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M.Sc. Felix Christopher Helmut Ludwig Brandt

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Referent:	Prof. Dr. Matthias Hieber
Korreferenten:	Prof. Dr. Moritz Egert
	Prof. Dr. Hideo Kozono

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Für meine Eltern

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Preface

The field of geophysical fluid dynamics is a very active area of research. One of the reasons for this is its importance for weather prediction and climate science. We mention here for example the primitive equations of the atmosphere and the ocean. The latter were introduced by Richardson in 1922, but at first, they were considered too complicated, so research focused on simpler submodels instead. The mathematical analysis of the primitive equations then only started in the series of articles of Lions, Temam and Wang [93–96].

The equations of geophysical fluid dynamics result from the conservation equations of physics such as conservation of mass, momentum, energy, or also salt in the case of the ocean or humidity for the atmosphere. In that respect, we also mention the presence of characteristic length and time scales in geophysical fluid dynamics. Typically, the horizontal scales span several thousand kilometers, while the vertical scales are much smaller.

Another characteristic feature of the equations of geophysical fluid dynamics is the hierarchy of models with regard to physical relevance and the level of complexity of the described physical phenomena as also explained in the book chapter of Temam and Ziane [128]. Let us mention that the primitive equations are physically simpler than the three-dimensional Navier-Stokes equations, owing to the so-called hydrostatic balance. The latter means that the conservation of momentum in the vertical direction is replaced by the hydrostatic equation as expressed in (2.34). However, the primitive equations were considered more mathematically demanding than the three-dimensional Navier-Stokes equations in view of the shape of the nonlinearity. The nonlinear term in the Navier-Stokes equations is of the form “velocity times first-order derivatives of the velocity", whereas the hydrostatic approximation causes that the nonlinear term in the primitive equations takes the shape "first-order derivatives of the horizontal velocity times first order derivatives of the horizontal velocity". In this context, the groundbreaking global strong well-posedness result of the three-dimensional viscous incompressible primitive equations for arbitrary large initial data in H^1 due to Cao and Titi [24] seems even more surprising as the corresponding problem for the three-dimensional Navier-Stokes equations remains open until today. This example illustrates that geophysical flow models call for some proper investigation in order to analyze and understand their behavior. Moreover, the characteristic scales may result in surprising phenomena that one might not expect from classical mathematical fluid dynamics.

The present thesis consists of two parts: the analysis of several geophysical flow models by means of quasilinear evolution equations and approaches to time periodic quasilinear problems. All geophysical flow models under consideration here are related to sea ice. The first and second part of the thesis are linked by the application of one of the time periodic frameworks to a problem related to sea ice.

In the sequel, we describe the content of the two parts of the thesis. The introduction of the problems is fairly brief, and the results presented in the following are simplified versions. Hence, we refer to the respective chapters for more thorough introductions of the underlying models and complete versions of the results.

Geophysical Flow Models

Recent years have witnessed an increase in research of mathematical climate models in general and sea ice models in particular. The reason for this is probably the role of sea ice in climate science. In fact, the climate system is constituted by the atmosphere, the oceans, the biosphere and the cryosphere. The latter includes of all types of frozen water, so sea ice makes part of it, see also the work of Kreyscher et al. [82]. Sea ice thus has multiple effects on the climate system. In the following, we mention some of these aspects described in more detail in the PhD thesis of Harder [59]. More information on the importance of sea ice models in climate study can also be found in the survey paper of Hunke, Lipscomb and Turner [72].

First, the ocean absorbs roughly 90% of the sunlight, whereas sea ice reflects 80%. This is significant for the heating of the ocean surface and thereby also for the evaporation of water, its transport in the atmosphere and the formation of clouds and rain. If the ice is additionally covered by snow, then the aforementioned effect is even stronger. Another important related aspect is the so-called ice-albedo effect, meaning that a decrease of the ice layer results in a decline in reflection, which in turn reduces the areal extent of ice. Sea ice represents a thin layer between the atmosphere and the ocean and thus acts as an insulator, since it reduces the heat exchange between the latter two. Furthermore, due to the much lower portion of salt in sea ice compared to ocean water, sea ice plays the role of fresh water in a climate system. Indeed, when oceanic water freezes, the added salt in the water underneath the sea ice results in an increase in density. In other words, the layers in the ocean destabilize, and one can observe convection. Conversely, melting sea ice has a stabilizing effect and reduces convection.

Among the sea ice models under consideration, the *viscous-plastic dynamic-thermodynamic model* introduced by William D. Hibler III in the seminal article [60] in 1979 has become one of the most frequently used ones in simulation and numerical analysis. It captures many characteristic features of sea ice as a material. Sea ice results from freezing sea water and consists of pure ice, liquid brine, air pockets and solid salt. Moreover, Hibler's model is a large-scale 2D model. This means that the dynamics of sea ice is described on a large scale of tens or hundreds of kilometers instead of tracking the behavior of individual ice floes, and the vertical motion of sea ice is neglected. Therefore, sea ice can be regarded as a highly fractured continuum. The model is constituted by a momentum equation for the horizontal sea ice velocity, where the forces are the internal ice stress due to friction as well as external forces such as the Coriolis force or atmospheric wind and ocean stresses. The momentum equation is coupled to balance laws for the mean ice thickness and the ice compactness via the ice strength appearing in the stress tensor. An essential aspect of the model is that sea ice is viewed as a viscous-plastic material. However, this leads to further mathematical difficulties, since the shear and bulk viscosities in the associated stress tensor degenerate.

With regard to the importance of sea ice in climate science as expressed above, it is no surprise that there is a vast literature on Hibler's sea ice model as well as related models in numerics and computation, see also the short bibliographic overview in Section 3.1. On the other hand, the rigorous mathematical analysis of Hibler's model has remained a terra incognita until quite recently. In fact, the mathematical analysis of regularized versions of this model started in the joint work with Disser, Haller-Dintelmann and Hieber [18] as well as independently by Liu, Thomas and Titi [97].

It is the purpose of the first part of this thesis to present the state-of-the-art concerning *strong* solutions to some geophysical flow models related to Hibler's model. More precisely, we first consider a fully parabolic regularized version of Hibler's model. In a second step, we investigate the interaction problem of sea ice with a rigid body. Another focal point is the analysis of a coupled atmosphere-sea ice-ocean model, which emphasizes that sea ice represents a thin layer between the atmosphere and the ocean. Finally, we study a more physical parabolic-hyperbolic variant of Hibler's model.

Analysis of the Fully Parabolic Regularized Hibler Model

Sea ice is considered on a bounded domain $\Omega \subset \mathbb{R}^2$ with sufficiently smooth boundary. The model variables are the horizontal sea ice velocity $v_{\text{ice}} \in \mathbb{R}^2$, the mean ice thickness $h \in [\kappa, \infty)$, where $\kappa > 0$ is small parameter, and the ice compactness $a \in (0,1)$. For $m_{\text{ice}} = \rho_{\text{ice}} h$ denoting the ice mass for a constant density $\rho_{\text{ice}} > 0$, the sea ice dynamics are described by the momentum equation

$$m_{\text{ice}}(\partial_t v_{\text{ice}} + (v_{\text{ice}} \cdot \nabla_{\text{H}}) v_{\text{ice}}) = F.$$

The right-hand side F encompasses the internal ice stress $\operatorname{div}_{\text{H}} \sigma$, with σ representing the stress tensor as made precise in (3.3), and the aforementioned external forcing terms. For the time being, these terms are summarized in F^{ice}. Since the stress tensor σ degenerates, we will investigate a regularized version σ_δ as introduced in (3.5) throughout this thesis. Hence, we study the modified sea ice momentum equation

$$\text{(ME)} \qquad m_{\text{ice}}(\partial_t v_{\text{ice}} + (v_{\text{ice}} \cdot \nabla_{\text{H}}) v_{\text{ice}}) = \operatorname{div}_{\text{H}} \sigma_\delta + F^{\text{ice}}.$$

The sea ice model is completed by balance laws for the mean ice thickness and the ice compactness given by

$$\text{(BLP)} \qquad \begin{cases} \partial_t h + \operatorname{div}_{\text{H}} (v_{\text{ice}} h) = S_{\text{h}} + d_{\text{h}} \Delta_{\text{H}} h, \\ \partial_t a + \operatorname{div}_{\text{H}} (v_{\text{ice}} a) = S_{\text{a}} + d_{\text{a}} \Delta_{\text{H}} a, \end{cases}$$

where S_{h} and S_{a} represent thermodynamic source terms, see (3.8). In most chapters, we take into account the balance laws of the above form. Hibler [60] already introduced diffusive terms in the balance laws for numerical stability. We further assume homogeneous Dirichlet boundary conditions for v_{ice} and take into account homogeneous Neumann boundary conditions for h and a. Moreover, initial conditions $(v_{\mathrm{ice},0}, h_0, a_0)$ are considered.

Chapter 3 is centered around the analysis of the system given by (ME) and (BLP), which is also referred to as the *fully parabolic regularized model*. The starting point is the analysis of the quasilinear operator corresponding to the internal ice stress $\mathrm{div}_{\mathrm{H}}\, \sigma_\delta$. More precisely, we establish a representation of the operator in non-divergence form and verify certain ellipticity properties. By means of the theory of parabolic boundary value problems as developed in the seminal work of Denk, Hieber and Prüss [37], this leads to the *maximal* L^p*-regularity* of the linearized version of this operator. In the next step, we rewrite (ME)–(BLP) as a quasilinear abstract Cauchy problem. A more thorough analysis of the associated linearized operator matrix, which is based on the above maximal L^p-regularity, and Lipschitz estimates of the nonlinear terms then pave the way for the first main result on the local strong well-posedness thanks to quasilinear existence theory as stated in the monograph of Prüss and Simonett [115, Chapter 5]. We provide a simplified version of this first main result in the case $p = q$.

Theorem. *Let $p > 4$, and consider $(v_{\mathrm{ice},0}, h_0, a_0) \in V \subset \mathrm{W}^{2-2/p,p}(\Omega)^4$ subject to compatibility conditions, where $u = (v_{\mathrm{ice}}, h, a) \in V$ guarantees $h > \kappa$ and $a \in (0,1)$. Then there exists $T > 0$ such that* (ME)–(BLP) *has a unique strong solution $u = (v_{\mathrm{ice}}, h, a)$ on $(0,T)$, i. e.,*

$$u \in \mathrm{W}^{1,p}(0,T;\mathrm{L}^p(\Omega)^4) \cap \mathrm{L}^p(0,T;\mathrm{W}^{2,p}(\Omega)^4) \cap \mathrm{C}([0,T];V).$$

When neglecting (most of) the external forcing terms, i. e., setting $F^{\mathrm{ice}} = 0$ and $S_{\mathrm{h}} = S_{\mathrm{a}} = 0$, we find that $(0, h_*, a_*)$, with $h_* > \kappa$ and $a_* \in (0,1)$ constant in time and space, is an equilibrium solution to

$$\text{(SIS)} \qquad \begin{cases} m_{\mathrm{ice}}(\partial_t v_{\mathrm{ice}} + (v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice}}) = \mathrm{div}_{\mathrm{H}}\, \sigma_\delta, \\ \partial_t h + \mathrm{div}_{\mathrm{H}}\, (v_{\mathrm{ice}} h) = d_{\mathrm{h}} \Delta_{\mathrm{H}} h, \\ \partial_t a + \mathrm{div}_{\mathrm{H}}\, (v_{\mathrm{ice}} a) = d_{\mathrm{a}} \Delta_{\mathrm{H}} a. \end{cases}$$

A refined analysis of the simplified system exhibits the global strong well-posedness for initial data close to such equilibria thanks to the *generalized principle of linearized stability* due to Prüss, Simonett and Zacher [117]. In a simplified form, this result can be rephrased as follows.

Theorem. *Let $p > 4$, and consider $(v_{\mathrm{ice},0}, h_0, a_0) \in V \subset \mathrm{W}^{2-2/p,p}(\Omega)^4$ close to a constant equilibrium $u_* = (0, h_*, a_*) \in V$ of* (SIS) *and subject to compatibility conditions. Then* (SIS) *admits a unique solution $u = (v_{\mathrm{ice}}, h, a)$ in $\mathbb{R}_+$ which converges to an equilibrium solution at an exponential rate.*

These results, which are contained in Chapter 3, have been obtained in a joint article with Karoline Disser, Robert Haller-Dintelmann and Matthias Hieber [18]. Moreover, parts of the linear theory of the operator emerging from $\operatorname{div}_{\mathrm{H}} \sigma_\delta$ are included in the master thesis [17]. The chapter is of essential importance in this thesis, because the following chapters in the context of the sea ice equations build upon the linear theory and the estimates of the nonlinear terms established here.

Interaction of Sea Ice with a Rigid Body

The investigation of interaction problems of fluids with rigid bodies is a classical topic in mathematical fluid mechanics. We refer here only to the survey article of Galdi [45]. The focal point of Chapter 4 is the analysis of the interaction of sea ice, described by the fully parabolic regularized model, with rigid structures. The physical motivation for this is the motion of large structures such as ships in ice floe fields as described in the article of Zhan et al. [134] or the survey article [131] by Tuhkuri and Polojärvi on ice-structure interaction.

From a mathematical point of view, the above system (ME)–(BLP) is complemented by equations for the translational velocity $\xi \in \mathbb{R}^2$ and the rotational velocity denoted by $\omega \in \mathbb{R}$ for convenience. The domain of the interaction problem is time-dependent, since it varies with the motion of the rigid body, and it will be denoted by $\mathcal{F}(t)$. The equations satisfied by ξ and ω follow from Newton's laws. For the rigid body mass $m_{\mathcal{S}}$, an inertia tensor J_0, the position of the center of mass of the rigid body $x_{\mathrm{c}}(t)$, the ice-structure interface $\partial\mathcal{S}(t)$ as well as the unit outward normal vector $\nu(t)$, these equations read as

$$\text{(RBE)} \qquad \begin{cases} m_{\mathcal{S}}\xi'(t) = -\displaystyle\int_{\partial\mathcal{S}(t)} \sigma_\delta(u)\nu(t)\,\mathrm{d}\Gamma, \\ J_0\omega'(t) = -\displaystyle\int_{\partial\mathcal{S}(t)} (x_{\mathrm{H}} - x_{\mathrm{c}}(t))^\perp \sigma_\delta(u)\nu(t)\,\mathrm{d}\Gamma. \end{cases}$$

The quasilinear nature of the stress tensor σ_δ in the surface integrals is challenging. Another mathematical difficulty arises from the interface condition

$$\text{(IC)} \qquad v_{\mathrm{ice}} = \xi + \omega(x_{\mathrm{H}} - x_{\mathrm{c}})^\perp$$

on $\partial\mathcal{S}(t)$, guaranteeing the equality of the velocities of sea ice and the rigid body on their interface.

In order to overcome the time-dependence of the domain, we employ a local change of coordinates capturing the motion of the rigid body. This transform has first been employed by Inoue and Wakimoto [73]. It leads to a more complicated shape of the equations in exchange for the time-independence of the domain. In a next step, we investigate the linearized interaction problem corresponding to the transformed versions of (ME), (BLP), (RBE) and (IC) by a “cascade approach”. This means that we first solve the equations of the rigid body and the parabolic equations associated to the balance laws and then interpret the sea ice momentum equation as an inhomogeneous boundary value problem in the framework of Denk, Hieber and Prüss [38]. This allows us to reformulate the task of finding a strong solution to the transformed interaction problem as a fixed point problem. The estimates of the nonlinear terms are then established thanks to a thorough analysis of the transform. Finally, we use the inverse coordinate transform to get back the solution to the original problem on the moving domain. Below, we state a simplified version of the local strong well-posedness result.

Theorem. *Let $p > 4$, and consider $(v_{\mathrm{ice},0}, h_0, a_0, \xi_0, \omega_0) \subset \mathrm{W}^{2-2/p,p}(\mathcal{F}_0)^4 \times \mathbb{R}^3$ such that $h_0 > \kappa$ and $a_0 \in (0,1)$, and subject to compatibility conditions. If the rigid body starts with a strictly positive distance from the outer sea ice boundary, then there is $T > 0$ such that the interaction problem* (ME), (BLP), (RBE) *and* (IC) *has a unique strong solution $(v_{\mathrm{ice}}, h, a, \xi, \omega)$ on $(0,T)$, so*

$$(v_{\mathrm{ice}}, h, a) \in \mathrm{W}^{1,p}(0,T;\mathrm{L}^p(\mathcal{F}(\cdot))^4) \cap \mathrm{L}^p(0,T;\mathrm{W}^{2,p}(\mathcal{F}(\cdot))^4),$$
$$\xi \in \mathrm{W}^{1,p}(0,T)^2 \ \text{ and } \ \omega \in \mathrm{W}^{1,p}(0,T).$$

The main result in Chapter 4 of the above shape has also been obtained in a joint article with Tim Binz and Matthias Hieber [11]. However, the strategy in [11] is quite different from the one employed in this thesis. The article relies on a non-autonomous version of a quasilinear existence result, and the estimates of the nonlinear terms are deduced directly from those of the original terms together with regularity properties of the transform. In contrast, in Chapter 4, we show the estimates of the transformed terms “by hand”. Also, the linear theory in [11] is based on a “monolithic” approach and a decoupling argument.

A Coupled Atmosphere-Sea Ice-Ocean Model

Lions, Temam and Wang [95, 96] introduced a coupled atmosphere-ocean model and provided numerical as well as mathematical analysis for it. The rigorous mathematical analysis of coupled models in the context of atmosphere and ocean dynamics has been an active area of research ever since. In

Chapter 5 of this thesis, we take into account a coupled atmosphere-sea ice-ocean model, where the atmosphere and ocean dynamics are modeled by the viscous incompressible primitive equations, while the fully parabolic regularized version of Hibler's model is used for the sea ice part. A slight difference with Chapter 3 is that the sea ice equations are considered on the square $G = (0,1) \times (0,1)$ and subject to periodic boundary conditions.

The primitive equations are given on cylindrical domains based on the square G, and for the respective full and horizontal velocities u_i and v_i as well as the pressure terms π_i, with $i \in \{\mathrm{atm}, \mathrm{ocn}\}$, they take the shape

$$\text{(PE)} \qquad \begin{cases} \partial_t v_{\mathrm{atm}} - \Delta v_{\mathrm{atm}} + (u_{\mathrm{atm}} \cdot \nabla) v_{\mathrm{atm}} + \nabla_{\mathrm{H}} \pi_{\mathrm{atm}} = 0, \\ \partial_z \pi_{\mathrm{atm}} = 0, \\ \operatorname{div} u_{\mathrm{atm}} = 0, \\ \partial_t v_{\mathrm{ocn}} - \Delta v_{\mathrm{ocn}} + (u_{\mathrm{ocn}} \cdot \nabla) v_{\mathrm{ocn}} + \nabla_{\mathrm{H}} \pi_{\mathrm{ocn}} = 0, \\ \partial_z \pi_{\mathrm{ocn}} = 0, \\ \operatorname{div} u_{\mathrm{ocn}} = 0. \end{cases}$$

We assume the sea ice to occupy a layer in between the atmosphere and the ocean, so $h \in (\kappa_1, \kappa_2)$ for $0 < \kappa_1 < \kappa_2$. As a consequence, the atmosphere exerts a force on the sea ice via atmospheric wind. On the other hand, we suppose the ocean stress on the sea ice to be proportional to the shear rate as for a plane Couette flow. In addition, we assume that the velocity of the ice and the ocean coincide on their interface. Denoting by τ_{atm} and τ_{ocn} the respective forcing terms appearing in F^{ice} from (ME), up to densities, drag coefficients and rotation matrices, the coupling conditions are

$$\text{(CC)} \qquad \tau_{\mathrm{atm}} = |v_{\mathrm{atm}}| v_{\mathrm{atm}}, \;\; \tau_{\mathrm{ocn}} = -\partial_z v_{\mathrm{ocn}} \text{ and } v_{\mathrm{ocn}} = v_{\mathrm{ice}}, \text{ on } G.$$

The last coupling condition is the mathematically most challenging one. In fact, we reformulate the coupled system (SIS), (PE) and (CC) as a quasilinear abstract Cauchy problem, and the domain of the resulting linearized operator matrix is non-diagonal. In order to bypass this, we investigate the stationary hydrostatic Stokes problem with inhomogeneous boundary conditions. We thereby construct a lifting operator for the interface condition of the ice and the ocean. This allows us to perform a similarity transform of the operator matrix with non-diagonal domain to an operator matrix with diagonal domain, but of a more complicated shape. We verify the bounded $\mathcal{H}^\infty$-calculus of the operator matrix with diagonal domain thereafter and exploit that it is preserved under similarity transforms. In particular, we obtain the maximal L^p-regularity of the original linearized operator matrix with non-diagonal domain. Together with estimates of the nonlinear terms, this results in the local strong well-posedness of the complete coupled system in view of quasilinear existence theory. The result in this context can be paraphrased in the following simplified way.

Theorem. *Let $p > 4$, and consider $(v_{\mathrm{atm},0}, v_{\mathrm{ocn},0}, v_{\mathrm{ice},0}, h_0, a_0) \subset (\mathrm{W}^{2-2/p,p})^8$ subject to compatibility conditions and such that $h_0 \in (\kappa_1, \kappa_2)$ and $a_0 \in (0,1)$. Then there is $T > 0$ such that the coupled model* (ME)*,* (BLP)*,* (PE)*,* (CC) *admits a unique strong solution $v = (v_{\mathrm{atm}}, v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a)$ on $(0,T)$, so*

$$v \in \mathrm{W}^{1,p}(0,T;(\mathrm{L}^p)^8) \cap \mathrm{L}^p(0,T;(\mathrm{W}^{2,p})^8) \cap \mathrm{C}([0,T];V),$$

where $V \subset \mathrm{W}^{2-2/p,p}(\Omega)^8$ is again such that $h \in (\kappa_1, \kappa_2)$ and $a \in (0,1)$.

In the special situation corresponding to $\tau_{\mathrm{atm}} = 0$ and $S_{\mathrm{h}} = S_{\mathrm{a}} = 0$, we observe that $(0,0,0,h_*,a_*)$, for $h_* \in (\kappa_1,\kappa_2)$ and $a_* \in (0,1)$ constant in time and space, is an equilibrium solution to (ME), (BLP), (PE) and (CC). A similar strategy as in the context of the fully parabolic regularized sea ice model, based on the generalized principle of linearized stability, also leads to the global strong well-posedness of this simplified model for initial data in the vicinity of constant equilibria. The result can be sketched as follows.

Theorem. *Let $p > 4$, and let $(v_{\mathrm{atm},0}, v_{\mathrm{ocn},0}, v_{\mathrm{ice},0}, h_0, a_0) \subset (\mathrm{W}^{2-2/p,p})^8$ subject to compatibility conditions and close to constant equilibria $(0,0,0,h_*,a_*)$, with $h_* \in (\kappa_1,\kappa_2)$ and $a_* \in (0,1)$. Then there exists a unique strong solution $v = (v_{\mathrm{atm}}, v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a)$ in $\mathbb{R}_+$ to* (ME)*,* (BLP)*,* (PE) *and* (CC) *in the situation $\tau_{\mathrm{atm}} = 0$ and $S_{\mathrm{h}} = S_{\mathrm{a}} = 0$. Moreover, v converges to an equilibrium of the simplified coupled system at an exponential rate.*

Chapter 5 is based on a joint work with Tim Binz and Matthias Hieber [12].

The Parabolic-Hyperbolic Regularized Hibler Model

As we have already partially indicated in the above, the introduction of viscous terms in the balance laws (BLP) was motivated by the resulting increase in numerical stability. However, it is more physical to consider the situation without these regularizing terms. This is precisely addressed in Chapter 6 of this thesis, where instead of (BLP), we take into account

$$\text{(BLH)} \qquad \begin{cases} \partial_t h + \mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} h) = S_{\mathrm{h}}, \\ \partial_t a + \mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} a) = S_{\mathrm{a}}. \end{cases}$$

The work of Liu, Thomas and Titi [97] tackles a problem which is similar to (ME)–(BLH). Their stress tensor is not only regularized, but it is also modified significantly. In contrast, the regularization of σ used in this thesis agrees with common regularizations employed in the numerical analysis of Hibler's model.

System (ME)–(BLH) is of parabolic-hyperbolic nature which results in additional mathematical difficulties. We circumvent the hyperbolic effects in the balance laws by introducing Lagrangian coordinates. Another ingredient is the choice of an anisotropic ground space. More precisely, the underlying space for the sea ice momentum equation is L^p, while we study the balance laws in $\mathrm{W}^{1,p}$. The maximal L^p-regularity of the linearized system allows us to reformulate the existence of a unique local-in-time strong solution as a fixed point problem. With regard to the contraction mapping principle, good estimates of the nonlinear terms are required. Due to the change of coordinates, the nonlinear terms take a more involved shape.

The associated main result asserts the local strong well-posedness of the *parabolic-hyperbolic regularized sea ice model* and can be paraphrased as follows.

Theorem. *Let $p > 4$ and $(v_{\mathrm{ice},0}, h_0, a_0) \in V \subset \mathrm{W}^{2-2/p,p}(\Omega)^2 \times \mathrm{W}^{1,p}(\Omega)^2$ subject to compatibility conditions, and $(v_{\mathrm{ice},0}, h_0, a_0) \in V$ satisfies $h_0 > \kappa$ as well as $a_0 \in (0,1)$. Then there is $T > 0$ such that system* (ME)–(BLH) *has a unique strong solution $v = (v_{\mathrm{ice}}, h, a)$, meaning that*

$$v \in \mathrm{W}^{1,p}(0,T;\mathrm{L}^p(\Omega)^2 \times \mathrm{W}^{1,p}(\Omega)^2) \cap \mathrm{L}^p(0,T;\mathrm{W}^{2,p}(\Omega)^2 \times \mathrm{W}^{1,p}(\Omega)^2).$$

The results in this chapter have not been published yet.

Time Periodic Quasilinear Evolution Equations

In the second part of this thesis, we concentrate on time periodic quasilinear problems. The study of time periodic problems is classical, see for example the works of Serrin [123], Judovič [74] or Prodi [112]. Unlike the first part of the thesis, which addresses concrete problems in the context of geophysical flow models, the second part is rather centered around the presentation of general frameworks. Applications of these frameworks are provided in a second step. One of these applications also concerns a time periodic problem associated to Hibler's sea ice model, providing a link with the first part.

Time Periodic Quasilinear Evolution Equations by the Arendt-Bu Theorem

For a Banach space X, a linear operator $A\colon \mathrm{D}(A) \subset \mathrm{X} \to \mathrm{X}$ and a term on the right-hand side $f \in \mathrm{L}^p(0, 2\pi; \mathrm{X})$, consider

$$\text{(PACP)} \qquad \begin{cases} u'(t) + Au(t) = f(t), & \text{for } t \in (0, 2\pi), \\ \qquad u(0) = u(2\pi). \end{cases}$$

Then A admits *maximal periodic* L^p*-regularity* if for any $f \in \mathrm{L}^p(0, 2\pi; \mathrm{X})$, there exists a unique solution

$$u \in \mathrm{W}^{1,p}(0, 2\pi; \mathrm{X}) \cap \mathrm{L}^p(0, 2\pi; \mathrm{D}(A))$$

to (PACP). Arendt and Bu [7] provided a characterization of this property by $\mathrm{i}\mathbb{Z} \subset \rho(-A)$ and the $\mathcal{R}$-boundedness of $(kR(\mathrm{i}k, -A))_{k\in\mathbb{Z}}$. This is in turn equivalent to the maximal L^p-regularity of the initial value problem on the interval $(0, 2\pi)$ and $1 \in \rho(\mathrm{e}^{-2\pi A})$, which is referred to as the *Arendt-Bu theorem*. The latter theorem proves useful for the investigation of time periodic problems. It is the purpose of Chapter 7 to provide several frameworks to time periodic quasilinear abstract Cauchy problems by means of the Arendt-Bu theorem. In fact, we investigate problems of the shape

$$\text{(QACP)} \qquad \begin{cases} u'(t) + A(u(t))u(t) = F_1(u(t)) + F_2(t, u(t)), & \text{for } t \in (0, T), \\ \qquad u(0) = u(T), \end{cases}$$

on a Banach space X_0, and we study the existence and uniqueness of solutions to (QACP) close to equilibrium solutions u_* of $A(u_*)u_* = F_1(u_*)$. Under certain Lipschitz assumptions on the nonlinear terms A, F_1 and F_2, and assuming that the linearization $A_* := A(u_*)$ with domain $\mathrm{D}(A_*) = \mathrm{X}_1$ fits in the framework of the Arendt-Bu theorem, we deduce the existence of a unique strong solution u to the problem (QACP). The main result in this context roughly reads as follows.

Theorem. *Let u_* be an equilibrium to the autonomous part of* (QACP)*, consider an open neighborhood V of u_*, and suppose that*

$$A \in \mathrm{C}^{0,1}(V; \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0)) \;\; \text{and} \;\; F_1,\, F_2 \in \mathrm{C}^{0,1}(V; \mathrm{X}_0),$$

where the Lipschitz constants of A at the equilibrium and of F_1 as well as F_2 are assumed to shrink to zero with the radius of the ball on which the Lipschitz *continuity is investigated. Moreover, suppose that* $0 \in \rho(A_*)$, *and that the operator* A_* *has maximal* L^p*-regularity on* $(0,T)$.

Then if $F_2(\cdot, u_*)$ *is small in* $\mathrm{L}^p(0,T;\mathrm{X}_0)$, *there exists a strong solution* $u \in \mathrm{W}^{1,p}(0,T;\mathrm{X}_0) \cap \mathrm{L}^p(0,T;\mathrm{X}_1)$ *to* (QACP), *which is unique in a small neighborhood of* u_*.

In addition, we elaborate on time periodic solutions in a neighborhood of zero, and we especially consider the case of bilinear nonlinearities.

The other part of Chapter 7 is dedicated to the investigation of the time periodic sea ice problem, so the problem (ME)–(BLP) is considered in the time periodic framework and subject to time periodic forcing terms f_{ice}, f_{h} and f_{a}. More precisely, we take into account

$$
\text{(SIP)} \quad \begin{cases}
\partial_t v_{\mathrm{ice}} + (v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice}} = \dfrac{1}{m_{\mathrm{ice}}} \mathrm{div}_{\mathrm{H}}\, \sigma_\delta + \dfrac{1}{m_{\mathrm{ice}}} F^{\mathrm{ice}} + f_{\mathrm{ice}}, & \text{in } \mathbb{R} \times \Omega, \\
\partial_t h + \mathrm{div}_{\mathrm{H}}\, (v_{\mathrm{ice}} h) = S_{\mathrm{h}} + d_{\mathrm{h}} \Delta_{\mathrm{H}} h + f_{\mathrm{h}}, & \text{in } \mathbb{R} \times \Omega, \\
\partial_t a + \mathrm{div}_{\mathrm{H}}\, (v_{\mathrm{ice}} a) = S_{\mathrm{a}} + d_{\mathrm{a}} \Delta_{\mathrm{H}} a + f_{\mathrm{a}}, & \text{in } \mathbb{R} \times \Omega, \\
u(t) = u(t+T), & \text{for } t \in \mathbb{R},
\end{cases}
$$

completed by Dirichlet boundary conditions for v_{ice} and Neumann boundary conditions for h and a. We study the existence of time periodic solutions in a neighborhood of constant equilibrium solutions $u_* = (0, h_*, a_*)$, with $h_* > \kappa$ and $a_* \in (0,1)$, to the simplified problem (SIS). For $F_{\mathrm{p}}^{\mathrm{ice}} := F^{\mathrm{ice}} + f_{\mathrm{ice}}$, the difference $\tilde{u} := u - u_* = (\tilde{v}_{\mathrm{ice}}, \tilde{h}, \tilde{a})$ satisfies

$$
\text{(SIPR)} \quad \begin{cases}
\partial_t \tilde{v}_{\mathrm{ice}} - \dfrac{1}{m_{\mathrm{ice}}} \mathrm{div}_{\mathrm{H}}\, \sigma_\delta = -(\tilde{v}_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) \tilde{v}_{\mathrm{ice}} + F_{\mathrm{p}}^{\mathrm{ice}}, & \text{in } \mathbb{R} \times \Omega, \\
\partial_t \tilde{h} + h_* \mathrm{div}_{\mathrm{H}}\, \tilde{v}_{\mathrm{ice}} - d_{\mathrm{h}} \Delta_{\mathrm{H}} \tilde{h} = S_{\mathrm{h}} - \mathrm{div}_{\mathrm{H}}\, (\tilde{v}_{\mathrm{ice}} \tilde{h}) + f_{\mathrm{h}}, & \text{in } \mathbb{R} \times \Omega, \\
\partial_t \tilde{a} + a_* \mathrm{div}_{\mathrm{H}}\, \tilde{v}_{\mathrm{ice}} - d_{\mathrm{a}} \Delta_{\mathrm{H}} \tilde{a} = S_{\mathrm{a}} - \mathrm{div}_{\mathrm{H}}\, (\tilde{v}_{\mathrm{ice}} \tilde{a}) + f_{\mathrm{a}}, & \text{in } \mathbb{R} \times \Omega, \\
\tilde{u}(t) = \tilde{u}(t+T), & \text{for } t \in \mathbb{R},
\end{cases}
$$

and the boundary conditions are unaffected. In the next step, we reformulate (SIPR) as a time periodic quasilinear abstract Cauchy problem.

A major obstacle in the analysis is the lack of invertibility due to the presence of Neumann Laplacian operators. We overcome this by taking into account L^p-functions with spatial average zero, denoted by $\mathrm{L}_0^p(\Omega)$, in the h- and a-component. The new ground space thus takes the shape

$$
\mathrm{X}_0 = \mathrm{L}^p(\Omega)^2 \times \mathrm{L}_0^p(\Omega) \times \mathrm{L}_0^p(\Omega).
$$

The equations satisfied by $\tilde{h}$ and $\tilde{a}$ preserve this property provided the thermodynamic terms and the periodic forcing terms also have spatial average zero. Another ingredient is that (SIPR) can be studied on the time interval $(0, T)$, and the solution is then extended thanks to the periodicity. Additionally verifying the maximal L^p-regularity of the linearized operator matrix by means of perturbation theory, and showing estimates of the nonlinear terms, we find that (SIPR) lies in the scope of the framework developed in the first part of the chapter. The emerging result asserts the existence of a strong time periodic solution to (SIPR) which is unique in a neighborhood of zero. Below, we state a simplified version of this result.

Theorem. *Consider* $p > 4$ *and a constant equilibrium solution* $u_* = (0, h_*, a_*)$ *to* (SIS)*, where* $h_* > \kappa$ *and* $a_* \in (0, 1)$*. Let also* $f = (f_{\mathrm{ice}}, f_{\mathrm{h}}, f_{\mathrm{a}}) \colon \mathbb{R} \to \mathrm{X}_0$ *be* T*-periodic such that* $f|_{(0,T)} \in \mathrm{L}^p(0, T; \mathrm{X}_0)$*. Then if* f *and the other external forcing terms are sufficiently small, there exists a strong* T*-periodic solution* $\tilde{u}$ *to* (SIPR)*, i. e.,*

$$\tilde{u}|_{(0,T)} \in \mathrm{W}^{1,p}(0, T; \mathrm{X}_0) \cap \mathrm{L}^p(0, T; \mathrm{W}^{2,p}(\Omega)^2 \times (\mathrm{W}^{2,p}(\Omega) \cap \mathrm{L}^p_0(\Omega))^2).$$

Moreover, the solution is unique in a small neighborhood of zero.

Let us observe that $u = \tilde{u} + u_*$ solves the original time periodic sea ice problem (SIP). A similar result based on a significantly different strategy has been obtained in a joint work with Matthias Hieber [20].

Time Periodic Quasilinear Evolution Equations in Real Interpolation Spaces

The above Arendt-Bu theorem requires maximal L^p-regularity of the associated initial value problem, which can also be deduced from the $\mathcal{R}$-boundedness of resolvents in applications in view of a famous result due to Weis [132]. In the context of initial value problems, a classical theorem due to Da Prato and Grisvard [32] asserts the maximal regularity of initial value problems in real interpolation spaces when only assuming that the underlying linear operator generates a bounded analytic semigroup and is invertible. A time periodic analogue of this result has been provided by Hieber et al. [64].

Chapter 8 is devoted to the study of time periodic quasilinear problems of the shape

$$\text{(QACP-I)} \qquad \begin{cases} u'(t) + A(u(t))u(t) = F(t, u(t)) + f(t), & \text{for } t \in \mathbb{R}, \\ u(t) = u(t+T), & \text{for } t \in \mathbb{R}. \end{cases}$$

For A_0 denoting the linearization of A at zero as well as an underlying Banach space X, the spaces for the maximal periodic L^p-regularity read as

$$\mathrm{D}_{A_0}(\theta,p) := \left\{ x \in \mathrm{X} : [x]_{\theta,p} := \left(\int_0^\infty \left\| t^{1-\theta} A_0 \mathrm{e}^{-tA_0} x \right\|_{\mathrm{X}}^p \frac{\mathrm{d}t}{t} \right)^{1/p} < \infty \right\}.$$

These spaces coincide with the real interpolation spaces $(\mathrm{X}, \mathrm{D}(A_0))_{\theta,p}$ under mild assumptions on A_0. Besides, we denote by E_1 the domain of the $\mathrm{D}_{A_0}(\theta,p)$-realization of the operator A_0. The result on the existence of a unique solution to (QACP-I) is described in a simplified way below.

Theorem. *Assume that A and F are locally Lipschitz continuous from*

$$\mathbb{E}_{1,\theta} := \mathrm{W}^{1,p}(0,T;\mathrm{D}_{A_0}(\theta,p)) \cap \mathrm{L}^p(0,T;\mathrm{E}_1)$$

into $\mathrm{L}^p(0,T;\mathrm{D}_{A_0}(\theta,p))$, and the Lipschitz constant of F shrinks with the radius of the balls. In addition, suppose that $-A_0$ generates a bounded analytic semigroup and satisfies $0 \in \rho(A_0)$.

Then if $F(\cdot,0)|_{(0,T)}$ and $f|_{(0,T)}$ are sufficiently small in $\mathrm{L}^p(0,T;\mathrm{D}_{A_0}(\theta,p))$, there exists a strong solution u with $u|_{(0,T)} \in \mathbb{E}_{1,\theta}$ to (QACP-I)*, and u is unique in a small neighborhood of zero.*

The second part of the Chapter 8 is concerned with applications of the general framework. In contrast to the previous considerations, these applications do *not* concern geophysical flow models, but we show that the framework can also be applied to different examples. More precisely, we investigate the time periodic problems associated to quasilinear Keller-Segel systems, and to a Nernst-Planck-Poisson type problem from electrochemistry.

For a bounded and sufficiently regular domain $\Omega \subset \mathbb{R}^d$, with $d \geq 2$, the density of a cell population $n\colon \mathbb{R} \times \Omega \to \mathbb{R}$, the concentration of a chemoattractant $c\colon \mathbb{R} \times \Omega \to \mathbb{R}$ and a T-periodic function $f = (f_\mathrm{n}, f_\mathrm{c})$, the time periodic quaslinear Keller-Segel system under investigation is

$$\text{(KSP)} \qquad \begin{cases} \partial_t n - \nabla \cdot ((n+1)^m \nabla n) = -\nabla \cdot (n \nabla c) + f_\mathrm{n}, & \text{in } \mathbb{R} \times \Omega, \\ \partial_t c - \Delta c + c - n = f_\mathrm{c}, & \text{in } \mathbb{R} \times \Omega, \\ n(t) = n(t+T), \;\; c(t) = c(t+T), & \text{in } \mathbb{R} \times \Omega. \end{cases}$$

Moreover, n and c are assumed to satisfy homogeneous Neumann boundary conditions. With regard to the first part of the chapter, we rewrite (KSP) as a time periodic quasilinear abstract Cauchy problem. In a similar way as for the sea ice equations in the periodic setting, we consider L^p-functions with spatial average zero in the n-equation in order to obtain invertibility. In view of the shape of the nonlinear term, we consider $\mathrm{W}^{1,p}$ as the ground space of the c-component, so

$$\mathrm{X}_0 = \mathrm{L}^p_0(\Omega) \times \mathrm{W}^{1,p}(\Omega).$$

The sectoriality of the linearized operator matrix A_0 with associated trace space $\mathrm{D}_{A_0}(\theta, p)$ is a consequence of the fact that the restriction of the Neumann Laplacian operator to its injective part inherits the well known properties from the Neumann Laplacian operator, while the argument in the c-component relies on a Banach scale argument for the shifted Neumann Laplacian. In order to estimate the nonlinear terms, we employ a newly developed version of the mixed derivative theorem in real interpolation spaces. This yields that the time periodic abstract Cauchy problem corresponding to (KSP) fits in the framework from the first part of the chapter. The result on the existence of a time periodic strong solution in the real interpolation space can be paraphrased as follows.

Theorem. *Let $p > d+2$ and $\theta \in (0,1)$ with $\theta \in (d/2p, 1/2+1/2p)$, and consider a T-periodic forcing term $f : \mathbb{R} \to \mathrm{D}_{A_0}(\theta, p)$, where $f|_{(0,T)} \in \mathrm{L}^p(0,T; \mathrm{D}_{A_0}(\theta, p))$. Then if f is sufficiently small, there exists a T-periodic strong solution (n, c) to* (KSP)*, and the solution is unique in a neighborhood of zero.*

This chapter is based on a joint article with Matthias Hieber [19].

Outline of the Thesis

In order to prepare for the further analysis in this thesis, which is intended to be as self-contained as possible, we recall selected concepts in Chapter 1 and Chapter 2. In that respect, Chapter 1 settles some notation, introduces basic interpolation theory, discusses function spaces and their properties such as traces as well as embedding or interpolation relations and collects further useful analytical tools such as the Poincaré inequality or the Rellich-Kondrachov theorem.

Chapter 2 is dedicated to the presentation of some abstract theory. Most importantly, we recall the notion of maximal L^p-regularity of abstract Cauchy problems and discuss its relation with other operator theoretic concepts as e. g. the bounded $\mathcal{H}^\infty$-calculus, and we study its application to parabolic boundary value problems. Another purpose of Chapter 2 is to set up a toolbox for the nonlinear analysis carried out in the following chapters. This means that we collect useful embedding relations, and we also invoke general quasilinear existence theory on the local well-posedness of quasilinear abstract Cauchy problems, or even on global strong well-posedness close to equilibria. Finally, we discuss the viscous primitive equations and recall the hydrostatic Stokes operator.

In Chapter 3, we investigate the fully parabolic regularized sea model, including a short bibliographic overview as well as the introduction of the system of equations. Moreover, we study the operator emerging from the internal ice stress in detail and rewrite the sea ice equations as a quasilinear abstract Cauchy problem. This culminates in the local strong well-posedness of the model. We also establish the global strong well-posedness of a simplified version of the model in the absence of external forcing terms for initial data close to constant equilibria.

Chapter 4 addresses the interaction problem of sea ice with a rigid body. This moving domain problem is reduced to a problem on a fixed domain by employing a local coordinate transform. We then tackle the linearized problem. Complemented by estimates of the nonlinear terms, this results in the local strong well-posedness of the interaction problem.

The focal point of Chapter 5 is the investigation of a coupled atmosphere-sea ice-ocean model. Again, we reformulate the complete system as a quasilinear abstract Cauchy problem. The linear theory is handled by the study of the stationary hydrostatic Stokes problem, finally leading to the maximal L^p-regularity of the linearized operator matrix. This allows us to show the local strong well-posedness of the complete coupled model, and to establish the global strong well-posedness close to equilibria for a simplified model.

The last chapter completely dedicated to geophysical flow models is Chapter 6. It discusses a parabolic-hyperbolic variant of Hibler's sea ice model. To this end, we use the Lagrangian change of coordinates, establish the maximal L^p-regularity of the linearized problem and show estimates of the nonlinear terms to derive the existence of a unique strong solution from an application of the contraction mapping principle. In the end of Chapter 6, we also provide a brief outlook on questions still open in the mathematical analysis of Hibler's sea ice model.

Chapter 7 provides frameworks to time periodic quasilinear evolution equations based on the Arendt-Bu theorem on maximal periodic L^p-regularity. Moreover, we especially address the situation of bilinear nonlinearities. In a second step, we apply the general frameworks to the time periodic problem corresponding to the fully parabolic regularized sea ice model.

In the final Chapter 8, we present a framework to time periodic quasilinear problems in real interpolation spaces by means of a time periodic version of the Da Prato-Grisvard theorem. The second part of this chapter is concerned with applications of the general framework to quasilinear Keller-Segel systems and to a parabolic Nernst-Planck-Poisson type system.

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Zusammenfassung in Deutscher Sprache

In dieser Arbeit werden geophysikalische Strömungsmodelle mithilfe der Theorie quasilinearer Evolutionsgleichungen untersucht. Ferner werden zeitperiodische quasilineare parabolische Gleichungen studiert.

Der erste Teil dieser Arbeit beschäftigt sich mit der rigorosen mathematischen Analyse von Modellen im Zusammenhang eines viskoplastischen Meereismodells. Dieses wurde 1979 durch den Geophysiker William D. Hibler III in einem viel beachteten Artikel eingeführt und entwickelte sich in den vergangenen Jahrzehnten zu einem häufig verwendeten Modell, um die Meereisdynamik in Klimamodellen abzubilden. Vor diesem Hintergrund mutet es erstaunlich an, dass die rigorose mathematische Untersuchung von Hiblers Modell erst in der jüngsten Vergangenheit begann. Ferner steht dies in klarem Kontrast zur großen Anzahl an wissenschaftlichen Artikeln aus der Geophysik, Simulation und numerischen Analyse zu Hiblers Meereismodell.

Hiblers Modell stellt ein großskaliges Modell dar, was bedeutet, dass nicht die Bewegung einzelner Eisschollen betrachtet wird, sondern das Verhalten von Meereis auf großen Skalen von vielen hundert Quadratkilometern beschrieben wird. Zudem bildet Hiblers Modell dynamische wie thermodynamische Aspekte ab.

Die Modellvariablen in Hiblers Modell sind die horizontale Eisgeschwindigkeit, die Eisdicke sowie die Eiskompaktheit. Letztere gibt den Anteil von dickem Eis in einem Kontrollgebiet an. Das zugrunde liegende Gebiet ist ein beschränktes Gebiet in zwei Raumdimensionen, dessen Rand als hinreichend regulär angenommen wird. Weiter besteht das Modell aus einer Impulsgleichung für die horizontale Eisgeschwindigkeit sowie Bilanzgleichungen für die

Eisdicke und die Eiskompaktheit. Die wesentlichen Kraftterme in der Impulsgleichung sind die aus der Reibung des Eises entstehenden inneren Eiskräfte, Kräfte aufgrund atmosphärischer Winde und ozeanischer Strömungen, Coriolis-Kräfte und eine Kraft aufgrund der wechselnden Neigung des Eises. Präziser werden die inneren Eiskräfte mithilfe einer viskoplastischen Rheologie modelliert. Andererseits enthalten die Bilanzgleichungen thermodynamische Quell- und Senkenterme.

Im ersten Schritt wird der Spannungstensor regularisiert, um eine Degeneration der Impulsgleichung zu vermeiden. Diese Regularisierung ist durch bestehende Ansätze aus der numerischen Analysis motiviert. Eine weitere Vereinfachung besteht darin, zusätzliche viskose Terme in den Bilanzgleichungen einzuführen. Bereits Hibler führte solche Terme ein, um die Stabilität der numerischen Algorithmen zu erhöhen. Das intensive Studium des zu den internen Eiskräften korrespondierenden Differentialoperators stellt einen weiteren bedeutenden Schritt in der rigorosen mathematischen Analysis dar. Über gewisse Elliptizitätseigenschaften führt dies schließlich zur sogenannten *maximalen* L^p*-Regularität* des zugehörigen linearisierten Operators. Anschließend wird das komplette System als quasilineare Evolutionsgleichung aufgefasst und es wird die maximale L^p-Regularität für die assoziierte linearisierte Operatormatrix bewiesen. Gemeinsam mit Lipschitz-Abschätzungen und quasilinearer Existenztheorie resultiert dies schließlich in der *lokalen* Existenz und Eindeutigkeit einer *starken* Lösung des regularisierten parabolisch-parabolischen Systems für hinreichend reguläre Anfangswerte.

Daraufhin wird ein vereinfachtes Modell ohne äußere Kräfte studiert. Dieses Modell besitzt sogar eindeutige starke Lösungen, die *global* in der Zeit existieren, sofern die Anfangswerte nahe genug an konstanten Gleichgewichtspunkten in Betracht gezogen werden.

Ein weiteres Kapitel widmet sich der Untersuchung der Fluid-Struktur-Interaktion von Meereis mit einem (großen) Festkörper, wobei Hiblers regularisiertes viskoplastisches Modell als Basis dient und Newtons Gesetze für die Bewegung des Festkörpers angenommen werden. Das Gebiet im entstehenden Problem ist zeitabhängig, sodass in einem ersten Schritt eine Transformation angewandt wird, die lokal im Ort agiert und auf ein zeitunabhängiges Gebiet führt. Auf der anderen Seite nehmen die transformierten Terme eine kompliziertere Gestalt an. Eine weitere Schwierigkeit ergibt sich aus der Bedingung, dass die Geschwindigkeit des Eises mit der aus Translation und Rotation resultierenden Geschwindigkeit des Festkörpers am gemeinsamen Rand übereinstimmt. Diesem Umstand wird beim Studium des linearisierten Modells

Rechnung getragen, indem das Modell schrittweise analysiert und letztlich als inhomogenes parabolisches Randwertproblem aufgefasst wird. Für die Fluid-Struktur-Interaktion wird die lokale starke Wohlgestelltheit für hinreichend reguläre Anfangswerte schließlich mithilfe eines Fixpunktarguments gezeigt. Neben der geeigneten Linearisierung basiert letzteres auf geeigneten Lipschitz-Abschätzungen.

Im nächsten Schritt stehen die zu den atmosphärischen Winden und ozeanischen Strömungen assoziierten Kräfte im Vordergrund. Präziser werden die Wind- und Ozeangeschwindigkeit in Hiblers viskoplastischem Modell internalisiert, wobei beide durch die *primitiven Gleichungen* beschrieben werden. Daher wird als zugrunde liegendes Gebiet für das Meereis $G = (0,1) \times (0,1)$ betrachtet und es werden periodische Randbedingungen angenommen. Eine weitere Bedingung ist, dass die Geschwindigkeiten des Ozeans und des Eises an ihrem gemeinsamen Rand identisch sind. Als Konsequenz handelt es sich um ein gekoppeltes Modell, das wieder als quasilineare Evolutionsgleichung behandelt wird. Von zentraler Bedeutung ist in diesem Kapitel die Untersuchung des stationären hydrostatischen Stokes-Problems. Dies erlaubt, die Analyse des linearen Problems auf ein entkoppeltes Problem zurückzuführen, um schließlich Eigenschaften wie die maximale L^p-Regularität zu etablieren. Mithilfe von Lipschitz-Abschätzungen und quasilinearer Existenztheorie münden diese Untersuchungen wieder in der lokalen starken Wohlgestelltheit für Anfangswerte von ausreichend hoher Regularität. Ebenso lässt sich die Existenz und Eindeutigkeit von globalen starken Lösungen nahe konstanten Gleichgewichtspunkten beweisen.

Den Schlusspunkt des ersten Teils der Dissertation markiert ein Kapitel zu einer parabolisch-hyperbolischen Variante von Hiblers Modell, die physikalisch realitätsgetreuer erscheint. Im Vergleich zu vorher werden dazu *keine viskosen Terme in den Bilanzgleichungen* in Betracht gezogen. Eine Schwierigkeit besteht dann in der Handhabung der hyperbolischen Effekte. Dies wird durch den Übergang zu Lagrange-Koordinaten gelöst, was eine Analyse des linearisierten Problems ermöglicht. Dagegen sind die Terme infolge der Transformation von einer komplizierteren Gestalt. Wieder wird die lokale starke Wohlgestelltheit mithilfe eines Fixpunktarguments bewiesen, wobei sorgfältig Lipschitz-Abschätzungen der nichtlinearen Terme hergeleitet werden. Weiter müssen hinreichend reguläre Anfangswerte betrachtet werden.

Gegenstand des zweiten Teils der vorliegenden Arbeit sind zeitperiodische quasilineare Probleme. Die Frage nach zeitperiodischen Lösungen für Probleme mit zeitperiodischen äußeren Kräften stellt ein klassisches Problem für die mathematische Analysis im Allgemeinen und die mathematische Fluiddynamik im Speziellen dar. In einem ersten Schritt entwickeln wir verschiedene theoretische Resultate für die starke Lösbarkeit zeitperiodischer Probleme in unterschiedlichen Zusammenhängen, die auf der Betrachtung als zeitperiodische quasilineare Evolutionsgleichungen basieren. Hinsichtlich der linearen Theorie wird an dieser Stelle der Begriff der *maximalen periodischen* L^p*-Regularität* genutzt. Zugrunde liegt dabei ein Theorem, das auf Arendt und Bu zurückgeht. Es gestattet eine Charakterisierung der maximalen periodischen L^p-Regularität über die gewöhnliche maximale L^p-Regularität parabolischer Anfangswertprobleme und Spektraleigenschaften der korrespondierenden erzeugten C_0-Halbgruppe. Die allgemeine Theorie wird daraufhin auf die regularisierte parabolisch-parabolische Variante von Hiblers Modell angewendet. In dieser Hinsicht stellt der zugehörige Abschnitt ein Bindeglied der beiden Teile der Dissertation dar. Eine wesentliche Schwierigkeit besteht in der fehlenden Invertierbarkeit der linearisierten Operatormatrix. Um diese Hürde zu überwinden, wird das Modell nahe Gleichgewichtspunkten betrachtet und entsprechend umformuliert. Eine Modifikation des Grundraums ebnet schlussendlich den Weg für die Invertierbarkeit der linearisierten Operatormatrix. Als Resultat kann die zuvor entwickelte allgemeine Theorie verwendet werden, um die Existenz einer zeitperiodischen starken Lösung für hinreichend kleine zeitperiodische äußere Kräfte sicherzustellen.

Das zweite Kapitel des zweiten Teils dieser Arbeit basiert auf einer periodischen Variante eines klassischen Resultats von Da Prato und Grisvard, das maximale Regularität in reellen Interpolationsräumen impliziert, sofern Sektorialität und Invertierbarkeit des unterliegenden Operators angenommen werden. Diese Variante liefert die starke Wohlgestelltheit von zeitperiodischen quasilinearen Evolutionsgleichungen für hinreichend kleine zeitperiodische äußere Kräfte. Die Anwendungen in diesem Kapitel betreffen ein quasilineares *Keller-Segel Modell*, welches die direkte Bewegung von Zellen und Organismen als Reaktion auf chemische Gradienten modelliert, sowie ein parabolisches *Nernst-Planck-Poisson Modell* aus der Elektrochemie. In beiden Fällen wird die Existenz zeitperiodischer starker Lösungen gezeigt.

Preliminaries

CHAPTER 1

Interpolation, Function Spaces and Analytical Tools

In this chapter, we introduce some basic concepts which will appear throughout the thesis. More precisely, after settling some notation and conventions in Section 1.1, we briefly discuss the real and complex interpolation method in Section 1.2, invoke a variety of function spaces in the scalar- and vector-valued setting and elaborate on the trace as well as interpolation and embedding relations of these spaces in Section 1.3 and collect some useful analytical tools in Section 1.4.

1.1. Basic Notation and Conventions

First, we make precise the notation of some sets and spaces.

(i) For $\theta \in (0, \pi]$, we denote by Σ_θ the sector of angle θ in the complex plane $\mathbb{C}$, i. e., $\Sigma_\theta \coloneqq \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$.

(ii) For a Banach space X, an element $x_0 \in \mathrm{X}$ and $r > 0$, we denote the open ball in X with center x_0 and radius r by $\mathbb{B}_{\mathrm{X}}(x_0, r)$, while $\overline{\mathbb{B}}_{\mathrm{X}}(x_0, r)$ represents the corresponding closed ball.

(iii) Given two Banach spaces X and Y, the space of bounded linear operators from X to Y is denoted by $\mathcal{L}(\mathrm{X}, \mathrm{Y})$. In the sequel, we use the convention $\mathcal{L}(\mathrm{X}, \mathrm{X}) \eqqcolon \mathcal{L}(\mathrm{X})$.

We proceed with some notation in the context of derivatives. In fact, ∂_j denotes the j-th partial derivative in the classical, weak or distributional sense, depending on the precise situation. Moreover, D_j abbreviates $-\mathrm{i}\partial_j$, and ∂^α represents $\partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$ for a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$. In that respect, we also mention the shorthand D^α for $\mathrm{D}_1^{\alpha_1} \dots \mathrm{D}_d^{\alpha_d}$. Concerning the chapters on sea ice, we also remark that we use x_H to denote the horizontal variables, i. e., $x_\mathrm{H} = (x, y)$, while z usually represents the vertical variable. In addition, we indicate objects associated to the horizontal variables with a subscript $_\mathrm{H}$, so div_H designates the horizontal divergence, while ∇_H is the horizontal gradient.

In this thesis, we use the concept of generic constants. In that respect, $C > 0$ typically represents a generic constant in the sequel. However, in proofs, we number the constants, i. e., we use $C_1, C_2, \dots$ in order to simplify the tracking of constants, and at the beginning of each proof, the numbering starts again from $C_1 > 0$.

Furthermore, for a Banach space X with norm $\| \cdot \|_\mathrm{X}$, we denote by X^d the d-product space as e. g. the $\mathbb{R}^d$- or $\mathbb{C}^d$-valued space of X and still denote the norm of this space by $\| \cdot \|_\mathrm{X}$ instead of $\| \cdot \|_{\mathrm{X}^d}$ for simplicity.

1.2. Interpolation Theory

In this section, we briefly recall the real and complex interpolation method. We mainly follow [2, Chapter 7] as well as [100, Section 1.1 and 2.1] here.

Before describing the two methods, we begin with some basics. Consider two Banach spaces X and Y, and assume that $\mathrm{X}, \mathrm{Y} \hookrightarrow \mathrm{Z}$ is valid for some Hausdorff topological vector space Z. Moreover, suppose that $\mathrm{X} \cap \mathrm{Y} \neq \emptyset$. Then (X, Y) is referred to as an *interpolation couple*. We observe that $\mathrm{X} \cap \mathrm{Y}$ and the algebraic sum $\mathrm{X} + \mathrm{Y}$ defined by

$$\mathrm{X} + \mathrm{Y} \coloneqq \{u = x + y : x \in \mathrm{X}, \;\; y \in \mathrm{Y}\}$$

become Banach spaces when endowed with the norms

$$\begin{aligned} \|u\|_{\mathrm{X}\cap\mathrm{Y}} &\coloneqq \max \{\|u\|_\mathrm{X}, \|u\|_\mathrm{Y}\} \;\; \text{and} \\ \|u\|_{\mathrm{X}+\mathrm{Y}} &\coloneqq \inf \{\|x\|_\mathrm{X} + \|y\|_\mathrm{Y} : u = x + y, \;\; \text{with } \; x \in \mathrm{X}, \;\; y \in \mathrm{Y}\} . \end{aligned}$$

In addition, we obtain $\mathrm{X} \cap \mathrm{Y} \hookrightarrow \mathrm{X}, \mathrm{Y} \hookrightarrow \mathrm{X} + \mathrm{Y}$. The above properties are classical, and we refer e. g. to [2, Chapter 7]. In general, a Banach space $\tilde{\mathrm{Z}}$ such that $\mathrm{X} \cap \mathrm{Y} \hookrightarrow \tilde{\mathrm{Z}} \hookrightarrow \mathrm{X} + \mathrm{Y}$ is called *intermediate space*.

In this context, we also mention the concept of (exact) interpolation spaces. To this end, consider two interpolation couples $(\mathrm{X}_0, \mathrm{Y}_0)$ and $(\mathrm{X}_1, \mathrm{Y}_1)$ as well as a bounded linear operator $T\colon \mathrm{X}_0 + \mathrm{Y}_0 \to \mathrm{X}_1 + \mathrm{Y}_1$ such that $T\colon \mathrm{X}_i \to \mathrm{Y}_i$ is also bounded with norm at most M_i for $i = 0, 1$. Two intermediate spaces Z_0 for $(\mathrm{X}_0, \mathrm{Y}_0)$ and Z_1 for $(\mathrm{X}_1, \mathrm{Y}_1)$ are called *interpolation spaces of type* θ for $(\mathrm{X}_0, \mathrm{Y}_0)$ and $(\mathrm{X}_1, \mathrm{Y}_1)$ if every operator T of the above form maps Z_0 into Z_1, with norm M satisfying $M \leq C M_0^{1-\theta} M_1^{\theta}$. In the above, $C \geq 1$ is independent of T. If the previous relation is fulfilled with $C = 1$, then Z_0 and Z_1 are referred to as *exact interpolation spaces*.

Real Interpolation

Let X and Y be as introduced at the beginning of this section. For $t > 0$ fixed, we set

$$K(t; u) \coloneqq \inf \{\|x\|_{\mathrm{X}} + t\|y\|_{\mathrm{Y}} : u = x + y, \text{ with } x \in \mathrm{X} \text{ and } y \in \mathrm{Y}\}.$$

This defines a norm on $\mathrm{X} + \mathrm{Y}$ which is equivalent to the above norm. Furthermore, for an interval $J \subset \mathbb{R}_+$ and $p \in [1, \infty)$, we define

$$\mathrm{L}_*^p(J) \coloneqq \left\{ f \in \mathrm{L}^p(J) : \|f\|_{\mathrm{L}_*^p(J)} < \infty \right\}, \text{ with } \|f\|_{\mathrm{L}_*^p(J)}^p \coloneqq \int_J |f(t)|^p \, \mathrm{d}t/t.$$

In the case $p = \infty$, we set $\mathrm{L}_*^\infty(J) \coloneqq \mathrm{L}^\infty(J)$. For $\theta \in (0, 1)$ and $p \in [1, \infty]$, we introduce the *real interpolation space* $(\mathrm{X}, \mathrm{Y})_{\theta,p}$ defined by

$$\begin{aligned} (\mathrm{X}, \mathrm{Y})_{\theta,p} &\coloneqq \left\{ u \in \mathrm{X} + \mathrm{Y} : t \mapsto t^{-\theta} K(t; u) \in \mathrm{L}_*^p(\mathbb{R}_+) \right\}, \text{ where} \\ \|u\|_{(\mathrm{X},\mathrm{Y})_{\theta,p}} &\coloneqq \|t^{-\theta} K(t; u)\|_{\mathrm{L}_*^p(\mathbb{R}_+)}. \end{aligned}$$

Let us observe that $(\mathrm{X}, \mathrm{Y})_{\theta,p}$ is indeed an intermediate space between X and Y, see for instance [2, Theorem 7.10]. We also observe that for two interpolation couples $(\mathrm{X}_0, \mathrm{Y}_0)$ and $(\mathrm{X}_1, \mathrm{Y}_1)$, the real interpolation spaces $(\mathrm{X}_0, \mathrm{Y}_0)_{\theta,p}$ and $(\mathrm{X}_1, \mathrm{Y}_1)_{\theta,p}$ are exact interpolation spaces of type θ. For a proof of this property, we refer for example to [100, Theorem 1.6]. In the following, we also use the term *real interpolation functor* for $(\cdot, \cdot)_{\theta,p}$, and we say that the real interpolation method is *functorial*.

If one space is contained in the other in an interpolation couple, we can establish a relation of the real interpolation spaces as stated below. The result is well known, see for instance [100, Proposition 1.4].

Lemma 1.2.1. *Let* X*,* Y *be an interpolation couple such that* $\mathrm{Y} \subset \mathrm{X}$*. Then for* θ_1*,* $\theta_2 \in (0, 1)$ *with* $\theta_1 < \theta_2$*, we have* $(\mathrm{X}, \mathrm{Y})_{\theta_2,\infty} \subset (\mathrm{X}, \mathrm{Y})_{\theta_1,1}$*. Moreover, for all* $p, q \in [1, \infty]$*, it holds that* $(\mathrm{X}, \mathrm{Y})_{\theta_2,p} \subset (\mathrm{X}, \mathrm{Y})_{\theta_1,q}$*.*

Complex Interpolation

We take into consideration X and Y as made precise at the beginning of the section. In addition, we introduce the set $\mathcal{F} = \mathcal{F}(\mathrm{X}, \mathrm{Y})$ given by all functions f of $\zeta = \theta + \mathrm{i}\tau$ taking values in $\mathrm{X} + \mathrm{Y}$ and satisfying that

(a) f is bounded and continuous on the strip $0 \leq \theta \leq 1$ into $\mathrm{X} + \mathrm{Y}$,

(b) f is holomorphic from $0 < \theta < 1$ into $\mathrm{X} + \mathrm{Y}$,

(c) f is continuous on the line $\theta = 0$ into X with $\|f(\mathrm{i}\tau)\|_{\mathrm{X}} \to 0$ as $|\tau| \to \infty$, and

(d) f is continuous on the line $\theta = 1$ into Y so that $\|f(1 + \mathrm{i}\tau)\|_{\mathrm{Y}} \to 0$ as $|\tau| \to \infty$.

It turns out that $\mathcal{F}$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{F}} := \max \left\{ \sup_{\tau \in \mathbb{R}} \|f(\mathrm{i}\tau)\|_{\mathrm{X}}, \sup_{\tau \in \mathbb{R}} \|f(1 + \mathrm{i}\tau)\|_{\mathrm{Y}} \right\},$$

see for example [100, Section 2.1]. The above space $\mathcal{F}$ allows us to define the complex interpolation method. For $\theta \in (0, 1)$, we define the *complex interpolation space* $[\mathrm{X}, \mathrm{Y}]_\theta$ by

$$\begin{aligned} [\mathrm{X}, \mathrm{Y}]_\theta &:= \{u \in \mathrm{X} + \mathrm{Y} : u = f(\theta) \ \text{ for some } \ f \in \mathcal{F}\}, \ \text{ with} \\ \|u\|_{[\mathrm{X},\mathrm{Y}]_\theta} &:= \inf \{\|f\|_{\mathcal{F}} : f(\theta) = u\}. \end{aligned}$$

Again, it follows that (X, Y) is an intermediate space between X and Y, and for two interpolation couples $(\mathrm{X}_0, \mathrm{Y}_0)$ and $(\mathrm{X}_1, \mathrm{Y}_1)$, the spaces $[\mathrm{X}_0, \mathrm{Y}_0]_\theta$ as well as $[\mathrm{X}_1, \mathrm{Y}_1]_\theta$ are exact interpolation spaces of type θ, see [100, Theorem 2.6]. Besides, we call $[\cdot, \cdot]_\theta$ the *complex interpolation functor* and refer to the complex interpolation method as *functorial*.

The following lemma establishes a link between real and complex interpolation spaces. The result is also well known, and for convenience, we refer for instance to [100, Proposition 2.10].

Lemma 1.2.2. *Let* X, Y *be an interpolation couple. Then for all* $\theta \in (0, 1)$, *we have* $[\mathrm{X}, \mathrm{Y}]_\theta \hookrightarrow (\mathrm{X}, \mathrm{Y})_{\theta,\infty}$.

Combining Lemma 1.2.1 and Lemma 1.2.2, we find the following relation of real and complex interpolation spaces in the situation of nested spaces.

Lemma 1.2.3. *Let* X, Y *be an interpolation couple such that* $\mathrm{Y} \subset \mathrm{X}$. *Then for* $\theta_1, \theta_2 \in (0, 1)$ *with* $\theta_1 < \theta_2$ *and* $p \in [1, \infty]$, *we have* $[\mathrm{X}, \mathrm{Y}]_{\theta_2} \hookrightarrow (\mathrm{X}, \mathrm{Y})_{\theta_1,p}$.

Interpolation of Closed Subspaces

We conclude the section on interpolation theory by commenting on the interpolation of closed subspaces. It is a consequence of [130, Theorem 1.17.1] together with the property of real and complex interpolation being exact interpolation functors of type θ.

Lemma 1.2.4. *Let* X*,* Y *be an interpolation couple, and consider a complemented subspace* C *of* $\mathrm{X} + \mathrm{Y}$ *such that the projection* P *satisfies* $P \in \mathcal{L}(\mathrm{X})$ *and* $P \in \mathcal{L}(\mathrm{Y})$*. Then* $\mathrm{X} \cap \mathrm{C}$*,* $\mathrm{Y} \cap \mathrm{C}$ *is an interpolation couple as well, and for* $\theta \in (0,1)$ *and* $p \in [1,\infty]$*, it holds that*

$$(\mathrm{X} \cap \mathrm{C}, \mathrm{Y} \cap \mathrm{C})_{\theta,p} = (\mathrm{X}, \mathrm{Y})_{\theta,p} \cap \mathrm{C} \ \text{ and } \ [\mathrm{X} \cap \mathrm{C}, \mathrm{Y} \cap \mathrm{C}]_\theta = [\mathrm{X}, \mathrm{Y}]_\theta \cap \mathrm{C}.$$

1.3. Function Spaces, Traces and Embeddings

In this section, we introduce the function spaces required in this thesis. Apart from classical function spaces, we also invoke Lebesgue spaces, Sobolev spaces, Bessel potential spaces, Besov spaces, Sobolev-Slobodeckij spaces as well as Triebel-Lizorkin spaces. Moreover, we discuss properties of the trace and normal derivative on some of these spaces, and we also recall interpolation and embedding relations. Since we require vector-valued spaces, we make this concept precise as well. In this context, we also discuss time weights in vector-valued Lebesgue and Sobolev spaces. Another topic is the introduction of functions on time-dependent domains. For this section, we mainly follow the monograph of Triebel [130].

Basic Function Spaces

Throughout this section, we consider $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}_0$, open, and m denotes a non-negative integer. We start with some classical functions spaces and variants of these.

(a) By $\mathrm{C}^m(\Omega)$, we denote the space of all functions f so that f and all its partial derivatives $\partial^\beta f$ of orders $|\beta| \le m$ are continuous on Ω. The space $\mathrm{C}^m(\overline{\Omega})$ is defined analogously, with Ω replaced by $\overline{\Omega}$.

(b) The space $\mathrm{C}^m_\mathrm{b}(\Omega)$ consists of all functions $f \in \mathrm{C}^m(\Omega)$ such that $\partial^\beta f$ is additionally bounded on Ω for all $0 \le |\beta| \le m$. Again, $\mathrm{C}^m_\mathrm{b}(\overline{\Omega})$ is defined likewise, and we observe that $\mathrm{C}^m_\mathrm{b}(\overline{\Omega}) = \mathrm{C}^m_\mathrm{b}(\Omega)$ provided Ω is bounded.

(c) For $\alpha \in (0,1]$, $\mathrm{C}^{m,\alpha}(\overline{\Omega})$ represents the subspace of $\mathrm{C}^m(\overline{\Omega})$ so that $\partial^\beta f$ satisfies a Hölder condition of exponent α in $\overline{\Omega}$ for $|\beta| = m$, i. e., there exists a constant $C > 0$ with

$$\left|\partial^\beta f(x_1) - \partial^\beta f(x_2)\right| \le C \cdot |x-y|^\alpha, \text{ for } x_1, x_2 \in \overline{\Omega}.$$

(d) We denote by $\mathrm{BUC}(\overline{\Omega})$ the *bounded and uniformly continuous functions* on $\overline{\Omega}$, while $\mathrm{BUC}^\alpha(\overline{\Omega})$ represents the *bounded and uniformly α-Hölder continuous functions* on $\overline{\Omega}$.

(e) Another subspace needed in the sequel is the space $\mathrm{C}_0(\mathbb{R}_+)$ consisting of the functions in $\mathrm{C}(\mathbb{R}_+)$ with $\lim_{x\to\infty} f(x) = 0$.

We proceed with the usual *Lebesgue* and *Sobolev spaces*.

(a) For $p \in [1,\infty)$, we denote by $\mathrm{L}^p(\Omega)$ the class of all measurable functions f on Ω for which

$$\|f\|^p_{\mathrm{L}^p(\Omega)} \coloneqq \int_\Omega |f(x)|^p \,\mathrm{d}x < \infty.$$

In the $p = \infty$-case, $\mathrm{L}^\infty(\Omega)$ designates the space of all essentially bounded functions on Ω, so $f \in \mathrm{L}^\infty(\Omega)$ if and only if

$$\|f\|_{\mathrm{L}^\infty(\Omega)} \coloneqq \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| < \infty.$$

In both situations, $p \in [1,\infty)$ and $p = \infty$, we identify functions if they are equal almost everywhere on Ω.

(b) We also define the *space of functions with average zero on* Ω. Indeed, for $\Omega \subset \mathbb{R}^d$ bounded and $p \in [1,\infty)$, we set

$$\mathrm{L}^p_0(\Omega) \coloneqq \left\{ f \in \mathrm{L}^p_0(\Omega) : \int_\Omega f(x)\,\mathrm{d}x = 0 \right\}. \tag{1.1}$$

Furthermore, for $f \in \mathrm{L}^p(\Omega)$, we introduce the splitting

$$f = f_{\mathrm{m}} + f_{\mathrm{avg}}, \text{ where } f_{\mathrm{avg}} \coloneqq \frac{1}{|\Omega|}\int_\Omega f(x)\,\mathrm{d}x \text{ and } f_{\mathrm{m}} \coloneqq f - f_{\mathrm{avg}}, \tag{1.2}$$

leading to $f_{\mathrm{m}} \in \mathrm{L}^p_0(\Omega)$.

(c) For $m \in \mathbb{Z}_+$ and $p \in [1, \infty]$, we recall the *Sobolev space*

$$\mathrm{W}^{m,p}(\Omega) := \{f \in \mathrm{L}^p(\Omega) : \partial^\beta f \in \mathrm{L}^p(\Omega), \text{ for } 0 \leq |\beta| \leq m\},$$

and $\partial^\beta f$ denotes the weak partial derivative here. For $p = 2$, we also use the notation $\mathrm{H}^m(\Omega) := \mathrm{W}^{m,2}(\Omega)$.

We finish the consideration of the basic function spaces with an elementary lemma yielding T-powers in estimates of Lebesgue and Sobolev functions on the time interval $(0, T)$. It is a direct consequence of Hölder's inequality.

Lemma 1.3.1. *Let $p \in (1, \infty)$ with associated Hölder conjugate $p' \in (1, \infty)$, so $1/p + 1/p' = 1$, and consider $T \in (0, \infty)$. Then for $f \in \mathrm{L}^\infty(0, T)$, we have*

$$\|f\|_{\mathrm{L}^p(0,T)} \leq T^{1/p} \cdot \|f\|_{\mathrm{L}^\infty(0,T)}.$$

A function $f \in \mathrm{W}^{1,p}(0, T)$ with $f(0) = 0$ satisfies the estimate

$$\|f\|_{\mathrm{L}^\infty(0,T)} \leq T^{1/p'} \cdot \|f\|_{\mathrm{W}^{1,p}(0,T)}.$$

Bessel Potential, Besov, Sobolev-Slobodeckij and Triebel-Lizorkin Spaces

The aforementioned spaces do not suffice in order to fully exploit the methods used in this thesis. Instead, a wider range of function spaces is needed. As we shall see later, the spaces presented in the sequel appear naturally as interpolation spaces.

Let us recall the *Schwartz space* $\mathrm{S}(\mathbb{R}^d)$ and its dual space $\mathrm{S}'(\mathbb{R}^d)$, the so-called space of *tempered distributions*. We denote by $\mathcal{F}$ as well as $\mathcal{F}^{-1}$ the Fourier transform and its inverse, and we recall that $\mathcal{F}, \mathcal{F}^{-1} \colon \mathrm{S}(\mathbb{R}^d) \to \mathrm{S}(\mathbb{R}^d)$ and $\mathcal{F}, \mathcal{F}^{-1} \colon \mathrm{S}'(\mathbb{R}^d) \to \mathrm{S}'(\mathbb{R}^d)$ are bounded.

As the following spaces are also defined via Fourier methods, it is natural to distinguish the situation of the whole space and to comment on the situation of more general domains $\Omega \subset \mathbb{R}^d$ in a second step.

(a) For $p \in (1, \infty)$ and $s \in \mathbb{R}$, we define the *Bessel potential space* $\mathrm{H}^{s,p}(\mathbb{R}^d)$ to be given by the elements in $\mathrm{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{\mathrm{H}^{s,p}(\mathbb{R}^d)} := \left\|\mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} f\right\|_{\mathrm{L}^p(\mathbb{R}^d)} < \infty.$$

(b) Let $p \in (1,\infty)$, $q \in [1,\infty]$, $s \in \mathbb{R}$, and consider a dyadic partition of unity $(\varphi_j)_{j\in\mathbb{N}}$. Then the *Besov spaces* $\mathrm{B}^s_{pq}(\mathbb{R}^d)$ are the elements in $\mathrm{S}'(\mathbb{R}^d)$ with $\|f\|_{\mathrm{B}^s_{pq}(\mathbb{R}^d)} < \infty$, where

$$\|f\|_{\mathrm{B}^s_{pq}(\mathbb{R}^d)} := \left(\sum_{j=0}^{\infty} 2^{jsq} \cdot \left\|\mathcal{F}^{-1}\varphi_j\mathcal{F}f\right\|_{\mathrm{L}^p(\mathbb{R}^d)}\right)^{1/q}, \quad \text{if } q < \infty, \text{ and}$$
$$\|f\|_{\mathrm{B}^s_{p\infty}(\mathbb{R}^d)} := \sup_{j\in\mathbb{N}_0} 2^{js} \cdot \left\|\mathcal{F}^{-1}\varphi_j\mathcal{F}f\right\|_{\mathrm{L}^p(\mathbb{R}^d)}, \quad \text{if } q = \infty.$$

(c) We are now in the position to introduce the *Sobolev-Slobodeckij spaces*. For $p \in (1,\infty)$ and $s \geq 0$, they are defined by

$$\mathrm{W}^{s,p}(\mathbb{R}^d) := \begin{cases} \mathrm{H}^{s,p}(\mathbb{R}^d), & \text{if } s \in \mathbb{N}_0, \\ \mathrm{B}^s_{pp}(\mathbb{R}^d), & \text{if } s \notin \mathbb{N}_0. \end{cases}$$

(d) The last class of function spaces to be presented in this context are the *Triebel-Lizorkin spaces* denoted by $\mathrm{F}^s_{pq}(\mathbb{R}^d)$. In fact, for $p \in (1,\infty)$, $q \in [1,\infty]$, $s \in \mathbb{R}$ and a dyadic partition of unity $(\varphi_j)_{j\in\mathbb{N}}$, these are the elements in $\mathrm{S}'(\mathbb{R}^d)$ such that $\|f\|_{\mathrm{F}^s_{pq}(\mathbb{R}^d)} < \infty$, with

$$\|f\|_{\mathrm{F}^s_{pq}(\mathbb{R}^d)} := \left\|\left(\sum_{j=0}^{\infty} 2^{jsq}\mathcal{F}^{-1}\varphi_j\mathcal{F}f\right)^{1/q}\right\|_{\mathrm{L}^p(\mathbb{R}^d)}, \quad \text{if } q < \infty, \text{ and}$$
$$\|f\|_{\mathrm{F}^s_{p\infty}(\mathbb{R}^d)} := \left\|\sup_{j\in\mathbb{N}_0} 2^{js}\mathcal{F}^{-1}\varphi_j\mathcal{F}f\right\|_{\mathrm{L}^p(\mathbb{R}^d)}, \quad \text{if } q = \infty.$$

For $s \in \mathbb{N}_0$, it is well known that the Bessel potential spaces $\mathrm{H}^{s,p}(\mathbb{R}^n)$ and thus also the Sobolev-Slobodeckij spaces $\mathrm{W}^{s,p}(\mathbb{R}^n)$ coincide with the usual Sobolev spaces as introduced above.

Next, we comment on the definition of the aforementioned function spaces on more general domains $\Omega \subset \mathbb{R}^d$.

(a) Let $p \in (1,\infty)$ as well as $s \in \mathbb{R}$. The Bessel potential space $\mathrm{H}^{s,p}(\Omega)$ is defined by restriction, so $\mathrm{H}^{s,p}(\Omega)$ is the restriction of $\mathrm{H}^{s,p}(\mathbb{R}^d)$ to Ω, and

$$\|f\|_{\mathrm{H}^{s,p}(\Omega)} = \inf_{g|_\Omega = f,\, g\in\mathrm{H}^{s,p}(\mathbb{R}^d)} \|g\|_{\mathrm{H}^{s,p}(\mathbb{R}^d)}.$$

(b) For $p \in (1,\infty)$, $q \in [1,\infty]$ and $s \in \mathbb{R}$, the Besov space $\mathrm{B}^s_{pq}(\Omega)$ is also defined via restriction to Ω, and the norm $\|f\|_{\mathrm{B}^s_{pq}(\Omega)}$ is obtained as in (a).

(c) For $p \in (1, \infty)$ and $s \geq 0$, we set

$$\mathrm{W}^{s,p}(\Omega) := \begin{cases} \mathrm{H}^{s,p}(\Omega), & \text{if } s \in \mathbb{N}_0, \\ \mathrm{B}^s_{pp}(\Omega), & \text{if } s \notin \mathbb{N}_0. \end{cases}$$

(d) Given $p \in (1, \infty)$, $q \in [1, \infty]$ as well as $s \in \mathbb{R}$, the Triebel-Lizorkin space $\mathrm{F}^s_{pq}(\Omega)$ is defined by restricting $\mathrm{F}^s_{pq}(\mathbb{R}^d)$ to Ω again, and the norm can be deduced as in (a).

It is known that for $s \in \mathbb{N}_0$, the Bessel potential spaces $\mathrm{H}^{s,p}(\Omega)$ and then also Sobolev-Slobodeckij spaces $\mathrm{W}^{s,p}(\Omega)$ are equal to the usual Sobolev spaces provided the domain Ω satisfies the so-called *uniform* C^1*-condition*. Instead of elaborating on this condition, we only mention that it is especially satisfied for bounded domains with C^2-boundary as usually considered throughout this thesis. In this context, we mention that $\Omega \subset \mathbb{R}^d$ *having a* C^k*-domain*, with $k \in \mathbb{N}_0$, means that $\partial\Omega$ can be locally represented as the graph of a C^k-diffeomorphism. More precisely, for all $x \in \partial\Omega$, there are $r_x > 0$ and a bijective map $\psi_x \colon \mathbb{B}_{\mathbb{R}^d}(x, r_x) \to D \subset \mathbb{R}^d$ so that

(a) $\psi_x(\mathbb{B}_{\mathbb{R}^d}(x, r_x) \cap \Omega) \to \mathbb{R}^{d-1} \times (0, \infty)$,

(b) $\psi_x(\mathbb{B}_{\mathbb{R}^d}(x, r_x) \cap \partial\Omega) \to \mathbb{R}^{d-1} \times \{0\}$, and

(c) $\psi_x \in \mathrm{C}^k(\mathbb{B}_{\mathbb{R}^d}(x, r_x))$ as well as $\psi_x^{-1} \in \mathrm{C}^k(D)$.

Moreover, let us observe that for $s \in \mathbb{R}$ and $p \in (1, \infty)$, the Bessel potential spaces $\mathrm{H}^{s,p}(\Omega)$ and the Triebel-Lizorkin spaces $\mathrm{F}^s_{p2}(\Omega)$ with parameter $q = 2$ coincide, so $\mathrm{H}^{s,p}(\Omega) = \mathrm{F}^s_{p2}(\Omega)$. In particular, for $m \in \mathbb{N}$, we have the relation

$$\mathrm{W}^{m,p}(\Omega) = \mathrm{F}^m_{p2}(\Omega). \tag{1.3}$$

The Trace and the Normal Derivative

We proceed with the trace operator as well as the normal derivative, and we mainly follow [130, Section 2.9 and Section 4.7] for this. In a first step, we focus on the situation of the trace acting on functions defined on the whole space $\mathbb{R}^d$ or half space $\mathbb{R}^d_+$, $d \geq 2$, and taking values in $\mathbb{R}^{d-1}$, while the focus in the second part is on bounded domains $\Omega \subset \mathbb{R}^d$ with sufficiently regular boundary. As in this thesis, we mainly investigate parabolic problems of second order, we will often consider Ω such that the boundary $\partial\Omega$ is of class C^2. Let us observe that in [130], it is assumed that $\partial\Omega$ is of class C^∞ for technical reasons, but the results carry over to the situation of less regular domains in case there exists a uniform extension operator. This is especially valid in the instances we are interested in.

We recall that for $p \in (1,\infty)$, $q \in [1,\infty]$ and $s \geq 0$, the spaces $\mathrm{H}^{s,p}(\mathbb{R}^d_+)$, $\mathrm{B}^s_{pq}(\mathbb{R}^d_+)$ and $\mathrm{W}^{s,p}(\mathbb{R}^d_+)$ are defined via restriction of the respective spaces on the whole space $\mathbb{R}^d$. In this case, for $x = (x', x_d) = (x_1, \dots, x_{d-1}, x_d)$, the *trace operator* γ and the *normal derivative* ∂_ν are defined by

$$\gamma f(x') := f(x', 0) \quad \text{and} \quad \partial_\nu f(x') := \frac{\partial}{\partial x_d} f(x', 0).$$

An operator R is referred to as *retraction* between Banach spaces X and Y if $R \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$, and if there exists a right-inverse $S \in \mathcal{L}(\mathrm{Y}, \mathrm{X})$, i. e., $RS = \mathrm{Id}$ on X. In this case, S is called *coretraction*. Note that a retraction is in particular surjective. For the result below on properties of the trace and the normal derivative on the whole space and the half space, we refer for example to [130, Theorem 2.9.3].

Lemma 1.3.2. *Let $p \in (1,\infty)$, $q \in [1,\infty]$.*

(a) If $s > 1/p$, then γ is a retraction
- *(i) from $\mathrm{H}^{s,p}(\mathbb{R}^d)$ and $\mathrm{H}^{s,p}(\mathbb{R}^d_+)$ onto $\mathrm{B}^{s-1/p}_{pp}(\mathbb{R}^{d-1})$,*
- *(ii) from $\mathrm{B}^s_{pq}(\mathbb{R}^d)$ and $\mathrm{B}^s_{pq}(\mathbb{R}^d_+)$ onto $\mathrm{B}^{s-1/p}_{pq}(\mathbb{R}^{d-1})$, and*
- *(iii) from $\mathrm{W}^{s,p}(\mathbb{R}^d)$ and $\mathrm{W}^{s,p}(\mathbb{R}^d_+)$ onto $\mathrm{W}^{s-1/p,p}(\mathbb{R}^{d-1})$.*

(b) In the case $s > 1 + 1/p$, the normal derivative ∂_ν is a retraction
- *(i) from $\mathrm{H}^{s,p}(\mathbb{R}^d)$ and $\mathrm{H}^{s,p}(\mathbb{R}^d_+)$ onto $\mathrm{B}^{s-1/p-1}_{pp}(\mathbb{R}^{d-1})$,*
- *(ii) from $\mathrm{B}^s_{pq}(\mathbb{R}^d)$ and $\mathrm{B}^s_{pq}(\mathbb{R}^d_+)$ onto $\mathrm{B}^{s-1/p-1}_{pq}(\mathbb{R}^{d-1})$, and*
- *(iii) from $\mathrm{W}^{s,p}(\mathbb{R}^d)$ and $\mathrm{W}^{s,p}(\mathbb{R}^d_+)$ onto $\mathrm{W}^{s-1/p-1,p}(\mathbb{R}^{d-1})$.*

The latter lemma also reveals that Besov and Sobolev-Slobodeckij spaces seem to be more “natural” spaces when dealing with traces and normal derivatives. As we shall see below, the same remains valid when considering bounded domains $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with sufficient regular domain. In particular, the result remains valid in the instances which we consider.

In the latter situation, for $p \in (1,\infty)$, $q \in [1,\infty]$ and $s \in \mathbb{R}$, the boundary spaces $\mathrm{H}^{s,p}(\partial\Omega)$ and $\mathrm{B}^s_{pq}(\partial\Omega)$ are defined via the respective spaces on $\mathbb{R}^{d-1}$ via localization and restriction, see for example [130, Definition 3.6.1] for more details. For $p \in (1,\infty)$ and $s \geq 0$, the Sobolev-Slobodeckij spaces $\mathrm{W}^{s,p}(\partial\Omega)$ can be set to coincide with $\mathrm{H}^{s,p}(\partial\Omega)$ for integer s, and to be equal to $\mathrm{B}^s_{pp}(\partial\Omega)$ for non-integer s as above. Moreover, the trace and the normal derivative are defined as the restriction to the boundary value and the derivative in the normal direction restricted to the boundary, respectively. This means that

$$\gamma f = f|_{\partial\Omega} \text{ as well as } \partial_\nu f = \frac{\partial f}{\partial \nu}\bigg|_{\partial\Omega}.$$

The result below can for example be found in [130, Theorem 4.7.1].

Lemma 1.3.3. *Consider a bounded domain $\Omega \subset \mathbb{R}^d$ with sufficiently smooth boundary, and let $p \in (1,\infty)$ as well as $q \in [1,\infty]$.*

(a) For $s > 1/p$, the trace γ is a retraction from $\mathrm{H}^{s,p}(\Omega)$ onto $\mathrm{B}^{s-1/p}_{pp}(\partial\Omega)$, from $\mathrm{B}^s_{pq}(\Omega)$ onto $\mathrm{B}^{s-1/p}_{pq}(\partial\Omega)$, and from $\mathrm{W}^{s,p}(\Omega)$ onto $\mathrm{W}^{s-1/p,p}(\partial\Omega)$.

(b) If $s > 1 + 1/p$, then ∂_ν is a retraction from $\mathrm{H}^{s,p}(\Omega)$ onto $\mathrm{B}^{s-1/p-1}_{pp}(\partial\Omega)$, from $\mathrm{B}^s_{pq}(\Omega)$ onto $\mathrm{B}^{s-1/p-1}_{pq}(\partial\Omega)$, and from $\mathrm{W}^{s,p}(\Omega)$ onto $\mathrm{W}^{s-1/p-1,p}(\partial\Omega)$.

In the sequel, we also use the subscripts $_\mathrm{D}$ and $_\mathrm{N}$ to denote function spaces with Dirichlet or Neumann boundary conditions on $\partial\Omega$, i. e., if $\gamma f = 0$ or if $\partial_\nu f = 0$, respectively. In addition, we fill also use $\mathrm{W}^{1,p}_0(\Omega)$ to denote the functions in $\mathrm{W}^{1,p}(\Omega)$ such that $\gamma f = 0$.

Function Spaces with Periodic Boundary Conditions

We also briefly elaborate on function spaces defined on $G = (0,1) \times (0,1)$ and subject to periodic boundary conditions on ∂G. A smooth function $f\colon \overline{G} \to \mathbb{R}$ is called *periodic of order m* if

$$\frac{\partial^\alpha}{\partial x^\alpha} f(0,y) = \frac{\partial^\alpha}{\partial x^\alpha} f(1,y) \text{ and } \frac{\partial^\alpha}{\partial y^\alpha} f(x,0) = \frac{\partial^\alpha}{\partial y^\alpha} f(x,1)$$

for all $\alpha = 0,\dots,m$. We then set

$$\mathrm{C}^\infty_{\mathrm{per}}(\overline{G}) := \Big\{ f \in \mathrm{C}^\infty(\overline{G}) : f \text{ is periodic of arbitrary order on } \partial G \Big\}.$$

Moreover, we define the *Sobolev spaces, Bessel potential spaces, Besov spaces and Sobolev-Slobodeckij spaces with periodic boundary conditions* as the respective closure of $\mathrm{C}^\infty_{\mathrm{per}}(\overline{G})$, so for $p \in (1,\infty)$, $q \in [1,\infty]$, $m \in \mathbb{N}_0$ and $s \in \mathbb{R}_+$, we define

$$\mathrm{W}^{m,p}_{\mathrm{per}}(G) := \overline{\mathrm{C}^\infty_{\mathrm{per}}(\overline{G})}^{\|\cdot\|_{\mathrm{W}^{m,p}}}, \quad \mathrm{H}^{s,p}_{\mathrm{per}}(G) := \overline{\mathrm{C}^\infty_{\mathrm{per}}(\overline{G})}^{\|\cdot\|_{\mathrm{H}^{s,p}}},$$
$$\mathrm{B}^s_{pq,per}(G) := \overline{\mathrm{C}^\infty_{\mathrm{per}}(\overline{G})}^{\|\cdot\|_{\mathrm{B}^s_{pq}}} \text{ and } \mathrm{W}^{s,p}_{\mathrm{per}}(G) := \overline{\mathrm{C}^\infty_{\mathrm{per}}(\overline{G})}^{\|\cdot\|_{\mathrm{W}^{s,p}}}.$$

The function spaces on cylindrical domains $\Omega = G \times (a, b)$ with periodic boundary conditions on the lateral boundary $\partial G \times (a, b)$, $-\infty < a < b < \infty$, are defined analogously, see also Section 2.7.

Interpolation and Embedding Relations

In the sequel, we unveil several interpolation and embedding relations of the spaces introduced before. The lemma below reveals that the Bessel potential spaces fit in the complex interpolation scale of the Lebesgue and Sobolev spaces, whereas the Besov spaces also arise as real interpolation spaces of Lebesgue and Sobolev spaces, and they fit in the complex interpolation scale. These relations can e. g. be deduced from [130, Section 2.4, 2.10.1 and 4.3.1].

Lemma 1.3.4. *Let $\Omega \subset \mathbb{R}^d$ denote the whole space, the half space, or a bounded domain with sufficiently regular boundary. Furthermore, consider the parameters $p \in (1, \infty)$, $q \in [1, \infty]$, $s \in \mathbb{R}$ and $\theta \in (0, 1)$. Then we have*

$$[\mathrm{L}^p(\Omega), \mathrm{W}^{m,p}(\Omega)]_\theta = \mathrm{H}^{\theta m,p}(\Omega), \quad (\mathrm{L}^p(\Omega), \mathrm{W}^{m,p}(\Omega))_{\theta,q} = \mathrm{B}^{\theta m}_{pq}(\Omega) \text{ and}$$
$$\left[\mathrm{B}^s_{pq}(\Omega), \mathrm{B}^{s+2}_{pq}(\Omega)\right]_\theta = \mathrm{B}^{s+2\theta}_{pq}(\Omega).$$

As a consequence of Lemma 1.2.4, applied to the projection onto $\mathrm{L}^p_0(\Omega)$ induced by (1.2), and Lemma 1.3.4, we especially obtain the following interpolation result.

Lemma 1.3.5. *For a bounded domain $\Omega \subset \mathbb{R}^d$ with sufficiently smooth boundary, $p, q \in (1, \infty)$, θ in $(0, 1)$, $m \in \mathbb{N}$ and $\mathrm{L}^p_0(\Omega)$ from (1.1), it holds that*

$$[\mathrm{L}^p_0(\Omega), \mathrm{W}^{m,p}(\Omega) \cap \mathrm{L}^p_0(\Omega)]_\theta = \mathrm{H}^{\theta m,p}(\Omega) \cap \mathrm{L}^p_0(\Omega) \text{ and}$$
$$(\mathrm{L}^p_0(\Omega), \mathrm{W}^{m,p}(\Omega) \cap \mathrm{L}^p_0(\Omega))_{\theta,q} = \mathrm{B}^{\theta m}_{pq}(\Omega) \cap \mathrm{L}^p_0(\Omega).$$

We are not only interested in the interpolation of the average zero condition as discussed in the previous lemma, but we are also inclined to find out more about the interpolation of boundary conditions. As we have already indicated, we also use the subscripts $_\mathrm{D}$ and $_\mathrm{N}$ to denote Dirichlet and Neumann boundary conditions. In that respect, for $s > 1/p$, we denote by $\mathrm{W}^{s,q}_\mathrm{D}(\Omega)$ the space of functions f in $\mathrm{W}^{s,q}(\Omega)$ such that $\gamma f = 0$, while $\mathrm{W}^{s,q}_\mathrm{N}(\Omega)$, $s > 1 + 1/p$, represents the functions f in $\mathrm{W}^{s,q}(\Omega)$ with $\partial_\nu f = 0$. The spaces $\mathrm{H}^{s,p}_\mathrm{D}(\Omega)$ and $\mathrm{H}^{s,p}_\mathrm{N}(\Omega)$ as well as $\mathrm{B}^s_{qp,\mathrm{D}}(\Omega)$ and $\mathrm{B}^s_{qp,\mathrm{N}}(\Omega)$ are defined analogously.

The result below can be deduced as a special instance of [4, Theorem 5.2]. For the (complex) interpolation of boundary conditions, we also refer to the article of Guidetti [56].

Lemma 1.3.6. *Consider a bounded domain $\Omega \subset \mathbb{R}^d$ with sufficiently regular boundary, and let $p \in (1,\infty)$, $q \in [1,\infty]$ and $\theta \in (0,1)$. Then*

$$[\mathrm{L}^p(\Omega), \mathrm{W}^{2,p}(\Omega) \cap \mathrm{W}_0^{1,p}(\Omega)]_\theta = \begin{cases} \mathrm{H}^{2\theta,p}(\Omega), & \text{if } \theta < 1/2p, \\ \mathrm{H}_{\mathrm{D}}^{2\theta,p}(\Omega), & \text{if } \theta > 1/2p, \end{cases}$$

$$[\mathrm{L}^p(\Omega), \mathrm{W}_{\mathrm{N}}^{2,p}(\Omega)]_\theta = \begin{cases} \mathrm{H}^{2\theta,p}(\Omega), & \text{if } \theta < 1/2 + 1/2p, \\ \mathrm{H}_{\mathrm{N}}^{2\theta,p}(\Omega), & \text{if } \theta > 1/2 + 1/2p, \end{cases}$$

and

$$(\mathrm{L}^p(\Omega), \mathrm{W}^{2,p}(\Omega) \cap \mathrm{W}_0^{1,p}(\Omega))_{\theta,q} = \begin{cases} \mathrm{B}_{pq}^{2\theta}(\Omega), & \text{if } \theta < 1/2p, \\ \mathrm{B}_{qp,\mathrm{D}}^{2\theta}(\Omega), & \text{if } \theta > 1/2p, \end{cases}$$

$$(\mathrm{L}^p(\Omega), \mathrm{W}_{\mathrm{N}}^{2,p}(\Omega))_{\theta,q} = \begin{cases} \mathrm{B}_{pq}^{2\theta}(\Omega), & \text{if } \theta < 1/2 + 1/2p, \\ \mathrm{B}_{pq,\mathrm{N}}^{2\theta}(\Omega), & \text{if } \theta > 1/2 + 1/2p. \end{cases}$$

In the above lemma, it is also reflected that the (strong) trace is only defined provided the regularity parameter exceeds $1/p$, whereas the normal derivative requires the regularity parameter to be larger than $1 + 1/p$. There are also certain ways to handle the situations of $\theta = 1/2q$ for Dirichlet boundary conditions and $\theta = 1/2 + 1/2q$ in the case of Neumann boundary conditions. However, we do not comment on these possibilities as this is not needed in the following.

Below, we provide a complex interpolation result of Besov spaces with boundary conditions, and this result can be obtained in a similar way as Lemma 1.3.6.

Lemma 1.3.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with sufficiently regular boundary, and consider $p \in (1,\infty)$, $q \in (1,\infty)$ and $s > 0$.*

If $s < 1/p$, then for all $\theta \in (0,1)$ with $\theta \neq 1/2p - s/2p$, we have

$$[\mathrm{B}_{pq}^s(\Omega), \mathrm{B}_{pq,\mathrm{D}}^{s+2}(\Omega)]_\theta = \begin{cases} \mathrm{B}_{pq}^{s+2\theta}(\Omega), & \text{if } s/2 + \theta < 1/2p, \\ \mathrm{B}_{pq,\mathrm{D}}^{s+2\theta}(\Omega), & \text{if } s/2 + \theta > 1/2p. \end{cases}$$

If $s < 1 + 1/p$, then for all $\theta \in (0,1)$ such that $\theta \neq 1/2 + 1/2p - s/2p$, we get

$$[\mathrm{B}_{pq}^s(\Omega), \mathrm{B}_{pq,\mathrm{N}}^{s+2}(\Omega)]_\theta = \begin{cases} \mathrm{B}_{pq}^{s+2\theta}(\Omega), & \text{if } s/2 + \theta < 1/2 + 1/2p, \\ \mathrm{B}_{pq,\mathrm{N}}^{s+2\theta}(\Omega), & \text{if } s/2 + \theta > 1/2 + 1/2p. \end{cases}$$

Let now $\Omega \subset \mathbb{R}^d$ be a bounded domain with a sufficiently smooth boundary. Again, we remark that $\partial\Omega \in \mathrm{C}^2$ as considered in this thesis is typically sufficient. For the following *embeddings*, we refer for example to [130, Theorem 4.6.1 and 4.6.2], and the relations of the Triebel-Lizorkin spaces can be obtained from [130, Section 2.3.2] upon invoking a suitable extension operator.

(a) For $p \in (1,\infty)$, $q_1, q_2 \in [1,\infty]$ with $q_1 \le q_2$, $s \in \mathbb{R}$ and $\varepsilon > 0$, we have

$$\mathrm{B}^{s+\varepsilon}_{p\infty}(\Omega) \hookrightarrow \mathrm{B}^{s}_{p1}(\Omega) \hookrightarrow \mathrm{B}^{s}_{pq_1}(\Omega) \hookrightarrow \mathrm{B}^{s}_{pq_2}(\Omega) \hookrightarrow \mathrm{B}^{s}_{p\infty}(\Omega). \tag{1.4}$$

(b) Let $p, q \in (1,\infty)$, $r \in [1,\infty]$ as well as $s, t \in \mathbb{R}$ such that $t \le s$, and consider $s - d/p \ge t - d/q$. Then

$$\mathrm{B}^{s}_{pr}(\Omega) \hookrightarrow \mathrm{B}^{t}_{qr}(\Omega), \ \ \mathrm{H}^{s,p}(\Omega) \hookrightarrow \mathrm{H}^{t,q}(\Omega), \ \ \mathrm{W}^{s,p}(\Omega) \hookrightarrow \mathrm{W}^{t,q}(\Omega), \tag{1.5}$$

and

$$\mathrm{H}^{s,p}(\Omega) \hookrightarrow \mathrm{B}^{t}_{qp}(\Omega) \ \text{ as well as } \ \mathrm{B}^{s}_{pq}(\Omega) \hookrightarrow \mathrm{H}^{t,q}(\Omega). \tag{1.6}$$

Moreover, if $s, t \ge 0$, by definition of the Sobolev-Slobodeckij spaces, we deduce from (1.5) and (1.6) that

$$\mathrm{W}^{s,p}(\Omega) \hookrightarrow \mathrm{B}^{t}_{qp}(\Omega) \ \text{ and } \ \mathrm{B}^{s}_{pq}(\Omega) \hookrightarrow \mathrm{W}^{t,q}(\Omega). \tag{1.7}$$

(c) Let $p \in (1,\infty)$, $r \in [1,\infty]$, $t \in \mathbb{R}_+$ and $s > t + d/p$. Then

$$\begin{aligned} \mathrm{B}^{s}_{pr}(\Omega) \hookrightarrow \mathrm{C}^{\lfloor t\rfloor, t-\lfloor t\rfloor}(\overline{\Omega}), \ \ \mathrm{H}^{s,p}(\Omega) \hookrightarrow \mathrm{C}^{\lfloor t\rfloor, t-\lfloor t\rfloor}(\overline{\Omega}) \ \text{ and} \\ \mathrm{W}^{s,p}(\Omega) \hookrightarrow \mathrm{C}^{\lfloor t\rfloor, t-\lfloor t\rfloor}(\overline{\Omega}), \end{aligned} \tag{1.8}$$

where we identify $\mathrm{C}^{t,0}(\overline{\Omega})$ with $\mathrm{C}^{t}(\overline{\Omega})$. Moreover, if $t > 0$ is non-integer, then (1.8) is also valid for $s = t + d/p$.

(d) For $p \in (1,\infty)$, $q_1, q_2 \in [1,\infty]$, $s \in \mathbb{R}$ and $\varepsilon > 0$, we have

$$\mathrm{F}^{s+\varepsilon}_{pq_0}(\Omega) \hookrightarrow \mathrm{F}^{s}_{pq_1}(\Omega). \tag{1.9}$$

Below, we state a result on the Banach algebra structure of Besov spaces for which we refer to [64, Lemma 5.2], see also [8, Cor. 2.86 and Prop. 2.39].

Lemma 1.3.8. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary of class C^2, and consider $s > 0$ as well as $p \in (1,\infty)$ and $q \in [1,\infty)$. If $s > d/p$, or, in the case $q = 1$ even for $s \ge d/p$, then $\mathrm{B}^{s}_{pq}(\Omega)$ is a Banach algebra.*

In particular, in the situation that $p = q$, we obtain the Banach algebra property of the Sobolev-Slobodeckij space $\mathrm{W}^{s,p}(\Omega)$ provided $s > d/p$.

Vector-Valued Function Spaces

With regard to the parabolic setting, vector-valued function spaces come into picture very naturally. These spaces rely on the *Bochner integral.* More precisely, consider an interval $J \subset \mathbb{R}$, a Banach space X and a measurable function $f\colon J \to \mathrm{X}$. It is well known that such a function is Bochner integrable if and only if it is measurable and $\|f\|_{\mathrm{X}}$ is integrable, i. e., the integral

$$\int_J \|f(t)\|_{\mathrm{X}} \,\mathrm{d}t$$

exists. More information on the Bochner integral and vector-valued distributions can for example be found in [6, Chapter 1]. For $p \in [1, \infty)$, the *vector-valued Lebesgue space* $\mathrm{L}^p(J; \mathrm{X})$ is defined to be the space of all measurable functions $f\colon J \to \mathrm{X}$ with

$$\|f\|_{\mathrm{L}^p(J;\mathrm{X})} \coloneqq \left(\int_J \|f(t)\|_{\mathrm{X}}^p \,\mathrm{d}t\right)^{1/p} < \infty.$$

The space $\mathrm{L}^\infty(J; \mathrm{X})$ consists of all measurable functions $f\colon J \to \mathrm{X}$ with

$$\|f\|_{\mathrm{L}^\infty(J;\mathrm{X})} \coloneqq \operatorname*{ess\,sup}_{t\in J} \|f(t)\|_{\mathrm{X}} < \infty.$$

The aforementioned spaces become Banach spaces with the common identification of functions coinciding almost everywhere on J. In an analogous manner, for $k \in \mathbb{N}_0$, $p \in [1, \infty]$ and $f^{(j)}$ representing the j-th weak derivative, we define the *vector-valued Sobolev spaces* $\mathrm{W}^{k,p}(J; \mathrm{X})$ by

$$\mathrm{W}^{k,p}(J;\mathrm{X}) \coloneqq \left\{ f \in \mathrm{W}^{k,1}_{\mathrm{loc}}(J;\mathrm{X}) : f^{(j)} \in \mathrm{L}^p(J;\mathrm{X}), \text{ for } j = 0, \dots, k \right\}, \text{ with}$$

$$\|f\|_{\mathrm{W}^{k,p}(J;\mathrm{X})} \coloneqq \left(\sum_{j=0}^{k} \|f^{(j)}\|_{\mathrm{L}^p(J;\mathrm{X})}^p \right)^{1/p}.$$

For $p \in (1, \infty)$ and $s \in \mathbb{N}$, the *vector-valued Bessel potential spaces* $\mathrm{H}^{s,p}(J; \mathrm{X})$ and *vector-valued Sobolev-Slobodeckij spaces* $\mathrm{W}^{s,p}(J; \mathrm{X})$ are defined as the corresponding Sobolev space, while for $s \notin \mathbb{N}$, they are introduced via complex and real interpolation. In fact, for $s = \lfloor s \rfloor + s_*$ and $p \in (1, \infty)$, we set

$$\mathrm{H}^{s,p}(J;\mathrm{X}) \coloneqq [\mathrm{W}^{\lfloor s \rfloor,p}(J;\mathrm{X}), \mathrm{W}^{\lfloor s \rfloor+1,p}(J;\mathrm{X})]_{s_*} \text{ and}$$
$$\mathrm{W}^{s,p}(J;\mathrm{X}) \coloneqq (\mathrm{W}^{\lfloor s \rfloor,p}(J;\mathrm{X}), \mathrm{W}^{\lfloor s \rfloor+1,p}(J;\mathrm{X}))_{s_*,p}.$$

Below, we comment on the Banach algebra property of some of the vector-valued Sobolev spaces. The lemma can be deduced from Lemma 1.3.8 together with the Sobolev embedding $\mathrm{W}^{1,p}(J) \hookrightarrow \mathrm{L}^\infty(J)$ for a finite interval $J \subset \mathbb{R}$ and the well-known Banach algebra structure of the space L^∞.

Lemma 1.3.9. *Consider a bounded interval* $J \subset \mathbb{R}$ *as well as a bounded domain* $\Omega \subset \mathbb{R}^d$ *with boundary of class* C^2*. Then for* $p \in (1,\infty)$ *and* $q > d$*, the spaces* $\mathrm{W}^{1,p}(J;\mathrm{W}^{1,q}(\Omega))$ *and* $\mathrm{L}^\infty(J;\mathrm{W}^{1,q}(\Omega))$ *are Banach algebras.*

Time Weights

Next, in order to allow for a larger class of initial data in the general framework, and to fully exploit the parabolic regularization, we introduce *weighted vector-valued Lebesgue* and *Sobolev spaces.* Consider a Banach space X as well as $p \in (1,\infty)$ and $\mu \in (1/p, 1]$. We define

$$\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X}) \coloneqq \left\{ f\colon \mathbb{R}_+ \to \mathrm{X} : t^{1-\mu} f \in \mathrm{L}^p(\mathbb{R}_+;\mathrm{X}) \right\}, \quad \text{with}$$
$$\|f\|_{\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})} \coloneqq \left(\int_0^\infty \|t^{1-\mu} f(t)\|_{\mathrm{X}}^p \,\mathrm{d}t \right)^{1/p},$$

rendering $\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})$ a Banach space. Moreover, we set

$$\mathrm{W}^{1,p}_\mu(\mathbb{R}_+;\mathrm{X}) \coloneqq \left\{ f \in \mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X}) \cap \mathrm{W}^{1,1}_{\mathrm{loc}}((0,\infty);\mathrm{X}) : \partial_t f \in \mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X}) \right\} \quad \text{and}$$
$$\|f\|_{\mathrm{W}^{1,p}_\mu(\mathbb{R}_+;\mathrm{X})} \coloneqq \left(\|f\|^p_{\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})} + \|\partial_t f\|^p_{\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})} \right)^{1/p}.$$

Equipped with the above norm, $\mathrm{W}^{1,p}_\mu(\mathbb{R}_+;\mathrm{X})$ also becomes a Banach space. The above definitions can be adapted easily to finite time intervals of the shape $J = (0,T)$, $0 < T < \infty$.

Function Spaces on Time-Dependent Domains

When facing moving domain problems, it is necessary to introduce function spaces on time-dependent domains. Indeed, let $(0,T)$, with $T \in (0,\infty]$, and for $t \in (0,T)$, consider $\Omega(t) \subset \mathbb{R}^d$. We define

$$\Omega_T \coloneqq \{(t,x) : t \in (0,T),\ x \in \Omega(t)\} \ \text{ and } \ \Omega_0 \coloneqq \Omega(0).$$

Moreover, let $X\colon \Omega_T \to \Omega_0$ be a map such that

$$\varphi\colon \Omega_T \to (0,T) \times \Omega_0, \ \ (t,x) \mapsto (t, X(t,x))$$

is a C^1-diffeomorphism, and $X(\tau,\cdot)\colon \Omega(\tau) \to \Omega_0$ are C^2-diffeomorphisms for all $\tau \in [0,T]$. For $p,q \in (1,\infty)$, $s \in \{0,1\}$ and $l \in \{0,1,2\}$, we then define the *function spaces on time-dependent domains* by

$$\mathrm{W}^{s,p}(0,T;\mathrm{W}^{l,q}(\Omega(\cdot))) \coloneqq \left\{ f(t,\cdot)\colon \Omega(t) \to \mathbb{R} : f \circ \varphi \in \mathrm{W}^{s,p}(0,T;\mathrm{W}^{l,q}(\Omega_0)) \right\}.$$

1.4. Further Analytical Tools

The last section of this chapter is dedicated to collecting further useful analytical tools such as the Poincaré and Korn inequality or the Rellich-Kondrachov theorem. We start with the classical *Poincaré inequality* allowing for an estimate of a Sobolev function with homogeneous Dirichlet boundary conditions by its gradient, and we refer for instance to [2, Theorem 6.30].

Lemma 1.4.1 (Poincaré's inequality)**.** *Consider a bounded domain* $\Omega \subset \mathbb{R}^d$ *with Lipschitz boundary. Then there exists a constant* $C > 0$ *such that*

$$\|f\|_{\mathrm{L}^p(\Omega)} \le C \cdot \|\nabla f\|_{\mathrm{L}^p(\Omega)}$$

for all $f \in \mathrm{W}_0^{1,p}(\Omega)$*, i. e., for all* $f \in \mathrm{W}^{1,q}(\Omega)$ *with* $\gamma f = 0$*.*

For a sufficiently smooth function f, we define the symmetric part of the gradient $\varepsilon = \varepsilon(f)$ by

$$\varepsilon := \frac{1}{2}\left(\nabla f + (\nabla f)^\top\right).$$

It readily follows that the L^p-norm of ε can be estimated by the L^p-norm of ∇f. *Korn's inequality* asserts that the converse is also true provided we take into account functions with homogeneous Dirichlet boundary conditions. For a more thorough discussion of Korn's inequality and variants of it under different assumptions, we also refer to [1, Section 3.1] and the references therein.

Lemma 1.4.2 (Korn's inequality)**.** *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain with Lipschitz boundary. Then there is a constant* $C > 0$ *with*

$$\|\nabla f\|_{\mathrm{L}^p(\Omega)} \le C \cdot \|\varepsilon(f)\|_{\mathrm{L}^p(\Omega)}$$

for all $f \in \mathrm{W}_0^{1,p}(\Omega)$*.*

The following result on compactness of Sobolev embeddings also proves powerful, especially with regard to the spectral analysis of differential operators on bounded domains. It is known as the *Rellich-Kondrachov theorem*, see for example [2, Theorem 6.3].

Lemma 1.4.3 (Rellich-Kondrachov theorem)**.** *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain with Lipschitz boundary, and consider* $p, q \in [1, \infty)$*,* $j \in \mathbb{N}_0$ *and* $m \in \mathbb{N}$*.*

(a) If $d > mp$ *as well as* $q < {}^{dp}/_{(d-mp)}$*, we obtain the compactness of the embedding* $\mathrm{W}^{j+m,p}(\Omega) \hookrightarrow \mathrm{W}^{j,q}(\Omega)$*.*

(b) If $mp \ge d$*, then the embedding* $\mathrm{W}^{j+m,p}(\Omega) \hookrightarrow \mathrm{W}^{j,q}(\Omega)$ *is even compact for all* $q \in [1, \infty)$*.*

CHAPTER 2

Abstract Theory

In this chapter, we collect some abstract theory that will be used throughout this thesis. Generally, we only recall definitions and state results. However, we also provide proofs if the results seem to be new or can be deduced directly from previous results. In Section 2.1, we first introduce the concept of sectorial operators, discuss the connection with the generators of analytic semigroups and deal with so-called trace spaces as the suitable spaces for the initial data of abstract Cauchy problems. The later parts of this section are centered around maximal L^p-regularity, including the definition, related properties and the characterization by the $\mathcal{R}$-boundedness of the resolvents due to Weis [132]. Section 2.2 is about maximal periodic regularity with characterizations due to Arendt and Bu [7]. Another topic in this section is the Da Prato-Grisvard theorem on maximal regularity in real interpolation spaces together with a time periodic version. In Section 2.3, we invoke the concept of operators with bounded imaginary powers and the bounded $\mathcal{H}^\infty$-calculus, and we also present several applications such as fractional power domains or the bounded $\mathcal{H}^\infty$-calculus of block operator matrices and relations of these properties. Section 2.4 is dedicated to establishing embeddings of the parabolic spaces, also based on the previous concepts. Elliptic and parabolic boundary value problems are the focal point of Section 2.5. We present an approach to quasilinear evolution equations by maximal regularity in the Section 2.6. To be more precise, we discuss the local strong well-posedness and also elaborate on the global strong well-posedness close to equilibria. In the last section of this chapter, Section 2.7, we recall the viscous primitive equations, and a special focal point is the corresponding hydrostatic Stokes operator.

The most important references for this chapter are the memoir of Denk, Hieber and Prüss [37] as well as the monograph of Prüss and Simonett [115].

Throughout this chapter, denote by X a Banach space with norm $\|\cdot\|_{\mathrm{X}}$. For simplicity, we will mostly denote the latter by $\|\cdot\|$. For a closed linear operator A on X, the space X_A represents the domain of A equipped with the graph norm, so

$$\mathrm{X}_A = (\mathrm{D}(A), \|\cdot\|_A), \quad \text{where } \|x\|_A := \|x\| + \|Ax\|. \tag{2.1}$$

We also recall the sector in the complex plane Σ_θ, with $\theta \in (0, \pi]$ from Section 1.1 as it will appear frequently in this chapter.

2.1. Sectorial Operators, Abstract Cauchy Problems and Maximal Regularity

This section is dedicated to the introduction of *sectorial operators*, to the presentation of so-called *trace spaces* for abstract Cauchy problems as well as to the concept of *maximal* L^p*-regularity*.

Sectorial Operators and Analytic Semigroups

In the following, we define the notion of a *sectorial operator*.

Definition 2.1.1. *A closed linear operator* $A\colon \mathrm{D}(A) \subset \mathrm{X} \to \mathrm{X}$ *on a Banach space* X *is called* sectorial *if*

- *(a) it satisfies* $\overline{\mathrm{D}(A)} = \overline{\mathrm{R}(A)} = \mathrm{X}$*, i. e., it is densely defined and has dense range, as well as* $(-\infty, 0) \subset \rho(A)$*, and*
- *(b) there is* $M \in (0, \infty)$ *such that* $\|t(t + A)^{-1}\|_{\mathcal{L}(\mathrm{X})} \le M$ *for all* $t > 0$.

We denote the class of sectorial operators by $\mathcal{S}(\mathrm{X})$*. If only* $(-\infty, 0) \subset \rho(A)$ *and (b) are valid, then we refer to* A *as* pseudo-sectorial*, and* $\mathcal{PS}(\mathrm{X})$ *represents the corresponding class of operators. An operator* $A \in \mathcal{PS}(\mathrm{X})$ *has the property that* $\rho(-A) \supset \Sigma_\theta$ *for some* $\theta > 0$*, and it follows that*

$$\sup\left\{\|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(\mathrm{X})} : |\arg \lambda| < \theta\right\} < \infty.$$

Hence, the spectral angle ϕ_A *of* $A \in \mathcal{PS}(\mathrm{X})$ *given by*

$$\phi_A := \inf \left\{ \phi : \rho(-A) \supset \Sigma_{\pi-\phi}, \sup_{\lambda \in \Sigma_{\pi-\phi}} \|\lambda(\lambda + A)^{-1}\| < \infty \right\}$$

is well-defined.

It turns out that injectivity of a pseudo-sectorial operator is sufficient for sectoriality provided X is reflexive. For a proof of the lemma below, we refer to [115, Theorem 3.1.2].

Lemma 2.1.2. *For a reflexive Banach space* X *as well as* $A \in \mathcal{PS}(\mathrm{X})$ *such that* $\mathrm{N}(A) = \{0\}$*, we have* $A \in \mathcal{S}(\mathrm{X})$*.*

In the following, we recall the notion of an analytic semigroup and investigate the relation of (pseudo-)sectorial operators and the generators of analytic semigroups. We start with the definition of C_0-semigroups and (bounded) analytic semigroups.

Definition 2.1.3. *We call a family of operators* $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(\mathrm{X})$ *a* semigroup *provided* $T(t+s) = T(t)T(s)$ *holds for* $t, s > 0$*, and* $T(0) = \mathrm{Id}$*.*
The semigroup is referred to as C_0-semigroup *if it also holds that*

$$\lim_{t\to 0+} T(t)x = x, \quad \text{for } x \in \mathrm{X}.$$

The semigroup is called analytic *of angle* $\theta \in (0, \pi/2]$ *if it has an analytic extension to the sector* Σ_θ *which is bounded on*

$$\Sigma_{\theta'} \cap \{z \in \mathbb{C} : |z| \leq 1\} \quad \text{for all } \theta' \in (0, \theta).$$

Besides, the semigroup is bounded analytic *of angle* $\theta \in (0, \pi/2]$ *if* T *has a bounded analytic extension to* $\Sigma_{\theta'}$ *for all* $\theta' \in (0, \theta)$*.*

The result below sheds light on the link of generators of bounded analytic semigroup and (pseudo-)sectorial operators. We refer e. g. to [115, Theorem 3.3.2].

