entry of a normalized matrix γ automatically equals $gcg - (c^*)^{-1}$ which easily follows from the condition $\mathrm{pdet}(\gamma) = 1$.

We now give an equivalent characterization of parabolic, loxodromic and elliptic elements based on their conjugacy classes. This classification is a consequence of the considerations in [Ahl85b].

Proposition 2.5.5. *Let $\gamma \in \mathrm{SL}_2(C_{n-1})$ with $\gamma \neq \pm I$. Then the following assertions hold true.*

(a) γ is parabolic if and only if it is conjugate in $\mathrm{SL}_2(C_{n-1})$ to a normalized matrix with fixed point 0, i.e. to a matrix of the form

$$\begin{pmatrix} gc & 0 \\ c & cg \end{pmatrix}$$

with $c \in \Gamma_{n-1}$ and $g \in V_{n-1} \setminus \{0\}$.

(b) γ is loxodromic if and only if it is conjugate in $\mathrm{SL}_2(C_{n-1})$ to a matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & (\lambda^*)^{-1} \end{pmatrix}$$

with $\lambda \in \Gamma_{n-1}$ and $|\lambda| \neq 1$.

(c) γ is elliptic if and only if it is conjugate in $\mathrm{SL}_2(C_n)$ to a matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}$$

with $\lambda \in \Gamma_n$, $|\lambda| = 1$ and $\lambda \neq \pm 1$.

For a parabolic element there is another equivalent characterization in terms of its conjugacy class.

Lemma 2.5.6. *Let $\gamma \in \mathrm{SL}_2(C_{n-1})$ with $\gamma \neq \pm I$. Then γ is parabolic if and only if it is conjugate in $\mathrm{SL}_2(C_{n-1})$ to a matrix of the form*

$$\begin{pmatrix} \lambda & \lambda\mu \\ 0 & \lambda' \end{pmatrix}$$

with $\lambda \in \Gamma_{n-1}$, $|\lambda| = 1$, $\mu \in V_{n-1} \setminus \{0\}$ and $\lambda\mu = \mu\lambda'$.

Proof. By Proposition 2.5.5 (a) we know that γ is parabolic if and only if it is conjugate in $\mathrm{SL}_2(C_{n-1})$ to a normalized matrix with fixed point 0. Since being conjugate in $\mathrm{SL}_2(C_{n-1})$ is an equivalence relation, it suffices to show that any normalized matrix with fixed point 0 is conjugate in $\mathrm{SL}_2(C_{n-1})$ to a matrix of the form $\begin{pmatrix} \lambda & \lambda\mu \\ 0 & \lambda' \end{pmatrix}$ with $\lambda \in \Gamma_{n-1}$, $|\lambda| = 1$, $\mu \in V_{n-1} \setminus \{0\}$ and

$\lambda\mu = \mu\lambda'$.

Let $\gamma_0 := \begin{pmatrix} gc & 0 \\ c & cg \end{pmatrix} \in \mathrm{SL}_2(C_{n-1})$ with $c \in \Gamma_{n-1}$ and $g \in V_{n-1} \setminus \{0\}$ be a normalized matrix with fixed point 0. We note that $1 = \mathrm{pdet}(\gamma_0) = gc(cg)^*$ yields

$$1 = |gc(cg)^*| = |gc|\,|(cg)^*| = |gc|\,|cg| = |cg|^2 \quad \Longrightarrow \quad |cg| = 1$$

and $gc = ((cg)^*)^{-1} = (cg)'$. Letting $\sigma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(C_{n-1})$, we obtain

$$\sigma\gamma_0\sigma^{-1} = \begin{pmatrix} -c & -cg \\ gc & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} cg & -c \\ 0 & gc \end{pmatrix}.$$

Now we set $\lambda := cg$ and $\mu := -g^{-1}$. Then we have $\lambda \in \Gamma_{n-1}$, satisfying $|\lambda| = |cg| = 1$ and $\lambda' = (cg)' = gc$, $\mu \in V_{n-1} \setminus \{0\}$ and $\lambda\mu = -cgg^{-1} = -c = -g^{-1}gc = \mu\lambda'$. Hence, $\sigma\gamma_0\sigma^{-1}$ is indeed of the form $\begin{pmatrix} \lambda & \lambda\mu \\ 0 & \lambda' \end{pmatrix}$ with $\lambda \in \Gamma_{n-1}$, $|\lambda| = 1$, $\mu \in V_{n-1} \setminus \{0\}$ and $\lambda\mu = \mu\lambda'$.

□

Using the characterization of loxodromic elements in Proposition 2.5.5 (b), we are now able to define hyperbolic elements of $\mathrm{SL}_2(C_{n-1})$ and $\mathrm{PSL}_2(C_{n-1})$.

Definition 2.5.7. Let $\gamma \in \mathrm{SL}_2(C_{n-1})$, $\gamma \neq \pm I$, be loxodromic. Then γ is called *hyperbolic* if the Clifford number λ in Proposition 2.5.5 (b) satisfies $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$. An element of $\mathrm{PSL}_2(C_{n-1})$ is called *hyperbolic* if its preimages in $\mathrm{SL}_2(C_{n-1})$ have this property.

In other words: A matrix $\gamma \in \mathrm{SL}_2(C_{n-1})$, $\gamma \neq \pm I$, is hyperbolic if and only if it is conjugate in $\mathrm{SL}_2(C_{n-1})$ to a matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

with $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$.

Remark 2.5.8. The classification of an element $\gamma \in \mathrm{SL}_2(C_{n-1})$, $\gamma \neq \pm I$, as hyperbolic in Definition 2.5.7 is invariant under conjugation in $\mathrm{SL}_2(C_{n-1})$. Thus, it only depends on the conjugacy class $[\gamma]$.

Notation 2.5.9. From now on we call a Clifford matrix $\gamma \in \mathrm{SL}_2(C_{n-1})$, $\gamma \neq \pm I$, *loxodromic* if it is loxodromic but not hyperbolic. An element of $\mathrm{PSL}_2(C_{n-1})$ is called *loxodromic* if its preimages in $\mathrm{SL}_2(C_{n-1})$ are loxodromic but not hyperbolic.

Remark 2.5.10. For $n = 2$, the group $\mathrm{SL}_2(C_{n-1}) = \mathrm{SL}_2(\mathbb{R})$ contains no loxodromic element as any $\lambda \in \Gamma_{n-1} = \mathbb{R} \setminus \{0\}$ with $|\lambda| \neq 1$ immediately satisfies $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$. So any matrix $\gamma \in \mathrm{SL}_2(\mathbb{R})$ with $\gamma \neq \pm I$ is either parabolic, hyperbolic or elliptic.

Note that in the characterization of an elliptic element $\gamma \in \mathrm{SL}_2(C_{n-1})$, $\gamma \neq \pm I$, in Proposition 2.5.5 (c) we have required that γ is conjugate in $\mathrm{SL}_2(C_n) \supseteq \mathrm{SL}_2(C_{n-1})$ to a matrix of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}$ with $\lambda \in \Gamma_n$, $|\lambda| = 1$ and $\lambda \neq \pm 1$, instead of being conjugate to such a matrix in $\mathrm{SL}_2(C_{n-1})$. The next lemma shows what happens if we restrict to conjugacy in $\mathrm{SL}_2(C_{n-1})$.

Lemma 2.5.11. *Let $n \geq 3$ and let $\gamma \in \mathrm{SL}_2(C_{n-1})$ be conjugate in $\mathrm{SL}_2(C_{n-1})$ to a matrix of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}$ with $\lambda \in \Gamma_{n-1}$, $|\lambda| = 1$ and $\lambda \neq \pm 1$. Then γ is non-regular elliptic.*

Proof. That $\gamma \in \mathrm{SL}_2(C_{n-1})$ is elliptic follows already from Proposition 2.5.5 (c). It remains to prove that γ has more than one fixed point in $\mathbb{H}^n$.

Let $\sigma \in \mathrm{SL}_2(C_{n-1})$ such that $\sigma\gamma\sigma^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}$. Then for any $x_{n-1} \in \mathbb{R}$ with $x_{n-1} > 0$ we have

$$\sigma\gamma\sigma^{-1}(x_{n-1}i_{n-1}) = \lambda x_{n-1}i_{n-1}\lambda^* = \lambda\overline{\lambda}x_{n-1}i_{n-1} = |\lambda|^2 x_{n-1}i_{n-1} = x_{n-1}i_{n-1},$$

where for the second equality we used Lemma 2.1.18. Therefore, $x_{n-1}i_{n-1} \in \mathbb{H}^n$ is a fixed point of $\sigma\gamma\sigma^{-1}$ for any $x_{n-1} \in \mathbb{R}$, $x_{n-1} > 0$. This implies that $\sigma^{-1}(x_{n-1}i_{n-1}) \in \mathbb{H}^n$ is a fixed point of γ for any $x_{n-1} \in \mathbb{R}$, $x_{n-1} > 0$.

□

Example 2.5.12. We classify the matrices from Example 2.3.13 and give an example of a loxodromic element.

(a) Any dilation $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$ is hyperbolic with the fixed points 0 and ∞.

(b) Any translation $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ with $\mu \in V_{n-1} \setminus \{0\}$ is parabolic with the fixed point ∞.

(c) Any matrix $\begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}$ with $a \in \Gamma_{n-1}$, $|a| = 1$, $a \neq \pm 1$, is a non-regular elliptic element which leaves any point of the form $x_{n-1}i_{n-1} \in \mathbb{H}^n$ with $x_{n-1} \in \mathbb{R}$, $x_{n-1} > 0$, fixed.

(d) The inversion $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is elliptic with fixed point $i_{n-1} \in \mathbb{H}^n$.
For $n = 2$ it is well-known that elliptic elements have exactly one fixed point in $\mathbb{H}$. Hence, in this case $i_{n-1} = i$ is the unique fixed point in $\mathbb{H}$ and the inversion is a regular elliptic element. If $n \geq 3$, we have

$$\sigma := \frac{1}{2}\begin{pmatrix} 1+i_1 & 1-i_1 \\ -1-i_1 & 1-i_1 \end{pmatrix} \in \mathrm{SL}_2(C_{n-1}) \quad \text{with} \quad \sigma^{-1} = \frac{1}{2}\begin{pmatrix} 1-i_1 & -1+i_1 \\ 1+i_1 & 1+i_1 \end{pmatrix}$$

and

$$\sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sigma^{-1} = \frac{1}{4}\begin{pmatrix} 1+i_1 & 1-i_1 \\ -1-i_1 & 1-i_1 \end{pmatrix}\begin{pmatrix} 1+i_1 & 1+i_1 \\ -1+i_1 & 1-i_1 \end{pmatrix} = \begin{pmatrix} i_1 & 0 \\ 0 & -i_1 \end{pmatrix} = \begin{pmatrix} i_1 & 0 \\ 0 & i_1' \end{pmatrix}.$$

Thus, the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is conjugate in $\mathrm{SL}_2(C_{n-1})$ to a matrix of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}$ with $\lambda \in \Gamma_{n-1}$, $|\lambda| = 1$ and $\lambda \neq \pm 1$. By Lemma 2.5.11, for $n \geq 3$ the inversion is non-regular elliptic. It leaves any point of the form $\sigma^{-1}(x_{n-1}i_{n-1}) \in \mathbb{H}^n$ with $x_{n-1} \in \mathbb{R}$, $x_{n-1} > 0$, fixed.

(e) For $n \geq 3$ the matrix $\begin{pmatrix} 1+i_1 & 0 \\ 0 & \frac{1}{2}(1-i_1) \end{pmatrix} \in \mathrm{SL}_2(C_{n-1})$ is loxodromic with the fixed points 0 and ∞.

We want to understand the action of a parabolic, hyperbolic, elliptic or loxodromic element geometrically. The next lemma gives us this geometric illustration for an elliptic element.

Lemma 2.5.13. *Let $\gamma \in \mathrm{SL}_2(C_{n-1})$ be elliptic. Then γ is conjugate in $\mathrm{SL}_2(C_n)$ to a matrix of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}$ with*

$$\lambda = \prod_{m=0}^{\lfloor \frac{n-3}{2} \rfloor} r_m,$$

where

$$r_m = \cos(\theta_m) + \sin(\theta_m)\, i_{2m}i_{2m+1} = \frac{1}{2}\,(1+i_{2m})\,(\cos(\theta_m) + \sin(\theta_m)\, i_{2m+1})\,(1-i_{2m}) \in \Gamma_n$$

and $\theta_m \in [0, 2\pi)$ $\left(m = 0, \ldots, \lfloor \frac{n-3}{2} \rfloor\right)$. For $m = 0, \ldots, \lfloor \frac{n-3}{2} \rfloor$ the matrix $\begin{pmatrix} r_m & 0 \\ 0 & r_m' \end{pmatrix}$ acts on $\mathbb{H}^n \cup \mathbb{R}^{n-1}$ as a rotation in the i_{2m}-i_{2m+1}-plane by $2\theta_m$. Further, the r_m $\left(m = 0, \ldots, \lfloor \frac{n-3}{2} \rfloor\right)$ commute and so do λ, λ', λ^* and $\overline{\lambda}$.

Proof. See, e.g., [Wat93], Lemma 13.

□

Remark 2.5.14. We can now deduce the following geometric properties of the action of parabolic, hyperbolic, elliptic and loxodromic elements.

(a) Let $\gamma \in \mathrm{SL}_2(C_{n-1})$ be parabolic. By Lemma 2.5.6, γ is conjugate to a matrix of the form

$$\begin{pmatrix} \lambda & \lambda\mu \\ 0 & \lambda' \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}$$

with $\lambda \in \Gamma_{n-1}$, $|\lambda| = 1$, $\mu \in V_{n-1} \setminus \{0\}$ and $\lambda\mu = \mu\lambda'$. Therefore, up to conjugation, γ acts on $\mathbb{H}^n \cup \mathbb{R}^{n-1}$ as a composition of a translation and finitely many rotations.

(b) Let $\gamma \in \mathrm{SL}_2(C_{n-1})$ be hyperbolic. By Proposition 2.5.5 (b) and Definition 2.5.7, up to conjugation, γ acts on $\mathbb{H}^n \cup \mathbb{R}^{n-1}$ as a dilation.

(c) Let $\gamma \in \mathrm{SL}_2(C_{n-1})$ be elliptic. By Lemma 2.5.13, up to conjugation, γ acts on $\mathbb{H}^n \cup \mathbb{R}^{n-1}$ as a composition of finitely many rotations.

(d) Let $\gamma \in \mathrm{SL}_2(C_{n-1})$ be loxodromic. By Proposition 2.5.5 (b) and Notation 2.5.9, γ is conjugate to a matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & (\lambda^*)^{-1} \end{pmatrix} = \begin{pmatrix} |\lambda| & 0 \\ 0 & |\lambda|^{-1} \end{pmatrix} \begin{pmatrix} \lambda/|\lambda| & 0 \\ 0 & \lambda'/|\lambda| \end{pmatrix} = \begin{pmatrix} \lambda/|\lambda| & 0 \\ 0 & \lambda'/|\lambda| \end{pmatrix} \begin{pmatrix} |\lambda| & 0 \\ 0 & |\lambda|^{-1} \end{pmatrix}$$

with $\lambda \in \Gamma_{n-1}$, $|\lambda| \neq 1$ and $\lambda \notin \mathbb{R}$. Consequently, up to conjugation, γ acts on $\mathbb{H}^n \cup \mathbb{R}^{n-1}$ as a composition of a dilation and finitely many rotations.

In [Ahl85b] Ahlfors gave a more detailed characterization of parabolic, hyperbolic and elliptic elements $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(C_{n-1})$ in terms of the entries a, b, c, d, and also determined their fixed points. We do not include these results here and simply refer to [Ahl85b]. Instead we finish this section with a remark about the characterization of parabolic, hyperbolic, elliptic and loxodromic elements in terms of their entries in the cases $n = 2$ and $n = 3$.

Remark 2.5.15. For $n = 2$ and $n = 3$ the classification of an element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(C_{n-1})$, $\gamma \neq \pm I$, simplifies significantly in the respect that it only depends on the trace $\mathrm{tr}(\gamma) = a + d$.

(a) In the case $n = 2$ an element $\gamma \in \mathrm{SL}_2(C_{n-1}) = \mathrm{SL}_2(\mathbb{R})$, $\gamma \neq \pm I$, is parabolic, hyperbolic or elliptic if and only if its trace satisfies $|\mathrm{tr}(\gamma)| = 2$, $|\mathrm{tr}(\gamma)| > 2$ or $|\mathrm{tr}(\gamma)| < 2$, respectively.

(b) If $n = 3$, an element $\gamma \in \mathrm{SL}_2(C_{n-1}) = \mathrm{SL}_2(\mathbb{C})$, $\gamma \neq \pm I$, is parabolic if and only if $|\mathrm{tr}(\gamma)| = 2$ and $\mathrm{tr}(\gamma) \in \mathbb{R}$, hyperbolic if and only if $|\mathrm{tr}(\gamma)| > 2$ and $\mathrm{tr}(\gamma) \in \mathbb{R}$, elliptic if and only if $|\mathrm{tr}(\gamma)| < 2$ and $\mathrm{tr}(\gamma) \in \mathbb{R}$, and loxodromic if and only if $\mathrm{tr}(\gamma) \notin \mathbb{R}$.

2.6. Fixed points and stabilizer subgroups

Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. We want to study the points in $\mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ that are fixed by parabolic, hyperbolic, elliptic or loxodromic elements of Γ. Further, we are also interested in their respective stabilizer subgroups.

Definition 2.6.1. A point $P \in \mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ is called a *parabolic*, *hyperbolic*, *elliptic* or *loxodromic fixed point* of Γ if it is a fixed point of a parabolic, hyperbolic, elliptic or loxodromic element of Γ, respectively.

Definition 2.6.2. By P_Γ and E_Γ we denote a complete set of Γ-inequivalent parabolic and elliptic fixed points of Γ, respectively, and by H_Γ and L_Γ we denote a complete set of Γ-inequivalent pairs of hyperbolic and loxodromic fixed points of Γ, respectively.

Lemma 2.6.3. *For $P \in \widehat{\mathbb{R}}^{n-1}$ the following assertions hold true.*

(a) P cannot be a parabolic and a hyperbolic fixed point of Γ simultaneously.

(b) P cannot be a parabolic and a loxodromic fixed point of Γ simultaneously.

Proof.

(a) Suppose $P \in \widehat{\mathbb{R}}^{n-1}$ is a parabolic and a hyperbolic fixed point of Γ simultaneously, i.e. there are a parabolic element $\gamma_1 \in \Gamma$ and a hyperbolic element $\gamma_2 \in \Gamma$ with $\gamma_1 P = \gamma_2 P = P$. Since P must be the unique fixed point of γ_1, the two matrices have exactly one fixed point in common. But then by Theorem 5.5.4 in [Rat94] the subgroup $\langle \gamma_1, \gamma_2 \rangle \subseteq \Gamma$, which is generated by γ_1 and γ_2, is not discrete. This contradicts that $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ is a discrete subgroup.

(b) Assuming that $P \in \widehat{\mathbb{R}}^{n-1}$ is a parabolic and a loxodromic fixed point of Γ simultaneously, the contradiction follows in the same way as in (a).

□

2.6.1. The parabolic case

In this subsection we first determine the stabilizer subgroup of a point $\eta \in \widehat{\mathbb{R}}^{n-1}$ in $\mathrm{PSL}_2(C_{n-1})$. After that we introduce cusps of Γ and determine the stabilizer subgroup and the translational stabilizer subgroup of a cusp $\eta \in \widehat{\mathbb{R}}^{n-1}$ in Γ.

We start by considering the special case $\eta = \infty$.

Lemma 2.6.4. *The stabilizer subgroup $\mathrm{PSL}_2(C_{n-1})_\infty$ of the point $\infty \in \widehat{\mathbb{R}}^{n-1}$ in $\mathrm{PSL}_2(C_{n-1})$ is given by*

$$\mathrm{PSL}_2(C_{n-1})_\infty = \left\{ \begin{pmatrix} a & a\beta \\ 0 & (a^*)^{-1} \end{pmatrix} \;\middle|\; a \in \Gamma_{n-1},\, \beta \in V_{n-1} \right\} / \{\pm I\}.$$

Proof. Using Proposition 2.1.17, we see that any matrix $\gamma = \begin{pmatrix} a & a\beta \\ 0 & (a^*)^{-1} \end{pmatrix}$ with $a \in \Gamma_{n-1}$ and $\beta \in V_{n-1}$ is an element of $\mathrm{PSL}_2(C_{n-1})$ and clearly fixes ∞.
Conversely, let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(C_{n-1})$ with $\gamma\infty = \infty$. Then $\gamma\infty = ac^{-1}$ gives us $c = 0$, and the condition $1 = \mathrm{pdet}(\gamma) = ad^* - bc^* = ad^*$ implies $a \neq 0$ and $d = (a^*)^{-1}$. Moreover, from $ab^* \in V_{n-1}$ and Proposition 2.1.17 we conclude that $a^{-1}b \in V_{n-1}$. Therefore, we have $\gamma = \begin{pmatrix} a & a\beta \\ 0 & (a^*)^{-1} \end{pmatrix}$ for some $a \in \Gamma_{n-1}$ and $\beta \in V_{n-1}$.

□

Now let $\eta \in \widehat{\mathbb{R}}^{n-1}$ be an arbitrary point on the boundary of $\mathbb{H}^n$. As $\mathrm{PSL}_2(C_{n-1})$ acts (doubly) transitively on $\widehat{\mathbb{R}}^{n-1}$ by Proposition 2.3.14 (b), there exists $\sigma \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma\infty = \eta$. This matrix is unique up to multiplication on the right by elements of $\mathrm{PSL}_2(C_{n-1})_\infty$, since $\sigma_1, \sigma_2 \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma_1\infty = \sigma_2\infty = \eta$ implies $\sigma_2^{-1}\sigma_1\infty = \infty$ and $\sigma_2^{-1}\sigma_1 \in \mathrm{PSL}_2(C_{n-1})_\infty$.

We see now that the stabilizer subgroup $\mathrm{PSL}_2(C_{n-1})_\eta$ is obtained by conjugating the stabilizer subgroup $\mathrm{PSL}_2(C_{n-1})_\infty$ by a matrix $\sigma \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma\infty = \eta$.

Lemma 2.6.5. *Let $\eta \in \widehat{\mathbb{R}}^{n-1}$ and $\sigma \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma\infty = \eta$. Then the stabilizer subgroup $\mathrm{PSL}_2(C_{n-1})_\eta$ of the point η in $\mathrm{PSL}_2(C_{n-1})$ is given by*

$$\mathrm{PSL}_2(C_{n-1})_\eta = \sigma\, \mathrm{PSL}_2(C_{n-1})_\infty\, \sigma^{-1}.$$

In particular, $\mathrm{PSL}_2(C_{n-1})_\eta$ is isomorphic to $\mathrm{PSL}_2(C_{n-1})_\infty$.

Proof. Let $\gamma \in \sigma\, \mathrm{PSL}_2(C_{n-1})_\infty\, \sigma^{-1}$, i.e. $\gamma = \sigma\delta\sigma^{-1}$ for some $\delta \in \mathrm{PSL}_2(C_{n-1})_\infty$. Then we have $\gamma \in \mathrm{PSL}_2(C_{n-1})$ and

$$\gamma\eta = \sigma\delta\sigma^{-1}\eta = \sigma\delta\infty = \sigma\infty = \eta,$$

so that $\gamma \in \mathrm{PSL}_2(C_{n-1})_\eta$. Conversely, if $\gamma \in \mathrm{PSL}_2(C_{n-1})_\eta$, then $\sigma^{-1}\gamma\sigma \in \mathrm{PSL}_2(C_{n-1})$ and

$$\sigma^{-1}\gamma\sigma\infty = \sigma^{-1}\gamma\eta = \sigma^{-1}\eta = \infty.$$

This yields $\sigma^{-1}\gamma\sigma \in \mathrm{PSL}_2(C_{n-1})_\infty$ and $\gamma \in \sigma\, \mathrm{PSL}_2(C_{n-1})_\infty\, \sigma^{-1}$.

Clearly, the subgroups $\mathrm{PSL}_2(C_{n-1})_\eta$ and $\mathrm{PSL}_2(C_{n-1})_\infty$ are isomorphic as they are conjugate to each other.

□

We note that the stabilizer group $\mathrm{PSL}_2(C_{n-1})_\infty$ particularly contains any translation in $\mathrm{PSL}_2(C_{n-1})$, i.e. any matrix of the form $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ with $\beta \in V_{n-1}$, and that the translations form an abelian subgroup of $\mathrm{PSL}_2(C_{n-1})_\infty$. This leads to the following definition.

Definition 2.6.6. We define the *translational subgroup* $N(C_{n-1})$ of $\mathrm{PSL}_2(C_{n-1})$ as

$$N(C_{n-1}) := \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \;\middle|\; \beta \in V_{n-1} \right\} / \{\pm I\}.$$

Considering the group isomorphism

$$N(C_{n-1}) \;\to\; V_{n-1}, \quad \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \cdot \{\pm I\} \;\mapsto\; \beta,$$

onto the additive group V_{n-1} and recalling the identification of V_{n-1} and $\mathbb{R}^{n-1}$, the translational subgroup $N(C_{n-1})$ is isomorphic to the additive group $\mathbb{R}^{n-1}$.

Now let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup and $\eta \in P_\Gamma$ be a parabolic fixed point of Γ. For $\sigma \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma\infty = \eta$ we can look at the conjugate group $\sigma\, N(C_{n-1})\, \sigma^{-1}$ and its intersection with Γ.

First we note that for any $\gamma_1 := \begin{pmatrix} a & a\beta_1 \\ 0 & (a^*)^{-1} \end{pmatrix} \in \mathrm{PSL}_2(C_{n-1})_\infty$ and $\gamma_2 := \begin{pmatrix} 1 & \beta_2 \\ 0 & 1 \end{pmatrix} \in N(C_{n-1})$, where $a \in \Gamma_{n-1}$ and $\beta_1, \beta_2 \in V_{n-1}$, we have

$$\begin{aligned} \gamma_1\gamma_2\gamma_1^{-1} &= \begin{pmatrix} a & a\beta_1 \\ 0 & (a^*)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \beta_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & -\beta_1 a^* \\ 0 & a^* \end{pmatrix} \\ &= \begin{pmatrix} a & a(\beta_1+\beta_2) \\ 0 & (a^*)^{-1} \end{pmatrix} \begin{pmatrix} a^{-1} & -\beta_1 a^* \\ 0 & a^* \end{pmatrix} = \begin{pmatrix} 1 & a\beta_2 a^* \\ 0 & 1 \end{pmatrix} \in N(C_{n-1}). \end{aligned}$$

Here we used that $a\beta_2 a^* = |a|^2\, a\beta_2(a')^{-1} \in V_{n-1}$ by Proposition 2.1.16. Since the matrix σ is unique up to multiplication on the right by elements of $\mathrm{PSL}_2(C_{n-1})_\infty$, we can deduce from the above calculation that the group $\sigma\, N(C_{n-1})\, \sigma^{-1}$ does not depend on the exact choice of σ.

Definition 2.6.7. Let $\eta \in P_\Gamma$ be a parabolic fixed point of Γ and $\sigma \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma\infty = \eta$. We define the *translational stabilizer subgroup* Γ'_η of η in Γ as

$$\Gamma'_\eta := \Gamma \cap \sigma\, N(C_{n-1})\, \sigma^{-1}.$$

Remark 2.6.8. From the definition it is immediate that $\Gamma_\eta \cap \sigma\, N(C_{n-1})\, \sigma^{-1} \subseteq \Gamma'_\eta$. Moreover, for $\gamma \in \Gamma'_\eta$ we have

$$\gamma \in \Gamma \cap \sigma\, N(C_{n-1})\, \sigma^{-1} \subseteq \Gamma \cap \sigma\, \mathrm{PSL}_2(C_{n-1})_\infty\, \sigma^{-1} = \Gamma \cap \mathrm{PSL}_2(C_{n-1})_\eta = \Gamma_\eta,$$

i.e. $\gamma \in \Gamma_\eta \cap \sigma\, N(C_{n-1})\, \sigma^{-1}$, so that $\Gamma'_\eta = \Gamma_\eta \cap \sigma\, N(C_{n-1})\, \sigma^{-1}$. Together with

$$(\sigma^{-1}\, \Gamma\, \sigma)_\infty = \{\sigma^{-1}\gamma\sigma \mid \gamma \in \Gamma,\, \sigma^{-1}\gamma\sigma\infty = \infty\} = \{\sigma^{-1}\gamma\sigma \mid \gamma \in \Gamma,\, \gamma\eta = \eta\} = \sigma^{-1}\, \Gamma_\eta\, \sigma$$

this gives us the identity

$$\sigma^{-1}\, \Gamma'_\eta\, \sigma = \sigma^{-1}\, \Gamma_\eta\, \sigma \cap N(C_{n-1}) = (\sigma^{-1}\, \Gamma\, \sigma)_\infty \cap N(C_{n-1}) = (\sigma^{-1}\, \Gamma\, \sigma)'_\infty.$$

Now $\sigma^{-1}\, \Gamma'_\eta\, \sigma = (\sigma^{-1}\, \Gamma\, \sigma)'_\infty$ is a discrete subgroup of $N(C_{n-1})$, which implies that Γ'_η is isomorphic to a discrete additive subgroup of $\mathbb{R}^{n-1}$, i.e. to a lattice $\Lambda_\eta \subseteq \mathbb{R}^{n-1}$. This lattice is a free abelian group of rank $\leq n-1$.

We want to introduce a notion for a parabolic fixed point $\eta \in P_\Gamma$ for which the free abelian group Γ'_η has full rank $n-1$.

Definition 2.6.9.

(a) Let $\eta \in P_\Gamma$ be a parabolic fixed point of Γ and $\sigma \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma\infty = \eta$. We call η a *cusp* of Γ, if the translational stabilizer subgroup Γ'_η is a free abelian group of rank $n-1$, i.e. if there exists a lattice $\Lambda_\eta \subseteq V_{n-1} \cong \mathbb{R}^{n-1}$ of full rank $n-1$ such that

$$\Gamma'_\eta = \left\{ \sigma \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \sigma^{-1} \;\middle|\; \mu \in \Lambda_\eta \right\} / \{\pm I\}.$$

(b) A *cusp* of $\Gamma\backslash\mathbb{H}^n$ is defined as the Γ-orbit of a cusp of Γ.

Definition 2.6.10. By C_Γ we denote a complete set of Γ-inequivalent cusps of Γ and we set $c_\Gamma := |C_\Gamma|$.

Having made this definition, the quotient $\Gamma\backslash\mathbb{H}^n$ admits c_Γ cusps.

While for discrete subgroups $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ of infinite volume, not necessarily any parabolic fixed point has to be a cusp, for cofinite subgroups we have the following result.

Lemma 2.6.11. *Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup and let $\eta \in P_\Gamma$ be a parabolic fixed point of Γ. Then the translational stabilizer subgroup Γ'_η is a free abelian group of rank $n-1$ and has finite index in the full stabilizer subgroup Γ_η. In particular, η is a cusp of Γ.*

Proof. See, e.g., [Her93], pp. 471-472.

□

Remark 2.6.12. By the last lemma, in our study of discrete and cofinite subgroups $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ in this thesis we can use the terms "parabolic fixed point" and "cusp" synonymously. For brevity we will usually just employ the term "cusp".

Remark 2.6.13. Identifying the quotient $\Gamma\backslash\mathbb{H}^n$ with a convex fundamental polyhedron $\mathcal{P}_\Gamma$ for Γ, we can view the cusps of $\Gamma\backslash\mathbb{H}^n$ as the parabolic fixed points of Γ in $\overline{\mathcal{P}_\Gamma} \cap \widehat{\mathbb{R}}^{n-1}$. Then $\overline{\mathcal{P}_\Gamma}$ consists of a subset of $\mathbb{H}^n$ together with c_Γ vertices on the boundary $\partial\mathbb{H}^n = \widehat{\mathbb{R}}^{n-1}$. The group Γ is cocompact if and only if $c_\Gamma = 0$, i.e. if and only if $\Gamma\backslash\mathbb{H}^n$ has no cusp.

Proposition 2.6.14. *Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. Then $\Gamma\backslash\mathbb{H}^n$ has only finitely many cusps, i.e. the number c_Γ is finite.*

Proof. For torsion-free Γ see, e.g., [Wie77], Theorem 1 (a) (see also [Kel95], Satz 2.2.3).
Now suppose that Γ is not torsion-free. By a theorem of Selberg [Sel60] any discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ has a torsion-free subgroup $\overline{\Gamma}$ of finite index (see also [Bor63], Theorem B (ii)). As Γ is a discrete subgroup of $\mathrm{PSL}_2(C_{n-1})$, also $\overline{\Gamma} \subseteq \Gamma$ is. Moreover, by Theorem 6.7.3 in [Rat94] we have

$$\mathrm{vol}\left(\overline{\Gamma}\backslash\mathbb{H}^n\right) = \left[\Gamma : \overline{\Gamma}\right] \cdot \mathrm{vol}(\Gamma\backslash\mathbb{H}^n) < \infty.$$

This shows that $\overline{\Gamma} \subseteq \mathrm{PSL}_2(C_{n-1})$ is a discrete and cofinite subgroup.
By the result in the torsion-free case, the quotient $\overline{\Gamma}\backslash\mathbb{H}^n$, respectively a convex fundamental polyhedron $\mathcal{P}_{\overline{\Gamma}}$ for $\overline{\Gamma}$, has only finitely many cusps. Since $\overline{\Gamma}$ is a subgroup of Γ, a convex fundamental polyhedron $\mathcal{P}_\Gamma$ for Γ can be chosen as a subset of $\mathcal{P}_{\overline{\Gamma}}$. Hence, also the polyhedron $\mathcal{P}_\Gamma$, respectively the quotient $\Gamma\backslash\mathbb{H}^n$, has only finitely many cusps.
□

Notation 2.6.15. In the following the cusps of Γ are frequently denoted by η_j $(j = 1, \ldots, c_\Gamma)$.

Let $\eta_j \in C_\Gamma$ be a cusp of Γ.

Definition 2.6.16. An element $\sigma_{\eta_j} \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma_{\eta_j}\infty = \eta_j$ is called *parabolic scaling matrix* of η_j if a fundamental parallelotope for the action of $\sigma_{\eta_j}^{-1}\,\Gamma'_{\eta_j}\,\sigma_{\eta_j}$ on $\mathbb{R}^{n-1}$, i.e. a fundamental parallelotope for the lattice Λ_{η_j}, has Euclidean volume 1.

Remark 2.6.17. Let $\sigma_{\eta_j}, \sigma'_{\eta_j} \in \mathrm{PSL}_2(C_{n-1})$ be two parabolic scaling matrices of $\eta_j \in C_\Gamma$, i.e. $\sigma_{\eta_j}\infty = \sigma'_{\eta_j}\infty = \eta_j$ and

$$\Gamma'_{\eta_j} = \left\{ \sigma_{\eta_j} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \sigma_{\eta_j}^{-1} \;\middle|\; \mu \in \Lambda_{\eta_j} \right\} / \{\pm I\} = \left\{ \sigma'_{\eta_j} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \sigma'^{-1}_{\eta_j} \;\middle|\; \mu \in \Lambda'_{\eta_j} \right\} / \{\pm I\},$$

where $\Lambda_{\eta_j}, \Lambda'_{\eta_j} \subseteq V_{n-1} \cong \mathbb{R}^{n-1}$ are lattices of rank $n-1$ whose fundamental parallelotopes both have Euclidean volume 1.
We have already seen that $\sigma_{\eta_j} = \sigma'_{\eta_j}\delta$ for some $\delta \in \mathrm{PSL}_2(C_{n-1})_\infty$. Let $\delta = \begin{pmatrix} \alpha & \alpha\beta \\ 0 & (\alpha^*)^{-1} \end{pmatrix}$ with $\alpha \in \Gamma_{n-1}$ and $\beta \in V_{n-1}$, then for $\mu \in \Lambda_{\eta_j}$ we have

$$\delta \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \delta^{-1} = \begin{pmatrix} \alpha & \alpha\beta \\ 0 & (\alpha^*)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1} & -\beta\alpha^* \\ 0 & \alpha^* \end{pmatrix} = \begin{pmatrix} 1 & \alpha\mu\alpha^* \\ 0 & 1 \end{pmatrix},$$

so that

$$\Gamma'_{\eta_j} = \left\{ \sigma'_{\eta_j}\delta \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \delta^{-1}\sigma'^{-1}_{\eta_j} \;\middle|\; \mu \in \Lambda_{\eta_j} \right\} / \{\pm I\} = \left\{ \sigma'_{\eta_j} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \sigma'^{-1}_{\eta_j} \;\middle|\; \mu \in \alpha\Lambda_{\eta_j}\alpha^* \right\} / \{\pm I\}$$

and $\Lambda'_{\eta_j} = \alpha\Lambda_{\eta_j}\alpha^*$. By the identification $V_{n-1} \cong \mathbb{R}^{n-1}$ and Proposition 2.1.16 the map

$$\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}, \quad \mu \mapsto \alpha\mu\alpha^* = |\alpha'|^2\alpha\mu(\alpha')^{-1},$$

is an isometry if and only if $|\alpha'| = |\alpha| = 1$. Therefore, the lattice $\Lambda'_{\eta_j} = \alpha\Lambda_{\eta_j}\alpha^*$ has a fundamental parallelotope of Euclidean volume 1 if and only if $|\alpha| = 1$.
This shows that a parabolic scaling of η_j is unique up to multiplication on the right by elements of the form $\begin{pmatrix} \alpha & \alpha\beta \\ 0 & \alpha' \end{pmatrix}$ with $\alpha \in \Gamma_{n-1}$, $|\alpha| = 1$, and $\beta \in V_{n-1}$.

We can state the following result about the structure of the stabilizer subgroup and the translational stabilizer subgroup of a cusp.

Proposition 2.6.18. *Let $\sigma_{\eta_j} \in \mathrm{PSL}_2(C_{n-1})$ be a parabolic scaling matrix of the cusp $\eta_j \in C_\Gamma$. Then the following assertions hold true.*

(a) The stabilizer subgroup Γ_{η_j} of η_j in Γ is isomorphic to a discrete subgroup of $\mathrm{PSL}_2(C_{n-1})_\infty$. Moreover, we have

$$\Gamma_{\eta_j} \subseteq \left\{ \sigma_{\eta_j} \begin{pmatrix} \alpha & \alpha\beta \\ 0 & \alpha' \end{pmatrix} \sigma_{\eta_j}^{-1} \;\middle|\; \alpha \in \Gamma_{n-1},\, |\alpha| = 1,\, \beta \in V_{n-1} \right\} / \{\pm I\}.$$

(b) The translational stabilizer subgroup Γ'_{η_j} of η_j in Γ is given by

$$\Gamma'_{\eta_j} = \left\{ \sigma_{\eta_j} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \sigma_{\eta_j}^{-1} \;\middle|\; \mu \in \Lambda_{\eta_j} \right\} / \{\pm I\},$$

where $\Lambda_{\eta_j} \subseteq V_{n-1} \cong \mathbb{R}^{n-1}$ is a lattice of full rank $n-1$. In particular, Γ'_{η_j} is isomorphic to the additive group $\mathbb{Z}^{n-1}$.

Proof.

(a) Because of $\Gamma_{\eta_j} = \Gamma \cap \mathrm{PSL}_2(C_{n-1})_{\eta_j}$ it is immediate that Γ_{η_j} is a discrete subgroup of $\mathrm{PSL}_2(C_{n-1})_{\eta_j}$. This group is, in turn, isomorphic to $\mathrm{PSL}_2(C_{n-1})_\infty$ by Lemma 2.6.5.
Now we have

$$\sigma_{\eta_j}^{-1}\, \Gamma_{\eta_j}\, \sigma_{\eta_j} \subseteq \sigma_{\eta_j}^{-1}\, \mathrm{PSL}_2(C_{n-1})_{\eta_j}\, \sigma_{\eta_j} = \mathrm{PSL}_2(C_{n-1})_\infty,$$

so that any $\gamma \in \sigma_{\eta_j}^{-1}\, \Gamma_{\eta_j}\, \sigma_{\eta_j}$ is of the form $\begin{pmatrix} \alpha & \alpha\beta \\ 0 & (\alpha^*)^{-1} \end{pmatrix}$ with $\alpha \in \Gamma_{n-1}$ and $\beta \in V_{n-1}$. It only remains to show that $|\alpha| = 1$.
Any non-identity element of Γ_{η_j} is either parabolic or elliptic by Lemma 2.6.3. By the conjugacy invariance of the classification of elements, the same is true for the group $\sigma_{\eta_j}^{-1}\, \Gamma_{\eta_j}\, \sigma_{\eta_j}$.
From section 2.5 we know that there exists a matrix $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(C_n)$ such that $\delta\gamma\delta^{-1} = \begin{pmatrix} \lambda & \lambda\mu \\ 0 & \lambda' \end{pmatrix}$ for some $\lambda \in \Gamma_n$ with $|\lambda| = 1$ and $\mu \in V_{n-1}$. This means

$$\begin{aligned} \begin{pmatrix} a\alpha & a\alpha\beta + b(\alpha^*)^{-1} \\ c\alpha & c\alpha\beta + d(\alpha^*)^{-1} \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \alpha\beta \\ 0 & (\alpha^*)^{-1} \end{pmatrix} = \begin{pmatrix} \lambda & \lambda\mu \\ 0 & \lambda' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} \lambda a + \lambda\mu c & \lambda b + \lambda\mu d \\ \lambda' c & \lambda' d \end{pmatrix}. \end{aligned}$$

If $c \neq 0$, then the identity $c\alpha = \lambda' c$ implies $|c|\,|\alpha| = |\lambda'|\,|c|$ and $|\alpha| = |\lambda'| = 1$. If $c = 0$, then $a \neq 0$ must hold and the equality $a\alpha = \lambda a + \lambda\mu c = \lambda a$ yields $|a|\,|\alpha| = |\lambda|\,|a|$ and $|\alpha| = |\lambda| = 1$.

(b) It already follows from the definition of a cusp that

$$\Gamma'_{\eta_j} = \left\{ \sigma_{\eta_j} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \sigma_{\eta_j}^{-1} \;\middle|\; \mu \in \Lambda_{\eta_j} \right\} / \{\pm I\}$$

for some lattice $\Lambda_{\eta_j} \subseteq V_{n-1} \cong \mathbb{R}^{n-1}$ of full rank $n-1$. Then Γ'_{η_j} is isomorphic to the group

$$\left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \;\middle|\; \mu \in \Lambda_{\eta_j} \right\} / \{\pm I\}$$

since they are conjugate by the matrix σ_{η_j}. Further, the map

$$\left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \;\middle|\; \mu \in \Lambda_{\eta_j} \right\} / \{\pm I\} \;\to\; \Lambda_{\eta_j}, \qquad \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \cdot \{\pm I\} \;\mapsto\; \mu,$$

onto the additive group Λ_{η_j} is also a well-defined group isomorphism, and the free abelian group Λ_{η_j} of rank $n-1$ is isomorphic to the additive group $\mathbb{Z}^{n-1}$.

□

Example 2.6.19. We give some examples for the (translational) stabilizer subgroup of a cusp in the 2- and the 3-dimensional case.

(a) Let $n = 2$ and $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind. If $\eta_j \in C_\Gamma$ is a cusp, its stabilizer subgroup Γ_{η_j} is an infinite cyclic group which is generated by some parabolic element $\gamma_{\eta_j} \in \Gamma$, called a *primitive parabolic element*. There exists a parabolic scaling matrix $\sigma_{\eta_j} \in \mathrm{PSL}_2(\mathbb{R})$ such that $\sigma_{\eta_j}\infty = \eta_j$,

$$\gamma_{\eta_j} = \sigma_{\eta_j} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sigma_{\eta_j}^{-1}$$

and

$$\Gamma_{\eta_j} = \left\{ \sigma_{\eta_j} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \sigma_{\eta_j}^{-1} \;\middle|\; m \in \mathbb{Z} \right\} / \{\pm I\}.$$

Because of

$$\Gamma_{\eta_j} \subseteq \left\{ \sigma_{\eta_j} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \sigma_{\eta_j}^{-1} \;\middle|\; \beta \in \mathbb{R} \right\} = \sigma_{\eta_j} N(\mathbb{R}) \, \sigma_{\eta_j}^{-1}$$

we further get $\Gamma'_{\eta_j} = \Gamma_{\eta_j}$.

(b) Let $n = 3$ and $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{C})$ be a discrete and cofinite subgroup. The translational subgroup

$$N(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \;\middle|\; \beta \in \mathbb{C} \right\} / \{\pm I\}$$

of $\mathrm{PSL}_2(\mathbb{C})$ is isomorphic to the additive group $\mathbb{C}$. It is the maximal unipotent subgroup, i.e. the unipotent radical, of the stabilizer subgroup

$$\mathrm{PSL}_2(\mathbb{C})_\infty = \left\{ \begin{pmatrix} a & \beta \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in \mathbb{C} \setminus \{0\},\, \beta \in \mathbb{C} \right\} / \{\pm I\}$$

of ∞ in $\mathrm{PSL}_2(\mathbb{C})$. Let $\eta_j \in C_\Gamma$ be a cusp with parabolic scaling matrix $\sigma_{\eta_j} \in \mathrm{PSL}_2(\mathbb{C})$. There are several possibilities for the stabilizer subgroup Γ_{η_j}. For the case $\eta_j = \infty$ they are summarized in Theorem 2.1.8 in [EGM13]. The translational stabilizer subgroup

$$\Gamma'_{\eta_j} = \Gamma \cap \sigma_{\eta_j} N(\mathbb{C}) \, \sigma_{\eta_j}^{-1} = \Gamma_{\eta_j} \cap \sigma_{\eta_j} N(\mathbb{C}) \, \sigma_{\eta_j}^{-1}$$

is the maximal unipotent subgroup of Γ_{η_j} and consists of the identity and the parabolic elements of Γ_{η_j}. There is a lattice $\Lambda_{\eta_j} \subseteq \mathbb{C}$ of full rank 2 such that

$$\Gamma'_{\eta_j} = \left\{ \sigma_{\eta_j} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \sigma_{\eta_j}^{-1} \;\middle|\; \mu \in \Lambda_{\eta_j} \right\} / \{\pm I\}.$$

The index $[\Gamma_{\eta_j} : \Gamma'_{\eta_j}]$ is finite and restricted to the values $1, 2, 3, 4, 6$.

Now let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$ and class number h_K and let $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K)$. Then Γ has h_K cusps. For $\eta_j \in C_\Gamma$ there is a parabolic scaling matrix $\sigma_{\eta_j} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(K)$ such that $\sigma_{\eta_j}\infty = \eta_j$ and

$$\Gamma_{\eta_j} = \left\{ \sigma_{\eta_j} \begin{pmatrix} u & \mu \\ 0 & u^{-1} \end{pmatrix} \sigma_{\eta_j}^{-1} \;\middle|\; u \in \mathcal{O}_K^\times,\, \mu \in \Lambda_{\eta_j} \right\} / \{\pm I\}$$

with the lattice $\Lambda_{\eta_j} = (a\,\mathcal{O}_K + c\,\mathcal{O}_K)^{-2} \subseteq \mathbb{C}$ of full rank. Moreover, its maximal unipotent subgroup Γ'_{η_j}, consisting of the identity and the parabolic elements of Γ_{η_j}, satisfies

$$\Gamma'_{\eta_j} = \left\{ \sigma_{\eta_j} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \sigma_{\eta_j}^{-1} \;\middle|\; \mu \in \Lambda_{\eta_j} \right\} / \{\pm I\},$$

where Λ_{η_j} is as above. From this we can deduce that $[\Gamma_{\eta_j} : \Gamma'_{\eta_j}] = \frac{1}{2}\,|\mathcal{O}_K^\times|$.

To close this subsection, we note an important fact that will be useful in later chapters to derive results for a discrete and cofinite subgroup with an arbitrary cusp η_j from the special case $\eta_j = \infty$.

Remark 2.6.20. If $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ is a discrete and cofinite subgroup with the cusp η_j and $\sigma \in \mathrm{PSL}_2(C_{n-1})$, then $\sigma^{-1}\,\Gamma\,\sigma \subseteq \mathrm{PSL}_2(C_{n-1})$ is a discrete and cofinite subgroup with the cusp $\sigma^{-1}\eta_j$, the stabilizer subgroup

$$(\sigma^{-1}\,\Gamma\,\sigma)_{\sigma^{-1}\eta_j} = \sigma^{-1}\,\Gamma_{\eta_j}\,\sigma$$

and the translational stabilizer subgroup

$$(\sigma^{-1}\,\Gamma\,\sigma)'_{\sigma^{-1}\eta_j} = \sigma^{-1}\,\Gamma'_{\eta_j}\,\sigma.$$

In particular, if σ_{η_j} is a parabolic scaling matrix of η_j, then $\sigma_{\eta_j}^{-1}\,\Gamma\,\sigma_{\eta_j}$ has the cusp ∞, the stabilizer subgroup

$$\left(\sigma_{\eta_j}^{-1}\,\Gamma\,\sigma_{\eta_j}\right)_\infty = \sigma_{\eta_j}^{-1}\,\Gamma_{\eta_j}\,\sigma_{\eta_j}$$

and the translational stabilizer subgroup

$$\left(\sigma_{\eta_j}^{-1}\,\Gamma\,\sigma_{\eta_j}\right)'_\infty = \sigma_{\eta_j}^{-1}\,\Gamma'_{\eta_j}\,\sigma_{\eta_j}.$$

2.6.2. The hyperbolic case

Now we determine the stabilizer subgroup and the hyperbolic stabilizer subgroup of a pair (Q_1, Q_2) of distinct points $Q_1, Q_2 \in \widehat{\mathbb{R}}^{n-1}$ in $\mathrm{PSL}_2(C_{n-1})$, and of a pair $(Q_1, Q_2) \in H_\Gamma$ of hyperbolic fixed points $Q_1, Q_2 \in \widehat{\mathbb{R}}^{n-1}$ in Γ

At first we treat the special case $Q_1 = 0$, $Q_2 = \infty$.

Lemma 2.6.21. *The stabilizer subgroup* $\mathrm{PSL}_2(C_{n-1})_{(0,\infty)}$ *of the pair* $(0,\infty)$ *of points in* $\mathrm{PSL}_2(C_{n-1})$ *is given by*

$$\mathrm{PSL}_2(C_{n-1})_{(0,\infty)} = \left\{ \begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix} \;\middle|\; a \in \Gamma_{n-1} \right\} / \{\pm I\}.$$

Proof. Clearly, any matrix $\gamma = \begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix}$ with $a \in \Gamma_{n-1}$ is an element of $\mathrm{PSL}_2(C_{n-1})$ and fixes both 0 and ∞.

Conversely, let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(C_{n-1})$ with $\gamma 0 = 0$ and $\gamma\infty = \infty$. Because of $\gamma 0 = bd^{-1}$ and $\gamma\infty = ac^{-1}$ this yields $b = 0$ and $c = 0$. The condition $1 = \mathrm{pdet}(\gamma) = ad^* - bc^* = ad^*$ now implies $a \neq 0$ and $d = (a^*)^{-1}$, so that $\gamma = \begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix}$ with $a \in \Gamma_{n-1}$.

$\square$

An element $\begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix}$ of $\mathrm{PSL}_2(C_{n-1})_{(0,\infty)}$ is hyperbolic if and only if $a \in \mathbb{R} \setminus \{0, \pm 1\}$, which leads to the following corollary.

Corollary 2.6.22. *We have the identity*

$$\{\gamma \in \mathrm{PSL}_2(C_{n-1})_{(0,\infty)} \mid \gamma = I \text{ or } \gamma \text{ is hyperbolic}\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in \mathbb{R} \setminus \{0\} \right\} / \{\pm I\}.$$

Obviously, the right-hand side in the last corollary is again a group.

Definition 2.6.23. We define the *hyperbolic stabilizer subgroup* $\mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(0,\infty)}$ of the pair $(0, \infty)$ of points in $\mathrm{PSL}_2(C_{n-1})$ as

$$\mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(0,\infty)} := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in \mathbb{R} \setminus \{0\} \right\} / \{\pm I\}.$$

Now let $Q_1, Q_2 \in \widehat{\mathbb{R}}^{n-1}$ with $Q_1 \neq Q_2$ be two arbitrary distinct points. As $\mathrm{PSL}_2(C_{n-1})$ acts doubly transitively on $\widehat{\mathbb{R}}^{n-1}$ by Proposition 2.3.14 (b), there exists $\sigma \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma 0 = Q_1$ and $\sigma\infty = Q_2$. This matrix is unique up to multiplication on the right by elements of $\mathrm{PSL}_2(C_{n-1})_{(0,\infty)}$, because for $\sigma_1, \sigma_2 \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma_1 0 = \sigma_2 0 = Q_1$ and $\sigma_1\infty = \sigma_2\infty = Q_2$ we obtain $\sigma_2^{-1}\sigma_1 0 = 0$ and $\sigma_2^{-1}\sigma_1\infty = \infty$, hence $\sigma_2^{-1}\sigma_1 \in \mathrm{PSL}_2(C_{n-1})_{(0,\infty)}$.

Next we prove that we obtain the stabilizer subgroup $\mathrm{PSL}_2(C_{n-1})_{(Q_1,Q_2)}$ of (Q_1, Q_2) by conjugating $\mathrm{PSL}_2(C_{n-1})_{(0,\infty)}$ by such a matrix $\sigma \in \mathrm{PSL}_2(C_{n-1})$.

Lemma 2.6.24. *Let $Q_1, Q_2 \in \widehat{\mathbb{R}}^{n-1}$ with $Q_1 \neq Q_2$ and $\sigma \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma 0 = Q_1$ and $\sigma\infty = Q_2$. Then the stabilizer subgroup $\mathrm{PSL}_2(C_{n-1})_{(Q_1,Q_2)}$ of the pair (Q_1, Q_2) of points in $\mathrm{PSL}_2(C_{n-1})$ is given by*

$$\mathrm{PSL}_2(C_{n-1})_{(Q_1,Q_2)} = \sigma\, \mathrm{PSL}_2(C_{n-1})_{(0,\infty)}\, \sigma^{-1} = \left\{ \sigma \begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix} \sigma^{-1} \;\middle|\; a \in \Gamma_{n-1} \right\} / \{\pm I\}.$$

In particular, $\mathrm{PSL}_2(C_{n-1})_{(Q_1,Q_2)}$ is isomorphic to $\Gamma_{n-1}/\{\pm 1\}$.

Proof. Let $\gamma \in \sigma\, \mathrm{PSL}_2(C_{n-1})_{(0,\infty)}\, \sigma^{-1}$, i.e. $\gamma = \sigma\delta\sigma^{-1}$ for some $\delta \in \mathrm{PSL}_2(C_{n-1})_{(0,\infty)}$. Then we have $\gamma \in \mathrm{PSL}_2(C_{n-1})$ with

$$\gamma Q_1 = \sigma\delta\sigma^{-1}Q_1 = \sigma\delta 0 = \sigma 0 = Q_1 \qquad \text{and} \qquad \gamma Q_2 = \sigma\delta\sigma^{-1}Q_2 = \sigma\delta\infty = \sigma\infty = Q_2,$$

so that $\gamma \in \mathrm{PSL}_2(C_{n-1})_{(Q_1,Q_2)}$.
Conversely, if $\gamma \in \mathrm{PSL}_2(C_{n-1})_{(Q_1,Q_2)}$, then clearly $\sigma^{-1}\gamma\sigma \in \mathrm{PSL}_2(C_{n-1})$. Further, we have

$$\sigma^{-1}\gamma\sigma 0 = \sigma^{-1}\gamma Q_1 = \sigma^{-1}Q_1 = 0 \qquad \text{and} \qquad \sigma^{-1}\gamma\sigma\infty = \sigma^{-1}\gamma Q_2 = \sigma^{-1}Q_2 = \infty,$$

which implies $\sigma^{-1}\gamma\sigma \in \mathrm{PSL}_2(C_{n-1})_{(0,\infty)}$. This shows that $\gamma \in \sigma\, \mathrm{PSL}_2(C_{n-1})_{(0,\infty)}\, \sigma^{-1}$.

The conjugate subgroups $\mathrm{PSL}_2(C_{n-1})_{(Q_1,Q_2)}$ and $\mathrm{PSL}_2(C_{n-1})_{(0,\infty)}$ are clearly isomorphic, and the map

$$\mathrm{PSL}_2(C_{n-1})_{(0,\infty)} \;\to\; \Gamma_{n-1}/\{\pm 1\}, \qquad \begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix} \cdot \{\pm I\} \;\mapsto\; a \cdot \{\pm 1\},$$

is also a well-defined group isomorphism.

□

Definition 2.6.25. Let $Q_1, Q_2 \in \widehat{\mathbb{R}}^{n-1}$ with $Q_1 \neq Q_2$ and $\sigma \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma 0 = Q_1$ and $\sigma\infty = Q_2$. We define the *hyperbolic stabilizer subgroup* $\mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(Q_1,Q_2)}$ of the pair (Q_1, Q_2) of points in $\mathrm{PSL}_2(C_{n-1})$ as

$$\mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(Q_1,Q_2)} := \sigma\, \mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(0,\infty)}\, \sigma^{-1}.$$

As the matrix σ is unique up to multiplication on the right by elements of $\mathrm{PSL}_2(C_{n-1})_{(0,\infty)}$, this definition does not depend on the exact choice of σ. The next lemma shows that it indeed makes sense to denote the introduced group as "hyperbolic stabilizer subgroup" of (Q_1, Q_2).

Lemma 2.6.26. *We have the identity*

$$\mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(Q_1,Q_2)} = \left\{\gamma \in \mathrm{PSL}_2(C_{n-1})_{(Q_1,Q_2)} \;\middle|\; \gamma = I \textit{ or } \gamma \textit{ is hyperbolic}\right\}.$$

Proof. Let $\gamma \in \mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(Q_1,Q_2)}$, then we have $\gamma = \sigma\delta\sigma^{-1}$ for some $\delta \in \mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(0,\infty)}$, i.e. $\delta = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ for some $a \in \mathbb{R}\backslash\{0\}$. In particular, $\gamma \in \sigma\, \mathrm{PSL}_2(C_{n-1})_{(0,\infty)}\, \sigma^{-1} = \mathrm{PSL}_2(C_{n-1})_{(Q_1,Q_2)}$ holds true. If $a = \pm 1$, then clearly $\gamma = \pm I$. And if $a \neq \pm 1$, then γ is hyperbolic as it is conjugate to the matrix $\delta = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $a \in \mathbb{R} \setminus \{0, \pm 1\}$. This proves that γ is contained in the right-hand side.
Conversely, let γ be an element of the right-hand side. As in this case $\gamma \in \mathrm{PSL}_2(C_{n-1})_{(Q_1,Q_2)} = \sigma\, \mathrm{PSL}_2(C_{n-1})_{(0,\infty)}\, \sigma^{-1}$, we know that $\sigma^{-1}\gamma\sigma \in \mathrm{PSL}_2(C_{n-1})_{(0,\infty)}$, i.e. we have $\sigma^{-1}\gamma\sigma = \begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix}$ for some $a \in \Gamma_{n-1}$. If $\gamma = I$, then $\sigma^{-1}\gamma\sigma = I$ and $a = 1$. And if γ is hyperbolic, also $\sigma^{-1}\gamma\sigma$ must be hyperbolic, so that $a \in \mathbb{R} \setminus \{0, \pm 1\}$. In both cases $a \in \mathbb{R} \setminus \{0\}$ holds true, which implies $\sigma^{-1}\gamma\sigma \in \mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(0,\infty)}$ and $\gamma \in \sigma\, \mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(0,\infty)}\, \sigma^{-1} = \mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(Q_1,Q_2)}$. $\square$

Lemma 2.6.27. *Let $Q_1, Q_2 \in \widehat{\mathbb{R}}^{n-1}$ with $Q_1 \neq Q_2$. Then the hyperbolic stabilizer subgroup $\mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(Q_1,Q_2)}$ is isomorphic to $\mathbb{R}^\times/\{\pm 1\}$.*

Proof. We directly see that the groups $\mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(Q_1,Q_2)}$ and $\mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(0,\infty)}$ are isomorphic, since they are conjugate to each other. Moreover, the map

$$\mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(0,\infty)} \;\to\; \mathbb{R}^\times/\{\pm 1\}, \qquad \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \{\pm I\} \;\mapsto\; a \cdot \{\pm 1\},$$

is a well-defined isomorphism of groups. $\square$

Now let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup and $(Q_1, Q_2) \in H_\Gamma$ be a pair of hyperbolic fixed points $Q_1, Q_2 \in \widehat{\mathbb{R}}^{n-1}$ of Γ.

Definition 2.6.28. An element $\sigma_{(Q_1,Q_2)} \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma_{(Q_1,Q_2)} 0 = Q_1$ and $\sigma_{(Q_1,Q_2)}\infty = Q_2$ is called *hyperbolic scaling matrix* of the pair (Q_1, Q_2) of hyperbolic fixed points.

Definition 2.6.29. We define the *hyperbolic stabilizer subgroup* $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$ of (Q_1, Q_2) in Γ as

$$\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} := \Gamma \cap \mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(Q_1,Q_2)}.$$

From Lemma 2.6.26 we immediately derive

$$\begin{aligned}\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} &= \left\{\gamma \in \Gamma \cap \mathrm{PSL}_2(C_{n-1})_{(Q_1,Q_2)} \;\middle|\; \gamma = I \text{ or } \gamma \text{ is hyperbolic}\right\} \\ &= \left\{\gamma \in \Gamma_{(Q_1,Q_2)} \;\middle|\; \gamma = I \text{ or } \gamma \text{ is hyperbolic}\right\}.\end{aligned}$$

Remark 2.6.30. Both the stabilizer subgroup $\Gamma_{(Q_1,Q_2)}$ and the hyperbolic stabilizer subgroup $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$ of a pair $(Q_1,Q_2) \in H_\Gamma$ of hyperbolic fixed points of Γ are non-trivial since there is a hyperbolic element $\gamma \in \Gamma$ with the fixed points Q_1 and Q_2.

We can now prove the following important result.

Proposition 2.6.31. *The following assertions hold true.*

(a) The stabilizer subgroup $\Gamma_{(Q_1,Q_2)}$ of the pair $(Q_1,Q_2) \in H_\Gamma$ in Γ is isomorphic to a non-trivial, discrete subgroup of $\Gamma_{n-1}/\{\pm 1\}$.

(b) The hyperbolic stabilizer subgroup $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$ of the pair $(Q_1,Q_2) \in H_\Gamma$ in Γ is isomorphic to the additive group $\mathbb{Z}$. If $\sigma_{(Q_1,Q_2)} \in \mathrm{PSL}_2(C_{n-1})$ is a hyperbolic scaling matrix of (Q_1,Q_2), then there exists $\mu \in \mathbb{R}$, $\mu > 1$, such that

$$\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} = \left\{ \sigma_{(Q_1,Q_2)} \begin{pmatrix} \mu^m & 0 \\ 0 & \mu^{-m} \end{pmatrix} \sigma^{-1}_{(Q_1,Q_2)} \;\middle|\; m \in \mathbb{Z} \right\} / \{\pm I\}.$$

Proof.

(a) We see that $\Gamma_{(Q_1,Q_2)} = \Gamma \cap \mathrm{PSL}_2(C_{n-1})_{(Q_1,Q_2)}$ is a discrete subgroup of $\mathrm{PSL}_2(C_{n-1})_{(Q_1,Q_2)}$, and the latter is isomorphic to $\Gamma_{n-1}/\{\pm 1\}$ by Lemma 2.6.24. That $\Gamma_{(Q_1,Q_2)}$ is non-trivial has already been noted in the previous remark.

(b) It is immediate that $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$ is a discrete subgroup of $\mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(Q_1,Q_2)}$, with the latter being isomorphic to $\mathbb{R}^\times/\{\pm 1\}$ by Lemma 2.6.27. Since discrete subgroups of $\mathbb{R}^\times/\{\pm 1\}$ are cyclic, $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$ is a cyclic group. Moreover, by the previous remark the hyperbolic stabilizer subgroup $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$ is non-trivial. As any $a \in \mathbb{R}$ with $a \neq \pm 1$ has infinite order in $\mathbb{R}^\times/\{\pm 1\}$, the group $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$ is infinite. Hence, it is isomorphic to the additive group $\mathbb{Z}$.
Let $\sigma_{(Q_1,Q_2)} \in \mathrm{PSL}_2(C_{n-1})$ be a hyperbolic scaling matrix of (Q_1,Q_2). Using

$$\begin{aligned}\sigma^{-1}_{(Q_1,Q_2)}\, \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\, \sigma_{(Q_1,Q_2)} &\subseteq \sigma^{-1}_{(Q_1,Q_2)}\, \mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(Q_1,Q_2)}\, \sigma_{(Q_1,Q_2)} \\ &= \mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(0,\infty)} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in \mathbb{R} \setminus \{0\} \right\} / \{\pm I\},\end{aligned}$$

we find that $\sigma^{-1}_{(Q_1,Q_2)}\, \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\, \sigma_{(Q_1,Q_2)}$ is generated by a matrix of the form $\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$ for some $\mu \in \mathbb{R}$ with $\mu > 0$, i.e.

$$\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} = \left\{ \sigma_{(Q_1,Q_2)} \begin{pmatrix} \mu^m & 0 \\ 0 & \mu^{-m} \end{pmatrix} \sigma^{-1}_{(Q_1,Q_2)} \;\middle|\; m \in \mathbb{Z} \right\} / \{\pm I\}.$$

Since m runs through all integers, it is no restriction to assume $\mu > 1$.

$\square$

Definition 2.6.32. Let $(Q_1,Q_2) \in H_\Gamma$ be a pair of hyperbolic fixed points of Γ. A matrix $\gamma_{(Q_1,Q_2)} \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$ that generates the cyclic group $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$ is called *primitive hyperbolic element.*

Proposition 2.6.31 (b) yields that a primitive hyperbolic element is of the form

$$\gamma_{(Q_1,Q_2)} = \sigma_{(Q_1,Q_2)} \begin{pmatrix} \mu_{(Q_1,Q_2)} & 0 \\ 0 & \mu^{-1}_{(Q_1,Q_2)} \end{pmatrix} \sigma^{-1}_{(Q_1,Q_2)}$$

for some $\mu_{(Q_1,Q_2)} \in \mathbb{R}$ with $\mu_{(Q_1,Q_2)} > 1$, and a hyperbolic scaling matrix $\sigma_{(Q_1,Q_2)} \in \mathrm{PSL}_2(C_{n-1})$ of (Q_1,Q_2).

Notation 2.6.33. Let $(Q_1, Q_2) \in H_\Gamma$ be a pair of hyperbolic fixed points of Γ. By $\mathcal{L}_{(Q_1,Q_2)}$ we denote the unique geodesic in $\mathbb{H}^n$ connecting $Q_1 \in \widehat{\mathbb{R}}^{n-1}$ and $Q_2 \in \widehat{\mathbb{R}}^{n-1}$.

Example 2.6.34. The unique geodesic $\mathcal{L}_{(0,\infty)}$ in $\mathbb{H}^n$ which connects the points 0 and ∞ is the positive x_{n-1}-axis, respectively i_{n-1}-axis, in $\mathbb{H}^n$, i.e.

$$\mathcal{L}_{(0,\infty)} = \{(0, \ldots, 0, x_{n-1}) \mid x_{n-1} > 0\} = \{x_{n-1} i_{n-1} \mid x_{n-1} > 0\}.$$

A hyperbolic scaling matrix $\sigma_{(Q_1,Q_2)} \in \mathrm{PSL}_2(C_{n-1})$ of (Q_1, Q_2) satisfies $\sigma_{(Q_1,Q_2)} 0 = Q_1$ and $\sigma_{(Q_1,Q_2)} \infty = Q_2$, thus, it maps $\mathcal{L}_{(0,\infty)}$ to the geodesic $\mathcal{L}_{(Q_1,Q_2)}$.

Remark 2.6.35. The elements of $\mathrm{PSL}_2(C_{n-1})_{(Q_1,Q_2)}$ and $\mathrm{PSL}_2(C_{n-1})^{\mathrm{hyp}}_{(Q_1,Q_2)}$ leave the geodesic $\mathcal{L}_{(Q_1,Q_2)}$ fixed, but move points $P \in \mathcal{L}_{(Q_1,Q_2)}$ along this geodesic.

Remark 2.6.36. Let $(Q_1, Q_2) \in H_\Gamma$ be a pair of hyperbolic fixed points with hyperbolic scaling matrix $\sigma_{(Q_1,Q_2)} \in \mathrm{PSL}_2(C_{n-1})$ and primitive hyperbolic element

$$\gamma_{(Q_1,Q_2)} = \sigma_{(Q_1,Q_2)} \begin{pmatrix} \mu_{(Q_1,Q_2)} & 0 \\ 0 & \mu^{-1}_{(Q_1,Q_2)} \end{pmatrix} \sigma^{-1}_{(Q_1,Q_2)} \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$$

for some $\mu_{(Q_1,Q_2)} \in \mathbb{R}$ with $\mu_{(Q_1,Q_2)} > 1$. For a point $P \in \mathcal{L}_{(Q_1,Q_2)}$ we have

$$\begin{aligned} l_{(Q_1,Q_2)} :&= d_{\mathbb{H}^n}\left(P, \gamma_{(Q_1,Q_2)} P\right) = d_{\mathbb{H}^n}\left(\sigma^{-1}_{(Q_1,Q_2)} P, \, \mu^2_{(Q_1,Q_2)}\left(\sigma^{-1}_{(Q_1,Q_2)} P\right)\right) \\ &= \log\left(\mu^2_{(Q_1,Q_2)}\right) = 2\log\left(\mu_{(Q_1,Q_2)}\right), \end{aligned}$$

where we used that $\sigma^{-1}_{(Q_1,Q_2)} P \in \mathcal{L}_{(0,\infty)}$. Therefore, we can rewrite $\gamma_{(Q_1,Q_2)}$ as

$$\gamma_{(Q_1,Q_2)} = \sigma_{(Q_1,Q_2)} \begin{pmatrix} \exp\left(\frac{1}{2} l_{(Q_1,Q_2)}\right) & 0 \\ 0 & \exp\left(-\frac{1}{2} l_{(Q_1,Q_2)}\right) \end{pmatrix} \sigma^{-1}_{(Q_1,Q_2)}.$$

Since the geodesic $\mathcal{L}_{(Q_1,Q_2)}$ is mapped to itself under the primitive hyperbolic element $\gamma_{(Q_1,Q_2)}$, the points $P \in \mathcal{L}_{(Q_1,Q_2)}$ and $\gamma_{(Q_1,Q_2)} P \in \mathcal{L}_{(Q_1,Q_2)}$ are identified in $\Gamma \backslash \mathbb{H}^n$ and $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \mathbb{H}^n$. This means that the image $\pi_{\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}}(\mathcal{L}_{(Q_1,Q_2)})$ under the natural projection $\pi_{\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}} : \mathbb{H}^n \to \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \mathbb{H}^n$ is a closed geodesic in $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \mathbb{H}^n$ of hyperbolic length $l_{(Q_1,Q_2)}$.

Notation 2.6.37. We denote the closed geodesic $\pi_{\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}}(\mathcal{L}_{(Q_1,Q_2)})$ in $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \mathbb{H}^n$ by $L_{(Q_1,Q_2)}$.

Definition 2.6.38. The number $l_{(Q_1,Q_2)}$ is called the *length* of the pair (Q_1, Q_2) of hyperbolic fixed points, of the geodesic $\mathcal{L}_{(Q_1,Q_2)}$, or of the closed geodesic $L_{(Q_1,Q_2)}$, respectively.

Finally, we note an important fact that will be useful in the coming chapters to deduce results for a discrete and cofinite subgroup with an arbitrary pair (Q_1, Q_2) of hyperbolic fixed points from the special case $Q_1 = 0$, $Q_2 = \infty$.

Remark 2.6.39. If $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ is a discrete and cofinite subgroup with the pair (Q_1, Q_2) of hyperbolic fixed points and $\sigma \in \mathrm{PSL}_2(C_{n-1})$, then $\sigma^{-1}\,\Gamma\,\sigma \subseteq \mathrm{PSL}_2(C_{n-1})$ is a discrete and cofinite subgroup with the pair $(\sigma^{-1} Q_1, \sigma^{-1} Q_2)$ of hyperbolic fixed points, the stabilizer subgroup

$$(\sigma^{-1}\,\Gamma\,\sigma)_{(\sigma^{-1} Q_1, \sigma^{-1} Q_2)} = \sigma^{-1}\,\Gamma_{(Q_1,Q_2)}\,\sigma$$

and the hyperbolic stabilizer subgroup

$$(\sigma^{-1}\,\Gamma\,\sigma)^{\mathrm{hyp}}_{(\sigma^{-1} Q_1, \sigma^{-1} Q_2)} = \sigma^{-1}\,\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\,\sigma.$$

In particular, if $\sigma_{(Q_1,Q_2)}$ is a hyperbolic scaling matrix of (Q_1, Q_2), then $\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)}$ has the pair $(0,\infty)$ of hyperbolic fixed points, the stabilizer subgroup

$$\left(\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)}\right)_{(0,\infty)} = \sigma^{-1}_{(Q_1,Q_2)}\,\Gamma_{(Q_1,Q_2)}\,\sigma_{(Q_1,Q_2)}$$

and the hyperbolic stabilizer subgroup

$$\left(\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)}\right)^{\mathrm{hyp}}_{(0,\infty)} = \sigma^{-1}_{(Q_1,Q_2)}\,\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\,\sigma_{(Q_1,Q_2)}.$$

2.6.3. The elliptic case

We now turn to determine the stabilizer subgroup of a point $Q \in \mathbb{H}^n$ in $\mathrm{PSL}_2(C_{n-1})$ and Γ. Since we will define elliptic Eisenstein series associated to arbitrary points in $\mathbb{H}^n$ in section 4.2, we do not restrict to Q being an elliptic fixed point of Γ. We start with the special case $Q = i_{n-1} \in \mathbb{H}^n$. The assertion of the following lemma was also noted in [Wat93], p. 97.

Lemma 2.6.40. *The stabilizer subgroup* $\mathrm{PSL}_2(C_{n-1})_{i_{n-1}}$ *of the point* $i_{n-1} \in \mathbb{H}^n$ *in* $\mathrm{PSL}_2(C_{n-1})$ *is given by*

$$\mathrm{PSL}_2(C_{n-1})_{i_{n-1}} = \mathrm{PSU}_2(C_{n-1}) := \mathrm{SU}_2(C_{n-1})/\{\pm I\},$$

where

$$\mathrm{SU}_2(C_{n-1}) := \left\{ \begin{pmatrix} a & b \\ -b' & a' \end{pmatrix} \;\middle|\; a, b \in \Gamma_{n-1} \cup \{0\},\ ab^* \in V_{n-1},\ |a|^2 + |b|^2 = 1 \right\}.$$

Proof. First we recall from Theorem 2.3.11 that the action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(C_{n-1})$ on a point $P = x_0 + x_1 i_1 + \cdots + x_{n-1} i_{n-1} = P_0 + x_{n-1} i_{n-1} \in \mathbb{H}^n$ is given by

$$P \mapsto \gamma P = y_0 + y_1 i_1 + \cdots + y_{n-1} i_{n-1} = Q_0 + y_{n-1} i_{n-1},$$

where

$$Q_0 = \frac{(aP_0 + b)\,(\overline{cP_0 + d}) + a\bar{c}x_{n-1}^2}{|cP_0 + d|^2 + |c|^2 x_{n-1}^2}, \qquad y_{n-1} = \frac{x_{n-1}}{|cP_0 + d|^2 + |c|^2 x_{n-1}^2}.$$

If $\gamma = \begin{pmatrix} a & b \\ -b' & a' \end{pmatrix} \in \mathrm{PSU}_2(C_{n-1})$, then $-b'(a')^* = -(ba^*)' = -((ab^*)^*)' = -(ab^*)' \in V_{n-1}$ and $\mathrm{pdet}(\gamma) = a\bar{a} + b\bar{b} = |a|^2 + |b|^2 = 1$, so that $\gamma \in \mathrm{PSL}_2(C_{n-1})$. We have $\gamma i_{n-1} = Q_0 + y_{n-1} i_{n-1}$ with

$$\begin{aligned} Q_0 &= \frac{b\overline{a'} - a\overline{b'}}{|a'|^2 + |-b'|^2} = \frac{ba^* - ab^*}{|a|^2 + |b|^2} = ba^* - ab^* = (ab^*)^* - ab^* = 0, \\ y_{n-1} &= \frac{1}{|a'|^2 + |-b'|^2} = \frac{1}{|a|^2 + |b|^2} = 1, \end{aligned}$$

i.e. $\gamma i_{n-1} = i_{n-1}$. This proves that $\gamma \in \mathrm{PSL}_2(C_{n-1})_{i_{n-1}}$.

Conversely, let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(C_{n-1})$ with $\gamma i_{n-1} = Q_0 + y_{n-1} i_{n-1} = i_{n-1}$. By the formulas

$$0 = Q_0 = \frac{b\bar{d} + a\bar{c}}{|d|^2 + |c|^2} \qquad \text{and} \qquad 1 = y_{n-1} = \frac{1}{|d|^2 + |c|^2},$$

this implies $a\bar{c} + b\bar{d} = 0$ and $|c|^2 + |d|^2 = 1$. Using these conditions and $ad^* - bc^* = 1$, we compute

$$\begin{aligned} 0 &= (a\bar{c} + b\bar{d})\,cd^* = a\bar{c}cd^* + b\bar{d}cd^* = a\bar{c}cd^* + b\bar{d}(cd^*)^* = a\bar{c}cd^* + b\bar{d}dc^* \\ &= a|c|^2 d^* + b|d|^2 c^* = ad^*|c|^2 + bc^*|d|^2 = ad^*\left(1 - |d|^2\right) + bc^*|d|^2 \\ &= ad^* - (ad^* - bc^*)\,|d|^2 = ad^* - |d|^2 = (a - d')\,d^*. \end{aligned}$$

Hence, we have $a - d' = 0$ or $d = 0$. The first case means $d = a'$. In the latter case $c \neq 0$ must hold, so $0 = a\bar{c} + b\bar{d} = a\bar{c}$ gives us $a = 0$ and again $d = a'$ is satisfied. Now we employ this to obtain

$$0 = 1 - ad^* + bc^* = 1 - d'd^* + bc^* = 1 - |d|^2 + bc^* = |c|^2 + bc^* = (c' + b)\, c^*,$$

i.e. $c' + b = 0$ or $c = 0$. In the first case we have $c = -b'$. In the latter case $d \neq 0$ must hold, and from $0 = a\bar{c} + b\bar{d} = b\bar{d}$ we obtain $b = 0$, which shows that again $c = -b'$.
Moreover, we have

$$1 = |c|^2 + |d|^2 = |-b'|^2 + |a'|^2 = |a|^2 + |b|^2.$$

The conditions $a, b \in \Gamma_{n-1} \cup \{0\}$ and $ab^* \in V_{n-1}$ follow immediately from the definition of $\mathrm{PSL}_2(C_{n-1})$. Overall, this yields $\gamma \in \mathrm{PSU}_2(C_{n-1})$.

□

Definition 2.6.41. We call $\mathrm{SU}_2(C_{n-1})$ the *special unitary group* over the Clifford numbers C_{n-1}.

Now let $Q \in \mathbb{H}^n$ be an arbitrary point. Since $\mathrm{PSL}_2(C_{n-1})$ acts transitively on $\mathbb{H}^n$ by Proposition 2.3.14 (a), there exists $\sigma \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma i_{n-1} = Q$. This matrix is unique up to multiplication on the right by elements of $\mathrm{PSU}_2(C_{n-1})$, as for $\sigma_1, \sigma_2 \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma_1 i_{n-1} = \sigma_2 i_{n-1} = Q$ we get $\sigma_2^{-1}\sigma_1 i_{n-1} = i_{n-1}$ and so $\sigma_2^{-1}\sigma_1 \in \mathrm{PSL}_2(C_{n-1})_{i_{n-1}} = \mathrm{PSU}_2(C_{n-1})$.

The next lemma shows that the stabilizer subgroup $\mathrm{PSL}_2(C_{n-1})_Q$ of Q is simply obtained by conjugating $\mathrm{PSU}_2(C_{n-1})$ by such an element σ.

Lemma 2.6.42. *Let $Q \in \mathbb{H}^n$ and $\sigma \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma i_{n-1} = Q$. Then the stabilizer subgroup $\mathrm{PSL}_2(C_{n-1})_Q$ of the point Q in $\mathrm{PSL}_2(C_{n-1})$ is given by*

$$\mathrm{PSL}_2(C_{n-1})_Q = \sigma\, \mathrm{PSU}_2(C_{n-1})\, \sigma^{-1}.$$

In particular, $\mathrm{PSL}_2(C_{n-1})_Q$ is isomorphic to $\mathrm{PSU}_2(C_{n-1})$.

Proof. If $\gamma \in \sigma\, \mathrm{PSU}_2(C_{n-1})\, \sigma^{-1}$, then $\gamma = \sigma\delta\sigma^{-1}$ for some $\delta \in \mathrm{PSU}_2(C_{n-1}) = \mathrm{PSL}_2(C_{n-1})_{i_{n-1}}$. This implies that $\gamma \in \mathrm{PSL}_2(C_{n-1})$ with

$$\gamma Q = \sigma\delta\sigma^{-1} Q = \sigma\delta i_{n-1} = \sigma i_{n-1} = Q,$$

which shows that $\gamma \in \mathrm{PSL}_2(C_{n-1})_Q$.
Conversely, let $\gamma \in \mathrm{PSL}_2(C_{n-1})_Q$. Then we have $\sigma^{-1}\gamma\sigma \in \mathrm{PSL}_2(C_{n-1})$ with

$$\sigma^{-1}\gamma\sigma i_{n-1} = \sigma^{-1}\gamma Q = \sigma^{-1} Q = i_{n-1}.$$

This means $\sigma^{-1}\gamma\sigma \in \mathrm{PSL}_2(C_{n-1})_{i_{n-1}} = \mathrm{PSU}_2(C_{n-1})$, so that $\gamma \in \sigma\, \mathrm{PSU}_2(C_{n-1})\, \sigma^{-1}$.
The isomorphy of $\mathrm{PSL}_2(C_{n-1})_Q$ and $\mathrm{PSU}_2(C_{n-1})$ is immediate.

□

Remark 2.6.43. For $Q \in \mathbb{H}^n$ the elements of the stabilizer subgroup $\mathrm{PSL}_2(C_{n-1})_Q$ act on $\mathbb{H}^n$ as hyperbolic rotations around Q, since for $P \in \mathbb{H}^n$ and $\gamma \in \mathrm{PSL}_2(C_{n-1})_Q$ we have

$$d_{\mathbb{H}^n}(\gamma P, Q) = d_{\mathbb{H}^n}(P, \gamma^{-1} Q) = d_{\mathbb{H}^n}(P, Q).$$

Now let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup and $Q \in \mathbb{H}^n$ be a point which is not necessarily an elliptic fixed point of Γ.

Definition 2.6.44. An element $\sigma_Q \in \mathrm{PSL}_2(C_{n-1})$ with $\sigma_Q i_{n-1} = Q$ is called *elliptic scaling matrix* of the point Q .

Considering the stabilizer subgroup of Q in Γ, we can now derive the following result.

Proposition 2.6.45. *The stabilizer subgroup Γ_Q of the point $Q \in \mathbb{H}^n$ in Γ is isomorphic to a finite subgroup of $\mathrm{PSU}_2(C_{n-1})$. Moreover, any non-trivial element of Γ_Q is elliptic.*

Proof. First we note that $\Gamma_Q = \Gamma \cap \mathrm{PSL}_2(C_{n-1})_Q$ is a discrete subgroup of $\mathrm{PSL}_2(C_{n-1})_Q$. Thus, Γ_Q is isomorphic to a discrete subgroup of $\mathrm{PSU}_2(C_{n-1})$ by Lemma 2.6.42. From Corollary 2.4.5 we conclude that Γ_Q is even a finite group.
If $\gamma \in \Gamma_Q$, $\gamma \neq I$, is a non-trivial matrix which has the fixed point $Q \in \mathbb{H}^n$, then γ is elliptic by Definition 2.5.2 (c).

□

Remark 2.6.46. The stabilizer subgroup Γ_Q is non-trivial if and only if there exists an element $\gamma \in \Gamma$, $\gamma \neq I$, with $\gamma Q = Q$, i.e. if and only if Q is an elliptic fixed point of Γ.

Example 2.6.47. Let $n = 2$ and $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind. The stabilizer subgroup Γ_w of an elliptic fixed point $w \in E_\Gamma$ is a finite cyclic group which is generated by some elliptic element $\gamma_w \in \Gamma$. The matrix γ_w is called *primitive elliptic element.* If $\sigma_w \in \mathrm{PSL}_2(\mathbb{R})$ is an elliptic scaling matrix, i.e. $\sigma_w i = w$, then γ_w is of the form

$$\gamma_w = \sigma_w \begin{pmatrix} \cos\left(\frac{\pi}{n_w}\right) & \sin\left(\frac{\pi}{n_w}\right) \\ -\sin\left(\frac{\pi}{n_w}\right) & \cos\left(\frac{\pi}{n_w}\right) \end{pmatrix} \sigma_w^{-1},$$

where the number $n_w = |\Gamma_w| \in \mathbb{N}$ denotes the order of the elliptic fixed point w in Γ.

3. Linear operators and automorphic functions

This chapter acts as a collection of several topics and results that we will need in the further course of this thesis. We begin by determining the radial eigenfunctions of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ in elliptic coordinates in the first section. After that, we consider $\mathrm{PSL}_2(C_{n-1})$-invariant integral operators in the second section, and derive a result about eigenfunctions of these operators. In the chapter's third section we introduce automorphic functions in the upper half-space $\mathbb{H}^n$. Important examples for these functions are parabolic Eisenstein series whose definition and basic properties we give in the fourth section. In the fifth and final section we deal with the spectral expansion of a square-integrable automorphic function on $\Gamma\backslash\mathbb{H}^n$ and give some conditions for its convergence.

3.1. Radial eigenfunctions of the hyperbolic Laplace operator

At several points in this thesis we will need the radial eigenfunctions of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ in elliptic coordinates, so we determine them in this section. This was done before, e.g. by Awonusika in section 4.1 of his PhD thesis [Awo16].

Definition 3.1.1.

(a) A function $f : \mathbb{H}^n \to \mathbb{C}$, $P \mapsto f(P)$, is called *radial at* $Q \in \mathbb{H}^n$ if it only depends on the hyperbolic distance $d_{\mathbb{H}^n}(P,Q)$, i.e. if it can be written as $f(d_{\mathbb{H}^n}(P,Q))$.

(b) A function $f : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{C}$, $(P,Q) \mapsto f(P,Q)$, is called *radial at* $Q \in \mathbb{H}^n$ if, as a function of P, it only depends on the hyperbolic distance $d_{\mathbb{H}^n}(P,Q)$, i.e. if it can be written as $f(d_{\mathbb{H}^n}(P,Q),Q)$.

The eigenvalue problem in $\mathbb{H}^n$ is the equation

$$\Delta_{\mathbb{H}^n} G = \lambda G, \tag{3.1}$$

where $G(P)$ is an eigenfunction of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ with eigenvalue $\lambda \in \mathbb{C}$. From Lemma 1.2.8 (c) we know that in elliptic coordinates $\varrho := \varrho_Q(P)$, $\zeta := \zeta_Q(P)$ centered at some point $Q \in \mathbb{H}^n$ the hyperbolic Laplace operator takes the form

$$\Delta_{\mathbb{H}^n} = -\frac{\partial^2}{\partial\varrho^2} - (n-1)\,\frac{1}{\tanh(\varrho)}\,\frac{\partial}{\partial\varrho} - \frac{1}{\sinh(\varrho)^2}\,\Delta_{\mathbb{S}^{n-1}},$$

with its radial part $\Delta_{\mathbb{H}^n,\mathrm{rad}}$ given by

$$\Delta_{\mathbb{H}^n,\mathrm{rad}} = -\frac{\partial^2}{\partial\varrho^2} - (n-1)\,\frac{1}{\tanh(\varrho)}\,\frac{\partial}{\partial\varrho}.$$

If we assume a product solution of (3.1) of the form $G(\varrho,\zeta) = \Theta(\varrho)\,\Psi(\zeta)$, we obtain

$$\Psi\,\frac{\partial^2\Theta}{\partial\varrho^2} + (n-1)\,\Psi\,\frac{1}{\tanh(\varrho)}\,\frac{\partial\Theta}{\partial\varrho} + \frac{\Theta}{\sinh(\varrho)^2}\,\Delta_{\mathbb{S}^{n-1}}\Psi = -\lambda\,\Theta\,\Psi.$$

By multiplication with $\sinh(\varrho)^2/(\Theta\,\Psi)$ on both sides, this can be rewritten as

$$\frac{\sinh(\varrho)^2}{\Theta}\,\frac{\partial^2\Theta}{\partial\varrho^2} + \frac{n-1}{\Theta}\,\cosh(\varrho)\,\sinh(\varrho)\,\frac{\partial\Theta}{\partial\varrho} + \lambda\,\sinh(\varrho)^2 = -\frac{1}{\Psi}\,\Delta_{\mathbb{S}^{n-1}}\Psi.$$

Now the left-hand side depends only on ϱ, while the right-hand side depends only on ζ. Thus, both sides must be equal to some constant σ^2, resulting in the two separate equations

$$\frac{\sinh(\varrho)^2}{\Theta}\,\frac{d^2\Theta}{d\varrho^2} + \frac{n-1}{\Theta}\cosh(\varrho)\,\sinh(\varrho)\,\frac{d\Theta}{d\varrho} + \lambda\,\sinh(\varrho)^2 = \sigma^2 \tag{3.2}$$

and

$$\Delta_{\mathbb{S}^{n-1}}\Psi = -\sigma^2\,\Psi, \tag{3.3}$$

where

$$\sigma^2 = m(m+n-2) \qquad (m \in \mathbb{N}_0)$$

are the eigenvalues of $\Delta_{\mathbb{S}^{n-1}}$ corresponding to the eigenfunctions Ψ. The equation (3.2) is an ordinary differential equation.

Remark 3.1.2. In the special case that G is radial at Q, i.e. that $G(P) = G(\varrho)$ only depends on $\varrho = d_{\mathbb{H}^n}(P,Q)$, we have $\Psi \equiv 1$ and $\sigma^2 = 0$. Hence, equation (3.3) becomes redundant and the eigenvalue problem (3.1) reduces to the ordinary differential equation (3.2) with right-hand side equal to zero.

If we multiply equation (3.2) by $\Theta\,\sinh(\varrho)^{-2}$, we get

$$\frac{d^2\Theta}{d\varrho^2} + (n-1)\,\frac{1}{\tanh(\varrho)}\,\frac{d\Theta}{d\varrho} + \left(\lambda - \frac{\sigma^2}{\sinh(\varrho)^2}\right)\Theta = 0.$$

This homogeneous second-order linear differential equation is solved in the following lemma.