Lemma 2.1.4. *Let* A *be a closed densely defined operator on a Banach space* X*. Then* $A \in \mathcal{PS}(\mathrm{X})$ *with spectral angle* $\phi_A < \pi/2$ *if and only if* $-A$ *generates a bounded analytic semigroup of angle* $\pi/2 - \phi_A$*.*

In particular, if $A \in \mathcal{S}(\mathrm{X})$ *with* $\phi_A < \pi/2$*, then* $-A$ *generates a bounded analytic semigroup of angle* $\pi/2 - \phi_A$*.*

For a linear operator A on X, we define the *spectral bound* $s(A)$ by

$$s(A) \coloneqq \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}.$$

Given a C_0-semigroup $\{T(t)\}_{t\geq 0}$, the *growth bound* ω_{sg} is defined by

$$\omega_{\mathrm{sg}} \coloneqq \inf\left\{\omega \in \mathbb{R} : \exists M_\omega \geq 1 \text{ with } \|T(t)\|_{\mathcal{L}(\mathrm{X})} \leq M_\omega \mathrm{e}^{\omega t}\,\forall t \geq 0\right\}.$$

It readily follows that the growth bound of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ and the spectral bound $s(A)$ of its generator satisfy the relation $s(A) \leq \omega_{\mathrm{sg}}$, see for example [42, Corollary II.1.13]. Even more can be deduced in the particular situation of analytic semigroups. More precisely, we obtain the following *spectral mapping theorem* for analytic semigroups, for which we refer for instance to [42, Corollary IV.3.12].

Lemma 2.1.5. *Let $\{T(t)\}_{t\geq 0}$ be an analytic semigroup with generator A. Then the* spectral mapping theorem

$$\mathrm{e}^{t\sigma(A)} = \sigma(T(t)) \setminus \{0\}, \quad \textit{for } t \geq 0,$$

holds true. Moreover, the spectral bound $s(A)$ of A and the growth bound ω_{sg} of the semigroup $\{T(t)\}_{t\geq 0}$ coincide, i. e., $s(A) = \omega_{\mathrm{sg}}$.

In the sequel, we investigate operators with compact resolvents. The purpose of the following lemma is twofold. On the one hand, we provide a handy characterization of this property whose proof can e. g. be found in [42, Proposition II.4.25]. On the other hand, we collect useful properties of operators with compact resolvent. The assertion in (b) on the spectrum only consisting of eigenvalues is classical and can be found in [42, Corollary IV.1.19] for instance, while (c) follows from [35, Theorem 1.6.1 and Corollary 1.6.7].

Lemma 2.1.6. *Let $A\colon \mathrm{D}(A) \subset \mathrm{X} \to \mathrm{X}$ be an operator with $\rho(A) \neq \emptyset$, and consider X_A as introduced in* (2.1)*.*

(a) The operator A has compact resolvent*, i. e., the resolvent $R(\lambda, A)$ is compact for one (and thus for all) $\lambda \in \rho(A)$, if and only if the canonical injection $\mathrm{X}_A \hookrightarrow \mathrm{X}$ is compact.*

(b) If the operator A has compact resolvent, then

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda - A \ \textit{is not injective}\},$$

so all spectral values are eigenvalues.

(c) Let $\Omega \subset \mathbb{R}^d$ be open, and for $p \in [1,\infty)$, assume that there are consistent C_0-*semigroups* $\{T_p(t)\}_{t\geq 0} \in \mathcal{L}(\mathrm{L}^p(\Omega))$, *i. e.,*

$$T_p(t)f = T_q(t)f, \text{ for } f \in \mathrm{L}^p(\Omega) \cap \mathrm{L}^q(\Omega) \text{ and } t > 0,$$

and with $T_2(t) = T(t)$. Moreover, denote by A_p the associated generators. If A_2 has compact resolvent, then $\sigma(A_2) = \sigma(A_p)$ for $p \in (1,\infty)$, so the spectrum is p-independent.

We shall not be concerned about the term of consistent C_0-semigroup appearing in Lemma 2.1.6(c), because the objects studied in this thesis typically satisfy this property.

Trace Spaces and Real Interpolation

Next, we investigate the so-called *time trace spaces* and their relation with real interpolation spaces. For the generator $-A$ of a bounded analytic semigroup $\{T(t)\}_{t\geq 0}$ with $T(t) = \mathrm{e}^{-tA}$ on the Banach space X, we consider the *homogeneous abstract Cauchy problem*

$$\begin{cases} u'(t) + Au(t) = 0, & \text{for } t \in \mathbb{R}_+, \\ u(0) = x, \end{cases} \tag{2.2}$$

in X. The solution $u(t)$ to (2.2) is given by $\mathrm{e}^{-tA}x$. A natural question is which additional properties $x \in \mathrm{X}$ has to satisfy to guarantee $u(t) \in \mathrm{D}(A)$ for almost all $t > 0$ as well as $Au \in \mathrm{L}^p(\mathbb{R}_+;\mathrm{X})$. This explains the term *trace space* or *time trace space*.

The following definition gives rise to a candidate for the trace space. The assumption on A being pseudo-sectorial with spectral angle $\phi_A < \pi/2$ is equivalent to the generation of a bounded analytic semigroup with regard to Lemma 2.1.4.

Definition 2.1.7. *Consider a densely defined operator $A \in \mathcal{PS}(\mathrm{X})$ with spectral angle $\phi_A < \pi/2$. For $\theta \in (0,1)$ and $p \in [1,\infty)$, the spaces $\mathrm{D}_A(\theta,p)$, defined by*

$$\mathrm{D}_A(\theta,p) := \left\{ x \in \mathrm{X} : [x]_{\theta,p} := \left(\int_0^\infty \left\| t^{1-\theta} A\mathrm{e}^{-tA}x \right\|_{\mathrm{X}}^p \frac{\mathrm{d}t}{t} \right)^{1/p} < \infty \right\}, \tag{2.3}$$

and endowed with the norm $\|x\|_{\theta,p} := \|x\| + [x]_{\theta,p}$, are referred to as trace spaces. *The choice of the norm renders them Banach spaces.*

The following lemma reveals that the trace spaces introduced in Definition 2.1.7 are indeed natural spaces for the initial data when asking for the above regularity properties of the solution to the initial value problem (2.2). The result is classical, and we refer for instance to [115, Proposition 3.4.2]. For later use, we include the time-weighted setting.

Lemma 2.1.8. *Let $A \in \mathcal{S}(\mathrm{X})$ with spectral angle $\phi_A < \pi/2$ be invertible, consider $p \in (1, \infty)$ as well as $\mu \in (1/p, 1]$, and recall the weighted Lebesgue and Sobolev spaces from Section 1.3. Then for the solution u of (2.2), the following assertions are equivalent:*

(a) $u(t) \in \mathrm{D}(A)$ *for almost all* $t > 0$*, and* $u \in \mathrm{L}^p_\mu(\mathbb{R}_+; \mathrm{X}_A)$*,*

(b) $u \in \mathrm{W}^{1,p}_\mu(\mathbb{R}_+; \mathrm{X})$*, and*

(c) $x \in \mathrm{D}_A(\mu - 1/p, p)$.

In this case, there is a constant $C > 0$ such that

$$\|u'\|_{\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})} + \|Au\|_{\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})} \leq C \cdot \|x\|_{\mu-1/p,p}$$

for all $x \in \mathrm{D}_A(\mu - 1/p, p)$.

Below, we provide a useful characterization of the time trace space by a real interpolation space. This result is well known, see e. g. [99, Proposition 2.2.2].

Lemma 2.1.9. *Let $A \in \mathcal{PS}(\mathrm{X})$ be densely defined with $\phi_A < \pi/2$, and consider $\theta \in (0, 1)$ and $p \in [1, \infty)$. Then $\mathrm{D}_A(\theta, p) = (\mathrm{X}, \mathrm{X}_A)_{\theta,p}$, and $\| \cdot \|_{\theta,p}$ as well as $\| \cdot \|_{(\mathrm{X},\mathrm{X}_A)_{\theta,p}}$ are equivalent.*

If the underlying operator is invertible, then $[\cdot]_{\theta,p}$ from Definition 2.1.7 already gives rise to an equivalent norm of the real interpolation space. For this result, we refer to [58, Corollary 6.5.5].

Lemma 2.1.10. *Let $A \in \mathcal{S}(\mathrm{X})$ with spectral angle $\phi_A < \pi/2$ be invertible. Then for $\theta \in (0, 1)$ and $p \in [1, \infty)$, the space $\mathrm{D}_A(\theta, p)$ endowed with $[\cdot]_{\theta,p}$ is a Banach space, and it coincides with the real interpolation space $(\mathrm{X}, \mathrm{X}_A)_{\theta,p}$ with equivalent norms.*

Maximal Regularity

Next, we elaborate on the concept of maximal L^p-regularity. To this end, for a time interval $J = \mathbb{R}_+$ or $J = (0, a)$ with $a > 0$, let $f \colon J \to X$, and consider the inhomogeneous initial value problem

$$\begin{cases} u'(t) + Au(t) = f(t), & \text{for } t \in J, \\ \qquad\quad u(0) = u_0, \end{cases} \tag{2.4}$$

in $\mathrm{L}^p(J; \mathrm{X})$ for $p \in (1, \infty)$. The term *maximal* $\mathrm{L}^p(J)$*-regularity* for (2.4) is defined as follows.

Definition 2.1.11. *Let $A \colon \mathrm{D}(A) \subset \mathrm{X} \to \mathrm{X}$ be a closed and densely defined operator. Then we say that there is* maximal $\mathrm{L}^p(J)$-regularity *for* (2.4) *if for every $f \in \mathrm{L}^p(J; \mathrm{X})$, there is a unique $u \in \mathrm{W}^{1,p}(J; \mathrm{X}) \cap \mathrm{L}^p(J; \mathrm{X}_A)$ which satisfies* (2.4) *almost everywhere in J with $u_0 = 0$. In this case, A is said to belong to the class $\mathcal{MR}_p(J; \mathrm{X})$. For simplicity, we use the notation $\mathcal{MR}_p(\mathrm{X}) := \mathcal{MR}_p(\mathbb{R}_+; \mathrm{X})$. Unless specified otherwise,* maximal L^p-regularity *means maximal $\mathrm{L}^p(\mathbb{R}_+)$-regularity in the sequel.*

If there is maximal $\mathrm{L}^p(J)$-regularity for (2.4)*, then the closed graph theorem yields the existence of a constant $C > 0$ with*

$$\|u\|_{\mathrm{L}^p(J;\mathrm{X})} + \|u'\|_{\mathrm{L}^p(J;\mathrm{X})} + \|Au\|_{\mathrm{L}^p(J;\mathrm{X})} \le C \cdot \|f\|_{\mathrm{L}^p(J;\mathrm{X})}$$

for all $f \in \mathrm{L}^p(J; \mathrm{X})$. Additionally invoking Lemma 2.1.8 in the case $\mu = 1$ for the situation of inhomogeneous initial values, we even get

$$\|u\|_{\mathrm{L}^p(J;\mathrm{X})} + \|u'\|_{\mathrm{L}^p(J;\mathrm{X})} + \|Au\|_{\mathrm{L}^p(J;\mathrm{X})} \le C \cdot \Big(\|u_0\|_{\mathrm{D}_A(1-1/p,p)} + \|f\|_{\mathrm{L}^p(J;\mathrm{X})} \Big).$$

The lemma below discusses properties of operators in the class $\mathcal{MR}_p(J; \mathrm{X})$. In particular, it shows that invertibility is even necessary for $A \in \mathcal{MR}_p(\mathrm{X})$. We refer e. g. to [113, Proposition 1.2] or also [115, Proposition 3.5.2].

Lemma 2.1.12. *Let $A \in \mathcal{MR}_p(J; \mathrm{X})$ for some $p \in (1, \infty)$. Then*

(a) if $J = (0, a)$, there is $\omega \ge 0$ such that $\omega + A$ is sectorial with spectral angle $\phi_A < \pi/2$, and

(b) if $J = \mathbb{R}_+$, the operator A is sectorial with spectral angle $\phi_A < \pi/2$ as well as $0 \in \rho(A)$.

In view of Lemma 2.1.12, the property of maximal L^p-regularity on $\mathbb{R}_+$ is quite restrictive. We thus introduce a weaker notion which we will also refer to as *maximal regularity of* L^p*-type.*

Definition 2.1.13. *Let* $A\colon \mathrm{D}(A) \subset X \to X$ *be a closed and densely defined operator. Then we say that there is* maximal regularity of L^p-type *for* (2.4) *if for every* $f \in \mathrm{L}^p(\mathbb{R}_+;\mathrm{X})$*, there is a unique* $u \in \mathrm{C}(\mathbb{R}_+;\mathrm{X})$ *with* $u' \in \mathrm{L}^p(\mathbb{R}_+;\mathrm{X})$ *as well as* $Au \in \mathrm{L}^p(\mathbb{R}_+;\mathrm{X})$*, and solving* (2.4) *a. e. in* $\mathbb{R}_+$ *with* $u_0 = 0$*. Moreover, A is said to belong to the class* ${}_0\mathcal{MR}_p(\mathrm{X})$*. By the closed graph theorem, there is a constant* $C > 0$ *such that for all* $f \in \mathrm{L}^p(\mathbb{R}_+;\mathrm{X})$*, we have*

$$\|u'\|_{\mathrm{L}^p(\mathbb{R}_+;\mathrm{X})} + \|Au\|_{\mathrm{L}^p(\mathbb{R}_+;\mathrm{X})} \le C \cdot \|f\|_{\mathrm{L}^p(\mathbb{R}_+;\mathrm{X})}.$$

With regard to Lemma 2.1.8, we also obtain the estimate

$$\|u'\|_{\mathrm{L}^p(\mathbb{R}_+;\mathrm{X})} + \|Au\|_{\mathrm{L}^p(\mathbb{R}_+;\mathrm{X})} \le C \cdot \Big(\|u_0\|_{\mathrm{D}_A(1-1/p,p)} + \|f\|_{\mathrm{L}^p(\mathbb{R}_+;\mathrm{X})}\Big)$$

in the case of inhomogeneous initial values.

The next lemma can be found in [113, Corollary 1.3] or [115, Corollary 3.5.3] and establishes a link between the classes $\mathcal{MR}_p(\mathrm{X})$ and ${}_0\mathcal{MR}_p(\mathrm{X})$.

Lemma 2.1.14. *Consider* $A \in {}_0\mathcal{MR}_p(\mathrm{X})$*. Then A is pseudo-sectorial in* X *with spectral angle* $\Phi_A < \pi/2$*. Moreover, we have* $A \in \mathcal{MR}_p(\mathrm{X})$ *if and only if it holds that* $A \in {}_0\mathcal{MR}_p(\mathrm{X})$ *and* $0 \in \rho(A)$*.*

The preceding Lemma 2.1.14 also reveals that given $A \in {}_0\mathcal{MR}(\mathrm{X})$, we get maximal L^p-regularity by shifting the operator A to achieve invertibility. The next lemma takes a closer look at the shift and the possibility to omit it if invertibility is known. We refer here to the discussion in [115, Section 6.3.4] for the general idea of the proof.

Lemma 2.1.15. *Suppose that there is* $\omega_0 \in \mathbb{R}$ *such that for all* $\omega > \omega_0$*, we have* $A+\omega \in {}_0\mathcal{MR}_p(\mathrm{X})$*, and consider the spectral bound* $s(-A)$ *of the negative operator* $-A$*. Then it follows that* $A + \omega \in {}_0\mathcal{MR}_p(\mathrm{X})$ *for all* $\omega > s(-A)$*. Especially, if* $0 \in \rho(A)$*, then* $A \in \mathcal{MR}_p(\mathrm{X}) \subset {}_0\mathcal{MR}_p(\mathrm{X})$*.*

$\mathcal{R}$-Boundedness

We now introduce the notion of $\mathcal{R}$-boundedness of operator families. This concept is of fundamental importance with regard to the characterization of maximal regularity, and it has first been introduced by Bourgain [16].

Definition 2.1.16. *Consider Banach spaces* X *and* Y*. We refer to a family of operators* $\mathcal{T}(\mathrm{X}.\mathrm{Y})$ *as* $\mathcal{R}$-bounded *if there are* $C > 0$ *and* $p \in [1,\infty)$ *such* *that for all* $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in \mathrm{X}$, *and for every independent, symmetric,* $\{-1, 1\}$*-valued random variable* ε_j *on a probability space* $(\Omega, \mathcal{A}, \mu)$*, we have*

$$\left\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \right\|_{\mathrm{L}^p(\Omega;\mathrm{Y})} \leq C \cdot \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|_{\mathrm{L}^p(\Omega;\mathrm{X})}.$$

Moreover, the smallest $C > 0$ *for which the inequality holds true is called the* $\mathcal{R}$-bound *of* $\mathcal{T}$*, and it is denoted by* $\mathcal{R}(\mathcal{T})$*.*

Let us remark that $\mathcal{R}$-boundedness of an operator family is a stronger property than uniform boundedness. However, if X and Y are Hilbert spaces, then both terms coincide, see e. g. [37, Remark 3.1]. For particular instances of X and Y such as $\mathrm{X} = \mathrm{Y} = \mathrm{L}^q(G)$ for an open set $G \subset \mathbb{R}^d$, the $\mathcal{R}$-boundedness admits more handy characterizations, see for instance [115, Section 4.1].

Banach Spaces of Class $\mathcal{HT}$ and UMD Spaces

Next, we discuss two equivalent properties of Banach spaces, and these notions are of prime importance in applications of the maximal regularity. For the definition of the first property, we require the so-called *Hilbert transform* H. It is defined by

$$Hu(t) := \lim_{R\to\infty} \int_{R^{-1}\leq|s|\leq R} f(t-s)\, \mathrm{d}s/\pi s. \tag{2.5}$$

We now introduce Banach spaces of class $\mathcal{HT}$.

Definition 2.1.17. *A Banach space* X *is said to be* of class $\mathcal{HT}$ *if the Hilbert transform* H *as defined in* (2.5) *is bounded on* $\mathrm{L}^2(\mathbb{R};\mathrm{X})$*.*

In the sequel, we invoke unconditional martingale spaces, or, in short, UMD spaces.

Definition 2.1.18. *The Banach space* X *is called* unconditional martingale space*, which we abbreviate by the term* UMD space*, if for every* $p \in (1,\infty)$*, there is* $C_p > 0$ *so that for any* X*-valued martingale* $(f_k)_{k\geq 0}$ *on a probability space* $(\Omega, \mathcal{A}, \mu)$*,* $N \in \mathbb{N}$*, and each choice of signs* $(\varepsilon_n)_{n\in\mathbb{N}} \subset \{-1, 1\}$*, we have*

$$\left\| f_0 + \sum_{k=1}^{N} \varepsilon_k (f_k - f_{k-1}) \right\|_{\mathrm{L}^p(\Omega;\mathrm{X})} \leq C_p \cdot \|f_N\|_{\mathrm{L}^p(\Omega;\mathrm{X})}.$$

It is well known that the latter two properties are equivalent, see for instance [16] or [22]. For further details, we also refer to the survey article of Burkholder [23]. In view of the equivalence of both notions, we only use the term UMD space in the remainder of this thesis.

In the following, we recall useful properties of UMD spaces from [5, Theorem III.4.5.2].

Lemma 2.1.19. *(a) Every Banach space which is isomorphic to a UMD space is a UMD space.*

(b) Every Hilbert space is a UMD space.

(c) Every finite-dimensional Banach space is a UMD space.

(d) Finite products of UMD spaces are UMD spaces.

(e) For a UMD space X *as well as a* σ*-finite measure space* (Ω, μ)*, the space* $\mathrm{L}^p(\Omega, \mu; \mathrm{X})$ *is a UMD space for* $p \in (1, \infty)$*.*

(f) For an interpolation couple $(\mathrm{X}_0, \mathrm{X}_1)$ *of UMD spaces, the spaces* $[\mathrm{X}_0, \mathrm{X}_1]_\theta$ *and* $(\mathrm{X}_0, \mathrm{X}_1)_{\theta,p}$ *are UMD spaces for* $\theta \in (0, 1)$ *and* $p \in (1, \infty)$*.*

(g) Closed linear subspaces of UMD spaces are UMD spaces.

We will use the above properties of UMD spaces freely in the sequel. In particular, we often make use of the properties (c), (d) and (e) without explicitly mentioning them.

$\mathcal{R}$-Sectorial Operators and Maximal L^p-Regularity

We are approaching the characterization of the maximal L^p-regularity. The last ingredient is the subsequent notion of $\mathcal{R}$-sectoriality, which is a stronger property than the usual sectoriality from Definition 2.1.1, since it requires the $\mathcal{R}$-boundedness of the resolvents.

Definition 2.1.20. *We say that a sectorial operator* A *is* $\mathcal{R}$-sectorial *provided*

$$\mathcal{R}_A(0) := \mathcal{R}\left\{t(t + A)^{-1} : t > 0\right\} < \infty.$$

Moreover, the $\mathcal{R}$-angle $\phi_A^{\mathcal{R}}$ *of an* $\mathcal{R}$*-sectorial operator* A *is defined by*

$$\begin{aligned} \phi_A^{\mathcal{R}} &:= \inf\left\{\theta \in (0, \pi) : \mathcal{R}_A(\pi - \theta) < \infty\right\}, \quad \textit{where} \\ \mathcal{R}_A(\theta) &:= \mathcal{R}\left\{\lambda(\lambda + A)^{-1} : |\arg \lambda| \leq \theta\right\}. \end{aligned}$$

We denote the class of $\mathcal{R}$*-sectorial operators by* $\mathcal{RS}(\mathrm{X})$*.*

The next proposition asserts the celebrated characterization of ${}_0\mathcal{MR}_p$ from Definition 2.1.13 by $\mathcal{R}$-sectoriality. It is due to Weis [132]. Let us also refer to [37, Theorem 4.4], where the result is stated in terms of $\mathcal{R}$-sectoriality as below.

Proposition 2.1.21 ([132, Theorem 4.2])**.** *Let* X *be a UMD space,* $p \in (1,\infty)$*, and consider* $A \in \mathcal{S}(\mathrm{X})$ *with spectral angle* $\phi_A < \pi/2$*. Then* $A \in {}_0\mathcal{MR}_p(\mathrm{X})$ *if and only if* $A \in \mathcal{RS}(\mathrm{X})$ *with* $\mathcal{R}$*-angle* $\phi_A^{\mathcal{R}} < \pi/2$*.*

We remark that Proposition 2.1.21 provides a characterization of the weaker maximal regularity of L^p-type. However, if there is further information on the spectral bound of the operator, or by employing a shift, the usual maximal L^p-regularity can be deduced from there, see Lemma 2.1.14 or Lemma 2.1.15.

A Glimpse of Perturbation Theory

A nice feature of the notion of $\mathcal{R}$-sectoriality, and thus also of the maximal regularity of L^p-type in the situation of UMD spaces, is its behavior under perturbation. This is the next focal point.

We consider the situation of an $\mathcal{R}$-sectorial operator perturbed by a relatively bounded operator. For $A\colon \mathrm{D}(A) \subset \mathrm{X} \to \mathrm{X}$ and $B\colon \mathrm{D}(B) \subset \mathrm{X} \to \mathrm{X}$, we say that B *is relatively* A*-bounded* if there exist α, $\beta \in \mathbb{R}_+$ such that

$$\mathrm{D}(A) \subset \mathrm{D}(B) \text{ and } \|Bx\| \le \alpha \cdot \|Ax\| + \beta \cdot \|x\|, \text{ for all } x \in \mathrm{D}(A). \tag{2.6}$$

The $\mathcal{R}$-sectoriality is preserved provided the relative bound is sufficiently small and a shift is taken into consideration. This is precisely the assertion of the following lemma, which has been obtained in [37, Proposition 4.3] or [84, Corollary 2].

Lemma 2.1.22. *Let* A *be a sectorial operator on* X*, consider a linear operator* B *such that* (2.6) *holds for some* α, $\beta \in \mathbb{R}_+$*, assume*

$$\mathcal{R}\left\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_\theta\right\} =: a < \infty,$$

and set $C_A := \sup_{\mu>0} \left\|A(\mu + A)^{-1}\right\|$ *and* $M_A := \sup_{\mu>0} \left\|\mu(\mu + A)^{-1}\right\|$*. If it is valid that* $\alpha < 1/((1+a)C_A)$ *and* $\mu > (\beta M_A(1+a))/(1-\alpha C_A(1+a))$*, then*

$$\mathcal{R}\left\{\lambda(\lambda + \mu + A + B)^{-1} : \lambda \in \Sigma_\theta\right\} < \infty.$$

The latter abstract result calls for an application in the context of maximal L^p-regularity. The corollary below can be deduced from Lemma 2.1.22 together with Proposition 2.1.21 and Lemma 2.1.14.

Corollary 2.1.23. *Let* X *be a Banach space with the UMD property, consider an operator* $A \in {}_0\mathcal{MR}_p(\mathrm{X})$*, and suppose that for every* $\alpha > 0$*, there is* $\beta \geq 0$ *so that* A *and* B *fulfill* (2.6)*. Then there is* $\omega \geq 0$ *with* $A + B + \omega \in \mathcal{MR}_p(\mathrm{X})$*.*

Maximal Regularity in the Weighted Setting

We only provide this section for completeness. In fact, the result stated in this section is not used explicitly in this thesis, but it is underlying in the general framework in Section 2.6.

First, we generalize the notion of maximal L^p-regularity to the situation of weighted spaces. To this end, for $f\colon \mathbb{R} \to \mathrm{X}$, let us recall the inhomogeneous initial value problem

$$\begin{cases} u'(t) + Au(t) = f(t), & \text{for } t \in \mathbb{R}_+, \\ \qquad\qquad u(0) = u_0. \end{cases} \tag{2.7}$$

With regard to Definition 2.1.11, the definition of maximal L^p_μ-regularity below is natural.

Definition 2.1.24. *Consider* $A\colon \mathrm{D}(A) \subset \mathrm{X} \to \mathrm{X}$ *closed and densely defined. We say that* (2.7) *has* maximal L^p_μ-regularity *if for every* $f \in \mathrm{L}^p_\mu(\mathrm{X})$*, there exists a unique* $u \in \mathrm{W}^{1,p}_\mu(\mathbb{R}_+;\mathrm{X}) \cap \mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X}_A)$ *solving* (2.7) *almost everywhere in* $\mathbb{R}_+$ *with* $u_0 = 0$*. We denote the class of such operators by* $\mathcal{MR}_{p,\mu}(\mathrm{X})$*.*

Furthermore, the closed graph theorem implies the existence of some constant $C > 0$ *such that for all* $f \in \mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})$

$$\|u\|_{\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})} + \|u'\|_{\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})} + \|Au\|_{\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})} \leq C \cdot \|f\|_{\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})}.$$

As a consequence of Lemma 2.1.8, we also obtain the estimate

$$\|u\|_{\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})} + \|u'\|_{\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})} + \|Au\|_{\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})} \leq C \cdot \Big(\|u_0\|_{\mathrm{D}_A(\mu - 1/p, p)} + \|f\|_{\mathrm{L}^p_\mu(\mathbb{R}_+;\mathrm{X})}\Big)$$

for non-homogeneous initial data.

The next lemma reveals the equivalence of maximal L^p_μ-regularity with usual maximal L^p-regularity. It is due to Prüss and Simonett [114, Theorem 2.4].

Lemma 2.1.25. *For $p \in (1, \infty)$ and $\mu \in (1/p, 1]$, it holds that $A \in \mathcal{MR}_p(\mathrm{X})$ if and only if $A \in \mathcal{MR}_{p,\mu}(\mathrm{X})$.*

As already revealed in Definition 2.1.24, the "correct" space for the initial data, i. e., the *trace space* in the present weighted setting, takes the shape $\mathrm{D}_A(\mu - 1/p, p)$. From Lemma 2.1.9, we recall that it coincides with the real interpolation space $(\mathrm{X}, \mathrm{X}_A)_{\mu - 1/p, p}$ with equivalent norms provided the underlying operator is pseudo-sectorial with spectral angle $\phi_A < \pi/2$.

2.2. Maximal Periodic Regularity and Maximal Regularity in Real Interpolation Spaces

In this section, we are interested in *maximal periodic* L^p*-regularity.* In particular, we provide characterizations of this property due to Arendt and Bu [7]. We also discuss maximal regularity in real interpolation spaces via the Da Prato-Grisvard theorem [32] and mention a time periodic version.

Maximal Periodic Regularity - the Arendt-Bu Theorem

Let X be a Banach space, and consider a linear operator $A\colon \mathrm{D}(A) \subset \mathrm{X} \to \mathrm{X}$. For $f \in \mathrm{L}^p(0, 2\pi; \mathrm{X})$, we investigate the *time periodic abstract Cauchy problem*

$$\begin{cases} u'(t) + Au(t) = f(t), & \text{for } t \in (0, 2\pi), \\ u(0) = u(2\pi). \end{cases} \tag{2.8}$$

We introduce the notion of *maximal periodic* L^p*-regularity.*

Definition 2.2.1. *Let $A\colon \mathrm{D}(A) \subset \mathrm{X} \to \mathrm{X}$ be closed and densely defined, and $p \in (1, \infty)$. Then A admits* maximal periodic L^p-regularity *if for every $f \in \mathrm{L}^p(0, 2\pi; \mathrm{X})$, there is a unique solution $u \in \mathrm{W}^{1,p}(0, 2\pi; \mathrm{X}) \cap \mathrm{L}^p(0, 2\pi; \mathrm{X}_A)$ to* (2.8)*. We denote the class of operators with maximal periodic L^p-regularity by $\mathcal{MR}_{\mathrm{per},p}(\mathrm{X})$. Moreover, the closed graph theorem yields the existence of a constant $C > 0$ such that for all $f \in \mathrm{L}^p(0, 2\pi; \mathrm{X})$, we get*

$$\|u\|_{\mathrm{L}^p(0,2\pi;\mathrm{X})} + \|u'\|_{\mathrm{L}^p(0,2\pi;\mathrm{X})} + \|Au\|_{\mathrm{L}^p(0,2\pi;\mathrm{X})} \le C \cdot \|f\|_{\mathrm{L}^p(0,2\pi;\mathrm{X})}.$$

By a rescaling argument, we can also consider a general time interval $(0, T)$ for $0 < T < \infty$ in (2.8) and Definition 2.2.1. However, in order to keep the notation as simple as possible, we concentrate on the interval $(0, 2\pi)$ in this section.

The characterization of the maximal periodic L^p-regularity below can be regarded as a periodic analogue of the famous Weis theorem as recalled in Proposition 2.1.21. It was established by Arendt and Bu [7].

Proposition 2.2.2 ([7, Theorem 2.3])**.** *Let* X *be a UMD Banach space, and consider* $p \in (1, \infty)$*. Then* $A \in \mathcal{MR}_{\mathrm{per},p}(\mathrm{X})$ *if and only if* $\mathrm{i}\mathbb{Z} \subset \rho(-A)$*, and the sequence* $(kR(\mathrm{i}k, -A))_{k\in\mathbb{Z}}$ *is* $\mathcal{R}$*-bounded.*

Interestingly, even a characterization of the maximal periodic L^p-regularity in terms of the usual maximal $\mathrm{L}^p(0, 2\pi)$-regularity as introduced in Definition 2.1.11 can be obtained. The following result will be referred to as the *Arendt-Bu theorem.*

Proposition 2.2.3 ([7, Theorem 5.1])**.** *Consider a Banach space* X *and the generator of a* C_0*-semigroup* $-A\colon \mathrm{D}(A) \subset \mathrm{X} \to \mathrm{X}$*, and let* $p \in (1, \infty)$*. Then we have* $A \in \mathcal{MR}_{\mathrm{per},p}(\mathrm{X})$*, i. e.,* A *admits maximal periodic* L^p*-regularity, if and only if* $A \in \mathcal{MR}_p(0, 2\pi; \mathrm{X})$ *and* $1 \in \rho(\mathrm{e}^{-2\pi A})$*.*

The equivalent condition for the maximal periodic L^p-regularity means that A has maximal $\mathrm{L}^p(0, 2\pi)$-regularity, and the semigroup satisfies the spectral condition. Let us provide an equivalent property of the latter spectral condition of the semigroup. In fact, Lemma 2.1.12 and the maximal L^p-regularity on $(0, 2\pi)$ yield the sectoriality of A with spectral angle $\phi_A < \pi/2$, up to a shift. Lemma 2.1.4 then implies that the operator $-A$, possibly subject to a shift, generates a bounded analytic semigroup. This means that $-A$ generates an analytic semigroup, see for instance [6, Proposition 3.7.4]. Hence, from the spectral mapping theorem for analytic semigroups as stated in Lemma 2.1.5, it follows that $1 \in \rho(\mathrm{e}^{-2\pi A})$ if and only if $0 \in \rho(A)$.

We summarize the preceding discussion in the corollary below on the maximal periodic L^p-regularity and also invoke Lemma 2.1.15 and Lemma 2.1.14.

Corollary 2.2.4. *Suppose that* $A \in {}_0\mathcal{MR}_p(X)$*, and assume that* $0 \in \rho(A)$*. Then we have* $A \in \mathcal{MR}_{\mathrm{per},p}(\mathrm{X})$*.*

Maximal Regularity in Real Interpolation Spaces - the Da Prato-Grisvard Theorem and a Time Periodic Version

Now, we recall the *Da Prato-Grisvard theorem* on the maximal regularity in real interpolation spaces for sectorial and invertible underlying linear operators. This is a way to circumvent the task of showing $\mathcal{R}$-boundedness of the resolvents. Another significant aspect is that maximal L^1-regularity can be obtained as well.

Let us also recall from Lemma 2.1.9 the equality of $\mathrm{D}_A(\theta, p)$ and the real interpolation space $(\mathrm{X}, \mathrm{X}_A)_{\theta,p}$, justifying the term *maximal regularity in real interpolation spaces*.

Proposition 2.2.5 (Da Prato-Grisvard [32])**.** *Let $A \in \mathcal{S}(\mathrm{X})$ with spectral angle $\phi_A < \pi/2$ be invertible, and let $\theta \in (0,1)$ as well as $p \in [1,\infty)$. Then we have $A \in \mathcal{MR}_p(\mathrm{D}_A(\theta,p))$.*

Below, we discuss a time periodic version of the aforementioned Da Prato-Grisvard theorem. It can be viewed as a counterpart to the Arendt-Bu theorem presented in Proposition 2.2.3. For convenience, we also briefly introduce the setting. For a pseudo-sectorial and densely defined operator A with spectral angle $\phi_A < \pi/2$, the trace space $\mathrm{D}_A(\theta,p)$, for $\theta \in (0,1)$ and $p \in [1,\infty)$, a fixed time period $T > 0$ and $f \colon \mathbb{R} \to \mathrm{D}_A(\theta,p)$ T-periodic, we consider the *time periodic abstract Cauchy problem on the whole real line*

$$\begin{cases} u'(t) + Au(t) = f(t), & \text{for } t \in \mathbb{R}, \\ \qquad\qquad u(t) = u(t+T), & \text{for } t \in \mathbb{R}. \end{cases} \tag{2.9}$$

From a formal perspective, a candidate for a solution to (2.9) is

$$u(t) \coloneqq \int_{-\infty}^{t} \mathrm{e}^{-(t-s)A} f(s) \,\mathrm{d}s. \tag{2.10}$$

To shorten notation, we set $\mathbb{E}_{0,\theta} \coloneqq \mathrm{L}^p(0,T;\mathrm{D}_A(\theta,p))$, and $\mathbb{E}_{1,\theta}$ is defined by

$$\mathbb{E}_{1,\theta} \coloneqq \Big\{ u \in \mathrm{W}^{1,p}(0,T;\mathrm{D}_A(\theta,p)) : Au \in \mathrm{L}^p(0,T;\mathrm{D}_A(\theta,p)),\ \ u(0) = u(T) \Big\}.$$

The following result has been obtained in [64, Section 2].

Proposition 2.2.6. *Let $A \in \mathcal{S}(\mathrm{X})$ with spectral angle $\phi_A < \pi/2$ be invertible. Moreover, let $\theta \in (0,1)$ as well as $p \in [1,\infty)$, and consider a fixed time period $T > 0$ as well as $f \colon \mathbb{R} \to \mathrm{D}_A(\theta,p)$ with $f|_{(0,T)} \in \mathbb{E}_{0,\theta}$. Then u from (2.10) is the unique, strong solution to (2.9), meaning that u is the unique T-periodic function in $\mathrm{C}(\mathbb{R};\mathrm{X})$ which is differentiable for almost every $t \in \mathbb{R}$, and it fulfills $u \in \mathrm{D}(A)$, $Au \in \mathrm{L}^p(0,T;\mathrm{X})$ and solves $u'(t) + Au(t) = f(t)$. Furthermore, there exists a constant $C > 0$ such that for all $f \in \mathbb{E}_{0,\theta}$, we have*

$$\|u\|_{\mathbb{E}_{1,\theta}} \le C \cdot \|f\|_{\mathbb{E}_{0,\theta}}. \tag{2.11}$$

2.3. Bounded Imaginary Powers and Bounded $\mathcal{H}^\infty$-Calculus

In this section, we introduce the concept of *bounded imaginary powers* together with its applications to fractional power domains and interpolation-extrapolation scales. Moreover, we discuss the so-called *bounded $\mathcal{H}^\infty$-calculus* and point out relations with the concepts from Section 2.1.

Bounded Imaginary Powers and Banach Scales

The term of bounded imaginary powers of a sectorial operator is classical. As a preparation, for $z \in \mathbb{C}$, we consider the functions $h_z(\lambda) = \lambda^z$ and observe that they are holomorphic on the sector Σ_π, with estimate

$$|h_z(\lambda)| \leq |\lambda|^{\operatorname{Re} z} \mathrm{e}^{\phi |\operatorname{Im} z|}, \text{ for } \lambda \in \Sigma_\phi.$$

This justifies that the extended functional calculus as described in detail in [115, Section 3.3.2] and also introduced later in this section can be applied, leading to the following result. We also refer to [37, Proposition 2.2].

Lemma 2.3.1. *Let $A \in \mathcal{S}(\mathrm{X})$, and define A^z by $A^z := h_z(A)$. Then*

(a) $A^z x$ is holomorphic on the strip $|\operatorname{Re} z| < k$ for all $x \in \mathrm{D}(A^k) \cap \mathrm{R}(A^k)$,

(b) A^z is closed for every $z \in \mathbb{C}$,

(c) $A^{z+w}x = A^s A^w x$ for all z, $w \in \mathbb{C}$ and $x \in \mathrm{D}(A^k) \cap \mathrm{R}(A^k)$, where $k > |\operatorname{Re} z|$, $|\operatorname{Re} w|$, $|\operatorname{Re}(z+w)|$, and

(d) $A^z x = \lim_{\varepsilon \to 0} A^z_\varepsilon x$ for $x \in \mathrm{D}(A^k) \cap \mathrm{R}(A^k)$ as well as $|\operatorname{Re} z| \leq k$, with A^z_ε defined by $A_\varepsilon = (\varepsilon + A)(1 + \varepsilon A)^{-1}$.

The class $\mathcal{BIP}(\mathrm{X})$ defined below has first been considered by Prüss and Sohr [118]. It is well-defined thanks to Lemma 2.3.1.

Definition 2.3.2. *Let $A \in \mathcal{S}(\mathrm{X})$. Then A has* bounded imaginary powers *on* X *provided $A^{\mathrm{i}s} \in \mathcal{L}(\mathrm{X})$ is valid for all $s \in \mathbb{R}$, and there is a constant $C > 0$ with $\|A^{\mathrm{i}s}\|_{\mathcal{L}(\mathrm{X})} \leq C$ for $|s| \leq 1$.*

By Lemma 2.3.1, $A^{\mathrm{i}s}$ has the group property, and the bounded imaginary powers of A are thus equivalent to $\{A^{\mathrm{i}s}\}_{s \in \mathbb{R}}$ being a strongly continuous group *of bounded linear operators in* X*. The growth bound* θ_A *of this group is referred to as* power angle *of* A*, and it is given by*

$$\theta_A = \limsup_{|s|\to\infty} \frac{1}{|s|} \log |A^{\mathrm{i}s}|.$$

We observe that for every $\omega > \theta_A$, there exists a constant $M \geq 1$ with

$$\|A^{\mathrm{i}t}\|_{\mathcal{L}(\mathrm{X})} \leq M\mathrm{e}^{\omega|t|}, \quad \text{for } t \in \mathbb{R}.$$

The class of operators with bounded imaginary powers proves particularly useful when studying the so-called *fractional power spaces.*

Definition 2.3.3. *Let* $A \in \mathcal{S}(\mathrm{X})$ *as well as* $\alpha \in (0, \infty)$*. We define the space* X_{A^α} *by* $\mathrm{X}_{A^\alpha} := (\mathrm{D}(A^\alpha), \|\cdot\|_\alpha)$*, with* $\|x\|_\alpha := \|x\| + \|A^\alpha\|$*, and we also call this the* fractional power scale generated by (X, A)*.*

A proof of the following result on the characterization of the fractional power spaces in terms of complex interpolation spaces for the class $\mathcal{BIP}$ can e. g. be found in [130, pp. 103–104].

Lemma 2.3.4. *Let* $A \in \mathcal{BIP}(\mathrm{X})$*, and recall the spaces* X_A *and* X_{A^θ} *from* (2.1) *and Definition 2.3.3, respectively. Then for* $\theta \in (0, 1)$ *and* $[\cdot,\cdot]_\theta$ *denoting the complex interpolation functor, it holds that* $\mathrm{X}_{A^\theta} \cong [\mathrm{X}, \mathrm{X}_A]_\theta$*.*

Another important application of the theory of operators with bounded imaginary powers concerns the so-called *interpolation-extrapolation scales.* In the sequel, we follow the monograph of Amann [5, Section V.1.5] and [116, Appendix A.1] for the introduction of this concept. We start with the underlying notion of a *Banach scale.* Let $I \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}_+, \mathbb{R}\}$, and for every $\alpha \in I$, consider a Banach space $\mathrm{X}_\alpha := (\mathrm{X}_\alpha, \|\cdot\|_\alpha)$ and a linear isomorphism $A_\alpha \colon \mathrm{X}_{\alpha+1} \to \mathrm{X}_\alpha$.

Definition 2.3.5. *We call* $[(\mathrm{X}_\alpha, A_\alpha) : \alpha \in I]$ *a* Banach scale *provided the operator* $A_\alpha \colon \mathrm{X}_{\alpha+1} \to \mathrm{X}_\alpha$ *is a linear isomorphism for every* $\alpha \in I$*, it holds that* $\mathrm{X}_\alpha \hookrightarrow \mathrm{X}_\beta$ *for every pair* $\alpha, \beta \in I$ *with* $\alpha > \beta$*, and the diagram*

$$\begin{array}{ccccccc}
\dots & \hookrightarrow & \mathrm{X}_{\alpha+1} & \hookrightarrow & \mathrm{X}_{\beta+1} & \hookrightarrow & \dots \\
 & & \downarrow{\scriptstyle A_\alpha} & & \downarrow{\scriptstyle A_\beta} & & \\
\dots & \hookrightarrow & \mathrm{X}_\alpha & \hookrightarrow & \mathrm{X}_\beta & \hookrightarrow & \dots
\end{array}$$

is commutative.

Let now X be a Banach space, and consider a closed linear and densely defined operator A on X with $0 \in \rho(A)$. Moreover, for $j \in \mathbb{N}$, denote by X_j the domain of the operator A^j endowed with the norm $\|A^j \cdot\|$ which is in turn equivalent to the graph norm of A^j. This allows us to introduce A_j as the X_j-realization of the operator A. By virtue of [5, Theorem V.1.2.1], the family $[(\mathrm{X}_j, A_j) : j \in \mathbb{N}]$ is a Banach scale, so the definition below makes sense.

Definition 2.3.6. *For a closed linear and densely defined operator A on* X *with $0 \in \rho(A)$, consider $\mathrm{X}_j = (\mathrm{D}(A^j), \|A^j \cdot\|)$, and denote by A_j the X_j-realization of A. Then the Banach scale $[(\mathrm{X}_j, A_j) : j \in \mathbb{N}]$ is called* power scale generated by (X, A).

For a more thorough discussion of the fractional power scale as well as the extrapolation scale, we also refer to [5, Sections V.1.2 and V.1.3]. Now, consider a closed linear densely defined operator $A\colon \mathrm{X}_1 \to \mathrm{X}_0$ on a reflexive Banach space. The next notion is well-defined thanks to [5, Theorem V.1.5.1].

Definition 2.3.7. *Let $A\colon \mathrm{X}_1 \to \mathrm{X}_0$ denote a closed linear densely defined operator on a reflexive Banach space. For $[\cdot,\cdot]_\theta$, $\theta \in (0,1)$, denoting the complex interpolation functor, the pair (A, X_0) gives rise to the Banach scale $(\mathrm{X}_\alpha, A_\alpha)$, where $\mathrm{X}_\alpha \coloneqq [\mathrm{X}_j, \mathrm{X}_{j+1}]_{\alpha-j}$ for $j < \alpha < j+1$, and A_α is the X_α-realization of A. More precisely, for $\alpha \geq 0$, A_α represents the maximal restriction of A to X_α, while for $\alpha < 0$, the operator A_α denotes the closure of A in X_α.*

The Banach scale $(\mathrm{X}_\alpha, A_\alpha)$ is referred to as the interpolation-extrapolation scale generated by (X_0, A) and $[\cdot,\cdot]_\theta$.

The following lemma demonstrates the strength of the bounded imaginary powers of an operator, and it is implied directly by [5, Theorem V.1.5.4].

Lemma 2.3.8. *Under the assumptions on A as in Definition 2.3.7, suppose in addition that $A \in \mathcal{BIP}(\mathrm{X}_0)$. Then the interpolation-extrapolation scale $(\mathrm{X}_\alpha, A_\alpha)$ introduced in Definition 2.3.7 is equivalent to the fractional power scale generated by (X_0, A).*

In particular, recalling X_{A^α} from Definition 2.3.3, for $\alpha > 0$, we find the equality $\mathrm{X}_\alpha = \mathrm{X}_{A^\alpha}$ up to norm equivalence.

Another important feature of the class of operators with bounded imaginary powers is that this property is preserved in the interpolation-extrapolation scale generated by the complex interpolation functor. A proof of this result can be found in [5, Proposition V.1.5.5]

Lemma 2.3.9. *Let $A \in \mathcal{BIP}(\mathrm{X}_0)$ be a densely defined linear operator on a reflexive Banach space X_0 with $0 \in \rho(A)$ and power angle $\theta_A < \pi$, and recall the interpolation-extrapolation scale $(\mathrm{X}_\alpha, A_\alpha)$ from Definition 2.3.7. Then it holds that $A_\alpha \in \mathcal{BIP}(\mathrm{X}_\alpha)$ with power angle $\theta_{A_\alpha} = \theta_A < \pi$.*

Let us briefly comment on the invertibility assumption on A. If $0 \notin \rho(A)$, then it is possible to choose $\omega > 0$ such that $0 \in \rho(A + \omega)$ and to investigate the interpolation-extrapolation scale generated by $A + \omega$ instead.

We conclude this section with a lemma resulting from Lemma 2.3.4 in conjunction with Lemma 2.3.8 and Lemma 2.3.9.

Lemma 2.3.10. *Let $A \in \mathcal{BIP}(\mathrm{X}_0)$, and take into account the interpolation-extrapolation scale $(\mathrm{X}_\alpha, A_\alpha)$ generated by (X_0, A), or, in the case $0 \notin \rho(A)$, by $(\mathrm{X}_0, A+\omega)$ for $\omega > 0$ sufficiently large. For $k \in \mathbb{N}$, the spaces X_k and $\mathrm{X}_{k,A}$ are as introduced in Definition 2.3.6 and (2.1), respectively. Then for $\theta > 0$ with $\theta \notin \mathbb{N}$, we get $\mathrm{X}_{A^\theta} \cong [\mathrm{X}_{\lfloor\theta\rfloor}, \mathrm{X}_{\lfloor\theta\rfloor,A}]_{\theta-\lfloor\theta\rfloor}$, while $\mathrm{X}_{A^\theta} \cong \mathrm{X}_\theta$ for $\theta \in \mathbb{N}$.*

Bounded $\mathcal{H}^\infty$-Calculus

Another important concept for the upcoming considerations is the bounded $\mathcal{H}^\infty$-calculus which we discuss in the following. This term was first introduced by McIntosh [102]. Here, we mainly follow [67, Section 1.4] for a brief introduction of the concept. Let us also mention [85, Section 9] for more details. First, for $\theta \in (0, \pi]$, we set

$$\mathrm{H}^\infty(\Sigma_\theta) := \{f \colon \Sigma_\theta \to \mathbb{C} : f \text{ is holomorphic and bounded}\},$$

and $\mathrm{H}^\infty(\Sigma_\theta)$ becomes a Banach algebra when equipped with the norm

$$\|f\|_\infty^\theta := \sup\{|f(\lambda)| : |\arg \lambda| < \theta\}.$$

If $A \in \mathcal{S}(\mathrm{X})$ with spectral angle ϕ_A is bounded and invertible, then for $\theta > \phi_A$, the usual Dunford calculus for bounded linear operators can be used. More precisely, the spectrum $\sigma(A)$ then is a compact subset of Σ_θ. Thus, the choice of a closed path Γ_A in Σ_θ surrounding $\sigma(A)$ counterclockwise allows us to define

$$f(A) := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_A} f(\lambda) R(\lambda, A) \, \mathrm{d}\lambda, \quad \text{for } f \in \mathrm{H}^\infty(\Sigma_\theta),$$

because the integral is well-defined by the compactness of Γ_A.

Based on the above definition of $f(A)$, the bounded holomorphic functional calculus for operators in $\mathcal{S}(\mathrm{X})$ is constructed. For $\lambda \in \Sigma_\theta$, we introduce

$$\varrho(\lambda) \coloneqq \frac{\lambda}{(1+\lambda)^2},$$

and we suppose that there exist $C > 0$ and $\varepsilon > 0$ such that

$$|f(\lambda)| \leq C \cdot |\varrho(\lambda)|^\varepsilon, \quad \text{for } \lambda \in \Sigma_\theta. \tag{2.12}$$

Given $\theta \in (0, \pi)$, we also define

$$\mathrm{H}_0^\infty(\Sigma_\theta) \coloneqq \{f \in \mathrm{H}^\infty(\Sigma_\theta) : \text{there exist } C, \varepsilon > 0 \text{ so that (2.12) is valid}\}.$$

For $A \in \mathcal{S}(\mathrm{X})$ with spectral angle ϕ_A, $\theta > \phi_A$ and $f \in \mathrm{H}_0^\infty(\Sigma_\theta)$, we set

$$f(A) \coloneqq \frac{1}{2\pi \mathrm{i}} \int_{\partial \Sigma_{\theta'}} f(\lambda) R(\lambda, A) \, \mathrm{d}\lambda, \quad \text{for } \theta' \in (\phi_A, \theta). \tag{2.13}$$

As revealed in the next lemma, for which we refer e. g. to [37, Section 2.1], the above definition gives rise to an extended functional calculus on $\mathrm{H}_0^\infty(\Sigma_\theta)$.

Lemma 2.3.11. *Let $A \in \mathcal{S}(\mathrm{X})$ with spectral angle ϕ_A. For $\theta' \in (\phi_A, \theta)$, $f \in \mathrm{H}_0^\infty(\Sigma_\theta)$ and $f(A)$ from* (2.13)*, we define $\Phi_A(f) \coloneqq f(A)$. Then the mapping $\Phi_A \colon \mathrm{H}_0^\infty(\Sigma_\theta) \to \mathcal{L}(\mathrm{X})$ is linear, and*

(a) the integral in (2.13) *is independent of the choice of $\theta' \in (\phi_A, \theta)$,*

(b) $\Phi_A(f \cdot g) = \Phi_A(f)\Phi_A(g)$, for f, $g \in \mathrm{H}_0^\infty(\Sigma_\theta)$,

(c) if (f_n), $f \in \mathrm{H}_0^\infty(\Sigma_\theta)$ are uniformly bounded, and $f_n(\lambda) \to f(\lambda)$ is valid for all $\lambda \in \Sigma_\theta$, then for every $g \in \mathrm{H}_0^\infty(\Sigma_\theta)$, we have

$$\lim_{n\to\infty} \Phi_A(f_n \cdot g) = \Phi_A(f \cdot g), \quad \textit{in } \mathcal{L}(\mathrm{X}), \quad \textit{and}$$

(d) if $f(\lambda) = \lambda(\mu_1 - \lambda)^{-1}(\mu_2 - \lambda)^{-1}$ with μ_1, $\mu_2 \notin \overline{\Sigma_\theta}$, then

$$f(A) = AR(\mu_1, A)R(\mu_2, A).$$

Let us emphasize that so far, functions such as $\lambda \mapsto (\mu - \lambda)^{-1}$ are not within the scope of the functional calculus Φ_A as introduced in Lemma 2.3.11. The aim is thus to further generalize Φ_A so that its extension is defined for all $f \in \mathrm{H}^\infty(\Sigma_\theta)$ such that $f(A)$ gives rise to a bounded operator on X. Thus, for $A \in \mathcal{S}(\mathrm{X})$ with spectral angle ϕ_A, $\theta > \phi_A$ and $f \in \mathrm{H}_0^\infty(\Sigma_\theta)$, we define

$$\|f\|_A := \|f\|_{\mathrm{H}^\infty(\Sigma_\theta)} + \|f(A)\|_{\mathcal{L}(\mathrm{X})},$$

and we set $\mathrm{H}_A^\infty(\Sigma_\theta)$ to consist of the functions $f \in \mathrm{H}^\infty(\Sigma_\theta)$ for which there is a sequence $(f_n) \subset \mathrm{H}_0^\infty(\Sigma_\theta)$ such that $f_n(\lambda) \to f(\lambda)$ for all $\lambda \in \Sigma_\theta$ as well as $\sup_{n\in\mathbb{N}} \|f_n(A)\|_A < \infty$. Let us observe that $\mathrm{H}_A^\infty(\Sigma_\theta)$ is a subalgebra of $\mathrm{H}^\infty(\Sigma_\theta)$. The lemma below, also referred to as *McIntosh's convergence lemma*, guarantees the existence of the limit

$$\overline{\Phi_A}(f)(x) := \lim_{n\to\infty} \Phi_A(f_n)(x), \text{ for } f \in \mathrm{H}_A^\infty(\Sigma_\theta) \text{ and } x \in \mathrm{X},$$

and thus induces a functional calculus on $\mathrm{H}_A^\infty(\Sigma_\theta)$. We refer here to [102] or also [85, Theorem 9.6].

Lemma 2.3.12. *Consider $A \in \mathcal{S}(\mathrm{X})$ with spectral angle ϕ_A and $\theta > \phi_A$. Then the functional calculus Φ_A introduced in Lemma 2.3.11 admits an extension $\overline{\Phi_A}\colon \mathrm{H}_A^\infty(\Sigma_\theta) \to \mathcal{L}(\mathrm{X})$ such that*

(a) $\overline{\Phi_A}$ is linear and multiplicative,

(b) for $\mu \notin \overline{\Sigma_\theta}$, we have $r_\mu(\lambda) = (\mu-\lambda)^{-1} \in \mathrm{H}_A^\infty(\Sigma_\theta)$ and $\overline{\Phi_A}(r_\mu) = R(\mu, A)$, and

(c) if $f \in \mathrm{H}^\infty(\Sigma_\theta)$, $(f_n) \subset \mathrm{H}_A^\infty(\Sigma)$ with $f_n(\lambda) \to f(\lambda)$ for all $\lambda \in \Sigma_\theta$ and $\|f_n\|_A < C$ for all $n \in \mathbb{N}$ and some $C > 0$, then $f \in \mathrm{H}_A^\infty(\Sigma_\theta)$,

$$\lim_{n\to\infty} \overline{\Phi_A}(f_n)(x) = \overline{\Phi_A}(f)(x) \text{ for all } x \in \mathrm{X}, \text{ and } \|\overline{\Phi_A}\| \le C.$$

By virtue of Lemma 2.3.12, we are able to define the notion of an operator possessing a bounded $\mathcal{H}^\infty$-calculus.

Definition 2.3.13. *An operator $A \in \mathcal{S}(\mathrm{X})$ with spectral angle $\phi_A \in [0, \pi)$ and $\theta \in (\phi_A, \pi)$ is referred to have a* bounded $\mathcal{H}^\infty$-calculus *if there exists a constant $C > 0$ such that*

$$\|f(A)\|_{\mathcal{L}(\mathrm{X})} \le C \cdot \|f\|_{\mathrm{H}^\infty(\Sigma_\theta)}, \text{ for all } f \in \mathrm{H}_0^\infty(\Sigma_\theta).$$

Furthermore, ϕ_A^∞ designates the $\mathcal{H}^\infty$-angle *of A with this property, and it is defined as the infimum of such θ. We denote the class of sectorial operators admitting a bounded $\mathcal{H}^\infty$-calculus for some angle $\theta \in (0, \pi)$ on X by $\mathcal{H}^\infty(\mathrm{X})$.*

The class of operators with a bounded $\mathcal{H}^\infty$-calculus has the following handy permanence property, for which we refer e. g. to [37, Proposition 2.11(vi)].

Lemma 2.3.14. *Let* X *and* Y *be two Banach spaces and* $T \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ *bijective. Then* $A \in \mathcal{H}^\infty(\mathrm{X})$ *if and only if* $A_1 = TAT^{-1} \in \mathcal{H}^\infty(\mathrm{Y})$ *and* $\phi_A^\infty = \phi_{A_1}^\infty$.

The result below proves useful when dealing with restrictions of operators that can be obtained from a projection.

Lemma 2.3.15. *Consider* $A \in \mathcal{H}^\infty(\mathrm{X})$ *with* $\Phi_A^\infty \in [0,\pi)$, *and let* $\mathrm{Y} \subset \mathrm{X}$ *be a closed subspace resulting from a projection* $P \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ *such that for all* $x \in \mathrm{D}(A)$, *we have* $Px \in \mathrm{D}(A)$ *and* $PAx = APx$. *Then* $B := PA$ *has the property that* $B \in \mathcal{H}^\infty(\mathrm{Y})$ *with* $\Phi_B^\infty \leq \Phi_A^\infty$.

Proof. First, we observe that $B = A|_\mathrm{Y}$, so $\mathrm{D}(B) = \mathrm{D}(A|_\mathrm{Y})$, $\mathrm{R}(B) = \mathrm{R}(A|_\mathrm{Y})$ and $R(\lambda, B) = R(\lambda, A)|_\mathrm{Y}$ for all $\lambda \in \rho(A)$. Hence, the sectoriality of A on X implies that B is sectorial on Y. We also observe that $f(B) = f(A)|_\mathrm{Y}$ for all $f \in \mathrm{H}_0^\infty(\Sigma_\theta)$ with $\theta > \phi_A^\infty$. Thus, for such f and θ, we have

$$\|f(B)\|_{\mathcal{L}(\mathrm{Y})} = \|f(A)|_\mathrm{Y}\|_{\mathcal{L}(\mathrm{Y})} \leq C \cdot \|f\|_{\mathrm{H}^\infty(\Sigma_\theta)}$$

by $A \in \mathcal{H}^\infty(\mathrm{X})$ with $\mathcal{H}^\infty$-angle ϕ_A^∞. With regard to Definition 2.3.13, this yields $B \in \mathcal{H}^\infty(\mathrm{Y})$ with $\phi_B^\infty \leq \phi_A^\infty$. □

Let us invoke the following relations of the notions introduced so far, and for which we refer for example to [37, Section 4.4]. It holds that

$$A \in \mathcal{H}^\infty(\mathrm{X}) \subset \mathcal{BIP}(\mathrm{X}) \subset \mathcal{RS}(\mathrm{X}) \subset \mathcal{S}(\mathrm{X}), \text{ with} \\ \phi_A^\infty \geq \theta_A \geq \phi_A^\mathcal{R} \geq \phi_A. \tag{2.14}$$

In the sequel, we discuss a generalization of the Da Prato-Grisvard theorem as presented in Proposition 2.2.5. It is due to Dore [39].

Lemma 2.3.16. *Let* $A \in \mathcal{S}(\mathrm{X})$ *with* $\phi_A \in [0,\pi)$ *be invertible, and let* $\theta \in (0,1)$ *and* $p \in [1,\infty)$. *Then* $A \in \mathcal{H}^\infty(\mathrm{D}_A(\theta,p))$ *with* $\mathcal{H}^\infty$*-angle* $\phi_A^\infty = \phi_A$.

With regard to applications, we also present two perturbation results of the bounded $\mathcal{H}^\infty$-calculus in the sequel. In contrast to the perturbation result of the $\mathcal{R}$-sectoriality from Lemma 2.1.22 or also Corollary 2.1.23, the first result concerns perturbations of lower (fractional) order. It can be found in [85, Proposition 13.1].

Lemma 2.3.17. *Assume that* $A \in \mathcal{H}^\infty(\mathrm{X})$ *with* $\mathcal{H}^\infty$*-angle* $\phi_A^\infty \in [0,\pi)$ *satisfies* $0 \in \rho(A)$*. Moreover, consider* $\delta \in (0,1)$*, and suppose that the linear operator* B *satisfies* $\mathrm{D}(A) \subset \mathrm{D}(B)$*, and that there is a constant* $C > 0$ *with*

$$\|Bx\| \leq C \cdot \|A^{1-\delta}x\|$$

for all $x \in \mathrm{D}(A)$*. Then there exists* $\omega > 0$ *such that* $A + B + \omega \in \mathcal{H}^\infty(\mathrm{X})$ *with* $\mathcal{H}^\infty$*-angle* $\phi_{A+B+\omega}^\infty = \phi_A^\infty$*.*

The second perturbation result regards relatively A^α-bounded perturbations for some $\alpha \in [0,1)$. We refer here to [115, Corollary 3.3.15].

Lemma 2.3.18. *Let* $A \in \mathcal{H}^\infty(\mathrm{X})$ *with* $\phi_A^\infty \in [0,\pi)$*, and consider a linear operator* B *in* X *such that* $\mathrm{D}(A^\alpha) \subset \mathrm{D}(B)$ *for some* $\alpha \in [0,1)$*, and*

$$\|Bx\| \leq a \cdot \|x\| + b \cdot \|A^\alpha x\|, \text{ for } x \in \mathrm{D}(A^\alpha),$$

is valid for some constants $a, b > 0$*. If* $A + B \in \mathcal{S}(\mathrm{X})$ *with* $0 \in \rho(A+B)$ *and spectral angle* ϕ_{A+B}*, then* $A + B \in \mathcal{H}^\infty(\mathrm{X})$ *with* $\phi_{A+B}^\infty \leq \max\{\phi_A^\infty, \phi_{A+B}\}$*.*

Application to the Laplacian Operator

Next, we collect properties of the Laplacian operator on bounded domains with regard to the concepts introduced in the preceding sections. We first recall that the Laplacian operator is defined by

$$\Delta u \coloneqq \partial_1^2 u + \ldots + \partial_d^2 u.$$

Now, let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain with boundary of class C^2. The *Dirichlet Laplacian operator* on $\mathrm{L}^q(\Omega)$ is defined by

$$\Delta_{\mathrm{D}} u \coloneqq \Delta u, \text{ with } \mathrm{D}(\Delta_{\mathrm{D}}) = \mathrm{W}^{2,q}(\Omega) \cap \mathrm{W}_0^{1,q}(\Omega). \tag{2.15}$$

On the other hand, the *Neumann Laplacian operator* on $\mathrm{L}^q(\Omega)$ is defined by

$$\Delta_{\mathrm{N}} u \coloneqq \Delta u, \text{ with } \mathrm{D}(\Delta_{\mathrm{N}}) = \mathrm{W}_{\mathrm{N}}^{2,q}(\Omega), \tag{2.16}$$

where the subscript $_{\mathrm{N}}$ indicates homogeneous Neumann boundary conditions as introduced in Section 1.3.

The first assertion of the result can be found in [67, Proposition 1.4.12], or it can be deduced from the considerations in Section 2.5, while the other ones are implied by (2.14), Proposition 2.1.21 as well as Lemma 2.1.14 in conjunction with $0 \in \rho(-\Delta_{\mathrm{D}})$ and $\sigma(-\Delta_{\mathrm{N}}) \subset [0,\infty)$. Moreover, we invoke the UMD property of $\mathrm{L}^q(\Omega)$ for the last aspect.

Lemma 2.3.19. *Let $p, q \in (1,\infty)$, and denote by Δ_{D} and Δ_{N} the Dirichlet and Neumann Laplacian operators on $\mathrm{L}^q(\Omega)$ as introduced in (2.15) and (2.16), respectively. Then for every $\omega > 0$, it holds that*

(a) $-\Delta_{\mathrm{D}}$, $-\Delta_{\mathrm{N}} + \omega \in \mathcal{H}^\infty(\mathrm{L}^q(\Omega))$ *with* $\mathcal{H}^\infty$*-angles* $\phi^\infty_{-\Delta_{\mathrm{D}}} = \phi^\infty_{-\Delta_{\mathrm{N}}+\omega} = 0$,

(b) $-\Delta_{\mathrm{D}}$, $-\Delta_{\mathrm{N}} + \omega \in \mathcal{BIP}(\mathrm{L}^q(\Omega))$ *with power angles* $\theta_{-\Delta_{\mathrm{D}}} = \theta_{-\Delta_{\mathrm{N}}+\omega} = 0$,

(c) $-\Delta_{\mathrm{D}}$, $-\Delta_{\mathrm{N}} + \omega \in \mathcal{RS}(\mathrm{L}^q(\Omega))$ *with* $\mathcal{R}$*-angles* $\phi^{\mathcal{R}}_{-\Delta_{\mathrm{D}}} = \phi^{\mathcal{R}}_{-\Delta_{\mathrm{N}}+\omega} = 0$, *and*

(d) $-\Delta_{\mathrm{D}}$, $-\Delta_{\mathrm{N}} + \omega \in \mathcal{S}(\mathrm{L}^q(\Omega))$ *with spectral angles* $\phi_{-\Delta_{\mathrm{D}}} = \phi_{-\Delta_{\mathrm{N}}+\omega} = 0$.

(e) In particular, $-\Delta_{\mathrm{D}}$, $-\Delta_{\mathrm{N}} + \omega \in {}_0\mathcal{MR}_p(\mathrm{L}^q(\Omega))$, *and it is even valid that* $-\Delta_{\mathrm{D}} \in \mathcal{MR}_p(\mathrm{L}^q(\Omega))$ *and* $-\Delta_{\mathrm{N}} + \omega \in \mathcal{MR}_p(\mathrm{L}^q(\Omega))$.

We still denote by $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, a bounded domain with C^2-boundary. Another important application concerns the $\mathrm{L}^q_0(\Omega)$*-realization of the Laplacian operator subject to Neumann boundary conditions*. In fact, for $\mathrm{L}^q_0(\Omega)$ denoting the space of functions with average zero in Ω as introduced in (1.1), we set

$$\Delta_{\mathrm{N,m}} u := \Delta u, \quad \text{for } u \in \mathrm{D}(\Delta_{\mathrm{N,m}}) = \mathrm{W}^{2,q}_{\mathrm{N}}(\Omega) \cap \mathrm{L}^q_0(\Omega). \tag{2.17}$$

In contrast to the Neumann Laplacian operator Δ_{N} from (2.16), the operator $\Delta_{\mathrm{N,m}}$ is invertible thanks underlying space of functions with average zero. We collect the properties of $\Delta_{\mathrm{N,m}}$ in the lemma below.

Lemma 2.3.20. *Let $p, q \in (1,\infty)$, and consider $\Delta_{\mathrm{N,m}}$ from (2.17). Then we have $\sigma(-\Delta_{\mathrm{N,m}}) \subset (0,\infty)$ as well as $-\Delta_{\mathrm{N,m}} \in \mathcal{H}^\infty(\mathrm{L}^q_0(\Omega))$ with $\mathcal{H}^\infty$-angle $\phi^\infty_{-\Delta_{\mathrm{N,m}}} = 0$.*

As a consequence, we get $-\Delta_{\mathrm{N,m}} \in \mathcal{BIP}(\mathrm{L}^q_0(\Omega)) \subset \mathcal{RS}(\mathrm{L}^q_0(\Omega)) \subset \mathcal{S}(\mathrm{L}^q_0(\Omega))$ with $\theta_{-\Delta_{\mathrm{N,m}}} = \phi^{\mathcal{R}}_{-\Delta_{\mathrm{N,m}}} = \phi_{-\Delta_{\mathrm{N,m}}} = 0$. In particular, $-\Delta_{\mathrm{N,m}} \in \mathcal{MR}_p(\mathrm{L}^q_0(\Omega))$.

Proof. The idea is to deduce the bounded $\mathcal{H}^\infty$-calculus of $-\Delta_{\mathrm{N,m}}$ from the one of $-\Delta_{\mathrm{N}}$ by Lemma 2.3.15. First, we define $\mathrm{P}\colon \mathrm{L}^q(\Omega) \to \mathrm{L}^q_0(\Omega)$ by

$$\mathrm{P}u := u - \frac{1}{|\Omega|}\int_\Omega u \,\mathrm{d}x.$$

We clearly have $\mathrm{P}u \in \mathrm{D}(\Delta_{\mathrm{N}})$ for all $u \in \mathrm{D}(\Delta_{\mathrm{N}})$. Moreover, the divergence theorem together with the Neumann boundary conditions of $u \in \mathrm{D}(\Delta_{\mathrm{N}})$ imply

$$\mathrm{P}(\Delta_{\mathrm{N}} u) = \Delta u - \frac{1}{|\Omega|}\int_\Omega \Delta u \,\mathrm{d}x = \Delta(\mathrm{P}u) - \int_{\partial\Omega} \partial_\nu u \,\mathrm{d}S = \Delta_{\mathrm{N}}(\mathrm{P}u)$$

for all $u \in \mathrm{D}(\Delta_\mathrm{N})$. Thus, $\Delta_{\mathrm{N,m}} = \mathrm{P}\Delta_\mathrm{N}$ results from Δ_N by restriction to the subspace $\mathrm{L}^q_0(\Omega)$, so $\sigma(-\Delta_{\mathrm{N,m}}) \subset \sigma(-\Delta_\mathrm{N})$. On the other hand, $0 \in \rho(\Delta_{\mathrm{N,m}})$, so $\sigma(-\Delta_{\mathrm{N,m}}) \subset (0,\infty)$ is implied. As $\Delta_{\mathrm{N,m}} + \omega\,\mathrm{Id} = \mathrm{P}(\Delta_\mathrm{N} + \omega\,\mathrm{Id})$ on $\mathrm{L}^q_0(\Omega)$, we derive from Lemma 2.3.15 and Lemma 2.3.19(a) that $-\Delta_{\mathrm{N,m}}+\omega \in \mathcal{H}^\infty(\mathrm{L}^q_0(\Omega))$ with $\phi^\infty_{-\Delta_{\mathrm{N,m}}+\omega} = 0$ for all $\omega > 0$.

In the remainder of the proof, we show that the last property is also valid without shift. The strategy here is to view $-\Delta_{\mathrm{N,m}}$ as a perturbation of the shifted operator, and to employ the invertibility of $\Delta_{\mathrm{N,m}}$ joint with Lemma 2.3.18. By (2.14), we especially have $-\Delta_{\mathrm{N,m}} + \omega \in \mathcal{RS}(\mathrm{L}^q_0(\Omega))$ with $\phi^\mathcal{R}_{-\Delta_{\mathrm{N,m}}+\omega} = 0$. Thanks to the UMD property of $\mathrm{L}^q_0(\Omega)$ by Lemma 2.1.19(g), it follows that $-\Delta_{\mathrm{N,m}} + \omega \in {}_0\mathcal{MR}_p(\mathrm{L}^q_0(\Omega))$ with regard to Proposition 2.1.21. Lemma 2.1.15 together with $0 \in \rho(-\Delta_{\mathrm{N,m}})$ yields $-\Delta_{\mathrm{N,m}} \in \mathcal{MR}_p(\mathrm{L}^q_0(\Omega))$. From Lemma 2.1.12(b), we conclude $-\Delta_{\mathrm{N,m}} \in \mathcal{S}(\mathrm{L}^q_0(\Omega))$ with spectral angle $\phi_{-\Delta_{\mathrm{N,m}}} = 0$. Finally, an application of Lemma 2.3.18 to the operator $(-\Delta_{\mathrm{N,m}} + \omega\,\mathrm{Id}) - \omega\,\mathrm{Id}$ leads to $-\Delta_{\mathrm{N,m}} \in \mathcal{H}^\infty(\mathrm{L}^q_0(\Omega))$ with $\phi^\infty_{-\Delta_{\mathrm{N,m}}} = 0$. □

Next, we investigate the Laplacian operator on the set $G = (0,1) \times (0,1)$ with periodic boundary conditions on ∂G. In fact, for $\mathrm{W}^{2,q}_{\mathrm{per}}(G)$ as introduced in Section 1.3, we denote the operator by $\Delta_\mathrm{H} \colon \mathrm{W}^{2,q}_{\mathrm{per}}(G) \to \mathrm{L}^q(G)$.

From [108, Theorem 4.2], we derive the following result on the bounded $\mathcal{H}^\infty$-calculus of $-\Delta_\mathrm{H}$ up to a shift. By (2.14), this also yields the boundedness of the imaginary powers and thus leads to the asserted shape of the fractional power domains by Lemma 1.3.4 on the shape of the complex interpolation spaces in conjunction with Lemma 2.3.4. The periodic boundary conditions are preserved by interpolation, see also [63, Section 4].

Lemma 2.3.21. *Let $q \in (1,\infty)$. Then it holds that $-\Delta_\mathrm{H} + \omega \in \mathcal{H}^\infty(\mathrm{L}^q(G))$ for all $\omega > 0$. In particular, it is valid that $-\Delta_\mathrm{H} + \omega \in \mathcal{BIP}(\mathrm{L}^q(G))$, and for $\beta \in (0,1)$, we have $\mathrm{D}((-\Delta_\mathrm{H} + \omega)^\beta) \cong \mathrm{H}^{2\beta,q}_{\mathrm{per}}(G)$.*

Bounded $\mathcal{H}^\infty$-Calculus of Block Operator Matrices

As a next step, we investigate the bounded $\mathcal{H}^\infty$-calculus of block operator matrices. For this, we follow the paper of Agresti and Hussein [3]. More precisely, we denote by X and Y Banach spaces, and A, B, C and D represent linear operators such that $A \colon \mathrm{D}(A) \subset \mathrm{X} \to \mathrm{X}$, $D \colon \mathrm{D}(D) \subset \mathrm{Y} \to \mathrm{Y}$, $B \colon \mathrm{D}(B) \subset \mathrm{Y} \to \mathrm{X}$ and $C \colon \mathrm{D}(C) \subset \mathrm{X} \to \mathrm{Y}$ Furthermore, we assume that A and D are densely defined and closed operators. Setting $\mathrm{Z} := \mathrm{X} \times \mathrm{Y}$, we define the *block operator matrix*

$$K\colon \mathrm{D}(K) = \mathrm{D}(A) \times \mathrm{D}(D) \subset \mathrm{Z} \to \mathrm{Z}, \quad \text{where}$$

$$K\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{D}(K). \tag{2.18}$$

The following notion of diagonal dominance proves useful when studying block operator matrices with regard to bounded $\mathcal{H}^\infty$-calculus.

Definition 2.3.22. *The block operator matrix K on Z as introduced in* (2.18) *is called* diagonally dominant *provided* $\mathrm{D}(D) \subset \mathrm{D}(B)$ *and* $\mathrm{D}(A) \subset \mathrm{D}(C)$*, and if there exist c_A, c_D as well as $L \geq 0$ such that*

$$\|Cx\|_{\mathrm{Y}} \leq c_A \cdot \|Ax\|_{\mathrm{X}} + L \cdot \|x\|_{\mathrm{X}}, \ \textit{for } x \in \mathrm{D}(A), \ \textit{and}$$
$$\|By\|_{\mathrm{X}} \leq c_D \cdot \|Dy\|_{\mathrm{Y}} + L \cdot \|y\|_{\mathrm{Y}}, \ \textit{for } y \in \mathrm{D}(D).$$

Below, we discuss a sufficient condition for the relative A-boundedness of C with arbitrarily small constant $c_A > 0$, see also [3, Corollary 5.7].

Remark 2.3.23. *The relative A-boundedness of C as in Definition 2.3.22 can be established for every $c_A > 0$ if there is $\gamma \in (0,1)$ with $C \in \mathcal{L}(\mathrm{D}(A^\gamma), \mathrm{Y})$.*

The following result by Agresti and Hussein [3, Corollary 5.7] asserts the bounded $\mathcal{H}^\infty$-calculus of diagonally dominant block operator matrices.

Proposition 2.3.24. *Let K as defined in* (2.18) *be a diagonally dominant block operator matrix in the sense of Definition 2.3.22. Moreover, assume that $A \in \mathcal{H}^\infty(\mathrm{X})$ and $D \in \mathcal{H}^\infty(\mathrm{Y})$ with $\mathcal{H}^\infty$-angles $\phi_A^\infty \in [0,\pi)$ and $\phi_D^\infty \in [0,\pi)$, respectively, and suppose the existence of $\delta \in (0,1)$ so that for $c > 0$, we have*

$$\begin{aligned} &C(\mathrm{D}(A^{1+\delta})) \subset \mathrm{D}(D^\delta), \ \textit{and} \ \|D^\delta Cx\|_{\mathrm{Y}} \leq c \cdot \|A^{1+\delta}x\|_{\mathrm{X}}, \\ &B(\mathrm{D}(D^{1+\delta})) \subset \mathrm{D}(A^\delta), \ \textit{and} \ \|A^\delta By\|_{\mathrm{X}} \leq c \cdot \|D^{1+\delta}y\|_{\mathrm{Y}}, \end{aligned} \tag{2.19}$$

for all $x \in \mathrm{D}(A^{1+\delta})$ and $y \in \mathrm{D}(D^{1+\delta})$. Then for every $\phi \in (\max\{\phi_A^\infty, \phi_D^\infty\}, \pi)$, there are $\varepsilon_0 \geq 0$ and $\omega_0 \in \mathbb{R}$ so that if $c_A < \varepsilon_0$, it follows that $K + \omega \in \mathcal{H}^\infty(\mathrm{Z})$ with $\mathcal{H}^\infty$-angle $\phi_K^\infty \leq \phi$ for all $\omega > \omega_0$.

By Remark 2.3.23, the smallness of c_A in Proposition 2.3.24 is especially valid in the situation that $C \in \mathcal{L}(\mathrm{D}(A^\gamma), \mathrm{Y})$ for some $\gamma \in (0,1)$.

2.4. Embeddings of the Parabolic Spaces

In this section, we present tools for embedding results of the function spaces in the parabolic setting.

Extension Operators

We usually denote *extension operators* by $\mathfrak{E}$, while the associated *restriction operators* are denoted by $\mathfrak{R}$. The following lemma discusses the existence of continuous extension operators on UMD Banach space-valued Bessel potential and Sobolev-Slobodeckij spaces, and we refer to [107, Lemma 2.5].

Lemma 2.4.1. *Let $J = (0, T)$ be a finite interval and $p \in (1, \infty)$, and consider a UMD space* X. *Then there exists an extension operator $\mathfrak{E}_J$ from J to $\mathbb{R}_+$ such that for all $s \in [0, 1]$, we have*

$$\mathfrak{E}_J \in \mathcal{L}(\mathrm{H}^{s,p}(J; \mathrm{X}), \mathrm{H}^{s,p}(\mathbb{R}_+; \mathrm{X})) \cap \mathcal{L}(\mathrm{W}^{s,p}(J; \mathrm{X}), \mathrm{W}^{s,p}(\mathbb{R}_+; \mathrm{X})).$$

It turns out that the operator norm of the extension operator is even independent of T if we invoke the spaces ${}_0\mathrm{H}^{s,p}$ and ${}_0\mathrm{W}^{s,p}$ with zero time trace for $s > 1/p$. Again, we refer to [107, Lemma 2.5].

Lemma 2.4.2. *Let $J = (0, T)$ be a finite interval and $p \in (1, \infty)$, and consider a UMD space* X. *Then there exists an extension operator $\mathfrak{E}_J^0$ from J to $\mathbb{R}_+$ such that for all $s \in [0, 1/p)$, we have*

$$\mathfrak{E}_J^0 \in \mathcal{L}(\mathrm{H}^{s,p}(J; \mathrm{X}), \mathrm{H}^{s,p}(\mathbb{R}_+; \mathrm{X})) \cap \mathcal{L}(\mathrm{W}^{s,p}(J; \mathrm{X}), \mathrm{W}^{s,p}(\mathbb{R}_+; \mathrm{X})),$$

and for $s \in (1/p, 1]$, we get

$$\mathfrak{E}_J^0 \in \mathcal{L}\Big({}_0\mathrm{H}^{s,p}(J; \mathrm{X}), {}_0\mathrm{H}^{s,p}(\mathbb{R}_+; \mathrm{X})\Big) \cap \mathcal{L}\Big({}_0\mathrm{W}^{s,p}(J; \mathrm{X}), {}_0\mathrm{W}^{s,p}(\mathbb{R}_+; \mathrm{X})\Big).$$

In addition, the operator norm of $\mathfrak{E}_J^0$ is independent of T.

The Derivative Operator

We now introduce the derivative operator on various subintervals of $\mathbb{R}$ and study its properties. The first setting under consideration is the whole real line. Recall the Banach space X, and let $p \in (1, \infty)$ in the sequel. For $t \in \mathbb{R}$, we define the *derivative operator on* $\mathrm{L}^p(\mathbb{R}; \mathrm{X})$ by

$$(B_p u)(t) := u'(t), \quad \text{for } u \in \mathrm{D}(B_p) := \mathrm{W}^{1,p}(\mathbb{R}; \mathrm{X}). \tag{2.20}$$

Next, for $t \in \mathbb{R}_+$ we introduce the *derivative operator on* $\mathrm{L}^p(\mathbb{R}_+;\mathrm{X})$ given by

$$(2.21)\qquad (B_{\mathbb{R}_+,p}u)(t) := u'(t), \text{ for } u \in \mathrm{D}(B_{\mathbb{R}_+,p}) := {}_0\mathrm{W}^{1,p}(\mathbb{R}_+;\mathrm{X}).$$

The subscript 0 in (2.21) means that the time trace equals zero, i. e., $u(0) = 0$ for $u \in \mathrm{D}(B_{\mathbb{R}_+,p})$.

Last, we consider a finite interval $J = (0,T)$, where $0 < T < \infty$. The *derivative operator on* $\mathrm{L}^p(J;\mathrm{X})$ takes the shape

$$(2.22)\qquad (B_{J,p}u)(t) := u'(t), \text{ for } u \in \mathrm{D}(B_{J,p}) := {}_0\mathrm{W}^{1,p}(J;\mathrm{X}),$$

where $t \in J$, and the subscript 0 again means that $u \in \mathrm{D}(B_{J,p})$ has homogeneous initial values.

The result below establishes the bounded $\mathcal{H}^\infty$-calculus of B_p, $B_{\mathbb{R}_+,p}$ as well as $B_{J,p}$ under the additional assumption that the underlying Banach space X is a UMD space in the sense of Definition 2.1.18. It can for example be found in [115, Corollary 4.3.12 and Theorem 4.3.14].

Lemma 2.4.3. *Let $p \in (1,\infty)$, and suppose that* X *is a UMD space. Then the derivative operators B_p, $B_{\mathbb{R}_+,p}$ and $B_{J,p}$ from* (2.20), (2.21) *and* (2.22) *satisfy* $B_p \in \mathcal{H}^\infty(\mathrm{L}^p(\mathbb{R};\mathrm{X}))$, $B_{\mathbb{R}_+,p} \in \mathcal{H}^\infty(\mathrm{L}^p(\mathbb{R}_+;\mathrm{X}))$ *and* $B_{J,p} \in \mathcal{H}^\infty(\mathrm{L}^p(J;\mathrm{X}))$.

In particular, for $B_{\mathbb{R}_+,p}$, the $\mathcal{H}^\infty$-angle is equal to $\pi/2$, i. e., $\phi_{B^\infty_{\mathbb{R}_+,p}} = \pi/2$. For $\alpha \in (0, 1/p)$, we have $\mathrm{D}(B^\alpha_{\mathbb{R}_+,p}) = \mathrm{H}^{\alpha,p}(\mathbb{R}_+;\mathrm{X})$, while for $\alpha \in (1/p, 1)$, it holds that $\mathrm{D}(B^\alpha_{\mathbb{R}_+,p}) = {}_0\mathrm{H}^{\alpha,p}(\mathbb{R}_+;\mathrm{X})$.

For completeness, we also invoke the negative derivative operator on the half real line and on finite intervals. In fact, we do not impose a trace zero condition in this case. On the one hand, this makes it harder to apply the results from the whole line case. Fortunately, we still have the $\mathcal{H}^\infty$-calculus for the negative derivative operator on $\mathrm{L}^p(\mathbb{R}_+;\mathrm{X})$ and can extend this property to finite intervals by extension and restriction. For $p \in (1,\infty)$ and $t \in \mathbb{R}_+$, we define the *negative derivative operator on* $\mathrm{L}^p(\mathbb{R}_+;\mathrm{X})$ by

$$(2.23)\qquad (-B_{\mathbb{R}_+,p}u)(t) := -u'(t), \text{ for } u \in \mathrm{D}(B_{\mathbb{R}_+,p}) := \mathrm{W}^{1,p}(\mathbb{R}_+;\mathrm{X}).$$

Analogously, for $t \in J$ and $J = (0,T)$, $0 < T < \infty$, we introduce the *negative derivative operator on* $\mathrm{L}^p(J;\mathrm{X})$ given by

$$(2.24)\qquad (-B_{J,p}u)(t) := -u'(t), \text{ for } u \in \mathrm{D}(-B_{J,p}) := \mathrm{W}^{1,p}(J;\mathrm{X}).$$

The corresponding result on the bounded $\mathcal{H}^\infty$-calculus of $-B_{\mathbb{R}_+,p}$ and $-B_{J,p}$ reads as follows.

Lemma 2.4.4. *Let $p \in (1,\infty)$, and assume that* X *is a UMD space. Then the negative derivative operators* $-B_{\mathbb{R}_+,p}$ *and* $-B_{J,p}$ *from* (2.23) *and* (2.24) *satisfy* $-B_{\mathbb{R}_+,p} \in \mathcal{H}^\infty(\mathrm{L}^p(\mathbb{R}_+;\mathrm{X}))$ *and* $-B_{J,p} \in \mathcal{H}^\infty(\mathrm{L}^p(J;\mathrm{X}))$ *with* $\mathcal{H}^\infty$*-angles* $\phi^\infty_{-B_{\mathbb{R}_+,p}} = \phi^\infty_{-B_{J,p}} = \pi/2$. *Furthermore, for* $\alpha \in (0,1)$*, we especially have* $\mathrm{D}(-B^\alpha_{\mathbb{R}_+,p}) = \mathrm{H}^{\alpha,p}(\mathbb{R}_+;\mathrm{X})$.

Proof. For the assertion on $-B_{\mathbb{R}_+,p}$, we refer to [107, Theorem 2.7]. Let us also mention [91, Theorem 6.8] for the shape of the fractional power domain of $-B_{\mathbb{R}_+,p}$.

With regard to $-B_{J,p}$, we use an argument involving extensions and restrictions. In fact, for $J = (0,T)$, we recall the continuous extension operator $\mathfrak{E}_J$ from $\mathrm{L}^p(J;\mathrm{X})$ to $\mathrm{L}^p(\mathbb{R}_+;\mathrm{X})$ from Lemma 2.4.1 and denote the corresponding restriction operator by $\mathfrak{R}_J$. Then for $\lambda \in \mathbb{C}$ such that $|\arg\lambda| > \pi/2$, the resolvent $R(\lambda,-B_{J,p})$ of $-B_{J,p}$ admits the representation

$$R(\lambda,-B_{J,p}) = \mathfrak{R}_J R(\lambda,-B_{\mathbb{R}_+,p})\mathfrak{E}_J.$$

Together with the identity $f(-B_{J,p})u = \mathfrak{R}_J f(-B_{\mathbb{R}_+,p})\mathfrak{E}_J u$, for $u \in \mathrm{L}^p(J;\mathrm{X})$ and $f \in \mathcal{H}^\infty_0(\Sigma_\theta)$, this implies the assertion on $-B_{J,p}$ as well. □

The Mixed Derivative Theorem

In the next step, we provide the abstract version of the mixed derivative theorem, which is based on a result due to Kalton and Weis [75] on the extension of the scalar $\mathcal{H}^\infty$-calculus of a sectorial operator to the $\mathcal{R}$-bounded operator-valued case. In addition, we recall the *Dore-Venni theorem* on the sum of two operators before stating the mixed derivative theorem. Thereafter, we present more concrete instances in which properties of the derivation operator are used. In particular, we discuss a variant of the mixed derivative theorem in real interpolation spaces which seems to be new.

We start with the *Kalton-Weis theorem* paving the way for an extension of the bounded $\mathcal{H}^\infty$-calculus to the operator-valued case.

Proposition 2.4.5 ([75, Section 4])**.** *Consider* $A \in \mathcal{H}^\infty(\mathrm{X})$ *with* $\mathcal{H}^\infty$*-angle* $\phi^\infty_A \in [0,\pi)$*, and for* $\phi > \phi^\infty_A$*, let* $\mathcal{F} \subset \mathrm{H}^\infty(\Sigma_\phi;\mathcal{L}(\mathrm{X}))$ *denote an operator-valued family such that for* $\mu \in \rho(A)$*,* $\lambda \in \Sigma_\phi$ *and* $F \in \mathcal{F}$*, we have*

$$F(\lambda)(\mu - A)^{-1} = (\mu - A)^{-1}F(\lambda).$$

Then there is a constant $C_A > 0$ *so that* $\sup_{F\in\mathcal{F}} \mathcal{R}(F(\Sigma_\phi)) < \infty$ *implies the relation* $\mathcal{F}(A) := \{F(A) : F \in \mathcal{F}\} \subset \mathcal{L}(\mathrm{X})$*, and it holds that*

$$\|F(A)\|_{\mathcal{L}(\mathrm{X})} \le C_A \mathcal{R}(F(\Sigma_\phi)), \quad \text{for } F \in \mathcal{F}.$$

Let us also briefly recall the classical *Dore-Venni theorem* on the sum of two operators. As revealed in [75], it can also be deduced from Proposition 2.4.5 by considering $F(\lambda)$ of the form $F(\lambda) = f(\lambda, B)$ and setting $f(\lambda, \mu) = {}^{\mu}/_{(\lambda+\mu)}$.

Proposition 2.4.6 (Dore and Venni,[40]). *Let* $A \in \mathcal{H}^\infty(\mathrm{X})$ *and* $B \in \mathcal{RS}(\mathrm{X})$ *be commuting such that* $\phi_A^\infty + \phi_B^{\mathcal{R}} < \pi$. *Then the operator* $A+B$ *with domain* $\mathrm{D}(A+B) = \mathrm{D}(A) \cap \mathrm{D}(B)$ *is closed, and* $A+B \in \mathcal{S}(\mathrm{X})$ *is valid with spectral angle* $\phi_{A+B} \leq \max\{\phi_A^\infty, \phi_B^{\mathcal{R}}\}$, *and for a constant* $C > 0$, *the estimate*

$$\|Ax\| + \|Bx\| \leq C \cdot \|(A+B)x\|$$

holds true for all $x \in \mathrm{D}(A) \cap \mathrm{D}(B)$. *Moreover, the operator* $A+B$ *is invertible provided* A *or* B *have this property.*

The following *mixed derivative theorem* in the abstract formulation can be obtained by investigating the situation of

$$F(\lambda) = f(\lambda, B) = \lambda^\alpha B^{1-\alpha}(\lambda + B)^{-1}$$

in Proposition 2.4.5 and concluding the $\mathcal{R}$-boundedness of $F(\Sigma_\theta)$ from a contour integral representation of $F(\lambda)$ together with the $\mathcal{R}$-sectoriality of B as well as the convexity of $\mathcal{R}$-bounds. For details, we refer to [115, Section 4.5.2], while the result can be found in [115, Corollary 4.5.10].

Proposition 2.4.7. *Consider* $A \in \mathcal{RS}(\mathrm{X})$ *as well as* $B \in \mathcal{H}^\infty(\mathrm{X})$ *such that* A *and* B *commute and* $\phi_A^{\mathcal{R}} + \phi_B^\infty < \pi$. *Then for every* $\alpha \in (0,1)$, *the operator* $A^\alpha B^{1-\alpha}(A+B)^{-1}$ *is bounded, and we especially obtain*

$$\mathrm{D}(A) \cap \mathrm{D}(B) = \mathrm{D}(A+B) \hookrightarrow \mathrm{D}(A^\alpha B^{1-\alpha}).$$

The results on the derivative operator discussed before allow us to exploit the mixed derivative theorem Proposition 2.4.7. As a consequence, the UMD property of X as introduced in Definition 2.1.18 is required.

Proposition 2.4.8. *Let* X *be a UMD Banach space, and let* $A\colon \mathrm{D}(A) \to \mathrm{X}$ *be densely defined with* $A \in \mathcal{RS}(\mathrm{X})$ *and* $\phi_A^{\mathcal{R}} < {}^{\pi}/_{2}$. *Recall* X_A *from* (2.1), *and denote by* $\mathrm{D}(A^\beta)$, $\beta \in (0,1)$, *the fractional power domain of* A.

(a) For every $\beta \in (0,1)$, *we obtain the embeddings*

$$\begin{aligned}
\mathrm{W}^{1,p}(\mathbb{R}_+;\mathrm{X}) \cap \mathrm{L}^p(\mathbb{R}_+;\mathrm{X}_A) &\hookrightarrow \mathrm{H}^{1-\beta,p}(\mathbb{R}_+;\mathrm{D}(A^\beta)),\\
{}_0\mathrm{W}^{1,p}(\mathbb{R}_+;\mathrm{X}) \cap \mathrm{L}^p(\mathbb{R}_+;\mathrm{X}_A) &\hookrightarrow \mathrm{H}^{1-\beta,p}(\mathbb{R}_+;\mathrm{D}(A^\beta)), \quad \text{if } 1-\beta < {}^{1}/_{p} \text{ and}\\
{}_0\mathrm{W}^{1,p}(\mathbb{R}_+;\mathrm{X}) \cap \mathrm{L}^p(\mathbb{R}_+;\mathrm{X}_A) &\hookrightarrow {}_0\mathrm{H}^{1-\beta,p}(\mathbb{R}_+;\mathrm{D}(A^\beta)), \quad \text{if } 1-\beta > {}^{1}/_{p}.
\end{aligned}$$

(b) *For all* $\beta \in (0,1)$ *and* $J = (0,T)$, *we infer that*

$$\mathrm{W}^{1,p}(J;\mathrm{X}) \cap \mathrm{L}^p(J;\mathrm{X}_A) \hookrightarrow \mathrm{H}^{1-\beta,p}(J;\mathrm{D}(A^\beta)),$$
$$_0\mathrm{W}^{1,p}(J;\mathrm{X}) \cap \mathrm{L}^p(J;\mathrm{X}_A) \hookrightarrow \mathrm{H}^{1-\beta,p}(J;\mathrm{D}(A^\beta)), \quad \text{if } 1-\beta < 1/p \text{ and}$$
$$_0\mathrm{W}^{1,p}(J;\mathrm{X}) \cap \mathrm{L}^p(J;\mathrm{X}_A) \hookrightarrow {_0\mathrm{H}}^{1-\beta,p}(J;\mathrm{D}(A^\beta)) \quad \text{if } 1-\beta > 1/p.$$

In particular, the embedding constants in the second and third embedding can be chosen independent of T.

Proof. The idea is to exploit the properties of the derivative operator and then apply the abstract mixed derivative theorem in the formulation of Proposition 2.4.7. As X is a UMD space by assumption, Lemma 2.4.4 is applicable and yields $-B_{\mathbb{R}_+,p} \in \mathcal{H}^\infty(\mathrm{L}^p(\mathbb{R}_+;\mathrm{X}))$ with $\mathcal{H}^\infty$-angle $\phi_{-B_{\mathbb{R}_+,p}} = \pi/2$. The domain of $-B_{\mathbb{R}_+,p}$ is given by $\mathrm{W}^{1,p}(\mathbb{R}_+;\mathrm{X})$. The $\mathcal{R}$-sectoriality of A with $\mathcal{R}$-angle $\phi_A^{\mathcal{R}} < \pi/2$ then yields that $\phi_A^{\mathcal{R}} + \phi_{-B_{\mathbb{R}_+,p}}^\infty < \pi$. Therefore, the first part of the assertion of (a) is a consequence of Proposition 2.4.7 upon invoking the shape of the fractional powers of the negative derivative operator from Lemma 2.4.4.

The second and third part of the assertion of (a) follow from completely analogous arguments, where we replace Lemma 2.4.4 by Lemma 2.4.3 in order to get the $\mathcal{H}^\infty$-calculus of the derivative operator $B_{\mathbb{R}_+,p}$ on $\mathrm{L}^p(\mathbb{R}_+;\mathrm{X})$ with domain $_0\mathrm{W}^{1,p}(\mathbb{R}_+;\mathrm{X})$.

Concerning (b), we make use of (a) in conjunction with the extension operators $\mathfrak{E}_J$ and $\mathfrak{E}_J^0$ from Lemma 2.4.1 and Lemma 2.4.2. Indeed, for the first embedding, given

$$u \in \mathrm{W}^{1,p}(J;\mathrm{X}) \cap \mathrm{L}^p(J;\mathrm{X}_A),$$

and using that X_A also enjoys the UMD property as a closed subspace of X, see Lemma 2.1.19(g), we deduce from Lemma 2.4.1 and (a) that

$$\mathfrak{E}_J u \in \mathrm{W}^{1,p}(\mathbb{R}_+;\mathrm{X}) \cap \mathrm{L}^p(\mathbb{R}_+;\mathrm{X}_A) \hookrightarrow \mathrm{H}^{1-\beta,p}(\mathbb{R}_+;\mathrm{X}_\beta).$$

Since by Lemma 2.1.19(f), the space X_β has the UMD property as an interpolation space of X and X_A, the extension operator is also continuous on $\mathrm{H}^{1-\beta,p}(\mathbb{R}_+;\mathrm{X}_\beta)$ in view of Lemma 2.4.1, and the same is valid for the restriction operator. Thus, we conclude

$$u = \mathfrak{R}_J \mathfrak{E}_J u \in \mathrm{H}^{1-\beta,p}(J;\mathrm{X}_\beta).$$

The second and third embedding of (b) can be obtained likewise. This time, the extension operator $\mathfrak{E}_J^0$ from Lemma 2.4.2 is used. The T-independence of the emerging embedding constant holds thanks to the fact that the operator norm of $\mathfrak{E}_J^0$ is T-independent as stated in Lemma 2.4.2. □

By virtue of the UMD property of X and Proposition 2.1.21, the assertion of Proposition 2.4.8 remains valid provided we assume $A \in {}_0\mathcal{MR}_p(\mathrm{X})$ instead of $A \in \mathcal{RS}(\mathrm{X})$ with $\mathcal{R}$-angle $\phi_A^{\mathcal{R}} < \pi/2$.

If the assumption $A \in \mathcal{RS}(\mathrm{X})$ with $\phi_A^{\mathcal{R}} < \pi/2$ is replaced by $A \in \mathcal{BIP}(\mathrm{X})$ with $\theta_A < \pi/2$, then Lemma 2.3.4 yields $D(A^\beta) = [\mathrm{X}, \mathrm{X}_A]_\beta =: \mathrm{X}_\beta$, $\beta \in (0,1)$.

In view of the relation of the classes $\mathcal{RS}(\mathrm{X})$ and $\mathcal{BIP}(\mathrm{X})$ as stated in (2.14), the following embeddings are implied by Proposition 2.4.8.

Corollary 2.4.9. *Let* X *be a UMD Banach space, and consider a densely defined operator* $A\colon \mathrm{D}(A) \to \mathrm{X}$ *with* $A \in \mathcal{BIP}(\mathrm{X})$ *and power angle* $\theta_A < \pi/2$. *Denote by* X_β, $\beta \in (0,1)$, *the complex interpolation space* $[\mathrm{X}, \mathrm{X}_A]_\beta$.

(a) For every $\beta \in (0,1)$, *we get the embeddings*

$$\begin{aligned} \mathrm{W}^{1,p}(\mathbb{R}_+;\mathrm{X}) \cap \mathrm{L}^p(\mathbb{R}_+;\mathrm{X}_A) &\hookrightarrow \mathrm{H}^{1-\beta,p}(\mathbb{R}_+;\mathrm{X}_\beta), \\ {}_0\mathrm{W}^{1,p}(\mathbb{R}_+;\mathrm{X}) \cap \mathrm{L}^p(\mathbb{R}_+;\mathrm{X}_A) &\hookrightarrow \mathrm{H}^{1-\beta,p}(\mathbb{R}_+;\mathrm{X}_\beta), \quad \textit{if } 1-\beta < 1/p \textit{ and} \\ {}_0\mathrm{W}^{1,p}(\mathbb{R}_+;\mathrm{X}) \cap \mathrm{L}^p(\mathbb{R}_+;\mathrm{X}_A) &\hookrightarrow {}_0\mathrm{H}^{1-\beta,p}(\mathbb{R}_+;\mathrm{X}_\beta), \quad \textit{if } 1-\beta > 1/p. \end{aligned}$$

(b) For all $\beta \in (0,1)$ *and* $J = (0,T)$, *it holds that*

$$\begin{aligned} \mathrm{W}^{1,p}(J;\mathrm{X}) \cap \mathrm{L}^p(J;\mathrm{X}_A) &\hookrightarrow \mathrm{H}^{1-\beta,p}(J;\mathrm{X}_\beta), \\ {}_0\mathrm{W}^{1,p}(J;\mathrm{X}) \cap \mathrm{L}^p(J;\mathrm{X}_A) &\hookrightarrow \mathrm{H}^{1-\beta,p}(J;\mathrm{X}_\beta), \quad \textit{if } 1-\beta < 1/p \textit{ and} \\ {}_0\mathrm{W}^{1,p}(J;\mathrm{X}) \cap \mathrm{L}^p(J;\mathrm{X}_A) &\hookrightarrow {}_0\mathrm{H}^{1-\beta,p}(J;\mathrm{X}_\beta), \quad \textit{if } 1-\beta > 1/p. \end{aligned}$$

In particular, the embedding constant in the second and third embedding can be chosen independent of T *in the second embedding.*

By (2.14), the property that $A \in \mathcal{H}^\infty(\mathrm{X})$ with $\mathcal{H}^\infty$-angle $\phi_A^\infty < \pi/2$ is another sufficient condition to get the assertion of Corollary 2.4.9.

Finally, we discuss the mixed derivative theorem for evolution equations in real interpolation spaces. For this purpose, let us recall the trace spaces $\mathrm{D}_A(\theta,p)$, $\theta \in (0,1)$ and $p \in [1,\infty)$ from (2.3). We also invoke the domain E_1 of the realization of the operator A on the trace space $\mathrm{D}_A(\theta,p)$, so

$$\mathrm{E}_1 \coloneqq \{u \in \mathrm{D}(A) : Au \in \mathrm{D}_A(\theta,p)\}.$$

The result then reads as follows. We remark that it has already been obtained in a joint work with Matthias Hieber, see [19, Theorem 3.1].

Proposition 2.4.10. *Let* X *be a UMD space, and consider* $A \in \mathcal{S}(\mathrm{X})$ *invertible with spectral angle* $\phi_A < \pi/2$. *Moreover, recall the negative derivative operator* $-B_{\mathbb{R}_+,p}$ *and* $-B_{J,p}$ *on* $\mathbb{R}_+$ *and* $J = (0,T)$, *respectively, and assume that the operator* A *on* $\mathrm{L}^p(\mathbb{R}_+; \mathrm{D}_A(\theta,p))$ *or* $\mathrm{L}^p(J; \mathrm{D}_A(\theta,p))$ *and* $-B_{\mathbb{R}_+,p}$ *or* $-B_{J,p}$ *commute. Then it holds that*

$$\mathrm{W}^{1,p}(\mathbb{R}_+; \mathrm{D}_A(\theta,p)) \cap \mathrm{L}^p(\mathbb{R}_+; \mathrm{E}_1) \hookrightarrow \mathrm{H}^{1-\beta,p}(\mathbb{R}_+; [\mathrm{D}_A(\theta,p), \mathrm{E}_1]_\beta) \text{ and}$$
$$\mathrm{W}^{1,p}(J; \mathrm{D}_A(\theta,p)) \cap \mathrm{L}^p(J; \mathrm{E}_1) \hookrightarrow \mathrm{H}^{1-\beta,p}(J; [\mathrm{D}_A(\theta,p), \mathrm{E}_1]_\beta).$$

Proof. Let us recall from Lemma 2.1.9 that $\mathrm{D}_A(\theta,p) = (\mathrm{X}, \mathrm{X}_A)_{\theta,p}$ with equivalent norms. Moreover, thanks to $0 \in \rho(A)$ and Lemma 2.1.10, we get the equivalence of the norm of the real interpolation space with the homogeneous norm $[\cdot]_{\theta,p}$ from (2.3). We observe that X_A is also a UMD space in view of the isomorphism $A^{-1} \colon \mathrm{X} \to \mathrm{X}_A$ thanks to Lemma 2.1.19(a). This in turn reveals that $\mathrm{D}_A(\theta,p) = (\mathrm{X}, \mathrm{X}_A)_{\theta,p}$ has the UMD property by virtue of Lemma 2.1.19(f). From Lemma 2.4.4, we now recall that

$$-B_{\mathbb{R}_+,p} \in \mathcal{H}^\infty(\mathbb{R}_+; \mathrm{D}_A(\theta,p)) \text{ and } -B_{J,p} \in \mathcal{H}^\infty(J; \mathrm{D}_A(\theta,p))$$

for $J = (0,T)$, and with $\mathcal{H}^\infty$-angles $\phi^\infty_{-B_{\mathbb{R}_+,p}} = \phi^\infty_{-B_{J,p}} = \pi/2$. On the other hand, the Dore result as stated in Lemma 2.3.16 implies $A \in \mathcal{H}^\infty(\mathrm{D}_A(\theta,p))$ with $\mathcal{H}^\infty$-angle $\phi^\infty_A = \phi_A < \pi/2$ by assumption. From (2.14), we further deduce

$$A \in \mathcal{BIP}(\mathrm{D}_A(\theta,p)) \text{ as well as } A \in \mathcal{RS}(\mathrm{D}_A(\theta,p)),$$

with power angle θ_A and $\mathcal{R}$-angle $\phi^\mathcal{R}_A$ satisfying $\phi^\mathcal{R}_A \le \theta_A \le \phi^\infty_A < \pi/2$. The desired embeddings now follow from the general result Proposition 2.4.7 in the same way as Proposition 2.4.8 and Corollary 2.4.9. □

Trace Space Embeddings

Let X_0 be a Banach space, and consider another densely embedded Banach space X_1. Moreover, for $p \in (1,\infty)$, denote by X_γ the real interpolation space

$$\mathrm{X}_\gamma := (\mathrm{X}_0, \mathrm{X}_1)_{1-1/p,p}. \tag{2.25}$$

Recall that $\mathrm{BUC}(J; \mathrm{X}_\gamma)$ denotes the bounded and uniformly continuous functions on an interval $J \subset \mathbb{R}_+$ with values in X_γ. The next proposition discusses an embedding of the solution space in the maximal L^p-regularity setting into $\mathrm{BUC}(J; \mathrm{X}_\gamma)$. It can be found in [5, Theorem III.4.10.2].

Proposition 2.4.11. *Let* X_1 *be densely embedded into* X_0*, and consider an interval* $J \subset \mathbb{R}_+$*. Then for* X_γ *as introduced in* (2.25)*, we have*

$$W^{1,p}(J; X_0) \cap L^p(J; X_1) \hookrightarrow BUC(J; X_\gamma).$$

Let us comment on the particular situation of a finite interval $J = (0, T)$.

Remark 2.4.12. *In general, the embedding constant resulting from Proposition 2.4.11 depends on* T *when considering* $J = (0, T)$*, where* $0 < T < \infty$*. However, if we consider functions with homogeneous initial values, then we can invoke the extension operator* $\mathfrak{E}_J^0$ *from Lemma 2.4.2 and proceed as in the proof of Proposition 2.4.8(b). More precisely, the extension operator allows us to reduce the assertion on* J *to the one of* $\mathbb{R}_+$*, and the time-independence of the operator norm of* $\mathfrak{E}_J^0$ *implies that the resulting embedding constant can also be chosen independent of* T*.*

For completeness, we provide the analogue of Proposition 2.4.11 in the situation of time-weighted spaces, along with a result on instantaneous smoothing. In this context, for $\mu \in (1/p, 1]$, we set

$$X_{\gamma,\mu} \coloneqq (X_0, X_1)_{\mu - 1/p, p}. \tag{2.26}$$

We refer to [114, Proposition 3.1] for this result in the case $J = \mathbb{R}_+$ and remark that its extension to general time intervals $J = (0, T)$ follows by invoking suitable extension operators in the weighted setting, see [107, Section 2].

Lemma 2.4.13. *Let* X_1 *be densely embedded into* X_0*,* $J \subset \mathbb{R}_+$*, and recall* $X_{\gamma,\mu}$*, with* $p \in (1, \infty)$ *and* $\mu \in (1/p, 1]$*, from* (2.26)*. Then we have*

(a) $W^{1,p}_\mu(J; X_0) \cap L^p_\mu(J; X_1) \hookrightarrow BUC(J; X_{\gamma,\mu})$*, and*

(b) $W^{1,p}_\mu(\mathbb{R}_+; X_0) \cap L^p_\mu(\mathbb{R}_+; X_1) \hookrightarrow C((0, \infty); X_\gamma)$*.*

We also provide an estimate of a function in $BUC([0, T]; X_\gamma)$ by the initial values and the solution, where the special feature is that the constant in the estimate is independent of $T > 0$. Hence, it is particularly useful in applications to functions with non-homogeneous initial values. For a proof, we refer e. g. to [124, Section 3].

Lemma 2.4.14. *Let* X_1 *be densely embedded into* X_0*, consider* X_γ *as defined in* (2.25)*, and let*

$$u \in W^{1,p}(0, T; X_0) \cap L^p(0, T; X_1) =: \mathbb{E}_1.$$

Then there exists a constant $C > 0$*, independent of* T*, such that*

$$\sup_{t \in [0,T]} \|u(t)\|_{X_\gamma} \leq C \cdot \left(\|u(0)\|_{X_\gamma} + \|u\|_{\mathbb{E}_1} \right).$$

Application to the Laplacian Operator

We now exploit the properties of the Dirichlet and Neumann Laplacian on bounded domains to get embeddings for the spatial components being Besov spaces. First, we recall from Lemma 2.3.19 that $-\Delta_{\mathrm{D}} \in \mathcal{S}(\mathrm{L}^q(\Omega))$, and shifting the Neumann Laplacian, we get $\mathrm{Id} - \Delta_{\mathrm{N}} \in \mathcal{S}(\mathrm{L}^q(\Omega)$. For the spectral angles, we have $\phi_{-\Delta_{\mathrm{D}}} = \phi_{\mathrm{Id}-\Delta_{\mathrm{N}}} = 0$. Besides, $0 \in \rho(\Delta_{\mathrm{D}})$ and $0 \in \rho(\mathrm{Id} - \Delta_{\mathrm{N}})$. In addition, in view of Lemma 2.1.9 and Lemma 1.3.6, it holds that

$$\begin{aligned} \mathrm{D}_{-\Delta_{\mathrm{D}}}(\theta, p) &= \mathrm{B}^{2\theta}_{qp}(\Omega), \quad \text{if } \theta < \frac{1}{2q} \text{ and} \\ \mathrm{D}_{\mathrm{Id}-\Delta_{\mathrm{N}}}(\theta, p) &= \mathrm{B}^{2\theta}_{qp}(\Omega), \quad \text{if } \theta < \frac{1}{2} + \frac{1}{2q}. \end{aligned}$$

The domains of the respective realizations on the trace spaces $\mathrm{E}_1(-\Delta_{\mathrm{D}})$ as well as $\mathrm{E}_1(\mathrm{Id} - \Delta_{\mathrm{N}})$ are given by

$$\mathrm{E}_1(-\Delta_{\mathrm{D}}) = \mathrm{B}^{2\theta+2}_{qp,\mathrm{D}}(\Omega) \text{ and } \mathrm{E}_1(\mathrm{Id} - \Delta_{\mathrm{N}}) = \mathrm{B}^{2\theta+2}_{qp,\mathrm{N}}(\Omega),$$

and the subscripts $_{\mathrm{D}}$ and $_{\mathrm{N}}$ represent Dirichlet and Neumann boundary conditions on $\partial\Omega$, respectively. Hence, the below lemma follows from Proposition 2.4.10 in conjunction with the shape of the interpolation spaces as made precise in Lemma 1.3.4 and Lemma 1.3.7.

Lemma 2.4.15. *Consider $J = (0, T)$ with $T \in (0, \infty)$ or $J = \mathbb{R}_+$ as well as parameters $\theta \in (0, 1)$, $p \in (1, \infty)$ and $q \in (1, \infty)$.*

(a) If $\theta < 1/2q$, then for all $\beta \in (0, 1)$ with $\beta \neq 1/2q - \theta$, we have

$$\begin{aligned} &\mathrm{W}^{1,p}(J; \mathrm{B}^{2\theta}_{qp}(\Omega)) \cap \mathrm{L}^p(J; \mathrm{B}^{2\theta+2}_{qp,\mathrm{D}}(\Omega)) \\ &\hookrightarrow \begin{cases} \mathrm{H}^{1-\beta,p}(J; \mathrm{B}^{2\theta+2\beta}_{qp}(\Omega)), & \text{if } \theta + \beta < 1/2q, \\ \mathrm{H}^{1-\beta,p}(J; \mathrm{B}^{2\theta+2\beta}_{qp,\mathrm{D}}(\Omega)), & \text{if } \theta + \beta > 1/2q. \end{cases} \end{aligned}$$

(b) If $\theta < 1/2 + 1/2q$, then for all $\beta \in (0, 1)$ with $\beta \neq 1/2 + 1/2q - \theta$, we obtain

$$\begin{aligned} &\mathrm{W}^{1,p}(J; \mathrm{B}^{2\theta}_{qp}(\Omega)) \cap \mathrm{L}^p(J; \mathrm{B}^{2\theta+2}_{qp,\mathrm{N}}(\Omega)) \\ &\hookrightarrow \begin{cases} \mathrm{H}^{1-\beta,p}(J; \mathrm{B}^{2\theta+2\beta}_{qp}(\Omega)), & \text{if } \theta + \beta < 1/2 + 1/2q, \\ \mathrm{H}^{1-\beta,p}(J; \mathrm{B}^{2\theta+2\beta}_{qp,\mathrm{N}}(\Omega)), & \text{if } \theta + \beta > 1/2 + 1/2q. \end{cases} \end{aligned}$$

2.5. Elliptic and Parabolic Problems

This section is dedicated to the investigation of elliptic and parabolic boundary value problems by means of maximal L^p-regularity, bounded $\mathcal{H}^\infty$-calculus as well as optimal L^p-L^q estimates. For this, we mainly follow the articles by Denk, Hieber and Prüss [37, 38], Denk et al. [36] as well as Chapter 6 in the monograph of Prüss and Simonett [115].

In the following, let E be an arbitrary Hilbert space with inner product $(\cdot,\cdot)_{\mathrm{E}}$. For $d \in \mathbb{N}$, we consider a bounded domain $\Omega \subset \mathbb{R}^d$ with boundary of class C^2. The notions introduced below can be defined analogously in the context of the whole space or the half space. Moreover, we employ the notation $\mathrm{D} = -\mathrm{i}(\partial_1, \dots, \partial_d)$, and we typically take into account $x \in \Omega$.

Ellipticity of Differential Operators

First, we discuss several notions of ellipticity for second order *differential operators* of the form

$$\mathcal{A}(x,\mathrm{D}) = \sum_{|\alpha|\le 2} a_\alpha(x)\mathrm{D}^\alpha, \tag{2.27}$$

where $a_\alpha \in \mathcal{L}(\mathrm{E})$, and the highest-order coefficients are continuous. We denote the *principal part* by $\mathcal{A}_\#(x,\mathrm{D})$. The associated *symbol* is given by

$$\mathcal{A}_\#(x,\xi) = \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha. \tag{2.28}$$

In the particular situation of *differential operators acting on* $\mathbb{C}^d$*-valued functions*, the principal part takes the shape

$$[\mathcal{A}_\#(x,\mathrm{D})v(x)]_i = \sum_{j,k,l=1}^d a_{ij}^{kl}(x)\mathrm{D}_k\mathrm{D}_l v_j(x), \text{ for } x \in \Omega, \text{ with symbol}$$
$$(\mathcal{A}_\#(x,\xi))_{ij} = \sum_{k,l=1}^d a_{ij}^{kl}(x)\xi_k\xi_l, \text{ for } x \in \Omega.$$

Definition 2.5.1. *Let* $\mathcal{A}(x,\mathrm{D})$ *denote a second order differential operator as in* (2.27)*, with associated symbol of the principal part* $\mathcal{A}_\#(x,\xi)$ *as in* (2.28)*. Then* $\mathcal{A}(x,\mathrm{D})$ *is said to be*

(i) parameter-elliptic *of angle* $\phi \in (0,\pi]$ *if* $\sigma(\mathcal{A}_\#(x,\xi)) \subset \Sigma_\phi$ *for all* $x \in \overline{\Omega}$ *and for all* $\xi \in \mathbb{R}^d$ *such that* $|\xi| = 1$*, and the* angle of ellipticity *of* $\mathcal{A}$ *is*

$$\phi_{\mathcal{A}} := \inf\{\phi \in (0,\pi] : \sigma(\mathcal{A}_\#(x,\xi)) \subset \Sigma_\phi\},$$

(ii) normally elliptic *if it is parameter-elliptic of angle* $\phi_{\mathcal{A}} < {}^{\pi}/_{2}$*, and*

(iii) strongly elliptic *if there is a constant* $c > 0$ *such that*

$$\mathrm{Re}(\mathcal{A}_{\#}(x,\xi)w|w)_{\mathrm{E}} \geq c \cdot \|w\|_{\mathrm{E}}^2$$

is valid for all $x \in \overline{\Omega}$*, for all* $\xi \in \mathbb{R}^d$ *with* $|\xi| = 1$*, and for all* $w \in \mathrm{E}$*.*

Strong ellipticity of $\mathcal{A}$ implies parameter-ellipticity of angle $\phi_{\mathcal{A}} < {}^{\pi}/_{2}$ as the spectrum $\sigma(L)$ of a linear operator L is in particular contained in the *numerical range* $n(L)$ defined by

$$n(L) \coloneqq \overline{\{z \in \mathbb{C} : z = (Lw|w)_{\mathrm{E}} \text{ for some } w \in \mathrm{E} \text{ with } \|w\|_{\mathrm{E}} = 1\}}.$$

The Lopatinskii-Shapiro Condition

As the underlying domain $\Omega \subset \mathbb{R}^d$ has a boundary, it is natural to investigate the resulting boundary value problem with boundary differential operator. For $m \in \{0, 1\}$, $b_\beta \in \mathcal{L}(\mathrm{E})$ and $x \in \partial\Omega$, we consider *boundary differential operators* of the shape

$$\mathcal{B}(x,\mathrm{D}) = \sum_{|\beta| \leq m} b_\beta(x)\mathrm{D}^\beta, \tag{2.29}$$

with *principal part* given by

$$\mathcal{B}_{\#}(x,\mathrm{D}) = \sum_{|\beta| = m} b_\beta(x)\mathrm{D}^\beta.$$

At this stage, the *Lopatinskii-Shapiro condition*, a compatibility condition for the boundary value problem, comes into picture.

Definition 2.5.2. *Let* $\mathcal{A}(x,\mathrm{D})$ *as in* (2.27) *be a parameter-elliptic differential operator with angle of ellipticity* $\phi_{\mathcal{A}} \in [0,\pi)$*, and let* $\mathcal{B}(x,\mathrm{D})$ *denote a boundary differential operator as in* (2.29)*. Then the* Lopatinskii-Shapiro condition *is satisfied if for every* $x_0 \in \partial\Omega$*, the ODE problem in* $\mathbb{R}_+$ *given by*

$$\begin{cases} (\lambda + \mathcal{A}_{\#}(x_0,\xi',\mathrm{D}_y))v(y) = 0, & y > 0, \\ \mathcal{B}_{\#}(x_0,\xi',\mathrm{D}_y)v(0) = g, \end{cases}$$

admits a unique solution $v \in \mathrm{C}_0(\mathbb{R}_+;\mathrm{E})$ *for all* $g \in \mathrm{E}$ *and every* $\lambda \in \overline{\Sigma}_{\phi_{\mathcal{A}}}$ *as well as* $\xi' \in \mathbb{R}^d$ *such that* $|\xi'| + |\lambda| \neq 0$*.*

In the $\mathbb{C}^d$-valued situation, there is another type of ellipticity that guarantees the validity of the Lopatinskii-Shapiro condition when considering Dirichlet or Neumann boundary conditions. This stronger property is called *strong normal ellipticity* and was first introduced by Bothe and Prüss [14].

Definition 2.5.3. *Let $\mathcal{A}(x, \mathrm{D})$ be a differential operator acting on $\mathbb{C}^d$-valued functions with symbol of the principal part $\mathcal{A}_\#(x, \xi)$, so*

$$[\mathcal{A}_\#(x, \mathrm{D})v(x)]_i = \sum_{j,k,l=1}^{d} a_{ij}^{kl}(x)\mathrm{D}_k\mathrm{D}_l v_j(x) \ \ \text{and} \ \ (\mathcal{A}_\#(x, \xi))_{ij} = \sum_{k,l=1}^{d} a_{ij}^{kl}(x)\xi_k\xi_l$$

for $x \in \Omega$. Then the differential operator $\mathcal{A}(x, \mathrm{D})$ is referred to as strongly normally elliptic *if $\mathcal{A}(x, \mathrm{D})$ is strongly elliptic, and if it additionally holds that*

$$\mathrm{Re} \sum_{i,j,k,l=1}^{d} a_{ij}^{kl}(x)(\xi_l u_j - \nu_l v_j)\overline{(\xi_k u_i - \nu_k v_i)} > 0$$

for all $x \in \overline{\Omega}$, for all ξ, $\nu \in \mathbb{R}^d$ with $|\xi| = |\nu| = 1$ as well as $(\xi|\nu) = 0$, and for all u, $v \in \mathbb{C}^d$ such that $\mathrm{Im}(u|v) \neq 0$.

For a proof of the lemma below asserting the validity of the Lopatinskii-Shapiro condition for a strongly normally elliptic $\mathbb{C}^d$-valued differential operator with Dirichlet or Neumann boundary conditions, we refer to [14, Section 3], see also the discussion in [115, Section 6.2.5].

Lemma 2.5.4. *Let $\mathcal{A}(x, \mathrm{D})$ denote a strongly normally elliptic $\mathbb{C}^d$-valued operator, and consider Dirichlet or Neumann boundary conditions, meaning that $\mathcal{B}(x, \mathrm{D}) = \gamma$ or $\mathcal{B}(x, \mathrm{D}) = \partial_\nu$. Then $(\mathcal{A}, \mathcal{B})$ satisfies the Lopatinskii-Shapiro condition for all $x \in \partial\Omega$.*

Maximal Regularity and Bounded $\mathcal{H}^\infty$-Calculus

We are in the position to discuss properties of the L^q-realizations of differential operators of the above form. For this purpose, we require some smoothness and ellipticity conditions.

(S) Let a_α and b_β denote the coefficients of the differential operator $\mathcal{A}$ and the boundary differential operator $\mathcal{B}$ as introduced in (2.27) and (2.29), respectively. We assume that

  (i) $a_\alpha \in \mathrm{C}(\overline{\Omega}; \mathcal{L}(\mathrm{E}))$ for $|\alpha| = 2$,

(ii) $a_\alpha \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{E}))$ for $|\alpha| < 2$, and

(iii) $b_\beta \in \mathrm{C}^{2-m}(\partial\Omega; \mathcal{L}(\mathrm{E}))$ for $|\beta| \leq m$, where $m \in \{0, 1\}$.

(E) We assume the existence of $\phi_{\mathcal{A}} \in [0, \pi)$ such that

(i) $\mathcal{A}$ is parameter-elliptic of angle $\phi_{\mathcal{A}}$ for all $x \in \overline{\Omega}$, and

(ii) $(\mathcal{A}, \mathcal{B})$ satisfies the Lopatinskii-Shapiro condition for all $x \in \partial\Omega$.

In the sequel, we denote by A_B the $\mathrm{L}^q(\Omega; \mathrm{E})$-*realization of* $\mathcal{A}(x, \mathrm{D})$, so

$$\mathrm{D}(A_B) = \left\{ u \in \mathrm{W}^{2,q}(\Omega; \mathrm{E}) : \mathcal{B}(x, \mathrm{D})u = 0 \right\}. \tag{2.30}$$

The next result, which is due to Denk, Hieber and Prüss [37], establishes maximal L^p-regularity of A_B.

Proposition 2.5.5 ([37, Theorem 8.2]). *Consider a Hilbert space* E*,* $d \in \mathbb{N}$ *and* $q \in (1, \infty)$*, and let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain with* C^2*-boundary. Moreover, suppose that the boundary value problem* $(\mathcal{A}, \mathcal{B})$ *satisfies the smoothness and ellipticity conditions (S) and (E) from above for some* $\phi_{\mathcal{A}} \in [0, \pi)$*.*

Then for every $\phi > \phi_{\mathcal{A}}$*, there is* μ_ϕ *such that* $\mu_\phi + A_B$ *is* $\mathcal{R}$*-sectorial with* $\mathcal{R}$*-angle* $\phi_{\mu_\phi + A_B} \leq \phi_{\mathcal{A}}$*. If* $\phi_{\mathcal{A}} < \pi/2$*, then* $\mu_\phi + A_B$ *has the property of maximal regularity in* $\mathrm{L}^p(\mathbb{R}_+; \mathrm{L}^q(\Omega; \mathrm{E}))$ *for every* $p \in (1, \infty)$*.*

If we make slightly stronger smoothness assumptions on the coefficients of the principal part, we can even establish the bounded $\mathcal{H}^\infty$-calculus of A_B up to a shift. The stronger smoothness assumption reads as follows.

(S+) For the coefficients a_α of $\mathcal{A}$, it holds that $a_\alpha \in \mathrm{BUC}^\rho(\overline{\Omega}; \mathcal{L}(\mathrm{E}))$ for some $\rho \in (0, 1)$ and for all α with $|\alpha| = 2$.

The corresponding result by Denk et al. [36] reads as follows.

Proposition 2.5.6 ([36, Theorem 2.3]). *Let* E *be a Hilbert space,* $d \in \mathbb{N}$ *as well as* $q \in (1, \infty)$*, and let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain with* C^2*-boundary. Besides, suppose that the boundary value problem* $(\mathcal{A}, \mathcal{B})$ *satisfies the smoothness and ellipticity conditions (S), (E) and (S+) for some* $\phi_{\mathcal{A}} \in [0, \pi)$*.*

Then for every $\phi > \phi_{\mathcal{A}}$*, there is* μ_ϕ *such that* $\mu_\phi + A_B \in \mathcal{H}^\infty(\mathrm{L}^q(\Omega; \mathrm{E}))$ *with* $\mathcal{H}^\infty$*-angle* $\phi^\infty_{\mu_\phi + A_B} \leq \phi_{\mathcal{A}}$*.*

Optimal L^p-L^q Estimates

The purpose of the following discussion is twofold. On the one hand, we extend the previous considerations from homogeneous to non-homogeneous boundary data. On the other hand, we will see that the assumptions on the data are not only sufficient, but they are in fact necessary to obtain maximal L^p-regularity.

For a bounded domain $\Omega \subset \mathbb{R}^d$ with boundary of class C^2, a second order differential operator $\mathcal{A}(x, D)$ as defined in (2.27), a boundary differential operator $\mathcal{B}(x, D)$ as introduced in (2.29), and for given data (f, g, u_0), we study the parabolic problem

$$\begin{cases} \partial_t u + \omega u + \mathcal{A}(x, D)u = f, & \text{in } \Omega, \\ \mathcal{B}(x, D)u = g, & \text{on } \partial\Omega, \\ u(0) = u_0, & \text{in } \Omega. \end{cases} \tag{2.31}$$

We proceed with a notion of ellipticity for the optimal L^p-L^q estimates.

Definition 2.5.7. *We say that* $(\mathcal{A}(x, D), \mathcal{B}(x, D))$ *is* uniformly normally elliptic *provided* $\mathcal{A}(t, x, D)$ *is normally elliptic for all* $x \in \Omega$*, and the Lopatinskii-Shapiro condition is satisfied.*

We also invoke the following regularity assumptions on the coefficients.

(S') Let a_α and b_β denote the coefficients of the differential operator $\mathcal{A}$ and the boundary differential operator $\mathcal{B}$ as introduced in (2.27) and (2.29), respectively. We assume that

- (i) $a_\alpha \in C(\overline{\Omega}; \mathcal{L}(E))$ for $|\alpha| = 2$,
- (ii) $a_\alpha \in L^\infty(\Omega; \mathcal{L}(E))$ for $|\alpha| < 2$, and
- (iii) $b_\beta \in B^{2\kappa}_{r_j q}(\partial\Omega; \mathcal{L}(E))$ for $|\beta| = j \le m$ and $m \in \{0, 1\}$, and where $r_j \ge q$ as well as $2\kappa > {}^{(d-1)}/_{r_j}$, with $\kappa = 1 - {}^1/_{2q}$ if $m = 0$ and $\kappa = {}^1/_2 - {}^1/_{2q}$ in the situation of $m = 1$.

We remark that (S')(iii) is always satisfied in the case of constant coefficients of the boundary differential operator as for usual Dirichlet or Neumann boundary conditions.

The following result, see [115, Theorem 6.3.2] as well as [115, Section 6.3.4], discusses the sufficiency and necessity of the assumptions on the data (f, g, u_0). For a version with time-dependent coefficients, we refer to [38, Theorem 2.3].

Proposition 2.5.8. *Let* E *be a Hilbert space,* $d \in \mathbb{N}$ *and* $p, q \in (1, \infty)$, *consider a bounded domain* $\Omega \subset \mathbb{R}^d$ *with boundary of class* C^2, *and suppose that the boundary value problem* $(\mathcal{A}, \mathcal{B})$ *is uniformly normally elliptic in the sense of Definition 2.5.7 and satisfies the smoothness assumptions on the coefficients from (S'). Besides, assume that* $\kappa \neq 1/p$ *for* κ *from (S')(iii).*

Then there is $\omega_0 \in \mathbb{R}$ *such that for all* $\omega > \omega_0$, *the problem* (2.31) *admits a unique solution*

$$u \in \mathrm{W}^{1,p}(\mathbb{R}_+; \mathrm{L}^q(\Omega; \mathrm{E})) \cap \mathrm{L}^p(\mathbb{R}_+; \mathrm{W}^{2,q}(\Omega; \mathrm{E})) =: \mathbb{E}_1$$

if and only if the data (f, g, u_0) *satisfy*

(i) $f \in \mathrm{L}^p(\mathbb{R}_+; \mathrm{L}^q(\Omega; \mathrm{E})) =: \mathbb{E}_0$,

(ii) $u_0 \in \mathrm{B}^{2-2/p}_{qp}(\Omega; \mathrm{E}) =: \mathrm{X}_\gamma$,

(iii) $g \in \mathrm{F}^{\kappa}_{pq}(\mathbb{R}_+; \mathrm{L}^q(\partial\Omega; \mathrm{E})) \cap \mathrm{L}^p(\mathbb{R}_+; \mathrm{B}^{2\kappa}_{qq}(\partial\Omega; \mathrm{E})) =: \mathbb{F}$, *where* $\kappa = 1 - 1/2q$ *if* $m = 0$ *in* $\mathcal{B}(x, \mathrm{D})$ *from* (2.29) *as well as* $\kappa = 1/2 - 1/2q$ *if* $m = 1$, *and*

(iv) $\mathcal{B}(x, \mathrm{D})u_0 = g(0)$ *if* $\kappa > 1/p$.

Moreover, the closed graph theorem yields the existence of a constant $C > 0$ *such that for all* $f \in \mathbb{E}_0$, $u_0 \in \mathrm{X}_\gamma$ *and* $g \in \mathbb{F}$, *we have*

$$\|u\|_{\mathbb{E}_1} \leq C \cdot \Big(\|f\|_{\mathbb{E}_0} + \|u_0\|_{\mathrm{X}_\gamma} + \|g\|_{\mathbb{F}}\Big).$$

Furthermore, for A_B *denoting the* $\mathrm{L}^q(\Omega; \mathrm{E})$*-realization of* $\mathcal{A}(x, \mathrm{D})$ *subject to homogeneous boundary conditions as introduced in* (2.30), *the minimal shift* ω_0 *can be chosen equal to the spectral bound of* $-A_B$, *so* $\omega_0 = s(-A_B)$.

The above result remains valid when considering finite intervals $(0, T)$ so that the assumptions on the data also reduce to the case of a finite interval. Again, we refer to the article of Denk, Hieber and Prüss [38] for further details.

$\mathcal{H}^\infty$-Calculus of Elliptic Operators on Closed Manifolds

Next, we focus on the bounded $\mathcal{H}^\infty$-calculus of elliptic operators on closed manifolds. The importance of this paragraph for us becomes apparent when dealing with functions with periodic boundary conditions on the torus, because they can be identified with functions on a closed manifold.

Let X be a compact closed d-dimensional C^2-manifold and $G := (G, \pi, X)$ be a complex C^2-vector bundle over X of rank N with fiber H. Let $\mathcal{A}$ be a linear differential operator of second order with continuous coefficients and principal part $\mathcal{A}_\#$ whose symbol reads as

$$\mathcal{A}\colon T(X)^* \to \mathcal{L}(G).$$

In addition, the L^q-realization is given by

$$A\colon \mathrm{W}^{2,q}(X,G) \to \mathrm{L}^q(X,G).$$

The operator $\mathcal{A}$ is referred to as *ω-elliptic* if for the principal part, we have

$$\sigma(\mathcal{A}_\#) \subset \Sigma_\omega, \text{ for } \xi_x^* \in [T_x(X)^*] \text{ and } x \in X.$$

The following result is due to Duong and Simonett, see [41, Theorem 7.1].

Proposition 2.5.9. *Let $q \in (1,\infty)$, and assume that for $0 \le \omega < \phi < \pi$, the differential operator $\mathcal{A}$ is ω-elliptic. Then there is $\mu > 0$ with the property that $A + \mu \in \mathcal{H}^\infty(\mathrm{L}^q(X,G))$ and $\mathcal{H}^\infty$-angle $\phi^\infty_{A+\mu} = \phi$.*

2.6. Quasilinear Parabolic Evolution Equations

This section is devoted to recalling some theory with regard to quasilinear parabolic evolution equations. We mainly follow Sections 5.1 and 5.3 in the monograph of Prüss and Simonett [115]. With regard to an approach to quasilinear parabolic evolution equations by means of maximal L^p-regularity, let us also mention the works of Clément and Li [28], Prüss [113] as well as the series of articles by Köhne, Prüss and Wilke [78], LeCrone, Prüss and Wilke [89] and Prüss and Wilke [119] in which a framework to quasilinear parabolic evolution equations in weighted L^p-spaces has been developed. The generalized principle of linearized stability as presented in the second part of this section is due to Prüss, Simonett and Zacher [117].

General Setting and Local Strong Well-Posedness

We start by describing the general setting. In fact, let X_0 and X_1 denote Banach spaces such that the embedding $\mathrm{X}_1 \hookrightarrow \mathrm{X}_0$ is dense. The space X_0 will be referred to as *ground space*, while we also call X_1 *regularity space*. Moreover, as already introduced in (2.26), for $p \in (1,\infty)$ and $\mu \in (1/p, 1]$, denote by $\mathrm{X}_{\gamma,\mu}$ the real interpolation space

$$\mathrm{X}_{\gamma,\mu} \coloneqq (\mathrm{X}_0, \mathrm{X}_1)_{\mu-1/p,p}.$$

It is common to use the term *trace space* for the latter interpolation space. In addition, $V_\mu \subset \mathrm{X}_{\gamma,\mu}$ represents an open subset of the trace space $\mathrm{X}_{\gamma,\mu}$.

For $A\colon V_\mu \to \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0)$ and $F\colon V_\mu \to \mathrm{X}_0$, initial data $u_0 \in V_\mu$ and a time interval $J = (0, T)$, $0 < T \leq \infty$, we aim for the local strong well-posedness of the quasilinear abstract Cauchy problem

$$\begin{cases} u'(t) + A(u(t))u(t) = F(u(t)), & \text{for } t \in J, \\ u(0) = u_0, \end{cases} \tag{2.32}$$

on the ground space X_0. More precisely, we are particularly interested in solutions in the so-called *maximal regularity space*

$$\mathbb{E}_{1,\mu} \coloneqq \mathrm{W}^{1,p}_\mu(J; \mathrm{X}_0) \cap \mathrm{L}^p_\mu(J; \mathrm{X}_1),$$

where the weighted L^p- and Sobolev space are as introduced in Section 1.3. We refer to solutions u to (2.32) such that $u \in \mathbb{E}_{1,\mu}$ as *strong solutions*. The suitable space for data in the present framework is the *data space*

$$\mathbb{E}_{0,\mu} \coloneqq \mathrm{L}^p_\mu(J; \mathrm{X}_0).$$

The following proposition asserts the existence of a unique strong local solution to (2.32) under additional assumptions on the operator A as well as the nonlinear term on the right-hand side F. It can for example be found in [115, Theorem 5.1.1].

Proposition 2.6.1. *Consider $p \in (1, \infty)$, $\mu \in (1/p, 1]$ and $u_0 \in V_\mu$, and assume that $(A, F) \in \mathrm{C}^{0,1}(V_\mu; \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0) \times \mathrm{X}_0)$, meaning that there exist constants C_A, $C_F > 0$ such that*

$$\begin{aligned} \|(A(v_1) - A(v_2))w\|_{\mathrm{X}_0} &\leq C_A \cdot \|v_1 - v_2\|_{\mathrm{X}_{\gamma,\mu}} \cdot \|w\|_{\mathrm{X}_1}, \quad \textit{and} \\ \|F(v_1) - F(v_2)\|_{\mathrm{X}_0} &\leq C_F \cdot \|v_1 - v_2\|_{\mathrm{X}_{\gamma,\mu}} \end{aligned}$$

for all v_1, $v_2 \in V_\mu$ and $w \in \mathrm{X}_1$. Moreover, suppose $A(u_0) \in \mathcal{MR}_p(\mathrm{X}_0)$.

Then there are $T = T(u_0) > 0$ and $r = r(u_0) > 0$ with $\overline{\mathbb{B}}_{\mathrm{X}_{\gamma,\mu}}(u_0, r) \subset V_\mu$ such that (2.32) has a unique solution

$$u(\cdot, u_1) \in \mathbb{E}_{1,\mu} \cap \mathrm{C}([0, T]; V_\mu)$$

on $[0, T]$. Moreover, there exists a constant $C = C(u_0) > 0$ such that

$$\|u(\cdot, u_1) - u(\cdot, u_2)\|_{\mathbb{E}_{1,\mu}} \leq C \cdot \|u_1 - u_2\|_{\mathrm{X}_{\gamma,\mu}}$$

for all u_1, $u_2 \in \overline{\mathbb{B}}_{\mathrm{X}_{\gamma,\mu}}(u_0, r)$. For every $\delta \in (0, T)$, we additionally obtain

$$u \in \mathbb{E}_1(\delta, T) \coloneqq \mathrm{W}^{1,p}((\delta, T); \mathrm{X}_0) \cap \mathrm{L}^p((\delta, T); \mathrm{X}_1) \hookrightarrow \mathrm{C}([\delta, T]; \mathrm{X}_{\gamma,\mu}).$$

In other words, the solution regularizes instantly in time.

With regard to applications of the latter theorem, we briefly comment on the Lipschitz properties whose shape in Proposition 2.6.1 was mainly chosen for simplicity of notation.

Remark 2.6.2. *The Lipschitz properties of A and F are only needed locally. This can be recovered from inspecting the proof of [115, Theorem 5.1.1], see also [113, Section 3]. More precisely, it is sufficient to make the following assumptions: Given $u_0 \in V_\mu$, for every $r > 0$ with $\mathbb{B}_{\mathrm{X}_{\gamma,\mu}}(u_0, r) \subset V_\mu$, there exist $C_A(r) > 0$ and $C_F(r) > 0$ such that*

$$\begin{aligned} \|(A(v_1) - A(v_2))w\|_{\mathrm{X}_0} &\le C_A(r) \cdot \|v_1 - v_2\|_{\mathrm{X}_{\gamma,\mu}} \cdot \|w\|_{\mathrm{X}_1}, \quad \textit{and} \\ \|F(v_1) - F(v_2)\|_{\mathrm{X}_0} &\le C_F(r) \cdot \|v_1 - v_2\|_{\mathrm{X}_{\gamma,\mu}} \end{aligned}$$

for all v_1, $v_2 \in \mathbb{B}_{\mathrm{X}_{\gamma,\mu}}(u_0, r)$ and $w \in \mathrm{X}_1$.

The question on the continuation of the solution obtained in Proposition 2.6.1 arises naturally. The following result provides an answer and further yields a characterization of the maximal time interval of existence of the solution. We refer for instance to [115, Corollary 5.1.2].

Corollary 2.6.3. *Under the assumptions of Proposition 2.6.1, suppose in addition that $A(v) \in \mathcal{MR}_p(\mathrm{X}_0)$ for every $v \in V_\mu$. Then the solution u to* (2.32) *resulting from Proposition 2.6.1 has a maximal time interval of existence $J(u_0) = [0, t_+(u_0))$, with the characterization*

(a) global existence, i. e., $t_+(u_0) = \infty$,

(b) $\liminf_{t \to t_+(u_0)} \mathrm{dist}_{\mathrm{X}_{\gamma,\mu}}(u(t), \partial V_\mu) = 0$, or

(c) $\lim_{t \to t_+(u_0)} u(t)$ does not exist in $\mathrm{X}_{\gamma,\mu}$.

The Generalized Principle of Linearized Stability

The remainder of this section is devoted to the study of the stability of equilibrium solutions to (2.32). More precisely, we will introduce the so-called *generalized principle of linearized stability.* Throughout this section, we consider the unweighted situation, i. e., $\mu = 1$. In order to simplify the notation, we will use X_γ to denote the resulting trace space $(\mathrm{X}_0, \mathrm{X}_1)_{1-1/p,p}$, and we will also omit the subscript μ in the open subset $V \subset \mathrm{X}_\gamma$ as well as in the data and maximal regularity space. Moreover, $\mathcal{E} \subset V \cap \mathrm{X}_1$ represents the set of *equilibrium solutions to* (2.32), where $u \in \mathcal{E}$ if and only if

$$u \in V \cap \mathrm{X}_1 \text{ and } A(u)u = F(u).$$

In comparison with the investigation of the local strong well-posedness, we slightly strengthen the regularity assumption on the nonlinear terms (A, F). In fact, we demand that $(A, F) \in \mathrm{C}^1(V; \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0) \times \mathrm{X}_0)$, since the linearization considered in the sequel involves the Fréchet derivatives of A and F.

Let u_* denote an equilibrium solution to (2.32). Then for $u \in \mathrm{X}_1$, the *total linearization around the equilibrium* u_* is defined by

$$A_0 u \coloneqq A(u_*)u + (A'(u_*)u)u_* - F'(u_*)u. \tag{2.33}$$

The generalized principle of linearized stability is closely linked to the following notion of *normal stability* of an equilibrium.

Definition 2.6.4. *Let $u_* \in \mathcal{E}$, assume $(A, F) \in \mathrm{C}^1(V; \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0) \times \mathrm{X}_0)$, and recall A_0 from* (2.33)*. Then u_* is called* normally stable *if*

(a) near u_*, the set of equilibria $\mathcal{E}$ is an m-dimensional C^1-manifold in X_1,

(b) the tangent space for $\mathcal{E}$ at u_* is isomorphic to $\mathrm{N}(A_0)$,

(c) zero is a semi-simple eigenvalue of A_0, i. e., $\mathrm{N}(A_0) \oplus \mathrm{R}(A_0) = \mathrm{X}_0$, and

(d) it holds that $\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+$.

We now state the *generalized principle of linearized stability* due to Prüss, Simonett and Zacher [117], see also [115, Theorem 5.3.1].

Proposition 2.6.5 ([117, Theorem 2.1])**.** *Let $p \in (1, \infty)$, assume that the nonlinearities satisfy $(A, F) \in \mathrm{C}^1(V; \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0) \times \mathrm{X}_0)$, and let $u_* \in \mathcal{E}$ be normally stable in the sense of Definition 2.6.4 such that $A(u_*) \in {}_0\mathcal{MR}_p(\mathrm{X}_0)$.*

Then u_* is stable in X_γ, and there is $\delta > 0$ such that the unique solution u to (2.32) *for initial data $u_0 \in \mathrm{X}_\gamma$ fulfilling $\|u_0 - u_*\|_{\mathrm{X}_\gamma} < \delta$ exists on $\mathbb{R}_+$ and converges to some $u_\infty \in \mathcal{E}$ in X_γ at an exponential rate as $t \to \infty$.*

Below, we comment on the assumption $A(u_*) \in {}_0\mathcal{MR}_p(\mathrm{X}_0)$.

Remark 2.6.6. *The assumption $A(u_*) \in {}_0\mathcal{MR}_p(\mathrm{X}_0)$ in Proposition 2.6.5 can be relaxed to $A(u_*) + \omega \in {}_0\mathcal{MR}_p(\mathrm{X}_0)$ for some $\omega \in \mathbb{R}$ if the underlying space X_0 is a UMD Banach space. We can see this as follows. The idea in the proof* *of [115, Theorem 5.3.1] is to decompose the spectrum $\sigma(A_0)$ into the center part $\sigma_c = \{0\}$ and the stable part $\sigma_s \subset \mathbb{C}_+$. Next, we invoke the corresponding spectral projections $\mathcal{P}^l$, $l \in \{c, s\}$. We denote the part of A_0 in $\mathrm{X}_0^s := \mathcal{P}^s \mathrm{X}_0$ by A_s and observe that $\sigma(A_s) = \sigma_s \subset \mathbb{C}_+$ as well as $0 \in \rho(A_s)$. An inspection of the proof of [115, Theorem 5.3.1] shows that the gist is $A_s \in \mathcal{MR}_p(\mathrm{X}_0^s)$. From Corollary 2.1.23 and the shape of A_0, it follows that $A_0 + \omega_0 \in \mathcal{MR}_p(\mathrm{X}_0)$ for possibly larger $\omega_0 \in \mathbb{R}$ under the present assumptions. On the other hand, we have $A_s + \omega_0 \in \mathcal{MR}_p(\mathrm{X}_0^s)$. Thanks to the relation $\sigma(A_s) = \sigma_s \subset \mathbb{C}_+$, we deduce from Lemma 2.1.15 the validity of $A_0 \in \mathcal{MR}_p(\mathrm{X}_0^s)$.*

2.7. The Viscous Primitive Equations and the Hydrostatic Stokes Operator

This section is devoted to the introduction of the viscous incompressible primitive equations. In particular, we specify on the so-called *hydrostatic Stokes operator* appearing in the linearization of the primitive equations.

The mathematical analysis of the primitive equations was pioneered by Lions, Temam and Wang in a series of articles [93, 94, 96]. The global strong well-posedness of the viscous incompressible primitive equations was shown by Cao and Titi [24]. We also refer to the articles of Kukavica and Ziane [83] or Hieber and Kashiwabara [65] for the consideration of different boundary conditions or the global strong well-posedness in the L^p-L^q framework. For a survey of results and further references, we also refer to [92].

We consider $G = (0, 1) \times (0, 1)$ and $\Omega = G \times (a, b)$, where $-\infty < a < b < \infty$. The full velocity is denoted by $u \colon \Omega \to \mathbb{R}^3$, and we have $u = (v, w)$, for v and w representing the horizontal and vertical velocity, respectively. Moreover, $\pi \colon \Omega \to \mathbb{R}$ is the pressure. For a time interval $(0, T)$, $0 < T \le \infty$, and ∇_{H} representing the horizontal gradient as before, the *viscous incompressible primitive equations* take the shape

$$\begin{cases} \partial_t v + (u \cdot \nabla) v - \Delta v + \nabla_{\mathrm{H}} \pi = 0, & \text{in } (0, T) \times \Omega, \\ \partial_z \pi = 0, & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0, & \text{in } (0, T) \times \Omega, \\ v(0) = v_0, & \text{in } \Omega. \end{cases} \tag{2.34}$$

System (2.34) is completed by boundary conditions. Indeed, v, w and π are assumed to be periodic on the lateral boundary $\Gamma_l = \partial G \times (a, b)$, and w is supposed to satisfy homogeneous Dirichlet boundary conditions on the lower and upper boundary $\Gamma_a := G \times \{a\}$ and $\Gamma_b := G \times \{b\}$, so $w = 0$ on $\Gamma_a \cup \Gamma_b$. Moreover, homogeneous Dirichlet or Neumann boundary conditions on the upper and lower boundary are usually considered for v. The respective parts of the boundary are denoted by Γ_{D} and Γ_{N}, respectively. Hence, $v = 0$ on Γ_{D} and $\partial_z v = 0$ on Γ_{N}, where $\Gamma_{\mathrm{D}} \in \{\emptyset, \Gamma_a, \Gamma_b, \Gamma_a \cup \Gamma_b\}$ and $\Gamma_{\mathrm{N}} = (\Gamma_a \cup \Gamma_b) \setminus \Gamma_{\mathrm{D}}$.

The characteristic feature of the primitive equations is that no evolution equation for w is considered. Instead, due to the typically large horizontal scales and small vertical scales, the so-called *hydrostatic approximation* comes into picture. It reads as $\partial_z \pi = 0$.

Next, we comment on some of the consequences of the hydrostatic approximation on the mathematical analysis. With regard to the homogeneous boundary conditions of w as well as the divergence free condition, it is possible to recover w from the horizontal divergence of v by

$$w(x_{\mathrm{H}}, z) = w(v)(x_{\mathrm{H}}, z) = -\int_a^z \operatorname{div}_{\mathrm{H}} v(x_{\mathrm{H}}, \xi) \,\mathrm{d}\xi. \tag{2.35}$$

As a result of (2.35), the term $w\partial_z v$ appearing in the nonlinearity $u \cdot \nabla v$ in (2.34) has derivatives in both factors.

In the sequel, we use $\overline{v}$ to denote the vertical average of v, so

$$\overline{v}(\cdot) := \frac{1}{b-a} \int_a^b v(\cdot, \xi) \,\mathrm{d}\xi.$$

In that respect, we also introduce the *fracturing part* $\tilde{v}$ defined by $\tilde{v} := v - \overline{v}$.

In view of the homogeneous Dirichlet boundary conditions of w on the upper boundary Γ_b, another consequence of (2.35) is $\operatorname{div}_{\mathrm{H}} \overline{v} = 0$. In other words, the horizontal divergence of the vertical average of v vanishes.

In the second part of this section, we focus on the hydrostatic Stokes operator. This operator arises in the study of the *hydrostatic Stokes equations*

$$\begin{cases} \partial_t v - \Delta v + \nabla_{\mathrm{H}} \pi = 0, & \text{in } (0,T) \times \Omega, \\ \operatorname{div}_{\mathrm{H}} \overline{v} = 0, & \text{in } (0,T) \times \Omega, \\ v(0) = v_0, & \text{in } \Omega. \end{cases} \tag{2.36}$$

With regard to the analysis of (2.36), it is natural to introduce the *hydrostatic solenoidal* L^q*-functions*. The resulting function space is given by the L^q-closure of the smooth hydrostatic functions. This means that we introduce

$$\mathrm{L}^q_{\overline{\sigma}}(\Omega) := \overline{\{v \in \mathrm{C}^\infty_{\mathrm{per}}(\overline{\Omega})^2 \colon \operatorname{div}_{\mathrm{H}} \overline{v} = 0\}}^{\|\cdot\|_{\mathrm{L}^q(\Omega)}}, \tag{2.37}$$

and the subscript $_{\mathrm{per}}$ indicates the periodic boundary conditions on the lateral boundaries for the function spaces $\mathrm{C}^\infty_{\mathrm{per}}(\overline{\Omega})$ and $\mathrm{C}^\infty_{\mathrm{per}}(\overline{G})$. As the classical two-dimensional Helmholtz projection will be relevant in the following, we also recall the space of solenoidal vector fields in $\mathrm{L}^q(G)$ defined by

$$\mathrm{L}^q_\sigma(G) = \overline{\{\overline{v} \in \mathrm{C}^\infty_{\mathrm{per}}(\overline{G})^2 \colon \mathrm{div}_{\mathrm{H}}\, \overline{v} = 0\}}^{\|\cdot\|_{\mathrm{L}^q(G)}}.$$

Let us recall the *classical two-dimensional Helmholtz projection* $\mathbb{P}_{\mathrm{H}}$ on G, so

$$\mathbb{P}_{\mathrm{H}} \colon \mathrm{L}^q(G)^2 \to \mathrm{L}^q_\sigma(G), \ \text{ with } \ \mathbb{P}_{\mathrm{H}}\overline{v} := \overline{v} - \nabla_{\mathrm{H}}\pi.$$

For further information regarding the classical Helmholtz projection, we also refer to [68, Section 2]. We proceed with the *hydrostatic Helmholtz projection*

$$\mathbb{P}v := \mathbb{P}_{\mathrm{H}}\overline{v} + \tilde{v}. \tag{2.38}$$

We observe that $\mathbb{P}$ annihilates the pressure in the v-equation, so $\mathbb{P}(\nabla_{\mathrm{H}}\pi) = 0$, and the space $\mathrm{L}^q(\Omega)^2$ admits the decomposition

$$\mathrm{L}^q(\Omega)^2 = \mathrm{L}^q_\sigma(\Omega) \oplus \left\{\nabla_{\mathrm{H}}\pi : \pi \in \widehat{\mathrm{W}}^{1,q}(G)\right\},$$

where by $\widehat{\mathrm{W}}^{1,q}(G)$, we denote the functions in $\mathrm{L}^1_{\mathrm{loc}}(G)$ such that the (horizontal) gradient lies in $\mathrm{L}^q(G)$. Let us also refer here to [65, Section 4] as well as [62, Section 1.5.1] for further details.

For $p, q \in (1, \infty)$ and $s \in [0, \infty)$, similarly as in Section 1.3, we define the Bessel potential spaces with horizontally periodic boundary conditions by

$$\mathrm{H}^{s,q}_{\mathrm{per}}(\Omega) := \overline{\mathrm{C}^\infty_{\mathrm{per}}(\overline{\Omega})}^{\|\cdot\|_{\mathrm{H}^{s,q}(\Omega)}} \ \text{ and } \ \mathrm{H}^{s,q}_{\mathrm{per}}(G) := \overline{\mathrm{C}^\infty_{\mathrm{per}}(\overline{G})}^{\|\cdot\|_{\mathrm{H}^{s,q}(G)}},$$

and the Sobolev-Slobodeckij spaces $\mathrm{W}^{s,q}_{\mathrm{per}}$ and Besov spaces $\mathrm{B}^s_{qp,\mathrm{per}}$ with periodic boundary conditions on the lateral boundary are defined analogously. For $q \in (1, \infty)$ and $s \in [0, \infty)$, we also define

$$\mathrm{H}^{s,q}_{\mathrm{per,b.c.}}(\Omega) := \begin{cases} \left\{v \in \mathrm{H}^{s,q}_{\mathrm{per}}(\Omega)^2 \colon v|_{\Gamma_{\mathrm{D}}} = 0,\, \partial_z v|_{\Gamma_{\mathrm{N}}} = 0\right\}, & \text{for } s \in (1 + 1/q, 2], \\ \left\{v \in \mathrm{H}^{s,q}_{\mathrm{per}}(\Omega)^2 \colon v|_{\Gamma_{\mathrm{D}}} = 0\right\}, & \text{for } s \in (1/q, 1 + 1/q), \\ \mathrm{H}^{s,q}_{\mathrm{per}}(\Omega)^2, & \text{for } s \in (0, 1/q), \end{cases}$$

and $\mathrm{W}^{s,q}_{\mathrm{per,b.c.}}(\Omega)$ as well as $\mathrm{B}^s_{qp,\mathrm{per,b.c.}}(\Omega)$ are defined likewise.

With the hydrostatic Helmholtz projection $\mathbb{P}$ from (2.38), we introduce the *hydrostatic Stokes operator*

$$A_{\mathrm{b.c.}}v := \mathbb{P}\Delta v, \ \text{ where } \ \mathrm{D}(A_{\mathrm{b.c.}}) := \mathrm{W}^{2,q}_{\mathrm{per,b.c.}}(\Omega) \cap \mathrm{L}^q_\sigma(\Omega). \tag{2.39}$$

In the following, we recall the *bounded* $\mathcal{H}^\infty$*-calculus of the hydrostatic Stokes operator* and the *invertibility* in the case $\Gamma_{\mathrm{D}} \neq \emptyset$ from [50, Theorem 3.1] and [65, Sections 3 and 4], respectively. For the last part of the lemma below, we refer to [50, Section 4].

Lemma 2.7.1. *Let* $A_{\mathrm{b.c.}}$ *denote the hydrostatic Stokes operator as defined in* (2.39)*, and consider* $q \in (1, \infty)$ *as well as* $\mu \geq 0$.

(a) The operator $-A_{\mathrm{b.c.}} + \mu$ *admits a bounded* $\mathcal{H}^\infty$*-calculus on* $\mathrm{L}^q_{\overline{\sigma}}(\Omega)$ *with* $\mathcal{H}^\infty$*-angle* $\phi^\infty_{-A_{\mathrm{b.c.}}+\mu} = 0$ *if* $\mu > 0$.

(b) In the case $\Gamma_{\mathrm{D}} \neq \emptyset$*, we even have* $-A_{\mathrm{b.c.}} \in \mathcal{H}^\infty(\mathrm{L}^q_{\overline{\sigma}}(\Omega))$ *with* $\phi^\infty_{A_{\mathrm{b.c.}}} = 0$*, and* $0 \in \rho(A_{\mathrm{b.c.}})$ *also holds in this situation.*

(c) If $\Gamma_{\mathrm{D}} = \emptyset$*, then* $A_{\mathrm{b.c.}}$ *results from the restriction of* Δ *to* $\mathrm{L}^q_{\overline{\sigma}}(\Omega)$.

In particular, from the relations in (2.14), we deduce the lemma below.

Lemma 2.7.2. *Recall the hydrostatic Stokes operator* $A_{\mathrm{b.c.}}$ *from* (2.39)*, and let* $q \in (1, \infty)$ *as well as* $\mu \geq 0$.

(a) If $\mu > 0$*, then*

  (i) $-A_{\mathrm{b.c.}} + \mu \in \mathcal{BIP}(\mathrm{L}^q_{\overline{\sigma}}(\Omega))$ *with power angle* $\theta_{-A_{\mathrm{b.c.}}+\mu} = 0$,

  (ii) $-A_{\mathrm{b.c.}} + \mu \in \mathcal{RS}(\mathrm{L}^q_{\overline{\sigma}}(\Omega))$ *with* $\phi^{\mathcal{R}}_{-A_{\mathrm{b.c.}}+\mu} = 0$ *and*

  (iii) $-A_{\mathrm{b.c.}} + \mu \in \mathcal{S}(\mathrm{L}^q_{\overline{\sigma}}(\Omega))$ *with* $\phi_{-A_{\mathrm{b.c.}}+\mu} = 0$.

(b) If $\Gamma_{\mathrm{D}} \neq \emptyset$*, then the assertions from (a) remain valid for* $\mu = 0$.

Combining Lemma 2.7.2(a) with Lemma 2.3.4, the observation that the shift does not affect the domain, the shape of the complex interpolation spaces from Lemma 1.3.4 and the interpolation of closed subspaces from Lemma 1.2.4, we derive the following result on the fractional power spaces of the hydrostatic Stokes operator. The values $1/2q$ and $1/2 + 1/2q$ are avoided because of the subtlety in the boundary conditions, see also Section 1.3. For the interpolation of periodic boundary conditions, we refer to [63, Section 4].

Lemma 2.7.3. *Let* $A_{\mathrm{b.c.}}$ *be as in* (2.39) *as well as* $q \in (1, \infty)$*, consider* $\beta \in (0, 1) \setminus \{1/2q, 1/2 + 1/2q\}$*, and let* $\mu > 0$*, or if* $\Gamma_{\mathrm{D}} \neq \emptyset$*, let* $\mu = 0$*. Then*

$$\mathrm{D}((-A_{\mathrm{b.c.}} + \mu)^\beta) \cong \mathrm{H}^{2\beta,q}_{\mathrm{per,b.c.}}(\Omega) \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega) \hookrightarrow \mathrm{H}^{2\beta,q}_{\mathrm{per}}(\Omega) \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega).$$

Similarly, this time applying Lemma 2.3.10, we also get the following characterization of the higher fractional powers.

Lemma 2.7.4. *Recall* $A_{\mathrm{b.c.}}$ *from* (2.39)*, consider* $q \in (1,\infty)$*, and for* $\Gamma_{\mathrm{N}} = \emptyset$*, let* $\beta \in [1, 1 + 1/2q)$*, while for* $\Gamma_{\mathrm{D}} = \emptyset$*, let* $\beta \in [1, 3/2 + 1/2q)$*. Then for* $\mu > 0$*, or* $\mu = 0$ *if* $\Gamma_{\mathrm{N}} = \emptyset$*, it holds that*

$$\mathrm{D}((-A_{\mathrm{b.c.}} + \mu)^{\beta}) \cong \mathrm{H}^{2\beta,q}_{\mathrm{per,b.c.}}(\Omega) \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega) \hookrightarrow \mathrm{H}^{2\beta,q}_{\mathrm{per}}(\Omega) \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega).$$

When considering the resolvent problem associated to (2.36) and taking the vertical average, we obtain

$$\lambda\overline{v} - \Delta_{\mathrm{H}}\overline{v} + \nabla_{\mathrm{H}}\pi = \frac{1}{b-a} \cdot \partial_z v\Big|_{\Gamma_{\mathrm{D}}} \quad \text{and} \quad \mathrm{div}_{\mathrm{H}}\,\overline{v} = 0.$$

An application of $\mathrm{div}_{\mathrm{H}}$ then yields

$$\nabla_{\mathrm{H}}\pi = \frac{1}{b-a} \cdot \nabla_{\mathrm{H}}\Delta_{\mathrm{H}}^{-1}\mathrm{div}_{\mathrm{H}} \cdot \partial_z v\Big|_{\Gamma_{\mathrm{D}}}. \tag{2.40}$$

We also refer to [50, (4.3)] here.

Concerning the linear theory, we also invoke the following relation of the adjoint of the hydrostatic Stokes operator. For clarity, we denote it by $A_{\mathrm{b.c.},q}$. Let us refer here to [65, Remark 4.5(c)] for a proof.

Lemma 2.7.5. *Let* $q \in (1,\infty)$ *and recall* $A_{\mathrm{b.c.},q}$ *from* (2.39)*. Then the adjoint* $A'_{\mathrm{b.c.},q}$ *of* $A_{\mathrm{b.c.},q}$ *satisfies* $A'_{\mathrm{b.c.},q} = A_{\mathrm{b.c.},q'}$*, where* $1/q + 1/q' = 1$*.*

The last part of this section is dedicated to the estimate of the bilinear term appearing in the primitive equations in (2.34). This requires some further preparation in terms of notation, namely so-called *anisotropic function spaces.* For s, $t \geq 0$ and $1 \leq p, q \leq \infty$, we consider the spaces

$$\begin{aligned} \mathrm{W}^{r,q}_z\mathrm{W}^{s,p}_{xy} &:= \mathrm{W}^{r,q}((a,b);\mathrm{W}^{s,p}(G)), \ \text{ endowed with the norms} \\ \|v\|_{\mathrm{W}^{r,q}_z\mathrm{W}^{s,p}_{xy}} &:= \Big\| \|v(\cdot,z)\|_{\mathrm{W}^{s,p}(G)} \Big\|_{\mathrm{W}^{r,q}(a,b)}. \end{aligned}$$

This renders the anisotropic function spaces Banach spaces. Hölder inequality applied independently with respect to z and $x_{\mathrm{H}} = (x,y)$ yields

$$\|fg\|_{\mathrm{L}^q_z\mathrm{L}^p_{xy}} \leq \|f\|_{\mathrm{L}^{q_1}_z\mathrm{L}^{p_1}_{xy}} \cdot \|g\|_{\mathrm{L}^{q_2}_z\mathrm{L}^{p_2}_{xy}} \tag{2.41}$$

for p, p_1, p_2 as well as q, q_1 and q_2 with $1/p_1 + 1/p_2 = 1/p$ and $1/q_1 + 1/q_2 = 1/q$. In addition, it proves useful to employ the embedding relations separately in z and (x,y) to infer that

$$\mathrm{W}_z^{r,q}\mathrm{W}_{xy}^{s,p} \hookrightarrow \mathrm{W}_z^{r_1,q_1}\mathrm{W}_{xy}^{s,p}, \quad \text{for } \mathrm{W}_z^{r,q}(a,b) \hookrightarrow \mathrm{W}_z^{r_1,q_1}(a,b) \text{ and}$$
$$\mathrm{W}_z^{r,q}\mathrm{W}_{xy}^{s,p} \hookrightarrow \mathrm{W}_z^{r,q}\mathrm{W}_{xy}^{s_1,p_1}, \quad \text{for } \mathrm{W}_{xy}^{s,p}(G) \hookrightarrow \mathrm{W}_{xy}^{s_1,p_1}(G).$$

In the case $p = q$, it especially follows that $\mathrm{W}^{r+s,q}(\Omega) \subset \mathrm{W}_z^{r,q}\mathrm{W}_{xy}^{s,q}$. The same relations also hold true if the Sobolev-Slobodeckij spaces are replaced by Bessel potential spaces.

In a similar manner, for s, $t \geq 0$ and $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, we define the anisotropic Besov spaces by $\mathrm{B}^r_{q_0p_0,z}\mathrm{B}^s_{q_1p_1,xy} := \mathrm{B}^r_{q_0p_0}((a,b);\mathrm{B}^s_{q_1p_1}(G))$ and equip them with the analogous norm as above. Again, the above identities remain valid. In particular, we have $\mathrm{B}^{r+s}_{qp}(\Omega) \subset \mathrm{B}^r_{qp,z}\mathrm{B}^s_{qp,xy}$.

Now, for the hydrostatic Helmholtz projection $\mathbb{P}$ from (2.38), we set

$$F(v_1, v_2) := \mathbb{P}((v_1 \cdot \nabla_{\mathrm{H}})v_2 + w(v_1) \cdot \partial_z v_2), \text{ with } F(v) := F(v,v). \tag{2.42}$$

This is precisely the *bilinearity* of the primitive equations after applying the hydrostatic Helmholtz projection. The lemma below addresses the continuity and resulting Lipschitz estimates of this bilinearity. It is analogous to [65, Lemma 5.1] or [51, Lemma 6.1]. Let us stress that the latter two are in different functional analytic set-ups. For convenience, we introduce the abbreviation $\mathrm{X}_\gamma^{\mathrm{pe}} := \mathrm{B}^{2-2/p}_{qp}(\Omega)^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega)$.

Lemma 2.7.6. *Let p, $q \in (1,\infty)$ be such that $2/p + 1/q < 1$, and consider the bilinearity F from* (2.42)*. Then there is a constant $C > 0$ such that*

(a) $\|F(v)\|_{\mathrm{L}^q_{\overline{\sigma}}(\Omega)} \leq C \cdot \|v\|^2_{\mathrm{X}_\gamma^{\mathrm{pe}}}$ for all $v \in \mathrm{X}_\gamma^{\mathrm{pe}}$, and

(b) for all v_1, $v_2 \in \mathrm{X}_\gamma^{\mathrm{pe}}$, it is valid that

$$\|F(v_1) - F(v_2)\|_{\mathrm{L}^q_{\overline{\sigma}}(\Omega)} \leq C \cdot \Big(\|v_1\|_{\mathrm{X}_\gamma^{\mathrm{pe}}} + \|v_2\|_{\mathrm{X}_\gamma^{\mathrm{pe}}}\Big) \cdot \|v_1 - v_2\|_{\mathrm{X}_\gamma^{\mathrm{pe}}}.$$

Proof. First, we note that the assertion of (b) readily follows from (a) since

$$\begin{aligned}\|F(v_1) - F(v_2)\|_{\mathrm{L}^q_{\overline{\sigma}}(\Omega)} &\leq \|F(v_1, v_1 - v_2)\|_{\mathrm{L}^q_{\overline{\sigma}}(\Omega)} + \|F(v_1 - v_2, v_2)\|_{\mathrm{L}^q_{\overline{\sigma}}(\Omega)}\\ &\leq C \cdot \Big(\|v_1\|_{\mathrm{X}_\gamma^{\mathrm{pe}}} + \|v_2\|_{\mathrm{X}_\gamma^{\mathrm{pe}}}\Big) \cdot \|v_1 - v_2\|_{\mathrm{X}_\gamma^{\mathrm{pe}}}.\end{aligned}$$

By the boundedness of $\mathbb{P}\colon \mathrm{L}^q(\Omega)^2 \to \mathrm{L}^q_{\overline{\sigma}}(\Omega)$ in conjunction with the inclusion $\mathrm{X}_\gamma^{\mathrm{pe}} \subset \mathrm{B}^{2-2/p}_{qp}(\Omega)^2$, it is sufficient to estimate $(v \cdot \nabla_{\mathrm{H}})v$ and $w(v) \cdot \partial_z v$ in $\mathrm{L}^q(\Omega)$ by $\|v\|^2_{\mathrm{B}^{2-2/p}_{qp}(\Omega)}$.

For this purpose, we deduce from the assumption on p and q the existence of $\varepsilon > 0$ with $2 - {}^2\!/\!_p - \varepsilon - {}^3\!/\!_q \geq -{}^3\!/\!_{3q}$ and $2 - {}^2\!/\!_p - \varepsilon - {}^3\!/\!_q \geq 1 - {}^2\!/\!_q$. Hence, the embeddings from (1.4) and (1.6) yield

$$\begin{aligned}&\mathrm{B}_{qp}^{2-2/p}(\Omega) \hookrightarrow \mathrm{B}_{q3q}^{2-2/p-\varepsilon}(\Omega) \hookrightarrow \mathrm{L}^{3q}(\Omega) \text{ and}\\ &\mathrm{B}_{qp}^{2-2/p}(\Omega) \hookrightarrow \mathrm{B}_{q3q}^{2-2/p-\varepsilon/2}(\Omega) \hookrightarrow \mathrm{H}^{1,3q/2+\varepsilon/2}(\Omega) \hookrightarrow \mathrm{W}^{1,3q/2}(\Omega).\end{aligned}$$

Together with Hölder's inequality, the latter embeddings lead to

$$\|(v \cdot \nabla_{\mathrm{H}})v\|_{\mathrm{L}^q(\Omega)} \leq C_1 \cdot \|v\|_{\mathrm{L}^{3q}(\Omega)} \|v\|_{\mathrm{W}^{1,3q/2}(\Omega)} \leq C_2 \cdot \|v\|^2_{\mathrm{B}_{qp}^{2-2/p}(\Omega)},$$

establishing the desired estimate of the first addend.

The treatment of the second addend requires slightly more effort. First, we make use of Hölder's inequality in anisotropic spaces (2.41) to obtain

$$\|w \cdot \partial_z v\|_{\mathrm{L}^q(\Omega)} \leq \|w\|_{\mathrm{L}_z^\infty \mathrm{L}_{xy}^{2q}} \cdot \|\partial_z v\|_{\mathrm{L}_z^q \mathrm{L}_{xy}^{2q}}.$$

In the sequel, we provide separate estimates of the two factors. From the embedding $\mathrm{W}^{1,q}(a,b) \hookrightarrow \mathrm{L}^\infty(a,b)$, Poincaré's inequality applied to $\partial_z w$ thanks to $w = 0$ on $\Gamma_a \cup \Gamma_b$, the embedding $\mathrm{B}_{qp}^{1-2/p-\varepsilon/2}(G) \hookrightarrow \mathrm{H}^{1+\varepsilon/2,2q}(G) \hookrightarrow \mathrm{W}^{1,2q}(G)$, which follows in a similar way as above, the aforementioned relations of the anisotropic function spaces and the condition $\operatorname{div}_{\mathrm{H}} v + \partial_z w = 0$, we conclude

$$\begin{aligned}\|w\|_{\mathrm{L}_z^\infty \mathrm{L}_{xy}^{2q}} &\leq C_3 \cdot \|w\|_{\mathrm{W}_z^{1,q} \mathrm{L}_{xy}^{2q}} \leq C_4 \cdot \|\partial_z w\|_{\mathrm{L}_z^q \mathrm{L}_{xy}^{2q}} \leq C_5 \cdot \|\operatorname{div}_{\mathrm{H}} v\|_{\mathrm{L}_z^q \mathrm{L}_{xy}^{2q}}\\ &\leq C_6 \cdot \|v\|_{\mathrm{L}_z^q \mathrm{W}_{xy}^{1,2q}} \leq C_7 \cdot \|v\|_{\mathrm{B}_{qp,z}^{\varepsilon/2} \mathrm{B}_{qp,xy}^{2-2/p-\varepsilon/2}} \leq C_8 \cdot \|v\|_{\mathrm{B}_{qp}^{2-2/p}(\Omega)}.\end{aligned}$$

On the other hand, the conditions on p and q again imply that there exists $\varepsilon > 0$ sufficiently small such that $1 - {}^2\!/\!_p - \varepsilon - {}^2\!/\!_q \geq -{}^2\!/\!_{2q}$, so (1.4) and (1.6) result in $\mathrm{B}_{qp}^{1-2/p-\varepsilon/2}(G) \hookrightarrow \mathrm{B}_{q2q}^{1-2/p-\varepsilon}(G) \hookrightarrow \mathrm{L}^{2q}(G)$. Combining this embedding and the above relations of the anisotropic function spaces, we get

$$\|\partial_z v\|_{\mathrm{L}_z^q \mathrm{L}_{xy}^{2q}} \leq C_9 \cdot \|v\|_{\mathrm{W}_z^{1,q} \mathrm{L}_{xy}^{2q}} \leq C_{10} \cdot \|v\|_{\mathrm{B}_{qp,z}^{1+\varepsilon/2} \mathrm{B}_{qp,xy}^{1-2/q-\varepsilon/2}} \leq C_{11} \cdot \|v\|_{\mathrm{B}_{qp}^{2-2/p}(\Omega)}.$$

A concatenation of the previous two estimates finishes the proof. □

The following lemma discusses a cancellation law which proves useful when establishing energy estimates of the primitive equations.

Lemma 2.7.7. *Let $u = (v, w)$ be a solution to the primitive equations* (2.34), *so $v \in \mathrm{W}^{2,q}_{\mathrm{per}}(\Omega)$ and $w = w(v) = 0$ on the upper and lower boundary. Then*

$$\int_\Omega (u \cdot \nabla)v \cdot v \,\mathrm{d}(x_{\mathrm{H}}, z) = 0.$$

Proof. Making use of $\operatorname{div} u = 0$ as well as the divergence theorem, and invoking the periodic boundary conditions and $w = 0$ on the upper and lower boundary, we derive

$$\int_\Omega (u \cdot \nabla) v \cdot v \, \mathrm{d}(x_\mathrm{H}, z) = \frac{1}{2} \int_\Omega \operatorname{div}\left(|v|^2 u\right) \mathrm{d}(x_\mathrm{H}, z) = \frac{1}{2} \int_{\partial\Omega} |v|^2 \binom{v}{w} \cdot \nu \, \mathrm{dS} = 0,$$

for ν denoting the outer normal vector. This proves the claim. □

Geophysical Flow Models

CHAPTER 3

Analysis of the Fully Parabolic Regularized Hibler Model

This chapter presents the rigorous analysis of the fully parabolic regularized Hibler model. The first steps for this are the introduction of the model with all its features, the analysis of the operator emerging from the internal ice stress, the reformulation as an abstract Cauchy problem as well as the maximal regularity of the associated linearized operator matrix. In a second step, the local strong well-posedness and finally also the global strong well-posedness of a simplified version of the model without external forces and close to constant equilibria are established. This chapter is essential in this thesis, because it not only settles the strong well-posedness, but also contains many results on the linearized operator matrix, especially with regard to the *Hibler operator* corresponding to the internal ice stress, and the nonlinear estimates are used throughout this thesis as well.

The structure of this chapter is as follows. In Section 3.1, we give an overview of the literature on Hibler's sea ice model with regard to the modeling, simulation, numerical and mathematical analysis. Section 3.2 is dedicated to introducing the model variables, the stress tensor σ in (3.3) as well as its regularized version σ_δ in (3.5), the momentum equations, the balance laws and finally the complete system in (3.10). The focal point of Section 3.3 is the analysis of the differential operator related to $\operatorname{div}_{\mathrm{H}} \sigma_\delta$. We briefly elaborate on the derivation of the operator and then discuss its ellipticity properties. From there, we also deduce the bounded $\mathcal{H}^\infty$-calculus in Proposition 3.3.4 and the maximal regularity as well as further consequences in Corollary 3.3.5. It is the purpose of Section 3.4 to rewrite the fully parabolic regularized Hibler model in operator form in (3.26) so that it fits into the general framework presented in Section 2.6. Moreover, we discuss the maximal regularity of the linearized operator matrix in Proposition 3.4.1. Section 3.5 is devoted to establishing the first main result of this chapter, Theorem 3.5.2, on the local strong well-posedness by showing estimates of the nonlinear terms and invoking the maximal regularity in order to apply the general local well-posedness result, Proposition 2.6.1. The last section, Section 3.6, introduces a simplified model without external forces in (3.35), which is also rewritten as a quasilinear abstract Cauchy problem in (3.39). Thereafter, the normal stability of constant equilibria is shown, leading to the second main result of the chapter, Theorem 3.6.6, on the global strong well-posedness of the simplified model for initial data close to such equilibria.

The results obtained in this chapter have partially been obtained in the master thesis [17] and published in the joint article with Karoline Disser, Robert Haller-Dintelmann and Matthias Hieber [18]. More precisely, the derivation and ellipticity properties of Hibler's operator are studied in detail in [17, Chapter 4 and 5], while the maximal regularity of the linearized operator matrix, the local strong well-posedness and the global strong well-posedness can also be found in [18, Section 6 and 7].

3.1. A Short Bibliographic Overview

The large-scale dynamic-thermodynamic viscous-plastic sea ice model under consideration in this thesis was introduced by William D. Hibler III in the seminal article [60]. Since then, this model has been used frequently for describing sea ice on large scales in climate science. For further information and literature on the role of sea ice models in climate science, we refer to the preface and the survey article of Hunke, Lipscomb and Turner [72].

One of the main features of Hibler's model is the underlying rheology, leading to a description of sea ice as a viscous-plastic material. This is related to the complex mechanical and thermodynamic behavior of sea ice, resulting from freezing sea water and consisting of pure ice, liquid brine, air pockets and solid salt. The precise formation of sea ice is highly dependent on the laminar or turbulent environmental conditions. In addition to the aforementioned article by Hibler [60], we also refer to the articles of Feltham [44] and Golden [52]. A survey of the modeling of sea ice can be found in the overview paper of Golden et al. [53].

Even though the rigorous mathematical analysis of Hibler's sea ice model has only recently started, there is a plethora of literature on simulation and numerical analysis by various communities. We do not provide an exhaustive literature overview here, but we mention the work of Kreyscher et al. [82], Lemieux and Tremblay [90], Losch and Danilov [98], Bouchat and Tremblay [15], Danilov et al. [34], Kimmritz, Danilov and Losch [77], Mehlmann and Richter [106], Seinen and Khouider [122], Mehlmann [103], Mehlmann and Korn [105], Mehlmann et al. [104], Yaremchuk and Panteleev [133], Shih [125], Shih et al. [126] or Bertrand and Schneider [10] as well as the references therein. In some of the above articles, an *elastic-viscous-plastic sea ice model* with additional elasticity is used for the numerical analysis. This model has been first introduced by Hunke and Dukowicz [71] in order to circumvent the degeneration of the stress tensor in Hibler's *viscous-plastic model*. We discuss this property of the stress tensor as well as our way to bypass in detail in Section 3.2. As demonstrated in [98], the original viscous-plastic sea ice model and its elastic-viscous-plastic counterpart exhibit a different behavior in terms of numerical analysis. We also mention the recent article of Ringeisen, Losch and Tremblay [120] using teardrop and parabolic lens yield curves for the numerical analysis of viscous-plastic sea ice models.

Concerning the mathematical analysis of Hibler's model, the first works in this direction are due to Gray [54] and Guba, Lorenz and Sulsky [55]. In fact, the authors investigated particular simplified submodels in the context of hyperbolic systems and reported ill-posedness. Recently, Chatta, Khouider and Kesri [25] established the linear well-posedness of one-dimensional viscous-plastic equations, where the Heaviside function cut-off for the viscosity coefficient is replaced by a hyperbolic tangent. The first works to address the complete system, though employing certain regularizations, are the joint article with Disser, Haller-Dintelmann and Hieber [18] and the paper of Liu, Thomas and Titi [97]. While the first article uses a common regularization of the stress tensor as also present in [82] or [105] as well as diffusion terms in the balance laws for the mean ice thickness and the ice compactness, the second article relies on a different regularization of the stress tensor, which seems to be further away from the original viscous-plastic one, and does not include parabolicity in the balance laws. Moreover, in [18], local strong well-posedness of the fully parabolic regularized model and global strong well-posedness for initial data close to constant equilibria in the absence of external forces are shown by means of the theory of quasilinear evolution equations. On the other hand, Liu, Thomas and Titi [97] prove local strong well-posedness of their model under consideration by a direct approximation.

Let us also mention the recent paper of Piersanti and Temam [111] introducing and analyzing a model for the thickness evolution of grounded shallow ice sheet by means of a semi-discrete scheme and the penalty method.

3.2. Introduction of the Model

For the introduction of the model, we follow the article of Hibler [60]. In this chapter and also in Chapter 6 and Section 7.2, we assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\Omega$ of class C^2. Moreover, for $T \in (0, \infty)$, we fix a time interval $(0, T)$. The first model variable is the *horizontal sea ice velocity*

$$v_{\mathrm{ice}} \colon (0,T) \times \Omega \to \mathbb{R}^2.$$

The other model variables are the *mean ice thickness*

$$h \colon (0,T) \times \Omega \to [\kappa, \infty),$$

where $\kappa > 0$ denotes a small parameter, and the *ice compactness*

$$a \colon (0,T) \times \Omega \to (0,1).$$

It describes the horizontal average of area covered by thick ice, where ice is considered *thick* if $h \geq h_\bullet$ for some specific $h_\bullet > 0$.

Let us briefly comment on the ranges of the latter two variables. The assumption on h means that the average ice volume per control area is bounded from below by κ. This amounts to saying that there is at least some sea ice in each control area, which is also indicated by the variable a only attaining strictly positive values. At the same time, as $a < 1$, it is implicit that each control area is not completely covered by thick ice.

One of the most characteristic features of Hibler's sea ice model is the stress tensor with its viscous-plastic rheology, giving rise to a quasilinear degenerate problem. In the sequel, we introduce this stress tensor. By

$$\varepsilon = \varepsilon(v_{\mathrm{ice}}) = \frac{1}{2}\left(\nabla_{\mathrm{H}} v_{\mathrm{ice}} + (\nabla_{\mathrm{H}} v_{\mathrm{ice}})^\top\right),$$

we denote the deformation tensor associated to the horizontal sea ice velocity. Next, P represents the *ice strength*, and for given constants p^*, $c_\bullet > 0$, it takes the explicit shape

$$P = P(h,a) = p^* h \mathrm{e}^{-c_\bullet(1-a)}. \tag{3.1}$$

Moreover, $e > 1$ is the ratio of the major to minor axes of the elliptical yield curve on which the principle components of the stress lie. We set

$$\triangle^2(\varepsilon) := \left(\varepsilon_{11}^2 + \varepsilon_{22}^2\right)\left(1 + \frac{1}{e^2}\right) + \frac{4}{e^2}\varepsilon_{12}^2 + 2\varepsilon_{11}\varepsilon_{22}\left(1 - \frac{1}{e^2}\right).$$

The *bulk* and *shear viscosities* are then given by

$$\zeta(\varepsilon, P) = \frac{P(h,a)}{2\triangle(\varepsilon)} \quad \text{and} \quad \eta(\varepsilon, P) = e^{-2}\zeta(\varepsilon, P). \tag{3.2}$$

The *stress tensor* σ is constituted by the aforementioned bulk and shear viscosities, the deformation tensor as well as the ice strength, so

$$\sigma = 2\eta(\varepsilon, P)\varepsilon + [\zeta(\varepsilon, P) - \eta(\varepsilon, P)]\,\mathrm{tr}(\varepsilon)\,\mathrm{Id}_2 - \frac{P}{2}\,\mathrm{Id}_2, \tag{3.3}$$

where Id_2 represents the identity matrix in two dimensions.

The bulk and shear viscosities ζ and η from (3.2) exhibit a singularity for $\triangle(\varepsilon)$ tending to zero. Hence, the stress tensor also degenerates even though the above law describes an idealized viscous-plastic material. With regard to the rigorous and numerical analysis, this phenomenon calls for a remedy in a first step. To this end, similarly as in [82] and [105], we set

$$\triangle_\delta(\varepsilon) := \sqrt{\delta + \triangle^2(\varepsilon)} \tag{3.4}$$

for $\delta > 0$. Equipped with $\triangle_\delta(\varepsilon)$, we introduce the regularized viscosities

$$\zeta_\delta(\varepsilon, P) := \frac{P(h,a)}{2\triangle_\delta(\varepsilon)} \quad \text{and} \quad \eta_\delta(\varepsilon, P) := e^{-2}\zeta_\delta(\varepsilon, P).$$

For $\delta > 0$, this leads to the *regularized stress tensor*

$$\sigma_\delta := 2\eta_\delta(\varepsilon, P)\varepsilon + [\zeta_\delta(\varepsilon, P) - \eta_\delta(\varepsilon, P)]\,\mathrm{tr}(\varepsilon)\,\mathrm{Id}_2 - \frac{P}{2}\,\mathrm{Id}_2\,. \tag{3.5}$$

In the sequel, we will assume that the *ice density* $\rho_{\mathrm{ice}} > 0$ is constant. The *ice mass* m_{ice} is given by $m_{\mathrm{ice}} = \rho_{\mathrm{ice}} h$. As in [60], we take into account the momentum equation of the form "mass times acceleration equals force", i. e.,

$$m_{\mathrm{ice}} \frac{\mathrm{D}v_{\mathrm{ice}}}{\mathrm{D}t} = F, \text{ with material derivative } \frac{\mathrm{D}v_{\mathrm{ice}}}{\mathrm{D}t} = \partial_t v_{\mathrm{ice}} + (v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}})v_{\mathrm{ice}},$$

and F represents the forcing term. The latter consists of the internal ice stress as well as the external forcing terms F^{ice}, so for σ as introduced in (3.3), it takes the shape

$$F = \mathrm{div}_{\mathrm{H}}\,\sigma + F^{\mathrm{ice}}.$$

In the remainder of this thesis, we focus on the regularized version of the model. Hence, we will replace σ by σ_δ from (3.5) in the above.

We now discuss the external forcing terms. For a positive Coriolis parameter $c_{\mathrm{cor}} > 0$, the term representing the Coriolis force is given by $-m_{\mathrm{ice}} c_{\mathrm{cor}} v_{\mathrm{ice}}^{\perp}$, where $v^{\perp} = (-v_2, v_1)^{\top}$ for $v = (v_1, v_2) \in \mathbb{R}^2$. Besides, for the gravity g and $H \colon (0,T) \times \Omega \to [0,\infty)$ denoting the sea surface dynamic height, the force resulting from the varying sea surface tilt is $-m_{\mathrm{ice}} g \nabla_{\mathrm{H}} H$. The remaining external forcing terms are due to atmospheric winds and ocean currents. Throughout this chapter as well as Chapter 4, Chapter 6 and Section 7.2, we assume that the atmospheric wind and ocean currents are external, whereas Chapter 5 is centered around the analysis of a coupled model with internalized air and ocean velocities. In the sequel, V_{atm} and V_{ocn} denote the externally given velocities of the surface winds of the atmosphere and the surface velocity of the ocean. With the air and ocean drag coefficients C_{atm} and C_{ocn}, the densities for air and sea water ρ_{atm} and ρ_{ocn} as well as the rotation matrices R_{atm} and R_{ocn}, the *atmospheric wind* and *ocean force* are

$$
\begin{aligned}
\tau_{\mathrm{atm}} &= \rho_{\mathrm{atm}} C_{\mathrm{atm}} |V_{\mathrm{atm}}| R_{\mathrm{atm}} V_{\mathrm{atm}} \quad \text{and} \\
\tau_{\mathrm{ocn}}(v_{\mathrm{ice}}) &= \rho_{\mathrm{ocn}} C_{\mathrm{ocn}} |V_{\mathrm{ocn}} - v_{\mathrm{ice}}| R_{\mathrm{ocn}} (V_{\mathrm{ocn}} - v_{\mathrm{ice}}).
\end{aligned} \tag{3.6}
$$

The latter two take the shape of drag conditions, where the assumption that the atmospheric wind velocity V_{atm} is typically much higher than the sea ice velocity v_{ice} is underlying. For brevity, we also write $\tau_{\mathrm{ice}} := \tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(v_{\mathrm{ice}})$.

In summary, the *momentum equation* investigated in the sequel is given by

$$
m_{\mathrm{ice}} \frac{\mathrm{D} v_{\mathrm{ice}}}{\mathrm{D} t} = \operatorname{div}_{\mathrm{H}} \sigma_\delta - m_{\mathrm{ice}} c_{\mathrm{cor}} v_{\mathrm{ice}}^{\perp} - m_{\mathrm{ice}} g \nabla_{\mathrm{H}} H + \tau_{\mathrm{ice}}. \tag{3.7}
$$

The sea ice model is completed by balance laws for the mean ice thickness and the ice compactness. Before providing these equations, we introduce the *thermodynamic source terms* S_{h} and S_{a}. For a function $f_{\mathrm{gr}} \in \mathrm{C}_{\mathrm{b}}^1([0,\infty))$ describing the ice growth rate, see for example the one suggested by Hibler [60], and the parameter $\kappa > 0$, the thermodynamic terms take the shape

$$
\begin{aligned}
S_{\mathrm{h}}(h,a) &= f_{\mathrm{gr}}\left(\frac{h}{a}\right) a + (1-a) f_{\mathrm{gr}}(0) \quad \text{and} \\
S_{\mathrm{a}}(h,a) &= \begin{cases} \frac{f_{\mathrm{gr}}(0)}{\kappa}(1-a), & \text{if } f_{\mathrm{gr}}(0) > 0, \\ 0, & \text{if } f_{\mathrm{gr}}(0) < 0, \end{cases} + \begin{cases} 0, & \text{if } S_{\mathrm{h}} > 0, \\ \frac{a}{2h} S_{\mathrm{h}}, & \text{if } S_{\mathrm{h}} < 0. \end{cases}
\end{aligned} \tag{3.8}
$$

For simplicity of the presentation, we assume the ice growth rate f_{gr} to be independent of time here.

In this chapter as well as in Chapter 4, Chapter 5 and Section 7.2, we consider the fully parabolic variant of Hibler's model as introduced in [60] for numerical stabilization, while Chapter 6 is dedicated to the study of the parabolic-hyperbolic problem. Therefore, at this stage, for d_{h}, $d_{\mathrm{a}} > 0$ constant, we consider the *balance laws*

$$\begin{cases} \partial_t h + \mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} h) = S_{\mathrm{h}}(h,a) + d_{\mathrm{h}} \Delta_{\mathrm{H}} h, \\ \partial_t a + \mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} a) = S_{\mathrm{a}}(h,a) + d_{\mathrm{a}} \Delta_{\mathrm{H}} a. \end{cases} \tag{3.9}$$

The sea ice model consists of the momentum equation as introduced in (3.7) and the balance laws from (3.9). Furthermore, it is completed by boundary and initial conditions. More precisely, we assume that v_{ice} is subject to homogeneous Dirichlet boundary conditions, so

$$v_{\mathrm{ice}} = 0, \ \text{ on } \ (0,T) \times \partial\Omega,$$

meaning that the sea ice is supposed to be at rest at the boundary. On the other hand, h and a fulfill Neumann boundary conditions, i. e.,

$$\partial_\nu h = \partial_\nu a = 0, \ \text{ on } \ (0,T) \times \partial\Omega.$$

The initial conditions read as

$$v_{\mathrm{ice}}(0) = v_{\mathrm{ice},0}, \ \ h(0) = h_0 \ \text{ and } \ a(0) = a_0, \ \text{ on } \ \Omega.$$

Finally, we provide the complete system, referred to as *fully parabolic regularized model*. Thanks to $h \geq \kappa$, it is valid that $m_{\mathrm{ice}} > 0$, and we may divide the momentum equation by m_{ice}. The system is given by

$$\begin{cases} \partial_t v_{\mathrm{ice}} + (v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice}} = \frac{1}{m_{\mathrm{ice}}} \mathrm{div}_{\mathrm{H}}\, \sigma_\delta - c_{\mathrm{cor}} v_{\mathrm{ice}}^{\perp} & \\ \qquad\qquad - g \nabla_{\mathrm{H}} H + \frac{1}{m_{\mathrm{ice}}} \tau_{\mathrm{ice}}, & \text{in } (0,T) \times \Omega, \\ \partial_t h + \mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} h) = S_{\mathrm{h}}(h,a) + d_{\mathrm{h}} \Delta_{\mathrm{H}} h, & \text{in } (0,T) \times \Omega, \\ \partial_t a + \mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} a) = S_{\mathrm{a}}(h,a) + d_{\mathrm{a}} \Delta_{\mathrm{H}} a, & \text{in } (0,T) \times \Omega, \\ v_{\mathrm{ice}} = 0, \ \ \partial_\nu h = \partial_\nu a = 0, & \text{on } (0,T) \times \partial\Omega, \\ v_{\mathrm{ice}}(0) = v_{\mathrm{ice},0}, \ \ h(0) = h_0, \ \ a(0) = a_0, & \text{in } \Omega. \end{cases} \tag{3.10}$$

3.3. Hibler's Operator

This section is devoted to the differential operator emerging from the internal ice stress. Since parts of this section have already been included in the master thesis [17], we are rather brief here and refer to [17] for the proofs of some results. First, we introduce this operator and then investigate its L^q-realization by discussing ellipticity properties of the principle part and exploiting the theory on parabolic boundary value problems from Section 2.5.

In order to deduce the shape of the quasilinear second order operator arising from $\mathrm{div}_{\mathrm{H}}\, \sigma_\delta$, we introduce a matrix $\mathbb{S}\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ such that

$$\mathbb{S}\varepsilon = \begin{pmatrix} \left(1+\frac{1}{e^2}\right)\varepsilon_{11} + \left(1-\frac{1}{e^2}\right)\varepsilon_{22} & \frac{1}{e^2}\left(\varepsilon_{12}+\varepsilon_{21}\right) \\ \frac{1}{e^2}\left(\varepsilon_{12}+\varepsilon_{21}\right) & \left(1-\frac{1}{e^2}\right)\varepsilon_{11} + \left(1+\frac{1}{e^2}\right)\varepsilon_{22} \end{pmatrix}. \tag{3.11}$$

Upon identifying $\varepsilon \in \mathbb{R}^{2\times 2}$ with the vector $(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22})^\top \in \mathbb{R}^4$, the action of $\mathbb{S}$ to ε amounts to the multiplication by the matrix

$$\mathbb{S} = \left(\mathbb{S}_{ij}^{kl}\right) = \begin{pmatrix} 1+\frac{1}{e^2} & 0 & 0 & 1-\frac{1}{e^2} \\ 0 & \frac{1}{e^2} & \frac{1}{e^2} & 0 \\ 0 & \frac{1}{e^2} & \frac{1}{e^2} & 0 \\ 1-\frac{1}{e^2} & 0 & 0 & 1+\frac{1}{e^2} \end{pmatrix}.$$

We then rewrite

$$\begin{aligned} \triangle^2(\varepsilon) = \varepsilon^\top \mathbb{S}\varepsilon &= \sum_{i,j,k,l=1}^{2} \varepsilon_{ik}\mathbb{S}_{ij}^{kl}\varepsilon_{jl} \\ &= \left(\varepsilon_{11}+\varepsilon_{22}\right)^2 + \frac{1}{e^2}\left(\varepsilon_{11}-\varepsilon_{22}\right)^2 + \frac{1}{e^2}\left(\varepsilon_{12}+\varepsilon_{21}\right)^2. \end{aligned} \tag{3.12}$$

With $S(\varepsilon, P) := \frac{P}{2}\frac{\mathbb{S}\varepsilon}{\triangle(\varepsilon)}$, the stress tensor σ from (3.3) has the representation

$$\sigma(\varepsilon, P) = S(\varepsilon, P) - \frac{P}{2}\,\mathrm{Id}_2\,.$$

Correspondingly, for the regularized situation, we set

$$S_\delta = S_\delta(\varepsilon, P) := \frac{P}{2}\frac{\mathbb{S}\varepsilon}{\triangle_\delta(\varepsilon)}. \tag{3.13}$$

Hibler's operator is then defined by

$$\mathbb{A}^{\mathrm{H}} v_{\mathrm{ice}} := \frac{1}{\rho_{\mathrm{ice}} h} \cdot \mathrm{div}_{\mathrm{H}}\, S_\delta = \frac{1}{\rho_{\mathrm{ice}} h} \cdot \mathrm{div}_{\mathrm{H}} \left(\frac{P}{2}\frac{\mathbb{S}\varepsilon}{\sqrt{\delta + \varepsilon^\top \mathbb{S}\varepsilon}}\right).$$

Let us observe that the preceding definition of Hibler's operator slightly deviates from its introduction in [18, Section 3] and [17, Chapter 4]. In fact, for the sake of consistency of notation, we change the sign, and we also include the term $m_{\mathrm{ice}} = \rho_{\mathrm{ice}} h$ in the denominator.

From an application of the product rule and the chain rule together with the symmetries of $\mathbb{S}$ and ε, it follows that $\mathbb{A}^{\mathrm{H}} v_{\mathrm{ice}} = \mathbb{A}^{\mathrm{H}}(u) v_{\mathrm{ice}}$ takes the shape

$$
\begin{aligned}
(\mathbb{A}^{\mathrm{H}} v_{\mathrm{ice}})_i &= \sum_{j,k,l=1}^{2} \frac{P}{2\rho_{\mathrm{ice}} h} \frac{1}{\triangle_\delta(\varepsilon)} \left(\mathbb{S}_{ij}^{kl} - \frac{1}{\triangle_\delta^2(\varepsilon)} (\mathbb{S}\varepsilon)_{ik} (\mathbb{S}\varepsilon)_{lj} \right) \partial_k \varepsilon_{jl} \\
&\quad + \frac{1}{2\rho_{\mathrm{ice}} h \triangle_\delta(\varepsilon)} \sum_{j=1}^{2} (\partial_j P)(\mathbb{S}\varepsilon)_{ij} \\
&= - \sum_{j,k,l=1}^{2} \frac{P}{2\rho_{\mathrm{ice}} h} \frac{1}{\triangle_\delta(\varepsilon)} \left(\mathbb{S}_{ij}^{kl} - \frac{1}{\triangle_\delta^2(\varepsilon)} (\mathbb{S}\varepsilon)_{ik} (\mathbb{S}\varepsilon)_{jl} \right) \mathrm{D}_k \mathrm{D}_l v_{\mathrm{ice},j} \\
&\quad + \frac{1}{2\rho_{\mathrm{ice}} h \triangle_\delta(\varepsilon)} \sum_{j=1}^{2} (\partial_j P)(\mathbb{S}\varepsilon)_{ij},
\end{aligned}
\tag{3.14}
$$

where $i = 1, 2$ and $\mathrm{D}_m = -\mathrm{i}\partial_m$, see also [18, Section 3]. A more thorough derivation of Hibler's operator can also be found in [17, Chapter 4]. The coefficients of the principal part of $\mathbb{A}^{\mathrm{H}}$ are given by

$$
a_{ij}^{kl}(\varepsilon, P) := - \frac{P}{2\rho_{\mathrm{ice}} h} \frac{1}{\triangle_\delta(\varepsilon)} \left(\mathbb{S}_{ij}^{kl} - \frac{1}{\triangle_\delta^2(\varepsilon)} (\mathbb{S}\varepsilon)_{ik} (\mathbb{S}\varepsilon)_{jl} \right). \tag{3.15}
$$

In the sequel, we will denote the principle variable associated to the sea ice equations by $u = (v_{\mathrm{ice}}, h, a)$. For $u_0 = (v_{\mathrm{ice},0}, h_0, a_0) \in \mathrm{C}^1(\overline{\Omega})^4$ with $h_0 \geq \kappa$, the *linearized Hibler operator* takes the shape

$$
\begin{aligned}
[\mathbb{A}^{\mathrm{H}}(u_0) v_{\mathrm{ice}}]_i &= \sum_{j,k,l=1}^{2} a_{ij}^{kl}(\varepsilon(v_{\mathrm{ice},0}), P(h_0, a_0)) \mathrm{D}_k \mathrm{D}_l v_{\mathrm{ice},j} \\
&\quad + \frac{1}{2\rho_{\mathrm{ice}} h_0 \triangle_\delta(\varepsilon(v_{\mathrm{ice},0}))} \sum_{j=1}^{2} (\partial_j P(h_0, a_0))(\mathbb{S}\varepsilon(v_{\mathrm{ice}}))_{ij}.
\end{aligned}
\tag{3.16}
$$

Having introduced the Hibler operator $\mathbb{A}^{\mathrm{H}}$ and its linearization $\mathbb{A}^{\mathrm{H}}(u_0)$, we now discuss the ellipticity properties from Definition 2.5.1. For a detailed proof, we refer to [17, Chapter 5], see also [18, Proposition 4.1].

Proposition 3.3.1. *Let $u_0 \in \mathrm{C}^1(\overline{\Omega})^4$ be such that $h_0 \geq \kappa$. Then for all $x \in \overline{\Omega}$, the principal part of the negative linearized Hibler operator $-\mathbb{A}^{\mathrm{H}}(u_0)$ as introduced in* (3.16) *is strongly elliptic and parameter-elliptic of angle $\phi_{-\mathbb{A}^{\mathrm{H}}(u_0)} = 0$.*

We emphasize that the statement remains true even though the factor $1/\rho_{\mathrm{ice}} h_0$ is included in the present operator in contrast to the definition in [17] or [18]. This is due to the regularity assumptions on u_0 as well as the assumption that h_0 is bounded from below, preventing this factor from degenerating.

The following lemma is important for the verification of the Lopatinskii-Shapiro condition. Again, we do not give a proof here and refer to [17, Section 5.3] or [18, Lemma 4.2] instead.

Lemma 3.3.2. *Let $u_0 \in \mathrm{C}^1(\overline{\Omega})^4$ be such that $h_0 \geq \kappa$, and recall from* (3.15) *the coefficients $a_{ij}^{kl}(u_0)$ of the principal part of the linearized Hibler operator $\mathbb{A}^{\mathrm{H}}(u_0)$. Then for $x \in \partial\Omega$, ξ, $\nu \in \mathbb{R}^2$ such that $|\xi| = |\nu| = 1$ as well as $(\xi|\nu) = 0$, and u, $v \in \mathbb{C}^2$, it holds that*

$$\mathrm{Re}\left(\sum_{i,j,k,l=1}^{2} -a_{ij}^{kl}(u_0)(\xi_l u_j - \nu_l v_j)\overline{(\xi_k u_i - \nu_k v_i)} \right) \geq 0, \quad \textit{and}$$
$$\mathrm{Re}\left(\sum_{i,j,k,l=1}^{2} -a_{ij}^{kl}(u_0)(\xi_l u_j - \nu_l v_j)\overline{(\xi_k u_i - \nu_k v_i)} \right) > 0, \quad \textit{if} \;\; \mathrm{Im}(u|v) \neq 0.$$

In particular, the negative Hibler operator is strongly normally elliptic in the sense of Definition 2.5.3.

The proposition below on the validity of the Lopatinskii-Shapiro condition for the negative Hibler operator subject to homogeneous Dirichlet boundary conditions is a consequence of Lemma 3.3.2 together with Lemma 2.5.4, since Hibler's operator acts on $\mathbb{R}^2$-valued functions.

Proposition 3.3.3. *Let $u_0 \in \mathrm{C}^1(\overline{\Omega})^4$ with $h_0 \geq \kappa$. Then the principal part of the negative Hibler operator $-\mathbb{A}^{\mathrm{H}}(u_0)$ subject to homogeneous Dirichlet boundary conditions satisfies the Lopatinskii-Shapiro condition.*

After recalling the ellipticity properties of the Hibler operator, we invoke the resulting L^q-realization. For $u_0 \in \mathrm{C}^1(\overline{\Omega})^4$ with $h_0 \geq \kappa$, the *L^q-realization of the linearized Hibler operator $\mathbb{A}^{\mathrm{H}}(u_0)$ subject to Dirichlet boundary conditions on $\partial\Omega$* is defined by

$$\begin{aligned} \left[A_{\mathrm{D}}^{\mathrm{H}}(u_0)\right] v_{\mathrm{ice}} &:= \left[\mathbb{A}^{\mathrm{H}}(u_0)\right] v_{\mathrm{ice}}, \quad \text{with} \\ \mathrm{D}(A_{\mathrm{D}}^{\mathrm{H}}(u_0)) &:= \mathrm{W}^{2,q}(\Omega)^2 \cap \mathrm{W}_0^{1,q}(\Omega)^2. \end{aligned} \tag{3.17}$$

The next proposition on the bounded $\mathcal{H}^\infty$-calculus of $A_{\mathrm{D}}^{\mathrm{H}}(u_0)$ is the starting point for the further discussion of the Hibler operator and the investigation of the local and the global strong well-posedness. Let us observe that the regularity assumption on u_0 is slightly stronger than in the above introduction of the L^q-realization.

Proposition 3.3.4. *Let $q \in (1,\infty)$, consider $u_0 \in \mathrm{C}^{1,\alpha}(\overline{\Omega})^4$ for some $\alpha > 0$, with $h_0 \geq \kappa$, and recall $A^{\mathrm{H}}_{\mathrm{D}}(u_0)$ from* (3.17)*. Then there exists $\omega_0 \in \mathbb{R}$ such that for all $\omega > \omega_0$, the shifted operator $-A^{\mathrm{H}}_{\mathrm{D}}(u_0) + \omega$ admits a bounded $\mathcal{H}^\infty$-calculus on $\mathrm{L}^q(\Omega)^2$ with angle $\phi^\infty_{-A^{\mathrm{H}}_{\mathrm{D}}(u_0)+\omega} = 0$.*

Proof. The proof relies on an application of Proposition 2.5.6, so we need to verify the smoothness conditions (S) and (S+) as well as the ellipticity condition (E) from Section 2.5.

First, recalling the shape of the coefficients of the principal part of $\mathbb{A}^{\mathrm{H}}$ from (3.15), and invoking the assumptions $u_0 \in \mathrm{C}^{1,\alpha}(\overline{\Omega})^4$ and $h_0 \geq \kappa$, we find the existence of $\rho \in (0,1)$ such that $a^{kl}_{ij}(\varepsilon, P) \in \mathrm{BUC}^\rho(\overline{\Omega})$ for all $i,j,k,l = 1,2$. Similarly, with regard to (3.16), it follows that the lower order coefficients of Hibler's operator $\mathbb{A}^{\mathrm{H}}$ are in $\mathrm{L}^\infty(\Omega)$. The aspect (S)(iii) is satisfied as we consider Dirichlet boundary conditions so that the coefficients of the corresponding boundary differential operator are even constant. Thus, conditions (S) and (S+) are thus fulfilled.

On the other hand, we have seen in Proposition 3.3.1 that $-\mathbb{A}^{\mathrm{H}}(u_0)$ is in particular parameter-elliptic with angle $\phi_{\mathbb{A}^{\mathrm{H}}} = 0$, whereas Proposition 3.3.3 yields that $-\mathbb{A}^{\mathrm{H}}(u_0)$ subject to Dirichlet boundary conditions on $\partial\Omega$ satisfies the Lopatinskii-Shapiro condition. Hence, condition (E) from Section 2.5 is also valid, showing the assertion by the above argument. □

Next, we collect properties of the negative L^q-realization $-A^{\mathrm{H}}_{\mathrm{D}}$. In fact, the first two assertions result from Proposition 3.3.4 in conjunction with the relation of the properties as stated in (2.14). The other second part of (b) is implied by the characterization of the maximal regularity of L^p-type from Proposition 2.1.21 joint with the UMD property of $\mathrm{L}^q(\Omega)^2$ for $q \in (1,\infty)$ as well as Lemma 2.1.14 on the link with maximal L^p-regularity. The assertion of (c) follows from Lemma 2.1.4.

Corollary 3.3.5. *Let $p, q \in (1,\infty)$, and consider $u_0 \in \mathrm{C}^{1,\alpha}(\overline{\Omega})^4$ for some $\alpha > 0$ such that $h_0 \geq \kappa$, and consider $A^{\mathrm{H}}_{\mathrm{D}}(u_0)$ as introduced in* (3.17)*. Then there is $\omega_0 \in \mathbb{R}$ such that for all $\omega > \omega_0$, it follows that*

(a) $-A^{\mathrm{H}}_{\mathrm{D}}(u_0) + \omega \in \mathcal{BIP}(\mathrm{L}^q(\Omega)^2)$ *with* $\theta_{-A^{\mathrm{H}}_{\mathrm{D}}(u_0)+\omega} = 0$, *and*

(b) $-A^{\mathrm{H}}_{\mathrm{D}}(u_0) + \omega \in \mathcal{RS}(\mathrm{L}^q(\Omega)^2)$ *with* $\phi^{\mathcal{R}}_{-A^{\mathrm{H}}_{\mathrm{D}}(u_0)+\omega} = 0$.

In particular, we have $-A^{\mathrm{H}}_{\mathrm{D}}(u_0) + \omega \in {}_0\mathcal{MR}_p(\mathrm{L}^q(\Omega)^2)$. *For* $\omega_1 \geq \omega_0$ *sufficiently large, we further obtain* $-A^{\mathrm{H}}_{\mathrm{D}}(u_0) + \omega \in \mathcal{MR}_p(\mathrm{L}^q(\Omega)^2)$ *for all* $\omega > \omega_1$. *In other words,* $-A^{\mathrm{H}}_{\mathrm{D}}(u_0) + \omega$ *has the property of maximal* L^p*-regularity on* $\mathrm{L}^q(\Omega)^2$.

(c) *The operator* $A^{\mathrm{H}}_{\mathrm{D}}(u_0)$ *generates an analytic semigroup* $\mathrm{e}^{tA^{\mathrm{H}}_{\mathrm{D}}(u_0)}$ *on* $\mathrm{L}^q(\Omega)^2$, *referred to as the* Hibler semigroup.

For later use, we comment on the maximal regularity of Hibler's operator without making use of the bounded $\mathcal{H}^\infty$-calculus, and if we merely assume that $u_0 \in \mathrm{C}^1(\overline{\Omega})^2 \times \mathrm{C}(\overline{\Omega}) \times \mathrm{C}(\overline{\Omega})$. We proceed similarly as in the proof of Proposition 3.3.4, this time applying Proposition 2.5.5. The regularity assumption on u_0 guarantees the validity of (S) with regard to the coefficients of the Hibler operator as introduced in (3.15), because they depend smoothly on ε, h and a. This can be summarized in the proposition below.

Proposition 3.3.6. *Let* $p, q \in (1,\infty)$ *as well as* $u_0 \in \mathrm{C}^1(\overline{\Omega})^2 \times \mathrm{C}(\overline{\Omega}) \times \mathrm{C}(\overline{\Omega})$ *with* $h_0 \geq \kappa$, *and recall* $A^{\mathrm{H}}_{\mathrm{D}}(u_0)$ *from* (3.17). *Then there exists* $\omega_0 \in \mathbb{R}$ *such that for every* $\omega > \omega_0$, *it follows that* $-A^{\mathrm{H}}_{\mathrm{D}}(u_0) + \omega \in \mathcal{MR}_p(\mathrm{L}^q(\Omega)^2)$.

We conclude this section by a small, yet important lemma with regard to the analysis developed in the following sections.

Lemma 3.3.7. *Let* $q \in (1,\infty)$ *and* $u_0 \in \mathrm{C}^{1,\alpha}(\overline{\Omega})^4$ *for some* $\alpha > 0$ *with* $h_0 \geq \kappa$. *Then for the* L^q*-linearization* $A^{\mathrm{H}}_{\mathrm{D}}(u_0)$ *as defined in* (3.17), *it holds that*

(a) $A^{\mathrm{H}}_{\mathrm{D}}(u_0)$ *has a compact resolvent,*

(b) *the spectrum* $\sigma(A^{\mathrm{H}}_{\mathrm{D}}(u_0))$ *as an operator on* $\mathrm{L}^q(\Omega)^2$ *is q-independent, and*

(c) *the spectrum* $\sigma(A^{\mathrm{H}}_{\mathrm{D}}(u_0))$ *only consists of eigenvalues.*

Proof. The compactness of the resolvent is a consequence of Lemma 2.1.6(a) together with the compact embedding of $\mathrm{D}(A^{\mathrm{H}}_{\mathrm{D}}(u_0))$ into $\mathrm{L}^q(\Omega)^2$. In fact, the Rellich-Kondrachov theorem as recalled in Lemma 1.4.3 implies that the embedding $\mathrm{W}^{2,q}(\Omega) \hookrightarrow \mathrm{L}^q(\Omega)$ is compact, and this is also preserved for the closed subspace $\mathrm{W}^{2,q}(\Omega) \cap \mathrm{W}^{1,q}_0(\Omega)$. The assertions of (b) and (c) are then implied in view of Lemma 2.1.6(b) and (c). □

3.4. Reformulation in Operator Form and Linear Theory

In this section, we rewrite the complete system of equations from (3.10) as a quasilinear abstract Cauchy problem. Moreover, we investigate the resulting linearized operator matrix. For $q \in (1, \infty)$, we introduce the ground space

$$\mathrm{X}_0 \coloneqq \mathrm{L}^q(\Omega)^2 \times \mathrm{L}^q(\Omega) \times \mathrm{L}^q(\Omega). \tag{3.18}$$

Moreover, the regularity space takes the shape

$$\mathrm{X}_1 \coloneqq \mathrm{W}^{2,q}(\Omega)^2 \cap \mathrm{W}_0^{1,q}(\Omega)^2 \times \mathrm{W}_{\mathrm{N}}^{2,q}(\Omega) \times \mathrm{W}_{\mathrm{N}}^{2,q}(\Omega), \tag{3.19}$$

where the subscript $_\mathrm{N}$ indicates Neumann boundary conditions on $\partial\Omega$. For parameters $p \in (1, \infty)$ and $\mu \in (1/p, 1]$ with $2(\mu - 1/p) > 1 + 1/q$, it then follows from Lemma 1.3.6 that the trace space $\mathrm{X}_{\gamma,\mu} = (\mathrm{X}_0, \mathrm{X}_1)_{\mu-1/p,p}$ is given by

$$\mathrm{X}_{\gamma,\mu} = \mathrm{B}_{qp,\mathrm{D}}^{2(\mu-1/p)}(\Omega)^2 \times \mathrm{B}_{qp,\mathrm{N}}^{2(\mu-1/p)}(\Omega) \times \mathrm{B}_{qp,\mathrm{N}}^{2(\mu-1/p)}(\Omega) \hookrightarrow \mathrm{B}_{qp}^{2(\mu-1/p)}(\Omega)^4. \tag{3.20}$$

In the above, the subscript $_\mathrm{D}$ represents homogeneous Dirichlet boundary conditions on $\partial\Omega$. In view of (3.20) and the embedding (1.8), to guarantee

$$\mathrm{X}_{\gamma,\mu} \hookrightarrow \mathrm{C}^{1,\alpha}(\overline{\Omega})^4, \tag{3.21}$$

we will consider $p, q \in (1, \infty)$ and $\mu \in (1/p, 1]$ with

$$\frac{1}{2} + \frac{1}{p} + \frac{1}{q} < \mu \le 1. \tag{3.22}$$

Let us observe that (3.22) implies in particular that $2(\mu - 1/p) > 1 + 1/q$, so the trace space $\mathrm{X}_{\gamma,\mu}$ incorporates Dirichlet boundary conditions in the component of v_{ice} and Neumann boundary conditions in the h- and a-component.

Next, we define the open subset $V_\mu \subset \mathrm{X}_{\gamma,\mu}$ by

$$V_\mu \coloneqq \{ u = (v_{\mathrm{ice}}, h, a) \in \mathrm{X}_{\gamma,\mu} : h > \kappa \text{ and } a \in (0,1) \}. \tag{3.23}$$

By introducing the above set V_μ, we ensure that the mean ice thickness h and ice compactness a only attain physically reasonable values, and it is guaranteed that the terms in (3.10) do not degenerate when dividing by m_{ice}.

We are equipped with all the relevant pieces of notation to present the quasilinear operator A and the nonlinear right-hand side F. For $u \in V_\mu$, recalling the L^q-realization of the linearized Hibler operator $A_D^H(u_0)$ and the Neumann Laplacian operator Δ_N from (2.16), we first define $A\colon V_\mu \to \mathcal{L}(X_1, X_0)$ by

$$A(u) \coloneqq \begin{pmatrix} -A_D^H(u) & \frac{\partial_h P(h,a)}{2\rho_{\text{ice}} h}\nabla_H & \frac{\partial_a P(h,a)}{2\rho_{\text{ice}} h}\nabla_H \\ 0 & -d_h\Delta_N & 0 \\ 0 & 0 & -d_a\Delta_N \end{pmatrix}. \tag{3.24}$$

We remark that the terms

$$\frac{\partial_h P(h,a)}{2\rho_{\text{ice}} h}\nabla_H \text{ and } \frac{\partial_a P(h,a)}{2\rho_{\text{ice}} h}\nabla_H$$

result from $\operatorname{div}_H P(h,a)\,\mathrm{Id}_2$ which has not yet been captured.

Furthermore, for $u \in V_\mu$, and recalling $\tau_{\text{ice}} = \tau_{\text{atm}} + \tau_{\text{ocn}}(v_{\text{ice}})$, S_h and S_a from (3.6) as well as (3.8), we define $F\colon V_\mu \to X_0$ by

$$F(u) \coloneqq \begin{pmatrix} -(v_{\text{ice}} \cdot \nabla_H)v_{\text{ice}} - c_{\text{cor}} v_{\text{ice}}^{\perp} - g\nabla_H H + \frac{1}{\rho_{\text{ice}} h}\tau_{\text{ice}} \\ -\operatorname{div}_H(v_{\text{ice}} h) + S_h(h,a) \\ -\operatorname{div}_H(v_{\text{ice}} a) + S_a(h,a) \end{pmatrix}. \tag{3.25}$$

In total, for $A(u)$ and $F(u)$ from (3.24) and (3.25), we rewrite the fully parabolic regularized model (3.10) as the quasilinear abstract Cauchy problem

$$\begin{cases} u'(t) + A(u(t))u(t) = F(u(t)), & \text{for } t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{3.26}$$

on the ground space X_0 from (3.18). Next, we discuss the maximal L^p-regularity of the linearized operator matrix.

Proposition 3.4.1. *Let $p, q \in (1,\infty)$ and $\mu \in (1/p, 1]$ satisfy (3.22), $u_0 \in V_\mu$, and consider $A(u_0)$ as defined in (3.24). Then there exists $\omega_0 \in \mathbb{R}$ such that for all $\omega > \omega_0$, we have $A(u_0) + \omega \in \mathcal{MR}_p(X_0)$. In other words, $A(u_0) + \omega$ has maximal L^p-regularity on X_0 for all $\omega > \omega_0$.*

Proof. We collect the properties of the operators on the diagonal of $A(u_0)$ and then use a perturbation argument to derive the desired property for the complete operator matrix. More precisely, we first use the decomposition

$$A(u_0) \coloneqq A_1(u_0) + B(u_0), \text{ with } A_1(u_0) \coloneqq \operatorname{diag}(-A_D^H(u_0), -d_h\Delta_N, -d_a\Delta_N)$$

and

$$B(u_0) \coloneqq \begin{pmatrix} 0 & \frac{\partial_h P(h_0,a_0)}{2\rho_{\mathrm{ice}} h_0} \nabla_{\mathrm{H}} & \frac{\partial_a P(h_0,a_0)}{2\rho_{\mathrm{ice}} h_0} \nabla_{\mathrm{H}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thanks to the conditions on p, q and μ from (3.22) and $u_0 \in V_\mu$, we obtain the embedding (3.21) and are thus able to apply Corollary 3.3.5 for the respective property of the linearized Hibler operator. Additionally recalling Lemma 2.3.19 for the (shifted) Neumann Laplacian operators, we derive the existence of $\tilde{\omega}_0$ with $A_1(u_0) + \omega \in {}_0\mathcal{MR}(\mathrm{X}_0)$ for all $\omega > \tilde{\omega}_0$.

The next step is to show the relative $(A_1(u_0) + \omega)$-boundedness of $B(u_0)$ for such ω. To this end, recall from (3.1) that

$$\partial_h P(h_0, a_0) = p^* \mathrm{e}^{-c_\bullet(1-a_0)}, \text{ so } \frac{\partial_h P(h_0, a_0)}{2\rho_{\mathrm{ice}} h_0} = \frac{p^* \mathrm{e}^{-c_\bullet(1-a_0)}}{2\rho_{\mathrm{ice}} h_0}.$$

From the embedding (3.21) and the assumption $u_0 \in V_\mu$, it especially follows that $\partial_h P(h_0, a_0) \in \mathrm{L}^\infty(\Omega)$. Additionally, we recall that there is $C_1 > 0$ with

$$\|h\|_{\mathrm{W}^{2,q}(\Omega)} \le C_1 \cdot \|(-d_{\mathrm{h}}\Delta_{\mathrm{N}} + \omega)h\|_{\mathrm{L}^q(\Omega)}.$$

Hence, Hölder's inequality, the interpolation of $\mathrm{L}^q(\Omega)$ and $\mathrm{W}^{2,q}(\Omega)$ as revealed in Lemma 1.3.4 and Young's inequality yield

$$\begin{aligned} \left\| \frac{\partial_h P(h_0, a_0)}{2\rho_{\mathrm{ice}} h_0} \nabla_{\mathrm{H}} h \right\|_{\mathrm{L}^q(\Omega)} &\le C_2 \cdot \|h\|_{\mathrm{W}^{1,q}(\Omega)} \\ &\le \frac{C_3}{\alpha} \cdot \|h\|_{\mathrm{L}^q(\Omega)} + \frac{\alpha}{2C_1} \cdot \|h\|_{\mathrm{W}^{2,q}(\Omega)} \\ &\le \frac{C_3}{\alpha} \cdot \|u\|_{\mathrm{X}_0} + \frac{\alpha}{2} \cdot \|(-d_{\mathrm{h}}\Delta_{\mathrm{N}} + \omega)h\|_{\mathrm{L}^q(\Omega)} \\ &\le \frac{C_3}{\alpha} \cdot \|u\|_{\mathrm{X}_0} + \frac{\alpha}{2} \cdot \|(A_1(u_0) + \omega)u\|_{\mathrm{X}_0} \end{aligned} \tag{3.27}$$

for $u \in \mathrm{X}_1$, and for every $\alpha > 0$. Concerning the other term in $B(u_0)$, an analogous argument can be used, so for every $\alpha > 0$, there is $C > 0$ with

$$\|B(u_0)u\|_{\mathrm{X}_0} \le \alpha \cdot \|(A_1(u_0) + \omega)u\|_{\mathrm{X}_0} + C \cdot \|u\|_{\mathrm{X}_0}$$

for all $u \in \mathrm{X}_1$. By the above property of $A_1(u_0)$ and Corollary 2.1.23, there exists a possibly larger $\omega_0 \in \mathbb{R}$ such that for all $\omega > \omega_0$, we have

$$A(u_0) + \omega = A_1(u_0) + B(u_0) + \omega \in \mathcal{MR}_p(\mathrm{X}_0).$$

Thus, $A(u_0) + \omega$ has the property of maximal L^p-regularity for all $\omega > \omega_0$. □

3.5. Local Strong Well-Posedness

In this section, we present the first main result of this thesis, namely the local strong well-posedness of the fully parabolic regularized model from (3.10). For this, we first collect some assumptions on the external forcing terms.

Assumption 3.5.1. *Let $q \in (1, \infty)$. We make the following assumptions on the external forcing terms.*

(a) The wind velocity at the surface V_{atm} and the ocean velocity V_{ocn} have the property that V_{atm}, $V_{\mathrm{ocn}} \in \mathrm{L}^{2q}(\Omega)^2$.

(b) The sea surface dynamic height H satisfies $\nabla_{\mathrm{H}} H \in \mathrm{L}^q(\Omega)^2$.

(c) The ice growth rate f_{gr} fulfills $f_{\mathrm{gr}} \in \mathrm{C}_{\mathrm{b}}^1([0, \infty))$.

The terms in Assumption 3.5.1 are considered independent of time. It is a straightforward task to include time-dependence and adjust the assumptions accordingly. However, for simplicity of the presentation, we do not deal with this aspect here.

The following result not only captures the local strong well-posedness, but it also presents several other features of the emerging strong solution such as the continuous dependence on the initial data, the instantaneous smoothing in time or the characterization of the maximal time interval of existence of the solution.

Theorem 3.5.2. *Let $p, q \in (1, \infty)$ and $\mu \in (1/p, 1]$ be such that (3.22) holds true, let $u_0 \in V_\mu$, where V_μ is as defined in (3.23), suppose that the external terms V_{atm}, V_{ocn}, H and f_{gr} satisfy Assumption 3.5.1, and recall X_0, X_1 as well as $\mathrm{X}_{\gamma,\mu}$ from (3.18), (3.19) and (3.20).*

Then there exist $T = T(u_0) > 0$ and $r = r(u_0) > 0$ with $\overline{\mathbb{B}}_{\mathrm{X}_{\gamma,\mu}}(u_0, r) \subset V_\mu$ such that the quasilinear abstract Cauchy problem (3.26) associated to sea ice, i. e., the fully parabolic regularized model as in (3.10), has a unique solution

$$u(\cdot, u_1) \in \mathrm{W}_\mu^{1,p}(0, T; \mathrm{X}_0) \cap \mathrm{L}_\mu^p(0, T; \mathrm{X}_1) \cap \mathrm{C}([0, T]; V_\mu) =: \mathbb{E}_{1,\mu} \cap \mathrm{C}([0, T]; V_\mu)$$

for every initial value $u_1 \in \overline{\mathbb{B}}_{\mathrm{X}_{\gamma,\mu}}(u_0, r)$. In addition, the solution has the following properties.

(a) There is $C = C(u_0) > 0$ such that

$$\|u(\cdot, u_1) - u(\cdot, u_2)\|_{\mathbb{E}_{1,\mu}} \le C \cdot \|u_1 - u_2\|_{\mathrm{X}_{\gamma,\mu}}$$

for all $u_1, u_2 \in \overline{\mathbb{B}}_{\mathrm{X}_{\gamma,\mu}}(u_0, r)$. This means that the solution depends continuously on the initial data.

(b) *For every $\delta > 0$, we obtain*

$$u \in \mathbb{E}_1(\delta, T) = \mathrm{W}^{1,p}(\delta, T; \mathrm{X}_0) \cap \mathrm{L}^p(\delta, T; \mathrm{X}_1) \hookrightarrow \mathrm{C}([\delta, T]; \mathrm{X}_\gamma),$$

so the solution regularizes instantly in time.

(c) *The existence of the solution $u = u(u_0)$ is guaranteed on a maximal time interval $J(u_0) = [0, t_+(u_0))$, where $t_+(u_0)$ is characterized by*

(i) *global existence, i. e., $t_+(u_0) = \infty$,*

(ii) $\liminf_{t \to t_+(u_0)} \mathrm{dist}_{\mathrm{X}_{\gamma,\mu}}(u(t), \partial V_\mu) = 0$, *or*

(iii) $\lim_{t \to t_+(u_0)} u(t)$ *does not exist in* $\mathrm{X}_{\gamma,\mu}$.

Proof. The proof relies on an application of Proposition 2.6.1. With regard to the maximal L^p-regularity result, Proposition 3.4.1, for $\omega > \omega_0$ and $u \in V_\mu$, we first introduce the shifted operator matrix and right-hand side given by

$$A_\omega(u) := A(u) + \omega \, \mathrm{Id}_4 \quad \text{and} \quad F_\omega(u) := F(u) + \omega u,$$

where $A(u)$ and $F(u)$ have been defined in (3.24) and (3.25), respectively. Solving (3.26) is now equivalent to treating

$$\begin{cases} u'(t) + A_\omega(u(t))u(t) = F_\omega(u(t)), & \text{for } t \in (0, T), \\ u(0) = u_0. \end{cases}$$

Proposition 3.4.1 yields that $A_\omega(u_0) \in \mathcal{MR}_p(\mathrm{X}_0)$, i. e., $A_\omega(u_0)$ has maximal L^p-regularity on X_0.

In order to make use of Proposition 2.6.1, it remains to verify that the operator matrix and right-hand side $(A_\omega, F_\omega) \colon V_\mu \to \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0) \times \mathrm{X}_0$ satisfy suitable Lipschitz estimates. In view of Remark 2.6.2, let $r > 0$ with $\mathbb{B}_{\mathrm{X}_{\gamma,\mu}}(u_0, r) \subset V_\mu$, and consider u_1, $u_2 \in \mathbb{B}_{\mathrm{X}_{\gamma,\mu}}(u_0, r)$ as well as $w = (v_{\mathrm{ice}}, h, a) \in \mathrm{X}_1$. Concerning A_ω, it follows that

$$\begin{aligned} &\|A_\omega(u_1)w - A_\omega(u_2)w\|_{\mathrm{X}_0} \\ \leq &\left\| \left(A_{\mathrm{D}}^{\mathrm{H}}(u_1) - A_{\mathrm{D}}^{\mathrm{H}}(u_2) \right) v_{\mathrm{ice}} \right\|_{\mathrm{L}^q(\Omega)} + \left\| \left(\frac{\partial_h P(h_1, a_1)}{2\rho_{\mathrm{ice}} h_1} - \frac{\partial_h P(h_2, a_2)}{2\rho_{\mathrm{ice}} h_2} \right) \nabla_{\mathrm{H}} h \right\|_{\mathrm{L}^q(\Omega)} \\ &+ \left\| \left(\frac{\partial_a P(h_1, a_1)}{2\rho_{\mathrm{ice}} h_1} - \frac{\partial_a P(h_2, a_2)}{2\rho_{\mathrm{ice}} h_2} \right) \nabla_{\mathrm{H}} a \right\|_{\mathrm{L}^q(\Omega)}. \end{aligned}$$

In the sequel, we denote by $A_{\mathrm{D},\#}^{\mathrm{H}}$ the principal part of the L^q-realization of the linearized Hibler operator. Moreover, recalling the coefficients of the linearized Hibler operator from (3.15), and invoking u_1, $u_2 \in V_\mu$, we observe that the coefficients $a_{ij}^{kl}(\varepsilon_m, P_m) := a_{ij}^{kl}(\varepsilon(v_{\mathrm{ice},m}), h_m, a_m)$ depend in a smooth way on $\varepsilon_m := \varepsilon(v_{\mathrm{ice},m})$, h_m and a_m, where $m = 1, 2$. Therefore, Hölder's inequality, the mean value theorem, the embedding (3.21) and u_1, $u_2 \in V_\mu$ imply the existence of constants $C_i > 0$, $i \in \{1, 2\}$, such that

$$\begin{aligned}
&\left\| \left(A_{\mathrm{D},\#}^{\mathrm{H}}(u_1) - A_{\mathrm{D},\#}^{\mathrm{H}}(u_2) \right) v_{\mathrm{ice}} \right\|_{\mathrm{L}^q(\Omega)} \\
&\le \max_{i,j,k,l=1,2} \left\| a_{ij}^{kl}(\varepsilon_1, P_1) - a_{ij}^{kl}(\varepsilon_2, P_2) \right\|_{\mathrm{L}^\infty(\Omega)} \cdot \| v_{\mathrm{ice}} \|_{\mathrm{W}^{2,q}(\Omega)} \\
&\le C_1 \cdot \| (\varepsilon_1, h_1, a_1) - (\varepsilon_2, h_2, a_2) \|_{\mathrm{L}^\infty(\Omega)} \cdot \| w \|_{\mathrm{X}_1} \\
&\le C_2 \cdot \| u_1 - u_2 \|_{\mathrm{X}_{\gamma,\mu}} \cdot \| w \|_{\mathrm{X}_1}.
\end{aligned} \tag{3.28}$$

Similarly, we find that the remaining part of Hibler's operator and the off-diagonal terms in $A_\omega(u_i)$ satisfy Lipschitz estimates. In total, we conclude that $A_\omega \colon V_\mu \to \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0)$ is indeed valid. On the other hand, we deduce that for every $r > 0$ with $\mathbb{B}_{\mathrm{X}_{\gamma,\mu}}(u_0, r) \subset V_\mu$, there exists $C_A > 0$ with

$$\| A_\omega(u_1) w - A_\omega(u_2) w \|_{\mathrm{X}_0} \le C_A \cdot \| u_1 - u_2 \|_{\mathrm{X}_{\gamma,\mu}} \cdot \| w \|_{\mathrm{X}_1} \tag{3.29}$$

for all u_1, $u_2 \in \mathbb{B}_{\mathrm{X}_{\gamma,\mu}}(u_0, r)$ and $w \in \mathrm{X}_1$.

Next, we deal with the Lipschitz estimates of F_ω, and we will estimate all terms separately. As above, we restrict ourselves to the case of sufficiently small $r > 0$ such that $\mathbb{B}_{\mathrm{X}_{\gamma,\mu}}(u_0, r) \subset V_\mu$ and consider u_1, $u_2 \in \mathbb{B}_{\mathrm{X}_{\gamma,\mu}}(u_0, r)$. For small $\alpha > 0$, as in (3.21), we deduce from condition (3.22) on p, q and μ and the embedding in (1.8) that

$$\mathrm{X}_{\gamma,\mu} \hookrightarrow \mathrm{B}_{qp}^{2(\mu-1/p)}(\Omega)^4 \hookrightarrow \mathrm{C}^{1,\alpha}(\overline{\Omega})^4 \hookrightarrow \mathrm{L}^\infty(\Omega)^4 \cap \mathrm{W}^{1,q}(\Omega)^4. \tag{3.30}$$

Therefore, we first get by Hölder's inequality the estimate

$$\begin{aligned}
&\| (v_{\mathrm{ice},1} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice},1} - (v_{\mathrm{ice},2} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice},2} \|_{\mathrm{L}^q(\Omega)} \\
&\le \| ((v_{\mathrm{ice},1} - v_{\mathrm{ice},2}) \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice},1} \|_{\mathrm{L}^q(\Omega)} + \| (v_{\mathrm{ice},2} \cdot \nabla_{\mathrm{H}}) (v_{\mathrm{ice},1} - v_{\mathrm{ice},2}) \|_{\mathrm{L}^q(\Omega)} \\
&\le \| v_{\mathrm{ice},1} - v_{\mathrm{ice},2} \|_{\mathrm{L}^\infty(\Omega)} \cdot \| v_{\mathrm{ice},1} \|_{\mathrm{W}^{1,q}(\Omega)} + \| v_{\mathrm{ice},2} \|_{\mathrm{L}^\infty(\Omega)} \cdot \| v_{\mathrm{ice},1} - v_{\mathrm{ice},2} \|_{\mathrm{W}^{1,q}(\Omega)} \\
&\le C_3 \cdot \left(\| u_1 \|_{\mathrm{X}_{\gamma,\mu}} + \| u_2 \|_{\mathrm{X}_{\gamma,\mu}} \right) \cdot \| u_1 - u_2 \|_{\mathrm{X}_{\gamma,\mu}} \\
&\le C_4 \cdot \left(r + \| u_0 \|_{\mathrm{X}_{\gamma,\mu}} \right) \cdot \| u_1 - u_2 \|_{\mathrm{X}_{\gamma,\mu}}
\end{aligned}$$

for constants C_3, $C_4 > 0$. Recalling that $\mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} h) = v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}} h + h \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}}$, we find corresponding Lipschitz estimates of $\mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} h)$ and $\mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} a)$ in a completely analogous way. Thanks to $\mathrm{X}_{\gamma,\mu} \hookrightarrow \mathrm{X}_0$, for $C_5 > 0$, we also obtain

$$\left\| (-c_{\mathrm{cor}} v_{\mathrm{ice},1}^\perp, 0, 0)^\top + \omega u_1 - (-c_{\mathrm{cor}} v_{\mathrm{ice},1}^\perp, 0, 0)^\top - \omega u_2 \right\|_{\mathrm{X}_0} \le C_5 \cdot \| u_1 - u_2 \|_{\mathrm{X}_{\gamma,\mu}}.$$

Let us observe that the term $-g\nabla_{\mathrm{H}} H$ does not depend on u_i. On the other hand, Assumption 3.5.1(b) yields that $-g\nabla_{\mathrm{H}} H \in \mathrm{L}^q(\Omega)^2$.