Lemma 3.1.3. *Let $P_\nu^\mu(z)$ denote the associated Legendre function of the first kind of degree ν and order μ (see* (A.24)*), and let $Q_\nu^\mu(z)$ denote the associated Legendre function of the second kind of degree ν and order μ (see* (A.25)*). Then for $r := \sqrt{s(n-1-s) - \left(\frac{n-1}{2}\right)^2}$ the functions*

$$\Theta^{(1)}_{n,s,m}(\varrho) := \sinh(\varrho)^{1-\frac{n}{2}}\,P^{1-\frac{n}{2}-m}_{-\frac{1}{2}+ir}(\cosh(\varrho)), \qquad \Theta^{(2)}_{n,s,m}(\varrho) := \sinh(\varrho)^{1-\frac{n}{2}}\,Q^{1-\frac{n}{2}-m}_{-\frac{1}{2}+ir}(\cosh(\varrho))$$

form a fundamental system of solutions of the differential equation

$$\frac{d^2\Theta}{d\varrho^2} + (n-1)\,\frac{1}{\tanh(\varrho)}\,\frac{d\Theta}{d\varrho} + \left(s(n-1-s) - \frac{m(m+n-2)}{\sinh(\varrho)^2}\right)\Theta = 0. \tag{3.4}$$

In particular, the functions

$$\Theta^{(1)}_{n,s,0}(\varrho) = \sinh(\varrho)^{1-\frac{n}{2}}\,P^{1-\frac{n}{2}}_{-\frac{1}{2}+ir}(\cosh(\varrho)), \qquad \Theta^{(2)}_{n,s,0}(\varrho) = \sinh(\varrho)^{1-\frac{n}{2}}\,Q^{1-\frac{n}{2}}_{-\frac{1}{2}+ir}(\cosh(\varrho))$$

form a fundamental system of solutions of the differential equation

$$\frac{d^2\Theta}{d\varrho^2} + (n-1)\,\frac{1}{\tanh(\varrho)}\,\frac{d\Theta}{d\varrho} + s(n-1-s)\,\Theta = 0, \tag{3.5}$$

and the functions

$$\Theta^{(1)}_{n,0,m}(\varrho) = \sinh(\varrho)^{1-\frac{n}{2}}\,P^{1-\frac{n}{2}-m}_{-\frac{n}{2}}(\cosh(\varrho)), \qquad \Theta^{(2)}_{n,0,m}(\varrho) = \sinh(\varrho)^{1-\frac{n}{2}}\,Q^{1-\frac{n}{2}-m}_{-\frac{n}{2}}(\cosh(\varrho))$$

form a fundamental system of solutions of the differential equation

$$\frac{d^2\Theta}{d\varrho^2} + (n-1)\,\frac{1}{\tanh(\varrho)}\,\frac{d\Theta}{d\varrho} - \frac{m(m+n-2)}{\sinh(\varrho)^2}\,\Theta = 0. \tag{3.6}$$

Proof. Let $\lambda := s(n-1-s)$ and $\sigma^2 := m(m+n-2)$. We make the substitution

$$\Theta(\varrho) = \sinh(\varrho)^{1-\frac{n}{2}}\, f(\varrho)$$

with

$$\frac{d\Theta}{d\varrho} = \sinh(\varrho)^{1-\frac{n}{2}}\,\frac{df}{d\varrho} + \Big(1-\frac{n}{2}\Big)\cosh(\varrho)\,\sinh(\varrho)^{-\frac{n}{2}}\, f$$

and

$$\begin{aligned}
\frac{d^2\Theta}{d\varrho^2} &= \sinh(\varrho)^{1-\frac{n}{2}}\,\frac{d^2 f}{d\varrho^2} + \Big(1-\frac{n}{2}\Big)\cosh(\varrho)\,\sinh(\varrho)^{-\frac{n}{2}}\,\frac{df}{d\varrho}\\
&\quad + \Big(1-\frac{n}{2}\Big)\Big(\cosh(\varrho)\,\sinh(\varrho)^{-\frac{n}{2}}\,\frac{df}{d\varrho} + \sinh(\varrho)^{1-\frac{n}{2}}\, f - \frac{n}{2}\cosh(\varrho)^2\,\sinh(\varrho)^{-\frac{n}{2}-1}\, f\Big)\\
&= \sinh(\varrho)^{1-\frac{n}{2}}\,\frac{d^2 f}{d\varrho^2} + (2-n)\,\cosh(\varrho)\,\sinh(\varrho)^{-\frac{n}{2}}\,\frac{df}{d\varrho}\\
&\quad + \bigg(\Big(1-\frac{n}{2}\Big)\sinh(\varrho)^{1-\frac{n}{2}} + \Big(\frac{n^2}{4}-\frac{n}{2}\Big)\cosh(\varrho)^2\,\sinh(\varrho)^{-\frac{n}{2}-1}\bigg)\, f.
\end{aligned}$$

Inserting these identities into (3.4), we obtain

$$\begin{aligned}
0 &= \sinh(\varrho)^{1-\frac{n}{2}}\,\frac{d^2 f}{d\varrho^2} + (2-n)\,\cosh(\varrho)\,\sinh(\varrho)^{-\frac{n}{2}}\,\frac{df}{d\varrho}\\
&\quad + \bigg(\Big(1-\frac{n}{2}\Big)\sinh(\varrho)^{1-\frac{n}{2}} + \Big(\frac{n^2}{4}-\frac{n}{2}\Big)\cosh(\varrho)^2\,\sinh(\varrho)^{-\frac{n}{2}-1}\bigg)\, f\\
&\quad + (n-1)\,\frac{1}{\tanh(\varrho)}\Big(\sinh(\varrho)^{1-\frac{n}{2}}\,\frac{df}{d\varrho} + \Big(1-\frac{n}{2}\Big)\cosh(\varrho)\,\sinh(\varrho)^{-\frac{n}{2}}\, f\Big)\\
&\quad + \Big(\lambda - \frac{\sigma^2}{\sinh(\varrho)^2}\Big)\sinh(\varrho)^{1-\frac{n}{2}}\, f\\
&= \sinh(\varrho)^{1-\frac{n}{2}}\,\frac{d^2 f}{d\varrho^2} + \cosh(\varrho)\,\sinh(\varrho)^{-\frac{n}{2}}\,\frac{df}{d\varrho}\\
&\quad + \bigg(\Big(1-\frac{n}{2}\Big)\sinh(\varrho)^{1-\frac{n}{2}} + \Big(\frac{n^2}{4}-\frac{n}{2}\Big)\cosh(\varrho)^2\,\sinh(\varrho)^{-\frac{n}{2}-1}\bigg)\, f\\
&\quad + \bigg(\Big(-\frac{n^2}{2}+\frac{3n}{2}-1\Big)\cosh(\varrho)^2\,\sinh(\varrho)^{-\frac{n}{2}-1} + \lambda\,\sinh(\varrho)^{1-\frac{n}{2}} - \sigma^2\,\sinh(\varrho)^{-\frac{n}{2}-1}\bigg)\, f.
\end{aligned}$$

Multiplying this equation with $\sinh(\varrho)^{\frac{n}{2}-1}$ yields

$$\begin{aligned}
0 &= \frac{d^2 f}{d\varrho^2} + \frac{1}{\tanh(\varrho)}\,\frac{df}{d\varrho}\\
&\quad + \bigg(1-\frac{n}{2} + \Big(\frac{n^2}{4}-\frac{n}{2}\Big)\frac{1}{\tanh(\varrho)^2} + \Big(-\frac{n^2}{2}+\frac{3n}{2}-1\Big)\frac{1}{\tanh(\varrho)^2} + \lambda - \frac{\sigma^2}{\sinh(\varrho)^2}\bigg)\, f\\
&= \frac{d^2 f}{d\varrho^2} + \frac{1}{\tanh(\varrho)}\,\frac{df}{d\varrho} + \bigg(1-\frac{n}{2}+\lambda + \Big(-\frac{n^2}{4}+n-1\Big)\frac{1}{\tanh(\varrho)^2} - \frac{\sigma^2}{\sinh(\varrho)^2}\bigg)\, f.
\end{aligned}$$

If we now set $x := \cosh(\varrho)$ and $y(x) := f(\varrho)$, we have $x^2-1 = \sinh(\varrho)^2$,

$$\frac{df}{d\varrho} = \frac{dy}{d\varrho} = \frac{dy}{dx}\,\frac{dx}{d\varrho} = \sinh(\varrho)\,\frac{dy}{dx}$$

and

$$\begin{aligned}
\frac{d^2 f}{d\varrho^2} &= \sinh(\varrho)\,\frac{d^2 y}{dx^2}\,\frac{dx}{d\varrho} + \cosh(\varrho)\,\frac{dy}{dx} = \sinh(\varrho)^2\,\frac{d^2 y}{dx^2} + \cosh(\varrho)\,\frac{dy}{dx}\\
&= (x^2-1)\,\frac{d^2 y}{dx^2} + x\,\frac{dy}{dx}.
\end{aligned}$$

Plugging in these identities and multiplying by -1, the equation further simplifies to

$$\begin{aligned} 0 &= (1-x^2)\,\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} - \left(1-\frac{n}{2}+\lambda+\left(-\frac{n^2}{4}+n-1\right)\frac{x^2}{x^2-1}-\frac{\sigma^2}{x^2-1}\right)y \\ &= (1-x^2)\,\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + \left(-\lambda+\frac{n}{2}-1-\frac{\left(\frac{n^2}{4}-n+1\right)x^2+\sigma^2}{1-x^2}\right)y. \end{aligned} \tag{3.7}$$

In the next step we want to determine $\nu, \mu \in \mathbb{C}$ such that this becomes the associated Legendre equation

$$(1-x^2)\,\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + \left(\nu(\nu+1)-\frac{\mu^2}{1-x^2}\right)y = 0$$

of degree ν and order μ (see (A.23)). To do this, we first rewrite the bracket above as

$$\begin{aligned} -\lambda+\frac{n}{2}-1-\frac{\left(\frac{n^2}{4}-n+1\right)x^2+\sigma^2}{1-x^2} &= \frac{-\lambda+\frac{n}{2}-1+\lambda x^2-\frac{nx^2}{2}+x^2-\frac{n^2x^2}{4}+nx^2-x^2-\sigma^2}{1-x^2} \\ &= \frac{-\lambda+\frac{n}{2}-1+\lambda x^2+\frac{nx^2}{2}-\frac{n^2x^2}{4}-\sigma^2}{1-x^2} = \frac{\left(\frac{n^2}{4}-\frac{n}{2}-\lambda\right)(1-x^2)-\frac{n^2}{4}+n-1-\sigma^2}{1-x^2} \\ &= \frac{n^2}{4}-\frac{n}{2}-\lambda-\frac{\frac{n^2}{4}-n+1+\sigma^2}{1-x^2}. \end{aligned}$$

Thus, the degree ν must satisfy

$$\nu(\nu+1) = \frac{n^2}{4}-\frac{n}{2}-\lambda,$$

from which we derive the quadratic equation

$$\nu^2+\nu-\frac{n^2}{4}+\frac{n}{2}+\lambda = 0$$

with solutions

$$\nu = -\frac{1}{2}\pm\sqrt{\frac{1}{4}+\frac{n^2}{4}-\frac{n}{2}-\lambda} = -\frac{1}{2}\pm\sqrt{\left(\frac{n-1}{2}\right)^2-\lambda} = -\frac{1}{2}\pm ir,$$

where we have set $r := \sqrt{\lambda-\left(\frac{n-1}{2}\right)^2} = \sqrt{s(n-1-s)-\left(\frac{n-1}{2}\right)^2}$. Further, the order μ is given by

$$\begin{aligned} \mu &= \pm\sqrt{\frac{n^2}{4}-n+1+\sigma^2} = \pm\sqrt{\frac{(n-2)^2+4\sigma^2}{4}} = \pm\sqrt{\frac{(n-2)^2+4m^2+4m(n-2)}{4}} \\ &= \pm\sqrt{\frac{(n-2+2m)^2}{4}} = \pm\frac{n-2+2m}{2} = \pm\left(\frac{n}{2}-1+m\right). \end{aligned}$$

This shows that (3.7) is indeed the associated Legendre equation of degree $\nu = -\frac{1}{2}+ir$ and order $\mu = 1-\frac{n}{2}-m$, which has the fundamental system $P^{1-\frac{n}{2}-m}_{-\frac{1}{2}+ir}(x)$, $Q^{1-\frac{n}{2}-m}_{-\frac{1}{2}+ir}(x)$ of solutions. Recalling that we have made the substitutions $x = \cosh(\varrho)$, $y(x) = f(\varrho)$ and $\Theta(\varrho) = \sinh(\varrho)^{1-\frac{n}{2}}\,f(\varrho)$, we end up with the fundamental system

$$\Theta^{(1)}_{n,s,m}(\varrho) = \sinh(\varrho)^{1-\frac{n}{2}}\,P^{1-\frac{n}{2}-m}_{-\frac{1}{2}+ir}(\cosh(\varrho)), \qquad \Theta^{(2)}_{n,s,m}(\varrho) = \sinh(\varrho)^{1-\frac{n}{2}}\,Q^{1-\frac{n}{2}-m}_{-\frac{1}{2}+ir}(\cosh(\varrho))$$

of solutions of the eigenvalue problem (3.4).
If we have $m = 0$, we obtain the claimed fundamental system of solutions of (3.5), while for $s = 0$ (which implies $ir = -\frac{n-1}{2}$) we get the asserted fundamental system of solutions of (3.6).

□

Remark 3.1.4. In case that we are interested in the eigenfunctions $G(P) = G(\varrho)$ of $\Delta_{\mathbb{H}^n}$ that are radial at $Q \in \mathbb{H}^n$, we just have to solve equation (3.2) with the right-hand side zero which becomes equation (3.5) through multiplication by $\Theta \sinh(\varrho)^{-2}$. Hence, the eigenfunctions which are radial at Q are exactly given by the solutions of (3.5).

The first of the two linearly independent solutions of (3.5), namely the function $\Theta^{(1)}_{n,s,0}(\varrho)$, is bounded at $\varrho = 0$ and we can determine its value at this point.

Lemma 3.1.5. *The function $\Theta^{(1)}_{n,s,0}(\varrho)$ from Lemma 3.1.3 has the special value*

$$\Theta^{(1)}_{n,s,0}(0) = \frac{2^{1-\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

Proof. We make use of formula (A.26), i.e. of the integral representation

$$P_\nu^{-\mu}(z) = \frac{(z^2-1)^{\mu/2}}{2^\mu \sqrt{\pi}\,\Gamma\left(\mu+\frac{1}{2}\right)} \int_{-1}^{1} \frac{(1-t^2)^{\mu-1/2}}{\left(z+t\sqrt{z^2-1}\right)^{\mu-\nu}}\,dt,$$

which is valid for $\mathrm{Re}(\mu) > -\frac{1}{2}$ and $|\arg(z \pm 1)| < \pi$. Inserting $\mu = \frac{n}{2} - 1$, $\nu = -\frac{1}{2} + ir$ and $z = \cosh(\varrho)$ into this identity, we obtain

$$\begin{aligned}
\Theta^{(1)}_{n,s,0}(\varrho) &= \sinh(\varrho)^{1-\frac{n}{2}}\, P^{1-\frac{n}{2}}_{-\frac{1}{2}+ir}(\cosh(\varrho)) \\
&= \sinh(\varrho)^{1-\frac{n}{2}}\, \frac{(\cosh(\varrho)^2-1)^{\frac{n}{4}-\frac{1}{2}}}{2^{\frac{n}{2}-1}\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{\left(\cosh(\varrho)+t\sqrt{\cosh(\varrho)^2-1}\right)^{\frac{n-1}{2}-ir}}\,dt \\
&= \sinh(\varrho)^{1-\frac{n}{2}}\, \frac{\sinh(\varrho)^{\frac{n}{2}-1}}{2^{\frac{n}{2}-1}\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{(\cosh(\varrho)+t\,\sinh(\varrho))^{\frac{n-1}{2}-ir}}\,dt \\
&= \frac{2^{1-\frac{n}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{(\cosh(\varrho)+t\,\sinh(\varrho))^{\frac{n-1}{2}-ir}}\,dt.
\end{aligned}$$

For $\varrho = 0$ we have

$$(\cosh(\varrho)+t\,\sinh(\varrho))^{\frac{n-1}{2}-ir} = 1^{\frac{n-1}{2}-ir} = e^{\left(\frac{n-1}{2}-ir\right)\log(1)} = e^0 = 1,$$

so the function $\Theta^{(1)}_{n,s,0}(\varrho)$ simplifies to

$$\Theta^{(1)}_{n,s,0}(0) = \frac{2^{1-\frac{n}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}}\,dt.$$

Moreover, by formula (A.17), for $\mathrm{Re}(\alpha) > 0$ we have

$$\int_{-1}^{1} (1-t^2)^{\alpha-1}\,dt = 2\int_{0}^{1} (1-t^2)^{\alpha-1}\,dt = \mathrm{B}\left(\frac{1}{2},\alpha\right) = \frac{\sqrt{\pi}\,\Gamma(\alpha)}{\Gamma\left(\alpha+\frac{1}{2}\right)},$$

where $\mathrm{B}(a,b)$ is the beta function (see (A.16)). Choosing $\alpha = \frac{n-1}{2}$, this now gives us the asserted equality

$$\Theta^{(1)}_{n,s,0}(0) = \frac{2^{1-\frac{n}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}\,\frac{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = \frac{2^{1-\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

□

3.2. The $\mathrm{PSL}_2(C_{n-1})$-invariant integral operators

In this section we introduce $\mathrm{PSL}_2(C_{n-1})$-invariant integral operators and the notion of a point-pair invariant function. We proceed essentially as in section 1.8. in [Iwa02], where $\mathrm{PSL}_2(\mathbb{R})$-invariant integral operators are treated. Our main result on eigenfunctions of $\mathrm{PSL}_2(C_{n-1})$-invariant integral operators will be employed to use in proving the absolute and locally uniform convergence of hyperbolic Eisenstein series in section 4.1.

Definition 3.2.1.

(a) A *linear operator* L is an endomorphism of the space $\{f : \mathbb{H}^n \to \mathbb{C}\}$ of complex-valued functions on $\mathbb{H}^n$.

(b) A linear operator L is called $\mathrm{PSL}_2(C_{n-1})$-*invariant* if

$$L(f(\gamma P)) = (Lf)(\gamma P)$$

for any $\gamma \in \mathrm{PSL}_2(C_{n-1})$ and $P \in \mathbb{H}^n$.

Example 3.2.2. By Proposition 2.3.15 (c), the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ is a $\mathrm{PSL}_2(C_{n-1})$-invariant linear operator.

Definition 3.2.3. An *integral operator* L is given by

$$(Lf)(P) = \int_{\mathbb{H}^n} K(P,Q)\, f(Q)\ \mu_{\mathbb{H}^n}(Q), \tag{3.8}$$

where $f : \mathbb{H}^n \to \mathbb{C}$ and where $K : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{C}$ is a given function called the *kernel* of L.

In the following we always assume that the kernel $K(P,Q)$ and the function $f(Q)$ are such that the integral in (3.8) converges absolutely.

An integral operator L with kernel K defines a linear operator, and it is $\mathrm{PSL}_2(C_{n-1})$-invariant if and only if

$$\int_{\mathbb{H}^n} K(P,Q)\, f(\gamma Q)\ \mu_{\mathbb{H}^n}(Q) = \int_{\mathbb{H}^n} K(\gamma P,Q)\, f(Q)\ \mu_{\mathbb{H}^n}(Q)$$

for any $\gamma \in \mathrm{PSL}_2(C_{n-1})$ and $P \in \mathbb{H}^n$. For that it is necessary and sufficient that

$$K(\gamma P, \gamma Q) = K(P,Q) \tag{3.9}$$

for any $\gamma \in \mathrm{PSL}_2(C_{n-1})$ and $P, Q \in \mathbb{H}^n$.

Definition 3.2.4. A function $K : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{C}$ which satisfies (3.9) for any $\gamma \in \mathrm{PSL}_2(C_{n-1})$ and $P, Q \in \mathbb{H}^n$ is called *point-pair invariant.*

Lemma 3.2.5. *A point-pair invariant function $K : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{C}$, $(P,Q) \mapsto K(P,Q)$, depends only on the hyperbolic distance $d_{\mathbb{H}^n}(P,Q)$, i.e. it can be written as $K(d_{\mathbb{H}^n}(P,Q))$.*

Proof. Let $P_1, P_2, Q_1, Q_2 \in \mathbb{H}^n$ with $d_{\mathbb{H}^n}(P_1,Q_1) = d_{\mathbb{H}^n}(P_2,Q_2)$. We prove that there is a matrix $\gamma \in \mathrm{PSL}_2(C_{n-1})$ such that $\gamma P_1 = P_2$ and $\gamma Q_1 = Q_2$:
We let $\sigma_{P_1} \in \mathrm{PSL}_2(C_{n-1})$ be an elliptic scaling matrix of P_1, i.e. $\sigma_{P_1}^{-1} P_1 = i_{n-1}$. Furthermore, there exists a hyperbolic rotation around $i_{n-1} \in \mathbb{H}^n$ which maps $\sigma_{P_1}^{-1} Q_1$ onto the x_{n-1}-axis, i.e. we can choose $\delta \in \mathrm{PSL}_2(C_{n-1})_{i_{n-1}} = \mathrm{PSU}_2(C_{n-1})$ such that $\delta(\sigma_{P_1}^{-1} Q_1) = t i_{n-1}$ for some $t \in \mathbb{R}$, $t > 0$. After a possible application of the inversion $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{PSU}_2(C_{n-1})$ we can assume $t \geq 1$. Now the matrix $\gamma_1 := \delta \sigma_{P_1}^{-1} \in \mathrm{PSL}_2(C_{n-1})$ satisfies $\gamma_1 P_1 = i_{n-1}$ and $\gamma_1 Q_1 = t i_{n-1}$. Moreover, we have

$$\cosh(d_{\mathbb{H}^n}(P_1,Q_1)) = \cosh(d_{\mathbb{H}^n}(\gamma_1 P_1, \gamma_1 Q_1)) = \cosh(d_{\mathbb{H}^n}(i_{n-1}, t i_{n-1})) = \frac{1}{2}\left(t + \frac{1}{t}\right),$$

which yields $d_{\mathbb{H}^n}(P_1, Q_1) = \log(t)$ and $t = \exp(d_{\mathbb{H}^n}(P_1, Q_1))$.
Analogously, there exists an element $\gamma_2 \in \mathrm{PSL}_2(C_{n-1})$ with $\gamma_2 P_2 = i_{n-1}$ and $\gamma_2 Q_2 = t' i_{n-1}$ for some $t' \in \mathbb{R}$, $t' \geq 1$, and from $d_{\mathbb{H}^n}(P_2, Q_2) = d_{\mathbb{H}^n}(P_1, Q_1)$ we can deduce $t' = \exp(d_{\mathbb{H}^n}(P_2, Q_2)) = t$.
Thus, the matrix $\gamma := \gamma_2^{-1}\gamma_1 \in \mathrm{PSL}_2(C_{n-1})$ has the required properties $\gamma P_1 = P_2$ and $\gamma Q_1 = Q_2$.
Since the function K is point-pair invariant, we finally get

$$K(P_2, Q_2) = K(\gamma P_1, \gamma Q_1) = K(P_1, Q_1),$$

proving that $K(P, Q)$ only depends on $d_{\mathbb{H}^n}(P, Q)$.

□

Remark 3.2.6. For $P, Q \in \mathbb{H}^n$ we have $d_{\mathbb{H}^n}(P, Q) = \varrho_Q(P) =: \varrho(P, Q)$, where $\varrho_Q(P)$ is the first component of the elliptic coordinates of P centered at Q (see Definition 1.2.5). Thus, a point-pair invariant function $K(P, Q)$ can always be viewed as a function

$$K(P, Q) = K(\varrho(P, Q)) = K(\varrho)$$

in a single variable $\varrho \in [0, \infty)$.

We recall from Definition 3.1.1 (b) that a function $f : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{C}$, $(P, Q) \mapsto f(P, Q)$, is radial at $Q \in \mathbb{H}^n$ if, as a function of P, it only depends on the hyperbolic distance $d_{\mathbb{H}^n}(P, Q)$. A function $f : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{C}$ which is radial at some point $Q \in \mathbb{H}^n$ is in general not necessarily radial at other points. Though, if f is point-pair invariant, then it is always radial at any $Q \in \mathbb{H}^n$.

To a given function $f : \mathbb{H}^n \to \mathbb{C}$ we can associate a function $f_Q : \mathbb{H}^n \to \mathbb{C}$ which is radial at some point $Q \in \mathbb{H}^n$. This radial function can be obtained by averaging $f(P)$ over the stabilizer subgroup $\mathrm{PSL}_2(C_{n-1})_Q$. We formalize this in the following definition of the mean-value operator.

Definition 3.2.7. Let $f : \mathbb{H}^n \to \mathbb{C}$ and $Q \in \mathbb{H}^n$. We define the *mean-value operator* M_Q as the mapping $f \mapsto f_Q$ with

$$f_Q(P) := \int_{\mathrm{PSL}_2(C_{n-1})_Q} f(\gamma P)\, d\gamma. \tag{3.10}$$

We prove now that this indeed yields a function which is radial at Q.

Lemma 3.2.8. *The function $f_Q(P)$ from Definition 3.2.7 is radial at Q. Moreover, it satisfies $f_Q(Q) = f(Q)$.*

Proof. Let $P_1, P_2 \in \mathbb{H}^n$ with $d_{\mathbb{H}^n}(P_1, Q) = d_{\mathbb{H}^n}(P_2, Q)$. Then P_1 and P_2 both lie on the hyperbolic sphere with center Q and radius $d_{\mathbb{H}^n}(P_1, Q) = d_{\mathbb{H}^n}(P_2, Q)$, so there is a hyperbolic rotation around Q which maps P_1 to P_2, i.e. a matrix $\sigma \in \mathrm{PSL}_2(C_{n-1})_Q$ with $\sigma P_1 = P_2$. This implies

$$\begin{aligned} f_Q(P_2) &= \int_{\mathrm{PSL}_2(C_{n-1})_Q} f(\gamma P_2)\, d\gamma = \int_{\mathrm{PSL}_2(C_{n-1})_Q} f(\gamma\sigma P_1)\, d\gamma \\ &= \int_{\mathrm{PSL}_2(C_{n-1})_Q} f(\gamma P_1)\, d\gamma = f_Q(P_1), \end{aligned}$$

so $f_Q(P)$ depends only on the hyperbolic distance $d_{\mathbb{H}^n}(P, Q)$.
For the second assertion we use $\gamma Q = Q$ for any $\gamma \in \mathrm{PSL}_2(C_{n-1})_Q$ to obtain

$$f_Q(Q) = \int_{\mathrm{PSL}_2(C_{n-1})_Q} f(\gamma Q)\, d\gamma = \int_{\mathrm{PSL}_2(C_{n-1})_Q} f(Q)\, d\gamma = f(Q).$$

□

Remark 3.2.9. The mean-value operator M_Q can also be applied to a linear operator L by setting

$$(M_Q L)(f) := L f_Q$$

for a function $f : \mathbb{H}^n \to \mathbb{C}$.

Lemma 3.2.10. *A* $\mathrm{PSL}_2(C_{n-1})$*-invariant integral operator* L *is not altered by the mean-value operator, i.e. we have*

$$((M_P L)f)(P) = (Lf)(P)$$

for any $P \in \mathbb{H}^n$ *and* $f : \mathbb{H}^n \to \mathbb{C}$.

Proof. Let $K : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{C}$ be a kernel of L. For $f : \mathbb{H}^n \to \mathbb{C}$ and $P \in \mathbb{H}^n$ we compute

$$\begin{aligned}
((M_P L)f)(P) = (Lf_P)(P) &= \int_{\mathbb{H}^n} K(P,Q)\, f_P(Q)\ \mu_{\mathbb{H}^n}(Q) \\
&= \int_{\mathbb{H}^n} K(P,Q) \left(\int_{\mathrm{PSL}_2(C_{n-1})_P} f(\gamma Q)\ d\gamma \right) \mu_{\mathbb{H}^n}(Q) \\
&= \int_{\mathrm{PSL}_2(C_{n-1})_P} \left(\int_{\mathbb{H}^n} K(P,Q)\, f(\gamma Q)\ \mu_{\mathbb{H}^n}(Q) \right) d\gamma.
\end{aligned}$$

Since L is a $\mathrm{PSL}_2(C_{n-1})$-invariant integral operator, we have

$$\int_{\mathbb{H}^n} K(P,Q)\, f(\gamma Q)\ \mu_{\mathbb{H}^n}(Q) = \int_{\mathbb{H}^n} K(\gamma P,Q)\, f(Q)\ \mu_{\mathbb{H}^n}(Q),$$

from which we obtain

$$\begin{aligned}
((M_P L)f)(P) &= \int_{\mathrm{PSL}_2(C_{n-1})_P} \left(\int_{\mathbb{H}^n} K(\gamma P,Q)\, f(Q)\ \mu_{\mathbb{H}^n}(Q) \right) d\gamma \\
&= \left(\int_{\mathbb{H}^n} K(P,Q)\, f(Q)\ \mu_{\mathbb{H}^n}(Q) \right) \left(\int_{\mathrm{PSL}_2(C_{n-1})_P} d\gamma \right) \\
&= \int_{\mathbb{H}^n} K(P,Q)\, f(Q)\ \mu_{\mathbb{H}^n}(Q) = (Lf)(P).
\end{aligned}$$

□

In order to state our main result on $\mathrm{PSL}_2(C_{n-1})$-invariant integral operators, we need two more lemmas.

Lemma 3.2.11. *Let* $Q_0 \in \mathbb{H}^n$ *and* $\lambda = s(n-1-s) \in \mathbb{C}$ *be fixed. Then there exists a unique function* $\omega : \mathbb{H}^n \to \mathbb{C}$, $P \mapsto \omega(P, Q_0)$*, which satisfies the following properties:*

(i) $\omega(P, Q_0)$ *is radial at* Q_0,

(ii) $\Delta_{\mathbb{H}^n} \omega(P, Q_0) = \lambda\, \omega(P, Q_0)$,

(iii) $\omega(Q_0, Q_0) = 1$.

Proof. Let $\omega : \mathbb{H}^n \to \mathbb{C}$ be a function which satisfies the properties (i) and (ii). Using elliptic coordinates centered at Q_0, by (i) we can write $\omega(P, Q_0) = \omega(\varrho)$, where

$$\varrho = \varrho(P, Q_0) = \varrho_{Q_0}(P) = d_{\mathbb{H}^n}(P, Q_0).$$

On the other hand, we have seen in Lemma 3.1.3 and Remark 3.1.4 that the eigenfunctions of $\Delta_{\mathbb{H}^n}$ in elliptic coordinates which are radial at Q_0 are the solutions of the differential equation

$$\frac{d^2\Theta}{d\varrho^2} + (n-1)\, \frac{1}{\tanh(\varrho)}\, \frac{d\Theta}{d\varrho} + s(n-1-s)\, \Theta = 0,$$

which has the fundamental system

$$\Theta^{(1)}_{n,s,0}(\varrho) = \sinh(\varrho)^{1-\frac{n}{2}}\, P^{1-\frac{n}{2}}_{-\frac{1}{2}+ir}(\cosh(\varrho)), \qquad \Theta^{(2)}_{n,s,0}(\varrho) = \sinh(\varrho)^{1-\frac{n}{2}}\, Q^{1-\frac{n}{2}}_{-\frac{1}{2}+ir}(\cosh(\varrho))$$

of solutions, where $r = \sqrt{s(n-1-s) - \left(\frac{n-1}{2}\right)^2}$. Hence, $\omega(\varrho)$ is a linear combination of $\Theta^{(1)}_{n,s,0}(\varrho)$ and $\Theta^{(2)}_{n,s,0}(\varrho)$. The function $\Theta^{(1)}_{n,s,0}(\varrho)$ is bounded at $\varrho = 0$ by Lemma 3.1.5, while the function

$$\Theta^{(2)}_{n,s,0}(\varrho) = \sinh(\varrho)^{1-\frac{n}{2}}\, Q^{1-\frac{n}{2}}_{-\frac{1}{2}+ir}(\cosh(\varrho))$$

is unbounded at $\varrho = 0$. Thus, if the function $\omega(P, Q_0)$ should additionally satisfy property (iii), i.e.

$$\omega(0) = \omega(Q_0, Q_0) = 1,$$

using Lemma 3.1.5, we find

$$\omega(P, Q_0) = \omega(\varrho(P, Q_0)) = \frac{\Theta^{(1)}_{n,s,0}(\varrho(P, Q_0))}{\Theta^{(1)}_{n,s,0}(0)} = 2^{\frac{n}{2}-1}\,\Gamma\left(\frac{n}{2}\right)\,\Theta^{(1)}_{n,s,0}(\varrho(P, Q_0)).$$

This proves that the function $\omega(P, Q_0)$ exists and is uniquely determined by (i), (ii) and (iii). □

Lemma 3.2.12. *Let $f : \mathbb{H}^n \to \mathbb{C}$ be an eigenfunction of $\Delta_{\mathbb{H}^n}$ with eigenvalue $\lambda = s(n-1-s) \in \mathbb{C}$, and let $f_Q : \mathbb{H}^n \to \mathbb{C}$ be the function which is associated to f via* (3.10) *and radial at $Q \in \mathbb{H}^n$. Then we have*

$$f_Q(P) = \omega(P, Q)\, f(Q)$$

for any $P \in \mathbb{H}^n$ with $\omega(P, Q)$ as in Lemma 3.2.11.

Proof. The function $f_Q(P)$ is radial at $Q \in \mathbb{H}^n$ by construction, so it satisfies property (i) in Lemma 3.2.11. Using the $\mathrm{PSL}_2(C_{n-1})$-invariance of the hyperbolic Laplace operator, we further have

$$\begin{aligned} \Delta_{\mathbb{H}^n} f_Q(P) &= \Delta_{\mathbb{H}^n}\left(\int_{\mathrm{PSL}_2(C_{n-1})_Q} f(\gamma P)\, d\gamma\right) = \int_{\mathrm{PSL}_2(C_{n-1})_Q} \Delta_{\mathbb{H}^n}(f(\gamma P))\, d\gamma \\ &= \int_{\mathrm{PSL}_2(C_{n-1})_Q} (\Delta_{\mathbb{H}^n} f)(\gamma P)\, d\gamma = \int_{\mathrm{PSL}_2(C_{n-1})_Q} \lambda f(\gamma P)\, d\gamma \\ &= \lambda f_Q(P). \end{aligned}$$

Hence, $f_Q(P)$ also satisfies property (ii) in Lemma 3.2.11. Now the lemma and its proof yield that

$$f_Q(P) = C(Q)\,\omega(P, Q)$$

must hold, where $\omega(P, Q)$ is the function from Lemma 3.2.11 and $C(Q) \in \mathbb{C}$ is a constant which only depends on Q. Finally, we can determine this constant as

$$C(Q) = C(Q) \cdot 1 = C(Q)\,\omega(Q, Q) = f_Q(Q) = f(Q),$$

where the last equality follows from Lemma 3.2.8. This completes the proof. □

After these preliminaries we are now able to prove the following proposition.

Proposition 3.2.13. *Let $f : \mathbb{H}^n \to \mathbb{C}$ be an eigenfunction of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ with eigenvalue $\lambda = s(n-1-s) \in \mathbb{C}$, and let L be a $\mathrm{PSL}_2(C_{n-1})$-invariant integral operator on $\mathbb{H}^n$ with kernel K such that Lf exists. Then f is an eigenfunction of L. More precisely, there exists a constant $\Lambda(\lambda, K) \in \mathbb{C}$, depending only on λ and K, such that*

$$Lf = \Lambda(\lambda, K)\, f,$$

i.e.

$$\int_{\mathbb{H}^n} K(P, Q)\, f(Q)\ \mu_{\mathbb{H}^n}(Q) = \Lambda(\lambda, K)\, f(P)$$

for any $P \in \mathbb{H}^n$.

Proof. Making use of Lemma 3.2.10 and Lemma 3.2.12, for $P \in \mathbb{H}^n$ we compute

$$\begin{aligned}(Lf)(P) &= ((M_P L)f)(P) = (Lf_P)(P) = \int_{\mathbb{H}^n} K(P,Q)\, f_P(Q)\ \mu_{\mathbb{H}^n}(Q) \\ &= \int_{\mathbb{H}^n} K(P,Q)\,\omega(Q,P)\, f(P)\ \mu_{\mathbb{H}^n}(Q) = \left(\int_{\mathbb{H}^n} K(P,Q)\,\omega(Q,P)\ \mu_{\mathbb{H}^n}(Q)\right) f(P).\end{aligned}$$

Since $\omega(Q,P)$ is radial at any $P \in \mathbb{H}^n$, we have $\omega(Q,P) = \omega(d_{\mathbb{H}^n}(Q,P)) = \omega(P,Q)$. Moreover, we also write the point-pair invariant function $K(P,Q)$ as $K(d_{\mathbb{H}^n}(P,Q))$.
Let $\sigma_P \in \mathrm{PSL}_2(C_{n-1})$ be an elliptic scaling matrix of P, i.e. $\sigma_P i_{n-1} = P$. Using the $\mathrm{PSL}_2(C_{n-1})$-invariance of $\mu_{\mathbb{H}^n}(Q)$, we observe that the integral

$$\begin{aligned}\int_{\mathbb{H}^n} K(P,Q)\,\omega(Q,P)\ \mu_{\mathbb{H}^n}(Q) &= \int_{\mathbb{H}^n} K(d_{\mathbb{H}^n}(P,Q))\,\omega(d_{\mathbb{H}^n}(P,Q))\ \mu_{\mathbb{H}^n}(Q) \\ &= \int_{\mathbb{H}^n} K(d_{\mathbb{H}^n}(\sigma_P i_{n-1},Q))\,\omega(d_{\mathbb{H}^n}(\sigma_P i_{n-1},Q))\ \mu_{\mathbb{H}^n}(Q) \\ &= \int_{\mathbb{H}^n} K\left(d_{\mathbb{H}^n}\left(i_{n-1},\sigma_P^{-1}Q\right)\right)\ \omega\left(d_{\mathbb{H}^n}\left(i_{n-1},\sigma_P^{-1}Q\right)\right)\ \mu_{\mathbb{H}^n}(Q) \\ &= \int_{\mathbb{H}^n} K\left(d_{\mathbb{H}^n}(i_{n-1},Q)\right)\ \omega\left(d_{\mathbb{H}^n}(i_{n-1},Q)\right)\ \mu_{\mathbb{H}^n}(Q)\end{aligned}$$

is independent of P. If we take into account that $\omega(P,Q)$ depends only on λ and set

$$\Lambda(\lambda,K) := \int_{\mathbb{H}^n} K(P,Q)\,\omega(P,Q)\ \mu_{\mathbb{H}^n}(Q) = \int_{\mathbb{H}^n} K(P,Q)\,\omega(Q,P)\ \mu_{\mathbb{H}^n}(Q),$$

we obtain $(Lf)(P) = \Lambda(\lambda,K)\, f(P)$, where the constant $\Lambda(\lambda,K) \in \mathbb{C}$ depends only on λ and K. □

3.3. Automorphic functions in $\mathbb{H}^n$

Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. We now introduce the notion of automorphic functions in $\mathbb{H}^n$ with respect to Γ. These are complex-valued functions on $\mathbb{H}^n$ which are invariant under the action of Γ.

Definition 3.3.1. A function $f : \mathbb{H}^n \to \mathbb{C}$ with the property

$$f(\gamma P) = f(P)$$

for any $\gamma \in \Gamma$ and $P \in \mathbb{H}^n$ is called an *automorphic function with respect to* Γ.

Remark 3.3.2. An automorphic function $f : \mathbb{H}^n \to \mathbb{C}$ with respect to Γ yields a well-defined function $\Gamma\backslash\mathbb{H}^n \to \mathbb{C}$, $\Gamma P \mapsto f(P)$ on the quotient $\Gamma\backslash\mathbb{H}^n$.

It is immediate from the definition that the automorphic functions with respect to Γ, together with the usual addition and scalar multiplication of functions, form a complex vector space.

Notation 3.3.3. The complex vector space of automorphic functions with respect to Γ is denoted by $\mathcal{A}(\Gamma\backslash\mathbb{H}^n)$.

Remark 3.3.4. Let $\eta_j \in C_\Gamma$ be a cusp with parabolic scaling matrix $\sigma_{\eta_j} \in \mathrm{PSL}_2(C_{n-1})$ and translational stabilizer subgroup Γ'_{η_j}. Then

$$\sigma_{\eta_j}^{-1}\Gamma'_{\eta_j}\sigma_{\eta_j} = \left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \;\middle|\; \mu \in \Lambda_{\eta_j} \right\} / \{\pm I\}$$

for some lattice $\Lambda_{\eta_j} \subseteq V_{n-1} \cong \mathbb{R}^{n-1}$ of full rank $n-1$ (see Proposition 2.6.18 (b)). Thus, for any $\mu \in \Lambda_{\eta_j}$ there is a matrix $\gamma_\mu \in \Gamma'_{\eta_j}$ such that $\sigma_{\eta_j} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} = \gamma_\mu \sigma_{\eta_j}$.
Let f be an automorphic function with respect to Γ. For $\mu \in \Lambda_{\eta_j}$ and $P = P_0 + x_{n-1} i_{n-1} \in \mathbb{H}^n$ we have

$$f(\sigma_{\eta_j}(P+\mu)) = f\left(\sigma_{\eta_j} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} P\right) = f(\gamma_\mu \sigma_{\eta_j} P) = f(\sigma_{\eta_j} P),$$

so that the function $f(\sigma_{\eta_j} P)$ is periodic under the lattice Λ_{η_j} with respect to P_0. Therefore, a smooth automorphic function $f \in \mathcal{A}(\Gamma\backslash\mathbb{H}^n)$ admits a Fourier expansion with respect to the cusp η_j of the form

$$f(\sigma_{\eta_j} P) = \sum_{\mu \in \Lambda^*_{\eta_j}} a_{\mu;\eta_j}(x_{n-1})\, e^{2\pi i \langle \mu, P_0 \rangle} = a_{0,\eta_j}(x_{n-1}) + \sum_{\substack{\mu \in \Lambda^*_{\eta_j}, \\ \mu \neq 0}} a_{\mu;\eta_j}(x_{n-1})\, e^{2\pi i \langle \mu, P_0 \rangle}, \tag{3.11}$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product on $\mathbb{R}^{n-1}$ and

$$\Lambda^*_{\eta_j} = \{\mu \in \mathbb{R}^{n-1} \mid \langle \mu, \nu \rangle \in \mathbb{Z} \text{ for any } \nu \in \Lambda_{\eta_j}\}$$

denotes the dual lattice of Λ_{η_j}, and where the Fourier coefficients $a_{\mu;\eta_j}(x_{n-1})$ are given by

$$a_{\mu;\eta_j}(x_{n-1}) = \frac{1}{\operatorname{covol}(\Lambda_{\eta_j})} \int_{\mathbb{R}^{n-1}/\Lambda_{\eta_j}} f(\sigma_{\eta_j} P)\, e^{-2\pi i \langle \mu, P_0 \rangle}\, dP_0.$$

Example 3.3.5. Let $n = 2$ and $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind. Moreover, let $\eta_j \in C_\Gamma$ be a cusp with parabolic scaling matrix $\sigma_{\eta_j} \in \mathrm{PSL}_2(\mathbb{R})$ and stabilizer subgroup $\Gamma_{\eta_j} = \langle \gamma_{\eta_j} \rangle$ for some primitive parabolic element γ_{η_j} (see Example 2.6.19 (a)). For an automorphic function $f \in \mathcal{A}(\Gamma\backslash\mathbb{H})$ on the upper half-plane $\mathbb{H}$ and $z = x + iy \in \mathbb{H}$ we derive from $\gamma_{\eta_j} \in \Gamma$ the identity

$$f(\gamma_{\eta_j} \sigma_{\eta_j} z) = f(\sigma_{\eta_j} z).$$

Because of $\gamma_{\eta_j} \sigma_{\eta_j} = \sigma_{\eta_j} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z = z + 1$ this yields

$$f(\sigma_{\eta_j}(z+1)) = f(\sigma_{\eta_j} z),$$

so that the function $f(\sigma_{\eta_j} z)$ is 1-periodic with respect to $x = \mathrm{Re}(z)$. Hence, f has the Fourier expansion

$$f = \sum_{m \in \mathbb{Z}} a_{m;\eta_j}(y)\, e^{2\pi i m x} \tag{3.12}$$

with respect to the cusp η_j, where the coefficients $a_{m;\eta_j}(y)$ are given by

$$a_{m;\eta_j}(y) = \int_0^1 f(\sigma_{\eta_j} z)\, e^{-2\pi i m x}\, dx.$$

If $f \in \mathcal{A}(\Gamma\backslash\mathbb{H})$ is smooth, the series (3.12) converges absolutely and uniformly for z ranging over compact subsets $K \subseteq \mathbb{H}$.

Definition 3.3.6. For two automorphic functions $f_1, f_2 \in \mathcal{A}(\Gamma\backslash\mathbb{H}^n)$ we define their *inner product* $\langle f_1, f_2 \rangle$ by

$$\langle f_1, f_2 \rangle = \int_{\Gamma\backslash\mathbb{H}^n} f_1(P)\, \overline{f_2(P)}\, \mu_{\mathbb{H}^n}(P),$$

provided that the integral exists.

Now the $\mu_{\mathbb{H}^n}$-measurable functions $f \in \mathcal{A}(\Gamma\backslash\mathbb{H}^n)$ satisfying the property

$$\langle f, f\rangle = \int_{\Gamma\backslash\mathbb{H}^n} |f(P)|^2 \,\mu_{\mathbb{H}^n}(P) < \infty,$$

together with the inner product $\langle\cdot,\cdot\rangle$, form a complex Hilbert space.

Notation 3.3.7. The Hilbert space of $\mu_{\mathbb{H}^n}$-measurable functions $f \in \mathcal{A}(\Gamma\backslash\mathbb{H}^n)$ with $\langle f, f\rangle < \infty$ is denoted by $\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ and the norm on this space by

$$||f||_{\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)} = \sqrt{\langle f, f\rangle}.$$

Remark 3.3.8. Since $\Gamma\backslash\mathbb{H}^n$ has finite hyperbolic volume, every function $f \in \mathcal{A}(\Gamma\backslash\mathbb{H}^n)$ which is bounded on $\Gamma\backslash\mathbb{H}^n$ satisfies $f \in \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$.

3.4. Parabolic Eisenstein series

In this section we define an important class of automorphic functions in $\mathbb{H}^n$, namely parabolic Eisenstein series. These are a generalization of the classical non-holomorphic Eisenstein series on the upper half-plane $\mathbb{H}$. We give some properties of parabolic Eisenstein series, and treat the special case $n = 2$ in more detail as an example.
Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. Further, let $\eta_j \in C_\Gamma$ $(j \in \{1,\ldots,c_\Gamma\})$ be a cusp (see Definition 2.6.9) with parabolic scaling matrix $\sigma_{\eta_j} \in \mathrm{PSL}_2(C_{n-1})$ (see Definition 2.6.16), stabilizer subgroup Γ_{η_j} and translational stabilizer subgroup Γ'_{η_j} (see Definition 2.6.7).

Definition 3.4.1. For $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ we define the *parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_j}(P, s)$ associated to the cusp $\eta_j \in C_\Gamma$* by

$$E^{\mathrm{par}}_{\eta_j}(P, s) = \sum_{\gamma\in\Gamma_{\eta_j}\backslash\Gamma} x_{n-1}(\sigma^{-1}_{\eta_j}\gamma P)^s, \tag{3.13}$$

where $x_{n-1}(\sigma^{-1}_{\eta_j}\gamma P)$ denotes the x_{n-1}-coordinate of $\sigma^{-1}_{\eta_j}\gamma P$.

Notation 3.4.2. In case we want to refer explicitly to the dimension n, we write $E^{\mathrm{par}}_{n,\eta_j}(P, s)$ instead of $E^{\mathrm{par}}_{\eta_j}(P, s)$. If we want to refer explicitly to the underlying group Γ, we write $E^{\mathrm{par}}_{\Gamma,\eta_j}(P, s)$ or $E^{\mathrm{par}}_{n,\Gamma,\eta_j}(P, s)$.

Remark 3.4.3. We make two remarks on the above definition of parabolic Eisenstein series.

(a) If $\sigma'_{\eta_j} \in \mathrm{PSL}_2(C_{n-1})$ is another parabolic scaling matrix for the cusp η_j, then by Remark 2.6.17 we have $\sigma'_{\eta_j} = \sigma_{\eta_j}\delta$ for some $\delta = \begin{pmatrix}\alpha & \alpha\beta\\ 0 & \alpha'\end{pmatrix} \in \mathrm{PSL}_2(C_{n-1})_\infty$ with $\alpha \in \Gamma_{n-1}$, $|\alpha| = 1$, and $\beta \in V_{n-1}$. This leads to $\delta^{-1} = \begin{pmatrix}\alpha^{-1} & -\beta\alpha^*\\ 0 & \alpha^*\end{pmatrix}$ and

$$x_{n-1}(\sigma'^{-1}_{\eta_j}\gamma P) = x_{n-1}(\delta^{-1}\sigma^{-1}_{\eta_j}\gamma P) = \frac{x_{n-1}(\sigma^{-1}_{\eta_j}\gamma P)}{|\alpha^*|^2} = x_{n-1}(\sigma^{-1}_{\eta_j}\gamma P)$$

for $\gamma \in \Gamma_{\eta_j}\backslash\Gamma$. Hence, the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_j}(P, s)$ is independent of the exact choice of the parabolic scaling matrix σ_{η_j}.

(b) Some authors prefer to define the parabolic Eisenstein series associated to the cusp $\eta_j \in C_\Gamma$ as

$$\widehat{E}^{\mathrm{par}}_{\eta_j}(P, s) = \sum_{\gamma\in\Gamma'_{\eta_j}\backslash\Gamma} x_{n-1}(\sigma^{-1}_{\eta_j}\gamma P)^s \tag{3.14}$$

instead of (3.13). If $\delta \in \Gamma_{\eta_j}$, then by Proposition 2.6.18 (a) we have

$$\sigma_{\eta_j}^{-1}\delta = \begin{pmatrix} \alpha & \alpha\beta \\ 0 & \alpha' \end{pmatrix} \sigma_{\eta_j}^{-1}$$

for some $\alpha \in \Gamma_{n-1}$ with $|\alpha| = 1$ and $\beta \in V_{n-1}$. For any $\delta \in \Gamma'_{\eta_j}\backslash\Gamma_{\eta_j}$ this gives us

$$x_{n-1}(\sigma_{\eta_j}^{-1}\delta\gamma P) = x_{n-1}\left(\begin{pmatrix} \alpha & \alpha\beta \\ 0 & \alpha' \end{pmatrix} \sigma_{\eta_j}^{-1}\gamma P\right) = \frac{x_{n-1}(\sigma_{\eta_j}^{-1}\gamma P)}{|\alpha'|^2} = x_{n-1}(\sigma_{\eta_j}^{-1}\gamma P).$$

Since the index $[\Gamma_{\eta_j} : \Gamma'_{\eta_j}] = |\Gamma'_{\eta_j}\backslash\Gamma_{\eta_j}|$ is finite by Lemma 2.6.11, we can rewrite the parabolic Eisenstein series $E_{\eta_j}^{\mathrm{par}}(P, s)$ from Definition 3.4.1 as

$$\begin{aligned} E_{\eta_j}^{\mathrm{par}}(P, s) &= \sum_{\gamma \in \Gamma_{\eta_j}\backslash\Gamma} \frac{1}{[\Gamma_{\eta_j} : \Gamma'_{\eta_j}]} \sum_{\delta \in \Gamma'_{\eta_j}\backslash\Gamma_{\eta_j}} x_{n-1}(\sigma_{\eta_j}^{-1}\delta\gamma P)^s \\ &= \frac{1}{[\Gamma_{\eta_j} : \Gamma'_{\eta_j}]} \sum_{\gamma \in \Gamma'_{\eta_j}\backslash\Gamma} x_{n-1}(\sigma_{\eta_j}^{-1}\gamma P)^s. \end{aligned}$$

Therefore, the definitions (3.13) and (3.14) agree up to the factor $[\Gamma_{\eta_j} : \Gamma'_{\eta_j}]$, i.e. we have

$$\widehat{E}_{\eta_j}^{\mathrm{par}}(P, s) = [\Gamma_{\eta_j} : \Gamma'_{\eta_j}]\, E_{\eta_j}^{\mathrm{par}}(P, s).$$

We list several important properties of parabolic Eisenstein series. Throughout we omit the proofs and just refer, e.g., to [CS80], chapter 6, instead.

Lemma 3.4.4. *For $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the series (3.13), defining the parabolic Eisenstein series $E_{\eta_j}^{\mathrm{par}}(P, s)$, converges absolutely and locally uniformly. It is a holomorphic function for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$.*

Lemma 3.4.5. *The parabolic Eisenstein series $E_{\eta_j}^{\mathrm{par}}(P, s)$ is invariant in P under the action of Γ, i.e. we have*

$$E_{\eta_j}^{\mathrm{par}}(\gamma P, s) = E_{\eta_j}^{\mathrm{par}}(P, s)$$

for any $\gamma \in \Gamma$, $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$. Thus, we obtain $E_{\eta_j}^{\mathrm{par}}(P, s) \in \mathcal{A}(\Gamma\backslash\mathbb{H}^n)$.

Lemma 3.4.6. *For $P = (x_0, \ldots, x_{n-1}) \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the parabolic Eisenstein series $E_{\eta_j}^{\mathrm{par}}(P, s)$ is infinitely often continuously differentiable with respect to $x_0, \ldots, x_{n-1}$.*

Lemma 3.4.7. *For $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the parabolic Eisenstein series $E_{\eta_j}^{\mathrm{par}}(P, s)$ satisfies the differential equation*

$$(\Delta_{\mathbb{H}^n} - s(n-1-s))\, E_{\eta_j}^{\mathrm{par}}(P, s) = 0.$$

Hence, $E_{\eta_j}^{\mathrm{par}}(P, s)$ is an eigenfunction of $\Delta_{\mathbb{H}^n}$ with eigenvalue $s(n-1-s)$.

If $\eta_k \in C_\Gamma$ ($k \in \{1, \ldots, c_\Gamma\}$) is a cusp of Γ with parabolic scaling matrix $\sigma_{\eta_k} \in \mathrm{PSL}_2(C_{n-1})$ and $P = P_0 + x_{n-1}i_{n-1} \in \mathbb{H}^n$, then the function $E_{\eta_j}^{\mathrm{par}}(\sigma_{\eta_k}P, s)$ is periodic under the lattice Λ_{η_k}, corresponding to the translational stabilizer subgroup Γ'_{η_k}, with respect to P_0. This leads to the following Fourier expansion of $E_{\eta_j}^{\mathrm{par}}(P, s)$.

Theorem 3.4.8. *For $P = (x_0, \ldots, x_{n-1}) \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the parabolic Eisenstein series $E_{\eta_j}^{\mathrm{par}}(P, s)$ admits a Fourier expansion with respect to the cusp $\eta_k \in C_\Gamma$ ($k \in \{1, \ldots, c_\Gamma\}$) of the form*

$$E_{\eta_j}^{\mathrm{par}}(\sigma_{\eta_k}P, s) = \delta_{j,k}\, x_{n-1}^s + \varphi_{\eta_j,\eta_k}(s)\, x_{n-1}^{n-1-s} + \sum_{\substack{\mu \in \Lambda_{\eta_k}^*, \\ \mu \neq 0}} c_{\mu;\eta_j,\eta_k}(s)\, x_{n-1}^{\frac{n-1}{2}}\, K_{s-\frac{n-1}{2}}(2\pi\, |\mu|\, x_{n-1})\, e^{2\pi i \langle \mu, P_0\rangle},$$

where $\delta_{j,k}$ is the Kronecker delta, $\varphi_{\eta_j,\eta_k}(s)$ is a meromorphic function, $\Lambda_{\eta_k}^*$ denotes the dual lattice of Λ_{η_k}, $c_{\mu;\eta_j,\eta_k}(s) \in \mathbb{C}$ and $K_\nu(z)$ is the modified Bessel function of the second kind (see (A.29)).

Theorem 3.4.9. *For $P \in \mathbb{H}^n$ the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_j}(P,s)$ admits a meromorphic continuation in s to the whole complex plane. It has only finitely many poles with* $\mathrm{Re}(s) > \frac{n-1}{2}$*; they are located in the interval $(\frac{n-1}{2}, n-1]$ on the real axis and are simple. Moreover, there is always a simple pole at $s = n-1$.*

Definition 3.4.10. The meromorphic functions $\varphi_{\eta_j,\eta_k}(s)$ $(j,k = 1,\ldots,c_\Gamma)$ appearing in the Fourier expansion in Theorem 3.4.8 are called the *scattering constants*, and the matrix

$$\Phi(s) := (\varphi_{\eta_j,\eta_k}(s))_{j,k=1,\ldots,c_\Gamma}$$

with the scattering constants $\varphi_{\eta_j,\eta_k}(s)$ as entries is called the *scattering matrix*.

Writing all parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_j}(P,s)$ $(j = 1,\ldots,c_\Gamma)$ together in a vector

$$\mathbf{E}^{\mathrm{par}}(P,s) := \Big(E^{\mathrm{par}}_{\eta_1}(P,s),\ldots,E^{\mathrm{par}}_{\eta_{c_\Gamma}}(P,s)\Big)^T,$$

we obtain the following functional equation involving the scattering matrix.

Proposition 3.4.11. *For $s \in \mathbb{C}$ the vector $\mathbf{E}^{\mathrm{par}}(P,s)$ satisfies the functional equation*

$$\mathbf{E}^{\mathrm{par}}(P,n-1-s) = \Phi(n-1-s)\,\mathbf{E}^{\mathrm{par}}(P,s),$$

i.e. for $j = 1,\ldots,c_\Gamma$ we have the identity

$$E^{\mathrm{par}}_{\eta_j}(P,n-1-s) = \sum_{k=1}^{c_\Gamma} \varphi_{\eta_j,\eta_k}(n-1-s)\,E^{\mathrm{par}}_{\eta_k}(P,s). \tag{3.15}$$

Corollary 3.4.12. *For $s \in \mathbb{C}$ the scattering matrix $\Phi(s)$ satisfies the identity*

$$\Phi(n-1-s)\,\Phi(s) = I,$$

i.e. for $j,k = 1,\ldots,c_\Gamma$ we have

$$\sum_{l=1}^{c_\Gamma} \varphi_{\eta_j,\eta_l}(n-1-s)\,\varphi_{\eta_l,\eta_k}(s) = \delta_{j,k}. \tag{3.16}$$

We give a few further properties of the scattering matrix.

Proposition 3.4.13. *The following assertions hold true.*

(a) The scattering matrix $\Phi(s)$ is symmetric, i.e. for $j,k = 1,\ldots,c_\Gamma$ and $s \in \mathbb{C}$ we have $\varphi_{\eta_j,\eta_k}(s) = \varphi_{\eta_k,\eta_j}(s)$.

(b) For $s \in \mathbb{C}$ the scattering matrix $\Phi(s)$ satisfies $\Phi(\overline{s}) = \overline{\Phi(s)}$.

(c) The scattering matrix $\Phi(s)$ is holomorphic for $\mathrm{Re}(s) = \frac{n-1}{2}$*. It is a unitary matrix on this line, i.e. for $j,k = 1,\ldots,c_\Gamma$ and $t \in \mathbb{R}$ we have*

$$\sum_{l=1}^{c_\Gamma} \varphi_{\eta_j,\eta_l}\Big(\frac{n-1}{2}+it\Big)\,\overline{\varphi_{\eta_k,\eta_l}\Big(\frac{n-1}{2}+it\Big)} = \delta_{j,k}.$$

(d) If $\Phi(s)$ is holomorphic at some point $s_0 \in \mathbb{C}$, then also $\mathbf{E}^{\mathrm{par}}(P,s)$ is.

From parts (c) and (d) of the previous proposition one can draw the following fundamental conclusion.

Corollary 3.4.14. *The parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_j}(P,s)$ is holomorphic on the line* $\mathrm{Re}(s) = \frac{n-1}{2}$*. In particular, there are no poles with* $\mathrm{Re}(s) = \frac{n-1}{2}$*.*

Example 3.4.15. We want to consider the case $n = 2$ and particularly the case $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ in more detail. All the results stated in this example are well-known. A more extensive study of parabolic Eisenstein series on $\mathbb{H}$ including proofs could be found, e.g., in [Hej06], [Iwa02] or [Kub73].

(a) Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind, let $\eta_j \in C_\Gamma$ ($j \in \{1, \ldots, c_\Gamma\}$) be a cusp with parabolic scaling matrix $\sigma_{\eta_j} \in \mathrm{PSL}_2(\mathbb{R})$ and stabilizer subgroup Γ_{η_j}. Then for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ the parabolic Eisenstein $E^{\mathrm{par}}_{2,\eta_j}(z, s)$ series associated to the cusp $\eta_j \in C_\Gamma$ is denoted by $E^{\mathrm{par}}_{\eta_j}(z, s)$ and given by

$$E^{\mathrm{par}}_{\eta_j}(z, s) = \sum_{\gamma \in \Gamma_{\eta_j} \backslash \Gamma} \mathrm{Im}(\sigma^{-1}_{\eta_j} \gamma z)^s. \tag{3.17}$$

The series (3.17) converges absolutely and locally uniformly for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, and it is a holomorphic function for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. The parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_j}(z, s)$ is invariant in z under the action of Γ, so $E^{\mathrm{par}}_{\eta_j}(z, s) \in \mathcal{A}(\Gamma \backslash \mathbb{H})$. It satisfies the differential equation

$$\left(\Delta_{\mathbb{H}} - s(1 - s)\right) E^{\mathrm{par}}_{\eta_j}(z, s) = 0.$$

Therefore, $E^{\mathrm{par}}_{\eta_j}(z, s)$ is an eigenfunction of $\Delta_{\mathbb{H}}$ with eigenvalue $s(1 - s)$ which implies that it is a real-analytic function with respect to $z = x + iy$.

For $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_j}(z, s)$ admits the Fourier expansion

$$E^{\mathrm{par}}_{\eta_j}(\sigma_{\eta_k} z, s) = \sum_{m \in \mathbb{Z}} a_{m;\eta_j,\eta_k}(y, s)\, e^{2\pi i m x}$$

with respect to the cusp $\eta_k \in C_\Gamma$ ($k \in \{1, \ldots, c_\Gamma\}$), where the coefficients are given by

$$a_{0;\eta_j,\eta_k}(y, s) = \delta_{j,k}\, y^s + \frac{\sqrt{\pi}\,\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)}\, y^{1-s}\, \varphi_{0;\eta_j,\eta_k}(s),$$

$$a_{m;\eta_j,\eta_k}(y, s) = \frac{\pi^s\, |m|^{s-1}}{\Gamma(s)}\, y^{1/2}\, K_{s-\frac{1}{2}}(2\pi\, |m|\, y)\, \varphi_{m;\eta_j,\eta_k}(s) \qquad (m \neq 0).$$

Here, for $m \in \mathbb{Z}$ the function $\varphi_{m;\eta_j,\eta_k}(s)$ is given by

$$\varphi_{m;\eta_j,\eta_k}(s) = \sum_{c=1}^{\infty} \frac{1}{c^{2s}} \left(\sum_{\substack{d \bmod c, \\ \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \sigma^{-1}_{\eta_j} \Gamma \sigma_{\eta_k}}} \exp\left(2\pi i m \frac{d}{c}\right) \right).$$

The parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_j}(z, s)$ admits a meromorphic continuation to all $s \in \mathbb{C}$. It has no poles with $\mathrm{Re}(s) = \frac{1}{2}$ and only finitely many poles with $\mathrm{Re}(s) > \frac{1}{2}$; they are located in the interval $(\frac{1}{2}, 1]$ on the real axis and are simple. There is always a pole at the point $s = 1$ with residue

$$\mathrm{Res}_{s=1} E^{\mathrm{par}}_{\eta_j}(z, s) = \frac{1}{\mathrm{vol}(\Gamma \backslash \mathbb{H})}.$$

While there are Fuchsian subgroups of the first kind with parabolic Eisenstein series having many poles in the open interval $\left(\frac{1}{2}, 1\right)$, in case that $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ is a congruence subgroup, the only pole of $E^{\mathrm{par}}_{\eta_j}(z, s)$ with $\mathrm{Re}(s) \geq \frac{1}{2}$ is at $s = 1$.

Furthermore, the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_j}(z,s)$ satisfies the functional equation

$$E^{\mathrm{par}}_{\eta_j}(z,1-s) = \sum_{k=1}^{c_\Gamma} \varphi_{\eta_j,\eta_k}(1-s)\, E^{\mathrm{par}}_{\eta_k}(z,s), \tag{3.18}$$

where the scattering constants

$$\varphi_{\eta_j,\eta_k}(s) = \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, \varphi_{0;\eta_j,\eta_k}(s)$$

fulfil the identity

$$\sum_{l=1}^{c_\Gamma} \varphi_{\eta_j,\eta_l}(1-s)\, \varphi_{\eta_l,\eta_k}(s) = \delta_{j,k} \qquad (j,k = 1,\ldots,c_\Gamma).$$

(b) For $n=2$ and the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ there are further interesting results about parabolic Eisenstein series. In this case Γ has only one cusp $\eta_1 = \infty$ with $\sigma_\infty = I$, and for $z = x+iy \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ the parabolic Eisenstein series $E^{\mathrm{par}}_\infty(z,s)$ is given by

$$E^{\mathrm{par}}_\infty(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2,\\ \gcd(c,d)=1}} \frac{\mathrm{Im}(z)^s}{|cz+d|^{2s}}.$$

It admits the Fourier expansion

$$E^{\mathrm{par}}_\infty(z,s) = y^s + \varphi(s)\, y^{1-s} + \sum_{\substack{m\in\mathbb{Z},\\ m\neq 0}} \varphi_m(s)\, y^{1/2}\, K_{s-\frac{1}{2}}(2\pi\,|m|\,y)\; e^{2\pi i m x}$$

with respect to the cusp ∞, where

$$\varphi(s) = \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, \frac{\zeta(2s-1)}{\zeta(2s)}$$

and

$$\varphi_m(s) = \frac{2\pi^s\,|m|^{s-1/2}}{\Gamma(s)\,\zeta(2s)} \sum_{d|m} d^{1-2s}$$

for $m\in\mathbb{Z}$, $m\neq 0$, with $\zeta(s)$ denoting the Riemann zeta function (see (A.28)). Moreover, the functional equation (3.18) simplifies to

$$E^{\mathrm{par}}_\infty(z,1-s) = \varphi(1-s)\, E^{\mathrm{par}}_\infty(z,s). \tag{3.19}$$

The parabolic Eisenstein series $E^{\mathrm{par}}_\infty(z,s)$ has only one pole with $\mathrm{Re}(s) \geq \frac{1}{2}$; it is located at $s=1$ and the residue is $\mathrm{Res}_{s=1} E^{\mathrm{par}}_\infty(z,s) = 3/\pi$. The famous Kronecker limit formula provides information about the Laurent expansion at this pole (see e.g. [Sie80] or [Zag92]). It states that for $z\in\mathbb{H}$ the parabolic Eisenstein series $E^{\mathrm{par}}_\infty(z,s)$ admits a Laurent expansion at $s=1$ of the form

$$E^{\mathrm{par}}_\infty(z,s) = \frac{3}{\pi}\cdot\frac{1}{s-1} - \frac{1}{2\pi}\log\left(|\Delta(z)|\,\mathrm{Im}(z)^6\right) + \frac{6 - 72\,\zeta'(-1) - 6\,\log(4\pi)}{\pi} + \mathrm{O}(s-1).$$

In this formula

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728} \tag{3.20}$$

denotes the Delta function which is a cusp form of weight 12 with respect to $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, where for $k \in 2\mathbb{Z}$ with $k \geq 4$,

$$E_k(z) = \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} \frac{1}{(cz+d)^k} = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2, \\ \gcd(c,d)=1}} \frac{1}{(cz+d)^k} \tag{3.21}$$

is the normalized holomorphic Eisenstein series of weight k. As its name suggests, $E_k(z)$ is a modular form of weight k with respect to $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Further, from the functional equation (3.19) one can deduce that the parabolic Eisenstein series $E_\infty^{\mathrm{par}}(z,s)$ admits a Laurent expansion at $s = 0$ of the form

$$E_\infty^{\mathrm{par}}(z,s) = 1 + \log\Big(|\Delta(z)|^{1/6}\,\mathrm{Im}(z)\Big) \cdot s + \mathrm{O}(s^2).$$

3.5. Spectral expansion

Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup, and let $\Delta_{\mathbb{H}^n}$ be the hyperbolic Laplace operator on $\mathbb{H}^n$. In this section we consider the spectral expansion of a square-integrable automorphic function with respect to Γ in terms of eigenfunctions associated to the discrete and the continuous spectrum of $\Delta_{\mathbb{H}^n}$. Referring to [Söd12], we also give some conditions under which this expansion converges absolutely and locally uniformly on $\mathbb{H}^n$.

The hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ defined on a suitable domain in $\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ has a unique positive, self-adjoint extension to an operator on the whole space $\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$, also denoted by $\Delta_{\mathbb{H}^n}$.

Definition 3.5.1.

(a) A function $f \in \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ is called a *cuspidal function with respect to* Γ if in every cusp $\eta_j \in C_\Gamma$ $(j = 1, \ldots, c_\Gamma)$ it admits a Fourier expansion of the form (3.11) with $a_{0;\eta_j}(x_{n-1}) = 0$.

(b) We define $\mathcal{C}(\Gamma\backslash\mathbb{H}^n)$ as the subspace of $\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ of all cuspidal functions with respect to Γ.

(c) A cuspidal function $f \in \mathcal{C}(\Gamma\backslash\mathbb{H}^n)$ is called a *cusp form with respect to* Γ if it is an eigenfunction of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$.

Let $E_{\eta_k}^{\mathrm{par}}(P,s)$ be the parabolic Eisenstein series associated to the cusp $\eta_k \in C_\Gamma$ $(k = 1, \ldots, c_\Gamma)$ introduced in Definition 3.4.1. Every pole $s = s_j$ of $E_{\eta_k}^{\mathrm{par}}(P,s)$ with $\mathrm{Re}(s_j) > \frac{n-1}{2}$ is related to an eigenvalue λ_j of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ via $\lambda_j = s_j(n-1-s_j)$. By Theorem 3.4.9 the poles s_j are all located in the interval $\big(\frac{n-1}{2}, n-1\big]$ on the real axis. The residue of the parabolic Eisenstein series $E_{\eta_k}^{\mathrm{par}}(P,s)$ at $s = s_j$ is an automorphic function with respect to Γ which is an eigenfunction of $\Delta_{\mathbb{H}^n}$ and an element of $\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$.

Definition 3.5.2. We define $\mathcal{R}(\Gamma\backslash\mathbb{H}^n)$ as the complex vector space that is spanned by the residues of all parabolic Eisenstein series $E_{\eta_k}^{\mathrm{par}}(P,s)$ $(k = 1, \ldots, c_\Gamma)$ at all poles $s = s_j$ in the interval $\big(\frac{n-1}{2}, n-1\big]$.

As each $E_{\eta_k}^{\mathrm{par}}(P,s)$ $(k = 1, \ldots, c_\Gamma)$ has only finitely many poles in the interval $\big(\frac{n-1}{2}, n-1\big]$ by Theorem 3.4.9, the space $\mathcal{R}(\Gamma\backslash\mathbb{H}^n)$ is finite-dimensional.

Definition 3.5.3. We define $\mathcal{E}(\Gamma\backslash\mathbb{H}^n)$ as the complex vector space that is spanned by the parabolic Eisenstein series $E_{\eta_k}^{\mathrm{par}}(P,s)$ $(k = 1, \ldots, c_\Gamma)$ along the line $\mathrm{Re}(s) = \frac{n-1}{2}$, i.e. by $E_{\eta_k}^{\mathrm{par}}\big(P, \frac{n-1}{2} + it\big)$ $(k = 1, \ldots, c_\Gamma)$.

The space $\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ decomposes orthogonally into $\Delta_{\mathbb{H}^n}$-invariant subspaces

$$\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n) = \mathcal{C}(\Gamma\backslash\mathbb{H}^n) \oplus \mathcal{R}(\Gamma\backslash\mathbb{H}^n) \oplus \mathcal{E}(\Gamma\backslash\mathbb{H}^n).$$

The spectrum of $\Delta_{\mathbb{H}^n}$ in $\mathcal{C}(\Gamma\backslash\mathbb{H}^n) \oplus \mathcal{R}(\Gamma\backslash\mathbb{H}^n)$ is discrete, while the spectrum of $\Delta_{\mathbb{H}^n}$ in $\mathcal{E}(\Gamma\backslash\mathbb{H}^n)$ is absolutely continuous.

The discrete spectrum contains 0 and is a discrete subset of the non-negative real numbers $[0, \infty)$. We enumerate the eigenvalues of the discrete spectrum by

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$$

and write

$$\lambda_j = \left(\frac{n-1}{2}\right)^2 + r_j^2 = s_j(n-1-s_j),$$

i.e. $s_j = \frac{n-1}{2} + ir_j$ with $r_j \geq 0$ or $r_j \in \left[-i\,\frac{n-1}{2}, 0\right)$, so that either $s_j \in \left[\frac{n-1}{2}, \frac{n-1}{2} + i\infty\right)$ or $s_j \in \left(\frac{n-1}{2}, n-1\right]$.

Remark 3.5.4. For a discrete subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ which is *geometrically finite*, i.e. which admits a convex fundamental polyhedron with finitely many sides, Lax and Phillips proved in [LP82] that there are only finitely many eigenvalues λ_j of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ on $\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ such that $\lambda_j \in \left[0, \left(\frac{n-1}{2}\right)^2\right)$, each of finite multiplicity (see also, e.g., [GM12]). By [Wie77], Theorem 1 (a), every discrete, cofinite and torsion-free subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ is geometrically finite (see also, e.g., [Kel95], Satz 2.2.3). Further, if $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ is discrete and cofinite but not torsion-free, it has a torsion-free subgroup $\overline{\Gamma}$ of finite index which is also discrete and cofinite (see also the proof of Proposition 2.6.14). Then $\overline{\Gamma}$ admits a convex fundamental polyhedron $\mathcal{P}_{\overline{\Gamma}}$ with finitely many sides, and because of $\overline{\Gamma} \subseteq \Gamma$ a convex fundamental polyhedron $\mathcal{P}_\Gamma$ for Γ can be chosen as a subset of $\mathcal{P}_{\overline{\Gamma}}$. Consequently, also $\mathcal{P}_\Gamma$ has finitely many sides and Γ is geometrically finite.
Hence, in the setting of this section there are only finitely many $j \in \mathbb{N}_0$ with $\lambda_j \in \left[0, \left(\frac{n-1}{2}\right)^2\right)$, that is $s_j \in \left(\frac{n-1}{2}, n-1\right]$ and $r_j \in \left[-i\,\frac{n-1}{2}, 0\right)$.

Now we choose a complete orthonormal system of cusp forms in $\mathcal{C}(\Gamma\backslash\mathbb{H}^n)$ and an orthonormal basis in $\mathcal{R}(\Gamma\backslash\mathbb{H}^n)$ out of which we choose an orthonormal basis $\{\psi_j(P) \mid j \in \mathbb{N}_0\}$ of $\mathcal{C}(\Gamma\backslash\mathbb{H}^n) \oplus \mathcal{R}(\Gamma\backslash\mathbb{H}^n)$, where each $\psi_j(P)$ is an eigenfunction for the discrete eigenvalue λ_j.
In the case $j = 0$ we have $\lambda_0 = 0$, $s_0 = n-1$ and $r_0 = -i\,\frac{n-1}{2}$, and the eigenfunction $\psi_0(P)$ associated to the eigenvalue 0 is given by

$$\psi_0(P) = \frac{1}{\sqrt{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}}.$$

For $j \geq 1$ the eigenfunction $\psi_j(P)$ is a smooth function and admits a Fourier expansion with respect to the cusp $\eta_k \in C_\Gamma$ $(k = 1, \ldots, c_\Gamma)$ of the form (see, e.g., [Söd12], section 4.1)

$$\psi_j(\sigma_{\eta_k} P) = a_{j;0;\eta_k}\, x_{n-1}^{n-1-s_j} + \sum_{\substack{\mu \in \Lambda^*_{\eta_k},\\ \mu \neq 0}} a_{j;\mu;\eta_k}\, x_{n-1}^{\frac{n-1}{2}}\, K_{s_j - \frac{n-1}{2}}(2\pi|\mu|x_{n-1})\, e^{2\pi i \langle \mu, P_0\rangle},$$

where $P = (x_0, \ldots, x_{n-1}) = P_0 + x_{n-1}i_{n-1} \in \mathbb{H}^n$, $\Lambda^*_{\eta_k}$ denotes the dual lattice of Λ_{η_k} and $K_\nu(z)$ is the modified Bessel function of the second kind. Here $a_{j;0;\eta_k} = 0$ is satisfied if $\psi_j(P)$ is a cusp form. Otherwise $\psi_j(P)$ is a linear combination of the residues of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_l}(P, s)$ $(l = 1, \ldots, c_\Gamma)$ at $s = s_j \in \left(\frac{n-1}{2}, n-1\right]$. In the case $s_j \in \left[\frac{n-1}{2}, \frac{n-1}{2} + i\infty\right)$, i.e. $\lambda_j \geq (\frac{n-1}{2})^2$ and $r_j \geq 0$, the function $\psi_j(P)$ is always a cusp form.

The spectrum of $\Delta_{\mathbb{H}^n}$ in $\mathcal{E}(\Gamma\backslash\mathbb{H}^n)$ is absolutely continuous and covers the interval $\left[\left(\frac{n-1}{2}\right)^2, \infty\right)$ uniformly with multiplicity c_Γ. The eigenvalues of the continuous spectrum are of the form

$$\lambda = \left(\frac{n-1}{2}\right)^2 + t^2 = s(n-1-s),$$

i.e. $s = \frac{n-1}{2} + it$ with $t \in \mathbb{R}$, so that $s \in \left(\frac{n-1}{2} - i\infty, \frac{n-1}{2} + i\infty\right)$. The corresponding eigenfunctions are given by the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}\left(P, \frac{n-1}{2} + it\right)$ $(k = 1, \ldots, c_\Gamma)$.

For a function $f \in \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ there is a spectral expansion in terms of the eigenfunctions $\psi_j(P)$ associated to the discrete eigenvalues λ_j of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ and the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P, s)$ associated to the cusps $\eta_k \in C_\Gamma$ $(k = 1, \ldots, c_\Gamma)$ (see, e.g., [Söd12], section 4.1, and [CS80], chapter 7).

Theorem 3.5.5. *Every $f \in \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ admits the spectral expansion*

$$f(P) = \sum_{j=0}^{\infty} a_j\, \psi_j(P) + \sum_{k=1}^{c_\Gamma} \int_0^\infty g_{t,\eta_k}\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt, \tag{3.22}$$

where the coefficients a_j and g_{t,η_k} are given by

$$a_j = \langle f, \psi_j \rangle = \int_{\Gamma\backslash\mathbb{H}^n} f(Q)\, \overline{\psi_j(Q)}\, \mu_{\mathbb{H}^n}(Q),$$
$$g_{t,\eta_k} = \frac{1}{2\pi} \int_{\Gamma\backslash\mathbb{H}^n} f(Q)\, \overline{E^{\mathrm{par}}_{\eta_k}}\Big(Q, \frac{n-1}{2} + it\Big)\, \mu_{\mathbb{H}^n}(Q),$$

respectively. The series (3.22) converges in the $\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$-norm. Moreover, we have a corresponding Parseval's formula

$$||f||^2_{\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)} = \sum_{j=0}^{\infty} |a_j|^2 + 2\pi \sum_{k=1}^{c_\Gamma} \int_0^\infty |g_{t,\eta_k}|^2\, dt.$$

We want to give some conditions on f that are sufficient for the absolute and locally uniform convergence of its spectral expansion on $\mathbb{H}^n$, so that equation (3.22) holds true as a pointwise relation.

Proposition 3.5.6. *Let $k_0 = \left\lfloor \frac{n}{4} \right\rfloor + 1$ and $f \in C^{2k_0}(\mathbb{H}^n) \cap \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ such that $\Delta^l_{\mathbb{H}^n} f \in \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ for $l = 0, \ldots, k_0$. Then the spectral expansion of f converges absolutely and uniformly on compact subsets of $\mathbb{H}^n$.*

Proof. See, e.g., [Söd12], section 4.2.

□

Remark 3.5.7. By the Bessel inequality (see, e.g., [CS80], 7.3 and Corollary 7.7), applied with the full spectral expansion (3.22) (see, e.g., formula (4.9) in [Söd12], and also [Iwa02], Proposition 7.2, for the case $n = 2$), for $T \geq 1$ and $P \in \mathbb{H}^n$ we have

$$\sum_{\substack{j \in \mathbb{N}_0: \\ |r_j| \leq T}} |\psi_j(P)|^2 + \sum_{k=1}^{c_\Gamma} \int_0^T \left| E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big) \right|^2 dt = \mathrm{O}\left(T^n + T\, \mathcal{Y}_\Gamma(P)^{n-1}\right),$$

where

$$\mathcal{Y}_\Gamma(P) := \max_{k \in \{1, \ldots, c_\Gamma\}} \max_{\gamma \in \Gamma} x_{n-1}(\sigma^{-1}_{\eta_k} \gamma P)$$

denotes the *invariant height function* (see, e.g., [Söd12], section 2.3), and where the implied constant depends only on Γ. Particularly, for each $j \in \mathbb{N}_0$ with $|r_j| \leq T$ we have the bound

$$|\psi_j(P)|^2 = \mathrm{O}\left(T^n + T\,\mathcal{Y}_\Gamma(P)^{n-1}\right) = \mathrm{O}(T^n) \qquad (T \to \infty).$$

If $r_j \geq 1$, letting $T = r_j$, for $P \in \mathbb{H}^n$ we obtain

$$|\psi_j(P)|^2 = \mathrm{O}(r_j^n) \quad (r_j \to \infty) \quad \text{and} \quad |\psi_j(P)| = \mathrm{O}\left(r_j^{n/2}\right) \quad (r_j \to \infty).$$

This yields the bound

$$\sup_{P \in \mathbb{H}^n} |\psi_j(P)| = \mathrm{O}\left(r_j^{n/2}\right) \qquad (r_j \to \infty),$$

which will be essential in chapter 6, where we establish meromorphic continuations via spectral expansions.

Remark 3.5.8. We want to rewrite the "Eisenstein part"

$$\begin{aligned}
&\sum_{k=1}^{c_\Gamma} \int_0^\infty g_{t,\eta_k}\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt \\
&\quad = \frac{1}{2\pi} \sum_{k=1}^{c_\Gamma} \int_0^\infty \int_{\Gamma\backslash\mathbb{H}^n} f(Q)\, \overline{E^{\mathrm{par}}_{\eta_k}}\Big(Q, \frac{n-1}{2} + it\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, \mu_{\mathbb{H}^n}(Q)\, dt
\end{aligned}$$

in the spectral expansion (3.22) arising from the continuous spectrum, in order to bring it into the familiar form from the 2-dimensional case. Using the identity

$$\overline{E^{\mathrm{par}}_{\eta_k}}\Big(P, \frac{n-1}{2} + it\Big) = E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} - it\Big)$$

for $P \in \mathbb{H}^n$ and $t \in \mathbb{R}$ and the functional equation (3.15), we have

$$\begin{aligned}
&\sum_{k=1}^{c_\Gamma} \overline{E^{\mathrm{par}}_{\eta_k}}\Big(Q, \frac{n-1}{2} + i(-t)\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + i(-t)\Big) \\
&\quad = \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} + it\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} - it\Big) \\
&\quad = \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} + it\Big) \sum_{l=1}^{c_\Gamma} \varphi_{\eta_k,\eta_l}\Big(\frac{n-1}{2} - it\Big) E^{\mathrm{par}}_{\eta_l}\Big(P, \frac{n-1}{2} + it\Big) \\
&\quad = \sum_{l=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_l}\Big(P, \frac{n-1}{2} + it\Big) \sum_{k=1}^{c_\Gamma} \varphi_{\eta_k,\eta_l}\Big(\frac{n-1}{2} - it\Big) E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} + it\Big).
\end{aligned}$$

The symmetry of the scattering matrix (see Proposition 3.4.13 (a)) and a second application of (3.15) further give us

$$\begin{aligned}
&\sum_{l=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_l}\Big(P, \frac{n-1}{2} + it\Big) \sum_{k=1}^{c_\Gamma} \varphi_{\eta_k,\eta_l}\Big(\frac{n-1}{2} - it\Big) E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} + it\Big) \\
&\quad = \sum_{l=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_l}\Big(P, \frac{n-1}{2} + it\Big) \sum_{k=1}^{c_\Gamma} \varphi_{\eta_l,\eta_k}\Big(\frac{n-1}{2} - it\Big) E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} + it\Big) \\
&\quad = \sum_{l=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_l}\Big(P, \frac{n-1}{2} + it\Big)\, E^{\mathrm{par}}_{\eta_l}\Big(Q, \frac{n-1}{2} - it\Big) \\
&\quad = \sum_{k=1}^{c_\Gamma} \overline{E^{\mathrm{par}}_{\eta_k}}\Big(Q, \frac{n-1}{2} + it\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big).
\end{aligned}$$

This means that

$$\mathbb{R} \to \mathbb{C}, \quad t \mapsto \sum_{k=1}^{c_\Gamma} \overline{E^{\mathrm{par}}_{\eta_k}}\Big(Q, \frac{n-1}{2} + it\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)$$

defines an even function. Thus, provided that all integrals exist, which is particularly the case if the spectral expansion (3.22) converges absolutely and locally uniformly, we can write

$$\begin{aligned}
&\sum_{k=1}^{c_\Gamma} \int_0^\infty g_{t,\eta_k}\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt \\
&= \frac{1}{2\pi} \int_0^\infty \int_{\Gamma\backslash\mathbb{H}^n} f(Q) \left(\sum_{k=1}^{c_\Gamma} \overline{E^{\mathrm{par}}_{\eta_k}}\Big(Q, \frac{n-1}{2} + it\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\right) \mu_{\mathbb{H}^n}(Q)\; dt \\
&= \frac{1}{4\pi} \int_{-\infty}^\infty \int_{\Gamma\backslash\mathbb{H}^n} f(Q) \left(\sum_{k=1}^{c_\Gamma} \overline{E^{\mathrm{par}}_{\eta_k}}\Big(Q, \frac{n-1}{2} + it\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\right) \mu_{\mathbb{H}^n}(Q)\; dt \\
&= \frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^\infty \int_{\Gamma\backslash\mathbb{H}^n} f(Q)\, \overline{E^{\mathrm{par}}_{\eta_k}}\Big(Q, \frac{n-1}{2} + it\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\; \mu_{\mathbb{H}^n}(Q)\; dt \\
&= \frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^\infty a_{t,\eta_k}\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\; dt,
\end{aligned}$$

where the coefficient a_{t,η_k} is given by

$$a_{t,\eta_k} = \int_{\Gamma\backslash\mathbb{H}^n} f(Q)\, \overline{E^{\mathrm{par}}_{\eta_k}}\Big(Q, \frac{n-1}{2} + it\Big)\; \mu_{\mathbb{H}^n}(Q).$$

In chapter 5 we use this form of the spectral expansion of a function $f \in \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$.

4. Hyperbolic and elliptic Eisenstein series in $\mathbb{H}^n$

In this chapter we introduce the functions on the hyperbolic upper half-space $\mathbb{H}^n$ that are the main objects of this thesis. We define hyperbolic Eisenstein series associated to a pair $(Q_1, Q_2) \in H_\Gamma$ of hyperbolic fixed points of a discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ and elliptic Eisenstein series associated to a point $Q \in \mathbb{H}^n$ in the first and second section, respectively, and investigate their basic properties. These series are generalizations of hyperbolic and elliptic Eisenstein series on the upper half-plane which have been considered before. Moreover, in the third section we define the hyperbolic kernel function, that will act as a kind of auxiliary function, and study its properties.

4.1. Hyperbolic Eisenstein series

In [KM79] Kudla and Millson introduced form-valued non-holomorphic Eisenstein series on the upper half-plane $\mathbb{H}$ which are associated to hyperbolic elements of a Fuchsian subgroup of the first kind, and called them "hyperbolic Eisenstein series". Later, scalar-valued hyperbolic Eisenstein series on $\mathbb{H}$ were investigated, e.g., in [Ris04], [Fal07] and [JKP10]. Analogues in higher dimensions were considered, e.g., by Irie who studied hyperbolic Eisenstein series in $\mathbb{H}^3$ that are associated to hyperbolic or loxodromic elements of a cofinite Kleinian group $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{C})$ in [Iri19b], as well as hyperbolic Eisenstein series in the upper half-space $\mathbb{H}^n$ that are associated to an involution in [Iri19a]. We now introduce hyperbolic Eisenstein series in $\mathbb{H}^n$ that are associated to a pair of hyperbolic fixed points of a discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$. We determine its range of convergence and prove several other of its basic properties. To be more precise, we find that the hyperbolic Eisenstein series is invariant under the action Γ, bounded and square-integrable on $\Gamma\backslash\mathbb{H}^n$, a smooth function, and fulfils a certain differential equation under the hyperbolic Laplace operator.

Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. Further, let $(Q_1, Q_2) \in H_\Gamma$ be a pair of hyperbolic fixed points with hyperbolic scaling matrix $\sigma_{(Q_1,Q_2)} \in \mathrm{PSL}_2(C_{n-1})$ and hyperbolic stabilizer subgroup $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$, and let $\mathcal{L}_{(Q_1,Q_2)}$ be the unique geodesic in $\mathbb{H}^n$ connecting Q_1 and Q_2.

Definition 4.1.1. For $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ we define the *hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ associated to the pair $(Q_1,Q_2) \in H_\Gamma$ of hyperbolic fixed points* by

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(Q_1,Q_2)}))^{-s}. \tag{4.1}$$

Notation 4.1.2. In case we want to refer explicitly to the dimension n, we write $E^{\mathrm{hyp}}_{n,(Q_1,Q_2)}(P,s)$ instead of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$. If we want to refer explicitly to the underlying group Γ, we write $E^{\mathrm{hyp}}_{\Gamma,(Q_1,Q_2)}(P,s)$ or $E^{\mathrm{hyp}}_{n,\Gamma,(Q_1,Q_2)}(P,s)$ instead of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$.

Remark 4.1.3. We make two remarks on the definition of hyperbolic Eisenstein series.

(a) In Remark 1.2.2 we have noted that for $n \geq 3$ the hyperbolic distance of a point $P = (x_0, \ldots, x_{n-1}) \in \mathbb{H}^n$ to the positive x_{n-1}-axis $\mathcal{L}_{(0,\infty)}$ satisfies the identity

$$\cosh(d_{\mathbb{H}^n}(P, \mathcal{L}_{(0,\infty)})) = \cosh(d_{\mathbb{H}^n}(P, (0, \ldots, 0, |P|))) = \frac{|P|}{x_{n-1}} = \frac{1}{\cos(\theta_{n-1}(P))}$$

with $\theta_{n-1}(P)$ as in Definition 1.2.1 (b). As the element $\sigma_{(Q_1,Q_2)} \in \mathrm{PSL}_2(C_{n-1})$ maps $\mathcal{L}_{(0,\infty)}$ onto the geodesic $\mathcal{L}_{(Q_1,Q_2)}$, and the hyperbolic distance is $\mathrm{PSL}_2(C_{n-1})$-invariant, for $n \geq 3$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ can be written in terms of the hyperbolic coordinates $u, \theta_1, \ldots, \theta_{n-1}$ as

$$\begin{aligned} E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) &= \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma} \cosh\left(d_{\mathbb{H}^n}\left(\gamma P, \sigma_{(Q_1,Q_2)}\mathcal{L}_{(0,\infty)}\right)\right)^{-s} \\ &= \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma} \cosh\left(d_{\mathbb{H}^n}\left(\sigma^{-1}_{(Q_1,Q_2)}\gamma P, \mathcal{L}_{(0,\infty)}\right)\right)^{-s} \\ &= \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma} \left(\frac{x_{n-1}\left(\sigma^{-1}_{(Q_1,Q_2)}\gamma P\right)}{\left|\sigma^{-1}_{(Q_1,Q_2)}\gamma P\right|}\right)^{s} = \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma} \cos\left(\theta_{n-1}\left(\sigma^{-1}_{(Q_1,Q_2)}\gamma P\right)\right)^{s}. \end{aligned}$$

Analogously, in the case $n = 2$ we have by Remark 1.2.2

$$\cosh(d_{\mathbb{H}}(z, \mathcal{L}_{(0,\infty)})) = \cosh(d_{\mathbb{H}}(z, i|z|)) = \frac{|z|}{y} = \frac{1}{\sin(\theta(z))},$$

where $z = x + iy \in \mathbb{H}$ and $\theta(z)$ is given as in Definition 1.2.1 (a). Hence, the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{2,(Q_1,Q_2)}(z,s)$ can be written in terms of the hyperbolic coordinates u, θ as

$$\begin{aligned} E^{\mathrm{hyp}}_{2,(Q_1,Q_2)}(z,s) &= \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma} \cosh\left(d_{\mathbb{H}}\left(\sigma^{-1}_{(Q_1,Q_2)}\gamma z, \mathcal{L}_{(0,\infty)}\right)\right)^{-s} \\ &= \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma} \left(\frac{\mathrm{Im}\left(\sigma^{-1}_{(Q_1,Q_2)}\gamma z\right)}{\left|\sigma^{-1}_{(Q_1,Q_2)}\gamma z\right|}\right)^{s} = \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma} \sin\left(\theta\left(\sigma^{-1}_{(Q_1,Q_2)}\gamma z\right)\right)^{s}. \end{aligned}$$

(b) The hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ can be expressed as the Stieltjes integral

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = \int_0^\infty \cosh(u)^{-s}\, dN^{\mathrm{hyp}}_{(Q_1,Q_2)}(u;P), \tag{4.2}$$

where $N^{\mathrm{hyp}}_{(Q_1;Q_2)}(u;P)$ denotes the counting function

$$N^{\mathrm{hyp}}_{(Q_1,Q_2)}(u;P) := \left|\left\{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma \;\middle|\; d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(Q_1,Q_2)}) < u\right\}\right|.$$

In the rest of the section we give some elementary properties of hyperbolic Eisenstein series. We start by proving its absolute and locally uniform convergence.

Lemma 4.1.4. *The following assertions hold true.*

(a) For fixed $P \in \mathbb{H}^n$ the series (4.1)*, defining the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$, converges absolutely and locally uniformly for $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > n-1$*, hence it defines a holomorphic function there.*

(b) For fixed $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ *the series* (4.1)*, defining the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$, converges absolutely and uniformly for P ranging over compact subsets of $\mathbb{H}^n$.*

Proof. The proof is similar to the 2-dimensional case (see [Pip05], see also [KM79]), but we have to make some adaptations for general n.

(a) We write $s = \sigma + it \in \mathbb{C}$ and assume that $\sigma = \mathrm{Re}(s) > n-1$. First we suppose that the hyperbolic fixed points are given by $Q_1 = 0$ and $Q_2 = \infty$ and prove the assertion for the hyperbolic Eisenstein series

$$E^{\mathrm{hyp}}_{(0,\infty)}(P,s) = \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)} \backslash \Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(0,\infty)}))^{-s}.$$

Choosing $\sigma_{(Q_1,Q_2)} = \sigma_{(0,\infty)} = I$ as hyperbolic scaling matrix, by Remark 4.1.3 (a) we can write

$$E^{\mathrm{hyp}}_{(0,\infty)}(P,s) = \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)} \backslash \Gamma} \left(\frac{x_{n-1}(\gamma P)}{|\gamma P|} \right)^s .$$

Since

$$\Gamma^{\mathrm{hyp}}_{(0,\infty)} = \left\{ \begin{pmatrix} \mu^m & 0 \\ 0 & \mu^{-m} \end{pmatrix} \;\middle|\; m \in \mathbb{Z} \right\} / \{\pm I\}$$

for some $\mu \in \mathbb{R}$, $\mu > 1$, by Proposition 2.6.31 (b), the set

$$\mathcal{F}_{\Gamma^{\mathrm{hyp}}_{(0,\infty)}} = \left\{ P \in \mathbb{H}^n \mid 1 < |P| < \mu^2 \right\}$$

is a fundamental domain for $\Gamma^{\mathrm{hyp}}_{(0,\infty)}$.

Now let $P \in \mathbb{H}^n$ be fixed. Due to the definition of a fundamental domain, for any $\gamma \in \Gamma$ there exists an element $\gamma' \in \Gamma^{\mathrm{hyp}}_{(0,\infty)}$ such that $\gamma'\gamma P \in \overline{\mathcal{F}}_{\Gamma^{\mathrm{hyp}}_{(0,\infty)}}$, and $\gamma'\gamma \in \Gamma$ represents the same right coset in $\Gamma^{\mathrm{hyp}}_{(0,\infty)} \backslash \Gamma$ as γ. Therefore, the representatives $\gamma \in \Gamma$ of the right cosets $\Gamma^{\mathrm{hyp}}_{(0,\infty)} \backslash \Gamma$ can be chosen such that they all satisfy $\gamma P \in \overline{\mathcal{F}}_{\Gamma^{\mathrm{hyp}}_{(0,\infty)}}$, and even $1 \leq |\gamma P| < \mu^2$. This implies that $|\gamma P| \geq 1$ for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)} \backslash \Gamma$, so we obtain the bound

$$\left| E^{\mathrm{hyp}}_{(0,\infty)}(P,s) \right| \leq \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)} \backslash \Gamma} \left| \left(\frac{x_{n-1}(\gamma P)}{|\gamma P|} \right)^s \right| = \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)} \backslash \Gamma} \frac{|x_{n-1}(\gamma P)^s|}{|\gamma P|^\sigma} \leq \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)} \backslash \Gamma} |x_{n-1}(\gamma P)^s| \, .$$

The function $\mathbb{H}^n \to \mathbb{C}$, $R \mapsto x_{n-1}(R)^s$ is an eigenfunction of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ with eigenvalue $\lambda = s(n-1-s)$ as

$$\begin{aligned} \Delta_{\mathbb{H}^n} \, x_{n-1}(R)^s &= \left(-x_{n-1}^2 \, \frac{\partial^2}{\partial x_{n-1}^2} + (n-2) \, x_{n-1} \, \frac{\partial}{\partial x_{n-1}} \right) x_{n-1}^s \\ &= -s(s-1) \, x_{n-1}^s + (n-2) \, s \, x_{n-1}^s = s(n-1-s) \, x_{n-1}^s . \end{aligned}$$

Moreover, for $\varepsilon > 0$ we define the function

$$K_\varepsilon : \mathbb{H}^n \times \mathbb{H}^n \to \{0,1\}, \quad K_\varepsilon(R,Q) := \begin{cases} 1, & \text{if } d_{\mathbb{H}^n}(R,Q) < \varepsilon, \\ 0, & \text{otherwise,} \end{cases}$$

which clearly is a point-pair invariant function. Thus, it yields a $\mathrm{PSL}_2(C_{n-1})$-invariant integral operator L_ε which is given by

$$(L_\varepsilon f)(R) = \int_{\mathbb{H}^n} K_\varepsilon(R,Q) \, f(Q) \; \mu_{\mathbb{H}^n}(Q).$$

By Proposition 3.2.13 there is a constant $\Lambda_\varepsilon = \Lambda_\varepsilon(\lambda, K_\varepsilon) \in \mathbb{C}$, depending only on λ and K_ε, such that

$$L_\varepsilon \, x_{n-1}(R)^s = \Lambda_\varepsilon \, x_{n-1}(R)^s,$$

i.e. for any $R \in \mathbb{H}^n$ we have

$$x_{n-1}(R)^s = \frac{1}{\Lambda_\varepsilon} \int_{\mathbb{H}^n} K_\varepsilon(R,Q)\, x_{n-1}(Q)^s \ \mu_{\mathbb{H}^n}(Q) = \frac{1}{\Lambda_\varepsilon} \int_{B_\varepsilon(R)} x_{n-1}(Q)^s \ \mu_{\mathbb{H}^n}(Q),$$

where $B_\varepsilon(R)$ denotes the open hyperbolic ball with center R and radius ε.

Now let $\varepsilon = \varepsilon(P) > 0$ be chosen sufficiently small such that the open hyperbolic balls $B_{\varepsilon(P)}(\gamma P)$ with center γP and radius $\varepsilon(P)$ do not intersect for all $\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma$ and are all contained in the open box

$$(-\mu^2, \mu^2)^{n-1} \times (0, \mu^2) \subseteq \mathbb{H}^n.$$

Such an $\varepsilon(P)$ exists since Γ acts discontinuously on $\mathbb{H}^n$. Then by our above considerations we have

$$x_{n-1}(\gamma P)^s = \frac{1}{\Lambda_{\varepsilon(P)}} \int_{B_{\varepsilon(P)}(\gamma P)} x_{n-1}(Q)^s \ \mu_{\mathbb{H}^n}(Q)$$

for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma$. Hence, writing $Q = (y_0, \ldots, y_{n-1})$, we obtain the bound

$$\begin{aligned}
\sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma} |x_{n-1}(\gamma P)^s| &\leq \frac{1}{|\Lambda_{\varepsilon(P)}|} \sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma} \int_{B_{\varepsilon(P)}(\gamma P)} |x_{n-1}(Q)^s| \ \mu_{\mathbb{H}^n}(Q) \\
&= \frac{1}{|\Lambda_{\varepsilon(P)}|} \sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma} \int_{B_{\varepsilon(P)}(\gamma P)} x_{n-1}(Q)^\sigma \ \mu_{\mathbb{H}^n}(Q) \\
&\leq \frac{1}{|\Lambda_{\varepsilon(P)}|} \int_0^{\mu^2} \int_{-\mu^2}^{\mu^2} \cdots \int_{-\mu^2}^{\mu^2} y_{n-1}^{\sigma-n} \ dy_0 \ldots dy_{n-2} \ dy_{n-1} \\
&= \frac{(2\mu^2)^{n-1}}{|\Lambda_{\varepsilon(P)}|} \int_0^{\mu^2} y_{n-1}^{\sigma-n} \ dy_{n-1} = \frac{(2\mu^2)^{n-1}}{|\Lambda_{\varepsilon(P)}|} \, \frac{(\mu^2)^{\sigma-n+1}}{\sigma-n+1} \\
&= \frac{2^{n-1}\,\mu^{2\sigma}}{|\Lambda_{\varepsilon(P)}|\,(\sigma-n+1)},
\end{aligned}$$

where we used that $\sigma > n-1$. This shows the absolute and locally uniform convergence of the series $E^{\mathrm{hyp}}_{(0,\infty)}(P,s)$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$.