Next, thanks to u_1, $u_2 \in V_\mu$ and the mean value theorem in conjunction with the above embedding (3.30), we find constants C_6, $C_7 > 0$ such that

$$\left\| \frac{1}{\rho_{\mathrm{ice}} h_1} - \frac{1}{\rho_{\mathrm{ice}} h_2} \right\|_{\mathrm{L}^\infty(\Omega)} \le C_6 \cdot \|h_1 - h_2\|_{\mathrm{L}^\infty(\Omega)} \le C_7 \cdot \|u_1 - u_2\|_{\mathrm{X}_{\gamma,\mu}}. \tag{3.31}$$

The shape of τ_{atm} from (3.6) and (3.31) joint with Assumption 3.5.1(a) imply

$$\left\| \left(\frac{1}{\rho_{\mathrm{ice}} h_1} - \frac{1}{\rho_{\mathrm{ice}} h_2} \right) \tau_{\mathrm{atm}} \right\|_{\mathrm{L}^q(\Omega)} \le C_8 \cdot \|u_1 - u_2\|_{\mathrm{X}_{\gamma,\mu}}.$$

For τ_{ocn} as introduced in (3.6), setting $c_{\mathrm{ocn}} := \|V_{\mathrm{ocn}}\|_{\mathrm{L}^{2q}(\Omega)} < \infty$ thanks to Assumption 3.5.1(a), and employing Hölder's and Young's inequality as well as the embedding (3.30), which yields in particular $\mathrm{X}_{\gamma,\mu} \hookrightarrow \mathrm{L}^{2q}(\Omega)^2$, we get

$$\begin{aligned} \|\tau_{\mathrm{ocn}}(v_{\mathrm{ice},i})\|_{\mathrm{L}^q(\Omega)} &\le C_9 \cdot \|V_{\mathrm{ocn}} - v_{\mathrm{ice}}\|_{\mathrm{L}^{2q}(\Omega)}^2 \\ &\le C_{10} \cdot \left(\|V_{\mathrm{ocn}}\|_{\mathrm{L}^{2q}(\Omega)}^2 + \|v_{\mathrm{ice},i}\|_{\mathrm{L}^{2q}(\Omega)}^2 \right) \\ &\le C_{11} \cdot \left(c_{\mathrm{ocn}}^2 + \|u_i\|_{\mathrm{X}_{\gamma,\mu}}^2 \right) \\ &\le C_{12} \cdot \left(c_{\mathrm{ocn}}^2 + r^2 + \|u_0\|_{\mathrm{X}_{\gamma,\mu}}^2 \right). \end{aligned} \tag{3.32}$$

Similar arguments together with u_1, $u_2 \in V_\mu$ result in

$$\begin{aligned} &\left\| \frac{1}{\rho_{\mathrm{ice}} h_i} \left(\tau_{\mathrm{ocn}}(v_{\mathrm{ice},1}) - \tau_{\mathrm{ocn}}(v_{\mathrm{ice},2}) \right) \right\|_{\mathrm{L}^q(\Omega)} \\ &\le C_{13} \cdot \|(\tau_{\mathrm{ocn}}(v_{\mathrm{ice},1}) - \tau_{\mathrm{ocn}}(v_{\mathrm{ice},2})\|_{\mathrm{L}^q(\Omega)} \\ &\le C_{14} \cdot \left(c_{\mathrm{ocn}} + r + \|u_0\|_{\mathrm{X}_{\gamma,\mu}} \right) \cdot \|u_1 - u_2\|_{\mathrm{X}_{\gamma,\mu}}. \end{aligned} \tag{3.33}$$

Hence, a concatenation of (3.31), (3.32) and (3.33) leads to

$$\left\| \frac{1}{\rho_{\mathrm{ice}} h_1} \tau_{\mathrm{ocn}}(v_{\mathrm{ice},1}) - \frac{1}{\rho_{\mathrm{ice}} h_2} \tau_{\mathrm{ocn}}(v_{\mathrm{ice},2}) \right\|_{\mathrm{L}^q(\Omega)} \le C_{15} \left(r, \|u_0\|_{\mathrm{X}_{\gamma,\mu}} \right) \cdot \|u_1 - u_2\|_{\mathrm{X}_{\gamma,\mu}}.$$

It remains to estimate the thermodynamic terms $S_{\mathrm{h}}(h,a)$ and $S_{\mathrm{a}}(h,a)$ from (3.8). By u_1, $u_2 \in V_\mu$, Assumption 3.5.1(c) and the mean value theorem, we first get

$$\begin{aligned} &\|S_{\mathrm{h}}(h_1, a_1) - S_{\mathrm{h}}(h_2, a_2)\|_{\mathrm{L}^q(\Omega)} \\ &\le C_{16} \cdot \|u_1 - u_2\|_{\mathrm{X}_{\gamma,\mu}} + \left\| f_{\mathrm{gr}} \left(\frac{h_1}{a_1} \right) - f_{\mathrm{gr}} \left(\frac{h_2}{a_2} \right) \right\|_{\mathrm{L}^\infty(\Omega)} \cdot \|u_1\|_{\mathrm{X}_{\gamma,\mu}} \\ &\le C_{17} \left(r, \|u_0\|_{\mathrm{X}_{\gamma,\mu}} \right) \cdot \|u_1 - u_2\|_{\mathrm{X}_{\gamma,\mu}}. \end{aligned} \tag{3.34}$$

As $r > 0$ is assumed to be small enough, we may suppose that $S_{\mathrm{h}}(h_i, a_i) > 0$ or $S_{\mathrm{h}}(h_i, a_i) < 0$ for $i = 1, 2$. Consequently, with regard to the shape of S_{a}, the above estimate (3.34) as well as u_1, $u_2 \in V_\mu$ and the estimate

$$\begin{aligned}\left\| \frac{a_1}{2h_1} S_{\mathrm{h}}(h_1, a_1) - \frac{a_2}{2h_2} S_{\mathrm{h}}(h_2, a_2) \right\|_{\mathrm{L}^q(\Omega)} &\le \left\| \frac{a_1}{2h_1} (S_{\mathrm{h}}(h_1, a_1) - S_{\mathrm{h}}(h_2, a_2)) \right\|_{\mathrm{L}^q(\Omega)} \\ &\quad + \left\| \left(\frac{a_1}{2h_1} - \frac{a_2}{2h_2} \right) S_{\mathrm{h}}(h_2, a_2) \right\|_{\mathrm{L}^q(\Omega)},\end{aligned}$$

we conclude

$$\| S_{\mathrm{a}}(h_1, a_1) - S_{\mathrm{a}}(h_2, a_2) \|_{\mathrm{L}^q(\Omega)} \le C_{18} \left(r, \| u_0 \|_{\mathrm{X}_{\gamma,\mu}} \right) \cdot \| u_1 - u_2 \|_{\mathrm{X}_{\gamma,\mu}}.$$

In total, putting together the above arguments and estimates, we infer the validity of $F_\omega \colon V_\mu \to \mathrm{X}_0$. Besides, for every $r > 0$ sufficiently small such that $\mathbb{B}_{\mathrm{X}_{\gamma,\mu}}(u_0, r) \subset V_\mu$, there exists $C_F(r, \| u_0 \|_{\mathrm{X}_{\gamma,\mu}}) > 0$ with

$$\| F_\omega(u_1) - F_\omega(u_2) \|_{\mathrm{X}_0} \le C_F \left(r, \| u_0 \|_{\mathrm{X}_{\gamma,\mu}} \right) \cdot \| u_1 - u_2 \|_{\mathrm{X}_{\gamma,\mu}}.$$

The main assertion of the theorem as well as (a) and (b) are then implied by Proposition 2.6.1, where for the sufficiency of the previous Lipschitz estimates, we also refer to Remark 2.6.2. Finally, the assertion in (c) is a result of Corollary 2.6.3. □

3.6. Global Strong Well-Posedness close to Constant Equilibria

After elaborating on the local strong well-posedness of (3.10), it is natural to ask for results which are *global-in-time*. As we shall see in the sequel, when assuming the external forces to vanish, we obtain global strong well-posedness for initial data close to constant equilibria.

More precisely, we investigate the situation when there are no forces due to the changing sea surface height, the atmospheric wind and the ocean currents, so we assume

$$-g \nabla_{\mathrm{H}} H = \tau_{\mathrm{atm}} = \tau_{\mathrm{ocn}} = 0.$$

We also concentrate on the situation without thermodynamic effects, i. e.,

$$S_{\mathrm{h}} = S_{\mathrm{a}} = 0.$$

The resulting *simplified fully parabolic regularized model* reads as

$$
(3.35)\qquad \begin{cases} \partial_t v_{\mathrm{ice}} + (v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice}} = \dfrac{1}{m_{\mathrm{ice}}} \mathrm{div}_{\mathrm{H}}\, \sigma_\delta - c_{\mathrm{cor}} v_{\mathrm{ice}}^\perp, & \text{in } (0,T) \times \Omega, \\ \partial_t h + \mathrm{div}_{\mathrm{H}}\, (v_{\mathrm{ice}} h) = d_{\mathrm{h}} \Delta_{\mathrm{H}} h, & \text{in } (0,T) \times \Omega, \\ \partial_t a + \mathrm{div}_{\mathrm{H}}\, (v_{\mathrm{ice}} a) = d_{\mathrm{a}} \Delta_{\mathrm{H}} a, & \text{in } (0,T) \times \Omega, \\ v_{\mathrm{ice}} = 0, \;\; \partial_\nu h = \partial_\nu a = 0, & \text{on } (0,T) \times \partial\Omega, \\ v_{\mathrm{ice}}(0) = v_{\mathrm{ice},0}, \;\; h(0) = h_0, \;\; a(0) = a_0, & \text{in } \Omega. \end{cases}
$$

Next, we rewrite (3.35) as a quasilinear evolution equation on the ground space X_0 as introduced in (3.18). With regard to Proposition 2.6.5, we restrict ourselves to the setting without time weights, meaning that we focus on the case $\mu = 1$. As a consequence, we will denote the trace space by $\mathrm{X}_\gamma = \mathrm{X}_{\gamma,1}$. For X_0 and X_1 from (3.18) and (3.19), respectively, and for $2 - 2/p > 1 + 1/q$, it follows from (3.20) that the trace space is given by

$$
(3.36)\qquad \mathrm{X}_\gamma = (\mathrm{X}_0, \mathrm{X}_1)_{1-1/p,p} = \mathrm{B}_{qp,\mathrm{D}}^{2-2/p}(\Omega)^2 \times \mathrm{B}_{qp,\mathrm{N}}^{2-2/p}(\Omega) \times \mathrm{B}_{qp,\mathrm{N}}^{2-2/p}(\Omega),
$$

with the meaning of the subscripts as explained after (3.20). Besides, we denote the open set by $V := V_1$, and V_1 has been defined in (3.23). For convenience, we also recall the condition on p, $q \in (1,\infty)$ resulting from in the case $\mu = 1$. It takes the shape

$$
(3.37)\qquad \frac{1}{p} + \frac{1}{q} < \frac{1}{2}
$$

and especially implies $2 - 2/p > 1 + 1/q$.

Similarly as in (3.25) for the complete model, we define $F_{\mathrm{s}} \colon V \to \mathrm{X}_0$ by

$$
(3.38)\qquad F_{\mathrm{s}}(u) := \begin{pmatrix} -(v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice}} - c_{\mathrm{cor}} v_{\mathrm{ice}}^\perp \\ -\mathrm{div}_{\mathrm{H}}\, (v_{\mathrm{ice}} h) \\ -\mathrm{div}_{\mathrm{H}}\, (v_{\mathrm{ice}} a) \end{pmatrix}.
$$

For $A \colon V \to \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0)$ as defined in (3.24), the quasilinear abstract Cauchy problem associated to (3.35) is given by

$$
(3.39)\qquad \begin{cases} u'(t) + A(u(t))u(t) = F_{\mathrm{s}}(u(t)), \;\; \text{for } t \in (0,T), \\ u(0) = u_0. \end{cases}
$$

We will denote the set of equilibrium solutions to (3.39), or, equivalently, to (3.35), by $\mathcal{E}$. This set is given by

$$\mathcal{E} := \{u \in V \cap \mathrm{X}_1 : A(u)u = F_{\mathrm{s}}(u)\}.$$

Considering $h_* > \kappa$ as well as $a_* \in (0,1)$ constant in time and space, and inserting $u_* = (0, h_*, a_*) \in V \cap \mathrm{X}_1$ into the operator matrix $A(u)$ from (3.24) as well as into $F_{\mathrm{s}}(u)$ from (3.38), we find that $A(u_*)u_* = 0 = F_{\mathrm{s}}(u_*)$. This is summarized in the following lemma.

Lemma 3.6.1. *Let $u_* = (0, h_*, a_*)$, where $h_* > \kappa$ and $a_* \in (0,1)$ are constant in time and space. Then $u_* \in \mathcal{E}$, i. e., u_* is an equilibrium solution to* (3.39)*, or, equivalently, to* (3.35)*.*

Concerning the investigation of equilibria u_* of the above shape, we aim for an application of the generalized principle of linearized stability from Section 2.6. For this purpose, we need to calculate the total linearization of the problem given by

$$A_0 u = A(u_*)u + (A'(u_*)u)u_* - F_{\mathrm{s}}'(u_*)u$$

for $u \in \mathrm{X}_1$ in the present case. Before, in order to simplify the notation, we introduce the terms

$$\begin{aligned} P_* &:= \frac{P(h_*, a_*)}{2\rho_{\mathrm{ice}} h_*} = \frac{p^* \mathrm{e}^{-c_\bullet(1-a_*)}}{2\rho_{\mathrm{ice}}}, \\ P_{\mathrm{h},*} &:= \frac{\partial_h P(h_*, a_*)}{2\rho_{\mathrm{ice}} h_*} = \frac{p^* \mathrm{e}^{-c_\bullet(1-a_*)}}{2\rho_{\mathrm{ice}} h_*} \quad \text{and} \\ P_{\mathrm{a},*} &:= \frac{\partial_a P(h_*, a_*)}{2\rho_{\mathrm{ice}} h_*} = \frac{c_\bullet p^* \mathrm{e}^{-c_\bullet(1-a_*)}}{2\rho_{\mathrm{ice}}}. \end{aligned} \tag{3.40}$$

Lemma 3.6.2. *Let $p, q \in (1, \infty)$ be such that* (3.37) *holds true, and recall A and F_{s} from* (3.24) *and* (3.38)*. Then $(A, F_{\mathrm{s}}) \in \mathrm{C}^1(V; \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0) \times \mathrm{X}_0)$, and for $u \in \mathrm{X}_1$, the total linearization is given by*

$$A_0 u = \begin{pmatrix} -A_{\mathrm{D}}^{\mathrm{H}}(u_*) v_{\mathrm{ice}} + c_{\mathrm{cor}} v_{\mathrm{ice}}^\perp + P_{\mathrm{h},*} \nabla_{\mathrm{H}} h + P_{\mathrm{a},*} \nabla_{\mathrm{H}} a \\ -d_{\mathrm{h}} \Delta_{\mathrm{N}} h + h_* \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}} \\ -d_{\mathrm{a}} \Delta_{\mathrm{N}} a + a_* \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}} \end{pmatrix}. \tag{3.41}$$

Proof. Let us first observe that F_{s} from (3.38) consists of bilinear terms and the linear term $-c_{\mathrm{cor}} v_{\mathrm{ice}}^{\perp}$. Thus, it follows that $F_{\mathrm{s}} \colon V \to \mathrm{X}_0$ is Fréchet differentiable. In view of the shape of $u_* = (0, h_*, a_*)$ with $h_* > \kappa$ as well as $a_* \in (0,1)$ constant in time and space, for $u \in \mathrm{X}_1$, we find that

$$F_{\mathrm{s}}'(u_*)u = \begin{pmatrix} -c_{\mathrm{cor}} v_{\mathrm{ice}}^{\perp} \\ -h_* \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}} \\ -a_* \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}} \end{pmatrix}. \tag{3.42}$$

Next, with regard to $A(u_*)$, we insert u_* of the above shape into (3.24) and use $P_{\mathrm{h},*}$ and $P_{\mathrm{a},*}$ from (3.40) to obtain

$$A(u_*)u = \begin{pmatrix} -A_{\mathrm{D}}^{\mathrm{H}}(u_*) v_{\mathrm{ice}} + P_{\mathrm{h},*} \nabla_{\mathrm{H}} h + P_{\mathrm{a},*} \nabla_{\mathrm{H}} a \\ -d_{\mathrm{h}} \Delta_{\mathrm{N}} h \\ -d_{\mathrm{a}} \Delta_{\mathrm{N}} a \end{pmatrix} \tag{3.43}$$

for $u \in \mathrm{X}_1$. Concerning the h- and a-component in the operator matrix A, the Fréchet differentiability is immediate as $-d_{\mathrm{h}} \Delta_{\mathrm{N}}$ and $-d_{\mathrm{a}} \Delta_{\mathrm{N}}$ are linear operators. For the terms appearing in the equation of v_{ice}, we recall the shape of Hibler's operator $A_{\mathrm{D}}^{\mathrm{H}}$ from Section 3.3 and argue that the dependence of the coefficients on $\varepsilon(v_{\mathrm{ice}})$, h and a is smooth thanks to the latter variables being contained in V so that the operator does not degenerate. In total, it follows that A is also Fréchet differentiable. The shape of $u_* = (0, h_*, a_*)$, with h_* and a_* constant in time and space, then yields that $(A'(u_*)u)u_* = 0$.

Concatenating the latter observation with (3.42) and (3.43), we recover the shape of A_0 as asserted in (3.41). □

In the following, we verify that u_* as in Lemma 3.6.1 are normally stable equilibria in the sense of Definition 2.6.4. We start with spectral properties of the linearized Hibler operator $A_{\mathrm{D}}^{\mathrm{H}}(u_*)$ and the total linearization A_0.

Lemma 3.6.3. *Consider $u_* = (0, h_*, a_*)$, where $h_* > \kappa$ and $a_* \in (0,1)$ are constant in time and space, and recall the linearized Hibler operator $A_{\mathrm{D}}^{\mathrm{H}}(u_*)$ from* (3.17) *and the total linearization A_0 around u_* from* (3.41)*. Then*

(a) we have $0 \in \rho(A_{\mathrm{D}}^{\mathrm{H}}(u_*))$, $s(A_{\mathrm{D}}^{\mathrm{H}}(u_*)) < 0$ and $-A_{\mathrm{D}}^{\mathrm{H}}(u_*) \in \mathcal{MR}_p(\mathrm{L}^q(\Omega)^2)$,

(b) the operator A_0 has a compact resolvent on X_0. In particular, the spectrum of A_0 is q-independent and only consists of eigenvalues, and

(c) it holds that $\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+$ *and* $\mathrm{N}(A_0) = \{0\} \times \mathbb{R} \times \mathbb{R}$.

Proof. The shape of u_* implies $u_* \in \mathrm{C}^{1,\alpha}(\overline{\Omega})$ for $\alpha > 0$ and $h_* \geq \kappa$. Therefore, Corollary 3.3.5(b) and Lemma 2.1.15 lead to $-A_{\mathrm{D}}^{\mathrm{H}}(u_*) + \omega \in {}_0\mathcal{MR}_p(\mathrm{L}^q(\Omega)^2)$ for all $\omega > s(A_{\mathrm{D}}^{\mathrm{H}}(u_*))$. Moreover, by virtue of Lemma 3.3.7, the spectrum $\sigma(A_{\mathrm{D}}^{\mathrm{H}}(u_*))$ is q-independent and only consists of eigenvalues. Hence, for $v_{\mathrm{ice}} \in \mathrm{D}(A_{\mathrm{D}}^{\mathrm{H}}(u_*))$, it is sufficient to consider the eigenvalue equation for $A_{\mathrm{D}}^{\mathrm{H}}(u_*)$ which reads as

$$\lambda v_{\mathrm{ice}} - A_{\mathrm{D}}^{\mathrm{H}}(u_*) v_{\mathrm{ice}} = 0. \tag{3.44}$$

Before proceeding, we first elaborate on the precise shape of $A_{\mathrm{D}}^{\mathrm{H}}(u_*)$. Recalling the coefficients $a_{ij}^{kl}(\varepsilon, P)$ from (3.15), we find by the shape of u_* that

$$-A_{\mathrm{D}}^{\mathrm{H}}(u_*) v_{\mathrm{ice}} = -\frac{P_*}{\sqrt{\delta}} \sum_{j,k,l=1}^{2} \mathbb{S}_{ij}^{kl} \partial_k \partial_l v_{\mathrm{ice},j}. \tag{3.45}$$

Now, as a further preparation, we test (3.45) by v_{ice}, integrate by vector parts, where we invoke the Dirichlet boundary conditions of v_{ice}, and rediscover $\triangle^2(\nabla_{\mathrm{H}} v_{\mathrm{ice}})$ as introduced in (3.12) to get

$$\begin{aligned} -\int_\Omega A_{\mathrm{D}}^{\mathrm{H}}(u_*) v_{\mathrm{ice}} \cdot v_{\mathrm{ice}} \, \mathrm{d}x_{\mathrm{H}} &= -\frac{P_*}{\sqrt{\delta}} \sum_{i,j,k,l=1}^{2} \int_\Omega \mathbb{S}_{ij}^{kl} \partial_k \partial_l v_{\mathrm{ice},j} v_{\mathrm{ice},i} \, \mathrm{d}x_{\mathrm{H}} \\ &= \frac{P_*}{\sqrt{\delta}} \sum_{i,j,k,l=1}^{2} \int_\Omega \mathbb{S}_{ij}^{kl} \partial_k v_{\mathrm{ice},i} \partial_l v_{\mathrm{ice},j} \, \mathrm{d}x_{\mathrm{H}} \\ &= \frac{P_*}{\sqrt{\delta}} \int_\Omega \triangle^2(\nabla_{\mathrm{H}} v_{\mathrm{ice}}) \, \mathrm{d}x_{\mathrm{H}}. \end{aligned}$$

By the shape of $\triangle^2(\nabla_{\mathrm{H}} v_{\mathrm{ice}})$, we find the estimate

$$\triangle^2(\nabla_{\mathrm{H}} v_{\mathrm{ice}}) \geq \frac{1}{e^2} \cdot |\varepsilon(v_{\mathrm{ice}})|^2.$$

Together with Korn's inequality, see Lemma 1.4.2, and Poincaré's inequality as stated in Lemma 1.4.1, the latter inequality leads to

$$\begin{aligned} \frac{P_*}{\sqrt{\delta}} \int_\Omega \triangle^2(\nabla_{\mathrm{H}} v_{\mathrm{ice}}) \, \mathrm{d}x_{\mathrm{H}} &\geq \frac{C_1}{\sqrt{\delta}} \int_\Omega |\varepsilon(v_{\mathrm{ice}})|^2 \, \mathrm{d}x_{\mathrm{H}} \\ &\geq \frac{C_2}{\sqrt{\delta}} \cdot \|\nabla_{\mathrm{H}} v_{\mathrm{ice}}\|_{\mathrm{L}^2(\Omega)}^2 \\ &\geq \frac{C_3}{\sqrt{\delta}} \cdot \|v_{\mathrm{ice}}\|_{\mathrm{H}^1(\Omega)}^2. \end{aligned}$$

In summary, for some constant $C_4(\delta) > 0$, we have

$$-\int_\Omega A_{\mathrm{D}}^{\mathrm{H}}(u_*)v_{\mathrm{ice}} \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} \ge C_4(\delta) \cdot \|v_{\mathrm{ice}}\|_{\mathrm{H}^1(\Omega)}^2. \tag{3.46}$$

Consequently, testing equation (3.44) by v_{ice}, for $C_4(\delta) > 0$, we conclude

$$0 = \lambda\|v_{\mathrm{ice}}\|_{\mathrm{L}^2(\Omega)}^2 - \int_\Omega A_{\mathrm{D}}^{\mathrm{H}}(u_*)v_{\mathrm{ice}} \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} \ge \lambda\|v_{\mathrm{ice}}\|_{\mathrm{L}^2(\Omega)}^2 + C_4(\delta) \cdot \|v_{\mathrm{ice}}\|_{\mathrm{H}^1(\Omega)}^2.$$

It follows that $\lambda \in \mathbb{R}$ and $v_{\mathrm{ice}} = 0$ if $\lambda = 0$, so $0 \in \rho(A_{\mathrm{D}}^{\mathrm{H}}(u_*))$. If on the other hand $v_{\mathrm{ice}} \neq 0$, then $\lambda < 0$. Thus, the spectral bound of the $\mathrm{L}^2(\Omega)^2$-realization of $A_{\mathrm{D}}^{\mathrm{H}}(u_*)$ is negative. With regard to the q-independence of the spectrum, this carries over to every $q \in (1, \infty)$, i. e., we get $s(A_{\mathrm{D}}^{\mathrm{H}}(u_*)) < 0$. In particular, the argument from the beginning of the proof and Lemma 2.1.15 imply $-A_{\mathrm{D}}^{\mathrm{H}}(u_*) \in \mathcal{MR}_p(\mathrm{L}^q(\Omega)^2)$, finishing the proof of (a).

As in the proof of Lemma 3.3.7, we use the Rellich-Kondrachov theorem from Lemma 1.4.3 to argue that the embedding of $\mathrm{X}_1 = \mathrm{D}(A_0)$ from (3.19) into X_0 from (3.18) is compact, so the compactness of the resolvent of A_0 follows from Lemma 2.1.6(a). The q-independence of the spectrum as well as the fact that the latter only consists of eigenvalues are then a result of Lemma 2.1.6(b) and (c). This shows the assertion of (b).

Thanks to (b), for $u \in \mathrm{D}(A_0)$, it is sufficient to consider the eigenvalue equation

$$\lambda u + A_0 u = 0 \tag{3.47}$$

in order to determine the spectrum. Similarly as in the proof of (a), we start with the L^2-case which then extends to general q by (b). First, we set

$$c_1 := \frac{P_{\mathrm{h},*}}{h_*} > 0, \quad c_2 := \frac{P_{\mathrm{a},*}}{a_*} > 0 \text{ and } \tilde{u} := \left(v_{\mathrm{ice}}, c_1^{1/2} h, c_2^{1/2}\right).$$

Let us observe that

$$\int_\Omega u^\top \cdot (v_{\mathrm{ice}}, c_1 h, c_2 a)^\top \,\mathrm{d}x_{\mathrm{H}} = \|\tilde{u}\|_{\mathrm{L}^2(\Omega)}^2.$$

Moreover, we have $v_{\mathrm{ice}}^\perp \cdot v_{\mathrm{ice}} = 0$. An integration by parts based on the Neumann boundary conditions of h and a leads to

$$\begin{aligned} -\int_\Omega d_{\mathrm{h}} \Delta_{\mathrm{N}} h \cdot c_1 h \,\mathrm{d}x_{\mathrm{H}} &= d_{\mathrm{h}} c_1 \cdot \|\nabla_{\mathrm{H}} h\|_{\mathrm{L}^2(\Omega)}^2 \text{ and} \\ -\int_\Omega d_{\mathrm{a}} \Delta_{\mathrm{N}} a \cdot c_2 a \,\mathrm{d}x_{\mathrm{H}} &= d_{\mathrm{a}} c_2 \cdot \|\nabla_{\mathrm{H}} a\|_{\mathrm{L}^2(\Omega)}^2. \end{aligned} \tag{3.48}$$

Testing the eigenvalue equation (3.47) by $(v_{\text{ice}}, c_1 h, c_2 a)$, and integrating by vector parts, additionally invoking the preceding relations, we obtain

$$
\begin{aligned}
0 = \lambda \|\tilde{u}\|_{\mathrm{L}^2(\Omega)}^2 &- \int_\Omega A_{\mathrm{D}}^{\mathrm{H}}(u_*) v_{\text{ice}} \cdot v_{\text{ice}} \, \mathrm{d}x_{\mathrm{H}} \\
&+ P_{\mathrm{h},*} \int_\Omega \nabla_{\mathrm{H}} h \cdot v_{\text{ice}} \, \mathrm{d}x_{\mathrm{H}} + P_{\mathrm{a},*} \int_\Omega \nabla_{\mathrm{H}} a \cdot v_{\text{ice}} \, \mathrm{d}x_{\mathrm{H}} \\
&+ d_{\mathrm{h}} c_1 \cdot \|\nabla_{\mathrm{H}} h\|_{\mathrm{L}^2(\Omega)}^2 + c_1 h_* \int_\Omega h \cdot \operatorname{div}_{\mathrm{H}} v_{\text{ice}} \, \mathrm{d}x_{\mathrm{H}} \\
&+ d_{\mathrm{a}} c_2 \cdot \|\nabla_{\mathrm{H}} a\|_{\mathrm{L}^2(\Omega)}^2 + c_2 a_* \int_\Omega a \cdot \operatorname{div}_{\mathrm{H}} v_{\text{ice}} \, \mathrm{d}x_{\mathrm{H}}.
\end{aligned} \tag{3.49}
$$

Now, we treat the terms (3.49) separately. Let us observe that we already treated the term related to the linearized Hibler operator in (3.46). Another integration by parts as well as the shape of c_1 imply that

$$
P_{\mathrm{h},*} \int_\Omega \nabla_{\mathrm{H}} h \cdot v_{\text{ice}} \, \mathrm{d}x_{\mathrm{H}} + c_1 h_* \int_\Omega h \cdot \operatorname{div}_{\mathrm{H}} v_{\text{ice}} \, \mathrm{d}x_{\mathrm{H}} = (P_{\mathrm{h},*} - c_1 h_*) \int_\Omega \nabla_{\mathrm{H}} h \cdot v_{\text{ice}} \, \mathrm{d}x_{\mathrm{H}}
$$

equals zero, and likewise

$$
P_{\mathrm{a},*} \int_\Omega \nabla_{\mathrm{H}} a \cdot v_{\text{ice}} \, \mathrm{d}x_{\mathrm{H}} + c_2 a_* \int_\Omega a \cdot \operatorname{div}_{\mathrm{H}} v_{\text{ice}} \, \mathrm{d}x_{\mathrm{H}} = 0.
$$

Plugging these estimates and identities back into (3.49), we infer the existence of $C_5(\delta) > 0$ such that

$$
0 \geq \lambda \|\tilde{u}\|_{\mathrm{L}^2(\Omega)}^2 + C_5(\delta) \cdot \left(\|v_{\text{ice}}\|_{\mathrm{H}^1(\Omega)}^2 + \|\nabla_{\mathrm{H}} h\|_{\mathrm{L}^2(\Omega)}^2 + \|\nabla_{\mathrm{H}} h\|_{\mathrm{L}^2(\Omega)}^2 \right). \tag{3.50}
$$

It follows from (3.50) that $\lambda \in \mathbb{R}$ as well as $\lambda \leq 0$. In conjunction with the q-independence of the spectrum of A_0 by (b), we get $\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+$. For $\lambda = 0$ in (3.50), we can determine $\mathrm{N}(A_0)$. In fact, it follows that $v_{\text{ice}} = 0$, and h and a are constant, completing the proof of (c). □

The next lemma investigates the shape of the set of equilibria $\mathcal{E}$.

Lemma 3.6.4. *Let $p, q \in (1, \infty)$ be such that* (3.37) *holds true, and consider an equilibrium $u_* = (0, h_*, a_*)$, where $h_* > \kappa$ and $a_* \in (0,1)$ are constant in time and space. Near u_*, the set of equilibria $\mathcal{E}$ is a C^1-manifold in X_1. Moreover, the tangent space of $\mathcal{E}$ at u_* is isomorphic to $\mathrm{N}(A_0)$.*

Proof. Let $u = (v_{\text{ice}}, h, a) \in V \cap \mathrm{X}_1$ be an equilibrium with $\|u - u_*\|_{\mathrm{X}_\gamma} < r$ for given $r > 0$. Then u satisfies $A(u)u = F_{\mathrm{s}}(u)$. For the presentation below, it is advantageous to multiply the respective first equation by $2\rho_{\mathrm{ice}}h$. The resulting set of equations satisfied by u is given by

$$(3.51)\qquad 0 = \begin{pmatrix} 2\rho_{\mathrm{ice}}h(-A_{\mathrm{D}}^{\mathrm{H}}(u) + (v_{\mathrm{ice}}\cdot\nabla_{\mathrm{H}})v_{\mathrm{ice}} + c_{\mathrm{cor}}v_{\mathrm{ice}}^{\perp}) + \nabla_{\mathrm{H}}P(h,a) \\ -d_{\mathrm{h}}\Delta_{\mathrm{N}}h + \mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}}h) \\ -d_{\mathrm{a}}\Delta_{\mathrm{N}}a + \mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}}a) \end{pmatrix}.$$

The idea is again to test (3.51) by a suitable test function. In the sequel, we calculate and estimate some terms as a preparation. First, let us observe that $u \in V$ with $\|u - u_*\|_{\mathrm{X}_\gamma} < r$ especially yields

$$(3.52)\qquad P(h,a) \ge p^*\kappa\mathrm{e}^{-c_\bullet} =: P_{**} > 0 \ \text{ and } \ \frac{1}{\triangle_\delta(\varepsilon(v_{\mathrm{ice}}))} \ge \frac{1}{\sqrt{\delta + c_e r^2}}$$

for some constant $c_e > 0$. Hence, recalling the shape of Hibler's operator from (3.17), see also (3.16), using an integration by parts along with the Dirichlet boundary conditions of v_{ice}, invoking the shape of $\triangle^2(\varepsilon) = \varepsilon^\top\mathbb{S}\varepsilon$ from (3.12), and exploiting the above inequalities from (3.52), we find

$$\begin{aligned} -\int_\Omega 2\rho_{\mathrm{ice}}hA_{\mathrm{D}}^{\mathrm{H}}(u)v_{\mathrm{ice}}\cdot v_{\mathrm{ice}}\,\mathrm{d}x_{\mathrm{H}} &= -\int_\Omega \mathrm{div}_{\mathrm{H}}\left(P(h,a)\frac{\mathbb{S}\varepsilon}{\triangle_\delta(\varepsilon)}\right)\cdot v_{\mathrm{ice}}\,\mathrm{d}x_{\mathrm{H}} \\ &= \int_\Omega P(h,a)\frac{\varepsilon(v_{\mathrm{ice}})^\top\mathbb{S}\varepsilon(v_{\mathrm{ice}})}{\triangle_\delta(\varepsilon(v_{\mathrm{ice}}))}\,\mathrm{d}x_{\mathrm{H}} \\ &\ge \frac{P_{**}}{\sqrt{\delta + c_e r^2}}\int_\Omega \varepsilon(v_{\mathrm{ice}})^\top\mathbb{S}\varepsilon(v_{\mathrm{ice}})\,\mathrm{d}x_{\mathrm{H}} \\ &\ge \frac{P_{**}(1 - 1/e^2)}{\sqrt{\delta + c_e r^2}}\cdot\|\varepsilon(v_{\mathrm{ice}})\|_{\mathrm{L}^2(\Omega)}^2. \end{aligned}$$

Additionally using Korn's and Poincaré's inequality from Lemma 1.4.2 and Lemma 1.4.1, for some constants C_1, $C_2 > 0$, we derive the estimate

$$(3.53)\qquad \begin{aligned} -\int_\Omega 2\rho_{\mathrm{ice}}hA_{\mathrm{D}}^{\mathrm{H}}(u)v_{\mathrm{ice}}\cdot v_{\mathrm{ice}}\,\mathrm{d}x_{\mathrm{H}} &\ge \frac{C_1}{\sqrt{\delta + c_e r^2}}\cdot\|\nabla_{\mathrm{H}}v_{\mathrm{ice}}\|_{\mathrm{L}^2(\Omega)}^2 \\ &\ge \frac{C_2}{\sqrt{\delta + c_e r^2}}\cdot\|v_{\mathrm{ice}}\|_{\mathrm{H}^1(\Omega)}^2. \end{aligned}$$

Next, Hölder's inequality and the trace space embedding (3.21) joint with the shape of the equilibrium $u_* = (0, h_*, a_*)$ and $\|u - u_*\|_{X_\gamma} < r$ yield

$$
\begin{aligned}
&2\rho_{\mathrm{ice}} \int_\Omega h(v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice}} \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} \\
&\le C_3 \cdot \|h\|_{\mathrm{L}^\infty(\Omega)} \cdot \|\nabla_{\mathrm{H}} v_{\mathrm{ice}}\|_{\mathrm{L}^\infty(\Omega)} \cdot \|v_{\mathrm{ice}}\|_{\mathrm{L}^2(\Omega)}^2 \\
&\le C_4 \cdot \Bigl(h_* + \|h - h_*\|_{\mathrm{L}^\infty(\Omega)}\Bigr) \cdot \|v_{\mathrm{ice}}\|_{\mathrm{C}^1(\overline{\Omega})} \cdot \|v_{\mathrm{ice}}\|_{\mathrm{H}^1(\Omega)}^2 \\
&\le C_5 \cdot (1 + r) \cdot r \cdot \|v_{\mathrm{ice}}\|_{\mathrm{H}^1(\Omega)}^2
\end{aligned}
\tag{3.54}
$$

for some constant $C_5 > 0$. Next, we define $c_1 \coloneqq 2\rho_{\mathrm{ice}} P_{\mathrm{h},*}$ and $c_2 \coloneqq 2\rho_{\mathrm{ice}} P_{\mathrm{a},*}$. Adding and subtracting suitable terms, we then compute

$$
\begin{aligned}
&\int_\Omega \nabla_{\mathrm{H}} P(h, a) \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} \\
&= \int_\Omega (\partial_h P(h, a) - c_1 h) \nabla_{\mathrm{H}} h \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} + c_1 \int_\Omega h \nabla_{\mathrm{H}} h \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} \\
&\quad + \int_\Omega (\partial_a P(h, a) - c_2 a) \nabla_{\mathrm{H}} a \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} + c_2 \int_\Omega a \nabla_{\mathrm{H}} a \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}}.
\end{aligned}
\tag{3.55}
$$

From (3.55) and an integration by parts, it follows that

$$
\begin{aligned}
&\int_\Omega \nabla_{\mathrm{H}} P(h, a) \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} + c_1 \int_\Omega \mathrm{div}_{\mathrm{H}} \,(v_{\mathrm{ice}} h) h \,\mathrm{d}x_{\mathrm{H}} \\
&\quad + c_2 \int_\Omega \mathrm{div}_{\mathrm{H}} \,(v_{\mathrm{ice}} a) a \,\mathrm{d}x_{\mathrm{H}} \\
&= \int_\Omega \nabla_{\mathrm{H}} P(h, a) \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} - c_1 \int_\Omega h \nabla_{\mathrm{H}} h \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} \\
&\quad - c_2 \int_\Omega a \nabla_{\mathrm{H}} a \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} \\
&= \int_\Omega (\partial_h P(h, a) - c_1 h) \nabla_{\mathrm{H}} h \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} \\
&\quad + \int_\Omega (\partial_a P(h, a) - c_2 a) \nabla_{\mathrm{H}} a \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}}.
\end{aligned}
\tag{3.56}
$$

Finally, we test (3.51) by $(v_{\mathrm{ice}}, c_1 h, c_2 a)$. From the estimates and identities (3.53), (3.48), (3.54), (3.56) and $v_{\mathrm{ice}}^\perp \cdot v_{\mathrm{ice}} = 0$, we infer that

$$
\begin{aligned}
0 \ge &\left(\frac{C_2}{\sqrt{\delta + c_e r^2}} - C_5(1 + r) r\right) \cdot \|v_{\mathrm{ice}}\|_{\mathrm{H}^1(\Omega)}^2 + c_1 d_{\mathrm{h}} \cdot \|\nabla_{\mathrm{H}} h\|_{\mathrm{L}^2(\Omega)}^2 \\
&+ c_2 d_{\mathrm{a}} \cdot \|\nabla_{\mathrm{H}} a\|_{\mathrm{L}^2(\Omega)}^2 + \int_\Omega (\partial_h P(h, a) - c_1 h) \nabla_{\mathrm{H}} h \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} \\
&+ \int_\Omega (\partial_a P(h, a) - c_2 a) \nabla_{\mathrm{H}} a \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}}.
\end{aligned}
\tag{3.57}
$$

It remains to absorb the terms without sign in (3.57) into the other terms. To this end, we note that

$$
\partial_h P(h, a) - c_1 h = \frac{p^*}{h_*} \Bigl((h_* - h) \mathrm{e}^{-c_\bullet (1 - a)} + h \Bigl(\mathrm{e}^{-c_\bullet (1 - a)} - \mathrm{e}^{-c_\bullet (1 - a_*)}\Bigr)\Bigr).
$$

Moreover, let us observe that $\|h\|_{\mathrm{L}^\infty(\Omega)} \le h_* + r$ by $\|u - u_*\|_{\mathrm{X}_\gamma} < r$ as well as the embedding (3.21). Therefore, making use of the mean value theorem and the aforementioned relations as well as $u \in V$, we get

$$\begin{aligned} &\|\partial_h P(h,a) - c_1 h\|_{\mathrm{L}^\infty(\Omega)} \\ &\le C_6 \cdot \|h_* - h\|_{\mathrm{L}^\infty(\Omega)} + C_7 \cdot (1+r) \cdot \|a_* - a\|_{\mathrm{L}^\infty(\Omega)} \\ &\le C_8(1+r)r. \end{aligned} \tag{3.58}$$

Upon remarking that

$$\begin{aligned} \partial_a P(h,a) - c_2 a = \frac{c_\bullet p^*}{a_*}\Big((a_* - a)h\mathrm{e}^{-c_\bullet(1-a)} + a(h-h_*)\mathrm{e}^{-c_\bullet(1-a)} \\ + a h_* \left(\mathrm{e}^{-c_\bullet(1-a)} - \mathrm{e}^{-c_\bullet(1-a_*)}\right)\Big), \end{aligned}$$

we use an analogous strategy to find for some $C_9 > 0$ the estimate

$$\|\partial_a P(h,a) - c_2 a\|_{\mathrm{L}^\infty(\Omega)} \le C_9(1+r)r. \tag{3.59}$$

An application of Hölder's and Young's inequality in (3.57) in conjunction with (3.58) and (3.59) results in

$$\begin{aligned} 0 \ge &\left(\frac{C_2}{\sqrt{\delta + c_e r^2}} - C_{10}(1+r)r\right) \cdot \|v_{\mathrm{ice}}\|_{\mathrm{H}^1(\Omega)}^2 \\ &+ (c_1 d_{\mathrm{h}} - C_{11}(1+r)r) \cdot \|\nabla_{\mathrm{H}} h\|_{\mathrm{L}^2(\Omega)}^2 + (c_2 d_{\mathrm{a}} - C_{12}(1+r)r) \cdot \|\nabla_{\mathrm{H}} a\|_{\mathrm{L}^2(\Omega)}^2. \end{aligned}$$

Consequently, if we choose $r > 0$ sufficiently small, we get

$$0 \ge \|v_{\mathrm{ice}}\|_{\mathrm{H}^1(\Omega)}^2 + \|\nabla_{\mathrm{H}} h\|_{\mathrm{L}^2(\Omega)}^2 + \|\nabla_{\mathrm{H}} a\|_{\mathrm{L}^2(\Omega)}^2.$$

In summary, for an equilibrium $u \in V \cap \mathrm{X}_1$ with $\|u - u_*\| < r$ for such $r > 0$, we conclude that $v_{\mathrm{ice}} = 0$, and h as well as a are constant, i. e.,

$$\mathbb{B}_{\mathrm{X}_\gamma \cap \mathcal{E}}(u_*, r) = \{0\} \times \mathbb{R} \times \mathbb{R} = \mathrm{N}(A_0)$$

by Lemma 3.6.3(c). This also shows that near u_*, the set of equilibria is a two-dimensional C^1-manifold. Besides, the tangent space of the set of equilibria near u_* coincides with $\mathrm{N}(A_0)$ and is thus especially isomorphic to the latter one. This completes the proof of the lemma. □

Next, we study the spectral value zero in more detail.

Lemma 3.6.5. *For $u_* = (0, h_*, a_*)$, with $h_* > \kappa$ and $a_* \in (0,1)$ constant in time and space, consider the resulting A_0 as in (3.41). Then zero is a semi-simple eigenvalue of A_0, i. e., we have $\mathrm{N}(A_0) \oplus \mathrm{R}(A_0) = \mathrm{X}_0$.*

Proof. For the space of functions in $\mathrm{L}^q(\Omega)$ with average zero $\mathrm{L}^q_0(\Omega)$ as defined in (1.1), we set

$$\mathrm{X}_0^{\mathrm{m}} := \mathrm{L}^q(\Omega)^2 \times \mathrm{L}^q_0(\Omega) \times \mathrm{L}^q_0(\Omega).$$

Moreover, A_0^{m} represents the restriction of the total linearization to $\mathrm{X}_0^{\mathrm{m}}$, i. e.,

$$A_0^{\mathrm{m}} u := A_0 u, \quad \text{for } u \in \mathrm{D}(A_0^{\mathrm{m}}) := \mathrm{D}(A_0) \cap \mathrm{X}_0^{\mathrm{m}}. \tag{3.60}$$

Also in this case, we deduce from (3.50) the existence of $C_1 > 0$ with

$$0 \geq C_1 \cdot \left(\|v_{\mathrm{ice}}\|^2_{\mathrm{H}^1(\Omega)} + \|\nabla_{\mathrm{H}} h\|^2_{\mathrm{L}^2(\Omega)} + \|\nabla_{\mathrm{H}} a\|^2_{\mathrm{L}^2(\Omega)} \right) \tag{3.61}$$

when testing the equation $A_0 u = 0$ for $u \in \mathrm{D}(A_0^{\mathrm{m}})$. As h, $a \in \mathrm{L}^q_0(\Omega)$, we conclude from (3.61) that $v_{\mathrm{ice}} = 0$ as well as $h = a = 0$. The compact resolvent of A_0 also carries over to A_0^{m} by observing that the average zero condition is simply preserved by interpolation, see Lemma 1.3.5. Thus, the spectrum of A_0^{m} is also q-independent by Lemma 2.1.6(a). Therefore, it holds that $0 \in \rho(A_0^{\mathrm{m}})$. On the other hand, we recall from Lemma 3.6.3(c) that

$$\mathrm{N}(A_0) = \{0\} \times \mathbb{R} \times \mathbb{R},$$

so in order to show that $\mathrm{X}_0 = \mathrm{N}(A_0) + \mathrm{R}(A_0)$, it suffices to verify

$$\mathrm{L}^q(\Omega)^2 \times \mathrm{L}^q_0(\Omega) \times \mathrm{L}^q_0(\Omega) \subset \mathrm{R}(A_0). \tag{3.62}$$

For this, consider $f = (f_{\mathrm{ice}}, f_{\mathrm{h}}, f_{\mathrm{a}}) \in \mathrm{L}^q(\Omega)^2 \times \mathrm{L}^q_0(\Omega) \times \mathrm{L}^q_0(\Omega)$. From $0 \in \rho(A_0^{\mathrm{m}})$, it follows that there is $u \in \mathrm{D}(A_0^{\mathrm{m}})$ such that $A_0 u = A_0^{\mathrm{m}} u = f$. Therefore, we have (3.62) which in turns implies $\mathrm{X}_0 = \mathrm{N}(A_0) + \mathrm{R}(A_0)$.

It remains to verify $\mathrm{N}(A_0) \cap \mathrm{R}(A_0) = \{0\}$. Thus, let $u \in \mathrm{N}(A_0) \cap \mathrm{R}(A_0)$. From the above shape of $\mathrm{N}(A_0)$, it especially follows that $u = (0, c_{\mathrm{h}}, c_{\mathrm{a}})$ for two constants c_{h} and c_{a}. On the other hand, as $u \in \mathrm{R}(A_0)$, there is $\tilde{u} \in \mathrm{D}(A_0)$ such that $A_0 \tilde{u} = u$. Next, we invoke the splitting into mean value zero part and average part as introduced in (1.2) in the h- and a-component, i. e.,

$$\tilde{u} = \begin{pmatrix} \tilde{v}_{\mathrm{ice}} \\ \tilde{h} \\ \tilde{a} \end{pmatrix} = \begin{pmatrix} \tilde{v}_{\mathrm{ice}} \\ \tilde{h}_{\mathrm{m}} \\ \tilde{a}_{\mathrm{m}} \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{h}_{\mathrm{avg}} \\ \tilde{a}_{\mathrm{avg}} \end{pmatrix} =: \tilde{u}_{\mathrm{m}} + \tilde{u}_{\mathrm{avg}}.$$

In particular, we have $\tilde{u}_{\mathrm{m}} \in \mathrm{X}_0^{\mathrm{m}}$, yielding $\tilde{u}_{\mathrm{m}} \in \mathrm{D}(A_0^{\mathrm{m}})$, and $\tilde{u}_{\mathrm{avg}} \in \mathrm{N}(A_0)$. Consequently, it is valid that

$$u = A_0 \tilde{u} = A_0 \tilde{u}_{\mathrm{m}} + A_0 \tilde{u}_{\mathrm{avg}} = A_0^{\mathrm{m}} \tilde{u}_{\mathrm{m}}.$$

In other words, $u \in \mathrm{R}(A_0^{\mathrm{m}})$, so $c_{\mathrm{h}}, c_{\mathrm{a}} \in \mathrm{L}_0^q(\Omega)$, but this implies $c_{\mathrm{h}} = c_{\mathrm{a}} = 0$ and hence $u = 0$ in total. This proves that $\mathrm{N}(A_0) \cap \mathrm{R}(A_0) = \{0\}$. In conclusion, we have shown that $\mathrm{X}_0 = \mathrm{N}(A_0) \oplus \mathrm{R}(A_0)$ is indeed valid. □

Combining Lemma 3.6.2, Lemma 3.6.3, Lemma 3.6.4 and Lemma 3.6.5, for $p, q \in (1, \infty)$ such that (3.37), we deduce the normal stability of equilibria of the shape $u_* = (0, h_*, a_*)$, with $h_* > \kappa$ and $a_* \in (0,1)$ constant in time and space. With regard to the generalized principle of linearized stability as summed up in Proposition 2.6.5, it remains to argue that $A(u_*)$ admits maximal regularity of L^p-type. Thanks to Remark 2.6.6, it is sufficient to have this property up to a shift, which is guaranteed by Proposition 3.4.1.

The theorem below is the second main result of this chapter and asserts the global strong well-posedness of the simplified fully parabolic regularized model (3.35) for initial data close to constant equilibria of the aforementioned shape. By the above arguments, it is implied by an application of the generalized principle of linearized stability as stated in Proposition 2.6.5.

Theorem 3.6.6. *Let $p, q \in (1, \infty)$ satisfy (3.37), consider $u_* = (0, h_*, a_*)$, where $h_* > \kappa$ and $a_* \in (0,1)$ are constant in time and space, and recall X_γ from (3.36). Then u_* is stable in X_γ, and there is $r > 0$ so that the unique solution u to (3.39), and thus also to (3.35), for initial data $u_0 \in \mathrm{X}_\gamma$ with*

$$\|u_0 - u_*\|_{\mathrm{X}_\gamma} < r$$

exists on $\mathbb{R}_+$ and converges to some $u_\infty \in \mathcal{E}$ in X_γ at an exponential rate as $t \to \infty$.

In comparison with [18, Theorem 2.3], we do not impose a condition on the regularization parameter $\delta > 0$. This is due to a refined choice of the test function in the proof of Lemma 3.6.3.

CHAPTER 4

Interaction of Sea Ice with a Rigid Body

In this chapter, we investigate the interaction problem of sea ice, modeled by the fully parabolic regularized version of Hibler's model as discussed in Chapter 3, with a rigid body. The main result asserts the local strong well-posedness of the interaction problem.

The interaction of rigid structures with viscous fluids is a classical topic in mathematical fluid mechanics. We do not provide a full reference list here, but we only refer to the survey article of Galdi [45], the articles of Takahashi [127] and Geissert, Götze and Hieber [48] in the context of strong solutions to fluid-structure interaction problems with incompressible Newtonian and generalized Newtonian fluids, the work of Maity and Tucsnak [101] on a maximal regularity approach to fluid-structure interaction with incompressible fluids, or the papers of Hieber and Murata [66] and Haak et al. [57] on the interaction problem of a rigid body with viscous compressible fluids and compressible Navier-Stokes-Fourier fluids, respectively.

Let us observe that the physical motivation of the problem under consideration in this chapter lies in understanding the interaction of sea ice with large rigid structures with heavy mass such as ships. Concerning the numerical simulation of the motion of ships in an ice floe field, we refer for example to the article of Zhan et al. [134]. On the other hand, Tuhkuri and Polojärvi [131] provide a survey on ice-structure interaction.

Unlike most of the above references, the present underlying sea ice system is of *quasilinear* nature, leading to additional mathematical problems. The main result of the present chapter asserting the local strong well-posedness of the interaction problem has been obtained in a joint work with Tim Binz and Matthias Hieber [11]. However, the strategy in this chapter is completely different from the one in [11]. The linear theory there is obtained by a "monolithic" approach based on a decoupling argument, i. e., the coupling conditions are included in the domain of the operator matrix. In contrast, this chapter relies on a "cascade" approach. More precisely, the inhomogeneous boundary conditions resulting from the equality of the sea ice velocity and the rigid body velocity are handled by optimal L^p-L^q estimates as recalled in Section 2.5. Another difference is that abstract *non-autonomous* quasilinear existence theory is used in [11], whereas we employ a direct fixed point argument here.

The chapter is organized as follows. In Section 4.1, we present the interaction problem of sea ice with a rigid body on the moving domain, leading to the complete system in (4.8). Section 4.2 is dedicated to the transformation of the moving domain problem to a problem on the fixed domain by means of a local transform as used first by Inoue and Wakimoto [73]. The properties of this transform are also discussed in Appendix A. This transformation to the fixed domain comes at the cost of non-autonomous terms as visible in the transformed interaction problem (4.28). Section 4.3 discusses the resulting linearization (4.30). In fact, Proposition 4.3.3 establishes the maximal regularity of the linearized problem by first solving the equations of the rigid body and regarding the interface condition on the sea ice and rigid body velocity as an inhomogeneous boundary conditions, resulting in the maximal regularity by virtue of L^p-L^q estimates as in Section 2.5. The purpose of Section 4.4 is to prepare the main result on the local strong well-posedness, Theorem 4.4.13, by setting up the fixed point argument, estimating the nonlinear terms and then showing the local strong well-posedness on the fixed domain. From there, the main result is deduced by employing the inverse coordinate transform.

4.1. The Interaction Problem of Sea Ice

Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial\mathcal{O}$ of class C^2, and consider a positive time $0 < T \leq \infty$. In the sequel, we investigate the situation of a 2D rigid body immersed in sea ice, and we denote the time-dependent domain occupied by the rigid body at time $t \in (0, T)$ by $\mathcal{S}(t)$. The remaining part of the domain filled by sea ice, which is modeled by Hibler's viscous-plastic model as introduced in Chapter 3, is denoted by $\mathcal{F}(t) = \mathcal{O} \setminus \mathcal{S}(t)$. Consequently, the interface is $\partial\mathcal{S}(t)$. Moreover, $\mathcal{S}_0$ represents the initial domain of the rigid body, and we assume its boundary $\partial\mathcal{S}_0$ to be of class C^2. Accordingly, $\mathcal{F}_0 = \mathcal{O} \setminus \mathcal{S}_0$ designates the initial sea ice domain, and the interface at time zero is $\partial\mathcal{S}_0$.

We denote by $x_\mathrm{c}(t)$ the center of mass of the rigid body at time $t \in (0,T)$. For convenience, we suppose that $x_\mathrm{c}(0) = 0$. Moreover,

$$\xi\colon (0,T) \to \mathbb{R}^2$$

represents the *translational velocity* of the rigid body, i. e., $x_\mathrm{c}'(t) = \xi(t)$. It follows from $x_\mathrm{c}(0) = 0$ that x_c can be deduced from ξ by integration, i. e.,

$$x_\mathrm{c}(t) = \int_0^t \xi(s)\,\mathrm{d}s. \tag{4.1}$$

As the rigid body also rotates, we introduce the rotation angle $\beta(t)$, $t \in (0,T)$, and associate to it a special orthogonal matrix $Q(t) \in \mathrm{SO}(2)$ of the shape

$$Q(t) = \begin{pmatrix} \cos\beta(t) & -\sin\beta(t) \\ \sin\beta(t) & \cos\beta(t) \end{pmatrix}, \tag{4.2}$$

accounting for the rotation of the rigid body. Its *angular velocity*

$$\Omega\colon (0,T) \to \mathbb{R}$$

then represents the derivative of the rotation angle, i. e., $\beta'(t) = \Omega(t)$. When assuming $\beta(0) = 0$ without loss of generality, we can thus derive β from

$$\beta(t) = \int_0^t \Omega(s)\,\mathrm{d}s. \tag{4.3}$$

With $y_\mathrm{H}^\perp = (-y_2, y_1)^\top$ for $y_\mathrm{H} = (y_1, y_2)^\top \in \mathbb{R}^2$, we compute

$$Q'(t)Q^\top(t)y_\mathrm{H} = \Omega(t)y_\mathrm{H}^\perp, \text{ for all } t > 0 \text{ and } y_\mathrm{H} \in \mathbb{R}^2. \tag{4.4}$$

Hence, the rigid body domain $\mathcal{S}(t)$ at time t is determined by

$$\mathcal{S}(t) = \left\{ x_\mathrm{c}(t) + Q(t)y_\mathrm{H} : y_\mathrm{H} \in \mathcal{S}_0 \right\},$$

and the velocity of the rigid body is given by

$$v_\mathcal{S}(t, x_\mathrm{H}) = \xi(t) + \Omega(t) \left(x_\mathrm{H} - x_\mathrm{c}(t) \right)^\perp, \text{ for all } (t, x_\mathrm{H}) \in (0,T) \times \mathcal{S}(t).$$

To distinguish the model variables from the fixed domain case, we introduce

$$\overline{v}_\mathrm{ice}\colon (0,T) \times \mathcal{F}(t) \to \mathbb{R}^2, \ \overline{h}\colon (0,T) \times \mathcal{F}(t) \to [\kappa, \infty) \text{ and}$$
$$\overline{a}\colon (0,T) \times \mathcal{F}(t) \to (0,1),$$

representing the sea ice velocity, the mean ice thickness and the ice compactness on the present domain in spacetime $(0,T) \times \mathcal{F}(t)$. The ranges of the variables have been explained in detail in Section 3.2. From there, we also recall the ice mass $m_{\mathrm{ice}} = \rho_{\mathrm{ice}} h$ for $\rho_{\mathrm{ice}} > 0$ constant, the regularized stress tensor σ_δ from (3.5), the Coriolis parameter $c_{\mathrm{cor}} > 0$, the term $g \nabla_{\mathrm{H}} H$ associated to the sea surface dynamics height, the atmospheric wind and ocean current force $\tau_{\mathrm{ice}} = \tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(v_{\mathrm{ice}})$ for τ_{atm} and $\tau_{\mathrm{ocn}}(v_{\mathrm{ice}})$ as in (3.6), the thermodynamic source terms S_{h} and S_{a} from (3.8) as well as the constants d_{h}, $d_{\mathrm{a}} > 0$. Similarly as in Section 3.2, up to the adjustment to the time-dependent domain $(0,T) \times \mathcal{F}(t)$, the system of equations satisfied by $\overline{u} = (\overline{v}_{\mathrm{ice}}, \overline{h}, \overline{a})$ is

$$
(4.5) \quad \begin{cases}
\partial_t \overline{v}_{\mathrm{ice}} + (\overline{v}_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) \overline{v}_{\mathrm{ice}} = \frac{1}{m_{\mathrm{ice}}} \mathrm{div}_{\mathrm{H}}\, \sigma_\delta - c_{\mathrm{cor}} \overline{v}_{\mathrm{ice}}^{\perp} & \\
\qquad\qquad - g \nabla_{\mathrm{H}} H + \frac{1}{m_{\mathrm{ice}}} \tau_{\mathrm{ice}}, & \text{in } (0,T) \times \mathcal{F}(t), \\
\partial_t \overline{h} + \mathrm{div}_{\mathrm{H}}\, (\overline{v}_{\mathrm{ice}} \overline{h}) = S_{\mathrm{h}}(\overline{h}, \overline{a}) + d_{\mathrm{h}} \Delta_{\mathrm{H}} \overline{h}, & \text{in } (0,T) \times \mathcal{F}(t), \\
\partial_t \overline{a} + \mathrm{div}_{\mathrm{H}}\, (\overline{v}_{\mathrm{ice}} \overline{a}) = S_{\mathrm{a}}(\overline{h}, \overline{a}) + d_{\mathrm{a}} \Delta_{\mathrm{H}} \overline{a}, & \text{in } (0,T) \times \mathcal{F}(t).
\end{cases}
$$

We further assume that the velocities of the sea ice $\overline{v}_{\mathrm{ice}}$ and the rigid body $v_{\mathcal{S}}$ coincide on their common interface, i. e.,

$$
(4.6) \qquad \overline{v}_{\mathrm{ice}} = v_{\mathcal{S}}, \quad \text{on } (0,T) \times \partial \mathcal{S}(t).
$$

Moreover, we suppose that the sea ice satisfies a no-slip boundary condition on the outer boundary $\partial \mathcal{O}$, while the mean ice thickness $\overline{h}$ and the ice compactness $\overline{a}$ are assumed to satisfy homogeneous Neumann boundary conditions on the boundary of the sea ice domain $\partial \mathcal{F}(t)$. This reads as

$$
(4.7) \quad \overline{v}_{\mathrm{ice}} = 0, \quad \text{on } (0,T) \times \partial \mathcal{O}, \quad \text{and } \partial_\nu \overline{h} = \partial_\nu \overline{a} = 0, \quad \text{on } (0,T) \times \partial \mathcal{F}(t).
$$

After introducing the system of equations as well as the boundary conditions satisfied by the sea ice part in the interaction problem, we focus on the equations of the rigid body. By $\rho_{\mathcal{S}}$, we denote the *density of the rigid body*, and for simplicity, as it does not affect the analysis, we suppose that $\rho_{\mathcal{S}} \equiv 1$. As a result, the *mass of the rigid body* $m_{\mathcal{S}}$ is determined by

$$
m_{\mathcal{S}} = \int_{\mathcal{S}(t)} 1 \, \mathrm{d}x_{\mathrm{H}}.
$$

As revealed in [31, Section 1], since we are in the 2D case, for J denoting the *inertia tensor of the rigid body*, we obtain

$$
J = \int_{\mathcal{S}(t)} \rho_{\mathcal{S}} \left| x_{\mathrm{H}} - x_{\mathrm{c}}(t) \right|^2 \, \mathrm{d}x_{\mathrm{H}} = \int_{\mathcal{S}_0} \left| y_{\mathrm{H}} \right|^2 \, \mathrm{d}y_{\mathrm{H}} =: J_0.
$$

In particular, we deduce therefrom that $(J\Omega)'(t) = J_0\Omega'(t)$. We further introduce external forces and torques $\overline{F}\colon (0,T) \to \mathbb{R}^2$ and $\overline{N}\colon (0,T) \to \mathbb{R}$. Recalling the regularized stress tensor $\sigma_\delta = \sigma_\delta(\overline{u})$ from (3.5), and following Newton's laws, we find that the equations satisfied by the momentum and angular momentum of the rigid body are given by

$$\begin{cases} m_{\mathcal{S}}\xi'(t) = -\displaystyle\int_{\partial\mathcal{S}(t)} \sigma_\delta(\overline{u})\overline{\nu}(t)\,\mathrm{d}\Gamma + \overline{F}(t), & \text{for } t \in (0,T), \\ J_0\Omega'(t) = -\displaystyle\int_{\partial\mathcal{S}(t)} (x_{\mathrm{H}} - x_{\mathrm{c}}(t))^\perp \sigma_\delta(\overline{u})\overline{\nu}(t)\,\mathrm{d}\Gamma + \overline{N}(t), & \text{for } t \in (0,T), \end{cases}$$

where $\overline{\nu}(t)$ represents the unit outward normal to the boundary of $\mathcal{F}(t)$, so it is directed inwards to $\mathcal{S}(t)$. We observe that $\overline{\nu}(t)$ is generally time-dependent.

In total, combining the equations satisfied by the sea ice and the rigid body, invoking the boundary conditions as well as the interface condition as in (4.6), and completing the system by initial conditions $v_{\mathrm{ice},0}$, h_0 and a_0 in $\mathcal{F}_0$ for the sea ice variables as well as ℓ_0 and ω_0 for the translational and angular velocity, we obtain the *interaction problem of sea ice with a rigid body*

$$\begin{cases} \partial_t \overline{v}_{\mathrm{ice}} = \dfrac{1}{m_{\mathrm{ice}}} \mathrm{div}_{\mathrm{H}}\, \sigma_\delta(\overline{u}) - (\overline{v}_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}})\overline{v}_{\mathrm{ice}} & \\ \qquad - c_{\mathrm{cor}}\overline{v}_{\mathrm{ice}}^\perp - g\nabla_{\mathrm{H}} H + \dfrac{1}{m_{\mathrm{ice}}}\tau_{\mathrm{ice}}, & \text{in } (0,T)\times\mathcal{F}(t), \\ \partial_t \overline{h} = d_{\mathrm{h}}\Delta_{\mathrm{H}}\overline{h} & \\ \qquad - \mathrm{div}_{\mathrm{H}}\,(\overline{v}_{\mathrm{ice}}\overline{h}) + S_{\mathrm{h}}(\overline{h},\overline{a}), & \text{in } (0,T)\times\mathcal{F}(t), \\ \partial_t \overline{a} = d_{\mathrm{a}}\Delta_{\mathrm{H}}\overline{a} & \\ \qquad - \mathrm{div}_{\mathrm{H}}\,(\overline{v}_{\mathrm{ice}}\overline{a}) + S_{\mathrm{a}}(\overline{h},\overline{a}), & \text{in } (0,T)\times\mathcal{F}(t), \\ m_{\mathcal{S}}\xi'(t) = \overline{F}(t) & \\ \qquad - \displaystyle\int_{\partial\mathcal{S}(t)} \sigma_\delta(\overline{u})\overline{\nu}(t)\,\mathrm{d}\Gamma, & \text{for } t\in(0,T), \\ J_0\Omega'(t) = \overline{N}(t) & \\ \qquad - \displaystyle\int_{\partial\mathcal{S}(t)} (x_{\mathrm{H}} - x_{\mathrm{c}}(t))^\perp\sigma_\delta(\overline{u})\overline{\nu}(t)\,\mathrm{d}\Gamma, & \text{for } t\in(0,T), \end{cases} \tag{4.8}$$

and completed by the boundary conditions

$$\begin{aligned} \overline{v}_{\mathrm{ice}} &= v^{\mathcal{S}},\ \ \partial_\nu\overline{h} = \partial_\nu\overline{a} = 0, &&\text{on } (0,T)\times\partial\mathcal{S}(t), \text{ and} \\ \overline{v}_{\mathrm{ice}} &= 0,\ \ \partial_\nu\overline{h} = \partial_\nu\overline{a} = 0, &&\text{on } (0,T)\times\partial\mathcal{O}, \end{aligned}$$

as well as the initial conditions

$$\begin{aligned} &\overline{v}_{\mathrm{ice}}(0) = v_{\mathrm{ice},0},\ \ \overline{h}(0) = h_0,\ \ \overline{a}(0) = a_0,\ \ \text{in } \mathcal{F}_0, \text{ and} \\ &x_{\mathrm{c}}(0) = 0,\ \ \xi(0) = \ell_0,\ \ \beta(0) = 0,\ \ \Omega(0) = \omega_0. \end{aligned}$$

4.2. Transformation to a Fixed Domain

As the complete system of equations (4.8) is defined on a moving domain, the first step is the transformation to a fixed domain. This is addressed in the present section. The cost of the transformation to a time-independent domain is that the terms in the resulting system become more complicated.

The main idea is to transform the system of equations only in a suitable neighborhood of the rigid body. This type of transform was introduced by Inoue and Wakimoto [73].

Throughout this section, we consider fixed translational and angular velocities $\xi \in \mathrm{W}^{1,p}(0,T)^2$ and $\Omega \in \mathrm{W}^{1,p}(0,T)$, respectively. Next, we introduce

$$m(t) = \Omega(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ so } m(t)x_{\mathrm{H}} = \Omega(t)x_{\mathrm{H}}^{\perp}.$$

By (4.4), $m(t)$ acts as $Q'(t)Q^{\top}(t)$ for the rotation matrix Q from (4.2) and with angle $\beta(t)$ deduced from Ω by (4.3). For the center of mass of the rigid body $x_{\mathrm{c}}(t)$ recovered from (4.1), we investigate the initial value problem

$$\begin{cases} \partial_t X_0(t,y_{\mathrm{H}}) = m(t)(X_0(t,y_{\mathrm{H}}) - x_{\mathrm{c}}(t)) + \xi(t), & \text{in } (0,T) \times \mathbb{R}^2, \\ X_0(0,y_{\mathrm{H}}) = y_{\mathrm{H}}, & \text{for } y_{\mathrm{H}} \in \mathbb{R}^2. \end{cases} \tag{4.9}$$

With the aforementioned $Q \in \mathrm{SO}(2)$, the solution to (4.9) takes the shape

$$X_0(t,y_{\mathrm{H}}) = Q(t)y_{\mathrm{H}} + x_{\mathrm{c}}(t). \tag{4.10}$$

From the above regularity assumptions on ξ and Ω, we especially derive that $Q \in \mathrm{W}^{2,p}(0,T)^{2\times 2}$. In addition, the inverse Y_0 of X_0 is

$$Y_0(t,x_{\mathrm{H}}) = Q^{\top}(t)(x_{\mathrm{H}} - x_{\mathrm{c}}(t)). \tag{4.11}$$

For $M(t) := Q^{\top}(t)m(t)Q(t)$ and $\ell(t) := Q^{\top}(t)\xi(t)$, the inverse Y_0 of X_0 in turn solves the initial value problem

$$\begin{cases} \partial_t Y_0(t,x_{\mathrm{H}}) = -M(t)Y_0(t,x_{\mathrm{H}}) - \ell(t), & \text{in } (0,T) \times \mathbb{R}^2, \\ Y_0(0,x_{\mathrm{H}}) = x_{\mathrm{H}}, & \text{for } x_{\mathrm{H}} \in \mathbb{R}^2. \end{cases} \tag{4.12}$$

Currently, the diffeomorphisms X_0 and Y_0 solving (4.9) and (4.12), respectively, do *not* only act in a suitable neighborhood of the moving rigid body. Hence, the following step is to modify them so that the transform is only local.

In this aspect, we follow the method of Geissert, Götze and Hieber [48, Sections 3 and 7], but it is not necessary for us to preserve a divergence free condition. As a result, we do not require the Bogovskiĭ operator, so the transform is of a less complicated shape compared to [48].

More precisely, the modified diffeomorphism is defined implicitly as the solution to the initial value problem

$$\begin{cases} \partial_t X(t, y_{\mathrm{H}}) = b(t, X(t, y_{\mathrm{H}})), & \text{in } (0,T) \times \mathbb{R}^2, \\ X(0, y_{\mathrm{H}}) = y_{\mathrm{H}}, & \text{for } y_{\mathrm{H}} \in \mathbb{R}^2. \end{cases} \tag{4.13}$$

The right-hand side b accounting for the precise behavior of the transform will be determined in the sequel. The concrete description of b requires some further knowledge regarding the situation of the rigid body trapped in sea ice. In fact, we assume it to start with a strictly positive distance from the outer sea ice boundary, i. e., there exists $r > 0$ such that

$$\operatorname{dist}(\mathcal{S}_0, \partial\mathcal{O}) > r.$$

By continuity of the rigid body velocity, we can consider solutions up to a time such that a positive distance of the rigid body and $\partial\mathcal{O}$ is preserved. However, it is not desirable to affect the regularity by introducing b. Consequently, we invoke a smooth cut-off function $\chi \in \mathrm{C}^\infty(\mathbb{R}^2; [0,1])$ with

$$\chi(x_{\mathrm{H}}) \coloneqq \begin{cases} 1, & \text{if } \operatorname{dist}(x_{\mathrm{H}}, \partial\mathcal{O}) \geq r, \\ 0, & \text{if } \operatorname{dist}(x_{\mathrm{H}}, \partial\mathcal{O}) \leq \dfrac{r}{2}. \end{cases}$$

We then define the right-hand side $b \colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$b(t, x_{\mathrm{H}}) \coloneqq \chi(x_{\mathrm{H}} - x_{\mathrm{c}}(t))[m(t)(x_{\mathrm{H}} - x_{\mathrm{c}}(t)) + \xi(t)], \tag{4.14}$$

so it readily follows from the construction that $b \in \mathrm{W}^{1,p}(0,T; \mathrm{C}_{\mathrm{c}}^\infty(\mathbb{R}^2))$ and

$$b_{\partial\mathcal{S}_0} = \omega_0 x_{\mathrm{H}}^\perp + \ell_0.$$

The initial value problem (4.13) has a unique solution $X \in \mathrm{C}^1(0,T; \mathrm{C}^\infty(\mathbb{R}^2)^2)$ as a consequence of the Picard-Lindelöf theorem. Let us also note the continuity of the mixed partial derivatives

$$\frac{\partial^{|\alpha|+1} X}{\partial t (\partial y_{\mathrm{H},j})^\alpha} \text{ and } \frac{\partial^{|\alpha|} X}{(\partial y_{\mathrm{H},j})^\alpha},$$

where $\alpha \in \mathbb{N}_0^3$ is a multi-index. Furthermore, it follows from (4.13) and an integration in time that the Jacobian matrix J_X takes the shape

$$(\mathrm{J}_X)_{ij}(t, y_\mathrm{H}) = \partial_j X_i(t, y_\mathrm{H}) = \delta_{ij} + \int_0^t \frac{\partial b_i}{\partial y_{\mathrm{H},j}}(s, X(s, y_\mathrm{H}))\,\mathrm{d}s. \tag{4.15}$$

The next lemma discusses conditions for the invertibility of J_X. The proof relies on a Neumann series argument in conjunction with the regularity of b as revealed above. Let us also refer to [66, Section 2] for a proof of a similar result in the context of the interaction problem of a compressible fluid.

Lemma 4.2.1. *Consider the time interval* $(0, T)$*, where* $0 < T \le \infty$*.*

(a) If $T_0 \in (0, T]$ *is sufficiently small, or*

(b) if $\|\nabla_\mathrm{H} b\|_{\mathrm{L}^\infty((0,T)\times\mathbb{R}^2)} < \delta_0$*, for some small* $\delta_0 > 0$*,*

then the Jacobian matrix $\mathrm{J}_X(t, \cdot)$ *from* (4.15) *is invertible for every* $t \in (0, T_0)$*, or even for every* $t \in (0, T)$ *in the situation of (b).*

Thanks to Lemma 4.2.1, the term

$$b^{(Y)}(t, y_\mathrm{H}) := -\mathrm{J}_X^{-1}(t, y_\mathrm{H}) b(t, X(t, y_\mathrm{H})) \tag{4.16}$$

is well-defined for $t \in (0, T_0)$, or even for $t \in (0, T)$ provided the spatial gradient of b is small. Besides, the inverse transform Y of X solves

$$\begin{cases} \partial_t Y(t, x_\mathrm{H}) = b^{(Y)}(t, Y(t, x_\mathrm{H})), & \text{in } (0, T_0) \times \mathbb{R}^2, \\ Y(0, x_\mathrm{H}) = x_\mathrm{H}, & \text{for } x_\mathrm{H} \in \mathbb{R}^2. \end{cases} \tag{4.17}$$

Let us observe that $\mathrm{J}_X(t, y_\mathrm{H})\mathrm{J}_Y(t, X(t, y_\mathrm{H})) = \mathrm{Id}_2$. It is also worth mentioning that $b^{(Y)}$ and Y inherit the regularity in time and space from b and X by construction. Moreover, if the rigid body does not come close to the outer boundary of the sea ice $\partial\mathcal{O}$, then X and Y are identical with X_0 and Y_0 from (4.10) and (4.11), respectively. In contrast, when the rigid body approaches the outer boundary $\partial\mathcal{O}$, we find

$$\partial_t X(t, y_\mathrm{H}) = \partial_t Y(t, x_\mathrm{H}) = 0.$$

The preceding discussion enables us to perform the transformation to the fixed domain. For the solution $X\colon (0, T) \times \mathbb{R}^2 \to \mathbb{R}^2$ to (4.13), the rotation matrix Q from (4.2) and with angle $\beta = \beta(t)$ resulting from the rotational velocity Ω by (4.3), considering $(t, y_{\rm H}) \in (0,T)\times\mathbb{R}^2$, and denoting the principle sea ice variable by $u = (v_{\rm ice}, h, a)$, we set

$$
\begin{aligned}
v_{\rm ice}(t, y_{\rm H}) &:= \overline{v}_{\rm ice}(t, X(t, y_{\rm H})),\\
h(t, y_{\rm H}) &:= \overline{h}(t, X(t, y_{\rm H})),\\
a(t, y_{\rm H}) &:= \overline{a}(t, X(t, y_{\rm H})),\\
\ell(t) &:= Q^\top(t)\xi(t) \ \text{ and } \ \omega(t) := \Omega(t),\\
F(t) &:= Q^\top(t)\overline{F}(t) \ \text{ and } \ N(t) := \overline{N}(t),\\
\mathcal{T}_\delta(u) &:= Q^\top(t)\sigma_\delta(\overline{u})Q(t),\\
I &:= J \ \text{ and } \ \nu := Q^\top(t)\overline{\nu}(t).
\end{aligned} \tag{4.18}
$$

Given the objects on the fixed domain, we calculate the system of equations satisfied by the transformed quantities from (4.18). First, we use the chain rule to compute the transformed time derivative given by

$$\partial_t\overline{v}_{\rm ice} = \partial_t v_{\rm ice} + \sum_{j=1}^{2}(\partial_t Y_j)\partial_j v_{\rm ice} = \partial_t v_{\rm ice} + (\nabla_{\rm H} v_{\rm ice})\partial_t Y.$$

Similarly, we obtain

$$\partial_t\overline{h} = \partial_t h + \nabla_{\rm H} h \cdot \partial_t Y \ \text{ and } \ \partial_t\overline{a} = \partial_t a + \nabla_{\rm H} a \cdot \partial_t Y.$$

Another application of the chain rule leads to

$$((\overline{v}_{\rm ice}\cdot\nabla_{\rm H})\overline{v}_{\rm ice})_i = \sum_{j,k=1}^{2} v_{{\rm ice},j}(\partial_j Y_k)\partial_k v_{{\rm ice},i} =: (\mathcal{N}(v_{\rm ice}))_i. \tag{4.19}$$

The next terms to be transformed are $\mathrm{div}_{\rm H}\,(\overline{v}_{\rm ice}\overline{h})$ and $\mathrm{div}_{\rm H}\,(\overline{v}_{\rm ice}\overline{a})$. We get

$$\mathrm{div}_{\rm H}\,(\overline{v}_{\rm ice}\overline{h}) = \sum_{j,k=1}^{2}(\partial_j Y_k)(v_{{\rm ice},j}\partial_k h + h\partial_k v_{{\rm ice},j}) =: \mathcal{M}(v_{\rm ice}, h). \tag{4.20}$$

Likewise, it follows that

$$\mathrm{div}_{\rm H}\,(\overline{v}_{\rm ice}\overline{a}) = \mathcal{M}(v_{\rm ice}, a).$$

In order to compute the transformed Laplacian terms, we invoke the metric contravariant tensor g^{ij} given by

$$g^{ij} := \sum_{k=1}^{2}(\partial_k Y_i)(\partial_k Y_j). \tag{4.21}$$

We then exploit the chain rule again to derive

$$\Delta_{\mathrm{H}} \overline{h} = \sum_{j,k=1}^{2} g^{jk} \partial_k \partial_j h + \sum_{j=1}^{2} (\Delta_{\mathrm{H}} Y_j) \partial_j h =: \mathcal{L} h. \tag{4.22}$$

By analogy, we get

$$\Delta_{\mathrm{H}} \overline{a} = \mathcal{L} a.$$

Transforming the Hibler operator as well as the ice strength P associated with $\operatorname{div}_{\mathrm{H}} \sigma_\delta$ is more delicate. In a first step, we calculate the transformed symmetric part of the gradient given by

$$\begin{aligned} 2\varepsilon_{ij}(\overline{v}_{\mathrm{ice}}) &= \partial_i \overline{v}_{\mathrm{ice},j} + \partial_j \overline{v}_{\mathrm{ice},i} \\ &= \sum_{k=1}^{2} \left((\partial_i Y_k) \partial_k v_{\mathrm{ice},j} + (\partial_j Y_k) \partial_k v_{\mathrm{ice},i} \right) =: 2\tilde{\varepsilon}_{ij}(v_{\mathrm{ice}}). \end{aligned}$$

For simplicity, we will also denote the latter by $\tilde{\varepsilon}$ in the following. As a result, we obtain

$$\partial_k \varepsilon_{jl}(\overline{v}_{\mathrm{ice}}) = (\partial_k Y_m) \partial_m \tilde{\varepsilon}_{jl}.$$

Moreover, we have

$$\begin{aligned} \partial_m \tilde{\varepsilon}_{jl}(v_{\mathrm{ice}}) = \frac{1}{2} \sum_{n=1}^{2} \Big(&(\partial_m \partial_j Y_n) \partial_n v_{\mathrm{ice},l} + (\partial_j Y_n) \partial_m \partial_n v_{\mathrm{ice},l} \\ &+ (\partial_m \partial_l Y_n) \partial_n v_{\mathrm{ice},j} + (\partial_l Y_n) \partial_m \partial_n v_{\mathrm{ice},j} \Big). \end{aligned} \tag{4.23}$$

Besides, for the ice strength P as defined in (3.1), we calculate

$$\begin{aligned} \partial_j P(\overline{h}, \overline{a}) &= \sum_{k=1}^{2} p^* \mathrm{e}^{-c_\bullet (1-a)} (\partial_j Y_k)(\partial_k h + c_\bullet h \partial_k a) \\ &= \sum_{k=1}^{2} \partial_h P(h,a) (\partial_j Y_k)(\partial_k h + c_\bullet h \partial_k a). \end{aligned} \tag{4.24}$$

In the sequel, we consider the part of the momentum equation associated with S_δ as introduced in (3.13), i. e.,

$$\frac{1}{\rho_{\mathrm{ice}} \overline{h}} \cdot \operatorname{div}_{\mathrm{H}} S_\delta(\overline{u}) = \frac{1}{\rho_{\mathrm{ice}} \overline{h}} \cdot \operatorname{div}_{\mathrm{H}} \left(\frac{P(\overline{h}, \overline{a})}{2} \frac{\mathbb{S} \varepsilon(\overline{v}_{\mathrm{ice}})}{\sqrt{\delta + \varepsilon(\overline{v}_{\mathrm{ice}})^\top \mathbb{S} \varepsilon(\overline{v}_{\mathrm{ice}})}} \right).$$

This corresponds to $\mathbb{A}^{\mathrm{H}}$ from Section 3.3. The derivation of the operator is completely analogous to Section 3.3. Invoking $a_{ij}^{kl}(\varepsilon(\overline{v}_{\mathrm{ice}}, P(\overline{h}, \overline{a}))$ from (3.15), we find that the i-th component of the operator admits the representation

$$\left[\frac{1}{\rho_{\mathrm{ice}}\overline{h}} \cdot \mathrm{div}_{\mathrm{H}}\, S_\delta(\overline{u})\right]_i = \sum_{j,k,l=1}^{2} -a_{ij}^{kl}(\varepsilon(\overline{v}_{\mathrm{ice}}), P(\overline{h},\overline{a}))\partial_k \varepsilon_{jl}(\overline{v}_{\mathrm{ice}}) + \frac{1}{2\rho_{\mathrm{ice}}\overline{h}\triangle_\delta(\varepsilon(\overline{v}_{\mathrm{ice}}))}\sum_{j=1}^{2}(\partial_j P(\overline{h},\overline{a}))(\mathbb{S}\varepsilon(\overline{v}_{\mathrm{ice}}))_{ij}.$$

Making use of the above relations and setting

$$a_{ij}^{klm}(\varepsilon(v_{\mathrm{ice}}), P(h,a)) \coloneqq (\partial_k Y_m) a_{ij}^{kl}(\varepsilon(v_{\mathrm{ice}}), P(h,a)), \tag{4.25}$$

we conclude that the *transformed quasilinear Hibler operator* $\mathcal{A}^{\mathrm{H}}$ on the fixed domain is given by

$$\begin{aligned} \frac{1}{\rho_{\mathrm{ice}}\overline{h}} \cdot \mathrm{div}_{\mathrm{H}}\, S_\delta(\overline{u})_i &= \sum_{j,k,l,m=1}^{2} -a_{ij}^{klm}(\tilde{\varepsilon}, P)\partial_m \tilde{\varepsilon}_{jl} \\ &\quad - \frac{\partial_h P(h,a)}{2\rho_{\mathrm{ice}} h \triangle_\delta(\tilde{\varepsilon})}\sum_{j,k=1}^{2}(\partial_j Y_k)(\partial_k h + c_\bullet h \partial_k a) \\ &\eqqcolon [\mathcal{A}^{\mathrm{H}}(u) v_{\mathrm{ice}}]_i. \end{aligned} \tag{4.26}$$

With regard to $\mathrm{div}_{\mathrm{H}}\, \sigma_\delta$, it remains to deal with the ice strength P. From

$$\mathrm{div}_{\mathrm{H}} \frac{P(\overline{h},\overline{a})}{2}\mathrm{Id}_2 = \frac{1}{2}\Big(\partial_1 P(\overline{h},\overline{a}) + \partial_2 P(\overline{h},\overline{a})\Big)$$

and an insertion of (4.24), we conclude

$$\begin{aligned} \frac{1}{\rho_{\mathrm{ice}} h} \cdot \mathrm{div}_{\mathrm{H}} \frac{P(\overline{h},\overline{a})}{2}\mathrm{Id}_2 &= \frac{\partial_h P(h,a)}{2\rho_{\mathrm{ice}} h}\sum_{j,k=1}^{2}(\partial_j Y_k)(\partial_k h + c_\bullet h \partial_k a) \\ &\eqqcolon \mathcal{B}_{\mathrm{h}}(h,a) + \mathcal{B}_{\mathrm{a}}(h,a) = \mathcal{B}(h,a). \end{aligned} \tag{4.27}$$

The other terms from the sea ice part of the interaction problem (4.5) do not include derivatives, so the resulting transformed terms are simply obtained by plugging in the transformed variables from (4.18). More precisely, we get

$$-c_{\mathrm{cor}}\overline{v}_{\mathrm{ice}}^{\perp} - g\nabla_{\mathrm{H}} H + \frac{1}{\rho_{\mathrm{ice}}\overline{h}}\tau_{\mathrm{ice}} = -c_{\mathrm{cor}} v_{\mathrm{ice}}^{\perp} - g\nabla_{\mathrm{H}} H + \frac{1}{\rho_{\mathrm{ice}} h}\tau_{\mathrm{ice}}$$

as well as $S_{\mathrm{h}}(\overline{h},\overline{a}) = S_{\mathrm{h}}(h,a)$ and $S_{\mathrm{a}}(\overline{h},\overline{a}) = S_{\mathrm{a}}(h,a)$. In addition, by construction of the transform, the boundary conditions from (4.7) are not affected, while the interface condition from (4.6) becomes

$$v_{\mathrm{ice}} = \ell(t) + \omega(t) y_{\mathrm{H}}^{\perp}, \ \text{ on } \ (0,T) \times \partial\mathcal{S}_0.$$

Before providing the complete transformed system, we calculate the transformed terms in the context of the rigid body equations. From $Q(t) \in \mathrm{SO}(2)$ and $\omega(t) = \Omega(t)$, we first deduce that

$$M(t)x_{\mathrm{H}} = Q^\top(t)m(t)Q(t)x_{\mathrm{H}} = \Omega(t)\det(Q(t))\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} = \omega(t)x_{\mathrm{H}}^\perp.$$

As the inertia tensor is not changed in (4.18), the new quantity I inherits the time-independence from $J = J_0$. The property $Q(t) \in \mathrm{SO}(2)$ also yields that $y_{\mathrm{H}}^\perp Q(t)z_{\mathrm{H}} = (Q^\top(t)y_{\mathrm{H}}^\perp)z_{\mathrm{H}}$. Equipped with the latter identity, and recalling the transformed stress tensor $\mathcal{T}_\delta(u)$ and transformed normal vector ν from (4.18), we determine the surface integrals on the fixed domain to be

$$\int_{\partial\mathcal{S}(t)} \sigma_\delta(\overline{u})\overline{\nu}(t)\,\mathrm{d}\Gamma = Q(t)\int_{\partial\mathcal{S}_0} \mathcal{T}_\delta(u)\nu\,\mathrm{d}\Gamma \quad \text{and}$$
$$\int_{\partial\mathcal{S}(t)} (x_{\mathrm{H}} - x_{\mathrm{c}}(t))^\perp \sigma_\delta(\overline{u})\overline{\nu}(t)\,\mathrm{d}\Gamma = \int_{\partial\mathcal{S}_0} y_{\mathrm{H}}^\perp \mathcal{T}_\delta(u)\nu\,\mathrm{d}\Gamma.$$

Last, we remark that the definition of ℓ yields that the term $\omega(t)\ell^\perp(t)$ arises in the transformed system of equations.

Now, we can rewrite the complete system from (4.8) on the fixed domain. The *transformed system of the sea ice interaction problem* is given by

$$(4.28)\quad \begin{cases} \partial_t v_{\mathrm{ice}} = \mathcal{A}^{\mathrm{H}}(u)v_{\mathrm{ice}} - \mathcal{B}(h,a) - (\nabla_{\mathrm{H}} v_{\mathrm{ice}})\partial_t Y \\ \qquad - \mathcal{N}(v_{\mathrm{ice}}) - c_{\mathrm{cor}} v_{\mathrm{ice}}^\perp - g\nabla_{\mathrm{H}} H \\ \qquad + \dfrac{1}{m_{\mathrm{ice}}}(\tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(v_{\mathrm{ice}})), & \text{in } (0,T)\times\mathcal{F}_0, \\ \partial_t h = d_{\mathrm{h}}\mathcal{L}h - \nabla_{\mathrm{H}} h \cdot \partial_t Y \\ \qquad - \mathcal{M}(v_{\mathrm{ice}}, h) + S_{\mathrm{h}}(h,a), & \text{in } (0,T)\times\mathcal{F}_0, \\ \partial_t a = d_{\mathrm{a}}\mathcal{L}a - \nabla_{\mathrm{H}} a \cdot \partial_t Y \\ \qquad - \mathcal{M}(v_{\mathrm{ice}}, a) + S_{\mathrm{a}}(h,a), & \text{in } (0,T)\times\mathcal{F}_0, \\ m_{\mathcal{S}}\ell'(t) = F(t) + m_{\mathcal{S}}\omega(t)\ell^\perp(t) \\ \qquad - Q(t)\displaystyle\int_{\partial\mathcal{S}_0} \mathcal{T}_\delta(u)\nu\,\mathrm{d}\Gamma, & \text{for } t \in (0,T), \\ I\omega'(t) = N(t) - \displaystyle\int_{\partial\mathcal{S}_0} y_{\mathrm{H}}^\perp \mathcal{T}_\delta(u)\nu\,\mathrm{d}\Gamma, & \text{for } t \in (0,T). \end{cases}$$

In the above, the transformed Hibler operator $\mathcal{A}^{\mathrm{H}}(u)v_{\mathrm{ice}}$ is as introduced in (4.26), the transformed term $\mathcal{B}(h,a)$ corresponding to the ice strength P has been defined in (4.27), the transformed convective term $\mathcal{N}(v_{\mathrm{ice}})$ can be found in (4.19), the transformed Laplacian $\mathcal{L}$ has been made precise in (4.22), and the transformed bilinear term $\mathcal{M}$ has been derived in (4.20). Accordingly, the boundary conditions on the fixed domain read as

$$\begin{aligned} &v_{\mathrm{ice}} = \ell(t) + \omega(t) y_{\mathrm{H}}^{\perp}, \ \ \partial_\nu h = \partial_\nu a = 0, && \text{on } (0,T) \times \partial \mathcal{S}_0, \ \text{and} \\ &v_{\mathrm{ice}} = 0, \ \ \partial_\nu h = \partial_\nu a = 0, && \text{on } (0,T) \times \partial \mathcal{O}, \end{aligned}$$

and the system is complemented by the initial data

$$\begin{aligned} v_{\mathrm{ice}}(0) = v_{\mathrm{ice},0}, \ \ h(0) = h_0, \ \ a(0) = a_0, \ \ \text{in } \mathcal{F}_0, \ \ \text{and} \\ x_{\mathrm{c}}(0) = 0, \ \ \ell(0) = \ell_0, \ \ \beta(0) = 0, \ \ \omega(0) = \omega_0. \end{aligned} \tag{4.29}$$

4.3. Maximal Regularity of the Linearized Interaction Problem

This section is dedicated to the linear theory for the transformed interaction problem from (4.28)–(4.29). The first step is to provide a suitable linearization. Thereafter, we analyze the resulting linearized problem, for which we will use a so-called "cascade" approach, meaning that we first solve the equations for the rigid body and then insert them into the sea ice part. The maximal regularity is finally obtained by an application of an optimal L^p-L^q regularity result as discussed in Proposition 2.5.8.

As we have already indicated, we start with the linearization of the transformed system (4.28)–(4.29) on the fixed domain. In the present cascade approach, we linearize the sea ice equations similarly as in Section 3.4, while we consider the ODEs for ℓ and ω without the surface integrals. Thus, for appropriate terms on the right-hand side

$$\begin{aligned} &f_1 \colon (0,T) \times \mathcal{F}_0 \to \mathbb{R}^2, \ \ f_2, f_3 \colon (0,T) \times \mathcal{F}_0 \to \mathbb{R}, \\ &f_4 \colon (0,T) \to \mathbb{R}^2 \ \text{and} \ f_5 \colon (0,T) \to \mathbb{R}, \end{aligned}$$

suitable initial data

$$z_0 = (v_{\mathrm{ice},0}, h_0, a_0, \ell_0, \omega_0)$$

as well as

$$u_1 = (v_{\mathrm{ice},1}, h_1, a_1) \in \mathrm{C}^{1,\alpha}(\overline{\mathcal{F}_0})^4, \ \text{with } \alpha > 0, \ \ h_1 \ge \kappa \ \text{and} \ a_1 \in (0,1),$$

and for a shift $\mu > 0$ in the sea ice part, we investigate the *linearized problem*

$$
(4.30)\qquad \begin{cases}
\partial_t v_{\mathrm{ice}} - (\mathbb{A}^{\mathrm{H}}(u_1) - \mu) v_{\mathrm{ice}} + B_1(u_1)\begin{pmatrix} h \\ a \end{pmatrix} = f_1, & \text{in } (0,T) \times \mathcal{F}_0, \\
\partial_t h - (d_{\mathrm{h}} \Delta_{\mathrm{H}} - \mu) h = f_2, & \text{in } (0,T) \times \mathcal{F}_0, \\
\partial_t a - (d_{\mathrm{a}} \Delta_{\mathrm{H}} - \mu) a = f_3, & \text{in } (0,T) \times \mathcal{F}_0, \\
m_{\mathcal{S}} \ell'(t) = f_4, & \text{for } t \in (0,T), \\
I \omega'(t) = f_5, & \text{for } t \in (0,T), \\
v_{\mathrm{ice}} = \ell(t) + \omega(t) y_{\mathrm{H}}^{\perp}, \ \partial_\nu h = \partial_\nu a = 0, & \text{on } (0,T) \times \partial \mathcal{S}_0, \\
v_{\mathrm{ice}} = 0, \ \partial_\nu h = \partial_\nu a = 0, & \text{on } (0,T) \times \partial \mathcal{O}, \\
v_{\mathrm{ice}}(0) = v_{\mathrm{ice},0}, \ h(0) = h_0, \ a(0) = a_0, & \text{in } \mathcal{F}_0, \\
\ell(0) = \ell_0, \ \omega(0) = \omega_0. &
\end{cases}
$$

In (4.30), we recall the linearized Hibler operator $\mathbb{A}^{\mathrm{H}}(u_1)$ from (3.16), whereas the term $B_1(u_1)$ captures the off-diagonal part, so

$$
(4.31)\qquad B_1(u_1)\begin{pmatrix} h \\ a \end{pmatrix} = \frac{\partial_h P(h_1, a_1)}{2 \rho_{\mathrm{ice}} h_1} \nabla_{\mathrm{H}} h + \frac{\partial_a P(h_1, a_1)}{2 \rho_{\mathrm{ice}} h_1} \nabla_{\mathrm{H}} a.
$$

At this point, we also introduce the *compatibility condition* of the initial data.

Definition 4.3.1. *For $z_0 = (v_{\mathrm{ice},0}, h_0, a_0, \ell_0, \omega_0) \in \mathrm{B}_{qp}^{2-2/p}(\mathcal{F}_0)^4 \times \mathbb{R}^3$, we say that the compatibility condition is satisfied if for $2 - 2/p > 1/q$, we have*

$$
v_{\mathrm{ice},0} = \ell_0 + \omega_0 y_{\mathrm{H}}^{\perp}, \ \text{ on } \partial \mathcal{S}_0, \ \text{ and } v_{\mathrm{ice},0} = 0, \ \text{ on } \partial \mathcal{O},
$$

and if additionally for $2 - 2/p > 1 + 1/q$, it holds that

$$
\partial_\nu h_0 = \partial_\nu a_0 = 0, \ \text{ on } \partial \mathcal{F}_0.
$$

There is no condition on the initial data in the case $2 - 2/p < 1/q$.

In the sequel, we introduce some further spaces to shorten the notation. We start with the space X_0 playing the role of the *ground space*. For $q \in (1, \infty)$, it is defined by

$$
(4.32)\qquad \mathrm{X}_0 := \mathrm{L}^q(\mathcal{F}_0)^2 \times \mathrm{L}^q(\mathcal{F}_0) \times \mathrm{L}^q(\mathcal{F}_0) \times \mathbb{R}^2 \times \mathbb{R}.
$$

Next, for $z = (v_{\mathrm{ice}}, h, a, \ell, \omega)$ and $q \in (1, \infty)$, we define the space X_1 acting as the *regularity space* by

$$
(4.33)\qquad \begin{aligned}
\mathrm{X}_1 := \Big\{ & z \in \mathrm{W}^{2,q}(\mathcal{F}_0)^2 \times \mathrm{W}^{2,q}(\mathcal{F}_0) \times \mathrm{W}^{2,q}(\mathcal{F}_0) \times \mathbb{R}^2 \times \mathbb{R} : \\
& v_{\mathrm{ice}} = \ell + \omega y_{\mathrm{H}}^{\perp}, \ \text{on } \partial \mathcal{S}_0, \ v_{\mathrm{ice}} = 0, \ \text{on } \partial \mathcal{O}, \ \text{and} \\
& \partial_\nu h = \partial_\nu a = 0, \ \text{on } \partial \mathcal{F}_0 \Big\}.
\end{aligned}
$$

For $p, q \in (1,\infty)$ with $2/p + 1/q \notin \{1,2\}$, the space X_γ is given as follows. We set $z \in X_\gamma$ if and only if $z \in B_{qp}^{2-2/p}(\mathcal{F}_0)^2 \times B_{qp}^{2-2/p}(\mathcal{F}_0) \times B_{qp}^{2-2/p}(\mathcal{F}_0) \times \mathbb{R}^2 \times \mathbb{R}$, and for $2 - 2/p > 1/q$, we additionally demand that

$$\text{(4.34)} \qquad \begin{aligned} &v_{\mathrm{ice}} = \ell + \omega y_{\mathrm{H}}^{\perp}, \ \text{on } \partial\mathcal{S}_0, \ v_{\mathrm{ice}} = 0, \ \text{on } \partial\mathcal{O}, && \text{if } 2 - 2/p < 1 + 1/q, \\ &\partial_\nu h = \partial_\nu a = 0, \ \text{on } \partial\mathcal{F}_0, && \text{if } 2 - 2/p > 1 + 1/q. \end{aligned}$$

From the considerations in Section 1.3, it follows that the space X_γ can also be obtained from real interpolation of X_0 and X_1 by $(X_0, X_1)_{1-1/p,p}$. At this stage, we also refer to [11, Lemma 5.3], where this fact is proved by using the underlying monolithic approach.

The following remark is an immediate consequence of the definition of X_γ.

Remark 4.3.2. *It holds that $z_0 \in X_\gamma$ if and only if z_0 satisfies the compatibility condition as introduced in Definition 4.3.1.*

Before discussing the maximal L^p-regularity of (4.30), we present the spaces for the data and the solution. For the ground space X_0 from (4.32) and a time interval $(0,T)$, $0 < T \le \infty$, the *data space* $\mathbb{E}_0$ is defined by

$$\text{(4.35)} \qquad \mathbb{E}_0 := L^p(0,T;X_0).$$

With X_1 from (4.33), we introduce the *maximal regularity space*

$$\text{(4.36)} \qquad \mathbb{E}_1 := W^{1,p}(0,T;X_0) \cap L^p(0,T;X_1).$$

The next proposition on the maximal regularity of the linearized system is the starting point for the further analysis of the interaction problem.

Proposition 4.3.3. *Let $p, q \in (1,\infty)$ be such that $2/p + 1/q \notin \{1,2\}$, and assume $u_1 = (v_{\mathrm{ice},1}, h_1, a_1) \in C^{1,\alpha}(\overline{\mathcal{F}_0})^4$ for some $\alpha > 0$, and with $h_1 \ge \kappa$ as well as $a_1 \in (0,1)$. Moreover, for X_γ as in (4.34) and $\mathbb{E}_0$ as in (4.35), we consider $z_0 = (v_{\mathrm{ice},0}, h_0, a_0, \ell_0, \omega_0) \in X_\gamma$ and $(f_1, f_2, f_3, f_4, f_5) \in \mathbb{E}_0$.*

Then there is $\mu_0 \in \mathbb{R}$ such that for all $\mu > \mu_0$, the linearized problem (4.30) has a unique solution $z = (v_{\mathrm{ice}}, h, a, \ell, \omega) \in \mathbb{E}_1$ on $(0,T)$, with $\mathbb{E}_1$ as introduced in (4.36). Moreover, there exists a constant $C_{\mathrm{MR}} > 0$, which can be chosen independent of T in the case of homogeneous initial values, so that the unique solution z of (4.30) satisfies

$$\text{(4.37)} \qquad \|z\|_{\mathbb{E}_1} \le C_{\mathrm{MR}} \cdot \left(\|(f_1, f_2, f_3, f_4, f_5)\|_{\mathbb{E}_0} + \|z_0\|_{X_\gamma} \right).$$

Proof. As partially explained above, the main idea of the proof is to consider the linearized problem (4.30) in “cascades”. More precisely, we first consider the equations of the rigid body which we can solve immediately by ODE theory. In parallel, we can handle the linear equations related to the mean ice thickness h and the ice compactness a thanks to properties of the Neumann Laplacian operator as discussed in Lemma 2.3.19. We can then tackle the resulting linearized sea ice momentum equation in the interaction problem with the optimal L^p-L^q result Proposition 2.5.8.

The ODEs accounting for the motion of the rigid body read as

$$\begin{cases} m_{\mathcal{S}}\ell'(t) = f_4 \text{ and } I\omega'(t) = f_5, & \text{in } (0,T), \\ \ell(0) = \ell_0, \ \omega(0) = \omega_0. \end{cases} \tag{4.38}$$

As $m_{\mathcal{S}} > 0$ and I are constant, it follows from classical ODE theory that there exists a unique solution $(\ell,\omega) \in \mathrm{W}^{1,p}(0,T)^3$ to (4.38), and there is a constant $C_1 > 0$ such that

$$\|(\ell,\omega)\|_{\mathrm{W}^{1,p}(0,T)} \le C_1 \cdot \Big(\|(f_4,f_5)\|_{\mathrm{L}^p(0,T)} + |(\ell_0,\omega_0)|\Big). \tag{4.39}$$

Next, we address the equations for h and a in (4.30). In fact, we are interested in the L^q-realization of the respective Neumann Laplacian operators. Thanks to the assumptions that $\mathcal{O}$ and $\partial\mathcal{S}_0$ are of class C^2, we may apply Lemma 2.3.19(e). The latter result yields that for every $\mu > 0$, there exists a unique solution $(h,a) \in \mathrm{W}^{1,p}(0,T;\mathrm{L}^q(\mathcal{F}_0)^2) \cap \mathrm{L}^p(0,T;\mathrm{W}^{2,q}_{\mathrm{N}}(\mathcal{F}_0)^2)$ to

$$\begin{cases} \partial_t h - (d_{\mathrm{h}}\Delta_{\mathrm{N}} - \mu)h = f_2, & \text{in } (0,T)\times\mathcal{F}_0, \\ \partial_t a - (d_{\mathrm{a}}\Delta_{\mathrm{N}} - \mu)a = f_3, & \text{in } (0,T)\times\mathcal{F}_0, \\ h(0) = h_0, \ a(0) = a_0, & \text{in } \mathcal{F}_0. \end{cases} \tag{4.40}$$

Moreover, for some constant $C_2 > 0$, this solution satisfies the estimate

$$\begin{aligned} &\|(h,a)\|_{\mathrm{W}^{1,p}(0,T;\mathrm{L}^q(\mathcal{F}_0))\cap\mathrm{L}^p(0,T;\mathrm{W}^{2,q}_{\mathrm{N}}(\mathcal{F}_0))} \\ &\le C_2 \cdot \Big(\|(f_2,f_3)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} + \|(h_0,a_0)\|_{\mathrm{B}^{2-2/p}_{qp}(\mathcal{F}_0)}\Big). \end{aligned} \tag{4.41}$$

It remains to study the linearized momentum equation part. Plugging the solutions (ℓ,ω) to (4.38) as well as (h,a) to (4.40) into (4.30), we obtain

$$\begin{cases} \partial_t v_{\mathrm{ice}} - (\mathbb{A}^{\mathrm{H}}(u_1) - \mu)v_{\mathrm{ice}} = \tilde{f}_1, & \text{in } (0,T)\times\mathcal{F}_0, \\ v_{\mathrm{ice}} = b_{\ell,\omega}, & \text{on } (0,T)\times\partial\mathcal{S}_0, \\ v_{\mathrm{ice}} = 0, & \text{on } (0,T)\times\partial\mathcal{O}, \\ v_{\mathrm{ice}}(0) = v_{\mathrm{ice},0}, & \text{in } \mathcal{F}_0, \end{cases} \tag{4.42}$$

where

$$\tilde{f}_1 = f_1 - B_1(u_1)\begin{pmatrix} h \\ a \end{pmatrix} \text{ and } b_{\ell,\omega} = \ell + \omega y_{\mathrm{H}}^{\perp}.$$

We need to verify the assumptions of Proposition 2.5.8 in the present setting. Let us first observe that v_{ice} in (4.42) is subject to Dirichlet boundary conditions, yielding that (S')(iii) holds true. Additionally invoking the assumption that $u_1 \in \mathrm{C}^{1,\alpha}(\overline{\mathcal{F}_0})^4$ for some $\alpha > 0$, we argue that (S')(i) as well as (S')(ii) are also valid in view of the shape of the linearized Hibler operator as in (3.16). Hence, in order to get the desired maximal L^p-regularity, we have to show that the data $(\tilde{f}_1, b_{\ell,\omega}, v_{\mathrm{ice},0})$ lie within the scope of Proposition 2.5.8. To this end, recall that $f_1 \in \mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0)^2)$. Next, making use of Hölder's inequality in conjunction with the assumption $u_1 \in \mathrm{C}^{1,\alpha}(\overline{\mathcal{F}_0})^4$ with $h_1 \ge \kappa$ and $a_1 \in (0,1)$, we find that $B_1(u_1)$ from (4.31) satisfies

$$\begin{aligned} &\left\| B_1(u_1)\begin{pmatrix} h \\ a \end{pmatrix} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\ &\le \left\| \frac{\partial_h P(h_1,a_1)}{2\rho_{\mathrm{ice}} h_1} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \cdot \|\nabla_{\mathrm{H}} h\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\ &\quad + \left\| \frac{\partial_a P(h_1,a_1)}{2\rho_{\mathrm{ice}} h_1} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \cdot \|\nabla_{\mathrm{H}} a\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\ &\le C_3 \cdot \|(h,a)\|_{\mathrm{W}^{1,p}(0,T;\mathrm{L}^q(\mathcal{F}_0)) \cap \mathrm{L}^p(0,T;\mathrm{W}^{2,q}_{\mathrm{N}}(\mathcal{F}_0))}, \end{aligned} \tag{4.43}$$

where $C_3 > 0$ is constant. Consequently, we have $\tilde{f}_1 \in \mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0)^2)$.

The next term under consideration is the boundary term $b_{\ell,\omega}$. By Proposition 2.5.8, we must check that

$$b_{\ell,\omega} \in \mathbb{F} := \mathrm{F}^{1-1/2q}_{pq}\left(0,T;\mathrm{L}^q(\partial\mathcal{S}_0)^2\right) \cap \mathrm{L}^p\left(0,T;\mathrm{B}^{2-2/q}_{qq}(\partial\mathcal{S}_0)^2\right). \tag{4.44}$$

From $(\ell,\omega) \in \mathrm{W}^{1,p}(0,T)^3$ and the fact that $b_{\ell,\omega}$ is smooth in the spatial variable, it readily follows that

$$b_{\ell,\omega} \in \mathrm{L}^p\left(0,T;\mathrm{B}^{2-2/q}_{qq}(\partial\mathcal{S}_0)^2\right).$$

As $b_{\ell,\omega}$ only depends on time via ℓ and ω, it suffices to verify the embedding

$$\mathrm{W}^{1,p}(0,T) \hookrightarrow \mathrm{F}^{1-1/2q}_{pq}(0,T).$$

In fact, by virtue of the identity (1.3) and the embedding relation (1.9), we get for $\varepsilon > 0$ sufficiently small that

$$\mathrm{W}^{1,p}(0,T) = \mathrm{F}^1_{p2}(0,T) \hookrightarrow \mathrm{F}^{1-\varepsilon}_{pq}(0,T) \hookrightarrow \mathrm{F}^{1-1/2q}_{pq}(0,T),$$

as desired. In total, (4.44) is satisfied, and for a constant $C_4 > 0$, we obtain

$$\|b_{\ell,\omega}\|_{\mathbb{F}} \le C_4 \cdot \|(\ell,\omega)\|_{\mathrm{W}^{1,p}(0,T)}. \tag{4.45}$$

It remains to study the initial data $v_{\mathrm{ice},0}$. From the above shape of X_γ, we deduce that $v_{\mathrm{ice},0} \in \mathrm{B}_{qp}^{2-2/p}(\mathcal{F}_0)^2$. Furthermore, using $z_0 \in \mathrm{X}_\gamma$ in conjunction with the equivalence of $1 - 1/2q > 1/p$ and $2 - 2/p > 1/q$, we conclude that

$$v_{\mathrm{ice},0} = \ell_0 + \omega_0 y_{\mathrm{H}}^{\perp} = b_{\ell,\omega}(0)$$

holds for $1 - 1/2q > 1/p$ by (4.34). Therefore, we can apply Proposition 2.5.8, and the latter result yields the existence of $\mu_0 > 0$ such that for all $\mu > \mu_0$, the remaining linearized momentum equation (4.42) has a unique solution

$$v_{\mathrm{ice}} \in \mathrm{W}^{1,p}(0,T;\mathrm{L}^q(\mathcal{F}_0)^2) \cap \mathrm{L}^p(0,T;\mathrm{W}^{2,q}(\mathcal{F}_0)^2).$$

In addition, the closed graph theorem together with the estimates (4.43) and (4.45) as well as (4.41) and (4.39) yields that

$$\begin{aligned}
&\|v_{\mathrm{ice}}\|_{\mathrm{W}^{1,p}(0,T;\mathrm{L}^q(\mathcal{F}_0))\cap\mathrm{L}^p(0,T;\mathrm{W}^{2,q}(\mathcal{F}_0))} \\
&\le C_5 \cdot \Big(\|\tilde{f}_1\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} + \|b_{\ell,\omega}\|_{\mathbb{F}} + \|v_{\mathrm{ice},0}\|_{\mathrm{B}_{qp}^{2-2/p}(\mathcal{F}_0)}\Big) \\
&\le C_6 \cdot \Big(\|f_1\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} + \|(h,a)\|_{\mathrm{W}^{1,p}(0,T;\mathrm{L}^q(\mathcal{F}_0))\cap\mathrm{L}^p(0,T;\mathrm{W}^{2,q}_{\mathrm{N}}(\mathcal{F}_0))} \\
&\quad + \|(\ell,\omega)\|_{\mathrm{W}^{1,p}(0,T)} + \|(v_{\mathrm{ice},0},h_0,a_0,\ell_0,\omega_0)\|_{\mathrm{X}_\gamma}\Big) \\
&\le C_7 \cdot \Big(\|(f_1,f_2,f_3,f_4,f_5)\|_{\mathbb{E}_0} + \|(v_{\mathrm{ice},0},h_0,a_0,\ell_0,\omega_0)\|_{\mathrm{X}_\gamma}\Big).
\end{aligned} \tag{4.46}$$

Collecting the solutions (ℓ,ω) to (4.38), (h,a) to (4.40) and the latter solution v_{ice} to (4.42), we infer that there is μ_0 such that for all $\mu > \mu_0$, the linearized problem (4.30) has a unique solution $z = (v_{\mathrm{ice}},h,a,\ell,\omega)$ with $z \in \mathbb{E}_1$. At the same time, the estimates in (4.46), (4.41) as well as (4.39) yield the existence of a constant $C_{\mathrm{MR}} > 0$ such that (4.37) holds true.

In the case of homogeneous initial values, the constants in the above estimates can be chosen independent of T, where we invoke that the result above can also be shown in the case of $\mathbb{R}_+$ thanks to Proposition 2.5.8, and we also employ the extension operator with T-independent norm from Lemma 2.4.2 as well as the maximal regularity constant C_{MR} in the situation of $\mathbb{R}_+$. □

4.4. Local Strong Well-Posedness

The aim of this section is to establish the local strong well-posedness of the interaction problem of sea ice (4.8). The general idea is to show first that the transformed system (4.28) admits a unique, strong, local-in-time solution, and to deduce the local strong well-posedness of (4.8) from there by performing the backward change of variables.

We start with the reformulation of the problem of finding a solution to (4.28) as a fixed point problem by virtue of the maximal regularity as seen in Proposition 4.3.3. Moreover, we provide estimates of the nonlinear terms in a second step. Finally, we are in the position to state and prove the local well-posedness result in the reference domain and in the moving domain.

The Fixed Point Argument

For the treatment of the initial conditions, we require the concept of a *reference solution* capturing the initial values, and this will also be the starting point in the process of reformulation.

Before rewriting the problem, we need some further preparation with regard to notation. We start by introducing an open subset $V \subset \mathrm{X}_\gamma$ of the trace space X_γ from (4.34), and this subset ensures that h and a take values in the physically reasonable range. More precisely, we define

$$V := \{z = (v_{\mathrm{ice}}, h, a, \ell, \omega) \in \mathrm{X}_\gamma : h > \kappa \text{ and } a \in (0,1)\}. \tag{4.47}$$

As already seen in the previous chapter, we impose an additional condition on p and q so that the trace space embeds into a classical space. In fact, we take into account p, $q \in (1, \infty)$ with

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}. \tag{4.48}$$

Recalling the shape of X_γ from (4.34), observing that this space especially embeds into the one without boundary and coupling conditions, and exploiting (4.48) for the embedding (1.8), we find that

$$\begin{aligned} \mathrm{X}_\gamma &\hookrightarrow \mathrm{B}_{qp}^{2-2/p}(\mathcal{F}_0)^2 \times \mathrm{B}_{qp}^{2-2/p}(\mathcal{F}_0) \times \mathrm{B}_{qp}^{2-2/p}(\mathcal{F}_0) \times \mathbb{R}^2 \times \mathbb{R} \\ &\hookrightarrow \mathrm{C}^{1,\alpha}(\overline{\mathcal{F}_0})^2 \times \mathrm{C}^{1,\alpha}(\overline{\mathcal{F}_0}) \times \mathrm{C}^{1,\alpha}(\overline{\mathcal{F}_0}) \times \mathbb{R}^2 \times \mathbb{R} \end{aligned} \tag{4.49}$$

for some $\alpha > 0$.

In the next step, we introduce the system for the reference solution. To this end, let us recall $\mu_0 \in \mathbb{R}$ from Proposition 4.3.3, and consider $\mu > \mu_0$. Given initial data $z_0 = (v_{\mathrm{ice},0}, h_0, a_0, \ell_0, \omega_0) \in V$, we then investigate the system

$$
\tag{4.50}
\begin{cases}
\partial_t v_{\mathrm{ice}} - (\mathbb{A}^{\mathrm{H}}(u_0) - \mu) v_{\mathrm{ice}} + B_1(u_0) \begin{pmatrix} h \\ a \end{pmatrix} = 0, & \text{in } (0,T) \times \mathcal{F}_0, \\
\partial_t h - (d_{\mathrm{h}} \Delta_{\mathrm{H}} - \mu) h = 0, & \text{in } (0,T) \times \mathcal{F}_0, \\
\partial_t a - (d_{\mathrm{a}} \Delta_{\mathrm{H}} - \mu) a = 0, & \text{in } (0,T) \times \mathcal{F}_0, \\
m_{\mathcal{S}} \ell'(t) = 0, & \text{for } t \in (0,T), \\
I \omega'(t) = 0, & \text{for } t \in (0,T), \\
v_{\mathrm{ice}} = \ell(t) + \omega(t) y_{\mathrm{H}}^{\perp}, \ \partial_\nu h = \partial_\nu a = 0, & \text{on } (0,T) \times \partial \mathcal{S}_0, \\
v_{\mathrm{ice}} = 0, \ \partial_\nu h = \partial_\nu a = 0, & \text{on } (0,T) \times \partial \mathcal{O}, \\
v_{\mathrm{ice}}(0) = v_{\mathrm{ice},0}, \ h(0) = h_0, \ a(0) = a_0, & \text{in } \mathcal{F}_0, \\
\ell(0) = \ell_0, \ \omega(0) = \omega_0.
\end{cases}
$$

We now discuss the existence of a unique *reference solution* to (4.50) for given initial data $z_0 \in V$. From $z_0 \in V$, it follows that $h_0 > \kappa$, $a_0 \in (0,1)$ and $z_0 \in \mathrm{X}_\gamma$. As a consequence, (4.49) yields that $u_0 = (v_{\mathrm{ice},0}, h_0, a_0)$ from z_0 satisfies $u_0 \in \mathrm{C}^{1,\alpha}(\overline{\mathcal{F}_0})^4$ for some $\alpha > 0$. We are thus in the scope of Proposition 4.3.3, leading to the existence of a unique solution $z_0^* \in \mathbb{E}_1$, with $\mathbb{E}_1$ as in (4.36), to (4.50). This is summarized in the proposition below.

Proposition 4.4.1. *Consider* $p, q \in (1, \infty)$ *with* (4.48)*, and let* $0 < T \le \infty$ *as well as* $z_0 = (v_{\mathrm{ice},0}, h_0, a_0, \ell_0, \omega_0) \in V$ *for* V *given in* (4.47)*. Then* (4.50) *admits a unique solution* $z_0^* = (v_{\mathrm{ice},0}^*, h_0^*, a_0^*, \ell_0^*, \omega_0^*) \in \mathbb{E}_1$.

The reference solution z_0^* emerging from Proposition 4.4.1 is of essential importance, because it allows us to reduce the analysis to a problem with homogeneous initial data. This is crucial in order to get the T-independence of embedding constants, allowing for a proof of the local strong well-posedness upon establishing suitable estimates of the nonlinear terms. Because of the role of the reference solution, some further remarks on it are in order.

Remark 4.4.2. *Under the assumptions of Proposition 4.4.1, consider the resulting reference solution* z_0^*.

(a) In the following, we denote by C_T^* *the norm of* z_0^**, so* $C_T^* := \|z_0^*\|_{\mathbb{E}_1}$.

(b) The reference solution $z_0^* \in \mathbb{E}_1$ *exists on* $(0,T)$*, where* $0 < T \le \infty$*, so its maximal regularity space norm shrinks to zero in time, i. e.,* $C_T^* \to 0$ *as* $T \to 0$.

(c) *We also obtain the convergence of the reference solution* z_0^* *to the initial datum* z_0 *in* $\mathrm{BUC}([0,T];\mathrm{X}_\gamma)$ *as time approaches zero, so*

$$\|z_0^* - z_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \to 0 \quad \text{as } T \to 0. \tag{4.51}$$

(d) *As* $V \subset \mathrm{X}_\gamma$ *is open and we have* $z_0 \in V$*, there exists* $r_0 > 0$ *sufficiently small with* $\overline{\mathbb{B}}_{\mathrm{X}_\gamma}(z_0, r_0) \subset V$*. Thanks to* (4.51)*, we conclude the existence of* $T_0 > 0$ *sufficiently small such that*

$$\sup_{t \in [0,T_0]} \|z_0^* - z_0\|_{\mathrm{X}_\gamma} \le \frac{r_0}{2}. \tag{4.52}$$

In particular, we have $z_0^*(t) \in V$ *for all* $t \in [0, T_0]$*.*

Equipped with the reference solution z_0^* from Proposition 4.4.1, we are able to reformulate the transformed interaction problem (4.28) as a linearized problem of the shape as (4.30) with homogeneous initial values. To this end, let $z = (v_{\mathrm{ice}}, h, a, \ell, \omega)$ denote a solution to the transformed interaction problem (4.28), and define $\widehat{z} = (\widehat{v}_{\mathrm{ice}}, \widehat{h}, \widehat{a}, \widehat{\ell}, \widehat{\omega})$ by $\widehat{z} := z - z_0^*$, so

$$\widehat{v}_{\mathrm{ice}} := v_{\mathrm{ice}} - v_{\mathrm{ice},0}^*, \;\; \widehat{h} := h - h_0^*, \;\; \widehat{a} := a - a_0^*, \;\; \widehat{\ell} := \ell - \ell_0^* \text{ and } \widehat{\omega} := \omega - \omega_0^*.$$

As $z_0 \in V$ is fixed, we also use the pieces of notation $\mathbb{A}^{\mathrm{H}} = \mathbb{A}^{\mathrm{H}}(u_0)$ as well as $B_1 = B_1(u_0)$ for simplicity. The above $\widehat{z}$ then solves

$$\begin{cases} \partial_t \widehat{v}_{\mathrm{ice}} - (\mathbb{A}^{\mathrm{H}} - \mu)\widehat{v}_{\mathrm{ice}} + B_1 \begin{pmatrix} \widehat{h} \\ \widehat{a} \end{pmatrix} = G_1(\widehat{z}), & \text{in } (0,T) \times \mathcal{F}_0, \\ \partial_t \widehat{h} - (d_{\mathrm{h}} \Delta_{\mathrm{H}} - \mu)\widehat{h} = G_2(\widehat{z}), & \text{in } (0,T) \times \mathcal{F}_0, \\ \partial_t \widehat{a} - (d_{\mathrm{a}} \Delta_{\mathrm{H}} - \mu)\widehat{a} = G_3(\widehat{z}), & \text{in } (0,T) \times \mathcal{F}_0, \\ m_{\mathcal{S}} \widehat{\ell}'(t) = G_4(\widehat{z}), & \text{for } t \in (0,T), \\ I \widehat{\omega}'(t) = G_5(\widehat{z}), & \text{for } t \in (0,T), \\ \widehat{v}_{\mathrm{ice}} = \widehat{\ell}(t) + \widehat{\omega}(t) y_{\mathrm{H}}^{\perp}, \;\; \partial_\nu \widehat{h} = \partial_\nu \widehat{a} = 0, & \text{on } (0,T) \times \partial\mathcal{S}_0, \\ \widehat{v}_{\mathrm{ice}} = 0, \;\; \partial_\nu \widehat{h} = \partial_\nu \widehat{a} = 0, & \text{on } (0,T) \times \partial\mathcal{O}, \\ \widehat{v}_{\mathrm{ice}}(0) = 0, \;\; \widehat{h}(0) = 0, \;\; \widehat{a}(0) = 0, & \text{in } \mathcal{F}_0, \\ \widehat{\ell}(0) = 0, \;\; \widehat{\omega}(0) = 0. \end{cases} \tag{4.53}$$

The terms on the right-hand side of (4.53) are given by

$$
\begin{aligned}
G_1(\widehat{z}) &:= \Big(\mathcal{A}^{\mathrm{H}}(\widehat{z}+z_0^*) - \mathbb{A}^{\mathrm{H}}(u_0)\Big)(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) \\
&\quad - (\mathcal{B}(\widehat{z}+z_0^*) - B_1(u_0))\begin{pmatrix}\widehat{h}+h_0^* \\ \widehat{a}+a_0^*\end{pmatrix} - \nabla_{\mathrm{H}}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)\partial_t Y \\
&\quad - \mathcal{N}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) + \mu(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) - c_{\mathrm{cor}}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)^{\perp} \\
&\quad - g\nabla_{\mathrm{H}} H + \frac{1}{\rho_{\mathrm{ice}}(\widehat{h}+h_0^*)}\Big(\tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)\Big)
\end{aligned}
\tag{4.54}
$$

in the transformed sea ice momentum equation, by

$$
\begin{aligned}
G_2(\widehat{z}) &:= d_{\mathrm{h}}(\mathcal{L} - \Delta_{\mathrm{H}})(\widehat{h}+h_0^*) + \mu(\widehat{h}+h_0^*) - \nabla_{\mathrm{H}}(\widehat{h}+h_0^*)\cdot\partial_t Y \\
&\quad - \mathcal{M}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*, \widehat{h}+h_0^*) + S_{\mathrm{h}}(\widehat{h}+h_0^*, \widehat{a}+a_0^*) \text{ and} \\
G_3(\widehat{z}) &:= d_{\mathrm{a}}(\mathcal{L} - \Delta_{\mathrm{H}})(\widehat{a}+a_0^*) + \mu(\widehat{a}+a_0^*) - \nabla_{\mathrm{H}}(\widehat{a}+a_0^*)\cdot\partial_t Y \\
&\quad - \mathcal{M}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*, \widehat{a}+a_0^*) + S_{\mathrm{a}}(\widehat{h}+h_0^*, \widehat{a}+a_0^*)
\end{aligned}
\tag{4.55}
$$

for the transformed balance laws for h and a, and by

$$
\begin{aligned}
G_4(\widehat{z}) &:= F + m_{\mathcal{S}}(\widehat{\omega}+\omega_0^*)(\widehat{\ell}+\ell_0^*)^{\perp} - Q\int_{\partial\mathcal{S}_0}\mathcal{T}_\delta(\widehat{u}+u_0^*)\nu\,\mathrm{d}\Gamma \text{ and} \\
G_5(\widehat{z}) &:= N - Q\int_{\partial\mathcal{S}_0} y_{\mathrm{H}}^{\perp}\mathcal{T}_\delta(\widehat{u}+u_0^*)\nu\,\mathrm{d}\Gamma
\end{aligned}
\tag{4.56}
$$

in the situation of the transformed rigid body equations.

With regard to (4.53), it is natural to introduce the maximal regularity space with homogeneous initial values, and we use ${}_0\mathbb{E}_1$ for this, i. e., $\widehat{z} \in {}_0\mathbb{E}_1$ satisfies $\widehat{z}(0) = 0$. Another important aspect is that the solution $z = \widehat{z} + z_0^*$ to the transformed system does not leave the physically relevant range V, and this also guarantees that $\rho_{\mathrm{ice}} h$ does not degenerate. For this, we fix $T_0 > 0$ from Remark 4.4.2(d). Not only do we assume an upper bound for the time, but we also demand that $R > 0$ is sufficiently small. Indeed, for $z_0 \in V$, there exists $r_0 > 0$ so that $\overline{\mathbb{B}}_{\mathrm{X}_\gamma}(z_0, r_0) \subset V$. The ultimate goal is to prove that $z(t) = \widehat{z}(t) + z_0^*(t)$ consisting of the unique fixed point of (4.53) and the reference solution z_0^* emerging from Proposition 4.4.1 is in V for sufficiently small t and R. Next, for the present time interval $(0, T_0)$, we recall

$$
{}_0\mathbb{E}_1 \hookrightarrow \mathrm{BUC}([0,T_0];\mathrm{X}_\gamma)
\tag{4.57}
$$

from Proposition 2.4.11 and invoke Remark 2.4.12 for the T-independence of the embedding constant $C > 0$ by virtue of the homogeneous initial values.

The estimate (4.52) on the time interval $[0, T_0]$ as well as $R_0 \leq {r_0}/{2C}$ yield that $\widehat{z} \in {}_0\mathbb{E}_1$ with $\|\widehat{z}\|_{\mathbb{E}_1} \leq R_0$ satisfies

$$\sup_{t\in[0,T_0]} \|z(t) - z_0\|_{\mathrm{X}_\gamma} \leq \sup_{t\in[0,T_0]} \Big(\|\widehat{z}\|_{\mathrm{X}_\gamma} + \|z_0^*(t) - z_0\|_{\mathrm{X}_\gamma}\Big)$$
$$\leq C \cdot \|\widehat{z}\|_{\mathbb{E}_1} + \sup_{t\in[0,T_0]} \|z_0^*(t) - z_0\|_{\mathrm{X}_\gamma} \leq \frac{r_0}{2} + \frac{r_0}{2} \leq r_0.$$

In summary, we obtain the lemma below guaranteeing that the solution to the transformed interaction problem stays in V when choosing the time and the norm of the solution sufficiently small.

Lemma 4.4.3. *Consider $z_0 \in V$ and recall $T_0 > 0$ from Remark 4.4.2. Besides, let $0 < R_0 \leq {r_0}/{2C}$, for $C > 0$ denoting the T-independent embedding constant from (4.57), and $r_0 > 0$ with $\overline{\mathbb{B}}_{\mathrm{X}_\gamma}(z_0, r_0) \subset V$. Take into account $T \in (0, T_0]$ and $R \in (0, R_0]$, and let $z := \widehat{z} + z_0^*$, with $\widehat{z} \in {}_0\mathbb{E}_1$ and $\|\widehat{z}\|_{\mathbb{E}_1} \leq R$ as well as $z_0^* \in \mathbb{E}_1$ representing the reference solution from Proposition 4.4.1. Then it follows that $z(t) \in V$ for all $t \in [0, T]$.*

Finally, we describe the precise fixed point argument in more details. In fact, let p, $q \in (1, \infty)$ satisfy (4.48), and for $T_0 > 0$ and $R_0 > 0$, consider the time $T \in (0, T_0]$ and the radius $R \in (0, R_0]$. We then set

$$\begin{aligned} &\mathcal{K}_T^R := \{\tilde{z} \in {}_0\mathbb{E}_1 : \|\tilde{z}\|_{\mathbb{E}_1} \leq R\} \quad \text{and} \\ &\Phi_T^R \colon \mathcal{K}_T^R \to {}_0\mathbb{E}_1, \quad \text{with} \quad \Phi_T^R(\tilde{z}) := \widehat{z}, \end{aligned} \tag{4.58}$$

where $\widehat{z}$ denotes the unique solution to (4.53) for the terms on the right-hand side $G_1(\tilde{z})$, $G_2(\tilde{z})$, $G_3(\tilde{z})$, $G_4(\tilde{z})$ and $G_5(\tilde{z})$ from (4.54), (4.55) as well as (4.56), with $\tilde{z} \in \mathcal{K}_T^R$. In view of the maximal regularity result Proposition 4.3.3, the map Φ_T^R is well-defined thanks to $z_0 \in V \subset \mathrm{X}_\gamma$ and the embedding (4.49) provided the terms on the right-hand side are contained in the data space. This will also be addressed below.

Estimates of the Nonlinear Terms

In this section, we consider $T_0 > 0$ and $R_0 > 0$ fixed as in Lemma 4.4.3 and let $T \in (0, T_0]$ as well as $R \in (0, R_0]$. Moreover, for $\tilde{z}$, $\tilde{z}_1$, $\tilde{z}_2 \in \mathcal{K}_T^R$ and the reference solution z_0^* from Proposition 4.4.1, we define

$$z := \tilde{z} + z_0^* = (\tilde{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*, \tilde{h} + h_0^*, \tilde{a} + a_0^*, \tilde{\ell} + \ell_0^*, \tilde{\omega} + \omega_0^*),$$

and we set $z_i := \tilde{z}_i + z_0^*$, $i = 1, 2$, likewise. From $\tilde{z} \in \mathcal{K}_T^R$ and the definition of C_T^* as the norm of the reference solution in Remark 4.4.2, we deduce that

$$\|z\|_{\mathbb{E}_1} \le R + C_T^*. \tag{4.59}$$

In the sequel, $C_0 > 0$ represents the norm of the initial values, so $C_0 := \|z_0\|_{\mathrm{X}_\gamma}$. Another estimate used frequently in the sequel concerns the reference solution in $\mathrm{BUC}([0,T];\mathrm{X}_\gamma)$. From Lemma 2.4.14, we deduce for a T-independent constant $C > 0$ as well as C_0 and C_T^* as explained above that

$$\|z_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \le C \cdot \Big(\|z_0\|_{\mathrm{X}_\gamma} + \|z_0^*\|_{\mathbb{E}_1}\Big) = C(C_0 + C_T^*). \tag{4.60}$$

The next focal point is the procedure to derive the diffeomorphisms X and Y from given rigid body velocities ℓ and ω. This is addressed below.

Remark 4.4.4. *Consider* $(\ell, \omega) \in \mathrm{W}^{1,p}(0,T)^3$.

(a) The original angular velocity Ω *coincides with* ω *by* (4.18)*, and the rotation angle* β *can then be deduced from* (4.3)*. This also leads to the rotation matrix* $Q \in \mathrm{W}^{2,p}(0,T)^{2\times 2}$ *with angle* β *as given in* (4.2)*.*

(b) After obtaining the rotation matrix Q*, we can recover the original translational velocity from* $\xi(t) = Q(t)\ell(t)$ *with regard to* (4.18)*. As revealed in* (4.1)*, the center of mass follows from the integral*

$$x_{\mathrm{c}}(t) = \int_0^t x_{\mathrm{c}}'(s)\,\mathrm{d}s = \int_0^t \xi(s)\,\mathrm{d}s.$$

(c) Consequently, $b(t, x_{\mathrm{H}})$ *can be reconstructed from* (4.14)*, and we plug it into* (4.13) *to get the diffeomorphism* X*.*

(d) Next, we conclude the diffeomorphism Y *by inserting the resulting right-hand side* $b^{(Y)}(t, y_{\mathrm{H}}) = \mathrm{J}_X^{-1}(t, y_{\mathrm{H}}) b(t, X(t, y_{\mathrm{H}}))$ *into* (4.17) *and solving the corresponding initial value problem. Let us observe that* $\mathrm{J}_X^{-1}(t, y_{\mathrm{H}})$ *is well-defined for sufficiently small time thanks to Lemma 4.2.1.*

In order to shorten the notation, we use the subscript $i \in \{1, 2\}$ to denote the dependence of objects on the rigid body velocities (ℓ_i, ω_i) in the sequel. In particular, X_i and Y_i represent the diffeomorphisms associated to (ℓ_i, ω_i) and deduced therefrom by the procedure described in Remark 4.4.4. Several properties of the diffeomorphisms X and Y can be found in Appendix A.

The lemma below provides ingredients for the estimate of the difference of the transformed Laplacian and the non-transformed Laplacian.

Lemma 4.4.5. *Consider $p, q \in (1, \infty)$ such that (4.48). Moreover, for the reference solution z_0^* from Proposition 4.4.1 with norm C_T^* and $\tilde{z} \in \mathcal{K}_T^R$, set $z := \tilde{z} + z_0^*$, and recall $C_0 = \|z_0\|_{\mathrm{X}_\gamma}$. Then for Y resulting from z as described in Remark 4.4.4 and the associated contravariant tensor g^{jk}, we have*

$$\|g^{jk} - \delta_{jk}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \le CT(R + C_0 + C_T^*) \text{ and}$$
$$\|\partial_j Y_k - \delta_{jk}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \le CT(R + C_0 + C_T^*).$$

Proof. First, the identity transform $X(t, y_{\mathrm{H}}) = y_{\mathrm{H}}$ for all $t > 0$ and $y_{\mathrm{H}} \in \mathbb{R}^2$ corresponds precisely to the situation of the body at rest, so $\ell = 0$ as well as $\omega = 0$. As a result, we can apply Lemma A.1.2 in this instance. Additionally, we split (ℓ, ω) into its component with initial value zero $(\tilde{\ell}, \tilde{\omega})$ and the part corresponding to the reference solution (ℓ_0^*, ω_0^*), exploit Lemma 1.3.1 for the first part and employ the estimate (4.60) for the second one to obtain

$$\begin{aligned}
\|g^{jk} - \delta_{jk}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} &\le C_1 T \cdot \Big(\|(\tilde{\ell}, \tilde{\omega})\|_{\mathrm{L}^\infty(0,T)} + \|(\ell_0^*, \omega_0^*)\|_{\mathrm{L}^\infty(0,T)}\Big) \\
&\le C_2 T \cdot \Big(T^{1/p'} \cdot \|(\tilde{\ell}, \tilde{\omega})\|_{\mathrm{W}^{1,p}(0,T)} + \|z_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}\Big) \\
&\le C_3 T \cdot \Big(T_0^{1/p'} \cdot \|\tilde{z}\|_{\mathbb{E}_1} + C_0 + C_T^*\Big) \\
&\le C_4 T(R + C_0 + C_T^*).
\end{aligned}$$

The second estimate follows in a similar way. □

The embeddings collected below equip us with an important tool in order to tackle the nonlinear estimates.

Lemma 4.4.6. *Let $p, q \in (1, \infty)$ fulfill (4.48), let $T \in (0, T_0]$, with $T_0 > 0$ from Remark 4.4.2, and recall the trace space X_γ from (4.34).*

(a) It holds that ${}_0\mathbb{E}_1 \hookrightarrow \mathrm{BUC}([0,T];\mathrm{X}_\gamma)$. Moreover, the embedding constant can be chosen independent of $T > 0$.

(b) We have $\mathrm{BUC}([0,T];\mathrm{X}_\gamma) \hookrightarrow \mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\mathcal{F}_0)^4 \times \mathbb{R}^3)$. In particular,

$${}_0\mathbb{E}_1 \hookrightarrow \mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\mathcal{F}_0)^4 \times \mathbb{R}^3) \hookrightarrow \mathrm{L}^{2p}(0,T;\mathrm{L}^{2q}(\mathcal{F}_0)^4 \times \mathbb{R}^3),$$

with T-independent embedding constants.

Proof. The embedding in (a) directly follows from Proposition 2.4.11, and the T-independence thanks to the homogeneous initial values is discussed in Remark 2.4.12. With regard to (b), the embedding in the space component is implied by the embedding of the trace space as revealed in (4.49) together with the boundedness of the initial domain $\mathcal{F}_0$. The second part of (b) is direct consequence of (a), the first embedding and $\mathrm{W}^{1,q}(\mathcal{F}_0) \hookrightarrow \mathrm{L}^{2q}(\mathcal{F}_0)$, following from the Sobolev embedding (1.5). $\square$

As a preparation of the nonlinear estimates, we first address some autonomous terms in the lemma below.

Lemma 4.4.7. *Let $p, q \in (1, \infty)$ satisfy (4.48), and let $z = \widehat{z} + z_0^*$, $z_i = \widehat{z}_i + z_0^*$, $i = 1, 2$, with $\widehat{z}$, $\widehat{z}_1$, $\widehat{z}_2 \in \mathcal{K}_T^R$ and $z_0^* = (u_0^*, \ell_0^*, \omega_0^*) = (v_{\mathrm{ice},0}^*, h_0^*, a_0^*, \ell_0^*, \omega_0^*)$ representing the reference solution from Proposition 4.4.1. Then there exists a T-independent constant $C > 0$ such that, using $B_1^* := B_1(u_0^*)$ for simplicity,*

$$\left\| \left(\mathbb{A}^{\mathrm{H}}(u_0^*) - \mathbb{A}^{\mathrm{H}}(u_0) \right) v_{\mathrm{ice}} - (B_1^* - B_1(u_0)) \begin{pmatrix} h \\ a \end{pmatrix} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))}$$
$$\le C \cdot \|z_0^* - z_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot (R + C_T^*),$$
$$\left\| \left(\mathbb{A}^{\mathrm{H}}(u_0^*) - \mathbb{A}^{\mathrm{H}}(u_0) \right) (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) - (B_1^* - B_1(u_0)) \begin{pmatrix} \widehat{h}_1 - \widehat{h}_2 \\ \widehat{a}_1 - \widehat{a}_2 \end{pmatrix} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))}$$
$$\le C \cdot \|z_0^* - z_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1},$$
$$\left\| \left(\mathbb{A}^{\mathrm{H}}(u) - \mathbb{A}^{\mathrm{H}}(u_0^*) \right) v_{\mathrm{ice}} - (B_1(u) - B_1^*) \begin{pmatrix} h \\ a \end{pmatrix} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))}$$
$$\le CR(R + C_T^*), \quad \text{and}$$

$$\left\| \left(\mathbb{A}^{\mathrm{H}}(u_2) - \mathbb{A}^{\mathrm{H}}(u_0^*) \right) (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) - (B_1(u_2) - B_1^*) \begin{pmatrix} \widehat{h}_1 - \widehat{h}_2 \\ \widehat{a}_1 - \widehat{a}_2 \end{pmatrix} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))}$$
$$\le CR \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.$$

Proof. We recall that $z = (v_{\mathrm{ice}}, h, a, \ell, \omega) \in V$ especially implies $h > \kappa$ as well as $a \in (0, 1)$ by the definition in (4.47). With regard to the shape of X_1 as in (4.33), we conclude from $z = (v_{\mathrm{ice}}, h, a, \ell, \omega) \in \mathrm{X}_1$ that $u = (v_{\mathrm{ice}}, h, a)$ possesses the required regularity properties as in the proof of Theorem 3.5.2. As a consequence, the same arguments as in the aforementioned proof as well as the embedding properties of the trace space from (4.49) reveal the existence of a constant $C_1 > 0$ related to the Lipschitz constant $C_A > 0$ from the proof of Theorem 3.5.2 with

$$\left\| \left(\mathbb{A}^{\mathrm{H}}(u_1) - \mathbb{A}^{\mathrm{H}}(u_2) \right) v_{\mathrm{ice}} - (B_1(u_1) - B_1(u_2)) \begin{pmatrix} h \\ a \end{pmatrix} \right\|_{\mathrm{L}^q(\mathcal{F}_0)} \tag{4.61}$$
$$\le C_1 \cdot \|z_1 - z_2\|_{\mathrm{X}_\gamma} \cdot \|z\|_{\mathrm{X}_1}$$

for $z_1, z_2 \in V$ and $z \in \mathrm{X}_1$. Thus, it remains to argue that the elements in our case are contained in V for the whole time interval $[0,T]$. By assumption, we have $z_0 \in V$. Moreover, it follows from Remark 4.4.2(d) that $z_0^*(t) \in V$ for all $t \in [0,T]$, whereas we recall from Lemma 4.4.3 that $z = \widehat{z} + z_0^*$ satisfies $z(t) \in V$ on $[0,T]$. An integration in time and an application of (4.61) as well as (4.59) yield, upon recalling the abbreviation $B_1^* = B_1(u_0^*)$, that

$$
\begin{aligned}
&\left\| \left(\mathbb{A}^{\mathrm{H}}(u_0^*) - \mathbb{A}^{\mathrm{H}}(u_0)\right) v_{\mathrm{ice}} - (B_1^* - B_1(u_0)) \begin{pmatrix} h \\ a \end{pmatrix} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\
&= \left(\int_0^T \left\| \left(\mathbb{A}^{\mathrm{H}}(u_0^*(t)) - \mathbb{A}^{\mathrm{H}}(u_0)\right) v_{\mathrm{ice}}(t) - (B_1^* - B_1(u_0)) \begin{pmatrix} h(t) \\ a(t) \end{pmatrix} \right\|_{\mathrm{L}^q(\mathcal{F}_0)}^p \mathrm{d}t \right)^{1/p} \\
&\le C_1 \left(\int_0^T \|z_0^*(t) - z_0\|_{\mathrm{X}_\gamma}^p \cdot \|z(t)\|_{\mathrm{X}_1}^p \,\mathrm{d}t \right)^{1/p} \\
&\le C_2 \cdot \|z_0^* - z_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot \|z\|_{\mathbb{E}_1} \\
&\le C_2 \cdot \|z_0^* - z_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot (R + C_T^*).
\end{aligned}
$$

The first estimate is thus shown. The other estimates follow in a similar way. At this stage, we also emphasize that the dependence of the constants on z_0 and R_0 is not critical as the latter objects are fixed. □

As in the previous chapter, we also make assumptions on the external data in order to obtain suitable estimates of the related terms.

Assumption 4.4.8. *Let $q \in (1,\infty)$, and recall $T_0 > 0$ from Remark 4.4.2. The external data are assumed to have the following properties.*

(a) For the surface wind velocity V_{atm} and the ocean velocity V_{ocn}, it is valid that $V_{\mathrm{atm}}, V_{\mathrm{ocn}} \in \mathrm{L}^\infty(0,T_0;\mathrm{L}^{2q}(\mathcal{F}_0)^2)$.

(b) The sea surface dynamic height H fulfills $\nabla_{\mathrm{H}} H \in \mathrm{L}^\infty(0,T_0;\mathrm{L}^q(\mathcal{F}_0)^2)$.

(c) For the ice growth rate function f_{gr}, we have $f_{\mathrm{gr}} \in \mathrm{C}_{\mathrm{b}}^1([0,\infty))$.

In the following lemmas, we collect the estimates of the nonlinear terms in order to ensure the self map and contraction property of the fixed point map Φ_T^R from (4.58) for $R > 0$ and $T > 0$ sufficiently small.

The first nonlinear term to be estimated is the term G_1 as determined in (4.54). As we also elaborate on its Lipschitz property in the sequel, we calculate the difference as well. In order to shorten the notation, we use z_1

and z_2, but we still observe that the dependence can be regarded as a dependence on $\widehat{z}_1$ and $\widehat{z}_2$. The difference is then given by

$$
\begin{aligned}
&G_1(\widehat{z}_1) - G_1(\widehat{z}_2) \\
&= \left(\mathcal{A}^{\mathrm{H}}(z_1) - \mathcal{A}^{\mathrm{H}}(z_2)\right) v_{\mathrm{ice},1} - (\mathcal{B}(z_1) - \mathcal{B}(z_2)) \begin{pmatrix} h_1 \\ a_1 \end{pmatrix} \\
&\quad + \left(\mathcal{A}^{\mathrm{H}}(z_2) - \mathbb{A}^{\mathrm{H}}(u_0)\right) (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) - (\mathcal{B}(z_2) - B_1(u_0)) \begin{pmatrix} \widehat{h}_1 - \widehat{h}_2 \\ \widehat{a}_1 - \widehat{a}_2 \end{pmatrix} \\
&\quad - \nabla_{\mathrm{H}}(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) \partial_t Y_1 - (\nabla_{\mathrm{H}} v_{\mathrm{ice},2}) \partial_t (Y_1 - Y_2) - (\mathcal{N}(v_{\mathrm{ice},1}) - \mathcal{N}(v_{\mathrm{ice},2})) \\
&\quad + \mu(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) - c_{\mathrm{cor}}(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})^{\perp} \\
&\quad + \left(\frac{1}{\rho_{\mathrm{ice}} h_1} - \frac{1}{\rho_{\mathrm{ice}} h_2}\right) (\tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(v_{\mathrm{ice},1})) + \frac{1}{\rho_{\mathrm{ice}} h_2} (\tau_{\mathrm{ocn}}(v_{\mathrm{ice},1}) - \tau_{\mathrm{ocn}}(v_{\mathrm{ice},2})) .
\end{aligned}
$$

The lemma below asserts that the L^p-L^q norm of $G_1(\widehat{z})$ becomes small as $R > 0$ and $T > 0$ tend to zero, and that G_1 also admits a Lipschitz estimate with shrinking Lipschitz constant.

Lemma 4.4.9. *Let $p, q \in (1, \infty)$ satisfy (4.48), and consider $z = \widehat{z} + z_0^*$, $z_1 = \widehat{z}_1 + z_0^*$ and $z_2 = \widehat{z}_2 + z_0^*$, with $\widehat{z}$, $\widehat{z}_1$, $\widehat{z}_2 \in \mathcal{K}_T^R$, and z_0^* representing the reference solution from Proposition 4.4.1. Moreover, suppose that V_{atm}, V_{ocn} and $\nabla_{\mathrm{H}} H$ lie within the scope of Assumption 4.4.8. In addition, recall the T-independent maximal regularity constant $C_{\mathrm{MR}} > 0$ from Proposition 4.3.3.*

Then there exist $C_{G_1}(R,T), L_{G_1}(R,T) > 0$ such that $C_{G_1}(R,T) < R/10C_{\mathrm{MR}}$ for $R > 0$ and $T > 0$ sufficiently small as well as $L_{G_1}(R,T) \to 0$ as $R \to 0$ and $T \to 0$, and with

$$
\begin{aligned}
\|G_1(\widehat{z})\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} &\le C_{G_1}(R,T) \quad \textit{and} \\
\|G_1(\widehat{z}_1) - G_1(\widehat{z}_2)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} &\le L_{G_1}(R,T) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.
\end{aligned}
$$

Proof. We first comment on the main ideas and tools used below. Throughout this proof, we rely on Lemma 1.3.1, yielding T-powers when estimating the norm $\|\cdot\|_{\mathrm{L}^p(0,T)}$ by $\|\cdot\|_{\mathrm{L}^\infty(0,T)}$, or when estimating $\|\cdot\|_{\mathrm{L}^\infty(0,T)}$ by $\|\cdot\|_{\mathrm{W}^{1,p}(0,T)}$ in the case of homogeneous initial values. We also frequently use that $z = \widehat{z} + z_0^*$, with $\widehat{z} \in \mathcal{K}_T^R$, satisfies $z(t) \in V$ for all $t \in [0,T]$ by virtue of Lemma 4.4.3 and $R \le R_0$ as well as $T \le T_0$. The same is valid for the analogously defined z_1 and z_2. Another important ingredient is the estimate of the $\mathbb{E}_1$-norm of such z, z_1 and z_2 by $R + C_T^*$ with regard to (4.59). In general, we often split the elements z, z_1 and z_2 into their respective parts with homogeneous initial values $\widehat{z}$, $\widehat{z}_1$, $\widehat{z}_2$ and the reference solution z_0^*. We then use embeddings with T-independent constants and the bound by R in $\mathbb{E}_1$ for the first parts, while for the reference solution z_0^*, we employ the estimate by $C(C_0 + C_T^*)$ in $\mathrm{BUC}([0,T];\mathrm{X}_\gamma)$ as settled in (4.60) or the $\mathbb{E}_1$-norm C_T^*. Concerning the terms in which the diffeomorphism Y appears, we make use of the estimates from Appendix A and Lemma 4.4.5.

By $\widehat{v}_{\mathrm{ice}}(0) = 0$ and Lemma 1.3.1, we get

$$\begin{aligned}
&\|\mu(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) - c_{\mathrm{cor}}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)^\perp\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\
&\le C_1 \left(T^{1/p} \cdot \|\widehat{v}_{\mathrm{ice}}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^q(\mathcal{F}_0))} + \|v_{\mathrm{ice},0}^*\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))}\right) \\
&\le C_2 \left(T \cdot \|\widehat{v}_{\mathrm{ice}}\|_{\mathrm{W}^{1,p}(0,T;\mathrm{L}^q(\mathcal{F}_0))} + \|z_0^*\|_{\mathbb{E}_1}\right) \\
&\le C_2 \left(T \cdot \|\widehat{z}\|_{\mathbb{E}_1} + C_T^*\right) \\
&\le C_3(TR + C_T^*).
\end{aligned} \tag{4.62}$$

From Lemma 1.3.1 and Assumption 4.4.8, we deduce that

$$\|-g\nabla_{\mathrm{H}} H\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le T^{1/p} \cdot \|-g\nabla_{\mathrm{H}} H\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_4 T^{1/p}.$$

Thanks to $z(t) \in V$ for all $t \in [0,T]$, we conclude $h(t) = \widehat{h}(t) + h_0^*(t) > \kappa$ on the time interval $[0,T]$. Therefore, there exists a constant $C_5 > 0$ such that

$$\left\| \frac{1}{\rho_{\mathrm{ice}}(\widehat{h} + h_0^*)} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \le C_5. \tag{4.63}$$

Together with Lemma 1.3.1, Hölder's inequality and Assumption 4.4.8(a), the estimate (4.63) yields that

$$\begin{aligned}
\left\| \frac{1}{\rho_{\mathrm{ice}}(\widehat{h} + h_0^*)} \tau_{\mathrm{atm}} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} &\le C_6 T^{1/p} \cdot \|\tau_{\mathrm{atm}}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\
&\le C_7 T^{1/p} \cdot \|V_{\mathrm{atm}}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^{2q}(\mathcal{F}_0))}^2 \\
&\le C_8 T^{1/p}.
\end{aligned} \tag{4.64}$$

The term accounting for the ocean force can be dealt with in a similar manner. Indeed, exploiting Lemma 1.3.1, the above estimate (4.63) of the fraction, the embedding from Lemma 4.4.6(b) and (4.60), we infer that

$$\begin{aligned}
&\left\| \frac{1}{\rho_{\mathrm{ice}}(\widehat{h} + h_0^*)} \tau_{\mathrm{ocn}}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\
&\le C_9 T^{1/p} \cdot \|\tau_{\mathrm{ocn}}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\
&\le C_{10} T^{1/p} \cdot \left(\|V_{\mathrm{ocn}} + \widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^{2q}(\mathcal{F}_0))}^2\right) \\
&\le C_{11} T^{1/p} \left(1 + \|\widehat{z}\|_{\mathbb{E}_1}^2 + \|z_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}^2\right) \\
&\le C_{12} T^{1/p} \left(1 + R^2 + C_0^2 + (C_T^*)^2\right).
\end{aligned} \tag{4.65}$$

Now, we address the terms related to the coordinate transform. In fact, Hölder's inequality, Lemma A.1.1(a) for the estimate of $\partial_t Y$, Lemma 1.3.1 and the embedding in Lemma 4.4.6(b) with T-independent embedding constant thanks to the homogeneous initial values of $\widehat{z}$ lead to

$$
\begin{aligned}
&\|\nabla_{\mathrm{H}}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)\partial_t Y\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\
&\le C_{13}\|\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\mathcal{F}_0))} \cdot \|\partial_t Y\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \\
&\le C_{14}\left(T^{1/p} \cdot \|\widehat{v}_{\mathrm{ice}}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\mathcal{F}_0))} + \|v_{\mathrm{ice},0}^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\mathcal{F}_0))}\right) \\
&\le C_{15}\left(T^{1/p} \cdot \|\widehat{z}\|_{\mathbb{E}_1} + \|z_0^*\|_{\mathbb{E}_1}\right) \\
&\le C_{15}\left(T^{1/p} R + C_T^*\right).
\end{aligned} \tag{4.66}
$$

Furthermore, Hölder's inequality, Lemma A.1.1(a) for the estimate of the coordinate transform, the embedding of the trace space from (4.49), the embedding in Lemma 4.4.6(a), the same arguments for $\|\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\mathcal{F}_0))}$ as in (4.66) and (4.60) imply

$$
\begin{aligned}
&\|\mathcal{N}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\
&\le C_{16} \cdot \|\partial_j Y_k\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \cdot \|\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \\
&\quad \cdot \|\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\mathcal{F}_0))} \\
&\le C_{17} \cdot \left(\|\widehat{z}\|_{\mathbb{E}_1} + \|z_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}\right) \\
&\quad \cdot \left(T^{1/p} \cdot \|\widehat{v}_{\mathrm{ice}}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\mathcal{F}_0))} + \|v_{\mathrm{ice},0}^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\mathcal{F}_0))}\right) \\
&\le C_{18}\left(R + C_0 + C_T^*\right)\left(T^{1/p} R + C_T^*\right).
\end{aligned} \tag{4.67}
$$

With regard to (4.54), it remains to control the difference

$$
\left(\mathcal{A}^{\mathrm{H}}(\widehat{z} + z_0^*) - \mathbb{A}^{\mathrm{H}}(u_0)\right)(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) - (\mathcal{B}(\widehat{z} + z_0^*) - B_1(u_0))\begin{pmatrix}\widehat{h} + h_0^* \\ \widehat{a} + a_0^*\end{pmatrix}.
$$

We reduce this task by introducing intermediate terms, so we consider

$$
\left(\mathcal{A}^{\mathrm{H}}(\widehat{z} + z_0^*) - \mathbb{A}^{\mathrm{H}}(\widehat{u} + u_0^*)\right)(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) - (\mathcal{B}(\widehat{z} + z_0^*) - B_1(\widehat{u} + u_0^*))\begin{pmatrix}\widehat{h} + h_0^* \\ \widehat{a} + a_0^*\end{pmatrix}
$$

as well as the resulting terms

$$
\left(\mathbb{A}^{\mathrm{H}}(\widehat{u} + u_0^*) - \mathbb{A}^{\mathrm{H}}(u_0^*)\right)(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) - (B_1(\widehat{u} + u_0^*) - B_1(u_0^*))\begin{pmatrix}\widehat{h} + h_0^* \\ \widehat{a} + a_0^*\end{pmatrix} \quad \text{and}
$$

$$
(\mathbb{A}^{\mathrm{H}}(u_0^*) - \mathbb{A}^{\mathrm{H}}(u_0))(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) - (B_1(u_0^*) - B_1(u_0))\begin{pmatrix}\widehat{h} + h_0^* \\ \widehat{a} + a_0^*\end{pmatrix}.
$$

For the third difference, with $z = \widehat{z} + z_0^*$, we recall from Lemma 4.4.7 that

$$\left\| \Big(\mathbb{A}^{\mathrm{H}}(u_0^*) - \mathbb{A}^{\mathrm{H}}(u_0) \Big) v_{\mathrm{ice}} - (B_1(u_0^*) - B_1(u_0)) \begin{pmatrix} h \\ a \end{pmatrix} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\ \le C_{19} \cdot \| z_0^* - z_0 \|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot (R + C_T^*).$$

Concerning the second difference, also using the abbreviation $B_1^* = B_1(u_0^*)$, we conclude from Lemma 4.4.7 the estimate

$$\left\| (\mathbb{A}^{\mathrm{H}}(u) - \mathbb{A}^{\mathrm{H}}(u_0^*)) v_{\mathrm{ice}} - (B_1(u) - B_1^*) \begin{pmatrix} h \\ a \end{pmatrix} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_{20} R (R + C_T^*).$$

Hence, it remains to estimate the term associated to the difference of the transformed problem and the non-transformed problem. In this context, we start with

$$\Big(\mathcal{A}^{\mathrm{H}}(\widehat{z} + z_0^*) - \mathbb{A}^{\mathrm{H}}(\widehat{u} + u_0^*) \Big) (\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*), \tag{4.68}$$

corresponding to the Hibler operator. With regard to the shape of the terms from (4.26) and (3.14), in the principal part, this amounts to estimating

$$\sum_{j,k,l,m=1}^{2} \Big(a_{ij}^{klm}(\tilde{\varepsilon}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*), P(\widehat{h} + h_0^*, \widehat{a} + a_0^*)) \partial_m \tilde{\varepsilon}_{jl}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) \\ - a_{ij}^{kl}(\varepsilon(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*), P(\widehat{h} + h_0^*, \widehat{a} + a_0^*)) \delta_{km} \partial_k \varepsilon_{jl}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) \Big) = \mathrm{I} + \mathrm{II}.$$

For simplicity of notation, we do not explicitly write the dependence of ε as well as $\tilde{\varepsilon}$ on $\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*$, and the dependence of P on $\widehat{h} + h_0^*$ and $\widehat{a} + a_0^*$. The terms in the above are defined by

$$\mathrm{I} := \sum_{j,k,l,m=1}^{2} \Big(a_{ij}^{klm}(\tilde{\varepsilon}, P) \partial_m \tilde{\varepsilon}_{jl} - a_{ij}^{kl}(\varepsilon, P) \delta_{km} \partial_k \tilde{\varepsilon}_{jl} \Big) \quad \text{and}$$

$$\mathrm{II} := \sum_{j,k,l=1}^{2} \Big(a_{ij}^{kl}(\varepsilon, P) \partial_k \tilde{\varepsilon}_{jl} - a_{ij}^{kl}(\varepsilon, P) \partial_k \varepsilon_{jl} \Big).$$

We start with the estimate of II, where the task is to estimate the difference of the transformed and the non-transformed symmetric part of the gradient. Invoking the derivatives of the transformed term as revealed in (4.23), we get, also using the notation $v_{\mathrm{ice}} = \widehat{v}_{\mathrm{ice}} + v^*_{\mathrm{ice},0}$, the identity

$$\begin{aligned}\partial_k \tilde{\varepsilon}_{jl} - \partial_k \varepsilon_{jl} &= \frac{1}{2} \sum_{n=1}^{2} \Big((\partial_k \partial_j Y_n) \partial_n v_{\mathrm{ice},j} + (\partial_j Y_n) \partial_k \partial_n v_{\mathrm{ice},l} + (\partial_k \partial_l Y_n) \partial_n v_{\mathrm{ice},j} \\ &\quad + (\partial_l Y_n) \partial_k \partial_n v_{\mathrm{ice},j} \Big) - \frac{1}{2} \partial_k (\partial_j v_{\mathrm{ice},l} + \partial_l v_{\mathrm{ice},j}) \\ &= \frac{1}{2} \sum_{n=1}^{2} \Big((\partial_j Y_n - \delta_{jn}) \partial_k \partial_n v_{\mathrm{ice},l} + (\partial_l Y_n - \delta_{ln}) \partial_k \partial_n v_{\mathrm{ice},j} \\ &\quad + (\partial_k \partial_j Y_n) \partial_n v_{\mathrm{ice},j} + (\partial_k \partial_l Y_n) \partial_n v_{\mathrm{ice},j} \Big).\end{aligned}$$

Lemma 4.4.5 yields estimates of terms as $\partial_j Y_n - \delta_{jn}$, so by (4.59), we get

$$\begin{aligned}&\|(\partial_j Y_n - \delta_{jn}) \partial_k \partial_n v_{\mathrm{ice},l}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\ &\le \|\partial_j Y_n - \delta_{jn}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \cdot \|v_{\mathrm{ice}}\|_{\mathrm{L}^p(0,T;\mathrm{W}^{2,q}(\mathcal{F}_0))} \\ &\le C_{21} T (R + C_0 + C_T^*) \cdot \|z\|_{\mathbb{E}_1} \\ &\le C_{22} T (R + C_0 + C_T^*)(R + C_T^*).\end{aligned}$$

The estimate of $(\partial_l Y_n - \delta_{ln}) \partial_k \partial_n v_{\mathrm{ice},j}$ follows in the same way. In order to complete the estimate of the terms in II, it remains to estimate the terms with two derivatives on Y. Indeed, Hölder's inequality together with Lemma A.1.1(a) and (4.59) result in

$$\begin{aligned}\|(\partial_k \partial_j Y_n) \partial_n v_{\mathrm{ice},j}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} &\le \|\partial_k \partial_j Y_n\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \cdot \|v_{\mathrm{ice}}\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\mathcal{F}_0))} \\ &\le C_{23} T (R + C_T^*) \cdot \|z\|_{\mathbb{E}_1} \\ &\le C_{24} T (R + C_T^*)^2.\end{aligned}$$

The term $(\partial_k \partial_l Y_n) \partial_n v_{\mathrm{ice},j}$ can again be treated analogously. We finish the estimate of II by deducing from the shape of the coefficients in (3.15) and from $z(t) \in V$ for all $t \in [0,T]$ the boundedness of the coefficients, i. e.,

$$\|a_{ij}^{kl}(\varepsilon, P)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \le C_{25}. \tag{4.69}$$

A concatenation of the previous estimates implies

$$\|\mathrm{II}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_{26} T (R + C_0 + C_T^*)(R + C_T^*).$$

The next step is the treatment of the term I tracking the difference of the transformed and non-transformed coefficients. In view of the transformed coefficients from (4.25), we expand I into

$$\mathrm{III} := \sum_{j,k,l,m=1}^{2} \Big(a_{ij}^{klm}(\tilde{\varepsilon}, P) - a_{ij}^{klm}(\varepsilon, P)\Big)\partial_m \tilde{\varepsilon}_{jl} \text{ and}$$

$$\mathrm{IV} := \sum_{j,k,l,m=1}^{2} a_{ij}^{kl}(\varepsilon, P)(\partial_k Y_m - \delta_{km})\partial_m \tilde{\varepsilon}_{jl}.$$

For IV, we recall (4.23) for the shape of $\partial_m \tilde{\varepsilon}_{jl}$ and make use of Lemma A.1.1(a) to estimate the resulting term related to the transform, whereas the derivatives of v_{ice} can be estimated in view of (4.59). In total, we find that

$$\|\partial_m \tilde{\varepsilon}_{jl}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_{27} \tag{4.70}$$

for some constant $C_{27} > 0$. Joint with the above boundedness of the coefficients and Lemma 4.4.5 for the estimate of $\partial_k Y_m - \delta_{km}$, this leads to

$$\|\mathrm{IV}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_{28} T (R + C_0 + C_T^*).$$

Concerning III, we first observe that (4.70) yields an estimate of the last factor. On the other hand, $\partial_k Y_m$ also appears in both terms, and it can be estimated in $\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))$ by virtue of Lemma A.1.1(a). Hence, it remains to estimate the difference $a_{ij}^{kl}(\tilde{\varepsilon}, P) - a_{ij}^{kl}(\varepsilon, P)$. Let us observe that the difference only regards the symmetric part of the gradient, so the coefficients have the common factor

$$\frac{P(\widehat{h} + h_0^*, \widehat{a} + a_0^*)}{2\rho_{\mathrm{ice}}(\widehat{h} + h_0^*)}.$$

This term can be treated by a combination of (4.63) as well as an estimate of $P(\widehat{h} + h_0^*, \widehat{a} + a_0^*)$ relying on $z(t) \in V$ for all $t \in [0,T]$. Thus, it is crucial to handle the difference

$$\frac{1}{\triangle_\delta(\tilde{\varepsilon})}\left(\mathbb{S}_{ij}^{kl} - \frac{1}{\triangle_\delta^2(\tilde{\varepsilon})}(\mathbb{S}\tilde{\varepsilon})_{ik}(\mathbb{S}\tilde{\varepsilon})_{jl}\right) - \frac{1}{\triangle_\delta(\varepsilon)}\left(\mathbb{S}_{ij}^{kl} - \frac{1}{\triangle_\delta^2(\varepsilon)}(\mathbb{S}\varepsilon)_{ik}(\mathbb{S}\varepsilon)_{jl}\right).$$

At this stage, we invoke the smooth dependence on $\tilde{\varepsilon}$ and ε thanks to the regularization by $\delta > 0$, so an application of the mean value theorem yields

$$\begin{aligned}&\left\| \frac{1}{\triangle_\delta(\tilde{\varepsilon})}\left(\mathbb{S}_{ij}^{kl} - \frac{1}{\triangle_\delta^2(\tilde{\varepsilon})}(\mathbb{S}\tilde{\varepsilon})_{ik}(\mathbb{S}\tilde{\varepsilon})_{jl}\right)\right.\\ &\left.\quad - \frac{1}{\triangle_\delta(\varepsilon)}\left(\mathbb{S}_{ij}^{kl} - \frac{1}{\triangle_\delta^2(\varepsilon)}(\mathbb{S}\varepsilon)_{ik}(\mathbb{S}\varepsilon)_{jl}\right)\right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))}\\ &\le C_{29} \cdot \|\tilde{\varepsilon} - \varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))}.\end{aligned}$$

The difference of the transformed symmetric part of the gradient and the original one admits the representation

$$\tilde{\varepsilon}_{ij} - \varepsilon_{ij} = \frac{1}{2} \sum_{k=1}^{2} \Big((\partial_i Y_k - \delta_{ik}) \partial_k v_{\mathrm{ice},j} + (\partial_j Y_k - \delta_{jk}) \partial_k v_{\mathrm{ice},i} \Big).$$

Making use of the trace space embedding (4.49), Lemma 4.4.6(a) thanks to the homogeneous initial values of $\widehat{z}$ and (4.60), we obtain

$$\begin{aligned} \|\partial_k v_{\mathrm{ice}}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} &\le C_{30} \cdot \|v_{\mathrm{ice}}\|_{\mathrm{L}^\infty(0,T;\mathrm{C}^1(\overline{\mathcal{F}_0}))} \\ &\le C_{31} \cdot \Big(\|\widehat{z}\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} + \|z_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \Big) \\ &\le C_{32} \cdot (\|\widehat{z}\|_{\mathbb{E}_1} + C_0 + C_T^*) \\ &\le C_{32}(R + C_0 + C_T^*). \end{aligned}$$

Invoking Lemma 4.4.5 for the estimates of $\partial_i Y_k - \delta_{ik}$ and $\partial_j Y_k - \delta_{jk}$, we derive

$$\|\mathrm{III}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_{33} T (R + C_0 + C_T^*)^2.$$

In total, we get

$$\|\mathrm{I}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_{34} T (R + C_0 + C_T^*)(1 + R + C_0 + C_T^*).$$

Let us observe that the non-principal part of Hibler's operator can be treated in the same way. The last part to be handled in order to get the estimate of $G_1(\widehat{z})$ is

$$(\mathcal{B}(\widehat{z} + z_0^*) - B_1(\widehat{u} + u_0^*)) \begin{pmatrix} \widehat{h} + h_0^* \\ \widehat{a} + a_0^* \end{pmatrix},$$

and we concentrate on the h-part of this difference. It takes the shape

$$\frac{\partial_h P(\widehat{h} + h_0^*, \widehat{a} + a_0^*)}{2\rho_{\mathrm{ice}}(\widehat{h} + h_0^*)} \sum_{k=1}^{2} (\partial_j Y_k - \delta_{jk}) \partial_k (\widehat{h} + h_0^*).$$

The denominator $2\rho_{\mathrm{ice}}(\widehat{h} + h_0^*)$ in the fraction can be estimated by using (4.63), while the derivative of the ice strength reads as

$$\partial_h P(\widehat{h} + h_0^*, \widehat{a} + a_0^*) = p^* \mathrm{e}^{-c_\bullet (1 - (\widehat{a} + a_0^*))}.$$

By virtue of $z(t) \in V$ for all $t \in [0,T]$, we get in particular that the ice compactness satisfies $a(t) = \widehat{a}(t) + a_0^*(t) \in (0,1)$ on $[0,T]$, and the result is

$$\left\| \partial_h P(\widehat{h} + h_0^*, \widehat{a} + a_0^*) \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \le C_{35}.$$

In conjunction with Lemma 4.4.5 in order to estimate $\partial_j Y_k - \delta_{jk}$ and the embedding in Lemma 4.4.6(b), we conclude

$$\begin{aligned} &\left\| \frac{\partial_h P(\widehat{h} + h_0^*, \widehat{a} + a_0^*)}{2\rho_{\mathrm{ice}}(\widehat{h} + h_0^*)} \sum_{k=1}^{2} (\partial_j Y_k - \delta_{jk}) \partial_j (\widehat{h} + h_0^*) \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\ &\le C_{36} \cdot \| \partial_j Y_k - \delta_{jk} \|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \cdot \| \widehat{h} + h_0^* \|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\mathcal{F}_0))} \\ &\le C_{37} T (R + C_0 + C_T^*) \left(T^{1/p} \cdot \| \widehat{h} \|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\mathcal{F}_0))} + \| z_0^* \|_{\mathbb{E}_1} \right) \\ &\le C_{38} T (R + C_0 + C_T^*) \left(T^{1/p} \cdot \| \widehat{z} \|_{\mathbb{E}_1} + C_T^* \right) \\ &\le C_{38} T (R + C_0 + C_T^*) \left(T^{1/p} R + C_T^* \right). \end{aligned} \tag{4.71}$$

Let us observe that the term corresponding to the a-part can be estimated in the exact same way.

Putting together all the above, we conclude the first part of the assertion for some $C_{G_1}(R, T) > 0$. Moreover, we first choose $R > 0$ sufficiently small and then let $T \to 0$ to infer that indeed, $C_{G_1}(R, T) < R/10C_{\mathrm{MR}}$, where we also exploit that $C_T^* \to 0$ for $T \to 0$ by Remark 4.4.2(b) and the convergence of the difference of z_0^* and z_0 in $\mathrm{BUC}([0, T]; \mathrm{X}_\gamma)$ to zero as $T \to 0$ by Remark 4.4.2(c).

The second part of the proof is dedicated to the Lipschitz estimate of G_1. Proceeding as in (4.62), additionally observing that $\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}$ has homogeneous initial values, we deduce that

$$\| \mu (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) - c_{\mathrm{cor}} (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})^\perp \|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_{39} T \cdot \| \widehat{z}_1 - \widehat{z}_2 \|_{\mathbb{E}_1}$$

for some constant $C_{39} > 0$. The mean value theorem and $\widehat{h}_i(t) + h_0^*(t) > \kappa$, thanks to $z_i(t) \in V$ for all $t \in [0, T]$ and $i = 1, 2$, yield

$$\left\| \frac{1}{\rho_{\mathrm{ice}}(\widehat{h}_1 + h_0^*)} - \frac{1}{\rho_{\mathrm{ice}}(\widehat{h}_2 + h_0^*)} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \le C_{40} \cdot \| \widehat{z}_1 - \widehat{z}_2 \|_{\mathbb{E}_1}.$$

Using the same arguments as in (4.64) and (4.65), we derive the estimate

$$\begin{aligned} &\left\| \left(\frac{1}{\rho_{\mathrm{ice}}(\widehat{h}_1 + h_0^*)} - \frac{1}{\rho_{\mathrm{ice}}(\widehat{h}_2 + h_0^*)} \right) \left(\tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*) \right) \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\ &\le C_{41} T^{1/p} \left(1 + R^2 + C_0^2 + (C_T^*)^2 \right) \cdot \| \widehat{z}_1 - \widehat{z}_2 \|_{\mathbb{E}_1}. \end{aligned}$$

For the other term related to the oceanic forcing term, we first observe the boundedness of the inverse of $\rho_{\mathrm{ice}}(\widehat{h}_2 + h_0^*)$ as argued above. On the other hand, employing the notation $v_{\mathrm{ice},i} = \widehat{v}_{\mathrm{ice},i} + v^*_{\mathrm{ice},0}$ for simplicity, we calculate

$$\begin{aligned}&\tau_{\mathrm{ocn}}(v_{\mathrm{ice},1}) - \tau_{\mathrm{ocn}}(v_{\mathrm{ice},2})\\ &= \rho_{\mathrm{ocn}} C_{\mathrm{ocn}} |V_{\mathrm{ocn}} - v_{\mathrm{ice},1}| R_{\mathrm{ocn}}(-(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}))\\ &\quad + \rho_{\mathrm{ocn}} C_{\mathrm{ocn}} (|V_{\mathrm{ocn}} - v_{\mathrm{ice},1}| - |V_{\mathrm{ocn}} - v_{\mathrm{ice},2}|)(V_{\mathrm{ocn}} - v_{\mathrm{ice},2}).\end{aligned}$$

The first resulting addend can be handled as in (4.65), so we get

$$\begin{aligned}&\|\rho_{\mathrm{ocn}} C_{\mathrm{ocn}} |V_{\mathrm{ocn}} - \widehat{v}_{\mathrm{ice},1} - v^*_{\mathrm{ice},0}| R_{\mathrm{ocn}}(-(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}))\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))}\\ &\le C_{42} T^{1/p}(1 + R + C_0 + C^*_T) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.\end{aligned}$$

The second addend can be treated similarly, so in total,

$$\begin{aligned}&\left\| \frac{1}{\rho_{\mathrm{ice}}(\widehat{h}_2 + h^*_0)} \Big(\tau_{\mathrm{ocn}}(\widehat{v}_{\mathrm{ice},1} + v^*_{\mathrm{ice},0}) - \tau_{\mathrm{ocn}}(\widehat{v}_{\mathrm{ice},2} + v^*_{\mathrm{ice},0})\Big)\right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))}\\ &\le C_{43} T^{1/p}(1 + R + C_0 + C^*_T) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.\end{aligned}$$

As in the first part of the proof, we next address the terms related to the coordinate transform. Mimicking the arguments from (4.66), and invoking again the homogeneous initial values of $\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}$, we find the estimate

$$\|\nabla_{\mathrm{H}}(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) \partial_t Y_1\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_{44} T^{1/p} \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}. \tag{4.72}$$

With regard to the other term related to the transformed time derivative, we also use a similar strategy as in (4.66), this time exploiting Lemma A.1.1(a) for the estimate of the $\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))$-norm of $\partial_t(Y_1 - Y_2)$ to obtain

$$\begin{aligned}&\|\nabla_{\mathrm{H}}(\widehat{v}_{\mathrm{ice},2} + v^*_{\mathrm{ice},0}) \partial_t (Y_1 - Y_2)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))}\\ &\le C_{45}(T^{1/p} R + C^*_T) T \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.\end{aligned} \tag{4.73}$$

The next task is to estimate the difference in the transformed term $\mathcal{N}$. First, we expand this term further so that it is easier to handle. More precisely, employing again $v_{\mathrm{ice},i} = \widehat{v}_{\mathrm{ice},i} + v^*_{\mathrm{ice},0}$ to simplify notation, we write

$$\begin{aligned}\mathcal{N}(v_{\mathrm{ice},1}) - \mathcal{N}(v_{\mathrm{ice},2}) = \sum_{j,k=1}^2 \Big(&(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})_j (\partial_j (Y_1)_k) \partial_k (v_{\mathrm{ice},1})_j\\ &+ (v_{\mathrm{ice},2})_j (\partial_j (Y_1 - Y_2)_k) \partial_k (v_{\mathrm{ice},1})_j\\ &+ (v_{\mathrm{ice},2})_j (\partial_j (Y_2)_k) \partial_k (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})_j\Big).\end{aligned}$$

A similar strategy as in (4.67), also based on Lemma A.1.1(a) in order to estimate the difference in Y in the middle term, yields

$$\begin{aligned}
&\|\mathcal{N}(\widehat{v}_{\mathrm{ice},1} + v^*_{\mathrm{ice},0}) - \mathcal{N}(\widehat{v}_{\mathrm{ice},2} + v^*_{\mathrm{ice},0})\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\
&\le C_{46}\Big((1 + T(R + C_0 + C^*_T))(T^{1/p}R + C^*_T) + T^{1/p}(R + C_0 + C^*_T)\Big) \\
&\cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.
\end{aligned} \tag{4.74}$$

It remains to estimate the terms related to the Hibler operator and the ice strength. In fact, we start with the difference

$$\begin{aligned}
&\left(\mathcal{A}^{\mathrm{H}}(\widehat{z}_2 + z^*_0) - \mathbb{A}^{\mathrm{H}}(u_0)\right)(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) \\
&- (\mathcal{B}(\widehat{z}_2 + z^*_0) - B_1(u_0))\begin{pmatrix}\widehat{h}_1 - \widehat{h}_2 \\ \widehat{a}_1 - \widehat{a}_2\end{pmatrix}.
\end{aligned} \tag{4.75}$$

As in the first part of the proof, we insert two intermediate term. In fact, Lemma 4.4.7 reveals that

$$\begin{aligned}
&\Bigg\|\left(\mathbb{A}^{\mathrm{H}}(\widehat{u}_2 + u^*_0) - \mathbb{A}^{\mathrm{H}}(u^*_0)\right)(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) \\
&\quad - (B_1(\widehat{u}_2 + u^*_0) - B_1(u^*_0))\begin{pmatrix}\widehat{h}_1 - \widehat{h}_2 \\ \widehat{a}_1 - \widehat{a}_2\end{pmatrix}\Bigg\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_{47}R \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}
\end{aligned}$$

as well as

$$\begin{aligned}
&\Bigg\|\left(\mathbb{A}^{\mathrm{H}}(u^*_0) - \mathbb{A}^{\mathrm{H}}(u_0)\right)(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) - (B^*_1 - B_1(u_0))\begin{pmatrix}\widehat{h}_1 - \widehat{h}_2 \\ \widehat{a}_1 - \widehat{a}_2\end{pmatrix}\Bigg\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\
&\le C_{48} \cdot \|z^*_0 - z_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.
\end{aligned}$$

Concerning the resulting terms in (4.75), similar arguments as for the estimate of (4.68) and (4.71) especially lead to

$$\begin{aligned}
&\left\|\left(\mathcal{A}^{\mathrm{H}}(\widehat{z}_2 + z^*_0) - \mathbb{A}^{\mathrm{H}}(\widehat{u}_2 + u^*_0)\right)(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})\right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\
&\le C_{49}T \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}
\end{aligned}$$

and

$$\left\|(\mathcal{B}(\widehat{z}_2 + z^*_0) - B_1(u^*_0))\begin{pmatrix}\widehat{h}_1 - \widehat{h}_2 \\ \widehat{a}_1 - \widehat{a}_2\end{pmatrix}\right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_{50}T \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.$$

The rest of the proof focuses on the Lipschitz continuity of

$$\begin{aligned}
&\left(\mathcal{A}^{\mathrm{H}}(\widehat{z}_1 + z^*_0) - \mathcal{A}^{\mathrm{H}}(\widehat{z}_2 + z^*_0)\right)(\widehat{v}_{\mathrm{ice},1} + v^*_{\mathrm{ice},0}) \\
&- (\mathcal{B}(\widehat{z}_1 + z^*_0) - \mathcal{B}(\widehat{z}_2 + z^*_0))\begin{pmatrix}\widehat{h}_1 + h^*_0 \\ \widehat{a}_1 + a^*_0\end{pmatrix}.
\end{aligned}$$

For this, we need to control the difference of the transformed terms in the arguments. The first step is to estimate the transformed Hibler operator. Moreover, we concentrate on the principal part and remark that the remaining part can be estimated in a similar way. In view of (4.26), this means that we need to estimate

$$\sum_{j,k,l,m=1}^{2} \Big(a_{ij}^{klm}\Big(\tilde{\varepsilon}(\widehat{v}_{\text{ice},1} + v_{\text{ice},0}^*), P(\widehat{h}_1 + h_0^*, \widehat{a}_1 + a_0^*)\Big) \\ - a_{ij}^{klm}\Big(\tilde{\varepsilon}(\widehat{v}_{\text{ice},2} + v_{\text{ice},0}^*), P(\widehat{h}_2 + h_0^*, \widehat{a}_2 + a_0^*)\Big)\Big)\partial_m \tilde{\varepsilon}_{jl}(\widehat{v}_{\text{ice},1} + v_{\text{ice},0}^*).$$

Compared to the above considerations, we require better estimates of the term $\partial_m \tilde{\varepsilon}_{jl}(\widehat{v}_{\text{ice},1} + v_{\text{ice},0}^*)$. In that respect, for $v_{\text{ice},1} = \widehat{v}_{\text{ice},1} + v_{\text{ice},0}^*$, we recall

$$\partial_m \tilde{\varepsilon}_{jl}(v_{\text{ice},1}) = \frac{1}{2}\sum_{n=1}^{2}\Big((\partial_m \partial_j (Y_1)_n)\partial_n (v_{\text{ice},1})_l + (\partial_j (Y_1)_n)\partial_m \partial_n (v_{\text{ice},1})_l \\ + (\partial_m \partial_l (Y_1)_n)\partial_n (v_{\text{ice},1})_j + (\partial_l (Y_1)_n)\partial_m \partial_n (v_{\text{ice},1})_j\Big).$$

We invoke Lemma 1.3.1, the trace space embedding (4.49), Lemma A.1.1(a) for the estimate of the second derivatives of Y, Lemma 4.4.6(a) and the estimate of $\|z_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}$ from (4.60) to get

$$\begin{aligned} &\|(\partial_m \partial_j (Y_1)_n)\partial_n (v_{\text{ice},1})_l\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\ &\le \|\partial_m \partial_j (Y_1)_n\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \cdot \|\partial_n (v_{\text{ice},1})_l\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \\ &\le C_{51} T^{1/p} \cdot \|\partial_m \partial_j (Y_1)_n\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \cdot \Big(\|\widehat{z}_1\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} + \|z_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}\Big) \\ &\le C_{52} T^{1+1/p}(R + C_T^*) \cdot (\|\widehat{z}_1\|_{\mathbb{E}_1} + C_0 + C_T^*) \\ &\le C_{53} T^{1+1/p}(R + C_T^*)(R + C_0 + C_T^*). \end{aligned}$$

At the same time, employing the boundedness of the gradient of Y_1 as stated in Lemma A.1.1(a) and (4.59), we find the estimate

$$\begin{aligned} &\|(\partial_j (Y_1)_n)\partial_m \partial_n (v_{\text{ice},1})_l\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\ &\le C_{54} \cdot \|\partial_j (Y_1)_n\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \cdot \|v_{\text{ice},1}\|_{\mathrm{L}^p(0,T;\mathrm{W}^{2,q}(\mathcal{F}_0))} \\ &\le C_{55} \cdot \|z_1\|_{\mathbb{E}_1} \le C_{55}(R + C_T^*). \end{aligned}$$

A concatenation of the preceding estimates shows that $\tilde{\varepsilon} = \tilde{\varepsilon}(v_{\text{ice},1})$ satisfies

$$\|\partial_m \tilde{\varepsilon}_{jl}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_{56}\left((R + C_T^*)(1 + T^{1+1/p}(R + C_0 + C_T^*))\right). \tag{4.76}$$

For $n = 1, 2$, we employ the notation $\tilde{\varepsilon}_n$ and P_n instead of $\tilde{\varepsilon}(\widehat{v}_{\mathrm{ice},n} + v^*_{\mathrm{ice},0})$ and $P(\widehat{h}_n + h^*_0, \widehat{a}_n + a^*_0)$ for simplicity. Moreover, we recall from (4.25) that the transformed coefficients are of the form

$$a^{klm}_{ij}(\tilde{\varepsilon}_n, P_n) = a^{kl}_{ij}(\tilde{\varepsilon}_n, P_n)(\partial_k(Y_n)_m).$$

Hence, when considering $a^{klm}_{ij}(\tilde{\varepsilon}_1, P_1) - a^{klm}_{ij}(\tilde{\varepsilon}_2, P_2)$, we observe that the difference is either in the original coefficients or in the diffeomorphism. With regard to (3.28), in the first case, we get the estimate

$$\begin{aligned} &\|a^{kl}_{ij}(\tilde{\varepsilon}_1, P_1) - a^{kl}_{ij}(\tilde{\varepsilon}_2, P_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \\ &\le C_{57} \cdot \|(\tilde{\varepsilon}_1, h_1, a_1) - (\tilde{\varepsilon}_2, h_2, a_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))}. \end{aligned}$$

The estimate of the resulting difference in h and a is straightforward, so we only elaborate on the estimate of the difference in the transformed symmetric part of the gradient. To this end, we compute

$$\begin{aligned} &(\tilde{\varepsilon}_1)_{ij} - (\tilde{\varepsilon}_2)_{ij} \\ &= \frac{1}{2}\sum_{k=1}^{2}\Big((\partial_i(Y_1)_k - \partial_i(Y_2)_k)\partial_k(\widehat{v}_{\mathrm{ice},1} + v^*_{\mathrm{ice},0})_j + (\partial_i(Y_2)_k)\partial_k(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})_j \\ &\quad + (\partial_j(Y_1)_k - \partial_j(Y_2)_k)\partial_k(\widehat{v}_{\mathrm{ice},1} + v^*_{\mathrm{ice},0})_i + (\partial_j(Y_2)_k)\partial_k(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})_i\Big). \end{aligned}$$

In order to estimate the resulting terms with the difference in the diffeomorphism, we exploit Lemma A.1.1(a) to get an estimate by the difference of the rigid body velocities. More precisely, we also use the embedding of the trace space from (4.49), the embedding from Lemma 4.4.6(a) and (4.60) to infer

$$\begin{aligned} &\|(\partial_i(Y_1)_k - \partial_i(Y_2)_k)\partial_k(\widehat{v}_{\mathrm{ice},1} + v^*_{\mathrm{ice},0})_j\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \\ &\le C_{58}T \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1} \cdot \|\widehat{v}_{\mathrm{ice},1} + v^*_{\mathrm{ice},0}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\mathcal{F}_0))} \\ &\le C_{59}T \cdot \Big(\|\widehat{z}_1\|_{\mathbb{E}_1} + \|z^*_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}\Big) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1} \\ &\le C_{60}T(R + C_0 + C^*_T) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}. \end{aligned}$$

Concerning the terms with the difference in the sea ice velocity, we conclude from Lemma A.1.1(a) the boundedness of the gradient of Y, so

$$\|(\partial_i(Y_2)_k)\partial_k(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})_j\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \le C_{61} \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.$$

In total, we get the estimate

$$\|\tilde{\varepsilon}_1 - \tilde{\varepsilon}_2\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \le C_{62}\left(T(R + C_0 + C^*_T) + 1\right) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}. \tag{4.77}$$

With regard to the above estimate of the difference of the coefficients, we derive that

$$\begin{aligned}&\|a_{ij}^{kl}(\tilde{\varepsilon}_1, P_1) - a_{ij}^{kl}(\tilde{\varepsilon}_2, P_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))}\\ &\le C_{63}\left(T(R + C_0 + C_T^*) + 1\right) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.\end{aligned}$$

Putting together the last estimate, the estimate of $\partial_m \tilde{\varepsilon}_{jl}$ from (4.76) and the boundedness of $\partial_k (Y_n)_m$ by virtue of Lemma A.1.1(a), we find that the difference term

$$(a_{ij}^{kl}(\tilde{\varepsilon}_1, P_1) - a_{ij}^{kl}(\tilde{\varepsilon}_2, P_2))(\partial_k (Y_1)_m)\partial_m \tilde{\varepsilon}_1$$

admits an estimate of the desired shape, i. e., it is Lipschitz continuous in $\widehat{z}$ with Lipschitz constant shrinking to zero as $R \to 0$ and $T \to 0$. On the other hand, because of

$$\|\partial_k (Y_1)_m - \partial_k (Y_2)_m\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \le C_{64} T \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1},$$

following from Lemma A.1.1(a) again, and invoking the boundedness of the coefficients as revealed in (4.69), we deduce a suitable estimate of the term

$$a_{ij}^{kl}(\tilde{\varepsilon}_2, P_2)\left(\partial_k (Y_1)_m - \partial_k (Y_2)_m\right)\partial_m \tilde{\varepsilon}_1.$$

This completes the treatment of the term

$$\Big(\mathcal{A}^{\mathrm{H}}(\widehat{z}_1 + z_0^*) - \mathcal{A}^{\mathrm{H}}(\widehat{z}_2 + z_0^*)\Big)\left(\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*\right)$$

upon observing that the non-principal part can be handled analogously.

The last term to be estimated is thus

$$(\mathcal{B}(\widehat{z}_1 + z_0^*) - \mathcal{B}(\widehat{z}_2 + z_0^*))\begin{pmatrix}\widehat{h}_1 + h_0^*\\ \widehat{a}_1 + a_0^*\end{pmatrix}.$$

In this context, we focus on the estimate of the h-part and stress that the a-part again allows a similar handling. As a first step, we provide a more advantageous representation, namely

$$\begin{aligned}&\Big((\mathcal{B}_{\mathrm{h}}(\widehat{z}_1 + z_0^*) - \mathcal{B}_{\mathrm{h}}(\widehat{z}_2 + z_0^*))\,(\widehat{h}_1 + h_0^*)\Big)_i\\ &= \left(\frac{\partial_h P_1}{2\rho_{\mathrm{ice}}(\widehat{h}_1 + h_0^*)} - \frac{\partial_h P_2}{2\rho_{\mathrm{ice}}(\widehat{h}_2 + h_0^*)}\right)\sum_{j=1}^2 (\partial_i (Y_1)_j)\partial_j(\widehat{h}_1 + h_0^*)\\ &\quad + \frac{\partial_h P_2}{2\rho_{\mathrm{ice}}(\widehat{h}_2 + h_0^*)}\sum_{j=1}^2 \left(\partial_i (Y_1)_j - \partial_i (Y_2)_j\right)\partial_j(\widehat{h}_1 + h_0^*).\end{aligned}$$

For the estimate of the second addend, we mimic the procedure from (4.71), where Lemma 4.4.5 has to be replaced by Lemma A.1.1(a), to estimate the difference of the diffeomorphisms, and we then get

$$\begin{aligned}&\left\| \frac{\partial_h P_2}{2\rho_{\mathrm{ice}}(\widehat{h}_2 + h_0^*)} \sum_{j=1}^{2} \left(\partial_i (Y_1)_j - \partial_i (Y_2)_j\right) \partial_j (\widehat{h}_1 + h_0^*) \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\ &\le C_{65} T (T^{1/p} R + C_T^*) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.\end{aligned}$$

With regard to the first addend in the above representation, a similar strategy can be used. Indeed, using the smoothness of the coefficients in h and a in conjunction with the mean value theorem in order to estimate the first factor, we end up with

$$\begin{aligned}&\left\| \left(\frac{\partial_h P_1}{2\rho_{\mathrm{ice}}(\widehat{h}_1 + h_0^*)} - \frac{\partial_h P_2}{2\rho_{\mathrm{ice}}(\widehat{h}_2 + h_0^*)} \right) \sum_{j=1}^{2} (\partial_i (Y_1)_j) \partial_j (\widehat{h}_1 + h_0^*) \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\ &\le C_{66} (T^{1/p} R + C_T^*) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.\end{aligned}$$

In summary, we have also verified the Lipschitz continuity of G_1 with shrinking Lipschitz constant $L_{G_1}(R, T)$. □

The next terms under consideration are G_2 and G_3 from (4.55). As for G_1, we provide the resulting differences as a preparation. In fact, employing the notation $z_i = \widehat{z}_i + z_0^*$, we have

$$\begin{aligned} G_2(\widehat{z}_1) - G_2(\widehat{z}_2) &= d_{\mathrm{h}} (\mathcal{L}_1 - \mathcal{L}_2) h_1 + d_{\mathrm{h}} (\mathcal{L}_2 - \Delta_{\mathrm{H}})(\widehat{h}_1 - \widehat{h}_2) + \mu (\widehat{h}_1 - \widehat{h}_2) \\ &\quad - \nabla_{\mathrm{H}} (\widehat{h}_1 - \widehat{h}_2) \cdot \partial_t Y_1 - \nabla_{\mathrm{H}} h_2 \cdot \partial_t (Y_1 - Y_2) - (\mathcal{M}(v_{\mathrm{ice},1}, h_1) \\ &\quad - \mathcal{M}(v_{\mathrm{ice},2}, h_2)) + S_{\mathrm{h}}(h_1, a_1) - S_{\mathrm{h}}(h_2, a_2) \end{aligned}$$

and

$$\begin{aligned} G_3(\widehat{z}_1) - G_3(\widehat{z}_2) &= d_{\mathrm{a}} (\mathcal{L}_1 - \mathcal{L}_2) a_1 + d_{\mathrm{a}} (\mathcal{L}_2 - \Delta_{\mathrm{H}})(\widehat{a}_1 - \widehat{a}_2) + \mu (\widehat{a}_1 - \widehat{a}_2) \\ &\quad - \nabla_{\mathrm{H}} (\widehat{a}_1 - \widehat{a}_2) \cdot \partial_t Y_1 - \nabla_{\mathrm{H}} a_2 \cdot \partial_t (Y_1 - Y_2) - \mathcal{M}(v_{\mathrm{ice},1}, a_1) \\ &\quad + \mathcal{M}(v_{\mathrm{ice},2}, a_2) + S_{\mathrm{a}}(h_1, a_1) - S_{\mathrm{a}}(h_2, a_2). \end{aligned}$$

Lemma 4.4.10. *Consider $p, q \in (1, \infty)$ such that (4.48), and let $z = \widehat{z} + z_0^*$, $z_1 = \widehat{z}_1 + z_0^*$ and $z_2 = \widehat{z}_2 + z_0^*$, where $\widehat{z}$, $\widehat{z}_1$, $\widehat{z}_2 \in \mathcal{K}_T^R$, while z_0^* denotes the reference solution from Proposition 4.4.1. In addition, suppose that f_{gr} satisfies Assumption 4.4.8(c), and recall the T-independent maximal regularity constant $C_{\mathrm{MR}} > 0$ from Proposition 4.3.3.*

Then there are constants $C_{G_2}(R,T), L_{G_2}(R,T), C_{G_3}(R,T), L_{G_3}(R,T) > 0$ *with* $C_{G_2}(R,T), C_{G_3}(R,T) < R/10C_{\mathrm{MR}}$ *for* $R > 0$ *and* $T > 0$ *sufficiently small and* $L_{G_2}(R,T)$, $L_{G_3}(R,T) \to 0$ *as* $R \to 0$ *and* $T \to 0$, *and such that*

$$\begin{aligned} \|G_2(\widehat{z})\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} &\le C_{G_2}(R,T), \\ \|G_3(\widehat{z})\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} &\le C_{G_3}(R,T), \\ \|G_2(\widehat{z}_1) - G_2(\widehat{z}_2)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} &\le L_{G_2}(R,T) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1} \quad \textit{and} \\ \|G_3(\widehat{z}_1) - G_3(\widehat{z}_2)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} &\le L_{G_3}(R,T) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}. \end{aligned}$$

Proof. The tools are the same as explained at the beginning of the proof of Lemma 4.4.9. Concerning the estimates, we start with the simple terms. As $\mu > 0$ is fixed, we first deduce from Lemma 1.3.1 and Lemma 4.4.6 that

$$\begin{aligned} \|\mu(\widehat{h} + h_0^*)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} &\le C_1 \cdot \left(\|\widehat{h}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} + \|h_0^*\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))}\right) \\ &\le C_2 \cdot \left(T^{1/p} \cdot \|\widehat{h}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^q(\mathcal{F}_0))} + \|z_0^*\|_{\mathbb{E}_1}\right) \\ &\le C_3 \cdot \left(T^{1/p} \cdot \|\widehat{z}\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} + C_T^*\right) \\ &\le C_4(T^{1/p} R + C_T^*). \end{aligned} \tag{4.78}$$

Next, the same arguments as in (4.66) reveal that

$$\|\nabla_{\mathrm{H}}(\widehat{h} + h_0^*) \cdot \partial_t Y\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_5 \left(T^{1/p} R + C_T^*\right).$$

Likewise, mimicking the procedure to obtain the estimate (4.67), and using that $\mathcal{M}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*, \widehat{h} + h_0^*)$ has a similar structure as $\mathcal{N}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)$, we infer

$$\|\mathcal{M}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*, \widehat{h} + h_0^*)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_6 \left(R + C_0 + C_T^*\right) \left(T^{1/p} R + C_T^*\right).$$

With regard to the thermodynamic term S_{h}, we employ Hölder's inequality, Assumption 4.4.8(c), Lemma 1.3.1 as well as Lemma 4.4.6 to deduce that

$$\begin{aligned} \|S_{\mathrm{h}}(\tilde{h}, \tilde{a})\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} &\le \left\| f_{\mathrm{gr}} \begin{pmatrix} \tilde{h} \\ \tilde{a} \end{pmatrix} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \cdot \|\widehat{a} + a_0^*\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\ &\quad + \|f_{\mathrm{gr}}(0)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} + \|f_{\mathrm{gr}}(0)(\widehat{a} + a_0^*)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\ &\le C_7 \cdot \left(T^{1/p} + \|\widehat{a}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} + \|a_0^*\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))}\right) \\ &\le C_8 \cdot \left(T^{1/p}(1 + \|\widehat{a}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^q(\mathcal{F}_0))}) + \|z_0^*\|_{\mathbb{E}_1}\right) \\ &\le C_9 \cdot \left(T^{1/p}(1 + \|\widehat{z}\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}) + C_T^*\right) \\ &\le C_{10}(T^{1/p}(1 + R) + C_T^*). \end{aligned}$$

Hence, the last part of G_2 to be estimated is $d_{\mathrm{h}}(\mathcal{L} - \Delta_{\mathrm{H}})(\widehat{h} + h_0^*)$ resulting from the transformed Laplacian. For this purpose, we recall the shape of $\mathcal{L}$ from (4.22) and employ the notation $h = \widehat{h} + h_0^*$ to get

$$(\mathcal{L} - \Delta_{\mathrm{H}})h = \sum_{j,k=1}^{2} (g^{jk} - \delta_{jk})\partial_k \partial_j h + \sum_{j=1}^{2} (\Delta_{\mathrm{H}} Y_j)\partial_j h.$$

Making use of this representation, and invoking Lemma A.1.1(a) for the estimate of $\Delta_{\mathrm{H}} Y_j$, Lemma 4.4.5 to estimate $g^{jk} - \delta_{jk}$ and (4.59), we derive

$$\begin{aligned}
&\|d_{\mathrm{h}}(\mathcal{L} - \Delta_{\mathrm{H}})(\widehat{h} + h_0^*)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\
&\le C_{11} \cdot \Big(\|g^{jk} - \delta_{jk}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \cdot \|\partial_k \partial_j h\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\
&\quad + \|\Delta_{\mathrm{H}} Y_j\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \cdot \|\partial_k h\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \Big) \\
&\le C_{12} T (R + C_0 + C_T^*) \cdot \|z\|_{\mathbb{E}_1} \\
&\le C_{13} T (R + C_0 + C_T^*)(R + C_T^*).
\end{aligned} \tag{4.79}$$

Upon recalling $C_T^* \to 0$ from Remark 4.4.2(b), we deduce the estimate of G_2 for some $C_{G_2}(R,T) > 0$ shrinking to zero as $R \to 0$ and $T \to 0$. The term G_3 can be estimated in a similar fashion.

For the Lipschitz estimates, we start again with the easiest terms. Similarly as in (4.78), also exploiting the homogeneous initial values of $\widehat{h}_1 - \widehat{h}_2$, we get

$$\|\mu(\widehat{h}_1 - \widehat{h}_2)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_{14} T^{1/p} \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.$$

Precisely as in (4.72) and (4.73) in the proof of Lemma 4.4.9, we deduce that

$$\begin{aligned}
&\|\nabla_{\mathrm{H}}(\widehat{h}_1 - \widehat{h}_2) \cdot \partial_t Y_1 + \nabla_{\mathrm{H}}(\widehat{h}_2 + h_0^*) \cdot \partial_t (Y_1 - Y_2)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\
&\le C_{15} \Big(T^{1/p} + (T^{1/p} R + C_T^*) T \Big) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.
\end{aligned}$$

Likewise, making use of the similar shape of $\mathcal{M}$ compared to $\mathcal{N}$, we can proceed as in (4.74) to establish

$$\begin{aligned}
&\|\mathcal{M}(\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*, \widehat{h}_1 + h_0^*) - \mathcal{M}(\widehat{v}_{\mathrm{ice},2} + v_{\mathrm{ice},0}^*, \widehat{h}_2 + h_0^*)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\
&\le C_{16} \Big((1 + T(R + C_0 + C_T^*))(T^{1/p} R + C_T^*) + T^{1/p}(R + C_0 + C_T^*) \Big) \\
&\quad \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.
\end{aligned}$$

For the estimate of the difference in the thermodynamic source term, we write

$$\begin{aligned}
&S_{\mathrm{h}}(\widehat{h}_1 + h_0^*, \widehat{a}_1 + a_0^*) - S_{\mathrm{h}}(\widehat{h}_2 + h_0^*, \widehat{a}_2 + a_0^*) \\
&= f_{\mathrm{gr}}\left(\frac{\widehat{h}_1 + h_0^*}{\widehat{a}_1 + a_0^*} \right)(\widehat{a}_1 + a_0^*) - f_{\mathrm{gr}}\left(\frac{\widehat{h}_2 + h_0^*}{\widehat{a}_2 + a_0^*} \right)(\widehat{a}_2 + a_0^*) - (\widehat{a}_1 - \widehat{a}_2) f_{\mathrm{gr}}(0).
\end{aligned}$$

Then the term $(\widehat{a}_1 - \widehat{a}_2) f_{\mathrm{gr}}(0)$ can be handled exactly as the term $\mu(\widehat{h}_1 - \widehat{h}_2)$ above. On the other hand, the remaining term can be expanded suitably in a similar manner as in the proof of Theorem 3.5.2, and it follows therefrom that the difference in the thermodynamic term can be estimated by

$$C_{17}(T^{1/p} + R + C_T^*) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.$$

For the Lipschitz estimates of G_2, it remains to deal with the objects associated to the transformed Laplacian. First, we derive the representation

$$\begin{aligned} &(\mathcal{L}_1 - \mathcal{L}_2)(\widehat{h}_1 + h_0^*) \\ &= \sum_{j,k=1}^{2} \left((g_1)^{jk} - (g_2)^{jk}\right) \partial_k \partial_j (\widehat{h}_1 + h_0^*) + \sum_{j=1}^{2} (\Delta_{\mathrm{H}}(Y_1 - Y_2)_j) \partial_j (\widehat{h}_1 + h_0^*). \end{aligned}$$

From Hölder's inequality joint with Lemma A.1.2 and Lemma A.1.1(a) to estimate the terms related to the coordinate transform and (4.59), we deduce

$$\begin{aligned} &\|d_{\mathrm{h}}(\mathcal{L}_1 - \mathcal{L}_2)(\widehat{h}_1 + h_0^*)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \\ &\le C_{18} \cdot \Big(\|(g_1)^{jk} - (g_2)^{jk}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \cdot \|\widehat{h}_1 + h_0^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{2,q}(\mathcal{F}_0))} \\ &\quad + \|\Delta_{\mathrm{H}}(Y_1 - Y_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \cdot \|\widehat{h}_1 + h_0^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\mathcal{F}_0))} \Big) \\ &\le C_{19} T \cdot \|z_1\|_{\mathbb{E}_1} \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1} \\ &\le C_{20} T (R + C_T^*) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}. \end{aligned}$$

Next, arguing as in (4.79), we find that

$$\|d_{\mathrm{h}}(\mathcal{L}_2 - \Delta_{\mathrm{H}})(\widehat{h}_1 - \widehat{h}_2)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathcal{F}_0))} \le C_{21} T (R + C_0 + C_T^*) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.$$

We observe again that the terms in G_3 can be handled likewise, completing the proof of the Lipschitz property. □

The last terms to be addressed are G_4 and G_5 from (4.56) corresponding to the rigid body equations. With regard to the desired Lipschitz estimates, we elaborate on the shape of the differences. They are given by

$$\begin{aligned} G_4(\widehat{z}_1) - G_4(\widehat{z}_2) &= m_{\mathcal{S}}(\widehat{\omega}_1 - \widehat{\omega}_2)(\widehat{\ell}_1 + \ell_0^*)^\perp + m_{\mathcal{S}}(\widehat{\omega}_2 + \widehat{\omega}_0^*)(\widehat{\ell}_1 - \widehat{\ell}_2)^\perp \\ &\quad + Q_1 \int_{\partial \mathcal{S}_0} \mathcal{T}_\delta(\widehat{u}_1 + u_0^*) \nu \,\mathrm{d}\Gamma - Q_2 \int_{\partial \mathcal{S}_0} \mathcal{T}_\delta(\widehat{u}_2 + u_0^*) \nu \,\mathrm{d}\Gamma \end{aligned}$$

as well as

$$G_5(\widehat{z}_1) - G_5(\widehat{z}_2) = Q_1 \int_{\partial \mathcal{S}_0} y_{\mathrm{H}}^\perp \mathcal{T}_\delta(\widehat{u}_1 + u_0^*) \nu \,\mathrm{d}\Gamma - Q_2 \int_{\partial \mathcal{S}_0} y_{\mathrm{H}}^\perp \mathcal{T}_\delta(\widehat{u}_2 + u_0^*) \nu \,\mathrm{d}\Gamma.$$

The lemma on the self map and Lipschitz estimate of the terms G_4 and G_5 reads as follows.

Lemma 4.4.11. *Let $p, q \in (1, \infty)$ such that (4.48), and consider $z = \widehat{z} + z_0^*$, $z_1 = \widehat{z}_1 + z_0^*$ and $z_2 = \widehat{z}_2 + z_0^*$, with $\widehat{z}$, $\widehat{z}_1$, $\widehat{z}_2 \in \mathcal{K}_T^R$ and the reference solution z_0^* from Proposition 4.4.1. Furthermore, assume $F \in \mathrm{L}^p(0, T_0)^2$ as well as $N \in \mathrm{L}^p(0, T_0)$ for $T_0 > 0$ from Remark 4.4.2, and recall the T-independent maximal regularity constant $C_{\mathrm{MR}} > 0$ from Proposition 4.3.3.*

Then there are constants $C_{G_4}(R, T), L_{G_4}(R, T), C_{G_5}(R, T), L_{G_5}(R, T) > 0$ with $C_{G_4}(R, T), C_{G_5}(R, T) < R/10C_{\mathrm{MR}}$ for $R > 0$ and $T > 0$ sufficiently small and $L_{G_4}(R, T)$, $L_{G_5}(R, T) \to 0$ as $R \to 0$ and $T \to 0$, and such that

$$\begin{aligned} \|G_4(\widehat{z})\|_{\mathrm{L}^p(0,T)} &\le C_{G_4}(R, T), \\ \|G_5(\widehat{z})\|_{\mathrm{L}^p(0,T)} &\le C_{G_5}(R, T), \\ \|G_4(\widehat{z}_1) - G_4(\widehat{z}_2)\|_{\mathrm{L}^p(0,T)} &\le L_{G_4}(R, T) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1} \quad \textit{and} \\ \|G_5(\widehat{z}_1) - G_5(\widehat{z}_2)\|_{\mathrm{L}^p(0,T)} &\le L_{G_5}(R, T) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}. \end{aligned}$$

Proof. Again, we use many of the tools as described at the beginning of the proof of Lemma 4.4.9. Concerning the estimates, we first observe that the external forcing terms F and N can be estimated thanks to the assumption, since the norms shrink to zero as $T \to 0$. Moreover, Hölder's inequality, the embedding from Lemma 4.4.6(a) and (4.60) yield

$$\begin{aligned} &\|(\widehat{\omega} + \omega_0^*)(\widehat{\ell} + \ell_0^*)^\perp\|_{\mathrm{L}^p(0,T)} \\ &\le \|\widehat{\omega} + \omega_0^*\|_{\mathrm{L}^p(0,T)} \cdot \|\widehat{\ell} + \ell_0^*\|_{\mathrm{L}^\infty(0,T)} \\ &\le C_1 T^{1/p} \cdot \left(\|\widehat{\omega}\|_{\mathrm{L}^\infty(0,T)} + \|\omega_0^*\|_{\mathrm{L}^\infty(0,T)}\right) \cdot \left(\|\widehat{\ell}\|_{\mathrm{L}^\infty(0,T)} + \|\ell_0^*\|_{\mathrm{L}^\infty(0,T)}\right) \\ &\le C_2 T^{1/p} \cdot \left(\|\widehat{z}\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} + \|z_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}\right)^2 \\ &\le C_3 T^{1/p} \cdot (\|\widehat{z}\|_{\mathbb{E}_1} + C_T^* + C_0)^2 \\ &\le C_3 T^{1/p} (R + C_T^* + C_0)^2. \end{aligned} \tag{4.80}$$

With regard to the surface integrals, we first recall the more compact representation of $\sigma_\delta(u)$ from Section 3.3. In fact, for $\mathbb{S}\tilde{\varepsilon}$ as introduced in (3.11) and $\triangle_\delta(\tilde{\varepsilon})$ as made precise in (3.4), the stress tensor takes the shape

$$\sigma_\delta(u) = \frac{P}{2} \left(\frac{\mathbb{S}\tilde{\varepsilon}}{\triangle_\delta(\tilde{\varepsilon})} - \mathrm{Id}_2 \right). \tag{4.81}$$

Thanks to $z(t) \in V$ for all $t \in [0, T]$ and the embeddings of the trace space from (4.49), for u with $z = (u, \ell, \omega) = (v_{\mathrm{ice}}, h, a, \ell, \omega)$, we have $u(t) \in \mathrm{C}^1(\overline{\mathcal{F}_0})^4$

for all $t \in [0,T]$. Besides, it also holds that $a(t) \in (0,1)$ for all $t \in [0,T]$ by definition of V. Consequently, for all $t \in [0,T]$, we get

$$\left\| \frac{P(h(t),a(t))}{2} \right\|_{\mathrm{L}^\infty(\partial\mathcal{F}_0)} \le C_4 \cdot \|u(t)\|_{\mathrm{C}^1(\overline{\mathcal{F}_0})} \le C_5 \cdot \|z(t)\|_{\mathrm{X}_\gamma}.$$

On the other hand, the definition of $\triangle_\delta(\tilde{\varepsilon})$ in (3.4), the shape of $\mathbb{S}\tilde{\varepsilon}$ in (3.11) and the aforementioned embedding of the trace space imply

$$\left\| \frac{1}{\triangle_\delta(\tilde{\varepsilon})} \right\|_{\mathrm{L}^\infty(\partial\mathcal{F}_0)} \le C_6 \quad \text{and} \quad \|\mathbb{S}\tilde{\varepsilon}\|_{\mathrm{L}^\infty(\partial\mathcal{F}_0)} \le C_7 \cdot \|\nabla_{\mathrm{H}} Y(t)\|_{\mathrm{L}^\infty(\partial\mathcal{F}_0)} \cdot \|z(t)\|_{\mathrm{X}_\gamma}$$

for some constants $C_6 > 0$ and $C_7 > 0$. Hence, putting together the previous estimates and the shape of the transformed stress tensor $\mathcal{T}_\delta$ from (4.18), and exploiting the compactness of $\partial\mathcal{S}_0$, we conclude

$$\begin{aligned}
&\left| \int_{\partial\mathcal{S}_0} \mathcal{T}_\delta(\widehat{u}(t) + u_0^*(t))\nu \,\mathrm{d}\Gamma \right| \\
&= \left| \int_{\partial\mathcal{S}_0} Q(t)^\top \sigma_\delta(\widehat{u}(t) + u_0^*(t)) Q(t)\nu \,\mathrm{d}\Gamma \right| \\
&\le |Q(t)|^2 \int_{\partial\mathcal{S}_0} |\sigma_\delta(\widehat{u}(t) + u_0^*(t))| \,\mathrm{d}\Gamma \\
&\le C_8 \cdot |Q(t)|^2 \cdot \Big(1 + \|\nabla_{\mathrm{H}} Y(t)\|_{\mathrm{L}^\infty(\partial\mathcal{F}_0)} \cdot \|z(t)\|_{\mathrm{X}_\gamma}\Big) \cdot \|z(t)\|_{\mathrm{X}_\gamma}.
\end{aligned} \tag{4.82}$$

Therefore, integrating in time, exploiting Hölder's inequality and using the estimate of Q from Lemma A.1.1(b), $Y \in \mathrm{C}^1((0,T);\mathrm{C}^\infty(\mathbb{R}^2)^2)$ and the estimate of $\nabla_{\mathrm{H}} Y$ from Lemma A.1.1(a), the embedding from Lemma 4.4.6(a) as well as (4.60), we infer that

$$\begin{aligned}
&\left\| Q \int_{\partial\mathcal{S}_0} \mathcal{T}_\delta(u)\nu \,\mathrm{d}\Gamma \right\|_{\mathrm{L}^p(0,T)} \\
&\le C_9 \Big(T^{1/2p} + \|z\|_{\mathrm{L}^{2p}(0,T;\mathrm{X}_\gamma)}\Big) \cdot \|z\|_{\mathrm{L}^{2p}(0,T;\mathrm{X}_\gamma)} \\
&\le C_{10} T^{1/p} \cdot \Big(1 + \|z\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}\Big) \cdot \|z\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \\
&\le C_{11} T^{1/p} \cdot \Big(1 + \|\widehat{z}\|_{\mathbb{E}_1} + \|z_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}\Big) \\
&\quad \cdot \Big(\|\widehat{z}\|_{\mathbb{E}_1} + \|z_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}\Big) \\
&\le C_{12} T^{1/p} \left(1 + R + C_0 + C_T^*\right)\left(R + C_0 + C_T^*\right).
\end{aligned} \tag{4.83}$$

The surface integral appearing in G_5 can be estimated in the same way, so the self map estimates of G_4 and G_5 are implied upon observing that $C_T^* \to 0$.

The remaining task is thus to show the Lipschitz estimates of G_4 and G_5. As in (4.80), we first derive the estimate

$$\begin{aligned}&\|m_{\mathcal{S}}(\widehat{\omega}_1 - \widehat{\omega}_2)(\widehat{\ell}_1 + \ell_0^*)^\perp + m_{\mathcal{S}}(\widehat{\omega}_2 + \widehat{\omega}_0^*)(\widehat{\ell}_1 - \widehat{\ell}_2)^\perp\|_{\mathrm{L}^p(0,T)}\\ &\le C_{13} T^{1/p}(R + C_T^* + C_0) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.\end{aligned}$$

In the sequel, we estimate the differences in the surface integrals. Concerning

$$(Q_1 - Q_2) \int_{\partial\mathcal{S}_0} \mathcal{T}_\delta(\widehat{u}_1 + u_0^*)\nu \,\mathrm{d}\Gamma,$$

we make use of Lemma A.1.1(b) to estimate $Q_1 - Q_2$ and proceed as in (4.83) to infer that

$$\begin{aligned}&\left\|(Q_1 - Q_2) \int_{\partial\mathcal{S}_0} \mathcal{T}_\delta(\widehat{u}_1 + u_0^*)\nu \,\mathrm{d}\Gamma\right\|_{\mathrm{L}^p(0,T)}\\ &\le \|Q_1 - Q_2\|_{\mathrm{L}^\infty(0,T)} \cdot \left\|\int_{\partial\mathcal{S}_0} \mathcal{T}_\delta(\widehat{u}_1 + u_0^*)\nu \,\mathrm{d}\Gamma\right\|_{\mathrm{L}^p(0,T)}\\ &\le C_{14} T^{1/p}\,(1 + R + C_0 + C_T^*)\,(R + C_0 + C_T^*) \cdot \|\widehat{z}_1 - \widehat{z}_2\|_{\mathbb{E}_1}.\end{aligned} \tag{4.84}$$

For G_4, it remains to deal with

$$Q_2 \int_{\partial\mathcal{S}_0} (\mathcal{T}_\delta(\widehat{u}_1 + u_0^*) - \mathcal{T}_\delta(\widehat{u}_2 + u_0^*))\nu \,\mathrm{d}\Gamma.$$

With regard to Lemma A.1.1(b) ensuring the boundedness of Q_2 in $\mathrm{L}^\infty(0,T)$, the task boils down to estimating the integral. Furthermore, we observe that

$$\begin{aligned}&\mathcal{T}_\delta(\widehat{u}_1 + u_0^*) - \mathcal{T}_\delta(\widehat{u}_2 + u_0^*)\\ &= Q_1^\top \sigma_\delta(\widehat{u}_1 + u_0^*) Q_1 - Q_2^\top \sigma_\delta(\widehat{u}_2 + u_0^*) Q_2\\ &= (Q_1^\top - Q_2^\top)\sigma_\delta(\widehat{u}_1 + u_0^*) Q_1 + Q_2^\top(\sigma_\delta(\widehat{u}_1 + u_0^*) - \sigma_\delta(\widehat{u}_2 + u_0^*)) Q_1\\ &\quad + Q_2^\top \sigma_\delta(\widehat{u}_2 + u_0^*)(Q_1 - Q_2).\end{aligned}$$

Thanks to Lemma A.1.1(b), we argue that the handling of the first and the third resulting addend parallels the one from (4.84). Thus, the last term to be analyzed is the second addend. In view of (4.81), we have

$$\begin{aligned}&\sigma_\delta(\widehat{u}_1 + u_0^*) - \sigma_\delta(\widehat{u}_2 + u_0^*)\\ &= \frac{P(h_1, a_1)}{2}\left(\frac{\mathbb{S}\tilde{\varepsilon}(v_{\mathrm{ice},1})}{\triangle_\delta(\tilde{\varepsilon}(v_{\mathrm{ice},1}))} - \mathrm{Id}_2\right) - \frac{P(h_2, a_2)}{2}\left(\frac{\mathbb{S}\tilde{\varepsilon}(v_{\mathrm{ice},2})}{\triangle_\delta(\tilde{\varepsilon}(v_{\mathrm{ice},2}))} - \mathrm{Id}_2\right).\end{aligned}$$

Given this representation, we exploit the smoothness of the terms in $\tilde{\varepsilon}(v_{\mathrm{ice}})$, h and a together with the mean value theorem as well as a similar estimate of the difference $\tilde{\varepsilon}(v_{\mathrm{ice},1}) - \tilde{\varepsilon}(v_{\mathrm{ice},2})$ as seen in (4.77) to derive an estimate of the above term by $\widehat{z}_1 - \widehat{z}_2$. We can then plug the resulting estimate into (4.82) to get an estimate by the difference. In the same way as in the step from (4.82) to (4.83), we also derive the presence of a T-power with positive exponent in the estimate.

The difference in the surface integral of G_5 can be handled in the same way, so the proof of the lemma is complete. □

The previous lemma completes the preparation for the proof of the main result. This is the topic of the subsequent paragraph.

Statement and Proof of the Local Strong Well-Posedness

First, we state and prove the local well-posedness result in the reference configuration, i. e., on the time-independent domain $\mathcal{F}_0$, thanks to the fixed point procedure described at the beginning of this section and fueled by the maximal regularity from Proposition 4.3.3 and the above nonlinear estimates. In a second step, we deduce from there the actual local well-posedness of the interaction problem of sea ice in the moving domain.

The local-in-time well-posedness result in the fixed domain is given below.

Proposition 4.4.12. *Let $p, q \in (1, \infty)$ be such that* (4.48) *is valid, let $z_0 \in V$, with V as defined in* (4.47)*, suppose that V_{atm}, V_{ocn}, $\nabla_{\mathrm{H}} H$ and f_{gr} satisfy Assumption 4.4.8, assume $F \in \mathrm{L}^p(0, T_0)^2$ and $N \in \mathrm{L}^p(0, T_0)$ for $T_0 > 0$ from Remark 4.4.2, and recall the spaces X_0 and X_1 from* (4.32) *and* (4.33)*.*

Then there is $T \in (0, T_0]$ such that the transformed interaction problem on $\mathcal{F}_0$ as given in (4.28) *admits a unique strong solution z with*

$$z \in \mathrm{W}^{1,p}(0, T; \mathrm{X}_0) \cap \mathrm{L}^p(0, T; \mathrm{X}_1) \cap \mathrm{C}([0, T]; V).$$

Proof. The proof is based on the fixed point argument as described in detail at the beginning of the present section. More precisely, from the uniqueness of the reference solution obtained in Proposition 4.4.1, it follows that (4.28) has a unique solution if and only if (4.53) has a unique fixed point $\widehat{z}$, and the solution is then given by $z = \widehat{z} + z_0^*$ for the reference solution z_0^* from Proposition 4.4.1.

Hence, the idea is to show that the map Φ_T^R from (4.58) is a self map and contraction on $\mathcal{K}_T^R$ for $R > 0$ and $T > 0$ sufficiently small. Proposition 4.3.3 implies that system (4.53) with right-hand sides $G_1(\tilde{z})$, $G_2(\tilde{z})$, $G_3(\tilde{z})$, $G_4(\tilde{z})$

and $G_5(\tilde{z})$ admits a unique solution $\widehat{z} \in {}_0\mathbb{E}_1$. Moreover, in view of Proposition 4.3.3, and also invoking Lemma 4.4.9, Lemma 4.4.10 and Lemma 4.4.11, we infer that

$$\|\Phi_T^R(\tilde{z})\|_{\mathbb{E}_1} \le C_{\mathrm{MR}} \cdot \sum_{i=1}^5 C_{G_i}(R,T) \le C_{\mathrm{MR}} \cdot \frac{R}{2C_{\mathrm{MR}}} \le \frac{R}{2}$$

for $R > 0$ and $T > 0$ sufficiently small. Thus, Φ_T^R is indeed a self map for such R and T. In a similar way, considering $\tilde{z}_1$, $\tilde{z}_2 \in \mathcal{K}_T^R$, we obtain unique solutions $\widehat{z}_1 = \Phi(\tilde{z}_1)$ and $\widehat{z}_2 = \Phi(\tilde{z}_2)$ of the respective system (4.53) with corresponding terms on the right-hand side. In addition, from the maximal regularity in Proposition 4.3.3 as well as Lemma 4.4.9, Lemma 4.4.10 and Lemma 4.4.11, we deduce that

$$\|\Phi_T^R(\tilde{z}_1) - \Phi_T^R(\tilde{z}_2)\|_{\mathbb{E}_1} \le C_{\mathrm{MR}} \cdot \sum_{i=1}^5 L_{G_i}(R,T) \cdot \|\tilde{z}_1 - \tilde{z}_2\|_{\mathbb{E}_1} \le \frac{1}{2} \cdot \|\tilde{z}_1 - \tilde{z}_2\|_{\mathbb{E}_1}$$

for $R > 0$ and $T > 0$ sufficiently small again. In other words, Φ_T^R is a self map and contraction on $\mathcal{K}_T^R$ for such $R > 0$ and $T > 0$, and Φ_T^R thus has a unique fixed point $\widehat{z}$ thanks to the contraction mapping principle. By the above argument, this means precisely that there exists a unique solution $\widehat{z} \in {}_0\mathbb{E}_1$ to (4.53). Adding the reference solution z_0^*, we find that $z \coloneqq \widehat{z} + z_0^* \in \mathbb{E}_1$ is the desired solution to the transformed system (4.28). Furthermore, the solution lies in the asserted regularity class by definition of Φ_T^R on $\mathcal{K}_T^R$ as well as the regularity of the reference solution emerging from Proposition 4.4.1. The intersection with $\mathrm{C}([0,T];V)$ follows from the embedding of the maximal regularity space from Proposition 2.4.11 together with Lemma 4.4.3, because we especially consider $T \le T_0$. □

Now, we are in the position to state and prove the main theorem of this chapter. It asserts the existence of a unique strong solution to the sea ice interaction problem as introduced in (4.8). The result follows from Proposition 4.4.12 in conjunction with the inverse coordinate transform. The function spaces on time-dependent domains are defined via a pull-back induced by the coordinate transform, see also the respective paragraph in Section 1.3.

Theorem 4.4.13. *Let $p, q \in (1, \infty)$ satisfy* (4.48)*, consider $z_0 \in V$, with V as introduced in* (4.47)*, suppose that V_{atm}, V_{ocn}, $\nabla_{\mathrm{H}} H$ and f_{gr} fulfill Assumption 4.4.8, and assume $F \in \mathrm{L}^p(0,T_0)^2$ and $N \in \mathrm{L}^p(0,T_0)$ for $T_0 > 0$ from Remark 4.4.2. If the rigid body starts with a strictly positive distance from* *the outer boundary, i. e., for some* $r > 0$*, we have* $\operatorname{dist}(\mathcal{S}_0, \partial\mathcal{O}) > r$*, then there is* $T > 0$ *so that* $X \in \mathrm{C}^1([0,T];\mathrm{C}^2(\mathbb{R}^2)^2)$*, where* $X(\tau,\cdot)\colon \mathcal{F}(\tau) \to \mathcal{F}_0$ *are* C^2*-diffeomorphisms for all* $\tau \in [0,T]$*, and the interaction problem of sea ice* (4.8) *has a unique solution* $(\overline{v}_{\mathrm{ice}}, \overline{h}, \overline{a}, \xi, \Omega)$ *with*

$$\begin{aligned} \overline{v}_{\mathrm{ice}} &\in \mathrm{W}^{1,p}(0,T;\mathrm{L}^q(\mathcal{F}(\cdot)^2) \cap \mathrm{L}^p(0,T;\mathrm{W}^{2,q}(\mathcal{F}(\cdot)^2), \\ \overline{h}, \overline{a} &\in \mathrm{W}^{1,p}(0,T;\mathrm{L}^q(\mathcal{F}(\cdot)) \cap \mathrm{L}^p(0,T;\mathrm{W}^{2,q}(\mathcal{F}(\cdot)), \\ \xi &\in \mathrm{W}^{1,p}(0,T)^2 \text{ and } \Omega \in \mathrm{W}^{1,p}(0,T). \end{aligned}$$

Proof. As we have already indicated, the proof is based on Proposition 4.4.12 in the reference configuration. The latter yields the existence of a unique solution $z = (v_{\mathrm{ice}}, h, a, \ell, \omega)$ to (4.28). Given $(\ell, \omega) \in \mathrm{W}^{1,p}(0,T)^3$, we derive the original body velocities ξ and Ω together with the diffeomorphism X as expressed in Remark 4.4.4. In the next step, we perform the backward change of coordinates as revealed in Section 4.2. This leads to the solution to the interaction problem on the moving domain (4.8). The uniqueness is a consequence of the uniqueness obtained in Proposition 4.4.12. □

Finally, some remarks on the aforementioned main result of this chapter are in order.

Remark 4.4.14. *(a) A solution in the regularity class as in Theorem 4.4.13 is referred to as a* strong solution*.*

(b) The position of the center of mass x_{c} *as well as the rotation angle* β *can be deduced from the translational and angular velocities of the rigid body* ξ *and* Ω *by* (4.1) *and* (4.3)*, respectively.*

(c) In principle, it is possible to verify the aspects of the solution as in Theorem 3.5.2 such as the continuous dependence of the solution on the initial data or the characterization of the maximal time interval of existence of the solution. However, as the proof of Theorem 4.4.13 is rather "hands-on", based on a direct fixed point argument, the implementation of these aspects into the proof would require an adjustment of the proof.

(d) Furthermore, it is also possible to introduce time weights in order to lower the regularity of the initial data. Again, as the nonlinear estimates are handled by a direct approach, the inclusion of time weights would complicate the estimates.

CHAPTER 5

A Coupled Atmosphere-Sea Ice-Ocean Model

In this section, we investigate a coupled atmosphere-sea ice-ocean model. It results from internalizing the velocities of the atmosphere and the ocean appearing as terms on the right-hand side of the sea ice equations as introduced in (3.6). More precisely, the ocean and atmosphere dynamics are described by the viscous primitive equations, and the layer in between is occupied by sea ice, which is in turn modeled by Hibler's viscous-plastic model. The atmosphere and the ice are coupled via surface wind acting as a force in the sea ice momentum equation, while the velocities of the ice and the ocean are assumed to coincide at their common interface, and the ocean stress on the ice is considered to be proportional to the shear rate as in the situation of a plane Couette flow. The main results in this chapter are the local strong well-posedness and the global strong well-posedness close to constant equilibria for a simplified version of the model.

The investigation of coupled models in the context of the atmosphere, the ocean and sea ice has a long tradition. We refer here for instance to the ocean-ice model by Hibler and Bryan [61], where Hibler's sea ice model from [60] is coupled with a multilevel baroclinic ocean model introduced by Bryan [21]. In their model, the ocean and sea ice are coupled via heat and salt flux as well as momentum exchange. We also mention the article of Timmermann, Beckmann and Hellmer [129] on the coupling of a dynamic-thermodynamic sea ice model and the so called *S-coordinate primitive equations* (SPEM). Their model incorporates the coupling by heat and salt flux, transmission of solar radiation and momentum flux. Let us also mention a recent article of Constantin and Johnson [29] on the dynamics of the atmosphere, ocean and sea ice in near surface areas in the Arctic Ocean. The authors start with the Navier-Stokes equations rather than with the primitive equations, and they do not employ Hibler's sea ice model. With regard to coupled atmosphere-ocean models, we also refer to the work of Lions, Temam and Wang [95,96] on the introduction and numerical as well as mathematical analysis of a model coupling primitive equations of the atmosphere as introduced in [93] with primitive equations of the large-scale ocean [94] by nonlinear drag conditions.

The results of the present chapter are also contained in a recent preprint with Tim Binz and Matthias Hieber [12]. Concerning the general strategy, the chapter relies again on the abstract theory as introduced in Section 2.6. From that point of view, it is conceptually closer to Chapter 3 than to Chapter 4. On the other hand, it requires some effort to establish the linear theory in the present case. This is due to the (linear) coupling conditions of the ocean and the sea ice.

The outline of this chapter is described in the sequel. It is the purpose of Section 5.1 to settle some notation and to introduce the coupling conditions in (5.1), (5.2) and (5.3). We then summarize the complete coupled system (5.4) in Section 5.2, and we also present the reformulation as a quasilinear abstract Cauchy problem in this section in (5.16). The following Section 5.3 is of prime importance for the linear theory. In fact, the section is dedicated to the stationary hydrostatic Stokes problem corresponding to the coupling of the ocean and sea ice via the equality of velocities on the interface. In that respect, Proposition 5.3.9 and Proposition 5.3.10 on the emerging *hydrostatic Dirichlet operator* and its regularity properties are the starting point for the further linear theory. The latter is then addressed in Section 5.4, where we use a decoupling approach together with the bounded $\mathcal{H}^\infty$-calculus of block operator matrices as discussed in Section 2.3 to establish properties of the linearized operator matrix such as the bounded $\mathcal{H}^\infty$-calculus in Proposition 5.4.3 and Corollary 5.4.4. Section 5.5 contains the first main result of this chapter, Theorem 5.5.2, on the local strong well-posedness. The proof is based on the general framework to quasilinear evolution equations from Section 2.6. Finally, Section 5.6 tackles the global strong well-posedness of a simplified version of the coupled model in the case of initial data close to constant equilibria. For this, we establish the normal stability of such equilibria and then apply the generalized principle of linearized stability as revealed in Section 2.6 to get the second main result of this chapter, Theorem 5.6.6.

5.1. Notation and Coupling Conditions

This section is dedicated to settling some notation, and to introducing the coupling conditions. We recall that $x_{\mathrm{H}} = (x, y)$ denotes the horizontal coordinates, while z represents the vertical coordinate. Besides, the sub- or superscripts $_{\mathrm{atm}}$, $_{\mathrm{ocn}}$ and $_{\mathrm{ice}}$ indicate the correspondence of objects to the atmosphere, the ocean or the ice. In that respect, we denote by u_{atm}, v_{atm} and w_{atm} the full, horizontal and vertical velocity of the atmospheric wind. The pieces of notation u_{ocn}, v_{ocn} and w_{ocn} have an analogous meaning. As in Chapter 3, we denote by v_{ice} the horizontal sea ice velocity, h represents the mean ice thickness, and a is the ice compactness. We also keep the notation $u = (v_{\mathrm{ice}}, h, a)$ for the principle variable in the context of the sea ice equations. In contrast, the principle variable associated to the complete coupled system is denoted by $v = (v_{\mathrm{atm}}, v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a)$, while π_{atm} and π_{ocn} represent the respective pressure variables for the atmosphere and the ocean.

In comparison with Chapter 3, the sea ice domain changes. In fact, in order to guarantee compatibility with the primitive equations, which are typically investigated on cylindrical domains, we take into account $G = (0,1) \times (0,1)$ with boundary ∂G as the new sea ice domain throughout this chapter. As a consequence, the sea ice variables are assumed to have periodic boundary conditions on ∂G. Another difference to the aforementioned chapter is that we assume the mean ice thickness to be bounded by some sufficiently large parameter $\kappa_2 > 0$, so $h \in (\kappa_1, \kappa_2)$ is supposed, where $\kappa_1 > 0$ plays the role of the parameter $\kappa > 0$ from Section 3.2.

In the sequel, domains are typically denoted by Ω, whereas Γ usually represents an interface. These objects may be complemented by subscripts. For the parameter $\kappa_2 > 0$ and a fixed sufficiently large height $h_{\mathrm{atm}} > 0$, the domain occupied by the atmosphere is $\Omega_{\mathrm{atm}} = G \times (\kappa_2, h_{\mathrm{atm}})$, while $\Omega_{\mathrm{ocn}} = G \times (-h_{\mathrm{ocn}}, 0)$, for a fixed depth $h_{\mathrm{ocn}} > 0$, is the domain of the ocean. We continue with the boundaries and interfaces. In fact, we denote by $\Gamma_u = G \times \{h_{\mathrm{atm}}\}$ the upper boundary, and $\Gamma_b = G \times \{-h_{\mathrm{ocn}}\}$ represents the lower boundary, while the interfaces between the the atmosphere and the sea ice as well as the ocean and the sea ice are denoted by $\Gamma_i = G \times \{\kappa_2\}$ and $\Gamma_o = G \times \{0\}$, respectively. Finally, the lateral boundaries of the atmosphere and ocean are given by $\Gamma_{l,\mathrm{atm}} = \partial G \times (\kappa_2, h_{\mathrm{atm}})$ and $\Gamma_{l,\mathrm{ocn}} = \partial G \times (-h_{\mathrm{ocn}}, 0)$, respectively. For further aspects concerning notation, we also refer to Section 2.7.

We now discuss the present coupling and boundary conditions. Let us first recall the sea ice momentum equation from (3.7). On the right-hand side, the terms τ_{atm} and $\tau_{\text{ocn}}(v_{\text{ice}})$ appear. The velocities of the atmospheric wind and the ocean are internalized. With regard to the *force exerted by the atmosphere on the sea ice*, we assume that (3.6) remains valid, i. e.,

$$\tau_{\text{atm}} = \rho_{\text{atm}} C_{\text{atm}} |v_{\text{atm}}| R_{\text{atm}} v_{\text{atm}}, \quad \text{on } G. \tag{5.1}$$

Concerning the ocean force on the sea ice, we suppose that it is proportional to the shear rate. In the present set-up, for a viscosity parameter $\mu_{\text{ocn}} > 0$, this means that the *force of the ocean exerted on the sea ice* is given by

$$\tau_{\text{ocn}} = -\mu_{\text{ocn}} \partial_z v_{\text{ocn}}, \quad \text{on } G, \tag{5.2}$$

Let us observe that (5.2) is in accordance with the situation of a plane Couette flow for a Newtonian fluid. Finally, we also suppose that the velocities of the ocean and the sea ice coincide on their common interface Γ_o, so

$$v_{\text{ocn}} = v_{\text{ice}}, \quad \text{on } G. \tag{5.3}$$

5.2. The Complete Coupled System

In this section, we introduce the complete coupled atmosphere-sea ice-ocean model and provide its reformulation as a quasilinear abstract Cauchy problem.

In the sequel, we neglect the Coriolis terms in order to simplify the notation. We will comment on this at a later stage. Taking into account the sea ice equations as in Section 3.2, with the modifications pointed out in the previous section, and invoking the viscous incompressible primitive equations for the atmosphere and the ocean as introduced in Section 2.7, we get the *complete coupled atmosphere-sea ice-ocean model*

$$\begin{cases} \partial_t v_{\text{atm}} - \Delta v_{\text{atm}} + \nabla_{\text{H}} \pi_{\text{atm}} = F_{\text{atm}}, & \text{in } (0,T) \times \Omega_{\text{atm}}, \\ \partial_z \pi_{\text{atm}} = 0, & \text{in } (0,T) \times \Omega_{\text{atm}}, \\ \operatorname{div} u_{\text{atm}} = 0, & \text{in } (0,T) \times \Omega_{\text{atm}}, \\ \partial_t v_{\text{ocn}} - \Delta v_{\text{ocn}} + \nabla_{\text{H}} \pi_{\text{ocn}} = F_{\text{ocn}}, & \text{in } (0,T) \times \Omega_{\text{ocn}}, \\ \partial_z \pi_{\text{ocn}} = 0, & \text{in } (0,T) \times \Omega_{\text{ocn}}, \\ \operatorname{div} u_{\text{ocn}} = 0, & \text{in } (0,T) \times \Omega_{\text{ocn}}, \\ \partial_t v_{\text{ice}} - \dfrac{1}{m_{\text{ice}}} \cdot \operatorname{div}_{\text{H}} \sigma_\delta = F_{\text{ice}}, & \text{on } (0,T) \times G, \\ \partial_t h - d_{\text{h}} \Delta_{\text{H}} h = F_{\text{h}}, & \text{on } (0,T) \times G, \\ \partial_t a - d_{\text{a}} \Delta_{\text{H}} a = F_{\text{a}}, & \text{on } (0,T) \times G, \\ v_{\text{ocn}} = v_{\text{ice}}, & \text{on } (0,T) \times G, \end{cases} \tag{5.4}$$

where

$$F_{\mathrm{atm}} = -(u_{\mathrm{atm}} \cdot \nabla)v_{\mathrm{atm}} + f_{\mathrm{atm}}, \quad F_{\mathrm{ocn}} = -(u_{\mathrm{ocn}} \cdot \nabla)v_{\mathrm{ocn}} + f_{\mathrm{ocn}},$$
$$F_{\mathrm{ice}} = -(v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}})v_{\mathrm{ice}} - g\nabla_{\mathrm{H}} H + \frac{1}{m_{\mathrm{ice}}}(\tau_{\mathrm{atm}}(v_{\mathrm{atm}}) + \tau_{\mathrm{ocn}}(v_{\mathrm{ocn}})),$$
$$F_{\mathrm{h}} = -\mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} h) + S_{\mathrm{h}} \text{ and } F_{\mathrm{a}} = -\mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} a) + S_{\mathrm{a}}.$$

In the above, f_{atm} and f_{ocn} represent external forcing terms in the context of the atmosphere and ocean equations.

The coupled system is complemented by boundary conditions. More precisely, we suppose that the horizontal velocity of the atmosphere v_{atm} satisfies homogeneous Neumann boundary conditions on its upper and lower boundary Γ_u and Γ_i, while Dirichlet boundary conditions are considered for the horizontal velocity of the ocean v_{ocn} on the lower boundary Γ_b. In addition, we assume that the vertical velocities w_{atm} and w_{ocn} equal zero on all the boundaries. The preceding discussion can be summarized by

$$\begin{aligned} \partial_z v_{\mathrm{atm}} &= 0, \text{ on } \Gamma_u \cup \Gamma_i, \quad v_{\mathrm{ocn}} = 0, \text{ on } \Gamma_b, \text{ and} \\ w_{\mathrm{atm}} &= 0, \text{ on } \Gamma_u \cup \Gamma_i, \quad w_{\mathrm{ocn}} = 0, \text{ on } \Gamma_o \cup \Gamma_b. \end{aligned} \tag{5.5}$$

Finally, we consider periodic boundary conditions on all lateral boundaries, so v_{atm} and π_{atm} are periodic on $\Gamma_{l,\mathrm{atm}}$, v_{ocn} and π_{ocn} are periodic on $\Gamma_{l,\mathrm{ocn}}$, and v_{ice}, h and a are periodic on ∂G.