For the general case let $(Q_1, Q_2) \in H_\Gamma$ be an arbitrary pair of hyperbolic fixed points of Γ. Then by Remark 2.6.39 the group $\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)} \subseteq \mathrm{PSL}_2(C_{n-1})$ is a discrete and cofinite subgroup with the hyperbolic fixed points 0 and ∞ and

$$S := \left(\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)}\right)^{\mathrm{hyp}}_{(0,\infty)} = \sigma^{-1}_{(Q_1,Q_2)}\,\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\,\sigma_{(Q_1,Q_2)}.$$

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ and $Q := \sigma^{-1}_{(Q_1,Q_2)}P$ this gives us

$$\begin{aligned}
E^{\mathrm{hyp}}_{\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)},(0,\infty)}(Q,s) &= \sum_{\gamma\in S\backslash\left(\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)}\right)} \cosh(d_{\mathbb{H}^n}(\gamma Q, \mathcal{L}_{(0,\infty)}))^{-s} \\
&= \sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\Gamma} \cosh\left(d_{\mathbb{H}^n}\left(\sigma^{-1}_{(Q_1,Q_2)}\gamma\sigma_{(Q_1,Q_2)}Q, \mathcal{L}_{(0,\infty)}\right)\right)^{-s} \\
&= \sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\Gamma} \cosh\left(d_{\mathbb{H}^n}\left(\gamma P, \sigma_{(Q_1,Q_2)}\mathcal{L}_{(0,\infty)}\right)\right)^{-s} \\
&= \sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(Q_1,Q_2)}))^{-s} = E^{\mathrm{hyp}}_{\Gamma,(Q_1,Q_2)}(P,s).
\end{aligned}$$

Thus, the absolute and locally uniform convergence of $E^{\mathrm{hyp}}_{\sigma^{-1}_{(Q_1,Q_2)}\Gamma\,\sigma_{(Q_1,Q_2)},(0,\infty)}(Q,s)$ for fixed $Q\in\mathbb{H}^n$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$, proven above, implies the absolute and locally uniform convergence of $E^{\mathrm{hyp}}_{\Gamma,(Q_1,Q_2)}(P,s)$ for fixed $P\in\mathbb{H}^n$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$.
This proves the claim for an arbitrary discrete and cofinite subgroup $\Gamma\subseteq\mathrm{PSL}_2(C_{n-1})$ with the pair (Q_1,Q_2) of hyperbolic fixed points.

(b) Let $K\subseteq\mathbb{H}^n$ be a compact subset. Then the constant $\varepsilon(P)$ in part (a) of the proof can be chosen uniformly for all $P\in K$, i.e. as

$$\varepsilon := \min_{P\in K}\,\varepsilon(P).$$

Through this ε we also obtain a uniform constant $\Lambda_\varepsilon=\Lambda_{\varepsilon(P)}$ for all $P\in K$, satisfying

$$|\Lambda_\varepsilon| = \min_{P\in K}\,|\Lambda_{\varepsilon(P)}|.$$

Hence, for fixed $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ the series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ converges absolutely and uniformly on K.

□

We can easily conclude from its definition that the hyperbolic Eisenstein series is Γ-invariant.

Lemma 4.1.5. *The hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ is invariant in P under the action of Γ, i.e. we have*

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(\gamma P,s) = E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$$

for any $\gamma\in\Gamma$, $P\in\mathbb{H}^n$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$. Thus, we have $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)\in\mathcal{A}(\Gamma\backslash\mathbb{H}^n)$.

Proof. If $\gamma\in\Gamma$ is fixed, then $\eta\gamma$ runs through a system of representatives for $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\Gamma$ if and only if η does. This implies

$$\begin{aligned} E^{\mathrm{hyp}}_{(Q_1,Q_2)}(\gamma P,s) &= \sum_{\eta\in\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\Gamma} \cosh(d_{\mathbb{H}^n}(\eta\gamma P,\mathcal{L}_{(Q_1,Q_2)}))^{-s} \\ &= \sum_{\eta\in\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\Gamma} \cosh(d_{\mathbb{H}^n}(\eta P,\mathcal{L}_{(Q_1,Q_2)}))^{-s} = E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s). \end{aligned}$$

□

Now we show that the hyperbolic Eisenstein series is bounded on $\Gamma\backslash\mathbb{H}^n$ and therefore a square-integrable function.

Lemma 4.1.6. *For $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ is bounded on $\Gamma\backslash\mathbb{H}^n$ and satisfies $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)\in\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$.*

Proof. For $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ the series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ converges absolutely and locally uniformly on $\mathbb{H}^n$, so it only remains to prove that the hyperbolic Eisenstein series is also bounded at the cusps of $\Gamma\backslash\mathbb{H}^n$.

For that we first assume that the hyperbolic fixed points are given by $Q_1=0$ and $Q_2=\infty$ and consider the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(0,\infty)}(P,s)$. As ∞ is a hyperbolic fixed point of Γ, Lemma 2.6.3 (a) implies that ∞ is no cusp of Γ. Because of $\Gamma^{\mathrm{hyp}}_{(0,\infty)}\subseteq\Gamma$ we can choose the representatives $\eta_1,\ldots,\eta_{c_\Gamma}$ of the cusps such that $\eta_j\in\mathbb{R}^{n-1}$ with $1\le|\eta_j|<\mu^2$ $(j=1,\ldots,c_\Gamma)$. Now the bound

$$\left|E^{\mathrm{hyp}}_{(0,\infty)}(P,s)\right| \le \frac{2^{n-1}\,\mu^{2\sigma}}{|\Lambda_{\varepsilon(P)}|\,(\sigma-n+1)}$$

in the proof of Lemma 4.1.4 (a), where $\sigma = \mathrm{Re}(s)$, shows that $E^{\mathrm{hyp}}_{(0,\infty)}(P,s)$ is bounded for $P \to \eta_j$.

In the case of an arbitrary pair $(Q_1, Q_2) \in H_\Gamma$ of hyperbolic fixed points of Γ we have seen in the proof of Lemma 4.1.4 (a) that $\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)} \subseteq \mathrm{PSL}_2(C_{n-1})$ is a discrete and cofinite subgroup with the hyperbolic fixed points 0 and ∞, and for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ we have

$$E^{\mathrm{hyp}}_{\Gamma,(Q_1,Q_2)}(P,s) = E^{\mathrm{hyp}}_{\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)},(0,\infty)}\big(\sigma^{-1}_{(Q_1,Q_2)}P,s\big).$$

Moreover, if $\eta_j \in C_\Gamma$ is a cusp of Γ, then $\sigma^{-1}_{(Q_1,Q_2)}\eta_j$ is a cusp of $\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)}$ by Remark 2.6.20. Now the boundedness of $E^{\mathrm{hyp}}_{\Gamma,(Q_1,Q_2)}(P,s)$ as P tends to η_j is established by the boundedness of $E^{\mathrm{hyp}}_{\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)},(0,\infty)}\big(\sigma^{-1}_{(Q_1,Q_2)}P,s\big)$ as $\sigma^{-1}_{(Q_1,Q_2)}P$ tends to $\sigma^{-1}_{(Q_1,Q_2)}\eta_j$.

Since $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ is a cofinite subgroup, the hyperbolic volume $\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)$ is finite. Together with the boundedness this proves that $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) \in \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$.

□

The next lemma gives us that the hyperbolic Eisenstein series is a smooth function in $P \in \mathbb{H}^n$.

Lemma 4.1.7. *For $P = (x_0, \ldots, x_{n-1}) \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ is infinitely often continuously differentiable with respect to $x_0, \ldots, x_{n-1}$.*

Proof. For fixed $\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\Gamma$ we let $\sigma^{-1}_{(Q_1,Q_2)}\gamma =: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(C_{n-1})$ and write

$$\begin{aligned} g_\gamma(P) :&= \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(Q_1,Q_2)})) = \cosh\Big(d_{\mathbb{H}^n}\Big(\sigma^{-1}_{(Q_1,Q_2)}\gamma P, \mathcal{L}_{(0,\infty)}\Big)\Big) \\ &= \frac{\left|\sigma^{-1}_{(Q_1,Q_2)}\gamma P\right|}{x_{n-1}\left(\sigma^{-1}_{(Q_1,Q_2)}\gamma P\right)} = \frac{\left|(aP+b)\,(cP+d)^{-1}\right|}{x_{n-1}\,|cP+d|^{-2}} = \frac{|aP+b|\,|cP+d|}{x_{n-1}}. \end{aligned}$$

If $a \neq 0$, we can write $|aP+b| = |a(P+a^{-1}b)| = |a|\,|P+a^{-1}b|$, and $a^{-1}b \in V_{n-1} \cong \mathbb{R}^{n-1}$ is a vector by the definition of $\mathrm{PSL}_2(C_{n-1})$ and Proposition 2.1.17. Letting $a^{-1}b =: (y_0, \ldots, y_{n-2})$, we get

$$|aP+b| = |a|\sqrt{\sum_{j=0}^{n-2}(x_j+y_j)^2 + x_{n-1}^2}.$$

Analogously, if $c \neq 0$, we write $|cP+d| = |c(P+c^{-1}d)| = |c|\,|P+c^{-1}d|$, and $c^{-1}d \in V_{n-1} \cong \mathbb{R}^{n-1}$ is a vector. We set $c^{-1}d =: (z_0, \ldots, z_{n-2})$, so that

$$|cP+d| = |c|\sqrt{\sum_{j=0}^{n-2}(x_j+z_j)^2 + x_{n-1}^2}.$$

Note that the equalities $a = 0$ and $c = 0$ cannot be fulfilled simultaneously. Hence, we have

$$g_\gamma(P) = \begin{cases} \dfrac{|a|\,|c|\sqrt{\sum_{j=0}^{n-2}(x_j+y_j)^2 + x_{n-1}^2}\,\sqrt{\sum_{j=0}^{n-2}(x_j+z_j)^2 + x_{n-1}^2}}{x_{n-1}}, & \text{if } a \neq 0,\ c \neq 0, \\ \dfrac{|b|\,|c|\sqrt{\sum_{j=0}^{n-2}(x_j+z_j)^2 + x_{n-1}^2}}{x_{n-1}}, & \text{if } a = 0, \\ \dfrac{|a|\,|d|\sqrt{\sum_{j=0}^{n-2}(x_j+y_j)^2 + x_{n-1}^2}}{x_{n-1}}, & \text{if } c = 0. \end{cases} \tag{4.3}$$

Since $x_{n-1} > 0$, in each of the cases in (4.3) the function $g_\gamma(P)$ is infinitely often continuously differentiable with respect to the coordinates $x_0, \ldots, x_{n-1}$, and the same is true for $g_\gamma(P)^{-s}$, where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$. Therefore, for any multi-index $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{N}_0^n$ the derivative

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(Q_1,Q_2)}))^{-s} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_0^{\alpha_0} \ldots \partial x_{n-1}^{\alpha_{n-1}}} \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(Q_1,Q_2)}))^{-s}$$

exists and is continuous, and we are allowed to arbitrarily interchange the order of differentiation. We are left to prove that for any $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ the series of partial derivatives

$$\sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma} \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(Q_1,Q_2)}))^{-s} = \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma} \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} g_\gamma(P)^{-s} \tag{4.4}$$

converges absolutely and uniformly on compact subsets $K \subseteq \mathbb{H}^n$, provided that $\mathrm{Re}(s) > n-1$. This is done in eight steps.

We assume that $a \neq 0$ and set $h_\gamma(P) := \sum_{j=0}^{n-2}(x_j + y_j)^2 + x_{n-1}^2$, where $(y_0, \ldots, y_{n-2}) = a^{-1}b$.

In the first step we show that for any $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{N}_0^n$ there exists a constant $A_{\boldsymbol{\alpha}}(K) > 0$, depending only on the compact set K, such that

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} h_\gamma(P) \right| \leq A_{\boldsymbol{\alpha}}(K) \cdot h_\gamma(P) \tag{4.5}$$

for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma$ and $P \in K$. For $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ we find

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} h_\gamma(P) = \begin{cases} h_\gamma(P), \quad \text{if } |\boldsymbol{\alpha}| = \sum\limits_{j=0}^{n-1} \alpha_j = 0, \\ 2(x_k + y_k), \quad \text{if } \alpha_k = 1 \text{ for some } k \in \{0, \ldots, n-2\} \text{ and } \sum\limits_{\substack{j=0,\\ j \neq k}}^{n-1} \alpha_j = 0, \\ 2x_{n-1}, \quad \text{if } \alpha_{n-1} = 1 \text{ and } \sum\limits_{j=0}^{n-2} \alpha_j = 0, \\ 2, \quad \text{if } \alpha_k = 2 \text{ for some } k \in \{0, \ldots, n-1\} \text{ and } \sum\limits_{\substack{j=0,\\ j \neq k}}^{n-1} \alpha_j = 0, \\ 0, \quad \text{if } \alpha_k, \alpha_l \geq 1 \text{ for some } k, l \in \{0, \ldots, n-1\} \text{ with } k \neq l \\ \quad\;\; \text{or } \alpha_k \geq 3 \text{ for some } k \in \{0, \ldots, n-1\}. \end{cases}$$

We verify the bound (4.5) separately in each of these five cases. Both for $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} h_\gamma(P) = 0$ and $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} h_\gamma(P) = h_\gamma(P)$ the bound is immediate. If $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} h_\gamma(P) = 2$, we find

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} h_\gamma(P) \right| = 2 = \frac{2\, h_\gamma(P)}{\sum_{j=0}^{n-2}(x_j + y_j)^2 + x_{n-1}^2} \leq \frac{2}{x_{n-1}^2} \cdot h_\gamma(P).$$

In the case $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} h_\gamma(P) = 2x_{n-1}$ we obtain

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} h_\gamma(P) \right| = 2x_{n-1} = \frac{2x_{n-1}\, h_\gamma(P)}{\sum_{j=0}^{n-2}(x_j + y_j)^2 + x_{n-1}^2} \leq \frac{2}{x_{n-1}} \cdot h_\gamma(P).$$

And for $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} h_\gamma(P) = 2(x_k + y_k)$ for some $k \in \{0, \ldots, n-2\}$ we get

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} h_\gamma(P) \right| = 2\,|x_k + y_k| = \frac{2\,|x_k + y_k|\, h_\gamma(P)}{\sum_{j=0}^{n-2}(x_j + y_j)^2 + x_{n-1}^2} \leq \max\left(\frac{2}{x_{n-1}^2},\, 2 \right) \cdot h_\gamma(P),$$

where we used that

$$|x_k + y_k| \leq \max(1,\, (x_k + y_k)^2) \leq \max\left(1,\, \sum_{j=0}^{n-2}(x_j + y_j)^2 + x_{n-1}^2 \right).$$

This proves that in each case there is a constant $A_{\boldsymbol{\alpha}}(K) > 0$, depending only on K, such that the bound (4.5) is satisfied for any $\gamma \in \Gamma^{\text{hyp}}_{(Q_1,Q_2)} \backslash \Gamma$ and $P \in K$.

In the second step we show by induction over the order $|\boldsymbol{\alpha}|$ that for any $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{N}_0^n$ the partial derivative

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \sqrt{\sum_{j=0}^{n-2}(x_j + y_j)^2 + x_{n-1}^2} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \sqrt{h_\gamma(P)}$$

is a finite sum of summands of the form

$$t \cdot h_\gamma(P)^{1/2-r} \cdot \prod_{j=1}^{r} \frac{\partial^{\left|\boldsymbol{\beta}^{(j)}\right|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} h_\gamma(P), \tag{4.6}$$

where $t \in \mathbb{Q}$, $r \in \mathbb{N}_0$ with $r \leq |\boldsymbol{\alpha}|$ and $\boldsymbol{\beta}^{(j)} \in \mathbb{N}_0^n$ $(j = 1, \ldots, r)$ are multi-indices with $\sum_{j=1}^{r} \left|\boldsymbol{\beta}^{(j)}\right| = |\boldsymbol{\alpha}|$. In the case $|\boldsymbol{\alpha}| = 1$ this is a consequence of

$$\frac{\partial}{\partial x_k} \sqrt{h_\gamma(P)} = \frac{1}{2} \cdot h_\gamma(P)^{-1/2} \cdot \frac{\partial}{\partial x_k} h_\gamma(P)$$

for $k = 0, \ldots, n-1$. Now let $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ with $|\boldsymbol{\alpha}| = m+1$ for some $m \in \mathbb{N}$. Then there is a multi-index $\boldsymbol{\alpha'} = (\alpha'_0, \ldots, \alpha'_{n-1}) \in \mathbb{N}_0^n$ with $|\boldsymbol{\alpha'}| = m$ such that $\alpha_k = \alpha'_k + 1$ for some $k \in \{0, \ldots, n-1\}$ and $\alpha_j = \alpha'_j$ for $j \neq k$, and we have

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \sqrt{h_\gamma(P)} = \frac{\partial}{\partial x_k} \left(\frac{\partial^{|\boldsymbol{\alpha'}|}}{\partial P^{\boldsymbol{\alpha'}}} \sqrt{h_\gamma(P)} \right).$$

Employing the induction hypothesis, the partial derivative in the bracket is a finite sum of summands of the form (4.6), where $t \in \mathbb{Q}$, $r \in \mathbb{N}_0$ with $r \leq m$, and $\boldsymbol{\beta}^{(j)} \in \mathbb{N}_0^n$ $(j = 1, \ldots, r)$ are multi-indices with $\sum_{j=1}^{r} \left|\boldsymbol{\beta}^{(j)}\right| = m$. If we differentiate such a summand with respect to the variable x_k, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial x_k} \left(t \cdot h_\gamma(P)^{1/2-r} \cdot \prod_{j=1}^{r} \frac{\partial^{\left|\boldsymbol{\beta}^{(j)}\right|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} h_\gamma(P) \right) \\
&\quad = t \cdot \left(\frac{1}{2} - r \right) \cdot h_\gamma(P)^{1/2-(r+1)} \cdot \frac{\partial}{\partial x_k} h_\gamma(P) \cdot \prod_{j=1}^{r} \frac{\partial^{\left|\boldsymbol{\beta}^{(j)}\right|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} h_\gamma(P) \\
&\qquad + \sum_{j=1}^{r} \left(t \cdot h_\gamma(P)^{1/2-r} \cdot \frac{\partial}{\partial x_k} \left(\frac{\partial^{\left|\boldsymbol{\beta}^{(j)}\right|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} h_\gamma(P) \right) \cdot \prod_{\substack{l=1,\\ l\neq j}}^{r} \frac{\partial^{\left|\boldsymbol{\beta}^{(l)}\right|}}{\partial P^{\boldsymbol{\beta}^{(l)}}} h_\gamma(P) \right),
\end{aligned}$$

which is again a finite sum of summands of the form

$$t' \cdot h_\gamma(P)^{1/2-r'} \cdot \prod_{j=1}^{r'} \frac{\partial^{|\boldsymbol{\beta'}^{(j)}|}}{\partial P^{\boldsymbol{\beta'}^{(j)}}} h_\gamma(P),$$

where $t' \in \mathbb{Q}$, $r' \in \mathbb{N}_0$ with $r' \leq m+1 = |\boldsymbol{\alpha}|$, and $\boldsymbol{\beta'}^{(j)} \in \mathbb{N}_0^n$ $(j = 1, \ldots, r')$ are multi-indices with $\sum_{j=1}^{r'} |\boldsymbol{\beta'}^{(j)}| = m+1 = |\boldsymbol{\alpha}|$. As the derivation is linear, also the derivative $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \sqrt{h_\gamma(P)}$ is a sum of summands of the asserted form, completing the second step of the proof.

In the third step we deduce from the first two steps that for any $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ there exists a constant $B_{\boldsymbol{\alpha}}(K) > 0$, depending only on K, such that

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \sqrt{h_\gamma(P)} \right| \leq B_{\boldsymbol{\alpha}}(K) \cdot \sqrt{h_\gamma(P)} \tag{4.7}$$

for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma$ and $P \in K$. In the second step we have seen that the derivative $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \sqrt{h_\gamma(P)}$ is a finite sum of summands of the form (4.6). Moreover, by the bound (4.5), for each of these summands there are constants $A_{\boldsymbol{\beta}^{(j)}}(K) > 0$ $(j = 1, \ldots, r)$, depending only on K, such that

$$\begin{aligned} \left| t \cdot h_\gamma(P)^{1/2-r} \cdot \prod_{j=1}^{r} \frac{\partial^{|\boldsymbol{\beta}^{(j)}|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} h_\gamma(P) \right| &= |t| \cdot h_\gamma(P)^{1/2-r} \cdot \prod_{j=1}^{r} \left| \frac{\partial^{|\boldsymbol{\beta}^{(j)}|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} h_\gamma(P) \right| \\ &\leq |t| \cdot h_\gamma(P)^{1/2-r} \cdot \prod_{j=1}^{r} \big(A_{\boldsymbol{\beta}^{(j)}}(K) \cdot h_\gamma(P) \big) = |t| \cdot \sqrt{h_\gamma(P)} \cdot \prod_{j=1}^{r} A_{\boldsymbol{\beta}^{(j)}}(K) \end{aligned}$$

for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma$ and $P \in K$. Now the sum of the terms $|t| \cdot \prod_{j=1}^{r} A_{\boldsymbol{\beta}^{(j)}}(K)$ in the finitely many summands of $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \sqrt{h_\gamma(P)}$ is a constant $B_{\boldsymbol{\alpha}}(K) > 0$, depending only on K, such that the bound (4.7) holds true for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma$ and $P \in K$.

In the fourth step we prove that for any $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{N}_0^n$ there is a constant $C_{\boldsymbol{\alpha}}(K) > 0$, depending only on K, such that

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \frac{1}{x_{n-1}} \right| \leq C_{\boldsymbol{\alpha}}(K) \cdot \frac{1}{x_{n-1}} \tag{4.8}$$

for any $P \in K$. For $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ we easily see that

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \frac{1}{x_{n-1}} = \begin{cases} \dfrac{(-1)^{\alpha_{n-1}}\, \alpha_{n-1}!}{x_{n-1}^{1+\alpha_{n-1}}}, & \text{if } \displaystyle\sum_{j=0}^{n-2} \alpha_j = 0, \\ 0, \quad \text{if } \displaystyle\sum_{j=0}^{n-2} \alpha_j \geq 1. \end{cases}$$

In the second case the bound (4.8) is obvious, while in the first case it follows from

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \frac{1}{x_{n-1}} \right| = \frac{\alpha_{n-1}!}{x_{n-1}^{1+\alpha_{n-1}}} = \frac{\alpha_{n-1}!}{x_{n-1}^{\alpha_{n-1}}} \cdot \frac{1}{x_{n-1}}.$$

This finishes the fourth step of the proof.

In the fifth step we infer from the previous two steps that for any $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ there is a constant $D_{\boldsymbol{\alpha}}(K) > 0$, depending only on K, such that

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \frac{\sqrt{h_\gamma(P)}}{x_{n-1}} \right| \leq D_{\boldsymbol{\alpha}}(K) \cdot \frac{\sqrt{h_\gamma(P)}}{x_{n-1}} \tag{4.9}$$

for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma$ and $P \in K$.

Recall that for two multi-indices $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{n-1}), \boldsymbol{\beta} = (\beta_0, \ldots, \beta_{n-1}) \in \mathbb{N}_0^n$ we have

$$\boldsymbol{\beta} \leq \boldsymbol{\alpha} \iff \beta_j \leq \alpha_j \quad \text{for all } j = 0, \ldots, n-1$$

and

$$\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!\,(\boldsymbol{\alpha} - \boldsymbol{\beta})!} = \prod_{j=0}^{n-1} \frac{\alpha_j!}{\beta_j!\,(\alpha_j - \beta_j)!} = \prod_{j=0}^{n-1} \binom{\alpha_j}{\beta_j}.$$

Now the product rule for partial derivatives yields

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \frac{\sqrt{h_\gamma(P)}}{x_{n-1}} = \sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^n, \\ \boldsymbol{\beta} \leq \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \cdot \frac{\partial^{|\boldsymbol{\beta}|}}{\partial P^{\boldsymbol{\beta}}} \sqrt{h_\gamma(P)} \cdot \frac{\partial^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}}{\partial P^{\boldsymbol{\alpha}-\boldsymbol{\beta}}} \frac{1}{x_{n-1}}.$$

Consequently, by (4.7) and (4.8) there are constants $B_{\boldsymbol{\beta}}(K), C_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(K) > 0$, depending only on K, such that

$$\begin{aligned}
\left| \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \frac{\sqrt{h_\gamma(P)}}{x_{n-1}} \right| &\leq \sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^n, \\ \boldsymbol{\beta} \leq \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \cdot \left| \frac{\partial^{|\boldsymbol{\beta}|}}{\partial P^{\boldsymbol{\beta}}} \sqrt{h_\gamma(P)} \right| \cdot \left| \frac{\partial^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}}{\partial P^{\boldsymbol{\alpha}-\boldsymbol{\beta}}} \frac{1}{x_{n-1}} \right| \\
&\leq \sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^n, \\ \boldsymbol{\beta} \leq \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \cdot B_{\boldsymbol{\beta}}(K) \cdot \sqrt{h_\gamma(P)} \cdot C_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(K) \cdot \frac{1}{x_{n-1}} \\
&= \frac{\sqrt{h_\gamma(P)}}{x_{n-1}} \cdot \left(\sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^n, \\ \boldsymbol{\beta} \leq \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \cdot B_{\boldsymbol{\beta}}(K) \cdot C_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(K) \right)
\end{aligned}$$

for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma$ and $P \in K$. This proves the asserted bound (4.9), where the constant

$$D_{\boldsymbol{\alpha}}(K) := \sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^n, \\ \boldsymbol{\beta} \leq \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \cdot B_{\boldsymbol{\beta}}(K) \cdot C_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(K)$$

depends only on K.

In the sixth step we conclude in each of the three cases in (4.3) that for any $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ there exists a constant $E_{\boldsymbol{\alpha}}(K) > 0$, depending only on the compact set K, such that

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} g_\gamma(P) \right| \leq E_{\boldsymbol{\alpha}}(K) \cdot g_\gamma(P) \tag{4.10}$$

for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma$ and $P \in K$.

In case that $c = 0$, we have $a \neq 0$ and

$$g_\gamma(P) = \frac{|a|\,|d|\,\sqrt{\sum_{j=0}^{n-2}(x_j + y_j)^2 + x_{n-1}^2}}{x_{n-1}} = \frac{|a|\,|d|\,\sqrt{h_\gamma(P)}}{x_{n-1}}.$$

Hence, the bound (4.9) from the fifth step immediately yields

$$\left|\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} g_\gamma(P)\right| = |a|\,|d| \left|\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \frac{\sqrt{h_\gamma(P)}}{x_{n-1}}\right| \leq |a| \cdot |d| \cdot D_{\boldsymbol{\alpha}}(K) \cdot \frac{\sqrt{h_\gamma(P)}}{x_{n-1}} = D_{\boldsymbol{\alpha}}(K) \cdot g_\gamma(P)$$

for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma$ and $P \in K$.

If $a = 0$, then $c \neq 0$ and

$$g_\gamma(P) = \frac{|b|\,|c| \sqrt{\sum_{j=0}^{n-2}(x_j + z_j)^2 + x_{n-1}^2}}{x_{n-1}} = \frac{|b|\,|c| \sqrt{\widetilde{h}_\gamma(P)}}{x_{n-1}}$$

with $\widetilde{h}_\gamma(P) := \sum_{j=0}^{n-2}(x_j + z_j)^2 + x_{n-1}^2$, where $(z_0, \ldots, z_{n-2}) = c^{-1}d$. If we replace y_j by z_j for $j = 0, \ldots, n-2$, then completely analogous to the first five steps of the proof we obtain the bound

$$\left|\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \frac{\sqrt{\widetilde{h}_\gamma(P)}}{x_{n-1}}\right| \leq D_{\boldsymbol{\alpha}}(K) \cdot \frac{\sqrt{\widetilde{h}_\gamma(P)}}{x_{n-1}} \tag{4.11}$$

for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma$ and $P \in K$, with $D_{\boldsymbol{\alpha}}(K)$ as in (4.9). This gives us

$$\left|\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} g_\gamma(P)\right| = |b|\,|c| \left|\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \frac{\sqrt{\widetilde{h}_\gamma(P)}}{x_{n-1}}\right| \leq |b| \cdot |c| \cdot D_{\boldsymbol{\alpha}}(K) \cdot \frac{\sqrt{\widetilde{h}_\gamma(P)}}{x_{n-1}} = D_{\boldsymbol{\alpha}}(K) \cdot g_\gamma(P)$$

for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma$ and $P \in K$.

Moreover, for $a \neq 0$ and $c \neq 0$ we have

$$g_\gamma(P) = \frac{|a|\,|c| \sqrt{\sum_{j=0}^{n-2}(x_j + y_j)^2 + x_{n-1}^2}\, \sqrt{\sum_{j=0}^{n-2}(x_j + z_j)^2 + x_{n-1}^2}}{x_{n-1}} = \frac{|a|\,|c| \sqrt{h_\gamma(P)}\, \sqrt{\widetilde{h}_\gamma(P)}}{x_{n-1}}.$$

From the product rule and the bounds (4.7) and (4.11) we derive the existence of constants $B_{\boldsymbol{\beta}}(K), D_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(K) > 0$, depending only on K, such that

$$\begin{aligned}
\left|\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} g_\gamma(P)\right| &= |a|\,|c| \left|\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \frac{\sqrt{h_\gamma(P)}\, \sqrt{\widetilde{h}_\gamma(P)}}{x_{n-1}}\right| \\
&\leq |a|\,|c| \sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^n, \\ \boldsymbol{\beta} \leq \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \cdot \left|\frac{\partial^{|\boldsymbol{\beta}|}}{\partial P^{\boldsymbol{\beta}}} \sqrt{h_\gamma(P)}\right| \cdot \left|\frac{\partial^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}}{\partial P^{\boldsymbol{\alpha}-\boldsymbol{\beta}}} \frac{\sqrt{\widetilde{h}_\gamma(P)}}{x_{n-1}}\right| \\
&\leq |a|\,|c| \sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^n, \\ \boldsymbol{\beta} \leq \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \cdot B_{\boldsymbol{\beta}}(K) \cdot \sqrt{h_\gamma(P)} \cdot D_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(K) \cdot \frac{\sqrt{\widetilde{h}_\gamma(P)}}{x_{n-1}} \\
&= g_\gamma(P) \cdot \left(\sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^n, \\ \boldsymbol{\beta} \leq \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \cdot B_{\boldsymbol{\beta}}(K) \cdot D_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(K) \right)
\end{aligned}$$

for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma$ and $P \in K$. This proves the bound (4.10), where the constant

$$E_{\boldsymbol{\alpha}}(K) := \sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^n, \\ \boldsymbol{\beta} \leq \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \cdot B_{\boldsymbol{\beta}}(K) \cdot D_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(K)$$

depends only on K, and completes the sixth step of the proof.

In the seventh step we show by induction over $|\boldsymbol{\alpha}|$ that for any $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{n-1}) \in \mathbb{N}_0^n$ the partial derivative

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(Q_1,Q_2)}))^{-s} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} g_\gamma(P)^{-s}$$

is a finite sum of summands of the form

$$p(s) \cdot g_\gamma(P)^{-s-r} \cdot \prod_{j=1}^{r} \frac{\partial^{|\boldsymbol{\beta}^{(j)}|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} g_\gamma(P), \tag{4.12}$$

where $p \in \mathbb{Z}[X]$ is a polynomial with integer coefficients and $\deg(p) \leq |\boldsymbol{\alpha}|$, $r \in \mathbb{N}_0$ with $r \leq |\boldsymbol{\alpha}|$, and $\boldsymbol{\beta}^{(j)} \in \mathbb{N}_0^n$ $(j = 1, \dots, r)$ are multi-indices with $\sum_{j=1}^{r} \left|\boldsymbol{\beta}^{(j)}\right| = |\boldsymbol{\alpha}|$. In the case $|\boldsymbol{\alpha}| = 1$ this follows from

$$\frac{\partial}{\partial x_k} g_\gamma(P)^{-s} = -s \cdot g_\gamma(P)^{-s-1} \cdot \frac{\partial}{\partial x_k} g_\gamma(P)$$

for $k = 0, \dots, n-1$. Now let $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ with $|\boldsymbol{\alpha}| = m+1$ for some $m \in \mathbb{N}$. Then we can choose a multi-index $\boldsymbol{\alpha'} = (\alpha'_0, \dots, \alpha'_{n-1}) \in \mathbb{N}_0^n$ with $|\boldsymbol{\alpha'}| = m$ such that $\alpha_k = \alpha'_k + 1$ for some $k \in \{0, \dots, n-1\}$ and $\alpha_j = \alpha'_j$ for $j \neq k$, and we write

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} g_\gamma(P)^{-s} = \frac{\partial}{\partial x_k} \left(\frac{\partial^{|\boldsymbol{\alpha'}|}}{\partial P^{\boldsymbol{\alpha'}}} g_\gamma(P)^{-s} \right).$$

By the induction hypothesis the partial derivative in the bracket is a finite sum of summands of the form (4.12), where $p \in \mathbb{Z}[X]$ is a polynomial with integer coefficients and $\deg(p) \leq m$, $r \in \mathbb{N}_0$ with $r \leq m$, and $\boldsymbol{\beta}^{(j)} \in \mathbb{N}_0^n$ $(j = 1, \dots, r)$ are multi-indices with $\sum_{j=1}^{r} \left|\boldsymbol{\beta}^{(j)}\right| = m$. Differentiating such a summand with respect to the variable x_k gives us

$$\begin{aligned}
&\frac{\partial}{\partial x_k} \left(p(s) \cdot g_\gamma(P)^{-s-r} \cdot \prod_{j=1}^{r} \frac{\partial^{|\boldsymbol{\beta}^{(j)}|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} g_\gamma(P) \right) \\
&\quad = p(s) \cdot (-s-r) \cdot g_\gamma(P)^{-s-(r+1)} \cdot \frac{\partial}{\partial x_k} g_\gamma(P) \cdot \prod_{j=1}^{r} \frac{\partial^{|\boldsymbol{\beta}^{(j)}|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} g_\gamma(P) \\
&\quad\quad + \sum_{j=1}^{r} \left(p(s) \cdot g_\gamma(P)^{-s-r} \cdot \frac{\partial}{\partial x_k} \left(\frac{\partial^{|\boldsymbol{\beta}^{(j)}|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} g_\gamma(P) \right) \cdot \prod_{\substack{l=1, \\ l \neq j}}^{r} \frac{\partial^{|\boldsymbol{\beta}^{(l)}|}}{\partial P^{\boldsymbol{\beta}^{(l)}}} g_\gamma(P) \right),
\end{aligned}$$

which is again a finite sum of summands of the form

$$p'(s) \cdot g_\gamma(P)^{-s-r'} \cdot \prod_{j=1}^{r'} \frac{\partial^{|\boldsymbol{\beta'}^{(j)}|}}{\partial P^{\boldsymbol{\beta'}^{(j)}}} g_\gamma(P),$$

where $p' \in \mathbb{Z}[X]$ is a polynomial with integer coefficients and $\deg(p') \leq m+1 = |\boldsymbol{\alpha}|$, $r' \in \mathbb{N}_0$ with $r' \leq m+1 = |\boldsymbol{\alpha}|$, and $\boldsymbol{\beta'}^{(j)} \in \mathbb{N}_0^n$ $(j = 1, \dots, r')$ are multi-indices with $\sum_{j=1}^{r'} \left|\boldsymbol{\beta'}^{(j)}\right| = m+1 = |\boldsymbol{\alpha}|$. The linearity of the derivation implies that the same is true for the derivative $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} g_\gamma(P)^{-s}$.

In the eighth and last step we finally conclude that any series (4.4) converges absolutely and uniformly on compact subsets $K \subseteq \mathbb{H}^n$, provided that $\sigma := \mathrm{Re}(s) > n-1$. From the previous step we know that the derivative

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(Q_1,Q_2)}))^{-s} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} g_\gamma(P)^{-s}$$

is a finite sum of summands of the form (4.12). By the bound (4.10) from the sixth step, for each of these summands there are constants $E_{\boldsymbol{\beta}^{(j)}}(K) > 0$ ($j = 1, \ldots, r$), depending only on K, such that

$$\begin{aligned}\left| p(s) \cdot g_\gamma(P)^{-s-r} \cdot \prod_{j=1}^{r} \frac{\partial^{|\boldsymbol{\beta}^{(j)}|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} g_\gamma(P) \right| &= |p(s)| \cdot g_\gamma(P)^{-\sigma-r} \cdot \prod_{j=1}^{r} \left| \frac{\partial^{|\boldsymbol{\beta}^{(j)}|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} g_\gamma(P) \right| \\ \leq |p(s)| \cdot g_\gamma(P)^{-\sigma-r} \cdot \prod_{j=1}^{r} \big(E_{\boldsymbol{\beta}^{(j)}}(K) \cdot g_\gamma(P) \big) &= |p(s)| \cdot g_\gamma(P)^{-\sigma} \cdot \prod_{j=1}^{r} E_{\boldsymbol{\beta}^{(j)}}(K)\end{aligned}$$

for any $\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma$ and $P \in K$. The absolute and uniform convergence of the series

$$\sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma} |p(s)| \cdot g_\gamma(P)^{-\sigma} \cdot \prod_{j=1}^{r} E_{\boldsymbol{\beta}^{(j)}}(K) = |p(s)| \cdot \prod_{j=1}^{r} E_{\boldsymbol{\beta}^{(j)}}(K) \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(Q_1,Q_2)}))^{-\sigma}$$

on K now follows from Lemma 4.1.4 (b), provided that $\sigma > n-1$. This proves the absolute and uniform convergence of the series (4.4) on compact subsets $K \subseteq \mathbb{H}^n$ for $\sigma > n-1$.

□

In contrast to parabolic Eisenstein series, the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ is no eigenfunction of the hyperbolic Laplace operator. However, it still fulfils a certain differential equation under $\Delta_{\mathbb{H}^n}$ involving the shifted function $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s+2)$.

Lemma 4.1.8. *For $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ satisfies the differential equation*

$$(\Delta_{\mathbb{H}^n} - s(n-1-s))\, E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = s^2\, E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s+2).$$

Proof. We first assume that $n \geq 3$ and use the hyperbolic Laplace operator

$$\begin{aligned}\Delta_{\mathbb{H}^n} = &-\cos(\theta_{n-1})^2 \frac{\partial^2}{\partial u^2} - \sum_{k=1}^{n-2} \frac{1}{\left(\prod_{j=k+1}^{n-2} \sin(\theta_j)^2\right) \tan(\theta_{n-1})^2} \frac{\partial^2}{\partial \theta_k^2} - \cos(\theta_{n-1})^2 \frac{\partial^2}{\partial \theta_{n-1}^2} \\ &- \sum_{k=2}^{n-2} \frac{(k-1)\cos(\theta_{n-1})^2}{\left(\prod_{j=k+1}^{n-1} \sin(\theta_j)^2\right) \tan(\theta_k)} \frac{\partial}{\partial \theta_k} - \frac{n-2}{\tan(\theta_{n-1})} \frac{\partial}{\partial \theta_{n-1}}\end{aligned}$$

in hyperbolic coordinates, computed in Lemma 1.2.4 (c), and the representation

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \Gamma} \cos\left(\theta_{n-1}\left(\sigma^{-1}_{(Q_1,Q_2)} \gamma P\right)\right)^s.$$

Since $\Delta_{\mathbb{H}^n}$ is invariant under the action of $\mathrm{PSL}_2(C_{n-1})$ and $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) \in C^2(\mathbb{H}^n)$ by Lemma 4.1.7, it is sufficient to prove the identity

$$(\Delta_{\mathbb{H}^n} - s(n-1-s))\, \cos(\theta_{n-1})^s = s^2\, \cos(\theta_{n-1})^{s+2}.$$

This follows from the straightforward calculation

$$\begin{aligned}
\Delta_{\mathbb{H}^n} \cos(\theta_{n-1})^s &= \left(-\cos(\theta_{n-1})^2 \frac{\partial^2}{\partial\theta_{n-1}^2} - \frac{n-2}{\tan(\theta_{n-1})}\frac{\partial}{\partial\theta_{n-1}}\right)\cos(\theta_{n-1})^s \\
&= -\cos(\theta_{n-1})^2 \left(s(s-1)\cos(\theta_{n-1})^{s-2}\sin(\theta_{n-1})^2 - s\cos(\theta_{n-1})^s\right) \\
&\quad - \frac{n-2}{\tan(\theta_{n-1})}\left(-s\cos(\theta_{n-1})^{s-1}\sin(\theta_{n-1})\right) \\
&= -s(s-1)\cos(\theta_{n-1})^s\sin(\theta_{n-1})^2 + s\cos(\theta_{n-1})^{s+2} + s(n-2)\cos(\theta_{n-1})^s \\
&= -s^2\cos(\theta_{n-1})^s\left(1-\cos(\theta_{n-1})^2\right) + s\cos(\theta_{n-1})^s\left(\sin(\theta_{n-1})^2+\cos(\theta_{n-1})^2\right) \\
&\quad + s(n-2)\cos(\theta_{n-1})^s \\
&= -s^2\cos(\theta_{n-1})^s + s^2\cos(\theta_{n-1})^{s+2} + s\cos(\theta_{n-1})^s + s(n-2)\cos(\theta_{n-1})^s \\
&= s(n-1-s)\cos(\theta_{n-1})^s + s^2\cos(\theta_{n-1})^{s+2}.
\end{aligned}$$

It remains to consider the case $n = 2$. We make use of the hyperbolic Laplace operator

$$\Delta_{\mathbb{H}} = -\sin(\theta)^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial\theta^2}\right)$$

in hyperbolic coordinates from Example 1.2.3 and the representation

$$E^{\mathrm{hyp}}_{2,(Q_1,Q_2)}(z,s) = \sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\Gamma} \sin\left(\theta\left(\sigma^{-1}_{(Q_1,Q_2)}\gamma z\right)\right)^s.$$

Taking into account that $\Delta_{\mathbb{H}}$ is $\mathrm{PSL}_2(C_{n-1})$-invariant and $E^{\mathrm{hyp}}_{2,(Q_1,Q_2)}(z,s)\in C^2(\mathbb{H})$, it suffices to prove the identity

$$(\Delta_{\mathbb{H}} - s(1-s))\sin(\theta)^s = s^2\sin(\theta)^{s+2}.$$

This is a consequence of the straightforward calculation

$$\begin{aligned}
\Delta_{\mathbb{H}}\sin(\theta)^s &= -\sin(\theta)^2\frac{\partial^2}{\partial\theta^2}\sin(\theta)^s = -\sin(\theta)^2\left(s(s-1)\sin(\theta)^{s-2}\cos(\theta)^2 - s\sin(\theta)^s\right) \\
&= -s(s-1)\sin(\theta)^s\cos(\theta)^2 + s\sin(\theta)^{s+2} \\
&= -s^2\sin(\theta)^s\left(1-\sin(\theta)^2\right) + s\sin(\theta)^s\left(\cos(\theta)^2+\sin(\theta)^2\right) \\
&= -s^2\sin(\theta)^s + s^2\sin(\theta)^{s+2} + s\sin(\theta)^s \\
&= s(1-s)\sin(\theta)^s + s^2\sin(\theta)^{s+2}.
\end{aligned}$$

□

We shortly consider the case $n = 2$ as an example. For a more comprehensive study of hyperbolic Eisenstein series on the upper half-plane $\mathbb{H}$ we refer, for instance, to [Fal07], [GJM08], [KM79], [Ris04] or [Pip05].

Example 4.1.9. Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind, and let $(Q_1,Q_2)\in H_\Gamma$ be a pair of hyperbolic fixed points with hyperbolic scaling matrix $\sigma_{(Q_1,Q_2)}\in\mathrm{PSL}_2(\mathbb{R})$ and hyperbolic stabilizer subgroup $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$. Note that $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$ agrees with the full stabilizer subgroup $\Gamma_{(Q_1,Q_2)}$ because the latter contains no elliptic and no loxodromic elements in the 2-dimensional case. Further, let $\mathcal{L}_{(Q_1,Q_2)}$ be the unique geodesic in $\mathbb{H}$ connecting Q_1 and Q_2.
For $z\in\mathbb{H}$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{2,(Q_1,Q_2)}(z,s)$ associated to the pair of hyperbolic fixed points $(Q_1,Q_2)\in H_\Gamma$ is given by

$$E^{\mathrm{hyp}}_{2,(Q_1,Q_2)}(z,s) = \sum_{\gamma\in\Gamma_{(Q_1,Q_2)}\backslash\Gamma}\cosh(d_{\mathbb{H}}(\gamma z,\mathcal{L}_{(Q_1,Q_2)}))^{-s}. \tag{4.13}$$

The series (4.13) converges absolutely and locally uniformly for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, and is a holomorphic function for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Moreover, the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{2,(Q_1,Q_2)}(z,s)$ is invariant in z under the action of Γ and bounded on $\Gamma\backslash\mathbb{H}$, therefore we have $E^{\mathrm{hyp}}_{2,(Q_1,Q_2)}(z,s) \in \mathcal{L}^2(\Gamma\backslash\mathbb{H})$. It satisfies the differential equation

$$(\Delta_{\mathbb{H}} - s(1-s))\, E^{\mathrm{hyp}}_{2,(Q_1,Q_2)}(z,s) = s^2\, E^{\mathrm{hyp}}_{2,(Q_1,Q_2)}(z,s+2).$$

4.2. Elliptic Eisenstein series

Non-holomorphic Eisenstein series on the upper half-plane $\mathbb{H}$ that are associated to elliptic fixed points of a Fuchsian subgroup of the first kind were introduced by Jorgenson and Kramer in 2004 in their unpublished work [JK04] (see also, e.g., [JK11]). Later, their student von Pippich studied these elliptic Eisenstein series in detail in her PhD thesis [Pip10]. In this section we define elliptic Eisenstein series in the upper half-space $\mathbb{H}^n$ that are associated to a discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ and a point $Q \in \mathbb{H}^n$ which is not necessarily an elliptic fixed point of Γ. We prove some of its basic properties, including its range of convergence. Moreover, we see that the elliptic Eisenstein series is invariant under Γ, bounded at the cusps, and satisfies a certain differential equation under the hyperbolic Laplace operator.
Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. Further, let $Q \in \mathbb{H}^n$ be a point with elliptic scaling matrix $\sigma_Q \in \mathrm{PSL}_2(C_{n-1})$ and stabilizer subgroup Γ_Q.

Definition 4.2.1. For $P \in \mathbb{H}^n$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ we define the *elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$ associated to the point $Q \in \mathbb{H}^n$* by

$$E^{\mathrm{ell}}_Q(P,s) = \sum_{\gamma \in \Gamma_Q\backslash\Gamma} \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s}. \tag{4.14}$$

Notation 4.2.2. In case we want to refer explicitly to the dimension n, we write $E^{\mathrm{ell}}_{n,Q}(P,s)$ instead of $E^{\mathrm{ell}}_Q(P,s)$. If we want to refer explicitly to the underlying group Γ, we write $E^{\mathrm{ell}}_{\Gamma,Q}(P,s)$ or $E^{\mathrm{ell}}_{n,\Gamma,Q}(P,s)$ instead of $E^{\mathrm{ell}}_Q(P,s)$.

Remark 4.2.3. We make some remarks on the above definition.

(a) For any $\eta \in \Gamma_Q$ we have $d_{\mathbb{H}^n}(\eta\gamma P, Q) = d_{\mathbb{H}^n}(\gamma P, \eta^{-1}Q) = d_{\mathbb{H}^n}(\gamma P, Q)$, and Γ_Q is a finite subgroup of Γ of order $|\Gamma_Q|$. Hence, we can rewrite the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$ as

$$E^{\mathrm{ell}}_Q(P,s) = \sum_{\gamma \in \Gamma_Q\backslash\Gamma} \frac{1}{|\Gamma_Q|} \sum_{\eta \in \Gamma_Q} \sinh(d_{\mathbb{H}^n}(\eta\gamma P, Q))^{-s} = \frac{1}{|\Gamma_Q|} \sum_{\gamma \in \Gamma} \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s}.$$

(b) If Q is not an elliptic fixed point of Γ, then the stabilizer subgroup Γ_Q is trivial. In this case we have

$$E^{\mathrm{ell}}_Q(P,s) = \sum_{\gamma \in \Gamma} \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s}.$$

(c) The elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$ can be expressed as the Stieltjes integral

$$E^{\mathrm{ell}}_Q(P,s) = \int_0^\infty \sinh(u)^{-s}\, dN^{\mathrm{ell}}_Q(u;P), \tag{4.15}$$

where $N^{\mathrm{ell}}_Q(u;P)$ denotes the counting function

$$N^{\mathrm{ell}}_Q(u;P) := \big|\{\gamma \in \Gamma_Q\backslash\Gamma \mid d_{\mathbb{H}^n}(\gamma P, Q) < u\}\big|. \tag{4.16}$$

In the following we give a few basic properties of elliptic Eisenstein series, beginning with their convergence.

Lemma 4.2.4. *The following assertions hold true.*

(a) For fixed $P \in \mathbb{H}^n$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ the series (4.14)*, defining the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$, converges absolutely and locally uniformly for $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > n-1$*, hence it defines a holomorphic function there.*

(b) For fixed $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ *the series* (4.14)*, defining the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$, converges absolutely and uniformly for P ranging over compact subsets of $\mathbb{H}^n$ not containing any translate γQ of Q by $\gamma \in \Gamma$.*

Proof. We mimic the proof of Lemma 3.1.3 (i), (iii) in [Pip10] for $n = 2$, but make the necessary adaptations to the general n-dimensional case.

(a) We write $s = \sigma + it \in \mathbb{C}$ and assume that $\sigma = \mathrm{Re}(s) > n-1$.
Let $P \in \mathbb{H}^n$ be fixed with $P \neq \gamma Q$ for any $\gamma \in \Gamma$. As the group Γ acts discontinuously on $\mathbb{H}^n$ and $\gamma P \neq Q$ for any $\gamma \in \Gamma$ by the choice of P, the minimum

$$R_{\min}(P) := \min_{\gamma \in \Gamma} d_{\mathbb{H}^n}(\gamma P, Q)$$

exists and is greater than zero. Hence, setting

$$C_1(P) := \frac{1-\exp(-2R_{\min}(P))}{2},$$

we have $C_1(P) > 0$ and

$$\frac{1-\exp(-2\, d_{\mathbb{H}^n}(\gamma P, Q))}{2} \geq C_1(P)$$

for any $\gamma \in \Gamma$. This yields the estimate

$$\begin{aligned}\sinh(d_{\mathbb{H}^n}(\gamma P, Q)) &= \exp(d_{\mathbb{H}^n}(\gamma P, Q)) \cdot \frac{1-\exp(-2\, d_{\mathbb{H}^n}(\gamma P, Q))}{2} \\ &\geq C_1(P) \cdot \exp(d_{\mathbb{H}^n}(\gamma P, Q))\end{aligned}$$

for any $\gamma \in \Gamma$. From this we obtain the bound

$$\begin{aligned}\left|E_Q^{\mathrm{ell}}(P,s)\right| &\leq \sum_{\gamma \in \Gamma_Q \backslash \Gamma} \left|\sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s}\right| = \sum_{\gamma \in \Gamma_Q \backslash \Gamma} \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-\sigma} \\ &\leq C_1(P)^{-\sigma} \sum_{\gamma \in \Gamma_Q \backslash \Gamma} \exp(-\sigma\, d_{\mathbb{H}^n}(\gamma P, Q)).\end{aligned}$$

So it remains to prove that the series

$$\sum_{\gamma \in \Gamma_Q \backslash \Gamma} \exp(-\sigma\, d_{\mathbb{H}^n}(\gamma P, Q))$$

converges locally uniformly for $\sigma > n-1$.

In order to do this, for $r \in \mathbb{R}$, $r > 0$, we consider the counting function $N_Q^{\mathrm{ell}}(r;P)$ given as in (4.16), i.e.

$$N_Q^{\mathrm{ell}}(r;P) = |\mathcal{N}_Q(r;P)| \quad \text{with} \quad \mathcal{N}_Q(r;P) := \{\gamma \in \Gamma_Q \backslash \Gamma \mid d_{\mathbb{H}^n}(\gamma P, Q) < r\}.$$

Recalling Corollary 2.4.6 and Corollary 2.4.7, we note that the number $N_Q^{\mathrm{ell}}(r;P)$ is always finite. Moreover, by the definition of $R_{\min}(P)$ we have $N_Q^{\mathrm{ell}}(r;P) = 0$ for $r \leq R_{\min}(P)$.

Next we derive an estimate for the number $N_Q^{\mathrm{ell}}(r;P)$ for fixed $r \in \mathbb{R}$, $r > 0$. We choose $\varepsilon = \varepsilon(P) > 0$ sufficiently small such that the open hyperbolic balls $B_{\varepsilon(P)}(\gamma P)$ with center γP and radius $\varepsilon(P)$ do not intersect for all $\gamma \in \Gamma_Q \backslash \Gamma$. Then the finitely many balls $B_{\varepsilon(P)}(\gamma P)$ around the translates γP of P for $\gamma \in \mathcal{N}_Q(r;P)$ are all contained in $B_{r+\varepsilon(P)}(Q)$, which gives us

$$N_Q^{\mathrm{ell}}(r;P) \cdot \mathrm{vol}_{\mathbb{H}^n}(B_{\varepsilon(P)}(\gamma P)) \leq \mathrm{vol}_{\mathbb{H}^n}(B_{r+\varepsilon(P)}(Q))$$

for any $\gamma \in \mathcal{N}_Q(r;P)$. This implies the bound

$$\begin{aligned} N_Q^{\mathrm{ell}}(r;P) \leq \frac{\mathrm{vol}_{\mathbb{H}^n}(B_{r+\varepsilon(P)}(Q))}{\mathrm{vol}_{\mathbb{H}^n}(B_{\varepsilon(P)}(\gamma P))} &= \frac{2\,\pi^{n/2}\,\Gamma\left(\frac{n}{2}\right)^{-1} \int_0^{r+\varepsilon(P)} \sinh(t)^{n-1}\,dt}{2\,\pi^{n/2}\,\Gamma\left(\frac{n}{2}\right)^{-1} \int_0^{\varepsilon(P)} \sinh(t)^{n-1}\,dt} \\ &= \frac{\int_0^{r+\varepsilon(P)} \sinh(t)^{n-1}\,dt}{\int_0^{\varepsilon(P)} \sinh(t)^{n-1}\,dt}. \end{aligned}$$

Using the inequality $\sinh(t) \leq \frac{1}{2}\exp(t)$, we can bound the numerator as

$$\begin{aligned} \int_0^{r+\varepsilon(P)} \sinh(t)^{n-1}\,dt \leq 2^{1-n} \int_0^{r+\varepsilon(P)} \exp((n-1)\,t)\,dt &= \frac{2^{1-n}}{n-1}\left(\exp((n-1)\,(r+\varepsilon(P))) - 1\right) \\ &\leq \frac{2^{1-n}}{n-1}\exp((n-1)\,\varepsilon(P))\,\exp((n-1)\,r). \end{aligned}$$

Therefore, we obtain the estimate

$$N_Q^{\mathrm{ell}}(r;P) \leq C_2(P) \cdot \exp((n-1)\,r), \tag{4.17}$$

where the constant $C_2(P) > 0$ is given by

$$C_2(P) := \frac{2^{1-n}\,\exp((n-1)\,\varepsilon(P))}{(n-1)\,\int_0^{\varepsilon(P)} \sinh(t)^{n-1}\,dt}.$$

For fixed $R \in \mathbb{R}$, $R > 0$, the monotonically increasing step function $N_Q^{\mathrm{ell}} : [0,R] \to \mathbb{N}_0$ induces a Stieltjes measure $dN_Q^{\mathrm{ell}}(r;P)$ on the interval $[0,R]$. Since $N_Q^{\mathrm{ell}}(r;P)$ is of bounded variation and the function $\exp(-\sigma r) : [0,R] \to (0,\infty)$ is continuous, the function $\exp(-\sigma r)$ is Riemann-Stieltjes integrable with respect to $N_Q^{\mathrm{ell}}(r;P)$ on the interval $[0,R]$. Moreover, both functions are bounded on $[0,R]$. Thus, applying the theorem of partial integration we obtain

$$\begin{aligned} \sum_{\gamma \in \mathcal{N}_Q(R;P)} \exp(-\sigma\, d_{\mathbb{H}^n}(\gamma P, Q)) &= \int_0^R \exp(-\sigma r)\, dN_Q^{\mathrm{ell}}(r;P) \\ &= \Big[N_Q^{\mathrm{ell}}(r;P)\,\exp(-\sigma r)\Big]_0^R - \int_0^R N_Q^{\mathrm{ell}}(r;P)\, d(\exp(-\sigma r)) \\ &= \Big[N_Q^{\mathrm{ell}}(r;P)\,\exp(-\sigma r)\Big]_0^R + \sigma \int_0^R N_Q^{\mathrm{ell}}(r;P)\,\exp(-\sigma r)\,dr. \end{aligned} \tag{4.18}$$

Using the estimate (4.17), the two summands in (4.18) can be bounded as

$$\Big[N_Q^{\mathrm{ell}}(r;P)\,\exp(-\sigma r)\Big]_0^R = N_Q^{\mathrm{ell}}(R;P) \cdot \exp(-\sigma R) \leq C_2(P) \cdot \exp((n-1-\sigma)\,R)$$

and

$$\begin{aligned} \sigma \int_0^R N_Q^{\mathrm{ell}}(r;P)\,\exp(-\sigma r)\,dr &\leq \sigma\, C_2(P) \int_0^R \exp((n-1-\sigma)\,r)\,dr \\ &= \frac{\sigma\, C_2(P)}{n-1-\sigma} \cdot \left(\exp((n-1-\sigma)\,R) - 1\right), \end{aligned}$$

respectively. These two bounds now imply

$$\begin{aligned}\sum_{\gamma\in\Gamma_Q\backslash\Gamma}\exp(-\sigma\, d_{\mathbb{H}^n}(\gamma P,Q)) &= \lim_{R\to\infty}\sum_{\gamma\in\mathcal{N}_Q(R;P)}\exp(-\sigma\, d_{\mathbb{H}^n}(\gamma P,Q))\\ &\leq \lim_{R\to\infty}\left(C_2(P)\cdot\exp((n-1-\sigma)\,R)+\frac{\sigma\,C_2(P)}{n-1-\sigma}\cdot\big(\exp((n-1-\sigma)\,R)-1\big)\right)\\ &= \frac{\sigma\,C_2(P)}{\sigma-n+1},\end{aligned}$$

where we used $\sigma > n-1$, so that $\lim_{R\to\infty}\exp((n-1-\sigma)\,R)=0$. This gives us the asserted absolute and locally uniform convergence of the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$.

(b) If $K\subseteq\mathbb{H}^n$ is a compact subset not containing any translate γQ of Q by $\gamma\in\Gamma$, then the constants $C_1(P)$ and $C_2(P)$ from part (a) of the proof can be chosen uniformly for all $P\in K$, i.e. we can set

$$C_1 := \min_{P\in K} C_1(P) \quad\text{ and }\quad C_2 := \max_{P\in K} C_2(P).$$

Therefore, for fixed $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ the series $E_Q^{\mathrm{ell}}(P,s)$ converges absolutely and uniformly on K.

$\square$

An easy consequence of the definition of the elliptic Eisenstein series is its Γ-invariance.

Lemma 4.2.5. *The elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ is invariant in P under the action of Γ, i.e. we have*

$$E_Q^{\mathrm{ell}}(\gamma P,s)=E_Q^{\mathrm{ell}}(P,s)$$

for any $\gamma\in\Gamma$, $P\in\mathbb{H}^n$ with $P\neq\eta Q$ for any $\eta\in\Gamma$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$.

Proof. As for a fixed $\gamma\in\Gamma$, $\eta\gamma$ runs through a system of representatives for $\Gamma_Q\backslash\Gamma$ if and only if η does, we have

$$E_Q^{\mathrm{ell}}(\gamma P,s)=\sum_{\eta\in\Gamma_Q\backslash\Gamma}\sinh(d_{\mathbb{H}^n}(\eta\gamma P,Q))^{-s}=\sum_{\eta\in\Gamma_Q\backslash\Gamma}\sinh(d_{\mathbb{H}^n}(\eta P,Q))^{-s}=E_Q^{\mathrm{ell}}(P,s).$$

$\square$

Though the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ is unbounded near the translates γQ of Q by $\gamma\in\Gamma$, it is bounded at the cusps of Γ.

Lemma 4.2.6. *For $P\in\mathbb{H}^n$ with $P\neq\gamma Q$ for any $\gamma\in\Gamma$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ is bounded as $P\to\eta_j$, where $\eta_j\in C_\Gamma$ $(j=1,\ldots,c_\Gamma)$.*

Proof. For $P\in\mathbb{H}^n$ with $P\neq\gamma Q$ for any $\gamma\in\Gamma$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ we have seen in the proof of Lemma 4.2.4 (a) that

$$\left|E_Q^{\mathrm{ell}}(P,s)\right|\leq C_1(P)^{-\sigma}\,\frac{\sigma\,C_2(P)}{\sigma-n+1},$$

where $\sigma=\mathrm{Re}(s)$. This proves that $E_Q^{\mathrm{ell}}(P,s)$ is bounded for $P\to\eta_j$.

$\square$

Remark 4.2.7. For $P=(x_0,\ldots,x_{n-1})\in\mathbb{H}^n$ with $P\neq\gamma Q$ for any $\gamma\in\Gamma$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ is infinitely often continuously differentiable with respect to $x_0,\ldots,x_{n-1}$. We will prove this result as Lemma 4.3.8 in the next section.

Similar to hyperbolic Eisenstein series, also the elliptic Eisenstein series satisfies a certain shift equation under the hyperbolic Laplace operator.

Lemma 4.2.8. *For $P \in \mathbb{H}^n$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ satisfies the differential equation*

$$(\Delta_{\mathbb{H}^n} - s(n-1-s))\, E_Q^{\mathrm{ell}}(P,s) = s(n-2-s)\, E_Q^{\mathrm{ell}}(P,s+2).$$

Proof. Using elliptic coordinates centered at $Q \in \mathbb{H}^n$ (see Definition 1.2.5), by Lemma 1.2.8 (c) the hyperbolic Laplace operator is given by

$$\Delta_{\mathbb{H}^n} = -\frac{\partial^2}{\partial \varrho_Q^2} - (n-1)\,\frac{1}{\tanh(\varrho_Q)}\,\frac{\partial}{\partial \varrho_Q} - \frac{1}{\sinh(\varrho_Q)^2}\,\Delta_{\mathbb{S}^{n-1}}.$$

Further, we have $\varrho_Q(\gamma P) = d_{\mathbb{H}^n}(\gamma P, Q)$ and

$$E_Q^{\mathrm{ell}}(P,s) = \sum_{\gamma \in \Gamma_Q \backslash \Gamma} \sinh(\varrho_Q(\gamma P))^{-s}.$$

Since the differential operator $\Delta_{\mathbb{H}^n}$ is invariant under the action of $\mathrm{PSL}_2(C_{n-1})$ and $E_Q^{\mathrm{ell}}(P,s) \in C^2(\mathbb{H}^n)$, as we will see in Lemma 4.3.8, it suffices to prove the identity

$$(\Delta_{\mathbb{H}^n} - s(n-1-s))\, \sinh(\varrho_Q)^{-s} = s(n-2-s)\, \sinh(\varrho_Q)^{-s-2}.$$

This follows immediately from the calculation

$$\begin{aligned}
\Delta_{\mathbb{H}^n} \sinh(\varrho_Q)^{-s} &= \left(-\frac{\partial^2}{\partial \varrho_Q^2} - (n-1)\,\frac{1}{\tanh(\varrho_Q)}\,\frac{\partial}{\partial \varrho_Q} \right) \sinh(\varrho_Q)^{-s} \\
&= -s(s+1)\, \sinh(\varrho_Q)^{-s-2}\, \cosh(\varrho_Q)^2 + s\, \sinh(\varrho_Q)^{-s} \\
&\quad + s(n-1)\, \sinh(\varrho_Q)^{-s-2}\, \cosh(\varrho_Q)^2 \\
&= (-s(s+1) + s(n-1))\, \sinh(\varrho_Q)^{-s-2}\, (1 + \sinh(\varrho_Q)^2) + s\, \sinh(\varrho_Q)^{-s} \\
&= s(n-2-s)\, \sinh(\varrho_Q)^{-s-2} + s(n-2-s)\, \sinh(\varrho_Q)^{-s} + s\, \sinh(\varrho_Q)^{-s} \\
&= s(n-2-s)\, \sinh(\varrho_Q)^{-s-2} + s(n-1-s)\, \sinh(\varrho_Q)^{-s}.
\end{aligned}$$

□

In the following example we briefly take a closer look at elliptic Eisenstein series on the upper half-plane $\mathbb{H}$ that are associated to an elliptic fixed point of a Fuchsian subgroup of the first kind.

Example 4.2.9. Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind, and let $e_j \in E_\Gamma$ ($j \in \{1, \ldots, e_\Gamma\}$) be an elliptic fixed point of order n_{e_j} with elliptic scaling matrix $\sigma_{e_j} \in \mathrm{PSL}_2(\mathbb{R})$ and stabilizer subgroup Γ_{e_j}. Using elliptic coordinates $\varrho = \varrho(z)$ and $\vartheta = \vartheta(z)$ centered at i (see Example 1.2.7), for $z \in \mathbb{H}$ with $z \neq \gamma e_j$ for any $\gamma \in \Gamma$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ the elliptic Eisenstein series $E_{2,e_j}^{\mathrm{ell}}(z,s)$ associated to the elliptic fixed point $e_j \in E_\Gamma$ is given by

$$E_{2,e_j}^{\mathrm{ell}}(z,s) = \sum_{\gamma \in \Gamma_{e_j} \backslash \Gamma} \sinh(\varrho(\sigma_{e_j}^{-1} \gamma z))^{-s} = \sum_{\gamma \in \Gamma_{e_j} \backslash \Gamma} \sinh(d_{\mathbb{H}}(\gamma z, e_j))^{-s}. \tag{4.19}$$

For $z \in \mathbb{H}$ with $z \neq \gamma e_j$ for any $\gamma \in \Gamma$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ the series (4.19) converges absolutely and locally uniformly, and it defines a holomorphic function for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. The elliptic Eisenstein series $E_{2,e_j}^{\mathrm{ell}}(z,s)$ is invariant in z under the action of Γ. It satisfies the differential equation

$$(\Delta_{\mathbb{H}} - s(1-s))\, E_{2,e_j}^{\mathrm{ell}}(z,s) = -s^2\, E_{2,e_j}^{\mathrm{ell}}(z,s+2)$$

and is a real-analytic function with respect to $z = x + iy$.
In [Pip10], von Pippich further computed the parabolic, hyperbolic and elliptic Fourier expansion of $E^{\mathrm{ell}}_{2,e_j}(z,s)$. Moreover, she proved that the elliptic Eisenstein series $E^{\mathrm{ell}}_{2,e_j}(z,s)$ admits a meromorphic continuation in s to the whole complex plane and computed its possible poles and residues. There is always a simple pole at the point $s = 1$ with residue

$$\mathrm{Res}_{s=1}\, E^{\mathrm{ell}}_{2,e_j}(z,s) = \frac{2\pi}{n_{e_j}\,\mathrm{vol}(\Gamma\backslash\mathbb{H})}.$$

In [Pip10] (see also [Pip16]), von Pippich further determined a Kronecker limit type formula for elliptic Eisenstein series which we will recall in section 8.2.

4.3. The hyperbolic kernel function

Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. Having introduced hyperbolic and elliptic Eisenstein series, we now define a third function called the hyperbolic kernel function. We determine its range of convergence and present some other elementary properties. We see that the hyperbolic kernel function is symmetric in the variables $P, Q \in \mathbb{H}^n$ and Γ-invariant in both of them, bounded and square-integrable on $\Gamma\backslash\mathbb{H}^n$ and a smooth function with respect to P, and that it fulfils a certain differential equation under the hyperbolic Laplace operator. Additionally, we prove that the elliptic Eisenstein series associated to a point in $\mathbb{H}^n$ is also a smooth function. The hyperbolic kernel function will be of great importance in the upcoming chapters, in which we will see that both hyperbolic and elliptic Eisenstein series can be expressed in terms of this function. In the case $n = 2$, this idea was first given in [Pip10] for elliptic Eisenstein series, while in [JPS16] the idea was realized for hyperbolic Eisenstein series.

Definition 4.3.1. For $P, Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ we define the *hyperbolic kernel function* $K^{\mathrm{hyp}}(P,Q,s)$ by

$$K^{\mathrm{hyp}}(P,Q,s) = \sum_{\gamma\in\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s}. \tag{4.20}$$

Notation 4.3.2. In case we want to refer explicitly to the dimension n, we write $K^{\mathrm{hyp}}_n(P,Q,s)$ instead of $K^{\mathrm{hyp}}(P,Q,s)$. If we want to refer explicitly to the underlying group Γ, we write $K^{\mathrm{hyp}}_{\Gamma}(P,Q,s)$ or $K^{\mathrm{hyp}}_{n,\Gamma}(P,Q,s)$ instead of $K^{\mathrm{hyp}}(P,Q,s)$.

Remark 4.3.3. The hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ can be expressed as the Stieltjes integral

$$K^{\mathrm{hyp}}(P,Q,s) = \int_0^\infty \cosh(u)^{-s}\, dN^{\mathrm{hyp}}(u;P,Q), \tag{4.21}$$

where $N^{\mathrm{hyp}}(u;P,Q)$ denotes the counting function

$$N^{\mathrm{hyp}}(u;P,Q) := |\{\gamma \in \Gamma \mid d_{\mathbb{H}^n}(\gamma P, Q) < u\}|.$$

We give some elementary properties of the hyperbolic kernel function in the form of several lemmas.

Lemma 4.3.4. *The following assertions hold true.*

(a) For fixed $P, Q \in \mathbb{H}^n$ the series (4.20), defining the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$, converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$, hence it defines a holomorphic function there.

(b) For fixed $Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the series (4.20), defining the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$, converges absolutely and uniformly for P ranging over compact subsets of $\mathbb{H}^n$.

(c) *For fixed $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the series (4.20), defining the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$, converges absolutely and uniformly for Q ranging over compact subsets of $\mathbb{H}^n$.*

Proof.

(a) By the bound

$$\left|K^{\mathrm{hyp}}(P,Q,s)\right| \leq \sum_{\gamma\in\Gamma} \left|\cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s}\right| = \sum_{\gamma\in\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-\mathrm{Re}(s)}$$

and the inequality $\cosh(x) > \sinh(x)$ ($x \in \mathbb{R}$), for $P, Q \in \mathbb{H}^n$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ the function $K^{\mathrm{hyp}}(P,Q,s)$ can be majorized by the series

$$\sum_{\gamma\in\Gamma} \sinh(d_{\mathbb{H}^n}(\gamma P,Q))^{-\mathrm{Re}(s)} = |\Gamma_Q|\, E_Q^{\mathrm{ell}}(P, \mathrm{Re}(s)).$$

Using Lemma 4.2.4 (a), for fixed $P, Q \in \mathbb{H}^n$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ we obtain the absolute and locally uniform convergence of the series $K^{\mathrm{hyp}}(P,Q,s)$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$.

Now suppose that $P, Q \in \mathbb{H}^n$ with $\gamma_1 P = Q$, i.e. $P = \gamma_1^{-1}Q$, for some $\gamma_1 \in \Gamma$. Then for any $\gamma_2 \in \Gamma$ with $\gamma_2 P = Q$, i.e. $P = \gamma_2^{-1}Q$, we have $\gamma_2\gamma_1^{-1}Q = Q$ and $\gamma_2\gamma_1^{-1} \in \Gamma_Q$, so that γ_1 and γ_2 represent the same right coset in $\Gamma_Q\backslash\Gamma$. Taking into account that $d_{\mathbb{H}^n}(\eta\gamma P, Q) = d_{\mathbb{H}^n}(\gamma P, Q)$ for any $\eta \in \Gamma_Q$, we can write

$$\begin{aligned}
K^{\mathrm{hyp}}(P,Q,s) &= \sum_{\gamma\in\Gamma_Q\backslash\Gamma} \sum_{\eta\in\Gamma_Q} \cosh(d_{\mathbb{H}^n}(\eta\gamma P,Q))^{-s} = |\Gamma_Q| \sum_{\gamma\in\Gamma_Q\backslash\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s} \\
&= |\Gamma_Q| \cosh(d_{\mathbb{H}^n}(\gamma_1 P,Q))^{-s} + |\Gamma_Q| \sum_{\substack{\gamma\in\Gamma_Q\backslash\Gamma,\\ \gamma\neq\gamma_1}} \cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s} \\
&= |\Gamma_Q| + |\Gamma_Q| \sum_{\substack{\gamma\in\Gamma_Q\backslash\Gamma,\\ \gamma\neq\gamma_1}} \cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s} \\
&\leq |\Gamma_Q| + |\Gamma_Q| \sum_{\substack{\gamma\in\Gamma_Q\backslash\Gamma,\\ \gamma\neq\gamma_1}} \sinh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s}.
\end{aligned}$$

The absolute and locally uniform convergence of the latter series for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ can be verified analogously to the proof of Lemma 4.2.4 (a). This proves that the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$.

(b) Analogous to the proof of part (a), depending on whether $\gamma_1 P = Q$ for some $\gamma_1 \in \Gamma$ or not, we can majorize the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ either by the series $|\Gamma_Q|\, E_Q^{\mathrm{ell}}(P, \mathrm{Re}(s))$ or by

$$|\Gamma_Q| + |\Gamma_Q| \sum_{\substack{\gamma\in\Gamma_Q\backslash\Gamma,\\ \gamma\neq\gamma_1}} \sinh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s}.$$

The first series converges absolutely and locally uniformly for P ranging over compact subsets of $\mathbb{H}^n$ by Lemma 4.2.4 (b), while the absolute and uniform convergence of the latter series on compact subsets of $\mathbb{H}^n$ follows analogously to the proof of Lemma 4.2.4 (b). For fixed $Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ this shows that the series $K^{\mathrm{hyp}}(P,Q,s)$ converges absolutely and uniformly for P ranging over compact subsets of $\mathbb{H}^n$.

(c) As the map $\Gamma \to \Gamma$, $\gamma \mapsto \gamma^{-1}$, is a bijection, we obtain

$$\begin{aligned} K^{\mathrm{hyp}}(P,Q,s) &= \sum_{\gamma\in\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s} = \sum_{\gamma\in\Gamma} \cosh(d_{\mathbb{H}^n}(Q,\gamma P))^{-s} \\ &= \sum_{\gamma\in\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma^{-1}Q,P))^{-s} = \sum_{\gamma\in\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma Q,P))^{-s} = K^{\mathrm{hyp}}(Q,P,s). \end{aligned}$$

Hence, the assertion follows immediately from part (b).

□

Lemma 4.3.5. *The following assertions hold true.*

(a) The hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ is symmetric in the variables P and Q, i.e. we have

$$K^{\mathrm{hyp}}(Q,P,s) = K^{\mathrm{hyp}}(P,Q,s)$$

for any $P,Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$.

(b) The hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ is invariant under the action of Γ in both P and Q i.e. we have

$$K^{\mathrm{hyp}}(\gamma P,Q,s) = K^{\mathrm{hyp}}(P,\gamma Q,s) = K^{\mathrm{hyp}}(P,Q,s)$$

for any $\gamma \in \Gamma$, $P,Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$. Hence, for fixed $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$, it is a well-defined function on $(\Gamma\backslash\mathbb{H}^n) \times (\Gamma\backslash\mathbb{H}^n)$.

Proof.

(a) This was already shown in the proof of Lemma 4.3.4 (c).

(b) For fixed $\gamma \in \Gamma$ the map $\eta \mapsto \eta\gamma$ defines a bijection $\Gamma \to \Gamma$, so that

$$K^{\mathrm{hyp}}(\gamma P,Q,s) = \sum_{\eta\in\Gamma} \cosh(d_{\mathbb{H}^n}(\eta\gamma P,Q))^{-s} = \sum_{\eta\in\Gamma} \cosh(d_{\mathbb{H}^n}(\eta P,Q))^{-s} = K^{\mathrm{hyp}}(P,Q,s).$$

Likewise, for fixed $\gamma \in \Gamma$ the map $\eta \mapsto \gamma^{-1}\eta$ defines a bijection $\Gamma \to \Gamma$, which gives us

$$\begin{aligned} K^{\mathrm{hyp}}(P,\gamma Q,s) &= \sum_{\eta\in\Gamma} \cosh(d_{\mathbb{H}^n}(\eta P,\gamma Q))^{-s} = \sum_{\eta\in\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma^{-1}\eta P,Q))^{-s} \\ &= \sum_{\eta\in\Gamma} \cosh(d_{\mathbb{H}^n}(\eta P,Q))^{-s} = K^{\mathrm{hyp}}(P,Q,s). \end{aligned}$$

□

As for the hyperbolic Eisenstein series, we find that the hyperbolic kernel function is bounded on $\Gamma\backslash\mathbb{H}^n$ and an element of $\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$.

Lemma 4.3.6. *For $Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$, as a function in P, is bounded on $\Gamma\backslash\mathbb{H}^n$ and satisfies $K^{\mathrm{hyp}}(P,Q,s) \in \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$.*

Proof. Since for $Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the series $K^{\mathrm{hyp}}(P,Q,s)$ converges absolutely and locally uniformly on $\mathbb{H}^n$, it remains to show that the hyperbolic kernel function is bounded at the cusps of $\Gamma\backslash\mathbb{H}^n$.
If P is sufficiently close to a cusp $\eta_j \in C_\Gamma$ ($j = 1,\dots,c_\Gamma$), then $P \neq \gamma Q$ for any $\gamma \in \Gamma$ and we have seen in the proof of Lemma 4.3.4 (a) that $K^{\mathrm{hyp}}(P,Q,s)$ can be majorized by the elliptic Eisenstein series $|\Gamma_Q|\, E_Q^{\mathrm{ell}}(P,\mathrm{Re}(s))$. Now Lemma 4.2.6 implies that $K^{\mathrm{hyp}}(P,Q,s)$ is bounded as $P \to \eta_j$.

Because the group $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ has finite hyperbolic volume, we deduce from the boundedness that $K^{\mathrm{hyp}}(P,Q,s) \in \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$.

□

Examining the differentiability of the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ with respect to the coordinates of $P=(x_0,\dots,x_{n-1})\in\mathbb{H}^n$, we have the following result.

Lemma 4.3.7. *For $P=(x_0,\dots,x_{n-1}),Q\in\mathbb{H}^n$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ is infinitely often continuously differentiable with respect to $x_0,\dots,x_{n-1}$.*

Proof. For fixed $Q\in\mathbb{H}^n$ and $\gamma\in\Gamma$ we let $\gamma^{-1}Q=:(y_0,\dots,y_{n-1})\in\mathbb{H}^n$ and write

$$f_\gamma(P):=\cosh(d_{\mathbb{H}^n}(\gamma P,Q))=\cosh(d_{\mathbb{H}^n}(P,\gamma^{-1}Q))=\frac{\sum_{j=0}^{n-2}(x_j-y_j)^2+x_{n-1}^2+y_{n-1}^2}{2\,x_{n-1}\,y_{n-1}}. \tag{4.22}$$

As $x_{n-1},y_{n-1}>0$, this function is infinitely often continuously differentiable with respect to the coordinates $x_0,\dots,x_{n-1}$, and the same is true for $f_\gamma(P)^{-s}$, where $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$. Hence, for any multi-index $\boldsymbol{\alpha}=(\alpha_0,\dots,\alpha_{n-1})\in\mathbb{N}_0^n$ the derivative

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}}\cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s}=\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_0^{\alpha_0}\dots\partial x_{n-1}^{\alpha_{n-1}}}\cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s}$$

exists and is continuous, and we are allowed to arbitrarily interchange the order of differentiation. It remains to prove that for any $\boldsymbol{\alpha}\in\mathbb{N}_0^n$ the series of partial derivatives

$$\sum_{\gamma\in\Gamma}\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}}\cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s}=\sum_{\gamma\in\Gamma}\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}}f_\gamma(P)^{-s} \tag{4.23}$$

converges absolutely and uniformly on compact subsets $K\subseteq\mathbb{H}^n$, provided that $\mathrm{Re}(s)>n-1$. We do this in four steps.

In the first step we compute the partial derivative

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}}\cosh(d_{\mathbb{H}^n}(\gamma P,Q))=\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}}f_\gamma(P)$$

for an arbitrary multi-index $\boldsymbol{\alpha}=(\alpha_0,\dots,\alpha_{n-1})\in\mathbb{N}_0^n$. Considering only the derivatives with respect to $x_0,\dots,x_{n-2}$ first, for any $\boldsymbol{\alpha}\in\mathbb{N}_0^n$ with $\alpha_{n-1}=0$ we find

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}}f_\gamma(P)=\begin{cases} f_\gamma(P), \quad \text{if } |\boldsymbol{\alpha}|=\displaystyle\sum_{j=0}^{n-2}\alpha_j=0,\\[2ex] \dfrac{x_k-y_k}{x_{n-1}\,y_{n-1}}, \quad \text{if } \alpha_k=1 \text{ for some } k\in\{0,\dots,n-2\} \text{ and } \displaystyle\sum_{\substack{j=0,\\ j\neq k}}^{n-2}\alpha_j=0,\\[2ex] \dfrac{1}{x_{n-1}\,y_{n-1}}, \quad \text{if } \alpha_k=2 \text{ for some } k\in\{0,\dots,n-2\} \text{ and } \displaystyle\sum_{\substack{j=0,\\ j\neq k}}^{n-2}\alpha_j=0,\\[2ex] 0, \quad \text{if } \alpha_k,\alpha_l\geq 1 \text{ for some } k,l\in\{0,\dots,n-2\} \text{ with } k\neq l\\ \qquad \text{or } \alpha_k\geq 3 \text{ for some } k\in\{0,\dots,n-2\}.\end{cases} \tag{4.24}$$

Now we compute

$$\frac{\partial}{\partial x_{n-1}}f_\gamma(P)=\frac{-\sum_{j=0}^{n-2}(x_j-y_j)^2+x_{n-1}^2-y_{n-1}^2}{2\,x_{n-1}^2\,y_{n-1}},\qquad \frac{\partial^2}{\partial x_{n-1}^2}f_\gamma(P)=\frac{\sum_{j=0}^{n-2}(x_j-y_j)^2+y_{n-1}^2}{x_{n-1}^3\,y_{n-1}},$$

and generally

$$\frac{\partial^{\alpha_{n-1}}}{\partial x_{n-1}^{\alpha_{n-1}}}f_\gamma(P)=\frac{(-1)^{\alpha_{n-1}}\,\alpha_{n-1}!\,\left(\sum_{j=0}^{n-2}(x_j-y_j)^2+y_{n-1}^2\right)}{2\,x_{n-1}^{1+\alpha_{n-1}}\,y_{n-1}}\qquad(\alpha_{n-1}\geq 2),$$

as well as

$$\frac{\partial^{\alpha_{n-1}}}{\partial x_{n-1}^{\alpha_{n-1}}}\left(\frac{x_k-y_k}{x_{n-1}\,y_{n-1}}\right)=\frac{(-1)^{\alpha_{n-1}}\,\alpha_{n-1}!\,(x_k-y_k)}{x_{n-1}^{1+\alpha_{n-1}}\,y_{n-1}}\qquad(k=0,\ldots,n-2,\ \alpha_{n-1}\geq 0),$$
$$\frac{\partial^{\alpha_{n-1}}}{\partial x_{n-1}^{\alpha_{n-1}}}\left(\frac{1}{x_{n-1}\,y_{n-1}}\right)=\frac{(-1)^{\alpha_{n-1}}\,\alpha_{n-1}!}{x_{n-1}^{1+\alpha_{n-1}}\,y_{n-1}}\qquad(\alpha_{n-1}\geq 0).$$

Therefore, from (4.24) we can deduce for any $\boldsymbol{\alpha}=(\alpha_0,\ldots,\alpha_{n-1})\in\mathbb{N}_0^n$ that

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}}f_\gamma(P)=\begin{cases} f_\gamma(P), \quad \text{if } |\boldsymbol{\alpha}|=\sum\limits_{j=0}^{n-1}\alpha_j=0,\\ \dfrac{-\sum_{j=0}^{n-2}(x_j-y_j)^2+x_{n-1}^2-y_{n-1}^2}{2\,x_{n-1}^2\,y_{n-1}}, \quad \text{if } \sum\limits_{j=0}^{n-2}\alpha_j=0 \text{ and } \alpha_{n-1}=1,\\ \dfrac{(-1)^{\alpha_{n-1}}\,\alpha_{n-1}!\left(\sum_{j=0}^{n-2}(x_j-y_j)^2+y_{n-1}^2\right)}{2\,x_{n-1}^{1+\alpha_{n-1}}\,y_{n-1}}, \quad \text{if } \sum\limits_{j=0}^{n-2}\alpha_j=0 \text{ and } \alpha_{n-1}\geq 2,\\ \dfrac{(-1)^{\alpha_{n-1}}\,\alpha_{n-1}!\,(x_k-y_k)}{x_{n-1}^{1+\alpha_{n-1}}\,y_{n-1}}, \quad \text{if } \alpha_k=1 \text{ for some } k\in\{0,\ldots,n-2\} \text{ and } \sum\limits_{\substack{j=0,\\ j\neq k}}^{n-2}\alpha_j=0,\\ \dfrac{(-1)^{\alpha_{n-1}}\,\alpha_{n-1}!}{x_{n-1}^{1+\alpha_{n-1}}\,y_{n-1}}, \quad \text{if } \alpha_k=2 \text{ for some } k\in\{0,\ldots,n-2\} \text{ and } \sum\limits_{\substack{j=0,\\ j\neq k}}^{n-2}\alpha_j=0,\\ 0, \quad \text{if } \alpha_k,\alpha_l\geq 1 \text{ for some } k,l\in\{0,\ldots,n-2\} \text{ with } k\neq l\\ \qquad \text{or } \alpha_k\geq 3 \text{ for some } k\in\{0,\ldots,n-2\}. \end{cases} \tag{4.25}$$

This finishes the first part of the proof.

In the second step we prove that for any $\boldsymbol{\alpha}=(\alpha_0,\ldots,\alpha_{n-1})\in\mathbb{N}_0^n$ there is a constant $C_{\boldsymbol{\alpha}}(K)>0$, depending only on the compact set K, such that

$$\left|\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}}f_\gamma(P)\right|\leq C_{\boldsymbol{\alpha}}(K)\cdot f_\gamma(P) \tag{4.26}$$

for any $\gamma\in\Gamma$ and $P\in K$. We verify this bound separately in each of the six cases in (4.25). In the cases $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}}f_\gamma(P)=0$ and $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}}f_\gamma(P)=f_\gamma(P)$ the bound (4.26) is obvious. For

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}}f_\gamma(P)=\frac{(-1)^{\alpha_{n-1}}\,\alpha_{n-1}!}{x_{n-1}^{1+\alpha_{n-1}}\,y_{n-1}}$$

we have

$$\left|\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}}f_\gamma(P)\right|=\frac{\alpha_{n-1}!}{x_{n-1}^{1+\alpha_{n-1}}\,y_{n-1}}=\frac{2\,\alpha_{n-1}!\,f_\gamma(P)}{x_{n-1}^{\alpha_{n-1}}\left(\sum_{j=0}^{n-2}(x_j-y_j)^2+x_{n-1}^2+y_{n-1}^2\right)}\leq\frac{2\,\alpha_{n-1}!}{x_{n-1}^{2+\alpha_{n-1}}}\cdot f_\gamma(P).$$

If

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}}f_\gamma(P)=\frac{(-1)^{\alpha_{n-1}}\,\alpha_{n-1}!\,(x_k-y_k)}{x_{n-1}^{1+\alpha_{n-1}}\,y_{n-1}}$$

for some $k \in \{0, \ldots, n-2\}$, we find

$$\left|\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} f_\gamma(P)\right| = \frac{\alpha_{n-1}!\,|x_k - y_k|}{x_{n-1}^{1+\alpha_{n-1}}\, y_{n-1}} = \frac{2\,\alpha_{n-1}!\,|x_k - y_k|\, f_\gamma(P)}{x_{n-1}^{\alpha_{n-1}} \left(\sum_{j=0}^{n-2}(x_j - y_j)^2 + x_{n-1}^2 + y_{n-1}^2\right)}$$
$$\leq \max\left(\frac{2\,\alpha_{n-1}!}{x_{n-1}^{2+\alpha_{n-1}}},\, \frac{2\,\alpha_{n-1}!}{x_{n-1}^{\alpha_{n-1}}}\right) \cdot f_\gamma(P),$$

where we made use of

$$|x_k - y_k| \leq \max(1,\, (x_k - y_k)^2) \leq \max\left(1,\, \sum_{j=0}^{n-2}(x_j - y_j)^2 + x_{n-1}^2 + y_{n-1}^2\right).$$

In the case

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} f_\gamma(P) = \frac{(-1)^{\alpha_{n-1}}\,\alpha_{n-1}!\,\left(\sum_{j=0}^{n-2}(x_j - y_j)^2 + y_{n-1}^2\right)}{2\,x_{n-1}^{1+\alpha_{n-1}}\, y_{n-1}}$$

we obtain

$$\left|\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} f_\gamma(P)\right| = \frac{\alpha_{n-1}!\,\left(\sum_{j=0}^{n-2}(x_j - y_j)^2 + y_{n-1}^2\right)}{2\,x_{n-1}^{1+\alpha_{n-1}}\, y_{n-1}} = \frac{\alpha_{n-1}!\,\left(\sum_{j=0}^{n-2}(x_j - y_j)^2 + y_{n-1}^2\right) f_\gamma(P)}{x_{n-1}^{\alpha_{n-1}} \left(\sum_{j=0}^{n-2}(x_j - y_j)^2 + x_{n-1}^2 + y_{n-1}^2\right)}$$
$$\leq \frac{\alpha_{n-1}!}{x_{n-1}^{\alpha_{n-1}}} \cdot f_\gamma(P).$$

And for

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} f_\gamma(P) = \frac{-\sum_{j=0}^{n-2}(x_j - y_j)^2 + x_{n-1}^2 - y_{n-1}^2}{2\,x_{n-1}^2\, y_{n-1}}$$

we get

$$\left|\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} f_\gamma(P)\right| = \frac{\left|\sum_{j=0}^{n-2}(x_j - y_j)^2 - x_{n-1}^2 + y_{n-1}^2\right|}{2\,x_{n-1}^2\, y_{n-1}} = \frac{\left|\sum_{j=0}^{n-2}(x_j - y_j)^2 - x_{n-1}^2 + y_{n-1}^2\right| f_\gamma(P)}{x_{n-1} \left(\sum_{j=0}^{n-2}(x_j - y_j)^2 + x_{n-1}^2 + y_{n-1}^2\right)}$$
$$\leq \frac{1}{x_{n-1}} \cdot f_\gamma(P),$$

where we used that

$$\left|\sum_{j=0}^{n-2}(x_j - y_j)^2 - x_{n-1}^2 + y_{n-1}^2\right| \leq \sum_{j=0}^{n-2}(x_j - y_j)^2 + x_{n-1}^2 + y_{n-1}^2.$$

This proves that in each case there exists a constant $C_{\boldsymbol{\alpha}}(K) > 0$, depending only on K, such that the bound (4.26) is satisfied for any $\gamma \in \Gamma$ and $P \in K$, completing the second part of the proof.

As a third step we conclude, exactly analogous to the seventh step in the proof of Lemma 4.1.7, that for any $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ the partial derivative

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \cosh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} f_\gamma(P)^{-s}$$

is a finite sum of summands of the form

$$p(s) \cdot f_\gamma(P)^{-s-r} \cdot \prod_{j=1}^{r} \frac{\partial^{|\boldsymbol{\beta}^{(j)}|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} f_\gamma(P), \tag{4.27}$$

where $p \in \mathbb{Z}[X]$ is a polynomial with integer coefficients and $\deg(p) \leq |\boldsymbol{\alpha}|$, $r \in \mathbb{N}_0$ with $r \leq |\boldsymbol{\alpha}|$, and $\boldsymbol{\beta}^{(j)} \in \mathbb{N}_0^n$ $(j = 1, \ldots, r)$ are multi-indices with $\sum_{j=1}^{r} \left|\boldsymbol{\beta}^{(j)}\right| = |\boldsymbol{\alpha}|$.

In the fourth and final step we now conclude that any series (4.23) converges absolutely and uniformly on compact subsets $K \subseteq \mathbb{H}^n$, provided that $\sigma := \operatorname{Re}(s) > n-1$.
By the third step of the proof the derivative

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \cosh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} f_\gamma(P)^{-s}$$

is a finite sum of summands of the form (4.27). Further, from the second step of the proof we infer that for each of these summands there are constants $C_{\boldsymbol{\beta}^{(j)}}(K) > 0$ $(j = 1, \ldots, r)$, depending only on K, such that

$$\begin{aligned} \left| p(s) \cdot f_\gamma(P)^{-s-r} \cdot \prod_{j=1}^{r} \frac{\partial^{|\boldsymbol{\beta}^{(j)}|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} f_\gamma(P) \right| &= |p(s)| \cdot f_\gamma(P)^{-\sigma-r} \cdot \prod_{j=1}^{r} \left| \frac{\partial^{|\boldsymbol{\beta}^{(j)}|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} f_\gamma(P) \right| \\ \leq |p(s)| \cdot f_\gamma(P)^{-\sigma-r} \cdot \prod_{j=1}^{r} \big(C_{\boldsymbol{\beta}^{(j)}}(K) \cdot f_\gamma(P) \big) &= |p(s)| \cdot f_\gamma(P)^{-\sigma} \cdot \prod_{j=1}^{r} C_{\boldsymbol{\beta}^{(j)}}(K) \end{aligned}$$

for any $\gamma \in \Gamma$ and $P \in K$. Provided that $\sigma > n-1$, the absolute and locally uniform convergence of the series

$$\sum_{\gamma \in \Gamma} |p(s)| \cdot f_\gamma(P)^{-\sigma} \cdot \prod_{j=1}^{r} C_{\boldsymbol{\beta}^{(j)}}(K) = |p(s)| \cdot \prod_{j=1}^{r} C_{\boldsymbol{\beta}^{(j)}}(K) \sum_{\gamma \in \Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, Q))^{-\sigma}$$

follows from Lemma 4.3.4 (b). For $\sigma > n-1$ this proves the absolute and uniform convergence of the series (4.23) on compact subsets $K \subseteq \mathbb{H}^n$.

$\square$

Using our considerations in the proof of Lemma 4.3.7, for $P = (x_0, \ldots, x_{n-1}) \in \mathbb{H}^n$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > n-1$ we now study the differentiability of the elliptic Eisenstein series

$$E_Q^{\mathrm{ell}}(P, s) = \sum_{\gamma \in \Gamma_Q \backslash \Gamma} \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s} = \frac{1}{|\Gamma_Q|} \sum_{\gamma \in \Gamma} \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s}$$

with respect to $x_0, \ldots, x_{n-1}$. As we already mentioned in Remark 4.2.7, it is also infinitely often continuously differentiable.

Lemma 4.3.8. *For $P = (x_0, \ldots, x_{n-1}) \in \mathbb{H}^n$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > n-1$ the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P, s)$ is infinitely often continuously differentiable with respect to $x_0, \ldots, x_{n-1}$.*

Proof. For fixed $\gamma \in \Gamma$ we let $\gamma^{-1} Q =: (y_0, \ldots, y_{n-1}) \in \mathbb{H}^n$ and $f_\gamma(P)$ as in (4.22) and write

$$\sinh(d_{\mathbb{H}^n}(\gamma P, Q)) = \sqrt{f_\gamma(P)^2 - 1}. \tag{4.28}$$

Provided that $P \neq \gamma Q$ for any $\gamma \in \Gamma$, we have $f_\gamma(P) > 1$. This implies that the function $\sinh(d_{\mathbb{H}^n}(\gamma P, Q))$ is infinitely often continuously differentiable with respect to $x_0, \ldots, x_{n-1}$ in any such point P, and the same is true for $\sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s}$, where $s \in \mathbb{C}$ with $\operatorname{Re}(s) > n-1$. Thus, for any multi-index $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{N}_0^n$ the derivative

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_0^{\alpha_0} \ldots \partial x_{n-1}^{\alpha_{n-1}}} \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s}$$

exists and is continuous, and we are allowed to arbitrarily interchange the order of differentiation. It remains to prove that for any $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ the series of partial derivatives

$$\sum_{\gamma \in \Gamma} \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s} \tag{4.29}$$

converges absolutely and uniformly on compact subsets $K \subseteq \mathbb{H}^n$ not containing any translate γQ of Q by $\gamma \in \Gamma$, provided that $\sigma := \mathrm{Re}(s) > n-1$.