The next step is to rewrite the complete coupled system (5.4) completed by the boundary conditions in (5.5) in operator form. For this purpose, we first invoke the underlying operators. Let $\mathbb{P}$ denote the hydrostatic Helmholtz projection in the context of the primitive equations as introduced in (2.38). In order to distinguish the objects in the situation of the atmosphere and the ocean, we denote by $\mathbb{P}_{\mathrm{atm}}$ the hydrostatic Helmholtz projection for the atmosphere, while $\mathbb{P}_{\mathrm{ocn}}$ represents the hydrostatic Helmholtz projection for the ocean. As a special case of the hydrostatic Stokes operator as defined in (2.39), we set the *hydrostatic Stokes operator for the atmosphere* to be

$$A^{\mathrm{atm}} v_{\mathrm{atm}} := \mathbb{P}_{\mathrm{atm}} \Delta v_{\mathrm{atm}}, \text{ with} \tag{5.6}$$

$$\mathrm{D}(A^{\mathrm{atm}}) := \left\{ v_{\mathrm{atm}} \in \mathrm{W}^{2,q}_{\mathrm{per}}(\Omega_{\mathrm{atm}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{atm}}) : \partial_z v_{\mathrm{atm}} = 0, \text{ on } \Gamma_u \cup \Gamma_i \right\}.$$

In the above $\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{atm}})$, denotes the space of hydrostatic solenoidal vector fields whose definition can be found in (2.37) in Section 2.7. On the other hand, with regard to the inhomogeneous boundary conditions of the ocean equations, we also introduce the so-called *maximal hydrostatic Stokes operator*. Its name is related to the fact that the domain of the operator is chosen "maximal". There are no boundary conditions incorporated into its domain, so

$$\begin{aligned} &A_{\mathrm{m}}^{\mathrm{ocn}} v_{\mathrm{ocn}} := \mathbb{P}_{\mathrm{ocn}} \Delta v_{\mathrm{ocn}}, \quad \text{with} \\ &\mathrm{D}(A_{\mathrm{m}}^{\mathrm{ocn}}) := \{v_{\mathrm{ocn}} \in \mathrm{L}_{\overline{\sigma}}^q(\Omega_{\mathrm{ocn}}) : \mathbb{P}_{\mathrm{ocn}} \Delta v_{\mathrm{ocn}} \in \mathrm{L}_{\overline{\sigma}}^q(\Omega_{\mathrm{ocn}})\}\,. \end{aligned} \tag{5.7}$$

Since we will use a similarity transform leading to the investigation of the hydrostatic Stokes operator for the ocean, we also introduce the *oceanic hydrostatic Stokes operator with homogeneous boundary conditions*, so we set

$$\begin{aligned} &A_0^{\mathrm{ocn}} v_{\mathrm{ocn}} := \mathbb{P}_{\mathrm{ocn}} \Delta v_{\mathrm{ocn}}, \\ &\mathrm{D}(A_0^{\mathrm{ocn}}) := \left\{v_{\mathrm{ocn}} \in \mathrm{W}_{\mathrm{per}}^{2,q}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}_{\overline{\sigma}}^q(\Omega_{\mathrm{ocn}}) : v_{\mathrm{ocn}} = 0, \ \text{ on } \Gamma_o \cup \Gamma_b\right\}. \end{aligned} \tag{5.8}$$

For convenience, we briefly recall the Hibler operator on G. By a slight abuse of notation, we still denote the differential form of Hibler's operator on G by $\mathbb{A}^{\mathrm{H}}$. As in (3.14), the operator $\mathbb{A}^{\mathrm{H}} v_{\mathrm{ice}} = \mathbb{A}^{\mathrm{H}}(u) v_{\mathrm{ice}}$ takes the shape

$$\begin{aligned} (\mathbb{A}^{\mathrm{H}} v_{\mathrm{ice}})_i = &- \sum_{j,k,l=1}^{2} \frac{P}{2\rho_{\mathrm{ice}} h} \frac{1}{\triangle_\delta(\varepsilon)} \left(\mathbb{S}_{ij}^{kl} - \frac{1}{\triangle_\delta^2(\varepsilon)} (\mathbb{S}\varepsilon)_{ik} (\mathbb{S}\varepsilon)_{jl}\right) \mathrm{D}_k \mathrm{D}_l v_{\mathrm{ice},j} \\ &+ \frac{1}{2\rho_{\mathrm{ice}} h \triangle_\delta(\varepsilon)} \sum_{j=1}^{2} (\partial_j P)(\mathbb{S}\varepsilon)_{ij}, \end{aligned} \tag{5.9}$$

where $i = 1, 2$ and $\mathrm{D}_m = -\mathrm{i}\partial_m$. In contrast to Section 3.3, no Dirichlet boundary conditions are considered, but we assume periodic boundary conditions instead. For sufficiently regular $u_0 = (v_{\mathrm{ice},0}, h_0, a_0)$, we define the L^q*-realization of the linearized Hibler operator with periodic boundary conditions* A^{H} by

$$[A^{\mathrm{H}}(u_0)] v_{\mathrm{ice}} := [\mathbb{A}^{\mathrm{H}}(u_0)] v_{\mathrm{ice}}, \ \ \mathrm{D}(A^{\mathrm{H}}(u_0)) := \mathrm{W}_{\mathrm{per}}^{2,q}(G)^2. \tag{5.10}$$

In order to shorten the notation, we also abbreviate the off-diagonal terms appearing in the momentum equation of v_{ice} and acting on h and a by

$$B_{\mathrm{h}}(h_0, a_0) h := \frac{\partial_h P(h_0, a_0)}{2\rho_{\mathrm{ice}} h_0} \nabla_{\mathrm{H}} h \ \text{ and } \ B_{\mathrm{a}}(h_0, a_0) a := \frac{\partial_a P(h_0, a_0)}{2\rho_{\mathrm{ice}} h_0} \nabla_{\mathrm{H}} a. \tag{5.11}$$

Finally, by Δ_{H}, we denote the L^q-realization of the Laplacian operator on G subject to periodic boundary conditions, so $\mathrm{D}(\Delta_{\mathrm{H}}) := \mathrm{W}_{\mathrm{per}}^{2,q}(G)$.

Having settled all the relevant pieces of notation, we are now able to introduce the objects in order to rewrite the complete coupled system (5.4) subject to the boundary conditions (5.5) as a quasilinear abstract Cauchy problem. First, the *ground space* is defined by

$$X_0 \coloneqq L^q_\sigma(\Omega_{\mathrm{atm}}) \times L^q_\sigma(\Omega_{\mathrm{ocn}}) \times L^q(G)^2 \times L^q(G) \times L^q(G). \tag{5.12}$$

For the principle variable $v = (v_{\mathrm{atm}}, v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a)$, the *regularity space*, which coincides with the domain of the linearized operator matrix, takes the shape

$$\begin{aligned} X_1 \coloneqq \Big\{ & v \in D(A^{\mathrm{atm}}) \times D(A^{\mathrm{ocn}}_{\mathrm{m}}) \times D(A^{\mathrm{H}}) \times D(\Delta_{\mathrm{H}}) \times D(\Delta_{\mathrm{H}}) : \\ & v_{\mathrm{ocn}} = 0, \ \text{ on } \Gamma_b, \ \text{ and } v_{\mathrm{ocn}} = v_{\mathrm{ice}}, \ \text{ on } \Gamma_o \Big\}. \end{aligned} \tag{5.13}$$

The decoupling argument together with regularity theory will reveal that the v_{ocn}-component of an element $v \in X_1$ also enjoys $W^{2,q}$-regularity.

With $C_{\mathrm{o,i}}(h) := \mu_{\mathrm{ocn}}/\rho_{\mathrm{ice}} h$, it follows that $1/m_{\mathrm{ice}} \tau_{\mathrm{ocn}} = C_{\mathrm{o,i}}(h) \partial_z v_{\mathrm{ocn}}$ for τ_{ocn} as introduced in (5.2). In comparison with the complete coupled system (5.4), we apply the respective Helmholtz projection associated to the atmosphere and the ocean in the first two components to handle the pressure. We observe that all nonlinear terms in the operator matrix only depend on the sea ice variables. For the corresponding principle variable $u_0 = (v_{\mathrm{ice},0}, h_0, a_0)$ of the sea ice equations, we then define the *linearized operator matrix* $A(u_0)$ by

$$A(u_0) \coloneqq \begin{pmatrix} -A^{\mathrm{atm}} & 0 & 0 & 0 & 0 \\ 0 & -A^{\mathrm{ocn}}_{\mathrm{m}} & 0 & 0 & 0 \\ 0 & C_{\mathrm{o,i}}(h_0)\partial_z & -A^{\mathrm{H}}(u_0) & B_{\mathrm{h}}(h_0, a_0) & B_{\mathrm{a}}(h_0, a_0) \\ 0 & 0 & 0 & -d_{\mathrm{h}}\Delta_{\mathrm{H}} & 0 \\ 0 & 0 & 0 & 0 & -d_{\mathrm{a}}\Delta_{\mathrm{H}} \end{pmatrix}, \tag{5.14}$$

with domain $D(A(u_0)) \coloneqq X_1$. The last step is to capture the remaining inhomogeneous and nonlinear terms in the *right-hand side* F. In fact, we set

$$F(v) \coloneqq \begin{pmatrix} -\mathbb{P}_{\mathrm{atm}}\left((v_{\mathrm{atm}} \cdot \nabla_{\mathrm{H}})v_{\mathrm{atm}} + w_{\mathrm{atm}}(v_{\mathrm{atm}}) \cdot \partial_z v_{\mathrm{atm}} - f_{\mathrm{atm}}\right) \\ -\mathbb{P}_{\mathrm{ocn}}\left((v_{\mathrm{ocn}} \cdot \nabla_{\mathrm{H}})v_{\mathrm{ocn}} + w_{\mathrm{ocn}}(v_{\mathrm{ocn}}) \cdot \partial_z v_{\mathrm{ocn}} - f_{\mathrm{ocn}}\right) \\ -(v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}})v_{\mathrm{ice}} - g\nabla_{\mathrm{H}} H + \frac{1}{\rho_{\mathrm{ice}} h}\tau_{\mathrm{atm}}(v_{\mathrm{atm}}) \\ -\mathrm{div}_{\mathrm{H}}\left(v_{\mathrm{ice}} h\right) + S_{\mathrm{h}} \\ -\mathrm{div}_{\mathrm{H}}\left(v_{\mathrm{ice}} a\right) + S_{\mathrm{a}} \end{pmatrix}. \tag{5.15}$$

We remark that w_{atm} and w_{ocn} can be derived from v_{atm} as well as v_{ocn} as described in (2.35). In total, the coupled system (5.4) subject to the boundary conditions (5.5) admits the reformulation in *operator form* as

$$(5.16)\qquad \begin{cases} v'(t) + A(v(t))v(t) = F(v(t)), & \text{for } t \in (0,T), \\ v(0) = v_0, \end{cases}$$

on the ground space X_0 from (5.12).

5.3. The Stationary Hydrostatic Stokes Problem

In this section, we investigate the *stationary hydrostatic Stokes problem* with inhomogeneous boundary conditions. This is associated to the coupling condition via the equality of the velocities as in (5.3), and to the Dirichlet boundary conditions of the horizontal velocity of the ocean on the lower boundary as in (5.5). More precisely, for the maximal hydrostatic Stokes operator $A_{\mathrm{m}}^{\mathrm{ocn}}$ from (5.7) and $\varphi \in \mathrm{L}^q(G)^2$, we investigate the stationary problem

$$(5.17)\qquad \begin{cases} A_{\mathrm{m}}^{\mathrm{ocn}} v_{\mathrm{ocn}} = 0, & \text{on } \Omega_{\mathrm{ocn}}, \\ v_{\mathrm{ocn}} = \varphi, & \text{on } \Gamma_o, \\ v_{\mathrm{ocn}} = 0, & \text{on } \Gamma_b, \end{cases}$$

on the hydrostatic solenoidal space of the ocean $\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$. In the sequel, the corresponding solution operator of (5.17) is referred to as *hydrostatic Dirichlet operator* and denoted by

$$(5.18)\qquad L_0 \colon \mathrm{L}^q(\Gamma_o)^2 \to \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}).$$

We stress that $\Gamma_o = G \times \{0\}$ will be identified with G in the following, so we will also write $\mathrm{L}^q(G)^2$ instead of $\mathrm{L}^q(\Gamma_o)^2$ in some instances.

In order to prepare for the investigation of the stationary hydrostatic Stokes problem, we first collect some properties of the hydrostatic Stokes operators in the present setting, and we also discuss some properties of the Hibler operator and the Laplacian operator on G subject to periodic boundary conditions.

The assertions of the lemmas below follow directly from Lemma 2.7.1, Lemma 2.7.2 and Lemma 2.7.3. We start with the hydrostatic Stokes operators of the atmosphere A^{atm}.

Lemma 5.3.1. *Recall* A^{atm} *from* (5.6)*, and let* $q \in (1,\infty)$*. Then for* $\mu > 0$*, we have* $-A^{\mathrm{atm}} + \mu \in \mathcal{H}^\infty(\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{atm}}))$ *with* $\mathcal{H}^\infty$*-angle* $\phi^\infty_{-A^{\mathrm{atm}}+\mu} = 0$*. In particular,*

(a) $-A^{\mathrm{atm}} + \mu \in \mathcal{BIP}(\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{atm}}))$ *with power angle* $\theta_{-A^{\mathrm{atm}}+\mu} = 0$,

(b) $-A^{\mathrm{atm}} + \mu \in \mathcal{RS}(\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{atm}}))$ *with* $\mathcal{R}$*-angle* $\phi^{\mathcal{R}}_{-A^{\mathrm{atm}}+\mu} = 0$, *and*

(c) *for the fractional powers of* A^{atm}, *it holds that*

$$\mathrm{D}((-A^{\mathrm{atm}}+\mu)^{\beta}) \cong \mathrm{H}^{2\beta,q}_{\mathrm{per,N}}(\Omega_{\mathrm{atm}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{atm}}) \hookrightarrow \mathrm{H}^{2\beta,q}_{\mathrm{per}}(\Omega_{\mathrm{atm}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{atm}})$$

for $\beta \in (1/2+1/2q, 3/2+1/2q)$, *where the subscript* $_{\mathrm{N}}$ *indicates homogeneous Neumann boundary conditions on* $\Gamma_u \cup \Gamma_i$, *and*

$$\mathrm{D}((-A^{\mathrm{atm}}+\mu)^{\beta}) \cong \mathrm{H}^{2\beta,q}_{\mathrm{per}}(\Omega_{\mathrm{atm}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{atm}})$$

in the case $\beta < 1/2 + 1/2q$.

In the lemma below, we deal with the hydrostatic Stokes operator A_0^{ocn} corresponding to the ocean and with homogeneous boundary conditions. Additionally, we discuss the higher fractional powers. It is a consequence of Lemma 2.7.4.

Lemma 5.3.2. *Consider* A_0^{ocn} *as introduced in* (5.8), *and let* $q \in (1,\infty)$. *Then we have* $0 \in \rho(A_0^{\mathrm{ocn}})$, *and* $-A_0^{\mathrm{ocn}} \in \mathcal{H}^\infty(\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}))$ *with* $\mathcal{H}^\infty$*-angle* $\phi^\infty_{A_0^{\mathrm{ocn}}} = 0$. *Moreover, it especially holds that*

(a) $-A_0^{\mathrm{ocn}} \in \mathcal{BIP}(\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}))$ *with power angle* $\theta_{-A_0^{\mathrm{ocn}}} = 0$,

(b) $-A_0^{\mathrm{ocn}} \in \mathcal{RS}(\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}))$ *with* $\mathcal{R}$*-angle* $\phi^{\mathcal{R}}_{-A_0^{\mathrm{ocn}}} = 0$, *and*

(c) *for the fractional powers of* A_0^{ocn}, *we have*

$$\mathrm{D}((-A_0^{\mathrm{ocn}})^{\beta}) \cong \mathrm{H}^{2\beta,q}_{\mathrm{per,D}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}) \hookrightarrow \mathrm{H}^{2\beta,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$$

for $\beta \in (1/2q, 1 + 1/2q)$, *for* $_{\mathrm{D}}$ *denoting homogeneous Dirichlet boundary conditions on* $\Gamma_b \cup \Gamma_o$, *whereas for* $\beta < 1/2q$, *it holds that*

$$\mathrm{D}((-A_0^{\mathrm{ocn}})^{\beta}) \cong \mathrm{H}^{2\beta,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}).$$

Let us remark that the shape of the fractional power domain of $-A_0^{\mathrm{ocn}}$ is not affected by a shift by $\omega > 0$ as the operator A_0^{ocn} already admits bounded imaginary powers without shift, and the property is preserved under shifts in the correct direction.

Next, we focus on Hibler's operator on G subject to periodic boundary conditions. We get the following result.

Lemma 5.3.3. *Let* $q \in (1,\infty)$ *as well as* $u_0 = (v_{ice,0}, h_0, a_0) \in \mathrm{C}^1(G)^4$ *such that* $h_0 \geq \kappa_1$*, and recall the linearized operator* $A^{\mathrm{H}}(u_0)$ *from* (5.10)*. Then there is* $\omega_0 \in \mathbb{R}$ *so that for all* $\omega > \omega_0$*, the shifted operator* $-A^{\mathrm{H}}(u_0) + \omega$ *has a bounded* $\mathcal{H}^\infty$*-calculus on* $\mathrm{L}^q(G)^2$ *with angle* $\phi_{-A^{\mathrm{H}}(u_0)+\omega} < \pi/2$.

Proof. The proof is based on Proposition 2.5.9 upon identifying functions on the domain $G = (0,1) \times (0,1)$ subject to periodic boundary conditions with functions on the two-dimensional torus $\mathbb{T}^2$. Moreover, the ellipticity properties as stated in Proposition 3.3.1 can be shown to be valid as well in the present setting, yielding ω-ellipticity in the sense of Section 2.5 for $\omega = 0$. In addition, recalling the shape of the coefficients of Hibler's operator on G from (5.9), we observe the smooth dependence on $\nabla_{\mathrm{H}} v_{\mathrm{ice},0}$, h_0 and a_0, so the regularity assumption on u_0 implies the continuity of the coefficients. The assertion then follows from Proposition 2.5.9. □

After establishing the bounded $\mathcal{H}^\infty$-calculus, we also discuss several further properties of the shifted Hibler operator. They follow from Lemma 5.3.3 in conjunction with the relation (2.14) as well as Lemma 2.3.4 and Lemma 2.3.10 for the fractional power domains. For the interpolation of the periodic boundary conditions, we also refer to the discussion preceding Lemma 2.7.3.

Lemma 5.3.4. *Let* $p, q \in (1,\infty)$ *as well as* $u_0 = (v_{\mathrm{ice},0}, h_0, a_0) \in \mathrm{C}^1(G)^4$ *such that* $h_0 \geq \kappa_1$*, and consider the linearized Hibler operator* $A^{\mathrm{H}}(u_0)$ *as introduced in* (5.10)*. Then there exists* $\omega_0 \in \mathbb{R}$ *so that for all* $\omega > \omega_0$*, it holds that*

(a) $-A^{\mathrm{H}}(u_0) + \omega \in \mathcal{BIP}(\mathrm{L}^q(G)^2)$ *with power angle* $\theta_{-A^{\mathrm{H}}(u_0)+\omega} < \pi/2$,

(b) $-A^{\mathrm{H}}(u_0) + \omega \in \mathcal{RS}(\mathrm{L}^q(G)^2)$ *with* $\mathcal{R}$*-angle* $\phi^{\mathcal{R}}_{-A^{\mathrm{H}}(u_0)+\omega} < \pi/2$*, and*

(c) $\mathrm{D}((-A^{\mathrm{H}}(u_0) + \omega)^\beta) \cong \mathrm{H}^{2\beta,q}_{\mathrm{per}}(G)^2$ *for* $\beta > 0$.

Next, we verify that the hydrostatic Dirichlet operator L_0 from (5.18) is well-defined and bounded. The first step in this direction is the following result on an extension with vertical average zero. Denoting by $\overline{g}$ the vertical average of g, we consider

$$\begin{cases} \overline{g} = 0, & \text{on } \Omega_{\mathrm{ocn}}, \\ g = \varphi, & \text{on } \Gamma_o, \\ g = 0, & \text{on } \Gamma_b. \end{cases} \tag{5.19}$$

The result on the solvability of (5.19) reads as follows.

Lemma 5.3.5. *Let* $\varphi \in \mathrm{C}^\infty_{\mathrm{per}}(G)^2$. *Then the extension problem* (5.19) *admits a smooth solution* $g \in \mathrm{C}^\infty_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2$.

Proof. Let us recall that $\Omega_{\mathrm{ocn}} = G \times (-h_{\mathrm{ocn}}, 0)$. The idea is to split g into a product of φ and a function depending only on z. More precisely, we consider $g(x_{\mathrm{H}}, z) = r(z) \cdot \varphi(x_{\mathrm{H}})$. As a result, (5.19) reduces to the study of

$$\begin{cases} \overline{r} = 0, & \text{on } (-h_{\mathrm{ocn}}, 0), \\ r(0) = 1, & \\ r(-h_{\mathrm{ocn}}) = 0. & \end{cases} \tag{5.20}$$

Now, (5.20) can be solved explicitly, namely by $r(z) = {}^3\!/\!{}_{h_{\mathrm{ocn}}^2} \cdot z^2 + {}^4\!/\!{}_{h_{\mathrm{ocn}}} \cdot z + 1$. As r is especially smooth, and $\varphi \in \mathrm{C}^\infty_{\mathrm{per}}(G)^2$ is assumed, it follows in particular that $g = r \cdot \varphi \in \mathrm{C}^\infty_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2$ is a solution to (5.19). □

Let us comment on the extension problem (5.19). The solutions to (5.19) are generally not unique. We observe that $\mathrm{div_H}\, \overline{g} = 0$ holds for all $\varphi \in \mathrm{C}^\infty_{\mathrm{per}}(G)^2$, since $g(x_{\mathrm{H}}, z) = r(z) \cdot \varphi(x_{\mathrm{H}})$ satisfying $\mathrm{div_H}\, \overline{g} = 0$ for all $\varphi \in \mathrm{C}^\infty_{\mathrm{per}}(G)^2$ is equivalent to $\overline{r} = 0$.

From $\overline{g} = 0$ and the regularity of g, it follows that g lies in the domain of the maximal hydrostatic Stokes operator $A_{\mathrm{m}}^{\mathrm{ocn}}$ from (5.7), so $g \in \mathrm{D}(A_{\mathrm{m}}^{\mathrm{ocn}})$. Therefore, $v_g := v_{\mathrm{ocn}} - g \in \mathrm{D}(A_{\mathrm{m}}^{\mathrm{ocn}})$ is valid, and v_g solves

$$\begin{cases} A_{\mathrm{m}}^{\mathrm{ocn}} v_g = f, & \text{on } \Omega_{\mathrm{ocn}}, \\ v_g = 0, & \text{on } \Gamma_o, \\ v_g = 0, & \text{on } \Gamma_b, \end{cases} \tag{5.21}$$

where $f := -A_{\mathrm{m}}^{\mathrm{ocn}} g \in \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$. As a result, (5.17) has a unique solution if and only if the lifted problem (5.21) has this property. We note that (5.21) means precisely that $A_0^{\mathrm{ocn}} v_g = f$ thanks to classical regularity theory. From Lemma 2.7.1, we recall in particular that $0 \in \rho(A_0^{\mathrm{ocn}})$. Together with the previous observation, this results in the existence and uniqueness of a solution to (5.17) as stated in the lemma below.

Lemma 5.3.6. *Given* $\varphi \in \mathrm{C}^\infty_{\mathrm{per}}(G)^2$, *there exists a unique* $u \in \mathrm{D}(A_{\mathrm{m}}^{\mathrm{ocn}})$ *solving* (5.17).

Thanks to Lemma 5.3.6, L_0 is well-defined on the dense subspace $\mathrm{C}^\infty_{\mathrm{per}}(G)^2$ of $\mathrm{L}^q(G)^2$. Thus, we may define a unique adjoint

$$L_0' \colon \mathrm{D}(L_0') \subset (\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}))' \to (\mathrm{L}^q(G)^2)',$$

and we have $(\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}))' \cong \mathrm{L}^{q'}_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$ and $(\mathrm{L}^q(G)^2)' \cong \mathrm{L}^{q'}(G)^2$ for ${}^1/_q + {}^1/_{q'} = 1$. In the sequel, we use ∂_z^r to denote the distributional normal derivative on Γ_o, so for $r \in (1,\infty)$, we consider

$$\partial_z^r \colon \mathrm{D}(\partial_z^r) := \Bigl\{ f \in \mathrm{L}^r_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}) : \partial_z^r f \in \mathrm{L}^r(G)^2 \Bigr\} \subset \mathrm{L}^r_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}) \to \mathrm{L}^r(G)^2.$$

Furthermore, $A_{0,r}^{\mathrm{ocn}}$ represents the hydrostatic Stokes operator for the ocean on $\mathrm{L}^r_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$ and subject to homogeneous boundary conditions as introduced in (5.8). We recall from Lemma 2.7.5 that $(A_{0,q}^{\mathrm{ocn}})' = A_{0,q'}^{\mathrm{ocn}}$ for ${}^1/_q + {}^1/_{q'} = 1$.

The adjoint L_0' of the hydrostatic Dirichlet operator L_0 is linked to the hydrostatic Stokes operator A_0^{ocn} as follows.

Lemma 5.3.7. *Let $q, q' \in (1,\infty)$ with ${}^1/_q + {}^1/_{q'} = 1$. Then for the adjoint L_0' of L_0, it is valid that*

$$L_0' = \partial_z^{q'} R(0, A_{0,q}^{\mathrm{ocn}})' = \partial_z^{q'} R(0, A_{0,q'}^{\mathrm{ocn}}). \tag{5.22}$$

Proof. Consider $\varphi \in \mathrm{C}^\infty_{\mathrm{per}}(G)^2$ and $k \in \mathrm{L}^{q'}_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$. We have $(A_{0,q}^{\mathrm{ocn}})' = A_{0,q'}^{\mathrm{ocn}}$ and further define $f := L_0 \varphi$ and $g := (A_{0,q'}^{\mathrm{ocn}})^{-1} k$. The function g is well-defined in view of Lemma 2.7.1(b). Moreover, we obtain $f \in \mathrm{D}(A_{\mathrm{m}}^{\mathrm{ocn}}) \subset \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$ and $g \in \mathrm{D}(A_{0,q'}^{\mathrm{ocn}}) \subset \mathrm{L}^{q'}_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$. Recalling the periodic boundary conditions on the lateral boundary in conjunction with an integration by parts, we deduce

$$\begin{aligned}
\langle \nabla_{\mathrm{H}} \pi_{\mathrm{ocn}}, g \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} &= \int_G \int_{-h_{\mathrm{ocn}}}^0 \nabla_{\mathrm{H}} \pi_{\mathrm{ocn}} \cdot g \,\mathrm{d}z \,\mathrm{d}x_{\mathrm{H}} \\
&= h_{\mathrm{ocn}} \int_G \nabla_{\mathrm{H}} \pi_{\mathrm{ocn}} \cdot \overline{g} \,\mathrm{d}x_{\mathrm{H}} \\
&= -h_{\mathrm{ocn}} \int_G \pi_{\mathrm{ocn}} \cdot \mathrm{div}_{\mathrm{H}} \, \overline{g} \,\mathrm{d}x_{\mathrm{H}} = 0.
\end{aligned}$$

In a similar way, it follows that $\langle f, \nabla_{\mathrm{H}} \pi_{\mathrm{ocn}} \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} = 0$. Combining the latter two identities and additionally making use of Green's second identity and the horizontal periodicity, we infer

$$\begin{aligned}
&\langle \Delta f + \nabla_{\mathrm{H}} \pi_{\mathrm{ocn}}, g \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} - \langle f, \Delta g + \nabla_{\mathrm{H}} \pi_{\mathrm{ocn}} \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} \\
&= \langle \Delta f, g \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} - \langle f, \Delta g \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} \\
&= \int_G \partial_z f(x_{\mathrm{H}}, 0) g(x_{\mathrm{H}}, 0) \,\mathrm{d}x_{\mathrm{H}} + \int_G \partial_z f(x_{\mathrm{H}}, -h_{\mathrm{ocn}}) g(x_{\mathrm{H}}, -h_{\mathrm{ocn}}) \,\mathrm{d}x_{\mathrm{H}} \\
&\quad - \int_G f(x_{\mathrm{H}}, 0) \partial_z g(x_{\mathrm{H}}, 0) \,\mathrm{d}x_{\mathrm{H}} - \int_G f(x_{\mathrm{H}}, -h_{\mathrm{ocn}}) \partial_z g(x_{\mathrm{H}}, -h_{\mathrm{ocn}}) \,\mathrm{d}x_{\mathrm{H}}.
\end{aligned}$$

By virtue of $g \in \mathrm{D}(A_{0,q'}^{\mathrm{ocn}})$, we have $g(x_{\mathrm{H}}, 0) = g(x_{\mathrm{H}}, -h_{\mathrm{ocn}}) = 0$. On the other hand, by construction, it holds that $f(x_{\mathrm{H}}, -h_{\mathrm{ocn}}) = 0$ and $f(x_{\mathrm{H}}, 0) = \varphi$. As a result, we obtain

$$\langle f, \Delta g + \nabla_{\mathrm{H}} \pi_{\mathrm{ocn}} \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} = \langle \Delta f + \nabla_{\mathrm{H}} \pi_{\mathrm{ocn}}, g \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} + \int_G \varphi(x_{\mathrm{H}}) \partial_z g(x_{\mathrm{H}}, 0) \,\mathrm{d}x_{\mathrm{H}}.$$

Together with the definition of the hydrostatic Stokes operator A_0^{ocn} from (5.8) and $A_{\mathrm{m}}^{\mathrm{ocn}} f = 0$ on Ω_{ocn}, which in turn follows by construction, the latter identity leads to

$$\begin{aligned}
\langle L_0 \varphi, k \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} &= \langle f, A_{0,q'}^{\mathrm{ocn}} g \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} \\
&= \langle f, \Delta g + \nabla_{\mathrm{H}} \pi_{\mathrm{ocn}} \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} \\
&= \langle \Delta f + \nabla_{\mathrm{H}} \pi_{\mathrm{ocn}}, g \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} + \int_G \varphi(x_{\mathrm{H}}) \partial_z g(x_{\mathrm{H}}, 0) \,\mathrm{d}x_{\mathrm{H}} \\
&= \langle A_{\mathrm{m}}^{\mathrm{ocn}} f, g \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} + \int_G \varphi(x_{\mathrm{H}}) \partial_z g(x_{\mathrm{H}}, 0) \,\mathrm{d}x_{\mathrm{H}} \\
&= \langle \varphi, \partial_z ((A_{0,q'}^{\mathrm{ocn}})^{-1} k)_{\Gamma_o} \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})}.
\end{aligned}$$

This proves the assertion. □

Now, we justify that the right-hand side of the identity (5.22) of the adjoint L_0' can be extended to a bounded operator.

Lemma 5.3.8. *The normal derivative ∂_z^r is relatively $(-A_{0,r}^{\mathrm{ocn}})^\delta$-bounded provided $\delta > 1/2 + 1/2r$.*

Proof. We recall from Lemma 1.3.2 as well as $\mathrm{B}_{qq}^s \hookrightarrow \mathrm{L}^q$ for all $s > 0$ that in the half-space case, the normal derivative is bounded from $\mathrm{H}^{r,q}$ to L^q provided $r > 1 + 1/q$. Let us observe that horizontally periodic functions can be extended periodically onto a layer and then be cut off, so the latter result also holds in the present situation. This yields

$$\mathrm{H}_{\mathrm{per}}^{r,q}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}_{\overline{\sigma}}^q(\Omega_{\mathrm{ocn}}) \hookrightarrow \mathrm{D}(\partial_z^r)$$

in the case $r > 1 + 1/q$. As on the other hand, Lemma 5.3.2(c) implies that $\mathrm{D}((A_0^{\mathrm{ocn}})^\delta) \hookrightarrow \mathrm{H}_{\mathrm{per}}^{2\delta,q}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}_{\overline{\sigma}}^q(\Omega_{\mathrm{ocn}})$, the assertion follows. □

Putting together Lemma 5.3.7 and Lemma 5.3.8, we find that the adjoint L_0' of L_0 can be extended to a bounded operator from $\mathrm{L}_{\overline{\sigma}}^{q'}(\Omega_{\mathrm{ocn}})$ to $\mathrm{L}^{q'}(G)^2$, paving the way to the result below on the hydrostatic Dirichlet operator.

Proposition 5.3.9. *The operator L_0 admits an extension to a bounded operator from $\mathrm{L}^q(G)^2$ to $\mathrm{L}_{\overline{\sigma}}^q(\Omega_{\mathrm{ocn}})$, still denoted by L_0 by abuse of notation. Furthermore, for $C > 0$, the unique solution $v_{\mathrm{ocn}} = L_0 \varphi$ to (5.17) satisfies*

$$\|v_{\mathrm{ocn}}\|_{\mathrm{L}_{\overline{\sigma}}^q(\Omega_{\mathrm{ocn}})} \le C \cdot \|\varphi\|_{\mathrm{L}^q(G)}.$$

In the following, we discuss that higher regularity of the boundary data also results in an increase in regularity of the solution to the stationary hydrostatic Stokes problem (5.17).

Proposition 5.3.10. *Let $p, q \in (1, \infty)$ as well as $r < 1/q$.*

(a) For $\varphi \in \mathrm{L}^q(G)^2$, the unique solution v_{ocn} to (5.17) already has the property that $v_{\mathrm{ocn}} \in \mathrm{H}^{r,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$.

(b) If $\varphi \in \mathrm{H}^{s,q}_{\mathrm{per}}(G)^2$ for $s \in (0, 2]$, then the unique solution v_{ocn} to (5.17) fulfills $v_{\mathrm{ocn}} \in \mathrm{H}^{s+r,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$.

(c) For $s \in (0, 2]$ and $\varphi \in \mathrm{B}^s_{qp,\mathrm{per}}(\Omega_{\mathrm{ocn}})^2$, the unique solution v_{ocn} to (5.17) satisfies $v_{\mathrm{ocn}} \in \mathrm{B}^{s+r}_{qp,\mathrm{per}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$.

(d) In particular, L_0 is bounded from $\mathrm{H}^{s,q}_{\mathrm{per}}(G)^2$ to $\mathrm{H}^{s,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$ and from $\mathrm{B}^s_{qp,\mathrm{per}}(G)^2$ to $\mathrm{B}^s_{qp,\mathrm{per}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$, where $s \in (0, 2]$, so the solution v_{ocn} to (5.17) admits the estimates

$$\|v_{\mathrm{ocn}}\|_{\mathrm{H}^{s,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})} \le C \cdot \|\varphi\|_{\mathrm{H}^{s,q}_{\mathrm{per}}(G)} \quad \text{and} \quad \|v_{\mathrm{ocn}}\|_{\mathrm{B}^s_{qp,\mathrm{per}}(\Omega_{\mathrm{ocn}})} \le C \cdot \|\varphi\|_{\mathrm{B}^s_{qp,\mathrm{per}}(G)}.$$

Proof. First, we investigate the case $s \in [0, 2 - 1/q]$ as well as $\varphi \in \mathrm{H}^{s,q}_{\mathrm{per}}(G)^2$. With regard to Proposition 5.3.9, we deduce the existence of a unique solution $v_{\mathrm{ocn}} \in \mathrm{D}(A^{\mathrm{ocn}}_{\mathrm{m}}) \subset \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$ to (5.17). This solution especially satisfies

$$\mathbb{P}_{\mathrm{ocn}} \Delta v_{\mathrm{ocn}} = A^{\mathrm{ocn}}_{\mathrm{m}} v_{\mathrm{ocn}} = 0 \in \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}).$$

We then deduce from the surjectivity of the hydrostatic Helmholtz projection $\mathbb{P}_{\mathrm{ocn}} \colon \mathrm{L}^q(\Omega_{\mathrm{ocn}})^2 \to \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$ that $\Delta v_{\mathrm{ocn}} \in \mathrm{L}^q(\Omega_{\mathrm{ocn}})^2$. As v_{ocn} solves the inhomogeneous problem

$$\begin{cases} \Delta v_{\mathrm{ocn}} = \nabla_{\mathrm{H}} \pi_{\mathrm{ocn}}, & \text{on } \Omega_{\mathrm{ocn}}, \\ v_{\mathrm{ocn}} = \varphi, & \text{on } \Gamma_o, \\ v_{\mathrm{ocn}} = 0, & \text{on } \Gamma_b, \end{cases}$$

and we have $\Delta v_{\mathrm{ocn}} \in \mathrm{L}^q(\Omega_{\mathrm{ocn}})^2$, we also find $\nabla_{\mathrm{H}} \pi_{\mathrm{ocn}} \in \mathrm{L}^q(\Omega_{\mathrm{ocn}})^2$. By standard regularity theory of the Laplacian, we obtain $v_{\mathrm{ocn}} \in \mathrm{H}^{s+r,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2$ for $r < 1/q$. In particular, the assertion of (a) follows.

Next, we focus on the case $s \ge 2 - 1/q$ and consider $\varphi \in \mathrm{H}^{s,q}_{\mathrm{per}}(G)^2$. The previous step then yields $v_{\mathrm{ocn}} \in \mathrm{H}^{2,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$. In addition, employing the representation of $\nabla_{\mathrm{H}} \pi_{\mathrm{ocn}}$ as discussed in (2.40), we find that

$$\nabla_{\mathrm{H}} \pi_{\mathrm{ocn}} = \frac{1}{h_{\mathrm{ocn}}} \nabla_{\mathrm{H}} \Delta_{\mathrm{H}}^{-1} \mathrm{div}_{\mathrm{H}}\, \partial_z v_{\mathrm{ocn}},$$

resulting in $\nabla_{\mathrm{H}}\pi_{\mathrm{ocn}} \in \mathrm{H}^{1-1/q,q}_{\mathrm{per}}(G)^2 \hookrightarrow \mathrm{H}^{1-1/q,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2$. The case $s \in (2-1/q, 2]$ can then be obtained by a bootstrap argument, showing (b). Finally, (c) follows by real interpolation, while (d) is a consequence of (b) and (c). □

We also comment on the situation of the general hydrostatic Stokes problem

$$\begin{cases} A^{\mathrm{ocn}}_{\mathrm{m}} v_{\mathrm{ocn}} = f, & \text{on } \Omega_{\mathrm{ocn}}, \\ v_{\mathrm{ocn}} = \varphi, & \text{on } \Gamma_o, \\ v_{\mathrm{ocn}} = 0, & \text{on } \Gamma_b, \end{cases}$$

where $f \in \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})$ and $\varphi \in \mathrm{L}^q(G)^2$. By virtue of the invertibility of A^{ocn}_0 from Lemma 5.3.2, Proposition 5.3.9 and the linearity of the hydrostatic Stokes operator, the above problem admits a unique solution $v_{\mathrm{ocn}} \in \mathrm{D}(A^{\mathrm{ocn}}_{\mathrm{m}})$ which takes the shape $v_{\mathrm{ocn}} = -R(0, A^{\mathrm{ocn}}_0)f + L_0\varphi$.

Based on the considerations on the hydrostatic Dirichlet operator, we now introduce the so-called *hydrostatic Dirichlet-to-Neumann operator*. For the hydrostatic Dirichlet operator L_0 from (5.18), it is defined by

$$N_0\varphi := \partial^q_z L_0\varphi, \text{ with } \mathrm{D}(N_0) := \left\{\varphi \in \mathrm{L}^q(G)^2 : L_0\varphi \in \mathrm{D}(\partial^q_z)\right\}. \tag{5.23}$$

With regard to the regularity theory of the inhomogeneous stationary hydrostatic Stokes problem, we infer the following result on the domain of the hydrostatic Dirichlet-to-Neumann operator.

Proposition 5.3.11. *Let N_0 be the hydrostatic Dirichlet-to-Neumann operator as introduced in* (5.23). *Then $\mathrm{H}^{s,q}_{\mathrm{per}}(G)^2 \subset \mathrm{D}(N_0)$ holds for all $s > 1$.*

Proof. Similarly as in the proof of Lemma 5.3.8, we find that

$$\mathrm{H}^{r',q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}) \subset \mathrm{D}(\partial^q_z)$$

for $r' > 1 + 1/q$. On the other hand, for $r < 1/q$, Proposition 5.3.10 implies

$$L_0\left(\mathrm{H}^{s,q}_{\mathrm{per}}(G)^2\right) \subset \mathrm{H}^{s+r,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}) \subset \mathrm{D}(\partial^q_z)$$

provided $s + r > 1 + 1/q$, i. e., $s > 1$. This yields $\mathrm{H}^{s,q}_{\mathrm{per}}(G)^2 \subset \mathrm{D}(N_0)$. □

The next result on the relative boundedness of the hydrostatic Dirichlet-to-Neumann operator N_0 with respect to fractional powers of the Hibler operator on G is a direct consequence of Proposition 5.3.11 and the shape of the fractional power domains of $-A^{\mathrm{H}}(u_0) + \omega$ as revealed in Lemma 5.3.4(c).

Corollary 5.3.12. *Let $q \in (1, \infty)$ and $u_0 \in \mathrm{C}^1(G)^4$ with $h_0 \geq \kappa_1$, and recall Hibler's operator on G from* (5.10) *as well as $\omega > \omega_0$ from Lemma 5.3.4. Then the hydrostatic Dirichlet-to-Neumann operator N_0 from* (5.23) *is relatively $(-A^{\mathrm{H}} + \omega)^\delta$-bounded for $\delta > 1/2$.*

5.4. Linear Theory of the Coupled System

This section discusses properties of the operator matrix $A(u_0)$ from (5.14). The investigation is based on a decoupling argument and the consideration of the operator as a block operator matrix, so it remains to verify that it lies within the scope of Proposition 2.3.24. Before proceeding, let us remark that we omit writing the dependence on $u_0 = (v_{\mathrm{ice},0}, h_0, a_0)$ explicitly in the remainder of this section as $u_0 \in \mathrm{C}^1(G)^4$ with $h_0 \geq \kappa_1$ is fixed.

We recall the hydrostatic Dirichlet operator L_0 as well as the hydrostatic Dirichlet-to-Neumann operator N_0 from (5.18) and (5.23), while the other operators are as introduced in Section 5.2. As a first step, we define the operator matrix $\tilde{A}$ of a more complicated shape than the one from (5.14) by

$$\tag{5.24} \begin{pmatrix} -A^{\mathrm{atm}} & 0 & 0 & 0 & 0 \\ 0 & -A_0^{\mathrm{ocn}} - L_0 C_{\mathrm{o,i}} \partial_z & L_0(A^{\mathrm{H}} - C_{\mathrm{o,i}} N_0) & 0 & 0 \\ 0 & C_{\mathrm{o,i}} \partial_z & -A^{\mathrm{H}} + C_{\mathrm{o,i}} N_0 & B_{\mathrm{h}} & B_{\mathrm{a}} \\ 0 & 0 & 0 & -d_{\mathrm{h}} \Delta_{\mathrm{H}} & 0 \\ 0 & 0 & 0 & 0 & -d_{\mathrm{a}} \Delta_{\mathrm{H}} \end{pmatrix},$$

with domain $\mathrm{D}(\tilde{A}) \coloneqq \mathrm{D}(A^{\mathrm{atm}}) \times \mathrm{D}(A_0^{\mathrm{ocn}}) \times \mathrm{D}(A^{\mathrm{H}}) \times \mathrm{D}(\Delta_{\mathrm{H}}) \times \mathrm{D}(\Delta_{\mathrm{H}})$, and on the ground space X_0 from (5.12). The advantage of $\tilde{A}$ over A is the diagonal domain. Below, we reveal the link of the operator matrices A and $\tilde{A}$.

Lemma 5.4.1. *The operator matrix A from* (5.14) *and the above operator matrix $\tilde{A}$ from* (5.24) *are isomorphic on* X_0.

Proof. We provide the similarity transform explicitly. In fact, it is given by

$$S = \begin{pmatrix} \mathrm{Id} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Id} & -L_0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id} \end{pmatrix} \quad \text{and} \quad S^{-1} = \begin{pmatrix} \mathrm{Id} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Id} & L_0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id} \end{pmatrix}.$$

Let us recall from Proposition 5.3.9 that indeed, $L_0 \in \mathcal{L}(\mathrm{L}^q(G)^2, \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}))$, yielding $S \in \mathcal{L}(\mathrm{X}_0)$ and $S^{-1} \in \mathcal{L}(\mathrm{X}_0)$. A direct computation further exhibits $\tilde{A} = SAS^{-1}$. Moreover, by standard regularity theory, it follows that $\tilde{v}_{\mathrm{ocn}} \in \mathrm{D}(A_{\mathrm{m}}^{\mathrm{ocn}})$ with homogeneous boundary conditions already satisfies $\tilde{v}_{\mathrm{ocn}} \in \mathrm{D}(A_0^{\mathrm{ocn}})$. Hence, as $\tilde{v}_{\mathrm{ocn}} \coloneqq v_{\mathrm{ocn}} - L_0 v_{\mathrm{ice}}$ for $v \in \mathrm{D}(A) = \mathrm{X}_1$ has precisely these properties, it follows that $\mathrm{D}(\tilde{A}) = S\mathrm{D}(A)$. In total, A and $\tilde{A}$ are isomorphic, as desired. □

For $v = (v_{\mathrm{atm}}, v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a) \in \mathrm{D}(A)$, with $\mathrm{D}(A) = \mathrm{X}_1$ as introduced in (5.13), we deduce from Lemma 5.4.1 and its proof that $v_{\mathrm{ocn}} = \tilde{v}_{\mathrm{ocn}} + L_0 \tilde{v}_{\mathrm{ice}}$, where $\tilde{v}_{\mathrm{ocn}} \in \mathrm{D}(A_0^{\mathrm{ocn}})$ and $\tilde{v}_{\mathrm{ice}} \in \mathrm{W}_{\mathrm{per}}^{2,q}(G)^2$. In conjunction with Proposition 5.3.10, this yields $v_{\mathrm{ocn}} \in \mathrm{W}_{\mathrm{per}}^{2,q}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}_{\overline{\sigma}}^q(\Omega_{\mathrm{ocn}}) =: \mathrm{D}(A^{\mathrm{ocn}})$. Hence,

$$
\begin{aligned}
\mathrm{X}_1 = \Big\{ & v \in \mathrm{D}(A^{\mathrm{atm}}) \times \mathrm{D}(A^{\mathrm{ocn}}) \times \mathrm{D}(A^{\mathrm{H}}) \times \mathrm{D}(\Delta_{\mathrm{H}}) \times \mathrm{D}(\Delta_{\mathrm{H}}) : \\
& v_{\mathrm{ocn}} = 0, \text{ on } \Gamma_b, \text{ and } v_{\mathrm{ocn}} = v_{\mathrm{ice}}, \text{ on } \Gamma_o \Big\}.
\end{aligned} \tag{5.25}
$$

Thanks to the diagonal domain of the operator matrix $\tilde{A}$, it is possible to investigate its individual components. Regarding the bounded $\mathcal{H}^\infty$-calculus, it also proves useful to split the complete matrix $\tilde{A}$ into smaller blocks. First, we separate the hydrostatic Stokes operator for the atmosphere, so

$$
\tilde{A} = \begin{pmatrix} -A^{\mathrm{atm}} & 0 \\ 0 & J \end{pmatrix}, \text{ with } \mathrm{D}(\tilde{A}) = \mathrm{D}(A^{\mathrm{atm}}) \times \mathrm{D}(J).
$$

The smaller block J then has the structure

$$
J = \begin{pmatrix} J_1 & B' \\ 0 & J_2 \end{pmatrix}, \text{ with } \mathrm{D}(J) = \mathrm{D}(J_1) \times \mathrm{D}(J_2) \text{ and}
$$

$$
B' = \begin{pmatrix} 0 & 0 \\ B_{\mathrm{h}} & B_{\mathrm{a}} \end{pmatrix} \text{ as well as } J_2 = \operatorname{diag}(-d_{\mathrm{h}}\Delta_{\mathrm{H}}, -d_{\mathrm{a}}\Delta_{\mathrm{H}}).
$$

The resulting block operator matrix J_1 takes the shape

$$
J_1 = \begin{pmatrix} -A_0^{\mathrm{ocn}} - L_0 C_{\mathrm{o,i}} \partial_z & L_0(A^{\mathrm{H}} - C_{\mathrm{o,i}} N_0) \\ C_{\mathrm{o,i}} \partial_z & -A^{\mathrm{H}} + C_{\mathrm{o,i}} N_0 \end{pmatrix}, \text{ where}
$$
$$
\mathrm{D}(J_1) = \mathrm{D}(A_0^{\mathrm{ocn}}) \times \mathrm{D}(A^{\mathrm{H}}).
$$

Concerning the bounded $\mathcal{H}^\infty$-calculus, we first verify this property for J_1.

Lemma 5.4.2. *Let $q \in (1,\infty)$ as well as $u_0 \in \mathrm{C}^1(G)^4$ with $h_0 \geq \kappa_1$. Then there is $\omega_0 \in \mathbb{R}$ such that for all $\omega > \omega_0$, the shifted operator matrix $J_1 + \omega$ satisfies $J_1 + \omega \in \mathcal{H}^\infty(\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}) \times \mathrm{L}^q(G)^2)$ with $\mathcal{H}^\infty$-angle $\phi^\infty_{J_1+\omega} < \pi/2$.*

Proof. As we have already indicated, the main idea is to use the theory on bounded $\mathcal{H}^\infty$-calculus for block operator matrices as presented in Section 2.3. More precisely, we first apply Proposition 2.3.24 to the simplified version $\tilde{J}_1$ of J_1, defined by

$$\tilde{J}_1 := \begin{pmatrix} -A_0^{\mathrm{ocn}} & L_0 A^{\mathrm{H}} \\ C_{\mathrm{o,i}}\partial_z & -A^{\mathrm{H}} \end{pmatrix}, \quad \text{with } \mathrm{D}(\tilde{J}_1) = \mathrm{D}(A_0^{\mathrm{ocn}}) \times \mathrm{D}(A^{\mathrm{H}}),$$

and then exploit perturbation theory.

For $\omega_0 \in \mathbb{R}$ from Lemma 5.3.3 and $\omega > \omega_0$, we study $\tilde{J}_{1,\omega} := \tilde{J}_1 + \omega \,\mathrm{Id}$ in the sequel. First, we establish the diagonal dominance of $\tilde{J}_{1,\omega}$. From Lemma 5.3.2 and Lemma 5.3.3, we recall that $-A_0^{\mathrm{ocn}} \in \mathcal{H}^\infty(\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}))$ with $\phi^\infty_{-A_0^{\mathrm{ocn}}} = 0$ and $-A^{\mathrm{H}} + \omega \in \mathcal{H}^\infty(\mathrm{L}^q(G)^2)$ with $\phi^\infty_{-A^{\mathrm{H}}+\omega} < \pi/2$, so $-A_0^{\mathrm{ocn}} + \omega$ and $-A^{\mathrm{H}} + \omega$ are especially closed and densely defined. Moreover, for $\delta > 1/2 + 1/2q$, we deduce from Lemma 5.3.8 that

$$C_{\mathrm{o,i}}\partial_z \in \mathcal{L}\left(\mathrm{D}((-A_0^{\mathrm{ocn}})^\delta), \mathrm{L}^q(G)^2\right).$$

The last assertion remains valid for $-A_0^{\mathrm{ocn}} + \omega$. Hence, Remark 2.3.23 yields the relative $(-A_0^{\mathrm{ocn}} + \omega)$-boundedness of $C_{\mathrm{o,i}}\partial_z$, and the $(-A_0^{\mathrm{ocn}} + \omega)$-bound is arbitrarily small. On the other hand, we conclude from Proposition 5.3.9 the existence of a constant $C_1 > 0$ such that

$$\|L_0 A^{\mathrm{H}} v_{\mathrm{ice}}\|_{\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})} \leq C_1 \cdot \left(\|(-A^{\mathrm{H}} + \omega) v_{\mathrm{ice}}\|_{\mathrm{L}^q(G)} + \|v_{\mathrm{ice}}\|_{\mathrm{L}^q(G)}\right)$$

for all $v_{\mathrm{ice}} \in \mathrm{D}(A^{\mathrm{H}})$. Therefore, the diagonal dominance of $\tilde{J}_{1,\omega}$ as presented in Definition 2.3.22 indeed follows.

In order to show the bounded $\mathcal{H}^\infty$-calculus of $\tilde{J}_1$, it remains to verify (2.19). For this purpose, consider $\delta \in (0, 1/2q)$. From Lemma 5.3.2(c) as well as Lemma 5.3.4(c), we deduce the embeddings and identities

$$\begin{aligned} \mathrm{D}((-A_0^{\mathrm{ocn}})^{1+\delta}) &\hookrightarrow \mathrm{H}^{2+2\delta,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}), \\ \mathrm{D}((-A_0^{\mathrm{ocn}})^{\delta}) &= \mathrm{H}^{2\delta,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}), \\ \mathrm{D}((-A^{\mathrm{H}} + \omega)^{1+\delta}) &\hookrightarrow \mathrm{H}^{2+2\delta,q}_{\mathrm{per}}(G)^2 \text{ and } \mathrm{D}((-A^{\mathrm{H}} + \omega)^{\delta}) \hookrightarrow \mathrm{H}^{2\delta,q}_{\mathrm{per}}(G)^2. \end{aligned}$$

Let us observe that the fractional power domains of $-A_0^{\mathrm{ocn}}+\omega$ take the same shape, see also the remark after Lemma 5.3.2. Besides, as in the proof of Lemma 5.3.8, it especially follows that

$$C_{\mathrm{o,i}}\partial_z\left(\mathrm{D}((-A_0^{\mathrm{ocn}}+\omega)^{1+\delta})\right)\subset \mathrm{H}^{2\delta,q}_{\mathrm{per}}(G)^2=\mathrm{D}((-A^{\mathrm{H}}+\omega)^{\delta}).$$

On the other hand, as a consequence of Proposition 5.3.10, we obtain

$$\begin{aligned}L_0A^{\mathrm{H}}\left(\mathrm{D}((-A^{\mathrm{H}}+\omega)^{1+\delta})\right)&=L_0\left(\mathrm{D}((-A^{\mathrm{H}}+\omega)^{\delta})\right)=L_0\left(\mathrm{H}^{2\delta,q}_{\mathrm{per}}(G)^2\right)\\&\subset \mathrm{H}^{2\delta,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2\cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})=\mathrm{D}((-A_0^{\mathrm{ocn}}+\omega)^{\delta}).\end{aligned}$$

In addition, thanks to the operators L_0A^{H} as well as $C_{\mathrm{o,i}}\partial_z$ being closed, the estimates in (2.19) follow from the closed graph theorem. Hence, Proposition 2.3.24 yields the existence of $\omega_1>0$ such that for every $\omega>\omega_1$, it holds that $\tilde{J}_1+\omega\in\mathcal{H}^\infty(\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})\times\mathrm{L}^q(G)^2)$ with $\mathcal{H}^\infty$-angle $\phi^\infty_{\tilde{J}_1+\omega}<\pi/2$.

With regard to the perturbation argument, we consider $\omega>0$ sufficiently large such that $0\in\rho(\tilde{J}_1+\omega)$. Considering $v_{\mathrm{ocn}}\in\mathrm{D}(A_0^{\mathrm{ocn}})$ and concatenating Proposition 5.3.9 for the boundedness of L_0 and Lemma 5.3.8 for the relative boundedness of ∂_z with respect to $-A_0^{\mathrm{ocn}}$, for $\delta>1/2+1/2q$, we first get

$$\begin{aligned}\|-L_0C_{\mathrm{o,i}}\partial_z v_{\mathrm{ocn}}\|_{\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})}&\le C_2\cdot\|\partial_z v_{\mathrm{ocn}}\|_{\mathrm{L}^q(G)}\\&\le C_3\cdot\|(-A_0^{\mathrm{ocn}}+\omega)^{\delta}v_{\mathrm{ocn}}\|_{\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})}.\end{aligned}$$

At the same time, Proposition 5.3.9 and the relative $(-A^{\mathrm{H}}+\omega)$-boundedness of N_0 as asserted in Corollary 5.3.12 imply

$$\begin{aligned}\|-L_0C_{\mathrm{o,i}}N_0 v_{\mathrm{ice}}\|_{\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})}&\le C_4\cdot\|N_0 v_{\mathrm{ice}}\|_{\mathrm{L}^q(G)}\\&\le C_5\cdot\|(-A^{\mathrm{H}}+\omega)^{\delta}v_{\mathrm{ice}}\|_{\mathrm{L}^q(G)}\end{aligned}$$

for $v_{\mathrm{ice}}\in\mathrm{D}(A^{\mathrm{H}})$ and $\delta>1/2$. In total, setting

$$B:=\begin{pmatrix}-L_0C_{\mathrm{o,i}}\partial_z & -L_0C_{\mathrm{o,i}}N_0\\ 0 & C_{\mathrm{o,i}}N_0\end{pmatrix},$$

and taking into account $(v_{\mathrm{ocn}},v_{\mathrm{ice}})\in\mathrm{D}(\tilde{J}_1)$, for $\delta>1/2+1/2q$, we derive that

$$\left\|B\begin{pmatrix}v_{\mathrm{ocn}}\\ v_{\mathrm{ice}}\end{pmatrix}\right\|_{\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})\times\mathrm{L}^q(G)}\le C_6\cdot\left\|(\tilde{J}_1+\omega)^{\delta}\begin{pmatrix}v_{\mathrm{ocn}}\\ v_{\mathrm{ice}}\end{pmatrix}\right\|_{\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})\times\mathrm{L}^q(G)}.$$

The assertion of the lemma is then implied by the perturbation result on the bounded $\mathcal{H}^\infty$-calculus, Lemma 2.3.17. □

The preceding lemma enables us to state and prove the main result of this section on the bounded $\mathcal{H}^\infty$-calculus of the complete operator matrix A with non-diagonal domain from (5.14).

Proposition 5.4.3. *Let* $q \in (1, \infty)$*, and consider* $u_0 \in \mathrm{C}^1(G)^4$ *with* $h_0 \geq \kappa_1$*. Then there exists* $\omega_0 \in \mathbb{R}$ *so that* $A + \omega$*, with* A *from* (5.14)*, admits a bounded* $\mathcal{H}^\infty$*-calculus for all* $\omega > \omega_0$*, so* $A + \omega \in \mathcal{H}^\infty(\mathrm{X}_0)$ *with* $\mathcal{H}^\infty$*-angle* $\phi_{A+\omega} < \pi/2$*.*

Proof. The idea is to establish first the bounded $\mathcal{H}^\infty$-calculus of the operator matrix with diagonal domain $\tilde{A}$ from (5.24) by making use of the previous lemma and by invoking the respective properties of the hydrostatic Stokes operator of the atmosphere and the Laplacian operators with periodic boundary conditions. In a second step, we exploit that A and $\tilde{A}$ are isomorphic.

From Lemma 2.3.21, we first recall that $J_2 + \omega \in \mathcal{H}^\infty(\mathrm{L}^q(G) \times \mathrm{L}^q(G))$ for all $\omega > 0$. Together with Lemma 5.4.2, we thus get the bounded $\mathcal{H}^\infty$-calculus of $\operatorname{diag}(J_1, J_2) + \omega$ on $\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}) \times \mathrm{L}^q(G)^2 \times \mathrm{L}^q(G) \times \mathrm{L}^q(G)$ for all $\omega > \omega_0$, where $\omega_0 \in \mathbb{R}$, from Lemma 5.4.2. Similarly as in (3.27), also invoking the fractional powers of the negative Laplacian from Lemma 2.3.21, it follows that

$$\left\| \frac{\partial_h P(h_0, a_0)}{2\rho_{\mathrm{ice}} h_0} \nabla_{\mathrm{H}} h \right\|_{\mathrm{L}^q(G)} \leq C_1 \cdot \|h\|_{\mathrm{W}^{1,q}_{\mathrm{per}}(G)} \leq C_2 \cdot \|(-\Delta_{\mathrm{H}} + \omega)^{1/2} h\|_{\mathrm{L}^q(G)}.$$

The other term B_{a} from (5.11) can be dealt with likewise, so it follows that

$$\left\| B' \begin{pmatrix} h \\ a \end{pmatrix} \right\|_{\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}) \times \mathrm{L}^q(G)} \leq C_3 \cdot \left\| (J_2 + \omega)^{1/2} \right\|_{\mathrm{L}^q(G) \times \mathrm{L}^q(G)}.$$

Hence, choosing $\omega > 0$ sufficiently large such that $0 \in \rho(\operatorname{diag}(J_1, J_2) + \omega)$, we conclude from Lemma 2.3.17 that

$$J + \omega \in \mathcal{H}^\infty \left(\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}) \times \mathrm{L}^q(G)^2 \times \mathrm{L}^q(G) \times \mathrm{L}^q(G) \right)$$

for $\omega > \omega_0$, where $\omega_0 \in \mathbb{R}$ is possibly larger than before. Moreover, we get the $\mathcal{H}^\infty$-angle $\phi^\infty_{J+\omega} < \pi/2$. By virtue of the bounded $\mathcal{H}^\infty$-calculus of A^{atm}, up to a shift, as asserted in Lemma 5.3.1, we infer that $\tilde{A} + \omega \in \mathcal{H}^\infty(\mathrm{X}_0)$ for $\omega > \omega_0$, and $\phi^\infty_{\tilde{A}+\omega} < \pi/2$.

Finally, the claim of the proposition follows from $\tilde{A}$ and A being isomorphic on X_0, as revealed in Lemma 5.4.1, and the preservation of the bounded $\mathcal{H}^\infty$-calculus and the $\mathcal{H}^\infty$-angle by similarity transforms as discussed in Lemma 2.3.14. □

From the preceding proposition, we draw further conclusions with regard to the linear theory of the operator matrix. As a preparation, and in order to shorten the notation in the remainder of this section, we introduce the spaces

$$\begin{aligned} \mathrm{Y}_0 &:= \mathrm{X}_0 = \mathrm{L}^q_\sigma(\Omega_{\mathrm{atm}}) \times \mathrm{L}^q_\sigma(\Omega_{\mathrm{ocn}}) \times \mathrm{L}^q(G)^4 \quad \text{and} \\ \mathrm{Y}_1 &:= \mathrm{W}^{2,q}_{\mathrm{per}}(\Omega_{\mathrm{atm}})^2 \cap \mathrm{L}^q_\sigma(\Omega_{\mathrm{atm}}) \times \mathrm{W}^{2,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_\sigma(\Omega_{\mathrm{ocn}}) \times \mathrm{W}^{2,q}_{\mathrm{per}}(G)^4, \end{aligned}$$

where no boundary conditions on the respective upper and lower boundaries are taken into account. In a similar manner as in Section 1.3, upon invoking the brief discussion before Lemma 2.7.3 for the interpolation of periodic boundary conditions, we get

$$\begin{aligned} \mathrm{Y}_\beta &:= [\mathrm{Y}_0, \mathrm{Y}_1]_\beta \\ &= \mathrm{H}^{2\beta,q}_{\mathrm{per}}(\Omega_{\mathrm{atm}})^2 \cap \mathrm{L}^q_\sigma(\Omega_{\mathrm{atm}}) \times \mathrm{H}^{2\beta,q}_{\mathrm{per}}(\Omega_{\mathrm{ocn}})^2 \cap \mathrm{L}^q_\sigma(\Omega_{\mathrm{ocn}}) \times \mathrm{H}^{2\beta,q}_{\mathrm{per}}(G)^4 \end{aligned}$$

for $\beta \in (0,1)$, while for $\theta \in (0,1)$ and $p \in (1,\infty)$, we obtain

$$\begin{aligned} \mathrm{Y}_{\theta,p} &:= (\mathrm{Y}_0, \mathrm{Y}_1)_{\theta,p} \\ &= \mathrm{B}^{2\theta}_{qp,\mathrm{per}}(\Omega_{\mathrm{atm}}) \cap \mathrm{L}^q_\sigma(\Omega_{\mathrm{atm}}) \times \mathrm{B}^{2\theta}_{qp,\mathrm{per}}(\Omega_{\mathrm{ocn}}) \cap \mathrm{L}^q_\sigma(\Omega_{\mathrm{ocn}}) \times \mathrm{B}^{2\theta}_{qp,\mathrm{per}}(G)^4. \end{aligned}$$

In particular, we use Y_γ to denote the space $\mathrm{Y}_{1-1/p,p} = (\mathrm{Y}_0, \mathrm{Y}_1)_{1-1/p,p}$. In the above, no certain values of the regularity parameter need to be omitted as no Dirichlet or Neumann boundary conditions are taken into consideration.

The collection of useful properties of the operator matrix A resulting from Proposition 5.4.3 is given in the following.

Corollary 5.4.4. *Let p, $q \in (1,\infty)$, and consider $u_0 \in \mathrm{C}^1(G)^4$ with $h_0 \ge \kappa_1$. Besides, we recall the notation $A = A(u_0)$ for simplicity.*

(a) Then there is $\omega_0 \in \mathbb{R}$ such that for all $\omega > \omega_0$, we have

(i) $A + \omega \in \mathcal{BIP}(\mathrm{X}_0)$ with power angle $\theta_{A+\omega} < \pi/2$,

(ii) $A + \omega \in \mathcal{RS}(\mathrm{X}_0)$ with $\mathcal{R}$-angle $\phi^{\mathcal{R}}_{A+\omega} < \pi/2$. In particular, $\omega > 0$ can be chosen sufficiently large such that $A + \omega \in \mathcal{MR}_p(\mathrm{X}_0)$,

(iii) $\mathrm{D}((A+\omega)^\beta) \cong [\mathrm{X}_0, \mathrm{X}_1]_\beta =: \mathrm{X}_\beta$ for $\beta \in (0,1) \setminus \{1/2q, 1/2 + 1/2q\}$. Moreover, we have $\mathrm{X}_\beta = \mathrm{Y}_\beta$ if $\beta \in (0, 1/2q)$,

$$\mathrm{X}_\beta = \{v \in \mathrm{Y}_\beta : v_{\mathrm{ocn}} = 0 \ \text{ on } \Gamma_b, \ \text{ and } v_{\mathrm{ocn}} = v_{\mathrm{ice}} \ \text{ on } \Gamma_o\}$$

provided $\beta \in (1/2q, 1/2 + 1/2q)$, and

$$\begin{aligned} \mathrm{X}_\beta = \{v \in \mathrm{Y}_\beta : \partial_z v_{\mathrm{atm}} = 0 \ \text{ on } \Gamma_u \cup \Gamma_i, \ v_{\mathrm{ocn}} = 0 \ \text{ on } \Gamma_b, \ \text{ and } \\ v_{\mathrm{ocn}} = v_{\mathrm{ice}} \ \text{ on } \Gamma_o\}, \end{aligned}$$

in the case $\beta \in (1/2 + 1/2q, 1)$, and *(iv)* $(\mathrm{X}_0, \mathrm{X}_1)_{\theta,p} = \mathrm{Y}_{\theta,p}$ *for* $\theta \in (0, 1/2q)$, *whereas for* $\theta \in (1/2q, 1/2 + 1/2q)$, *it is valid that*

$$(\mathrm{X}_0, \mathrm{X}_1)_{\theta,p} = \{v \in \mathrm{Y}_{\theta,p} : v_{\mathrm{ocn}} = 0 \ \text{ on } \Gamma_b, \ \text{ and } v_{\mathrm{ocn}} = v_{\mathrm{ice}} \ \text{ on } \Gamma_o\},$$

and for $\theta \in (1/2 + 1/2q, 1)$, *we get*

$$(\mathrm{X}_0, \mathrm{X}_1)_{\theta,p} = \{v \in \mathrm{Y}_{\theta,p} : \partial_z v_{\mathrm{atm}} = 0 \ \text{ on } \Gamma_u \cup \Gamma_i, \ v_{\mathrm{ocn}} = 0 \ \text{ on } \Gamma_b, \ \text{ and } v_{\mathrm{ocn}} = v_{\mathrm{ice}} \ \text{ on } \Gamma_o\}.$$

(b) The operator matrix A *has a compact resolvent, and the spectrum* $\sigma(A)$ *of* A *is* q*-independent and only consists of eigenvalues.*

Proof. The assertions of (a)(i) and (ii) are immediate consequences of the relations of the above concepts as made precise in (2.14), Proposition 2.1.21 and Lemma 2.1.14. The first relation in (iii) is implied by (i) in conjunction with Lemma 2.3.4. For the shape of the resulting interpolation spaces, we first deduce from Proposition 5.4.3 the bounded $\mathcal{H}^\infty$-calculus and thus also the boundedness of the imaginary powers of $\tilde{A}$ with diagonal domain $\mathrm{D}(\tilde{A})$ from (5.24). Thus, we derive from Lemma 2.3.4 together with the considerations in Section 1.3 and Section 2.7 that

$$\mathrm{D}((\tilde{A} + \omega)^\beta) \cong [\mathrm{X}_0, \mathrm{D}(\tilde{A})]_\beta =: \tilde{\mathrm{X}}_\beta, \ \ \beta \in (0, 1),$$

and $\tilde{\mathrm{X}}_\beta$ coincides with Y_β complemented by homogeneous Dirichlet boundary in the v_{ocn}-component if $\beta > 1/2q$, and by homogeneous Neumann boundary conditions for v_{atm} in the case $\beta > 1/2 + 1/2q$. The isomorphism S from the proof of Lemma 5.4.1 is also an isomorphism in the category of Banach couples. By the functoriality of the complex interpolation as explained in Section 1.2, we thus infer that

$$[\mathrm{X}_0, \mathrm{X}_1]_\beta = \left\{(v_{\mathrm{atm}}, v_{\mathrm{ocn}} + L_0 v_{\mathrm{ice}}, v_{\mathrm{ice}}, h, a) : (v_{\mathrm{atm}}, v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a) \in \tilde{\mathrm{X}}_\beta\right\}.$$

The concrete shape of $\mathrm{X}_\beta = [\mathrm{X}_0, \mathrm{X}_1]_\beta$ then follows from the above arguments together with the regularity properties of L_0 as shown in Proposition 5.3.10. This shows (a)(iii), and the assertion of (a)(iv) can be obtained analogously.

With regard to (b), we observe that the operator $\tilde{A} \colon \mathrm{D}(\tilde{A}) \to \mathrm{X}_0$ has a compact resolvent thanks to the compact embedding $\mathrm{D}(\tilde{A}) \hookrightarrow \mathrm{X}_0$ as a result of the Rellich-Kondrachov theorem from Lemma 1.4.3. At this stage, we recall the boundedness of the domains in the present study. In view of

$$(\lambda - A) = S^{-1}(\lambda - \tilde{A})S$$

for S as introduced in the proof of Lemma 5.4.1, we find that $\rho(A) = \rho(\tilde{A})$. Furthermore, the compactness of $(\lambda - \tilde{A})^{-1} \colon \mathrm{X}_0 \to \mathrm{D}(\tilde{A})$ implies that

$$(\lambda - A)^{-1} = S^{-1}(\lambda - \tilde{A})^{-1}S \colon \mathrm{X}_0 \to \mathrm{D}(A) = \mathrm{X}_1$$

is also compact for $\lambda \in \rho(A) = \rho(\tilde{A})$. The other two assertions of (b) then follow from Lemma 2.1.6(b) and (c). □

5.5. Local Strong Well-Posedness

In this section, we present the first main result of this chapter on the local strong well-posedness of the coupled system from (5.4), or, equivalently in operator form, from (5.16).

As in Chapter 3, we introduce an open set $V \subset \mathrm{X}_\gamma := (\mathrm{X}_0, \mathrm{X}_1)_{1-1/p,p}$ to make sure that the initial data attain physically relevant values. We set

$$V := \{v \in \mathrm{X}_\gamma : h_0 \in (\kappa_1, \kappa_2) \text{ and } a_0 \in (0,1)\}. \tag{5.26}$$

With regard to the results established in the previous sections, it is natural to impose constraints on p and q such that the ice component embeds into the space $\mathrm{C}^1(G)^4$. Indeed, the assumption

$$1/p + 1/q < 1/2 \tag{5.27}$$

yields $1 - 1/p > 1/2 + 1/2q$, so from the embedding relation (1.8) and Corollary 5.4.4(a)(iv), we deduce that u_0 from $v_0 = (v_{\mathrm{atm},0}, v_{\mathrm{ocn},0}, u_0) \in V$ is contained in C^1, i. e.,

$$u_0 \in \mathrm{C}^1(G)^4. \tag{5.28}$$

Next, we make some assumptions on the external terms. As the atmosphere and ocean velocity are internalized in the present situation, no assumptions on these quantities are required in contrast to Chapter 3.

Assumption 5.5.1. *Let p, $q \in (1, \infty)$ be such that* (5.27) *holds true.*

(a) The external forcing terms f_{atm} and f_{ocn} satisfy $f_{\mathrm{atm}} \in \mathrm{L}^q(\Omega_{\mathrm{atm}})^2$ as well as $f_{\mathrm{ocn}} \in \mathrm{L}^q(\Omega_{\mathrm{ocn}})^2$.

(b) The sea surface dynamic height H fulfills $\nabla_{\mathrm{H}} H \in \mathrm{L}^q(G)^2$.

(c) For the ice growth rate function f_{gr}, it is valid that $f_{\mathrm{gr}} \in \mathrm{C}^1_{\mathrm{b}}([0, \infty))$.

It would also be possible to take into consideration time-dependent external forcing terms. However, for simplicity, and as it fits better into the framework presented in Section 2.6, we focus on the autonomous case in this chapter.

The result below asserts the local strong well-posedness of the coupled system. We do not only state the existence of a unique local strong solution, but we also discuss several features of the resulting solution such as the continuous dependence on the initial data or the characterization of the maximal time interval of existence of the solution.

Theorem 5.5.2. *Let* $p, q \in (1, \infty)$ *be such that* (5.27) *is valid, consider initial data* $v_0 = (v_{\mathrm{atm},0}, v_{\mathrm{ocn},0}, u_0) \in V$*, with* $u_0 = (v_{\mathrm{ice},0}, h_0, a_0)$ *and for* V *as introduced in* (5.26)*, and suppose that the external terms* f_{atm}*,* f_{ocn}*,* H *and* f_{gr} *fulfill Assumption 5.5.1. Moreover, recall the spaces* X_0 *and* X_1 *from* (5.12) *and* (5.13)*, respectively, and take into account* $\mathrm{X}_\gamma = (\mathrm{X}_0, \mathrm{X}_1)_{1-1/p,p}$.

Then there are $T = T(v_0) > 0$ *and* $r = r(v_0) > 0$ *with* $\overline{\mathbb{B}}_{\mathrm{X}_\gamma}(v_0, r) \subset V$*, and for all* $v_1 \in \overline{\mathbb{B}}_{\mathrm{X}_\gamma}(v_0, r)$*, the quasilinear abstract Cauchy problem* (5.16)*, or, equivalently, the complete coupled system of equations* (5.4) *complemented by the boundary conditions* (5.5)*, admits a unique solution*

$$v(\cdot, v_1) \in \mathrm{W}^{1,p}(0, T; \mathrm{X}_0) \cap \mathrm{L}^p(0, T; \mathrm{X}_1) \cap \mathrm{C}([0, T]; V) =: \mathbb{E}_1 \cap \mathrm{C}([0, T]; V).$$

For the solution, we also get the properties stated below.

(a) There is a constant $C = C(u_0) > 0$ *with*

$$\|v(\cdot, v_1) - v(\cdot, v_2)\|_{\mathbb{E}_1} \le C \cdot \|v_1 - v_2\|_{\mathrm{X}_\gamma}$$

for all v_1*,* $v_2 \in \overline{\mathbb{B}}_{\mathrm{X}_\gamma}(v_0, r)$.

(b) The solution exists on a maximal time interval $J(v_0) = [0, t_+(v_0))$*, where* $t_+(v_0)$ *is characterized by*

(i) global existence, so $t_+(v_0) = \infty$*,*

(ii) $\liminf_{t \to t_+(v_0)} \mathrm{dist}_{\mathrm{X}_\gamma}(v(t), \partial V) = 0$*, or*

(iii) $\lim_{t \to t_+(v_0)} v(t)$ *does not exist in* X_γ.

Proof. The idea of the proof is again to apply the abstract result, Proposition 2.6.1. The hardest part of the proof in this case, namely the linear theory, has already been established. More precisely, in Corollary 5.4.4(a)(ii), we have proved that $A(u_0) + \omega \in \mathcal{MR}_p(\mathrm{X}_0)$ for $\omega > \omega_0$ for some $\omega_0 \in \mathbb{R}$. Therefore, for $\omega > \omega_0$, we introduce $A_\omega(u_0)v := A(u_0)v + \omega v$ and $F_\omega(v) := F(v) + \omega v$. The remainder of the proof is dedicated to showing the Lipschitz estimates.

With regard to notation, we will omit the subscript $_{\mathrm{per}}$ throughout this proof as the spaces with such boundary conditions embed into the ones without.

From the previous section together with the shape of V from (5.26) and the smoothness of u_0 as revealed in (5.28), we conclude as in the proof of Theorem 3.5.2 that $A_\omega \colon V \to \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0)$.

Again, we invoke Remark 2.6.2, and for $r > 0$ such that $\overline{\mathbb{B}}_{\mathrm{X}_\gamma}(v_0, r) \subset V$, we consider v_1, $v_2 \in \overline{\mathbb{B}}_{\mathrm{X}_\gamma}(v_0, r)$ and $v \in \mathrm{X}_1$. The only nonlinearity in the present operator matrix which does not emerge from the sea ice equations is $C_{\mathrm{o,i}}(h)\partial_z$, where ∂_z is understood as the normal derivative on Γ_o. Thanks to the condition on p and q given in (5.27), we deduce from (1.4) and (1.8) the embedding

$$\mathrm{B}_{qp}^{2-2/p}(G) \hookrightarrow \mathrm{B}_{q2q}^{2-2/p-\varepsilon}(G) \hookrightarrow \mathrm{L}^\infty(G) \tag{5.29}$$

for $\varepsilon > 0$ is sufficiently small. The continuity of ∂_z from $\mathrm{W}^{2,q}(\Omega_{\mathrm{ocn}})$ to $\mathrm{L}^q(G)$ and the mean value theorem in conjunction with v_1, $v_2 \in V$, the aforementioned embedding relation and the shape of X_γ as revealed in Corollary 5.4.4(a)(iv) then yield

$$\begin{aligned}
&\|C_{\mathrm{o,i}}(h_1)\partial_z v_{\mathrm{ocn}} - C_{\mathrm{o,i}}(h_2)\partial_z v_{\mathrm{ocn}}\|_{\mathrm{L}^q(G)} \\
&\le \|C_{\mathrm{o,i}}(h_1) - C_{\mathrm{o,i}}(h_2)\|_{\mathrm{L}^\infty(G)} \cdot \|\partial_z v_{\mathrm{ocn}}\|_{\mathrm{L}^q(G)} \\
&\le C_1 \cdot \left\| \frac{1}{h_1} - \frac{1}{h_2} \right\|_{\mathrm{L}^\infty(G)} \cdot \|v_{\mathrm{ocn}}\|_{\mathrm{W}^{2,q}(\Omega_{\mathrm{ocn}})} \\
&\le C_2 \cdot \|h_1 - h_2\|_{\mathrm{B}_{qp}^{2-2/p}(G)} \cdot \|v\|_{\mathrm{X}_1} \\
&\le C_3 \cdot \|v_1 - v_2\|_{\mathrm{X}_\gamma} \cdot \|v\|_{\mathrm{X}_1}.
\end{aligned}$$

For the other nonlinear terms in A_ω, we argue similarly as in the proof of Theorem 3.5.2 to get the existence of a constant $C_A > 0$ with

$$\begin{aligned}
\|A_\omega(u_1)v - A_\omega(u_2)v\|_{\mathrm{X}_0} &\le C_A \cdot \|v_1 - v_2\|_{\mathrm{Y}_\gamma} \cdot \|v\|_{\mathrm{Y}_1} \\
&\le C_A' \cdot \|v_1 - v_2\|_{\mathrm{X}_\gamma} \cdot \|v\|_{\mathrm{X}_1},
\end{aligned}$$

where we additionally made use of the embeddings $\mathrm{X}_1 \hookrightarrow \mathrm{Y}_1$ and $\mathrm{X}_\gamma \hookrightarrow \mathrm{Y}_\gamma$, following from Corollary 5.4.4(a)(iv).

Having established the estimate of the operator matrix, we now discuss the estimates of the shifted right-hand side F_ω, with F from (5.15). For this purpose, we recall the estimates of the bilinear terms of the primitive equations from Lemma 2.7.6. In the sequel, we use the more compact notation of the bilinearity, namely $(u_i \cdot \nabla)v_i = (v_i \cdot \nabla)v_i + w_i(v_i) \cdot \partial_z v_i$, where $i \in \{\mathrm{atm}, \mathrm{ocn}\}$. Making use of the embedding $\mathrm{X}_\gamma \hookrightarrow \mathrm{Y}_\gamma$ from Corollary 5.4.4(a)(iv) again, for v_1, $v_2 \in \overline{\mathbb{B}}_{\mathrm{X}_\gamma}(v_0, r)$, we find

$$
\begin{aligned}
&\|\mathbb{P}_{\mathrm{atm}}((u_{\mathrm{atm},1} \cdot \nabla)v_{\mathrm{atm},1} - (u_{\mathrm{atm},2} \cdot \nabla)v_{\mathrm{atm},2})\|_{\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{atm}})} \\
&\le C_4 \cdot \Big(\|v_{\mathrm{atm},1}\|_{\mathrm{B}^{2-2/p}_{qp}(\Omega_{\mathrm{atm}}) \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{atm}})} + \|v_{\mathrm{atm},2}\|_{\mathrm{B}^{2-2/p}_{qp}(\Omega_{\mathrm{atm}}) \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{atm}})}\Big) \\
&\quad \cdot \|v_{\mathrm{atm},1} - v_{\mathrm{atm},2}\|_{\mathrm{B}^{2-2/p}_{qp}(\Omega_{\mathrm{atm}}) \cap \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{atm}})} \\
&\le C_5 \cdot \Big(\|v_1\|_{\mathrm{Y}_\gamma} + \|v_2\|_{\mathrm{Y}_\gamma}\Big) \cdot \|v_1 - v_2\|_{\mathrm{Y}_\gamma} \\
&\le C_6(r, \|v_0\|_{\mathrm{X}_\gamma}) \cdot \|v_1 - v_2\|_{\mathrm{X}_\gamma}.
\end{aligned}
$$

Completely analogously, we obtain

$$
\begin{aligned}
&\|\mathbb{P}_{\mathrm{ocn}}((u_{\mathrm{ocn},1} \cdot \nabla)v_{\mathrm{ocn},1} - (u_{\mathrm{ocn},2} \cdot \nabla)v_{\mathrm{ocn},2})\|_{\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}})} \\
&\le C_7(r, \|v_0\|_{\mathrm{X}_\gamma}) \cdot \|v_1 - v_2\|_{\mathrm{X}_\gamma}.
\end{aligned}
$$

Concerning the nonlinear terms in the context of the sea ice equations, we observe that the forcing term associated to the ocean, $C_{\mathrm{o,i}}(h)\partial_z v_{\mathrm{ocn}}$, has already been incorporated into the operator matrix, whereas the forcing term related to the atmosphere takes a different shape compared to Section 3.5. For the estimate of the atmospheric forcing term

$$
\tau_{\mathrm{atm}} = \rho_{\mathrm{atm}} C_{\mathrm{atm}} |v_{\mathrm{atm}}| R_{\mathrm{atm}} v_{\mathrm{atm}},
$$

we first observe the estimate

$$
\||g_1|g_1 - |g_2|g_2\|_{\mathrm{L}^q(G)} \le \Big(\|g_1\|_{\mathrm{L}^{2q}(G)} + \|g_2\|_{\mathrm{L}^{2q}(G)}\Big) \|g_1 - g_2\|_{\mathrm{L}^{2q}(G)}.
$$

Besides, we conclude from (5.27) the existence of some small $\varepsilon > 0$ with

$$
2 - \frac{2}{p} - \frac{\varepsilon}{2} - \frac{3}{q} \ge \frac{1}{2q} + \frac{\varepsilon}{2} - \frac{3}{2q}.
$$

The embedding relations (1.4) and (1.7) then imply

$$
\mathrm{B}^{2-2/p}_{qp}(\Omega_{\mathrm{atm}}) \hookrightarrow \mathrm{B}^{2-2/p-\varepsilon/2}_{q2q}(\Omega_{\mathrm{atm}}) \hookrightarrow \mathrm{W}^{1/2q+\varepsilon/2,2q}(\Omega_{\mathrm{atm}}). \tag{5.30}
$$

Making use of v_1, $v_2 \in V$, employing the above basic estimate, invoking the continuity of the trace as an operator from $\mathrm{W}^{1/2q+\varepsilon/2}(\Omega_{\mathrm{ocn}})$ to $\mathrm{L}^{2q}(G)$ in the spirit of Lemma 1.3.3, and making use of the embedding (5.30), we obtain

$$
\begin{aligned}
&\left\| \frac{\rho_{\mathrm{atm}} C_{\mathrm{atm}}}{\rho_{\mathrm{ice}} h_1} (|v_{\mathrm{atm},1}| R_{\mathrm{atm}} v_{\mathrm{atm},1} - |v_{\mathrm{atm},2}| R_{\mathrm{atm}} v_{\mathrm{atm},2}) \right\|_{\mathrm{L}^q(G)} \\
&\le C_8 \cdot \Big(\|v_{\mathrm{atm},1}\|_{\mathrm{L}^{2q}(G)} + \|v_{\mathrm{atm},2}\|_{\mathrm{L}^{2q}(G)} \Big) \cdot \|v_{\mathrm{atm},1} - v_{\mathrm{atm},2}\|_{\mathrm{L}^{2q}(G)} \\
&\le C_9 \cdot \Big(\|v_{\mathrm{atm},1}\|_{\mathrm{W}^{1/2q+\varepsilon/2,2q}(\Omega_{\mathrm{ocn}})} + \|v_{\mathrm{atm},2}\|_{\mathrm{W}^{1/2q+\varepsilon/2,2q}(\Omega_{\mathrm{ocn}})} \Big) \\
&\quad \cdot \|v_{\mathrm{atm},1} - v_{\mathrm{atm},2}\|_{\mathrm{W}^{1/2q+\varepsilon/2,2q}(\Omega_{\mathrm{ocn}})} \\
&\le C_{10} \cdot \left(\|v_{\mathrm{atm},1}\|_{\mathrm{B}_{qp}^{2-2/p}(\Omega_{\mathrm{ocn}})} + \|v_{\mathrm{atm},2}\|_{\mathrm{B}_{qp}^{2-2/p}(\Omega_{\mathrm{ocn}})} \right) \cdot \|v_{\mathrm{atm},1} - v_{\mathrm{atm},2}\|_{\mathrm{B}_{qp}^{2-2/p}(\Omega_{\mathrm{ocn}})} \\
&\le C_{11} \cdot \Big(\|v_1\|_{\mathrm{X}_\gamma} + \|v_2\|_{\mathrm{X}_\gamma} \Big) \cdot \|v_1 - v_2\|_{\mathrm{X}_\gamma} \\
&\le C_{12}(r, \|v_0\|_{\mathrm{X}_\gamma}) \cdot \|v_1 - v_2\|_{\mathrm{X}_\gamma}.
\end{aligned}
$$

Likewise, proceeding in a similar way as for the estimate of $C_{\mathrm{o,i}}$, we derive

$$
\begin{aligned}
&\left\| \left(\frac{\rho_{\mathrm{atm}} C_{\mathrm{atm}}}{\rho_{\mathrm{ice}} h_1} - \frac{\rho_{\mathrm{atm}} C_{\mathrm{atm}}}{\rho_{\mathrm{ice}} h_2} \right) |v_{\mathrm{atm},2}| R_{\mathrm{atm}} v_{\mathrm{atm},2} \right\|_{\mathrm{L}^q(G)} \\
&\le C_{13} \cdot \left\| \frac{1}{h_1} - \frac{1}{h_2} \right\|_{\mathrm{L}^\infty(G)} \cdot \|v_{\mathrm{atm},2}\|^2_{\mathrm{W}^{1/2q+\varepsilon/2,2q}(\Omega_{\mathrm{atm}})} \\
&\le C_{14} \cdot \|v_2\|^2_{\mathrm{X}_\gamma} \cdot \|v_1 - v_2\|_{\mathrm{X}_\gamma} \\
&\le C_{15}(r, \|v_0\|_{\mathrm{X}_\gamma}) \cdot \|v_1 - v_2\|_{\mathrm{X}_\gamma}.
\end{aligned}
$$

For the remaining terms of the nonlinear right-hand side, we can mimic the procedure from the proof of Theorem 3.5.2. Therefore, we get the existence of a constant $C_F = C_F(r, \|v_0\|_{\mathrm{X}_\gamma}) > 0$ with

$$\|F_\omega(v_1) - F_\omega(v_2)\|_{\mathrm{X}_0} \le C_F \cdot \|v_1 - v_2\|_{\mathrm{X}_\gamma}.$$

In total, taking into consideration Assumption 5.5.1, the assertion of the theorem is a consequence of Proposition 2.6.1 and Corollary 2.6.3. □

We also comment on some possible generalizations of Theorem 5.5.2.

Remark 5.5.3. *(a) It is also possible to introduce time weights* $\mu \in (1/p, 1]$ *in order to exploit the parabolic regularization in the* L^p*-setting. In this case, the space for the initial data becomes*

$$\mathrm{X}_{\gamma,\mu} = (\mathrm{X}_0, \mathrm{X}_1)_{\mu-1/p,p},$$

and condition (5.27) *is then replaced by*

$$1/2 + 1/p + 1/q < \mu \le 1,$$

ensuring classical regularity of functions in the time trace space $X_{\gamma,\mu}$. *Accordingly, an open subset* $V_\mu \subset X_{\gamma,\mu}$ *in the spirit of* (5.26) *is introduced. An analogue of Theorem 5.5.2 can be shown upon slightly adjusting Assumption 5.5.1. The resulting solution* v *then satisfies*

$$v \in W^{1,p}_\mu(0,T;X_0) \cap L^p_\mu(0,T;X_1) =: \mathbb{E}_{1,\mu}.$$

As in Theorem 3.5.2, for every $\delta > 0$*, the solution* v *fulfills*

$$v \in \mathbb{E}_1(\delta,T) \hookrightarrow C([\delta,T];X_\gamma).$$

(b) *The assertion of Theorem 5.5.2 remains valid when including Coriolis terms for the atmosphere, the ocean and the ice.*

(c) *We can also consider anisotropic horizontal and vertical Reynolds numbers* $\mathrm{Re}_{\mathrm{H,atm}}$, $\mathrm{Re}_{z,\mathrm{atm}}$, $\mathrm{Re}_{\mathrm{H,ocn}}$ *and* $\mathrm{Re}_{z,\mathrm{ocn}}$ *for the atmosphere and the ocean. The viscous terms* Δv_{atm} *and* Δv_{ocn} *are then substituted by*

$$\frac{1}{\mathrm{Re}_{\mathrm{H,atm}}}\Delta_{\mathrm{H}} v_{\mathrm{atm}} + \frac{1}{\mathrm{Re}_{z,\mathrm{atm}}}\partial_z^2 v_{\mathrm{atm}} \quad \textit{and} \quad \frac{1}{\mathrm{Re}_{\mathrm{H,ocn}}}\Delta_{\mathrm{H}} v_{\mathrm{ocn}} + \frac{1}{\mathrm{Re}_{z,\mathrm{ocn}}}\partial_z^2 v_{\mathrm{ocn}}.$$

The statement of Theorem 5.5.2 is also still true in this situation. Indeed, one can consider the transformed velocity $U = (V,W)$*, the transformed pressure* Π *and the transformed right-hand side* F *of the shape*

$$\begin{aligned} V(t,x_{\mathrm{H}},z) &:= \mathrm{Re}_{\mathrm{H}} \cdot v\left(\mathrm{Re}_{\mathrm{H}} \cdot t, x_{\mathrm{H}}, 1/(\mathrm{Re}_{\mathrm{H}}^{3/2}\mathrm{Re}_z) \cdot z\right), \\ W(t,x_{\mathrm{H}},z) &:= \sqrt{\mathrm{Re}_{\mathrm{H}}^5 \mathrm{Re}_z} \cdot v\left(\mathrm{Re}_{\mathrm{H}} \cdot t, x_{\mathrm{H}}, 1/(\mathrm{Re}_{\mathrm{H}}^{3/2}\mathrm{Re}_z) \cdot z\right), \\ \Pi(t,x_{\mathrm{H}},z) &:= \mathrm{Re}_{\mathrm{H}}^2 \cdot \pi\left(\mathrm{Re}_{\mathrm{H}} \cdot t, x_{\mathrm{H}}, 1/(\mathrm{Re}_{\mathrm{H}}^{3/2}\mathrm{Re}_z) \cdot z\right) \quad \textit{and} \\ F(t,x_{\mathrm{H}},z) &:= f\left(\mathrm{Re}_{\mathrm{H}} \cdot t, x_{\mathrm{H}}, 1/(\mathrm{Re}_{\mathrm{H}}^{3/2}\mathrm{Re}_z) \cdot z\right). \end{aligned}$$

This yields usual primitive equations on $\tilde{\Omega} = G \times \left(a/\sqrt{\mathrm{Re}_{\mathrm{H}}^3 \mathrm{Re}_z}, b/\sqrt{\mathrm{Re}_{\mathrm{H}}^3 \mathrm{Re}_z}\right)$.

5.6. Global Strong Well-Posedness close to Equilibria

This final section of the chapter discusses the global strong well-posedness of a simplified version of the coupled model provided the initial data are chosen sufficiently close to constant equilibria. Conceptually, we proceed in a similar way as in Section 3.6. This means that we first introduce the simplified system and reformulate it as a quasilinear abstract Cauchy problem, determine the constant equilibria, provide the total linearization and finally verify the normal stability in order to apply the generalized principle of linearized stability as stated in Proposition 2.6.5.

We consider the situation that the external forces are absent, i. e.,

$$f_{\mathrm{atm}} = f_{\mathrm{ocn}} = g\nabla_{\mathrm{H}} H = 0,$$

and the thermodynamic terms are neglected, so

$$S_{\mathrm{h}} = S_{\mathrm{a}} = 0.$$

Moreover, we assume that the atmosphere has no effect on the sea ice, i. e.,

$$\tau_{\mathrm{atm}} = 0.$$

The emerging *simplified coupled atmosphere-sea ice-ocean model* is given by

$$\begin{cases} \partial_t v_{\mathrm{atm}} - \Delta v_{\mathrm{atm}} + \nabla_{\mathrm{H}} \pi_{\mathrm{atm}} = -(u_{\mathrm{atm}} \cdot \nabla) v_{\mathrm{atm}}, & \text{in } (0,T) \times \Omega_{\mathrm{atm}}, \\ \partial_z \pi_{\mathrm{atm}} = 0, & \text{in } (0,T) \times \Omega_{\mathrm{atm}}, \\ \operatorname{div} u_{\mathrm{atm}} = 0, & \text{in } (0,T) \times \Omega_{\mathrm{atm}}, \\ \partial_t v_{\mathrm{ocn}} - \Delta v_{\mathrm{ocn}} + \nabla_{\mathrm{H}} \pi_{\mathrm{ocn}} = -(u_{\mathrm{ocn}} \cdot \nabla) v_{\mathrm{ocn}}, & \text{in } (0,T) \times \Omega_{\mathrm{ocn}}, \\ \partial_z \pi_{\mathrm{ocn}} = 0, & \text{in } (0,T) \times \Omega_{\mathrm{ocn}}, \\ \operatorname{div} u_{\mathrm{ocn}} = 0, & \text{in } (0,T) \times \Omega_{\mathrm{ocn}}, \\ \partial_t v_{\mathrm{ice}} - \frac{1}{m_{\mathrm{ice}}} \cdot \operatorname{div}_{\mathrm{H}} \sigma_\delta = -(v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice}} & \\ \qquad + \frac{1}{m_{\mathrm{ice}}} \tau_{\mathrm{ocn}}(v_{\mathrm{ocn}}), & \text{on } (0,T) \times G, \\ \partial_t h - d_{\mathrm{h}} \Delta_{\mathrm{H}} h = -\operatorname{div}_{\mathrm{H}} (v_{\mathrm{ice}} h), & \text{on } (0,T) \times G, \\ \partial_t a - d_{\mathrm{a}} \Delta_{\mathrm{H}} a = -\operatorname{div}_{\mathrm{H}} (v_{\mathrm{ice}} a), & \text{on } (0,T) \times G, \\ v_{\mathrm{ocn}} = v_{\mathrm{ice}}, & \text{on } (0,T) \times G, \end{cases} \tag{5.31}$$

and the system is again completed by the boundary conditions as revealed in (5.5) as well as periodic boundary conditions on the lateral boundaries.

Compared to Section 5.2, the functional analytic set-up remains unchanged, so we consider X_0 as in (5.12), X_1 as in (5.13) and $\mathrm{X}_\gamma = (\mathrm{X}_0, \mathrm{X}_1)_{1-1/p,p}$ as examined in Corollary 5.4.4(a)(iv), where p, $q \in (1,\infty)$ satisfy (5.27). In addition, we invoke the open subset $V \subset \mathrm{X}_\gamma$ of the time trace space from (5.26). We define $F_{\mathrm{s}} \colon V \to \mathrm{X}_0$ in the present context by

$$F_{\mathrm{s}}(v) \coloneqq \begin{pmatrix} -\mathbb{P}_{\mathrm{atm}}((v_{\mathrm{atm}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{atm}} + w_{\mathrm{atm}}(v_{\mathrm{atm}}) \cdot \partial_z v_{\mathrm{atm}}) \\ -\mathbb{P}_{\mathrm{ocn}}((v_{\mathrm{ocn}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ocn}} + w_{\mathrm{ocn}}(v_{\mathrm{ocn}}) \cdot \partial_z v_{\mathrm{ocn}}) \\ -(v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice}} \\ -\mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} h) \\ -\mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} a) \end{pmatrix}, \tag{5.32}$$

while the operator matrix $A \colon V \to \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0)$ from (5.14) is unaffected. Thus, the quasilinear abstract Cauchy problem on the ground space X_0 corresponding to (5.31) reads as

$$\begin{cases} v'(t) + A(v(t))v(t) = F_{\mathrm{s}}(v(t)), \ \text{ for } \ t \in (0,T), \\ \qquad\qquad\qquad v(0) = v_0. \end{cases} \tag{5.33}$$

The set of equilibrium solutions to (5.33), or, equivalently, to (5.31) is

$$\mathcal{E} \coloneqq \{v \in V \cap \mathrm{X}_1 : A(v)v = F_{\mathrm{s}}(v)\}.$$

For the right-hand side F_{s} from (5.32) and the operator matrix A from (5.14), it readily follows that

$$A(v_*)v_* = 0 = F_{\mathrm{s}}(v_*)$$

for $v_* = (0, 0, 0, h_*, a_*)$, with $h_* \in (\kappa_1, \kappa_2)$ and $a_* \in (0, 1)$ constant in time and space. In particular, such v_* satisfies $v_* \in V \cap \mathrm{X}_1$. We summarize the previous findings in the lemma below.

Lemma 5.6.1. *Consider $v_* = (0, 0, 0, h_*, a_*)$ for $h_* \in (\kappa_1, \kappa_2)$ and $a_* \in (0, 1)$ constant in time and space. Then $v_* \in \mathcal{E}$, so v_* is an equilibrium solution to* (5.33)*, or, equivalently, to* (5.31)*.*

Next, we take into consideration the total linearization in the present context and the required underlying Fréchet-differentiability of A and F_{s}. For this purpose, let us recall the notation $P_{\mathrm{h},*}$ and $P_{\mathrm{a},*}$ from (3.40), namely

$$P_{\mathrm{h},*} \coloneqq \frac{\partial_h P(h_*, a_*)}{2\rho_{\mathrm{ice}} h_*} = \frac{p^* \mathrm{e}^{-c_\bullet(1-a_*)}}{2\rho_{\mathrm{ice}} h_*} \ \text{ and } \ P_{\mathrm{a},*} \coloneqq \frac{\partial_a P(h_*, a_*)}{2\rho_{\mathrm{ice}} h_*} = \frac{c_\bullet p^* \mathrm{e}^{-c_\bullet(1-a_*)}}{2\rho_{\mathrm{ice}}}.$$

The total linearization is determined in the lemma below.

Lemma 5.6.2. *For $p, q \in (1,\infty)$ satisfying* (5.27)*, the operator matrix A from* (5.14) *and F_s from* (5.32)*, we have $(A, F_\mathrm{s}) \in \mathrm{C}^1(V; \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0) \times \mathrm{X}_0)$. Moreover, for $v \in \mathrm{X}_1$, the total linearization at an equilibrium v_* of the shape in Lemma 5.6.1 reads as*

$$A_0 v = \begin{pmatrix} -\mathrm{A}^{\mathrm{atm}} v_{\mathrm{atm}} \\ -A_{\mathrm{m}}^{\mathrm{ocn}} v_{\mathrm{ocn}} \\ C_{\mathrm{o,i}}(h_*)\partial_z v_{\mathrm{ocn}} - A^{\mathrm{H}}(u_*) v_{\mathrm{ice}} + P_{\mathrm{h},*} \nabla_{\mathrm{H}} h + P_{\mathrm{a},*} \nabla_{\mathrm{H}} a \\ -d_{\mathrm{h}} \Delta_{\mathrm{H}} h + h_* \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}} \\ -d_{\mathrm{a}} \Delta_{\mathrm{H}} a + a_* \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}} \end{pmatrix}. \tag{5.34}$$

Proof. The Fréchet-differentiability of the term $F_\mathrm{s} \colon V \to \mathrm{X}_0$ is a consequence of its bilinear structure. Invoking the shape of $v_* = (0,0,0,h_*,a_*)$, and using the bilinear shape of F_s, we first argue that

$$F_\mathrm{s}'(v_*)v = (0,0,0,h_* \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}}, a_* \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}})^\top.$$

On the other hand, we find that

$$A(v_*) v = \begin{pmatrix} -\mathrm{A}^{\mathrm{atm}} v_{\mathrm{atm}} \\ -A_{\mathrm{m}}^{\mathrm{ocn}} v_{\mathrm{ocn}} \\ C_{\mathrm{o,i}}(h_*)\partial_z v_{\mathrm{ocn}} - A^{\mathrm{H}}(v_*) v_{\mathrm{ice}} + P_{\mathrm{h},*} \nabla_{\mathrm{H}} h + P_{\mathrm{a},*} \nabla_{\mathrm{H}} a \\ -d_{\mathrm{h}} \Delta_{\mathrm{H}} h + h_* \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}} \\ -d_{\mathrm{a}} \Delta_{\mathrm{H}} a + a_* \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}} \end{pmatrix}.$$

From the shape of v_*, it results that $(A'(v_*)v)v_* = 0$, so the assertion of the lemma is implied by $A_0 v = A(v_*)v + F_\mathrm{s}'(v_*)v$. □

Having determined the total linearization around constant equilibria, we now address the normal stability of the equilibrium solutions in the sense of Definition 2.6.4. Before proceeding, let us recall from (5.25) that $v \in \mathrm{X}_1$ especially fulfills $v_{\mathrm{ocn}} \in \mathrm{W}^{2,q}(\Omega_{\mathrm{ocn}})^2$. In the remainder of this section, we will use this improved regularity.

Next, we discuss the spectral properties of A_0.

Lemma 5.6.3. *Let $v_* = (0,0,0,h_*,a_*)$, with $h_* \in (\kappa_1, \kappa_2)$ and $a_* \in (0,1)$ constant in time and space, and recall A_0 from* (5.34)*. Then*

(a) *the operator A_0 has a compact resolvent on X_0, and the spectrum of A_0 is q-independent and only consists of eigenvalues, and*

(b) *we have $\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+$ and $\mathrm{N}(A_0) = \mathbb{R}^2 \times \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R}$.*

Proof. For the proof of the compactness of the resolvent of A_0, we can proceed as in the proof of this property in Corollary 5.4.4(b), i. e., we can invoke the similarity transform from Lemma 5.4.1 and use that the decoupled operator matrix with diagonal domain has this property by virtue of the compact embedding of the domain into the ground space. The q-independence of the spectrum and the fact that it is formed by eigenvalues are then implied by Lemma 2.1.6, showing (a).

With regard to (b), the first step consists of computing several integrals for $v = (v_{\mathrm{atm}}, v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a) \in \mathrm{X}_1$ to be used later on. An integration by parts, the periodic boundary conditions on the lateral boundary and the Neumann boundary conditions on the upper and lower boundary as well as $\mathrm{div}_{\mathrm{H}}\, \overline{v}_{\mathrm{atm}} = 0$ for $v_{\mathrm{atm}} \in \mathrm{D}(\mathrm{A}^{\mathrm{atm}})$ yield

$$
\begin{aligned}
&- \langle \mathrm{A}^{\mathrm{atm}} v_{\mathrm{atm}}, v_{\mathrm{atm}} \rangle_{\mathrm{L}^2(\Omega_{\mathrm{atm}})} \\
&= \langle -\Delta v_{\mathrm{atm}} + \nabla_{\mathrm{H}} \pi, v_{\mathrm{atm}} \rangle_{\mathrm{L}^2(\Omega_{\mathrm{atm}})} \\
&= \int_{\Omega_{\mathrm{atm}}} |\nabla v_{\mathrm{atm}}|^2 \,\mathrm{d}(x_{\mathrm{H}}, z) - (h_{\mathrm{atm}} - \kappa_2) \int_G \pi \cdot \mathrm{div}_{\mathrm{H}}\, \overline{v}_{\mathrm{atm}} \,\mathrm{d}x_{\mathrm{H}} \\
&= \|\nabla v_{\mathrm{atm}}\|^2_{\mathrm{L}^2(\Omega_{\mathrm{atm}})}.
\end{aligned} \tag{5.35}
$$

Likewise, employing $v_{\mathrm{ocn}} = v_{\mathrm{ice}}$ on Γ_o, which can be identified with G, the other boundary condition $v_{\mathrm{ocn}} = 0$ on Γ_b and Poincaré's inequality as stated in Lemma 1.4.1, we deduce

$$
\begin{aligned}
&- \langle A^{\mathrm{ocn}}_{\mathrm{m}} v_{\mathrm{ocn}}, v_{\mathrm{ocn}} \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} \\
&= \int_{\Omega_{\mathrm{ocn}}} |\nabla v_{\mathrm{ocn}}|^2 \,\mathrm{d}(x_{\mathrm{H}}, z) - \int_{\Gamma_o} \partial_z v_{\mathrm{ocn}} \cdot v_{\mathrm{ocn}} \,\mathrm{d}x_{\mathrm{H}} \\
&= \|\nabla v_{\mathrm{ocn}}\|^2_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} - \int_G \partial_z v_{\mathrm{ocn}} \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} \\
&\geq C_1 \cdot \|v_{\mathrm{ocn}}\|^2_{\mathrm{H}^1(\Omega_{\mathrm{ocn}})} - \langle \partial_z v_{\mathrm{ocn}}, v_{\mathrm{ice}} \rangle_{\mathrm{L}^2(G)}
\end{aligned} \tag{5.36}
$$

for some constant $C_1 > 0$. Furthermore, as in the proof of Lemma 3.6.3, recalling the shape of $\triangle^2(\nabla_{\mathrm{H}} v_{\mathrm{ice}})$ from (3.12), we establish the estimate

$$
\sum_{i,j,k,l=1}^{2} \mathbb{S}^{kl}_{ij} \partial_l v_{\mathrm{ice},j} \partial_k v_{\mathrm{ice},i} = \triangle^2(\nabla_{\mathrm{H}} v_{\mathrm{ice}}) \geq \frac{2}{e^2} \cdot |\varepsilon(v_{\mathrm{ice}})|^2.
$$

Next, as in (3.40), we introduce

$$P_* := \frac{P(h_*, a_*)}{2\rho_{\mathrm{ice}} h_*} = \frac{p^* \mathrm{e}^{-c_\bullet(1-a_*)}}{2\rho_{\mathrm{ice}}}, \quad P_{\mathrm{h},*} := \frac{\partial_h P(h_*, a_*)}{2\rho_{\mathrm{ice}} h_*} \text{ and } P_{\mathrm{a},*} := \frac{\partial_a P(h_*, a_*)}{2\rho_{\mathrm{ice}} h_*}.$$

Concatenating the previous estimate with an integration by parts, the periodic boundary conditions and Korn's inequality as in Lemma 1.4.2, we conclude

$$\begin{aligned} -\langle A^{\mathrm{H}}(u_*) v_{\mathrm{ice}}, v_{\mathrm{ice}} \rangle_{\mathrm{L}^2(G)} &= -\frac{P_*}{\delta^{1/2}} \int_G \sum_{i,j,k,l=1}^{2} \mathbb{S}_{ij}^{kl} \partial_k \partial_l v_{\mathrm{ice},j} v_{\mathrm{ice},i} \, \mathrm{d}x_{\mathrm{H}} \\ &= \frac{P_*}{\delta^{1/2}} \int_G \sum_{i,j,k,l=1}^{2} \mathbb{S}_{ij}^{kl} \partial_l v_{\mathrm{ice},j} \partial_k v_{\mathrm{ice},i} \, \mathrm{d}x_{\mathrm{H}} \\ &\ge C_2 \frac{P_*}{\delta^{1/2}} \cdot \| \nabla_{\mathrm{H}} v_{\mathrm{ice}} \|_{\mathrm{L}^2(G)}^2 \end{aligned} \tag{5.37}$$

for some constant $C_2 > 0$. The preparation of the proof is finished by the observation that

$$-\langle \Delta_{\mathrm{H}} h, h \rangle_{\mathrm{L}^2(G)} = \| \nabla_{\mathrm{H}} h \|_{\mathrm{L}^2(G)}^2 \text{ and } -\langle \Delta_{\mathrm{H}} a, a \rangle_{\mathrm{L}^2(G)} = \| \nabla_{\mathrm{H}} a \|_{\mathrm{L}^2(G)}^2 \tag{5.38}$$

thanks to another integration by parts joint with the periodic boundary conditions.

By virtue of the q-independence of the spectrum and its property of consisting of eigenvalues as revealed in (a), it is sufficient to test the eigenvalue equation $\lambda v + A_0 v = 0$ of $-A_0$ to locate the spectrum. We choose a suitable test function to exploit the cancellation of some terms. In fact, for an element $v = (v_{\mathrm{atm}}, v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a) \in \mathrm{X}_1$, we test the aforementioned eigenvalue equation by $(v_{\mathrm{atm}}, v_{\mathrm{ocn}}, c_3 v_{\mathrm{ice}}, c_4 h, c_5 a)$, where

$$c_3 = \frac{1}{C_{\mathrm{o,i}}(h_*)}, \quad c_4 = c_3 \frac{P_{\mathrm{h},*}}{h_*} \text{ and } c_5 = c_3 \frac{P_{\mathrm{a},*}}{a_*}.$$

In view of $h_* \in (\kappa_1, \kappa_2)$ as well as $a_* \in (0,1)$, it holds that c_3, c_4, $c_5 > 0$. Thus, testing the eigenvalue equation by the above test function, integrating by vector parts, making use of the periodic boundary conditions on the lateral boundary and plugging in (5.35), (5.36), (5.37) as well as (5.38), we obtain

$$
\begin{aligned}
0 &= \lambda \cdot \|v_{\mathrm{atm}}\|^2_{\mathrm{L}^2(\Omega_{\mathrm{atm}})} + \lambda \cdot \|v_{\mathrm{ocn}}\|^2_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} + \lambda c_3 \cdot \|v_{\mathrm{ice}}\|^2_{\mathrm{L}^2(G)} + \lambda c_4 \cdot \|h\|^2_{\mathrm{L}^2(G)} \\
&\quad + \lambda c_5 \cdot \|a\|^2_{\mathrm{L}^2(G)} - \langle \mathrm{A}^{\mathrm{atm}} v_{\mathrm{atm}}, v_{\mathrm{atm}} \rangle_{\mathrm{L}^2(\Omega_{\mathrm{atm}})} - \langle A^{\mathrm{ocn}}_{\mathrm{m}} v_{\mathrm{ocn}}, v_{\mathrm{ocn}} \rangle_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} \\
&\quad + C_{\mathrm{o,i}}(h_*) c_3 \langle \partial_z v_{\mathrm{ocn}}, v_{\mathrm{ice}} \rangle_{\mathrm{L}^2(G)} - c_3 \langle A^{\mathrm{H}}(u_*) v_{\mathrm{ice}}, v_{\mathrm{ice}} \rangle_{\mathrm{L}^2(G)} \\
&\quad - c_3 \langle P_{\mathrm{h},*} h, v_{\mathrm{ice}} \rangle_{\mathrm{L}^2(G)} - c_3 \langle P_{\mathrm{a},*} a, v_{\mathrm{ice}} \rangle_{\mathrm{L}^2(G)} - c_4 d_{\mathrm{h}} \langle \Delta_{\mathrm{H}} h, h \rangle_{\mathrm{L}^2(G)} \\
&\quad + c_4 h_* \langle \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}}, h \rangle_{\mathrm{L}^2(G)} - c_5 d_{\mathrm{a}} \langle \Delta_{\mathrm{H}} a, a \rangle_{\mathrm{L}^2(G)} + c_5 a_* \langle \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}}, a \rangle_{\mathrm{L}^2(G)} \\
&\geq \lambda \cdot \|v_{\mathrm{atm}}\|^2_{\mathrm{L}^2(\Omega_{\mathrm{atm}})} + \lambda \cdot \|v_{\mathrm{ocn}}\|^2_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} + \lambda c_3 \cdot \|v_{\mathrm{ice}}\|^2_{\mathrm{L}^2(G)} + \lambda c_4 \cdot \|h\|^2_{\mathrm{L}^2(G)} \\
&\quad + \lambda c_5 \cdot \|a\|^2_{\mathrm{L}^2(G)} + C_3 \cdot \Big(\|\nabla v_{\mathrm{atm}}\|^2_{\mathrm{L}^2(\Omega_{\mathrm{atm}})} + \|v_{\mathrm{ocn}}\|^2_{\mathrm{H}^1(\Omega_{\mathrm{ocn}})} + \|\nabla_{\mathrm{H}} v_{\mathrm{ice}}\|^2_{\mathrm{L}^2(G)} \\
&\quad + \|\nabla_{\mathrm{H}} h\|^2_{\mathrm{L}^2(G)} + \|\nabla_{\mathrm{H}} a\|^2_{\mathrm{L}^2(G)} \Big) + \left(C_{\mathrm{o,i}}(h_*) c_3 - 1 \right) \langle \partial_z v_{\mathrm{ocn}}, v_{\mathrm{ice}} \rangle_{\mathrm{L}^2(G)} \\
&\quad + \left(c_3 \frac{\partial_h P_*}{2 \rho_{\mathrm{ice}} h_*} - c_4 h_* \right) \langle \nabla_{\mathrm{H}} h, v_{\mathrm{ice}} \rangle_{\mathrm{L}^2(G)} \\
&\quad + \left(c_3 \frac{\partial_a P_*}{2 \rho_{\mathrm{ice}} h_*} - c_5 a_* \right) \langle \nabla_{\mathrm{H}} a, v_{\mathrm{ice}} \rangle_{\mathrm{L}^2(G)}.
\end{aligned}
$$

Inserting the above choice of c_3, c_4 and c_5, we infer that

$$
\begin{aligned}
0 &= \lambda \|v_{\mathrm{atm}}\|^2_{\mathrm{L}^2(\Omega_{\mathrm{atm}})} + \lambda \|v_{\mathrm{ocn}}\|^2_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})} + \lambda c_3 \|v_{\mathrm{ice}}\|^2_{\mathrm{L}^2(G)} + \lambda c_4 \|h\|^2_{\mathrm{L}^2(G)} \\
&\quad + \lambda c_5 \|a\|^2_{\mathrm{L}^2(G)} + C_3 \cdot \Big(\|\nabla v_{\mathrm{atm}}\|^2_{\mathrm{L}^2(\Omega_{\mathrm{atm}})} + \|v_{\mathrm{ocn}}\|^2_{\mathrm{H}^1(\Omega_{\mathrm{ocn}})} + \|\nabla_{\mathrm{H}} v_{\mathrm{ice}}\|^2_{\mathrm{L}^2(G)} \\
&\quad + \|\nabla_{\mathrm{H}} h\|^2_{\mathrm{L}^2(G)} + \|\nabla_{\mathrm{H}} a\|^2_{\mathrm{L}^2(G)} \Big).
\end{aligned}
$$

From the preceding inequality, we deduce that $\lambda \in \mathbb{R}$ with $\lambda \leq 0$. The q-independence of the spectrum then yields $\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+$. In particular, considering $\lambda = 0$, we are able to determine $\mathrm{N}(A_0)$. In this case, it follows that $\nabla v_{\mathrm{atm}} = 0$, $v_{\mathrm{ocn}} = 0$, $\nabla_{\mathrm{H}} v_{\mathrm{ice}} = 0$ and $\nabla_{\mathrm{H}} h = \nabla_{\mathrm{H}} a = 0$. Hence, the variables v_{atm}, v_{ice}, h and a are constant. With regard to v_{ice}, we additionally invoke the boundary condition $v_{\mathrm{ocn}} = v_{\mathrm{ice}}$ on G, yielding $v_{\mathrm{ice}} = 0$ as well. As a result, the kernel of A_0 is of the asserted shape. □

After settling the spectral properties of the total linearization A_0, we now elaborate on the shape of $\mathcal{E}$ near constant equilibria.

Lemma 5.6.4. *Consider $p, q \in (1, \infty)$ such that* (5.27) *is valid, and take into account an equilibrium v_* of the shape $v_* = (0, 0, 0, h_*, a_*)$, with $h_* \in (\kappa_1, \kappa_2)$ and $a_* \in (0, 1)$ constant in time and space. Near v_*, the set of equilibria $\mathcal{E}$ is a C^1-manifold in X_1, and the tangent space of $\mathcal{E}$ at v_* is isomorphic to* $\mathrm{N}(A_0)$.

Proof. Let $v = (v_{\mathrm{atm}}, v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a) \in V \cap \mathrm{X}_1$ be an equilibrium solution such that $\|v - v_*\|_{\mathrm{X}_\gamma} < r$ for some given $r > 0$. Hence, we obtain the equality $0 = A(v)v - F_{\mathrm{s}}(v)$. Multiplying the sea ice momentum equation by $2\rho_{\mathrm{ice}} h$, and employing the notation $u = (v_{\mathrm{ice}}, h, a)$, we then get

$$0 = \begin{pmatrix} -\mathrm{A}^{\mathrm{atm}} v_{\mathrm{atm}} + (v_{\mathrm{atm}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{atm}} + w_{\mathrm{atm}}(v_{\mathrm{atm}}) \cdot \partial_z v_{\mathrm{atm}} \\ -A_{\mathrm{m}}^{\mathrm{ocn}} v_{\mathrm{ocn}} + (v_{\mathrm{ocn}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ocn}} + w_{\mathrm{ocn}}(v_{\mathrm{ocn}}) \cdot \partial_z v_{\mathrm{ocn}} \\ 2\rho_{\mathrm{ice}} h \left(C_{\mathrm{o,i}}(h) \partial_z v_{\mathrm{ocn}} - A^{\mathrm{H}}(u) + (v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice}} \right) + \nabla_{\mathrm{H}} P(h,a) \\ -d_{\mathrm{h}} \Delta_{\mathrm{H}} h + \mathrm{div}_{\mathrm{H}} \, (v_{\mathrm{ice}} h) \\ -d_{\mathrm{a}} \Delta_{\mathrm{H}} a + \mathrm{div}_{\mathrm{H}} \, (v_{\mathrm{ice}} a) \end{pmatrix}. \tag{5.39}$$

In the sequel, we will also test (5.39) by a suitable test function.

For $i \in \{\mathrm{atm}, \mathrm{ocn}\}$, we observe that $(u_i \cdot \nabla) v_i = (v_i \cdot \nabla_{\mathrm{H}}) v_i + w_i(v_i) \cdot \partial_z v_i$ on Ω_i. Thus, Lemma 2.7.7 implies

$$\begin{aligned} &\int_{\Omega_{\mathrm{atm}}} ((v_{\mathrm{atm}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{atm}} + w_{\mathrm{atm}}(v_{\mathrm{atm}}) \cdot \partial_z v_{\mathrm{atm}}) \cdot v_{\mathrm{atm}} \, \mathrm{d}(x_{\mathrm{H}}, z) = 0 \text{ and} \\ &\int_{\Omega_{\mathrm{ocn}}} ((v_{\mathrm{ocn}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ocn}} + w_{\mathrm{ocn}}(v_{\mathrm{ocn}}) \cdot \partial_z v_{\mathrm{ocn}}) \cdot v_{\mathrm{ocn}} \, \mathrm{d}(x_{\mathrm{H}}, z) = 0. \end{aligned} \tag{5.40}$$

In particular, inspecting the proof of Lemma 2.7.7, we observe that the coupling conditions do not come into picture in view of the shape of the normal vector. Instead, it is only important that the vertical velocities w_i are zero on the boundary.