By the product rule for partial derivatives we have

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} f_\gamma(P)^2 = \sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^n, \\ \boldsymbol{\beta} \leq \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \cdot \frac{\partial^{|\boldsymbol{\beta}|}}{\partial P^{\boldsymbol{\beta}}} f_\gamma(P) \cdot \frac{\partial^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}}{\partial P^{\boldsymbol{\alpha}-\boldsymbol{\beta}}} f_\gamma(P).$$

So the bound (4.26) in the proof of Lemma 4.3.7 gives us

$$\begin{aligned}\left|\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} f_\gamma(P)^2\right| &\leq \sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^n, \\ \boldsymbol{\beta} \leq \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \cdot \left|\frac{\partial^{|\boldsymbol{\beta}|}}{\partial P^{\boldsymbol{\beta}}} f_\gamma(P)\right| \cdot \left|\frac{\partial^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}}{\partial P^{\boldsymbol{\alpha}-\boldsymbol{\beta}}} f_\gamma(P)\right| \\ &\leq \sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^n, \\ \boldsymbol{\beta} \leq \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \cdot C_{\boldsymbol{\beta}}(K) \cdot f_\gamma(P) \cdot C_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(K) \cdot f_\gamma(P) = f_\gamma(P)^2 \cdot \left(\sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^n, \\ \boldsymbol{\beta} \leq \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \cdot C_{\boldsymbol{\beta}}(K) \cdot C_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(K)\right)\end{aligned}$$

for any $\gamma \in \Gamma$ and $P \in K$, where the constants $C_{\boldsymbol{\beta}}(K), C_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(K) > 0$ ($\boldsymbol{\beta} \in \mathbb{N}_0^n$, $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$) depend only on the compact set K and are as in the proof of Lemma 4.3.7. Further, as for $P \in K$ the minimum $\min_{\gamma \in \Gamma} d_{\mathbb{H}^n}(\gamma P, Q)$ exists and is greater than zero, there is a constant $C'(K) > 0$, depending only on K, such that

$$\cosh(d_{\mathbb{H}^n}(\gamma P, Q))^2 \leq C'(K) \cdot \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^2,$$

i.e. $f_\gamma(P)^2 \leq C'(K) \cdot (f_\gamma(P)^2 - 1)$, for any $\gamma \in \Gamma$ and $P \in K$. Thus, we have the bound

$$\left|\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} f_\gamma(P)^2\right| \leq B_{\boldsymbol{\alpha}}(K) \cdot (f_\gamma(P)^2 - 1) \tag{4.30}$$

for any $\gamma \in \Gamma$ and $P \in K$, where the constant

$$B_{\boldsymbol{\alpha}}(K) := C'(K) \cdot \left(\sum_{\substack{\boldsymbol{\beta} \in \mathbb{N}_0^n, \\ \boldsymbol{\beta} \leq \boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \cdot C_{\boldsymbol{\beta}}(K) \cdot C_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(K)\right)$$

depends only on K.

Next, we find, analogous to the second step in the proof of Lemma 4.1.7, that for any $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{N}_0^n$ the partial derivative

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \sinh(d_{\mathbb{H}^n}(\gamma P, Q)) = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \sqrt{f_\gamma(P)^2 - 1}$$

is a finite sum of summands of the form

$$t \cdot (f_\gamma(P)^2 - 1)^{1/2-r} \cdot \prod_{j=1}^{r} \frac{\partial^{|\boldsymbol{\beta}^{(j)}|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} (f_\gamma(P)^2 - 1) = t \cdot (f_\gamma(P)^2 - 1)^{1/2-r} \cdot \prod_{j=1}^{r} \frac{\partial^{|\boldsymbol{\beta}^{(j)}|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} f_\gamma(P)^2,$$

where $t \in \mathbb{Q}$, $r \in \mathbb{N}_0$ with $r \leq |\boldsymbol{\alpha}|$ and $\boldsymbol{\beta}^{(j)} \in \mathbb{N}_0^n$ ($j = 1, \ldots, r$) are multi-indices with $\sum_{j=1}^r \left|\boldsymbol{\beta}^{(j)}\right| = |\boldsymbol{\alpha}|$. Using the bound (4.30), for each of these summands we have constants $B_{\boldsymbol{\beta}^{(j)}}(K) > 0$ ($j = 1, \ldots, r$), depending only on K, such that

$$\left| t \cdot (f_\gamma(P)^2 - 1)^{1/2-r} \cdot \prod_{j=1}^r \frac{\partial^{\left|\boldsymbol{\beta}^{(j)}\right|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} f_\gamma(P)^2 \right| = |t| \cdot (f_\gamma(P)^2 - 1)^{1/2-r} \cdot \prod_{j=1}^r \left| \frac{\partial^{\left|\boldsymbol{\beta}^{(j)}\right|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} f_\gamma(P)^2 \right|$$
$$\leq |t| \cdot (f_\gamma(P)^2 - 1)^{1/2-r} \cdot \prod_{j=1}^r \left(B_{\boldsymbol{\beta}^{(j)}}(K) \cdot (f_\gamma(P)^2 - 1) \right) = |t| \cdot \sqrt{f_\gamma(P)^2 - 1} \cdot \prod_{j=1}^r B_{\boldsymbol{\beta}^{(j)}}(K)$$

for any $\gamma \in \Gamma$ and $P \in K$. The sum of the terms $|t| \cdot \prod_{j=1}^r B_{\boldsymbol{\beta}^{(j)}}(K)$ in the finitely many summands of $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \sqrt{f_\gamma(P)^2 - 1}$ is a constant $A_{\boldsymbol{\alpha}}(K) > 0$, depending only on K, such that the bound

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \sqrt{f_\gamma(P)^2 - 1} \right| \leq A_{\boldsymbol{\alpha}}(K) \cdot \sqrt{f_\gamma(P)^2 - 1} \tag{4.31}$$

is satisfied for any $\gamma \in \Gamma$ and $P \in K$.

As in the seventh step in the proof of Lemma 4.1.7 and the third step in the proof of Lemma 4.3.7, we find that for any $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ the partial derivative

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial P^{\boldsymbol{\alpha}}} \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s}$$

is a finite sum of summands of the form

$$p(s) \cdot \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s-r} \cdot \prod_{j=1}^r \frac{\partial^{\left|\boldsymbol{\beta}^{(j)}\right|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} \sinh(d_{\mathbb{H}^n}(\gamma P, Q)),$$

where $p \in \mathbb{Z}[X]$ is a polynomial with integer coefficients and $\deg(p) \leq |\boldsymbol{\alpha}|$, $r \in \mathbb{N}_0$ with $r \leq |\boldsymbol{\alpha}|$, and $\boldsymbol{\beta}^{(j)} \in \mathbb{N}_0^n$ ($j = 1, \ldots, r$) are multi-indices with $\sum_{j=1}^r \left|\boldsymbol{\beta}^{(j)}\right| = |\boldsymbol{\alpha}|$.
Recalling (4.28) and using (4.31), for each of these summands there are constants $A_{\boldsymbol{\beta}^{(j)}}(K) > 0$ ($j = 1, \ldots, r$), depending only on K, such that

$$\left| p(s) \cdot \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s-r} \cdot \prod_{j=1}^r \frac{\partial^{\left|\boldsymbol{\beta}^{(j)}\right|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} \sinh(d_{\mathbb{H}^n}(\gamma P, Q)) \right|$$
$$= |p(s)| \cdot \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-\sigma-r} \cdot \prod_{j=1}^r \left| \frac{\partial^{\left|\boldsymbol{\beta}^{(j)}\right|}}{\partial P^{\boldsymbol{\beta}^{(j)}}} \sinh(d_{\mathbb{H}^n}(\gamma P, Q)) \right|$$
$$\leq |p(s)| \cdot \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-\sigma-r} \cdot \prod_{j=1}^r \left(A_{\boldsymbol{\beta}^{(j)}}(K) \cdot \sinh(d_{\mathbb{H}^n}(\gamma P, Q)) \right)$$
$$= |p(s)| \cdot \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-\sigma} \cdot \prod_{j=1}^r A_{\boldsymbol{\beta}^{(j)}}(K)$$

for any $\gamma \in \Gamma$ and $P \in K$. The absolute and locally uniform convergence of the series

$$\sum_{\gamma \in \Gamma} |p(s)| \cdot \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-\sigma} \cdot \prod_{j=1}^r A_{\boldsymbol{\beta}^{(j)}}(K) = |p(s)| \cdot \prod_{j=1}^r A_{\boldsymbol{\beta}^{(j)}}(K) \sum_{\gamma \in \Gamma} \sinh(d_{\mathbb{H}^n}(\gamma P, Q))^{-\sigma}$$

on K follows from Lemma 4.2.4 (b), provided that $\sigma > n - 1$. This implies the absolute and uniform convergence of the series (4.29) on compact subsets $K \subseteq \mathbb{H}^n$ not containing any translate γQ of Q by $\gamma \in \Gamma$, provided that $\sigma > n - 1$.

$\square$

To finish this section we establish the differential equation of the hyperbolic kernel function under the hyperbolic Laplace operator. Again we find a certain shift equation.

Lemma 4.3.9. *For $P, Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ satisfies the differential equation*

$$(\Delta_{\mathbb{H}^n} - s(n-1-s))\, K^{\mathrm{hyp}}(P,Q,s) = s(s+1)\, K^{\mathrm{hyp}}(P,Q,s+2),$$

where $\Delta_{\mathbb{H}^n}$ is the hyperbolic Laplace operator with respect to P.

Proof. If we use elliptic coordinates centered at Q (see Definition 1.2.5), we have $\varrho_Q(\gamma P) = d_{\mathbb{H}^n}(\gamma P, Q)$ and

$$K^{\mathrm{hyp}}(P,Q,s) = \sum_{\gamma \in \Gamma} \cosh(\varrho_Q(\gamma P))^{-s},$$

and by Lemma 1.2.8 (c) the hyperbolic Laplace operator is given by

$$\Delta_{\mathbb{H}^n} = -\frac{\partial^2}{\partial \varrho_Q^2} - (n-1)\,\frac{1}{\tanh(\varrho_Q)}\,\frac{\partial}{\partial \varrho_Q} - \frac{1}{\sinh(\varrho_Q)^2}\,\Delta_{\mathbb{S}^{n-1}}.$$

Since $\Delta_{\mathbb{H}^n}$ is invariant under the action of $\mathrm{PSL}_2(C_{n-1})$ and $K^{\mathrm{hyp}}(P,Q,s) \in C^2(\mathbb{H}^n)$ by Lemma 4.3.7, it it sufficient to prove that

$$(\Delta_{\mathbb{H}^n} - s(n-1-s))\, \cosh(\varrho_Q)^{-s} = s(s+1)\, \cosh(\varrho_Q)^{-s-2}.$$

This is an immediate consequence of the calculation

$$\begin{aligned}
\Delta_{\mathbb{H}^n} \cosh(\varrho_Q)^{-s} &= \left(-\frac{\partial^2}{\partial \varrho_Q^2} - (n-1)\,\frac{1}{\tanh(\varrho_Q)}\,\frac{\partial}{\partial \varrho_Q} \right) \cosh(\varrho_Q)^{-s} \\
&= -s(s+1)\, \cosh(\varrho_Q)^{-s-2}\, \sinh(\varrho_Q)^2 + s\, \cosh(\varrho_Q)^{-s} + s(n-1)\, \cosh(\varrho_Q)^{-s} \\
&= -s(s+1)\, \cosh(\varrho_Q)^{-s-2}\, (\cosh(\varrho_Q)^2 - 1) + s\,n\, \cosh(\varrho_Q)^{-s} \\
&= -s(s+1)\, \cosh(\varrho_Q)^{-s} + s(s+1)\, \cosh(\varrho_Q)^{-s-2} + s\,n\, \cosh(\varrho_Q)^{-s} \\
&= s(s+1)\, \cosh(\varrho_Q)^{-s-2} + s(n-1-s)\, \cosh(\varrho_Q)^{-s}.
\end{aligned}$$

□

5. Spectral expansions

In this chapter we compute the spectral expansion of the hyperbolic Eisenstein series in $\mathbb{H}^n$ which will be fundamental in establishing its meromorphic continuation in s to the whole complex plane in the next chapter. We start by determining the spectral expansion of the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ in the first section. After that, we derive the spectral expansion of the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ in the second section, using a representation of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ as an integral of $K^{\mathrm{hyp}}(P,Q,s)$ along a certain closed geodesic. This generalizes the spectral expansion of hyperbolic Eisenstein series on the upper half-plane which was derived in [JKP10].

5.1. Spectral expansion of the hyperbolic kernel function

Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. In this section we determine the spectral expansion of the hyperbolic kernel function in terms of the eigenfunctions $\psi_j(P)$ associated to the discrete eigenvalues λ_j of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ and the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,s)$ associated to the cusps $\eta_k \in C_\Gamma$ $(k=1,\ldots,c_\Gamma)$, as well as its range of convergence.

Theorem 5.1.1. *For $P,Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ admits the spectral expansion*

$$K^{\mathrm{hyp}}(P,Q,s) = \sum_{j=0}^{\infty} a_{j,Q}(s)\,\psi_j(P) + \frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2}+it\Big)\,dt, \tag{5.1}$$

where the coefficients $a_{j,Q}(s)$ and $a_{t,\eta_k,Q}(s)$ are given by

$$\begin{aligned} a_{j,Q}(s) &= \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big)\,\overline{\psi_j}(Q),\\ a_{t,\eta_k,Q}(s) &= \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+it}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-it}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2}-it\Big), \end{aligned}$$

respectively. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the spectral expansion (5.1) converges absolutely and uniformly for P ranging over compact subsets of $\mathbb{H}^n$.

Proof. For fixed $Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$, viewed as a function in P, is infinitely often continuously differentiable on $\mathbb{H}^n$ by Lemma 4.3.7, and an element of $\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ by Lemma 4.3.6. From the differential equation

$$\Delta_{\mathbb{H}^n} K^{\mathrm{hyp}}(P,Q,s) = s(n-1-s)\,K^{\mathrm{hyp}}(P,Q,s) + s(s+1)\,K^{\mathrm{hyp}}(P,Q,s+2)$$

(see Lemma 4.3.9), where $\Delta_{\mathbb{H}^n}$ is the hyperbolic Laplace operator with respect to P, it follows inductively that

$$\Delta^l_{\mathbb{H}^n} K^{\mathrm{hyp}}(P,Q,s) = \sum_{k=0}^{l} p_k(s)\cdot K^{\mathrm{hyp}}(P,Q,s+2k)$$

for $l \in \mathbb{N}_0$, where $p_k \in \mathbb{Z}[X]$ $(k=0,\ldots,l)$ are polynomials with integer coefficients. This implies that $\Delta^l_{\mathbb{H}^n} K^{\mathrm{hyp}}(P,Q,s) \in \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ for any $l \in \mathbb{N}_0$.
Hence, by Theorem 3.5.5 and Remark 3.5.8 the hyperbolic kernel function admits a spectral expansion of the form (5.1), which is absolutely and uniformly convergent on compact subsets of $\mathbb{H}^n$ by Proposition 3.5.6.

It remains to compute the coefficients in the spectral expansion. Using the definition of $K^{\mathrm{hyp}}(P,Q,s)$ and identifying $\Gamma\backslash\mathbb{H}^n$ with a fundamental domain $\mathcal{F}_\Gamma$ for Γ, for $j\in\mathbb{N}_0$ the coefficient $a_{j,Q}(s)$ arising from the discrete spectrum is given by

$$\begin{aligned} a_{j,Q}(s) &= \int_{\mathcal{F}_\Gamma} K^{\mathrm{hyp}}(P,Q,s)\,\overline{\psi_j}(P)\;\mu_{\mathbb{H}^n}(P) = \int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma}\cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s}\,\overline{\psi_j}(P)\;\mu_{\mathbb{H}^n}(P) \\ &= \sum_{\gamma\in\Gamma}\int_{\mathcal{F}_\Gamma}\cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s}\,\overline{\psi_j}(P)\;\mu_{\mathbb{H}^n}(P). \end{aligned}$$

We note that interchanging integration and summation is justified because the hyperbolic kernel function converges absolutely and locally uniformly for $P,Q\in\mathbb{H}^n$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$. Since $\overline{\psi_j}(P)$ and the hyperbolic volume element $\mu_{\mathbb{H}^n}(P)$ are invariant under the action of Γ, and the union of all translated fundamental domains $\gamma\mathcal{F}_\Gamma$ for $\gamma\in\Gamma$ covers the whole upper half-space $\mathbb{H}^n$, we further obtain

$$\begin{aligned} a_{j,Q}(s) &= \sum_{\gamma\in\Gamma}\int_{\gamma\mathcal{F}_\Gamma}\cosh(d_{\mathbb{H}^n}(P,Q))^{-s}\,\overline{\psi_j}(\gamma^{-1}P)\;\mu_{\mathbb{H}^n}(\gamma^{-1}P) \\ &= \sum_{\gamma\in\Gamma}\int_{\gamma\mathcal{F}_\Gamma}\cosh(d_{\mathbb{H}^n}(P,Q))^{-s}\,\overline{\psi_j}(P)\;\mu_{\mathbb{H}^n}(P) \\ &= \int_{\mathbb{H}^n}\cosh(d_{\mathbb{H}^n}(P,Q))^{-s}\,\overline{\psi_j}(P)\;\mu_{\mathbb{H}^n}(P). \end{aligned}$$

Using elliptic coordinates $\varrho:=\varrho_Q(P)$, $\zeta:=\zeta_Q(P)$ centered at Q (see Definition 1.2.5) and the corresponding volume element

$$\mu_{\mathbb{H}^n}(P)=\sinh(\varrho)^{n-1}\;d\varrho\;d\nu_{n-1}(\zeta)$$

from Lemma 1.2.8 (b), this becomes

$$\begin{aligned} a_{j,Q}(s) &= \int_0^\infty\cosh(\varrho)^{-s}\left(\int_{\mathbb{S}^{n-1}}\overline{\psi_j}(P)\;d\nu_{n-1}(\zeta)\right)\sinh(\varrho)^{n-1}\;d\varrho \\ &= \int_0^\infty\cosh(\varrho)^{-s}\,f_j(P)\,\sinh(\varrho)^{n-1}\;d\varrho, \end{aligned}\tag{5.2}$$

where we have set

$$f_j(P):=\int_{\mathbb{S}^{n-1}}\overline{\psi_j}(P)\;d\nu_{n-1}(\zeta).$$

It remains to compute the integral $f_j(P)$. As we integrate over the whole sphere $\mathbb{S}^{n-1}$, the function $f_j(P)$ is independent of the angle $\zeta=\zeta_Q(P)$. Thus, $f_j(P)$ solely depends on the hyperbolic distance $d_{\mathbb{H}^n}(P,Q)$, which means that it is radial at Q (see Definition 3.1.1). Moreover, since $\psi_j(P)$ is an eigenfunction of $\Delta_{\mathbb{H}^n}$ with eigenvalue $\lambda_j=s_j(n-1-s_j)=\left(\frac{n-1}{2}\right)^2+r_j^2\in\mathbb{R}$, the same holds true for the function $f_j(P)$. This implies that $f_j(P)=\widehat{f_j}(d_{\mathbb{H}^n}(P,Q))$ is a radial eigenfunction of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ and we have

$$\widehat{f_j}(0)=f_j(Q)=\int_{\mathbb{S}^{n-1}}\overline{\psi_j}(Q)\;d\nu_{n-1}(\zeta)=\frac{2\,\pi^{n/2}}{\Gamma(\frac{n}{2})}\,\overline{\psi_j}(Q).\tag{5.3}$$

On the other hand, the radial eigenfunctions of $\Delta_{\mathbb{H}^n}$ were determined in Lemma 3.1.3, i.e. the functions

$$\Theta^{(1)}_{n,s_j,0}(\varrho)=\sinh(\varrho)^{1-\frac{n}{2}}\,P^{1-\frac{n}{2}}_{-\frac12+ir_j}(\cosh(\varrho)),\qquad \Theta^{(2)}_{n,s_j,0}(\varrho)=\sinh(\varrho)^{1-\frac{n}{2}}\,Q^{1-\frac{n}{2}}_{-\frac12+ir_j}(\cosh(\varrho))$$

are a fundamental system of solutions of the differential equation

$$-\frac{d^2\Theta}{d\varrho^2} - (n-1)\,\frac{1}{\tanh(\varrho)}\,\frac{d\Theta}{d\varrho} = s_j(n-1-s_j)\,\Theta, \tag{5.4}$$

where $s_j = \frac{n-1}{2} + ir_j$ and

$$-\frac{\partial^2}{\partial\varrho^2} - (n-1)\,\frac{1}{\tanh(\varrho)}\,\frac{\partial}{\partial\varrho}$$

is the radial part of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ in elliptic coordinates centered at Q. The first of these two linearly independent solutions of (5.4), i.e. the function $\Theta^{(1)}_{n,s_j,0}(\varrho)$, is bounded at $\varrho = 0$ by Lemma 3.1.5, while the second solution, given by

$$\Theta^{(2)}_{n,s_j,0}(\varrho) = \sinh(\varrho)^{1-\frac{n}{2}}\,Q^{1-\frac{n}{2}}_{-\frac{1}{2}+ir_j}(\cosh(\varrho)),$$

is unbounded at $\varrho = 0$. Hence, we can drop the function $\Theta^{(2)}_{n,s_j,0}(\varrho)$ here, and up to a constant factor the function $\Theta^{(1)}_{n,s_j,0}(\varrho)$ is the unique radial eigenfunction of the hyperbolic Laplace operator that is bounded at $\varrho = 0$. The condition (5.3) and the value $\Theta^{(1)}_{n,s_j,0}(0) = 2^{1-\frac{n}{2}}\,\Gamma\left(\frac{n}{2}\right)^{-1}$ from Lemma 3.1.5 now yield

$$\begin{aligned} f_j(P) = \frac{\widehat{f_j}(0)}{\Theta^{(1)}_{n,s_j,0}(0)}\,\Theta^{(1)}_{n,s_j,0}(\varrho) &= \frac{2\,\pi^{n/2}}{\Gamma(\frac{n}{2})}\,\overline{\psi_j}(Q)\,2^{\frac{n}{2}-1}\,\Gamma\Big(\frac{n}{2}\Big)\,\sinh(\varrho)^{1-\frac{n}{2}}\,P^{1-\frac{n}{2}}_{-\frac{1}{2}+ir_j}(\cosh(\varrho)) \\ &= (2\pi)^{n/2}\,\overline{\psi_j}(Q)\,\sinh(\varrho)^{1-\frac{n}{2}}\,P^{1-\frac{n}{2}}_{-\frac{1}{2}+ir_j}(\cosh(\varrho)). \end{aligned}$$

Inserting this identity into (5.2) we obtain

$$a_{j,Q}(s) = (2\pi)^{n/2}\,\overline{\psi_j}(Q)\int_0^\infty \cosh(\varrho)^{-s}\,\sinh(\varrho)^{n/2}\,P^{1-\frac{n}{2}}_{-\frac{1}{2}+ir_j}(\cosh(\varrho))\,d\varrho.$$

It remains to compute the integral

$$\begin{aligned} &\int_0^\infty \cosh(\varrho)^{-s}\,\sinh(\varrho)^{n/2}\,P^{1-\frac{n}{2}}_{-\frac{1}{2}+ir_j}(\cosh(\varrho))\,d\varrho \\ &\quad= \int_0^\infty \cosh(\varrho)^{-s}\,(\cosh(\varrho)^2-1)^{\frac{n}{4}-\frac{1}{2}}\,\sinh(\varrho)\,P^{1-\frac{n}{2}}_{-\frac{1}{2}+ir_j}(\cosh(\varrho))\,d\varrho \\ &\quad= \int_1^\infty t^{-s}\,(t^2-1)^{\frac{n}{4}-\frac{1}{2}}\,P^{1-\frac{n}{2}}_{-\frac{1}{2}+ir_j}(t)\,dt, \end{aligned}$$

where we have substituted $t = \cosh(\varrho)$. Finally, we use the identity (A.27), i.e.

$$\int_1^\infty t^{-s}\,(t^2-1)^{-\mu/2}\,P^\mu_\nu(t)\,dt = \frac{2^{s+\mu-2}}{\sqrt{\pi}\,\Gamma(s)}\,\Gamma\Big(\frac{s+\mu+\nu}{2}\Big)\,\Gamma\Big(\frac{s+\mu-\nu-1}{2}\Big),$$

which is valid for $\mathrm{Re}(\mu) < 1$, $\mathrm{Re}(s+\mu+\nu) > 0$ and $\mathrm{Re}(s+\mu-\nu) > 1$, with $\mu = 1-\frac{n}{2}$ and $\nu = -\frac{1}{2} + ir_j$ to get

$$\int_1^\infty t^{-s}\,(t^2-1)^{\frac{n}{4}-\frac{1}{2}}\,P^{1-\frac{n}{2}}_{-\frac{1}{2}+ir_j}(t)\,dt = \frac{2^{s-1-\frac{n}{2}}}{\sqrt{\pi}\,\Gamma(s)}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big).$$

All in all, we end up with the coefficient

$$\begin{aligned} a_{j,Q}(s) &= (2\pi)^{n/2}\,\overline{\psi_j}(Q)\,\frac{2^{s-1-\frac{n}{2}}}{\sqrt{\pi}\,\Gamma(s)}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big) \\ &= \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big)\,\overline{\psi_j}(Q), \end{aligned}$$

which completes the proof for the discrete part.

The coefficient

$$a_{t,\eta_k,Q}(s) = \int_{\mathcal{F}_\Gamma} K^{\mathrm{hyp}}(P,Q,s)\, \overline{E^{\mathrm{par}}_{\eta_k}}\Big(P, \frac{n-1}{2} + it\Big)\, \mu_{\mathbb{H}^n}(P)$$

arising from the continuous part can be computed analogously by replacing $s_j = \frac{n-1}{2} + ir_j$ by $\frac{n-1}{2} + it$ and $\psi_j(P)$ by $E^{\mathrm{par}}_{\eta_k}(P, \frac{n-1}{2} + it)$, and taking into account that $\overline{E^{\mathrm{par}}_{\eta_k}}(P, \frac{n-1}{2} + it) = E^{\mathrm{par}}_{\eta_k}(P, \frac{n-1}{2} - it)$ for any $t \in \mathbb{R}$. This gives us

$$a_{t,\eta_k,Q}(s) = \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\,\Gamma\Big(\frac{s - \frac{n-1}{2} + it}{2}\Big)\,\Gamma\Big(\frac{s - \frac{n-1}{2} - it}{2}\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} - it\Big),$$

which completes the proof.

□

Remark 5.1.2. The continuous part

$$\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt$$

in the spectral expansion (5.1) in Theorem 5.1.1 does not appear in the case $c_\Gamma = 0$, i.e. if the discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ contains no parabolic element.

5.2. Spectral expansion of hyperbolic Eisenstein series

Now we turn to determine the spectral expansion of the hyperbolic Eisenstein series and its range of convergence. To do this, we first establish an integral representation of the hyperbolic Eisenstein series involving the hyperbolic kernel function, and then apply the results from the previous section. This gives us a generalization of the result of [JKP10] for $n = 2$.
Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. Further, let $(Q_1, Q_2) \in H_\Gamma$ be a pair of hyperbolic fixed points with hyperbolic scaling matrix $\sigma_{(Q_1,Q_2)} \in \mathrm{PSL}_2(C_{n-1})$ and hyperbolic stabilizer subgroup $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$. Let $\mathcal{L}_{(Q_1,Q_2)}$ be the unique geodesic in $\mathbb{H}^n$ connecting Q_1 and Q_2, and let $L_{(Q_1,Q_2)} = \pi_{\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}}(\mathcal{L}_{(Q_1,Q_2)})$ denote its image under the natural projection $\pi_{\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}} : \mathbb{H}^n \to \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\mathbb{H}^n$, which is a closed geodesic in $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\mathbb{H}^n$ of hyperbolic length $l_{(Q_1,Q_2)}$.

The hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ associated to the pair $(Q_1,Q_2) \in H_\Gamma$ of hyperbolic fixed points can essentially be written as a line integral of the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ along the closed geodesic $L_{(Q_1,Q_2)}$ as follows.

Proposition 5.2.1. *For $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ we have the relation*

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = \frac{2^{1-s}\,\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)^2}\int_{L_{(Q_1,Q_2)}} K^{\mathrm{hyp}}(P,Q,s)\, ds_{\mathbb{H}^n}(Q). \tag{5.5}$$

Proof. We first assume that the hyperbolic fixed points are $Q_1 = 0$ and $Q_2 = \infty$ and the hyperbolic scaling matrix is given by $\sigma_{(Q_1,Q_2)} = \sigma_{(0,\infty)} = I$.
From Proposition 2.6.31 (b) and Remark 2.6.36 we know that the hyperbolic stabilizer subgroup $\Gamma^{\mathrm{hyp}}_{(0,\infty)}$ of the pair $(0,\infty) \in H_\Gamma$ of hyperbolic fixed points is an infinite cyclic group which is generated by the primitive hyperbolic element

$$\gamma_{(0,\infty)} = \begin{pmatrix} \exp\left(\frac{1}{2}l_{(0,\infty)}\right) & 0 \\ 0 & \exp\left(-\frac{1}{2}l_{(0,\infty)}\right) \end{pmatrix},$$

where $l_{(0,\infty)}$ is the hyperbolic length of the closed geodesic $L_{(0,\infty)}$. For $m \in \mathbb{Z}$ the matrix $\gamma_{(0,\infty)}^m$ acts on $P \in \mathbb{H}^n$ as

$$\gamma_{(0,\infty)}^m P = \exp(m\, l_{(0,\infty)})P,$$

so we can identify $L_{(0,\infty)}$ with the subset

$$\begin{aligned}&\big\{(0,\ldots,0,x_{n-1}) \in \mathbb{H}^n \mid x_{n-1} \in \big[1, \exp(l_{(0,\infty)})\big)\big\} \\ &\quad\cong \big\{x_{n-1} i_{n-1} \mid x_{n-1} \in \big[1, \exp(l_{(0,\infty)})\big)\big\} = \big\{e^u\, i_{n-1} \mid u \in [0, l_{(0,\infty)})\big\} \subseteq \mathbb{H}^n.\end{aligned}$$

By its absolute and locally uniform convergence, for $P, Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ we can rewrite the hyperbolic kernel function as

$$\begin{aligned}K^{\mathrm{hyp}}(P,Q,s) &= \sum_{\gamma\in\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, Q))^{-s} = \sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma}\ \sum_{\eta\in\Gamma^{\mathrm{hyp}}_{(0,\infty)}} \cosh(d_{\mathbb{H}^n}(\eta\gamma P, Q))^{-s} \\ &= \sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma}\ \sum_{\eta\in\Gamma^{\mathrm{hyp}}_{(0,\infty)}} \cosh\big(d_{\mathbb{H}^n}(\gamma P, \eta^{-1}Q)\big)^{-s} \\ &= \sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma}\ \sum_{m\in\mathbb{Z}} \cosh\Big(d_{\mathbb{H}^n}(\gamma P, \gamma_{(0,\infty)}^{-m}Q)\Big)^{-s} \\ &= \sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma}\ \sum_{m\in\mathbb{Z}} \cosh\big(d_{\mathbb{H}^n}(\gamma P, \exp(-m\, l_{(0,\infty)})Q)\big)^{-s} \\ &= \sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma}\ \sum_{m\in\mathbb{Z}} \cosh\big(d_{\mathbb{H}^n}(\gamma P, \exp(m\, l_{(0,\infty)})Q)\big)^{-s},\end{aligned}$$

and the series in the last line is again absolutely and locally uniformly convergent. For the line integral of $K^{\mathrm{hyp}}(P,Q,s)$ along $L_{(0,\infty)}$ we obtain

$$\int_{L_{(0,\infty)}} K^{\mathrm{hyp}}(P,Q,s)\, ds_{\mathbb{H}^n}(Q) = \sum_{\gamma\in\Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma}\ \sum_{m\in\mathbb{Z}} \int_{L_{(0,\infty)}} \cosh\big(d_{\mathbb{H}^n}(\gamma P, \exp(m\, l_{(0,\infty)})Q)\big)^{-s}\, ds_{\mathbb{H}^n}(Q). \tag{5.6}$$

Employing hyperbolic coordinates (see Definition 1.2.1), for $n \geq 3$ the line of integration

$$\big\{Q = (0,\ldots,0,y_{n-1}) = y_{n-1} i_{n-1} \mid y_{n-1} \in \big[1, \exp(l_{(0,\infty)})\big)\big\}$$

becomes

$$\Big\{Q = \Big(u, 0, \frac{\pi}{2}, \ldots, \frac{\pi}{2}, 0\Big)\ \Big|\ u = \log(|Q|) \in \big[0, l_{(0,\infty)}\big)\Big\},$$

where $Q = (u, \theta_1, \ldots, \theta_{n-1})$ with $u = u(Q)$, $\theta_1 = \theta_1(Q)$, $\ldots$, $\theta_{n-1} = \theta_{n-1}(Q)$. On this line $ds_{\mathbb{H}^n}(Q)$ reduces to du, while $y_{n-1} = e^u \cos(\theta_{n-1})$ simplifies to $e^u \cos(0) = e^u$. Moreover, for $n = 2$ the line of integration

$$\big\{w = (0, y) = iy \mid y \in \big[1, \exp(l_{(0,\infty)})\big)\big\}$$

becomes

$$\Big\{w = \Big(u, \frac{\pi}{2}\Big)\ \Big|\ u = \log(|w|) \in \big[0, l_{(0,\infty)}\big)\Big\},$$

where $w = (u, \theta)$ with the hyperbolic coordinates $u = u(w)$, $\theta = \theta(w)$. On this line $ds_{\mathbb{H}}(w)$ reduces to du, while $y = e^u \sin(\theta)$ simplifies to $e^u \sin\left(\frac{\pi}{2}\right) = e^u$.

Using these considerations, for $\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma$ and $m \in \mathbb{Z}$ we write

$$\begin{aligned}\int_{L_{(0,\infty)}} &\cosh\left(d_{\mathbb{H}^n}(\gamma P, \exp(m\, l_{(0,\infty)})Q)\right)^{-s} \; ds_{\mathbb{H}^n}(Q) \\ &= \int_0^{l_{(0,\infty)}} \cosh\left(d_{\mathbb{H}^n}(\gamma P, \exp(m\, l_{(0,\infty)} + u)\, i_{n-1})\right)^{-s} \; du \\ &= \int_{m\, l_{(0,\infty)}}^{(m+1)\, l_{(0,\infty)}} \cosh\left(d_{\mathbb{H}^n}(\gamma P, e^u\, i_{n-1})\right)^{-s} \; du.\end{aligned}$$

Now we consider the hyperbolic triangle with vertices γP, $e^u\, i_{n-1}$ and $|\gamma P|\, i_{n-1}$. As the geodesic through γP and $|\gamma P|\, i_{n-1}$ is perpendicular to the positive i_{n-1}-axis $\mathcal{L}_{(0,\infty)}$, this triangle has a right-angle at $|\gamma P|\, i_{n-1}$. Applying the first law of cosines for right-angled hyperbolic triangles (see (A.31)) gives us the identity

$$\begin{aligned}\cosh(d_{\mathbb{H}^n}(\gamma P, e^u\, i_{n-1})) &= \cosh(d_{\mathbb{H}^n}(\gamma P, |\gamma P|\, i_{n-1})) \cdot \cosh(d_{\mathbb{H}^n}(|\gamma P|\, i_{n-1}, e^u\, i_{n-1})) \\ &= \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(0,\infty)})) \cdot \cosh(u - \log(|\gamma P|)).\end{aligned}$$

This yields the equality

$$\begin{aligned}\int_{L_{(0,\infty)}} &\cosh\left(d_{\mathbb{H}^n}(\gamma P, \exp(m\, l_{(0,\infty)})Q)\right)^{-s} \; ds_{\mathbb{H}^n}(Q) \\ &= \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(0,\infty)}))^{-s} \int_{m\, l_{(0,\infty)}}^{(m+1)\, l_{(0,\infty)}} \cosh(u - \log(|\gamma P|))^{-s} \; du. \end{aligned} \tag{5.7}$$

Inserting (5.7) into equation (5.6), we get

$$\begin{aligned}\int_{L_{(0,\infty)}} & K^{\mathrm{hyp}}(P,Q,s) \; ds_{\mathbb{H}^n}(Q) \\ &= \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(0,\infty)}))^{-s} \sum_{m\in\mathbb{Z}} \int_{m\, l_{(0,\infty)}}^{(m+1)\, l_{(0,\infty)}} \cosh(u - \log(|\gamma P|))^{-s} \; du \\ &= \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(0,\infty)}))^{-s} \int_{-\infty}^{\infty} \cosh(u - \log(|\gamma P|))^{-s} \; du \\ &= \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(0,\infty)}))^{-s} \int_{-\infty}^{\infty} \cosh(u)^{-s} \; du.\end{aligned}$$

Using the identity

$$\int_{-\infty}^{\infty} \cosh(u)^{-s} \; du = 2\int_0^{\infty} \cosh(u)^{-s} \; du = \frac{2^{s-1}\,\Gamma\left(\frac{s}{2}\right)^2}{\Gamma(s)}$$

for $\mathrm{Re}(s) > 0$ (see (A.18)), we finally obtain

$$\begin{aligned}\int_{L_{(0,\infty)}} K^{\mathrm{hyp}}(P,Q,s) \; ds_{\mathbb{H}^n}(Q) &= \frac{2^{s-1}\,\Gamma\left(\frac{s}{2}\right)^2}{\Gamma(s)} \sum_{\gamma \in \Gamma^{\mathrm{hyp}}_{(0,\infty)}\backslash\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, \mathcal{L}_{(0,\infty)}))^{-s} \\ &= \frac{2^{s-1}\,\Gamma\left(\frac{s}{2}\right)^2}{\Gamma(s)}\, E^{\mathrm{hyp}}_{(0,\infty)}(P,s),\end{aligned}$$

i.e. the integral representation

$$E^{\mathrm{hyp}}_{(0,\infty)}(P,s) = \frac{2^{1-s}\,\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)^2} \int_{L_{(0,\infty)}} K^{\mathrm{hyp}}(P,Q,s) \; ds_{\mathbb{H}^n}(Q).$$

This proves the assertion in the special case $Q_1 = 0$, $Q_2 = \infty$.

Now let $(Q_1, Q_2) \in H_\Gamma$ be an arbitrary pair of hyperbolic fixed points of Γ. As in the proof of Lemma 4.1.4 (a), the discrete and cofinite subgroup $\sigma_{(Q_1,Q_2)}^{-1}\,\Gamma\,\sigma_{(Q_1,Q_2)} \subseteq \mathrm{PSL}_2(C_{n-1})$ has the hyperbolic fixed points 0 and ∞ and

$$E^{\mathrm{hyp}}_{\Gamma,(Q_1,Q_2)}(P,s) = E^{\mathrm{hyp}}_{\sigma^{-1}_{(Q_1,Q_2)}\Gamma\sigma_{(Q_1,Q_2)},(0,\infty)}\left(\sigma^{-1}_{(Q_1,Q_2)}P,s\right)$$

for $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$. Setting

$$L'_{(0,\infty)} := \pi_{\left(\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)}\right)^{\mathrm{hyp}}_{(0,\infty)}}\left(\mathcal{L}_{(0,\infty)}\right),$$

the first part of the proof yields

$$E^{\mathrm{hyp}}_{\Gamma,(Q_1,Q_2)}(P,s) = \frac{2^{1-s}\,\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)^2}\int_{L'_{(0,\infty)}} K^{\mathrm{hyp}}_{\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)}}\left(\sigma^{-1}_{(Q_1,Q_2)}P,Q,s\right)\,ds_{\mathbb{H}^n}(Q).$$

The hyperbolic kernel function $K^{\mathrm{hyp}}_{\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)}}\left(\sigma^{-1}_{(Q_1,Q_2)}P,Q,s\right)$ can be rewritten as

$$\begin{aligned}
K^{\mathrm{hyp}}_{\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)}}\left(\sigma^{-1}_{(Q_1,Q_2)}P,Q,s\right) &= \sum_{\gamma\in\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)}} \cosh\left(d_{\mathbb{H}^n}\left(\gamma\sigma^{-1}_{(Q_1,Q_2)}P,Q\right)\right)^{-s} \\
&= \sum_{\gamma\in\Gamma}\cosh\left(d_{\mathbb{H}^n}\left(\sigma^{-1}_{(Q_1,Q_2)}\gamma\sigma_{(Q_1,Q_2)}\sigma^{-1}_{(Q_1,Q_2)}P,Q\right)\right)^{-s} = \sum_{\gamma\in\Gamma}\cosh\left(d_{\mathbb{H}^n}\left(\gamma P,\sigma_{(Q_1,Q_2)}Q\right)\right)^{-s} \\
&= K^{\mathrm{hyp}}_{\Gamma}\left(P,\sigma_{(Q_1,Q_2)}Q,s\right).
\end{aligned}$$

From this identity and the $\mathrm{PSL}_2(C_{n-1})$-invariance of the hyperbolic line element we infer that

$$\begin{aligned}
E^{\mathrm{hyp}}_{\Gamma,(Q_1,Q_2)}(P,s) &= \frac{2^{1-s}\,\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)^2}\int_{L'_{(0,\infty)}} K^{\mathrm{hyp}}_{\Gamma}\left(P,\sigma_{(Q_1,Q_2)}Q,s\right)\,ds_{\mathbb{H}^n}(Q) \\
&= \frac{2^{1-s}\,\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)^2}\int_{\sigma_{(Q_1,Q_2)}L'_{(0,\infty)}} K^{\mathrm{hyp}}_{\Gamma}\left(P,Q,s\right)\,ds_{\mathbb{H}^n}(Q).
\end{aligned}$$

Making use of

$$\left(\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma\,\sigma_{(Q_1,Q_2)}\right)^{\mathrm{hyp}}_{(0,\infty)} = \sigma^{-1}_{(Q_1,Q_2)}\,\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\,\sigma_{(Q_1,Q_2)}$$

and $\sigma_{(Q_1,Q_2)}\mathcal{L}_{(0,\infty)} = \mathcal{L}_{(Q_1,Q_2)}$, we find

$$\begin{aligned}
\sigma_{(Q_1,Q_2)}L'_{(0,\infty)} &= \sigma_{(Q_1,Q_2)}\left(\left\{\sigma^{-1}_{(Q_1,Q_2)}\,\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\,\sigma_{(Q_1,Q_2)}P \;\middle|\; P\in\mathcal{L}_{(0,\infty)}\right\}\right) \\
&= \left\{\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\,\sigma_{(Q_1,Q_2)}P \;\middle|\; P\in\mathcal{L}_{(0,\infty)}\right\} = \left\{\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}P \mid P\in\mathcal{L}_{(Q_1,Q_2)}\right\} \\
&= L_{(Q_1,Q_2)}.
\end{aligned}$$

This implies

$$E^{\mathrm{hyp}}_{\Gamma,(Q_1,Q_2)}(P,s) = \frac{2^{1-s}\,\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)^2}\int_{L_{(Q_1,Q_2)}} K^{\mathrm{hyp}}_{\Gamma}\left(P,Q,s\right)\,ds_{\mathbb{H}^n}(Q),$$

which proves the claim for an arbitrary pair $(Q_1, Q_2) \in H_\Gamma$ of hyperbolic fixed points.

$\square$

Theorem 5.1.1 and Proposition 5.2.1 together give us the following spectral expansion of the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ in terms of the eigenfunctions $\psi_j(P)$ associated to the discrete eigenvalues λ_j of the hyperbolic Laplace operator $\Delta_{\mathbb{H}^n}$ and the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,s)$ associated to the cusps $\eta_k \in C_\Gamma$ $(k = 1,\ldots,c_\Gamma)$.

Theorem 5.2.2. *For $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits the spectral expansion*

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = \sum_{j=0}^{\infty} b_j(s)\,\psi_j(P) + \frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} b_{t,\eta_k}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2}+it\Big)\,dt, \tag{5.8}$$

where the coefficients $b_j(s)$ and $b_{t,\eta_k}(s)$ are given by

$$b_j(s) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big)\int_{L_{(Q_1,Q_2)}} \overline{\psi_j}(Q)\,ds_{\mathbb{H}^n}(Q),$$

$$b_{t,\eta_k}(s) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+it}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-it}{2}\Big)\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2}-it\Big)\,ds_{\mathbb{H}^n}(Q),$$

respectively. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the spectral expansion (5.8) converges absolutely and uniformly for P ranging over compact subsets of $\mathbb{H}^n$.

Proof. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ is infinitely often continuously differentiable on $\mathbb{H}^n$ by Lemma 4.1.7, and an element of $\mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ by Lemma 4.1.6. From the differential equation

$$\Delta_{\mathbb{H}^n} E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = s(n-1-s)\,E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) + s^2\,E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s+2)$$

(see Lemma 4.1.8) we conclude inductively that

$$\Delta^l_{\mathbb{H}^n} E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = \sum_{k=0}^{l} p_k(s)\cdot E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s+2k)$$

for $l \in \mathbb{N}_0$, where $p_k \in \mathbb{Z}[X]$ $(k=0,\ldots,l)$ are polynomials with integer coefficients. Thus, we have $\Delta^l_{\mathbb{H}^n} E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) \in \mathcal{L}^2(\Gamma\backslash\mathbb{H}^n)$ for any $l \in \mathbb{N}_0$.
Now Theorem 3.5.5 and Remark 3.5.8 imply that the hyperbolic Eisenstein series admits a spectral expansion of the form (5.8), which converges absolutely and uniformly on compact subsets of $\mathbb{H}^n$ by Proposition 3.5.6.

The coefficients in the spectral expansion can now be derived from Theorem 5.1.1 and Proposition 5.2.1: Substituting the spectral expansion (5.1) of $K^{\mathrm{hyp}}(P,Q,s)$ into the identity (5.5), we obtain

$$\begin{aligned}E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) &= \sum_{j=0}^{\infty} b_j(s)\,\psi_j(P) + \frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} b_{t,\eta_k}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt \\ &= \frac{2^{1-s}\,\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)^2}\left(\sum_{j=0}^{\infty}\left(\int_{L_{(Q_1,Q_2)}} a_{j,Q}(s)\,ds_{\mathbb{H}^n}(Q)\right)\psi_j(P)\right. \\ &\qquad \left. + \frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty}\left(\int_{L_{(Q_1,Q_2)}} a_{t,\eta_k,Q}(s)\,ds_{\mathbb{H}^n}(Q)\right)E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt\right).\end{aligned}$$

Comparing coefficients leads to the formulas

$$\begin{aligned}b_j(s) &= \frac{2^{1-s}\,\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)^2}\int_{L_{(Q_1,Q_2)}} a_{j,Q}(s)\,ds_{\mathbb{H}^n}(Q) \\ &= \frac{2^{1-s}\,\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)^2}\int_{L_{(Q_1,Q_2)}} \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big)\,\overline{\psi_j}(Q)\,ds_{\mathbb{H}^n}(Q) \\ &= \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big)\int_{L_{(Q_1,Q_2)}} \overline{\psi_j}(Q)\,ds_{\mathbb{H}^n}(Q)\end{aligned}$$

and

$$\begin{aligned} b_{t,\eta_k}(s) &= \frac{2^{1-s}\,\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)^2} \int_{L_{(Q_1,Q_2)}} a_{t,\eta_k,Q}(s)\; ds_{\mathbb{H}^n}(Q) \\ &= \frac{2^{1-s}\,\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)^2} \int_{L_{(Q_1,Q_2)}} \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\, \Gamma\Big(\frac{s-\frac{n-1}{2}+it}{2}\Big)\, \Gamma\Big(\frac{s-\frac{n-1}{2}-it}{2}\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2}-it\Big)\; ds_{\mathbb{H}^n}(Q) \\ &= \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\, \Gamma\Big(\frac{s-\frac{n-1}{2}+it}{2}\Big)\, \Gamma\Big(\frac{s-\frac{n-1}{2}-it}{2}\Big) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2}-it\Big)\; ds_{\mathbb{H}^n}(Q). \end{aligned}$$

This finishes the proof of the theorem.

□

Remark 5.2.3. The continuous part

$$\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} b_{t,\eta_k}(s)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2}+it\Big)\; dt$$

in the spectral expansion (5.8) in Theorem 5.2.2 does not appear in the case $c_\Gamma = 0$, i.e. if the discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ contains no parabolic element.

6. Meromorphic continuations

After having determined the spectral expansions of the hyperbolic kernel function and the hyperbolic Eisenstein series in the last chapter, we now use our knowledge of the coefficients in these expansions to establish the meromorphic continuations of the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$, the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ and the elliptic Eisenstein series $E^{\mathrm{ell}}_{Q}(P,s)$ in s to the whole complex plane. In the first and second section we do this for the hyperbolic kernel function and the hyperbolic Eisenstein series, respectively. In the chapter's third section we derive a relation between the elliptic Eisenstein series and the hyperbolic kernel function which enables us to establish the meromorphic continuation of the elliptic Eisenstein series as well. These methods follow the ideas that were employed for $n = 2$ to establish the meromorphic continuations of hyperbolic Eisenstein series in [JKP10] and elliptic Eisenstein series in [Pip10].

6.1. Meromorphic continuation of the hyperbolic kernel function

Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. In this section we use the spectral expansion of the hyperbolic kernel function computed in section 5.1 to establish its meromorphic continuation in s to the whole complex plane. Moreover, we also determine its possible poles.

Theorem 6.1.1. *For $P, Q \in \mathbb{H}^n$ the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ admits a meromorphic continuation in s to the whole complex plane. For any $m \in \mathbb{N}_0$ the possible poles of the function*

$$\frac{\Gamma(s)\, K^{\mathrm{hyp}}(P,Q,s)}{\Gamma\left(s-\frac{n-1}{2}\right)}$$

in the strip $\left\{ s \in \mathbb{C} \,\middle|\, \frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) \leq \frac{n-1}{2} - 2m \right\}$ are located at the following points:

(i) $s = \frac{n-1}{2} \pm ir_j - 2N$, where $j \in \mathbb{N}_0$ and $\lambda_j = s_j(n-1-s_j) = \left(\frac{n-1}{2}\right)^2 + r_j^2$ is the eigenvalue of the eigenfunction $\psi_j(P)$, and where $N := \left\lceil m \pm \frac{\mathrm{Re}(ir_j)}{2} \right\rceil$ has to satisfy $N \geq 0$.
If $N \mp ir_j \notin \mathbb{N}_0$ and $2N \mp ir_j \notin \mathbb{N}_0$, it is a simple pole with residue

$$\begin{aligned}
&\mathrm{Res}_{s=\frac{n-1}{2}\pm ir_j-2N}\left[\frac{\Gamma(s)\, K^{\mathrm{hyp}}(P,Q,s)}{\Gamma\left(s-\frac{n-1}{2}\right)}\right] \\
&\quad = \frac{(-1)^N\, 2^{\frac{n-1}{2}\pm ir_j-2N}\, \pi^{\frac{n-1}{2}}\, \Gamma(\pm ir_j - N)}{N!\, \Gamma\left(\pm ir_j - 2N\right)} \sum_{\substack{l\in\mathbb{N}_0:\\ r_l=r_j}} \psi_l(P)\, \overline{\psi_l}(Q);
\end{aligned}$$

and if $N \mp ir_j \in \mathbb{N}_0$, it is a simple pole with residue

$$\mathrm{Res}_{s=\frac{n-1}{2}\pm ir_j-2N}\left[\frac{\Gamma(s)\, K^{\mathrm{hyp}}(P,Q,s)}{\Gamma\left(s-\frac{n-1}{2}\right)}\right] = \frac{2^{\frac{n+1}{2}\pm ir_j-2N}\, \pi^{\frac{n-1}{2}}\, (2N\mp ir_j)!}{N!\, (N\mp ir_j)!} \sum_{\substack{l\in\mathbb{N}_0:\\ r_l=r_j}} \psi_l(P)\, \overline{\psi_l}(Q).$$

In case that $N \mp ir_j \notin \mathbb{N}_0$ and $2N \mp ir_j \in \mathbb{N}_0$, it is no pole but a removable singularity.

(ii) $s = n-1-\rho-2N$, where $N \in \left\{\max\left(m - \left\lfloor\frac{n-1}{4}\right\rfloor, 0\right), \ldots, m\right\}$, and where $w = \rho$ is a pole of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,w)$ for some $k \in \{1, \ldots, c_\Gamma\}$ with

$$\rho \in \left[\frac{n-1}{2} + 2(m-N),\ \frac{n-1}{2} + 2(m+1-N)\right);$$

in particular, $\rho \in \left(\frac{n-1}{2}, n-1\right]$.

(iii) $s = \rho - 2N$*, where* $N \in \{0, \ldots, m\}$*, and where* $w = \rho$ *is a pole of the parabolic Eisenstein series* $E^{\mathrm{par}}_{\eta_k}(P, w)$ *for some* $k \in \{1, \ldots, c_\Gamma\}$ *with*

$$\mathrm{Re}(\rho) \in \Big(\frac{n-1}{2} - 2(m+1-N),\ \frac{n-1}{2} - 2(m-N)\Big];$$

in particular, $\mathrm{Re}(\rho) < \frac{n-1}{2}$.

The poles given in the cases (i), (ii), (iii) might coincide in parts.

Moreover, the possible poles of the function

$$\frac{\Gamma(s)\, K^{\mathrm{hyp}}(P, Q, s)}{\Gamma\left(s - \frac{n-1}{2}\right)}$$

in the half-plane $\left\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \frac{n-1}{2}\right\}$ *are located at the points* $s = \frac{n-1}{2} + ir_j - 2N$*, where* $j \in \mathbb{N}_0$ *and* $\lambda_j = s_j(n-1-s_j) = \left(\frac{n-1}{2}\right)^2 + r_j^2$ *is the eigenvalue of the eigenfunction* $\psi_j(P)$ *with* $r_j \in \left[-i\,\frac{n-1}{2}, 0\right)$*, and where* $N \in \mathbb{N}_0$ *with* $N < \frac{ir_j}{2}$*. The orders and residues are as in the case (i) above.*

Proof. We use the spectral expansion of $K^{\mathrm{hyp}}(P, Q, s)$ to prove its meromorphic continuation. By Theorem 5.1.1, for $P, Q \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ the hyperbolic kernel function admits the spectral expansion

$$K^{\mathrm{hyp}}(P, Q, s) = \sum_{j=0}^{\infty} a_{j,Q}(s)\, \psi_j(P) + \frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt, \tag{6.1}$$

where $a_{j,Q}(s)$ and $a_{t,\eta_k,Q}(s)$ are given by

$$a_{j,Q}(s) = \frac{2^{s-1}\, \pi^{\frac{n-1}{2}}}{\Gamma(s)}\, \Gamma\Big(\frac{s - \frac{n-1}{2} + ir_j}{2}\Big)\, \Gamma\Big(\frac{s - \frac{n-1}{2} - ir_j}{2}\Big)\, \overline{\psi_j}(Q), \tag{6.2}$$

$$a_{t,\eta_k,Q}(s) = \frac{2^{s-1}\, \pi^{\frac{n-1}{2}}}{\Gamma(s)}\, \Gamma\Big(\frac{s - \frac{n-1}{2} + it}{2}\Big)\, \Gamma\Big(\frac{s - \frac{n-1}{2} - it}{2}\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} - it\Big), \tag{6.3}$$

respectively.

First we give the meromorphic continuation of the series

$$\sum_{j=0}^{\infty} a_{j,Q}(s)\, \psi_j(P) \tag{6.4}$$

that arises from the discrete spectrum. The explicit formula (6.2) already proves the meromorphic continuation of the coefficient $a_{j,Q}(s)$ in s to the whole complex plane. We are left to show that the series (6.4) converges absolutely and locally uniformly for all $s \in \mathbb{C}$.
For this we use Stirling's asymptotic formula (A.5) for the gamma function, which for fixed $\sigma \in \mathbb{R}$ gives us the asymptotics

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}\, |t|^{\sigma - 1/2} \exp\Big(-\frac{\pi |t|}{2}\Big) \qquad (|t| \to \infty)$$

with an implied constant depending on σ. Using this formula, for fixed $s \in \mathbb{C}$ and any $j \in \mathbb{N}_0$ with $r_j \geq 0$ we obtain the bound

$$\begin{aligned}
\left|\Gamma\Big(\frac{s - \frac{n-1}{2} \pm ir_j}{2}\Big)\right| &= \mathrm{O}\left(\sqrt{2\pi}\, \Big(\frac{|\mathrm{Im}(s) \pm r_j|}{2}\Big)^{\frac{\mathrm{Re}(s)}{2} - \frac{n+1}{4}} \exp\Big(-\frac{\pi\, |\mathrm{Im}(s) \pm r_j|}{4}\Big)\right) \\
&= \mathrm{O}\left(|\mathrm{Im}(s) \pm r_j|^{\frac{\mathrm{Re}(s)}{2} - \frac{n+1}{4}} \exp\Big(-\frac{\pi\, |\mathrm{Im}(s) \pm r_j|}{4}\Big)\right) = \mathrm{O}\left(r_j^{\frac{\mathrm{Re}(s)}{2} - \frac{n+1}{4}} \exp\Big(-\frac{\pi\, r_j}{4}\Big)\right)
\end{aligned}$$

as $r_j \to \infty$, with an implied constant depending on s. This yields

$$\left|\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big)\right| = \mathrm{O}\left(r_j^{\mathrm{Re}(s)-\frac{n+1}{2}}\,\exp\Big(-\frac{\pi\, r_j}{2}\Big)\right) \qquad (r_j\to\infty) \tag{6.5}$$

with an implied constant depending on s. In Remark 3.5.7 we have seen that the eigenfunction $\psi_j(P)$ satisfies the bound

$$\sup_{P\in\mathbb{H}^n}|\psi_j(P)| = \mathrm{O}\left(r_j^{n/2}\right) \qquad (r_j\to\infty).$$

Further, by Remark 3.5.4 there are only finitely many $j\in\mathbb{N}_0$ with $\lambda_j\in\Big[0,\big(\frac{n-1}{2}\big)^2\Big)$, i.e. with $r_j\in\big[-i\,\frac{n-1}{2},0\big)$. Hence, for all but these finitely many $j\in\mathbb{N}_0$ we obtain the bound

$$|a_{j,Q}(s)\,\psi_j(P)| = \mathrm{O}\left(r_j^{\mathrm{Re}(s)-\frac{n+1}{2}}\,\exp\Big(-\frac{\pi\, r_j}{2}\Big)\,r_j^{n}\right) = \mathrm{O}\left(r_j^{\mathrm{Re}(s)+\frac{n-1}{2}}\,\exp\Big(-\frac{\pi\, r_j}{2}\Big)\right) \quad (r_j\to\infty)$$

with an implied constant depending on s. This shows that the series (6.4) arising from the discrete spectrum converges absolutely and locally uniformly for all $s\in\mathbb{C}$. Consequently, it defines a holomorphic function away from the poles of $a_{j,Q}(s)$.

Now we can determine the poles of the series (6.4) after multiplication by $\Gamma(s)\,\Gamma\left(s-\frac{n-1}{2}\right)^{-1}$, i.e. the poles of

$$\frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma\left(s-\frac{n-1}{2}\right)}\sum_{j=0}^{\infty}\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big)\,\overline{\psi_j}(Q)\,\psi_j(P), \tag{6.6}$$

which can only arise from the two gamma factors in the sum. Recall from section A.1 that $z\mapsto\Gamma\left(\frac{z}{2}\right)$ is a meromorphic function for all $z\in\mathbb{C}$ whose poles are located at the points $z=-2N$ ($N\in\mathbb{N}_0$) and are simple. This gives us the possible poles $s=\frac{n-1}{2}\pm ir_j-2N$ of (6.6), where $j,N\in\mathbb{N}_0$.
We note that $\frac{n-1}{2}-2(m+1)<\mathrm{Re}\left(\frac{n-1}{2}\pm ir_j-2N\right)\le\frac{n-1}{2}-2m$ holds true for some $m\in\mathbb{N}_0$ if and only if $m\pm\frac{\mathrm{Re}(ir_j)}{2}\le N<m+1\pm\frac{\mathrm{Re}(ir_j)}{2}$. As this interval contains exactly one integer, this yields $N=\left\lceil m\pm\frac{\mathrm{Re}(ir_j)}{2}\right\rceil$, while, due to the condition $N\in\mathbb{N}_0$, also $N\ge 0$ must hold.
Further, $\mathrm{Re}\left(\frac{n-1}{2}\pm ir_j-2N\right)>\frac{n-1}{2}$ is equivalent to $N<\pm\frac{\mathrm{Re}(ir_j)}{2}$. Since either $r_j\ge 0$ or $r_j\in\big[-i\,\frac{n-1}{2},0\big)$, the condition $N<-\frac{\mathrm{Re}(ir_j)}{2}$ cannot be satisfied, and there is no pole of the form $s=\frac{n-1}{2}-ir_j-2N$ with $\mathrm{Re}(s)>\frac{n-1}{2}$. On the other hand, the condition $N<\frac{\mathrm{Re}(ir_j)}{2}$, respectively a pole of the form $s=\frac{n-1}{2}+ir_j-2N$ with $\mathrm{Re}(s)>\frac{n-1}{2}$, is only possible for $r_j\in\big[-i\,\frac{n-1}{2},0\big)$.

If $N\mp ir_j\notin\mathbb{N}_0$ and $2N\mp ir_j\notin\mathbb{N}_0$, the point $s=\frac{n-1}{2}\pm ir_j-2N$ is a simple pole of exactly one of the gamma factors in the sum and no pole of $\Gamma\left(s-\frac{n-1}{2}\right)$. Thus, it is a simple pole of (6.6). The summands that contribute to the residue are all these for which $l\in\mathbb{N}_0$ satisfies $r_l=r_j$. Using the fact that

$$\mathrm{Res}_{z=\frac{n-1}{2}\pm ir_j-2N}\,\Gamma\Big(\frac{z-\frac{n-1}{2}\mp ir_j}{2}\Big) = \mathrm{Res}_{z=-2N}\,\Gamma\Big(\frac{z}{2}\Big) = \frac{2\,(-1)^N}{N!} \tag{6.7}$$

and inserting $s=\frac{n-1}{2}\pm ir_j-2N$ into the terms that are holomorphic at this point, we obtain the asserted residue

$$\frac{(-1)^N\,2^{\frac{n-1}{2}\pm ir_j-2N}\,\pi^{\frac{n-1}{2}}\,\Gamma(\pm ir_j-N)}{N!\,\Gamma\left(\pm ir_j-2N\right)}\sum_{\substack{l\in\mathbb{N}_0:\\ r_l=r_j}}\psi_l(P)\,\overline{\psi_l}(Q).$$

If $N\mp ir_j\in\mathbb{N}_0$ holds true, then also $2N\mp ir_j\in\mathbb{N}_0$, and $s=\frac{n-1}{2}\pm ir_j-2N$ is a simple pole of both gamma factors in the sum and a simple pole of $\Gamma\left(s-\frac{n-1}{2}\right)$ as well. Hence, it is a simple pole of (6.6), and again all the summands with $l \in \mathbb{N}_0$ such that $r_l = r_j$ contribute to the residue. Setting $M := N \mp ir_j$, we have

$$\mathrm{Res}_{z=\frac{n-1}{2}\pm ir_j-2N}\,\Gamma\Big(\frac{z-\frac{n-1}{2}\pm ir_j}{2}\Big) = \mathrm{Res}_{z=-2M}\,\Gamma\Big(\frac{z}{2}\Big) = \frac{2\,(-1)^M}{M!} = \frac{2\,(-1)^{N\mp ir_j}}{(N\mp ir_j)!}$$

and

$$\mathrm{Res}_{z=\frac{n-1}{2}\pm ir_j-2N}\,\Gamma\Big(z-\frac{n-1}{2}\Big) = \mathrm{Res}_{z=-(M+N)}\,\Gamma(z) = \frac{(-1)^{M+N}}{(M+N)!} = \frac{(-1)^{2N\mp ir_j}}{(2N\mp ir_j)!}.$$

Together with (6.7) this yields the residue

$$\frac{2^{\frac{n+1}{2}\pm ir_j-2N}\,\pi^{\frac{n-1}{2}}\,(2N\mp ir_j)!}{N!\,(N\mp ir_j)!}\sum_{\substack{l\in\mathbb{N}_0:\\ r_l=r_j}}\psi_l(P)\,\overline{\psi_l}(Q).$$

If $N \mp ir_j \notin \mathbb{N}_0$ and $2N \mp ir_j \in \mathbb{N}_0$ hold, the point $s = \frac{n-1}{2} \pm ir_j - 2N$ is a simple pole of exactly one of the gamma factors in the sum but also a simple pole of $\Gamma\left(s-\frac{n-1}{2}\right)$. Consequently, these two poles cancel each other and it is a removable singularity of (6.6).
This finishes the treatment of the discrete part of (6.1).

For a cusp η_k ($k = 1,\ldots,c_\Gamma$) of Γ we now give the meromorphic continuation of the integral

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt$$
$$= \frac{2^{s-1}\,\pi^{\frac{n-3}{2}}}{4\,\Gamma(s)}\int_{-\infty}^{\infty}\Gamma\Big(\frac{s-\frac{n-1}{2}+it}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-it}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-it\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt$$

that arises from the continuous part of the spectral expansion (6.1). Through the substitution $w := \frac{n-1}{2} + it$ the integral becomes

$$I_{\eta_k}(s) := \frac{2^{s-1}\,\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma(s)}\int_{\mathrm{Re}(w)=\frac{n-1}{2}}\Gamma\Big(\frac{s-n+1+w}{2}\Big)\,\Gamma\Big(\frac{s-w}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}(Q,n-1-w)\,E^{\mathrm{par}}_{\eta_k}(P,w)\,dw,$$

where the line of integration goes from $\frac{n-1}{2}-i\infty$ to $\frac{n-1}{2}+i\infty$. The function $I_{\eta_k}(s)$ is holomorphic for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$. In fact, it is even holomorphic for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \neq \frac{n-1}{2} - 2m$ for any $m \in \mathbb{N}_0$, so that in particular $I^{(0)}_{\eta_k}(s) = I_{\eta_k}(s)$ is a meromorphic function on the half-plane $\mathrm{Re}(s) > \frac{n-1}{2}$.

Now let $\varepsilon \in (0,1)$ sufficiently small such that the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,s)$ has no poles in the strip $\frac{n-1}{2} - 2\varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} + 2\varepsilon$. Such an ε exists by Corollary 3.4.14. For fixed $s \in \mathbb{C}$ with $\frac{n-1}{2} < \mathrm{Re}(s) < \frac{n-1}{2} + \varepsilon$ we set

$$f_{\eta_k,s}(w) := \frac{2^{s-1}\,\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma(s)}\,\Gamma\Big(\frac{s-n+1+w}{2}\Big)\,\Gamma\Big(\frac{s-w}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}(Q,n-1-w)\,E^{\mathrm{par}}_{\eta_k}(P,w).$$

Let $y \in \mathbb{R}$ with $y > |\mathrm{Im}(s)|$ and let $W_{y,\varepsilon}$ denote the following piecewise linear path: the vertical line from $\frac{n-1}{2} - i\infty$ to $\frac{n-1}{2} - iy$, the horizontal line segment from $\frac{n-1}{2} - iy$ to $\frac{n-1}{2} + \varepsilon - iy$, the vertical line segment from $\frac{n-1}{2}+\varepsilon-iy$ to $\frac{n-1}{2}+\varepsilon+iy$, the horizontal line segment from $\frac{n-1}{2}+\varepsilon+iy$ to $\frac{n-1}{2} + iy$, and the vertical line from $\frac{n-1}{2} + iy$ to $\frac{n-1}{2} + i\infty$.
Then the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(Q, n-1-w)$, $E^{\mathrm{par}}_{\eta_k}(P,w)$ have no poles on $W_{y,\varepsilon}$ by the choice of ε. The factor $\Gamma\left(\frac{s-w}{2}\right)$ of $f_{\eta_k,s}(w)$ has no poles on the vertical parts of $W_{y,\varepsilon}$, since on these we have $0 < \mathrm{Re}\left(\frac{s-w}{2}\right) < \varepsilon/2$ and $-\varepsilon/2 < \mathrm{Re}\left(\frac{s-w}{2}\right) < 0$, respectively, and we chose

$\varepsilon \in (0,1)$. On the horizontal parts of $W_{y,\varepsilon}$ it has no poles as well, because $|\mathrm{Im}(w)| = y > |\mathrm{Im}(s)|$ and thus $\mathrm{Im}\left(\frac{s-w}{2}\right) \neq 0$ holds true there by the choice of y. Likewise, the factor $\Gamma\left(\frac{s-n+1+w}{2}\right)$ of $f_{\eta_k,s}(w)$ has no poles on the vertical parts of $W_{y,\varepsilon}$, as on these we have $0 < \mathrm{Re}\left(\frac{s-n+1+w}{2}\right) < \varepsilon/2$ and $\varepsilon/2 < \mathrm{Re}\left(\frac{s-n+1+w}{2}\right) < \varepsilon$, respectively. On the horizontal parts of $W_{y,\varepsilon}$ it also has no poles, since $|\mathrm{Im}(w)| = y > |\mathrm{Im}(s)|$ and $\mathrm{Im}\left(\frac{s-n+1+w}{2}\right) = \mathrm{Im}\left(\frac{s+w}{2}\right) \neq 0$ holds true there by the choice of y.

Overall, $f_{\eta_k,s}(w)$ is a meromorphic function on a suitable domain containing $W_{y,\varepsilon}$ and has no poles on this path. Setting

$$\widetilde{I}_{\eta_k,y,\varepsilon}(s) := \int_{W_{y,\varepsilon}} f_{\eta_k,s}(w)\; dw,$$

we get by the residue theorem

$$\begin{aligned}\widetilde{I}_{\eta_k,y,\varepsilon}(s) - I_{\eta_k}(s) &= \int_{\frac{n-1}{2}-iy}^{\frac{n-1}{2}+\varepsilon-iy} f_{\eta_k,s}(w)\; dw + \int_{\frac{n-1}{2}+\varepsilon-iy}^{\frac{n-1}{2}+\varepsilon+iy} f_{\eta_k,s}(w)\; dw \\ &\quad + \int_{\frac{n-1}{2}+\varepsilon+iy}^{\frac{n-1}{2}+iy} f_{\eta_k,s}(w)\; dw + \int_{\frac{n-1}{2}+iy}^{\frac{n-1}{2}-iy} f_{\eta_k,s}(w)\; dw \\ &= 2\pi i \sum_a \mathrm{Res}_{w=a}\, f_{\eta_k,s}(w),\end{aligned}$$

where we sum over all poles $a \in \mathbb{C}$ that are contained in the box $\frac{n-1}{2} < \mathrm{Re}(a) < \frac{n-1}{2} + \varepsilon$, $-y < \mathrm{Im}(a) < y$. Hence, for $s \in \mathbb{C}$ with $\frac{n-1}{2} < \mathrm{Re}(s) < \frac{n-1}{2} + \varepsilon$ we get

$$\begin{aligned}I_{\eta_k}(s) = \widetilde{I}_{\eta_k,y,\varepsilon}(s) - 2\pi i \cdot \frac{2^{s-1}\,\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma(s)} \\ \cdot \sum_a \mathrm{Res}_{w=a}\left[\Gamma\Big(\frac{s-n+1+w}{2}\Big)\,\Gamma\Big(\frac{s-w}{2}\Big)\, E^{\mathrm{par}}_{\eta_k}(Q, n-1-w)\, E^{\mathrm{par}}_{\eta_k}(P,w)\right],\end{aligned}$$

where the sum runs over all poles $a \in \mathbb{C}$ with $\frac{n-1}{2} < \mathrm{Re}(a) < \frac{n-1}{2} + \varepsilon$ and $-y < \mathrm{Im}(a) < y$.
By the choice of ε the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(Q, n-1-w)$, $E^{\mathrm{par}}_{\eta_k}(P,w)$ have no poles in this box. The function $\Gamma\left(\frac{s-n+1+w}{2}\right)$ also has no poles there, since in the considered box $0 < \mathrm{Re}\left(\frac{s-n+1+w}{2}\right) < \varepsilon$ holds true. Further, the function $\Gamma\left(\frac{s-w}{2}\right)$ has only one pole in this box at $w = s$, because $-\varepsilon/2 < \mathrm{Re}\left(\frac{s-w}{2}\right) < \varepsilon/2$ holds true there and we chose $\varepsilon \in (0,1)$, and the corresponding residue is equal to -2. This implies

$$I_{\eta_k}(s) = \widetilde{I}_{\eta_k,y,\varepsilon}(s) + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\,\Gamma\Big(s - \frac{n-1}{2}\Big)\, E^{\mathrm{par}}_{\eta_k}(Q, n-1-s)\, E^{\mathrm{par}}_{\eta_k}(P,s). \tag{6.8}$$

The right-hand side of (6.8) is a meromorphic function for $\frac{n-1}{2} - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} + \varepsilon$ and $-y < \mathrm{Im}(s) < y$. Since y can be chosen arbitrarily large, this gives the meromorphic continuation $I^{(0,1)}_{\eta_k}(s)$ of the integral $I_{\eta_k}(s)$ to the whole strip $\frac{n-1}{2} - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} + \varepsilon$.

Now we assume $\frac{n-1}{2} - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2}$. Let $y \in \mathbb{R}$ with $y > |\mathrm{Im}(s)|$ and the path $W_{y,\varepsilon}$ as above. The parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(Q, n-1-w)$, $E^{\mathrm{par}}_{\eta_k}(P,w)$ have no poles on $W_{y,\varepsilon}$. The function $\Gamma\left(\frac{s-w}{2}\right)$ has no poles on the vertical parts of $W_{y,\varepsilon}$, as on these we have $-\varepsilon/2 < \mathrm{Re}\left(\frac{s-w}{2}\right) < 0$ and $-\varepsilon < \mathrm{Re}\left(\frac{s-w}{2}\right) < -\varepsilon/2$, respectively, and we chose $\varepsilon \in (0,1)$. That it has no poles on the horizontal parts of $W_{y,\varepsilon}$ follows from the choice of y, as before. Likewise, $\Gamma\left(\frac{s-n+1+w}{2}\right)$ has no poles on the vertical parts of $W_{y,\varepsilon}$, since on these we have $-\varepsilon/2 < \mathrm{Re}\left(\frac{s-n+1+w}{2}\right) < 0$ and $0 < \mathrm{Re}\left(\frac{s-n+1+w}{2}\right) < \varepsilon/2$, respectively, and we chose $\varepsilon \in (0,1)$. From the choice of y we conclude that it has no poles on the horizontal parts of $W_{y,\varepsilon}$ as well.

So, using the residue theorem as above, for $s \in \mathbb{C}$ with $\frac{n-1}{2} - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2}$ and $-y < \mathrm{Im}(s) < y$ we obtain

$$\begin{aligned} I_{\eta_k}^{(0,1)}(s) = I_{\eta_k}(s) + 2\pi i \cdot \frac{2^{s-1}\,\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma(s)} \\ \cdot \sum_a \mathrm{Res}_{w=a} \Big[\Gamma\Big(\frac{s-n+1+w}{2}\Big)\Gamma\Big(\frac{s-w}{2}\Big) E_{\eta_k}^{\mathrm{par}}(Q, n-1-w)\, E_{\eta_k}^{\mathrm{par}}(P, w)\Big] \\ + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\Gamma\Big(s - \frac{n-1}{2}\Big) E_{\eta_k}^{\mathrm{par}}(Q, n-1-s)\, E_{\eta_k}^{\mathrm{par}}(P, s), \end{aligned}$$

where we sum over all poles $a \in \mathbb{C}$ with $\frac{n-1}{2} < \mathrm{Re}(a) < \frac{n-1}{2} + \varepsilon$ and $-y < \mathrm{Im}(a) < y$ again.
As above, the parabolic Eisenstein series $E_{\eta_k}^{\mathrm{par}}(Q, n-1-w)$, $E_{\eta_k}^{\mathrm{par}}(P, w)$ have no poles in this box. Also the function $\Gamma\left(\frac{s-w}{2}\right)$ has no poles in the considered box, as $-\varepsilon < \mathrm{Re}\left(\frac{s-w}{2}\right) < 0$ holds true there and we chose $\varepsilon \in (0,1)$. Moreover, the function $\Gamma\left(\frac{s-n+1+w}{2}\right)$ has only one pole in this box at $w = n-1-s$, since $-\varepsilon/2 < \mathrm{Re}\left(\frac{s-n+1+w}{2}\right) < \varepsilon/2$ holds true there and $\varepsilon \in (0,1)$, and the corresponding residue is equal to 2. So we obtain

$$\begin{aligned} I_{\eta_k}^{(0,1)}(s) = I_{\eta_k}(s) + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\Gamma\Big(s - \frac{n-1}{2}\Big) E_{\eta_k}^{\mathrm{par}}(Q, s)\, E_{\eta_k}^{\mathrm{par}}(P, n-1-s) \\ + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\Gamma\Big(s - \frac{n-1}{2}\Big) E_{\eta_k}^{\mathrm{par}}(Q, n-1-s)\, E_{\eta_k}^{\mathrm{par}}(P, s). \end{aligned} \tag{6.9}$$

The right-hand side of (6.9) is now a meromorphic function for $\frac{n-1}{2} - 2 < \mathrm{Re}(s) < \frac{n-1}{2}$, giving the meromorphic continuation $I_{\eta_k}^{(0,2)}(s)$ of the integral $I_{\eta_k}^{(0,1)}(s)$ to this strip.
On the line $\mathrm{Re}(s) = \frac{n-1}{2}$ the meromorphic continuation of the integral $I_{\eta_k}(s)$ is given by the right hand side of (6.8), where y has to be chosen such that $y > |\mathrm{Im}(s)|$. Together these formulas provide the meromorphic continuation $I_{\eta_k}^{(1)}(s)$ of the integral $I_{\eta_k}(s)$ to the strip $\frac{n-1}{2} - 2 < \mathrm{Re}(s) \leq \frac{n-1}{2}$.

In the following we continue this two-step process and show inductively that for $m \in \mathbb{N}_0$ the meromorphic continuation $I_{\eta_k}^{(m+1)}(s)$ of the integral $I_{\eta_k}(s)$ to the strip $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) \leq \frac{n-1}{2} - 2m$ is given by

$$\begin{aligned} I_{\eta_k}(s) + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m} \frac{(-1)^l}{l!}\Gamma\Big(s - \frac{n-1}{2} + l\Big) E_{\eta_k}^{\mathrm{par}}(Q, n-1-s-2l)\, E_{\eta_k}^{\mathrm{par}}(P, s+2l) \\ + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m} \frac{(-1)^l}{l!}\Gamma\Big(s - \frac{n-1}{2} + l\Big) E_{\eta_k}^{\mathrm{par}}(Q, s+2l)\, E_{\eta_k}^{\mathrm{par}}(P, n-1-s-2l) \end{aligned} \tag{6.10}$$

for $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) < \frac{n-1}{2} - 2m$, while on the line $\mathrm{Re}(s) = \frac{n-1}{2} - 2m$ we have to take

$$\begin{aligned} \widetilde{I}_{\eta_k,y,\varepsilon}(s) + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m} \frac{(-1)^l}{l!}\Gamma\Big(s - \frac{n-1}{2} + l\Big) E_{\eta_k}^{\mathrm{par}}(Q, n-1-s-2l)\, E_{\eta_k}^{\mathrm{par}}(P, s+2l) \\ + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m-1} \frac{(-1)^l}{l!}\Gamma\Big(s - \frac{n-1}{2} + l\Big) E_{\eta_k}^{\mathrm{par}}(Q, s+2l)\, E_{\eta_k}^{\mathrm{par}}(P, n-1-s-2l), \end{aligned} \tag{6.11}$$

where y is chosen sufficiently large such that $y > |\mathrm{Im}(s)|$.

Suppose, for some $m \in \mathbb{N}_0$ we have established the meromorphic continuation $I_{\eta_k}^{(m+1)}(s)$ of the integral $I_{\eta_k}(s)$ to the strip $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) \leq \frac{n-1}{2} - 2m$ via the formulas (6.10) and (6.11). Let $s \in \mathbb{C}$ with $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) < \frac{n-1}{2} - 2(m+1) + \varepsilon$ and $y > |\mathrm{Im}(s)|$, and let the path

$W_{y,\varepsilon}$ as before.
The parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(Q, n-1-w)$, $E^{\mathrm{par}}_{\eta_k}(P, w)$ have no poles on $W_{y,\varepsilon}$. The function $\Gamma\left(\frac{s-w}{2}\right)$ has no poles on the vertical parts of $W_{y,\varepsilon}$, because on these we have $-(m+1) < \mathrm{Re}\left(\frac{s-w}{2}\right) < -(m+1)+\varepsilon/2$ and $-(m+1)-\varepsilon/2 < \mathrm{Re}\left(\frac{s-w}{2}\right) < -(m+1)$, respectively, and we chose $\varepsilon \in (0,1)$. On the horizontal parts of $W_{y,\varepsilon}$ it has no poles, since $|\mathrm{Im}(w)| = y > |\mathrm{Im}(s)|$ and thus $\mathrm{Im}\left(\frac{s-w}{2}\right) \neq 0$ holds true there by the choice of y. Similarly, the function $\Gamma\left(\frac{s-n+1+w}{2}\right)$ has no poles on the vertical parts of $W_{y,\varepsilon}$, as on these we have $-(m+1) < \mathrm{Re}\left(\frac{s-n+1+w}{2}\right) < -(m+1)+\varepsilon/2$ and $-(m+1)+\varepsilon/2 < \mathrm{Re}\left(\frac{s-n+1+w}{2}\right) < -(m+1)+\varepsilon$, respectively, and $\varepsilon \in (0,1)$. On the horizontal parts of $W_{y,\varepsilon}$ it also has no poles, because $|\mathrm{Im}(w)| = y > |\mathrm{Im}(s)|$ and $\mathrm{Im}\left(\frac{s-n+1+w}{2}\right) = \mathrm{Im}\left(\frac{s+w}{2}\right) \neq 0$ holds true there by the choice of y.

Now for $s \in \mathbb{C}$ with $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) < \frac{n-1}{2} - 2(m+1) + \varepsilon$ the residue theorem yields

$$I_{\eta_k}(s) = \widetilde{I}_{\eta_k,y,\varepsilon}(s) - 2\pi i \cdot \frac{2^{s-1}\,\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma(s)} \cdot \sum_a \mathrm{Res}_{w=a}\left[\Gamma\Big(\frac{s-n+1+w}{2}\Big)\,\Gamma\Big(\frac{s-w}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}(Q, n-1-w)\,E^{\mathrm{par}}_{\eta_k}(P, w)\right],$$

where we sum over all poles $a \in \mathbb{C}$ with $\frac{n-1}{2} < \mathrm{Re}(a) < \frac{n-1}{2} + \varepsilon$ and $-y < \mathrm{Im}(a) < y$.
By the choice of ε the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(Q, n-1-w)$, $E^{\mathrm{par}}_{\eta_k}(P, w)$ have no poles in this box. Also the function $\Gamma\left(\frac{s-n+1+w}{2}\right)$ has no poles in the considered box, as $-(m+1) < \mathrm{Re}\left(\frac{s-n+1+w}{2}\right) < -(m+1)+\varepsilon$ holds true there and we chose $\varepsilon \in (0,1)$. Moreover, the function $\Gamma\left(\frac{s-w}{2}\right)$ has only one pole in this box at $w = s + 2(m+1)$, because $-(m+1)-\varepsilon/2 < \mathrm{Re}\left(\frac{s-w}{2}\right) < -(m+1)+\varepsilon/2$ holds true there and $\varepsilon \in (0,1)$, and the corresponding residue is equal to $-2\,(-1)^{m+1}/(m+1)!$. We obtain

$$I_{\eta_k}(s) = \widetilde{I}_{\eta_k,y,\varepsilon}(s) + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\,\frac{(-1)^{m+1}}{(m+1)!}\,\Gamma\Big(s - \frac{n-1}{2} + m + 1\Big) \cdot E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2(m+1))\,E^{\mathrm{par}}_{\eta_k}(P, s+2(m+1))$$

and

$$\begin{aligned} I_{\eta_k}(s) &+ \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m} \frac{(-1)^l}{l!}\,\Gamma\Big(s - \frac{n-1}{2} + l\Big)\,E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2l)\,E^{\mathrm{par}}_{\eta_k}(P, s+2l) \\ &+ \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m} \frac{(-1)^l}{l!}\,\Gamma\Big(s - \frac{n-1}{2} + l\Big)\,E^{\mathrm{par}}_{\eta_k}(Q, s+2l)\,E^{\mathrm{par}}_{\eta_k}(P, n-1-s-2l) \\ = \widetilde{I}_{\eta_k,y,\varepsilon}(s) &+ \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m+1} \frac{(-1)^l}{l!}\,\Gamma\Big(s - \frac{n-1}{2} + l\Big)\,E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2l)\,E^{\mathrm{par}}_{\eta_k}(P, s+2l) \\ &+ \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m} \frac{(-1)^l}{l!}\,\Gamma\Big(s - \frac{n-1}{2} + l\Big)\,E^{\mathrm{par}}_{\eta_k}(Q, s+2l)\,E^{\mathrm{par}}_{\eta_k}(P, n-1-s-2l). \end{aligned} \tag{6.12}$$

The right-hand side of (6.12) is a meromorphic function for $\frac{n-1}{2} - 2(m+1) - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} - 2(m+1) + \varepsilon$ and $-y < \mathrm{Im}(s) < y$. Thus, as we can choose y arbitrarily large, this gives the meromorphic continuation $I^{(m+1,1)}_{\eta_k}(s)$ of the integral $I_{\eta_k}(s)$ to the whole strip $\frac{n-1}{2} - 2(m+1) - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} - 2(m+1) + \varepsilon$.

Now we assume $\frac{n-1}{2} - 2(m+1) - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} - 2(m+1)$ and let $y > |\mathrm{Im}(s)|$ and $W_{y,\varepsilon}$ as before. The parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(Q, n-1-w)$, $E^{\mathrm{par}}_{\eta_k}(P, w)$ have no poles on $W_{y,\varepsilon}$. The function $\Gamma\left(\frac{s-w}{2}\right)$ has no poles on the vertical parts of $W_{y,\varepsilon}$, as on these $-(m+1)-\varepsilon/2 <$

$\mathrm{Re}\left(\frac{s-w}{2}\right) < -(m+1)$ and $-(m+1)-\varepsilon < \mathrm{Re}\left(\frac{s-w}{2}\right) < -(m+1)-\varepsilon/2$ holds true, respectively, and we chose $\varepsilon \in (0,1)$. That it has no poles on the horizontal parts of $W_{y,\varepsilon}$ follows from the choice of y, as above. Similarly, $\Gamma\left(\frac{s-n+1+w}{2}\right)$ has no poles on the vertical parts of $W_{y,\varepsilon}$, since on these $-(m+1)-\varepsilon/2 < \mathrm{Re}\left(\frac{s-n+1+w}{2}\right) < -(m+1)$ and $-(m+1) < \mathrm{Re}\left(\frac{s-n+1+w}{2}\right) < -(m+1)+\varepsilon/2$ holds true, respectively, and $\varepsilon \in (0,1)$. From the choice of y it follows that it also has no poles on the horizontal parts of $W_{y,\varepsilon}$.

Hence, for $s \in \mathbb{C}$ with $\frac{n-1}{2} - 2(m+1) - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} - 2(m+1)$ and $-y < \mathrm{Im}(s) < y$ the residue theorem gives us

$$\begin{aligned}
I_{\eta_k}^{(m+1,1)}(s) = I_{\eta_k}(s) + 2\pi i \cdot \frac{2^{s-1}\,\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma(s)} \\
\cdot \sum_a \mathrm{Res}_{w=a}\left[\Gamma\Big(\frac{s-n+1+w}{2}\Big)\Gamma\Big(\frac{s-w}{2}\Big)\,E_{\eta_k}^{\mathrm{par}}(Q,n-1-w)\,E_{\eta_k}^{\mathrm{par}}(P,w)\right] \\
+ \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m+1} \frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big)\,E_{\eta_k}^{\mathrm{par}}(Q,n-1-s-2l)\,E_{\eta_k}^{\mathrm{par}}(P,s+2l) \\
+ \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m} \frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big)\,E_{\eta_k}^{\mathrm{par}}(Q,s+2l)\,E_{\eta_k}^{\mathrm{par}}(P,n-1-s-2l),
\end{aligned}$$

where the sum runs over all poles $a \in \mathbb{C}$ with $\frac{n-1}{2} < \mathrm{Re}(a) < \frac{n-1}{2} + \varepsilon$ and $-y < \mathrm{Im}(a) < y$ again. As above, the parabolic Eisenstein series $E_{\eta_k}^{\mathrm{par}}(Q,n-1-w)$, $E_{\eta_k}^{\mathrm{par}}(P,w)$ have no poles there. The function $\Gamma\left(\frac{s-w}{2}\right)$ also has no poles in this box, because $-(m+1)-\varepsilon < \mathrm{Re}\left(\frac{s-w}{2}\right) < -(m+1)$ holds true there and we chose $\varepsilon \in (0,1)$. Further, the function $\Gamma\left(\frac{s-n+1+w}{2}\right)$ has only one pole there at $w = n-1-s-2(m+1)$ with residue equal to $2\,(-1)^{m+1}/(m+1)!$, as in the considered box $-(m+1)-\varepsilon/2 < \mathrm{Re}\left(\frac{s-n+1+w}{2}\right) < -(m+1)+\varepsilon/2$ holds true and $\varepsilon \in (0,1)$. This implies

$$\begin{aligned}
I_{\eta_k}^{(m+1,1)}(s) &= I_{\eta_k}(s) + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\,\frac{(-1)^{m+1}}{(m+1)!}\,\Gamma\Big(s-\frac{n-1}{2}+m+1\Big) \\
&\quad \cdot E_{\eta_k}^{\mathrm{par}}(Q,s+2(m+1))\,E_{\eta_k}^{\mathrm{par}}(P,n-1-s-2(m+1)) \\
&\quad + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m+1} \frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big)\,E_{\eta_k}^{\mathrm{par}}(Q,n-1-s-2l)\,E_{\eta_k}^{\mathrm{par}}(P,s+2l) \\
&\quad + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m} \frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big)\,E_{\eta_k}^{\mathrm{par}}(Q,s+2l)\,E_{\eta_k}^{\mathrm{par}}(P,n-1-s-2l) \\
&= I_{\eta_k}(s) + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m+1} \frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big)\,E_{\eta_k}^{\mathrm{par}}(Q,n-1-s-2l)\,E_{\eta_k}^{\mathrm{par}}(P,s+2l) \\
&\quad + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m+1} \frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big)\,E_{\eta_k}^{\mathrm{par}}(Q,s+2l)\,E_{\eta_k}^{\mathrm{par}}(P,n-1-s-2l).
\end{aligned} \tag{6.13}$$

The right-hand side of (6.13) is a meromorphic function for $\frac{n-1}{2} - 2(m+2) < \mathrm{Re}(s) < \frac{n-1}{2} - 2(m+1)$, hence it gives the meromorphic continuation $I_{\eta_k}^{(m+1,2)}(s)$ of the integral $I_{\eta_k}^{(m+1,1)}(s)$ to this strip.
On the line $\mathrm{Re}(s) = \frac{n-1}{2} - 2(m+1)$ we have to take the right hand side of (6.12) as the meromorphic continuation of the integral $I_{\eta_k}(s)$, where y has to be chosen such that $y > |\mathrm{Im}(s)|$. Together these formulas give us the meromorphic continuation $I_{\eta_k}^{(m+2)}(s)$ of the integral $I_{\eta_k}(s)$ to the strip $\frac{n-1}{2} - 2(m+2) < \mathrm{Re}(s) \leq \frac{n-1}{2} - 2(m+1)$.

We have proven that for $m \in \mathbb{N}_0$ the meromorphic continuation of the integral $I_{\eta_k}(s)$ to the strip $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) \leq \frac{n-1}{2} - 2m$ is given by the formulas (6.10) and (6.11). Overall, we obtain the meromorphic continuation of the integral $I_{\eta_k}(s)$ in s to the whole complex plane.

From the functional equation (3.15) of the parabolic Eisenstein series and Proposition 3.4.13 (a), i.e. the symmetry of the scattering matrix, for $w \in \mathbb{C}$ we obtain

$$\begin{aligned}
\sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q, n-1-w)\, E^{\mathrm{par}}_{\eta_k}(P, w) &= \sum_{k=1}^{c_\Gamma}\sum_{j=1}^{c_\Gamma} \varphi_{\eta_k,\eta_j}(n-1-w)\, E^{\mathrm{par}}_{\eta_j}(Q, w)\, E^{\mathrm{par}}_{\eta_k}(P, w) \\
&= \sum_{j=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_j}(Q, w) \sum_{k=1}^{c_\Gamma} \varphi_{\eta_k,\eta_j}(n-1-w)\, E^{\mathrm{par}}_{\eta_k}(P, w) \\
&= \sum_{j=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_j}(Q, w) \sum_{k=1}^{c_\Gamma} \varphi_{\eta_j,\eta_k}(n-1-w)\, E^{\mathrm{par}}_{\eta_k}(P, w) \\
&= \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q, w) \sum_{j=1}^{c_\Gamma} \varphi_{\eta_k,\eta_j}(n-1-w)\, E^{\mathrm{par}}_{\eta_j}(P, w) \\
&= \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q, w)\, E^{\mathrm{par}}_{\eta_k}(P, n-1-w).
\end{aligned}$$

Using this identity, the meromorphic continuation of the continuous part

$$\frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt$$

of the spectral expansion (6.1) to the strip $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) \leq \frac{n-1}{2} - 2m$ is given by

$$\sum_{k=1}^{c_\Gamma} I_{\eta_k}(s) + \frac{2^s\, \pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m} \frac{(-1)^l}{l!}\, \Gamma\Big(s - \frac{n-1}{2} + l\Big) \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2l)\, E^{\mathrm{par}}_{\eta_k}(P, s+2l) \quad (6.14)$$

for $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) < \frac{n-1}{2} - 2m$, while on the line $\mathrm{Re}(s) = \frac{n-1}{2} - 2m$ we have to take

$$\begin{aligned}
&\sum_{k=1}^{c_\Gamma} \widetilde{I}_{\eta_k,y,\varepsilon}(s) + \frac{2^s\, \pi^{\frac{n-1}{2}}}{\Gamma(s)} \sum_{l=0}^{m-1} \frac{(-1)^l}{l!}\, \Gamma\Big(s - \frac{n-1}{2} + l\Big) \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2l)\, E^{\mathrm{par}}_{\eta_k}(P, s+2l) \\
&\qquad + \frac{2^{s-1}\, \pi^{\frac{n-1}{2}}}{\Gamma(s)}\, \frac{(-1)^m}{m!}\, \Gamma\Big(s - \frac{n-1}{2} + m\Big) \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2m)\, E^{\mathrm{par}}_{\eta_k}(P, s+2m),
\end{aligned} \quad (6.15)$$

where $\varepsilon \in (0,1)$ is chosen sufficiently small such that all parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,s)$ $(k = 1, \ldots, c_\Gamma)$ have no poles in the strip $\frac{n-1}{2} - 2\varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} + 2\varepsilon$, and y is chosen sufficiently large such that $y > |\mathrm{Im}(s)|$. In this way we get the meromorphic continuation of the continuous part of the spectral expansion (6.1) in s to the whole complex plane.

Now we turn to the computation of the poles arising from the continuous part after multiplication by $\Gamma(s)\,\Gamma\big(s - \frac{n-1}{2}\big)^{-1}$. Using formulas (6.14) and (6.15), valid in the strip $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) \leq \frac{n-1}{2} - 2m$, we have to determine the poles of

$$\begin{aligned}
&\frac{2^{s-1}\, \pi^{\frac{n-3}{2}}}{4\, i\, \Gamma\big(s - \frac{n-1}{2}\big)} \sum_{k=1}^{c_\Gamma} \int_{\mathrm{Re}(w)=\frac{n-1}{2}} \Gamma\Big(\frac{s-n+1+w}{2}\Big)\, \Gamma\Big(\frac{s-w}{2}\Big)\, E^{\mathrm{par}}_{\eta_k}(Q, n-1-w)\, E^{\mathrm{par}}_{\eta_k}(P, w)\, dw \\
&\qquad + 2^s\, \pi^{\frac{n-1}{2}} \sum_{l=0}^{m} \frac{(-1)^l}{l!} \Big(s - \frac{n-1}{2}\Big)_l \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2l)\, E^{\mathrm{par}}_{\eta_k}(P, s+2l)
\end{aligned}$$

or

$$\frac{2^{s-1}\,\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma\left(s-\frac{n-1}{2}\right)}\sum_{k=1}^{c_\Gamma}\int_{W_{y,\varepsilon}}\Gamma\Big(\frac{s-n+1+w}{2}\Big)\,\Gamma\Big(\frac{s-w}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}(Q,n-1-w)\,E^{\mathrm{par}}_{\eta_k}(P,w)\,dw$$
$$+\,2^{s}\,\pi^{\frac{n-1}{2}}\sum_{l=0}^{m-1}\frac{(-1)^l}{l!}\left(s-\frac{n-1}{2}\right)_l\sum_{k=1}^{c_\Gamma}E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2l)\,E^{\mathrm{par}}_{\eta_k}(P,s+2l)$$
$$+\,2^{s-1}\,\pi^{\frac{n-1}{2}}\,\frac{(-1)^m}{m!}\left(s-\frac{n-1}{2}\right)_m\sum_{k=1}^{c_\Gamma}E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2m)\,E^{\mathrm{par}}_{\eta_k}(P,s+2m),$$

respectively. The poles of both can only arise from the functions $E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2l)$ and $E^{\mathrm{par}}_{\eta_k}(P,s+2l)$, where $l=0,\dots,m$ and $k=1,\dots,c_\Gamma$.