Next, analogously as in (3.52) in the proof of Lemma 3.6.4, we argue that $v \in V$ with $\|v - v_*\|_{\mathrm{X}_\gamma} < r$ implies

$$P(h,a) \geq p^* \kappa \mathrm{e}^{-c_\bullet} =: P_{**} > 0 \text{ and } \frac{1}{\triangle_\delta(\varepsilon(v_{\mathrm{ice}}))} \geq \frac{1}{\sqrt{\delta + c_e r^2}}$$

for a constant $c_e > 0$. Let us stress the independence of P_{**} from u, δ and r. In the same way as in (3.53) in the proof of Lemma 3.6.4 up to the step where the Poincaré inequality is used, which is not applicable in our situation, we obtain the estimate

$$-\langle 2\rho_{\mathrm{ice}} h A^{\mathrm{H}}(v_{\mathrm{ice}}, h, a) v_{\mathrm{ice}}, v_{\mathrm{ice}} \rangle_{\mathrm{L}^2(G)} \geq \frac{C_1}{\sqrt{\delta + c_e r^2}} \cdot \|\nabla_{\mathrm{H}} v_{\mathrm{ice}}\|_{\mathrm{L}^2(G)}^2 \tag{5.41}$$

for some constant $C_1 > 0$. Now, for $v \in V \cap \mathrm{X}_1$ such that $\|v - v_*\|_{\mathrm{X}_\gamma} < r$, we test equation (5.39) by $(v_{\mathrm{atm}}, v_{\mathrm{ocn}}, c_3 v_{\mathrm{ice}}, c_4 h, c_5 a)$, where this time, we set

$$c_3 = \frac{1}{\mu_{\mathrm{ocn}}}, \quad c_4 = 2\rho_{\mathrm{ice}} P_{\mathrm{h},*} c_3 \text{ and } c_5 = 2\rho_{\mathrm{ice}} P_{\mathrm{a},*} c_3. \tag{5.42}$$

The choice of the test function as revealed in (5.42) in particular leads to

$$-\langle \partial_z v_{\mathrm{ocn}}, v_{\mathrm{ice}} \rangle_{\mathrm{L}^2(G)} + c_3 \mu_{\mathrm{ocn}} \langle \partial_z v_{\mathrm{ocn}}, v_{\mathrm{ice}} \rangle_{\mathrm{L}^2(G)} = 0.$$

On the other hand, an integration by parts joint with the shape of $\nabla_{\mathrm{H}} P(h,a)$ also result in

$$\begin{aligned}
&c_3 \int_G \nabla_{\mathrm{H}} P(h,a) \cdot v_{\mathrm{ice}} \, \mathrm{d}x_{\mathrm{H}} + c_4 \int_G \mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} h) h \, \mathrm{d}x_{\mathrm{H}} + c_5 \int_G \mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} a) a \, \mathrm{d}x_{\mathrm{H}} \\
&= c_3 \int_G \nabla_{\mathrm{H}} P(h,a) \cdot v_{\mathrm{ice}} \, \mathrm{d}x_{\mathrm{H}} - c_4 \int_G h \nabla_{\mathrm{H}} h \cdot v_{\mathrm{ice}} \, \mathrm{d}x_{\mathrm{H}} - c_5 \int_G a \nabla_{\mathrm{H}} a \cdot v_{\mathrm{ice}} \, \mathrm{d}x_{\mathrm{H}} \\
&= \int_G (c_3 \partial_h P(h,a) - c_4 h) \, \nabla_{\mathrm{H}} h \cdot v_{\mathrm{ice}} \, \mathrm{d}x_{\mathrm{H}} \\
&\quad + \int_G (c_3 \partial_a P(h,a) - c_5 a) \, \nabla_{\mathrm{H}} a \cdot v_{\mathrm{ice}} \, \mathrm{d}x_{\mathrm{H}}.
\end{aligned}$$

Therefore, testing (5.39) by $(v_{\mathrm{atm}}, v_{\mathrm{ocn}}, c_3 v_{\mathrm{ice}}, c_4 h, c_5 a)$ and employing the relations (5.35), (5.36), (5.38), (5.40) as well as (5.41), we get

$$\begin{aligned}
0 &\ge \|\nabla v_{\mathrm{atm}}\|_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})}^2 + C_2 \cdot \|v_{\mathrm{ocn}}\|_{\mathrm{H}^1(\Omega_{\mathrm{ocn}})}^2 + \frac{c_3 C_1}{\sqrt{\delta + c_e r^2}} \cdot \|\nabla_{\mathrm{H}} v_{\mathrm{ice}}\|_{\mathrm{L}^2(G)}^2 \\
&\quad + c_4 d_{\mathrm{h}} \cdot \|\nabla_{\mathrm{H}} h\|_{\mathrm{L}^2(G)}^2 + c_5 d_{\mathrm{a}} \cdot \|\nabla_{\mathrm{H}} a\|_{\mathrm{L}^2(G)}^2 \\
&\quad + 2 c_3 \rho_{\mathrm{ice}} \int_G h((v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice}}) \cdot v_{\mathrm{ice}} \, \mathrm{d}x_{\mathrm{H}} \\
&\quad + \int_G (c_3 \partial_h P(h,a) - c_4 h) \, \nabla_{\mathrm{H}} h \cdot v_{\mathrm{ice}} \, \mathrm{d}x_{\mathrm{H}} \\
&\quad + \int_G (c_3 \partial_a P(h,a) - c_5 a) \, \nabla_{\mathrm{H}} a \cdot v_{\mathrm{ice}} \, \mathrm{d}x_{\mathrm{H}}
\end{aligned} \tag{5.43}$$

for another constant $C_2 > 0$ as seen in the proof of Lemma 5.6.3. The remaining task is to absorb the terms without a sign in (5.43). Similarly as in (3.54), making use of the shape of the equilibrium together with $\|v - v_*\|_{\mathrm{X}_\gamma} < r$ and the embedding from (5.29), and invoking $v_{\mathrm{ice}} = v_{\mathrm{ocn}}$ on $\Gamma_o = G$ joint with the continuity of the trace as an operator from $\mathrm{H}^1(\Omega_{\mathrm{ocn}})$ to $\mathrm{L}^2(\Gamma_o)$, we derive

$$\begin{aligned}
&2 c_3 \rho_{\mathrm{ice}} \int_G h(v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice}} \cdot v_{\mathrm{ice}} \, \mathrm{d}x_{\mathrm{H}} \\
&\le C_3 \cdot \|h\|_{\mathrm{L}^\infty(G)} \cdot \|v_{\mathrm{ice}}\|_{\mathrm{L}^\infty(G)} \cdot \|v_{\mathrm{ice}}\|_{\mathrm{L}^2(G)} \cdot \|\nabla_{\mathrm{H}} v_{\mathrm{ice}}\|_{\mathrm{L}^2(G)} \\
&\le C_4 \, (1 + r) \, r \cdot \left(\|\nabla_{\mathrm{H}} v_{\mathrm{ice}}\|_{\mathrm{L}^2(G)}^2 + \|v_{\mathrm{ocn}}\|_{\mathrm{H}^1(\Omega_{\mathrm{ocn}})}^2 \right),
\end{aligned} \tag{5.44}$$

where $C_4 > 0$ represents a suitable constant. In an analogous manner as in (3.58) and (3.59), we further conclude that

$$\begin{aligned}
&\|c_3 \partial_h P(h,a) - c_4 h\|_{\mathrm{L}^\infty(G)} \le C_5 (1 + r) r \;\text{ and} \\
&\|c_3 \partial_a P(h,a) - c_5 a\|_{\mathrm{L}^\infty(G)} \le C_5 (1 + r) r
\end{aligned}$$

for a constant $C_5 > 0$. Combining these estimates with $v_{\mathrm{ocn}} = v_{\mathrm{ice}}$ on the interface $G = \Gamma_o$, the continuity of the trace operator from $\mathrm{H}^1(\Omega_{\mathrm{ocn}})$ to $\mathrm{L}^2(\Gamma_o)$, Hölder's inequality and Young's inequality, we infer that

$$
\begin{aligned}
&\int_G (c_3 \partial_h P(h,a) - c_4 h)\, \nabla_{\mathrm{H}} h \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} \\
&+ \int_G (c_3 \partial_a P(h,a) - c_5 a)\, \nabla_{\mathrm{H}} a \cdot v_{\mathrm{ice}} \,\mathrm{d}x_{\mathrm{H}} \\
&\geq -C_6 (1+r) r \cdot \left(\|\nabla_{\mathrm{H}} h\|_{\mathrm{L}^2(G)}^2 + \|\nabla_{\mathrm{H}} a\|_{\mathrm{L}^2(G)}^2 + \|v_{\mathrm{ocn}}\|_{\mathrm{H}^1(\Omega_{\mathrm{ocn}})}^2 \right).
\end{aligned} \tag{5.45}
$$

Hence, inserting the estimates (5.44) and (5.45) into (5.43), we conclude

$$
\begin{aligned}
0 \geq\ &\|\nabla v_{\mathrm{atm}}\|_{\mathrm{L}^2(\Omega_{\mathrm{ocn}})}^2 + (C_2 - (C_4 + C_6)(1+r)r) \cdot \|v_{\mathrm{ocn}}\|_{\mathrm{H}^1(\Omega_{\mathrm{ocn}})}^2 \\
&+ \left(\frac{c_3 C_1}{\sqrt{\delta + c_e r^2}} - C_4 (1+r) r \right) \cdot \|\nabla_{\mathrm{H}} v_{\mathrm{ice}}\|_{\mathrm{L}^2(G)}^2 \\
&+ (c_4 d_{\mathrm{h}} - C_6(1+r)r) \cdot \|\nabla_{\mathrm{H}} h\|_{\mathrm{L}^2(G)}^2 + (c_5 d_{\mathrm{a}} - C_6 (1+r) r) \cdot \|\nabla_{\mathrm{H}} a\|_{\mathrm{L}^2(G)}^2.
\end{aligned}
$$

As a consequence, when choosing $r > 0$ sufficiently small, we deduce the existence of a constant $C_7 > 0$ such that

$$
\begin{aligned}
0 \geq C_7 \cdot \Big(&\|\nabla v_{\mathrm{atm}}\|_{\mathrm{L}^2(\Omega_{\mathrm{atm}})}^2 + \|v_{\mathrm{ocn}}\|_{\mathrm{H}^1(\Omega_{\mathrm{ocn}})}^2 + \|\nabla_{\mathrm{H}} v_{\mathrm{ice}}\|_{\mathrm{L}^2(G)}^2 \\
&+ \|\nabla_{\mathrm{H}} h\|_{\mathrm{L}^2(G)}^2 + \|\nabla_{\mathrm{H}} a\|_{\mathrm{L}^2(G)}^2 \Big),
\end{aligned}
$$

so v_{atm}, v_{ice}, h and a are constant, whereas $v_{\mathrm{ocn}} = 0$. Because of $v_{\mathrm{ocn}} = v_{\mathrm{ice}}$ on $\Gamma_o = G$, we also have $v_{\mathrm{ice}} = 0$. Therefore, for $v = (v_{\mathrm{atm}}, v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a) \in \mathcal{E}$ with $v \in V \cap \mathrm{X}_1$ and $\|v - v_*\|_{\mathrm{X}_\gamma} < r$ for $r > 0$ sufficiently small, v_{atm}, h and a are constant, while $v_{\mathrm{ocn}} = v_{\mathrm{ice}} = 0$. In other words,

$$
\mathbb{B}_{\mathrm{X}_\gamma \cap \mathcal{E}}(v_*, r) = \mathbb{R}^2 \times \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R} = \mathrm{N}(A_0),
$$

and the last equality is due to Lemma 5.6.3(b). From this, we especially deduce that the set of equilibria near v_* is a C^1-manifold of dimension 4, and that the tangent space of $\mathcal{E}$ near v_* even coincides with $\mathrm{N}(A_0)$. Hence, the latter two are in particular isomorphic, finishing the proof. □

The remaining aspect in the verification of the normal stability of the equilibria is the property of zero being a semi-simple eigenvalue of the total linearization A_0 from (5.34). This is precisely addressed in the lemma below.

Lemma 5.6.5. *Let $v_* = (0, 0, 0, h_*, a_*)$, where $h_* \in (\kappa_1, \kappa_2)$ and $a_* \in (0, 1)$ are constant in time and space. Then zero is a semi-simple eigenvalue of the total linearization A_0 from* (5.34)*, meaning that* $\mathrm{N}(A_0) \oplus \mathrm{R}(A_0) = \mathrm{X}_0$.

Proof. The idea of the proof is similar to the one of Lemma 3.6.5. Before, we provide some arguments to simplify the proof. First, we invoke Lemma 2.7.1(c) to argue that $\mathrm{A}^{\mathrm{atm}}$ is simply the restriction of the Laplacian operator in view of the pure Neumann boundary conditions on the upper and lower boundary. Therefore, $\mathrm{A}^{\mathrm{atm}}$ inherits the properties of the underlying Laplacian operator with periodic boundary conditions on the lateral boundary and Neumann boundary conditions on the upper and lower boundary. It also follows that zero is a semi-simple eigenvalue of $\mathrm{A}^{\mathrm{atm}}$ on $\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{atm}})$. In view of the block structure of A_0, with $\mathrm{A}^{\mathrm{atm}}$ representing a decoupled block, we conclude that the remaining part of A_0 can be investigated separately.

We denote the reduced part of A_0 by $A_{0,\mathrm{r}}$, and it is given by

$$A_{0,\mathrm{r}} \begin{pmatrix} v_{\mathrm{ocn}} \\ v_{\mathrm{ice}} \\ h \\ a \end{pmatrix} = \begin{pmatrix} -A_{\mathrm{m}}^{\mathrm{ocn}} v_{\mathrm{ocn}} \\ C_{\mathrm{o,i}}(h_*)\partial_z v_{\mathrm{ocn}} - A^{\mathrm{H}}(u_*) v_{\mathrm{ice}} + P_{\mathrm{h},*}\nabla_{\mathrm{H}} h + P_{\mathrm{a},*}\nabla_{\mathrm{H}} a \\ -d_{\mathrm{h}}\Delta_{\mathrm{H}} h + h_* \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}} \\ -d_{\mathrm{a}}\Delta_{\mathrm{H}} a + a_* \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}} \end{pmatrix}.$$

The associated ground space $\mathrm{X}_{0,\mathrm{r}}$ takes the shape

$$\mathrm{X}_{0,\mathrm{r}} = \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}) \times \mathrm{L}^q(G)^2 \times \mathrm{L}^q(G) \times \mathrm{L}^q(G),$$

and the domain $\mathrm{D}(A_{0,\mathrm{r}})$ is defined accordingly. Furthermore, for $\mathrm{L}^q_0(G)$ representing the space of $\mathrm{L}^q(G)$-functions with mean value zero on G, we define

$$\mathrm{X}^{\mathrm{m}}_{0,\mathrm{r}} := \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}) \times \mathrm{L}^q(G)^2 \times \mathrm{L}^q_0(G) \times \mathrm{L}^q_0(G),$$

and we denote by $A^{\mathrm{m}}_{0,\mathrm{r}}$ the restriction of $A_{0,\mathrm{r}}$ to $\mathrm{X}^{\mathrm{m}}_{0,\mathrm{r}}$. The same arguments as in the proof of Lemma 5.6.3 for $\lambda = 0$ show that

$$(5.46)\quad 0 \ge C_1 \cdot \Big(\|v_{\mathrm{ocn}}\|^2_{\mathrm{H}^1(\Omega_{\mathrm{ocn}})} + \|\nabla_{\mathrm{H}} v_{\mathrm{ice}}\|^2_{\mathrm{L}^q(G)} + \|\nabla_{\mathrm{H}} h\|^2_{\mathrm{L}^q(G)} + \|\nabla_{\mathrm{H}} a\|^2_{\mathrm{L}^q(G)}\Big)$$

when testing the eigenvalue equation $A_{0,\mathrm{r}}(v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a) = 0$ by a suitable test function for $(v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a) \in \mathrm{D}(A^{\mathrm{m}}_{0,\mathrm{r}}) = \mathrm{D}(A_{0,\mathrm{r}}) \cap \mathrm{X}^{\mathrm{m}}_{0,\mathrm{r}}$. In a first step, it follows therefrom that $v_{\mathrm{ocn}} = 0$, while v_{ice}, h and a are constant. Because of $v_{\mathrm{ocn}} = v_{\mathrm{ice}}$ on $\Gamma_o = G$ as well as h, $a \in \mathrm{L}^q_0(G)$, we derive that $v_{\mathrm{ice}} = 0$ and also $h = a = 0$. As a result, zero is not an eigenvalue of $A^{\mathrm{m}}_{0,\mathrm{r}}$. The compact resolvent of A_0 as well as its q-independent spectrum only consisting of eigenvalues carry over from Lemma 5.6.3 to the present situation, leading to $0 \in \rho(A^{\mathrm{m}}_{0,\mathrm{r}})$.

From (5.46), applied in the situation of $\mathrm{X}_{0,\mathrm{r}}$, it follows that

$$\mathrm{N}(A_{0,\mathrm{r}}) = \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R}.$$

In order to verify $\mathrm{X}_{0,\mathrm{r}} = \mathrm{N}(A_{0,\mathrm{r}}) + \mathrm{R}(A_{0,\mathrm{r}})$, it then suffices to prove that

$$\mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}) \times \mathrm{L}^q(G)^2 \times \mathrm{L}^q_0(G) \times \mathrm{L}^q_0(G) \subset \mathrm{R}(A_{0,\mathrm{r}}).$$

For this purpose, let $f = (f_{\mathrm{ocn}}, f_{\mathrm{ice}}, f_{\mathrm{h}}, f_{\mathrm{a}}) \in \mathrm{L}^q(G)^2 \times \mathrm{L}^q_0(G) \times \mathrm{L}^q_0(G)$. Thanks to $0 \in \rho(A^{\mathrm{m}}_{0,\mathrm{r}})$, we find $(v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a) \in \mathrm{D}(A^{\mathrm{m}}_{0,\mathrm{r}})$ with

$$A_{0,\mathrm{r}}(v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a)^\top = A^{\mathrm{m}}_{0,\mathrm{r}}(v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a)^\top = f,$$

showing $\mathrm{R}(A_{0,\mathrm{r}}) \subset \mathrm{X}^{\mathrm{m}}_{0,\mathrm{r}}$ as well as $\mathrm{X}_{0,\mathrm{r}} = \mathrm{N}(A_{0,\mathrm{r}}) + \mathrm{R}(A_{0,\mathrm{r}})$.

The last step is thus to establish that $\mathrm{N}(A_{0,\mathrm{r}}) \cap \mathrm{R}(A_{0,\mathrm{r}}) = \{0\}$. In fact, for $\tilde{v} = (v_{\mathrm{ocn},0}, v_{\mathrm{ice},0}, h_0, a_0) \in \mathrm{N}(A_{0,\mathrm{r}}) \cap \mathrm{R}(A_{0,\mathrm{r}})$, we conclude from the above shape of $\mathrm{N}(A_{0,\mathrm{r}})$ that $\tilde{v} = (0, 0, c_{\mathrm{h}}, c_{\mathrm{a}})$, where c_{h} and c_{a} are constant. In contrast, $\tilde{v} \in \mathrm{R}(A_{0,\mathrm{r}})$ yields the existence of $\widehat{v} = (v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a) \in \mathrm{D}(A_{0,\mathrm{r}})$ with $A_{0,\mathrm{r}}\widehat{v} = \tilde{v}$. For h_{m} and a_{m} as well as h_{avg} and a_{avg} designating the respective mean value zero and average parts as also already used in the proof of Lemma 3.6.5, we consider

$$\widehat{v} = (v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h, a)^\top = (v_{\mathrm{ocn}}, v_{\mathrm{ice}}, h_{\mathrm{m}} + h_{\mathrm{avg}}, a_{\mathrm{m}} + a_{\mathrm{avg}})^\top.$$

In particular, we observe that $(0, 0, h_{\mathrm{avg}}, a_{\mathrm{avg}}) \in \mathrm{N}(A_{0,\mathrm{r}})$ since h_{avg} and a_{avg} are constant. Therefore, for $\tilde{v} = (0, 0, c_{\mathrm{h}}, c_{\mathrm{a}})^\top$, we have $\tilde{v} = A_{0,\mathrm{r}}\widehat{v}$, where

$$A_{0,\mathrm{r}}\widehat{v} = A_{0,\mathrm{r}} \begin{pmatrix} v_{\mathrm{ocn}} \\ v_{\mathrm{ice}} \\ h_{\mathrm{m}} \\ a_{\mathrm{m}} \end{pmatrix} + A_{0,\mathrm{r}} \begin{pmatrix} 0 \\ 0 \\ h_{\mathrm{avg}} \\ a_{\mathrm{avg}} \end{pmatrix} = A_{0,\mathrm{r}} \begin{pmatrix} v_{\mathrm{ocn}} \\ v_{\mathrm{ice}} \\ h_{\mathrm{m}} \\ a_{\mathrm{m}} \end{pmatrix} = A^{\mathrm{m}}_{0,\mathrm{r}} \begin{pmatrix} v_{\mathrm{ocn}} \\ v_{\mathrm{ice}} \\ h_{\mathrm{m}} \\ a_{\mathrm{m}} \end{pmatrix}.$$

As a result, $\tilde{v} \in \mathrm{R}(A^{\mathrm{m}}_{0,\mathrm{r}}) = \mathrm{L}^q_{\overline{\sigma}}(\Omega_{\mathrm{ocn}}) \times \mathrm{L}^q(G)^2 \times \mathrm{L}^q(G) \times \mathrm{L}^q(G)$, so we deduce that $c_{\mathrm{h}} = c_{\mathrm{a}} = 0$. This shows that indeed, $\tilde{v} = 0$. In total, we have thus verified $\mathrm{N}(A_{0,\mathrm{r}}) \oplus \mathrm{R}(A_{0,\mathrm{r}})$, as desired. □

Concatenating Lemma 5.6.3, Lemma 5.6.4 and Lemma 5.6.5, we conclude that equilibria of the shape as introduced in Lemma 5.6.1 are normally stable in the sense of Definition 2.6.4. On the other hand, it follows from Corollary 5.4.4(a)(ii) that the linearized operator matrix at an equilibrium v_* admits maximal L^p-regularity up to a shift. The following theorem is hence obtained by an application of Proposition 2.6.5 together with Remark 2.6.6. It asserts the global strong well-posedness of the simplified coupled system for initial data close to constant equilibria.

Theorem 5.6.6. *Let* $p, q \in (1, \infty)$ *be such that* (5.27)*, take into account an equilibrium* $v_* = (0, 0, 0, h_*, a_*)$*, with* $h_* \in (\kappa_1, \kappa_2)$ *and* $a_* \in (0, 1)$ *constant in time and space, and recall* $\mathrm{X}_\gamma = (\mathrm{X}_0, \mathrm{X}_1)_{1-1/p,p}$ *from Corollary 5.4.4(a)(iv). Then* v_* *is stable in* X_γ*, and there exists* $r > 0$ *so that the unique solution* v *to* (5.33)*, or, equivalently, to* (5.31)*, for initial data* $v_0 \in \mathrm{X}_\gamma$ *with*

$$\|v_0 - v_*\|_{\mathrm{X}_\gamma} < r$$

exists on $\mathbb{R}_+$ *and converges to some* $v_\infty \in \mathcal{E}$ *in* X_γ *at an exponential rate as* $t \to \infty$*.*

The next remark on extensions of Theorem 5.6.6 completes the chapter.

Remark 5.6.7. *The assertion of Theorem 5.6.6 remains valid when considering Coriolis forcing terms and anisotropic Reynolds numbers as explained in Remark 5.5.3(b) and (c) in the context of the local strong well-posedness. In fact, the treatment of the Coriolis terms parallels Section 3.6, while the reduction argument for the anisotropic Reynolds numbers has already been provided in Remark 5.5.3(c).*

CHAPTER 6

The Parabolic-Hyperbolic Regularized Hibler Model

In this chapter, we focus on a parabolic-hyperbolic variant of Hibler's model, meaning that we drop the viscous terms in the balance laws. In (3.10), this corresponds precisely to the situation of $d_{\mathrm{h}} = d_{\mathrm{a}} = 0$. Let us remark that Hibler included diffusion terms in his original model [60], but the latter were added for numerical stability, and it is more physical to avoid these terms. We establish the local strong well-posedness of the resulting system with regularized stress tensor. In comparison with the previous chapters, the general strategy is closer to the one in Chapter 4, i. e., we use a direct approach, so we set up a fixed point argument instead of applying the abstract theory from Section 2.6. The results of this chapter have not been published so far.

It is worth pointing out that the considerations in this chapter differ significantly from the work of Liu, Thomas and Titi [97]. In fact, they established local strong well-posedness of a parabolic-hyperbolic version of Hibler's model with strongly regularized stress tensor. In contrast, as we have already expressed in Chapter 3, the regularization used in this thesis and especially in this chapter agrees with one of the most common regularizations used in numerical analysis, see for instance [82] or [105].

The chapter has the following structure. In Section 6.1, we provide the parabolic-hyperbolic system in Eulerian and Lagrangian coordinates in (6.1) and (6.8), respectively. The reason for using Lagrangian coordinates is that it allows us to handle the hyperbolic terms. This becomes apparent in Section 6.2, where we establish the maximal regularity of the linearized system (6.9) in an anisotropic ground space in Proposition 6.2.1. The maximal regularity is also the starting point for the fixed point argument presented in Section 6.3. There is no such thing as a free lunch, especially not in mathematics, so employing the Lagrangian coordinates is not for free. In fact, the differential operators involved take more complicated shapes as a result. It is the purpose of Section 6.4 to provide estimates of the nonlinear terms. At this stage, the choice of the anisotropic ground space proves useful once more, since the Sobolev space $W^{1,q}(\Omega)$ is a Banach algebra for $q > 2$. In Section 6.5, we finally state the main result, Theorem 6.5.1, on the local strong well-posedness of the parabolic-hyperbolic regularized model. The proof is based on the fixed point argument and the contraction mapping principle which in turn rely on the maximal regularity and the nonlinear estimates. The final Section 6.6 discusses some remaining open problems concerning the mathematical analysis of Hibler's sea ice model.

6.1. The System in Eulerian and Lagrangian Coordinates

Our strategy to attack the parabolic-hyperbolic model relies on the transform from Eulerian to Lagrangian coordinates in order to circumvent the hyperbolic effects in the balance laws. We elaborate on this in detail in Section 6.2. For this reason, we provide the system in both configurations in this section.

For convenience, we recall the general setting. Except for the absence of the dissipative terms, the set-up does not change in comparison with Chapter 3. In that respect, we consider a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$ of class C^2 as well as a time interval $(0, T)$, where $0 < T \le \infty$. Moreover, by $v_{\mathrm{ice}} \colon (0,T) \times \Omega \to \mathbb{R}^2$, $h \colon (0,T) \times \Omega \to [\kappa, \infty)$ and $a \colon (0,T) \times \Omega \to (0,1)$, we denote the horizontal ice velocity, the mean ice thickness and the ice compactness, respectively. In contrast to the consideration in Chapter 3, we do not include the diffusion terms $d_{\mathrm{h}} \Delta_{\mathrm{H}} h$ and $d_{\mathrm{a}} \Delta_{\mathrm{H}} a$ in the balance laws. Therefore, with the thermodynamic terms S_{h} and S_{a} as made precise in (3.8), the adjusted *hyperbolic balance laws* are given by

$$\begin{cases} \partial_t h + \mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} h) = S_{\mathrm{h}}(h,a), \\ \partial_t a + \mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} a) = S_{\mathrm{a}}(h,a). \end{cases}$$

In this new setting, the mean ice thickness h and the ice compactness a are not subject to any boundary conditions. On the other hand, we still assume that the horizontal ice velocity v_{ice} satisfies Dirichlet boundary conditions on $\partial\Omega$, so $v_{\mathrm{ice}} = 0$ on $(0,T)\times\partial\Omega$. Recalling the momentum equation from (3.7), using the abbreviation $\tau_{\mathrm{ice}} \coloneqq \tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(v_{\mathrm{ice}})$, and invoking the initial conditions, we obtain the *parabolic-hyperbolic regularized model*

$$
(6.1)\qquad \begin{cases}
\partial_t v_{\mathrm{ice}} + (v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice}} = \dfrac{1}{m_{\mathrm{ice}}} \mathrm{div}_{\mathrm{H}}\, \sigma_\delta - c_{\mathrm{cor}} v_{\mathrm{ice}}^{\perp} & \\
\qquad\qquad\qquad - g\nabla_{\mathrm{H}} H + \dfrac{1}{m_{\mathrm{ice}}} \tau_{\mathrm{ice}}, & \text{in } (0,T)\times\Omega, \\
\partial_t h + \mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} h) = S_{\mathrm{h}}(h,a), & \text{in } (0,T)\times\Omega, \\
\partial_t a + \mathrm{div}_{\mathrm{H}}\,(v_{\mathrm{ice}} a) = S_{\mathrm{a}}(h,a), & \text{in } (0,T)\times\Omega, \\
v_{\mathrm{ice}} = 0, & \text{on } (0,T)\times\partial\Omega, \\
v_{\mathrm{ice}}(0) = v_{\mathrm{ice},0}, \;\; h(0) = h_0, \;\; a(0) = a_0, & \text{in } \Omega.
\end{cases}
$$

The next step is to transform (6.1) to Lagrangian coordinates. The underlying coordinate transform requires the characteristics X corresponding to the ice velocity v_{ice}. This means that X is the solution to the Cauchy problem

$$
(6.2)\qquad \begin{cases}
\partial_t X(t,y_{\mathrm{H}}) = v_{\mathrm{ice}}(t, X(t,y_{\mathrm{H}})), & \text{for } t > 0, \\
X(0,y_{\mathrm{H}}) = y_{\mathrm{H}}, & \text{for } y_{\mathrm{H}} \in \mathbb{R}^2.
\end{cases}
$$

For $t \in (0,T)$, we denote by $Y(t,\cdot) = [X(t,\cdot)]^{-1}$ the inverse of $X(t,\cdot)$, and we will comment on the invertibility at a later stage. The corresponding change of variables to Lagrangian coordinates is given by

$$
(6.3)\qquad \begin{aligned}
\tilde{v}_{\mathrm{ice}}(t,y_{\mathrm{H}}) &\coloneqq v_{\mathrm{ice}}(t, X(t,y_{\mathrm{H}})), \\
\tilde{h}(t,y_{\mathrm{H}}) &\coloneqq h(t, X(t,y_{\mathrm{H}})) \;\text{ and} \\
\tilde{a}(t,y_{\mathrm{H}}) &\coloneqq a(t, X(t,y_{\mathrm{H}})).
\end{aligned}
$$

In addition, the principal variable will be denoted by $\tilde{u} = (\tilde{v}_{\mathrm{ice}}, \tilde{h}, \tilde{a})$. With Y representing the above transform, we especially obtain

$$
u(t,x_{\mathrm{H}}) = \tilde{u}(t, Y(t,x_{\mathrm{H}})).
$$

An integration in time in (6.2) and the change of variables from (6.3) lead to

$$
X(t,y_{\mathrm{H}}) = y_{\mathrm{H}} + \int_0^t \tilde{v}_{\mathrm{ice}}(s,y_{\mathrm{H}}) \,\mathrm{d}s.
$$

Equipped with the variables in Lagrangian coordinates, we are interested in the emerging transformed system of equations. It is well known, or can directly be derived from the change of variables, that the time derivative of the variables in Lagrangian coordinates coincides with the material derivative in Eulerian coordinates, so

$$\begin{aligned}\partial_t \tilde{v}_{\mathrm{ice}}(t, y_{\mathrm{H}}) &= \partial_t v_{\mathrm{ice}}(t, X(t, y_{\mathrm{H}})) + (v_{\mathrm{ice}}(t, X(t, y_{\mathrm{H}})) \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice}}(t, X(t, y_{\mathrm{H}})),\\ \partial_t \tilde{h}(t, y_{\mathrm{H}}) &= \partial_t h(t, X(t, y_{\mathrm{H}})) + v_{\mathrm{ice}}(t, X(t, y_{\mathrm{H}})) \cdot \nabla_{\mathrm{H}} h(t, X(t, y_{\mathrm{H}})) \text{ and}\\ \partial_t \tilde{a}(t, y_{\mathrm{H}}) &= \partial_t a(t, X(t, y_{\mathrm{H}})) + v_{\mathrm{ice}}(t, X(t, y_{\mathrm{H}})) \cdot \nabla_{\mathrm{H}} a(t, X(t, y_{\mathrm{H}})).\end{aligned}$$

Similarly as for the interaction problem in Section 4.2, the transformed symmetric part of the gradient takes the shape

$$\begin{aligned}2\varepsilon_{ij}(v_{\mathrm{ice}}) &= \partial_i v_{\mathrm{ice},j} + \partial_j v_{\mathrm{ice},i}\\ &= \sum_{k=1}^{2} (\partial_i Y_k) \partial_k \tilde{v}_{\mathrm{ice},j} + (\partial_j Y_k) \partial_k \tilde{v}_{\mathrm{ice},i} =: 2\tilde{\varepsilon}_{ij}(\tilde{v}_{\mathrm{ice}}).\end{aligned} \tag{6.4}$$

Before dealing with the transformed Hibler operator, we introduce

$$a_{ij}^{klm}(\tilde{\varepsilon}(\tilde{v}_{\mathrm{ice}}), P(\tilde{h}, \tilde{a})) := (\partial_k Y_m) a_{ij}^{kl}(\tilde{\varepsilon}(\tilde{v}_{\mathrm{ice}}), P(\tilde{h}, \tilde{a})) \tag{6.5}$$

and calculate

$$\begin{aligned}\partial_m \tilde{\varepsilon}_{jl}(v_{\mathrm{ice}}) = \frac{1}{2} \sum_{n=1}^{2} \Big(&(\partial_m \partial_j Y_n) \partial_n \tilde{v}_{ice,l} + (\partial_j Y_n) \partial_m \partial_n \tilde{v}_{\mathrm{ice},l}\\ &+ (\partial_m \partial_l Y_n) \partial_n \tilde{v}_{\mathrm{ice},j} + (\partial_l Y_n) \partial_m \partial_n \tilde{v}_{\mathrm{ice},j} \Big).\end{aligned}$$

With regard to the Hibler operator in differential form as introduced in (3.14), we recover the transformed Hibler operator $\tilde{\mathbb{A}}^{\mathrm{H}}$ of the shape

$$\begin{aligned}[\tilde{\mathbb{A}}^{\mathrm{H}}(\tilde{u}) \tilde{v}_{\mathrm{ice}}]_i = &\sum_{j,k,l,m=1}^{2} a_{ij}^{klm}(\tilde{\varepsilon}(\tilde{v}_{\mathrm{ice}}), P(\tilde{h}, \tilde{a})) \partial_m \tilde{\varepsilon}_{jl}(\tilde{v}_{\mathrm{ice}})\\ &+ \frac{1}{2\rho_{\mathrm{ice}} \tilde{h} \triangle_\delta(\tilde{\varepsilon}(\tilde{v}_{\mathrm{ice}}))} \sum_{j,k=1}^{2} (\partial_j Y_k)(\partial_k \tilde{h} + c_\bullet \partial_k \tilde{a})(\mathbb{S} \tilde{\varepsilon}(\tilde{v}_{\mathrm{ice}}))_{ij}.\end{aligned} \tag{6.6}$$

We also comment on the transformation of terms resulting from the horizontal divergence of the ice strength P. Their respective i-th components read as

$$\begin{aligned}(\tilde{B}_{\mathrm{h}}(\tilde{u}) \tilde{h})_i &:= \frac{\partial_h P(\tilde{h}, \tilde{a})}{2\rho_{\mathrm{ice}} \tilde{h}} \sum_{j=1}^{2} (\partial_i Y_j) \partial_j \tilde{h} \text{ and}\\ (\tilde{B}_{\mathrm{a}}(\tilde{u}) \tilde{a})_i &:= \frac{\partial_a P(\tilde{h}, \tilde{a})}{2\rho_{\mathrm{ice}} \tilde{h}} \sum_{j=1}^{2} (\partial_i Y_j) \partial_j \tilde{a}.\end{aligned} \tag{6.7}$$

For convenience, we also introduce the notation

$$\tilde{B}(\tilde{u})\begin{pmatrix}\tilde{h}\\ \tilde{a}\end{pmatrix} := \tilde{B}_{\mathrm{h}}(\tilde{u})\tilde{h} + \tilde{B}_{\mathrm{a}}(\tilde{u})\tilde{a}.$$

Concerning the terms involving differential operators, it remains to calculate the transformed (horizontal) divergence of v_{ice}. In fact, we obtain

$$\operatorname{div}_{\mathrm{H}} v_{\mathrm{ice}} = \sum_{j,k=1}^{2} (\partial_j Y_k)\partial_k \tilde{v}_{\mathrm{ice},j}.$$

The remaining terms from the complete system (6.1) do not involve derivatives, so their respective transformed versions can be deduced from an insertion of $\tilde{u}$ instead of $u = (v_{\mathrm{ice}}, h, a)$. Consequently, the *system of equations in Lagrangian coordinates* is given by

$$(6.8)\qquad \begin{cases} \partial_t \tilde{v}_{\mathrm{ice}} = \tilde{\mathbb{A}}^{\mathrm{H}}(\tilde{u})\tilde{v}_{\mathrm{ice}} - \tilde{B}(\tilde{u})\begin{pmatrix}\tilde{h}\\ \tilde{a}\end{pmatrix} - c_{\mathrm{cor}}\tilde{v}_{\mathrm{ice}}^{\perp} & \\ \qquad - g\nabla_{\mathrm{H}} H + \dfrac{1}{\rho_{\mathrm{ice}}\tilde{h}}(\tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(\tilde{v}_{\mathrm{ice}})), & \text{in } (0,T)\times\Omega, \\ \partial_t \tilde{h} = -\tilde{h}\displaystyle\sum_{j,k=1}^{2}(\partial_j Y_k)\partial_k \tilde{v}_{\mathrm{ice},j} + S_{\mathrm{h}}(\tilde{h},\tilde{a}), & \text{in } (0,T)\times\Omega, \\ \partial_t \tilde{a} = -\tilde{a}\displaystyle\sum_{j,k=1}^{2}(\partial_j Y_k)\partial_k \tilde{v}_{\mathrm{ice},j} + S_{\mathrm{a}}(\tilde{h},\tilde{a}), & \text{in } (0,T)\times\Omega, \\ \tilde{v}_{\mathrm{ice}} = 0, & \text{on } (0,T)\times\partial\Omega, \\ \tilde{v}_{\mathrm{ice}}(0) = v_{\mathrm{ice},0},\ \ \tilde{h}(0) = h_0,\ \ \tilde{a}(0) = a_0, & \text{in } \Omega. \end{cases}$$

The initial conditions result from the fact that for time zero, the transform from Eulerian to Lagrangian coordinates is simply the identity.

6.2. Maximal Regularity of the Linearized Problem

With regard to the linearized problem, we consider $f_1\colon (0,T)\times\Omega \to \mathbb{R}^2$ as well as f_2, $f_3\colon (0,T)\times\Omega\to\mathbb{R}$. Moreover, we take into account suitable initial data $u_0 = (v_{\mathrm{ice},0}, h_0, a_0)$ and invoke $u_1 \in \mathrm{C}^1(\overline{\Omega})^2 \times \mathrm{C}(\overline{\Omega}) \times \mathrm{C}(\overline{\Omega})$, where $h_1 \geq \kappa$ and $a_1 \in (0,1)$. For $\omega \geq 0$, the Hibler operator $\mathbb{A}^{\mathrm{H}}(u_1)$ as defined in (3.16), and the term B_1 related to $\mathrm{div}_{\mathrm{H}}\, P$ and as made precise in (4.31), the *linearized problem* investigated in the sequel is given by

$$
(6.9)\qquad
\begin{cases}
\partial_t \tilde{v}_{\mathrm{ice}} - (\mathbb{A}^{\mathrm{H}}(u_1) - \omega)\tilde{v}_{\mathrm{ice}} + B_1(u_1)\begin{pmatrix}\tilde{h}\\ \tilde{a}\end{pmatrix} = f_1, & \text{in } (0,T)\times\Omega,\\
\partial_t \tilde{h} + h_1 \mathrm{div}_{\mathrm{H}}\, \tilde{v}_{\mathrm{ice}} + \omega \tilde{h} = f_2, & \text{in } (0,T)\times\Omega,\\
\partial_t \tilde{a} + a_1 \mathrm{div}_{\mathrm{H}}\, \tilde{v}_{\mathrm{ice}} + \omega \tilde{a} = f_3, & \text{in } (0,T)\times\Omega,\\
\tilde{v}_{\mathrm{ice}} = 0, & \text{on } (0,T)\times\partial\Omega,\\
\tilde{v}_{\mathrm{ice}}(0) = v_{\mathrm{ice},0},\ \ \tilde{h}(0) = h_0,\ \ \tilde{a}(0) = a_0, & \text{in } \Omega.
\end{cases}
$$

Again, before stating the linear result, we comment on the functional analytic set-up. It deviates significantly from the one introduced in Section 3.4 in the context of the parabolic-hyperbolic model. In fact, for $q \in (1,\infty)$, we consider the *ground space*

$$
(6.10)\qquad \mathrm{X}_0 := \mathrm{L}^q(\Omega)^2 \times \mathrm{W}^{1,q}(\Omega) \times \mathrm{W}^{1,q}(\Omega),
$$

while the *regularity space* takes the shape

$$
(6.11)\qquad \mathrm{X}_1 := \mathrm{W}^{2,q}(\Omega)^2 \cap \mathrm{W}_0^{1,q}(\Omega)^2 \times \mathrm{W}^{1,q}(\Omega) \times \mathrm{W}^{1,q}(\Omega).
$$

The natural space for the initial data is again $\mathrm{X}_\gamma = (\mathrm{X}_0, \mathrm{X}_1)_{1-1/p,p}$, and we refer to this space as *trace space*. For $2 - 2/p > 1/q$, it follows from Lemma 1.3.4 and Lemma 1.3.6 that it is given by

$$
(6.12)\qquad \mathrm{X}_\gamma = \mathrm{B}_{qp,\mathrm{D}}^{2-2/p}(\Omega)^2 \times \mathrm{W}^{1,q}(\Omega) \times \mathrm{W}^{1,q}(\Omega),
$$

where the subscript $_\mathrm{D}$ represents homogeneous Dirichlet boundary conditions. If $2 - 2/p < 1/q$, then we have $\mathrm{X}_\gamma = \mathrm{B}_{qp}^{2-2/p}(\Omega)^2 \times \mathrm{W}^{1,q}(\Omega) \times \mathrm{W}^{1,q}(\Omega)$.

The corresponding *data space* takes the shape

$$
(6.13)\qquad \mathbb{E}_0 := \mathrm{L}^p(0,T;\mathrm{X}_0),
$$

and the *maximal regularity space* reads as

$$
(6.14)\qquad \mathbb{E}_1 := \mathrm{W}^{1,p}(0,T;\mathrm{X}_0) \cap \mathrm{L}^p(0,T;\mathrm{X}_1).
$$

We are now in the position to state the maximal regularity result for (6.9).

Proposition 6.2.1. *Let $p \in (1,\infty)$ and $q \in (2,\infty)$ be such that $1/p + 1/2q \neq 1$, and for $u_1 = (v_{\mathrm{ice},1}, h_1, a_1)$, assume $u_1 \in \mathrm{C}^1(\overline{\Omega})^2 \times \mathrm{W}^{1,q}(\Omega) \times \mathrm{W}^{1,q}(\Omega)$ such* *that* $h_1 \geq \kappa$ *and* $a_1 \in (0,1)$. *Moreover, for the time trace space* X_γ *and the data space* $\mathbb{E}_0$ *as defined in* (6.12) *and* (6.13), *let* $u_0 = (v_{\mathrm{ice},0}, h_0, a_0) \in \mathrm{X}_\gamma$, *and consider* $(f_1, f_2, f_3) \in \mathbb{E}_0$.

Then there exists $\omega_0 \in \mathbb{R}$ *such that for all* $\omega > \omega_0$, *the linearized problem* (6.9) *admits a unique solution* $\tilde{u} = (\tilde{v}_{\mathrm{ice}}, \tilde{h}, \tilde{a}) \in \mathbb{E}_1$, *with* $\mathbb{E}_1$ *as introduced in* (6.14). *In addition, there exists a constant* $C_{\mathrm{MR}} > 0$, *which can be chosen independent of* T *provided* $u_0 = 0$, *such that*

$$\|\tilde{u}\|_{\mathbb{E}_1} \leq C_{\mathrm{MR}} \cdot \Big(\|(f_1, f_2, f_3)\|_{\mathbb{E}_0} + \|u_0\|_{\mathrm{X}_\gamma} \Big).$$

Proof. The operator matrix associated to (6.9) for $\omega = 0$ reads as

$$\begin{aligned} A(u_1) &= \begin{pmatrix} -A_{\mathrm{D}}^{\mathrm{H}}(u_1) & \frac{\partial_h P(h_1,a_1)}{2\rho_{\mathrm{ice}} h_1} \nabla_{\mathrm{H}} & \frac{\partial_a P(h_1,a_1)}{2\rho_{\mathrm{ice}} h_1} \nabla_{\mathrm{H}} \\ h_1 \mathrm{div}_{\mathrm{H}} & 0 & 0 \\ a_1 \mathrm{div}_{\mathrm{H}} & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -A_{\mathrm{D}}^{\mathrm{H}}(u_1) & 0 & 0 \\ h_1 \mathrm{div}_{\mathrm{H}} & 0 & 0 \\ a_1 \mathrm{div}_{\mathrm{H}} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{\partial_h P(h_1,a_1)}{2\rho_{\mathrm{ice}} h_1} \nabla_{\mathrm{H}} & \frac{\partial_a P(h_1,a_1)}{2\rho_{\mathrm{ice}} h_1} \nabla_{\mathrm{H}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &=: A_1(u_1) + B(u_1). \end{aligned}$$

As $q > 2$, we derive from (1.8) that the embedding $\mathrm{W}^{1,q}(\Omega) \hookrightarrow \mathrm{C}(\overline{\Omega})$ is valid. Therefore, invoking $u_1 \in \mathrm{C}^1(\overline{\Omega}) \times \mathrm{W}^{1,q}(\Omega) \times \mathrm{W}^{1,q}(\Omega)$, we first deduce from Proposition 3.3.6 the existence of $\omega_0' \in \mathbb{R}$ such that for all $\omega > \omega_0'$, we have $-A_{\mathrm{D}}^{\mathrm{H}}(u_1) + \omega \in \mathcal{MR}_p(\mathrm{L}^q(\Omega)^2)$. In other words, the parabolic problem

$$\partial_t \tilde{v}_{\mathrm{ice}} - (A_{\mathrm{D}}^{\mathrm{H}}(u_1) - \omega)\tilde{v}_{\mathrm{ice}} = f_1, \text{ in } (0,T) \times \Omega, \text{ and } \tilde{v}_{\mathrm{ice}}(0) = v_{\mathrm{ice},0} \tag{6.15}$$

admits a unique solution

$$\tilde{v}_{\mathrm{ice}} \in \mathrm{W}^{1,p}(0,T;\mathrm{L}^q(\Omega)^2) \cap \mathrm{L}^p(0,T;\mathrm{W}^{2,q}(\Omega)^2 \cap \mathrm{W}_0^{1,q}(\Omega)^2) =: \mathbb{E}_1^{\mathrm{ice}}.$$

On the other hand, the operator $\omega \, \mathrm{Id}$ on $\mathrm{W}^{1,q}(\Omega)$ is $\mathcal{R}$-sectorial in the sense of Definition 2.1.20 with $\mathcal{R}$-angle $\phi_{\omega \, \mathrm{Id}}^{\mathcal{R}} < \pi/2$. By the UMD property of $\mathrm{W}^{1,q}(\Omega)$, Proposition 2.1.21 implies $\omega \, \mathrm{Id} \in {}_0\mathcal{MR}_p(\mathrm{W}^{1,q}(\Omega))$. The fact that $0 \in \rho(\omega \, \mathrm{Id})$ together with Lemma 2.1.14 yields $\omega \, \mathrm{Id} \in \mathcal{MR}_p(\mathrm{W}^{1,q}(\Omega))$. Moreover, the solution $\tilde{v}_{\mathrm{ice}}$ to (6.15) satisfies $\mathrm{div}_{\mathrm{H}} \, \tilde{v}_{\mathrm{ice}} \in \mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))$. In conjunction with $h_1 \in \mathrm{W}^{1,q}(\Omega)$ being constant in time, the Banach algebra structure of $\mathrm{W}^{1,q}(\Omega)$ for $q > 2$, Lemma 1.3.8 and $f_2 \in \mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))$, this yields

$$\tilde{f}_2 := f_2 - h_1 \mathrm{div}_{\mathrm{H}}\, \tilde{v}_{\mathrm{ice}} \in \mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega)),$$

and likewise $\tilde{f}_3 := f_3 - a_1 \mathrm{div}_{\mathrm{H}}\, \tilde{v}_{\mathrm{ice}} \in \mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))$. Hence, the maximal regularity of $\omega\,\mathrm{Id}$ for $\omega > \max\{0,\omega_0'\}$ implies the existence of a unique solution $(\tilde{h},\tilde{a})$ with $\tilde{h}$, $\tilde{a} \in \mathrm{W}^{1,p}(0,T;\mathrm{W}^{1,q}(\Omega))$ to

$$\begin{aligned} &\partial_t \tilde{h} + \omega\,\mathrm{Id}\,\tilde{h} = \tilde{f}_2, \ \text{ in } \ (0,T)\times\Omega, \ \ \tilde{h}(0) = h_0, \ \text{ and} \\ &\partial_t \tilde{a} + \omega\,\mathrm{Id}\,\tilde{a} = \tilde{f}_3, \ \text{ in } \ (0,T)\times\Omega, \ \ \tilde{a}(0) = a_0. \end{aligned}$$

In summary, $A_1(u_1) + \omega$ has maximal L^p-regularity for $\omega > \max\{0,\omega_0'\}$.

It remains to argue that $B(u_1)$ can be handled as a perturbation term. For $u = (v_{\mathrm{ice}},h,a) \in \mathrm{X}_1$, as in (3.27), we find the existence of $C_1 > 0$ with

$$\left\| \frac{\partial_h P(h_1,a_1)}{2\rho_{\mathrm{ice}} h_1} \nabla_{\mathrm{H}} h \right\|_{\mathrm{L}^q(\Omega)} \le C_1 \cdot \|h\|_{\mathrm{W}^{1,q}(\Omega)}.$$

Thus, performing the same estimate for the corresponding a-term, we infer that $B(u_1)$ is a bounded perturbation of $A_1(u_1)$ in the present setting. Consequently, for $\omega > \omega_0$, where $\omega_0 \in \mathbb{R}$ is possibly larger than the previous ω_0', we derive the maximal L^p-regularity of $A(u_1) + \omega$ from Corollary 2.1.23. The maximal regularity estimate then follows from the closed graph theorem. Concerning the T-independence of C_{MR} in the case of homogeneous initial values, we observe that we can use the maximal $\mathrm{L}^p(\mathbb{R}_+)$-regularity joint with the extension operators with T-independent operator norm from Lemma 2.4.2. $\square$

In the sequel, we fix $\omega > \omega_0$, where $\omega_0 \in \mathbb{R}$ is the shift resulting from Proposition 6.2.1.

6.3. The Fixed Point Argument

In this section, we rewrite the process of finding a solution to the transformed problem as a fixed point problem. In order to handle the initial conditions, we first invoke a so-called *reference solution*. It solves the linearized problem with homogeneous right-hand side and captures the initial values. As in the previous chapters, we also introduce an open subset $V \subset \mathrm{X}_\gamma$ of the time trace space X_γ defined precisely in (6.12). In fact, we set

$$V := \{u = (v_{\mathrm{ice}},h,a) \in \mathrm{X}_\gamma : h > \kappa \ \text{ and } \ a \in (0,1)\}. \tag{6.16}$$

In the sequel, we consider $p, q \in (1, \infty)$ such that

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}. \tag{6.17}$$

With regard to (1.8), condition (6.17) yields the embedding

$$\mathrm{X}_\gamma \hookrightarrow \mathrm{B}_{qp}^{2-2/p}(\Omega)^2 \times \mathrm{W}^{1,q}(\Omega) \times \mathrm{W}^{1,q}(\Omega) \hookrightarrow \mathrm{C}^1(\overline{\Omega})^2 \times \mathrm{C}(\overline{\Omega}) \times \mathrm{C}(\overline{\Omega}). \tag{6.18}$$

For $\omega > \omega_0$ as described in Section 6.2 and $u_0 = (v_{\mathrm{ice},0}, h_0, a_0) \in V$, the system under consideration reads as

$$\begin{cases} \partial_t \tilde{v}_{\mathrm{ice}} - (\mathbb{A}^{\mathrm{H}}(u_0) - \omega)\tilde{v}_{\mathrm{ice}} + B_1(u_0)\begin{pmatrix} \tilde{h} \\ \tilde{a} \end{pmatrix} = 0, & \text{in } (0,T) \times \Omega, \\ \partial_t \tilde{h} + h_0 \mathrm{div}_{\mathrm{H}}\, \tilde{v}_{\mathrm{ice}} + \omega \tilde{h} = 0, & \text{in } (0,T) \times \Omega, \\ \partial_t \tilde{a} + a_0 \mathrm{div}_{\mathrm{H}}\, \tilde{v}_{\mathrm{ice}} + \omega \tilde{a} = 0, & \text{in } (0,T) \times \Omega, \\ \tilde{v}_{\mathrm{ice}} = 0, & \text{on } (0,T) \times \partial\Omega, \\ \tilde{v}_{\mathrm{ice}}(0) = v_{\mathrm{ice},0}, \ \tilde{h}(0) = h_0, \ \tilde{a}(0) = a_0, & \text{in } \Omega. \end{cases} \tag{6.19}$$

The following result discusses the existence of a unique reference solution.

Proposition 6.3.1. *Let $p, q \in (1, \infty)$ fulfill* (6.17)*, consider $0 < T \le \infty$, and let $u_0 = (v_{\mathrm{ice},0}, h_0, a_0) \in V$, where V was introduced in* (6.16)*. Then there exists a unique solution $u_0^* = (v_{\mathrm{ice},0}^*, h_0^*, a_0^*) \in \mathbb{E}_1$ to* (6.19)*, with $\mathbb{E}_1$ as in* (6.14)*.*

Proof. Let us first observe that (6.17) especially yields that $q \in (2, \infty)$ as well as $2/p + 1/q \neq 2$. Thanks to the embedding (6.18) and the definition of V, we deduce from $u_0 \in V \subset \mathrm{X}_\gamma$ that $u_0 \in \mathrm{C}^1(\overline{\Omega})^2 \times \mathrm{W}^{1,q}(\Omega) \times \mathrm{W}^{1,q}(\Omega)$ with $h_0 > \kappa$ and $a_0 \in (0,1)$. The choice of the shift larger than ω_0 guarantees the existence of a unique solution u_0^* to (6.19) by Proposition 6.2.1. □

Some additional remarks on the reference solution are in order now.

Remark 6.3.2. *(a) The reference solution u_0^* is simply obtained by applying the semigroup generated by $A(u_0) + \omega$ as invoked in the proof of Proposition 6.2.1, so $u_0^*(t) = \mathrm{e}^{-(A(u_0)+\omega)(t)} u_0$. For convenience, we will denote the norm of u_0^* by C_T^* in the sequel, i. e., $C_T^* := \|u_0^*\|_{\mathbb{E}_1}$.*

(b) As the reference solution exists on any time interval $(0,T)$, $0 < T \le \infty$, we obtain the convergence of C_T^* to zero as $T \to 0$, so $C_T^* \to 0$ as $T \to 0$.

(c) *The representation of the reference solution from (a) also yields*

$$\|u_0^* - u_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \to 0 \quad as \ T \to 0. \tag{6.20}$$

On the other hand, since $V \subset \mathrm{X}_\gamma$ is open and $u_0 \in V$, there is $r_0 > 0$ sufficiently small such that $\overline{\mathbb{B}}_{\mathrm{X}_\gamma}(u_0, r_0) \subset V$. By virtue of (6.20), there is T_0 sufficiently small such that

$$\sup_{t \in [0,T_0]} \|u_0^*(t) - u_0\|_{\mathrm{X}_\gamma} \le \frac{r_0}{2}, \tag{6.21}$$

implying $u_0^*(t) \in V$ for all $t \in [0, T_0]$.

In the sequel, we restrict ourselves to the situation of $T \le T_0$, where $T_0 > 0$ is the time chosen sufficiently small in Remark 6.3.2(c) in order to guarantee that $u_0^*(t) \in V$ on the complete time interval $[0, T_0]$, i. e., the reference solution only attains values in the physically relevant range.

In view of the maximal regularity result, we are inclined to reformulate the transformed system of equations (6.8) as a linearized problem based on the linearization (6.9) for which we have the maximal regularity result Proposition 6.2.1 at hand. As the reference solution u_0^* captures the initial values, we consider homogeneous initial values in the subsequent linearized problem. Indeed, recalling the reference solution $u_0^* = (v_{\mathrm{ice},0}^*, h_0^*, a_0^*)$ from Proposition 6.3.1, and considering a solution $\tilde{u} = (\tilde{v}_{\mathrm{ice}}, \tilde{h}, \tilde{a})$ to the transformed system (6.8), we define $\widehat{u} = (\widehat{v}_{\mathrm{ice}}, \widehat{h}, \widehat{a})$ by

$$\widehat{v}_{\mathrm{ice}} := \tilde{v}_{\mathrm{ice}} - v_{\mathrm{ice},0}^*, \ \ \widehat{h} := \tilde{h} - h_0^* \ \text{and} \ \widehat{a} := \tilde{a} - a_0^*.$$

Then $\widehat{u}$ solves

$$\begin{cases} \partial_t \widehat{v}_{\mathrm{ice}} - (\mathbb{A}^{\mathrm{H}}(u_0) - \omega)\widehat{v}_{\mathrm{ice}} + B_1(u_0)\begin{pmatrix} \widehat{h} \\ \widehat{a} \end{pmatrix} = F_1(\widehat{u}), & \text{in } (0,T) \times \Omega, \\ \partial_t \widehat{h} + h_0 \mathrm{div}_{\mathrm{H}}\, \widehat{v}_{\mathrm{ice}} + \omega \widehat{h} = F_2(\widehat{u}), & \text{in } (0,T) \times \Omega, \\ \partial_t \widehat{a} + a_0 \mathrm{div}_{\mathrm{H}}\, \widehat{v}_{\mathrm{ice}} + \omega \widehat{a} = F_3(\widehat{u}), & \text{in } (0,T) \times \Omega, \\ \widehat{v}_{\mathrm{ice}} = 0, & \text{on } (0,T) \times \partial\Omega, \\ \widehat{v}_{\mathrm{ice}}(0) = 0, \ \widehat{h}(0) = 0, \ \widehat{a}(0) = 0, & \text{in } \Omega, \end{cases} \tag{6.22}$$

where, also using the abbreviation $\tau_{\mathrm{ice}} = \tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)$, we have

$$\begin{aligned} F_1(\widehat{u}) := {} & \left(\tilde{\mathbb{A}}^{\mathrm{H}}(\widehat{u} + u_0^*) - \mathbb{A}^{\mathrm{H}}(u_0)\right)(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) \\ & - \left(\tilde{B}(\widehat{u} + u_0^*) - B_1(u_0)\right)\begin{pmatrix} \widehat{h} + h_0^* \\ \widehat{a} + a_0^* \end{pmatrix} + \omega(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) \\ & - c_{\mathrm{cor}}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)^\perp - g\nabla_{\mathrm{H}} H + \frac{1}{\rho_{\mathrm{ice}}(\widehat{h} + h_0^*)}\tau_{\mathrm{ice}} \end{aligned} \tag{6.23}$$

as well as

$$
\begin{aligned}
F_2(\widehat{u}) &\coloneqq h_0 \mathrm{div}_{\mathrm{H}}\,(\widehat{v}_{\mathrm{ice}} + v^*_{\mathrm{ice},0}) - (\widehat{h} + h^*_0) \sum_{j,k=1}^{2} (\partial_j Y_k) \partial_k (\widehat{v}_{\mathrm{ice}} + v^*_{\mathrm{ice},0})_j \\
&\quad + \omega(\widehat{h} + h^*_0) + S_{\mathrm{h}}(\widehat{h} + h^*_0, \widehat{a} + a^*_0) \text{ and} \\
F_3(\widehat{u}) &\coloneqq a_0 \mathrm{div}_{\mathrm{H}}\,(\widehat{v}_{\mathrm{ice}} + v^*_{\mathrm{ice},0}) - (\widehat{a} + a^*_0) \sum_{j,k=1}^{2} (\partial_j Y_k) \partial_k (\widehat{v}_{\mathrm{ice}} + v^*_{\mathrm{ice},0})_j \\
&\quad + \omega(\widehat{a} + a^*_0) + S_{\mathrm{a}}(\widehat{h} + h^*_0, \widehat{a} + a^*_0).
\end{aligned} \tag{6.24}
$$

By ${}_0\mathbb{E}_1$, we denote the elements in $\mathbb{E}_1$ with homogeneous initial values, that means $\widehat{u} \in {}_0\mathbb{E}_1$ satisfies $\widehat{u}(0) = 0$. We fix $T_0 > 0$ as in Remark 6.3.2(c). Concerning an upper bound R_0 for $R > 0$, we also impose some condition. In fact, recall that given $u_0 \in V$, there is r_0 with $\overline{\mathbb{B}}_{\mathrm{X}_\gamma}(u_0, r_0) \subset V$. The aim is to ensure that the solution $\tilde{u} = \widehat{u} + u^*_0$ given by the sum of the unique fixed point $\widehat{u}$ of (6.22) and the reference solution u^*_0 from Proposition 6.3.1 is still contained in the open ball V for a suitable choice of R_0. For the time interval $(0, T_0)$, we thus recall from Proposition 2.4.11 the embedding

$$
{}_0\mathbb{E}_1 \hookrightarrow \mathrm{BUC}([0, T_0]; \mathrm{X}_\gamma). \tag{6.25}
$$

The associated embedding constant $C > 0$ is independent of T by the homogeneous initial values, see Remark 2.4.12. Thus, making use of (6.21) by the choice of T_0, and choosing $R_0 \le r_0/2C$, for $\widehat{u} \in {}_0\mathbb{E}_1$ with $\|\widehat{u}\|_{\mathbb{E}_1} \le R_0$, we infer

$$
\begin{aligned}
\sup_{t \in [0,T_0]} \|\tilde{u}(t) - u_0\|_{\mathrm{X}_\gamma} &\le \sup_{t \in [0,T_0]} \Big(\|\widehat{u}\|_{\mathrm{X}_\gamma} + \|u^*_0(t) - u_0\|_{\mathrm{X}_\gamma} \Big) \\
&\le C \cdot \|\widehat{u}\|_{\mathbb{E}_1} + \sup_{t \in [0,T_0]} \|u^*_0(t) - u_0\|_{\mathrm{X}_\gamma} \le \frac{r_0}{2} + \frac{r_0}{2} \le r_0.
\end{aligned}
$$

We summarize the preceding discussion in the following lemma.

Lemma 6.3.3. *Let $u_0 \in V$, recall $T_0 > 0$ from Remark 6.3.2, and for the T-independent embedding constant C from* (6.25) *as well as $r_0 > 0$ such that $\overline{\mathbb{B}}_{\mathrm{X}_\gamma}(u_0, r_0) \subset V$, set $0 < R_0 \le r_0/2C$. Consider $T \in (0, T_0]$, $R \in (0, R_0]$ and $\tilde{u} \coloneqq \widehat{u} + u^*_0$, where $\widehat{u} \in {}_0\mathbb{E}_1$ with $\|\widehat{u}\|_{\mathbb{E}_1} \le R$, and $u^*_0 \in \mathbb{E}_1$ denotes the reference solution from Proposition 6.3.1. Then $\tilde{u}(t) \in V$ for all $t \in [0, T]$.*

We are now in the position to elaborate on the fixed point argument. Consider $p, q \in (1, \infty)$ fulfilling (6.17), and for $T_0 > 0$ and $R_0 > 0$ as in Lemma 6.3.3, let $T \in (0, T_0]$ as well as $R \in (0, R_0]$. From there, we define

$$
\begin{aligned}
&\mathcal{K}^R_T \coloneqq \{\overline{u} \in {}_0\mathbb{E}_1 : \|\overline{u}\|_{\mathbb{E}_1} \le R\} \text{ and} \\
&\Phi^R_T \colon \mathcal{K}^R_T \to {}_0\mathbb{E}_1, \text{ with } \Phi^R_T(\overline{u}) \coloneqq \widehat{u}.
\end{aligned} \tag{6.26}
$$

In the above, $\widehat{u}$ represents the unique solution to (6.22), where we consider the right-hand sides $F_1(\overline{u})$, $F_2(\overline{u})$ and $F_3(\overline{u})$ as introduced in (6.23) as well as (6.24), and $\overline{u} \in \mathcal{K}_T^R$ is assumed. If the terms on the right-hand side are contained in the data space, then the map Φ_T^R is indeed well-defined thanks to $u_0 \in V \subset \mathrm{X}_\gamma$, the embedding (6.18) of the trace space and the maximal regularity from Proposition 6.2.1.

6.4. Nonlinear Estimates

This section is dedicated to establishing suitable estimates of the terms F_1, F_2 and F_3 defined precisely in (6.23) and (6.24). The latter task requires some preparation which we address below.

We start with the coordinate transform from Eulerian to Lagrangian coordinates. At first, let us recall from Section 6.1 that

$$X(t, y_\mathrm{H}) = y_\mathrm{H} + \int_0^t \tilde{v}_\mathrm{ice}(s, y_\mathrm{H}) \,\mathrm{d}s, \tag{6.27}$$

and we denote by $Y(t, \cdot)$ the inverse of $X(t, \cdot)$, also leading to

$$\nabla_\mathrm{H} Y(t, X(t, y_\mathrm{H})) = [\nabla_\mathrm{H} X]^{-1}(t, y_\mathrm{H}).$$

We observe that the diffeomorphisms X and Y depend on $\tilde{v}_\mathrm{ice}$. In particular, we are interested in the situation of $\tilde{v}_\mathrm{ice}$ of the shape described in Section 6.3, so $\tilde{v}_\mathrm{ice}$ being the sea ice component of $\tilde{u} = \widehat{u} + u_0^*$, where u_0^* is the reference solution from Proposition 6.3.1 with norm C_T^*, while $\widehat{u} \in {}_0\mathbb{E}_1$ solves (6.22) and satisfies $\|\widehat{u}\|_{\mathbb{E}_1} \le R$, so $\widehat{u} \in \mathcal{K}_T^R$. For such $\tilde{u}$, writing $\mathbb{E}_1 = \mathbb{E}_1^\mathrm{ice} \times \mathbb{E}_1^\mathrm{h} \times \mathbb{E}_1^\mathrm{a}$, we deduce

$$\|\tilde{u}\|_{\mathbb{E}_1} \le R + C_T^* \le R_0 + C_{T_0}^*, \quad \text{so} \quad \|\tilde{v}_\mathrm{ice}\|_{\mathbb{E}_1^\mathrm{ice}} \le R_0 + C_{T_0}^*. \tag{6.28}$$

Denoting by C_0 the norm of the initial values, so $C_0 \coloneqq \|u_0\|_{\mathrm{X}_\gamma}$, we deduce from Lemma 2.4.14 the estimate

$$\|u_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \le C\left(\|u_0\|_{\mathrm{X}_\gamma} + \|u_0^*\|_{\mathbb{E}_1}\right) \le C(C_0 + C_T^*), \tag{6.29}$$

where the constant $C > 0$ is independent of T. In the subsequent discussion, we do not number the constants and remark instead that $C > 0$ represents a generic constant. For X as in (6.27) and p' denoting the Hölder conjugate of p, we first get by Hölder's inequality

$$
\begin{aligned}
\sup_{t\in[0,T]} \|\nabla_{\mathrm{H}} X - \mathrm{Id}_2\|_{\mathrm{W}^{1,q}(\Omega)} &\le C \int_0^T \|\nabla_{\mathrm{H}} \tilde{v}_{\mathrm{ice}}(t,\cdot)\|_{\mathrm{W}^{1,q}(\Omega)} \\
&\le C T^{1/p'} \cdot \|\tilde{v}_{\mathrm{ice}}\|_{\mathbb{E}_1^{\mathrm{ice}}} \\
&\le C T^{1/p'} (R_0 + C_{T_0}^*).
\end{aligned}
\tag{6.30}
$$

In view of the embedding $\mathrm{W}^{1,q}(\Omega) \hookrightarrow \mathrm{L}^\infty(\Omega)$ resulting from the Sobolev embedding (1.8) thanks to condition (6.17) on p and q, this also yields

$$
\sup_{t\in[0,T]} \|\nabla_{\mathrm{H}} X - \mathrm{Id}_2\|_{\mathrm{L}^\infty(\Omega)} \le C T^{1/p'} (R_0 + C_{T_0}^*).
$$

Hence, we may choose T_0' sufficiently small such that

$$
\sup_{t\in[0,T_0']} \|\nabla_{\mathrm{H}} X - \mathrm{Id}_2\|_{\mathrm{L}^\infty(\Omega)} \le \frac{1}{2}.
\tag{6.31}
$$

A Neumann series argument thus guarantees the invertibility of $\nabla_{\mathrm{H}} X(t,\cdot)$ for all $t \in [0,T_0']$, and $\nabla_{\mathrm{H}} Y(t,\cdot)$ thus exists on $[0,T_0']$. At the same time, it follows from (6.27) that $\partial_t \nabla_{\mathrm{H}} X(t,y_{\mathrm{H}}) = \nabla_{\mathrm{H}} \tilde{v}_{\mathrm{ice}}(t,y_{\mathrm{H}})$. By (6.28), we get

$$
\|\partial_t \nabla_{\mathrm{H}} X(t,\cdot)\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \le C \cdot \|\tilde{v}_{\mathrm{ice}}\|_{\mathbb{E}_1^{\mathrm{ice}}} \le C(R_0 + C_{T_0}^*).
$$

In summary, as $R_0 + C_{T_0}^*$ is fixed, there exists a constant $C > 0$ such that

$$
\|\nabla_{\mathrm{H}} X\|_{\mathrm{W}^{1,p}(0,T;\mathrm{W}^{1,q}(\Omega))} + \|\nabla_{\mathrm{H}} X\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \le C.
\tag{6.32}
$$

With regard to the condition on p and q from (6.17), we find that the spaces $\mathrm{W}^{1,p}(0,T;\mathrm{W}^{1,q}(\Omega))$ as well as $\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))$ are Banach algebras by virtue of Lemma 1.3.9, meaning that the norm of a product can be estimated by the product of the norm. As a consequence of (6.32) and the definition of det and Cof, we conclude the existence of a constant $C > 0$ with

$$
\begin{aligned}
\|\det \nabla_{\mathrm{H}} X\|_{\mathrm{W}^{1,p}(0,T;\mathrm{W}^{1,q}(\Omega))} + \|\det \nabla_{\mathrm{H}} X\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} &\le C \quad \text{and} \\
\|\operatorname{Cof} \nabla_{\mathrm{H}} X\|_{\mathrm{W}^{1,p}(0,T;\mathrm{W}^{1,q}(\Omega))} + \|\operatorname{Cof} \nabla_{\mathrm{H}} X\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} &\le C.
\end{aligned}
$$

Thanks to (6.31), we find that $\det \nabla_{\mathrm{H}} X \ge C > 0$ on $(0,T) \times \Omega$ for some constant $C > 0$ provided $T \le T_0'$. The representation

$$
\nabla_{\mathrm{H}} Y = [\nabla_{\mathrm{H}} X]^{-1} = \frac{1}{\det \nabla_{\mathrm{H}} X} (\operatorname{Cof} \nabla_{\mathrm{H}} X)^\top
\tag{6.33}
$$

then results in

$$
\|\nabla_{\mathrm{H}} Y\|_{\mathrm{W}^{1,p}(0,T;\mathrm{W}^{1,q}(\Omega))} + \|\nabla_{\mathrm{H}} Y\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \le C.
$$

We collect the preceding observations in the lemma below.

Lemma 6.4.1. *Let $p, q \in (1, \infty)$ be such that* (6.17), *and let $T \in (0, T_1]$, with $T_1 := \min\{T_0, T_0'\} > 0$, where $T_0 > 0$ comes from* (6.21), *and $T_0' > 0$ is related to* (6.31). *Besides, for $R_0 > 0$ as in Lemma 6.3.3, consider $R \in (0, R_0]$ and $\tilde{u} := \widehat{u} + u_0^*$, where $\widehat{u} \in \mathcal{K}_T^R$, and u_0^* denotes the reference solution from Proposition 6.3.1. Then for X as in* (6.27) *and $\nabla_{\mathrm{H}} Y := [\nabla_{\mathrm{H}} X]^{-1}$, whose existence is guaranteed by* (6.33), *we get*

$$\|\nabla_{\mathrm{H}} X\|_{\mathrm{W}^{1,p}(0,T;\mathrm{W}^{1,q}(\Omega))} + \|\nabla_{\mathrm{H}} X\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \le C \quad \text{and}$$
$$\|\nabla_{\mathrm{H}} Y\|_{\mathrm{W}^{1,p}(0,T;\mathrm{W}^{1,q}(\Omega))} + \|\nabla_{\mathrm{H}} Y\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \le C$$

for some constant $C > 0$.

In the remainder of this section, we consider $T \in (0, T_1]$ and $R \in (0, R_0]$, where $T_1 > 0$ is as determined in Lemma 6.4.1, and $R_0 > 0$ is as made precise in Lemma 6.3.3. Combining Lemma 6.4.1 with the T-powers resulting from Hölder's inequality as made precise in Lemma 1.3.1, we derive the following estimate of $\mathrm{Id}_2 - \nabla_{\mathrm{H}} Y$ upon observing $\nabla_{\mathrm{H}} Y(0, \cdot) = \mathrm{Id}_2$.

Lemma 6.4.2. *Let $p, q \in (1, \infty)$ be such that* (6.17) *is valid, $T \in (0, T_1]$ as well as $R \in (0, R_0]$. Let further $\tilde{u} := \widehat{u} + u_0^*$, where $\widehat{u} \in \mathcal{K}_T^R$, and $u_0^* \in \mathbb{E}_1$ represents the reference solution from Proposition 6.3.1. Then for X from* (6.27) *and $\nabla_{\mathrm{H}} Y = [\nabla_{\mathrm{H}} X]^{-1}$, we have*

$$\| \mathrm{Id}_2 - \nabla_{\mathrm{H}} Y\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \le C T^{1/p'}$$

for a constant $C > 0$ and $p' \in (1, \infty)$ with $1/p + 1/p' = 1$.

With regard to the Lipschitz estimates, we also discuss estimates of differences of the diffeomorphisms. More precisely, we investigate the transforms associated to $\tilde{v}_{\mathrm{ice},1}$ and $\tilde{v}_{\mathrm{ice},2}$, where $\tilde{u}_i = \widehat{u}_i + u_0^*$, for $\widehat{u}_i \in {}_0\mathbb{E}_1$ and the reference solution u_0^* from Proposition 6.3.1. The diffeomorphisms related to $\tilde{v}_{\mathrm{ice},i}$ are denoted by X_i and Y_i, $i = 1, 2$. As in (6.30), we get

$$\sup_{t \in [0,T]} \|\nabla_{\mathrm{H}} X_1 - \nabla_{\mathrm{H}} X_2\|_{\mathrm{W}^{1,q}(\Omega)} \le C T^{1/p'} \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1},$$

also implying

$$\sup_{t \in [0,T]} \|\nabla_{\mathrm{H}} X_1 - \nabla_{\mathrm{H}} X_2\|_{\mathrm{L}^\infty(\Omega)} \le C T^{1/p'} \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}.$$

On the other hand, from $\partial_t (\nabla_{\mathrm{H}} X_1 - \nabla_{\mathrm{H}} X_2) = \nabla_{\mathrm{H}} (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})$, we conclude

$$\|\partial_t \nabla_{\mathrm{H}} (X_1 - X_2)\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \le C \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}.$$

In summary, we thus obtain

$$\begin{aligned}(6.34)\qquad &\|\nabla_{\mathrm{H}}(X_1 - X_2)\|_{\mathrm{W}^{1,p}(0,T;\mathrm{W}^{1,q}(\Omega))}\\ &+ \|\nabla_{\mathrm{H}}(X_1 - X_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \le C \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}.\end{aligned}$$

For the treatment of the difference $\nabla_{\mathrm{H}}(Y_1 - Y_2) = \nabla_{\mathrm{H}} Y_1 - \nabla_{\mathrm{H}} Y_2$, we invoke

$$\nabla_{\mathrm{H}} Y_1 - \nabla_{\mathrm{H}} Y_2 = -\nabla_{\mathrm{H}} Y_1 \left(\nabla_{\mathrm{H}}(X_1 - X_2)\right) \nabla_{\mathrm{H}} Y_2.$$

Hence, exploiting the Banach algebra structure of $\mathrm{W}^{1,p}(0,T;\mathrm{W}^{1,q}(\Omega))$ as well as $\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))$ as asserted in Lemma 1.3.9, and using (6.34) together with Lemma 6.4.1, we find the existence of a constant $C > 0$ with

$$\begin{aligned}&\|\nabla_{\mathrm{H}}(Y_1 - Y_2)\|_{\mathrm{W}^{1,p}(0,T;\mathrm{W}^{1,q}(\Omega))}\\ &+ \|\nabla_{\mathrm{H}}(Y_1 - Y_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \le C \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}.\end{aligned}$$

The above discussion is summarized in the lemma below.

Lemma 6.4.3. *Let $p, q \in (1,\infty)$ satisfy* (6.17)*, $T \in (0, T_1]$ and $R \in (0, R_0]$. Moreover, consider $\tilde{u}_i := \widehat{u}_i + u_0^*$, $i = 1, 2$, where $\widehat{u}_i \in \mathcal{K}_T^R$, and u_0^* denotes the reference solution from Proposition 6.3.1. Then for X_i as made precise in* (6.27)*, $\nabla_{\mathrm{H}} Y_i := [\nabla_{\mathrm{H}} X_i]^{-1}$ and $\nabla_{\mathrm{H}}(Y_1 - Y_2)$, we infer the existence of a constant $C > 0$ such that*

$$\begin{aligned}&\|\nabla_{\mathrm{H}}(X_1 - X_2)\|_{\mathrm{W}^{1,p}(0,T;\mathrm{W}^{1,q}(\Omega))}\\ &+ \|\nabla_{\mathrm{H}}(X_1 - X_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \le C \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1} \quad \text{and}\\ &\|\nabla_{\mathrm{H}}(Y_1 - Y_2)\|_{\mathrm{W}^{1,p}(0,T;\mathrm{W}^{1,q}(\Omega))}\\ &+ \|\nabla_{\mathrm{H}}(Y_1 - Y_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \le C \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}.\end{aligned}$$

The analogue of Lemma 6.4.2 for the estimate of differences in $\nabla_{\mathrm{H}} Y$ reads as follows. It is a consequence of Lemma 6.4.3 in conjunction with Lemma 1.3.1 as well as the observation that $\nabla_{\mathrm{H}} Y_1(0,\cdot) = \nabla_{\mathrm{H}} Y_2(0,\cdot) = \mathrm{Id}_2$.

Lemma 6.4.4. *Let $p, q \in (1,\infty)$ satisfy* (6.17)*, $T \in (0, T_1]$ and $R \in (0, R_0]$. Besides, consider $\tilde{u}_i := \widehat{u}_i + u_0^*$, $i = 1, 2$, where $\widehat{u}_i \in \mathcal{K}_T^R$, and u_0^* denotes the reference solution from Proposition 6.3.1. Then for X_i as in* (6.27) *as well as $\nabla_{\mathrm{H}} Y_i := [\nabla_{\mathrm{H}} X_i]^{-1}$ and $\nabla_{\mathrm{H}}(Y_1 - Y_2)$, we conclude that*

$$\|\nabla_{\mathrm{H}}(Y_1 - Y_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \le C T^{1/p'} \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}$$

for a constant $C > 0$ and the Hölder conjugate $p' \in (1,\infty)$ of p.

Before attacking the estimates of the right-hand sides, we establish some further handy embedding results.

Lemma 6.4.5. *Let $p, q \in (1,\infty)$ satisfy (6.17), let $T \in (0, T_1]$, and recall the trace space X_γ from (6.12), also denoted by $\mathrm{X}_\gamma = \mathrm{X}_\gamma^{\mathrm{ice}} \times \mathrm{X}_\gamma^{\mathrm{h}} \times \mathrm{X}_\gamma^{\mathrm{a}}$ in the sequel.*

(a) We have the embedding ${}_0\mathbb{E}_1 \hookrightarrow \mathrm{BUC}([0,T]; \mathrm{X}_\gamma)$, and the associated embedding constant can be chosen independent of $T > 0$.

(b) It holds that

$$\mathrm{BUC}([0,T]; \mathrm{X}_\gamma^{\mathrm{ice}}) \hookrightarrow \mathrm{L}^\infty(0,T; \mathrm{L}^{2q}(\Omega)^2) \hookrightarrow \mathrm{L}^{2p}(0,T; \mathrm{L}^{2q}(\Omega)^2).$$

In particular, for a T-independent embedding constant, we obtain

$${}_0\mathbb{E}_1 \hookrightarrow \mathrm{BUC}([0,T]; \mathrm{X}_\gamma^{\mathrm{ice}}) \hookrightarrow \mathrm{L}^{2p}(0,T; \mathrm{L}^{2q}(\Omega)^2).$$

Proof. The assertion of (a) is a direct consequence of Proposition 2.4.11, where Remark 2.4.12 implies the T-independence of the embedding constant. Concerning (b), the embedding in time directly follows from the time interval being finite, whereas the embedding in space is a result of the trace space embedding (6.18) and the boundedness of the domain $\Omega \subset \mathbb{R}^2$. The second assertion of (b) is then implied by (a). □

We successively estimate the nonlinear terms F_1, F_2 and F_3 from (6.23) and (6.24). Let us remark that we consider $T \in (0, T_1]$ and $R \in (0, R_0]$ throughout this section, where $T_1 > 0$ and $R_0 > 0$ are fixed and result from Lemma 6.4.1 and Lemma 6.3.3, respectively. The first terms to be estimated only concern the quasilinear terms from the non-transformed equation.

Lemma 6.4.6. *Let $p, q \in (1,\infty)$ be such that (6.17). For $T \in (0, T_1]$ as well as $R \in (0, R_0]$, consider $\tilde{u} := \widehat{u} + u_0^*$ and $\tilde{u}_i := \widehat{u}_i + u_0^*$, with $i = 1, 2$ and for $\widehat{u}$, $\widehat{u}_1$, $\widehat{u}_2 \in \mathcal{K}_T^R$ and the reference solution u_0^* from Proposition 6.3.1. Moreover, we set $B_1^* := B_1(u_0^*)$ for simplicity. Then there exists $C > 0$ with*

$$\begin{aligned}
&\left\| \left(\mathbb{A}^{\mathrm{H}}(u_0^*) - \mathbb{A}^{\mathrm{H}}(u_0)\right) \tilde{v}_{\mathrm{ice}} - (B_1^* - B_1(u_0)) \begin{pmatrix} \tilde{h} \\ \tilde{a} \end{pmatrix} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \\
&\le C \cdot \|u_0^* - u_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot (R + C_T^*), \\
&\left\| \left(\mathbb{A}^{\mathrm{H}}(u_0^*) - \mathbb{A}^{\mathrm{H}}(u_0)\right) (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) - (B_1^* - B_1(u_0)) \begin{pmatrix} \widehat{h}_1 - \widehat{h}_2 \\ \widehat{a}_1 - \widehat{a}_2 \end{pmatrix} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \\
&\le C \cdot \|u_0^* - u_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1},
\end{aligned}$$

$$\left\| \left(\mathbb{A}^{\mathrm{H}}(\tilde{u}) - \mathbb{A}^{\mathrm{H}}(u_0^*)\right) \tilde{v}_{\mathrm{ice}} - (B_1(\tilde{u}) - B_1^*) \begin{pmatrix} \tilde{h} \\ \tilde{a} \end{pmatrix} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))}$$
$\leq CR(R + C_T^*)$ *and*
$$\left\| \left(\mathbb{A}^{\mathrm{H}}(\tilde{u}_2) - \mathbb{A}^{\mathrm{H}}(u_0^*)\right) (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) - (B_1(\tilde{u}_2) - B_1^*) \begin{pmatrix} \widehat{h}_1 - \widehat{h}_2 \\ \widehat{a}_1 - \widehat{a}_2 \end{pmatrix} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))}$$
$\leq CR \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}$.

Proof. First, let us observe that the reduced regularity of the trace space X_γ as introduced in (6.12) in the h- and a-components does not affect the estimates from the considerations in Chapter 3 as the terms depend smoothly on h and a. Thus, it is sufficient to have continuity in the h- and a-component up to the boundary. As in the proof of Theorem 3.5.2, for u_1, $u_2 \in V$ as well as $w = (v_{\mathrm{ice}}, h, a) \in \mathrm{X}_1$, we find the existence of a constant $C_A > 0$ with

$$\begin{aligned} &\left\| \left(\mathbb{A}^{\mathrm{H}}(u_1) - \mathbb{A}^{\mathrm{H}}(u_2)\right) v_{\mathrm{ice}} - (B_1(u_1) - B_1(u_2)) \begin{pmatrix} h \\ a \end{pmatrix} \right\|_{\mathrm{L}^q(\Omega)} \\ &\leq C_A \cdot \|u_1 - u_2\|_{\mathrm{X}_\gamma} \cdot \|w\|_{\mathrm{X}_1}. \end{aligned} \tag{6.35}$$

We have $u_0^*(t) \in V$ for all $t \in [0,T]$ thanks to Remark 6.3.2(c), while Lemma 6.3.3 implies that $\tilde{u}(t) = \widehat{u}(t) + u_0^*(t) \in V$ for all $t \in [0,T]$. Therefore, an application of (6.35) and (6.28) in the last step yields

$$\begin{aligned} &\left\| \left(\mathbb{A}^{\mathrm{H}}(u_0^*) - \mathbb{A}^{\mathrm{H}}(u_0)\right) \tilde{v}_{\mathrm{ice}} - (B_1(u_0^*) - B_1(u_0)) \begin{pmatrix} \tilde{h} \\ \tilde{a} \end{pmatrix} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \\ &= \left(\int_0^T \left\| \left(\mathbb{A}^{\mathrm{H}}(u_0^*(t)) - \mathbb{A}^{\mathrm{H}}(u_0)\right) \tilde{v}_{\mathrm{ice}}(t) - (B_1^* - B_1(u_0)) \begin{pmatrix} \tilde{h}(t) \\ \tilde{a}(t) \end{pmatrix} \right\|_{\mathrm{L}^q(\Omega)}^p \mathrm{d}t \right)^{1/p} \\ &\leq C_A \left(\int_0^T \|u_0^*(t) - u_0\|_{\mathrm{X}_\gamma}^p \cdot \|\tilde{u}(t)\|_{\mathrm{X}_1}^p \, \mathrm{d}t \right)^{1/p} \\ &\leq C_A \cdot \|u_0^* - u_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot \|\tilde{u}(t)\|_{\mathrm{L}^p(0,T;\mathrm{X}_1)} \\ &\leq C_1 \cdot \|u_0^* - u_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot (R + C_T^*). \end{aligned}$$

This shows the first part of the assertion. The other estimates can be obtained in a similar way. Note that the dependence of the constant on u_0 and r_0 is not an issue, since the latter two quantities are fixed. □

For the estimates of the nonlinearities, we impose assumptions on the data.

Assumption 6.4.7. *Let $q \in (1, \infty)$, and consider $T_1 > 0$ as in Lemma 6.4.1. We then make the following assumptions on the data.*

(a) *For the velocities of the surface winds* V_{atm} *and the ocean* V_{ocn}*, it is valid that* V_{atm}*,* $V_{\mathrm{ocn}} \in \mathrm{L}^\infty(0,T_1;\mathrm{L}^{2q}(\Omega)^2)$*.*

(b) *The sea surface dynamic height* H *satisfies* $\nabla_{\mathrm{H}} H \in \mathrm{L}^\infty(0,T_1;\mathrm{L}^q(\Omega)^2)$*.*

(c) *The ice growth rate function* f_{gr} *fulfills* $f_{\mathrm{gr}} \in \mathrm{C}_{\mathrm{b}}^1([0,\infty))$*.*

With regard to the estimates of the nonlinear terms, we start with $F_1(\widehat{u})$ from (6.23). As we will also address Lipschitz estimates of this term, we compute the difference of $F_1(\widehat{u}_1)$ and $F_1(\widehat{u}_2)$ as a preparation. Employing the notation $\tilde{u}_i = \widehat{u}_i + u_0^*$ for simplicity, and observing that the terms thus still depend on $\widehat{u}_i$, we deduce that

$$\begin{aligned}
&F_1(\widehat{u}_1) - F_1(\widehat{u}_2) \\
&= \left(\tilde{\mathbb{A}}^{\mathrm{H}}(\tilde{u}_1) - \tilde{\mathbb{A}}^{\mathrm{H}}(\tilde{u}_2)\right) \tilde{v}_{\mathrm{ice},1} - \left(\tilde{B}(\tilde{u}_1) - \tilde{B}(\tilde{u}_2)\right) \begin{pmatrix} \tilde{h}_1 \\ \tilde{a}_1 \end{pmatrix} \\
&\quad + \left(\tilde{\mathbb{A}}^{\mathrm{H}}(\tilde{u}_2) - \mathbb{A}^{\mathrm{H}}(u_0)\right) (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) - \left(\tilde{B}(\tilde{u}_2) - B_1(u_0)\right) \begin{pmatrix} \widehat{h}_1 - \widehat{h}_2 \\ \widehat{a}_1 - \widehat{a}_2 \end{pmatrix} \\
&\quad + \omega(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) - c_{\mathrm{cor}}(\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})^\perp \\
&\quad + \left(\frac{1}{\rho_{\mathrm{ice}}\tilde{h}_1} - \frac{1}{\rho_{\mathrm{ice}}\tilde{h}_2}\right)(\tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(\tilde{v}_{\mathrm{ice},1})) + \frac{1}{\rho_{\mathrm{ice}}\tilde{h}_2}\left(\tau_{\mathrm{ocn}}(\tilde{v}_{\mathrm{ice},1}) - \tau_{\mathrm{ocn}}(\tilde{v}_{\mathrm{ice},2})\right).
\end{aligned}$$

The estimates of $F_1(\widehat{u})$ from (6.23) with regard to the self map and contraction property of the fixed point map Φ_T^R are collected in the lemma below.

Lemma 6.4.8. *Let* $p, q \in (1,\infty)$ *be such that* (6.17) *holds true, and consider* $\tilde{u} = \widehat{u} + u_0^*$*,* $\tilde{u}_1 = \widehat{u}_1 + u_0^*$ *and* $\tilde{u}_2 = \widehat{u}_2 + u_0^*$*, with* $\widehat{u}$*,* $\widehat{u}_1$*,* $\widehat{u}_2 \in \mathcal{K}_T^R$*, and* u_0^* *denoting the reference solution from Proposition 6.3.1. Besides, suppose that* V_{atm}*,* V_{ocn} *and* $\nabla_{\mathrm{H}} H$ *satisfy Assumption 6.4.7, and recall the* T*-independent maximal regularity constant* $C_{\mathrm{MR}} > 0$ *from Proposition 6.2.1.*

Then there are $C_{F_1}(R,T)$*,* $L_{F_1}(R,T) > 0$ *such that* $C_{F_1}(R,T) < R/6C_{\mathrm{MR}}$ *for* $R > 0$ *and* $T > 0$ *sufficiently small and* $L_{F_1}(R,T) \to 0$ *as* $R \to 0$ *and* $T \to 0$*, and we obtain the estimates*

$$\begin{aligned}
\|F_1(\widehat{u})\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} &\le C_{F_1}(R,T) \ \text{ and} \\
\|F_1(\widehat{u}_1) - F_1(\widehat{u}_2)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} &\le L_{F_1}(R,T) \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}.
\end{aligned}$$

Proof. We start by describing the essential ideas and auxiliary material for the present rather lengthy proof. As deducible from the statement, we are interested in estimates of the nonlinear terms by powers of T and R or shrinking terms in R and T. For this purpose, we often employ Lemma 1.3.1 in order to estimate L^p-norms in time by L^∞-norms or L^∞-norms in time by $\mathrm{W}^{1,p}$-norms in the case of homogeneous initial values. Another important ingredient is that $\tilde{u}(t) \in V$ for all $t \in [0,T]$, where $\tilde{u} = \widehat{u} + u_0^*$ for $\widehat{u} \in \mathcal{K}_T^R$ and the reference solution u_0^*, thanks to Lemma 6.3.3 and the choice of $R \in (0, R_0]$ and $T \in (0, T_1]$. This is also valid for $\tilde{u}_1$ and $\tilde{u}_2$ of the above shape. Moreover, we often exploit the estimate of the $\mathbb{E}_1$-norm of $\tilde{u}$, $\tilde{u}_1$ and $\tilde{u}_2$ by $R + C_T^*$, resulting from (6.28). In addition, we frequently split $\tilde{u}$, $\tilde{u}_1$ and $\tilde{u}_2$ in their respective parts with homogeneous initial values $\widehat{u}$, $\widehat{u}_1$ and $\widehat{u}_2$, allowing us to use the embeddings with T-independent constant as established in Lemma 6.4.5. On the other hand, with regard to the reference solution, (6.29) proves useful as it provides an estimate of u_0^* in $\mathrm{BUC}([0,T]; \mathrm{X}_\gamma)$ by the norm of the initial values and the $\mathbb{E}_1$-norm of u_0^*, and with T-independent constant. Concerning the transform Y, we heavily rely on Lemma 6.4.1, Lemma 6.4.2, Lemma 6.4.3 and Lemma 6.4.4. We also often use the embedding $\mathrm{W}^{1,q}(\Omega) \hookrightarrow \mathrm{L}^\infty(\Omega)$.

By virtue of $\widehat{v}_{\mathrm{ice}}(0) = 0$, we first deduce from Lemma 1.3.1 that

$$
\begin{aligned}
&\|\omega(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) - c_{\mathrm{cor}}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)^\perp\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \\
&\le C_1 \left(T^{1/p} \cdot \|\widehat{v}_{\mathrm{ice}}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^q(\Omega))} + \|v_{\mathrm{ice},0}^*\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))}\right) \\
&\le C_2 \left(T \cdot \|\widehat{v}_{\mathrm{ice}}\|_{\mathrm{W}^{1,p}(0,T;\mathrm{L}^q(\Omega))} + C_T^*\right) \\
&\le C_3(TR + C_T^*).
\end{aligned} \tag{6.36}
$$

Lemma 1.3.1 and Assumption 6.4.7(b) further yield

$$
\| -g\nabla_{\mathrm{H}} H\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \le T^{1/p} \cdot \| -g\nabla_{\mathrm{H}} H\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^q(\Omega))} \le C_4 T^{1/p}.
$$

As $\tilde{u}(t) \in V$ on $[0,T]$, we have $\tilde{h}(t) = \widehat{h}(t) + h_0^*(t) > \kappa$ for all $t \in [0,T]$, so

$$
\left\| \frac{1}{\rho_{\mathrm{ice}}(\widehat{h} + h_0^*)} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \le C_5 \tag{6.37}
$$

for some constant $C_5 > 0$. A concatenation of (6.37), Lemma 1.3.1, Hölder's inequality and Assumption 6.4.7 then results in

$$
\begin{aligned}
\left\| \frac{1}{\rho_{\mathrm{ice}}(\widehat{h} + h_0^*)} \tau_{\mathrm{atm}} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} &\le C_6 T^{1/p} \cdot \|\tau_{\mathrm{atm}}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^q(\Omega))} \\
&\le C_7 T^{1/p} \cdot \|V_{\mathrm{atm}}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^{2q}(\Omega))}^2 \\
&\le C_8 T^{1/p}.
\end{aligned} \tag{6.38}
$$

For the oceanic forcing term, we proceed in a similar way. More precisely, we use the above estimate (6.37), the elementary estimate Lemma 1.3.1, the embedding of the maximal regularity space from Lemma 6.4.5(b) and (6.29) for the estimate of $\|u_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}$ to argue that

$$
\begin{aligned}
&\left\| \frac{1}{\rho_{\mathrm{ice}}(\widehat{h}+h_0^*)} \tau_{\mathrm{ocn}}(\widehat{v}_{\mathrm{ice}}+v_{\mathrm{ice},0}^*) \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \\
&\le C_9 T^{1/p} \cdot \|\tau_{\mathrm{ocn}}(\widehat{v}_{\mathrm{ice}}+v_{\mathrm{ice},0}^*)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^q(\Omega))} \\
&\le C_{10} T^{1/p} \cdot \left(\|V_{\mathrm{ocn}}+\widehat{v}_{\mathrm{ice}}+v_{\mathrm{ice},0}^*\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^{2q}(\Omega))}^2 \right) \\
&\le C_{11} T^{1/p} \cdot \left(1 + \|\widehat{u}\|_{\mathbb{E}_1}^2 + \|u_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \right) \\
&\le C_{12} T^{1/p} \left(1 + R^2 + C_0^2 + (C_T^*)^2 \right).
\end{aligned} \tag{6.39}
$$

Regarding the first part of the assertion, it remains to handle the terms associated to the Hibler operator and the off-diagonal terms, i. e., we consider

$$
\left(\tilde{\mathbb{A}}^{\mathrm{H}}(\widehat{u}+u_0^*) - \mathbb{A}^{\mathrm{H}}(u_0) \right) (\widehat{v}_{\mathrm{ice}}+v_{\mathrm{ice},0}^*) - \left(\tilde{B}(\widehat{u}+u_0^*) - B_1(u_0) \right) \begin{pmatrix} \widehat{h}+h_0^* \\ \widehat{a}+a_0^* \end{pmatrix}.
$$

In view of Lemma 6.4.6, we can further reduce this task by adding and subtracting suitable terms. In fact, it comes down to estimating

$$
\begin{aligned}
&\left(\tilde{\mathbb{A}}^{\mathrm{H}}(\widehat{u}+u_0^*) - \mathbb{A}^{\mathrm{H}}(\widehat{u}+u_0^*) \right) (\widehat{v}_{\mathrm{ice}}+v_{\mathrm{ice},0}^*) \\
&\quad - \left(\tilde{B}(\widehat{u}+u_0^*) - B_1(\widehat{u}+u_0^*) \right) \begin{pmatrix} \widehat{h}+h_0^* \\ \widehat{a}+a_0^* \end{pmatrix}
\end{aligned} \tag{6.40}
$$

as well as

$$
\left(\mathbb{A}^{\mathrm{H}}(\widehat{u}+u_0^*) - \mathbb{A}^{\mathrm{H}}(u_0^*) \right) (\widehat{v}_{\mathrm{ice}}+v_{\mathrm{ice},0}^*) - (B_1(\widehat{u}+u_0^*) - B_1(u_0^*)) \begin{pmatrix} \widehat{h}+h_0^* \\ \widehat{a}+a_0^* \end{pmatrix} \quad \text{and}
$$

$$
\left(\mathbb{A}^{\mathrm{H}}(u_0^*) - \mathbb{A}^{\mathrm{H}}(u_0) \right) (\widehat{v}_{\mathrm{ice}}+v_{\mathrm{ice},0}^*) - (B_1(u_0^*) - B_1(u_0)) \begin{pmatrix} \widehat{h}+h_0^* \\ \widehat{a}+a_0^* \end{pmatrix}.
$$

By Lemma 6.4.6, the last two differences can be estimated in $\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega)^2)$, namely by $R(R+C_T^*)$ and $\|u_0^*-u_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot (R+C_T^*)$, respectively.

Hence, it remains to estimate (6.40). For this, we begin with

$$
\left(\tilde{\mathbb{A}}^{\mathrm{H}}(\widehat{u}+u_0^*) - \mathbb{A}^{\mathrm{H}}(\widehat{u}+u_0^*) \right) (\widehat{v}_{\mathrm{ice}}+v_{\mathrm{ice},0}^*).
$$

We will focus on the principal part and observe that the other part can be treated likewise. In view of (3.14) and (6.6), we need to estimate

$$\sum_{j,k,l,m=1}^{2} \Big(a_{ij}^{klm}(\tilde{\varepsilon}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*), P(\widehat{h} + h_0^*, \widehat{a} + a_0^*))\partial_m \tilde{\varepsilon}_{jl}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) \\ - a_{ij}^{kl}(\varepsilon(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*), P(\widehat{h} + h_0^*, \widehat{a} + a_0^*))\delta_{km}\partial_k \varepsilon_{jl}(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)\Big) = \mathrm{I} + \mathrm{II}.$$

Since ε and $\tilde{\varepsilon}$ both depend on $\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*$, while P always depends on $\widehat{h} + h_0^*$ and $\widehat{a} + a_0^*$, we omit writing the dependence explicitly in the sequel. In the above, I and II are given by

$$\begin{aligned} \mathrm{I} &:= \sum_{j,k,l,m=1}^{2} \Big(a_{ij}^{klm}(\tilde{\varepsilon}, P)\partial_m \tilde{\varepsilon}_{jl} - a_{ij}^{kl}(\varepsilon, P)\delta_{km}\partial_k \tilde{\varepsilon}_{jl}\Big) \quad \text{and} \\ \mathrm{II} &:= \sum_{j,k,l=1}^{2} \Big(a_{ij}^{kl}(\varepsilon, P)\partial_k \tilde{\varepsilon}_{jl} - a_{ij}^{kl}(\varepsilon, P)\partial_k \varepsilon_{jl}\Big). \end{aligned} \tag{6.41}$$

The main difficulty in the estimate of II is to control the difference of the transformed symmetric part of the gradient $\tilde{\varepsilon}$ and the original symmetric part of the gradient ε. In view of (6.4), we obtain

$$\begin{aligned} \partial_k \tilde{\varepsilon}_{jl} - \partial_k \varepsilon_{jl} &= \frac{1}{2}\sum_{n=1}^{2}\Big((\partial_k \partial_j Y_n)\partial_n \tilde{v}_{\mathrm{ice},j} + (\partial_j Y_n)\partial_k \partial_n \tilde{v}_{\mathrm{ice},l} + (\partial_k \partial_l Y_n)\partial_n \tilde{v}_{\mathrm{ice},j} \\ &\quad + (\partial_l Y_n)\partial_k \partial_n \tilde{v}_{\mathrm{ice},j}\Big) - \frac{1}{2}\partial_k(\partial_j \tilde{v}_{\mathrm{ice},l} + \partial_l \tilde{v}_{\mathrm{ice},j}) \\ &= \frac{1}{2}\sum_{n=1}^{2}\Big((\partial_j Y_n - \delta_{jn})\partial_k \partial_n \tilde{v}_{\mathrm{ice},l} + (\partial_l Y_n - \delta_{ln})\partial_k \partial_n \tilde{v}_{\mathrm{ice},j} \\ &\quad + (\partial_k \partial_j Y_n)\partial_n \tilde{v}_{\mathrm{ice},j} + (\partial_k \partial_l Y_n)\partial_n \tilde{v}_{\mathrm{ice},j}\Big). \end{aligned}$$

Employing the estimates on Y from Lemma 6.4.2 and (6.28), we find that

$$\begin{aligned} &\|(\partial_j Y_n - \delta_{jn})\partial_k \partial_n \tilde{v}_{\mathrm{ice},l}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \\ &\le \|\operatorname{Id}_2 - \nabla_{\mathrm{H}} Y\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \cdot \|\tilde{v}_{\mathrm{ice}}\|_{\mathrm{L}^p(0,T;\mathrm{W}^{2,q}(\Omega))} \\ &\le C_{13} \cdot \|\operatorname{Id}_2 - \nabla_{\mathrm{H}} Y\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \cdot \|\tilde{u}\|_{\mathbb{E}_1} \\ &\le C_{14} T^{1/p'}(R + C_T^*). \end{aligned}$$

The estimate of the term $(\partial_l Y_n - \delta_{ln})\partial_k \partial_n \tilde{v}_{\mathrm{ice},j}$ is completely analogous. Next, we treat the term $(\partial_k \partial_j Y_n)\partial_n \tilde{v}_{\mathrm{ice},j}$. In fact, we make use of Hölder's inequality, the embedding of the trace space X_γ (6.18), Lemma 1.3.1, Lemma 6.4.1 on the transform Y, the embedding from Lemma 6.4.5(a) with T-independent embedding constant and (6.29) to deduce

$$\begin{aligned}
&\|(\partial_k \partial_j Y_n)\partial_n \tilde{v}_{\mathrm{ice},j}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \\
&\le \|\partial_k \partial_j Y_n\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \cdot \|\partial_n \tilde{v}_{\mathrm{ice},j}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \\
&\le C_{15} \cdot \|\nabla_{\mathrm{H}} Y\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \cdot \Big(\|\widehat{v}_{\mathrm{ice}}\|_{\mathrm{L}^\infty(0,T;\mathrm{C}^1(\overline{\Omega}))} + \|v_{\mathrm{ice},0}^*\|_{\mathrm{L}^\infty(0,T;\mathrm{C}^1(\overline{\Omega}))}\Big) \\
&\le C_{16} T^{1/p} \cdot \|\nabla_{\mathrm{H}} Y\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \cdot \Big(\|\widehat{u}\|_{\mathrm{BUC}(0,T;\mathrm{X}_\gamma)} + \|u_0^*\|_{\mathrm{BUC}(0,T;\mathrm{X}_\gamma)}\Big) \\
&\le C_{17} T^{1/p} \cdot (\|\widehat{u}\|_{\mathbb{E}_1} + C_0 + C_T^*) \\
&\le C_{17} T^{1/p} (R + C_0 + C_T^*).
\end{aligned}$$

The estimate of $(\partial_k \partial_l Y_n)\partial_n \tilde{v}_{\mathrm{ice},j}$ is completely analogous. Recalling the coefficients a_{ij}^{kl} from (3.15), and invoking $\tilde{u}(t) \in V$ for all $t \in [0,T]$, we conclude

$$\|a_{ij}^{kl}(\varepsilon, P)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \le C_{18}. \tag{6.42}$$

Thus, concatenating the preceding estimates, we find that there exist a constant $C_{19} > 0$ and $\beta > 0$ such that

$$\|\mathrm{II}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \le C_{19} T^\beta (R + C_0 + C_T^*).$$

Concerning the Hibler operator and its transformed version, it remains to estimate I from (6.41). Exploiting $a_{ij}^{klm} = (\partial_k Y_m) a_{ij}^{kl}$ from (6.5), we split I into

$$\begin{aligned}
\mathrm{III} &:= \sum_{j,k,l,m=1}^{2} \Big(a_{ij}^{klm}(\tilde{\varepsilon}, P) - a_{ij}^{klm}(\varepsilon, P)\Big)\partial_m \tilde{\varepsilon}_{jl} \ \text{ and} \\
\mathrm{IV} &:= \sum_{j,k,l,m=1}^{2} a_{ij}^{kl}(\varepsilon, P)(\partial_k Y_m - \delta_{km})\partial_m \tilde{\varepsilon}_{jl}.
\end{aligned} \tag{6.43}$$

With regard to IV, we use Lemma 6.4.2 to estimate $\mathrm{Id}_2 - \nabla_{\mathrm{H}} Y$ and thus get

$$\|\partial_k Y_m - \delta_{km}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \le C_{20} \|\mathrm{Id}_2 - \nabla_{\mathrm{H}} Y\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \le C_{21} T^{1/p'} \tag{6.44}$$

for some constant $C_{21} > 0$. The term $\partial_m \tilde{\varepsilon}_{jl}$ can be estimated by a constant in a similar way as above, using Lemma 6.4.1 for estimating the terms related to the diffeomorphism Y and (6.28) in order to obtain estimates of the derivatives of $\tilde{v}_{\mathrm{ice}}$ which are uniform in R and T, so there is a constant $C_{22} > 0$ with

$$\|\partial_m \tilde{\varepsilon}_{jl}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \le C_{22}. \tag{6.45}$$

In conjunction with (6.42), this results in

$$\|\mathrm{IV}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \leq C_{23} T^{1/p'}. \tag{6.46}$$

For III from (6.43), we can make use of (6.45) to estimate the last term. Moreover, the $\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))$-norm of $(\partial_j Y_k)$ appearing in both terms in the difference can be bounded by a constant thanks to Lemma 6.4.1. The estimate thus comes down to investigating $a_{ij}^{kl}(\tilde{\varepsilon}, P) - a_{ij}^{kl}(\varepsilon, P)$. As the difference only concerns the symmetric part of the gradient, we observe that the first factor

$$\frac{P(\widehat{h} + h_0^*, \widehat{a} + a_0^*)}{2\rho_{\text{ice}}(\widehat{h} + h_0^*)}$$

can be bounded in $\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))$ by combining (6.37) with an estimate of $P(\widehat{h} + h_0^*, \widehat{a} + a_0^*)$ which is in turn based on $\tilde{u}(t) \in V$ for all $t \in [0,T]$. The remaining difference then is

$$\frac{1}{\triangle_\delta(\tilde{\varepsilon})}\left(\mathbb{S}_{ij}^{kl} - \frac{1}{\triangle_\delta^2(\tilde{\varepsilon})}(\mathbb{S}\tilde{\varepsilon})_{ik}(\mathbb{S}\tilde{\varepsilon})_{jl}\right) - \frac{1}{\triangle_\delta(\varepsilon)}\left(\mathbb{S}_{ij}^{kl} - \frac{1}{\triangle_\delta^2(\varepsilon)}(\mathbb{S}\varepsilon)_{ik}(\mathbb{S}\varepsilon)_{jl}\right).$$

Since the dependence on ε is smooth thanks to the regularization by $\delta > 0$, we can use the mean value theorem to infer that

$$\begin{aligned}&\left\|\frac{1}{\triangle_\delta(\tilde{\varepsilon})}\left(\mathbb{S}_{ij}^{kl} - \frac{1}{\triangle_\delta^2(\tilde{\varepsilon})}(\mathbb{S}\tilde{\varepsilon})_{ik}(\mathbb{S}\tilde{\varepsilon})_{jl}\right)\right.\\ &\quad \left.- \frac{1}{\triangle_\delta(\varepsilon)}\left(\mathbb{S}_{ij}^{kl} - \frac{1}{\triangle_\delta^2(\varepsilon)}(\mathbb{S}\varepsilon)_{ik}(\mathbb{S}\varepsilon)_{jl}\right)\right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))}\\ &\leq C_{24} \cdot \|\tilde{\varepsilon} - \varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))}.\end{aligned}$$

Next, we calculate

$$\tilde{\varepsilon}_{ij} - \varepsilon_{ij} = \frac{1}{2}\sum_{k=1}^{2}\Big((\partial_i Y_k - \delta_{ik})\partial_k \tilde{v}_{\text{ice},j} + (\partial_j Y_k - \delta_{jk})\partial_k \tilde{v}_{\text{ice},i}\Big).$$

For the differences $(\partial_i Y_k) - \delta_{ik}$ and $(\partial_j Y_k) - \delta_{jk}$, we can use (6.44) to get an estimate by a T-power. For the derivative of $\tilde{v}_{\text{ice}}$, we employ the embeddings from (6.18) and Lemma 6.4.5(a) as well as the estimate (6.29) to argue that

$$\begin{aligned}\|\partial_k \tilde{v}_{\text{ice}}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} &\leq C_{25} \cdot \|\tilde{v}_{\text{ice}}\|_{\mathrm{L}^\infty(0,T;\mathrm{C}^1(\overline{\Omega}))}\\ &\leq C_{26} \cdot \Big(\|\widehat{u}\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} + \|u_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}\Big)\\ &\leq C_{27} \cdot (\|\widehat{u}\|_{\mathbb{E}_1} + C_0 + C_T^*)\\ &\leq C_{27}(R + C_0 + C_T^*).\end{aligned}$$

In conclusion, there exists $C_{28} > 0$ such that

$$\|\mathrm{III}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \le C_{28} T^{1/p'} (R + C_0 + C_T^*). \tag{6.47}$$

Combining (6.46) and (6.47), we find the existence of a constant $C_{29} > 0$ with

$$\|\mathrm{I}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \le C_{29} T^{1/p'} (1 + R + C_0 + C_T^*).$$

In order to complete the proof of the first part of the assertion, we still need to deal with the off-diagonal part

$$\Big(\tilde{B}(\widehat{u} + u_0^*) - B_1(\widehat{u} + u_0^*)\Big) \begin{pmatrix} \widehat{h} + h_0^* \\ \widehat{a} + a_0^* \end{pmatrix},$$

which we can further split into the difference of the h- and a-component. The resulting difference in the h-component is given by

$$\frac{\partial_h P(\widehat{h} + h_0^*, \widehat{a} + a_0^*)}{2\rho_{\mathrm{ice}}(\widehat{h} + h_0^*)} \sum_{j=1}^{2} (\partial_i Y_j - \delta_{ij}) \partial_j (\widehat{h} + h_0^*). \tag{6.48}$$

In (6.48), note that the inverse of $2\rho_{\mathrm{ice}}(\widehat{h} + h_0^*)$ can be handled with (6.37). On the other hand, we have

$$\partial_h P(\widehat{h} + h_0^*, \widehat{a} + a_0^*) = p^* \mathrm{e}^{-c_\bullet (1 - (\widehat{a} + a_0^*))}.$$

As $u(t) \in V$ on $[0,T]$, we also have $\tilde{a}(t) = \widehat{a}(t) + a_0^*(t) \in (0,1)$ for all $t \in [0,T]$. This implies

$$\|\partial_h P(\widehat{h} + h_0^*, \widehat{a} + a_0^*)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \le C_{30}.$$

Putting together the above arguments, making use of Lemma 6.4.2 to estimate $\partial_i Y_j - \delta_{ij}$, and employing Lemma 1.3.1 as well as the embedding of ${}_0\mathbb{E}_1$ into $\mathrm{BUC}([0,T];\mathrm{X}_\gamma)$ with T-independent embedding constant from Lemma 6.4.5(a), we derive that

$$\begin{aligned}
&\left\| \frac{\partial_h P(\widehat{h} + h_0^*, \widehat{a} + a_0^*)}{2\rho_{\mathrm{ice}}(\widehat{h} + h_0^*)} \sum_{j=1}^{2} (\partial_i Y_j - \delta_{ij}) \partial_j (\widehat{h} + h_0^*) \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \\
&\le C_{31} \cdot \| \mathrm{Id}_2 - \nabla_{\mathrm{H}} Y \|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \cdot \Big(\|\widehat{h} + h_0^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \Big) \\
&\le C_{32} \cdot \| \mathrm{Id}_2 - \nabla_{\mathrm{H}} Y \|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \Big(T^{1/p} \cdot \|\widehat{h}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} + \|u_0^*\|_{\mathbb{E}_1} \Big) \\
&\le C_{33} T^{1/p'} \cdot \Big(T^{1/p} \cdot \|\widehat{u}\|_{\mathbb{E}_1} + C_T^* \Big) \\
&\le C_{33} T^{1/p'} \Big(T^{1/p} R + C_T^* \Big).
\end{aligned} \tag{6.49}$$

The corresponding term in the a-component

$$\frac{\partial_a P(\widehat{h} + h_0^*, \widehat{a} + a_0^*)}{2\rho_{\text{ice}}(\widehat{h} + h_0^*)} \sum_{j=1}^{2} (\partial_i Y_j - \delta_{ij}) \partial_j (\widehat{a} + a_0^*)$$

can be treated analogously.

In total, the first part of the assertion follows for some $C_{F_1}(R,T) > 0$. For $R > 0$ and $T > 0$ sufficiently small, we conclude that $C_{F_1}(R,T) < {}^{R}/_{6C_{\text{MR}}}$ thanks to $C_T^* \to 0$ by Remark 6.3.2(b), while Remark 6.3.2(c) yields that the difference of u_0^* and u_0 in $\text{BUC}([0,T]; \text{X}_\gamma)$ converges to zero as $T \to 0$.