The poles of $E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2l)$ are located at $s=n-1-\rho-2l$, where $w=\rho$ is a pole of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(Q,w)$. The condition $\frac{n-1}{2}-2(m+1)<\mathrm{Re}(s)\le\frac{n-1}{2}-2m$ implies that $\frac{n-1}{2}+2(m-l)\le\mathrm{Re}(\rho)<\frac{n-1}{2}+2(m+1-l)$, and particularly $\mathrm{Re}(\rho)\ge\frac{n-1}{2}$. By Theorem 3.4.9 there are only finitely many poles ρ with $\mathrm{Re}(\rho)\ge\frac{n-1}{2}$ which are located in the real interval $(\frac{n-1}{2},n-1]$ and are simple, and no poles with $\mathrm{Re}(\rho)>n-1$. Therefore, we can only get such a pole $s=n-1-\rho-2l$ if the inequality $\frac{n-1}{2}+2(m-l)\le n-1$ holds, which is equivalent to $l\ge m-\frac{n-1}{4}$. This implies $l\ge\left\lceil m-\frac{n-1}{4}\right\rceil=m-\left\lfloor\frac{n-1}{4}\right\rfloor$ and a pole can only occur for $l=\max\left(m-\left\lfloor\frac{n-1}{4}\right\rfloor,0\right),\dots,m$.
Thus, we obtain the poles $s=n-1-\rho-2N$, where $N\in\left\{\max\left(m-\left\lfloor\frac{n-1}{4}\right\rfloor,0\right),\dots,m\right\}$ and $w=\rho$ is a pole of $E^{\mathrm{par}}_{\eta_k}(Q,w)$ for some $k\in\{1,\dots,c_\Gamma\}$ with $\rho\in\left[\frac{n-1}{2}+2(m-N),\,\frac{n-1}{2}+2(m+1-N)\right)$, and we have seen that in particular $\rho\in\left(\frac{n-1}{2},n-1\right]$.

The poles arising from $E^{\mathrm{par}}_{\eta_k}(P,s+2l)$ are located at $s=\rho-2l$, where $w=\rho$ is a pole of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,w)$. From the condition $\frac{n-1}{2}-2(m+1)<\mathrm{Re}(s)\le\frac{n-1}{2}-2m$ we get the inequality $\frac{n-1}{2}-2(m+1-l)<\mathrm{Re}(\rho)\le\frac{n-1}{2}-2(m-l)$, and particularly $\mathrm{Re}(\rho)\le\frac{n-1}{2}$. By Corollary 3.4.14 there is no pole ρ with $\mathrm{Re}(\rho)=\frac{n-1}{2}$.
In this way we obtain the poles $s=\rho-2N$, where $N\in\{0,\dots,m\}$ and $w=\rho$ is a pole of $E^{\mathrm{par}}_{\eta_k}(P,w)$ for some $k\in\{1,\dots,c_\Gamma\}$ with $\mathrm{Re}(\rho)\in\left(\frac{n-1}{2}-2(m+1-N),\,\frac{n-1}{2}-2(m-N)\right]$, and we have seen that in particular $\mathrm{Re}(\rho)<\frac{n-1}{2}$.

Finally, we note that for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>\frac{n-1}{2}$ there are no poles arising from the continuous part of (6.1) after multiplication by $\Gamma(s)\,\Gamma\left(s-\frac{n-1}{2}\right)^{-1}$ since

$$\frac{\Gamma(s)}{\Gamma\left(s-\frac{n-1}{2}\right)}\sum_{k=1}^{c_\Gamma}I_{\eta_k}(s)$$

is a holomorphic function in the half-plane $\mathrm{Re}(s)>\frac{n-1}{2}$. This completes the proof of the theorem. □

Remark 6.1.2.

(a) The possible poles of the types (ii) and (iii) in Theorem 6.1.1 do not occur in the case $c_\Gamma=0$, i.e. if the discrete and cofinite subgroup $\Gamma\subseteq\mathrm{PSL}_2(C_{n-1})$ contains no parabolic element.

(b) The orders of the possible poles of the types (ii) and (iii) in Theorem 6.1.1 and the corresponding residues depend on the location and orders of the poles of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,w)$ $(k=1,\dots,c_\Gamma)$ and their corresponding residues.

As a corollary of Theorem 6.1.1 we obtain that the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ has a simple pole at $s=n-1$, and we can determine its residue.

Corollary 6.1.3. *For $P, Q \in \mathbb{H}^n$ the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ admits a simple pole at $s = n-1$ with residue*

$$\operatorname{Res}_{s=n-1} K^{\mathrm{hyp}}(P,Q,s) = \frac{2\,\pi^{n/2}}{\operatorname{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma\left(\frac{n}{2}\right)}.$$

Proof. By Theorem 6.1.1, the function $\Gamma(s)\,\Gamma\left(s-\frac{n-1}{2}\right)^{-1} K^{\mathrm{hyp}}(P,Q,s)$ has a possible pole in the half-plane $\left\{s \in \mathbb{C} \mid \operatorname{Re}(s) > \frac{n-1}{2}\right\}$ at any point $s = \frac{n-1}{2} + ir_j - 2N$, where $j \in \mathbb{N}_0$ and $\lambda_j = s_j(n-1-s_j) = \left(\frac{n-1}{2}\right)^2 + r_j^2$ is the eigenvalue of the eigenfunction $\psi_j(P)$ with $r_j \in \left[-i\,\frac{n-1}{2}, 0\right)$, and where $N \in \mathbb{N}_0$ with $N < \frac{ir_j}{2}$.
For $N = 0$ and $j = 0$, i.e. $\lambda_j = 0$, $r_j = -i\,\frac{n-1}{2}$ and $\psi_j(P) = \operatorname{vol}(\Gamma\backslash\mathbb{H}^n)^{-1/2}$, we obtain the point $s = n-1$. Since $N - ir_j = 2N - ir_j = -\frac{n-1}{2} \notin \mathbb{N}_0$, it is a simple pole with residue

$$\operatorname{Res}_{s=n-1}\left[\frac{\Gamma(s)\,K^{\mathrm{hyp}}(P,Q,s)}{\Gamma\left(s-\frac{n-1}{2}\right)}\right] = \frac{2^{n-1}\,\pi^{\frac{n-1}{2}}\,\Gamma(\frac{n-1}{2})}{\Gamma\left(\frac{n-1}{2}\right)}\,\frac{1}{\operatorname{vol}(\Gamma\backslash\mathbb{H}^n)} = \frac{2^{n-1}\,\pi^{\frac{n-1}{2}}}{\operatorname{vol}(\Gamma\backslash\mathbb{H}^n)}.$$

As $\Gamma(s)$ and $\Gamma\left(s-\frac{n-1}{2}\right)$ have the non-zero values $\Gamma(n-1)$ and $\Gamma\left(\frac{n-1}{2}\right)$ at $s = n-1$, we can infer that also the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ admits a simple pole at $s = n-1$ with residue

$$\operatorname{Res}_{s=n-1} K^{\mathrm{hyp}}(P,Q,s) = \frac{2^{n-1}\,\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{n-1}{2}\right)}{\operatorname{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma(n-1)} = \frac{2\,\pi^{n/2}}{\operatorname{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma\left(\frac{n}{2}\right)},$$

where for the last equality we made use of the duplication formula (A.4) with $s = \frac{n-1}{2}$. □

6.2. Meromorphic continuation of hyperbolic Eisenstein series

Analogous to the previous section, we now establish the meromorphic continuation of the hyperbolic Eisenstein series in s to the whole complex plane via its spectral expansion computed in section 5.2, and determine its possible poles.
Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. Further, let $(Q_1, Q_2) \in H_\Gamma$ be a pair of hyperbolic fixed points with hyperbolic scaling matrix $\sigma_{(Q_1,Q_2)} \in \mathrm{PSL}_2(C_{n-1})$ and hyperbolic stabilizer subgroup $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$. Let $\mathcal{L}_{(Q_1,Q_2)}$ be the unique geodesic in $\mathbb{H}^n$ connecting Q_1 and Q_2, and let $L_{(Q_1,Q_2)} = \pi_{\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}}(\mathcal{L}_{(Q_1,Q_2)})$ denote its image under the natural projection $\pi_{\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}} : \mathbb{H}^n \to \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\mathbb{H}^n$, which is a closed geodesic in $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\mathbb{H}^n$ of hyperbolic length $l_{(Q_1,Q_2)}$.

Theorem 6.2.1. *For $P \in \mathbb{H}^n$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits a meromorphic continuation in s to the whole complex plane. For any $m \in \mathbb{N}_0$ the possible poles of the function*

$$\frac{\Gamma\left(\frac{s}{2}\right)^2 E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)}{\Gamma\left(s-\frac{n-1}{2}\right)}$$

in the strip $\left\{s \in \mathbb{C} \,\middle|\, \frac{n-1}{2} - 2(m+1) < \operatorname{Re}(s) \leq \frac{n-1}{2} - 2m\right\}$ are located at the following points:

(i) $s = \frac{n-1}{2} \pm ir_j - 2N$, where $j \in \mathbb{N}_0$ and $\lambda_j = s_j(n-1-s_j) = \left(\frac{n-1}{2}\right)^2 + r_j^2$ is the eigenvalue of the eigenfunction $\psi_j(P)$, and where $N := \left\lceil m \pm \frac{\operatorname{Re}(ir_j)}{2}\right\rceil$ has to satisfy $N \geq 0$.

If $N \mp ir_j \notin \mathbb{N}_0$ and $2N \mp ir_j \notin \mathbb{N}_0$, it is a simple pole with residue

$$\operatorname{Res}_{s=\frac{n-1}{2}\pm ir_j-2N}\left[\frac{\Gamma\left(\frac{s}{2}\right)^2 E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)}{\Gamma\left(s-\frac{n-1}{2}\right)}\right] = \frac{(-1)^N\,2\,\pi^{\frac{n-1}{2}}\,\Gamma(\pm ir_j-N)}{N!\,\Gamma\left(\pm ir_j-2N\right)}\sum_{\substack{l\in\mathbb{N}_0:\\ r_l=r_j}}\psi_l(P)\int_{L_{(Q_1,Q_2)}}\overline{\psi_l}(Q)\;ds_{\mathbb{H}^n}(Q);$$

and if $N \mp ir_j \in \mathbb{N}_0$, it is a simple pole with residue

$$\operatorname{Res}_{s=\frac{n-1}{2}\pm ir_j-2N}\left[\frac{\Gamma\left(\frac{s}{2}\right)^2 E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)}{\Gamma\left(s-\frac{n-1}{2}\right)}\right] = \frac{4\,\pi^{\frac{n-1}{2}}\,(2N\mp ir_j)!}{N!\,(N\mp ir_j)!}\sum_{\substack{l\in\mathbb{N}_0:\\ r_l=r_j}}\psi_l(P)\int_{L_{(Q_1,Q_2)}}\overline{\psi_l}(Q)\;ds_{\mathbb{H}^n}(Q).$$

In case that $N \mp ir_j \notin \mathbb{N}_0$ and $2N \mp ir_j \in \mathbb{N}_0$, it is no pole but a removable singularity.

(ii) $s = n-1-\rho-2N$, where $N \in \left\{\max\left(m-\left\lfloor\frac{n-1}{4}\right\rfloor,0\right),\dots,m\right\}$, and where $w=\rho$ is a pole of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,w)$ for some $k \in \{1,\dots,c_\Gamma\}$ with

$$\rho \in \left[\frac{n-1}{2}+2(m-N),\;\frac{n-1}{2}+2(m+1-N)\right);$$

in particular, $\rho \in \left(\frac{n-1}{2},n-1\right]$.

(iii) $s=\rho-2N$, where $N \in \{0,\dots,m\}$, and where $w=\rho$ is a pole of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,w)$ for some $k\in\{1,\dots,c_\Gamma\}$ with

$$\mathrm{Re}(\rho) \in \left(\frac{n-1}{2}-2(m+1-N),\;\frac{n-1}{2}-2(m-N)\right];$$

in particular, $\mathrm{Re}(\rho) < \frac{n-1}{2}$.

The poles given in the cases (i), (ii), (iii) might coincide in parts.

Moreover, the possible poles of the function

$$\frac{\Gamma\left(\frac{s}{2}\right)^2 E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)}{\Gamma\left(s-\frac{n-1}{2}\right)}$$

in the half-plane $\left\{s\in\mathbb{C}\mid \mathrm{Re}(s)>\frac{n-1}{2}\right\}$ are located at the points $s=\frac{n-1}{2}+ir_j-2N$, where $j\in\mathbb{N}_0$ and $\lambda_j=s_j(n-1-s_j)=\left(\frac{n-1}{2}\right)^2+r_j^2$ is the eigenvalue of the eigenfunction $\psi_j(P)$ with $r_j\in\left[-i\,\frac{n-1}{2},0\right)$, and where $N\in\mathbb{N}_0$ with $N<\frac{ir_j}{2}$. The orders and residues are as in the case (i) above.

Proof. In order to derive the meromorphic continuation of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$, we use its spectral expansion. Theorem 5.2.2 shows that for $P\in\mathbb{H}^n$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits the spectral expansion

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)=\sum_{j=0}^{\infty}b_j(s)\,\psi_j(P)+\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty}b_{t,\eta_k}(s)\,E^{\mathrm{par}}_{\eta_k}\left(P,\frac{n-1}{2}+it\right)dt, \tag{6.16}$$

where the coefficients $b_j(s)$ and $b_{t,\eta_k}(s)$ are given by

$$b_j(s) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big)\int_{L_{(Q_1,Q_2)}}\overline{\psi_j}(Q)\;ds_{\mathbb{H}^n}(Q), \tag{6.17}$$

$$b_{t,\eta_k}(s) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+it}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-it}{2}\Big)\int_{L_{(Q_1,Q_2)}}E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-it\Big)\;ds_{\mathbb{H}^n}(Q), \tag{6.18}$$

respectively.

We start by giving the meromorphic continuation of the series

$$\sum_{j=0}^{\infty} b_j(s)\,\psi_j(P) \tag{6.19}$$

that arises from the discrete spectrum. The explicit formula (6.17) already gives us the meromorphic continuation of the coefficient $b_j(s)$ in s to the whole complex plane. It remains to show that the series (6.19) converges absolutely and locally uniformly for all $s \in \mathbb{C}$.
As formula (6.5) in the proof of Theorem 6.1.1 we have established the bound

$$\left|\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big)\right| = \mathrm{O}\left(r_j^{\mathrm{Re}(s)-\frac{n+1}{2}}\,\exp\left(-\frac{\pi\,r_j}{2}\right)\right) \qquad (r_j \to \infty)$$

with an implied constant depending on s. Moreover, the sup-norm bound

$$\sup_{P\in\mathbb{H}^n}|\psi_j(P)| = \mathrm{O}\left(r_j^{n/2}\right) \qquad (r_j \to \infty)$$

from Remark 3.5.7 also yields

$$\left|\int_{L_{(Q_1,Q_2)}}\overline{\psi_j}(Q)\;ds_{\mathbb{H}^n}(Q)\right| \leq \sup_{P\in\mathbb{H}^n}|\psi_j(P)|\cdot l_{(Q_1,Q_2)} = \mathrm{O}\left(r_j^{n/2}\right) \qquad (r_j \to \infty).$$

Together, for all but the finitely many $j \in \mathbb{N}_0$ with $\lambda_j \in \left[0, \left(\frac{n-1}{2}\right)^2\right)$ (see Remark 3.5.4), i.e. with $r_j \in \left[-i\,\frac{n-1}{2}, 0\right)$, this gives us

$$|b_j(s)\,\psi_j(P)| = \mathrm{O}\left(r_j^{\mathrm{Re}(s)-\frac{n+1}{2}}\,\exp\left(-\frac{\pi\,r_j}{2}\right)\,r_j^{n}\right) = \mathrm{O}\left(r_j^{\mathrm{Re}(s)+\frac{n-1}{2}}\,\exp\left(-\frac{\pi\,r_j}{2}\right)\right) \quad (r_j \to \infty)$$

with an implied constant depending on s. Thus, the series (6.19) converges absolutely and locally uniformly for all $s \in \mathbb{C}$ and defines a holomorphic function away from the poles of $b_j(s)$.

We determine the poles of the series (6.19) after multiplication by $\Gamma\left(\frac{s}{2}\right)^2\,\Gamma\left(s-\frac{n-1}{2}\right)^{-1}$, i.e. the poles of

$$\frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(s-\frac{n-1}{2}\right)}\sum_{j=0}^{\infty}\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big)\,\psi_j(P)\int_{L_{(Q_1,Q_2)}}\overline{\psi_j}(Q)\;ds_{\mathbb{H}^n}(Q). \tag{6.20}$$

These can only arise from the two gamma factors in the sum, so the series (6.20) has the possible poles $s = \frac{n-1}{2} \pm ir_j - 2N$, where $j, N \in \mathbb{N}_0$.
The condition $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}\left(\frac{n-1}{2} \pm ir_j - 2N\right) \leq \frac{n-1}{2} - 2m$ for some $m \in \mathbb{N}_0$ is equivalent to $m \pm \frac{\mathrm{Re}(ir_j)}{2} \leq N < m+1 \pm \frac{\mathrm{Re}(ir_j)}{2}$. This interval contains exactly one integer, which implies that $N = \left\lceil m \pm \frac{\mathrm{Re}(ir_j)}{2}\right\rceil$, while also $N \geq 0$ must hold.

Moreover, $\operatorname{Re}\big(\frac{n-1}{2} \pm ir_j - 2N\big) > \frac{n-1}{2}$ holds true if and only if $N < \pm\frac{\operatorname{Re}(ir_j)}{2}$. As we either have $r_j \geq 0$ or $r_j \in \big[-i\,\frac{n-1}{2}, 0\big)$, the inequality $N < -\frac{\operatorname{Re}(ir_j)}{2}$ cannot be fulfilled, so that there is no pole of the form $s = \frac{n-1}{2} - ir_j - 2N$ with $\operatorname{Re}(s) > \frac{n-1}{2}$. In turn, the condition $N < \frac{\operatorname{Re}(ir_j)}{2}$, respectively a pole of the form $s = \frac{n-1}{2} + ir_j - 2N$ with $\operatorname{Re}(s) > \frac{n-1}{2}$, is only possible if $r_j \in \big[-i\,\frac{n-1}{2}, 0\big)$.

The residues at the above poles can now be computed similarly to the proof of Theorem 6.1.1. If $N \mp ir_j \notin \mathbb{N}_0$ and $2N \mp ir_j \notin \mathbb{N}_0$, then $s = \frac{n-1}{2} \pm ir_j - 2N$ is a simple pole of (6.20) and the corresponding residue is given by

$$\frac{(-1)^N\, 2\,\pi^{\frac{n-1}{2}}\, \Gamma(\pm ir_j - N)}{N!\,\Gamma\left(\pm ir_j - 2N\right)} \sum_{\substack{l\in\mathbb{N}_0:\\ r_l = r_j}} \psi_l(P) \int_{L_{(Q_1,Q_2)}} \overline{\psi_l}(Q)\; ds_{\mathbb{H}^n}(Q).$$

In case that $N \mp ir_j \in \mathbb{N}_0$, also $2N \mp ir_j \in \mathbb{N}_0$ holds true, and the point $s = \frac{n-1}{2} \pm ir_j - 2N$ is a simple pole of (6.20) with residue

$$\frac{4\,\pi^{\frac{n-1}{2}}\,(2N \mp ir_j)!}{N!\,(N \mp ir_j)!} \sum_{\substack{l\in\mathbb{N}_0:\\ r_l = r_j}} \psi_l(P) \int_{L_{(Q_1,Q_2)}} \overline{\psi_l}(Q)\; ds_{\mathbb{H}^n}(Q).$$

If $N \mp ir_j \notin \mathbb{N}_0$ and $2N \mp ir_j \in \mathbb{N}_0$, the point $s = \frac{n-1}{2} \pm ir_j - 2N$ is a removable singularity of (6.20). In each of these three cases the details are analogous to the proof of Theorem 6.1.1.

For a cusp η_k ($k = 1, \ldots, c_\Gamma$) of Γ we now turn to give the meromorphic continuation of the integral

$$\begin{aligned}
&\frac{1}{4\pi}\int_{-\infty}^{\infty} b_{t,\eta_k}(s)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt \\
&\quad= \frac{\pi^{\frac{n-3}{2}}}{4\,\Gamma\left(\frac{s}{2}\right)^2} \int_{-\infty}^{\infty} \Gamma\Big(\frac{s - \frac{n-1}{2} + it}{2}\Big)\, \Gamma\Big(\frac{s - \frac{n-1}{2} - it}{2}\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big) \\
&\qquad\qquad \cdot \bigg(\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} - it\Big)\, ds_{\mathbb{H}^n}(Q)\bigg)\, dt
\end{aligned}$$

arising from the continuous part of the spectral expansion (6.16). Substituting $w := \frac{n-1}{2} + it$, we rewrite the integral as

$$\begin{aligned}
J_{\eta_k}(s) := \frac{\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma\left(\frac{s}{2}\right)^2} &\int_{\operatorname{Re}(w)=\frac{n-1}{2}} \Gamma\Big(\frac{s-n+1+w}{2}\Big)\, \Gamma\Big(\frac{s-w}{2}\Big)\, E^{\mathrm{par}}_{\eta_k}(P, w) \\
&\qquad \cdot \bigg(\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q, n-1-w)\; ds_{\mathbb{H}^n}(Q)\bigg)\, dw,
\end{aligned}$$

where the line of integration goes from $x - i\infty$ to $x + i\infty$. The function $J_{\eta_k}(s)$ is holomorphic for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > n-1$. Indeed, it is even holomorphic for $s \in \mathbb{C}$ with $\operatorname{Re}(s) \neq \frac{n-1}{2} - 2m$ for any $m \in \mathbb{N}_0$. In particular, $J^{(0)}_{\eta_k}(s) = J_{\eta_k}(s)$ is a meromorphic function on the half-plane $\operatorname{Re}(s) > \frac{n-1}{2}$.

Using Corollary 3.4.14, we choose $\varepsilon \in (0,1)$ sufficiently small such that the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,s)$ has no poles in the strip $\frac{n-1}{2} - 2\varepsilon < \operatorname{Re}(s) < \frac{n-1}{2} + 2\varepsilon$. Let $y \in \mathbb{R}$ with $y > |\operatorname{Im}(s)|$ and the piecewise linear path $W_{y,\varepsilon}$ as in the proof of Theorem 6.1.1: the vertical line from $\frac{n-1}{2} - i\infty$ to $\frac{n-1}{2} - iy$, the horizontal line segment from $\frac{n-1}{2} - iy$ to $\frac{n-1}{2} + \varepsilon - iy$, the vertical line segment from $\frac{n-1}{2} + \varepsilon - iy$ to $\frac{n-1}{2} + \varepsilon + iy$, the horizontal line segment from $\frac{n-1}{2} + \varepsilon + iy$ to $\frac{n-1}{2} + iy$, and the vertical line from $\frac{n-1}{2} + iy$ to $\frac{n-1}{2} + i\infty$. Further, we set

$$\widetilde{J}_{\eta_k,y,\varepsilon}(s) := \frac{\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma\left(\frac{s}{2}\right)^2} \int_{W_{y,\varepsilon}} \Gamma\Big(\frac{s-n+1+w}{2}\Big)\,\Gamma\Big(\frac{s-w}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}(P,w) \cdot \left(\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q,n-1-w)\;ds_{\mathbb{H}^n}(Q)\right) dw.$$

Then for $s \in \mathbb{C}$ with $\frac{n-1}{2} < \mathrm{Re}(s) < \frac{n-1}{2} + \varepsilon$, analogous to the proof of Theorem 6.1.1, we obtain from the residue theorem

$$J_{\eta_k}(s) = \widetilde{J}_{\eta_k,y,\varepsilon}(s) - 2\pi i \cdot \frac{\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma\left(\frac{s}{2}\right)^2} \cdot \sum_a \mathrm{Res}_{w=a}\left[\Gamma\Big(\frac{s-n+1+w}{2}\Big)\,\Gamma\Big(\frac{s-w}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}(P,w)\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q,n-1-w)\;ds_{\mathbb{H}^n}(Q)\right],$$

where the sum runs over all poles $a \in \mathbb{C}$ with $\frac{n-1}{2} < \mathrm{Re}(a) < \frac{n-1}{2} + \varepsilon$ and $-y < \mathrm{Im}(a) < y$.
By the same arguments as in the proof of Theorem 6.1.1 the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,w)$, $E^{\mathrm{par}}_{\eta_k}(Q,n-1-w)$ and the function $\Gamma\left(\frac{s-n+1+w}{2}\right)$ have no poles in this box, while the function $\Gamma\left(\frac{s-w}{2}\right)$ has only one pole there at $w = s$ and the corresponding residue is equal to -2. This yields

$$J_{\eta_k}(s) = \widetilde{J}_{\eta_k,y,\varepsilon}(s) + \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(s - \frac{n-1}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}(P,s)\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q,n-1-s)\;ds_{\mathbb{H}^n}(Q). \tag{6.21}$$

The right-hand side of (6.21) is a meromorphic function for $\frac{n-1}{2} - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} + \varepsilon$ and $-y < \mathrm{Im}(s) < y$. As y can be chosen arbitrarily large, this gives the meromorphic continuation $J^{(0,1)}_{\eta_k}(s)$ of the integral $J_{\eta_k}(s)$ to the whole strip $\frac{n-1}{2} - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} + \varepsilon$.

Now assuming $\frac{n-1}{2} - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2}$ and choosing $y > |\mathrm{Im}(s)|$ and $W_{y,\varepsilon}$ as before, the residue theorem gives us

$$\begin{aligned} J^{(0,1)}_{\eta_k}(s) = J_{\eta_k}(s) &+ 2\pi i \cdot \frac{\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma\left(\frac{s}{2}\right)^2} \cdot \sum_a \mathrm{Res}_{w=a}\left[\Gamma\Big(\frac{s-n+1+w}{2}\Big)\,\Gamma\Big(\frac{s-w}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}(P,w)\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q,n-1-w)\;ds_{\mathbb{H}^n}(Q)\right] \\ &+ \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(s - \frac{n-1}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}(P,s)\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q,n-1-s)\;ds_{\mathbb{H}^n}(Q), \end{aligned}$$

where we sum over all poles $a \in \mathbb{C}$ with $\frac{n-1}{2} < \mathrm{Re}(a) < \frac{n-1}{2} + \varepsilon$ and $-y < \mathrm{Im}(a) < y$ again.
As in the proof of Theorem 6.1.1, the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,w)$, $E^{\mathrm{par}}_{\eta_k}(Q,n-1-w)$ and the function $\Gamma\left(\frac{s-w}{2}\right)$ have no poles in the considered box, while the function $\Gamma\left(\frac{s-n+1+w}{2}\right)$ has only one pole there at $w = n-1-s$ and the residue is equal to 2. Thus, we obtain

$$\begin{aligned} J^{(0,1)}_{\eta_k}(s) = J_{\eta_k}(s) &+ \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(s - \frac{n-1}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}(P,n-1-s)\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q,s)\;ds_{\mathbb{H}^n}(Q) \\ &+ \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(s - \frac{n-1}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}(P,s)\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q,n-1-s)\;ds_{\mathbb{H}^n}(Q). \end{aligned} \tag{6.22}$$

Now the right-hand side of (6.22) is a meromorphic function for $\frac{n-1}{2}-2<\mathrm{Re}(s)<\frac{n-1}{2}$, giving the meromorphic continuation $J^{(0,2)}_{\eta_k}(s)$ of the integral $J^{(0,1)}_{\eta_k}(s)$ to this strip.
Moreover, on the line $\mathrm{Re}(s)=\frac{n-1}{2}$ the meromorphic continuation of the integral $J_{\eta_k}(s)$ is given by the right hand side of (6.21), where y has to be chosen such that $y>|\mathrm{Im}(s)|$. Together this provides the meromorphic continuation $J^{(1)}_{\eta_k}(s)$ of the integral $J_{\eta_k}(s)$ to the strip $\frac{n-1}{2}-2<\mathrm{Re}(s)\leq\frac{n-1}{2}$.

Continuing this two-step process, we now prove inductively that for $m\in\mathbb{N}_0$ the meromorphic continuation $J^{(m+1)}_{\eta_k}(s)$ of the integral $J_{\eta_k}(s)$ to the strip $\frac{n-1}{2}-2(m+1)<\mathrm{Re}(s)\leq\frac{n-1}{2}-2m$ is given by

$$\begin{aligned}J_{\eta_k}(s)&+\frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\sum_{l=0}^{m}\frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big)\,E^{\mathrm{par}}_{\eta_k}(P,s+2l)\\&\qquad\cdot\int_{L_{(Q_1,Q_2)}}E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2l)\;ds_{\mathbb{H}^n}(Q)\\&+\frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\sum_{l=0}^{m}\frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big)\,E^{\mathrm{par}}_{\eta_k}(P,n-1-s-2l)\int_{L_{(Q_1,Q_2)}}E^{\mathrm{par}}_{\eta_k}(Q,s+2l)\;ds_{\mathbb{H}^n}(Q)\end{aligned}\tag{6.23}$$

for $\frac{n-1}{2}-2(m+1)<\mathrm{Re}(s)<\frac{n-1}{2}-2m$, while on the line $\mathrm{Re}(s)=\frac{n-1}{2}-2m$ we have to take

$$\begin{aligned}\widetilde{J}_{\eta_k,y,\varepsilon}(s)&+\frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\sum_{l=0}^{m}\frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big)\,E^{\mathrm{par}}_{\eta_k}(P,s+2l)\\&\qquad\cdot\int_{L_{(Q_1,Q_2)}}E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2l)\;ds_{\mathbb{H}^n}(Q)\\&+\frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\sum_{l=0}^{m-1}\frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big)\,E^{\mathrm{par}}_{\eta_k}(P,n-1-s-2l)\int_{L_{(Q_1,Q_2)}}E^{\mathrm{par}}_{\eta_k}(Q,s+2l)\;ds_{\mathbb{H}^n}(Q),\end{aligned}\tag{6.24}$$

where y is chosen sufficiently large such that $y>|\mathrm{Im}(s)|$.

Suppose, for some $m\in\mathbb{N}_0$ we have established the meromorphic continuation $J^{(m+1)}_{\eta_k}(s)$ of the integral $J_{\eta_k}(s)$ to the strip $\frac{n-1}{2}-2(m+1)<\mathrm{Re}(s)\leq\frac{n-1}{2}-2m$ via (6.23) and (6.24). Then for $s\in\mathbb{C}$ with $\frac{n-1}{2}-2(m+1)<\mathrm{Re}(s)<\frac{n-1}{2}-2(m+1)+\varepsilon$ and $y>|\mathrm{Im}(s)|$, analogous to the proof of Theorem 6.1.1, the residue theorem gives us

$$\begin{aligned}J_{\eta_k}(s)&=\widetilde{J}_{\eta_k,y,\varepsilon}(s)-2\pi i\cdot\frac{\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma\left(\frac{s}{2}\right)^2}\\&\cdot\sum_a\mathrm{Res}_{w=a}\left[\Gamma\Big(\frac{s-n+1+w}{2}\Big)\,\Gamma\Big(\frac{s-w}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}(P,w)\int_{L_{(Q_1,Q_2)}}E^{\mathrm{par}}_{\eta_k}(Q,n-1-w)\;ds_{\mathbb{H}^n}(Q)\right],\end{aligned}$$

where we sum over all poles $a\in\mathbb{C}$ with $\frac{n-1}{2}<\mathrm{Re}(a)<\frac{n-1}{2}+\varepsilon$ and $-y<\mathrm{Im}(a)<y$.
The same arguments as in the proof of Theorem 6.1.1 show that the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,w)$, $E^{\mathrm{par}}_{\eta_k}(Q,n-1-w)$ and the function $\Gamma\left(\frac{s-n+1+w}{2}\right)$ have no poles in this box, whereas the function $\Gamma\left(\frac{s-w}{2}\right)$ has only one pole there at $w=s+2(m+1)$ and the corresponding residue is equal to $-2\,(-1)^{m+1}/(m+1)!$. This implies

$$\begin{aligned}J_{\eta_k}(s)=\widetilde{J}_{\eta_k,y,\varepsilon}(s)&+\frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\frac{(-1)^{m+1}}{(m+1)!}\,\Gamma\Big(s-\frac{n-1}{2}+m+1\Big)\,E^{\mathrm{par}}_{\eta_k}(P,s+2(m+1))\\&\cdot\int_{L_{(Q_1,Q_2)}}E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2(m+1))\;ds_{\mathbb{H}^n}(Q)\end{aligned}$$

and

$$
\begin{aligned}
J_{\eta_k}(s) &+ \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{m} \frac{(-1)^l}{l!} \Gamma\Big(s - \frac{n-1}{2} + l\Big) E_{\eta_k}^{\mathrm{par}}(P, s+2l) \\
&\qquad \cdot \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, n-1-s-2l)\; ds_{\mathbb{H}^n}(Q) \\
&+ \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{m} \frac{(-1)^l}{l!} \Gamma\Big(s - \frac{n-1}{2} + l\Big) E_{\eta_k}^{\mathrm{par}}(P, n-1-s-2l) \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, s+2l)\; ds_{\mathbb{H}^n}(Q) \\
= \widetilde{J}_{\eta_k,y,\varepsilon}(s) &+ \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{m+1} \frac{(-1)^l}{l!} \Gamma\Big(s - \frac{n-1}{2} + l\Big) E_{\eta_k}^{\mathrm{par}}(P, s+2l) \\
&\qquad \cdot \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, n-1-s-2l)\; ds_{\mathbb{H}^n}(Q) \\
&+ \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{m} \frac{(-1)^l}{l!} \Gamma\Big(s - \frac{n-1}{2} + l\Big) E_{\eta_k}^{\mathrm{par}}(P, n-1-s-2l) \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, s+2l)\; ds_{\mathbb{H}^n}(Q).
\end{aligned}
\tag{6.25}
$$

The right-hand side of (6.25) is a meromorphic function for $\frac{n-1}{2} - 2(m+1) - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} - 2(m+1) + \varepsilon$ and $-y < \mathrm{Im}(s) < y$. Since we can choose y arbitrarily large, this gives the meromorphic continuation $J_{\eta_k}^{(m+1,1)}(s)$ of the integral $J_{\eta_k}(s)$ to the whole strip $\frac{n-1}{2} - 2(m+1) - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} - 2(m+1) + \varepsilon$.

If we assume $\frac{n-1}{2} - 2(m+1) - \varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} - 2(m+1)$ and choose $y > |\mathrm{Im}(s)|$, as before we obtain from the residue theorem

$$
\begin{aligned}
J_{\eta_k}^{(m+1,1)}(s) &= J_{\eta_k}(s) + 2\pi i \cdot \frac{\pi^{\frac{n-3}{2}}}{4\, i\, \Gamma\left(\frac{s}{2}\right)^2} \\
&\quad \cdot \sum_a \mathrm{Res}_{w=a} \left[\Big(\frac{s-n+1+w}{2}\Big) \Gamma\Big(\frac{s-w}{2}\Big) E_{\eta_k}^{\mathrm{par}}(P, w) \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, n-1-w)\; ds_{\mathbb{H}^n}(Q) \right] \\
&+ \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{m+1} \frac{(-1)^l}{l!} \Gamma\Big(s - \frac{n-1}{2} + l\Big) E_{\eta_k}^{\mathrm{par}}(P, s+2l) \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, n-1-s-2l)\; ds_{\mathbb{H}^n}(Q) \\
&+ \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{m} \frac{(-1)^l}{l!} \Gamma\Big(s - \frac{n-1}{2} + l\Big) E_{\eta_k}^{\mathrm{par}}(P, n-1-s-2l) \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, s+2l)\; ds_{\mathbb{H}^n}(Q),
\end{aligned}
$$

where the sum runs over all poles $a \in \mathbb{C}$ with $\frac{n-1}{2} < \mathrm{Re}(a) < \frac{n-1}{2} + \varepsilon$ and $-y < \mathrm{Im}(a) < a$ again. As in the proof of Theorem 6.1.1, the parabolic Eisenstein series $E_{\eta_k}^{\mathrm{par}}(Q, n-1-w)$, $E_{\eta_k}^{\mathrm{par}}(P, w)$ and the function $\Gamma\left(\frac{s-w}{2}\right)$ have no poles in this box, whereas the function $\Gamma\left(\frac{s-n+1+w}{2}\right)$ has only one pole there at $w = n-1-s-2(m+1)$ with residue equal to $2\,(-1)^{m+1}/(m+1)!$. This yields

$$
\begin{aligned}
J_{\eta_k}^{(m+1,1)}(s) &= J_{\eta_k}(s) + \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \frac{(-1)^{m+1}}{(m+1)!} \Gamma\Big(s - \frac{n-1}{2} + m + 1\Big) E_{\eta_k}^{\mathrm{par}}(P, n-1-s-2(m+1)) \\
&\qquad \cdot \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, s+2(m+1))\; ds_{\mathbb{H}^n}(Q) \\
&+ \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{m+1} \frac{(-1)^l}{l!} \Gamma\Big(s - \frac{n-1}{2} + l\Big) E_{\eta_k}^{\mathrm{par}}(P, s+2l) \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, n-1-s-2l)\; ds_{\mathbb{H}^n}(Q) \\
&+ \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{m} \frac{(-1)^l}{l!} \Gamma\Big(s - \frac{n-1}{2} + l\Big) E_{\eta_k}^{\mathrm{par}}(P, n-1-s-2l) \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, s+2l)\; ds_{\mathbb{H}^n}(Q),
\end{aligned}
$$

i.e. we have

$$J_{\eta_k}^{(m+1,1)}(s) = J_{\eta_k}(s) + \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{m+1} \frac{(-1)^l}{l!}\, \Gamma\Big(s - \frac{n-1}{2} + l\Big)\, E_{\eta_k}^{\mathrm{par}}(P, s+2l)$$
$$\cdot \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, n-1-s-2l)\; ds_{\mathbb{H}^n}(Q)$$
$$+ \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{m+1} \frac{(-1)^l}{l!}\, \Gamma\Big(s - \frac{n-1}{2} + l\Big)\, E_{\eta_k}^{\mathrm{par}}(P, n-1-s-2l) \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, s+2l)\; ds_{\mathbb{H}^n}(Q). \tag{6.26}$$

As the right-hand side of (6.26) is a meromorphic function for $\frac{n-1}{2} - 2(m+2) < \mathrm{Re}(s) < \frac{n-1}{2} - 2(m+1)$, it gives the meromorphic continuation $J_{\eta_k}^{(m+1,2)}(s)$ of the integral $J_{\eta_k}^{(m+1,1)}(s)$ to this strip.
Furthermore, on the line $\mathrm{Re}(s) = \frac{n-1}{2} - 2(m+1)$ we have to take the right hand side of (6.25) as the meromorphic continuation of the integral $J_{\eta_k}(s)$, where y is chosen such that $y > |\mathrm{Im}(s)|$. Together these formulas provide the meromorphic continuation $J_{\eta_k}^{(m+2)}(s)$ of the integral $J_{\eta_k}(s)$ to the strip $\frac{n-1}{2} - 2(m+2) < \mathrm{Re}(s) \le \frac{n-1}{2} - 2(m+1)$.

Since we have seen that for $m \in \mathbb{N}_0$ the meromorphic continuation of the integral $J_{\eta_k}(s)$ to the strip $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) \le \frac{n-1}{2} - 2m$ is given by means of the formulas (6.23) and (6.24), we obtain the meromorphic continuation of the integral $J_{\eta_k}(s)$ in s to the whole complex plane.

Using the identity

$$\sum_{k=1}^{c_\Gamma} E_{\eta_k}^{\mathrm{par}}(Q, n-1-w)\, E_{\eta_k}^{\mathrm{par}}(P, w) = \sum_{k=1}^{c_\Gamma} E_{\eta_k}^{\mathrm{par}}(Q, w)\, E_{\eta_k}^{\mathrm{par}}(P, n-1-w)$$

for $w \in \mathbb{C}$, established in the proof of Theorem 6.1.1, the meromorphic continuation of the continuous part

$$\frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^{\infty} b_{t,\eta_k}(s)\, E_{\eta_k}^{\mathrm{par}}\Big(P, \frac{n-1}{2} + it\Big)\; dt$$

of the spectral expansion (6.16) to the strip $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) \le \frac{n-1}{2} - 2m$ is given by

$$\sum_{k=1}^{c_\Gamma} J_{\eta_k}(s) + \frac{2\,\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{m} \frac{(-1)^l}{l!}\, \Gamma\Big(s - \frac{n-1}{2} + l\Big)$$
$$\cdot \sum_{k=1}^{c_\Gamma} E_{\eta_k}^{\mathrm{par}}(P, s+2l) \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, n-1-s-2l)\; ds_{\mathbb{H}^n}(Q) \tag{6.27}$$

for $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) < \frac{n-1}{2} - 2m$, while on the line $\mathrm{Re}(s) = \frac{n-1}{2} - 2m$ we have to take

$$\sum_{k=1}^{c_\Gamma} \widetilde{J}_{\eta_k,y,\varepsilon}(s) + \frac{2\,\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{m-1} \frac{(-1)^l}{l!}\, \Gamma\Big(s - \frac{n-1}{2} + l\Big)$$
$$\cdot \sum_{k=1}^{c_\Gamma} E_{\eta_k}^{\mathrm{par}}(P, s+2l) \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, n-1-s-2l)\; ds_{\mathbb{H}^n}(Q)$$
$$+ \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \frac{(-1)^m}{m!}\, \Gamma\Big(s - \frac{n-1}{2} + m\Big)$$
$$\cdot \sum_{k=1}^{c_\Gamma} E_{\eta_k}^{\mathrm{par}}(P, s+2m) \int_{L_{(Q_1,Q_2)}} E_{\eta_k}^{\mathrm{par}}(Q, n-1-s-2m)\; ds_{\mathbb{H}^n}(Q), \tag{6.28}$$

where $\varepsilon \in (0,1)$ is chosen sufficiently small such that all parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,s)$ ($k = 1,\ldots,c_\Gamma$) have no poles in the strip $\frac{n-1}{2} - 2\varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} + 2\varepsilon$, and y is chosen sufficiently large such that $y > |\mathrm{Im}(s)|$. In this way we obtain the meromorphic continuation of the continuous part of the spectral expansion (6.16) in s to the whole complex plane.

Next we turn to compute the poles arising from the continuous part after multiplication by $\Gamma\left(\frac{s}{2}\right)^2 \Gamma\left(s - \frac{n-1}{2}\right)^{-1}$. Making use of formulas (6.27) and (6.28), valid in the strip $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) \leq \frac{n-1}{2} - 2m$, we have to determine the poles of

$$\frac{\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma\left(s - \frac{n-1}{2}\right)} \sum_{k=1}^{c_\Gamma} \int_{\mathrm{Re}(w)=\frac{n-1}{2}} \Gamma\Big(\frac{s-n+1+w}{2}\Big)\,\Gamma\Big(\frac{s-w}{2}\Big)\, E^{\mathrm{par}}_{\eta_k}(P,w)$$
$$\cdot \left(\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q, n-1-w)\; ds_{\mathbb{H}^n}(Q) \right) dw$$
$$+ 2\,\pi^{\frac{n-1}{2}} \sum_{l=0}^{m} \frac{(-1)^l}{l!} \Big(s - \frac{n-1}{2}\Big)_l \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(P, s+2l) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2l)\; ds_{\mathbb{H}^n}(Q)$$

or

$$\frac{\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma\left(s - \frac{n-1}{2}\right)} \sum_{k=1}^{c_\Gamma} \int_{W_{y,\varepsilon}} \Gamma\Big(\frac{s-n+1+w}{2}\Big)\,\Gamma\Big(\frac{s-w}{2}\Big)\, E^{\mathrm{par}}_{\eta_k}(P,w)$$
$$\cdot \left(\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q, n-1-w)\; ds_{\mathbb{H}^n}(Q) \right) dw$$
$$+ 2\,\pi^{\frac{n-1}{2}} \sum_{l=0}^{m-1} \frac{(-1)^l}{l!} \Big(s - \frac{n-1}{2}\Big)_l \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(P, s+2l) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2l)\; ds_{\mathbb{H}^n}(Q)$$
$$+ \pi^{\frac{n-1}{2}} \frac{(-1)^m}{m!} \Big(s - \frac{n-1}{2}\Big)_m \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(P, s+2m) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2m)\; ds_{\mathbb{H}^n}(Q),$$

respectively. The poles of both can only arise from the functions $E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2l)$ and $E^{\mathrm{par}}_{\eta_k}(P, s+2l)$, where $l = 0,\ldots,m$ and $k = 1,\ldots,c_\Gamma$.

The poles arising from $E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2l)$ are located at $s = n-1-\rho-2l$, where $w = \rho$ is a pole of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(Q,w)$. From $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) \leq \frac{n-1}{2} - 2m$ we deduce that $\frac{n-1}{2} + 2(m-l) \leq \mathrm{Re}(\rho) < \frac{n-1}{2} + 2(m+1-l)$. As in the proof of Theorem 6.1.1, we see that a pole can only occur for $l = \max\left(m - \left\lfloor \frac{n-1}{4} \right\rfloor, 0\right), \ldots, m$ and that $\rho \in \left(\frac{n-1}{2}, n-1\right]$ must hold.
Hence, we obtain the poles $s = n-1-\rho-2N$, where $N \in \left\{\max\left(m - \left\lfloor \frac{n-1}{4} \right\rfloor, 0\right), \ldots, m\right\}$ and $w = \rho$ is a pole of $E^{\mathrm{par}}_{\eta_k}(Q,w)$ for some $k \in \{1,\ldots,c_\Gamma\}$ with $\rho \in \left[\frac{n-1}{2} + 2(m-N),\ \frac{n-1}{2} + 2(m+1-N)\right)$, and particularly $\rho \in \left(\frac{n-1}{2}, n-1\right]$.

The poles of $E^{\mathrm{par}}_{\eta_k}(P, s+2l)$ are located at $s = \rho - 2l$, where $w = \rho$ is a pole of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,w)$. The inequality $\frac{n-1}{2} - 2(m+1) < \mathrm{Re}(s) \leq \frac{n-1}{2} - 2m$ implies that $\frac{n-1}{2} - 2(m+1-l) < \mathrm{Re}(\rho) \leq \frac{n-1}{2} - 2(m-l)$, especially we have $\mathrm{Re}(\rho) < \frac{n-1}{2}$.
We obtain the poles $s = \rho - 2N$, where $N \in \{0,\ldots,m\}$ and $w = \rho$ is a pole of $E^{\mathrm{par}}_{\eta_k}(P,w)$ for some $k \in \{1,\ldots,c_\Gamma\}$ with $\mathrm{Re}(\rho) \in \left(\frac{n-1}{2} - 2(m+1-N),\ \frac{n-1}{2} - 2(m-N)\right]$, and particularly $\mathrm{Re}(\rho) < \frac{n-1}{2}$.

Lastly, we observe that for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{n-1}{2}$ there are no poles arising from the continuous part of (6.16) after multiplication by $\Gamma\left(\frac{s}{2}\right)^2 \Gamma\left(s-\frac{n-1}{2}\right)^{-1}$ since

$$\frac{\Gamma\left(\frac{s}{2}\right)^2}{\Gamma\left(s-\frac{n-1}{2}\right)} \sum_{k=1}^{c_\Gamma} J_{\eta_k}(s)$$

is a holomorphic function in the half-plane $\mathrm{Re}(s) > \frac{n-1}{2}$. This finishes the proof.

□

Remark 6.2.2.

(a) The possible poles of the types (ii) and (iii) in Theorem 6.2.1 do not occur in the case $c_\Gamma = 0$, i.e. if the discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ contains no parabolic element.

(b) The orders of the possible poles of the types (ii) and (iii) in Theorem 6.2.1 and the corresponding residues depend on the location and orders of the poles of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,w)$ $(k=1,\ldots,c_\Gamma)$ and their corresponding residues.

From Theorem 6.2.1 we can conclude that the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ associated to the pair $(Q_1,Q_2) \in H_\Gamma$ of hyperbolic fixed points has a simple pole at $s=n-1$, and we can compute its residue.

Corollary 6.2.3. *For $P \in \mathbb{H}^n$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits a simple pole at $s=n-1$ with residue*

$$\mathrm{Res}_{s=n-1} E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = \frac{2\, l_{(Q_1,Q_2)}\, \pi^{\frac{n-1}{2}}}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\, \Gamma\left(\frac{n-1}{2}\right)}.$$

Proof. Theorem 6.2.1 shows that the function $\Gamma\left(\frac{s}{2}\right)^2 \Gamma\left(s-\frac{n-1}{2}\right)^{-1} E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ has a possible pole in the half-plane $\left\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \frac{n-1}{2}\right\}$ at any point $s = \frac{n-1}{2} + ir_j - 2N$, where $j \in \mathbb{N}_0$ and $\lambda_j = s_j(n-1-s_j) = \left(\frac{n-1}{2}\right)^2 + r_j^2$ is the eigenvalue of the eigenfunction $\psi_j(P)$ with $r_j \in \left[-i\,\frac{n-1}{2}, 0\right)$, and where $N \in \mathbb{N}_0$ with $N < \frac{ir_j}{2}$.
If $N=0$ and $j=0$, i.e. $\lambda_j = 0$, $r_j = -i\,\frac{n-1}{2}$ and $\psi_j(P) = \mathrm{vol}(\Gamma\backslash\mathbb{H}^n)^{-1/2}$, we get the point $s=n-1$. We have $N - ir_j = 2N - ir_j = -\frac{n-1}{2} \notin \mathbb{N}_0$, thus, it is a simple pole with residue

$$\begin{aligned}\mathrm{Res}_{s=n-1}\left[\frac{\Gamma\left(\frac{s}{2}\right)^2 E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)}{\Gamma\left(s-\frac{n-1}{2}\right)}\right] &= \frac{2\,\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \int_{L_{(Q_1,Q_2)}} \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\, ds_{\mathbb{H}^n}(Q) \\ &= \frac{2\, l_{(Q_1,Q_2)}\, \pi^{\frac{n-1}{2}}}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}.\end{aligned}$$

Since $\Gamma\left(s-\frac{n-1}{2}\right)$ and $\Gamma\left(\frac{s}{2}\right)^2$ have the non-zero values $\Gamma\left(\frac{n-1}{2}\right)$ and $\Gamma\left(\frac{n-1}{2}\right)^2$ at $s=n-1$, also the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits a simple pole at $s=n-1$, and the residue is given by

$$\mathrm{Res}_{s=n-1} E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = \frac{2\, l_{(Q_1,Q_2)}\, \pi^{\frac{n-1}{2}}\, \Gamma\left(\frac{n-1}{2}\right)}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\, \Gamma\left(\frac{n-1}{2}\right)^2} = \frac{2\, l_{(Q_1,Q_2)}\, \pi^{\frac{n-1}{2}}}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\, \Gamma\left(\frac{n-1}{2}\right)}.$$

□

6.3. Meromorphic continuation of elliptic Eisenstein series

In this section we first prove that the elliptic Eisenstein series has a representation as an infinite sum of shifted hyperbolic kernel functions. Subsequently, we employ this relation to determine its meromorphic continuation in s to the whole complex plane and its possible poles.
Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. Further, let $Q \in \mathbb{H}^n$ be a point with elliptic scaling matrix $\sigma_Q \in \mathrm{PSL}_2(C_{n-1})$ and stabilizer subgroup Γ_Q.

The elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ can be written in terms of the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ as follows.

Proposition 6.3.1. *For $P, Q \in \mathbb{H}^n$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ we have the relation*

$$E_Q^{\mathrm{ell}}(P,s) = \frac{1}{|\Gamma_Q|} \sum_{k=0}^{\infty} \frac{\left(\frac{s}{2}\right)_k}{k!} K^{\mathrm{hyp}}(P,Q,s+2k),$$

where $\left(\frac{s}{2}\right)_k$ is the Pochhammer symbol (see (A.14)*).*

Proof. The proof is similar to the proof of Lemma 3.3.8 in [Pip10]. Nevertheless, we carry it out in detail.

First we have to check the absolute and locally uniform convergence of the series

$$\sum_{k=0}^{\infty} \frac{\left(\frac{s}{2}\right)_k}{k!} K^{\mathrm{hyp}}(P,Q,s+2k) \tag{6.29}$$

for fixed $P, Q \in \mathbb{H}^n$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$. Because Γ acts discontinuously on $\mathbb{H}^n$ and we assume $P \neq \gamma Q$ for any $\gamma \in \Gamma$, the minimum $\min_{\gamma\in\Gamma} d_{\mathbb{H}^n}(\gamma P, Q)$ exists and is greater than zero. Thus, there is a constant $C = C(P,Q) > 1$, depending only on P and Q, such that

$$\cosh(d_{\mathbb{H}^n}(\gamma P, Q)) \geq C$$

for any $\gamma \in \Gamma$. Furthermore, for $s' \in \mathbb{C}$ with $\mathrm{Re}(s') > 0$ we have the estimate

$$|\Gamma(s')| \leq |\Gamma(\mathrm{Re}(s'))| = \Gamma(\mathrm{Re}(s')),$$

which implies that

$$\left|\left(\frac{s}{2}\right)_k\right| \leq \frac{\Gamma\left(\frac{\mathrm{Re}(s)}{2} + k\right)}{|\Gamma\left(\frac{s}{2}\right)|} = \frac{\Gamma\left(\frac{\mathrm{Re}(s)}{2}\right)\left(\frac{\mathrm{Re}(s)}{2}\right)_k}{|\Gamma\left(\frac{s}{2}\right)|}$$

for any $k \in \mathbb{N}_0$. Using these bounds and the identity (A.22), we get

$$\begin{aligned}
\left|\sum_{k=0}^{\infty} \frac{\left(\frac{s}{2}\right)_k}{k!} K^{\mathrm{hyp}}(P,Q,s+2k)\right| &\leq \sum_{k=0}^{\infty} \frac{\left|\left(\frac{s}{2}\right)_k\right|}{k!} \sum_{\gamma\in\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, Q))^{-\mathrm{Re}(s)-2k} \\
&\leq \frac{\Gamma\left(\frac{\mathrm{Re}(s)}{2}\right)}{|\Gamma\left(\frac{s}{2}\right)|} \sum_{k=0}^{\infty} \frac{\left(\frac{\mathrm{Re}(s)}{2}\right)_k}{k!} C^{-2k} \sum_{\gamma\in\Gamma} \cosh(d_{\mathbb{H}^n}(\gamma P, Q))^{-\mathrm{Re}(s)} \\
&= \frac{\Gamma\left(\frac{\mathrm{Re}(s)}{2}\right)}{|\Gamma\left(\frac{s}{2}\right)|} \left(1 - \frac{1}{C^2}\right)^{-\mathrm{Re}(s)/2} K^{\mathrm{hyp}}(P,Q,\mathrm{Re}(s)).
\end{aligned}$$

This shows that the series (6.29) indeed converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$.

Now we are allowed to change the order of summation and obtain

$$\begin{aligned}\frac{1}{|\Gamma_Q|}\sum_{k=0}^{\infty}\frac{\left(\frac{s}{2}\right)_k}{k!}\,K^{\mathrm{hyp}}(P,Q,s+2k) &= \frac{1}{|\Gamma_Q|}\sum_{k=0}^{\infty}\frac{\left(\frac{s}{2}\right)_k}{k!}\sum_{\gamma\in\Gamma}\cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s-2k}\\ &= \frac{1}{|\Gamma_Q|}\sum_{\gamma\in\Gamma}\cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s}\sum_{k=0}^{\infty}\frac{\left(\frac{s}{2}\right)_k}{k!}\,\cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-2k}.\end{aligned}$$

Finally, a second application of formula (A.22) gives us

$$\begin{aligned}\frac{1}{|\Gamma_Q|}\sum_{k=0}^{\infty}\frac{\left(\frac{s}{2}\right)_k}{k!}\,K^{\mathrm{hyp}}(P,Q,s+2k) &= \frac{1}{|\Gamma_Q|}\sum_{\gamma\in\Gamma}\cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s}\left(1-\frac{1}{\cosh(d_{\mathbb{H}^n}(\gamma P,Q))^2}\right)^{-s/2}\\ &= \frac{1}{|\Gamma_Q|}\sum_{\gamma\in\Gamma}\left(\cosh(d_{\mathbb{H}^n}(\gamma P,Q))^2-1\right)^{-s/2} = \frac{1}{|\Gamma_Q|}\sum_{\gamma\in\Gamma}\sinh(d_{\mathbb{H}^n}(\gamma P,Q))^{-s}\\ &= E_Q^{\mathrm{ell}}(P,s).\end{aligned}$$

This proves the asserted relation.

□

With Proposition 6.3.1 we are now able to derive the desired meromorphic continuation of the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ associated to the point $Q\in\mathbb{H}^n$ to the whole complex plane.

Theorem 6.3.2. *For $P\in\mathbb{H}^n$ with $P\neq\gamma Q$ for any $\gamma\in\Gamma$ the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ admits a meromorphic continuation in s to the whole complex plane. For any $m\in\mathbb{N}_0$ the possible poles of the function*

$$\frac{E_Q^{\mathrm{ell}}(P,s)}{\Gamma\left(s-\frac{n-1}{2}\right)}$$

in the strip

$$\mathcal{S}_m := \{s\in\mathbb{C}\mid n-1-2(m+1)<\mathrm{Re}(s)\leq n-1-2m\}$$

are located at the following points:

(i) $s=\frac{n-1}{2}\pm ir_j-2N$, *where* $j\in\mathbb{N}_0$ *and* $\lambda_j=s_j(n-1-s_j)=\left(\frac{n-1}{2}\right)^2+r_j^2$ *is the eigenvalue of the eigenfunction* $\psi_j(P)$, *and where* $N:=m-\left\lfloor\frac{n-1}{4}\mp\frac{\mathrm{Re}(ir_j)}{2}\right\rfloor$.

(ii) $s=n-1-\rho-2N$, *where* $w=\rho$ *is a pole of the parabolic Eisenstein series* $E_{\eta_k}^{\mathrm{par}}(P,w)$ *for some* $k\in\{1,\ldots,c_\Gamma\}$ *with* $\rho\in\left(\frac{n-1}{2},n-1\right]$, *and where* $N:=m-\left\lfloor\frac{\rho}{2}\right\rfloor$.

(iii) $s=\rho-2N$, *where* $w=\rho$ *is a pole of the parabolic Eisenstein series* $E_{\eta_k}^{\mathrm{par}}(P,w)$ *for some* $k\in\{1,\ldots,c_\Gamma\}$ *with* $\mathrm{Re}(\rho)<\frac{n-1}{2}$, *and where* $N:=m-\left\lfloor\frac{n-1-\mathrm{Re}(\rho)}{2}\right\rfloor$.

The poles given in the cases (i), (ii), (iii) might coincide in parts.

Proof. First we prove that for $P\in\mathbb{H}^n$ with $P\neq\gamma Q$ for any $\gamma\in\Gamma$ and any $m\in\mathbb{N}_0$ the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ has a meromorphic continuation in s to the half-plane

$$\mathcal{H}_m := \{s\in\mathbb{C}\mid\mathrm{Re}(s)>n-1-2(m+1)\}.$$

By Proposition 6.3.1, for $P\in\mathbb{H}^n$ with $P\neq\gamma Q$ for any $\gamma\in\Gamma$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1$ the elliptic Eisenstein series can be written in terms of the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ as

$$\begin{aligned}E_Q^{\mathrm{ell}}(P,s) &= \frac{1}{|\Gamma_Q|}\sum_{l=0}^{\infty}\frac{\left(\frac{s}{2}\right)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l)\\ &= \frac{1}{|\Gamma_Q|}\sum_{l=0}^{m}\frac{\left(\frac{s}{2}\right)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l)+\frac{1}{|\Gamma_Q|}\sum_{l=m+1}^{\infty}\frac{\left(\frac{s}{2}\right)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l). \qquad (6.30)\end{aligned}$$

Next we show that the series

$$\frac{1}{|\Gamma_Q|}\sum_{l=m+1}^{\infty}\frac{\left(\frac{s}{2}\right)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l) \tag{6.31}$$

converges absolutely and locally uniformly for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1-2(m+1)$. As in the proof of Proposition 6.3.1 there is a constant $C=C(P,Q)>1$, depending only on P and Q, such that

$$\cosh(d_{\mathbb{H}^n}(\gamma P,Q))\geq C$$

for any $\gamma\in\Gamma$. Using this constant, we find the bound

$$\begin{aligned}\sum_{l=m+1}^{\infty}\left|\frac{\left(\frac{s}{2}\right)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l)\right| &= \sum_{l=0}^{\infty}\left|\frac{\left(\frac{s}{2}\right)_{l+m+1}}{(l+m+1)!}\,K^{\mathrm{hyp}}(P,Q,s+2(l+m+1))\right|\\ &\leq \sum_{l=0}^{\infty}\frac{\left|\left(\frac{s}{2}\right)_{l+m+1}\right|}{(l+m+1)!}\sum_{\gamma\in\Gamma}\cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-\mathrm{Re}(s)-2(l+m+1)}\\ &\leq \sum_{l=0}^{\infty}\frac{\left|\left(\frac{s}{2}\right)_{l+m+1}\right|}{(l+m+1)!}\,C^{-2l}\sum_{\gamma\in\Gamma}\cosh(d_{\mathbb{H}^n}(\gamma P,Q))^{-\mathrm{Re}(s)-2(m+1)}\\ &= K^{\mathrm{hyp}}(P,Q,\mathrm{Re}(s)+2(m+1))\cdot\sum_{l=0}^{\infty}\frac{\left|\left(\frac{s}{2}\right)_{l+m+1}\right|}{(l+m+1)!}\,C^{-2l}.\end{aligned}$$

As $\mathrm{Re}(s)+2(m+1)>n-1$ holds true for any $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1-2(m+1)$, the series $K^{\mathrm{hyp}}(P,Q,\mathrm{Re}(s)+2(m+1))$ is absolutely and locally uniformly convergent for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1-2(m+1)$. Moreover, the absolute value of the ratio of successive terms in the series

$$\sum_{l=0}^{\infty}\frac{\left|\left(\frac{s}{2}\right)_{l+m+1}\right|}{(l+m+1)!}\,C^{-2l} \tag{6.32}$$

has the limit

$$\begin{aligned}\lim_{l\to\infty}\left|\frac{\left(\frac{s}{2}\right)_{l+m+2}}{(l+m+2)!}\,C^{-2(l+1)}\,\frac{(l+m+1)!}{\left(\frac{s}{2}\right)_{l+m+1}}\,C^{2l}\right| &= \lim_{l\to\infty}\left|\frac{\Gamma\left(\frac{s}{2}+l+m+2\right)}{(l+m+2)\,\Gamma\left(\frac{s}{2}+l+m+1\right)\,C^2}\right|\\ = \lim_{l\to\infty}\left|\frac{\frac{s}{2}+l+m+1}{(l+m+2)\,C^2}\right| = \frac{1}{C^2}<1.\end{aligned}$$

By the ratio test the series (6.32) converges absolutely and locally uniformly for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1-2(m+1)$. This implies that also the series (6.31) is absolutely and locally uniformly convergent for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1-2(m+1)$, so it defines a holomorphic function on the half-plane $\mathcal{H}_m$.

Since the finite sum

$$\frac{1}{|\Gamma_Q|}\sum_{l=0}^{m}\frac{\left(\frac{s}{2}\right)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l)$$

admits a meromorphic continuation in s to the whole complex plane by Theorem 6.1.1, this proves the meromorphic continuation of the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$ to the half-plane $\mathcal{H}_m$. As m was chosen arbitrarily, we can conclude that $E^{\mathrm{ell}}_Q(P,s)$ has a meromorphic continuation in s to the whole complex plane.

Now, in order to determine the poles of the function $\Gamma\left(s-\frac{n-1}{2}\right)^{-1}E^{\mathrm{ell}}_Q(P,s)$ in the strip

$$\mathcal{S}_m=\{s\in\mathbb{C}\mid n-1-2(m+1)<\mathrm{Re}(s)\leq n-1-2m\}$$

for any $m \in \mathbb{N}_0$, we consider the decomposition (6.30). Since the infinite sum (6.31) is holomorphic in this strip, the possible poles can only arise from the finite sum

$$\frac{1}{|\Gamma_Q|\,\Gamma\left(s - \frac{n-1}{2}\right)} \sum_{l=0}^{m} \frac{\left(\frac{s}{2}\right)_l}{l!} K^{\mathrm{hyp}}(P, Q, s + 2l),$$

respectively from the functions $\Gamma\left(s - \frac{n-1}{2}\right)^{-1} K^{\mathrm{hyp}}(P, Q, s + 2l)$ for $l = 0, \ldots, m$.

Writing

$$\frac{K^{\mathrm{hyp}}(P, Q, s + 2l)}{\Gamma\left(s - \frac{n-1}{2}\right)} = \frac{\left(s - \frac{n-1}{2}\right)_{2l}}{\Gamma(s + 2l)} \frac{\Gamma(s + 2l)\, K^{\mathrm{hyp}}(P, Q, s + 2l)}{\Gamma\left(s + 2l - \frac{n-1}{2}\right)},$$

observing that the function $\left(s - \frac{n-1}{2}\right)_{2l} \Gamma(s + 2l)^{-1}$ has no poles and using Theorem 6.1.1, we see that for any $l \in \{0, \ldots, m\}$ the possible poles of $\Gamma\left(s - \frac{n-1}{2}\right)^{-1} K^{\mathrm{hyp}}(P, Q, s + 2l)$ are located at the following points:

(i) $s = \frac{n-1}{2} \pm ir_j - 2(M + l)$, where $j, M \in \mathbb{N}_0$ and $\lambda_j = s_j(n - 1 - s_j) = \left(\frac{n-1}{2}\right)^2 + r_j^2$ is the eigenvalue of the eigenfunction $\psi_j(P)$,

(ii) $s = n - 1 - \rho - 2(M + l)$, where $M \in \mathbb{N}_0$ and $w = \rho$ is a pole of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P, w)$ for some $k \in \{1, \ldots, c_\Gamma\}$ with $\rho \in \left(\frac{n-1}{2}, n - 1\right]$,

(iii) $s = \rho - 2(M + l)$, where $M \in \mathbb{N}_0$ and $w = \rho$ is a pole of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P, w)$ for some $k \in \{1, \ldots, c_\Gamma\}$ with $\mathrm{Re}(\rho) < \frac{n-1}{2}$.

It remains to check which of these points are contained in the strip $\mathcal{S}_m$.

For a pole of type (i) the condition $\frac{n-1}{2} \pm ir_j - 2(M + l) \in \mathcal{S}_m$, i.e.

$$n - 1 - 2(m + 1) < \frac{n-1}{2} \pm \mathrm{Re}(ir_j) - 2(M + l) \leq n - 1 - 2m,$$

holds true if and only if

$$m - l - \frac{n-1}{4} \pm \frac{\mathrm{Re}(ir_j)}{2} \leq M < m + 1 - l - \frac{n-1}{4} \pm \frac{\mathrm{Re}(ir_j)}{2}.$$

This gives us $M = \left\lceil m - l - \frac{n-1}{4} \pm \frac{\mathrm{Re}(ir_j)}{2} \right\rceil = m - l - \left\lfloor \frac{n-1}{4} \mp \frac{\mathrm{Re}(ir_j)}{2} \right\rfloor$. So for any $l \in \{0, \ldots, m\}$ in the strip $\mathcal{S}_m$ the function $\Gamma\left(s - \frac{n-1}{2}\right)^{-1} K^{\mathrm{hyp}}(P, Q, s + 2l)$ has the possible poles $s = \frac{n-1}{2} \pm ir_j - 2N$, where $j \in \mathbb{N}_0$ and $\lambda_j = s_j(n - 1 - s_j) = \left(\frac{n-1}{2}\right)^2 + r_j^2$ is the eigenvalue of the eigenfunction $\psi_j(P)$, and where $N := M + l = m - \left\lfloor \frac{n-1}{4} \mp \frac{\mathrm{Re}(ir_j)}{2} \right\rfloor$.

Furthermore, for a pole of type (ii) we have $n - 1 - \rho - 2(M + l) \in \mathcal{S}_m$, i.e.

$$n - 1 - 2(m + 1) < n - 1 - \rho - 2(M + l) \leq n - 1 - 2m,$$

if and only if

$$m - l - \frac{\rho}{2} \leq M < m + 1 - l - \frac{\rho}{2}.$$

This yields $M = \left\lceil m - l - \frac{\rho}{2} \right\rceil = m - l - \left\lfloor \frac{\rho}{2} \right\rfloor$. Thus, for any $l \in \{0, \ldots, m\}$ in the strip $\mathcal{S}_m$ the function $\Gamma\left(s - \frac{n-1}{2}\right)^{-1} K^{\mathrm{hyp}}(P, Q, s + 2l)$ has the possible poles $s = n - 1 - \rho - 2N$, where $w = \rho$ is a pole of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P, w)$ for some $k \in \{1, \ldots, c_\Gamma\}$ with $\rho \in \left(\frac{n-1}{2}, n - 1\right]$, and where $N := M + l = m - \left\lfloor \frac{\rho}{2} \right\rfloor$.

Moreover, for a pole of type (iii) the condition $\rho - 2(M+l) \in \mathcal{S}_m$, i.e.

$$n - 1 - 2(m+1) < \mathrm{Re}(\rho) - 2(M+l) \leq n - 1 - 2m,$$

is equivalent to

$$m - l - \frac{n-1-\mathrm{Re}(\rho)}{2} \leq M < m + 1 - l - \frac{n-1-\mathrm{Re}(\rho)}{2}.$$

We obtain $M = \left\lceil m - l - \frac{n-1-\mathrm{Re}(\rho)}{2} \right\rceil = m - l - \left\lfloor \frac{n-1-\mathrm{Re}(\rho)}{2} \right\rfloor$, and for any $l \in \{0, \ldots, m\}$ in the strip $\mathcal{S}_m$ the function $\Gamma\left(s - \frac{n-1}{2}\right)^{-1} K^{\mathrm{hyp}}(P, Q, s+2l)$ has the possible poles $s = \rho - 2N$, where $w = \rho$ is a pole of the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P, w)$ for some $k \in \{1, \ldots, c_\Gamma\}$ with $\mathrm{Re}(\rho) < \frac{n-1}{2}$, and where $N := M + l = m - \left\lfloor \frac{n-1-\mathrm{Re}(\rho)}{2} \right\rfloor$.

$\square$

Remark 6.3.3. The possible poles of the types (ii) and (iii) in Theorem 6.3.2 do not occur in the case $c_\Gamma = 0$, i.e. if the discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ contains no parabolic element.

To close this section, we show that the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P, s)$ associated to the point $Q \in \mathbb{H}^n$ has a simple pole at $s = n - 1$ and determine the corresponding residue.

Corollary 6.3.4. *For $P \in \mathbb{H}^n$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P, s)$ admits a simple pole at $s = n - 1$ with residue*

$$\mathrm{Res}_{s=n-1} E^{\mathrm{ell}}_Q(P, s) = \frac{2\,\pi^{n/2}}{|\Gamma_Q| \operatorname{vol}(\Gamma \backslash \mathbb{H}^n)\, \Gamma\left(\frac{n}{2}\right)}.$$

Proof. By the proof of Theorem 6.3.2, in the strip $\mathcal{S}_0 = \{s \in \mathbb{C} \mid n - 3 < \mathrm{Re}(s) \leq n - 1\}$ we have the decomposition

$$E^{\mathrm{ell}}_Q(P, s) = \frac{1}{|\Gamma_Q|} K^{\mathrm{hyp}}(P, Q, s) + \frac{1}{|\Gamma_Q|} \sum_{l=1}^{\infty} \frac{\left(\frac{s}{2}\right)_l}{l!} K^{\mathrm{hyp}}(P, Q, s+2l),$$

where the infinite sum over l is holomorphic in s. Since by Corollary 6.1.3 the hyperbolic kernel function $K^{\mathrm{hyp}}(P, Q, s)$ has a simple pole at $s = n - 1$ with residue

$$\mathrm{Res}_{s=n-1} K^{\mathrm{hyp}}(P, Q, s) = \frac{2\,\pi^{n/2}}{\operatorname{vol}(\Gamma \backslash \mathbb{H}^n)\, \Gamma\left(\frac{n}{2}\right)},$$

we can conclude that also the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P, s)$ admits a simple pole at this point, and the residue is given by

$$\mathrm{Res}_{s=n-1} E^{\mathrm{ell}}_Q(P, s) = \frac{1}{|\Gamma_Q|} \mathrm{Res}_{s=n-1} K^{\mathrm{hyp}}(P, Q, s) = \frac{2\,\pi^{n/2}}{|\Gamma_Q| \operatorname{vol}(\Gamma \backslash \mathbb{H}^n)\, \Gamma\left(\frac{n}{2}\right)}.$$

$\square$

7. Kronecker limit formulas for hyperbolic Eisenstein series

Classically, in the case $n=2$ and $\Gamma=\mathrm{PSL}_2(\mathbb{Z})$, Kronecker's limit formula states the behaviour of the parabolic Eisenstein series $E_\infty^{\mathrm{par}}(z,s)$ at $s=1$, and explicitly gives the constant term in the Laurent expansion. In this chapter we are interested in an analogue for the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ in $\mathbb{H}^n$ at the point $s=0$, so we study its behaviour in terms of its Laurent expansion. We compute the first two terms in the Laurent expansion for arbitrary dimension n and an arbitrary discrete and cofinite subgroup $\Gamma\subseteq\mathrm{PSL}_2(C_{n-1})$ in the first section. Then we consider two examples of specific dimensions and groups in the subsequent sections, and prove a formula of Kronecker limit type for the hyperbolic Eisenstein series in both cases. In the second section we do this for $n=2$ and $\Gamma=\mathrm{PSL}_2(\mathbb{Z})$, while in the third section we treat the case $n=3$ and $\Gamma=\mathrm{PSL}_2(\mathbb{Z}[i])$.

7.1. The Laurent expansion at $s=0$

In this section we determine the Laurent expansion of the hyperbolic Eisenstein series at $s=0$ via its meromorphic continuation established in section 6.2. We find that the form of this expansion depends on $n \mod 4$ for the dimension n of the hyperbolic space $\mathbb{H}^n$. Therefore, we consider the cases $n\equiv 0 \mod 2$, $n\equiv 3 \mod 4$ and $n\equiv 1 \mod 4$ in three separate propositions.
Let $\Gamma\subseteq\mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. Further, let $(Q_1,Q_2)\in H_\Gamma$ be a pair of hyperbolic fixed points with hyperbolic scaling matrix $\sigma_{(Q_1,Q_2)}\in\mathrm{PSL}_2(C_{n-1})$ and hyperbolic stabilizer subgroup $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$. Let $\mathcal{L}_{(Q_1,Q_2)}$ be the unique geodesic in $\mathbb{H}^n$ connecting Q_1 and Q_2, and let $L_{(Q_1,Q_2)}=\pi_{\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}}(\mathcal{L}_{(Q_1,Q_2)})$ denote its image under the natural projection $\pi_{\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}}:\mathbb{H}^n\to\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\mathbb{H}^n$, which is a closed geodesic in $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}\backslash\mathbb{H}^n$ of hyperbolic length $l_{(Q_1,Q_2)}$.

Notation 7.1.1. To keep the notation simple, in this section we again omit the index n and write $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ for the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{n,(Q_1,Q_2)}(P,s)$ associated to the pair $(Q_1,Q_2)\in H_\Gamma$ of hyperbolic fixed points, and $E^{\mathrm{par}}_{\eta_k}(P,s)$ for the parabolic Eisenstein series $E^{\mathrm{par}}_{n,\eta_k}(P,s)$ associated to the cusp $\eta_k\in C_\Gamma$ $(k=1,\ldots,c_\Gamma)$.

In case that the dimension is even, i.e. $n\equiv 0 \mod 2$, we find the following result.

Proposition 7.1.2. *Let $n\equiv 0 \mod 2$. For $P\in\mathbb{H}^n$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits a Laurent expansion at $s=0$ of the form*

$$\begin{aligned}
&E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)-\mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),1}(P,s)\\
&\quad=\left(\frac{l_{(Q_1,Q_2)}\,\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1-n}{2}\right)}{2\operatorname{vol}(\Gamma\backslash\mathbb{H}^n)}+\sum_{l=1}^{2}G_{n,(Q_1,Q_2),l}(P)\right)\cdot s\\
&\qquad+\left(\frac{l_{(Q_1,Q_2)}\,\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1-n}{2}\right)\left(\gamma+\psi^{(0)}\left(\frac{1-n}{2}\right)\right)}{4\operatorname{vol}(\Gamma\backslash\mathbb{H}^n)}+\sum_{l=1}^{4}H_{n,(Q_1,Q_2),l}(P)\right)\cdot s^2+\mathrm{O}(s^3), \qquad (7.1)
\end{aligned}$$

where

$$\mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),1}(P,s) := \frac{2\,\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{\lfloor \frac{n-1}{4} \rfloor} \frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big) \cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(P,s+2l) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2l)\;ds_{\mathbb{H}^n}(Q),$$

γ denotes the Euler-Mascheroni constant (see (A.6)*), $\psi^{(0)}(s)$ is the digamma function (see* (A.8)*), and where the functions $G_{n,(Q_1,Q_2),l}(P)$ $(l=1,2)$ and $H_{n,(Q_1,Q_2),l}(P)$ $(l=1,2,3,4)$ are invariant under the action of Γ, and are given by the formulas* (7.3)*,* (7.4)*,* (7.5)*,* (7.6)*,* (7.7) *and* (7.8) *in the proof, respectively.*

Proof. For $P\in\mathbb{H}^n$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits a meromorphic continuation in s to the whole complex plane by Theorem 6.2.1. In the proof of this theorem we established that for $s\in\mathbb{C}$ with $\frac{n-1}{2}-2(m+1)<\mathrm{Re}(s)<\frac{n-1}{2}-2m$ $(m\in\mathbb{N}_0)$ this meromorphic continuation is given by means of

$$\begin{aligned} E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) &= \sum_{j=0}^{\infty} b_j(s)\,\psi_j(P) + \frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} b_{t,\eta_k}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt \\ &\quad + \frac{2\,\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\sum_{l=0}^{m}\frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big)\sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(P,s+2l) \\ &\qquad \cdot \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2l)\;ds_{\mathbb{H}^n}(Q), \end{aligned}$$

where the coefficients $b_j(s)$ and $b_{t,\eta_k}(s)$ are given by

$$\begin{aligned} b_j(s) &= \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big)\int_{L_{(Q_1,Q_2)}}\overline{\psi_j}(Q)\;ds_{\mathbb{H}^n}(Q), \\ b_{t,\eta_k}(s) &= \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+it}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-it}{2}\Big)\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-it\Big)\;ds_{\mathbb{H}^n}(Q), \end{aligned}$$

respectively. In particular, in the strip $\frac{n-1}{2}-2\left(\left\lfloor\frac{n-1}{4}\right\rfloor+1\right)<\mathrm{Re}(s)<\frac{n-1}{2}-2\left\lfloor\frac{n-1}{4}\right\rfloor$ the identity

$$\begin{aligned} &E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)-\mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),1}(P,s) \\ &\quad = \sum_{j=0}^{\infty} b_j(s)\,\psi_j(P) + \frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} b_{t,\eta_k}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt \end{aligned} \tag{7.2}$$

holds true. From the assumption $n\equiv 0 \mod 2$ we conclude that $\left\lfloor\frac{n-1}{4}\right\rfloor<\frac{n-1}{4}$, which implies that the point $s=0$ lies in the considered strip. Hence, to derive the Laurent expansion at $s=0$ we work from formula (7.2).

For $j=0$, i.e. $\lambda_j=0$, $r_j=-i\,\frac{n-1}{2}$ and $\psi_j(P)=\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)^{-1/2}$, the function $b_j(s)\,\psi_j(P)$ in the series in (7.2) arising from the discrete spectrum takes the form

$$b_0(s)\,\psi_0(P)=\frac{l_{(Q_1,Q_2)}\,\pi^{\frac{n-1}{2}}}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(\frac{s}{2}\Big)\,\Gamma\Big(\frac{s-n+1}{2}\Big)=\frac{l_{(Q_1,Q_2)}\,\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{s-n+1}{2}\right)}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma\left(\frac{s}{2}\right)}.$$

At $s=0$ we have the Laurent expansions

$$\begin{aligned} \frac{1}{\Gamma\left(\frac{s}{2}\right)} &= \frac{1}{2}\cdot s+\frac{\gamma}{4}\cdot s^2+\mathrm{O}(s^3), \\ \Gamma\Big(\frac{s-n+1}{2}\Big) &= \Gamma\Big(\frac{1-n}{2}\Big)+\frac{1}{2}\,\Gamma\Big(\frac{1-n}{2}\Big)\,\psi^{(0)}\Big(\frac{1-n}{2}\Big)\cdot s+\mathrm{O}(s^2), \end{aligned}$$

where γ denotes the Euler-Mascheroni constant (see (A.6)), $\psi^{(0)}(s)$ is the digamma function (see (A.8)), and where we used $\frac{n-1}{2} \notin \mathbb{N}_0$, (A.9), (A.12) and (A.13). This implies that the function $b_0(s)\,\psi_0(P)$ admits a Laurent expansion at $s = 0$ of the form

$$b_0(s)\,\psi_0(P) = \frac{l_{(Q_1,Q_2)}\,\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1-n}{2}\right)}{2\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} \cdot s + \frac{l_{(Q_1,Q_2)}\,\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1-n}{2}\right)\,\left(\gamma + \psi^{(0)}\left(\frac{1-n}{2}\right)\right)}{4\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} \cdot s^2 + \mathrm{O}(s^3).$$

Now we consider $j \geq 1$ so that either $r_j > 0$ or $r_j \in \left(-i\,\frac{n-1}{2}, 0\right]$ holds true. In the first case it is clear that $\frac{n-1}{4} \pm \frac{ir_j}{2} \notin \mathbb{N}_0$. In the latter case we have $\frac{n-1}{4} \pm \frac{ir_j}{2} \in \mathbb{R}$ with $\frac{n-1}{4} - \frac{ir_j}{2} \in \left(0, \frac{n-1}{4}\right]$ and $\frac{n-1}{4} + \frac{ir_j}{2} \in \left[\frac{n-1}{4}, \frac{n-1}{2}\right)$. The natural numbers contained in the interval $\left(0, \frac{n-1}{4}\right]$ are $1, \ldots, \left\lfloor \frac{n-1}{4} \right\rfloor$, while the interval $\left[\frac{n-1}{4}, \frac{n-1}{2}\right)$ contains the natural numbers $\left\lceil \frac{n-1}{4} \right\rceil, \ldots, \frac{n}{2} - 1$.
This implies that the function $\Gamma\left(\frac{s - \frac{n-1}{2} + ir_j}{2}\right)$ has a pole at $s = 0$ if and only if

$$\frac{n-1}{4} - \frac{ir_j}{2} \in \left\{1, \ldots, \left\lfloor \frac{n-1}{4} \right\rfloor\right\} \iff r_j \in \left\{ -i\left(\frac{n-1}{2} - 2N\right) \;\middle|\; N \in \left\{1, \ldots, \left\lfloor \frac{n-1}{4} \right\rfloor\right\}\right\},$$

while the function $\Gamma\left(\frac{s - \frac{n-1}{2} - ir_j}{2}\right)$ has a pole at $s = 0$ if and only if

$$\begin{aligned} &\frac{n-1}{4} + \frac{ir_j}{2} \in \left\{\left\lceil \frac{n-1}{4} \right\rceil, \ldots, \frac{n}{2} - 1\right\} \\ &\quad \iff r_j \in \left\{ -i\left(2N - \frac{n-1}{2}\right) \;\middle|\; N \in \left\{\left\lceil \frac{n-1}{4} \right\rceil, \ldots, \frac{n}{2} - 1\right\}\right\}. \end{aligned}$$

Further, if $\frac{n-1}{4} - \frac{ir_j}{2} = N$ for some $N \in \left\{1, \ldots, \left\lfloor \frac{n-1}{4} \right\rfloor\right\}$, then $\frac{n-1}{2} \notin \mathbb{Z}$ implies that

$$\frac{n-1}{4} + \frac{ir_j}{2} = \frac{n-1}{2} - N \;\notin \mathbb{Z}$$

cannot be an element of $\mathbb{N}_0$; and if conversely $\frac{n-1}{4} + \frac{ir_j}{2} = N$ for some $N \in \left\{\left\lceil \frac{n-1}{4} \right\rceil, \ldots, \frac{n}{2} - 1\right\}$, then $\frac{n-1}{2} \notin \mathbb{Z}$ yields that

$$\frac{n-1}{4} - \frac{ir_j}{2} = \frac{n-1}{2} - N \;\notin \mathbb{Z}$$

cannot be an element of $\mathbb{N}_0$. This shows that for any $j \geq 1$ at most one of the two functions $\Gamma\left(\frac{s - \frac{n-1}{2} + ir_j}{2}\right)$, $\Gamma\left(\frac{s - \frac{n-1}{2} - ir_j}{2}\right)$ has a pole at $s = 0$. Setting

$$\begin{aligned} M_1(n) &:= \left\{ -i\left(\frac{n-1}{2} - 2N\right) \;\middle|\; N \in \left\{1, \ldots, \left\lfloor \frac{n-1}{4} \right\rfloor\right\}\right\}, \\ M_2(n) &:= \left\{ -i\left(2N - \frac{n-1}{2}\right) \;\middle|\; N \in \left\{\left\lceil \frac{n-1}{4} \right\rceil, \ldots, \frac{n}{2} - 1\right\}\right\}, \end{aligned}$$

we have shown above that $M_1(n) \cap M_2(n) = \emptyset$.

At $s = 0$ the function $\Gamma\left(\frac{s}{2}\right)^{-2}$ has the Laurent expansion

$$\frac{1}{\Gamma\left(\frac{s}{2}\right)^2} = \frac{1}{4} \cdot s^2 + \frac{\gamma}{4} \cdot s^3 + \mathrm{O}(s^4).$$

If $r_j \in M_1(n)$, then $r_j \notin M_2(n)$, and, by (A.12) and (A.13), at $s = 0$ we have the Laurent expansions

$$\Gamma\left(\frac{s - \frac{n-1}{2} + ir_j}{2}\right) = \frac{2\,(-1)^{\frac{n-1}{4} - \frac{ir_j}{2}}}{\left(\frac{n-1}{4} - \frac{ir_j}{2}\right)!} \cdot \frac{1}{s} + \frac{(-1)^{\frac{n-1}{4} - \frac{ir_j}{2}}}{\left(\frac{n-1}{4} - \frac{ir_j}{2}\right)!} \left(\sum_{l=1}^{\frac{n-1}{4} - \frac{ir_j}{2}} \frac{1}{l} - \gamma\right) + \mathrm{O}(s),$$

$$\Gamma\left(\frac{s - \frac{n-1}{2} - ir_j}{2}\right) = \Gamma\left(-\frac{ir_j}{2} - \frac{n-1}{4}\right) + \frac{1}{2}\,\Gamma\left(-\frac{ir_j}{2} - \frac{n-1}{4}\right)\psi^{(0)}\left(-\frac{ir_j}{2} - \frac{n-1}{4}\right) \cdot s + \mathrm{O}(s^2).$$

Thus, setting

$$C_{j,(Q_1,Q_2)} := \int_{L_{(Q_1,Q_2)}} \overline{\psi_j}(Q)\; ds_{\mathbb{H}^n}(Q),$$

for $j \geq 1$ with $r_j \in M_1(n)$ the function $b_j(s)\,\psi_j(P)$ admits a Laurent expansion at $s=0$ of the form

$$\begin{aligned} b_j(s)\,\psi_j(P) &= \frac{(-1)^{\frac{n-1}{4}-\frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{2\left(\frac{n-1}{4}-\frac{ir_j}{2}\right)!}\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi_j(P)\,C_{j,(Q_1,Q_2)}\cdot s \\ &\quad + \frac{(-1)^{\frac{n-1}{4}-\frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{4\left(\frac{n-1}{4}-\frac{ir_j}{2}\right)!}\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}\Big)\left(\sum_{l=1}^{\frac{n-1}{4}-\frac{ir_j}{2}}\frac{1}{l}+\gamma+\psi^{(0)}\Big(-\frac{ir_j}{2}-\frac{n-1}{4}\Big)\right) \\ &\quad \cdot \psi_j(P)\,C_{j,(Q_1,Q_2)}\cdot s^2 + \mathrm{O}(s^3), \end{aligned}$$

and the respective part of the series in (7.2) arising from the discrete spectrum has a Laurent expansion at $s=0$ of the form

$$\sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_1(n)}} b_j(s)\,\psi_j(P) = G_{n,(Q_1,Q_2),1}(P)\cdot s + H_{n,(Q_1,Q_2),1}(P)\cdot s^2 + \mathrm{O}(s^3),$$

where

$$G_{n,(Q_1,Q_2),1}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_1(n)}} \frac{(-1)^{\frac{n-1}{4}-\frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{2\left(\frac{n-1}{4}-\frac{ir_j}{2}\right)!}\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi_j(P)\,C_{j,(Q_1,Q_2)}, \tag{7.3}$$

$$\begin{aligned} H_{n,(Q_1,Q_2),1}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_1(n)}} &\frac{(-1)^{\frac{n-1}{4}-\frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{4\left(\frac{n-1}{4}-\frac{ir_j}{2}\right)!}\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi_j(P)\,C_{j,(Q_1,Q_2)} \\ &\cdot\left(\sum_{l=1}^{\frac{n-1}{4}-\frac{ir_j}{2}}\frac{1}{l}+\gamma+\psi^{(0)}\Big(-\frac{ir_j}{2}-\frac{n-1}{4}\Big)\right). \end{aligned} \tag{7.4}$$

If $r_j \in M_2(n)$, then $r_j \notin M_1(n)$, and, again by (A.12) and (A.13), at $s=0$ we have the Laurent expansions

$$\begin{aligned} \Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big) &= \Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)+\frac{1}{2}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi^{(0)}\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\cdot s+\mathrm{O}(s^2), \\ \Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big) &= \frac{2\,(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}}}{\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!}\cdot\frac{1}{s}+\frac{(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}}}{\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!}\left(\sum_{l=1}^{\frac{n-1}{4}+\frac{ir_j}{2}}\frac{1}{l}-\gamma\right)+\mathrm{O}(s). \end{aligned}$$

Consequently, for $j \geq 1$ with $r_j \in M_2(n)$ the function $b_j(s)\,\psi_j(P)$ admits a Laurent expansion at $s=0$ of the form

$$\begin{aligned} b_j(s)\,\psi_j(P) &= \frac{(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{2\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi_j(P)\,C_{j,(Q_1,Q_2)}\cdot s \\ &\quad + \frac{(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{4\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\left(\sum_{l=1}^{\frac{n-1}{4}+\frac{ir_j}{2}}\frac{1}{l}+\gamma+\psi^{(0)}\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\right) \\ &\quad \cdot \psi_j(P)\,C_{j,(Q_1,Q_2)}\cdot s^2 + \mathrm{O}(s^3), \end{aligned}$$

which implies that the respective part of the series in (7.2) arising from the discrete spectrum has a Laurent expansion at $s = 0$ of the form

$$\sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_2(n)}} b_j(s)\,\psi_j(P) = G_{n,(Q_1,Q_2),2}(P)\cdot s + H_{n,(Q_1,Q_2),2}(P)\cdot s^2 + \mathrm{O}(s^3),$$

where

$$G_{n,(Q_1,Q_2),2}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_2(n)}} \frac{(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{2\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi_j(P)\,C_{j,(Q_1,Q_2)}, \tag{7.5}$$

$$\begin{aligned} H_{n,(Q_1,Q_2),2}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_2(n)}} &\frac{(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{4\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi_j(P)\,C_{j,(Q_1,Q_2)} \\ &\cdot\left(\sum_{l=1}^{\frac{n-1}{4}+\frac{ir_j}{2}}\frac{1}{l}+\gamma+\psi^{(0)}\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\right). \end{aligned} \tag{7.6}$$

If $r_j \notin M_1(n)\cup M_2(n)$, then (A.13) implies that at $s = 0$ we have the Laurent expansions

$$\Gamma\Big(\frac{s-\frac{n-1}{2}\pm ir_j}{2}\Big) = \Gamma\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}\Big)+\frac{1}{2}\Gamma\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi^{(0)}\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}\Big)\cdot s+\mathrm{O}(s^2).$$

Hence, for $j \geq 1$ with $r_j \notin M_1(n)\cup M_2(n)$ the function $b_j(s)\,\psi_j(P)$ admits a Laurent expansion at $s = 0$ of the form

$$b_j(s)\,\psi_j(P) = \frac{\pi^{\frac{n-1}{2}}}{4}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi_j(P)\,C_{j,(Q_1,Q_2)}\cdot s^2+\mathrm{O}(s^3),$$

so the respective part of the series in (7.2) arising from the discrete spectrum has a Laurent expansion at $s = 0$ of the form

$$\sum_{\substack{j\in\mathbb{N}:\\ r_j\notin M_1(n)\cup M_2(n)}} b_j(s)\,\psi_j(P) = H_{n,(Q_1,Q_2),3}(P)\cdot s^2+\mathrm{O}(s^3),$$

where

$$H_{n,(Q_1,Q_2),3}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\notin M_1(n)\cup M_2(n)}} \frac{\pi^{\frac{n-1}{2}}}{4}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi_j(P)\,C_{j,(Q_1,Q_2)}. \tag{7.7}$$

Moreover, for any $t \in \mathbb{R}$ at $s = 0$ we have the Laurent expansions

$$\Gamma\Big(\frac{s-\frac{n-1}{2}\pm it}{2}\Big) = \Gamma\Big(\pm\frac{it}{2}-\frac{n-1}{4}\Big)+\frac{1}{2}\Gamma\Big(\pm\frac{it}{2}-\frac{n-1}{4}\Big)\,\psi^{(0)}\Big(\pm\frac{it}{2}-\frac{n-1}{4}\Big)\cdot s+\mathrm{O}(s^2),$$

where we used $\frac{n-1}{4}\notin\mathbb{N}_0$ and (A.13). From this we conclude that for $k = 1,\ldots,c_\Gamma$ and any $t\in\mathbb{R}$ the function $b_{t,\eta_k}(s)$ in the integral in (7.2) arising from the continuous spectrum admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned} b_{t,\eta_k}(s) &= \frac{\pi^{\frac{n-1}{2}}}{4}\,\Gamma\Big(\frac{it}{2}-\frac{n-1}{4}\Big)\,\Gamma\Big(-\frac{it}{2}-\frac{n-1}{4}\Big)\,C_{t,\eta_k,(Q_1,Q_2)}\cdot s^2+\mathrm{O}(s^3) \\ &= \frac{\pi^{\frac{n-1}{2}}}{4}\left|\Gamma\Big(\frac{it}{2}-\frac{n-1}{4}\Big)\right|^2 C_{t,\eta_k,(Q_1,Q_2)}\cdot s^2+\mathrm{O}(s^3), \end{aligned}$$

where we have set

$$C_{t,\eta_k,(Q_1,Q_2)} := \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} - it\Big)\, ds_{\mathbb{H}^n}(Q).$$

Hence, at $s = 0$ we obtain the Laurent expansion

$$\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} b_{t,\eta_k}(s)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt = H_{n,(Q_1,Q_2),4}(P)\cdot s^2 + \mathrm{O}(s^3)$$

of the continuous part, where

$$H_{n,(Q_1,Q_2),4}(P) := \frac{\pi^{\frac{n-3}{2}}}{16}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty}\Big|\Gamma\Big(\frac{it}{2} - \frac{n-1}{4}\Big)\Big|^2 E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, C_{t,\eta_k,(Q_1,Q_2)}\, dt. \quad (7.8)$$

Summing up, for $P \in \mathbb{H}^n$ we obtain a Laurent expansion at $s = 0$ of the form

$$\begin{aligned} &E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) - \mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),1}(P,s) \\ &\quad= \left(\frac{l_{(Q_1,Q_2)}\,\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1-n}{2}\right)}{2\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \sum_{l=1}^{2} G_{n,(Q_1,Q_2),l}(P)\right)\cdot s \\ &\qquad + \left(\frac{l_{(Q_1,Q_2)}\,\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1-n}{2}\right)\,\left(\gamma + \psi^{(0)}\left(\frac{1-n}{2}\right)\right)}{4\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \sum_{l=1}^{4} H_{n,(Q_1,Q_2),l}(P)\right)\cdot s^2 + \mathrm{O}(s^3), \end{aligned}$$

where $G_{n,(Q_1,Q_2),l}(P)$ $(l = 1,2)$ and $H_{n,(Q_1,Q_2),l}(P)$ $(l = 1,2,3,4)$ are given by (7.3), (7.4), (7.5), (7.6), (7.7) and (7.8), respectively.

We know that the eigenfunctions $\psi_j(P)$ $(j \in \mathbb{N})$ and the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,s)$ $(k = 1,\ldots,c_\Gamma)$ are all Γ-invariant. Therefore, also the functions $G_{n,(Q_1,Q_2),l}(P)$ $(l = 1,2)$ and $H_{n,(Q_1,Q_2),l}(P)$ $(l = 1,2,3,4)$ are invariant under the action of Γ.

□

Remark 7.1.3. In the cases $n = 2$ and $n = 4$ the Laurent expansion of the hyperbolic Eisenstein series in Proposition 7.1.2 simplifies as follows:
If $n = 2$ or $n = 4$, there is no $N \in \mathbb{N}$ with $1 \le N \le \left\lfloor \frac{n-1}{4} \right\rfloor$, so the set

$$M_1(n) = \Big\{ -i\Big(\frac{n-1}{2} - 2N\Big) \;\Big|\; N \in \Big\{1,\ldots,\Big\lfloor\frac{n-1}{4}\Big\rfloor\Big\}\Big\}$$

in the proof of Proposition 7.1.2 is empty. This implies that both $G_{n,(Q_1,Q_2),1}(P) = 0$ and $H_{n,(Q_1,Q_2),1}(P) = 0$ as the sums defining these functions are empty, meaning that the functions $G_{n,(Q_1,Q_2),1}(P)$ and $H_{n,(Q_1,Q_2),1}(P)$ in the Laurent expansion (7.1) do not appear for $n = 2$ and $n = 4$.
Moreover, if $n = 2$, there is no $N \in \mathbb{N}$ with $\left\lceil \frac{n-1}{4} \right\rceil \le N \le \frac{n}{2} - 1$ which yields that the set

$$M_2(n) = \Big\{ -i\Big(2N - \frac{n-1}{2}\Big) \;\Big|\; N \in \Big\{\Big\lceil\frac{n-1}{4}\Big\rceil,\ldots,\frac{n}{2} - 1\Big\}\Big\}$$

in the proof of Proposition 7.1.2 is empty. Hence, $G_{n,(Q_1,Q_2),2}(P) = 0$ and $H_{n,(Q_1,Q_2),2}(P) = 0$ since the sums defining these functions are empty. This shows that also the functions $G_{n,(Q_1,Q_2),2}(P)$ and $H_{n,(Q_1,Q_2),2}(P)$ in the Laurent expansion (7.1) do not appear for $n = 2$.

In the next proposition we consider the case $n \equiv 3 \mod 4$.

Proposition 7.1.4. *Let $n \equiv 3 \mod 4$. For $P \in \mathbb{H}^n$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits a Laurent expansion at $s = 0$ of the form*

$$\begin{aligned}
&E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) - \mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),2}(P,s) \\
&\quad = \frac{l_{(Q_1,Q_2)}\,(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + F_{n,(Q_1,Q_2)}(P) + \left(\frac{l_{(Q_1,Q_2)}\,(-\pi)^{\frac{n-1}{2}}}{2\left(\frac{n-1}{2}\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\sum_{l=1}^{\frac{n-1}{2}}\frac{1}{l} + G_{n,(Q_1,Q_2)}(P)\right)\cdot s + \mathrm{O}(s^2),
\end{aligned} \tag{7.9}$$

where

$$\begin{aligned}
\mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),2}(P,s) := &\frac{2\,\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\sum_{l=0}^{\frac{n-3}{4}}\frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big) \\
&\cdot\sum_{k=1}^{c_\Gamma}E^{\mathrm{par}}_{\eta_k}(P,s+2l)\int_{L_{(Q_1,Q_2)}}E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2l)\;ds_{\mathbb{H}^n}(Q),
\end{aligned}$$

and where the functions $F_{n,(Q_1,Q_2)}(P)$ and $G_{n,(Q_1,Q_2)}(P)$ are invariant under the action of Γ, and are given by the formulas (7.11) *and* (7.12) *in the proof, respectively.*

Proof. The assumption $n \equiv 3 \mod 4$ gives us $\frac{n-3}{4} \in \mathbb{N}_0$. Then, similar to the proof of Proposition 7.1.2, we see that for $s \in \mathbb{C}$ with $\frac{n-1}{2} - 2\left(\frac{n-3}{4}+1\right) < \mathrm{Re}(s) < \frac{n-1}{2} - 2\cdot\frac{n-3}{4}$ the identity

$$\begin{aligned}
&E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) - \mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),2}(P,s) \\
&\quad = \sum_{j=0}^{\infty}b_j(s)\,\psi_j(P) + \frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty}b_{t,\eta_k}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\;dt
\end{aligned} \tag{7.10}$$

holds true, where the coefficients $b_j(s)$ and $b_{t,\eta_k}(s)$ are given by the formulas

$$\begin{aligned}
b_j(s) &= \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big)\int_{L_{(Q_1,Q_2)}}\overline{\psi_j}(Q)\;ds_{\mathbb{H}^n}(Q), \\
b_{t,\eta_k}(s) &= \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+it}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-it}{2}\Big)\int_{L_{(Q_1,Q_2)}}E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-it\Big)\;ds_{\mathbb{H}^n}(Q),
\end{aligned}$$

respectively. The equality $\frac{n-1}{2} - 2\cdot\frac{n-3}{4} = 1$ yields that the point $s = 0$ lies in the considered strip. To derive the Laurent expansion at $s = 0$ we work from formula (7.10).

Analogous to the proof of Proposition 7.1.2, for $j = 0$, i.e. $\lambda_j = 0$, $r_j = -i\,\frac{n-1}{2}$ and $\psi_j(P) = \mathrm{vol}(\Gamma\backslash\mathbb{H}^n)^{-1/2}$, the function $b_j(s)\,\psi_j(P)$ has the form

$$b_0(s)\,\psi_0(P) = \frac{l_{(Q_1,Q_2)}\,\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{s-n+1}{2}\right)}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma\left(\frac{s}{2}\right)}.$$

Using (A.12), at $s = 0$ we have the Laurent expansions

$$\begin{aligned}
\frac{1}{\Gamma\left(\frac{s}{2}\right)} &= \frac{1}{2}\cdot s + \frac{\gamma}{4}\cdot s^2 + \mathrm{O}(s^3), \\
\Gamma\Big(\frac{s-n+1}{2}\Big) &= \frac{2\,(-1)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!}\cdot\frac{1}{s} + \frac{(-1)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!}\left(\sum_{l=1}^{\frac{n-1}{2}}\frac{1}{l} - \gamma\right) + \mathrm{O}(s),
\end{aligned}$$

as now $\frac{n-1}{2} \in \mathbb{N}_0$. Consequently, the function $b_0(s)\,\psi_0(P)$ admits a Laurent expansion at $s=0$ of the form

$$b_0(s)\,\psi_0(P) = \frac{l_{(Q_1,Q_2)}\,(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{l_{(Q_1,Q_2)}\,(-\pi)^{\frac{n-1}{2}}}{2\left(\frac{n-1}{2}\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\left(\sum_{l=1}^{\frac{n-1}{2}}\frac{1}{l}\right)\cdot s + \mathrm{O}(s^2).$$

Now let $j \geq 1$, so that we either have $r_j > 0$ or $r_j \in \left(-i\,\frac{n-1}{2}, 0\right]$. In the first case $\frac{n-1}{4} \pm \frac{ir_j}{2} \notin \mathbb{N}_0$ holds true, while in the latter case $\frac{n-1}{4} \pm \frac{ir_j}{2} \in \mathbb{R}$ with $\frac{n-1}{4} - \frac{ir_j}{2} \in \left(0, \frac{n-1}{4}\right]$ and $\frac{n-1}{4} + \frac{ir_j}{2} \in \left[\frac{n-1}{4}, \frac{n-1}{2}\right)$. The interval $\left(0, \frac{n-1}{4}\right]$ contains the natural numbers $1, \ldots, \frac{n-3}{4}$, while the natural numbers contained in the interval $\left[\frac{n-1}{4}, \frac{n-1}{2}\right)$ are $\frac{n+1}{4}, \ldots, \frac{n-3}{2}$.
As in the proof of the previous proposition, this implies that the function $\Gamma\left(\frac{s-\frac{n-1}{2}+ir_j}{2}\right)$ has a pole at $s=0$ if and only if

$$\frac{n-1}{4} - \frac{ir_j}{2} \in \left\{1, \ldots, \frac{n-3}{4}\right\} \iff r_j \in \left\{-i\left(\frac{n-1}{2} - 2N\right) \,\Big|\, N \in \left\{1, \ldots, \frac{n-3}{4}\right\}\right\},$$

while the function $\Gamma\left(\frac{s-\frac{n-1}{2}-ir_j}{2}\right)$ has a pole at $s=0$ if and only if

$$\begin{aligned} &\frac{n-1}{4} + \frac{ir_j}{2} \in \left\{\frac{n+1}{4}, \ldots, \frac{n-3}{2}\right\} \\ &\iff r_j \in \left\{-i\left(2N - \frac{n-1}{2}\right) \,\Big|\, N \in \left\{\frac{n+1}{4}, \ldots, \frac{n-3}{2}\right\}\right\}. \end{aligned}$$

Moreover, if $\frac{n-1}{4} - \frac{ir_j}{2} = N$ for some $N \in \left\{1, \ldots, \frac{n-3}{4}\right\}$, then $\frac{n-1}{2} \in \mathbb{Z}$ yields that

$$\frac{n-1}{4} + \frac{ir_j}{2} = \frac{n-1}{2} - N \ \in \mathbb{Z} \quad \text{and} \quad \frac{n-1}{2} - N \in \left[\frac{n+1}{4}, \frac{n-3}{2}\right],$$

so also $\frac{n-1}{4} + \frac{ir_j}{2} \in \left\{\frac{n+1}{4}, \ldots, \frac{n-3}{2}\right\}$. Conversely, if we have $\frac{n-1}{4} + \frac{ir_j}{2} = N$ for some $N \in \left\{\frac{n+1}{4}, \ldots, \frac{n-3}{2}\right\}$, then $\frac{n-1}{2} \in \mathbb{Z}$ implies that

$$\frac{n-1}{4} - \frac{ir_j}{2} = \frac{n-1}{2} - N \ \in \mathbb{Z} \quad \text{and} \quad \frac{n-1}{2} - N \in \left[1, \frac{n-3}{4}\right],$$

so also $\frac{n-1}{4} - \frac{ir_j}{2} \in \left\{1, \ldots, \frac{n-3}{4}\right\}$ holds true. This proves that for any $j \geq 1$ either both of the two functions $\Gamma\left(\frac{s-\frac{n-1}{2}+ir_j}{2}\right)$, $\Gamma\left(\frac{s-\frac{n-1}{2}-ir_j}{2}\right)$ have a pole at $s=0$, or none of them has. If we set

$$\begin{aligned} M_1(n) &:= \left\{-i\left(\frac{n-1}{2} - 2N\right) \,\Big|\, N \in \left\{1, \ldots, \frac{n-3}{4}\right\}\right\}, \\ M_2(n) &:= \left\{-i\left(2N - \frac{n-1}{2}\right) \,\Big|\, N \in \left\{\frac{n+1}{4}, \ldots, \frac{n-3}{2}\right\}\right\}, \end{aligned}$$

the considerations above show that $M_1(n) = M_2(n) =: M(n)$.

We recall that at $s=0$ the function $\Gamma\left(\frac{s}{2}\right)^{-2}$ has the Laurent expansion

$$\frac{1}{\Gamma\left(\frac{s}{2}\right)^2} = \frac{1}{4}\cdot s^2 + \frac{\gamma}{4}\cdot s^3 + \mathrm{O}(s^4).$$

If $r_j \in M(n)$, then, using (A.12), at $s=0$ we have the Laurent expansions

$$\Gamma\Big(\frac{s-\frac{n-1}{2} \pm ir_j}{2}\Big) = \frac{2\,(-1)^{\frac{n-1}{4}\mp\frac{ir_j}{2}}}{\left(\frac{n-1}{4} \mp \frac{ir_j}{2}\right)!}\cdot\frac{1}{s} + \frac{(-1)^{\frac{n-1}{4}\mp\frac{ir_j}{2}}}{\left(\frac{n-1}{4} \mp \frac{ir_j}{2}\right)!}\left(\sum_{l=1}^{\frac{n-1}{4}\mp\frac{ir_j}{2}}\frac{1}{l} - \gamma\right) + \mathrm{O}(s).$$

This yields that for $j \geq 1$ with $r_j \in M(n)$ the function $b_j(s)\,\psi_j(P)$ admits a Laurent expansion at $s = 0$ of the form

$$
\begin{aligned}
b_j(s)\,\psi_j(P) = {} & \frac{(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!\left(\frac{n-1}{4}-\frac{ir_j}{2}\right)!}\,\psi_j(P)\,C_{j,(Q_1,Q_2)} \\
& + \frac{(-\pi)^{\frac{n-1}{2}}}{2\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!\left(\frac{n-1}{4}-\frac{ir_j}{2}\right)!}\left(\sum_{l=1}^{\frac{n-1}{4}+\frac{ir_j}{2}}\frac{1}{l}+\sum_{l=1}^{\frac{n-1}{4}-\frac{ir_j}{2}}\frac{1}{l}\right) \\
& \cdot \psi_j(P)\,C_{j,(Q_1,Q_2)}\cdot s + \mathrm{O}(s^2),
\end{aligned}
$$

where again

$$
C_{j,(Q_1,Q_2)} := \int_{L_{(Q_1,Q_2)}} \overline{\psi_j}(Q)\; ds_{\mathbb{H}^n}(Q).
$$

Thus, the respective part of the series in (7.10) arising from the discrete spectrum has a Laurent expansion at $s = 0$ of the form

$$
\sum_{\substack{j\in\mathbb{N}:\\ r_j\in M(n)}} b_j(s)\,\psi_j(P) = F_{n,(Q_1,Q_2)}(P) + G_{n,(Q_1,Q_2)}(P)\cdot s + \mathrm{O}(s^2),
$$

where

$$
F_{n,(Q_1,Q_2)}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M(n)}} \frac{(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!\left(\frac{n-1}{4}-\frac{ir_j}{2}\right)!}\,\psi_j(P)\,C_{j,(Q_1,Q_2)}, \tag{7.11}
$$

$$
\begin{aligned}
G_{n,(Q_1,Q_2)}(P) := {} & \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M(n)}} \frac{(-\pi)^{\frac{n-1}{2}}}{2\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!\left(\frac{n-1}{4}-\frac{ir_j}{2}\right)!}\left(\sum_{l=1}^{\frac{n-1}{4}+\frac{ir_j}{2}}\frac{1}{l}+\sum_{l=1}^{\frac{n-1}{4}-\frac{ir_j}{2}}\frac{1}{l}\right) \\
& \cdot \psi_j(P)\,C_{j,(Q_1,Q_2)}.
\end{aligned} \tag{7.12}
$$

If $r_j \notin M(n)$ holds true, then (A.13) yields that at $s = 0$ we have the Laurent expansions

$$
\Gamma\Big(\frac{s-\frac{n-1}{2}\pm ir_j}{2}\Big) = \Gamma\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}\Big) + \frac{1}{2}\,\Gamma\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi^{(0)}\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}\Big)\cdot s + \mathrm{O}(s^2).
$$

This implies that for $j \geq 1$ with $r_j \notin M(n)$ the function $b_j(s)\,\psi_j(P)$ admits a Laurent expansion at $s = 0$ of the form

$$
b_j(s)\,\psi_j(P) = \frac{\pi^{\frac{n-1}{2}}}{4}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi_j(P)\,C_{j,(Q_1,Q_2)}\cdot s^2 + \mathrm{O}(s^3).
$$

Hence, the respective part of the series in (7.10) arising from the discrete spectrum has a Laurent expansion at $s = 0$ of the form

$$
\sum_{\substack{j\in\mathbb{N}:\\ r_j\notin M(n)}} b_j(s)\,\psi_j(P) = \mathrm{O}(s^2).
$$

Further, as in the proof of Proposition 7.1.2, we have $\frac{n-1}{4} \notin \mathbb{N}_0$ and the continuous part in (7.10) admits a Laurent expansion at $s = 0$ of the form

$$
\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} b_{t,\eta_k}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt = \mathrm{O}(s^2).
$$

Putting everything together, for $P \in \mathbb{H}^n$ we obtain a Laurent expansion at $s = 0$ of the form

$$
\begin{aligned}
&E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) - \mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),2}(P,s) \\
&= \frac{l_{(Q_1,Q_2)}\,(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + F_{n,(Q_1,Q_2)}(P) + \left(\frac{l_{(Q_1,Q_2)}\,(-\pi)^{\frac{n-1}{2}}}{2\left(\frac{n-1}{2}\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\sum_{l=1}^{\frac{n-1}{2}}\frac{1}{l} + G_{n,(Q_1,Q_2)}(P)\right)\cdot s + \mathrm{O}(s^2),
\end{aligned}
$$

where $F_{n,(Q_1,Q_2)}(P)$ and $G_{n,(Q_1,Q_2)}(P)$ are given by (7.11) and (7.12), respectively.

Since the eigenfunctions $\psi_j(P)$ $(j \in \mathbb{N})$ are invariant under the action of Γ, the same is true for the functions $F_{n,(Q_1,Q_2)}(P)$ and $G_{n,(Q_1,Q_2)}(P)$.

$\square$

Remark 7.1.5. In the case $n = 3$ the Laurent expansion of the hyperbolic Eisenstein series in Proposition 7.1.4 simplifies significantly:
Since for $n = 3$ there is no $N \in \mathbb{N}$ with $1 \leq N \leq \frac{n-3}{4}$ and no $N \in \mathbb{N}$ with $\frac{n+1}{4} \leq N \leq \frac{n-3}{2}$, the set

$$
\begin{aligned}
M(n) &= \left\{ -i\left(\frac{n-1}{2} - 2N\right) \;\middle|\; N \in \left\{1,\ldots,\frac{n-3}{4}\right\}\right\} \\
&= \left\{ -i\left(2N - \frac{n-1}{2}\right) \;\middle|\; N \in \left\{\frac{n+1}{4},\ldots,\frac{n-3}{2}\right\}\right\},
\end{aligned}
$$

in the proof of Proposition 7.1.4 is empty. Thus, we have $F_{n,(Q_1,Q_2)}(P) = 0$ and $G_{n,(Q_1,Q_2)}(P) = 0$ as the sums defining these functions are empty. This shows that the functions $F_{n,(Q_1,Q_2)}(P)$ and $G_{n,(Q_1,Q_2)}(P)$ in the Laurent expansion (7.9) do not appear for $n = 3$.

Now we treat the remaining case $n \equiv 1 \mod 4$.

Proposition 7.1.6. *Let $n \equiv 1 \mod 4$. For $P \in \mathbb{H}^n$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits a Laurent expansion at $s = 0$ of the form*

$$
\begin{aligned}
&E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) - \mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),3}(P,s) \\
&= \frac{l_{(Q_1,Q_2)}\,(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + F_{n,(Q_1,Q_2)}(P) + \left(\frac{l_{(Q_1,Q_2)}\,(-\pi)^{\frac{n-1}{2}}}{2\left(\frac{n-1}{2}\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\sum_{l=1}^{\frac{n-1}{2}}\frac{1}{l} + \sum_{l=1}^{2} G_{n,(Q_1,Q_2),l}(P)\right)\cdot s \\
&\quad + \mathrm{O}(s^2), \qquad (7.13)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),3}(P,s) := \frac{2\,\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\sum_{l=0}^{\frac{n-5}{4}}\frac{(-1)^l}{l!}\,\Gamma\Big(s - \frac{n-1}{2} + l\Big) \\
\cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(P, s+2l)\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2l)\, ds_{\mathbb{H}^n}(Q),
\end{aligned}
$$

and where the functions $F_{n,(Q_1,Q_2)}(P)$ and $G_{n,(Q_1,Q_2),l}(P)$ $(l = 1,2)$ are invariant under the action of Γ, and are given by the formulas (7.15), (7.16) and (7.17) in the proof, respectively.

Proof. In the proof of Theorem 6.2.1 we have seen that for $P \in \mathbb{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) = \frac{n-1}{2} - 2m$ $(m \in \mathbb{N}_0)$ the meromorphic continuation of the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$

is given by

$$\begin{aligned} E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = {}& \sum_{j=0}^{\infty} b_j(s)\,\psi_j(P) + \frac{\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma\left(\frac{s}{2}\right)^2} \sum_{k=1}^{c_\Gamma} \int_{W_{y,\varepsilon}} \Gamma\Big(\frac{s-n+1+w}{2}\Big)\,\Gamma\Big(\frac{s-w}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}(P,w) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot\, C_{\eta_k,(Q_1,Q_2)}(n-1-w)\;dw \\ & + \frac{2\,\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{m-1} \frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big) \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(P,s+2l)\,C_{\eta_k,(Q_1,Q_2)}(n-1-s-2l) \\ & + \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\frac{(-1)^m}{m!}\,\Gamma\Big(s-\frac{n-1}{2}+m\Big) \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(P,s+2m)\,C_{\eta_k,(Q_1,Q_2)}(n-1-s-2m), \end{aligned}$$

where for $z \in \mathbb{C}$ we have set

$$C_{\eta_k,(Q_1,Q_2)}(z) := \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q,z)\;ds_{\mathbb{H}^n}(Q).$$

Here $W_{y,\varepsilon}$ denotes the following piecewise linear path: the vertical line from $\frac{n-1}{2} - i\infty$ to $\frac{n-1}{2} - iy$, the horizontal line segment from $\frac{n-1}{2} - iy$ to $\frac{n-1}{2} + \varepsilon - iy$, the vertical line segment from $\frac{n-1}{2} + \varepsilon - iy$ to $\frac{n-1}{2} + \varepsilon + iy$, the horizontal line segment from $\frac{n-1}{2} + \varepsilon + iy$ to $\frac{n-1}{2} + iy$, and the vertical line from $\frac{n-1}{2} + iy$ to $\frac{n-1}{2} + i\infty$, where $\varepsilon \in (0,1)$ is chosen sufficiently small such that all parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,s)$ $(k = 1, \ldots, c_\Gamma)$ have no poles in the strip $\frac{n-1}{2} - 2\varepsilon < \mathrm{Re}(s) < \frac{n-1}{2} + 2\varepsilon$, and y is chosen sufficiently large such that $y > |\mathrm{Im}(s)|$. Moreover, the coefficient $b_j(s)$ is given by

$$b_j(s) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-ir_j}{2}\Big) \int_{L_{(Q_1,Q_2)}} \overline{\psi_j}(Q)\;ds_{\mathbb{H}^n}(Q).$$

Substituting $w = \frac{n-1}{2} + it$ in the integral, the above meromorphic continuation on the line $\mathrm{Re}(s) = \frac{n-1}{2} - 2m$ $(m \in \mathbb{N}_0)$ can be written as

$$\begin{aligned} E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) = {}& \sum_{j=0}^{\infty} b_j(s)\,\psi_j(P) + \frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{W'_{y,\varepsilon}} b_{t,\eta_k}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\;dt \\ & + \frac{2\,\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{l=0}^{m-1} \frac{(-1)^l}{l!}\,\Gamma\Big(s-\frac{n-1}{2}+l\Big) \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(P,s+2l)\,C_{\eta_k,(Q_1,Q_2)}(n-1-s-2l) \\ & + \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\frac{(-1)^m}{m!}\,\Gamma\Big(s-\frac{n-1}{2}+m\Big) \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(P,s+2m)\,C_{\eta_k,(Q_1,Q_2)}(n-1-s-2m), \end{aligned}$$

where $W'_{y,\varepsilon}$ denotes the following piecewise linear path: the horizontal line from $-\infty$ to $-y$, the vertical line segment from $-y$ to $-y - i\varepsilon$, the horizontal line segment from $-y - i\varepsilon$ to $y - i\varepsilon$, the vertical line segment from $y - i\varepsilon$ to y, and the horizontal line from y to ∞, and where the coefficient $b_{t,\eta_k}(s)$ is given by

$$b_{t,\eta_k}(s) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\Gamma\Big(\frac{s-\frac{n-1}{2}+it}{2}\Big)\,\Gamma\Big(\frac{s-\frac{n-1}{2}-it}{2}\Big) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-it\Big)\;ds_{\mathbb{H}^n}(Q).$$

The assumption $n \equiv 1 \mod 4$ yields $\frac{n-1}{4} \in \mathbb{N}$, so particularly on the line $\mathrm{Re}(s) = \frac{n-1}{2} - 2 \cdot \frac{n-1}{4} = 0$ the identity

$$\begin{aligned} & E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) - \mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),3}(P,s) \\ & \quad = \sum_{j=0}^{\infty} b_j(s)\,\psi_j(P) + \frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{W'_{y,\varepsilon}} b_{t,\eta_k}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\;dt \\ & \qquad + \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\,\frac{(-1)^{\frac{n-1}{4}}}{\left(\frac{n-1}{4}\right)!}\,\Gamma\Big(s-\frac{n-1}{4}\Big) \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+s\Big)\,C_{\eta_k,(Q_1,Q_2)}\Big(\frac{n-1}{2}-s\Big) \end{aligned} \tag{7.14}$$

holds true. Thus, to derive the Laurent expansion at $s = 0$ we work from formula (7.14).

As in the proof of Proposition 7.1.4, for $j = 0$, i.e. $\lambda_j = 0$, $r_j = -i\,\frac{n-1}{2}$ and $\psi_j(P) = \mathrm{vol}(\Gamma\backslash\mathbb{H}^n)^{-1/2}$, the function

$$b_0(s)\,\psi_0(P) = \frac{l_{(Q_1,Q_2)}\,\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{s-n+1}{2}\right)}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma\left(\frac{s}{2}\right)}$$

admits a Laurent expansion at $s = 0$ of the form

$$b_0(s)\,\psi_0(P) = \frac{l_{(Q_1,Q_2)}\,(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{l_{(Q_1,Q_2)}\,(-\pi)^{\frac{n-1}{2}}}{2\left(\frac{n-1}{2}\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\left(\sum_{l=1}^{\frac{n-1}{2}}\frac{1}{l}\right)\cdot s + \mathrm{O}(s^2)$$

since we have $\frac{n-1}{2} \in \mathbb{N}_0$.

Now we let $j \geq 1$, so either $r_j > 0$ or $r_j \in \left(-i\,\frac{n-1}{2}, 0\right]$ holds true. In the first case clearly $\frac{n-1}{4} \pm \frac{ir_j}{2} \notin \mathbb{N}_0$, while in the second case we have $\frac{n-1}{4} \pm \frac{ir_j}{2} \in \mathbb{R}$ with $\frac{n-1}{4} - \frac{ir_j}{2} \in \left(0, \frac{n-1}{4}\right]$ and $\frac{n-1}{4} + \frac{ir_j}{2} \in \left[\frac{n-1}{4}, \frac{n-1}{2}\right)$. The natural numbers contained in the interval $\left(0, \frac{n-1}{4}\right]$ are $1, \ldots, \frac{n-1}{4}$, and the interval $\left[\frac{n-1}{4}, \frac{n-1}{2}\right)$ contains the natural numbers $\frac{n-1}{4}, \ldots, \frac{n-3}{2}$.
Analogous to the proofs of the previous two propositions we see that the function $\Gamma\left(\frac{s-\frac{n-1}{2}+ir_j}{2}\right)$ has a pole at $s = 0$ if and only if

$$\frac{n-1}{4} - \frac{ir_j}{2} \in \left\{1, \ldots, \frac{n-1}{4}\right\} \iff r_j \in \left\{-i\left(\frac{n-1}{2} - 2N\right) \;\Big|\; N \in \left\{1, \ldots, \frac{n-1}{4}\right\}\right\},$$

while the function $\Gamma\left(\frac{s-\frac{n-1}{2}-ir_j}{2}\right)$ has a pole at $s = 0$ if and only if

$$\begin{aligned}&\frac{n-1}{4} + \frac{ir_j}{2} \in \left\{\frac{n-1}{4}, \ldots, \frac{n-3}{2}\right\}\\ &\quad\iff r_j \in \left\{-i\left(2N - \frac{n-1}{2}\right) \;\Big|\; N \in \left\{\frac{n-1}{4}, \ldots, \frac{n-3}{2}\right\}\right\}.\end{aligned}$$

If $\frac{n-1}{4} - \frac{ir_j}{2} = N$ for some $N \in \left\{1, \ldots, \frac{n-1}{4}\right\}$, then from $\frac{n-1}{2} \in \mathbb{Z}$ we derive that

$$\frac{n-1}{4} + \frac{ir_j}{2} = \frac{n-1}{2} - N \;\in \mathbb{Z} \quad \text{and} \quad \frac{n-1}{2} - N \in \left[\frac{n-1}{4}, \frac{n-3}{2}\right],$$

so that also $\frac{n-1}{4} + \frac{ir_j}{2} \in \left\{\frac{n-1}{4}, \ldots, \frac{n-3}{2}\right\}$ holds true. Conversely, if $\frac{n-1}{4} + \frac{ir_j}{2} = N$ for some $N \in \left\{\frac{n-1}{4}, \ldots, \frac{n-3}{2}\right\}$, then $\frac{n-1}{2} \in \mathbb{Z}$ gives us that

$$\frac{n-1}{4} - \frac{ir_j}{2} = \frac{n-1}{2} - N \;\in \mathbb{Z} \quad \text{and} \quad \frac{n-1}{2} - N \in \left[1, \frac{n-1}{4}\right],$$

so that also $\frac{n-1}{4} - \frac{ir_j}{2} \in \left\{1, \ldots, \frac{n-1}{4}\right\}$. From this we can conclude that for any $j \geq 1$ either both of the two functions $\Gamma\left(\frac{s-\frac{n-1}{2}+ir_j}{2}\right)$, $\Gamma\left(\frac{s-\frac{n-1}{2}-ir_j}{2}\right)$ have a pole at $s = 0$, or none of them has. Setting

$$\begin{aligned}M_1(n) &:= \left\{-i\left(\frac{n-1}{2} - 2N\right) \;\Big|\; N \in \left\{1, \ldots, \frac{n-1}{4}\right\}\right\},\\ M_2(n) &:= \left\{-i\left(2N - \frac{n-1}{2}\right) \;\Big|\; N \in \left\{\frac{n-1}{4}, \ldots, \frac{n-3}{2}\right\}\right\},\end{aligned}$$

we have just proven that $M_1(n) = M_2(n) =: M(n)$.

As in the proof of Proposition 7.1.4, at $s = 0$ we find the Laurent expansion

$$\sum_{\substack{j\in\mathbb{N}:\\ r_j\in M(n)}} b_j(s)\,\psi_j(P) = F_{n,(Q_1,Q_2)}(P) + G_{n,(Q_1,Q_2),1}(P)\cdot s + \mathrm{O}(s^2),$$

where

$$F_{n,(Q_1,Q_2)}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M(n)}} \frac{(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!\left(\frac{n-1}{4}-\frac{ir_j}{2}\right)!}\,\psi_j(P)\,C_{j,(Q_1,Q_2)}, \tag{7.15}$$

$$G_{n,(Q_1,Q_2),1}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M(n)}} \frac{(-\pi)^{\frac{n-1}{2}}}{2\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!\left(\frac{n-1}{4}-\frac{ir_j}{2}\right)!}\left(\sum_{l=1}^{\frac{n-1}{4}+\frac{ir_j}{2}}\frac{1}{l}+\sum_{l=1}^{\frac{n-1}{4}-\frac{ir_j}{2}}\frac{1}{l}\right)\cdot\psi_j(P)\,C_{j,(Q_1,Q_2)} \tag{7.16}$$

with

$$C_{j,(Q_1,Q_2)} := \int_{L_{(Q_1,Q_2)}} \overline{\psi_j}(Q)\;ds_{\mathbb{H}^n}(Q),$$

as well as the Laurent expansion

$$\sum_{\substack{j\in\mathbb{N}:\\ r_j\notin M(n)}} b_j(s)\,\psi_j(P) = \mathrm{O}(s^2).$$

For any $t\in W'_{y,\varepsilon}$ with $\mathrm{Re}(t)\neq 0$ we have $\frac{n-1}{4}\pm it\notin\mathbb{R}$, so that in particular $\frac{n-1}{4}\pm it\notin\mathbb{N}_0$. Further, the only $t_0\in W'_{y,\varepsilon}$ with $\mathrm{Re}(t_0)=0$ is $t_0=-i\varepsilon$. Using $\frac{n-1}{4}\in\mathbb{N}_0$ and $\varepsilon\in(0,1)$, we also have $\frac{n-1}{4}\pm it_0=\frac{n-1}{4}\pm\varepsilon\notin\mathbb{N}_0$. Together with (A.13) this implies that for any $t\in W'_{y,\varepsilon}$ at $s=0$ we have the Laurent expansions

$$\frac{1}{\Gamma\left(\frac{s}{2}\right)^2} = \frac{1}{4}\cdot s^2+\frac{\gamma}{4}\cdot s^3+\mathrm{O}(s^4),$$

$$\Gamma\Big(\frac{s-\frac{n-1}{2}\pm it}{2}\Big) = \Gamma\Big(\pm\frac{it}{2}-\frac{n-1}{4}\Big)+\frac{1}{2}\Gamma\Big(\pm\frac{it}{2}-\frac{n-1}{4}\Big)\psi^{(0)}\Big(\pm\frac{it}{2}-\frac{n-1}{4}\Big)\cdot s+\mathrm{O}(s^2).$$

Thus, for $k=1,\ldots,c_\Gamma$ and any $t\in W'_{y,\varepsilon}$ the function $b_{t,\eta_k}(s)$ in the integral in (7.14) arising from the continuous spectrum admits a Laurent expansion at $s=0$ of the form

$$b_{t,\eta_k}(s) = \frac{\pi^{\frac{n-1}{2}}}{4}\,\Gamma\Big(\frac{it}{2}-\frac{n-1}{4}\Big)\,\Gamma\Big(-\frac{it}{2}-\frac{n-1}{4}\Big)\,C_{\eta_k,(Q_1,Q_2)}\Big(\frac{n-1}{2}-it\Big)\cdot s^2+\mathrm{O}(s^3).$$

This shows that at $s=0$ we have the Laurent expansion

$$\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{W'_{y,\varepsilon}} b_{t,\eta_k}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\;dt = \mathrm{O}(s^2).$$

Moreover, the fact that $\frac{n-1}{4}\in\mathbb{N}_0$ yields that at $s=0$ we have the Laurent expansion

$$\Gamma\Big(s-\frac{n-1}{4}\Big) = \frac{(-1)^{\frac{n-1}{4}}}{\left(\frac{n-1}{4}\right)!}\cdot\frac{1}{s}+\frac{(-1)^{\frac{n-1}{4}}}{\left(\frac{n-1}{4}\right)!}\left(\sum_{l=1}^{\frac{n-1}{4}}\frac{1}{l}-\gamma\right)+\mathrm{O}(s).$$

Since the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,w)$ $(k=1,\ldots,c_\Gamma)$ have no poles on the line $\mathrm{Re}(w)=\frac{n-1}{2}$, for $k=1,\ldots,c_\Gamma$ we find the Laurent expansions

$$E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+s\Big) = E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}\Big)+\mathrm{O}(s),$$

$$C_{\eta_k,(Q_1,Q_2)}\Big(\frac{n-1}{2}-s\Big) = C_{\eta_k,(Q_1,Q_2)}\Big(\frac{n-1}{2}\Big)+\mathrm{O}(s)$$

at $s=0$. Together this gives us that the last summand on the right-hand side of (7.14) admits a Laurent expansion at $s=0$ of the form

$$\frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{s}{2}\right)^2}\frac{(-1)^{\frac{n-1}{4}}}{\left(\frac{n-1}{4}\right)!}\Gamma\Big(s-\frac{n-1}{4}\Big)\sum_{k=1}^{c_\Gamma}E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+s\Big)\,C_{\eta_k,(Q_1,Q_2)}\Big(\frac{n-1}{2}-s\Big)$$
$$=G_{n,(Q_1,Q_2),2}(P)\cdot s+\mathrm{O}(s^2),$$

where

$$G_{n,(Q_1,Q_2),2}(P)=\frac{(-\pi)^{\frac{n-1}{2}}}{4\left(\left(\frac{n-1}{4}\right)!\right)^2}\sum_{k=1}^{c_\Gamma}E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}\Big)\,C_{\eta_k,(Q_1,Q_2)}\Big(\frac{n-1}{2}\Big). \tag{7.17}$$

Overall, for $P\in\mathbb{H}^n$ we obtain a Laurent expansion at $s=0$ of the form

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)-\mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),3}(P,s)$$
$$=\frac{l_{(Q_1,Q_2)}\,(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}+F_{n,(Q_1,Q_2)}(P)+\left(\frac{l_{(Q_1,Q_2)}\,(-\pi)^{\frac{n-1}{2}}}{2\left(\frac{n-1}{2}\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\sum_{l=1}^{\frac{n-1}{2}}\frac{1}{l}+\sum_{l=1}^{2}G_{n,(Q_1,Q_2),l}(P)\right)\cdot s$$
$$+\mathrm{O}(s^2),$$

where $F_{n,(Q_1,Q_2)}(P)$ and $G_{n,(Q_1,Q_2),l}(P)$ $(l=1,2)$ are given by (7.15), (7.16) and (7.17), respectively.

The eigenfunctions $\psi_j(P)$ $(j\in\mathbb{N})$ and the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,s)$ $(k=1,\ldots,c_\Gamma)$ are invariant under the action of Γ. Hence, also the functions $F_{n,(Q_1,Q_2)}(P)$ and $G_{n,(Q_1,Q_2),l}(P)$ $(l=1,2)$ are Γ-invariant.

□

Remark 7.1.7. The functions $\mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),1}(P,s)$, $\mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),2}(P,s)$ and $\mathcal{G}^{\mathrm{par}}_{(Q_1,Q_2),3}(P,s)$ that are subtracted from the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ in the Laurent expansions (7.1), (7.9) and (7.13), respectively, differ only by the upper limit of summation in the sum over l. This is because the explicit formulas for the meromorphic continuation of $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ to the point $s=0$ slightly vary in the different cases for n.

7.2. Example 1: The case $n=2$, $\Gamma=\mathrm{PSL}_2(\mathbb{Z})$

In this section we first give a Laurent expansion at $s=0$ of the hyperbolic Eisenstein series on the upper half-plane $\mathbb{H}$ for a general Fuchsian subgroup $\Gamma\subseteq\mathrm{PSL}_2(\mathbb{R})$ of the first kind. Then we consider the case $\Gamma=\mathrm{PSL}_2(\mathbb{Z})$ and use the Laurent expansions of the parabolic Eisenstein series associated to the cusp ∞ at the points $s=0$ and $s=1$ to derive a Kronecker limit formula for the hyperbolic Eisenstein series for $\mathrm{PSL}_2(\mathbb{Z})$.
Throughout the section we let $n=2$. Let $\Gamma\subseteq\mathrm{PSL}_2(\mathbb{R})$ be a discrete and cofinite subgroup, i.e. a Fuchsian subgroup of the first kind. Further, let $(Q_1,Q_2)\in H_\Gamma$ be a pair of hyperbolic fixed points with hyperbolic scaling matrix $\sigma_{(Q_1,Q_2)}\in\mathrm{PSL}_2(\mathbb{R})$ and hyperbolic stabilizer subgroup $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$. Then $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$ agrees with the full stabilizer subgroup $\Gamma_{(Q_1,Q_2)}$ of (Q_1,Q_2). Let $\mathcal{L}_{(Q_1,Q_2)}$ be the unique geodesic in $\mathbb{H}$ connecting Q_1 and Q_2, and let $L_{(Q_1,Q_2)}=\pi_{\Gamma_{(Q_1,Q_2)}}(\mathcal{L}_{(Q_1,Q_2)})$ denote its image under the natural projection $\pi_{\Gamma_{(Q_1,Q_2)}}:\mathbb{H}\to\Gamma_{(Q_1,Q_2)}\backslash\mathbb{H}$, which is a closed geodesic in $\Gamma_{(Q_1,Q_2)}\backslash\mathbb{H}$ of hyperbolic length $l_{(Q_1,Q_2)}$.

Notation 7.2.1. To keep the notation simple, in this section we again omit the index 2 and write $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s)$ for the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{2,(Q_1,Q_2)}(z,s)$ associated to the pair $(Q_1,Q_2)\in H_\Gamma$ of hyperbolic fixed points, and $E^{\mathrm{par}}_{\eta_k}(z,s)$ for the parabolic Eisenstein series $E^{\mathrm{par}}_{2,\eta_k}(z,s)$ associated to the cusp $\eta_k\in C_\Gamma$ $(k=1,\ldots,c_\Gamma)$.

Before we consider the specific case of the modular group $\mathrm{PSL}_2(\mathbb{Z})$, we first state a consequence of Proposition 7.1.2 for a general Fuchsian subgroup of the first kind.

Proposition 7.2.2. *For $z \in \mathbb{H}$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s)$ admits a Laurent expansion at $s = 0$ of the form*

$$\begin{aligned}E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s) &- \frac{2\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)^2}\sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(z,s)\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(w,1-s)\,ds_{\mathbb{H}}(w)\\ &= -\frac{l_{(Q_1,Q_2)}\,\pi}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\cdot s + \left(-\frac{l_{(Q_1,Q_2)}\,\pi\,(1-\log(2))}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \sum_{l=3}^{4} H_{2,(Q_1,Q_2),l}(z)\right)\cdot s^2 + \mathrm{O}(s^3),\end{aligned}$$

where the functions $H_{2,(Q_1,Q_2),l}(z)$ $(l = 3,4)$ are invariant under the action of Γ, and are given by the formulas (7.18) *and* (7.19) *in the proof, respectively. Moreover, for $z \in \mathbb{H}$ they satisfy the differential equation*

$$\Delta_{\mathbb{H}}\left(\sum_{l=3}^{4} H_{2,(Q_1,Q_2),l}(z)\right) = -\frac{l_{(Q_1,Q_2)}\,\pi}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} + E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,2).$$

Proof. From Proposition 7.1.2 we can deduce that for $z \in \mathbb{H}$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s)$ admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned}E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s) &- \frac{2\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)^2}\sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(z,s)\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(w,1-s)\,ds_{\mathbb{H}}(w)\\ = &\left(-\frac{l_{(Q_1,Q_2)}\,\pi}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \sum_{l=1}^{2} G_{2,(Q_1,Q_2),l}(z)\right)\cdot s + \left(-\frac{l_{(Q_1,Q_2)}\,\pi\,(1-\log(2))}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \sum_{l=1}^{4} H_{2,(Q_1,Q_2),l}(z)\right)\cdot s^2\\ &+ \mathrm{O}(s^3),\end{aligned}$$

where we used the special values $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$ and $\psi^{(0)}\left(-\frac{1}{2}\right) = 2 - \gamma - \log(4)$ (which follows from (A.10) and the recursion formula (A.11)). Further, in Remark 7.1.3 we have seen that the functions $G_{2,(Q_1,Q_2),l}(z)$ $(l = 1,2)$ and $H_{2,(Q_1,Q_2),l}(z)$ $(l = 1,2)$ vanish identically as the sums defining these functions are all empty. Consequently, we have a Laurent expansion at $s = 0$ of the form

$$\begin{aligned}E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s) &- \frac{2\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)^2}\sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(z,s)\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(w,1-s)\,ds_{\mathbb{H}}(w)\\ &= -\frac{l_{(Q_1,Q_2)}\,\pi}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\cdot s + \left(-\frac{l_{(Q_1,Q_2)}\,\pi\,(1-\log(2))}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \sum_{l=3}^{4} H_{2,(Q_1,Q_2),l}(z)\right)\cdot s^2 + \mathrm{O}(s^3),\end{aligned}$$

where the functions $H_{2,(Q_1,Q_2),l}(z)$ $(l = 3,4)$ are given by

$$H_{2,(Q_1,Q_2),3}(z) = \frac{\sqrt{\pi}}{4}\sum_{j=1}^{\infty}\Gamma\Big(\frac{ir_j}{2}-\frac{1}{4}\Big)\Gamma\Big(-\frac{ir_j}{2}-\frac{1}{4}\Big)\,\psi_j(z)\int_{L_{(Q_1,Q_2)}}\overline{\psi_j}(w)\,ds_{\mathbb{H}}(w), \tag{7.18}$$

$$\begin{aligned}H_{2,(Q_1,Q_2),4}(z) = \frac{1}{16\sqrt{\pi}}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty}&\Big|\Gamma\Big(\frac{it}{2}-\frac{1}{4}\Big)\Big|^2 E^{\mathrm{par}}_{\eta_k}\Big(z,\frac{1}{2}+it\Big)\\ &\cdot\left(\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}\Big(w,\frac{1}{2}-it\Big)\,ds_{\mathbb{H}}(w)\right)dt,\end{aligned} \tag{7.19}$$

respectively, and are invariant under the action of Γ. It remains to prove the claimed differential equation.

We write the above Laurent expansion at $s = 0$ as

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s) - \frac{2\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(z,s) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(w,1-s)\, ds_{\mathbb{H}}(w) = \sum_{r=1}^{\infty} c_{(Q_1,Q_2),r}(z) \cdot s^r \tag{7.20}$$

with

$$c_{(Q_1,Q_2),1}(z) = -\frac{l_{(Q_1,Q_2)}\,\pi}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}, \qquad c_{(Q_1,Q_2),2}(z) = -\frac{l_{(Q_1,Q_2)}\,\pi\,(1-\log(2))}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \sum_{l=3}^{4} H_{2,(Q_1,Q_2),l}(z).$$

Moreover, for $z \in \mathbb{H}$ the function $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s+2)$ is holomorphic at $s = 0$ and non-vanishing by the definition of the series. Thus, we have a Laurent expansion at $s = 0$ of the form

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s+2) = \sum_{r=0}^{\infty} d_{(Q_1,Q_2),r}(z) \cdot s^r \tag{7.21}$$

with $d_{(Q_1,Q_2),0}(z) = E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,2) \neq 0$. Making use of the differential equations

$$(\Delta_{\mathbb{H}} - s(1-s))\, E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s) = s^2\, E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s+2)$$

and

$$(\Delta_{\mathbb{H}} - s(1-s))\, E^{\mathrm{par}}_{\eta_k}(z,s) = 0 \qquad (k = 1,\ldots,c_\Gamma),$$

we obtain

$$\begin{aligned}(\Delta_{\mathbb{H}} - s(1-s)) &\left(E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s) - \frac{2\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(z,s) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(w,1-s)\, ds_{\mathbb{H}}(w) \right) \\ &= s^2\, E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s+2).\end{aligned}$$

Substituting the Laurent expansions (7.20) and (7.21) into both sides of this equality yields the identities

$$\sum_{r=1}^{\infty} \Delta_{\mathbb{H}}\, c_{(Q_1,Q_2),r}(z) \cdot s^r - \sum_{r=1}^{\infty} s(1-s)\, c_{(Q_1,Q_2),r}(z) \cdot s^r = \sum_{r=0}^{\infty} s^2\, d_{(Q_1,Q_2),r}(z) \cdot s^r$$

and

$$\sum_{r=1}^{\infty} \Delta_{\mathbb{H}}\, c_{(Q_1,Q_2),r}(z) \cdot s^r = \sum_{r=2}^{\infty} c_{(Q_1,Q_2),r-1}(z) \cdot s^r - \sum_{r=3}^{\infty} c_{(Q_1,Q_2),r-2}(z) \cdot s^r + \sum_{r=2}^{\infty} d_{(Q_1,Q_2),r-2}(z) \cdot s^r.$$

Comparing coefficients gives us the recurrence formula

$$\Delta_{\mathbb{H}}\, c_{(Q_1,Q_2),r}(z) = c_{(Q_1,Q_2),r-1}(z) - c_{(Q_1,Q_2),r-2}(z) + d_{(Q_1,Q_2),r-2}(z),$$

where $c_{(Q_1,Q_2),r}(z) = 0$ for $r < 1$ and $d_{(Q_1,Q_2),r}(z) = 0$ for $r < 0$. In particular, for $r = 2$ we get

$$\Delta_{\mathbb{H}}\, c_{(Q_1,Q_2),2}(z) = c_{(Q_1,Q_2),1}(z) + d_{(Q_1,Q_2),0}(z),$$

i.e.

$$\begin{aligned}\Delta_{\mathbb{H}} \left(\sum_{l=3}^{4} H_{2,(Q_1,Q_2),l}(z) \right) &= \Delta_{\mathbb{H}} \left(-\frac{l_{(Q_1,Q_2)}\,\pi\,(1-\log(2))}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \sum_{l=3}^{4} H_{2,(Q_1,Q_2),l}(z) \right) \\ &= -\frac{l_{(Q_1,Q_2)}\,\pi}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} + E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,2).\end{aligned}$$

This finishes the proof of the proposition.

□

Remark 7.2.3. Taking into account that

$$\Delta_{\mathbb{H}}\, \psi_j(z) = \lambda_j\, \psi_j(z) = \left(\frac{1}{2} + ir_j\right)\left(\frac{1}{2} - ir_j\right)\psi_j(z)$$

for $j \in \mathbb{N}$, and

$$\Delta_{\mathbb{H}}\, E^{\mathrm{par}}_{\eta_k}\Big(z, \frac{1}{2} + it\Big) = \Big(\frac{1}{2} + it\Big)\Big(\frac{1}{2} - it\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(z, \frac{1}{2} + it\Big)$$

for $k = 1, \ldots, c_\Gamma$ and $t \in \mathbb{R}$, and using the spectral expansion (5.8) with $n = 2$ and $s = 2$, the differential equation

$$\Delta_{\mathbb{H}}\left(\sum_{l=3}^{4} H_{2,(Q_1,Q_2),l}(z)\right) = -\frac{l_{(Q_1,Q_2)}\,\pi}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} + E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z, 2)$$

can also be obtained by a direct computation.

For the rest of this section we let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. In this case we have $C_\Gamma = \{c_1 = \infty\}$ and $\mathrm{vol}(\Gamma\backslash\mathbb{H}) = \pi/3$. From Example 3.4.15 (b) we recall that the parabolic Eisenstein series $E^{\mathrm{par}}_{\infty}(z, s)$ for Γ admits the Laurent expansions

$$E^{\mathrm{par}}_{\infty}(z, s) = 1 + \log\Big(|\Delta(z)|^{1/6}\,\mathrm{Im}(z)\Big)\cdot s + \mathrm{O}(s^2), \tag{7.22}$$

$$E^{\mathrm{par}}_{\infty}(z, s) = \frac{3}{\pi}\cdot\frac{1}{s-1} - \frac{1}{2\pi}\log\big(|\Delta(z)|\,\mathrm{Im}(z)^6\big) + \frac{6 - 72\,\zeta'(-1) - 6\,\log(4\pi)}{\pi} + \mathrm{O}(s-1) \tag{7.23}$$

at $s = 0$ and $s = 1$, respectively, where the Delta function $\Delta(z)$ is given by (3.20).

Using (7.22) and (7.23) together with Proposition 7.2.2, we find the following Kronecker limit formula for $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z, s)$.

Theorem 7.2.4. *Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. For $z \in \mathbb{H}$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z, s)$ admits a Laurent expansion at $s = 0$ of the form*

$$\begin{aligned} E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z, s) = \Bigg(&\frac{1}{2}\int_{L_{(Q_1,Q_2)}} \log\big(|\Delta(z)|\,|\Delta(w)|\,\mathrm{Im}(z)^6\,\mathrm{Im}(w)^6\big)\; ds_{\mathbb{H}}(w) \\ &+ 3\,l_{(Q_1,Q_2)}\big(24\,\zeta'(-1) + \log(8\pi^2) - 1\big) + \sum_{l=3}^{4} H_{2,(Q_1,Q_2),l}(z)\Bigg)\cdot s^2 + \mathrm{O}(s^3), \end{aligned}$$

where the functions $H_{2,(Q_1,Q_2),l}(z)$ ($l = 3, 4$) are invariant under the action of Γ, and are given by the formulas (7.24) and (7.25) in the proof, respectively. Moreover, for $z \in \mathbb{H}$ they satisfy the differential equation

$$\Delta_{\mathbb{H}}\left(\sum_{l=3}^{4} H_{2,(Q_1,Q_2),l}(z)\right) = -3\,l_{(Q_1,Q_2)} + E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z, 2).$$

Proof. By Proposition 7.2.2, for $z \in \mathbb{H}$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z, s)$ admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned} &E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z, s) - \frac{2\sqrt{\pi}\,\Gamma\big(s - \frac{1}{2}\big)}{\Gamma\big(\frac{s}{2}\big)^2}\, E^{\mathrm{par}}_{\infty}(z, s)\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\infty}(w, 1-s)\; ds_{\mathbb{H}}(w) \\ &\quad = -3\,l_{(Q_1,Q_2)}\cdot s + \left(-3\,l_{(Q_1,Q_2)}\,(1 - \log(2)) + \sum_{l=3}^{4} H_{2,(Q_1,Q_2),l}(z)\right)\cdot s^2 + \mathrm{O}(s^3), \end{aligned}$$

where the functions

$$H_{2,(Q_1,Q_2),3}(z) = \frac{\sqrt{\pi}}{4} \sum_{j=1}^{\infty} \Gamma\Big(\frac{ir_j}{2} - \frac{1}{4}\Big)\, \Gamma\Big(-\frac{ir_j}{2} - \frac{1}{4}\Big)\, \psi_j(z) \int_{L_{(Q_1,Q_2)}} \overline{\psi_j}(w)\, ds_{\mathbb{H}}(w), \tag{7.24}$$

$$H_{2,(Q_1,Q_2),4}(z) = \frac{1}{16\sqrt{\pi}} \int_{-\infty}^{\infty} \Big|\Gamma\Big(\frac{it}{2} - \frac{1}{4}\Big)\Big|^2 E^{\mathrm{par}}_{\infty}\Big(z, \frac{1}{2} + it\Big) \left(\int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\infty}\Big(w, \frac{1}{2} - it\Big)\, ds_{\mathbb{H}}(w)\right) dt \tag{7.25}$$

are invariant under the action of Γ and satisfy the asserted differential equation.

In order to derive the Laurent expansion of the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s)$ at $s=0$, we first determine the respective expansion of

$$\frac{2\sqrt{\pi}\,\Gamma\left(s - \frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)^2}\, E^{\mathrm{par}}_{\infty}(z,s) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\infty}(w, 1-s)\, ds_{\mathbb{H}}(w).$$

At $s=0$ we have the Laurent expansions

$$\begin{aligned}
\frac{1}{\Gamma\left(\frac{s}{2}\right)^2} &= \frac{1}{4} \cdot s^2 + \frac{\gamma}{4} \cdot s^3 + \mathrm{O}(s^4), \\
\Gamma\Big(s - \frac{1}{2}\Big) &= -2\sqrt{\pi} - 2\sqrt{\pi}\,(2 - \gamma - \log(4)) \cdot s + \mathrm{O}(s^2),
\end{aligned}$$

yielding the expansion

$$\frac{2\sqrt{\pi}\,\Gamma\left(s - \frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)^2} = -\pi \cdot s^2 - 2\pi\,(1 - \log(2)) \cdot s^3 + \mathrm{O}(s^4).$$

Further, from (7.22) and

$$E^{\mathrm{par}}_{\infty}(w, 1-s) = -\frac{3}{\pi} \cdot \frac{1}{s} - \frac{1}{2\pi} \log\left(|\Delta(w)|\, \mathrm{Im}(w)^6\right) + \frac{6 - 72\,\zeta'(-1) - 6\,\log(4\pi)}{\pi} + \mathrm{O}(s),$$

which is an immediate consequence of (7.23), at $s=0$ we obtain the Laurent expansion

$$\begin{aligned}
&E^{\mathrm{par}}_{\infty}(z,s) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\infty}(w, 1-s)\, ds_{\mathbb{H}}(w) \\
&= -\frac{3\, l_{(Q_1,Q_2)}}{\pi} \cdot \frac{1}{s} - \frac{3\, l_{(Q_1,Q_2)}}{\pi} \log\Big(|\Delta(z)|^{1/6}\, \mathrm{Im}(z)\Big) - \frac{1}{2\pi} \int_{L_{(Q_1,Q_2)}} \log\left(|\Delta(w)|\, \mathrm{Im}(w)^6\right) ds_{\mathbb{H}}(w) \\
&\quad + \frac{l_{(Q_1,Q_2)}\,(6 - 72\,\zeta'(-1) - 6\,\log(4\pi))}{\pi} + \mathrm{O}(s).
\end{aligned}$$

This implies that at $s=0$ we have a Laurent expansion of the form

$$\begin{aligned}
&\frac{2\sqrt{\pi}\,\Gamma\left(s - \frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)^2}\, E^{\mathrm{par}}_{\infty}(z,s) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\infty}(w, 1-s)\, ds_{\mathbb{H}}(w) \\
&= 3\, l_{(Q_1,Q_2)} \cdot s + \Big(6\, l_{(Q_1,Q_2)}\,(1 - \log(2)) + 3\, l_{(Q_1,Q_2)} \log\Big(|\Delta(z)|^{1/6}\, \mathrm{Im}(z)\Big) \\
&\qquad + \frac{1}{2} \int_{L_{(Q_1,Q_2)}} \log\left(|\Delta(w)|\, \mathrm{Im}(w)^6\right) ds_{\mathbb{H}}(w) - l_{(Q_1,Q_2)}\,(6 - 72\,\zeta'(-1) - 6\,\log(4\pi))\Big) \cdot s^2 + \mathrm{O}(s^3) \\
&= 3\, l_{(Q_1,Q_2)} \cdot s \\
&\quad + \left(6\, l_{(Q_1,Q_2)}\,(12\,\zeta'(-1) + \log(2\pi)) + \frac{1}{2} \int_{L_{(Q_1,Q_2)}} \log\left(|\Delta(z)|\, |\Delta(w)|\, \mathrm{Im}(z)^6\, \mathrm{Im}(w)^6\right) ds_{\mathbb{H}}(w)\right) \cdot s^2 \\
&\quad + \mathrm{O}(s^3).
\end{aligned}$$

Consequently, for $z \in \mathbb{H}$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s)$ admits a Laurent expansion at $s = 0$ of the form

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(z,s) = \Bigg(\frac{1}{2} \int_{L_{(Q_1,Q_2)}} \log\big(|\Delta(z)|\,|\Delta(w)| \operatorname{Im}(z)^6 \operatorname{Im}(w)^6 \big) \; ds_{\mathbb{H}}(w) \\ + 3\, l_{(Q_1,Q_2)} \big(24\, \zeta'(-1) + \log(8\pi^2) - 1 \big) + \sum_{l=3}^{4} H_{2,(Q_1,Q_2),l}(z) \Bigg) \cdot s^2 + \mathrm{O}(s^3).$$

□

Remark 7.2.5. A different Kronecker limit type formula at $s = 0$ for the hyperbolic Eisenstein series for $\mathrm{PSL}_2(\mathbb{Z})$ is proven by Matsusaka in Appendix B in the preprint [Mat20]. Instead of using spectral expansions he follows a different approach involving the automorphic Green's function. For his result we refer to [Mat20].

7.3. Example 2: The case $n = 3$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$

As a second example we treat the case $n = 3$ and $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$ in this section. First we give a Laurent expansion of the hyperbolic Eisenstein series at $s = 0$ for a general discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{C})$. Then we let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$ and examine the behaviour of the parabolic Eisenstein series associated to the only cusp ∞ of this group at the points $s = 0$ and $s = 2$. This leads to a Kronecker limit formula for the hyperbolic Eisenstein series for $\mathrm{PSL}_2(\mathbb{Z}[i])$.
Throughout the section we let $n = 3$. Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{C})$ be a discrete and cofinite subgroup. Further, let $(Q_1, Q_2) \in H_\Gamma$ be a pair of hyperbolic fixed points with hyperbolic scaling matrix $\sigma_{(Q_1,Q_2)} \in \mathrm{PSL}_2(\mathbb{C})$ and hyperbolic stabilizer subgroup $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}$. Let $\mathcal{L}_{(Q_1,Q_2)}$ be the unique geodesic in $\mathbb{H}^3$ connecting Q_1 and Q_2, and let $L_{(Q_1,Q_2)} = \pi_{\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}}(\mathcal{L}_{(Q_1,Q_2)})$ denote its image under the natural projection $\pi_{\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)}} : \mathbb{H}^3 \to \Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \mathbb{H}^3$, which is a closed geodesic in $\Gamma^{\mathrm{hyp}}_{(Q_1,Q_2)} \backslash \mathbb{H}^3$ of hyperbolic length $l_{(Q_1,Q_2)}$.

Notation 7.3.1. To keep the notation simple, in this section we again omit the index 3 and write $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ for the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{3,(Q_1,Q_2)}(P,s)$ associated to the pair $(Q_1, Q_2) \in H_\Gamma$ of hyperbolic fixed points, and $E^{\mathrm{par}}_{\eta_k}(P,s)$ for the parabolic Eisenstein series $E^{\mathrm{par}}_{3,\eta_k}(P,s)$ associated to the cusp $\eta_k \in C_\Gamma$ ($k = 1, \ldots, c_\Gamma$).

Before we consider the specific case $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$ later in this section, from Proposition 7.1.4 we can easily draw the following conclusion for a general discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{C})$.

Proposition 7.3.2. *For $P \in \mathbb{H}^3$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits a Laurent expansion at $s = 0$ of the form*

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) - \frac{2\,\pi\,\Gamma(s-1)}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(P,s) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q, 2-s)\; ds_{\mathbb{H}^3}(Q) \\ = -\frac{l_{(Q_1,Q_2)}\,\pi}{\operatorname{vol}(\Gamma\backslash\mathbb{H}^3)} - \frac{l_{(Q_1,Q_2)}\,\pi}{2\operatorname{vol}(\Gamma\backslash\mathbb{H}^3)} \cdot s + \mathrm{O}(s^2).$$

Proof. Proposition 7.1.4 implies that for $P \in \mathbb{H}^3$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits a Laurent expansion at $s = 0$ of the form

$$E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) - \frac{2\,\pi\,\Gamma(s-1)}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(P,s) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q, 2-s)\; ds_{\mathbb{H}^3}(Q) \\ = -\frac{l_{(Q_1,Q_2)}\,\pi}{\operatorname{vol}(\Gamma\backslash\mathbb{H}^3)} + F_{3,(Q_1,Q_2)}(P) + \left(-\frac{l_{(Q_1,Q_2)}\,\pi}{2\operatorname{vol}(\Gamma\backslash\mathbb{H}^3)} + G_{3,(Q_1,Q_2)}(P) \right) \cdot s + \mathrm{O}(s^2).$$

Moreover, we have noted in Remark 7.1.5 that the functions $F_{3,(Q_1,Q_2)}(P)$ and $G_{3,(Q_1,Q_2)}(P)$ vanish identically since the sums defining these two functions are empty. Thus, we obtain the claimed Laurent expansion at $s=0$ of the form

$$\begin{aligned} &E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) - \frac{2\,\pi\,\Gamma(s-1)}{\Gamma\left(\frac{s}{2}\right)^2} \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(P,s) \int_{L_{(Q_1,Q_2)}} E^{\mathrm{par}}_{\eta_k}(Q,2-s)\; ds_{\mathbb{H}^3}(Q) \\ &= -\frac{l_{(Q_1,Q_2)}\,\pi}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)} - \frac{l_{(Q_1,Q_2)}\,\pi}{2\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)} \cdot s + \mathrm{O}(s^2). \end{aligned}$$

□

Now let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, discriminant d_K and class number h_K. Here and throughout this section we assume that a fixed embedding $K \subseteq \mathbb{C}$ has been chosen. Recalling Example 2.4.17 (b) and Example 2.6.19 (b), the group $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K)$ is a discrete and cofinite subgroup of $\mathrm{PSL}_2(\mathbb{C})$. It admits h_K cusps and the finite hyperbolic volume

$$\mathrm{vol}(\Gamma\backslash\mathbb{H}^3) = \frac{|d_K|^{3/2}}{4\pi^2}\,\zeta_K(2)$$

with

$$\zeta_K(s) = \sum_{\substack{I\subseteq\mathcal{O}_K \text{ ideal},\\ I\neq\{0\}}} \mathcal{N}(I)^{-s}$$

denoting the Dedekind zeta function, where $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$ and $\mathcal{N}(I)$ is the norm of I.

We write elements of $\mathbb{H}^3$ in the form $P = z + jr = x + iy + jr$ with $z\in\mathbb{C}$, $x,y,r\in\mathbb{R}$ and $r>0$. Then for $P\in\mathbb{H}^3$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$ the Eisenstein series $E_\infty(P,s)$ associated to the cusp ∞ after Elstrodt, Grunewald and Mennicke (see, e.g., [EGM13], section 3.2) is defined by

$$E_\infty(P,s) = \sum_{\gamma\in\Gamma'_\infty\backslash\Gamma} r(\gamma P)^{s+1},$$

where

$$\Gamma'_\infty = \{\gamma\in\Gamma_\infty \,|\, \gamma = \mathrm{id} \text{ or } \gamma \text{ is parabolic}\}$$

is the maximal unipotent subgroup of the stabilizer subgroup

$$\Gamma_\infty = \{\gamma\in\Gamma \,|\, \gamma\infty = \infty\}$$

of ∞ in Γ. This Eisenstein series is invariant in P under the action of Γ, holomorphic in s, and admits a meromorphic continuation in s to the whole complex plane with a simple pole at $s=1$ with residue

$$R_K := \mathrm{Res}_{s=1}\, E_\infty(P,s) = \frac{2\pi^2}{|d_K|\,\zeta_K(2)}.$$

In this setting the Kronecker limit formula states

$$\lim_{s\to1}\left(E_\infty(P,s) - \frac{R_K}{s-1}\right) = -R_K\,\log(\eta_K(P)\,r(P)) + C_K$$

(see, e.g., [EGM13], chapter 8, or also [Her+19]), where $C_K\in\mathbb{C}$ is a constant depending only on K. The function $\eta_K : \mathbb{H}^3 \to \mathbb{R}$ satisfies $\eta_K(\gamma P) = ||cP+d||^2\,\eta_K(P)$ for any $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in\Gamma$ and can be seen as an analogue of the weight 2 real-analytic modular form $|\eta(z)|^4$.

We want to derive the respective residue and Kronecker limit formula of the parabolic Eisenstein series $E_\infty^{\mathrm{par}}(P, s)$ associated to the cusp ∞ from Definition 3.4.1, which is given for $P \in \mathbb{H}^3$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$ by

$$E_\infty^{\mathrm{par}}(P, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} r(\gamma P)^s.$$

Replacing the summation condition $\gamma \in \Gamma_\infty \backslash \Gamma$ by the condition $\gamma \in \Gamma'_\infty \backslash \Gamma$ results in a multiplication by the finite index $[\Gamma_\infty : \Gamma'_\infty]$ (see Remark 3.4.3 (b)) which is equal to $\frac{1}{2}\,|\mathcal{O}_K^\times|$ by Example 2.6.19 (b). This implies

$$E_\infty(P, s) = \frac{|\mathcal{O}_K^\times|}{2}\, E_\infty^{\mathrm{par}}(P, s+1), \qquad \text{respectively} \qquad E_\infty^{\mathrm{par}}(P, s) = \frac{2}{|\mathcal{O}_K^\times|}\, E_\infty(P, s-1).$$

Hence, the parabolic Eisenstein series $E_\infty^{\mathrm{par}}(P, s)$ admits a simple pole at $s = 2$ with residue

$$\mathrm{Res}_{s=2}\, E_\infty^{\mathrm{par}}(P, s) = \frac{2}{|\mathcal{O}_K^\times|}\, \mathrm{Res}_{s=1}\, E_\infty(P, s) = \frac{2\, R_K}{|\mathcal{O}_K^\times|} = \frac{4\pi^2}{|\mathcal{O}_K^\times|\, |d_K|\, \zeta_K(2)},$$

and the Kronecker limit formula takes the form

$$\lim_{s \to 2} \left(E_\infty^{\mathrm{par}}(P, s) - \frac{2\, R_K}{|\mathcal{O}_K^\times|\,(s-2)} \right) = -\frac{2\, R_K}{|\mathcal{O}_K^\times|}\, \log(\eta_K(P)\, r(P)) + \frac{2\, C_K}{|\mathcal{O}_K^\times|}$$

with R_K, C_K and $\eta_K(P)$ as above.

Now we let $K = \mathbb{Q}(i)$ which is an imaginary quadratic field with ring of integers $\mathcal{O}_K = \mathbb{Z}[i]$, discriminant $d_K = d_{\mathbb{Q}(i)} = -4$ and class number $h_K = h_{\mathbb{Q}(i)} = 1$. Then the group $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K) = \mathrm{PSL}_2(\mathbb{Z}[i])$ has only one cusp $\eta_1 = \infty$ and the hyperbolic volume

$$\mathrm{vol}(\Gamma \backslash \mathbb{H}^3) = \frac{|d_{\mathbb{Q}(i)}|^{3/2}}{4\pi^2}\, \zeta_{\mathbb{Q}(i)}(2) = \frac{2\, \zeta_{\mathbb{Q}(i)}(2)}{\pi^2}. \tag{7.26}$$

Moreover, we have $|\mathcal{O}_K^\times| = |\mathbb{Z}[i]^\times| = 4$, which gives us

$$E_\infty(P, s) = 2\, E_\infty^{\mathrm{par}}(P, s+1), \qquad \text{respectively} \qquad E_\infty^{\mathrm{par}}(P, s) = \frac{1}{2}\, E_\infty(P, s-1).$$

We obtain the residue

$$\mathrm{Res}_{s=2}\, E_\infty^{\mathrm{par}}(P, s) = \frac{R_{\mathbb{Q}(i)}}{2} = \frac{\pi^2}{4\, \zeta_{\mathbb{Q}(i)}(2)} = \frac{1}{2\, \mathrm{vol}(\Gamma \backslash \mathbb{H}^3)},$$

and at $s = 2$ we have the Laurent expansion

$$E_\infty^{\mathrm{par}}(P, s) = \frac{\pi^2}{4\, \zeta_{\mathbb{Q}(i)}(2)} \cdot \frac{1}{s-2} - \frac{\pi^2}{4\, \zeta_{\mathbb{Q}(i)}(2)}\, \log(\eta_{\mathbb{Q}(i)}(P)\, r(P)) + \frac{C_{\mathbb{Q}(i)}}{2} + \mathrm{O}(s-2), \tag{7.27}$$

where $C_{\mathbb{Q}(i)} \in \mathbb{C}$ is a constant and $\eta_{\mathbb{Q}(i)} : \mathbb{H}^3 \to \mathbb{R}$ satisfies $\eta_{\mathbb{Q}(i)}(\gamma P) = ||cP + d||^2\, \eta_{\mathbb{Q}(i)}(P)$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. For the rest of this section we let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$.

Introducing the *normalized Dedekind zeta function*

$$\zeta_{\mathbb{Q}(i)}^*(s) := 4\, \pi^{-s}\, \Gamma(s)\, \zeta_{\mathbb{Q}(i)}(s),$$

satisfying

$$\zeta_{\mathbb{Q}(i)}^*(s) = \zeta_{\mathbb{Q}(i)}^*(1-s),$$

and the *normalized parabolic Eisenstein series*

$$\widetilde{E}^{\mathrm{par}}_{\infty}(P,s) := \zeta^{*}_{\mathbb{Q}(i)}(s)\, E^{\mathrm{par}}_{\infty}(P,s) = 4\,\pi^{-s}\,\Gamma(s)\,\zeta_{\mathbb{Q}(i)}(s)\, E^{\mathrm{par}}_{\infty}(P,s),$$

we have the functional equation (see, e.g., [Szm83], p. 395)

$$\widetilde{E}^{\mathrm{par}}_{\infty}(P,s) = \widetilde{E}^{\mathrm{par}}_{\infty}(P,2-s).$$

This yields the identity

$$\begin{aligned} E^{\mathrm{par}}_{\infty}(P,s) &= \frac{4\,\pi^{s-2}\,\Gamma(2-s)\,\zeta_{\mathbb{Q}(i)}(2-s)}{4\,\pi^{-s}\,\Gamma(s)\,\zeta_{\mathbb{Q}(i)}(s)}\, E^{\mathrm{par}}_{\infty}(P,2-s) \\ &= \frac{4\,\pi^{s-2}\,\Gamma(2-s)\,\zeta_{\mathbb{Q}(i)}(2-s)}{\zeta^{*}_{\mathbb{Q}(i)}(s)}\, E^{\mathrm{par}}_{\infty}(P,2-s), \end{aligned} \tag{7.28}$$

which we will use together with (7.27) to derive the Laurent expansion of $E^{\mathrm{par}}_{\infty}(P,s)$ at $s=0$. Prior to that we need two further lemmas.

Lemma 7.3.3. *For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ the Dedekind zeta function $\zeta_{\mathbb{Q}(i)}(s)$ satisfies the identity*

$$\zeta_{\mathbb{Q}(i)}(s) = \frac{1}{2}\,\zeta(2s)\, E^{\mathrm{par}}_{2,\mathrm{PSL}_2(\mathbb{Z}),\infty}(i,s),$$

where $\zeta(s)$ is the Riemann zeta function (see (A.28)*) and $E^{\mathrm{par}}_{2,\mathrm{PSL}_2(\mathbb{Z}),\infty}(z,s)$ denotes the parabolic Eisenstein series on the upper half-plane $\mathbb{H}$ associated to the modular group $\mathrm{PSL}_2(\mathbb{Z})$ and the cusp ∞.*

Proof. As $\mathbb{Z}[i]$ is a principal ideal domain, any non-trivial ideal $\{0\} \neq I \subseteq \mathbb{Z}[i]$ is of the form $I = \langle ci + d \rangle$ for some pair $(c,d) \in \mathbb{Z}^2$ with $(c,d) \neq (0,0)$. Moreover, two elements $\omega_1, \omega_2 \in \mathbb{Z}[i]$ generate the same principal ideal if and only if $\omega_2 = \mu\,\omega_1$ for some $\mu \in \mathbb{Z}[i]^{\times} = \{\pm 1, \pm i\}$. Thus, for any ideal $\{0\} \neq I \subseteq \mathbb{Z}[i]$ there are exactly four pairs $(c_j, d_j) \in \mathbb{Z}^2$ ($j = 1,2,3,4$) with $(c_j, d_j) \neq (0,0)$ and $I = \langle c_j i + d_j \rangle$. Taking into account that

$$\mathcal{N}(\langle ci + d \rangle) = N_{\mathbb{Q}(i)/\mathbb{Q}}(ci + d) = c^2 + d^2,$$

this leads to

$$\zeta_{\mathbb{Q}(i)}(s) = \sum_{\substack{I \subseteq \mathbb{Z}[i] \text{ ideal}, \\ I \neq \{0\}}} \mathcal{N}(I)^{-s} = \frac{1}{4} \sum_{\substack{(c,d) \in \mathbb{Z}^2, \\ (c,d) \neq (0,0)}} \mathcal{N}(\langle ci + d \rangle)^{-s} = \frac{1}{4} \sum_{\substack{(c,d) \in \mathbb{Z}^2, \\ (c,d) \neq (0,0)}} \frac{1}{(c^2 + d^2)^s}.$$

On the other hand we can rewrite $E^{\mathrm{par}}_{2,\mathrm{PSL}_2(\mathbb{Z}),\infty}(i,s)$ as

$$\begin{aligned} E^{\mathrm{par}}_{2,\mathrm{PSL}_2(\mathbb{Z}),\infty}(i,s) &= \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2, \\ \gcd(c,d) = 1}} \frac{1}{|ci + d|^{2s}} = \frac{1}{2\,\zeta(2s)} \sum_{k=1}^{\infty} \frac{1}{|k|^{2s}} \sum_{\substack{(c,d) \in \mathbb{Z}^2, \\ \gcd(c,d) = 1}} \frac{1}{|ci + d|^{2s}} \\ &= \frac{1}{2\,\zeta(2s)} \sum_{k=1}^{\infty} \sum_{\substack{(c,d) \in \mathbb{Z}^2, \\ \gcd(c,d) = 1}} \frac{1}{|kci + kd|^{2s}} = \frac{1}{2\,\zeta(2s)} \sum_{\substack{(c,d) \in \mathbb{Z}^2, \\ (c,d) \neq (0,0)}} \frac{1}{|ci + d|^{2s}} \\ &= \frac{1}{2\,\zeta(2s)} \sum_{\substack{(c,d) \in \mathbb{Z}^2, \\ (c,d) \neq (0,0)}} \frac{1}{(c^2 + d^2)^s}. \end{aligned}$$

Consequently, we obtain

$$\zeta_{\mathbb{Q}(i)}(s) = \frac{1}{2}\,\zeta(2s)\, E^{\mathrm{par}}_{2,\mathrm{PSL}_2(\mathbb{Z}),\infty}(i,s).$$

□

Lemma 7.3.4. *At $s = 0$ we have the Laurent expansions*

$$\begin{aligned}
\zeta^*_{\mathbb{Q}(i)}(s) &= -\frac{1}{s} - \log\left(|\Delta(i)|^{1/6}\right) + \gamma - \log(4\pi) + \mathrm{O}(s),\\
\frac{1}{\zeta^*_{\mathbb{Q}(i)}(s)} &= -s + \left(\log\left(|\Delta(i)|^{1/6}\right) - \gamma + \log(4\pi)\right) \cdot s^2 + \mathrm{O}(s^3),
\end{aligned}$$

where $\Delta(z)$ is the Delta function given by (3.20) *and γ denotes the Euler-Mascheroni constant (see* (A.6)*).*

Proof. The previous lemma gives us the identity

$$\zeta^*_{\mathbb{Q}(i)}(s) = 4\,\pi^{-s}\,\Gamma(s)\,\zeta_{\mathbb{Q}(i)}(s) = 2\,\pi^{-s}\,\Gamma(s)\,\zeta(2s)\,E^{\mathrm{par}}_{2,\mathrm{PSL}_2(\mathbb{Z}),\infty}(i,s).$$

At $s = 0$ we have

$$\begin{aligned}
\pi^{-s} &= 1 - \log(\pi) \cdot s + \mathrm{O}(s^2),\\
\Gamma(s) &= \frac{1}{s} - \gamma + \mathrm{O}(s),\\
\zeta(2s) &= -\frac{1}{2} - \log(2\pi) \cdot s + \mathrm{O}(s^2),
\end{aligned}$$

so that we obtain the Laurent expansion

$$2\,\pi^{-s}\,\Gamma(s)\,\zeta(2s) = -\frac{1}{s} + \gamma - \log(4\pi) + \mathrm{O}(s)$$

at this point. Further, we recall from (7.22) that

$$E^{\mathrm{par}}_{2,\mathrm{PSL}_2(\mathbb{Z}),\infty}(i,s) = 1 + \log\left(|\Delta(i)|^{1/6}\right) \cdot s + \mathrm{O}(s^2).$$

Combining these two formulas, at $s = 0$ we get

$$\zeta^*_{\mathbb{Q}(i)}(s) = -\frac{1}{s} - \log\left(|\Delta(i)|^{1/6}\right) + \gamma - \log(4\pi) + \mathrm{O}(s),$$

as claimed. Setting $a := -1$ and

$$b := -\log\left(|\Delta(i)|^{1/6}\right) + \gamma - \log(4\pi),$$

the second of the asserted Laurent expansions follows from

$$\begin{aligned}
\frac{1}{\zeta^*_{\mathbb{Q}(i)}(s)} &= \frac{1}{\frac{a}{s} + b + \mathrm{O}(s)} = \frac{s}{a} - \frac{b}{a^2} \cdot s^2 + \mathrm{O}(s^3)\\
&= -s + \left(\log\left(|\Delta(i)|^{1/6}\right) - \gamma + \log(4\pi)\right) \cdot s^2 + \mathrm{O}(s^3).
\end{aligned}$$

□

Now we turn to determine the Laurent expansions of the parabolic Eisenstein series $E^{\mathrm{par}}_{\infty}(P,s)$ and of the product $E^{\mathrm{par}}_{\infty}(Q,2-s)\,E^{\mathrm{par}}_{\infty}(P,s)$ at $s = 0$, where $P, Q \in \mathbb{H}^3$.

Lemma 7.3.5. *Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$. For $P, Q \in \mathbb{H}^3$, at $s = 0$ we have the Laurent expansions*

$$\begin{aligned} E_\infty^{\mathrm{par}}(Q, 2-s) &= -\frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)} \cdot \frac{1}{s} - \frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)} \log(\eta_{\mathbb{Q}(i)}(Q)\, r(Q)) + \frac{C_{\mathbb{Q}(i)}}{2} + \mathrm{O}(s), \\ E_\infty^{\mathrm{par}}(P, s) &= 1 + \Bigg(\log(\eta_{\mathbb{Q}(i)}(P)\, r(P)) - \frac{2\, C_{\mathbb{Q}(i)}\, \zeta_{\mathbb{Q}(i)}(2)}{\pi^2} \\ &\qquad - \log\left(|\Delta(i)|^{1/6}\right) + 2\gamma - 1 - \log(4) - \frac{\zeta'_{\mathbb{Q}(i)}(2)}{\zeta_{\mathbb{Q}(i)}(2)} \Bigg) \cdot s + \mathrm{O}(s^2), \\ E_\infty^{\mathrm{par}}(Q, 2-s)\, E_\infty^{\mathrm{par}}(P, s) &= -\frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)} \cdot \frac{1}{s} - \frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)} \log\left(\eta_{\mathbb{Q}(i)}(P)\, \eta_{\mathbb{Q}(i)}(Q)\, r(P)\, r(Q)\right) + C_{\mathbb{Q}(i)} \\ &\quad + \frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)} \left(\log\left(|\Delta(i)|^{1/6}\right) + 1 - 2\gamma + \log(4) + \frac{\zeta'_{\mathbb{Q}(i)}(2)}{\zeta_{\mathbb{Q}(i)}(2)} \right) + \mathrm{O}(s). \end{aligned}$$

Proof. The first of the stated Laurent expansions, i.e.

$$E_\infty^{\mathrm{par}}(Q, 2-s) = -\frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)} \cdot \frac{1}{s} - \frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)} \log(\eta_{\mathbb{Q}(i)}(Q)\, r(Q)) + \frac{C_{\mathbb{Q}(i)}}{2} + \mathrm{O}(s),$$

is an immediate consequence of (7.27).

Next we derive the Laurent expansion of $E_\infty^{\mathrm{par}}(P, s)$ at $s = 0$ with the aid of relation (7.28). Using (A.9) and (A.13), at $s = 0$ we have

$$\begin{aligned} \pi^{s-2} &= \frac{1}{\pi^2} + \frac{\log(\pi)}{\pi^2} \cdot s + \mathrm{O}(s^2), \\ \Gamma(2-s) &= 1 + (\gamma - 1) \cdot s + \mathrm{O}(s^2), \end{aligned}$$

which implies the Laurent expansion

$$4\,\pi^{s-2}\,\Gamma(2-s) = \frac{4}{\pi^2} + \frac{4\,(\gamma - 1 + \log(\pi))}{\pi^2} \cdot s + \mathrm{O}(s^2).$$

Together with the Laurent expansion

$$\zeta_{\mathbb{Q}(i)}(2-s) = \zeta_{\mathbb{Q}(i)}(2) - \zeta'_{\mathbb{Q}(i)}(2) \cdot s + \mathrm{O}(s^2)$$

this yields

$$4\,\pi^{s-2}\,\Gamma(2-s)\,\zeta_{\mathbb{Q}(i)}(2-s) = \frac{4\,\zeta_{\mathbb{Q}(i)}(2)}{\pi^2} + \frac{4\left(\left(\gamma - 1 + \log(\pi)\right)\zeta_{\mathbb{Q}(i)}(2) - \zeta'_{\mathbb{Q}(i)}(2)\right)}{\pi^2} \cdot s + \mathrm{O}(s^2).$$

Further, by Lemma 7.3.4 we have

$$\frac{1}{\zeta^*_{\mathbb{Q}(i)}(s)} = -s + \left(\log\left(|\Delta(i)|^{1/6}\right) - \gamma + \log(4\pi) \right) \cdot s^2 + \mathrm{O}(s^3).$$

All in all, at $s = 0$ we find Laurent expansions of the form

$$\begin{aligned} &\frac{4\,\pi^{s-2}\,\Gamma(2-s)\,\zeta_{\mathbb{Q}(i)}(2-s)}{\zeta^*_{\mathbb{Q}(i)}(s)} \\ &= -\frac{4\,\zeta_{\mathbb{Q}(i)}(2)}{\pi^2} \cdot s + \frac{4\left(\left(\log\left(|\Delta(i)|^{1/6}\right) + 1 - 2\gamma + \log(4)\right)\zeta_{\mathbb{Q}(i)}(2) + \zeta'_{\mathbb{Q}(i)}(2)\right)}{\pi^2} \cdot s^2 + \mathrm{O}(s^3) \end{aligned}$$

and

$$\begin{aligned}
E_{\infty}^{\mathrm{par}}(P,s) &= \frac{4\,\pi^{s-2}\,\Gamma(2-s)\,\zeta_{\mathbb{Q}(i)}(2-s)}{\zeta^{*}_{\mathbb{Q}(i)}(s)}\,E_{\infty}^{\mathrm{par}}(P,2-s) \\
&= 1 + \left(\log(\eta_{\mathbb{Q}(i)}(P)\,r(P)) - \frac{2\,C_{\mathbb{Q}(i)}\,\zeta_{\mathbb{Q}(i)}(2)}{\pi^2} - \log\left(|\Delta(i)|^{1/6}\right) + 2\gamma - 1 - \log(4) - \frac{\zeta'_{\mathbb{Q}(i)}(2)}{\zeta_{\mathbb{Q}(i)}(2)}\right)\cdot s \\
&\quad + \mathrm{O}(s^2).
\end{aligned}$$

This proves the second Laurent expansion in the lemma.

Finally, combining the first two expansions, at $s = 0$ we get the Laurent expansion

$$\begin{aligned}
E_{\infty}^{\mathrm{par}}(Q,2-s)\,E_{\infty}^{\mathrm{par}}(P,s) &= -\frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)}\cdot\frac{1}{s} - \frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)}\,\log\left(\eta_{\mathbb{Q}(i)}(P)\,\eta_{\mathbb{Q}(i)}(Q)\,r(P)\,r(Q)\right) + C_{\mathbb{Q}(i)} \\
&\quad + \frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)}\left(\log\left(|\Delta(i)|^{1/6}\right) + 1 - 2\gamma + \log(4) + \frac{\zeta'_{\mathbb{Q}(i)}(2)}{\zeta_{\mathbb{Q}(i)}(2)}\right) + \mathrm{O}(s),
\end{aligned}$$

which finishes the proof.

□

With the aid of the last lemma we are now able to derive a Kronecker limit formula for the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$.

Theorem 7.3.6. *Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$. For $P \in \mathbb{H}^3$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits a Laurent expansion at $s = 0$ of the form*

$$\begin{aligned}
E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) &= -\frac{3\,l_{(Q_1,Q_2)}\,\pi^3}{8\,\zeta_{\mathbb{Q}(i)}(2)} \\
&\quad + \left(\frac{\pi^3}{8\,\zeta_{\mathbb{Q}(i)}(2)}\int_{L_{(Q_1,Q_2)}}\log\left(\eta_{\mathbb{Q}(i)}(P)\,\eta_{\mathbb{Q}(i)}(Q)\,r(P)\,r(Q)\right)\,ds_{\mathbb{H}^3}(Q) - \frac{l_{(Q_1,Q_2)}\,\pi\,C_{\mathbb{Q}(i)}}{2}\right. \\
&\quad \left. - \frac{l_{(Q_1,Q_2)}\,\pi^3}{8\,\zeta_{\mathbb{Q}(i)}(2)}\left(\log\left(|\Delta(i)|^{1/6}\right) + 2\,(1-\gamma+\log(2)) + \frac{\zeta'_{\mathbb{Q}(i)}(2)}{\zeta_{\mathbb{Q}(i)}(2)}\right)\right)\cdot s + \mathrm{O}(s^2).
\end{aligned}$$

Proof. From Proposition 7.3.2 we know that for $P \in \mathbb{H}^3$ the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned}
&E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) - \frac{2\,\pi\,\Gamma(s-1)}{\Gamma\left(\frac{s}{2}\right)^2}\,E_{\infty}^{\mathrm{par}}(P,s)\int_{L_{(Q_1,Q_2)}}E_{\infty}^{\mathrm{par}}(Q,2-s)\;ds_{\mathbb{H}^3}(Q) \\
&\quad = -\frac{l_{(Q_1,Q_2)}\,\pi^3}{2\,\zeta_{\mathbb{Q}(i)}(2)} - \frac{l_{(Q_1,Q_2)}\,\pi^3}{4\,\zeta_{\mathbb{Q}(i)}(2)}\cdot s + \mathrm{O}(s^2),
\end{aligned}$$

where we used (7.26). To derive the Laurent expansion of the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s)$ at $s = 0$, we first compute the respective expansion of

$$\frac{2\,\pi\,\Gamma(s-1)}{\Gamma\left(\frac{s}{2}\right)^2}\,E_{\infty}^{\mathrm{par}}(P,s)\int_{L_{(Q_1,Q_2)}}E_{\infty}^{\mathrm{par}}(Q,2-s)\;ds_{\mathbb{H}^3}(Q).$$

Using the Laurent expansions

$$\begin{aligned}
\frac{1}{\Gamma\left(\frac{s}{2}\right)^2} &= \frac{1}{4}\cdot s^2 + \frac{\gamma}{4}\cdot s^3 + \mathrm{O}(s^4), \\
\Gamma(s-1) &= -\frac{1}{s} + \gamma - 1 + \mathrm{O}(s),
\end{aligned}$$

at $s=0$ we obtain the Laurent expansion

$$\frac{2\,\pi\,\Gamma(s-1)}{\Gamma\left(\frac{s}{2}\right)^2} = -\frac{\pi}{2}\cdot s - \frac{\pi}{2}\cdot s^2 + \mathrm{O}(s^3).$$

From Lemma 7.3.5 we can conclude that at $s=0$ we have the Laurent expansion

$$\begin{aligned}
&E_\infty^{\mathrm{par}}(P,s)\int_{L_{(Q_1,Q_2)}} E_\infty^{\mathrm{par}}(Q,2-s)\;ds_{\mathbb{H}^3}(Q)\\
&\quad= -\frac{l_{(Q_1,Q_2)}\,\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)}\cdot\frac{1}{s} - \frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)}\int_{L_{(Q_1,Q_2)}}\log\left(\eta_{\mathbb{Q}(i)}(P)\,\eta_{\mathbb{Q}(i)}(Q)\,r(P)\,r(Q)\right)\;ds_{\mathbb{H}^3}(Q) + l_{(Q_1,Q_2)}\,C_{\mathbb{Q}(i)}\\
&\qquad+\frac{l_{(Q_1,Q_2)}\,\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)}\left(\log\left(|\Delta(i)|^{1/6}\right)+1-2\gamma+\log(4)+\frac{\zeta'_{\mathbb{Q}(i)}(2)}{\zeta_{\mathbb{Q}(i)}(2)}\right)+\mathrm{O}(s).
\end{aligned}$$

Hence, at $s=0$ we have

$$\begin{aligned}
&\frac{2\,\pi\,\Gamma(s-1)}{\Gamma\left(\frac{s}{2}\right)^2}\,E_\infty^{\mathrm{par}}(P,s)\int_{L_{(Q_1,Q_2)}} E_\infty^{\mathrm{par}}(Q,2-s)\;ds_{\mathbb{H}^3}(Q)\\
&\quad=\frac{l_{(Q_1,Q_2)}\,\pi^3}{8\,\zeta_{\mathbb{Q}(i)}(2)}+\left(\frac{\pi^3}{8\,\zeta_{\mathbb{Q}(i)}(2)}\int_{L_{(Q_1,Q_2)}}\log\left(\eta_{\mathbb{Q}(i)}(P)\,\eta_{\mathbb{Q}(i)}(Q)\,r(P)\,r(Q)\right)\;ds_{\mathbb{H}^3}(Q)-\frac{l_{(Q_1,Q_2)}\,\pi\,C_{\mathbb{Q}(i)}}{2}\right.\\
&\qquad\left.-\frac{l_{(Q_1,Q_2)}\,\pi^3}{8\,\zeta_{\mathbb{Q}(i)}(2)}\left(\log\left(|\Delta(i)|^{1/6}\right)-2\gamma+\log(4)+\frac{\zeta'_{\mathbb{Q}(i)}(2)}{\zeta_{\mathbb{Q}(i)}(2)}\right)\right)\cdot s+\mathrm{O}(s^2).
\end{aligned}$$

Adding up, at $s=0$ we get the Laurent expansion

$$\begin{aligned}
E^{\mathrm{hyp}}_{(Q_1,Q_2)}(P,s) &= -\frac{3\,l_{(Q_1,Q_2)}\,\pi^3}{8\,\zeta_{\mathbb{Q}(i)}(2)}\\
&\quad+\left(\frac{\pi^3}{8\,\zeta_{\mathbb{Q}(i)}(2)}\int_{L_{(Q_1,Q_2)}}\log\left(\eta_{\mathbb{Q}(i)}(P)\,\eta_{\mathbb{Q}(i)}(Q)\,r(P)\,r(Q)\right)\;ds_{\mathbb{H}^3}(Q)-\frac{l_{(Q_1,Q_2)}\,\pi\,C_{\mathbb{Q}(i)}}{2}\right.\\
&\qquad\left.-\frac{l_{(Q_1,Q_2)}\,\pi^3}{8\,\zeta_{\mathbb{Q}(i)}(2)}\left(\log\left(|\Delta(i)|^{1/6}\right)+2\,(1-\gamma+\log(2))+\frac{\zeta'_{\mathbb{Q}(i)}(2)}{\zeta_{\mathbb{Q}(i)}(2)}\right)\right)\cdot s+\mathrm{O}(s^2).
\end{aligned}$$

□

Remark 7.3.7. Similar to Theorem 7.3.6, one could also derive a Kronecker limit formula for hyperbolic Eisenstein series in $\mathbb{H}^3$ for $\Gamma=\mathrm{PSL}_2(\mathcal{O}_K)$, where K is some other imaginary quadratic field with ring of integers $\mathcal{O}_K$ and class number $h_K=1$.

8. Kronecker limit formulas for elliptic Eisenstein series

As we have done for the hyperbolic Eisenstein series, in this final chapter we study the behaviour of the elliptic Eisenstein series $E^{\mathrm{ell}}_{Q}(P,s)$ in $\mathbb{H}^n$ at the point $s=0$ in terms of its Laurent expansion. In the first section we determine the first two terms in this Laurent expansion for arbitrary dimension n and an arbitrary discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$. Afterwards we consider two examples of specific dimensions and groups. In the special case $n=2$ and $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ the Kronecker limit formula for the elliptic Eisenstein series was proven by von Pippich in [Pip10], and we recall her results in the second section. Finally, in the third section we derive a Kronecker limit formula for the elliptic Eisenstein series in the case $n=3$ and $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$.

8.1. The Laurent expansion at $s = 0$

In this section we use the meromorphic continuation of the elliptic Eisenstein series established in section 6.3 to compute its Laurent expansion at $s=0$. Since the form of this expansion again depends on the dimension n of the hyperbolic space $\mathbb{H}^n$, precisely on $n \mod 4$, we have to consider the cases $n \equiv 0 \mod 2$, $n \equiv 3 \mod 4$ and $n \equiv 1 \mod 4$ separately.
Let $\Gamma \subseteq \mathrm{PSL}_2(C_{n-1})$ be a discrete and cofinite subgroup. Further, let $Q \in \mathbb{H}^n$ be a point with elliptic scaling matrix $\sigma_Q \in \mathrm{PSL}_2(C_{n-1})$ and stabilizer subgroup Γ_Q.

Notation 8.1.1. To keep the notation simple, in this section we again omit the index n and write $E^{\mathrm{ell}}_{Q}(P,s)$ for the elliptic Eisenstein series $E^{\mathrm{ell}}_{n,Q}(P,s)$ associated to the point $Q \in \mathbb{H}^n$, and $E^{\mathrm{par}}_{\eta_k}(P,s)$ for the parabolic Eisenstein series $E^{\mathrm{par}}_{n,\eta_k}(P,s)$ associated to the cusp $\eta_k \in C_\Gamma$ $(k=1,\dots,c_\Gamma)$.

We first assume that the dimension is even, i.e. that $n \equiv 0 \mod 2$.

Proposition 8.1.2. *Let $n \equiv 0 \mod 2$. For $P \in \mathbb{H}^n$ with $P \neq \eta Q$ for any $\eta \in \Gamma$ the elliptic Eisenstein series $E^{\mathrm{ell}}_{Q}(P,s)$ admits a Laurent expansion at $s=0$ of the form*

$$\begin{aligned}
&E^{\mathrm{ell}}_{Q}(P,s) - \mathcal{G}^{\mathrm{par}}_{Q,1}(P,s) \\
&\quad = \frac{\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1-n}{2}\right)}{|\Gamma_Q|\operatorname{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|}\sum_{m=1}^{2} G_{n,Q,m}(P) + \frac{1}{|\Gamma_Q|}\sum_{l=1}^{\frac{n}{2}-1}\sum_{m=1}^{2} G_{n,Q,l,m}(P) \\
&\qquad + \left(\frac{\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1-n}{2}\right)\left(\gamma + \log(4) + \psi^{(0)}\left(\frac{1-n}{2}\right)\right)}{2\,|\Gamma_Q|\operatorname{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|}\sum_{m=1}^{5} H_{n,Q,m}(P)\right. \\
&\qquad\quad \left. + \sum_{l=1}^{\frac{n}{2}-1} \frac{4^{l-1}\,\pi^{\frac{n-1}{2}}\,(l-1)!\,\Gamma\left(\frac{1-n}{2}+l\right)}{l\,(2l-1)!\,|\Gamma_Q|\operatorname{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|}\sum_{l=1}^{\frac{n}{2}-1}\sum_{m=1}^{4} H_{n,Q,l,m}(P)\right)\cdot s + \mathrm{O}(s^2), \quad (8.1)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{G}^{\mathrm{par}}_{Q,1}(P,s) := \frac{1}{|\Gamma_Q|}\sum_{l=0}^{\frac{n}{2}-1} \frac{2^{s+2l}\,\pi^{\frac{n-1}{2}}\left(\frac{s}{2}\right)_l}{l!\,\Gamma(s+2l)} \sum_{l'=l}^{\left\lfloor\frac{n-1}{4}\right\rfloor} \frac{(-1)^{l'-l}}{(l'-l)!}\,\Gamma\Big(s - \frac{n-1}{2} + l + l'\Big) \\
\cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2l')\,E^{\mathrm{par}}_{\eta_k}(P, s+2l'),
\end{aligned}$$

γ denotes the Euler-Mascheroni constant (see (A.6)*), $\psi^{(0)}(s)$ is the digamma function (see* (A.8)*), and where the functions $G_{n,Q,m}(P)$ $(m=1,2)$, $G_{n,Q,l,m}(P)$ $\big(l=1,\ldots,\frac{n}{2}-1,\, m=1,2\big)$, $H_{n,Q,m}(P)$ $(m=1,2,3,4,5)$ and $H_{n,Q,l,m}(P)$ $\big(l=1,\ldots,\frac{n}{2}-1,\, m=1,2,3,4\big)$ are invariant under the action of Γ, and are given by the formulas* (8.8), (8.10), (8.11), (8.12), (8.13), (8.14), (8.15), (8.18), (8.19), (8.20), (8.21), (8.22) *and* (8.23) *in the proof, respectively.*

Proof. For $P\in\mathbb{H}^n$ with $P\neq\eta Q$ for any $\eta\in\Gamma$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1-2(N+1)$ $(N\in\mathbb{N}_0)$ the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$ is given by the decomposition (6.30) in the proof of Theorem 6.3.2, i.e. as

$$E^{\mathrm{ell}}_Q(P,s)=\frac{1}{|\Gamma_Q|}\sum_{l=0}^{N}\frac{\left(\frac{s}{2}\right)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l)+\frac{1}{|\Gamma_Q|}\sum_{l=N+1}^{\infty}\frac{\left(\frac{s}{2}\right)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l),$$

where the infinite sum is a holomorphic function in s. In particular, for $\mathrm{Re}(s)>n-1-2\cdot\frac{n}{2}$ we have

$$E^{\mathrm{ell}}_Q(P,s)=\frac{1}{|\Gamma_Q|}\sum_{l=0}^{\frac{n}{2}-1}\frac{\left(\frac{s}{2}\right)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l)+\frac{1}{|\Gamma_Q|}\sum_{l=\frac{n}{2}}^{\infty}\frac{\left(\frac{s}{2}\right)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l),\tag{8.2}$$

where the infinite sum is holomorphic for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>n-1-2\cdot\frac{n}{2}=-1$, and the point $s=0$ lies in this half-plane.

Moreover, for $P,Q\in\mathbb{H}^n$ and $l=0,\ldots,\frac{n}{2}-1$ the function $K^{\mathrm{hyp}}(P,Q,s+2l)$ admits a meromorphic continuation in s to the whole complex plane by Theorem 6.1.1. From the proof of this theorem we deduce that for $s\in\mathbb{C}$ with $\frac{n-1}{2}-2(m+1)<\mathrm{Re}(s)+2l<\frac{n-1}{2}-2m$ $(m\in\mathbb{N}_0)$ the meromorphic continuation is given by means of

$$\begin{aligned}K^{\mathrm{hyp}}(P,Q,s+2l)&=\sum_{j=0}^{\infty}a_{j,Q}(s+2l)\,\psi_j(P)+\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty}a_{t,\eta_k,Q}(s+2l)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt\\&\quad+\frac{2^{s+2l}\,\pi^{\frac{n-1}{2}}}{\Gamma(s+2l)}\sum_{l'=0}^{m}\frac{(-1)^{l'}}{l'!}\,\Gamma\Big(s+2l-\frac{n-1}{2}+l'\Big)\\&\qquad\cdot\sum_{k=1}^{c_\Gamma}E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2(l+l'))\,E^{\mathrm{par}}_{\eta_k}(P,s+2(l+l')),\end{aligned}\tag{8.3}$$

where the coefficients $a_{j,Q}(s+2l)$ and $a_{t,\eta_k,Q}(s+2l)$ are given by

$$\begin{aligned}a_{j,Q}(s+2l)&=\frac{2^{s+2l-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s+2l)}\,\Gamma\Big(\frac{s+2l-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s+2l-\frac{n-1}{2}-ir_j}{2}\Big)\,\overline{\psi_j}(Q),\\a_{t,\eta_k,Q}(s+2l)&=\frac{2^{s+2l-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s+2l)}\,\Gamma\Big(\frac{s+2l-\frac{n-1}{2}+it}{2}\Big)\,\Gamma\Big(\frac{s+2l-\frac{n-1}{2}-it}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-it\Big),\end{aligned}$$

respectively, while for $s\in\mathbb{C}$ with $\frac{n-1}{2}<\mathrm{Re}(s)+2l$ the meromorphic continuation is already given by

$$K^{\mathrm{hyp}}(P,Q,s+2l)=\sum_{j=0}^{\infty}a_{j,Q}(s+2l)\,\psi_j(P)+\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty}a_{t,\eta_k,Q}(s+2l)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt.$$

As $\sum_{l'=0}^{m}$ becomes the empty sum for $m<0$, formula (8.3) is indeed valid for $s\in\mathbb{C}$ with $\frac{n-1}{2}-2(m+1)<\mathrm{Re}(s)+2l<\frac{n-1}{2}-2m$ for any $m\in\mathbb{Z}$.

Especially, in the strip $\frac{n-1}{2}-2\left(\left\lfloor\frac{n-1}{4}\right\rfloor - l + 1\right) < \mathrm{Re}(s)+2l < \frac{n-1}{2}-2\left(\left\lfloor\frac{n-1}{4}\right\rfloor - l\right)$ $\left(l \in \left\{0, \ldots, \frac{n}{2} - 1\right\}\right)$ the identity

$$\begin{aligned}K^{\mathrm{hyp}}(P, Q, s + 2l) &= \sum_{j=0}^{\infty} a_{j,Q}(s + 2l)\, \psi_j(P) + \frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s + 2l)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt \\ &\quad + \frac{2^{s+2l}\, \pi^{\frac{n-1}{2}}}{\Gamma(s + 2l)} \sum_{l'=0}^{\left\lfloor\frac{n-1}{4}\right\rfloor - l} \frac{(-1)^{l'}}{l'!}\, \Gamma\Big(s + 2l - \frac{n-1}{2} + l'\Big) \\ &\qquad\qquad \cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q, n - 1 - s - 2(l + l'))\, E^{\mathrm{par}}_{\eta_k}(P, s + 2(l + l'))\end{aligned} \tag{8.4}$$

holds true, where we can rewrite the sum over l' as

$$\sum_{l'=l}^{\left\lfloor\frac{n-1}{4}\right\rfloor} \frac{(-1)^{l'-l}}{(l'-l)!}\, \Gamma\Big(s - \frac{n-1}{2} + l + l'\Big) \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q, n - 1 - s - 2l')\, E^{\mathrm{par}}_{\eta_k}(P, s + 2l').$$

The assumption $n \equiv 0 \mod 2$ gives us $\left\lfloor\frac{n-1}{4}\right\rfloor < \frac{n-1}{4}$, which yields that for $l = 0, \ldots, \frac{n}{2} - 1$ the point $s = 0$ lies in the considered strip.

Thus, inserting (8.4) into (8.2), at $s = 0$ we obtain the identity

$$\begin{aligned}E^{\mathrm{ell}}_{Q}(P, s) - \mathcal{G}^{\mathrm{par}}_{Q,1}(P, s) &= \frac{1}{|\Gamma_Q|} \sum_{l=\frac{n}{2}}^{\infty} \frac{\left(\frac{s}{2}\right)_l}{l!}\, K^{\mathrm{hyp}}(P, Q, s + 2l) + \frac{1}{|\Gamma_Q|} \sum_{l=0}^{\frac{n}{2}-1} \frac{\left(\frac{s}{2}\right)_l}{l!} \sum_{j=0}^{\infty} a_{j,Q}(s + 2l)\, \psi_j(P) \\ &\quad + \frac{1}{|\Gamma_Q|} \sum_{l=0}^{\frac{n}{2}-1} \frac{\left(\frac{s}{2}\right)_l}{l!}\, \frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s + 2l)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt,\end{aligned} \tag{8.5}$$

and to derive the Laurent expansion at this point we work from formula (8.5).

We start by determining the Laurent expansion at $s = 0$ of the series

$$\frac{1}{|\Gamma_Q|} \sum_{l=\frac{n}{2}}^{\infty} \frac{\left(\frac{s}{2}\right)_l}{l!}\, K^{\mathrm{hyp}}(P, Q, s + 2l).$$

By the proof of Theorem 6.3.2 this series is a holomorphic function for $P \in \mathbb{H}^n$ with $P \neq \eta Q$ for any $\eta \in \Gamma$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n - 1 - 2 \cdot \frac{n}{2} = -1$. For any $l \geq \frac{n}{2}$ the function $K^{\mathrm{hyp}}(P, Q, s+2l)$ admits a Laurent expansion at $s = 0$ of the form

$$K^{\mathrm{hyp}}(P, Q, s + 2l) = K^{\mathrm{hyp}}(P, Q, 2l) + \mathrm{O}(s),$$

while the Pochhammer symbol $\left(\frac{s}{2}\right)_l$ has the Laurent expansion

$$\left(\frac{s}{2}\right)_l = \frac{(l-1)!}{2} \cdot s + \frac{(l-1)!}{4} \left(\sum_{m=1}^{l-1} \frac{1}{m}\right) \cdot s^2 + \mathrm{O}(s^3) \tag{8.6}$$

at $s = 0$ (see (A.15)). Thus, at $s = 0$ we obtain the Laurent expansion

$$\frac{1}{|\Gamma_Q|} \sum_{l=\frac{n}{2}}^{\infty} \frac{\left(\frac{s}{2}\right)_l}{l!}\, K^{\mathrm{hyp}}(P, Q, s + 2l) = \frac{H_{n,Q,1}(P)}{|\Gamma_Q|} \cdot s + \mathrm{O}(s^2), \tag{8.7}$$

where

$$H_{n,Q,1}(P) := \sum_{l=\frac{n}{2}}^{\infty} \frac{1}{2l}\, K^{\mathrm{hyp}}(P,Q,2l). \tag{8.8}$$

Now we turn to the other summands on the right-hand side of (8.5), and first consider the case $l=0$.

For $j=0$, i.e. $\lambda_j=0$, $r_j=-i\,\frac{n-1}{2}$ and $\psi_j(P)=\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)^{-1/2}$, the function $a_{j,Q}(s)\,\psi_j(P)$ in the series

$$\sum_{j=0}^{\infty} a_{j,Q}(s)\,\psi_j(P) \tag{8.9}$$

arising from the discrete spectrum takes the form

$$a_{0,Q}(s)\,\psi_0(P) = \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma(s)}\,\Gamma\Big(\frac{s}{2}\Big)\,\Gamma\Big(\frac{s-n+1}{2}\Big).$$

At $s=0$ we have the Laurent expansions

$$\begin{aligned}
2^{s-1} &= \frac{1}{2} + \frac{\log(2)}{2}\cdot s + \mathrm{O}(s^2),\\
\frac{1}{\Gamma(s)} &= s + \gamma\cdot s^2 + \mathrm{O}(s^3),\\
\Gamma\Big(\frac{s}{2}\Big) &= 2\cdot\frac{1}{s} - \gamma + \mathrm{O}(s),\\
\Gamma\Big(\frac{s-n+1}{2}\Big) &= \Gamma\Big(\frac{1-n}{2}\Big) + \frac{1}{2}\,\Gamma\Big(\frac{1-n}{2}\Big)\,\psi^{(0)}\Big(\frac{1-n}{2}\Big)\cdot s + \mathrm{O}(s^2),
\end{aligned}$$

where γ denotes the Euler-Mascheroni constant (see (A.6)), $\psi^{(0)}(s)$ is the digamma function (see (A.8)), and where we used $\frac{n-1}{2}\notin\mathbb{N}_0$, (A.12) and (A.13). This implies that the function $a_{0,Q}(s)\,\psi_0(P)$ admits a Laurent expansion at $s=0$ of the form

$$a_{0,Q}(s)\,\psi_0(P) = \frac{\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1-n}{2}\right)}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1-n}{2}\right)\left(\gamma+\log(4)+\psi^{(0)}\left(\frac{1-n}{2}\right)\right)}{2\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\cdot s + \mathrm{O}(s^2).$$

Now we consider $j\geq 1$, so that either $r_j>0$ or $r_j\in\left(-i\,\frac{n-1}{2},0\right]$ holds true. In the proof of Proposition 7.1.2 we have shown that the function $\Gamma\left(\frac{s-\frac{n-1}{2}+ir_j}{2}\right)$ has a pole at $s=0$ if and only if

$$\begin{aligned}
&\frac{n-1}{4}-\frac{ir_j}{2}\in\Big\{1,\dots,\Big\lfloor\frac{n-1}{4}\Big\rfloor\Big\}\\
&\quad\Longleftrightarrow\; r_j\in\Big\{-i\Big(\frac{n-1}{2}-2N\Big)\;\Big|\;N\in\Big\{1,\dots,\Big\lfloor\frac{n-1}{4}\Big\rfloor\Big\}\Big\} =: M_1(n),
\end{aligned}$$

while the function $\Gamma\left(\frac{s-\frac{n-1}{2}-ir_j}{2}\right)$ has a pole at $s=0$ if and only if

$$\begin{aligned}
&\frac{n-1}{4}+\frac{ir_j}{2}\in\Big\{\Big\lceil\frac{n-1}{4}\Big\rceil,\dots,\frac{n}{2}-1\Big\}\\
&\quad\Longleftrightarrow\; r_j\in\Big\{-i\Big(2N-\frac{n-1}{2}\Big)\;\Big|\;N\in\Big\{\Big\lceil\frac{n-1}{4}\Big\rceil,\dots,\frac{n}{2}-1\Big\}\Big\} =: M_2(n),
\end{aligned}$$

and that $M_1(n)\cap M_2(n)=\emptyset$.

If $r_j \in M_1(n)$, then $r_j \notin M_2(n)$, and, using (A.12) and (A.13), at $s = 0$ we have the Laurent expansions

$$\Gamma\Big(\frac{s - \frac{n-1}{2} + ir_j}{2}\Big) = \frac{2\,(-1)^{\frac{n-1}{4} - \frac{ir_j}{2}}}{\Big(\frac{n-1}{4} - \frac{ir_j}{2}\Big)!} \cdot \frac{1}{s} + \frac{(-1)^{\frac{n-1}{4} - \frac{ir_j}{2}}}{\Big(\frac{n-1}{4} - \frac{ir_j}{2}\Big)!} \left(\sum_{m=1}^{\frac{n-1}{4} - \frac{ir_j}{2}} \frac{1}{m} - \gamma \right) + \mathrm{O}(s),$$

$$\Gamma\Big(\frac{s - \frac{n-1}{2} - ir_j}{2}\Big) = \Gamma\Big(- \frac{ir_j}{2} - \frac{n-1}{4}\Big) + \frac{1}{2}\,\Gamma\Big(- \frac{ir_j}{2} - \frac{n-1}{4}\Big)\,\psi^{(0)}\Big(- \frac{ir_j}{2} - \frac{n-1}{4}\Big) \cdot s + \mathrm{O}(s^2).$$

Hence, for $j \geq 1$ with $r_j \in M_1(n)$ the function $a_{j,Q}(s)\,\psi_j(P)$ admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned} a_{j,Q}(s)\,\psi_j(P) = {} & \frac{(-1)^{\frac{n-1}{4} - \frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{\Big(\frac{n-1}{4} - \frac{ir_j}{2}\Big)!}\,\Gamma\Big(- \frac{ir_j}{2} - \frac{n-1}{4}\Big)\,\psi_j(P)\,\overline{\psi_j}(Q) \\ & + \frac{(-1)^{\frac{n-1}{4} - \frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{2\Big(\frac{n-1}{4} - \frac{ir_j}{2}\Big)!}\,\Gamma\Big(- \frac{ir_j}{2} - \frac{n-1}{4}\Big) \left(\sum_{m=1}^{\frac{n-1}{4} - \frac{ir_j}{2}} \frac{1}{m} + \gamma + \log(4) + \psi^{(0)}\Big(- \frac{ir_j}{2} - \frac{n-1}{4}\Big) \right) \\ & \cdot \psi_j(P)\,\overline{\psi_j}(Q) \cdot s + \mathrm{O}(s^2), \end{aligned}$$

and the respective part of the series (8.9) arising from the discrete spectrum has a Laurent expansion at $s = 0$ of the form

$$\sum_{\substack{j \in \mathbb{N}:\\ r_j \in M_1(n)}} a_{j,Q}(s)\,\psi_j(P) = G_{n,Q,1}(P) + H_{n,Q,2}(P) \cdot s + \mathrm{O}(s^2),$$

where

$$G_{n,Q,1}(P) := \sum_{\substack{j \in \mathbb{N}:\\ r_j \in M_1(n)}} \frac{(-1)^{\frac{n-1}{4} - \frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{\Big(\frac{n-1}{4} - \frac{ir_j}{2}\Big)!}\,\Gamma\Big(- \frac{ir_j}{2} - \frac{n-1}{4}\Big)\,\psi_j(P)\,\overline{\psi_j}(Q), \tag{8.10}$$

$$\begin{aligned} H_{n,Q,2}(P) := {} & \sum_{\substack{j \in \mathbb{N}:\\ r_j \in M_1(n)}} \frac{(-1)^{\frac{n-1}{4} - \frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{2\Big(\frac{n-1}{4} - \frac{ir_j}{2}\Big)!}\,\Gamma\Big(- \frac{ir_j}{2} - \frac{n-1}{4}\Big) \\ & \cdot \left(\sum_{m=1}^{\frac{n-1}{4} - \frac{ir_j}{2}} \frac{1}{m} + \gamma + \log(4) + \psi^{(0)}\Big(- \frac{ir_j}{2} - \frac{n-1}{4}\Big) \right) \psi_j(P)\,\overline{\psi_j}(Q). \end{aligned} \tag{8.11}$$

If $r_j \in M_2(n)$, then $r_j \notin M_1(n)$, and, using (A.12) and (A.13) again, at $s = 0$ we have the Laurent expansions

$$\Gamma\Big(\frac{s - \frac{n-1}{2} + ir_j}{2}\Big) = \Gamma\Big(\frac{ir_j}{2} - \frac{n-1}{4}\Big) + \frac{1}{2}\,\Gamma\Big(\frac{ir_j}{2} - \frac{n-1}{4}\Big)\,\psi^{(0)}\Big(\frac{ir_j}{2} - \frac{n-1}{4}\Big) \cdot s + \mathrm{O}(s^2),$$

$$\Gamma\Big(\frac{s - \frac{n-1}{2} - ir_j}{2}\Big) = \frac{2\,(-1)^{\frac{n-1}{4} + \frac{ir_j}{2}}}{\Big(\frac{n-1}{4} + \frac{ir_j}{2}\Big)!} \cdot \frac{1}{s} + \frac{(-1)^{\frac{n-1}{4} + \frac{ir_j}{2}}}{\Big(\frac{n-1}{4} + \frac{ir_j}{2}\Big)!} \left(\sum_{m=1}^{\frac{n-1}{4} + \frac{ir_j}{2}} \frac{1}{m} - \gamma \right) + \mathrm{O}(s).$$

Thus, for $j \geq 1$ with $r_j \in M_2(n)$ the function $a_{j,Q}(s)\,\psi_j(P)$ admits a Laurent expansion at $s = 0$

of the form

$$\begin{aligned} a_{j,Q}(s)\,\psi_j(P) &= \frac{(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi_j(P)\,\overline{\psi_j}(Q) \\ &+ \frac{(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{2\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\left(\sum_{m=1}^{\frac{n-1}{4}+\frac{ir_j}{2}}\frac{1}{m}+\gamma+\log(4)+\psi^{(0)}\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\right) \\ &\quad\cdot\psi_j(P)\,\overline{\psi_j}(Q)\cdot s+\mathrm{O}(s^2), \end{aligned}$$

which implies that the respective part of the series (8.9) arising from the discrete spectrum has a Laurent expansion at $s=0$ of the form

$$\sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_2(n)}} a_{j,Q}(s)\,\psi_j(P) = G_{n,Q,2}(P)+H_{n,Q,3}(P)\cdot s+\mathrm{O}(s^2),$$

where

$$G_{n,Q,2}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_2(n)}} \frac{(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi_j(P)\,\overline{\psi_j}(Q), \tag{8.12}$$

$$\begin{aligned} H_{n,Q,3}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_2(n)}} &\frac{(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}}\,\pi^{\frac{n-1}{2}}}{2\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big) \\ &\cdot\left(\sum_{m=1}^{\frac{n-1}{4}+\frac{ir_j}{2}}\frac{1}{m}+\gamma+\log(4)+\psi^{(0)}\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\right)\psi_j(P)\,\overline{\psi_j}(Q). \end{aligned} \tag{8.13}$$

If $r_j\notin M_1(n)\cup M_2(n)$, then (A.13) yields that at $s=0$ we have the Laurent expansions

$$\Gamma\Big(\frac{s-\frac{n-1}{2}\pm ir_j}{2}\Big)=\Gamma\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}\Big)+\frac{1}{2}\Gamma\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi^{(0)}\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}\Big)\cdot s+\mathrm{O}(s^2).$$

Thus, for $j\geq 1$ with $r_j\notin M_1(n)\cup M_2(n)$ the function $a_{j,Q}(s)\,\psi_j(P)$ admits a Laurent expansion at $s=0$ of the form

$$a_{j,Q}(s)\,\psi_j(P)=\frac{\pi^{\frac{n-1}{2}}}{2}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi_j(P)\,\overline{\psi_j}(Q)\cdot s+\mathrm{O}(s^2),$$

so the respective part of the series (8.9) arising from the discrete spectrum has a Laurent expansion at $s=0$ of the form

$$\sum_{\substack{j\in\mathbb{N}:\\ r_j\notin M_1(n)\cup M_2(n)}} a_{j,Q}(s)\,\psi_j(P)=H_{n,Q,4}(P)\cdot s+\mathrm{O}(s^2),$$

where

$$H_{n,Q,4}(P):=\sum_{\substack{j\in\mathbb{N}:\\ r_j\notin M_1(n)\cup M_2(n)}}\frac{\pi^{\frac{n-1}{2}}}{2}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi_j(P)\,\overline{\psi_j}(Q). \tag{8.14}$$

Further, for any $t\in\mathbb{R}$ at $s=0$ we have the Laurent expansions

$$\Gamma\Big(\frac{s-\frac{n-1}{2}\pm it}{2}\Big)=\Gamma\Big(\pm\frac{it}{2}-\frac{n-1}{4}\Big)+\frac{1}{2}\Gamma\Big(\pm\frac{it}{2}-\frac{n-1}{4}\Big)\,\psi^{(0)}\Big(\pm\frac{it}{2}-\frac{n-1}{4}\Big)\cdot s+\mathrm{O}(s^2),$$

where we made use of $\frac{n-1}{4} \notin \mathbb{N}_0$ and (A.13). Hence, for $k = 1, \ldots, c_\Gamma$ and any $t \in \mathbb{R}$ the function $a_{t,\eta_k,Q}(s)$ in the integral

$$\int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt$$

arising from the continuous spectrum admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned} a_{t,\eta_k,Q}(s) &= \frac{\pi^{\frac{n-1}{2}}}{2}\, \Gamma\Big(\frac{it}{2} - \frac{n-1}{4}\Big)\, \Gamma\Big(-\frac{it}{2} - \frac{n-1}{4}\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} - it\Big) \cdot s + \mathrm{O}(s^2) \\ &= \frac{\pi^{\frac{n-1}{2}}}{2} \Big|\Gamma\Big(\frac{it}{2} - \frac{n-1}{4}\Big)\Big|^2 E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} - it\Big) \cdot s + \mathrm{O}(s^2). \end{aligned}$$

We conclude that at $s = 0$ we have the Laurent expansion

$$\frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt = H_{n,Q,5}(P) \cdot s + \mathrm{O}(s^2)$$

of the continuous part, where

$$H_{n,Q,5}(P) := \frac{\pi^{\frac{n-3}{2}}}{8} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^{\infty} \Big|\Gamma\Big(\frac{it}{2} - \frac{n-1}{4}\Big)\Big|^2 E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} - it\Big)\, dt. \quad (8.15)$$

Adding up and using $\left(\frac{s}{2}\right)_0 / 0! = 1$, the summand for $l = 0$ on the right-hand side of formula (8.5) admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned} &\frac{1}{|\Gamma_Q|} \sum_{j=0}^{\infty} a_{j,Q}(s)\, \psi_j(P) + \frac{1}{4\pi\, |\Gamma_Q|} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt \\ &= \frac{\pi^{\frac{n-1}{2}}\, \Gamma\left(\frac{1-n}{2}\right)}{|\Gamma_Q|\, \mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|} \sum_{m=1}^{2} G_{n,Q,m}(P) \\ &\quad + \left(\frac{\pi^{\frac{n-1}{2}}\, \Gamma\left(\frac{1-n}{2}\right) \left(\gamma + \log(4) + \psi^{(0)}\left(\frac{1-n}{2}\right)\right)}{2\, |\Gamma_Q|\, \mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|} \sum_{m=2}^{5} H_{n,Q,m}(P) \right) \cdot s + \mathrm{O}(s^2). \end{aligned} \quad (8.16)$$

Now we consider the case $l > 0$, i.e. $l \in \left\{1, \ldots, \frac{n}{2} - 1\right\}$.

If $j = 0$, i.e. $\lambda_j = 0$, $r_j = -i\,\frac{n-1}{2}$ and $\psi_j(P) = \mathrm{vol}(\Gamma\backslash\mathbb{H}^n)^{-1/2}$, the function $a_{j,Q}(s + 2l)\, \psi_j(P)$ in the series

$$\sum_{j=0}^{\infty} a_{j,Q}(s + 2l)\, \psi_j(P) \quad (8.17)$$

arising from the discrete spectrum takes the form

$$a_{0,Q}(s + 2l)\, \psi_0(P) = \frac{2^{s+2l-1}\, \pi^{\frac{n-1}{2}}}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\, \Gamma(s + 2l)}\, \Gamma\Big(\frac{s}{2} + l\Big)\, \Gamma\Big(\frac{s-n+1}{2} + l\Big).$$

At $s = 0$ we have the Laurent expansions

$$\begin{aligned} &2^{s+2l-1} = 2^{2l-1} + 2^{2l-1}\, \log(2) \cdot s + \mathrm{O}(s^2), \\ &\frac{1}{\Gamma(s+2l)} = \frac{1}{\Gamma(2l)} - \frac{\psi^{(0)}(2l)}{\Gamma(2l)} \cdot s + \mathrm{O}(s^2) = \frac{1}{(2l-1)!} - \frac{1}{(2l-1)!} \left(\sum_{m=1}^{2l-1} \frac{1}{m} - \gamma \right) \cdot s + \mathrm{O}(s^2), \\ &\Gamma\left(\frac{s}{2} + l\right) = \Gamma(l) + \frac{1}{2}\, \Gamma(l)\, \psi^{(0)}(l) \cdot s + \mathrm{O}(s^2) = (l-1)! + \frac{(l-1)!}{2} \left(\sum_{m=1}^{l-1} \frac{1}{m} - \gamma \right) \cdot s + \mathrm{O}(s^2), \\ &\Gamma\Big(\frac{s-n+1}{2} + l\Big) = \Gamma\Big(\frac{1-n}{2} + l\Big) + \frac{1}{2}\, \Gamma\Big(\frac{1-n}{2} + l\Big)\, \psi^{(0)}\Big(\frac{1-n}{2} + l\Big) \cdot s + \mathrm{O}(s^2), \end{aligned}$$

where we used $\frac{n-1}{2} \notin \mathbb{N}_0$ and the formulas (A.9) and (A.13). This implies that the function $a_{0,Q}(s+2l)\,\psi_0(P)$ admits a Laurent expansion at $s=0$ of the form

$$\begin{aligned} a_{0,Q}(s+2l)\,\psi_0(P) &= \frac{2^{2l-1}\,\pi^{\frac{n-1}{2}}\,(l-1)!\,\Gamma\left(\frac{1-n}{2}+l\right)}{(2l-1)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} \\ &+ \frac{4^{l-1}\,\pi^{\frac{n-1}{2}}\,(l-1)!\,\Gamma\left(\frac{1-n}{2}+l\right)}{(2l-1)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} \left(\sum_{m=1}^{l-1}\frac{1}{m} - 2\sum_{m=1}^{2l-1}\frac{1}{m} + \gamma + \log(4) + \psi^{(0)}\Big(\frac{1-n}{2}+l\Big)\right)\cdot s \\ &+ \mathrm{O}(s^2). \end{aligned}$$

Together with the Laurent expansion (8.6) of the Pochhammer symbol $\left(\frac{s}{2}\right)_l$, at $s=0$ we obtain the Laurent expansion

$$\frac{\left(\frac{s}{2}\right)_l}{l!}\,a_{0,Q}(s+2l)\,\psi_0(P) = \frac{4^{l-1}\,\pi^{\frac{n-1}{2}}\,(l-1)!\,\Gamma\left(\frac{1-n}{2}+l\right)}{l\,(2l-1)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\cdot s + \mathrm{O}(s^2).$$

Now we let $j \geq 1$ so that either $r_j > 0$ or $r_j \in \left(-i\,\frac{n-1}{2}, 0\right]$ holds true. In the first case it is clear that $\frac{n-1}{4} \pm \frac{ir_j}{2} - l \notin \mathbb{N}_0$. In the latter case we have $\frac{n-1}{4} \pm \frac{ir_j}{2} - l \in \mathbb{R}$ with $\frac{n-1}{4} - \frac{ir_j}{2} - l \in \left(-l, \frac{n-1}{4} - l\right]$ and $\frac{n-1}{4} + \frac{ir_j}{2} - l \in \left[\frac{n-1}{4} - l, \frac{n-1}{2} - l\right)$.
The elements of $\mathbb{N}_0$ contained in the interval $\left(-l, \frac{n-1}{4} - l\right]$ are $0, \ldots, \left\lfloor\frac{n-1}{4}\right\rfloor - l$. The interval $\left[\frac{n-1}{4} - l, \frac{n-1}{2} - l\right)$ contains the elements $\left\lceil\frac{n-1}{4}\right\rceil - l, \ldots, \frac{n}{2} - l - 1$ of $\mathbb{N}_0$ if $l \leq \frac{n-1}{4}$, and the elements $0, \ldots, \frac{n}{2} - l - 1$ of $\mathbb{N}_0$ if $l > \frac{n-1}{4}$, showing that the elements of $\mathbb{N}_0$ contained in the interval $\left[\frac{n-1}{4} - l, \frac{n-1}{2} - l\right)$ are $\max\left(\left\lceil\frac{n-1}{4}\right\rceil - l, 0\right), \ldots, \frac{n}{2} - l - 1$.
This implies that the function $\Gamma\left(\frac{s+2l-\frac{n-1}{2}+ir_j}{2}\right)$ has a pole at $s=0$ if and only if

$$\begin{aligned} &\frac{n-1}{4} - \frac{ir_j}{2} - l \in \Big\{0, \ldots, \Big\lfloor\frac{n-1}{4}\Big\rfloor - l\Big\} \iff \frac{n-1}{4} - \frac{ir_j}{2} \in \Big\{l, \ldots, \Big\lfloor\frac{n-1}{4}\Big\rfloor\Big\} \\ &\iff r_j \in \Big\{ -i\Big(\frac{n-1}{2} - 2N\Big) \;\Big|\; N \in \Big\{l, \ldots, \Big\lfloor\frac{n-1}{4}\Big\rfloor\Big\}\Big\}, \end{aligned}$$

while the function $\Gamma\left(\frac{s+2l-\frac{n-1}{2}-ir_j}{2}\right)$ has a pole at $s=0$ if and only if

$$\begin{aligned} &\frac{n-1}{4} + \frac{ir_j}{2} - l \in \Big\{\max\Big(\Big\lceil\frac{n-1}{4}\Big\rceil - l, 0\Big), \ldots, \frac{n}{2} - l - 1\Big\} \\ &\iff \frac{n-1}{4} + \frac{ir_j}{2} \in \Big\{\max\Big(\Big\lceil\frac{n-1}{4}\Big\rceil, l\Big), \ldots, \frac{n}{2} - 1\Big\} \\ &\iff r_j \in \Big\{ -i\Big(2N - \frac{n-1}{2}\Big) \;\Big|\; N \in \Big\{\max\Big(\Big\lceil\frac{n-1}{4}\Big\rceil, l\Big), \ldots, \frac{n}{2} - 1\Big\}\Big\}. \end{aligned}$$

Further, if $\frac{n-1}{4} - \frac{ir_j}{2} = N$ for some $N \in \left\{l, \ldots, \left\lfloor\frac{n-1}{4}\right\rfloor\right\}$, then $\frac{n-1}{2} \notin \mathbb{Z}$ implies that

$$\frac{n-1}{4} + \frac{ir_j}{2} = \frac{n-1}{2} - N \;\notin \mathbb{Z}$$

cannot be an element of $\mathbb{N}_0$; and if $\frac{n-1}{4} + \frac{ir_j}{2} = N$ for some $N \in \left\{\max\left(\left\lceil\frac{n-1}{4}\right\rceil, l\right), \ldots, \frac{n}{2} - 1\right\}$, then $\frac{n-1}{2} \notin \mathbb{Z}$ yields that

$$\frac{n-1}{4} - \frac{ir_j}{2} = \frac{n-1}{2} - N \;\notin \mathbb{Z}$$

cannot be an element of $\mathbb{N}_0$. This proves that for any $j \geq 1$ at most one of the two functions $\Gamma\left(\frac{s+2l-\frac{n-1}{2}+ir_j}{2}\right)$, $\Gamma\left(\frac{s+2l-\frac{n-1}{2}-ir_j}{2}\right)$ has a pole at $s=0$. Now we set

$$\begin{aligned} M_1(n,l) &:= \Big\{ -i\Big(\frac{n-1}{2} - 2N\Big) \;\Big|\; N \in \Big\{l, \ldots, \Big\lfloor\frac{n-1}{4}\Big\rfloor\Big\}\Big\}, \\ M_2(n,l) &:= \Big\{ -i\Big(2N - \frac{n-1}{2}\Big) \;\Big|\; N \in \Big\{\max\Big(\Big\lceil\frac{n-1}{4}\Big\rceil, l\Big), \ldots, \frac{n}{2} - 1\Big\}\Big\}, \end{aligned}$$

and we have just shown that $M_1(n,l) \cap M_2(n,l) = \emptyset$.

If $r_j \in M_1(n,l)$, then $r_j \notin M_2(n,l)$, and at $s = 0$ we have the Laurent expansions

$$\Gamma\Big(\frac{s+2l-\frac{n-1}{2}+ir_j}{2}\Big) = \frac{2\,(-1)^{\frac{n-1}{4}-\frac{ir_j}{2}-l}}{\Big(\frac{n-1}{4}-\frac{ir_j}{2}-l\Big)!}\cdot\frac{1}{s} + \frac{(-1)^{\frac{n-1}{4}-\frac{ir_j}{2}-l}}{\Big(\frac{n-1}{4}-\frac{ir_j}{2}-l\Big)!}\left(\sum_{m=1}^{\frac{n-1}{4}-\frac{ir_j}{2}-l}\frac{1}{m}-\gamma\right) + \mathrm{O}(s),$$

$$\begin{aligned}\Gamma\Big(\frac{s+2l-\frac{n-1}{2}-ir_j}{2}\Big) &= \Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\\ &\quad+\frac{1}{2}\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\psi^{(0)}\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\cdot s+\mathrm{O}(s^2).\end{aligned}$$

Together with the Laurent expansion

$$\frac{2^{s+2l-1}}{\Gamma(s+2l)} = \frac{2^{2l-1}}{(2l-1)!} + \frac{2^{2l-1}}{(2l-1)!}\left(-\sum_{m=1}^{2l-1}\frac{1}{m}+\gamma+\log(2)\right)\cdot s + \mathrm{O}(s^2)$$

this implies that for $j \geq 1$ with $r_j \in M_1(n,l)$ the function $a_{j,Q}(s+2l)\,\psi_j(P)$ admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned}a_{j,Q}(s+2l)\,\psi_j(P) &= \frac{4^l\,(-1)^{\frac{n-1}{4}-\frac{ir_j}{2}-l}\,\pi^{\frac{n-1}{2}}}{(2l-1)!\,\Big(\frac{n-1}{4}-\frac{ir_j}{2}-l\Big)!}\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q)\cdot\frac{1}{s}\\ &+\frac{2^{2l-1}\,(-1)^{\frac{n-1}{4}-\frac{ir_j}{2}-l}\,\pi^{\frac{n-1}{2}}}{(2l-1)!\,\Big(\frac{n-1}{4}-\frac{ir_j}{2}-l\Big)!}\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\\ &\cdot\left(\sum_{m=1}^{\frac{n-1}{4}-\frac{ir_j}{2}-l}\frac{1}{m}-2\sum_{m=1}^{2l-1}\frac{1}{m}+\gamma+\log(4)+\psi^{(0)}\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\right)\psi_j(P)\,\overline{\psi_j}(Q)+\mathrm{O}(s),\end{aligned}$$

and the respective part of the series (8.17) after multiplication by $\left(\frac{s}{2}\right)_l/l!$ has a Laurent expansion at $s = 0$ of the form

$$\frac{\left(\frac{s}{2}\right)_l}{l!}\sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_1(n,l)}} a_{j,Q}(s+2l)\,\psi_j(P) = G_{n,Q,l,1}(P) + H_{n,Q,l,1}(P)\cdot s + \mathrm{O}(s^2),$$

where

$$G_{n,Q,l,1}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_1(n,l)}} \frac{2^{2l-1}\,(-1)^{\frac{n-1}{4}-\frac{ir_j}{2}-l}\,\pi^{\frac{n-1}{2}}}{l\,(2l-1)!\,\Big(\frac{n-1}{4}-\frac{ir_j}{2}-l\Big)!}\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q), \tag{8.18}$$

$$\begin{aligned}H_{n,Q,l,1}(P) &:= \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_1(n,l)}} \frac{4^{l-1}\,(-1)^{\frac{n-1}{4}-\frac{ir_j}{2}-l}\,\pi^{\frac{n-1}{2}}}{l\,(2l-1)!\,\Big(\frac{n-1}{4}-\frac{ir_j}{2}-l\Big)!}\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q)\\ &\cdot\left(\sum_{m=1}^{\frac{n-1}{4}-\frac{ir_j}{2}-l}\frac{1}{m}-2\sum_{m=1}^{2l-1}\frac{1}{m}+\sum_{m=1}^{l-1}\frac{1}{m}+\gamma+\log(4)+\psi^{(0)}\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\right).\end{aligned} \tag{8.19}$$

In the case $r_j \in M_2(n,l)$ we have $r_j \notin M_1(n,l)$ and at $s=0$ we have the Laurent expansions

$$\begin{aligned}\Gamma\Big(\frac{s+2l-\frac{n-1}{2}+ir_j}{2}\Big) &= \Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\\ &\quad+\frac{1}{2}\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\psi^{(0)}\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\cdot s+\mathrm{O}(s^2),\\ \Gamma\Big(\frac{s+2l-\frac{n-1}{2}-ir_j}{2}\Big) &= \frac{2\,(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}-l}}{\Big(\frac{n-1}{4}+\frac{ir_j}{2}-l\Big)!}\cdot\frac{1}{s}+\frac{(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}-l}}{\Big(\frac{n-1}{4}+\frac{ir_j}{2}-l\Big)!}\left(\sum_{m=1}^{\frac{n-1}{4}+\frac{ir_j}{2}-l}\frac{1}{m}-\gamma\right)+\mathrm{O}(s).\end{aligned}$$

Hence, for $j\geq 1$ with $r_j \in M_2(n,l)$ the function $a_{j,Q}(s+2l)\,\psi_j(P)$ admits a Laurent expansion at $s=0$ of the form

$$\begin{aligned}a_{j,Q}(s+2l)\,\psi_j(P) &= \frac{4^l\,(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}-l}\,\pi^{\frac{n-1}{2}}}{(2l-1)!\,\Big(\frac{n-1}{4}+\frac{ir_j}{2}-l\Big)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q)\cdot\frac{1}{s}\\ &+\frac{2^{2l-1}\,(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}-l}\,\pi^{\frac{n-1}{2}}}{(2l-1)!\,\Big(\frac{n-1}{4}+\frac{ir_j}{2}-l\Big)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\\ &\cdot\left(\sum_{m=1}^{\frac{n-1}{4}+\frac{ir_j}{2}-l}\frac{1}{m}-2\sum_{m=1}^{2l-1}\frac{1}{m}+\gamma+\log(4)+\psi^{(0)}\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\right)\psi_j(P)\,\overline{\psi_j}(Q)+\mathrm{O}(s),\end{aligned}$$

which implies that the respective part of the series (8.17) after multiplication by $\big(\frac{s}{2}\big)_l/l!$ has a Laurent expansion at $s=0$ of the form

$$\frac{\big(\frac{s}{2}\big)_l}{l!}\sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_2(n,l)}} a_{j,Q}(s+2l)\,\psi_j(P)=G_{n,Q,l,2}(P)+H_{n,Q,l,2}(P)\cdot s+\mathrm{O}(s^2),$$

where

$$G_{n,Q,l,2}(P):=\sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_2(n,l)}}\frac{2^{2l-1}\,(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}-l}\,\pi^{\frac{n-1}{2}}}{l\,(2l-1)!\,\Big(\frac{n-1}{4}+\frac{ir_j}{2}-l\Big)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q),\tag{8.20}$$

$$\begin{aligned}H_{n,Q,l,2}(P):=&\sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_2(n,l)}}\frac{4^{l-1}\,(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}-l}\,\pi^{\frac{n-1}{2}}}{l\,(2l-1)!\,\Big(\frac{n-1}{4}+\frac{ir_j}{2}-l\Big)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q)\\ &\cdot\left(\sum_{m=1}^{\frac{n-1}{4}+\frac{ir_j}{2}-l}\frac{1}{m}-2\sum_{m=1}^{2l-1}\frac{1}{m}+\sum_{m=1}^{l-1}\frac{1}{m}+\gamma+\log(4)+\psi^{(0)}\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\right).\end{aligned}\tag{8.21}$$

If $r_j\notin M_1(n,l)\cup M_2(n,l)$, then at $s=0$ we have the Laurent expansions

$$\begin{aligned}\Gamma\Big(\frac{s+2l-\frac{n-1}{2}\pm ir_j}{2}\Big)&=\Gamma\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\\ &\quad+\frac{1}{2}\Gamma\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\psi^{(0)}\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\cdot s+\mathrm{O}(s^2).\end{aligned}$$

This yields that for $j\geq 1$ with $r_j\notin M_1(n,l)\cup M_2(n,l)$ the function $a_{j,Q}(s+2l)\,\psi_j(P)$ admits a

Laurent expansion at $s = 0$ of the form

$$\begin{aligned} a_{j,Q}(s+2l)\,\psi_j(P) &= \frac{2^{2l-1}\,\pi^{\frac{n-1}{2}}}{(2l-1)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q) \\ &+ \frac{4^{l-1}\,\pi^{\frac{n-1}{2}}}{(2l-1)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q) \\ &\cdot\Bigg(-2\sum_{m=1}^{2l-1}\frac{1}{m}+2\gamma+\log(4)+\psi^{(0)}\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)+\psi^{(0)}\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\Bigg)\cdot s+\mathrm{O}(s^2), \end{aligned}$$

so the respective part of the series (8.17) after multiplication by $\left(\frac{s}{2}\right)_l/l!$ has a Laurent expansion at $s = 0$ of the form

$$\frac{\left(\frac{s}{2}\right)_l}{l!}\sum_{\substack{j\in\mathbb{N}:\\ r_j\notin M_1(n,l)\cup M_2(n,l)}} a_{j,Q}(s+2l)\,\psi_j(P) = H_{n,Q,l,3}(P)\cdot s+\mathrm{O}(s^2),$$

where

$$H_{n,Q,l,3}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\notin M_1(n,l)\cup M_2(n,l)}} \frac{4^{l-1}\,\pi^{\frac{n-1}{2}}}{l\,(2l-1)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q). \tag{8.22}$$

Moreover, for any $t \in \mathbb{R}$ at $s = 0$ we have the Laurent expansions

$$\begin{aligned}\Gamma\Big(\frac{s+2l-\frac{n-1}{2}\pm it}{2}\Big) &= \Gamma\Big(\pm\frac{it}{2}-\frac{n-1}{4}+l\Big)+\frac{1}{2}\,\Gamma\Big(\pm\frac{it}{2}-\frac{n-1}{4}+l\Big)\,\psi^{(0)}\Big(\pm\frac{it}{2}-\frac{n-1}{4}+l\Big)\cdot s \\ &+\mathrm{O}(s^2),\end{aligned}$$

where we used $\frac{n-1}{4}\notin\mathbb{N}_0$ and (A.13) again. Hence, for $k = 1,\ldots,c_\Gamma$ and any $t \in \mathbb{R}$ the function $a_{t,\eta_k,Q}(s+2l)$ in the integral

$$\int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s+2l)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt$$

arising from the continuous spectrum admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned} a_{t,\eta_k,Q}(s+2l) &= \frac{2^{2l-1}\,\pi^{\frac{n-1}{2}}}{(2l-1)!}\,\Big|\Gamma\Big(\frac{it}{2}-\frac{n-1}{4}+l\Big)\Big|^2\,E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-it\Big) \\ &+\frac{4^{l-1}\,\pi^{\frac{n-1}{2}}}{(2l-1)!}\,\Big|\Gamma\Big(\frac{it}{2}-\frac{n-1}{4}+l\Big)\Big|^2\,E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-it\Big) \\ &\cdot\Bigg(-2\sum_{m=1}^{2l-1}\frac{1}{m}+2\gamma+\log(4)+\psi^{(0)}\Big(\frac{it}{2}-\frac{n-1}{4}+l\Big)+\psi^{(0)}\Big(-\frac{it}{2}-\frac{n-1}{4}+l\Big)\Bigg)\cdot s+\mathrm{O}(s^2). \end{aligned}$$

We conclude that, after multiplication by $\left(\frac{s}{2}\right)_l/l!$, at $s = 0$ we have the Laurent expansion

$$\frac{\left(\frac{s}{2}\right)_l}{l!}\,\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s+2l)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt = H_{n,Q,l,4}(P)\cdot s+\mathrm{O}(s^2),$$

where

$$H_{n,Q,l,4}(P) := \frac{4^{l-2}\,\pi^{\frac{n-3}{2}}}{l\,(2l-1)!}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty}\Big|\Gamma\Big(\frac{it}{2}-\frac{n-1}{4}+l\Big)\Big|^2\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-it\Big)\,dt. \tag{8.23}$$

Summing up, for $l = 1, \ldots, \frac{n}{2} - 1$ we obtain a Laurent expansion at $s = 0$ of the form

$$\frac{1}{|\Gamma_Q|}\frac{\left(\frac{s}{2}\right)_l}{l!}\sum_{j=0}^{\infty} a_{j,Q}(s+2l)\,\psi_j(P) + \frac{1}{|\Gamma_Q|}\frac{\left(\frac{s}{2}\right)_l}{l!}\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s+2l)\,E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2}+it\Big)\,dt$$
$$= \frac{1}{|\Gamma_Q|}\sum_{m=1}^{2} G_{n,Q,l,m}(P) + \left(\frac{4^{l-1}\,\pi^{\frac{n-1}{2}}\,(l-1)!\,\Gamma\left(\frac{1-n}{2}+l\right)}{l\,(2l-1)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|}\sum_{m=1}^{4} H_{n,Q,l,m}(P)\right)\cdot s$$
$$+ \mathrm{O}(s^2). \tag{8.24}$$

Putting (8.7), (8.16) and (8.24) together, for $P \in \mathbb{H}^n$, we finally obtain a Laurent expansion at $s = 0$ of the form

$$E^{\mathrm{ell}}_{Q}(P,s) - \mathcal{G}^{\mathrm{par}}_{Q,1}(P,s)$$
$$= \frac{\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1-n}{2}\right)}{|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|}\sum_{m=1}^{2} G_{n,Q,m}(P) + \frac{1}{|\Gamma_Q|}\sum_{l=1}^{\frac{n}{2}-1}\sum_{m=1}^{2} G_{n,Q,l,m}(P)$$
$$+ \left(\frac{\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1-n}{2}\right)\left(\gamma+\log(4)+\psi^{(0)}\left(\frac{1-n}{2}\right)\right)}{2\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|}\sum_{m=1}^{5} H_{n,Q,m}(P)\right.$$
$$\left. + \sum_{l=1}^{\frac{n}{2}-1}\frac{4^{l-1}\,\pi^{\frac{n-1}{2}}\,(l-1)!\,\Gamma\left(\frac{1-n}{2}+l\right)}{l\,(2l-1)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|}\sum_{l=1}^{\frac{n}{2}-1}\sum_{m=1}^{4} H_{n,Q,l,m}(P)\right)\cdot s + \mathrm{O}(s^2),$$

where $G_{n,Q,m}(P)$ $(m = 1,2)$, $G_{n,Q,l,m}(P)$ $\big(l = 1, \ldots, \frac{n}{2}-1,\ m = 1,2\big)$, $H_{n,Q,m}(P)$ $(m = 1,2,3,4,5)$ and $H_{n,Q,l,m}(P)$ $\big(l = 1, \ldots, \frac{n}{2}-1,\ m = 1,2,3,4\big)$ are given by (8.8), (8.10), (8.11), (8.12), (8.13), (8.14), (8.15), (8.18), (8.19), (8.20), (8.21), (8.22) and (8.23), respectively.

We know that the eigenfunctions $\psi_j(P)$ $(j \in \mathbb{N})$, the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,s)$ $(k = 1, \ldots, c_\Gamma)$ and the hyperbolic kernel function $K^{\mathrm{hyp}}(P,Q,s)$ are all Γ-invariant. Therefore, also the functions $G_{n,Q,m}(P)$ $(m = 1,2)$, $G_{n,Q,l,m}(P)$ $\big(l = 1, \ldots, \frac{n}{2}-1,\ m = 1,2\big)$, $H_{n,Q,m}(P)$ $(m = 1,2,3,4,5)$ and $H_{n,Q,l,m}(P)$ $\big(l = 1, \ldots, \frac{n}{2}-1,\ m = 1,2,3,4\big)$ are invariant under the action of Γ. $\square$

Remark 8.1.3.

(a) In the cases $n = 2$ and $n = 4$ the Laurent expansion of the elliptic Eisenstein series in Proposition 8.1.2 simplifies as follows:
For $n = 2$ or $n = 4$ there is no $N \in \mathbb{N}$ with $1 \leq N \leq \lfloor \frac{n-1}{4} \rfloor$, so the set

$$M_1(n) = \Big\{ -i\Big(\frac{n-1}{2} - 2N\Big) \;\Big|\; N \in \Big\{1, \ldots, \Big\lfloor \frac{n-1}{4} \Big\rfloor\Big\}\Big\}$$

in the proof of Proposition 8.1.2 is empty. This implies that both $G_{n,Q,1}(P) = 0$ and $H_{n,Q,2}(P) = 0$ as the sums defining these functions are empty. Hence, the functions $G_{n,Q,1}(P)$ and $H_{n,Q,2}(P)$ in the Laurent expansion (8.1) do not appear for $n = 2$ and $n = 4$.
Moreover, if $n = 2$ or $n = 4$, then for $l = 1, \ldots, \frac{n}{2} - 1$ there exists no $N \in \mathbb{N}$ with $l \leq N \leq \lfloor \frac{n-1}{4} \rfloor$, so that the set

$$M_1(n,l) = \Big\{ -i\Big(\frac{n-1}{2} - 2N\Big) \;\Big|\; N \in \Big\{l, \ldots, \Big\lfloor \frac{n-1}{4} \Big\rfloor\Big\}\Big\} \qquad \Big(l = 1, \ldots, \frac{n}{2} - 1\Big)$$

in the proof of Proposition 8.1.2 is also empty. Therefore, $G_{n,Q,l,1}(P) = 0$ $\big(l = 1, \ldots, \frac{n}{2} - 1\big)$ and $H_{n,Q,l,1}(P) = 0$ $\big(l = 1, \ldots, \frac{n}{2} - 1\big)$ because the sums defining these functions are empty, and the functions $G_{n,Q,l,1}(P)$ $\big(l = 1, \ldots, \frac{n}{2} - 1\big)$ and $H_{n,Q,l,1}(P)$ $\big(l = 1, \ldots, \frac{n}{2} - 1\big)$ in the

Laurent expansion (8.1) do not appear for $n = 2$ and $n = 4$.
Further, for $n = 2$ there is no $N \in \mathbb{N}$ with $\lceil \frac{n-1}{4} \rceil \leq N \leq \frac{n}{2} - 1$ which implies that the set

$$M_2(n) = \Big\{ -i\Big(2N - \frac{n-1}{2}\Big) \;\Big|\; N \in \Big\{\Big\lceil \frac{n-1}{4}\Big\rceil, \ldots, \frac{n}{2} - 1\Big\}\Big\}$$

in the proof of Proposition 8.1.2 is empty. Thus, we have $G_{n,Q,2}(P) = 0$ and $H_{n,Q,3}(P) = 0$ since the sums defining these functions are empty. We conclude that also the functions $G_{n,Q,2}(P)$ and $H_{n,Q,3}(P)$ in the Laurent expansion (8.1) do not appear for $n = 2$.
Additionally, in the case $n = 2$ the sum $\sum_{l=1}^{\frac{n}{2}-1}$ is empty. Hence, the functions $G_{n,Q,l,2}(P)$ $\big(l = 1, \ldots, \frac{n}{2} - 1\big)$ and $H_{n,Q,l,m}(P)$ $\big(l = 1, \ldots, \frac{n}{2} - 1,\ m = 2, 3, 4\big)$ in the Laurent expansion (8.1) do also not appear for $n = 2$.

(b) If $n \equiv 0 \mod 2$ and $l > \lfloor \frac{n-1}{4} \rfloor$, the set $M_1(n,l)$ in the proof of Proposition 8.1.2 is empty, so we always have $G_{n,Q,l,1}(P) = 0$ $\big(l = \lfloor \frac{n-1}{4} \rfloor + 1, \ldots, \frac{n}{2} - 1\big)$ and $H_{n,Q,l,1}(P) = 0$ $\big(l = \lfloor \frac{n-1}{4} \rfloor + 1, \ldots, \frac{n}{2} - 1\big)$ since the sums defining these functions are empty.

Next we treat the case $n \equiv 3 \mod 4$.

Proposition 8.1.4. *Let $n \equiv 3 \mod 4$. For $P \in \mathbb{H}^n$ with $P \neq \eta Q$ for any $\eta \in \Gamma$ the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$ admits a Laurent expansion at $s = 0$ of the form*

$$\begin{aligned} &E^{\mathrm{ell}}_Q(P,s) - \mathcal{G}^{\mathrm{par}}_{Q,2}(P,s) \\ &= \left(\frac{2\,(-\pi)^{\frac{n-1}{2}}}{\big(\frac{n-1}{2}\big)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{F_{n,Q}(P)}{|\Gamma_Q|} + \frac{1}{|\Gamma_Q|}\sum_{l=1}^{\frac{n-1}{2}} F_{n,Q,l}(P) \right) \cdot \frac{1}{s} \\ &\quad + \frac{(-\pi)^{\frac{n-1}{2}}}{\big(\frac{n-1}{2}\big)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} \left(\sum_{m=1}^{\frac{n-1}{2}} \frac{1}{m} + \log(4) \right) + \frac{G_{n,Q}(P)}{|\Gamma_Q|} \\ &\quad + \sum_{l=1}^{\frac{n-1}{2}} \frac{2^{2l-1}\,(-1)^{\frac{n-1}{2}-l}\,\pi^{\frac{n-1}{2}}\,(l-1)!}{l\,(2l-1)!\,\big(\frac{n-1}{2}-l\big)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|}\sum_{l=1}^{\frac{n-1}{2}}\sum_{m=1}^{2} G_{n,Q,l,m}(P) + \mathrm{O}(s), \end{aligned} \tag{8.25}$$

where

$$\begin{aligned} \mathcal{G}^{\mathrm{par}}_{Q,2}(P,s) := \frac{1}{|\Gamma_Q|}\sum_{l=0}^{\frac{n-1}{2}} \frac{2^{s+2l}\,\pi^{\frac{n-1}{2}}\,\big(\frac{s}{2}\big)_l}{l!\,\Gamma(s+2l)} \sum_{l'=l}^{\frac{n-3}{4}} \frac{(-1)^{l'-l}}{(l'-l)!}\,\Gamma\Big(s - \frac{n-1}{2} + l + l'\Big) \\ \cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2l')\,E^{\mathrm{par}}_{\eta_k}(P, s+2l'), \end{aligned}$$

and where the functions $F_{n,Q}(P)$, $F_{n,Q,l}(P)$ $\big(l = 1, \ldots, \frac{n-1}{2}\big)$, $G_{n,Q}(P)$ and $G_{n,Q,l,m}(P)$ $\big(l = 1, \ldots, \frac{n-1}{2},\ m = 1, 2\big)$ are invariant under the action of Γ, and are given by the formulas (8.29), (8.30), (8.33), (8.34) and (8.35) in the proof, respectively.

Proof. Similar to the proof of Proposition 8.1.2, for $P \in \mathbb{H}^n$ with $P \neq \eta Q$ for any $\eta \in \Gamma$ and $\mathrm{Re}(s) > n - 1 - 2\big(\frac{n-1}{2} + 1\big) = -2$ we have

$$E^{\mathrm{ell}}_Q(P,s) = \frac{1}{|\Gamma_Q|}\sum_{l=0}^{\frac{n-1}{2}} \frac{\big(\frac{s}{2}\big)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l) + \frac{1}{|\Gamma_Q|}\sum_{l=\frac{n+1}{2}}^{\infty} \frac{\big(\frac{s}{2}\big)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l),$$

where the infinite sum is holomorphic in s and we used that $\frac{n-1}{2} \in \mathbb{N}$.

Furthermore, making use of $\lfloor \frac{n-1}{4} \rfloor = \frac{n-3}{4}$, for $P, Q \in \mathbb{H}^n$, $l = 0, \ldots, \frac{n-1}{2}$ and $s \in \mathbb{C}$ with $\frac{n-1}{2} - 2\left(\frac{n-3}{4} - l + 1\right) < \mathrm{Re}(s) + 2l < \frac{n-1}{2} - 2\left(\frac{n-3}{4} - l\right)$ the identity

$$\begin{aligned} K^{\mathrm{hyp}}(P, Q, s+2l) &= \sum_{j=0}^{\infty} a_{j,Q}(s+2l)\, \psi_j(P) + \frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s+2l)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt \\ &\quad + \frac{2^{s+2l}\, \pi^{\frac{n-1}{2}}}{\Gamma(s+2l)} \sum_{l'=l}^{\frac{n-3}{4}} \frac{(-1)^{l'-l}}{(l'-l)!}\, \Gamma\Big(s - \frac{n-1}{2} + l + l'\Big) \\ &\qquad \cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q, n-1-s-2l')\, E^{\mathrm{par}}_{\eta_k}(P, s+2l') \end{aligned}$$

holds true, where the coefficients $a_{j,Q}(s+2l)$ and $a_{t,\eta_k,Q}(s+2l)$ are given by the formulas

$$\begin{aligned} a_{j,Q}(s+2l) &= \frac{2^{s+2l-1}\, \pi^{\frac{n-1}{2}}}{\Gamma(s+2l)}\, \Gamma\Big(\frac{s+2l-\frac{n-1}{2}+ir_j}{2}\Big)\, \Gamma\Big(\frac{s+2l-\frac{n-1}{2}-ir_j}{2}\Big)\, \overline{\psi_j}(Q), \\ a_{t,\eta_k,Q}(s+2l) &= \frac{2^{s+2l-1}\, \pi^{\frac{n-1}{2}}}{\Gamma(s+2l)}\, \Gamma\Big(\frac{s+2l-\frac{n-1}{2}+it}{2}\Big)\, \Gamma\Big(\frac{s+2l-\frac{n-1}{2}-it}{2}\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} - it\Big), \end{aligned}$$

respectively. Since $\frac{n-1}{2} - 2\left(\frac{n-3}{4} - l\right) = 1 + 2l$, for $l = 0, \ldots, \frac{n-1}{2}$ the point $s = 0$ lies in the considered strip.

Putting the above equalities together, at $s = 0$ we obtain the identity

$$\begin{aligned} E^{\mathrm{ell}}_Q(P, s) - \mathcal{G}^{\mathrm{par}}_{Q,2}(P, s) &= \frac{1}{|\Gamma_Q|} \sum_{l=\frac{n+1}{2}}^{\infty} \frac{\left(\frac{s}{2}\right)_l}{l!}\, K^{\mathrm{hyp}}(P, Q, s+2l) + \frac{1}{|\Gamma_Q|} \sum_{l=0}^{\frac{n-1}{2}} \frac{\left(\frac{s}{2}\right)_l}{l!} \sum_{j=0}^{\infty} a_{j,Q}(s+2l)\, \psi_j(P) \\ &\quad + \frac{1}{|\Gamma_Q|} \sum_{l=0}^{\frac{n-1}{2}} \frac{\left(\frac{s}{2}\right)_l}{l!}\, \frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s+2l)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt. \end{aligned} \tag{8.26}$$

To derive the Laurent expansion at $s = 0$ we work from formula (8.26).

As in the proof of Proposition 8.1.2, at $s = 0$ we obtain a Laurent expansion of the form

$$\frac{1}{|\Gamma_Q|} \sum_{l=\frac{n+1}{2}}^{\infty} \frac{\left(\frac{s}{2}\right)_l}{l!}\, K^{\mathrm{hyp}}(P, Q, s+2l) = \mathrm{O}(s). \tag{8.27}$$

Now we consider the other summands on the right-hand side of (8.26), and start with the case $l = 0$.

Analogous to the proof of Proposition 8.1.2, for $j = 0$, i.e. $\lambda_j = 0$, $r_j = -i\,\frac{n-1}{2}$ and $\psi_j(P) = \mathrm{vol}(\Gamma\backslash\mathbb{H}^n)^{-1/2}$, the function $a_{j,Q}(s)\, \psi_j(P)$ in the series

$$\sum_{j=0}^{\infty} a_{j,Q}(s)\, \psi_j(P) \tag{8.28}$$

arising from the discrete spectrum has the form

$$a_{0,Q}(s)\, \psi_0(P) = \frac{2^{s-1}\, \pi^{\frac{n-1}{2}}}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\, \Gamma(s)}\, \Gamma\Big(\frac{s}{2}\Big)\, \Gamma\Big(\frac{s-n+1}{2}\Big).$$

Using formula (A.12), at $s = 0$ we have the Laurent expansions

$$\begin{aligned}
2^{s-1} &= \frac{1}{2} + \frac{\log(2)}{2} \cdot s + \mathrm{O}(s^2), \\
\frac{1}{\Gamma(s)} &= s + \gamma \cdot s^2 + \mathrm{O}(s^3), \\
\Gamma\left(\frac{s}{2}\right) &= 2 \cdot \frac{1}{s} - \gamma + \mathrm{O}(s), \\
\Gamma\Big(\frac{s-n+1}{2}\Big) &= \frac{2\,(-1)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!} \cdot \frac{1}{s} + \frac{(-1)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!} \left(\sum_{m=1}^{\frac{n-1}{2}} \frac{1}{m} - \gamma \right) + \mathrm{O}(s),
\end{aligned}$$

since $\frac{n-1}{2} \in \mathbb{N}_0$ holds true. Consequently, the function $a_{0,Q}(s)\,\psi_0(P)$ admits a Laurent expansion at $s = 0$ of the form

$$a_{0,Q}(s)\,\psi_0(P) = \frac{2\,(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} \cdot \frac{1}{s} + \frac{(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} \left(\sum_{m=1}^{\frac{n-1}{2}} \frac{1}{m} + \log(4) \right) + \mathrm{O}(s).$$

Now we let $j \geq 1$, so that we either have $r_j > 0$ or $r_j \in \left(-i\,\frac{n-1}{2}, 0\right]$. In the proof of Proposition 7.1.4 we have verified that the function $\Gamma\left(\frac{s-\frac{n-1}{2}+ir_j}{2}\right)$ has a pole at $s = 0$ if and only if

$$\begin{aligned}
&\frac{n-1}{4} - \frac{ir_j}{2} \in \Big\{1, \ldots, \frac{n-3}{4}\Big\} \\
&\iff r_j \in \Big\{ -i\Big(\frac{n-1}{2} - 2N\Big) \;\Big|\; N \in \Big\{1, \ldots, \frac{n-3}{4}\Big\}\Big\} =: M_1(n),
\end{aligned}$$

while the function $\Gamma\left(\frac{s-\frac{n-1}{2}-ir_j}{2}\right)$ has a pole at $s = 0$ if and only if

$$\begin{aligned}
&\frac{n-1}{4} + \frac{ir_j}{2} \in \Big\{\frac{n+1}{4}, \ldots, \frac{n-3}{2}\Big\} \\
&\iff r_j \in \Big\{ -i\Big(2N - \frac{n-1}{2}\Big) \;\Big|\; N \in \Big\{\frac{n+1}{4}, \ldots, \frac{n-3}{2}\Big\}\Big\} =: M_2(n),
\end{aligned}$$

and that $M_1(n) = M_2(n) =: M(n)$.

If $r_j \in M(n)$, then, by (A.12), at $s = 0$ we have the Laurent expansions

$$\Gamma\Big(\frac{s - \frac{n-1}{2} \pm ir_j}{2}\Big) = \frac{2\,(-1)^{\frac{n-1}{4} \mp \frac{ir_j}{2}}}{\left(\frac{n-1}{4} \mp \frac{ir_j}{2}\right)!} \cdot \frac{1}{s} + \frac{(-1)^{\frac{n-1}{4} \mp \frac{ir_j}{2}}}{\left(\frac{n-1}{4} \mp \frac{ir_j}{2}\right)!} \left(\sum_{m=1}^{\frac{n-1}{4} \mp \frac{ir_j}{2}} \frac{1}{m} - \gamma \right) + \mathrm{O}(s).$$

This implies that for $j \geq 1$ with $r_j \in M(n)$ the function $a_{j,Q}(s)\,\psi_j(P)$ admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned}
a_{j,Q}(s)\,\psi_j(P) = {}& \frac{2\,(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{4} + \frac{ir_j}{2}\right)!\left(\frac{n-1}{4} - \frac{ir_j}{2}\right)!}\, \psi_j(P)\,\overline{\psi_j}(Q) \cdot \frac{1}{s} \\
&+ \frac{(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{4} + \frac{ir_j}{2}\right)!\left(\frac{n-1}{4} - \frac{ir_j}{2}\right)!} \left(\sum_{m=1}^{\frac{n-1}{4}+\frac{ir_j}{2}} \frac{1}{m} + \sum_{m=1}^{\frac{n-1}{4}-\frac{ir_j}{2}} \frac{1}{m} + \log(4) \right) \psi_j(P)\,\overline{\psi_j}(Q) + \mathrm{O}(s),
\end{aligned}$$

and the respective part of the series (8.28) arising from the discrete spectrum has a Laurent expansion at $s = 0$ of the form

$$\sum_{\substack{j \in \mathbb{N}: \\ r_j \in M(n)}} a_{j,Q}(s)\,\psi_j(P) = F_{n,Q}(P) \cdot \frac{1}{s} + G_{n,Q}(P) + \mathrm{O}(s),$$

where

$$F_{n,Q}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M(n)}} \frac{2\,(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!\left(\frac{n-1}{4}-\frac{ir_j}{2}\right)!}\,\psi_j(P)\,\overline{\psi_j}(Q), \tag{8.29}$$

$$G_{n,Q}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M(n)}} \frac{(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{4}+\frac{ir_j}{2}\right)!\left(\frac{n-1}{4}-\frac{ir_j}{2}\right)!}\left(\sum_{m=1}^{\frac{n-1}{4}+\frac{ir_j}{2}}\frac{1}{m}+\sum_{m=1}^{\frac{n-1}{4}-\frac{ir_j}{2}}\frac{1}{m}+\log(4)\right)\psi_j(P)\,\overline{\psi_j}(Q). \tag{8.30}$$

If $r_j \notin M(n)$ holds true, then (A.13) yields that at $s=0$ we have the Laurent expansions

$$\Gamma\Big(\frac{s-\frac{n-1}{2}\pm ir_j}{2}\Big) = \Gamma\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}\Big)+\frac{1}{2}\Gamma\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi^{(0)}\Big(\pm\frac{ir_j}{2}-\frac{n-1}{4}\Big)\cdot s+\mathrm{O}(s^2).$$

So for $j\geq 1$ with $r_j\notin M(n)$ the function $a_{j,Q}(s)\,\psi_j(P)$ admits a Laurent expansion at $s=0$ of the form

$$a_{j,Q}(s)\,\psi_j(P) = \frac{\pi^{\frac{n-1}{2}}}{2}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}\Big)\,\psi_j(P)\,\overline{\psi_j}(Q)\cdot s+\mathrm{O}(s^2).$$

Hence, the respective part of the series (8.28) arising from the discrete spectrum has a Laurent expansion at $s=0$ of the form

$$\sum_{\substack{j\in\mathbb{N}:\\ r_j\notin M(n)}} a_{j,Q}(s)\,\psi_j(P) = \mathrm{O}(s).$$

Further, since $\frac{n-1}{4}\notin\mathbb{N}_0$ holds true again, as in the proof of Proposition 8.1.2 we find the Laurent expansion

$$\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt = \mathrm{O}(s)$$

of the continuous part at $s=0$.

Adding up and making use of $\left(\frac{s}{2}\right)_0/0! = 1$, we obtain that the summand for $l=0$ on the right-hand side of formula (8.26) admits a Laurent expansion at $s=0$ of the form

$$\begin{aligned}
&\frac{1}{|\Gamma_Q|}\sum_{j=0}^{\infty} a_{j,Q}(s)\,\psi_j(P)+\frac{1}{4\pi\,|\Gamma_Q|}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt\\
&\quad=\left(\frac{2\,(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}+\frac{F_{n,Q}(P)}{|\Gamma_Q|}\right)\cdot\frac{1}{s}\\
&\qquad+\frac{(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\left(\sum_{m=1}^{\frac{n-1}{2}}\frac{1}{m}+\log(4)\right)+\frac{G_{n,Q}(P)}{|\Gamma_Q|}+\mathrm{O}(s).
\end{aligned} \tag{8.31}$$

Now we turn to the case $l>0$, i.e. $l\in\left\{1,\ldots,\frac{n-1}{2}\right\}$.

For $j=0$, i.e. $\lambda_j=0$, $r_j=-i\,\frac{n-1}{2}$ and $\psi_j(P)=\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)^{-1/2}$, the function $a_{j,Q}(s+2l)\,\psi_j(P)$ in the series

$$\sum_{j=0}^{\infty} a_{j,Q}(s+2l)\,\psi_j(P) \tag{8.32}$$

arising from the discrete spectrum has the form

$$a_{0,Q}(s+2l)\,\psi_0(P) = \frac{2^{s+2l-1}\,\pi^{\frac{n-1}{2}}}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma(s+2l)}\,\Gamma\Big(\frac{s}{2}+l\Big)\,\Gamma\Big(\frac{s-n+1}{2}+l\Big).$$

At $s = 0$ we have the Laurent expansions

$$\begin{aligned}
2^{s+2l-1} &= 2^{2l-1} + 2^{2l-1}\,\log(2)\cdot s + \mathrm{O}(s^2),\\
\frac{1}{\Gamma(s+2l)} &= \frac{1}{(2l-1)!} - \frac{1}{(2l-1)!}\left(\sum_{m=1}^{2l-1}\frac{1}{m} - \gamma\right)\cdot s + \mathrm{O}(s^2),\\
\Gamma\left(\frac{s}{2}+l\right) &= (l-1)! + \frac{(l-1)!}{2}\left(\sum_{m=1}^{l-1}\frac{1}{m} - \gamma\right)\cdot s + \mathrm{O}(s^2),\\
\Gamma\Big(\frac{s-n+1}{2}+l\Big) &= \frac{2\,(-1)^{\frac{n-1}{2}-l}}{\big(\frac{n-1}{2}-l\big)!}\cdot\frac{1}{s} + \frac{(-1)^{\frac{n-1}{2}-l}}{\big(\frac{n-1}{2}-l\big)!}\left(\sum_{m=1}^{\frac{n-1}{2}-l}\frac{1}{m} - \gamma\right) + \mathrm{O}(s),
\end{aligned}$$

where we used $\frac{n-1}{2}\in\mathbb{N}_0$ and $l \le \frac{n-1}{2}$, as well as the identities (A.9), (A.12) and (A.13). It follows that the function $a_{0,Q}(s+2l)\,\psi_0(P)$ admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned}
a_{0,Q}(s+2l)\,\psi_0(P) &= \frac{4^l\,(-1)^{\frac{n-1}{2}-l}\,\pi^{\frac{n-1}{2}}\,(l-1)!}{(2l-1)!\,\big(\frac{n-1}{2}-l\big)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\cdot\frac{1}{s}\\
&+ \frac{2^{2l-1}\,(-1)^{\frac{n-1}{2}-l}\,\pi^{\frac{n-1}{2}}\,(l-1)!}{(2l-1)!\,\big(\frac{n-1}{2}-l\big)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\left(\sum_{m=1}^{l-1}\frac{1}{m} - 2\sum_{m=1}^{2l-1}\frac{1}{m} + \sum_{m=1}^{\frac{n-1}{2}-l}\frac{1}{m} + \log(4)\right) + \mathrm{O}(s).
\end{aligned}$$

Together with the Laurent expansion of the Pochhammer symbol $\big(\frac{s}{2}\big)_l$, at $s = 0$ we obtain the Laurent expansion

$$\frac{\big(\frac{s}{2}\big)_l}{l!}\,a_{0,Q}(s+2l)\,\psi_0(P) = \frac{2^{2l-1}\,(-1)^{\frac{n-1}{2}-l}\,\pi^{\frac{n-1}{2}}\,(l-1)!}{l\,(2l-1)!\,\big(\frac{n-1}{2}-l\big)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \mathrm{O}(s).$$

Now let $j \ge 1$ so that we either have $r_j > 0$ or $r_j \in \big(-i\,\frac{n-1}{2}, 0\big]$. In the first case $\frac{n-1}{4} \pm \frac{ir_j}{2} - l \notin \mathbb{N}_0$ holds, while in the latter case $\frac{n-1}{4} \pm \frac{ir_j}{2} - l \in \mathbb{R}$ with $\frac{n-1}{4} - \frac{ir_j}{2} - l \in \big(-l, \frac{n-1}{4} - l\big]$ and $\frac{n-1}{4} + \frac{ir_j}{2} - l \in \big[\frac{n-1}{4} - l, \frac{n-1}{2} - l\big)$.
The interval $\big(-l, \frac{n-1}{4} - l\big]$ contains the elements $0, \dots, \frac{n-3}{4} - l$ of $\mathbb{N}_0$. The elements of $\mathbb{N}_0$ contained in the interval $\big[\frac{n-1}{4} - l, \frac{n-1}{2} - l\big)$ are $\frac{n+1}{4} - l, \dots, \frac{n-3}{2} - l$ if $l \le \frac{n-1}{4}$, and $0, \dots, \frac{n-3}{2} - l$ if $l > \frac{n-1}{4}$, proving that the elements of $\mathbb{N}_0$ contained in the interval $\big[\frac{n-1}{4} - l, \frac{n-1}{2} - l\big)$ are $\max\big(\frac{n+1}{4} - l, 0\big), \dots, \frac{n-3}{2} - l$.
This shows that the function $\Gamma\left(\frac{s+2l-\frac{n-1}{2}+ir_j}{2}\right)$ has a pole at $s = 0$ if and only if

$$\begin{aligned}
&\frac{n-1}{4} - \frac{ir_j}{2} - l \in \Big\{0, \dots, \frac{n-3}{4} - l\Big\} \iff \frac{n-1}{4} - \frac{ir_j}{2} \in \Big\{l, \dots, \frac{n-3}{4}\Big\}\\
&\iff r_j \in \Big\{ -i\Big(\frac{n-1}{2} - 2N\Big) \;\Big|\; N \in \Big\{l, \dots, \frac{n-3}{4}\Big\}\Big\} =: M_1(n,l),
\end{aligned}$$

while the function $\Gamma\left(\frac{s+2l-\frac{n-1}{2}-ir_j}{2}\right)$ has a pole at $s = 0$ if and only if

$$\begin{aligned}
&\frac{n-1}{4} + \frac{ir_j}{2} - l \in \Big\{\max\Big(\frac{n+1}{4} - l, 0\Big), \dots, \frac{n-3}{2} - l\Big\}\\
&\iff \frac{n-1}{4} + \frac{ir_j}{2} \in \Big\{\max\Big(\frac{n+1}{4}, l\Big), \dots, \frac{n-3}{2}\Big\}\\
&\iff r_j \in \Big\{ -i\Big(2N - \frac{n-1}{2}\Big) \;\Big|\; N \in \Big\{\max\Big(\frac{n+1}{4}, l\Big), \dots, \frac{n-3}{2}\Big\}\Big\} =: M_2(n,l).
\end{aligned}$$

If $l > \frac{n-3}{4}$, then $M_1(n,l) = \emptyset$ and trivially $M_1(n,l) \subseteq M_2(n,l)$. Moreover, if $l \leq \frac{n-3}{4}$ and $\frac{n-1}{4} - \frac{ir_j}{2} = N$ for some $N \in \left\{l, \ldots, \frac{n-3}{4}\right\}$, then $\max\left(\frac{n+1}{4}, l\right) = \frac{n+1}{4}$ and $\frac{n-1}{2} \in \mathbb{Z}$ gives us

$$\frac{n-1}{4} + \frac{ir_j}{2} = \frac{n-1}{2} - N \ \in \mathbb{Z} \quad \text{and} \quad \frac{n-1}{2} - N \in \left[\frac{n+1}{4}, \frac{n-1}{2} - l\right],$$

so also $\frac{n-1}{4} + \frac{ir_j}{2} \in \left\{\frac{n+1}{4}, \ldots, \frac{n-1}{2} - l\right\} \subseteq \left\{\max\left(\frac{n+1}{4}, l\right), \ldots, \frac{n-3}{2}\right\}$. Therefore, we have proven that in any case $M_1(n,l) \subseteq M_2(n,l)$ holds true.

Similar to the proof of Proposition 8.1.2, we see that for $j \geq 1$ with $r_j \in M_2(n,l) \setminus M_1(n,l)$ we obtain a Laurent expansion of the function $a_{j,Q}(s+2l)\, \psi_j(P)$ at $s = 0$ of the form

$$\begin{aligned} a_{j,Q}(s+2l)\, \psi_j(P) = {} & \frac{4^l \, (-1)^{\frac{n-1}{4} + \frac{ir_j}{2} - l} \, \pi^{\frac{n-1}{2}}}{(2l-1)! \left(\frac{n-1}{4} + \frac{ir_j}{2} - l\right)!} \, \Gamma\Big(\frac{ir_j}{2} - \frac{n-1}{4} + l\Big)\, \psi_j(P)\, \overline{\psi_j}(Q) \cdot \frac{1}{s} \\ & + \frac{2^{2l-1} \, (-1)^{\frac{n-1}{4} + \frac{ir_j}{2} - l} \, \pi^{\frac{n-1}{2}}}{(2l-1)! \left(\frac{n-1}{4} + \frac{ir_j}{2} - l\right)!} \, \Gamma\Big(\frac{ir_j}{2} - \frac{n-1}{4} + l\Big) \\ & \cdot \left(\sum_{m=1}^{\frac{n-1}{4} + \frac{ir_j}{2} - l} \frac{1}{m} - 2 \sum_{m=1}^{2l-1} \frac{1}{m} + \gamma + \log(4) + \psi^{(0)}\Big(\frac{ir_j}{2} - \frac{n-1}{4} + l\Big) \right) \psi_j(P)\, \overline{\psi_j}(Q) + \mathrm{O}(s), \end{aligned}$$

implying that the respective part of the series (8.32) after multiplication by $\left(\frac{s}{2}\right)_l / l!$ has a Laurent expansion at $s = 0$ of the form

$$\frac{\left(\frac{s}{2}\right)_l}{l!} \sum_{\substack{j \in \mathbb{N}: \\ r_j \in M_2(n,l) \setminus M_1(n,l)}} a_{j,Q}(s+2l)\, \psi_j(P) = G_{n,Q,l,1}(P) + \mathrm{O}(s),$$

where

$$G_{n,Q,l,1}(P) := \sum_{\substack{j \in \mathbb{N}: \\ r_j \in M_2(n,l) \setminus M_1(n,l)}} \frac{2^{2l-1} \, (-1)^{\frac{n-1}{4} + \frac{ir_j}{2} - l} \, \pi^{\frac{n-1}{2}}}{l\,(2l-1)! \left(\frac{n-1}{4} + \frac{ir_j}{2} - l\right)!} \, \Gamma\Big(\frac{ir_j}{2} - \frac{n-1}{4} + l\Big)\, \psi_j(P)\, \overline{\psi_j}(Q). \tag{8.33}$$

In the case $r_j \in M_1(n,l) \subseteq M_2(n,l)$ we have the Laurent expansions

$$\begin{aligned} \Gamma\Big(\frac{s + 2l - \frac{n-1}{2} + ir_j}{2}\Big) &= \frac{2\,(-1)^{\frac{n-1}{4} - \frac{ir_j}{2} - l}}{\left(\frac{n-1}{4} - \frac{ir_j}{2} - l\right)!} \cdot \frac{1}{s} + \frac{(-1)^{\frac{n-1}{4} - \frac{ir_j}{2} - l}}{\left(\frac{n-1}{4} - \frac{ir_j}{2} - l\right)!} \left(\sum_{m=1}^{\frac{n-1}{4} - \frac{ir_j}{2} - l} \frac{1}{m} - \gamma \right) + \mathrm{O}(s), \\ \Gamma\Big(\frac{s + 2l - \frac{n-1}{2} - ir_j}{2}\Big) &= \frac{2\,(-1)^{\frac{n-1}{4} + \frac{ir_j}{2} - l}}{\left(\frac{n-1}{4} + \frac{ir_j}{2} - l\right)!} \cdot \frac{1}{s} + \frac{(-1)^{\frac{n-1}{4} + \frac{ir_j}{2} - l}}{\left(\frac{n-1}{4} + \frac{ir_j}{2} - l\right)!} \left(\sum_{m=1}^{\frac{n-1}{4} + \frac{ir_j}{2} - l} \frac{1}{m} - \gamma \right) + \mathrm{O}(s) \end{aligned}$$

at $s = 0$. This implies that for $j \geq 1$ with $r_j \in M_1(n,l)$ the function $a_{j,Q}(s+2l)\, \psi_j(P)$ admits a

Laurent expansion at $s = 0$ of the form

$$\begin{aligned} a_{j,Q}(s+2l)\,\psi_j(P) = {}& \frac{2^{2l+1}\,(-\pi)^{\frac{n-1}{2}}}{(2l-1)!\left(\frac{n-1}{4}+\frac{ir_j}{2}-l\right)!\left(\frac{n-1}{4}-\frac{ir_j}{2}-l\right)!}\,\psi_j(P)\,\overline{\psi_j}(Q)\cdot\frac{1}{s^2} \\ &+ \frac{4^l\,(-\pi)^{\frac{n-1}{2}}}{(2l-1)!\left(\frac{n-1}{4}+\frac{ir_j}{2}-l\right)!\left(\frac{n-1}{4}-\frac{ir_j}{2}-l\right)!}\,\psi_j(P)\,\overline{\psi_j}(Q) \\ &\cdot\left(\sum_{m=1}^{\frac{n-1}{4}+\frac{ir_j}{2}-l}\frac{1}{m}+\sum_{m=1}^{\frac{n-1}{4}-\frac{ir_j}{2}-l}\frac{1}{m}-2\sum_{m=1}^{2l-1}\frac{1}{m}+\log(4)\right)\cdot\frac{1}{s}+\mathrm{O}(1), \end{aligned}$$

and the respective part of the series (8.32) after multiplication by $\left(\frac{s}{2}\right)_l/l!$ admits a Laurent expansion at $s = 0$ of the form

$$\frac{\left(\frac{s}{2}\right)_l}{l!}\sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_1(n,l)}} a_{j,Q}(s+2l)\,\psi_j(P) = F_{n,Q,l}(P)\cdot\frac{1}{s}+G_{n,Q,l,2}(P)+\mathrm{O}(s),$$

where

$$F_{n,Q,l}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_1(n,l)}} \frac{4^l\,(-\pi)^{\frac{n-1}{2}}}{l\,(2l-1)!\left(\frac{n-1}{4}+\frac{ir_j}{2}-l\right)!\left(\frac{n-1}{4}-\frac{ir_j}{2}-l\right)!}\,\psi_j(P)\,\overline{\psi_j}(Q), \tag{8.34}$$

$$\begin{aligned} G_{n,Q,l,2}(P) := {}& \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_1(n,l)}} \frac{2^{2l-1}\,(-\pi)^{\frac{n-1}{2}}}{l\,(2l-1)!\left(\frac{n-1}{4}+\frac{ir_j}{2}-l\right)!\left(\frac{n-1}{4}-\frac{ir_j}{2}-l\right)!}\,\psi_j(P)\,\overline{\psi_j}(Q) \\ &\cdot\left(\sum_{m=1}^{\frac{n-1}{4}+\frac{ir_j}{2}-l}\frac{1}{m}+\sum_{m=1}^{\frac{n-1}{4}-\frac{ir_j}{2}-l}\frac{1}{m}-2\sum_{m=1}^{2l-1}\frac{1}{m}+\sum_{m=1}^{l-1}\frac{1}{m}+\log(4)\right). \end{aligned} \tag{8.35}$$

As in the case $r_j \notin M_1(n,l)\cup M_2(n,l)$ in the proof of Proposition 8.1.2, for $j \geq 1$ with $r_j \notin M_2(n,l)$ the function $a_{j,Q}(s+2l)\,\psi_j(P)$ has a Laurent expansion at $s = 0$ of the form

$$\begin{aligned} a_{j,Q}(s+2l)\,\psi_j(P) = {}& \frac{2^{2l-1}\,\pi^{\frac{n-1}{2}}}{(2l-1)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q) \\ &+ \frac{4^{l-1}\,\pi^{\frac{n-1}{2}}}{(2l-1)!}\,\Gamma\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\Gamma\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q) \\ &\cdot\left(-2\sum_{m=1}^{2l-1}\frac{1}{m}+2\gamma+\log(4)+\psi^{(0)}\Big(\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)+\psi^{(0)}\Big(-\frac{ir_j}{2}-\frac{n-1}{4}+l\Big)\right)\cdot s+\mathrm{O}(s^2), \end{aligned}$$

so the respective part of the series (8.32) after multiplication by $\left(\frac{s}{2}\right)_l/l!$ has a Laurent expansion at $s = 0$ of the form

$$\frac{\left(\frac{s}{2}\right)_l}{l!}\sum_{\substack{j\in\mathbb{N}:\\ r_j\notin M_2(n,l)}} a_{j,Q}(s+2l)\,\psi_j(P) = \mathrm{O}(s).$$

Moreover, because of $\frac{n-1}{4} \notin \mathbb{N}_0$ we immediately obtain, analogous to the proof of Proposition 8.1.2, that at $s = 0$ we have the Laurent expansion

$$\frac{\left(\frac{s}{2}\right)_l}{l!}\,\frac{1}{4\pi}\sum_{k=1}^{\mathrm{cr}}\int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s+2l)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt = \mathrm{O}(s).$$

Summing up, for $l = 1, \ldots, \frac{n-1}{2}$ we get a Laurent expansion at $s = 0$ of the form

$$\frac{1}{|\Gamma_Q|} \frac{\left(\frac{s}{2}\right)_l}{l!} \sum_{j=0}^{\infty} a_{j,Q}(s+2l)\, \psi_j(P) + \frac{1}{|\Gamma_Q|} \frac{\left(\frac{s}{2}\right)_l}{l!} \frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s+2l)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\, dt$$
$$= \frac{F_{n,Q,l}(P)}{|\Gamma_Q|} \cdot \frac{1}{s} + \frac{2^{2l-1}\, (-1)^{\frac{n-1}{2}-l}\, \pi^{\frac{n-1}{2}}\, (l-1)!}{l\, (2l-1)!\, \big(\frac{n-1}{2} - l\big)!\, |\Gamma_Q| \operatorname{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|} \sum_{m=1}^{2} G_{n,Q,l,m}(P) + \mathrm{O}(s). \tag{8.36}$$

Putting (8.27), (8.31) and (8.36) together, for $P \in \mathbb{H}^n$ we finally obtain a Laurent expansion at $s = 0$ of the form

$$\begin{aligned} &E^{\mathrm{ell}}_Q(P,s) - \mathcal{G}^{\mathrm{par}}_{Q,2}(P,s) \\ &= \left(\frac{2\,(-\pi)^{\frac{n-1}{2}}}{\big(\frac{n-1}{2}\big)!\, |\Gamma_Q| \operatorname{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{F_{n,Q}(P)}{|\Gamma_Q|} + \frac{1}{|\Gamma_Q|} \sum_{l=1}^{\frac{n-1}{2}} F_{n,Q,l}(P) \right) \cdot \frac{1}{s} \\ &\quad + \frac{(-\pi)^{\frac{n-1}{2}}}{\big(\frac{n-1}{2}\big)!\, |\Gamma_Q| \operatorname{vol}(\Gamma\backslash\mathbb{H}^n)} \left(\sum_{m=1}^{\frac{n-1}{2}} \frac{1}{m} + \log(4) \right) + \frac{G_{n,Q}(P)}{|\Gamma_Q|} \\ &\quad + \sum_{l=1}^{\frac{n-1}{2}} \frac{2^{2l-1}\, (-1)^{\frac{n-1}{2}-l}\, \pi^{\frac{n-1}{2}}\, (l-1)!}{l\, (2l-1)!\, \big(\frac{n-1}{2} - l\big)!\, |\Gamma_Q| \operatorname{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|} \sum_{l=1}^{\frac{n-1}{2}} \sum_{m=1}^{2} G_{n,Q,l,m}(P) + \mathrm{O}(s), \end{aligned}$$

where $F_{n,Q}(P)$, $F_{n,Q,l}(P)$ $\big(l = 1, \ldots, \frac{n-1}{2}\big)$, $G_{n,Q}(P)$ and $G_{n,Q,l,m}(P)$ $\big(l = 1, \ldots, \frac{n-1}{2}, m = 1, 2\big)$ are given by (8.29), (8.30), (8.33), (8.34) and (8.35), respectively.

Since the eigenfunctions $\psi_j(P)$ $(j \in \mathbb{N})$ are invariant under the action of Γ, the same is true for the functions $F_{n,Q}(P)$, $F_{n,Q,l}(P)$ $\big(l = 1, \ldots, \frac{n-1}{2}\big)$, $G_{n,Q}(P)$ and $G_{n,Q,l,m}(P)$ $\big(l = 1, \ldots, \frac{n-1}{2}, m = 1, 2\big)$. □

Remark 8.1.5.

(a) In the case $n = 3$ the Laurent expansion of the elliptic Eisenstein series in Proposition 8.1.4 simplifies considerably:
Since for $n = 3$ there is no $N \in \mathbb{N}$ with $1 \le N \le \frac{n-3}{4}$ and no $N \in \mathbb{N}$ with $\frac{n+1}{4} \le N \le \frac{n-3}{2}$, the set

$$\begin{aligned} M(n) &= \Big\{ -i \Big(\frac{n-1}{2} - 2N\Big) \,\Big|\, N \in \Big\{1, \ldots, \frac{n-3}{4}\Big\}\Big\}, \\ &= \Big\{ -i \Big(2N - \frac{n-1}{2}\Big) \,\Big|\, N \in \Big\{\frac{n+1}{4}, \ldots, \frac{n-3}{2}\Big\}\Big\}, \end{aligned}$$

in the proof of Proposition 8.1.4 is empty. Thus, we have $F_{n,Q}(P) = 0$ and $G_{n,Q}(P) = 0$ as the sums defining these functions are empty. This shows that the functions $F_{n,Q}(P)$ and $G_{n,Q}(P)$ in the Laurent expansion (8.25) do not appear for $n = 3$.
Moreover, for $n = 3$ and $l = 1, \ldots, \frac{n-1}{2}$ there exists no $N \in \mathbb{N}$ with $\max\big(\frac{n+1}{4}, l\big) \le N \le \frac{n-3}{2}$, which implies that also the set

$$M_2(n,l) = \Big\{ -i \Big(2N - \frac{n-1}{2}\Big) \,\Big|\, N \in \Big\{\max\Big(\frac{n+1}{4}, l\Big), \ldots, \frac{n-3}{2}\Big\}\Big\}$$

in the proof of Proposition 8.1.4 is empty. Hence, we get $F_{n,Q,l}(P) = 0$ $\big(l = 1, \ldots, \frac{n-1}{2}\big)$ and $G_{n,Q,l,m}(P) = 0$ $\big(l = 1, \ldots, \frac{n-1}{2},\ m = 1, 2\big)$ since the sums defining these functions are empty, and the functions $F_{n,Q,l}(P)$ $\big(l = 1, \ldots, \frac{n-1}{2}\big)$ and $G_{n,Q,l,m}(P)$ $\big(l = 1, \ldots, \frac{n-1}{2},\ m = 1, 2\big)$ in the Laurent expansion (8.25) do also not appear for $n = 3$.

(b) If $n \equiv 3 \mod 4$ and $l > \frac{n-3}{4}$, the set

$$M_1(n,l) = \Big\{ -i\Big(\frac{n-1}{2} - 2N\Big) \;\Big|\; N \in \Big\{l, \ldots, \frac{n-3}{4}\Big\}\Big\}$$

in the proof of Proposition 8.1.4 is empty. We can conclude that we always have $F_{n,Q,l}(P) = 0$ $\big(l = \frac{n+1}{4}, \ldots, \frac{n-1}{2}\big)$ and $G_{n,Q,l,2}(P) = 0$ $\big(l = \frac{n+1}{4}, \ldots, \frac{n-1}{2}\big)$ since the sums defining these functions are empty.

It remains to consider the case $n \equiv 1 \mod 4$.

Proposition 8.1.6. *Let $n \equiv 1 \mod 4$. For $P \in \mathbb{H}^n$ with $P \neq \eta Q$ for any $\eta \in \Gamma$ the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ admits a Laurent expansion at $s = 0$ of the form*

$$\begin{aligned}
&E_Q^{\mathrm{ell}}(P,s) - \mathcal{G}_{Q,3}^{\mathrm{par}}(P,s) \\
&= \left(\frac{2\,(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{F_{n,Q}(P)}{|\Gamma_Q|} + \frac{1}{|\Gamma_Q|}\sum_{l=1}^{\frac{n-1}{4}} F_{n,Q,l}(P) \right) \cdot \frac{1}{s} \\
&\quad + \frac{(-\pi)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} \left(\sum_{m=1}^{\frac{n-1}{2}} \frac{1}{m} + \log(4) \right) + \frac{1}{|\Gamma_Q|}\sum_{m=1}^{2} G_{n,Q,m}(P) \\
&\quad + \sum_{l=1}^{\frac{n-1}{2}} \frac{2^{2l-1}\,(-1)^{\frac{n-1}{2}-l}\,\pi^{\frac{n-1}{2}}\,(l-1)!}{l\,(2l-1)!\,\left(\frac{n-1}{2}-l\right)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|}\sum_{l=1}^{\frac{n-1}{4}}\sum_{m=1}^{3} G_{n,Q,l,m}(P) \\
&\quad + \frac{1}{|\Gamma_Q|}\sum_{l=\frac{n+3}{4}}^{\frac{n-1}{2}} G_{n,Q,l,4}(P) + \mathrm{O}(s),
\end{aligned} \tag{8.37}$$

where

$$\begin{aligned}
\mathcal{G}_{Q,3}^{\mathrm{par}}(P,s) := \frac{1}{|\Gamma_Q|}\sum_{l=0}^{\frac{n-1}{4}} \frac{2^{s+2l}\,\pi^{\frac{n-1}{2}}\,\left(\frac{s}{2}\right)_l}{l!\,\Gamma(s+2l)} \sum_{l'=l}^{\frac{n-5}{4}} \frac{(-1)^{l'-l}}{(l'-l)!}\,\Gamma\Big(s - \frac{n-1}{2} + l + l'\Big) \\
\cdot \sum_{k=1}^{c_\Gamma} E_{\eta_k}^{\mathrm{par}}(Q, n-1-s-2l')\,E_{\eta_k}^{\mathrm{par}}(P, s+2l'),
\end{aligned}$$

and where the functions $F_{n,Q}(P)$, $F_{n,Q,l}(P)$ $\big(l = 1, \ldots, \frac{n-1}{4}\big)$, $G_{n,Q,m}(P)$ $(m = 1,2)$, $G_{n,Q,l,m}(P)$ $\big(l = 1, \ldots, \frac{n-1}{4}, m = 1,2,3\big)$ and $G_{n,Q,l,4}(P)$ $\big(l = \frac{n+3}{4}, \ldots, \frac{n-1}{2}\big)$ are invariant under the action of Γ, and are given by the formulas (8.43), (8.44), (8.45), (8.47), (8.48), (8.49), (8.50) *and* (8.52) *in the proof, respectively.*

Proof. As in the proof of Proposition 8.1.4, for $P \in \mathbb{H}^n$ with $P \neq \eta Q$ for any $\eta \in \Gamma$ and $\mathrm{Re}(s) > n - 1 - 2\left(\frac{n-1}{2} + 1\right) = -2$ we have

$$E_Q^{\mathrm{ell}}(P,s) = \frac{1}{|\Gamma_Q|}\sum_{l=0}^{\frac{n-1}{2}} \frac{\left(\frac{s}{2}\right)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l) + \frac{1}{|\Gamma_Q|}\sum_{l=\frac{n+1}{2}}^{\infty} \frac{\left(\frac{s}{2}\right)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l), \tag{8.38}$$

where the infinite sum is holomorphic in s and we made use of $\frac{n-1}{2} \in \mathbb{N}$ again.

Moreover, in the proof of Theorem 6.1.1 we have seen that for $P, Q \in \mathbb{H}^n$, $l = 0, \ldots, \frac{n-1}{2}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) + 2l = \frac{n-1}{2} - 2m$ $(m \in \mathbb{N}_0)$ the function $K^{\mathrm{hyp}}(P,Q,s+2l)$ admits the meromorphic continuation

$$
\begin{aligned}
K^{\mathrm{hyp}}(P,Q,s+2l) &= \sum_{j=0}^{\infty} a_{j,Q}(s+2l)\,\psi_j(P) \\
&+ \frac{2^{s+2l-1}\,\pi^{\frac{n-3}{2}}}{4\,i\,\Gamma(s+2l)} \sum_{k=1}^{c_\Gamma} \int_{W_{y,\varepsilon}} \Gamma\Big(\frac{s+2l-n+1+w}{2}\Big)\,\Gamma\Big(\frac{s+2l-w}{2}\Big)\, E^{\mathrm{par}}_{\eta_k}(Q,n-1-w)\,E^{\mathrm{par}}_{\eta_k}(P,w)\,dw \\
&+ \frac{2^{s+2l}\,\pi^{\frac{n-1}{2}}}{\Gamma(s+2l)} \sum_{l'=0}^{m-1} \frac{(-1)^{l'}}{l'!}\,\Gamma\Big(s+2l-\frac{n-1}{2}+l'\Big) \\
&\qquad \cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2(l+l'))\,E^{\mathrm{par}}_{\eta_k}(P,s+2(l+l')) \\
&+ \frac{2^{s+2l-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s+2l)}\,\frac{(-1)^m}{m!}\,\Gamma\Big(s+2l-\frac{n-1}{2}+m\Big) \\
&\qquad \cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2(l+m))\,E^{\mathrm{par}}_{\eta_k}(P,s+2(l+m)).
\end{aligned}
$$

Here $W_{y,\varepsilon}$ denotes the following piecewise linear path: the vertical line from $\frac{n-1}{2}-i\infty$ to $\frac{n-1}{2}-iy$, the horizontal line segment from $\frac{n-1}{2}-iy$ to $\frac{n-1}{2}+\varepsilon-iy$, the vertical line segment from $\frac{n-1}{2}+\varepsilon-iy$ to $\frac{n-1}{2}+\varepsilon+iy$, the horizontal line segment from $\frac{n-1}{2}+\varepsilon+iy$ to $\frac{n-1}{2}+iy$, and the vertical line from $\frac{n-1}{2}+iy$ to $\frac{n-1}{2}+i\infty$, where $\varepsilon\in(0,1)$ is chosen sufficiently small such that all parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P,s)$ ($k=1,\ldots,c_\Gamma$) have no poles in the strip $\frac{n-1}{2}-2\varepsilon<\mathrm{Re}(s)<\frac{n-1}{2}+2\varepsilon$, and y is chosen sufficiently large such that $y>|\mathrm{Im}(s)|$. Further, the coefficient $a_{j,Q}(s+2l)$ is given by

$$
a_{j,Q}(s+2l) = \frac{2^{s+2l-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s+2l)}\,\Gamma\Big(\frac{s+2l-\frac{n-1}{2}+ir_j}{2}\Big)\,\Gamma\Big(\frac{s+2l-\frac{n-1}{2}-ir_j}{2}\Big)\,\overline{\psi_j}(Q).
$$

Through the substitution $w=\frac{n-1}{2}+it$ in the integral the above meromorphic continuation on the line $\mathrm{Re}(s)+2l=\frac{n-1}{2}-2m$ ($m\in\mathbb{N}_0$) becomes

$$
\begin{aligned}
K^{\mathrm{hyp}}(P,Q,s+2l) = \sum_{j=0}^{\infty} a_{j,Q}(s+2l)\,\psi_j(P) &+ \frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{W'_{y,\varepsilon}} a_{t,Q,\eta_k}(s+2l)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt \\
&+ \frac{2^{s+2l}\,\pi^{\frac{n-1}{2}}}{\Gamma(s+2l)} \sum_{l'=0}^{m-1} \frac{(-1)^{l'}}{l'!}\,\Gamma\Big(s+2l-\frac{n-1}{2}+l'\Big) \\
&\qquad \cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2(l+l'))\,E^{\mathrm{par}}_{\eta_k}(P,s+2(l+l')) \\
&+ \frac{2^{s+2l-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s+2l)}\,\frac{(-1)^m}{m!}\,\Gamma\Big(s+2l-\frac{n-1}{2}+m\Big) \\
&\qquad \cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2(l+m))\,E^{\mathrm{par}}_{\eta_k}(P,s+2(l+m)),
\end{aligned}
$$

where $W'_{y,\varepsilon}$ denotes the following piecewise linear path: the horizontal line from $-\infty$ to $-y$, the vertical line segment from $-y$ to $-y-i\varepsilon$, the horizontal line segment from $-y-i\varepsilon$ to $y-i\varepsilon$, the vertical line segment from $y-i\varepsilon$ to y, and the horizontal line from y to ∞, and where the coefficient $a_{t,Q,\eta_k}(s+2l)$ is given by

$$
a_{t,\eta_k,Q}(s+2l) = \frac{2^{s+2l-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s+2l)}\,\Gamma\Big(\frac{s+2l-\frac{n-1}{2}+it}{2}\Big)\,\Gamma\Big(\frac{s+2l-\frac{n-1}{2}-it}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-it\Big).
$$

The assumption $n \equiv 1 \mod 4$ implies that $\frac{n-1}{4} \in \mathbb{N}$, so particularly on the line $\mathrm{Re}(s) + 2l = \frac{n-1}{2} - 2\left(\frac{n-1}{4} - l\right) = 2l$ $\left(l \in \left\{0, \ldots, \frac{n-1}{4}\right\}\right)$ the identity

$$\begin{aligned} K^{\mathrm{hyp}}(P,Q,s+2l) &= \sum_{j=0}^{\infty} a_{j,Q}(s+2l)\,\psi_j(P) + \frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{W'_{y,\varepsilon}} a_{t,Q,\eta_k}(s+2l)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt \\ &+ \frac{2^{s+2l}\,\pi^{\frac{n-1}{2}}}{\Gamma(s+2l)} \sum_{l'=0}^{\frac{n-5}{4}-l} \frac{(-1)^{l'}}{l'!}\,\Gamma\Big(s+2l-\frac{n-1}{2}+l'\Big) \\ &\qquad\cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2(l+l'))\,E^{\mathrm{par}}_{\eta_k}(P,s+2(l+l')) \\ &+ \frac{2^{s+2l-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s+2l)}\,\frac{(-1)^{\frac{n-1}{4}-l}}{\left(\frac{n-1}{4}-l\right)!}\,\Gamma\Big(s+l-\frac{n-1}{4}\Big) \\ &\qquad\cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-s\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+s\Big) \end{aligned} \tag{8.39}$$

holds true, and the sum over l' in (8.39) can be rewritten as

$$\sum_{l'=l}^{\frac{n-5}{4}} \frac{(-1)^{l'-l}}{(l'-l)!}\,\Gamma\Big(s-\frac{n-1}{2}+l+l'\Big)\sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(Q,n-1-s-2l')\,E^{\mathrm{par}}_{\eta_k}(P,s+2l').$$

On the line $\mathrm{Re}(s) + 2l = \frac{n-1}{2} - 2\left(\frac{n-1}{4} - l\right) = 2l$ $\left(l \in \left\{\frac{n+3}{4}, \ldots, \frac{n-1}{2}\right\}\right)$ we have $\mathrm{Re}(s) + 2l > \frac{n-1}{2}$, so the meromorphic continuation is already given by means of

$$K^{\mathrm{hyp}}(P,Q,s+2l) = \sum_{j=0}^{\infty} a_{j,Q}(s+2l)\,\psi_j(P) + \frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s+2l)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt. \tag{8.40}$$

Inserting (8.39) and (8.40) into (8.38), at $s = 0$ we obtain

$$\begin{aligned} E^{\mathrm{ell}}_Q(P,s) - \mathcal{G}^{\mathrm{par}}_{Q,3}(P,s) &= \frac{1}{|\Gamma_Q|}\sum_{l=\frac{n+1}{2}}^{\infty} \frac{\left(\frac{s}{2}\right)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l) + \frac{1}{|\Gamma_Q|}\sum_{l=0}^{\frac{n-1}{2}} \frac{\left(\frac{s}{2}\right)_l}{l!}\sum_{j=0}^{\infty} a_{j,Q}(s+2l)\,\psi_j(P) \\ &+ \frac{1}{|\Gamma_Q|}\sum_{l=0}^{\frac{n-1}{4}} \frac{\left(\frac{s}{2}\right)_l}{l!}\,\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{W'_{y,\varepsilon}} a_{t,\eta_k,Q}(s+2l)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt \\ &+ \frac{1}{|\Gamma_Q|}\sum_{l=\frac{n+3}{4}}^{\frac{n-1}{2}} \frac{\left(\frac{s}{2}\right)_l}{l!}\,\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s+2l)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt \\ &+ \frac{1}{|\Gamma_Q|}\sum_{l=0}^{\frac{n-1}{4}} \frac{2^{s+2l-1}\,\pi^{\frac{n-1}{2}}\left(\frac{s}{2}\right)_l}{l!\,\Gamma(s+2l)}\,\frac{(-1)^{\frac{n-1}{4}-l}}{\left(\frac{n-1}{4}-l\right)!}\,\Gamma\Big(s+l-\frac{n-1}{4}\Big) \\ &\qquad\cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-s\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+s\Big), \end{aligned} \tag{8.41}$$

and to derive the Laurent expansion at $s = 0$ we work from formula (8.41).

As in the proofs of Proposition 8.1.2 and Proposition 8.1.4, at $s = 0$ we have the Laurent expansion

$$\frac{1}{|\Gamma_Q|}\sum_{l=\frac{n+1}{2}}^{\infty} \frac{\left(\frac{s}{2}\right)_l}{l!}\,K^{\mathrm{hyp}}(P,Q,s+2l) = \mathrm{O}(s). \tag{8.42}$$

We start the treatment of the other summands on the right-hand side of (8.41) with the case $l = 0$.

As in the proof of Proposition 8.1.4, for $j = 0$, i.e. $\lambda_j = 0$, $r_j = -i\,\frac{n-1}{2}$ and $\psi_j(P) = \mathrm{vol}(\Gamma\backslash\mathbb{H}^n)^{-1/2}$, the function

$$a_{0,Q}(s)\,\psi_0(P) = \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma(s)}\,\Gamma\Big(\frac{s}{2}\Big)\,\Gamma\Big(\frac{s-n+1}{2}\Big)$$

admits a Laurent expansion at $s = 0$ of the form

$$a_{0,Q}(s)\,\psi_0(P) = \frac{2\,(-\pi)^{\frac{n-1}{2}}}{\big(\frac{n-1}{2}\big)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\cdot\frac{1}{s} + \frac{(-\pi)^{\frac{n-1}{2}}}{\big(\frac{n-1}{2}\big)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\left(\sum_{m=1}^{\frac{n-1}{2}}\frac{1}{m} + \log(4)\right) + \mathrm{O}(s)$$

since $\frac{n-1}{2} \in \mathbb{N}_0$ holds true.

Now we let $j \geq 1$, so either $r_j > 0$ or $r_j \in \big(-i\,\frac{n-1}{2}, 0\big]$ holds true. In the proof of Proposition 7.1.6 we have seen that the function $\Gamma\left(\frac{s-\frac{n-1}{2}+ir_j}{2}\right)$ has a pole at $s = 0$ if and only if

$$\frac{n-1}{4} - \frac{ir_j}{2} \in \Big\{1,\ldots,\frac{n-1}{4}\Big\} \quad\Longleftrightarrow\quad r_j \in \Big\{-i\Big(\frac{n-1}{2} - 2N\Big)\,\Big|\,N \in \Big\{1,\ldots,\frac{n-1}{4}\Big\}\Big\} =: M_1(n),$$

while the function $\Gamma\left(\frac{s-\frac{n-1}{2}-ir_j}{2}\right)$ has a pole at $s = 0$ if and only if

$$\begin{aligned}&\frac{n-1}{4} + \frac{ir_j}{2} \in \Big\{\frac{n-1}{4},\ldots,\frac{n-3}{2}\Big\}\\ &\quad\Longleftrightarrow\quad r_j \in \Big\{-i\Big(2N - \frac{n-1}{2}\Big)\,\Big|\,N \in \Big\{\frac{n-1}{4},\ldots,\frac{n-3}{2}\Big\}\Big\} =: M_2(n),\end{aligned}$$

and that $M_1(n) = M_2(n) =: M(n)$.

Completely analogous to the proof of Proposition 8.1.4, at $s = 0$ we find the Laurent expansion

$$\sum_{\substack{j\in\mathbb{N}:\\ r_j\in M(n)}} a_{j,Q}(s)\,\psi_j(P) = F_{n,Q}(P)\cdot\frac{1}{s} + G_{n,Q,1}(P) + \mathrm{O}(s),$$

where

$$F_{n,Q}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M(n)}} \frac{2\,(-\pi)^{\frac{n-1}{2}}}{\Big(\frac{n-1}{4}+\frac{ir_j}{2}\Big)!\,\Big(\frac{n-1}{4}-\frac{ir_j}{2}\Big)!}\,\psi_j(P)\,\overline{\psi_j}(Q), \tag{8.43}$$

$$G_{n,Q,1}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M(n)}} \frac{(-\pi)^{\frac{n-1}{2}}}{\Big(\frac{n-1}{4}+\frac{ir_j}{2}\Big)!\,\Big(\frac{n-1}{4}-\frac{ir_j}{2}\Big)!}\left(\sum_{m=1}^{\frac{n-1}{4}+\frac{ir_j}{2}}\frac{1}{m} + \sum_{m=1}^{\frac{n-1}{4}-\frac{ir_j}{2}}\frac{1}{m} + \log(4)\right)\psi_j(P)\,\overline{\psi_j}(Q), \tag{8.44}$$

as well as the Laurent expansion

$$\sum_{\substack{j\in\mathbb{N}:\\ r_j\notin M(n)}} a_{j,Q}(s)\,\psi_j(P) = \mathrm{O}(s).$$

Analogous to the proof of Proposition 7.1.6, the assumptions $\frac{n-1}{4} \in \mathbb{N}_0$ and $\varepsilon \in (0,1)$ imply that $\frac{n-1}{4} \pm it \notin \mathbb{N}_0$ for any $t \in W'_{y,\varepsilon}$. Together with (A.13) this yields that for any $t \in W'_{y,\varepsilon}$ at $s = 0$

we have the Laurent expansions

$$\begin{aligned}2^{s-1} &= \frac{1}{2} + \frac{\log(2)}{2} \cdot s + \mathrm{O}(s^2),\\ \frac{1}{\Gamma(s)} &= s + \gamma \cdot s^2 + \mathrm{O}(s^3),\\ \Gamma\Big(\frac{s - \frac{n-1}{2} \pm it}{2}\Big) &= \Gamma\Big(\pm\frac{it}{2} - \frac{n-1}{4}\Big) + \frac{1}{2}\,\Gamma\Big(\pm\frac{it}{2} - \frac{n-1}{4}\Big)\,\psi^{(0)}\Big(\pm\frac{it}{2} - \frac{n-1}{4}\Big) \cdot s + \mathrm{O}(s^2).\end{aligned}$$

Thus, for $k = 1, \ldots, c_\Gamma$ and any $t \in W'_{y,\varepsilon}$ the function $a_{t,\eta_k,Q}(s)$ admits a Laurent expansion at $s = 0$ of the form

$$a_{t,\eta_k,Q}(s) = \frac{\pi^{\frac{n-1}{2}}}{2}\,\Gamma\Big(\frac{it}{2} - \frac{n-1}{4}\Big)\,\Gamma\Big(-\frac{it}{2} - \frac{n-1}{4}\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} - it\Big) \cdot s + \mathrm{O}(s^2).$$

This shows that at $s = 0$ we have the Laurent expansion

$$\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{W'_{y,\varepsilon}} a_{t,\eta_k,Q}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\,dt = \mathrm{O}(s).$$

Moreover, the fact that $\frac{n-1}{4} \in \mathbb{N}_0$ implies that at $s = 0$ we have the Laurent expansion

$$\Gamma\Big(s - \frac{n-1}{4}\Big) = \frac{(-1)^{\frac{n-1}{4}}}{\big(\frac{n-1}{4}\big)!} \cdot \frac{1}{s} + \frac{(-1)^{\frac{n-1}{4}}}{\big(\frac{n-1}{4}\big)!}\left(\sum_{m=1}^{\frac{n-1}{4}}\frac{1}{m} - \gamma\right) + \mathrm{O}(s).$$

As the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P, w)$ ($k = 1, \ldots, c_\Gamma$) have no poles on the line $\mathrm{Re}(w) = \frac{n-1}{2}$, for $k = 1, \ldots, c_\Gamma$ we find the Laurent expansions

$$\begin{aligned}E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + s\Big) &= E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2}\Big) + \mathrm{O}(s),\\ E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} - s\Big) &= E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2}\Big) + \mathrm{O}(s)\end{aligned}$$

at $s = 0$. Together this gives us a Laurent expansion at $s = 0$ of the form

$$\frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(s)}\,\frac{(-1)^{\frac{n-1}{4}}}{\big(\frac{n-1}{4}\big)!}\,\Gamma\Big(s - \frac{n-1}{4}\Big)\sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} - s\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + s\Big) = G_{n,Q,2}(P) + \mathrm{O}(s),$$

where

$$G_{n,Q,2}(P) := \frac{(-\pi)^{\frac{n-1}{2}}}{2\,\big(\big(\frac{n-1}{4}\big)!\big)^2}\sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2}\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2}\Big). \tag{8.45}$$

Adding up and taking into account that $\big(\frac{s}{2}\big)_0/0! = 1$, the summand for $l = 0$ on the right-hand side of formula (8.41) admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned}&\frac{1}{|\Gamma_Q|}\sum_{j=0}^{\infty} a_{j,Q}(s)\,\psi_j(P) + \frac{1}{4\pi\,|\Gamma_Q|}\sum_{k=1}^{c_\Gamma}\int_{W'_{y,\varepsilon}} a_{t,\eta_k,Q}(s)\,E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\,dt\\ &\qquad + \frac{2^{s-1}\,\pi^{\frac{n-1}{2}}}{|\Gamma_Q|\,\Gamma(s)}\,\frac{(-1)^{\frac{n-1}{4}}}{\big(\frac{n-1}{4}\big)!}\,\Gamma\Big(s - \frac{n-1}{4}\Big)\sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} - s\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + s\Big)\\ &= \left(\frac{2\,(-\pi)^{\frac{n-1}{2}}}{\big(\frac{n-1}{2}\big)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{F_{n,Q}(P)}{|\Gamma_Q|}\right) \cdot \frac{1}{s}\\ &\qquad + \frac{(-\pi)^{\frac{n-1}{2}}}{\big(\frac{n-1}{2}\big)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\left(\sum_{m=1}^{\frac{n-1}{2}}\frac{1}{m} + \log(4)\right) + \frac{1}{|\Gamma_Q|}\sum_{m=1}^{2} G_{n,Q,m}(P) + \mathrm{O}(s).\end{aligned} \tag{8.46}$$

Next we turn to the case $l \in \left\{1, \ldots, \frac{n-1}{4}\right\}$.

If $j = 0$, i.e. $\lambda_j = 0$, $r_j = -i\,\frac{n-1}{2}$ and $\psi_j(P) = \mathrm{vol}(\Gamma\backslash\mathbb{H}^n)^{-1/2}$, analogous to the proof of Proposition 8.1.4, the function

$$\frac{\left(\frac{s}{2}\right)_l}{l!}\, a_{0,Q}(s+2l)\,\psi_0(P) = \frac{\left(\frac{s}{2}\right)_l}{l!}\, \frac{2^{s+2l-1}\,\pi^{\frac{n-1}{2}}}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma(s+2l)}\,\Gamma\Big(\frac{s}{2}+l\Big)\,\Gamma\Big(\frac{s-n+1}{2}+l\Big)$$

admits a Laurent expansion at $s = 0$ of the form

$$\frac{\left(\frac{s}{2}\right)_l}{l!}\, a_{0,Q}(s+2l)\,\psi_0(P) = \frac{2^{2l-1}\,(-1)^{\frac{n-1}{2}-l}\,\pi^{\frac{n-1}{2}}\,(l-1)!}{l\,(2l-1)!\,\left(\frac{n-1}{2}-l\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \mathrm{O}(s).$$

Now we let $j \geq 1$, so that either $r_j > 0$ or $r_j \in \left(-i\,\frac{n-1}{2}, 0\right]$ holds true. In the first case clearly $\frac{n-1}{4} \pm \frac{ir_j}{2} - l \notin \mathbb{N}_0$, and in the latter case $\frac{n-1}{4} \pm \frac{ir_j}{2} - l \in \mathbb{R}$ with $\frac{n-1}{4} - \frac{ir_j}{2} - l \in \left(-l, \frac{n-1}{4} - l\right]$ and $\frac{n-1}{4} + \frac{ir_j}{2} - l \in \left[\frac{n-1}{4} - l, \frac{n-1}{2} - l\right)$.
The elements of $\mathbb{N}_0$ contained in the interval $\left(-l, \frac{n-1}{4} - l\right]$ are $0, \ldots, \frac{n-1}{4} - l$, and the interval $\left[\frac{n-1}{4} - l, \frac{n-1}{2} - l\right)$ contains the elements $\frac{n-1}{4} - l, \ldots, \frac{n-3}{2} - l$ of $\mathbb{N}_0$.
Analogous to the proofs of the previous propositions, this implies that the function $\Gamma\left(\frac{s+2l-\frac{n-1}{2}+ir_j}{2}\right)$ has a pole at $s = 0$ if and only if

$$\begin{aligned}\frac{n-1}{4} - \frac{ir_j}{2} - l \in \Big\{0, \ldots, \frac{n-1}{4} - l\Big\} &\iff \frac{n-1}{4} - \frac{ir_j}{2} \in \Big\{l, \ldots, \frac{n-1}{4}\Big\} \\ \iff r_j \in \Big\{ -i\Big(\frac{n-1}{2} - 2N\Big) \;\Big|\; N \in \Big\{l, \ldots, \frac{n-1}{4}\Big\}\Big\} &=: M_1(n,l),\end{aligned}$$

while the function $\Gamma\left(\frac{s+2l-\frac{n-1}{2}-ir_j}{2}\right)$ has a pole at $s = 0$ if and only if

$$\begin{aligned}\frac{n-1}{4} + \frac{ir_j}{2} - l \in \Big\{\frac{n-1}{4} - l, \ldots, \frac{n-3}{2} - l\Big\} &\iff \frac{n-1}{4} + \frac{ir_j}{2} \in \Big\{\frac{n-1}{4}, \ldots, \frac{n-3}{2}\Big\} \\ \iff r_j \in \Big\{ -i\Big(2N - \frac{n-1}{2}\Big) \;\Big|\; N \in \Big\{\frac{n-1}{4}, \ldots, \frac{n-3}{2}\Big\}\Big\} &=: M_2(n,l).\end{aligned}$$

Moreover, if $\frac{n-1}{4} - \frac{ir_j}{2} = N$ for some $N \in \left\{l, \ldots, \frac{n-1}{4}\right\}$, then $\frac{n-1}{2} \in \mathbb{Z}$ gives us

$$\frac{n-1}{4} + \frac{ir_j}{2} = \frac{n-1}{2} - N \;\in \mathbb{Z} \quad \text{and} \quad \frac{n-1}{2} - N \in \Big[\frac{n-1}{4}, \frac{n-1}{2} - l\Big],$$

so also $\frac{n-1}{4} + \frac{ir_j}{2} \in \left\{\frac{n-1}{4}, \ldots, \frac{n-1}{2} - l\right\} \subseteq \left\{\frac{n-1}{4}, \ldots, \frac{n-3}{2}\right\}$. Therefore, we have proven that $M_1(n,l) \subseteq M_2(n,l)$.

Exactly as in the proof of Proposition 8.1.4, at $s = 0$ we obtain the Laurent expansion

$$\frac{\left(\frac{s}{2}\right)_l}{l!} \sum_{\substack{j\in\mathbb{N}: \\ r_j \in M_2(n,l)\setminus M_1(n,l)}} a_{j,Q}(s+2l)\,\psi_j(P) = G_{n,Q,l,1}(P) + \mathrm{O}(s),$$

where

$$G_{n,Q,l,1}(P) := \sum_{\substack{j\in\mathbb{N}: \\ r_j \in M_2(n,l)\setminus M_1(n,l)}} \frac{2^{2l-1}\,(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}-l}\,\pi^{\frac{n-1}{2}}}{l\,(2l-1)!\,\left(\frac{n-1}{4}+\frac{ir_j}{2}-l\right)!}\,\Gamma\Big(\frac{ir_j}{2} - \frac{n-1}{4} + l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q), \tag{8.47}$$

the Laurent expansion

$$\frac{\left(\frac{s}{2}\right)_l}{l!} \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_1(n,l)}} a_{j,Q}(s+2l)\,\psi_j(P) = F_{n,Q,l}(P)\cdot\frac{1}{s} + G_{n,Q,l,2}(P) + \mathrm{O}(s),$$

where

$$F_{n,Q,l}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_1(n,l)}} \frac{4^l\,(-\pi)^{\frac{n-1}{2}}}{l\,(2l-1)!\left(\frac{n-1}{4}+\frac{ir_j}{2}-l\right)!\left(\frac{n-1}{4}-\frac{ir_j}{2}-l\right)!}\,\psi_j(P)\,\overline{\psi_j}(Q), \tag{8.48}$$

$$\begin{aligned} G_{n,Q,l,2}(P) := &\sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_1(n,l)}} \frac{2^{2l-1}\,(-\pi)^{\frac{n-1}{2}}}{l\,(2l-1)!\left(\frac{n-1}{4}+\frac{ir_j}{2}-l\right)!\left(\frac{n-1}{4}-\frac{ir_j}{2}-l\right)!}\,\psi_j(P)\,\overline{\psi_j}(Q)\\ &\cdot\left(\sum_{m=1}^{\frac{n-1}{4}+\frac{ir_j}{2}-l}\frac{1}{m} + \sum_{m=1}^{\frac{n-1}{4}-\frac{ir_j}{2}-l}\frac{1}{m} - 2\sum_{m=1}^{2l-1}\frac{1}{m} + \sum_{m=1}^{l-1}\frac{1}{m} + \log(4)\right), \end{aligned} \tag{8.49}$$

and the Laurent expansion

$$\frac{\left(\frac{s}{2}\right)_l}{l!} \sum_{\substack{j\in\mathbb{N}:\\ r_j\notin M_2(n,l)}} a_{j,Q}(s+2l)\,\psi_j(P) = \mathrm{O}(s).$$

Moreover, from $\frac{n-1}{4}, l\in\mathbb{N}_0$ and $\varepsilon\in(0,1)$ we can conclude that $\frac{n-1}{4}-l\pm\frac{\varepsilon}{2}\notin\mathbb{N}_0$, yielding that $\frac{n-1}{4}-l\pm it\notin\mathbb{N}_0$ for any $t\in W'_{y,\varepsilon}$. Consequently, for any $t\in W'_{y,\varepsilon}$ at $s=0$ we have the Laurent expansions

$$\begin{aligned} 2^{s+2l-1} &= 2^{2l-1} + 2^{2l-1}\,\log(2)\cdot s + \mathrm{O}(s^2),\\ \frac{1}{\Gamma(s+2l)} &= \frac{1}{(2l-1)!} - \frac{1}{(2l-1)!}\left(\sum_{m=1}^{2l-1}\frac{1}{m} - \gamma\right)\cdot s + \mathrm{O}(s^2),\\ \Gamma\Big(\frac{s+2l-\frac{n-1}{2}\pm it}{2}\Big) &= \Gamma\Big(\pm\frac{it}{2}-\frac{n-1}{4}+l\Big)\\ &\quad+\frac{1}{2}\,\Gamma\Big(\pm\frac{it}{2}-\frac{n-1}{4}+l\Big)\,\psi^{(0)}\Big(\pm\frac{it}{2}-\frac{n-1}{4}+l\Big)\cdot s + \mathrm{O}(s^2). \end{aligned}$$

This implies that for $k=1,\ldots,c_\Gamma$ and any $t\in W'_{y,\varepsilon}$ the function $a_{t,\eta_k,Q}(s+2l)$ admits a Laurent expansion at $s=0$ of the form

$$a_{t,\eta_k,Q}(s+2l) = \frac{2^{2l-1}\,\pi^{\frac{n-1}{2}}}{(2l-1)!}\,\Gamma\Big(\frac{it}{2}-\frac{n-1}{4}+l\Big)\,\Gamma\Big(-\frac{it}{2}-\frac{n-1}{4}+l\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(Q,\frac{n-1}{2}-it\Big) + \mathrm{O}(s).$$

Together with the Laurent expansion

$$\left(\frac{s}{2}\right)_l = \frac{(l-1)!}{2}\cdot s + \frac{(l-1)!}{4}\left(\sum_{m=1}^{l-1}\frac{1}{m}\right)\cdot s^2 + \mathrm{O}(s^3)$$

at $s=0$ of the Pochhammer symbol $\left(\frac{s}{2}\right)_l$ this shows that after multiplication by $\left(\frac{s}{2}\right)_l/l!$ at $s=0$ we have the Laurent expansion

$$\frac{\left(\frac{s}{2}\right)_l}{l!}\,\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{W'_{y,\varepsilon}} a_{t,\eta_k,Q}(s+2l)\,E^{\mathrm{par}}_{\eta_k}\Big(P,\frac{n-1}{2}+it\Big)\,dt = \mathrm{O}(s).$$

Furthermore, $\frac{n-1}{4} \in \mathbb{N}_0$ and $l \leq \frac{n-1}{4}$ imply that at $s = 0$ we have the Laurent expansions

$$\begin{aligned} \Gamma\Big(s+l-\frac{n-1}{4}\Big) &= \frac{(-1)^{\frac{n-1}{4}-l}}{\left(\frac{n-1}{4}-l\right)!} \cdot \frac{1}{s} + \frac{(-1)^{\frac{n-1}{4}-l}}{\left(\frac{n-1}{4}-l\right)!} \left(\sum_{m=1}^{\frac{n-1}{4}-l} \frac{1}{m} - \gamma \right) + \mathrm{O}(s), \\ E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2}+s\Big) &= E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2}\Big) + \mathrm{O}(s), \\ E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2}-s\Big) &= E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2}\Big) + \mathrm{O}(s). \end{aligned}$$

Using this together with the Laurent expansion of $\left(\frac{s}{2}\right)_l$, we obtain a Laurent expansion at $s = 0$ of the form

$$\begin{aligned} &\frac{2^{s+2l-1}\,\pi^{\frac{n-1}{2}}\,\left(\frac{s}{2}\right)_l}{l!\,\Gamma(s+2l)} \frac{(-1)^{\frac{n-1}{4}-l}}{\left(\frac{n-1}{4}-l\right)!} \Gamma\Big(s+l-\frac{n-1}{4}\Big) \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2}-s\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2}+s\Big) \\ &\quad = G_{n,Q,l,3}(P) + \mathrm{O}(s), \end{aligned}$$

where

$$G_{n,Q,l,3}(P) := \frac{4^{l-1}\,(-\pi)^{\frac{n-1}{2}}}{l\,(2l-1)!\,\left(\left(\frac{n-1}{4}-l\right)!\right)^2} \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2}\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2}\Big). \tag{8.50}$$

Summing up, for $l = 1, \ldots, \frac{n-1}{4}$ we obtain a Laurent expansion at $s = 0$ of the form

$$\begin{aligned} &\frac{1}{|\Gamma_Q|} \frac{\left(\frac{s}{2}\right)_l}{l!} \sum_{j=0}^{\infty} a_{j,Q}(s+2l)\,\psi_j(P) + \frac{1}{|\Gamma_Q|} \frac{\left(\frac{s}{2}\right)_l}{l!} \frac{1}{4\pi} \sum_{k=1}^{c_\Gamma} \int_{W'_{y,\varepsilon}} a_{t,\eta_k,Q}(s+2l)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2}+it\Big)\, dt \\ &\qquad + \frac{1}{|\Gamma_Q|} \frac{2^{s+2l-1}\,\pi^{\frac{n-1}{2}}\,\left(\frac{s}{2}\right)_l}{l!\,\Gamma(s+2l)} \frac{(-1)^{\frac{n-1}{4}-l}}{\left(\frac{n-1}{4}-l\right)!} \Gamma\Big(s+l-\frac{n-1}{4}\Big) \\ &\qquad\qquad \cdot \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2}-s\Big)\, E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2}+s\Big) \\ &= \frac{F_{n,Q,l}(P)}{|\Gamma_Q|} \cdot \frac{1}{s} + \frac{2^{2l-1}\,(-1)^{\frac{n-1}{2}-l}\,\pi^{\frac{n-1}{2}}\,(l-1)!}{l\,(2l-1)!\,\left(\frac{n-1}{2}-l\right)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|} \sum_{m=1}^{3} G_{n,Q,l,m}(P) + \mathrm{O}(s). \end{aligned} \tag{8.51}$$

It remains to consider the case $l \in \left\{\frac{n+3}{4}, \ldots, \frac{n-1}{2}\right\}$.

As for $l \in \left\{1, \ldots, \frac{n-1}{4}\right\}$, if $j = 0$, i.e. $\lambda_j = 0$, $r_j = -i\,\frac{n-1}{2}$ and $\psi_j(P) = \mathrm{vol}(\Gamma\backslash\mathbb{H}^n)^{-1/2}$, then the function

$$\frac{\left(\frac{s}{2}\right)_l}{l!} a_{0,Q}(s+2l)\,\psi_0(P) = \frac{\left(\frac{s}{2}\right)_l}{l!} \frac{2^{s+2l-1}\,\pi^{\frac{n-1}{2}}}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)\,\Gamma(s+2l)} \Gamma\Big(\frac{s}{2}+l\Big)\,\Gamma\Big(\frac{s-n+1}{2}+l\Big)$$

admits a Laurent expansion at $s = 0$ of the form

$$\frac{\left(\frac{s}{2}\right)_l}{l!} a_{0,Q}(s+2l)\,\psi_0(P) = \frac{2^{2l-1}\,(-1)^{\frac{n-1}{2}-l}\,\pi^{\frac{n-1}{2}}\,(l-1)!}{l\,(2l-1)!\,\left(\frac{n-1}{2}-l\right)!\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \mathrm{O}(s).$$

Let $j \geq 1$, so we either have $r_j > 0$ or $r_j \in \left(-i\,\frac{n-1}{2}, 0\right]$. In the first case $\frac{n-1}{4} \pm \frac{ir_j}{2} - l \notin \mathbb{N}_0$, while in the latter case we have $\frac{n-1}{4} \pm \frac{ir_j}{2} - l \in \mathbb{R}$ with $\frac{n-1}{4} - \frac{ir_j}{2} - l \in \left(-l, \frac{n-1}{4} - l\right]$ and $\frac{n-1}{4} + \frac{ir_j}{2} - l \in \left[\frac{n-1}{4} - l, \frac{n-1}{2} - l\right)$.
The interval $\left(-l, \frac{n-1}{4} - l\right]$ contains no element of $\mathbb{N}_0$ because of $l > \frac{n-1}{4}$, whereas the interval

$\left[\frac{n-1}{4} - l, \frac{n-1}{2} - l\right)$ contains the elements $0, \dots, \frac{n-3}{2} - l$ of $\mathbb{N}_0$.
This shows that the function $\Gamma\left(\frac{s+2l-\frac{n-1}{2}+ir_j}{2}\right)$ cannot have a pole at $s = 0$, while the function $\Gamma\left(\frac{s+2l-\frac{n-1}{2}-ir_j}{2}\right)$ has a pole at $s = 0$ if and only if

$$\begin{aligned}\frac{n-1}{4} + \frac{ir_j}{2} - l \in \left\{0, \dots, \frac{n-3}{2} - l\right\} &\iff \frac{n-1}{4} + \frac{ir_j}{2} \in \left\{l, \dots, \frac{n-3}{2}\right\} \\ \iff r_j \in \left\{ -i\left(2N - \frac{n-1}{2}\right) \;\middle|\; N \in \left\{l, \dots, \frac{n-3}{2}\right\}\right\} &=: M_3(n,l).\end{aligned}$$

If $r_j \in M_3(n,l)$, then at $s = 0$ we have the Laurent expansions

$$\begin{aligned}\Gamma\Big(\frac{s + 2l - \frac{n-1}{2} + ir_j}{2}\Big) &= \Gamma\Big(\frac{ir_j}{2} - \frac{n-1}{4} + l\Big) \\ &\quad + \frac{1}{2}\Gamma\Big(\frac{ir_j}{2} - \frac{n-1}{4} + l\Big)\,\psi^{(0)}\Big(\frac{ir_j}{2} - \frac{n-1}{4} + l\Big)\cdot s + \mathrm{O}(s^2), \\ \Gamma\Big(\frac{s + 2l - \frac{n-1}{2} - ir_j}{2}\Big) &= \frac{2\,(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}-l}}{\left(\frac{n-1}{4} + \frac{ir_j}{2} - l\right)!}\cdot\frac{1}{s} + \frac{(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}-l}}{\left(\frac{n-1}{4} + \frac{ir_j}{2} - l\right)!}\left(\sum_{m=1}^{\frac{n-1}{4}+\frac{ir_j}{2}-l}\frac{1}{m} - \gamma\right) + \mathrm{O}(s).\end{aligned}$$

Therefore, for $j \geq 1$ with $r_j \in M_3(n,l)$ the function $a_{j,Q}(s+2l)\,\psi_j(P)$ admits a Laurent expansion at $s = 0$ of the form

$$a_{j,Q}(s + 2l)\,\psi_j(P) = \frac{4^l\,(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}-l}\,\pi^{\frac{n-1}{2}}}{(2l-1)!\left(\frac{n-1}{4} + \frac{ir_j}{2} - l\right)!}\Gamma\Big(\frac{ir_j}{2} - \frac{n-1}{4} + l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q)\cdot\frac{1}{s} + \mathrm{O}(1),$$

implying that the respective part of the series after multiplication by $\left(\frac{s}{2}\right)_l/l!$ admits a Laurent expansion at $s = 0$ of the form

$$\frac{\left(\frac{s}{2}\right)_l}{l!}\sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_3(n,l)}} a_{j,Q}(s + 2l)\,\psi_j(P) = G_{n,Q,l,4}(P) + \mathrm{O}(s),$$

where

$$G_{n,Q,l,4}(P) := \sum_{\substack{j\in\mathbb{N}:\\ r_j\in M_3(n,l)}} \frac{2^{2l-1}\,(-1)^{\frac{n-1}{4}+\frac{ir_j}{2}-l}\,\pi^{\frac{n-1}{2}}}{l\,(2l-1)!\left(\frac{n-1}{4} + \frac{ir_j}{2} - l\right)!}\Gamma\Big(\frac{ir_j}{2} - \frac{n-1}{4} + l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q). \tag{8.52}$$

In the case $r_j \notin M_3(n,l)$ we have the Laurent expansions

$$\begin{aligned}\Gamma\Big(\frac{s + 2l - \frac{n-1}{2} \pm ir_j}{2}\Big) &= \Gamma\Big(\pm\frac{ir_j}{2} - \frac{n-1}{4} + l\Big) \\ &\quad + \frac{1}{2}\Gamma\Big(\pm\frac{ir_j}{2} - \frac{n-1}{4} + l\Big)\,\psi^{(0)}\Big(\pm\frac{ir_j}{2} - \frac{n-1}{4} + l\Big)\cdot s + \mathrm{O}(s^2).\end{aligned}$$

at $s = 0$. For $j \geq 1$ with $r_j \notin M_3(n,l)$ this gives us a Laurent expansion of the function $a_{j,Q}(s + 2l)\,\psi_j(P)$ at $s = 0$ of the form

$$a_{j,Q}(s + 2l)\,\psi_j(P) = \frac{2^{2l-1}\,\pi^{\frac{n-1}{2}}}{(2l-1)!}\Gamma\Big(\frac{ir_j}{2} - \frac{n-1}{4} + l\Big)\,\Gamma\Big(-\frac{ir_j}{2} - \frac{n-1}{4} + l\Big)\,\psi_j(P)\,\overline{\psi_j}(Q) + \mathrm{O}(s),$$

and the respective part of the series after multiplication by $\left(\frac{s}{2}\right)_l/l!$ has a Laurent expansion at $s = 0$ of the form

$$\frac{\left(\frac{s}{2}\right)_l}{l!}\sum_{\substack{j\in\mathbb{N}:\\ r_j\notin M_3(n,l)}} a_{j,Q}(s + 2l)\,\psi_j(P) = \mathrm{O}(s).$$

Moreover, from the inequality $l > \frac{n-1}{4}$ we derive that for any $t \in \mathbb{R}$ at $s = 0$ we have the Laurent expansions

$$\Gamma\Big(\frac{s + 2l - \frac{n-1}{2} \pm it}{2}\Big) = \Gamma\Big(\pm \frac{it}{2} - \frac{n-1}{4} + l\Big) + \frac{1}{2}\,\Gamma\Big(\pm \frac{it}{2} - \frac{n-1}{4} + l\Big)\,\psi^{(0)}\Big(\pm \frac{it}{2} - \frac{n-1}{4} + l\Big) \cdot s + \mathrm{O}(s^2).$$

Thus, for $k = 1, \ldots, c_\Gamma$ and any $t \in \mathbb{R}$ the function $a_{t,\eta_k,Q}(s + 2l)$ admits a Laurent expansion at $s = 0$ of the form

$$a_{t,\eta_k,Q}(s + 2l) = \frac{2^{2l-1}\,\pi^{\frac{n-1}{2}}}{(2l-1)!}\,\Gamma\Big(\frac{it}{2} - \frac{n-1}{4} + l\Big)\,\Gamma\Big(-\frac{it}{2} - \frac{n-1}{4} + l\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(Q, \frac{n-1}{2} - it\Big) + \mathrm{O}(s).$$

This implies that after multiplication by $\big(\frac{s}{2}\big)_l / l!$ we have the Laurent expansion

$$\frac{\big(\frac{s}{2}\big)_l}{l!}\,\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s + 2l)\,E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\,dt = \mathrm{O}(s)$$

at $s = 0$.

Summing up, for $l = \frac{n+3}{4}, \ldots, \frac{n-1}{2}$ we get a Laurent expansion at $s = 0$ of the form

$$\begin{aligned}
&\frac{1}{|\Gamma_Q|}\,\frac{\big(\frac{s}{2}\big)_l}{l!}\sum_{j=0}^{\infty} a_{j,Q}(s + 2l)\,\psi_j(P) + \frac{1}{|\Gamma_Q|}\,\frac{\big(\frac{s}{2}\big)_l}{l!}\,\frac{1}{4\pi}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} a_{t,\eta_k,Q}(s + 2l)\,E^{\mathrm{par}}_{\eta_k}\Big(P, \frac{n-1}{2} + it\Big)\,dt \\
&= \frac{2^{2l-1}\,(-1)^{\frac{n-1}{2}-l}\,\pi^{\frac{n-1}{2}}\,(l-1)!}{l\,(2l-1)!\,\big(\frac{n-1}{2} - l\big)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{G_{n,Q,l,4}(P)}{|\Gamma_Q|} + \mathrm{O}(s).
\end{aligned} \tag{8.53}$$

If we finally put together (8.42), (8.46), (8.51) and (8.53), for $P \in \mathbb{H}^n$ we obtain a Laurent expansion at $s = 0$ of the form

$$\begin{aligned}
&E^{\mathrm{ell}}_Q(P, s) - \mathcal{G}^{\mathrm{par}}_{Q,3}(P, s) \\
&= \left(\frac{2\,(-\pi)^{\frac{n-1}{2}}}{\big(\frac{n-1}{2}\big)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{F_{n,Q}(P)}{|\Gamma_Q|} + \frac{1}{|\Gamma_Q|}\sum_{l=1}^{\frac{n-1}{4}} F_{n,Q,l}(P)\right) \cdot \frac{1}{s} \\
&\quad + \frac{(-\pi)^{\frac{n-1}{2}}}{\big(\frac{n-1}{2}\big)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}\left(\sum_{m=1}^{\frac{n-1}{2}} \frac{1}{m} + \log(4)\right) + \frac{1}{|\Gamma_Q|}\sum_{m=1}^{2} G_{n,Q,m}(P) \\
&\quad + \sum_{l=1}^{\frac{n-1}{2}} \frac{2^{2l-1}\,(-1)^{\frac{n-1}{2}-l}\,\pi^{\frac{n-1}{2}}\,(l-1)!}{l\,(2l-1)!\,\big(\frac{n-1}{2} - l\big)!\,|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)} + \frac{1}{|\Gamma_Q|}\sum_{l=1}^{\frac{n-1}{4}}\sum_{m=1}^{3} G_{n,Q,l,m}(P) \\
&\quad + \frac{1}{|\Gamma_Q|}\sum_{l=\frac{n+3}{4}}^{\frac{n-1}{2}} G_{n,Q,l,4}(P) + \mathrm{O}(s),
\end{aligned}$$

where $F_{n,Q}(P)$, $F_{n,Q,l}(P)$ $\big(l = 1, \ldots, \frac{n-1}{4}\big)$, $G_{n,Q,m}(P)$ $(m = 1, 2)$, $G_{n,Q,l,m}(P)$ $\big(l = 1, \ldots, \frac{n-1}{4}$, $m = 1, 2, 3\big)$ and $G_{n,Q,l,4}(P)$ $\big(l = \frac{n+3}{4}, \ldots, \frac{n-1}{2}\big)$ are given by (8.43), (8.44), (8.45), (8.47), (8.48), (8.49), (8.50) and (8.52), respectively.

The eigenfunctions $\psi_j(P)$ $(j \in \mathbb{N})$ and the parabolic Eisenstein series $E^{\mathrm{par}}_{\eta_k}(P, s)$ $(k = 1, \ldots, c_\Gamma)$ are invariant under the action of Γ. Hence, also the functions $F_{n,Q}(P)$, $F_{n,Q,l}(P)$ $\big(l = 1, \ldots, \frac{n-1}{4}\big)$, $G_{n,Q,m}(P)$ $(m = 1, 2)$, $G_{n,Q,l,m}(P)$ $\big(l = 1, \ldots, \frac{n-1}{4}$, $m = 1, 2, 3\big)$ and $G_{n,Q,l,4}(P)$ $\big(l = \frac{n+3}{4}, \ldots, \frac{n-1}{2}\big)$ are Γ-invariant.

□

Remark 8.1.7. For $n \equiv 1 \mod 4$ and $l > \frac{n-3}{2}$ the set

$$M_3(n,l) = \Big\{ -i\Big(2N - \frac{n-1}{2}\Big) \;\Big|\; N \in \Big\{l, \ldots, \frac{n-3}{2}\Big\}\Big\}.$$

in the proof of Proposition 8.1.6 is empty. Thus, we always have $G_{n,Q,l,4}(P) = 0$ for $l = \frac{n-1}{2}$ since the sum defining this function is empty.

Remark 8.1.8. The functions $\mathcal{G}^{\mathrm{par}}_{Q,1}(P,s)$, $\mathcal{G}^{\mathrm{par}}_{Q,2}(P,s)$ and $\mathcal{G}^{\mathrm{par}}_{Q,3}(P,s)$ that are subtracted from the elliptic Eisenstein series $E^{\mathrm{ell}}_Q(P,s)$ in the Laurent expansions (8.1), (8.25) and (8.37), respectively, differ only by the upper limits of summation in the sums over l and l'. This is because the explicit formulas for the meromorphic continuation of $E^{\mathrm{ell}}_Q(P,s)$ to the point $s = 0$ slightly vary in the different cases for n.

8.2. Example 1: The case $n = 2$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$

For elliptic Eisenstein series on the upper half-plane $\mathbb{H}$ the Laurent expansion at $s = 0$ and the Kronecker limit formula in the case $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ were determined by von Pippich in her PhD thesis [Pip10]. Therefore, the results stated in this section are already known. Nonetheless, we include them here for the sake of completeness.
Throughout the section we let $n = 2$. Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be a discrete and cofinite subgroup, i.e. a Fuchsian subgroup of the first kind. Further, let $w \in \mathbb{H}$ be a point with elliptic scaling matrix $\sigma_w \in \mathrm{PSL}_2(\mathbb{R})$ and stabilizer subgroup Γ_w.

Notation 8.2.1. To keep the notation simple, in this section we again omit the index 2 and write $E^{\mathrm{ell}}_w(z,s)$ for the elliptic Eisenstein series $E^{\mathrm{ell}}_{2,w}(z,s)$ associated to the point $w \in \mathbb{H}$, and $E^{\mathrm{par}}_{\eta_k}(z,s)$ for the parabolic Eisenstein series $E^{\mathrm{par}}_{2,\eta_k}(z,s)$ associated to the cusp $\eta_k \in C_\Gamma$ $(k = 1, \ldots, c_\Gamma)$.

As an application of Proposition 8.1.2 we can reprove the Laurent expansion of the elliptic Eisenstein series $E^{\mathrm{ell}}_w(z,s)$ on the upper half-plane $\mathbb{H}$ at $s = 0$, which was established as Proposition 6.1.1. in [Pip10] (see also Proposition 5.1 in [Pip16]). Note that our function $K^{\mathrm{hyp}}(z,w,s)$ differs from the function $P^{\mathrm{ell}}_w(z,s)$ in [Pip10] and [Pip16] by the factor $|\Gamma_w|$, i.e. $K^{\mathrm{hyp}}(z,w,s) = |\Gamma_w|\, P^{\mathrm{ell}}_w(z,s)$.

Proposition 8.2.2. *For $z \in \mathbb{H}$ with $z \neq \eta w$ for any $\eta \in \Gamma$ the elliptic Eisenstein series $E^{\mathrm{ell}}_w(z,s)$ admits a Laurent expansion at $s = 0$ of the form*

$$\begin{aligned} &E^{\mathrm{ell}}_w(z,s) - \frac{2^s\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{|\Gamma_w|\,\Gamma(s)} \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(w,1-s)\, E^{\mathrm{par}}_{\eta_k}(z,s) \\ &\quad = -\frac{2\pi}{|\Gamma_w|\,\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \left(-\frac{2\pi}{|\Gamma_w|\,\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \frac{1}{|\Gamma_w|}\left(H_{2,w,1}(z) + H_{2,w,4}(z) + H_{2,w,5}(z)\right)\right)\cdot s + \mathrm{O}(s^2), \end{aligned}$$

where the functions $H_{2,w,m}(z)$ $(m = 1,4,5)$ are invariant under the action of Γ, and are given by the formulas (8.54), (8.55) *and* (8.56) *in the proof, respectively. Moreover, for $z \in \mathbb{H}$ they satisfy the differential equation*

$$\Delta_{\mathbb{H}}\left(H_{2,w,1}(z) + H_{2,w,4}(z) + H_{2,w,5}(z)\right) = -\frac{2\pi}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}.$$

Proof. Proposition 8.1.2 gives us that for $z \in \mathbb{H}$ with $z \neq \eta w$ for any $\eta \in \Gamma$ the elliptic Eisenstein series $E^{\mathrm{ell}}_w(z,s)$ admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned} &E^{\mathrm{ell}}_w(z,s) - \frac{2^s\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{|\Gamma_w|\,\Gamma(s)} \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(w,1-s)\, E^{\mathrm{par}}_{\eta_k}(z,s) \\ &\quad = -\frac{2\pi}{|\Gamma_w|\,\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \frac{1}{|\Gamma_w|}\sum_{m=1}^{2} G_{2,w,m}(z) + \left(-\frac{2\pi}{|\Gamma_w|\,\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \frac{1}{|\Gamma_w|}\sum_{m=1}^{5} H_{2,w,m}(z)\right)\cdot s + \mathrm{O}(s^2), \end{aligned}$$

where we used the special values $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$ and $\psi^{(0)}\left(-\frac{1}{2}\right) = 2 - \gamma - \log(4)$. Moreover, in Remark 8.1.3 (a) we have noted that the functions $G_{2,w,m}(z)$ $(m = 1, 2)$ and $H_{2,w,m}(z)$ $(m = 2, 3)$ vanish identically since the sums defining these functions are all empty. Thus, we obtain a Laurent expansion at $s = 0$ of the form

$$\begin{aligned}
&E^{\mathrm{ell}}_w(z,s) - \frac{2^s\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{|\Gamma_w|\,\Gamma(s)} \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(w,1-s)\,E^{\mathrm{par}}_{\eta_k}(z,s) \\
&\quad = -\frac{2\pi}{|\Gamma_w|\,\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \left(-\frac{2\pi}{|\Gamma_w|\,\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \frac{1}{|\Gamma_w|}\left(H_{2,w,1}(z) + H_{2,w,4}(z) + H_{2,w,5}(z)\right)\right) \cdot s + \mathrm{O}(s^2),
\end{aligned}$$

where the functions $H_{2,w,m}(z)$ $(m = 1, 4, 5)$ are given by

$$H_{2,w,1}(z) = \sum_{l=1}^{\infty} \frac{1}{2l}\,K^{\mathrm{hyp}}(z,w,2l), \tag{8.54}$$

$$H_{2,w,4}(z) = \frac{\sqrt{\pi}}{2}\sum_{j=1}^{\infty} \Gamma\Big(\frac{ir_j}{2} - \frac{1}{4}\Big)\,\Gamma\Big(-\frac{ir_j}{2} - \frac{1}{4}\Big)\,\psi_j(z)\,\overline{\psi_j}(w), \tag{8.55}$$

$$H_{2,w,5}(z) = \frac{1}{8\sqrt{\pi}}\sum_{k=1}^{c_\Gamma}\int_{-\infty}^{\infty} \Big|\Gamma\Big(\frac{it}{2} - \frac{1}{4}\Big)\Big|^2 E^{\mathrm{par}}_{\eta_k}\Big(z, \frac{1}{2} + it\Big)\,E^{\mathrm{par}}_{\eta_k}\Big(w, \frac{1}{2} - it\Big)\,dt, \tag{8.56}$$

respectively, and are invariant under the action of Γ. We are left to prove the asserted differential equation.

We write the above Laurent expansion at $s = 0$ as

$$E^{\mathrm{ell}}_w(z,s) - \frac{2^s\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{|\Gamma_w|\,\Gamma(s)} \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(w,1-s)\,E^{\mathrm{par}}_{\eta_k}(z,s) = \sum_{r=0}^{\infty} c_{w,r}(z)\cdot s^r, \tag{8.57}$$

where

$$c_{w,0}(z) = -\frac{2\pi}{|\Gamma_w|\,\mathrm{vol}(\Gamma\backslash\mathbb{H})}, \quad c_{w,1}(z) = -\frac{2\pi}{|\Gamma_w|\,\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \frac{1}{|\Gamma_w|}\left(H_{2,w,1}(z) + H_{2,w,4}(z) + H_{2,w,5}(z)\right).$$

Further, for $z \in \mathbb{H}$ the function $E^{\mathrm{ell}}_w(z, s+2)$ is holomorphic at $s = 0$ and non-vanishing by the definition of the series. Hence, we have a Laurent expansion at $s = 0$ of the form

$$E^{\mathrm{ell}}_w(z,s+2) = \sum_{r=0}^{\infty} d_{w,r}(z)\cdot s^r, \tag{8.58}$$

where $d_{w,0}(z) = E^{\mathrm{ell}}_w(z,2) \neq 0$. Using the differential equations

$$\left(\Delta_{\mathbb{H}} - s(1-s)\right) E^{\mathrm{ell}}_w(z,s) = -s^2\,E^{\mathrm{ell}}_w(z,s+2)$$

and

$$\left(\Delta_{\mathbb{H}} - s(1-s)\right) E^{\mathrm{par}}_{\eta_k}(z,s) = 0 \qquad (k = 1, \ldots, c_\Gamma),$$

we obtain the identity

$$\left(\Delta_{\mathbb{H}} - s(1-s)\right)\left(E^{\mathrm{ell}}_w(z,s) - \frac{2^s\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{|\Gamma_w|\,\Gamma(s)} \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(w,1-s)\,E^{\mathrm{par}}_{\eta_k}(z,s)\right) = -s^2\,E^{\mathrm{ell}}_w(z,s+2).$$

Substituting the Laurent expansions (8.57) and (8.58) into both sides of this equation gives us the identities

$$\sum_{r=0}^{\infty} \Delta_{\mathbb{H}}\,c_{w,r}(z)\cdot s^r - \sum_{r=0}^{\infty} s(1-s)\,c_{w,r}(z)\cdot s^r = -\sum_{r=0}^{\infty} s^2\,d_{w,r}(z)\cdot s^r$$

and

$$\sum_{r=0}^{\infty} \Delta_{\mathbb{H}}\, c_{w,r}(z) \cdot s^r = \sum_{r=1}^{\infty} c_{w,r-1}(z) \cdot s^r - \sum_{r=2}^{\infty} c_{w,r-2}(z) \cdot s^r - \sum_{r=2}^{\infty} d_{w,r-2}(z) \cdot s^r.$$

Comparing coefficients yields the recurrence formula

$$\Delta_{\mathbb{H}}\, c_{w,r}(z) = c_{w,r-1}(z) - c_{w,r-2}(z) - d_{w,r-2}(z),$$

where $c_{w,r}(z) = d_{w,r}(z) = 0$ for $r < 0$. In particular, for $r = 1$ we obtain

$$\Delta_{\mathbb{H}}\, c_{w,1}(z) = c_{w,0}(z),$$

which leads to

$$\begin{aligned} &\Delta_{\mathbb{H}} \left(H_{2,w,1}(z) + H_{2,w,4}(z) + H_{2,w,5}(z)\right) \\ &\quad = |\Gamma_w|\, \Delta_{\mathbb{H}} \left(-\frac{2\pi}{|\Gamma_w| \operatorname{vol}(\Gamma\backslash\mathbb{H})} + \frac{1}{|\Gamma_w|} \left(H_{2,w,1}(z) + H_{2,w,4}(z) + H_{2,w,5}(z)\right) \right) = -\frac{2\pi}{\operatorname{vol}(\Gamma\backslash\mathbb{H})}. \end{aligned}$$

This finishes the proof. □

Remark 8.2.3. Observing that

$$\Delta_{\mathbb{H}}\, K^{\mathrm{hyp}}(z,w,2l) = 2l\,(1-2l)\, K^{\mathrm{hyp}}(z,w,2l) + 2l\,(2l+1)\, K^{\mathrm{hyp}}(z,w,2l+2)$$

for $l \in \mathbb{N}$,

$$\Delta_{\mathbb{H}}\, \psi_j(z) = \lambda_j\, \psi_j(z) = \left(\frac{1}{2} + ir_j\right) \left(\frac{1}{2} - ir_j\right) \psi_j(z)$$

for $j \in \mathbb{N}$, and

$$\Delta_{\mathbb{H}}\, E^{\mathrm{par}}_{\eta_k}\!\left(z, \frac{1}{2} + it\right) = \left(\frac{1}{2} + it\right) \left(\frac{1}{2} - it\right) E^{\mathrm{par}}_{\eta_k}\!\left(z, \frac{1}{2} + it\right)$$

for $k = 1, \ldots, c_\Gamma$ and $t \in \mathbb{R}$, and using the spectral expansion (5.1) with $n = 2$ and $s = 2$, the differential equation

$$\Delta_{\mathbb{H}} \left(H_{2,w,1}(z) + H_{2,w,4}(z) + H_{2,w,5}(z)\right) = -\frac{2\pi}{\operatorname{vol}(\Gamma\backslash\mathbb{H})}.$$

can also be verified by a direct computation.

In the remainder of this section we state the results from [Pip10] and [Pip16]. We omit the proofs here.

Theorem 8.2.4. *For $z \in \mathbb{H}$ with $z \neq \gamma w$ for any $\gamma \in \Gamma$ the elliptic Eisenstein series $E^{\mathrm{ell}}_w(z,s)$ admits a Laurent expansion at $s = 0$ of the form*

$$\begin{aligned} &E^{\mathrm{ell}}_w(z,s) - \frac{2^s \sqrt{\pi}\, \Gamma\left(s - \frac{1}{2}\right)}{|\Gamma_w|\, \Gamma(s)} \sum_{k=1}^{c_\Gamma} E^{\mathrm{par}}_{\eta_k}(w, 1-s)\, E^{\mathrm{par}}_{\eta_k}(z,s) \\ &\quad = -\frac{2\pi}{|\Gamma_w| \operatorname{vol}(\Gamma\backslash\mathbb{H})} - \log\left(|\mathcal{H}_w(z)| \operatorname{Im}(z)^{C_w}\right) \cdot s + \mathrm{O}(s^2), \end{aligned}$$

where $\mathcal{H}_w(z)$ is a holomorphic function, unique up to multiplication with a complex constant of absolute value 1, which vanishes if and only if $z = \gamma w$ for some $\gamma \in \Gamma$, and which satisfies

$$\mathcal{H}_w(\gamma z) = \varepsilon_w(\gamma)\, (cz+d)^{2C_w}\, \mathcal{H}_w(z)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Here $\varepsilon_w(\gamma) \in \mathbb{C}$ is a constant of absolute value 1 which depends only on w and γ but is independent of z, and

$$C_w := \frac{2\pi}{|\Gamma_w| \operatorname{vol}(\Gamma\backslash\mathbb{H})} \in \mathbb{Q}.$$

Proof. See [Pip10], Theorem 6.1.2. See also [Pip16], Theorem 5.2. □

For the specific group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and the elliptic Eisenstein series $E^{\mathrm{ell}}_{e_j}(z,s)$ associated to the elliptic fixed points $e_1 = i$ and $e_2 = \rho := \exp\left(\frac{2\pi i}{3}\right)$ of Γ, von Pippich has proven the following Kronecker limit formulas.

Proposition 8.2.5. *Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. For $z \in \mathbb{H}$ with $z \neq \gamma i$ for any $\gamma \in \Gamma$ the elliptic Eisenstein series $E^{\mathrm{ell}}_i(z,s)$ admits a Laurent expansion at $s = 0$ of the form*

$$\begin{aligned} E^{\mathrm{ell}}_i(z,s) &- \frac{2^{s-1}\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\,E^{\mathrm{par}}_\infty(i,1-s)\,E^{\mathrm{par}}_\infty(z,s) \\ &= -3 + \left(-\log(|E_6(z)|\,\mathrm{Im}(z)^3) + B_i\right)\cdot s + \mathrm{O}(s^2), \end{aligned}$$

where $B_i := -72\,\zeta'(-1) + 3\log(2\pi) - 12\log\left(\Gamma\left(\frac{1}{4}\right)\right)$, and where $E_6(z)$ is the holomorphic Eisenstein series of weight 6 given by (3.21).
Further, for $z \in \mathbb{H}$ with $z \neq \gamma\rho$ for any $\gamma \in \Gamma$ the elliptic Eisenstein series $E^{\mathrm{ell}}_\rho(z,s)$ admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned} E^{\mathrm{ell}}_\rho(z,s) &- \frac{2^{s}\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{3\,\Gamma(s)}\,E^{\mathrm{par}}_\infty(\rho,1-s)\,E^{\mathrm{par}}_\infty(z,s) \\ &= -2 + \left(-\log(|E_4(z)|\,\mathrm{Im}(z)^2) + B_\rho\right)\cdot s + \mathrm{O}(s^2), \end{aligned}$$

where $B_\rho := -48\,\zeta'(-1) + 4\log\left(\frac{2\pi}{\sqrt{3}}\right) - 12\log\left(\Gamma\left(\frac{1}{3}\right)\right)$, and where $E_4(z)$ is the holomorphic Eisenstein series of weight 4 given by (3.21).

Proof. See [Pip10], Proposition 6.2.2. See also [Pip16], Proposition 6.2. □

Corollary 8.2.6. *Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. For $z \in \mathbb{H}$ with $z \neq \gamma i$ for any $\gamma \in \Gamma$ the elliptic Eisenstein series $E^{\mathrm{ell}}_i(z,s)$ admits a Laurent expansion at $s = 0$ of the form*

$$E^{\mathrm{ell}}_i(z,s) = -\log\left(|E_6(z)|\,|\Delta(z)|^{-1/2}\right)\cdot s + \mathrm{O}(s^2).$$

Further, for $z \in \mathbb{H}$ with $z \neq \gamma\rho$ for any $\gamma \in \Gamma$ the elliptic Eisenstein series $E^{\mathrm{ell}}_\rho(z,s)$ admits a Laurent expansion at $s = 0$ of the form

$$E^{\mathrm{ell}}_\rho(z,s) = -\log\left(|E_4(z)|\,|\Delta(z)|^{-1/3}\right)\cdot s + \mathrm{O}(s^2).$$

Here $\Delta(z)$ is the Delta function given by (3.20).

Proof. See [Pip10], Corollary 6.2.3. See also [Pip16], Corollary 6.3. □

8.3. Example 2: The case $n = 3$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$

As in the previous chapter we now consider the specific case $n = 3$ and $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$ as a second example. We first give a Laurent expansion of the elliptic Eisenstein series at $s = 0$ for a general discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{C})$, and then use our knowledge about the parabolic Eisenstein series $E^{\mathrm{par}}_\infty(P,s)$ for $\mathrm{PSL}_2(\mathbb{Z}[i])$ to derive a Kronecker limit formula for the elliptic Eisenstein series for this group.
Throughout the section we let $n = 3$. Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{C})$ be a discrete and cofinite subgroup. Further, let $Q \in \mathbb{H}^3$ be a point with elliptic scaling matrix $\sigma_Q \in \mathrm{PSL}_2(\mathbb{C})$ and stabilizer subgroup Γ_Q.

Notation 8.3.1. To keep the notation simple, in this section we again omit the index 3 and write $E_Q^{\mathrm{ell}}(P,s)$ for the elliptic Eisenstein series $E_{3,Q}^{\mathrm{ell}}(P,s)$ associated to the point $Q \in \mathbb{H}^3$, and $E_{\eta_k}^{\mathrm{par}}(P,s)$ for the parabolic Eisenstein series $E_{3,\eta_k}^{\mathrm{par}}(P,s)$ associated to the cusp $\eta_k \in C_\Gamma$ $(k = 1, \ldots, c_\Gamma)$.

Before we treat the special case $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$, from Proposition 8.1.4 we derive the following Laurent expansion for a general discrete and cofinite subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{C})$.

Proposition 8.3.2. *For $P \in \mathbb{H}^3$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ admits a Laurent expansion at $s = 0$ of the form*

$$\begin{aligned}E_Q^{\mathrm{ell}}(P,s) &- \frac{2^s\,\pi}{|\Gamma_Q|\,(s-1)} \sum_{k=1}^{c_\Gamma} E_{\eta_k}^{\mathrm{par}}(Q, 2-s)\, E_{\eta_k}^{\mathrm{par}}(P,s) \\ &= -\frac{2\pi}{|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)} \cdot \frac{1}{s} + \frac{\pi\,(1-\log(4))}{|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)} + \mathrm{O}(s).\end{aligned}$$

Proof. Proposition 8.1.4 implies that for $P \in \mathbb{H}^3$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$ the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ admits a Laurent expansion at $s = 0$ of the form

$$\begin{aligned}E_Q^{\mathrm{ell}}(P,s) &- \frac{2^s\,\pi}{|\Gamma_Q|\,(s-1)} \sum_{k=1}^{c_\Gamma} E_{\eta_k}^{\mathrm{par}}(Q, 2-s)\, E_{\eta_k}^{\mathrm{par}}(P,s) \\ &= \left(-\frac{2\pi}{|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)} + \frac{F_{3,Q}(P)}{|\Gamma_Q|} + \frac{F_{3,Q,1}(P)}{|\Gamma_Q|}\right) \cdot \frac{1}{s} \\ &\quad - \frac{\pi\,(1+\log(4))}{|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)} + \frac{G_{3,Q}(P)}{|\Gamma_Q|} + \frac{2\pi}{|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)} + \frac{1}{|\Gamma_Q|} \sum_{m=1}^{2} G_{3,Q,1,m}(P) + \mathrm{O}(s),\end{aligned}$$

where we used that $\Gamma(s)^{-1}\,\Gamma(s-1) = (s-1)^{-1}$. Further, we have noted in Remark 8.1.5 (a) that the functions $F_{3,Q}(P)$, $G_{3,Q}(P)$, $F_{3,Q,1}(P)$ and $G_{3,Q,1,m}(P)$ $(m = 1, 2)$ all vanish identically since the sums defining these functions are empty. Thus, we obtain a Laurent expansion at $s = 0$ of the form

$$\begin{aligned}E_Q^{\mathrm{ell}}(P,s) &- \frac{2^s\,\pi}{|\Gamma_Q|\,(s-1)} \sum_{k=1}^{c_\Gamma} E_{\eta_k}^{\mathrm{par}}(Q, 2-s)\, E_{\eta_k}^{\mathrm{par}}(P,s) \\ &= -\frac{2\pi}{|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)} \cdot \frac{1}{s} + \frac{\pi\,(1-\log(4))}{|\Gamma_Q|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)} + \mathrm{O}(s).\end{aligned}$$

□

In the remainder of this section we consider the case $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$. To that aim we quickly recall the setting and the results from section 7.3.

The imaginary quadratic field $K = \mathbb{Q}(i)$ has the ring of integers $\mathcal{O}_K = \mathbb{Z}[i]$, discriminant $d_K = d_{\mathbb{Q}(i)} = -4$ and class number $h_K = h_{\mathbb{Q}(i)} = 1$. Further, the group $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K) = \mathrm{PSL}_2(\mathbb{Z}[i])$ is a discrete and cofinite subgroup of $\mathrm{PSL}_2(\mathbb{C})$ which admits only one cusp $\eta_1 = \infty$ and has the hyperbolic volume

$$\mathrm{vol}(\Gamma\backslash\mathbb{H}^3) = \frac{|d_{\mathbb{Q}(i)}|^{3/2}}{4\pi^2}\,\zeta_{\mathbb{Q}(i)}(2) = \frac{2\,\zeta_{\mathbb{Q}(i)}(2)}{\pi^2}. \tag{8.59}$$

The parabolic Eisenstein series $E_\infty^{\mathrm{par}}(P,s)$ associated to the cusp ∞ of Γ, which is given for $P \in \mathbb{H}^3$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$ by

$$E_\infty^{\mathrm{par}}(P,s) = \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} r(\gamma P)^s,$$

admits a simple pole at $s = 2$ with residue

$$\mathrm{Res}_{s=2}\, E_\infty^{\mathrm{par}}(P,s) = \frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)} = \frac{1}{2\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}.$$

At $s=2$ it has the Laurent expansion

$$E_\infty^{\mathrm{par}}(P,s)=\frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)}\cdot\frac{1}{s-2}-\frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)}\log(\eta_{\mathbb{Q}(i)}(P)\,r(P))+\frac{C_{\mathbb{Q}(i)}}{2}+\mathrm{O}(s-2),$$

where $C_{\mathbb{Q}(i)}\in\mathbb{C}$ is a constant and $\eta_{\mathbb{Q}(i)}:\mathbb{H}^3\to\mathbb{R}$ is a function which satisfies $\eta_{\mathbb{Q}(i)}(\gamma P)=||cP+d||^2\,\eta_{\mathbb{Q}(i)}(P)$ for any $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma$.

Using Proposition 8.3.2 and the results from section 7.3, in particular Lemma 7.3.5, we now find the following Kronecker limit formula for the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$.

Theorem 8.3.3. *Let $\Gamma=\mathrm{PSL}_2(\mathbb{Z}[i])$. For $P\in\mathbb{H}^3$ with $P\neq\eta Q$ for any $\eta\in\Gamma$ the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ admits a Laurent expansion at $s=0$ of the form*

$$\begin{aligned}E_Q^{\mathrm{ell}}(P,s)=&-\frac{3\,\pi^3}{4\,|\Gamma_Q|\,\zeta_{\mathbb{Q}(i)}(2)}\cdot\frac{1}{s}+\frac{\pi^3}{4\,|\Gamma_Q|\,\zeta_{\mathbb{Q}(i)}(2)}\log\left(\eta_{\mathbb{Q}(i)}(P)\,\eta_{\mathbb{Q}(i)}(Q)\,r(P)\,r(Q)\right)-\frac{\pi\,C_{\mathbb{Q}(i)}}{|\Gamma_Q|}\\&-\frac{\pi^3}{4\,|\Gamma_Q|\,\zeta_{\mathbb{Q}(i)}(2)}\left(\log\left(|\Delta(i)|^{1/6}\right)-2\,(1+\gamma)+\log(32)+\frac{\zeta'_{\mathbb{Q}(i)}(2)}{\zeta_{\mathbb{Q}(i)}(2)}\right)+\mathrm{O}(s),\end{aligned}$$

where $\zeta_{\mathbb{Q}(i)}(s)$ is the Dedekind zeta function of $\mathbb{Q}(i)$ and $\Delta(z)$ is the Delta function given by (3.20).

Proof. Proposition 8.3.2 yields that for $P\in\mathbb{H}^3$ with $P\neq\eta Q$ for any $\eta\in\Gamma$ the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ admits a Laurent expansion at $s=0$ of the form

$$\begin{aligned}E_Q^{\mathrm{ell}}(P,s)&-\frac{2^s\,\pi}{|\Gamma_Q|\,(s-1)}\,E_\infty^{\mathrm{par}}(Q,2-s)\,E_\infty^{\mathrm{par}}(P,s)\\&=-\frac{\pi^3}{|\Gamma_Q|\,\zeta_{\mathbb{Q}(i)}(2)}\cdot\frac{1}{s}+\frac{\pi^3\,(1-\log(4))}{2\,|\Gamma_Q|\,\zeta_{\mathbb{Q}(i)}(2)}+\mathrm{O}(s),\end{aligned}$$

where we inserted the hyperbolic volume (8.59). In order to obtain the Laurent expansion of the elliptic Eisenstein series $E_Q^{\mathrm{ell}}(P,s)$ at $s=0$, we first determine the respective expansion of

$$\frac{2^s\,\pi}{|\Gamma_Q|\,(s-1)}\,E_\infty^{\mathrm{par}}(Q,2-s)\,E_\infty^{\mathrm{par}}(P,s).$$

From the Laurent expansions

$$\begin{aligned}2^s&=1+\log(2)\cdot s+\mathrm{O}(s^2),\\\frac{1}{s-1}&=-1-s+\mathrm{O}(s^2),\end{aligned}$$

at $s=0$ we derive the Laurent expansion

$$\frac{2^s\,\pi}{|\Gamma_Q|\,(s-1)}=-\frac{\pi}{|\Gamma_Q|}-\frac{\pi\,(1+\log(2))}{|\Gamma_Q|}\cdot s+\mathrm{O}(s^2).$$

Moreover, by Lemma 7.3.5, at $s=0$ we have the Laurent expansion

$$\begin{aligned}E_\infty^{\mathrm{par}}(Q,2-s)\,E_\infty^{\mathrm{par}}(P,s)=&-\frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)}\cdot\frac{1}{s}-\frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)}\log\left(\eta_{\mathbb{Q}(i)}(P)\,\eta_{\mathbb{Q}(i)}(Q)\,r(P)\,r(Q)\right)+C_{\mathbb{Q}(i)}\\&+\frac{\pi^2}{4\,\zeta_{\mathbb{Q}(i)}(2)}\left(\log\left(|\Delta(i)|^{1/6}\right)+1-2\gamma+\log(4)+\frac{\zeta'_{\mathbb{Q}(i)}(2)}{\zeta_{\mathbb{Q}(i)}(2)}\right)+\mathrm{O}(s).\end{aligned}$$

Therefore, at $s = 0$ we obtain

$$\begin{aligned}
&\frac{2^s\,\pi}{|\Gamma_Q|\,(s-1)}\,E_\infty^{\mathrm{par}}(Q,2-s)\,E_\infty^{\mathrm{par}}(P,s) \\
&\quad= \frac{\pi^3}{4\,|\Gamma_Q|\,\zeta_{\mathbb{Q}(i)}(2)}\cdot\frac{1}{s} + \frac{\pi^3}{4\,|\Gamma_Q|\,\zeta_{\mathbb{Q}(i)}(2)}\,\log\big(\eta_{\mathbb{Q}(i)}(P)\,\eta_{\mathbb{Q}(i)}(Q)\,r(P)\,r(Q)\big) - \frac{\pi\,C_{\mathbb{Q}(i)}}{|\Gamma_Q|} \\
&\quad\quad - \frac{\pi^3}{4\,|\Gamma_Q|\,\zeta_{\mathbb{Q}(i)}(2)}\left(\log\left(|\Delta(i)|^{1/6}\right) - 2\gamma + \log(2) + \frac{\zeta'_{\mathbb{Q}(i)}(2)}{\zeta_{\mathbb{Q}(i)}(2)}\right) + \mathrm{O}(s).
\end{aligned}$$

Adding up, at $s = 0$ we finally get the Laurent expansion

$$\begin{aligned}
E_Q^{\mathrm{ell}}(P,s) &= -\frac{3\,\pi^3}{4\,|\Gamma_Q|\,\zeta_{\mathbb{Q}(i)}(2)}\cdot\frac{1}{s} + \frac{\pi^3}{4\,|\Gamma_Q|\,\zeta_{\mathbb{Q}(i)}(2)}\,\log\big(\eta_{\mathbb{Q}(i)}(P)\,\eta_{\mathbb{Q}(i)}(Q)\,r(P)\,r(Q)\big) - \frac{\pi\,C_{\mathbb{Q}(i)}}{|\Gamma_Q|} \\
&\quad - \frac{\pi^3}{4\,|\Gamma_Q|\,\zeta_{\mathbb{Q}(i)}(2)}\left(\log\left(|\Delta(i)|^{1/6}\right) - 2\,(1+\gamma) + \log(32) + \frac{\zeta'_{\mathbb{Q}(i)}(2)}{\zeta_{\mathbb{Q}(i)}(2)}\right) + \mathrm{O}(s).
\end{aligned}$$

□

Remark 8.3.4. Similar to Theorem 8.3.3, one could also derive a Kronecker limit formula for elliptic Eisenstein series in $\mathbb{H}^3$ for $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K)$, where K is some other imaginary quadratic field with ring of integers $\mathcal{O}_K$ and class number $h_K = 1$.

A. Appendix: Special functions and identities

The main references for this appendix are [GR14] and [AS48].

A.1. The gamma function and related functions

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ the *gamma function* $\Gamma(s)$ is defined as the integral

$$\Gamma(s) := \int_0^\infty \exp(-t)\, t^{s-1}\; dt.$$

It admits a meromorphic continuation in s to the whole complex plane, which is given by

$$\Gamma(s) = \int_0^\infty \exp(-t)\, t^{s-1}\; dt + \sum_{m=0}^\infty \frac{(-1)^m}{m!}\,\frac{1}{s+m}.$$

This implies that $\Gamma(s)$ has a simple pole at the point $s = -m$ for any $m \in \mathbb{N}_0$ with residue

$$\mathrm{Res}_{s=-m}\,\Gamma(s) = \frac{(-1)^m}{m!}. \tag{A.1}$$

The meromorphically continued function $\Gamma(s)$ has no zeros in $\mathbb{C}$, so that $1/\Gamma(s)$ is an entire function.

For any $m \in \mathbb{N}$ the gamma function has the special value

$$\Gamma(m) = (m-1)!,$$

while other important values are

$$\Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi} \quad\text{ and }\quad \Gamma\Big(-\frac{1}{2}\Big) = -2\sqrt{\pi}.$$

A further useful property is the identity

$$\Gamma(\overline{s}) = \overline{\Gamma(s)}. \tag{A.2}$$

The function $\Gamma(s)$ satisfies the well-known recursion formula

$$\Gamma(s+1) = s\,\Gamma(s), \tag{A.3}$$

the functional equation

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

and the duplication formula

$$\Gamma(s)\,\Gamma\Big(s+\frac{1}{2}\Big) = 2^{1-2s}\,\sqrt{\pi}\,\Gamma(2s). \tag{A.4}$$

Stirling's asymptotic formula states that

$$\log(\Gamma(s)) = \frac{1}{2}\,\log(2\pi) - \frac{1}{2}\,\log(s) + s\,\log(s) - s + \mathrm{o}(1)$$

as $s \to \infty$, provided that s remains in a sector of the form $|\arg(s)| < \pi - \varepsilon$ for some $\varepsilon > 0$. In particular, for fixed $\sigma \in \mathbb{R}$ and $|t| \to \infty$ we have the asymptotics

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}\, |t|^{\sigma-1/2} \exp\Big(- \frac{\pi|t|}{2}\Big) \tag{A.5}$$

with an implied constant depending on σ.

The *Euler-Mascheroni constant* γ is defined as the limit

$$\gamma := \lim_{n\to\infty} \left(\sum_{l=1}^{n} \frac{1}{l} - \log(n) \right). \tag{A.6}$$

It has the numerical value $\gamma = 0.57721....$

For $k \in \mathbb{N}_0$ the *polygamma function* $\psi^{(k)}(s)$ *of order* k is defined as the $(k+1)$-th derivative of the logarithm of the gamma function, i.e.

$$\psi^{(k)}(s) := \frac{d^{k+1}}{ds^{k+1}} \log(\Gamma(s)). \tag{A.7}$$

The function $\psi^{(k)}(s)$ is meromorphic on $\mathbb{C}$ and admits a pole of order $k+1$ at the point $s = -m$ for any $m \in \mathbb{N}_0$.
In particular, for $k = 0$ we get the *digamma function* $\psi^{(0)}(s)$, which is given by

$$\psi^{(0)}(s) = \frac{d}{ds} \log(\Gamma(s)) = \frac{\Gamma'(s)}{\Gamma(s)} \tag{A.8}$$

and satisfies the identity

$$\psi^{(k)}(s) = \frac{d^k}{ds^k} \psi^{(0)}(s).$$

Thus, the derivative $\Gamma'(s)$ of the gamma function can be written as

$$\Gamma'(s) = \Gamma(s)\, \psi^{(0)}(s) \qquad (k \in \mathbb{N}_0).$$

For any $m \in \mathbb{N}$ the function $\psi^{(0)}(s)$ has the special value

$$\psi^{(0)}(m) = \sum_{l=1}^{m-1} \frac{1}{l} - \gamma, \tag{A.9}$$

while for $m \in \mathbb{N}_0$ we have

$$\psi^{(0)}\Big(m + \frac{1}{2}\Big) = \sum_{l=1}^{m} \frac{2}{2l-1} - \gamma - 2\log(2). \tag{A.10}$$

The digamma function satisfies the recursion formula

$$\psi^{(0)}(s+1) = \psi^{(0)}(s) + \frac{1}{s}. \tag{A.11}$$

For any $m \in \mathbb{N}_0$, at the pole $s = -m$ the gamma function admits a Laurent expansion of the form

$$\begin{aligned}\Gamma(s) &= \frac{(-1)^m}{m!} \cdot \frac{1}{s+m} + \frac{(-1)^m}{m!} \psi^{(0)}(m+1) + \mathrm{O}(s+m) \\ &= \frac{(-1)^m}{m!} \cdot \frac{1}{s+m} + \frac{(-1)^m}{m!} \left(\sum_{l=1}^{m} \frac{1}{l} - \gamma \right) + \mathrm{O}(s+m). \end{aligned} \tag{A.12}$$

If $\Gamma(s)$ is holomorphic at some point $s_0 \in \mathbb{C}$, then at $s = s_0$ it admits the Laurent expansion (respectively, the Taylor expansion)

$$\Gamma(s) = \Gamma(s_0) + \Gamma(s_0)\,\psi^{(0)}(s_0) \cdot (s - s_0) + \mathrm{O}((s - s_0)^2). \tag{A.13}$$

For $s \in \mathbb{C}$ and $m \in \mathbb{N}_0$ the *Pochhammer symbol* $(s)_m$ is given by

$$(s)_m := \frac{\Gamma(s+m)}{\Gamma(s)}. \tag{A.14}$$

From (A.3) one derives the alternative formula

$$(s)_m = \prod_{j=0}^{m-1} (s+j).$$

At $s = 0$ the Pochhammer symbol $(s)_m$ admits the Laurent expansion

$$\begin{aligned}(s)_m &= \Gamma(m) \cdot s + \Gamma(m)\,(\psi^{(0)}(m) + \gamma) \cdot s^2 + \mathrm{O}(s^3)\\ &= (m-1)! \cdot s + (m-1)!\left(\sum_{l=1}^{m-1} \frac{1}{l}\right) \cdot s^2 + \mathrm{O}(s^3).\end{aligned} \tag{A.15}$$

For $a, b \in \mathbb{C}$ with $\mathrm{Re}(a), \mathrm{Re}(b) > 0$ the *beta function* $\mathrm{B}(a,b)$ is defined as

$$\mathrm{B}(a,b) := \int_0^1 t^{a-1}\,(1-t)^{b-1}\;dt. \tag{A.16}$$

It is related to the gamma function via the identity

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} = \mathrm{B}(b,a).$$

For $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$ we have (see, e.g., [GR14], formula 3.249, 5.)

$$\int_0^1 (1-t^2)^{\alpha-1}\;dt = \frac{1}{2}\,\mathrm{B}\Big(\frac{1}{2}, \alpha\Big) = \frac{\sqrt{\pi}\,\Gamma(\alpha)}{2\,\Gamma\left(\alpha + \frac{1}{2}\right)}. \tag{A.17}$$

Further, for $\mu, \nu \in \mathbb{C}$ with $\mathrm{Re}(\mu) > -1$ and $\mathrm{Re}(\nu - \mu) > 0$ the integral formula

$$\int_0^\infty \frac{\sinh(u)^\mu}{\cosh(u)^\nu}\;du = \frac{1}{2}\,\mathrm{B}\Big(\frac{\mu+1}{2}, \frac{\nu-\mu}{2}\Big)$$

holds true (see, e.g., [GR14], formula 3.512, 2.). Choosing $\mu = 0$, for $\mathrm{Re}(\nu) > 0$ we obtain

$$\int_0^\infty \cosh(u)^{-\nu}\;du = \frac{1}{2}\,\mathrm{B}\Big(\frac{1}{2}, \frac{\nu}{2}\Big) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)}{2\,\Gamma\left(\frac{\nu+1}{2}\right)}.$$

Applying the duplication formula (A.4) for the Gamma function now yields the identity

$$\int_0^\infty \cosh(u)^{-\nu}\;du = \frac{2^{\nu-2}\,\Gamma\left(\frac{\nu}{2}\right)^2}{\Gamma(\nu)}, \tag{A.18}$$

where $\mathrm{Re}(\nu) > 0$.

A.2. The Gauss hypergeometric function

The differential equation

$$z(1-z)\,f'' + (c-(a+b+1)z)\,f' - ab\,f = 0,$$

where $a, b, c \in \mathbb{C}$, $c \neq -m$ for any $m \in \mathbb{N}_0$ and $z \in \mathbb{C}$, has as solution the *Gauss hypergeometric function* ${}_2F_1(a,b;c;z)$. For $z \in \mathbb{C}$ with $|z| < 1$ it is given by the series

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{k!\,(c)_k}\, z^k. \tag{A.19}$$

For $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ the Gauss hypergeometric function has the integral representation (see, e.g., [GR14], formula 9.111)

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-tz)^{-a}\,dt.$$

This integral is an analytic function for z lying in the complex plane cut along the real axis from 1 to ∞, giving the analytic continuation of the series (A.19).

The function ${}_2F_1(a,b;c;z)$ satisfies a variety of transformation formulas, examples of these being the linear transformation rules

$$\begin{aligned} {}_2F_1(a,b;c;z) &= (1-z)^{-a}\,{}_2F_1\Big(a, c-b; c; \frac{z}{z-1}\Big) = (1-z)^{-b}\,{}_2F_1\Big(b, c-a; c; \frac{z}{z-1}\Big) \\ &= (1-z)^{c-a-b}\,{}_2F_1(c-a, c-b; c; z) \end{aligned} \tag{A.20}$$

(see, e.g., [GR14], formula 9.131, 1.), and

$$\begin{aligned} {}_2F_1(a,b;c;z) = {} & \frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)}\,(1-z)^{-a}\,{}_2F_1\Big(a, c-b; a-b+1; \frac{1}{1-z}\Big) \\ & + \frac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)}\,(1-z)^{-b}\,{}_2F_1\Big(b, c-a; b-a+1; \frac{1}{1-z}\Big), \end{aligned} \tag{A.21}$$

where $|\arg(1-z)| < \pi$ (see, e.g., [AS48], formula 15.3.8).

Moreover, for $z \in \mathbb{C}$ with $|z| < 1$ the identity

$$ {}_2F_1(a,b;b;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k = (1-z)^{-a} \tag{A.22}$$

holds true (see, e.g., [AS48], formula 15.1.8).

A.3. Associated Legendre functions

The differential equation

$$(1-z^2)\,f'' - 2z\,f' + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right) f = 0, \tag{A.23}$$

where the *degree* $\nu \in \mathbb{C}$ and the *order* $\mu \in \mathbb{C}$ are arbitrary complex numbers, is called *associated Legendre equation*. A fundamental system of solutions for this homogeneous linear differential equation of second order is given by the associated Legendre functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$.

For $z \in \mathbb{C}$ with $|\arg(z \pm 1)| < \pi$ the *associated Legendre function of the first kind* $P_\nu^\mu(z)$ *of degree* ν *and order* μ is given by

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1\Big(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\Big), \tag{A.24}$$

while for $z \in \mathbb{R}$ with $-1 < z < 1$ we have the representation

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{1+z}{1-z}\right)^{\mu/2} {}_2F_1\Big(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\Big).$$

Furthermore, for $z \in \mathbb{C}$ with $|\arg(z \pm 1)| < \pi$ the *associated Legendre function of the second kind* $Q_\nu^\mu(z)$ *of degree* ν *and order* μ is given by

$$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}\sqrt{\pi}\,\Gamma(\nu+\mu+1)}{2^{\nu+1}\,\Gamma\left(\nu+\frac{3}{2}\right)} (z^2-1)^{\mu/2}\, z^{-\nu-\mu-1}\, {}_2F_1\Big(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\Big), \tag{A.25}$$

and for $z \in \mathbb{R}$ with $-1 < z < 1$ we have the representation

$$Q_\nu^\mu(z) = \frac{\pi}{2\sin(\mu\pi)} \left(P_\nu^\mu(z)\cos(\mu\pi) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(z)\right).$$

In the case $\mu = 0$, equation (A.23) is called *Legendre equation* and its linearly independent solutions

$$P_\nu(z) := P_\nu^0(z) = {}_2F_1\Big(-\nu, \nu+1; 1; \frac{1-z}{2}\Big)$$

and $Q_\nu(z) := Q_\nu^0(z)$ are called *Legendre functions of the first and second kind*, respectively. For $\mu = 0$ and $\nu \in \mathbb{N}_0$, equation (A.23) reduces to the differential equation for Legendre polynomials.

For $\mathrm{Re}(\mu) > -\frac{1}{2}$ and $|\arg(z \pm 1)| < \pi$ the associated Legendre function of the first kind has the integral representation

$$P_\nu^{-\mu}(z) = \frac{(z^2-1)^{\mu/2}}{2^\mu\sqrt{\pi}\,\Gamma\left(\mu+\frac{1}{2}\right)} \int_{-1}^{1} \frac{(1-t^2)^{\mu-1/2}}{\left(z+t\sqrt{z^2-1}\right)^{\mu-\nu}}\, dt \tag{A.26}$$

(see, e.g., [GR14], formula 8.711, 1.).

Moreover, for $\mathrm{Re}(\mu) < 1$, $\mathrm{Re}(s+\mu+\nu) > 0$ and $\mathrm{Re}(s+\mu-\nu) > 1$ we have the identity (see, e.g., formula [GR14], 7.132, 7.)

$$\int_1^\infty t^{-s}\,(t^2-1)^{-\mu/2}\, P_\nu^\mu(t)\, dt = \frac{2^{s+\mu-2}}{\sqrt{\pi}\,\Gamma(s)}\, \Gamma\Big(\frac{s+\mu+\nu}{2}\Big)\, \Gamma\Big(\frac{s+\mu-\nu-1}{2}\Big). \tag{A.27}$$

A.4. Further functions and identities

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ the *Riemann zeta function* $\zeta(s)$ is defined as the Dirichlet series

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}. \tag{A.28}$$

It admits a meromorphic continuation in s to the whole complex plane, which is holomorphic on $\mathbb{C} \setminus \{1\}$ and has a simple pole at $s = 1$ with residue 1.

The function $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = 2^s \, \pi^{s-1} \, \sin\left(\frac{\pi\, s}{2}\right) \Gamma(1-s)\, \zeta(1-s),$$

which can also be written in the symmetric form

$$\Gamma\Big(\frac{s}{2}\Big)\, \pi^{-s/2}\, \zeta(s) = \Gamma\Big(\frac{1-s}{2}\Big)\, \pi^{(s-1)/2}\, \zeta(1-s).$$

The Riemann zeta function $\zeta(s)$ has no zero with $\mathrm{Re}(s) > 1$, and a zero at the point $s = -2m$ for any $m \in \mathbb{N}$. The remaining zeros of $\zeta(s)$ are all located in the strip $\{s \in \mathbb{C} \mid 0 < \mathrm{Re}(s) < 1\}$.

Well-known values of the Riemann zeta function include

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(0) = -\frac{1}{2} \quad \text{and} \quad \zeta(-1) = -\frac{1}{12}.$$

The differential equation

$$z^2\, f'' + z\, f' - (z^2 - \nu^2)\, f = 0$$

has as solutions the modified Bessel functions. The *modified Bessel function of the first kind* $I_\nu(z)$ is given by

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k},$$

while the *modified Bessel function of the second kind* $K_\nu(z)$ is given by

$$K_\nu(z) = \frac{\pi\,(I_{-\nu}(z) - I_\nu(z))}{2\sin(\pi\nu)}. \tag{A.29}$$

Finally, we state the first law of cosines in hyperbolic geometry, which we quote from Theorem 3.5.3. in [Rat94]: If α, β, γ are the angles of a hyperbolic triangle and a, b, c are the lengths of the opposite sides, then we have

$$\cos(\gamma) = \frac{\cosh(a)\, \cosh(b) - \cosh(c)}{\sinh(a)\, \sinh(b)}. \tag{A.30}$$

In case that γ is a right angle, we have $\cos(\gamma) = 0$, so (A.30) implies that the side lengths satisfy the identity

$$\cosh(c) = \cosh(a)\, \cosh(b). \tag{A.31}$$

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