Now, we address the Lipschitz estimate of F_1. First, as in (6.36), upon noting the homogeneous initial values of $\widehat{v}_{\text{ice},1} - \widehat{v}_{\text{ice},2}$, we find that

$$\|\omega(\widehat{v}_{\text{ice},1} - \widehat{v}_{\text{ice},2}) - c_{\text{cor}}(\widehat{v}_{\text{ice},1} - \widehat{v}_{\text{ice},2})^{\perp}\|_{\text{L}^p(0,T;\text{L}^q(\Omega))} \le C_{34} T \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}.$$

Moreover, we use the mean value theorem together with $\widehat{h}_i(t) + h_0^*(t) > \kappa$, for $i = 1, 2$, on $[0,T]$ as well as Lemma 1.3.1 for

$$\left\| \frac{1}{\rho_{\text{ice}}(\widehat{h}_1 + h_0^*)} - \frac{1}{\rho_{\text{ice}}(\widehat{h}_2 + h_0^*)} \right\|_{\text{L}^\infty(0,T;\text{L}^\infty(\Omega))} \le C_{35} T^{1/p} \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}.$$

The norm $\|\tau_{\text{atm}} + \tau_{\text{ocn}}(\widehat{v}_{\text{ice},1} + v_{\text{ice},0}^*)\|_{\text{L}^p(0,T;\text{L}^q(\Omega))}$ can be estimated as in (6.38) and (6.39) to conclude that

$$\begin{aligned} &\left\| \left(\frac{1}{\rho_{\text{ice}}(\widehat{h}_1 + h_0^*)} - \frac{1}{\rho_{\text{ice}}(\widehat{h}_2 + h_0^*)} \right) (\tau_{\text{atm}} + \tau_{\text{ocn}}(\widehat{v}_{\text{ice},1} + v_{\text{ice},0}^*)) \right\|_{\text{L}^p(0,T;\text{L}^q(\Omega))} \\ &\le C_{36} T \left(1 + R^2 + C_0^2 + (C_T^*)^2\right) \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}. \end{aligned}$$

As above, the inverse of $\rho_{\text{ice}}(\widehat{h}_2 + h_0^*)$ is bounded thanks to $\widehat{h}_2(t) + h_0^*(t) > \kappa$ for all $t \in [0,T]$. With $\tilde{v}_{\text{ice},i} = \widehat{v}_{\text{ice},i} + v_{\text{ice},0}^*$, the remaining contribution of the oceanic forcing term can be expanded as

$$\begin{aligned} &\tau_{\text{ocn}}(\tilde{v}_{\text{ice},1}) - \tau_{\text{ocn}}(\tilde{v}_{\text{ice},2}) \\ &= \rho_{\text{ocn}} C_{\text{ocn}} |V_{\text{ocn}} - \tilde{v}_{\text{ice},1}| R_{\text{ocn}}(-(\widehat{v}_{\text{ice},1} - \widehat{v}_{\text{ice},2})) \\ &\quad + \rho_{\text{ocn}} C_{\text{ocn}} (|V_{\text{ocn}} - \tilde{v}_{\text{ice},1}| - |V_{\text{ocn}} - \tilde{v}_{\text{ice},2}|)(V_{\text{ocn}} - \tilde{v}_{\text{ice},2}). \end{aligned}$$

Concerning the first addend, we proceed again as in (6.39) to obtain

$$\begin{aligned} &\|\rho_{\text{ocn}} C_{\text{ocn}} |V_{\text{ocn}} - \widehat{v}_{\text{ice},1} - v_{\text{ice},0}^*| R_{\text{ocn}}(-(\widehat{v}_{\text{ice},1} - \widehat{v}_{\text{ice},2}))\|_{\text{L}^p(0,T;\text{L}^q(\Omega))} \\ &\le C_{37} T^{1/p} \cdot \|V_{\text{ocn}} - \widehat{v}_{\text{ice},1} - v_{\text{ice},0}^*\|_{\text{L}^\infty(0,T;\text{L}^{2q}(\Omega))} \cdot \|\widehat{v}_{\text{ice},1} - \widehat{v}_{\text{ice},2}\|_{\text{L}^\infty(0,T;\text{L}^{2q}(\Omega))} \\ &\le C_{38} T^{1/p} (1 + R + C_0 + C_T^*) \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}. \end{aligned}$$

For the second addend, in an analogous way, we derive the estimate

$$\begin{aligned}&\Big\|\rho_{\mathrm{ocn}}C_{\mathrm{ocn}}\left(|V_{\mathrm{ocn}}-\widehat{v}_{\mathrm{ice},1}-v^*_{\mathrm{ice},0}|-|V_{\mathrm{ocn}}-\widehat{v}_{\mathrm{ice},2}-v^*_{\mathrm{ice},0}|\right)\\ &\quad\cdot\left(V_{\mathrm{ocn}}-\widehat{v}_{\mathrm{ice},2}-v^*_{\mathrm{ice},0}\right)\Big\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))}\\ &\le C_{39}T^{1/p}(1+R+C_0+C_T^*)\cdot\|\widehat{u}_1-\widehat{u}_2\|_{\mathbb{E}_1}.\end{aligned}$$

To finish the Lipschitz estimate of F_1, we need to handle the terms associated to the Hibler operator and the ice strength. In that respect, concerning

$$\begin{aligned}&\left(\tilde{\mathbb{A}}^{\mathrm{H}}(\widehat{u}_2+u_0^*)-\mathbb{A}^{\mathrm{H}}(u_0)\right)(\widehat{v}_{\mathrm{ice},1}-\widehat{v}_{\mathrm{ice},2})\\ &-\left(\tilde{B}(\widehat{u}_2+u_0^*)-B_1(u_0)\right)\begin{pmatrix}\widehat{h}_1-\widehat{h}_2\\ \widehat{a}_1-\widehat{a}_2\end{pmatrix},\end{aligned}$$

we proceed similarly as in the first part of the proof by plugging in an intermediate term. For B_1^* denoting $B_1(u_0^*)$ for simplicity, Lemma 6.4.6 yields

$$\begin{aligned}&\Bigg\|\left(\mathbb{A}^{\mathrm{H}}(\widehat{u}_2+u_0^*)-\mathbb{A}^{\mathrm{H}}(u_0^*)\right)(\widehat{v}_{\mathrm{ice},1}-\widehat{v}_{\mathrm{ice},2})\\ &\quad-\left(B_1(\widehat{u}_2+u_0^*)-B_1^*\right)\begin{pmatrix}\widehat{h}_1-\widehat{h}_2\\ \widehat{a}_1-\widehat{a}_2\end{pmatrix}\Bigg\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))}\le C_{40}R\cdot\|\widehat{u}_1-\widehat{u}_2\|_{\mathbb{E}_1}\end{aligned}$$

and

$$\begin{aligned}&\Bigg\|\left(\mathbb{A}^{\mathrm{H}}(u_0^*)-\mathbb{A}^{\mathrm{H}}(u_0)\right)(\widehat{v}_{\mathrm{ice},1}-\widehat{v}_{\mathrm{ice},2})-\left(B_1^*-B_1(u_0)\right)\begin{pmatrix}\widehat{h}_1-\widehat{h}_2\\ \widehat{a}_1-\widehat{a}_2\end{pmatrix}\Bigg\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))}\\ &\le C_{41}\cdot\|u_0^*-u_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}\cdot\|\widehat{u}_1-\widehat{u}_2\|_{\mathbb{E}_1}.\end{aligned}$$

Following the lines of the first part of the proof, we find $\beta>0$ such that

$$\begin{aligned}&\left\|\left(\tilde{\mathbb{A}}^{\mathrm{H}}(\widehat{u}_2+u_0^*)-\mathbb{A}^{\mathrm{H}}(\widehat{u}_2+u_0^*)\right)(\widehat{v}_{\mathrm{ice},1}-\widehat{v}_{\mathrm{ice},2})\right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))}\\ &\le C_{42}T^\beta\cdot\|\widehat{u}_1-\widehat{u}_2\|_{\mathbb{E}_1}\end{aligned}$$

and

$$\begin{aligned}&\Bigg\|(\tilde{B}(\widehat{u}_2+u_0^*)-B_1(\widehat{u}_2+u_0^*))\begin{pmatrix}\widehat{h}_1-\widehat{h}_2\\ \widehat{a}_1-\widehat{a}_2\end{pmatrix}\Bigg\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))}\\ &\le C_{43}T^\beta\cdot\|\widehat{u}_1-\widehat{u}_2\|_{\mathbb{E}_1}.\end{aligned}$$

It remains to show the Lipschitz continuity with shrinking constant of

$$\begin{aligned}&\left(\tilde{\mathbb{A}}^{\mathrm{H}}(\widehat{u}_1+u_0^*)-\tilde{\mathbb{A}}^{\mathrm{H}}(\tilde{u}_2+u_0^*)\right)(\widehat{v}_{\mathrm{ice},1}+v^*_{\mathrm{ice},0})\\ &-\left(\tilde{B}(\widehat{u}_1+u_0^*)-\tilde{B}(\widehat{u}_2+u_0^*)\right)\begin{pmatrix}\widehat{h}_1+h_0^*\\ \widehat{a}_1+a_0^*\end{pmatrix}.\end{aligned}\tag{6.50}$$

In the beginning, we focus on the difference in $\tilde{\mathbb{A}}^{\mathrm{H}}$, and there, we concentrate on the principal part as the other part can then be handled likewise. Thus, the first task is to estimate

$$\begin{aligned}\sum_{j,k,l,m=1}^{2} \Big(& a_{ij}^{klm}(\tilde{\varepsilon}(\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*), P(\widehat{h}_1 + h_0^*, \widehat{a}_1 + a_0^*)) \\ & - a_{ij}^{klm}(\tilde{\varepsilon}(\widehat{v}_{\mathrm{ice},2} + v_{\mathrm{ice},0}^*), P(\widehat{h}_2 + h_0^*, \widehat{a}_2 + a_0^*))\Big)\partial_m \tilde{\varepsilon}_{jl}(\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*).\end{aligned}$$

In contrast to the self map estimate, the boundedness of $\partial_m \tilde{\varepsilon}_{jl}(\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*)$ in $\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))$ is not sufficient. Thus, we revisit this estimate and recall

$$\begin{aligned}\partial_m \tilde{\varepsilon}_{jl}(\tilde{v}_{\mathrm{ice},1}) = \frac{1}{2}\sum_{n=1}^{2}\Big(& (\partial_m \partial_j (Y_1)_n)\partial_n(\tilde{v}_{ice,1})_l + (\partial_j (Y_1)_n)\partial_m\partial_n(\tilde{v}_{\mathrm{ice},1})_l \\ & + (\partial_m\partial_l(Y_1)_n)\partial_n(\tilde{v}_{\mathrm{ice},1})_j + (\partial_l(Y_1)_n)\partial_m\partial_n(\tilde{v}_{\mathrm{ice},1})_j\Big).\end{aligned}$$

Making use of the elementary estimate from Lemma 1.3.1, Lemma 6.4.1 in order to control the diffeomorphism, Lemma 6.4.5(a) for the embedding of the maximal regularity space with homogeneous initial values into $\mathrm{BUC}([0,T];\mathrm{X}_\gamma)$ and (6.29) to control $\|u_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}$, we find that

$$\begin{aligned}&\|(\partial_m\partial_j(Y_1)_n)\partial_n(\tilde{v}_{ice,1})_l\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \\ &\le C_{44}\cdot\|\nabla_{\mathrm{H}} Y_1\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))}\cdot\|\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*\|_{\mathrm{L}^\infty(0,T;\mathrm{X}_\gamma^{\mathrm{ice}})} \\ &\le C_{45}T^{1/p}\cdot\|\nabla_{\mathrm{H}} Y_1\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))}\cdot\Big(\|\widehat{u}_1\|_{\mathbb{E}_1} + \|u_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}\Big) \\ &\le C_{46}T^{1/p}(R + C_0 + C_T^*).\end{aligned}$$

On the other hand, Lemma 6.4.1 also yields

$$\begin{aligned}&\|(\partial_j(Y_1)_m)\partial_m\partial_n(\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*)_l\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \\ &\le C_{47}\cdot\|\nabla_{\mathrm{H}} Y_1\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))}\cdot(\|\widehat{u}_1\|_{\mathbb{E}_1} + \|u_0^*\|_{\mathbb{E}_1}) \\ &\le C_{48}(R + C_T^*).\end{aligned}$$

Combining the previous two estimates, we conclude the estimate

$$\|\partial_m\tilde{\varepsilon}_{jl}(\tilde{v}_{\mathrm{ice},1})\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \le C_{49}\left(T^{1/p}(R + C_0 + C_T^*) + R + C_T^*\right). \tag{6.51}$$

In order to keep the notation simpler, we use $\tilde{\varepsilon}_n$ and P_n, $n = 1,2$, to denote $\tilde{\varepsilon}(\widehat{v}_{\mathrm{ice},n} + v_{\mathrm{ice},0}^*)$ as well as $P(\widehat{h}_n + h_0^*, \widehat{a}_n + a_0^*)$ in the sequel. We invoke that the coefficients $a_{ij}^{klm}(\tilde{\varepsilon}_n, P_n)$ take the shape

$$a_{ij}^{kl}(\tilde{\varepsilon}_n, P_n)(\partial_k(Y_n)_m). \tag{6.52}$$

Regarding (6.52), and adding and subtracting suitable terms, we find that the difference can be considered in the original coefficients as introduced in (3.15) or in the diffeomorphism. In view of (3.28) in the local strong well-posedness proof in Chapter 3, we obtain

$$\begin{aligned} &\|a_{ij}^{kl}(\tilde{\varepsilon}_1, P_1) - a_{ij}^{kl}(\tilde{\varepsilon}_2, P_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \\ &\le C_{50} \cdot \|(\tilde{\varepsilon}_1, h_1, a_1) - (\tilde{\varepsilon}_2, h_2, a_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))}. \end{aligned} \tag{6.53}$$

For the difference in h and a, we get from Lemma 1.3.1 that

$$\begin{aligned} \|(\widehat{h}_1, \widehat{a}_1) - (\widehat{h}_2, \widehat{a}_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} &\le C_{51} \cdot \|(\widehat{h}_1, \widehat{a}_1) - (\widehat{h}_2, \widehat{a}_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \\ &\le C_{52} T^{1/p'} \|(\widehat{h}_1, \widehat{a}_1) - (\widehat{h}_2, \widehat{a}_2)\|_{\mathrm{W}^{1,p}(0,T;\mathrm{W}^{1,q}(\Omega))} \\ &\le C_{53} T^{1/p'} \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}. \end{aligned}$$

Concerning the difference in ε, using the simplification $\tilde{\varepsilon}_i = \varepsilon(\widehat{v}_{\mathrm{ice},i} + v_{\mathrm{ice},0}^*)$ for convenience, we calculate

$$\begin{aligned} &(\tilde{\varepsilon}_1)_{ij} - (\tilde{\varepsilon}_2)_{ij} \\ &= \frac{1}{2} \sum_{k=1}^{2} \Big((\partial_i (Y_1)_k - \partial_i (Y_2)_k) \partial_k (\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*)_j + (\partial_i (Y_2)_k) \partial_k (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})_j \\ &\quad + (\partial_j (Y_1)_k - \partial_j (Y_2)_k) \partial_k (\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*)_i + (\partial_j (Y_2)_k) \partial_k (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})_i \Big). \end{aligned}$$

For the estimate of the first term in the above, we exploit Lemma 6.4.4 to estimate the difference in the diffeomorphisms, Lemma 6.4.5(a) and (6.29), so

$$\begin{aligned} &\|(\partial_i (Y_1)_k - \partial_i (Y_2)_k) \partial_k (\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*)_j\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \\ &\le C_{54} T^{1/p'} \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1} \cdot \|\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*\|_{\mathrm{L}^\infty(0,T;\mathrm{X}_\gamma^{\mathrm{ice}})} \\ &\le C_{55} T^{1/p'} \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1} \cdot \Big(\|\widehat{u}_1\|_{\mathbb{E}_1} + \|u_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \Big) \\ &\le C_{56} T^{1/p'} (R + C_0 + C_T^*) \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}. \end{aligned}$$

On the other hand, we deduce that

$$\|(\partial_i (Y_2)_k) \partial_k (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})_j\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \le C_{57} \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1},$$

implying

$$\|\tilde{\varepsilon}_1 - \tilde{\varepsilon}_2\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \le C_{58} \Big(T^{1/p'} (R + C_0 + C_T^*) + 1 \Big) \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}.$$

In view of (6.51), (6.53) and the boundedness of $\partial_k (Y_n)_m$ by Lemma 6.4.1, we conclude a Lipschitz estimate of the term

$$\Big(a_{ij}^{kl}(\tilde{\varepsilon}_1, P_1) - a_{ij}^{kl}(\varepsilon_2, P_2) \Big) (\partial_k (Y_1)_m) \partial_m \tilde{\varepsilon}_{jl}(\tilde{v}_{\mathrm{ice},1})$$

in $\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))$, where the Lipschitz constant converges to zero as $R \to 0$ and $T \to 0$.

As the coefficients are especially bounded, we also deduce from

$$\|\partial_k (Y_1)_m - \partial_k (Y_2)_m\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \le C_{59} T^{1/p'} \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1},$$

following in turn from Lemma 6.4.4, the desired Lipschitz estimate of the term

$$a_{ij}^{kl}(\tilde{\varepsilon}_2, P_2)\, (\partial_k (Y_1)_m - \partial_k (Y_2)_m)\, \partial_m \tilde{\varepsilon}_{jl}(\tilde{v}_{\mathrm{ice},1}).$$

This also completes the Lipschitz estimate of the first part of (6.50).

The final term to be estimated is

$$\Bigl(\tilde{B}(\widehat{u}_1 + u_0^*) - \tilde{B}(\widehat{u}_2 + u_0^*)\Bigr) \begin{pmatrix} \widehat{h}_1 + h_0^* \\ \widehat{a}_1 + a_0^* \end{pmatrix}.$$

We only focus on the h-term and remark that the a-term can be treated by analogy. To this end, recalling $\tilde{B}_{\mathrm{h}}$ from (6.7), and using P_n, $n = 1, 2$, to denote the dependence on $\widehat{u}_n + u_0^*$, we calculate

$$\begin{aligned}
&\Bigl((\tilde{B}_{\mathrm{h}}(\widehat{u}_1 + u_0^*) - \tilde{B}_{\mathrm{h}}(\widehat{u}_2 + u_0^*)\Bigr)\, (\widehat{h}_1 + h_0^*))_i \\
&= \left(\frac{\partial_h P_1}{2\rho_{\mathrm{ice}}(\widehat{h}_1 + h_0^*)} - \frac{\partial_h P_2}{2\rho_{\mathrm{ice}}(\widehat{h}_2 + h_0^*)}\right) \sum_{j=1}^{2} (\partial_i (Y_1)_j) \partial_j (\widehat{h}_1 + h_0^*) \\
&\quad + \frac{\partial_h P_2}{2\rho_{\mathrm{ice}}(\widehat{h}_2 + h_0^*)} \sum_{j=1}^{2} (\partial_i (Y_1)_j - \partial_i (Y_2)_j)\, \partial_j (\widehat{h}_1 + h_0^*).
\end{aligned}$$

Concerning the second term, we proceed as in (6.49), with Lemma 6.4.2 replaced by Lemma 6.4.4 in order to control the difference in Y, so

$$\begin{aligned}
&\left\| \frac{\partial_h P_2}{2\rho_{\mathrm{ice}}(\widehat{h}_2 + h_0^*)} \sum_{j=1}^{2} (\partial_i (Y_1)_j - \partial_i (Y_2)_j) \partial_j (\widehat{h}_1 + h_0^*) \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \\
&\le C_{60} T^{1/p'} (T^{1/p} R + C_T^*) \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}.
\end{aligned}$$

For the first term, we can argue likewise, this time using the smoothness of the coefficients in $\tilde{h}$ and $\tilde{a}$ together with the mean value theorem to estimate the first factor. This results in

$$\begin{aligned}
&\left\| \left(\frac{\partial_h P_1}{2\rho_{\mathrm{ice}}(\widehat{h}_1 + h_0^*)} - \frac{\partial_h P_2}{2\rho_{\mathrm{ice}}(\widehat{h}_2 + h_0^*)}\right) \sum_{j=1}^{2} (\partial_i (Y_1)_j) \partial_j (\widehat{h}_1 + h_0^*) \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \\
&\le C_{61} (T^{1/p} R + C_T^*) \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}.
\end{aligned}$$

Concatenating the previous estimates, we deduce the desired Lipschitz estimate of (6.50), completing the proof of this lemma upon invoking Remark 6.3.2. □

In the next step, we establish estimates of the nonlinear terms F_2 and F_3 as defined in (6.24). As a preparation, for $\tilde{u}_i = \widehat{u}_i + u_0^*$, we calculate

$$\begin{aligned} F_2(\widehat{u}_1) - F_2(\widehat{u}_2) = &- \sum_{j,k=1}^{2} \Big(\tilde{h}_1 \partial_j (Y_1)_k - \tilde{h}_2 \partial_j (Y_2)_k\Big) \partial_k (\tilde{v}_{\mathrm{ice},1})_j \\ &- \tilde{h}_2 \sum_{j,k=1}^{2} (\partial_j (Y_2)_k) \partial_k (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})_j + h_0 \mathrm{div}_{\mathrm{H}} \, (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) \\ &+ \omega(\widehat{h}_1 - \widehat{h}_2) + S_{\mathrm{h}}(\tilde{h}_1, \tilde{a}_1) - S_{\mathrm{h}}(\tilde{h}_2, \tilde{a}_2) \end{aligned}$$

as well as

$$\begin{aligned} F_3(\widehat{u}_1) - F_3(\widehat{u}_2) = &- \sum_{j,k=1}^{2} \left(\tilde{a}_1 \partial_j (Y_1)_k - \tilde{a}_2 \partial_j (Y_2)_k\right) \partial_k (\tilde{v}_{\mathrm{ice},1})_j \\ &- \tilde{a}_2 \sum_{j,k=1}^{2} (\partial_j (Y_2)_k) \partial_k (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})_j + a_0 \mathrm{div}_{\mathrm{H}} \, (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) \\ &+ \omega(\widehat{a}_1 - \widehat{a}_2) + S_{\mathrm{a}}(\tilde{h}_1, \tilde{a}_1) - S_{\mathrm{a}}(\tilde{h}_2, \tilde{a}_2). \end{aligned}$$

The estimates of F_2 and F_3 are given below.

Lemma 6.4.9. *Let $p, q \in (1, \infty)$ fulfill (6.17), and consider $\tilde{u} = \widehat{u} + u_0^*$, $\tilde{u}_1 = \widehat{u}_1 + u_0^*$ and $\tilde{u}_2 = \widehat{u}_2 + u_0^*$, where $\widehat{u}$, $\widehat{u}_1$, $\widehat{u}_2 \in \mathcal{K}_T^R$, and u_0^* represents the reference solution from Proposition 6.3.1. Moreover, suppose that f_{gr} satisfies Assumption 6.4.7, and recall the T-independent maximal regularity constant from Proposition 6.2.1.*

Then there exist $C_{F_2}(R,T)$, $L_{F_2}(R,T)$, $C_{F_3}(R,T)$, $L_{F_3}(R,T) > 0$ such that $C_{F_2}(R,T)$, $C_{F_3}(R,T) < R/6C_{\mathrm{MR}}$ for $R > 0$ and $T > 0$ sufficiently small as well as $L_{F_2}(R,T)$, $L_{F_3}(R,T) \to 0$ as $R \to 0$ and $T \to 0$, and we get

$$\begin{aligned} \|F_2(\widehat{u})\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} &\le C_{F_2}(R,T), \\ \|F_3(\widehat{u})\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} &\le C_{F_3}(R,T), \\ \|F_2(\widehat{u}_1) - F_2(\widehat{u}_2)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} &\le L_{F_2}(R,T) \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1} \quad \text{and} \\ \|F_3(\widehat{u}_1) - F_3(\widehat{u}_2)\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} &\le L_{F_3}(R,T) \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}. \end{aligned}$$

Proof. The main strategy and tools remain unchanged in comparison with Lemma 6.4.8. In view of the present setting, with $\mathrm{W}^{1,q}(\Omega)$ being the ground space in the h- and a-component, we also heavily rely on the Banach algebra structure of $\mathrm{W}^{1,q}(\Omega)$ thanks to $q > 2$ which is in turn implied by (6.17).

Recalling that $\omega > 0$ is fixed, and employing Lemma 1.3.1 as well as the embedding from Lemma 6.4.5(a), we conclude

$$\begin{aligned} \|\omega\tilde{h}\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} &\le C_1 \cdot \Big(\|\widehat{h}\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} + \|h_0^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))}\Big) \\ &\le C_2 \cdot \Big(T^{1/p} \cdot \|\widehat{h}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} + \|u_0^*\|_{\mathbb{E}_1}\Big) \\ &\le C_3 \cdot \Big(T^{1/p} \cdot \|\widehat{u}\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} + C_T^*\Big) \\ &\le C_4(T^{1/p}R + C_T^*). \end{aligned} \tag{6.54}$$

With regard to the thermodynamic term S_{h}, we use Hölder's inequality, Assumption 6.4.7(c), Lemma 1.3.1 and Lemma 6.4.5(a) to get

$$\begin{aligned} &\|S_{\mathrm{h}}(\tilde{h},\tilde{a})\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \\ &\le \left\| f_{\mathrm{gr}}\left(\frac{\tilde{h}}{\tilde{a}}\right)\right\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \cdot \|\widehat{a} + a_0^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \\ &\quad + \|f_{\mathrm{gr}}(0)\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} + \|f_{\mathrm{gr}}(0)(\widehat{a} + a_0^*)\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \\ &\le C_5 \cdot \Big(T^{1/p} + \|\widehat{a}\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} + \|a_0^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))}\Big) \\ &\le C_6 \Big(T^{1/p} \cdot \Big(1 + \|\widehat{a}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))}\Big) + \|u_0^*\|_{\mathbb{E}_1}\Big) \\ &\le C_7 \Big(T^{1/p} \cdot \Big(1 + \|\widehat{u}\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}\Big) + C_T^*\Big) \\ &\le C_8 \Big(T^{1/p} \cdot (1 + \|\widehat{u}\|_{\mathbb{E}_1}) + C_T^*\Big) \\ &\le C_9(T^{1/p}(1+R) + C_T^*). \end{aligned}$$

The remaining term to be estimated is thus

$$h_0 \mathrm{div}_{\mathrm{H}}\,(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) - (\widehat{h} + h_0^*) \sum_{j,k=1}^{2} (\partial_j Y_k)\partial_k(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)_j,$$

and we handle this term by the addition and subtraction of the intermediate terms $(\widehat{h} + h_0^*)\mathrm{div}_{\mathrm{H}}\,(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)$ and $h_0^*\mathrm{div}_{\mathrm{H}}\,(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)$. We deduce from Hölder's inequality and (6.28) that

$$\begin{aligned} &\|(h_0 - h_0^*)\mathrm{div}_{\mathrm{H}}\,(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \\ &\le C_{10} \cdot \|h_0 - h_0^*\|_{\mathrm{BUC}([0,T];\mathrm{W}^{1,q}(\Omega))} \cdot \|\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{2,q}(\Omega))} \\ &\le C_{11} \cdot \|u_0 - u_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot \|\tilde{u}\|_{\mathbb{E}_1} \\ &\le C_{11} \cdot \|u_0 - u_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot (R + C_T^*). \end{aligned} \tag{6.55}$$

For the other intermediate term, we use Hölder's inequality, the embedding from Lemma 6.4.5(a) and (6.28) to infer that

$$
\begin{aligned}
&\|h_0^* \mathrm{div}_{\mathrm{H}}\,(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) - (\widehat{h} + h_0^*) \mathrm{div}_{\mathrm{H}}\,(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \\
&\le C_{12} \cdot \|\widehat{h}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \cdot (\|\widehat{v}_{\mathrm{ice}}\|_{\mathrm{L}^p(0,T;\mathrm{W}^{2,q}(\Omega))} + \|v_{\mathrm{ice},0}^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))}) \\
&\le C_{13} \cdot \|\widehat{u}\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot (\|\widehat{u}\|_{\mathbb{E}_1} + \|u_0^*\|_{\mathbb{E}_1}) \\
&\le C_{14} R (R + C_T^*).
\end{aligned} \tag{6.56}
$$

Let us observe that the last term in the context of F_2 can be written as

$$
\begin{aligned}
&(\widehat{h} + h_0^*) \mathrm{div}_{\mathrm{H}}\,(\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*) - (\widehat{h} + h_0^*) \sum_{j,k=1}^{2} (\partial_j Y_k) \partial_k (\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)_j \\
&= (\widehat{h} + h_0^*) \sum_{j,k=1}^{2} (\delta_{jk} - \partial_j Y_k)\, \partial_k (\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)_j .
\end{aligned}
$$

We then use Hölder's inequality, Lemma 6.4.2 for handling the transform, the embedding of ${}_0\mathbb{E}_1$ into $\mathrm{BUC}([0,T];\mathrm{X}_\gamma)$ with T-independent embedding constant from Lemma 6.4.5(a) and (6.29) to conclude

$$
\begin{aligned}
&\left\| (\widehat{h} + h_0^*) \sum_{j,k=1}^{2} (\delta_{jk} - \partial_j Y_k) \partial_k (\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*)_j \right\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \\
&\le C_{15} \cdot \|\widehat{h} + h_0^*\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \cdot \| \mathrm{Id}_2 - \nabla_{\mathrm{H}} Y \|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \\
&\quad \cdot \|\widehat{v}_{\mathrm{ice}} + v_{\mathrm{ice},0}^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{2,q}(\Omega))} \\
&\le C_{16} T^{1/p'} \cdot \Big(\|\widehat{u}\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} + \|u_0^*\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \Big) \cdot (\|\widehat{u}\|_{\mathbb{E}_1} + \|u_0^*\|_{\mathbb{E}_1}) \\
&\le C_{17} T^{1/p'} \cdot \Big(\|\widehat{u}\|_{\mathbb{E}_1} + \|u_0\|_{\mathrm{X}_\gamma} + \|u_0^*\|_{\mathbb{E}_1} \Big) (R + C_T^*) \\
&\le C_{17} T^{1/p'} (R + C_0 + C_T^*)(R + C_T^*).
\end{aligned} \tag{6.57}
$$

A concatenation of the estimates leads to the assertion for F_2 upon noting that Remark 6.3.2(b) and (6.20) ensure $C_T^* \to 0$ and $\|u_0^* - u_0\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \to 0$, respectively, in order to get $C_{F_2}(R,T) < R/6C_{\mathrm{MR}}$ for $R > 0$ sufficiently small and then letting $T \to 0$. The desired estimate of F_3 follows likewise.

The second part of the proof is dedicated to the Lipschitz estimates of F_2, and we remark again that F_3 can then be dealt with analogously. With regard to the linear term in F_2, we argue as in (6.54) to get

$$
\|\omega(\widehat{h}_1 - \widehat{h}_2)\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \le C_{18} T^{1/p} \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}.
$$

For the difference in S_h, we observe that it is given by

$$\begin{aligned} &S_\mathrm{h}(\widehat{h}_1 + h_0^*, \widehat{a}_1 + a_0^*) - S_\mathrm{h}(\widehat{h}_2 + h_0^*, \widehat{a}_2 + a_0^*) \\ &= f_\mathrm{gr}\left(\frac{\widehat{h}_1 + h_0^*}{\widehat{a}_1 + a_0^*}\right)(\widehat{a}_1 + a_0^*) - f_\mathrm{gr}\left(\frac{\widehat{h}_2 + h_0^*}{\widehat{a}_2 + a_0^*}\right)(\widehat{a}_2 + a_0^*) - (\widehat{a}_1 - \widehat{a}_2) f_\mathrm{gr}(0). \end{aligned}$$

The term $(\widehat{a}_1 - \widehat{a}_2) f_\mathrm{gr}(0)$ can be estimated as $\omega(\widehat{h}_1 - \widehat{h}_2)$ above, i. e.,

$$\|(\widehat{a}_1 - \widehat{a}_2) f_\mathrm{gr}(0)\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \le C_{19} T^{1/p} \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}.$$

We further expand the remaining term and make use of Assumption 6.4.7(c) to conclude an estimate by $C_{20}(T^{1/p} + R + C_T^*) \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}$.

The first line in the difference of F_2 can be expanded as

$$\begin{aligned} &(\widehat{h}_1 - \widehat{h}_2) \sum_{j,k=1}^{2} (\partial_j (Y_1)_k) \partial_k (\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*)_j \\ &\quad + (\widehat{h}_2 + h_0^*) \sum_{j,k=1}^{2} (\partial_j ((Y_1)_k - (Y_2)_k))\, \partial_k (\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*)_j. \end{aligned} \tag{6.58}$$

The first addend in (6.58) allows a similar treatment as (6.57). We replace Lemma 6.4.2 by Lemma 6.4.1 to bound the gradient of Y_1, so

$$\begin{aligned} &\left\| (\widehat{h}_1 - \widehat{h}_2) \sum_{j,k=1}^{2} (\partial_j (Y_1)_k) \partial_k (\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*)_j \right\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \\ &\le C_{21}(R + C_T^*) \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}. \end{aligned}$$

In a similar manner, additionally invoking Lemma 6.4.4 for the estimate of the difference in the diffeomorphisms and (6.29) to control $\|\widehat{h}_2 + h_0^*\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))}$, we find that

$$\begin{aligned} &\left\| (\widehat{h}_2 + h_0^*) \sum_{j,k=1}^{2} (\partial_j ((Y_1)_k - (Y_2)_k))\, \partial_k (\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*)_j \right\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \\ &\le C_{22} \cdot \|\widehat{h}_2 + h_0^*\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \cdot \|\nabla_\mathrm{H}(Y_1 - Y_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,q}(\Omega))} \\ &\quad \cdot \|\widehat{v}_{\mathrm{ice},1} + v_{\mathrm{ice},0}^*\|_{\mathrm{L}^p(0,T;\mathrm{W}^{2,q}(\Omega))} \\ &\le C_{23}(R + C_0 + C_T^*) T^{1/p'} (R + C_T^*) \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}. \end{aligned}$$

In total, this yields a suitable estimate of (6.58) in $\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))$.

Next, the negative of the second line of the difference of F_2 equals

$$(6.59)\qquad \begin{aligned}&(\widehat{h}_2 + h_0^*) \sum_{j,k=1}^{2} (\partial_j (\mathrm{Y}_2)_k - \delta_{jk})\, \partial_k (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})_j \\ &\quad + \widehat{h}_2 \mathrm{div}_{\mathrm{H}}\, (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) + (h_0^* - h_0) \mathrm{div}_{\mathrm{H}}\, (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}).\end{aligned}$$

In the same way as in (6.57), we argue that the first addend in (6.59) satisfies

$$\begin{aligned}&\left\| (\widehat{h}_2 + h_0^*) \sum_{j,k=1}^{2} (\partial_j (\mathrm{Y}_2)_k - \delta_{jk}) \partial_k (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2})_j \right\|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \\ &\le C_{24} T^{1/p'} (R + C_0 + C_T^*) \cdot \| \widehat{u}_1 - \widehat{u}_2 \|_{\mathbb{E}_1}.\end{aligned}$$

For the second term in (6.59), we can proceed as in (6.56) to derive

$$\| \widehat{h}_2 \mathrm{div}_{\mathrm{H}}\, (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) \|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \le C_{25} R \cdot \| \widehat{u}_1 - \widehat{u}_2 \|_{\mathbb{E}_1}.$$

As in (6.55), we establish the estimate

$$\begin{aligned}&\| (h_0^* - h_0) \mathrm{div}_{\mathrm{H}}\, (\widehat{v}_{\mathrm{ice},1} - \widehat{v}_{\mathrm{ice},2}) \|_{\mathrm{L}^p(0,T;\mathrm{W}^{1,q}(\Omega))} \\ &\le C_{26} \cdot \| u_0 - u_0^* \|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot \| \widehat{u}_1 - \widehat{u}_2 \|_{\mathbb{E}_1}.\end{aligned}$$

Putting together all the above estimates, and recalling again that $C_T^* \to 0$ and $\| u_0 - u_0^* \|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \to 0$ as $T \to 0$, we conclude the assertion. □

6.5. Local Strong Well-Posedness

In this section, we finally state and prove the main result of this chapter on the local strong well-posedness of the parabolic-hyperbolic regularized sea ice model as introduced in (6.1). We consider again $T \in (0, T_1]$ and $R \in (0, R_0]$, where $T_1 > 0$ from Lemma 6.4.1 and $R_0 > 0$ from Lemma 6.3.3 are fixed.

The local-in-time strong well-posedness result of the parabolic-hyperbolic regularized sea ice model reads as follows.

Theorem 6.5.1. *Let $p, q \in (1, \infty)$ be such that* (6.17) *holds true, let $u_0 \in V$, where V is as made precise in* (6.16)*, suppose that V_{atm}, V_{ocn}, H and f_{gr} fulfill Assumption 6.4.7, and recall the spaces X_0, X_1 and X_γ from* (6.10)*,* (6.11) *and* (6.12)*. Then there exists $T > 0$ such that* (6.1) *admits a unique solution*

$$u \in \mathrm{W}^{1,p}(0,T;\mathrm{X}_0) \cap \mathrm{L}^p(0,T;\mathrm{X}_1) \cap \mathrm{C}([0,T];V) = \mathbb{E}_1 \cap \mathrm{C}([0,T];V).$$

Proof. The strategy of this proof consists of finding a unique solution to the transformed system (6.8) in Lagrangian coordinates first and then to transform back to Eulerian coordinates.

Let us recall the set $\mathcal{K}_T^R$ and the map Φ_T^R from (6.26). For $\widehat{u} \in \mathcal{K}_T^R$, we deduce from the maximal regularity result Proposition 6.2.1 as well as Lemma 6.4.8 and Lemma 6.4.9 that

$$\begin{aligned}\|\Phi_T^R(\widehat{u})\|_{\mathbb{E}_1} &\le C_{\mathrm{MR}} \cdot \|(F_1(\widehat{u}), F_2(\widehat{u}), F_3(\widehat{u}))\|_{\mathbb{E}_0}\\ &\le C_{\mathrm{MR}} \cdot \left(\frac{R}{6C_{\mathrm{MR}}} + \frac{R}{6C_{\mathrm{MR}}} + \frac{R}{6C_{\mathrm{MR}}}\right) \le \frac{R}{2}\end{aligned}$$

upon exploiting the T-independence of C_{MR} and choosing $R > 0$ as well as $T > 0$ sufficiently small. In other words, Φ_T^R is a self map on $\mathcal{K}_T^R$ for such $R > 0$ and $T > 0$.

On the other hand, Lemma 6.4.8 and Lemma 6.4.9 yield the existence of $L(R,T) > 0$ with $L(R,T) \to 0$ as $R \to 0$ and $T \to 0$ such that

$$\begin{aligned}\|\Phi_T^R(\widehat{u}_1) - \Phi_T^R(\widehat{u}_2)\|_{\mathbb{E}_1} &\le C_{\mathrm{MR}} L(R,T) \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}\\ &\le \frac{1}{2} \cdot \|\widehat{u}_1 - \widehat{u}_2\|_{\mathbb{E}_1}\end{aligned}$$

for $\widehat{u}_1$, $\widehat{u}_2 \in \mathcal{K}_T^R$ as well as $R > 0$ and $T > 0$ sufficiently small. This means that Φ_T^R is also a contraction on $\mathcal{K}_T^R$. The contraction mapping principle then yields the existence of a unique fixed point $\widehat{u} \in \mathcal{K}_T^R$ of Φ_T^R.

By construction, $\tilde{u} := \widehat{u} + u_0^*$, for u_0^* representing the reference solution from Proposition 6.3.1, solves the transformed system (6.8). We also observe that

$$\tilde{u} = \widehat{u} + u_0^* \in \mathbb{E}_1,$$

while $\tilde{u} \in \mathrm{C}([0,T]; V)$ follows from the usual embedding of the maximal regularity space into $\mathrm{BUC}([0,T]; \mathrm{X}_\gamma)$ from Proposition 2.4.11 and Lemma 6.3.3.

Thanks to $T \le T_1$, the map $X(t, y_{\mathrm{H}})$ is especially bijective on $\overline{\Omega}$, with regularity $X \in \mathrm{C}^1([0,T]; \mathrm{W}^{2,q}(\Omega)^2)$, see also [43, Section 8]. The same is also valid for the inverse $Y(t,\cdot) = [X(t,\cdot)]^{-1}$. We can thus recover the Eulerian variables $u = (v_{\mathrm{ice}}, h, a)$, which also solve the original problem (6.1), from

$$\begin{aligned}v_{\mathrm{ice}}(t, x_{\mathrm{H}}) &:= \tilde{v}_{\mathrm{ice}}(t, Y(t, x_{\mathrm{H}})),\\ h(t, x_{\mathrm{H}}) &:= \tilde{h}(t, Y(t, x_{\mathrm{H}})) \text{ and}\\ a(t, y_{\mathrm{H}}) &:= \tilde{a}(t, Y(t, x_{\mathrm{H}})).\end{aligned}$$

The transform does not change the regularity properties, and it does not affect the property $u \in \mathrm{C}([0,T]; V)$ either. The uniqueness is a result of the uniqueness of the fixed point joint with the uniqueness of the transform. □

Generally, it would be possible to establish properties of the solution such as the continuous dependence on the initial data as collected in Theorem 3.5.2. This could e. g. be realized by an adjustment of the fixed point argument.

Moreover, it would also be possible to include time weights in order to lower the regularity of the initial data, and to exploit the instantaneous smoothing. As the terms become non-autonomous after invoking the Lagrangian coordinates, this would require refined estimates of the nonlinear terms.

6.6. Open Problems in Connection with Hibler's Sea Ice Model

In this last section of the first part of the thesis, we provide a brief overview of some remaining open problems in the context of the mathematical analysis of Hibler's viscous-plastic sea ice model.

One important problem concerns the existence of (global) *weak* solutions. Note that the results presented in this thesis as well as the work of Liu, Thomas and Titi [97] all concern *strong* solutions. In contrast, the investigation of weak solutions remains open so far. In that respect, it would also be interesting to study the relation between weak solutions and the results obtained in the numerical analysis. Since the results on the parabolic-hyperbolic regularized variants of Hibler's model from [97] and the present chapter are *local-in-time*, results with *global-in-time* character would be an interesting complement.

With regard to the shape of the original stress tensor from (3.3), it is a natural question whether the resulting problem in the situation of degenerate viscosities allows some well-posedness result. This task appears to be particularly challenging due to the quasilinear and degenerate nature of the problem.

Another significant aspect that has not been addressed to date is the analysis of the thermodynamic consistency of the model. In general, it seems that the thermodynamic aspect in Hibler's model might be subject to adjustments.

Finally, having in mind the results from the present chapter, one can ask if similar results as the ones established for the sea ice interaction problem in Chapter 4, the coupled atmosphere-sea ice-ocean model in Chapter 5 and also the time periodic problem in Section 7.2 can be proved for the hyperbolic-parabolic regularized variant of Hibler's model.

Time Periodic Quasilinear Evolution Equations

CHAPTER 7

Time Periodic Quasilinear Evolution Equations by the Arendt-Bu Theorem

In this chapter, we present different frameworks to time periodic quasilinear evolution equations based on the Arendt-Bu theorem on maximal periodic regularity as stated in Section 2.2. As an application, we consider the time periodic problem associated to Hibler's sea ice model, where the underlying model will be the fully parabolic regularized one from Chapter 3. The general frameworks have not been published, whereas the result in the application to Hibler's sea ice model has been obtained in a similar form in a joint article with Matthias Hieber [20].

The study of time periodic problems has a long history. In the context of mathematical fluid mechanics, the topic was pioneered by Serrin [123], Judovič [74] and Prodi [112]. In the past decades, Kozono and Nakao [80], Galdi and Sohr [47], Kozono, Mashiko and Takada [79], Geissert, Hieber and Nguyen [49] or Galdi and Kyed [46] further investigated time periodic problems in fluid mechanics and developed new techniques. With regard to strong solutions, the work of Kyed [86, 87] based on Fourier series together with a splitting of the time periodic problem into a stationary part and a part with mean value zero in time shed new light. Kyed and Sauer [88] extended the analysis to time periodic parabolic boundary value problems. In [69], Hieber and Stinner introduced a framework to time periodic quasilinear evolution equations, based on the Arendt-Bu theorem and with applications to quasilinear Keller-Segel systems. We slightly generalize their approach and also elaborate on the particular situation of bilinear nonlinearities in the following section. Another approach to time periodic solutions to semilinear evolution equations in real interpolation spaces was presented by Hieber, Kajiwara, Kress and Tolksdorf [64], and they applied this method to the time periodic problem of the bidomain equations. Their work is the inspiration for the considerations in Chapter 8, where we investigate time periodic quasilinear problems in real interpolation spaces.

This chapter is organized as follows. Section 7.1 is dedicated to the presentation of different frameworks to time periodic quasilinear problems, and the common aspect is that all frameworks rely on the Arendt-Bu theorem. The first framework, leading to Theorem 7.1.2, is the most general one and discusses the existence and uniqueness of strong time periodic solutions close to equilibria of the autonomous part of the evolution equation. The second framework as summarized in Corollary 7.1.4 is deduced from there. It is concerned with time periodic solutions in a neighborhood of zero. In the last part of this section, we elaborate on the semilinear situation with bilinear right-hand side, and the corresponding main results are Corollary 7.1.5 and Corollary 7.1.6. The second part of this chapter, Section 7.2, is centered around the application to Hibler's sea ice model. The main difficulty here is to circumvent the lack of invertibility of the underlying Neumann Laplacian operators. For this purpose, we focus on time periodic solutions to the system close to constant equilibria. By subtracting the equilibrium parts, we can adjust the ground space to get invertibility, finally resulting in Theorem 7.2.7.

7.1. General Frameworks

We provide various frameworks to time periodic quasilinear problems based on the Arendt-Bu theorem as recalled in Proposition 2.2.3.

Throughout this section, let X_0 and X_1 be Banach spaces such that X_1 is densely embedded into X_0. We start by providing a fairly general framework to time periodic quasilinear problems in the vicinity of equilibrium solutions. For a given time period $(0,T)$, $T \in (0,\infty)$, we consider problems of the shape

$$\begin{cases} u'(t) + A(u(t))u(t) = F_1(u(t)) + F_2(t,u(t)), & \text{for } t \in (0,T), \\ u(0) = u(T), \end{cases} \tag{7.1}$$

on some Banach space X_0. We also refer to (7.1) as the *general time periodic quasilinear abstract Cauchy problem*. Moreover, for X_0 and X_1 as described above and $p \in (1,\infty)$, we set $X_\gamma := (X_0, X_1)_{1-1/p,p}$, and we suppose the existence of an open subset $V \subset X_\gamma$ such that

$$A\colon V \to \mathcal{L}(X_1, X_0), \quad F_1\colon V \to X_0 \text{ and } F_2\colon [0,T]\times V \to X_0$$

are well-defined. Furthermore, let $u_* \in V \cap X_1$ denote a constant-in-time equilibrium solution to the associated autonomous problem, i. e.,

$$A(u_*)u_* = F_1(u_*). \tag{7.2}$$

The appropriate spaces for the *data* and the *solution* in this case are

$$\mathbb{E}_0 := L^p(0,T;X_0) \text{ and } \mathbb{E}_1 := W^{1,p}(0,T;X_0) \cap L^p(0,T;X_1). \tag{7.3}$$

Before making assumptions on the nonlinear terms and the underlying linearized operator, we discuss that $u_* \in \mathbb{E}_1 \cap V$ already implies $\overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, R_0) \subset V$ for some $R_0 > 0$. From Proposition 2.4.11, it follows that

$$\mathbb{E}_1 \hookrightarrow \mathrm{BUC}([0,T];X_\gamma). \tag{7.4}$$

Therefore, given $u \in \mathbb{E}_1$, we get $u(t) \in X_\gamma$ for every $t \in [0,T]$. Thanks to V being open, there is $R_1 > 0$ sufficiently small such that $\overline{\mathbb{B}}_{X_\gamma}(u_*, R_1) \subset V$. Hence, setting $R_0 := R_1/C$, where $C > 0$ represents the embedding constant from (7.4), it follows from $u \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, R_0)$ that

$$\sup_{t\in[0,T]} \|u(t) - u_*\|_{X_\gamma} \le C \cdot \|u - u_*\|_{\mathbb{E}_1} \le R_1.$$

In other words, $u(t) \in V$ for all $t \in [0,T]$.

The following assumption paves the way for a result on the existence and uniqueness of a solution to (7.1). Let us observe that $R_0 > 0$ as described below exists thanks to the preceding argument.

Assumption 7.1.1. *Let $u_* \in V \cap X_1$ be a solution to* (7.2)*, and consider a radius $R_0 > 0$ so that $u \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, R_0)$ satisfies $u(t) \in V$ for all $t \in [0,T]$.*

(i) The operators $A\colon V \to \mathcal{L}(X_1, X_0)$ are a family of closed linear operators. In addition, for every $R \in (0, R_0)$, there exists $L(R) > 0$ such that

$$\|(A(u_1(\cdot)) - A(u_2(\cdot)))v(\cdot)\|_{\mathbb{E}_0} \le L(R) \cdot \|u_1 - u_2\|_{\mathbb{E}_1} \cdot \|v\|_{\mathbb{E}_1}$$

for all u_1, $u_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, R)$ and $v \in \mathbb{E}_1$. Besides, for every $R \in (0, R_0)$, there is $L'(R) > 0$ with

$$\|(A(u_1(\cdot)) - A(u_2(\cdot)))u_*\|_{\mathbb{E}_0} \le L'(R) \cdot \|u_1 - u_2\|_{\mathbb{E}_1}$$

for all u_1, $u_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, R)$.

(ii) *The maps* $F_1\colon V \to \mathrm{X}_0$ *as well as* $F_2\colon [0,T]\times V \to \mathrm{X}_0$ *have the property that* $F_1(u(\cdot))$, $F_2(\cdot,u(\cdot)) \in \mathbb{E}_0$ *for all* $u \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*,R_0)$. *Furthermore, for every* $R \in (0,R_0)$, *there are* $C_1(R)$, $C_2(R) > 0$ *such that*

$$\|F_1(u_1(\cdot)) - F_1(u_2(\cdot))\|_{\mathbb{E}_0} \le C_1(R)\cdot \|u_1-u_2\|_{\mathbb{E}_1} \quad \text{and}$$
$$\|F_2(\cdot,u_1(\cdot)) - F_2(\cdot,u_2(\cdot))\|_{\mathbb{E}_0} \le C_2(R)\cdot \|u_1-u_2\|_{\mathbb{E}_1}$$

for all u_1, $u_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*,R)$. *Moreover, we have* $F_2(0,u) = F_2(T,u)$ *for all* $u \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*,R)$.

(iii) *For all* $p \in (1,\infty)$, *the operator* $A_* := A(u_*)$ *satisfies* $0 \in \rho(A_*)$ *as well as* $A_* \in {}_0\mathcal{MR}_p(\mathrm{X}_0)$.

The result on the existence and uniqueness of a solution to the T-periodic quasilinear abstract Cauchy problem (7.1) now reads as follows.

Theorem 7.1.2. *Let* $u_* \in V \cap \mathrm{X}_1$ *be a solution to* (7.2), *and suppose that Assumption 7.1.1 is satisfied. Assume further that for some* $R \in (0,R_0)$, *the constants* $L'(R)$, $C_1(R)$ *and* $C_2(R)$ *fulfill* $\max\{L'(R), C_1(R), C_2(R)\} < \delta_1$, *where* $\delta_1 > 0$ *is sufficiently small.*

Then there are $r \in (0,R)$ *and* $\delta_2 = \delta_2(r) > 0$ *so that if* $\|F_2(\cdot,u_*)\|_{\mathbb{E}_0} < \delta_2$, *the problem* (7.1) *has a solution* $u \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*,r)$ *which is unique in* $\overline{\mathbb{B}}_{\mathbb{E}_1}(u_*,r)$.

Proof. The proof is based on the Arendt-Bu theorem. First, we linearize (7.1) to reformulate the question of the existence and uniqueness of a solution as a fixed point problem. We observe that finding a unique solution to (7.1) is equivalent to establishing the existence of a unique fixed point of

$$\text{(7.5)} \qquad \begin{cases} u'(t) + A_*u(t) = (A(u_*) - A(v(t)))v(t) \\ \qquad\qquad\qquad + F_1(v(t)) + F_2(t,v(t)), \quad \text{for } t \in (0,T), \\ \qquad u(0) = u(T), \end{cases}$$

where $v \in \mathbb{E}_1$. For $r \in (0,R)$, with $R \in (0,R_0)$ as in the assertion, we obtain $\overline{\mathbb{B}}_{\mathbb{E}_1}(u_*,r) \subset V$. We define the solution map Φ to (7.5) by

$$\Phi\colon \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*,r) \to \mathbb{E}_1, \quad \Phi(v) := u,$$

where $u \in \mathbb{E}_1$ is the unique solution to (7.5). First, we show that Φ is well-defined. Thanks to Assumption 7.1.1(iii), we can apply Corollary 2.2.4, which is based on the Arendt-Bu theorem, to A_* and deduce $A_* \in \mathcal{MR}_{\mathrm{per},p}(\mathrm{X}_0)$.

Hence, A_* has maximal periodic L^p-regularity. As a consequence, there exists a unique solution to (7.5) if the right-hand side lies in the data space, i. e.,

$$(A(u_*) - A(v))v + F_1(v) + F_2(t, v) \in \mathbb{E}_0.$$

This is ensured by Assumption 7.1.1(i) and (ii), so Φ is indeed well-defined.

With regard to the above argument, the task now is to prove that Φ admits a unique fixed point. The first step here is the verification that Φ is also a self map for $r \in (0, R)$ sufficiently small. To this end, consider $v \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, r)$ as well as the resulting solution $u = \Phi(v)$ to the linearized problem (7.5), and set $w \coloneqq u - u_*$. From the time-independence of u_* and $A(u_*)u_* = F_1(u_*)$ by assumption, we find that w solves

$$\begin{aligned} w' + A_* w &= u' + A_* u - A(u_*)u_* \\ &= (A(u_*) - A(v))v + F_1(v) + F_2(t, v)) - A(u_*)u_* \\ &= (A(u_*) - A(v))(v - u_*) - (A(v) - A(u_*))u_* - A(u_*)u_* \\ &\quad + F_1(v) + F_2(t, v) - F_2(t, u_*) + F_2(t, u_*) \\ &= (A(u_*) - A(v))(v - u_*) - (A(v) - A(u_*))u_* + F_1(v) - F_1(u_*) \\ &\quad + F_2(t, v) - F_2(t, u_*) + F_2(t, u_*). \end{aligned}$$

As $u(0) = u(T)$, and u_* is time-independent, we get $w(0) = w(T)$. By virtue of $A_* \in \mathcal{MR}_{\mathrm{per},p}(\mathrm{X}_0)$, we deduce from Definition 2.2.1 the existence of a constant $C > 0$ such that

$$\begin{aligned} \|\Phi(v) - u_*\|_{\mathbb{E}_1} \le C \cdot \Big(&\|(A(u_*) - A(v(\cdot)))(v(\cdot) - u_*)\|_{\mathbb{E}_0} \\ &+ \|(A(v(\cdot)) - A(u_*))u_*\|_{\mathbb{E}_0} + \|F_1(v(\cdot)) - F_1(u_*)\|_{\mathbb{E}_0} \\ &+ \|F_2(\cdot, v(\cdot)) - F_2(\cdot, u_*)\|_{\mathbb{E}_0} + \|F_2(\cdot, u_*)\|_{\mathbb{E}_0}\Big). \end{aligned} \tag{7.6}$$

Thanks to $v,\ u_* \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, R)$ for every $R \in (0, R_0)$, we may plug the Lipschitz estimates of A, F_1 and F_2 from Assumption 7.1.1(i) and (ii) into (7.6). Invoking $\|F_2(\cdot, u_*)\|_{\mathbb{E}_0} < \delta_2$, and exploiting $v \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, r)$, we obtain

$$\begin{aligned} \|\Phi(v) - u_*\|_{\mathbb{E}_1} &\le C\Big(L(R) \cdot \|v(\cdot) - u_*\|_{\mathbb{E}_1} + L'(R) + C_1(R) \\ &\qquad + C_2(R)\Big) \cdot \|v(\cdot) - u_*\|_{\mathbb{E}_1} + C\delta_2 \\ &\le C\Big(L(R)r + L'(R) + C_1(R) + C_2(R)\Big)r + C\delta_2. \end{aligned} \tag{7.7}$$

Concerning the contraction property of Φ, we consider $v_1,\ v_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, r)$

and investigate $u_i = \Phi(v_i)$, where $i = 1, 2$. Then $u \coloneqq u_1 - u_2$ solves

$$\begin{aligned} u' + A_* u &= u_1' + A_* u_1 - (u_2' + A_* u_2) \\ &= (A(u_*) - A(v_1))v_1 + F_1(v_1) + F_2(t, v_1) \\ &\quad - (A(u_*) - A(v_2))v_2 - F_1(v_2) - F_2(t.v_2) \\ &= (A(u_*) - A(v_1))(v_1 - v_2) + (A(v_1) - A(v_2))(u_* - v_2) \\ &\quad - (A(v_1) - A(v_2))u_* + F_1(v_1) - F_1(v_2) + F_2(t, v_1) - F_2(t, v_2). \end{aligned}$$

In addition, we deduce $u(0) = u(T)$ from the respective property of u_1 and u_2. Thus, with the above maximal periodic regularity constant $C > 0$, we get

$$\begin{aligned} \|\Phi(v_1) - \Phi(v_2)\|_{\mathbb{E}_1} \leq C \cdot \Big(&\|(A(u_*) - A(v_1(\cdot)))(v_1(\cdot) - v_2(\cdot))\|_{\mathbb{E}_0} \\ &+ \|(A(v_1(\cdot)) - A(v_2(\cdot)))(u_* - v_2(\cdot))\|_{\mathbb{E}_0} \\ &+ \|(A(v_1(\cdot)) - A(v_2(\cdot)))u_*\|_{\mathbb{E}_0} \\ &+ \|F_1(v_1(\cdot)) - F_1(v_2(\cdot))\|_{\mathbb{E}_0} \\ &+ \|F_2(\cdot, v_1(\cdot)) - F_2(\cdot, v_2(\cdot))\|_{\mathbb{E}_0} \Big). \end{aligned}$$

As v_1, $v_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, r) \subset V$, it is again justified to use the Lipschitz estimates of A and F from Assumption 7.1.1(i) and (ii). This leads to

$$\begin{aligned} \|\Phi(v_1) - \Phi(v_2)\|_{\mathbb{E}_1} &\leq C\Big(L(R) \cdot \|v_1 - u_*\|_{\mathbb{E}_1} + L(R) \cdot \|v_2 - u_*\|_{\mathbb{E}_1} \\ &\qquad + L'(R) + C_1(R) + C_2(R)\Big) \cdot \|v_1 - v_2\|_{\mathbb{E}_1} \\ &\leq C\Big(2L(R)r + L'(R) + C_1(R) + C_2(R)\Big) \\ &\qquad \cdot \|v_1 - v_2\|_{\mathbb{E}_1}. \end{aligned} \tag{7.8}$$

If for some $R \in (0, R_0)$, we have $\max\{L'(R), C_1(R), C_2(R)\} < {}^1\!/_{12C} \eqqcolon \delta_1$, $r < \min\{{}^1\!/_{4CL(R)}, R\}$ and $\delta_2 \coloneqq {}^r\!/_{2C}$ with $\|F_2(\cdot, u_*)\|_{\mathbb{E}_0} < \delta_2$, for $v \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, r)$, we deduce from (7.7) that

$$\|\Phi(v) - u_*\|_{\mathbb{E}_1} < C\left(\frac{1}{4C} + \frac{1}{4C}\right) r + \frac{r}{2} = r.$$

In other words, Φ is a self map on $\overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, r)$. At the same time, (7.8) implies

$$\|\Phi(v_1) - \Phi(v_2)\|_{\mathbb{E}_1} < C\left(\frac{1}{2C} + \frac{1}{4C}\right) \cdot \|v_1 - v_2\|_{\mathbb{E}_1} = \frac{3}{4} \cdot \|v_1 - v_2\|_{\mathbb{E}_1}$$

for v_1, $v_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, r)$. Thus, Φ is also a contraction map on $\overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, r)$. The contraction mapping principle hence yields a unique fixed point $u \in \overline{\mathbb{B}}_{\mathbb{E}_1}(u_*, r)$ of Φ. By the reformulation from the beginning of the proof, this is in turn the unique solution to (7.1). □

After establishing a relatively general framework to time periodic quasilinear problems, we next focus on some more particular situations. In fact, the first more specific case deals with time periodic solutions close to zero. Thus, for an open set $V \subset \mathrm{X}_\gamma$ with $0 \in V$, we consider

$$\begin{cases} u'(t) + A(u(t))u(t) = F(t, u(t)), \text{ for } t \in (0, T), \\ u(0) = u(T). \end{cases} \tag{7.9}$$

Similarly as in the preceding framework, we suppose that

$$A\colon V \to \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0) \text{ and } F\colon [0, T] \times V \to \mathrm{X}_0.$$

We also still use the data space $\mathbb{E}_0$ as well as the solution space $\mathbb{E}_1$ as introduced in (7.3). Before stating the well-posedness result of (7.9), we provide assumptions tailored to the present situation.

Assumption 7.1.3. *Let $R_0 > 0$ be such that every $u \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0, R_0)$ has the property $u(t) \in V$ for all $t \in [0, T]$.*

(i) The operators $A\colon V \to \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0)$ are a family of closed linear operators, and for every $R \in (0, R_0)$, there exists $L(R) > 0$ such that

$$\|(A(u_1(\cdot)) - A(u_2(\cdot)))v(\cdot)\|_{\mathbb{E}_0} \le L(R) \cdot \|u_1 - u_2\|_{\mathbb{E}_1} \cdot \|v\|_{\mathbb{E}_1}$$

for all u_1, $u_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0, R)$ and $v \in \mathbb{E}_1$.

(ii) The map $F\colon [0, T] \times V \to \mathrm{X}_0$ has the property that $F(\cdot, u(\cdot)) \in \mathbb{E}_0$ for all $u \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0, R_0)$, and for every $R \in (0, R_0)$, there is $C(R) > 0$ with

$$\|F(\cdot, u_1(\cdot)) - F(\cdot, u_2(\cdot))\|_{\mathbb{E}_0} \le C(R) \cdot \|u_1 - u_2\|_{\mathbb{E}_1}$$

for all u_1, $u_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0, R)$. Furthermore, suppose that $F(0, u) = F(T, u)$ for all $u \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0, R)$.

(iii) For all $p \in (1, \infty)$, the operator $A_0 := A(0)$ fulfills $0 \in \rho(A_0)$ as well as $A_0 \in {}_0\mathcal{MR}_p(\mathrm{X}_0)$.

The result in the simplified situation of (7.9) is given as follows.

Corollary 7.1.4. *Suppose that Assumption 7.1.3 is satisfied. Assume further that for some $R \in (0, R_0)$, the constant $C(R) > 0$ fulfills $C(R) < \delta_1$ for $\delta_1 > 0$ sufficiently small. Then there exist $r \in (0, R)$ as well as $\delta_2 = \delta_2(r) > 0$ such that for $\|F(\cdot, 0)\|_{\mathbb{E}_0} < \delta_2$, there is a solution $u \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0, r)$ to (7.9), and u is unique in $\overline{\mathbb{B}}_{\mathbb{E}_1}(0, r)$.*

Proof. The assertion can be derived from Theorem 7.1.2. For this purpose, we consider $u_* = 0$ and set $F_1(u) = 0$ as well as $F_2(t, u) = F(t, u)$. It readily follows that $u_* = 0$ satisfies (7.2), and $0 \in V$ is also valid by assumption. Let us observe that the first part Assumption 7.1.1(i) is satisfied in view of Assumption 7.1.3(i), while the second part is an immediate consequence of

$$(A(u_1) - A(u_2))u_* = 0$$

for all $u_1,\ u_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0, R_0)$. The concrete choice of $F_1 = 0$ clearly has the properties demanded in Assumption 7.1.1(ii), whereas Assumption 7.1.3(ii) yields that the above F_2 lies within the scope of Assumption 7.1.1(ii) as well. Moreover, Assumption 7.1.1(iii) coincides with Assumption 7.1.3(iii) in the present case. We are thus in the framework of Theorem 7.1.2 upon observing that $L'(R)$ and $C_1(R)$ can be chosen equal to zero here, and $C(R)$ plays the role of $C_2(R)$. □

Since many time periodic problems, especially the ones arising from fluid mechanics, are of semilinear nature with bilinear nonlinearity, we also provide a well-posedness result for time periodic problems of this shape. For X_0 and X_1 as above and $\beta \in (0, 1)$, we consider the complex interpolation space

$$\mathrm{X}_\beta \coloneqq [\mathrm{X}_0, \mathrm{X}_1]_\beta.$$

In this set-up, we suppose that $A\colon \mathrm{X}_1 \to \mathrm{X}_0$ is a linear operator, while the term on the right-hand side $G\colon \mathrm{X}_\beta \times \mathrm{X}_\beta \to \mathrm{X}_0$ is assumed to be bilinear and bounded. For $f \in \mathbb{E}_0$ with $f(0) = f(T)$, the problem under consideration takes the shape

$$\begin{cases} u'(t) + Au(t) = G(u(t), u(t)) + f(t), & \text{for } t \in (0, T), \\ \qquad\qquad u(0) = u(T). \end{cases} \tag{7.10}$$

The well-posedness result associated to (7.10) is given below. Compared to the previous results, we make stronger assumptions on the ground space X_0 as well as on the operator A. Later, we will comment on possible relaxations.

Corollary 7.1.5. *Let X_0 be a UMD Banach space, consider $p \in (1, \infty)$, and let $A \in \mathcal{BIP}(\mathrm{X}_0)$ with power angle $\theta_A < \pi/2$ such that $0 \in \rho(A)$. Moreover, for $\beta \in (1 - 1/p, 1)$ with $2\beta - 1 \le 1 - 1/p$, assume that $G\colon \mathrm{X}_\beta \times \mathrm{X}_\beta \to \mathrm{X}_0$ is bilinear and bounded.*

Then there exist $r > 0$ and $\delta = \delta(r) > 0$ such that if $\|f\|_{\mathbb{E}_0} < \delta$, there is a solution $u \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0, r)$ to (7.10). In addition, u is unique in $\overline{\mathbb{B}}_{\mathbb{E}_1}(0, r)$.

Proof. The idea here is to reduce the assertion to Corollary 7.1.4. The task thus is to verify Assumption 7.1.3. From $A \in \mathcal{BIP}(\mathrm{X}_0)$ with $\theta_A < \pi/2$, it follows that $A \in \mathcal{RS}(\mathrm{X}_0)$ with $\phi_A^{\mathcal{R}} < \pi/2$ by virtue of (2.14). Hence, Proposition 2.1.21 implies that $A \in {}_0\mathcal{MR}_p(\mathrm{X}_0)$ thanks to the UMD property of X_0. The assumption $0 \in \rho(A)$ implies that Assumption 7.1.3(iii) holds true. As A is linear, it readily follows that Assumption 7.1.3(i) is satisfied.

It remains to show the validity of Assumption 7.1.3(ii). The variant of the mixed derivative theorem as in Corollary 2.4.9(b) implies

$$\mathbb{E}_1 = \mathrm{W}^{1,p}(0,T;\mathrm{X}_0) \cap \mathrm{L}^p(0,T;\mathrm{X}_1) \hookrightarrow \mathrm{H}^{1-\beta,p}(0,T;\mathrm{X}_\beta) \tag{7.11}$$

for all $\beta \in (0,1)$ and $p \in (1,\infty)$. Let us observe that $2\beta - 1 \le 1 - 1/p$ is equivalent to $1 - \beta - 1/p \ge -1/2p$. Thus, we conclude from the Sobolev embedding in (1.5) that $\mathrm{H}^{1-\beta,p}(0,T;\mathrm{X}_\beta) \hookrightarrow \mathrm{L}^{2p}(0,T;\mathrm{X}_\beta)$. Under the present assumptions on $\beta \in (0,1)$ and $p \in (1,\infty)$. Together with the above embedding (7.11), this results in

$$\mathbb{E}_1 \hookrightarrow \mathrm{L}^{2p}(0,T;\mathrm{X}_\beta). \tag{7.12}$$

We set $F(t,u) := G(u,u) + f(t)$. Let $R > 0$, and consider $u_1, u_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0,R)$. Making use of the bilinearity and boundedness of $G\colon \mathrm{X}_\beta \times \mathrm{X}_\beta \to \mathrm{X}_0$, applying Hölder's inequality, and exploiting (7.12) and $\|u_i\|_{\mathbb{E}_1} \le R$, $i = 1,2$, we find

$$\begin{aligned}
&\|F(\cdot,u_1(\cdot)) - F(\cdot,u_2(\cdot))\|_{\mathbb{E}_0} \\
&\le \|G(u_1(\cdot),u_1(\cdot)-u_2(\cdot))\|_{\mathbb{E}_0} + \|G(u_1(\cdot)-u_2(\cdot),u_2(\cdot))\|_{\mathbb{E}_0} \\
&= \Big(\int_0^T \|G(u_1(t),u_1(t)-u_2(t))\|_{\mathrm{X}_0}^p \,\mathrm{d}t\Big)^{1/p} \\
&\quad + \Big(\int_0^T \|G(u_1(t)-u_2(t),u_2(t))\|_{\mathrm{X}_0}^p \,\mathrm{d}t\Big)^{1/p} \\
&\le C_1\bigg(\Big(\int_0^T \|u_1(t)\|_{\mathrm{X}_\beta}^p \cdot \|u_1(t)-u_2(t)\|_{\mathrm{X}_\beta}^p \,\mathrm{d}t\Big)^{1/p} \\
&\quad + \Big(\int_0^T \|u_1(t)-u_2(t)\|_{\mathrm{X}_\beta}^p \cdot \|u_2(t)\|_{\mathrm{X}_\beta}^p \,\mathrm{d}t\Big)^{1/p}\bigg) \\
&\le C_2 \cdot \Big(\|u_1\|_{\mathrm{L}^{2p}(0,T;\mathrm{X}_\beta)} + \|u_2\|_{\mathrm{L}^{2p}(0,T;\mathrm{X}_\beta)}\Big) \cdot \|u_1 - u_2\|_{\mathrm{L}^{2p}(0,T;\mathrm{X}_\beta)} \\
&\le C_3 R \cdot \|u_1 - u_2\|_{\mathbb{E}_1}.
\end{aligned} \tag{7.13}$$

This yields that the Lipschitz estimate from Assumption 7.1.3(ii) holds true with $C(R) = C_3 R$. Additionally invoking the assumption on f, and observing that the only explicit time-dependence in $F(t,u) = G(u,u) + f(t)$

is via f, which satisfies $f(0) = f(T)$ by assumption, we conclude that Assumption 7.1.3(ii) is satisfied. Moreover, we observe that $G(0,0) = 0$, implying $F(\cdot, 0) = f(\cdot)$. The assertion thus follows from Corollary 7.1.4. $\square$

We briefly comment on the preceding proof of Corollary 7.1.5 and on a possibility to weaken the assumptions on the operator A. First, we observe that the embedding constant emerging from the mixed derivative embedding (7.11) above is generally not time-independent as we do not consider zero initial values. This does not pose any problems in the above proof as we consider a fixed time period $(0, T)$.

The proof of Corollary 7.1.5 also reveals that we only require the bounded imaginary powers of A in order to derive the embedding (7.11). The latter can also be concluded if there exists a densely defined operator B on X_0 such that $D(B) = \mathrm{X}_1$ and $B \in \mathcal{BIP}(\mathrm{X}_0)$ with $\theta_B < \pi/2$. By Corollary 2.4.9(b), this yields the desired embedding (7.11). With regard to A, it is sufficient to assume $A \in {}_0\mathcal{MR}_p(\mathrm{X}_0)$ with $0 \in \rho(A)$ to satisfy Assumption 7.1.3(iii). In particular, if there is $B \colon \mathrm{D}(B) = \mathrm{X}_1 \to \mathrm{X}_0$ with $B \in \mathcal{BIP}(\mathrm{X}_0)$ and $\theta_B < \pi/2$, then for $p \in (1, \infty)$ and $2\beta - 1 \le 1 - 1/p$, we have

$$\mathbb{E}_1 \hookrightarrow \mathrm{H}^{1-\beta,p}(0,T;\mathrm{X}_\beta) \hookrightarrow \mathrm{L}^{2p}(0,T;\mathrm{X}_\beta). \tag{7.14}$$

Next, we see that the structural assumptions can be relaxed provided we content ourselves with $2\beta - 1 < 1 - 1/p$.

Corollary 7.1.6. *Let $p \in (1, \infty)$ as well as $A \in {}_0\mathcal{MR}_p(\mathrm{X}_0)$ with $0 \in \rho(A)$. Besides, for $\beta \in (1 - 1/p, 1)$ such that $2\beta - 1 < 1 - 1/p$, suppose that the right-hand side $G \colon \mathrm{X}_\beta \times \mathrm{X}_\beta \to \mathrm{X}_0$ is bilinear and bounded.*

Then there are $r > 0$ and $\delta = \delta(r) > 0$ such that if $\|f\|_{\mathbb{E}_0} < \delta$, there exists a unique solution $u \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0, r)$ to (7.10).

Proof. Again, the plan is to show that the present corollary lies within the scope of Corollary 7.1.4. As in the proof of Corollary 7.1.5, it readily follows that Assumption 7.1.3(i) and (iii) are fulfilled.

For the verification of Assumption 7.1.3(ii), we set $F(t,u) \coloneqq G(u,u) + f(t)$. With regard to the below estimates, we first establish some embeddings. In fact, as $\mathrm{X}_1 \hookrightarrow \mathrm{X}_0$, thanks to $\beta - 1 + 1/p > 0$, we deduce from the embedding in Lemma 1.2.3 that

$$\mathrm{X}_\beta = [\mathrm{X}_0, \mathrm{X}_1]_\beta \hookrightarrow (\mathrm{X}_0, \mathrm{X}_1)_{1-1/p,p} = \mathrm{X}_\gamma. \tag{7.15}$$

As a result of (7.15), for $u \in \mathrm{X}_1$, we obtain the interpolation inequality

$$\|u\|_{\mathrm{X}_\beta} \le C_1 \cdot \|u\|_{\mathrm{X}_\gamma}^{1-\alpha} \cdot \|u\|_{\mathrm{X}_1}^{\alpha}, \tag{7.16}$$

where $\alpha/p = \beta - 1 + 1/p$. Let us also recall the embedding of the maximal regularity space from Proposition 2.4.11, yielding

$$\mathbb{E}_1 = \mathrm{W}^{1,p}(0,T;\mathrm{X}_0) \cap \mathrm{L}^p(0,T;\mathrm{X}_1) \hookrightarrow \mathrm{BUC}([0,T];\mathrm{X}_\gamma). \tag{7.17}$$

First, the bilinearity and boundedness of $G\colon \mathrm{X}_\beta \times \mathrm{X}_\beta \to \mathrm{X}_0$, the interpolation inequality from (7.16) as well as Hölder's inequality result in

$$\begin{aligned}
&\|G(u_1(\cdot), u_2(\cdot))\|_{\mathbb{E}_0} \\
&\le C_2 \left(\int_0^T (\|u_1(t)\|_{\mathrm{X}_\beta} \cdot \|u_2(t)\|_{\mathrm{X}_\beta})^p \,\mathrm{d}t \right)^{1/p} \\
&\le C_3 \left(\int_0^T \|u_1(t)\|_{\mathrm{X}_\gamma}^{(1-\alpha)p} \cdot \|u_1(t)\|_{\mathrm{X}_1}^{\alpha p} \cdot \|u_2(t)\|_{\mathrm{X}_\gamma}^{(1-\alpha)p} \cdot \|u_2(t)\|_{\mathrm{X}_1}^{\alpha p} \,\mathrm{d}t \right)^{1/p} \\
&\le C_4 \cdot \|u_1\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}^{1-\alpha} \cdot \|u_2\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)}^{1-\alpha} \cdot \left(\int_0^T \|u_1(t)\|_{\mathrm{X}_1}^{2\alpha p} \,\mathrm{d}t \right)^{\alpha/\alpha p} \\
&\quad \cdot \left(\int_0^T \|u_2(t)\|_{\mathrm{X}_1}^{2\alpha p} \,\mathrm{d}t \right)^{\alpha/\alpha p}.
\end{aligned} \tag{7.18}$$

Next, we consider $r \in (1,\infty)$ with $1/r = 1/2\alpha p - 1/p = (1-2\alpha)/2\alpha p > 0$, where the positivity is a consequence of $2\beta - 1 < 1 - 1/p$. Using Hölder's inequality, and employing the embedding (7.17), we derive from (7.18) that

$$\begin{aligned}
&\|G(u_1(\cdot), u_2(\cdot))\|_{\mathbb{E}_0} \\
&\le C_5 \cdot \|u_1\|_{\mathbb{E}_1}^{1-\alpha} \cdot \|u_2\|_{\mathbb{E}_1}^{1-\alpha} \cdot T^{2\alpha/r} \cdot \left(\int_0^T \|u_1(t)\|_{\mathrm{X}_1}^{p} \,\mathrm{d}t \right)^{\alpha/p} \cdot \left(\int_0^T \|u_2(t)\|_{\mathrm{X}_1}^{p} \,\mathrm{d}t \right)^{\alpha/p} \\
&\le C_6 T^{2\alpha/r} \|u_1\|_{\mathbb{E}_1} \cdot \|u_2\|_{\mathbb{E}_1}.
\end{aligned}$$

Similarly as in the proof of Corollary 7.1.5, in conjunction with the bilinearity of G, for all u_1, $u_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0,R)$, $R > 0$, the previous estimate leads to

$$\|F(\cdot, u_1(\cdot)) - F(\cdot, u_2(\cdot))\|_{\mathbb{E}_0} \le C_7 T^{2\alpha/r} R \cdot \|u_1 - u_2\|_{\mathbb{E}_1}.$$

Hence, the Lipschitz estimate from Assumption 7.1.3(ii) is verified, and the proof can be concluded in the same way as the one of Corollary 7.1.5. □

7.2. Application to Hibler's Sea Ice Model

This section is dedicated to the time periodic problem of the Hibler sea ice model as introduced and analyzed in detail in Chapter 3. We use the general framework to time periodic problems developed in Section 7.1 to show the existence of time periodic strong solutions to Hibler's model subject to time periodic forces. As in Chapter 3, $\Omega \subset \mathbb{R}^2$ denotes a bounded domain of class C^2, and $u = (v_{\mathrm{ice}}, h, a)$ represents the principle variable. For a fixed time period $T > 0$, we consider T-periodic forces f_{ice}, f_{h} and f_{a}. In contrast to Chapter 3, we take into account *generic* thermodynamic terms S_{h} and S_{a} with certain structural assumptions to be specified below. With Hibler's model as in Section 3.2 and summarized in (3.10), and with $\tau_{\mathrm{ice}} = \tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(v_{\mathrm{ice}})$, the *time periodic problem associated to Hibler's sea ice model* is given by

$$
(7.19) \quad \begin{cases}
\partial_t v_{\mathrm{ice}} + (v_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) v_{\mathrm{ice}} = \frac{1}{m_{\mathrm{ice}}} \mathrm{div}_{\mathrm{H}}\, \sigma_\delta - c_{\mathrm{cor}} v_{\mathrm{ice}}^\perp & \\
\qquad - g \nabla_{\mathrm{H}} H + \frac{1}{m_{\mathrm{ice}}} \tau_{\mathrm{ice}} + f_{\mathrm{ice}}, & \text{in } \mathbb{R} \times \Omega, \\
\partial_t h + \mathrm{div}_{\mathrm{H}}\, (v_{\mathrm{ice}} h) = S_{\mathrm{h}} + d_{\mathrm{h}} \Delta_{\mathrm{H}} h + f_{\mathrm{h}}, & \text{in } \mathbb{R} \times \Omega, \\
\partial_t a + \mathrm{div}_{\mathrm{H}}\, (v_{\mathrm{ice}} a) = S_{\mathrm{a}} + d_{\mathrm{a}} \Delta_{\mathrm{H}} a + f_{\mathrm{a}}, & \text{in } \mathbb{R} \times \Omega, \\
v_{\mathrm{ice}} = 0, \ \partial_\nu h = \partial_\nu a = 0, & \text{on } \mathbb{R} \times \partial\Omega, \\
u(t) = u(t+T), & \text{for } t \in \mathbb{R}.
\end{cases}
$$

Below, we consider time periodic solutions close to constant equilibria u_* of the simplified system from (3.35), where $-g\nabla_{\mathrm{H}} H = \tau_{\mathrm{atm}} = \tau_{\mathrm{ocn}} = 0$ as well as $S_{\mathrm{h}} = S_{\mathrm{a}} = 0$. Hence, $u_* = (0, h_*, a_*)$, with $h_* > \kappa$ and $a_* \in (0,1)$ constant in time and space. If $u = (v_{\mathrm{ice}}, h, a)$ solves (7.19), then $\tilde{u} := u - u_* := (\tilde{v}_{\mathrm{ice}}, \tilde{h}, \tilde{a})$ is a solution to

$$
(7.20) \quad \begin{cases}
\partial_t \tilde{v}_{\mathrm{ice}} - \frac{1}{m_{\mathrm{ice}}} \mathrm{div}_{\mathrm{H}}\, \sigma_\delta = -(\tilde{v}_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}) \tilde{v}_{\mathrm{ice}} - c_{\mathrm{cor}} \tilde{v}_{\mathrm{ice}}^\perp & \\
\qquad - g \nabla_{\mathrm{H}} H + \frac{1}{m_{\mathrm{ice}}} \tau_{\mathrm{ice}} + f_{\mathrm{ice}}, & \text{in } \mathbb{R} \times \Omega, \\
\partial_t \tilde{h} - d_{\mathrm{h}} \Delta_{\mathrm{H}} \tilde{h} = S_{\mathrm{h}} - h_* \mathrm{div}_{\mathrm{H}}\, \tilde{v}_{\mathrm{ice}} & \\
\qquad - \mathrm{div}_{\mathrm{H}}\, (\tilde{v}_{\mathrm{ice}} \tilde{h}) + f_{\mathrm{h}}, & \text{in } \mathbb{R} \times \Omega, \\
\partial_t \tilde{a} - d_{\mathrm{a}} \Delta_{\mathrm{H}} \tilde{a} = S_{\mathrm{a}} - a_* \mathrm{div}_{\mathrm{H}}\, \tilde{v}_{\mathrm{ice}} & \\
\qquad - \mathrm{div}_{\mathrm{H}}\, (\tilde{v}_{\mathrm{ice}} \tilde{a}) + f_{\mathrm{a}}, & \text{in } \mathbb{R} \times \Omega, \\
\tilde{v}_{\mathrm{ice}} = 0, \ \partial_\nu \tilde{h} = \partial_\nu \tilde{a} = 0, & \text{on } \mathbb{R} \times \partial\Omega, \\
\tilde{u}(t) = \tilde{u}(t+T), & \text{for } t \in \mathbb{R}.
\end{cases}
$$

We observe that $m_{\text{ice}} = \rho_{\text{ice}}(\tilde{h} + h_*)$, and the terms in σ_δ and τ_{ice} also depend on $v_{\text{ice}} = \tilde{v}_{\text{ice}}$, $h = \tilde{h} + h_*$ and $a = \tilde{a} + a_*$.

The next step is to rewrite (7.20) as a time periodic quasilinear evolution equation on a suitable ground space X_0. The ground space and regularity space slightly differ from the ones introduced in (3.18) and (3.19). In fact, in order to obtain invertibility of the Neumann Laplacian operators, we set

$$\begin{aligned} &X_0 = L^q(\Omega)^2 \times L^q_0(\Omega) \times L^q_0(\Omega) \text{ and} \\ &X_1 = W^{2,q}(\Omega)^2 \cap W^{1,q}_0(\Omega)^2 \times W^{2,q}_N(\Omega) \cap L^q_0(\Omega) \times W^{2,q}_N(\Omega) \cap L^q_0(\Omega), \end{aligned}$$

where the subscript $_N$ encodes Neumann boundary conditions, while $L^q_0(\Omega)$ represents the space of functions in $L^q(\Omega)$ with mean value zero as defined in (1.1). In view of the interpolation results Lemma 1.3.5 and Lemma 1.3.6, for $2 - 2/p > 1 + 1/q$, the space $X_\gamma = (X_0, X_1)_{1-1/p,p}$ takes the shape

$$X_\gamma = B^{2-2/p}_{qp,D}(\Omega)^2 \times B^{2-2/p}_{qp,N}(\Omega) \cap L^q_0(\Omega) \times B^{2-2/p}_{qp,N}(\Omega) \cap L^q_0(\Omega) \hookrightarrow B^{2-2/p}_{qp}(\Omega)^4. \tag{7.21}$$

In the above, the subscript $_D$ indicates the presence of Dirichlet boundary conditions on $\partial\Omega$, whereas $_N$ again represents Neumann boundary conditions. The validity of $2 - 2/p > 1 + 1/q$ is ensured by the assumptions on p, $q \in (1, \infty)$, namely we assume

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}. \tag{7.22}$$

Let us observe that this corresponds precisely to condition (3.37) from Section 3.6, which in turn coincides with condition (3.22) from Section 3.4 in the absence of weights, i. e., in the situation of $\mu = 1$. In view of (7.22), for small $\alpha > 0$, it also follows from the embeddings stated in (1.8) that

$$X_\gamma \hookrightarrow C^{1,\alpha}(\overline{\Omega})^4. \tag{7.23}$$

Similarly as in (3.23), we next define an open set $W \subset B^{2-2/p}_{qp}(\Omega)^4$ in order to avoid degeneration. Let us emphasize that W is not chosen to be a subset of X_γ as the h- and a-component of u_* do not lie in $L^q_0(\Omega)$. Thus, we set

$$W := \left\{ u = (v_{\text{ice}}, h, a) \in B^{2-2/p}_{qp}(\Omega)^4 : h > \kappa \text{ and } a \in (0,1) \right\}, \tag{7.24}$$

where $\kappa > 0$ represents the small parameter from Section 3.2. Next, we verify that $u = \tilde{u} + u_* \in W$ provided $u_* = (0, h_*, a_*) \in W$, and $\tilde{u} = (\tilde{v}_{\text{ice}}, \tilde{h}, \tilde{a})$ is sufficiently small in X_γ or in the maximal regularity space $\mathbb{E}_1$ given by

$$\mathbb{E}_1 := W^{1,p}(0, T; X_0) \cap L^p(0, T; X_1).$$

As $u_* \in W$ and W is open, there is $R_0' > 0$ with $\overline{\mathbb{B}}_{\mathrm{B}_{qp}^{2-2/p}(\Omega)^4}(u_*, R_0') \subset W$. In addition, from Proposition 2.4.11, we recall $\mathbb{E}_1 \hookrightarrow \mathrm{BUC}([0,T];\mathrm{X}_\gamma)$. Thus, the embedding of X_γ into $\mathrm{B}_{qp}^{2-2/p}(\Omega)^4$ as revealed in (7.21) yields

$$\sup_{t\in[0,T]} \|u(t) - u_*\|_{\mathrm{B}_{qp}^{2-2/p}(\Omega)} \le C \cdot \sup_{t\in[0,T]} \|\tilde{u}(t)\|_{\mathrm{X}_\gamma} \le C \cdot \|\tilde{u}\|_{\mathbb{E}_1}. \tag{7.25}$$

Hence, for $R_0 := R_0'/C$ and $\|\tilde{u}\|_{\mathbb{E}_1} \le R_0$, we deduce that $u(t) = \tilde{u}(t) + u_* \in W$ for all $t \in [0,T]$, where we also exploit $\mathrm{B}_{qp}^{2-2/p}(\Omega)^4 \hookrightarrow \mathrm{C}^{1,\alpha}(\overline{\Omega})^4$, following from (1.8) thanks to (7.22). This discussion is summarized below.

Lemma 7.2.1. *Let $p, q \in (1,\infty)$ satisfy (7.22), and consider $u_* \in W$ constant in time for W as in (7.24). Then there is some small $R_0 > 0$ such that if*

(a) $\tilde{u} \in \mathrm{X}_\gamma$ with $\|\tilde{u}\|_{\mathrm{X}_\gamma} \le R_0$, or

(b) $\tilde{u} \in \mathbb{E}_1$ with $\|\tilde{u}\|_{\mathbb{E}_1} \le R_0$,

then $u(t) = \tilde{u}(t) + u_* \in W$ for all $t \in [0,T]$.

Given $u_* = (0, h_*, a_*) \in W$ constant in time and space, and defining $V \subset \mathrm{X}_\gamma$ to be an open neighborhood of zero in X_γ, i. e.,

$$V := \mathbb{B}_{\mathrm{X}_\gamma}(0, R_0), \tag{7.26}$$

Lemma 7.2.1 especially implies $\tilde{h}(t) + h_* > \kappa$ for all $t \in [0,T]$, and for $\tilde{u} \in V$ and $u_* \in W$ as above. As we will see below, the objects under consideration are well-defined for $\tilde{u} \in V$.

In the following, we take into account some fixed $u_* = (0, h_*, a_*) \in W$ constant in time and space, and for $R_0 > 0$ resulting from Lemma 7.2.1, we consider V as defined in (7.26). Now, we reformulate (7.20) as a time periodic quasilinear evolution equation. For $\tilde{u} \in V$, we recall the L^q-realization of the Hibler operator $A_{\mathrm{D}}^{\mathrm{H}}(\tilde{u} + u_*)$ from (3.17), which is well-defined by Lemma 7.2.1 and the embedding from (7.23). Also, $\Delta_{\mathrm{N,m}}$ represents the $\mathrm{L}_0^q(\Omega)$-realization of the Laplacian operator subject to Neumann boundary conditions as introduced in (2.17). Writing $P(\tilde{u} + u_*) = P(\tilde{h} + h_*, \tilde{a} + a_*)$ for simplicity, we introduce the operator matrix $A\colon V \to \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0)$ taking the shape

$$A(\tilde{u}) := \begin{pmatrix} -A_{\mathrm{D}}^{\mathrm{H}}(\tilde{u} + u_*) + c_{\mathrm{cor}}(\cdot)^\perp & \frac{\partial_h P(\tilde{u}+u_*)}{2\rho_{\mathrm{ice}}(\tilde{h}+h_*)}\nabla_{\mathrm{H}} & \frac{\partial_a P(\tilde{u}+u_*)}{2\rho_{\mathrm{ice}}(\tilde{h}+h_*)}\nabla_{\mathrm{H}} \\ h_* \mathrm{div}_{\mathrm{H}} & -d_{\mathrm{h}}\Delta_{\mathrm{N,m}} & 0 \\ a_* \mathrm{div}_{\mathrm{H}} & 0 & -d_{\mathrm{a}}\Delta_{\mathrm{N,m}} \end{pmatrix}. \tag{7.27}$$

Moreover, we define $F\colon \mathbb{R} \times V \to \mathrm{X}_0$ corresponding to (7.20) by

$$F(t,\tilde{u}) \coloneqq \begin{pmatrix} -(\tilde{v}_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}})\tilde{v}_{\mathrm{ice}} - g\nabla_{\mathrm{H}} H + \frac{1}{m_{\mathrm{ice}}}(\tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(\tilde{v}_{\mathrm{ice}})) \\ -\mathrm{div}_{\mathrm{H}}\,(\tilde{v}_{\mathrm{ice}}\tilde{h}) + S_{\mathrm{h}} \\ -\mathrm{div}_{\mathrm{H}}\,(\tilde{v}_{\mathrm{ice}}\tilde{a}) + S_{\mathrm{a}} \end{pmatrix} + f(t),$$

where $m_{\mathrm{ice}} = \rho_{\mathrm{ice}}(\tilde{h}+h_*)$ and $f = (f_{\mathrm{ice}}, f_{\mathrm{h}}, f_{\mathrm{a}})^\top$. The above operator matrix A and right-hand side F allow us to rewrite the complete system (7.20) as the time periodic quasilinear abstract Cauchy problem

$$\begin{cases} \tilde{u}'(t) + A(\tilde{u}(t))\tilde{u}(t) = F(t,\tilde{u}(t)), & \text{for } t \in \mathbb{R}, \\ \tilde{u}(t) = \tilde{u}(t+T), & \text{for } t \in \mathbb{R}. \end{cases} \tag{7.28}$$

We now elaborate on the procedure to make the present problem fit into the framework of Section 7.1. We will restrict ourselves to the time period $(0,T)$ and then extend the resulting solution to the whole real line $\mathbb{R}$ by $\tilde{u}(0) = \tilde{u}(T)$. Instead of (7.28), we will thus analyze the problem on the period $(0,T)$. First, we show that the linearization of the operator matrix at zero fits in the framework. To this end, we define

$$A_0 \coloneqq A(0) = \begin{pmatrix} -A_{\mathrm{D}}^{\mathrm{H}}(u_*) + c_{\mathrm{cor}}(\cdot)^\perp & \frac{\partial_h P(u_*)}{2\rho_{\mathrm{ice}} h_*}\nabla_{\mathrm{H}} & \frac{\partial_a P(u_*)}{2\rho_{\mathrm{ice}} h_*}\nabla_{\mathrm{H}} \\ h_*\mathrm{div}_{\mathrm{H}} & -d_{\mathrm{h}}\Delta_{\mathrm{N,m}} & 0 \\ a_*\mathrm{div}_{\mathrm{H}} & 0 & -d_{\mathrm{a}}\Delta_{\mathrm{N,m}} \end{pmatrix}. \tag{7.29}$$

The operator A_0 from (7.29) coincides with the restriction of the total linearization from (3.41) to the closed subspace $\mathrm{L}^q(\Omega)^2 \times \mathrm{L}^q_0(\Omega) \times \mathrm{L}^q_0(\Omega)$. This has been made precise in (3.60) when verifying the global strong well-posedness close to constant equilibria.

Lemma 7.2.2. *Consider $u_* = (0, h_*, a_*)$, with $h_* > \kappa$ and $a_* \in (0,1)$ constant in time and space. Then A_0 from* (7.29) *satisfies Assumption 7.1.3(iii), meaning that $0 \in \rho(A_0)$ and $A_0 \in {}_0\mathcal{MR}_p(\mathrm{X}_0)$.*

Proof. From the proof of Lemma 3.6.5 and the above observation, it follows that $0 \in \rho(A_0)$ thanks to the choice of the ground space.

With regard to the second part of the assertion, we employ the splitting

$$A_0 = A_1 + A_2 \coloneqq \begin{pmatrix} -A_{\mathrm{D}}^{\mathrm{H}}(u_*) & \frac{\partial_h P(u_*)}{2\rho_{\mathrm{ice}} h_*}\nabla_{\mathrm{H}} & \frac{\partial_a P(u_*)}{2\rho_{\mathrm{ice}} h_*}\nabla_{\mathrm{H}} \\ 0 & -d_{\mathrm{h}}\Delta_{\mathrm{N,m}} & 0 \\ 0 & 0 & -d_{\mathrm{a}}\Delta_{\mathrm{N,m}} \end{pmatrix} + \begin{pmatrix} c_{\mathrm{cor}}(\cdot)^\perp & 0 & 0 \\ h_*\mathrm{div}_{\mathrm{H}} & 0 & 0 \\ a_*\mathrm{div}_{\mathrm{H}} & 0 & 0 \end{pmatrix}.$$

Similarly as in the proof of Proposition 3.4.1, using the maximal L^p-regularity of $-\Delta_{\mathrm{N,m}}$ on $\mathrm{L}^q_0(\Omega)$ as asserted in Lemma 2.3.20, and employing a perturbation argument again, we find that there is $\omega_0 \in \mathbb{R}$ such that $A_1 + \omega \in \mathcal{MR}_p(\mathrm{X}_0)$ for all $\omega > \omega_0$.

Next, we use a perturbation argument in order to derive this property for the complete operator A_0. Thus, we consider $\tilde{u} = (\tilde{v}_{\mathrm{ice}}, \tilde{h}, \tilde{a}) \in \mathrm{X}_1$ and get

$$\|c_{\mathrm{cor}} \tilde{v}_{\mathrm{ice}}^{\perp}\|_{\mathrm{L}^q(\Omega)} \le C_1 \cdot \|\tilde{u}\|_{\mathrm{X}_0}$$

for some constant $C_1 > 0$. From the divergence theorem, it follows that

$$\int_\Omega h_* \mathrm{div}_{\mathrm{H}}\, \tilde{v}_{\mathrm{ice}} \, \mathrm{d}x_{\mathrm{H}} = h_* \int_{\partial\Omega} \tilde{v}_{\mathrm{ice}} \cdot \nu \, \mathrm{d}S = 0,$$

because $\tilde{v}_{\mathrm{ice}} = 0$ on $\partial\Omega$. As a result, $h_* \mathrm{div}_{\mathrm{H}}\, \tilde{v}_{\mathrm{ice}} \in \mathrm{L}^q_0(\Omega)$, and for every $\alpha > 0$, we deduce from interpolation and Young's inequality that

$$\begin{aligned} \|h_* \mathrm{div}_{\mathrm{H}}\, \tilde{v}_{\mathrm{ice}}\|_{\mathrm{L}^q_0(\Omega)} &\le C_2 \cdot \|\tilde{v}_{\mathrm{ice}}\|_{\mathrm{W}^{1,q}(\Omega)} \\ &\le C_3 \cdot \|\tilde{v}_{\mathrm{ice}}\|_{\mathrm{L}^q(\Omega)}^{1/2} \cdot \|\tilde{v}_{\mathrm{ice}}\|_{\mathrm{W}^{2,q}(\Omega)}^{1/2} \\ &\le C_4(\alpha) \cdot \|\tilde{v}_{\mathrm{ice}}\|_{\mathrm{L}^q(\Omega)} + \alpha \cdot \|\tilde{v}_{\mathrm{ice}}\|_{\mathrm{W}^{2,q}(\Omega)} \\ &\le C_4(\alpha) \cdot \|\tilde{u}\|_{\mathrm{X}_0} + \alpha \cdot \|\tilde{u}\|_{\mathrm{X}_1}. \end{aligned}$$

Since the term $a_* \mathrm{div}_{\mathrm{H}}\, v_{\mathrm{ice}}$ allows a completely analogous treatment, and we have $\mathrm{X}_1 = \mathrm{D}(A_1 + \omega)$, it follows that

$$\|A_2 \tilde{u}\|_{\mathrm{X}_0} \le C_5(\alpha) \cdot \|\tilde{u}\|_{\mathrm{X}_0} + \alpha \cdot \|(A_1 + \omega)\tilde{u}\|_{\mathrm{X}_0}$$

for all $\alpha > 0$. In other words, A_2 is relatively $(A_1 + \omega)$-bounded with arbitrarily small $(A_1 + \omega)$-bound for all $\omega > \omega_0$. Corollary 2.1.23 then implies the existence of $\omega_1 \ge 0$ such that $A_0 + \omega = A_1 + A_2 + \omega \in {}_0\mathcal{MR}_p(\mathrm{X}_0)$ for all $\omega > \omega_1$. On the other hand, the spectral properties from Lemma 3.6.3, which carry over to the present situation of the restricted operator, and the proof of Lemma 3.6.5 yield $s(-A_0) < 0$. This leads to $0 \in \rho(A_0)$ and thus also $A_0 \in {}_0\mathcal{MR}_p(\mathrm{X}_0)$ by Lemma 2.1.15, completing the proof. □

The next task is the verification of Assumption 7.1.3(i), and this is precisely addressed in the lemma below.

Lemma 7.2.3. *Let $p, q \in (1, \infty)$ be such that* (7.22)*, let $u_* = (0, h_*, a_*)$, where $h_* > \kappa$ and $a_* \in (0, 1)$ are constant in time and space, and consider $R_0 > 0$ as in Lemma 7.2.1, so $u(t) = \tilde{u}(t) + u_* \in W$ for $t \in [0, T]$ holds for all $\tilde{u} \in V$ or $\tilde{u} \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0, R_0)$. Then* *(a) the operators* $A\colon V \to \mathcal{L}(\mathrm{X}_1, \mathrm{X}_0)$ *are closed and linear, and*

(b) for every $R \in (0, R_0)$*, there exists a constant* $L(R) > 0$ *such that*

$$\|(A(\tilde{u}_1(\cdot)) - A(\tilde{u}_2(\cdot)))\tilde{u}(\cdot)\|_{\mathbb{E}_0} \le L(R) \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{E}_1} \cdot \|\tilde{u}\|_{\mathbb{E}_1}$$

for all $\tilde{u}_1$*,* $\tilde{u}_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0, R)$ *and* $\tilde{u} \in \mathbb{E}_1$*.*

In summary, Assumption 7.1.3(i) is fulfilled for A *as introduced in* (7.27)*.*

Proof. In a similar fashion as in the proof of Lemma 7.2.2, we can show that there is $\omega_0 \in \mathbb{R}$ such that $A(\tilde{u}) + \omega \in \mathcal{MR}_p(\mathrm{X}_0)$ for all $\omega > \omega_0$ and $\tilde{u} \in V$, showing the assertion of (a).

The remaining task is to establish the Lipschitz estimate of A. First, we observe that the only nonlinear terms contributing to the Lipschitz estimate come from the momentum equation. We then follow the lines of the proof of Theorem 3.5.2, where we replace u_0 by u_* and observe that $\tilde{u}_i + u_*$, $i = 1, 2$, is contained in a neighborhood of u_* in $\mathrm{B}_{qp}^{2-2/p}(\Omega)^4$, see also (7.25). As in (3.29), we deduce the existence of a constant $C_A > 0$ such that

$$\|(A(\tilde{u}_1) - A(\tilde{u}_2))\tilde{u}\|_{\mathrm{X}_0} \le C_A \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathrm{X}_\gamma} \cdot \|\tilde{u}\|_{\mathrm{X}_1} \tag{7.30}$$

in the present setting. Making use of (7.30), and noting that the embedding of $\mathbb{E}_1$ into $\mathrm{BUC}([0, T]; \mathrm{X}_\gamma)$ from (7.4) is also valid in this context, we find

$$\begin{aligned} \|(A(\tilde{u}_1(\cdot)) - A(\tilde{u}_2(\cdot)))\tilde{u}(\cdot)\|_{\mathbb{E}_0} &= \left(\int_0^T \|(A(\tilde{u}_1(t)) - A(\tilde{u}_2(t)))\tilde{u}(t)\|_{\mathrm{X}_0}^p \,\mathrm{d}t \right)^{1/p} \\ &\le C_A \left(\int_0^T \|\tilde{u}_1(t) - \tilde{u}_2(t)\|_{\mathrm{X}_\gamma}^p \cdot \|\tilde{u}(t)\|_{\mathrm{X}_1}^p \,\mathrm{d}t \right)^{1/p} \\ &\le C_A \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \cdot \|\tilde{u}\|_{\mathbb{E}_1} \\ &\le L(R) \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{E}_1} \cdot \|\tilde{u}\|_{\mathbb{E}_1} \end{aligned}$$

for a constant $L(R) > 0$, finishing the proof. □

With regard to the last aspect in Assumption 7.1.3, the Lipschitz continuity of the right-hand side, we further split it into an autonomous and bilinear part $F_1\colon V \to \mathrm{X}_0$ as well as a remainder part $F_2\colon [0, T] \times V \to \mathrm{X}_0$. These right-hand sides take the precise shapes

$$F_1(\tilde{u}) = \begin{pmatrix} -(\tilde{v}_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}})\tilde{v}_{\mathrm{ice}} \\ -\mathrm{div}_{\mathrm{H}}\,(\tilde{v}_{\mathrm{ice}}\tilde{h}) \\ -\mathrm{div}_{\mathrm{H}}\,(\tilde{v}_{\mathrm{ice}}\tilde{a}) \end{pmatrix}$$

and

$$F_2(t,\tilde{u}) = \begin{pmatrix} -g\nabla_{\mathrm{H}} H + \frac{1}{\rho_{\mathrm{ice}}(\tilde{h}+h_*)}(\tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(\tilde{v}_{\mathrm{ice}})) + f_{\mathrm{ice}}(t) \\ S_{\mathrm{h}} + f_{\mathrm{h}}(t) \\ S_{\mathrm{a}} + f_{\mathrm{a}}(t) \end{pmatrix}.$$

The lemma below discusses the right-hand side F_1.

Lemma 7.2.4. *Consider $p, q \in (1,\infty)$ such that (7.22) and $u_* = (0, h_*, a_*)$, with $h_* > \kappa$ and $a_* \in (0,1)$ constant in time and space, and let $R_0 > 0$ be as in Lemma 7.2.1, so $u(t) = \tilde{u}(t) + u_* \in W$ on $t \in [0,T]$ is valid for all $\tilde{u} \in V$ or $\tilde{u} \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0,R_0)$. Then*

(a) $F_1(\tilde{u}(\cdot)) \in \mathbb{E}_0$ for all $\tilde{u} \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0,R_0)$, and

(b) for every $R \in (0,R_0)$, there is a constant $C_{F_1}(R) = CR > 0$ with

$$\|F_1(\tilde{u}_1(\cdot)) - F_1(\tilde{u}_2(\cdot))\|_{\mathbb{E}_0} \le C_{F_1}(R) \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{E}_1}$$

for all $\tilde{u}_1$, $\tilde{u}_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0,R)$.

Proof. We observe that F_1 is of the shape

$$F_1(\tilde{u}) = G(\tilde{u},\tilde{u}) \coloneqq \begin{pmatrix} -(\tilde{v}_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}})\tilde{v}_{\mathrm{ice}} \\ -\mathrm{div}_{\mathrm{H}}\,(\tilde{v}_{\mathrm{ice}}\tilde{h}) \\ -\mathrm{div}_{\mathrm{H}}\,(\tilde{v}_{\mathrm{ice}}\tilde{a}) \end{pmatrix} = \begin{pmatrix} -(\tilde{v}_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}})\tilde{v}_{\mathrm{ice}} \\ -\mathrm{div}_{\mathrm{H}}\,(\tilde{v}_{\mathrm{ice}})\tilde{h} - \tilde{v}_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}\tilde{h} \\ -\mathrm{div}_{\mathrm{H}}\,(\tilde{v}_{\mathrm{ice}})\tilde{a} - \tilde{v}_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}}\tilde{a} \end{pmatrix},$$

where G is bilinear. Moreover, we introduce the operator B on X_0 defined by

$$B \coloneqq \mathrm{diag}(-\Delta_{\mathrm{D}}, -\Delta_{\mathrm{N,m}}, -\Delta_{\mathrm{N,m}}), \text{ with } \mathrm{D}(B) = \mathrm{X}_1,$$

where we denote by Δ_{D} the Dirichlet Laplacian operator on $\mathrm{L}^q(\Omega)^2$ as presented in (2.15). From Lemma 2.3.19(b) and Lemma 2.3.20, we derive that B satisfies $B \in \mathcal{BIP}(\mathrm{X}_0)$ with $\theta_B = 0$. As introduced previously, X_β represents the complex interpolation space $[\mathrm{X}_0, \mathrm{X}_1]_\beta$ for $\beta \in (0,1)$. Hence, from (7.14), we conclude the embedding

$$\mathbb{E}_1 \hookrightarrow \mathrm{L}^{2p}(0,T;\mathrm{X}_\beta) \tag{7.31}$$

for all $p \in (1,\infty)$ and $2\beta - 1 \le 1 - 1/p$. If we now manage to show the boundedness of $G\colon \mathrm{X}_\beta \times \mathrm{X}_\beta \to \mathrm{X}_0$ for such $p \in (1,\infty)$ and $\beta \in (0,1)$, we can argue as in (7.13) and exploit $\tilde{u}_1, \tilde{u}_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0,R)$ for

$$\begin{aligned}\|F_1(\tilde{u}_1(\cdot)) - F_1(\tilde{u}_2(\cdot))\|_{\mathbb{E}_0} &\le C_1 \cdot \Big(\|\tilde{u}_1\|_{\mathrm{L}^{2p}(0,T;\mathrm{X}_\beta)} + \|\tilde{u}_2\|_{\mathrm{L}^{2p}(0,T;\mathrm{X}_\beta)}\Big) \\ &\quad \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathrm{L}^{2p}(0,T;\mathrm{X}_\beta)} \\ &\le C_2 \cdot (\|\tilde{u}_1\|_{\mathbb{E}_1} + \|\tilde{u}_2\|_{\mathbb{E}_1}) \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{E}_1} \\ &\le 2C_2 R \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{E}_1}.\end{aligned}$$

The last part of the proof thus consists of verifying the boundedness of the map $G\colon \mathrm{X}_\beta \times \mathrm{X}_\beta \to \mathrm{X}_0$ for some $\beta \in (0,1)$ with $2\beta - 1 \le 1 - 1/p$. In view of (7.22), we get by the Sobolev embedding (1.8) that $\mathrm{W}^{1,q}(\Omega) \hookrightarrow \mathrm{L}^\infty(\Omega)$. Thus, Hölder's inequality yields

$$\|(\tilde{v}_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}})\tilde{v}_{\mathrm{ice}}\|_{\mathrm{L}^q(\Omega)} \le \|\tilde{v}_{\mathrm{ice}}\|_{\mathrm{L}^\infty(\Omega)} \cdot \|\tilde{v}_{\mathrm{ice}}\|_{\mathrm{W}^{1,q}(\Omega)} \le C_3 \cdot \|\tilde{v}_{\mathrm{ice}}\|^2_{\mathrm{W}^{1,q}(\Omega)}.$$

The choice $\beta = 1/2$ means that $2\beta - 1 \le 1 - 1/p$ is satisfied for all $p \in (1,\infty)$. As a result, also invoking $\mathrm{X}_\beta \hookrightarrow \mathrm{H}^{2\beta,q}(\Omega)^2 \times \mathrm{H}^{2\beta,q}(\Omega) \times \mathrm{H}^{2\beta,q}(\Omega)$, which in turn follows from Lemma 1.3.5, we conclude for $\beta = 1/2$ that

$$\|(\tilde{v}_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}})\tilde{v}_{\mathrm{ice}}\|_{\mathrm{L}^q(\Omega)} \le C_4 \cdot \|\tilde{u}\|^2_{\mathrm{X}_\beta}. \tag{7.32}$$

From $\tilde{v}_{\mathrm{ice}} = 0$ on $\partial\Omega$ for $\tilde{u} = (\tilde{v}_{\mathrm{ice}}, \tilde{h}, \tilde{a}) \in \mathrm{X}_{1/2}$, we derive

$$\int_\Omega \mathrm{div}_{\mathrm{H}}\,(\tilde{v}_{\mathrm{ice}}\tilde{h})\,\mathrm{d}x_{\mathrm{H}} = \int_{\partial\Omega} \tilde{h}\tilde{v}_{\mathrm{ice}} \cdot \nu \,\mathrm{d}S = 0.$$

This implies $\mathrm{div}_{\mathrm{H}}\,(\tilde{v}_{\mathrm{ice}}\tilde{h}), \mathrm{div}_{\mathrm{H}}\,(\tilde{v}_{\mathrm{ice}}\tilde{a}) \in \mathrm{L}^q_0(\Omega)$. Hence, using the above shape of $G(\tilde{u},\tilde{u})$, making use of the fact that $\mathrm{L}^q_0(\Omega)$ is a closed subspace of $\mathrm{L}^q(\Omega)$, and proceeding in the same way as in (7.32), we obtain the estimates of the terms in the h- and a-component.

In total, the map $G\colon \mathrm{X}_\beta \times \mathrm{X}_\beta \to \mathrm{X}_0$ is bilinear and bounded for $\beta = 1/2$, completing the proof of the Lipschitz estimate. On the other hand, with regard to (7.31), we also get $F_1(\tilde{u}(\cdot)) \in \mathbb{E}_0$ for all $\tilde{u} \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0,R_0)$. □

Finally, we study of the remaining part of the right-hand side F_2. As this term also includes external terms, we make some assumptions below.

Assumption 7.2.5. *Let $p, q \in (1,\infty)$. We suppose that the data have the following properties.*

(a) The surface wind and ocean velocity V_{atm} as well as V_{ocn} satisfy V_{atm}, $V_{\mathrm{ocn}} \in \mathrm{L}^{2p}(0,T;\mathrm{L}^{2q}(\Omega)^2)$, $V_{\mathrm{atm}}(0) = V_{\mathrm{atm}}(T)$ and $V_{\mathrm{ocn}}(0) = V_{\mathrm{ocn}}(T)$.

(b) The sea surface dynamic height H fulfills $\nabla_{\mathrm{H}} H \in \mathrm{L}^p(0,T;\mathrm{L}^q(\Omega)^2)$ and additionally has the property that $H(0) = H(T)$.

(c) With regard to the thermodynamic terms S_{h} as well as S_{a}, it holds that S_{h}, $S_{\mathrm{a}} \in \mathrm{L}^p(0,T;\mathrm{L}^q_0(\Omega))$ with $S_{\mathrm{h}}(0) = S_{\mathrm{h}}(T)$ and $S_{\mathrm{a}}(0) = S_{\mathrm{a}}(T)$.

Moreover, assume the existence of $\delta_1 > 0$ with

$$\begin{aligned}&\|V_{\mathrm{atm}}\|_{\mathrm{L}^{2p}(0,T;\mathrm{L}^{2q}(\Omega))} + \|V_{\mathrm{ocn}}\|_{\mathrm{L}^{2p}(0,T;\mathrm{L}^{2q}(\Omega))}\\ &+ \|g\nabla_{\mathrm{H}} H\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} + \|S_{\mathrm{h}}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q_0(\Omega))} + \|S_{\mathrm{a}}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q_0(\Omega))} < \delta_1.\end{aligned}$$

The lemma on F_2 reads as follows.

Lemma 7.2.6. *Let $p, q \in (1,\infty)$ be such that (7.22), let $u_* = (0, h_*, a_*)$, where $h_* > \kappa$ and $a_* \in (0,1)$ are constant in time and space, and consider $R_0 > 0$ as in Lemma 7.2.1, yielding $u(t) = \tilde{u}(t) + u_* \in V$ for all $t \in [0,T]$ provided $\tilde{u} \in V$ or $\tilde{u} \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0,R_0)$. Moreover, suppose that V_{atm}, V_{ocn}, H, S_{h} and S_{a} satisfy Assumption 7.2.5, and $f \in \mathrm{L}^p(0,T;\mathrm{X}_0)$ holds true. Then*

(a) we have $F_2(\cdot, \tilde{u}(\cdot)) \in \mathbb{E}_0$ for all $\tilde{u} \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0,R_0)$, and

(b) for every $R \in (0, R_0)$, there exists $C_{F_2} = C_{F_2}(\delta_1, R) > 0$ with

$$\|F_2(\cdot, \tilde{u}_1(\cdot)) - F_2(\cdot, \tilde{u}_2(\cdot))\|_{\mathbb{E}_0} \le C_{F_2} \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{E}_1}$$

for all $\tilde{u}_1$, $\tilde{u}_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0,R)$.

Proof. By assumption, $g\nabla_{\mathrm{H}} H \in \mathrm{L}^p(0,T;\mathrm{L}^q(\Omega)^2)$, S_{h}, $S_{\mathrm{a}} \in \mathrm{L}^p(0,T;\mathrm{L}^q_0(\Omega))$ and $f \in \mathbb{E}_0$. Therefore, to prove (a), it remains to verify

$$\frac{1}{\rho_{\mathrm{ice}}(\tilde{h} + h_*)}(\tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(\tilde{v}_{\mathrm{ice}})) \in \mathrm{L}^p(0,T;\mathrm{L}^q(\Omega)^2) \tag{7.33}$$

for all $\tilde{u} = (\tilde{v}_{\mathrm{ice}}, \tilde{h}, \tilde{a}) \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0,R_0)$. By the choice of R_0, we especially have $\tilde{h}(t) + h_* > \kappa$ for all $t \in [0,T]$, so the inverse of $\rho_{\mathrm{ice}}(\tilde{h} + h_*)$ is in particular contained in $\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))$. Concerning τ_{atm}, we make use of Hölder's inequality and the assumption on V_{atm} from Assumption 7.2.5 to get

$$\|\tau_{\mathrm{atm}}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \le C_1 \cdot \|V_{\mathrm{atm}}\|^2_{\mathrm{L}^{2p}(0,T;\mathrm{L}^{2q}(\Omega))} < C_1 \delta_1^2. \tag{7.34}$$

Likewise, additionally invoking Young's inequality, we find that

$$\|\tau_{\mathrm{ocn}}(\tilde{v}_{\mathrm{ice}})\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \le C_2 \cdot \left(\delta_1^2 + \|\tilde{v}_{\mathrm{ice}}\|^2_{\mathrm{L}^{2p}(0,T;\mathrm{L}^{2q}(\Omega))}\right). \tag{7.35}$$

In view of the condition on p and q, we observe that $\mathrm{X}_\gamma \hookrightarrow \mathrm{C}^1(\overline{\Omega})^4$. In conjunction with $\mathbb{E}_1 \hookrightarrow \mathrm{BUC}([0,T];\mathrm{X}_\gamma)$, we thus conclude

$$\|\tilde{v}_{\mathrm{ice}}\|_{\mathrm{L}^{2p}(0,T;\mathrm{L}^{2q}(\Omega))} \le C_3 \cdot \|\tilde{u}\|_{\mathrm{BUC}([0,T];\mathrm{X}_\gamma)} \le C_4 \cdot \|\tilde{u}\|_{\mathbb{E}_1}. \tag{7.36}$$

A concatenation of the above arguments and estimates yields (7.33), thereby showing the first part of the assertion.

We now focus on the Lipschitz estimate. To this end, let $\tilde{u}_1$, $\tilde{u}_2 \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0,R)$. Concerning the term related to τ_{atm}, only the factor in front varies. In fact, the above arguments also imply

$$\mathbb{E}_1 \hookrightarrow \mathrm{BUC}([0,T];\mathrm{X}_\gamma) \hookrightarrow \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega)^4).$$

It then follows from the mean value theorem and $\tilde{h}_i(t) + h_* > \kappa$ on $[0,T]$ that

$$\begin{aligned} \left\| \frac{1}{\rho_{\mathrm{ice}}(\tilde{h}_1 + h_*)} - \frac{1}{\rho_{\mathrm{ice}}(\tilde{h}_2 + h_*)} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} &\le C_5 \cdot \|\tilde{h}_1 - \tilde{h}_2\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \\ &\le C_6 \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{E}_1}. \end{aligned}$$

Combining the previous estimate with (7.34), we find the estimate

$$\left\| \left(\frac{1}{\rho_{\mathrm{ice}}(\tilde{h}_1 + h_*)} - \frac{1}{\rho_{\mathrm{ice}}(\tilde{h}_2 + h_*)} \right) \tau_{\mathrm{atm}} \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \le C_7 \delta_1^2 \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{E}_1}.$$

With regard to the difference in the other thermodynamic term, we can split it into the difference in the factor in front as well as the difference in the second factor. For the difference in the first factor, we proceed as above, this time putting together (7.35) and (7.36), to infer

$$\begin{aligned} &\left\| \left(\frac{1}{\rho_{\mathrm{ice}}(\tilde{h}_1 + h_*)} - \frac{1}{\rho_{\mathrm{ice}}(\tilde{h}_2 + h_*)} \right) \tau_{\mathrm{ocn}}(\tilde{v}_{\mathrm{ice},1}) \right\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \\ &\le C_8 \cdot \left(\|V_{\mathrm{ocn}}\|_{\mathrm{L}^{2p}(0,T;\mathrm{L}^{2q}(\Omega))}^2 + \|\tilde{v}_{\mathrm{ice},1}\|_{\mathrm{L}^{2p}(0,T;\mathrm{L}^{2q}(\Omega))}^2 \right) \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{E}_1} \\ &\le C_9 \cdot \left(\delta_1^2 + \|\tilde{u}_1\|_{\mathbb{E}_1}^2 \right) \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{E}_1} \\ &\le C_{10}(\delta_1^2 + R^2) \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{E}_1}. \end{aligned}$$

The inverse of $\rho_{\mathrm{ice}}(\tilde{h}_2 + h_*)$ in the first factor of the remaining term can be estimated in $\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))$ by virtue of Lemma 7.2.1. Thus, the last task is to consider the difference in τ_{ocn}. By Hölder's inequality, Assumption 7.2.5(a) and (7.36), we get

$$
\begin{aligned}
&\|\tau_{\mathrm{ocn}}(\tilde{v}_{\mathrm{ice},1}) - \tau_{\mathrm{ocn}}(\tilde{v}_{\mathrm{ice},2})\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\Omega))} \\
&\le C_{11} \cdot \Big(\|V_{\mathrm{ocn}} - \tilde{v}_{\mathrm{ice},1}\|_{\mathrm{L}^{2p}(0,T;\mathrm{L}^{2q}(\Omega))} \cdot \|\tilde{v}_{\mathrm{ice},1} - \tilde{v}_{\mathrm{ice},2}\|_{\mathrm{L}^{2p}(0,T;\mathrm{L}^{2q}(\Omega))} \\
&\quad + \|\tilde{v}_{\mathrm{ice},1} - \tilde{v}_{\mathrm{ice},2}\|_{\mathrm{L}^{2p}(0,T;\mathrm{L}^{2q}(\Omega))} \cdot \|V_{\mathrm{ocn}} - \tilde{v}_{\mathrm{ice},2}\|_{\mathrm{L}^{2p}(0,T;\mathrm{L}^{2q}(\Omega))} \Big) \\
&\le C_{12} \cdot \Big(\|V_{\mathrm{ocn}}\|_{\mathrm{L}^{2p}(0,T;\mathrm{L}^{2q}(\Omega))} + \|\tilde{u}_1\|_{\mathbb{E}_1} + \|\tilde{u}_2\|_{\mathbb{E}_1} \Big) \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{E}_1} \\
&\le C_{13}(\delta_1 + R) \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{E}_1}.
\end{aligned}
$$

In total, there is $C_{14} > 0$ so that the difference can be estimated by

$$
C_{14}(\delta_1^2 + R^2 + \delta_1 + R) \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{E}_1},
$$

completing the proof. □

We are now in the position to state and prove the main result of this section.

Theorem 7.2.7. *Consider $p, q \in (1, \infty)$ such that (7.22), let $u_* = (0, h_*, a_*)$, with $h_* > \kappa$ and $a_* \in (0,1)$ constant in time and space, and recall $R_0 > 0$ from Lemma 7.2.1. Moreover, assume that $f = (f_{\mathrm{ice}}, f_{\mathrm{h}}, f_{\mathrm{a}}) \colon \mathbb{R} \to \mathrm{X}_0$ is T-periodic such that $f|_{(0,T)} \in \mathrm{L}^p(0,T;\mathrm{X}_0)$.*

Then there exist $R_1 > 0$ and $\delta_1 > 0$ such that for all $R \in (0, R_1)$, and provided V_{atm}, V_{ocn}, H, S_{h} and S_{a} satisfy Assumption 7.2.5 for $\delta_1 > 0$, there is $\delta_2 > 0$ so that if $\| f|_{(0,T)} \|_{\mathbb{E}_0} < \delta_2$, there exists a strong T-periodic solution $\tilde{u} \colon \mathbb{R} \to \mathrm{X}_0$ to (7.20) with $\tilde{u}|_{(0,T)} \in \overline{\mathbb{B}}_{\mathbb{E}_1}(0,R)$. Additionally, $\tilde{u}$ is unique in $\overline{\mathbb{B}}_{\mathbb{E}_1}(0,R)$.

Proof. As we have already indicated, the proof relies on an application of Corollary 7.1.4. Let us recall from Lemma 7.2.2 that A_0 from (7.29) lies within the scope of Assumption 7.1.3(iii). On the other hand, Lemma 7.2.3 yields the validity of Assumption 7.1.3(i) for the family of operator matrices A from (7.27). A combination of Lemma 7.2.4 and Lemma 7.2.6 further yields that the right-hand side F fulfills Assumption 7.1.3(ii). The latter two lemmas also reveal that for $R > 0$ and $\delta_1 > 0$ sufficiently small, the Lipschitz constant $C_F = C_F(R, \delta_1) > 0$ satisfies $C_F < \delta_1'$, where $\delta_1' > 0$ is small enough.

Furthermore, we observe

$$
F(0) = \begin{pmatrix} -g\nabla_{\mathrm{H}} H + \frac{1}{\rho_{\mathrm{ice}} h_*}(\tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(0)) \\ S_{\mathrm{h}}(0) \\ S_{\mathrm{a}}(0) \end{pmatrix} + f.
$$

Hence, possibly choosing $\delta_1 > 0$ even smaller and letting $\delta_2 > 0$ be sufficiently small, we find that $\|F(0)\|_{\mathbb{E}_0} < \delta_2'$ for some small $\delta_2' > 0$ thanks to Assumption 7.2.5 and the assumption on f. The assertion is thus implied by Corollary 7.1.4. □

Finally, we add a few comments on the last result.

Remark 7.2.8. *(a) By construction, for the solution* $\tilde{u} \in \mathbb{E}_1$ *from Theorem 7.2.7, it follows that* $u := \tilde{u} + u_* = (\tilde{v}_{\mathrm{ice}}, \tilde{h}, \tilde{a}) + (0, h_*, a_*)$ *solves the original problem* (7.19).

(b) It also holds that $u \in \mathbb{E}_1$, *and the embedding* $\mathbb{E}_1 \hookrightarrow \mathrm{BUC}([0,T]; \mathrm{X}_\gamma)$ *in conjunction with Lemma 7.2.1 yields that* $u(t) \in W$ *for all* $t \in [0,T]$ *for* W *as in* (7.24), *so it even holds that* $u \in \mathrm{BUC}([0,T]; W)$. *Let us stress that* W *plays the role of* V *from the previous chapter. The introduction of the two different open sets is required here, because elements in the subset* W, *ensuring that* h *and* a *take values in the physically meaningful ranges, can generally not be contained in the adjusted trace space* X_γ *from* (7.21).

CHAPTER 8

Time Periodic Quasilinear Evolution Equations in Real Interpolation Spaces

This final chapter of the thesis is also concerned with the investigation of time periodic quasilinear evolution equations. In contrast to Chapter 7, where the underlying linear result is the Arendt-Bu theorem, this chapter relies on a time periodic version of the Da Prato-Grisvard theorem as exposed in Proposition 2.2.6. In fact, merely assuming sectoriality instead of $\mathcal{R}$-sectoriality of the linearized operator, we obtain maximal periodic L^p-regularity in real interpolation spaces. Another important aspect is that the case $p = 1$ is included. The latter property has also been exploited by Danchin, Hieber, Mucha and Tolksdorf [33] in the context of global existence results for free boundary value problems in the critical space $\mathrm{L}^1(\mathbb{R}_+; \dot{\mathrm{B}}^s_{p1}(\mathbb{R}^n_+))$. In a second step, we apply the general framework to quasilinear Keller-Segel systems and to a Nernst-Planck-Poisson type system of equations in electrochemistry. The results in this chapter have been published together with Matthias Hieber [19].

The precise structure of this chapter is as follows. In Section 8.1, we present the framework to time periodic quasilinear evolution equations in real interpolation spaces, and the main result in that respect is Theorem 8.1.2. Section 8.2 is dedicated to the application to quasilinear Keller-Segel systems, resulting in Theorem 8.2.6. In the final Section 8.3, we apply the general framework to a Nernst-Planck-Poisson type system to obtain the existence of a unique time periodic strong solution in Theorem 8.3.5.

8.1. A Framework by the Da Prato-Grisvard Theorem

This section is devoted to developing a general framework to time periodic quasilinear evolution equations by means of the time periodic version of the Da Prato-Grisvard theorem due to Hieber, Kajiwara, Kress and Tolksdorf as presented in Proposition 2.2.6. By X, we denote a Banach space.

For a fixed time period $T \in (0, \infty)$, we consider

$$\begin{cases} u'(t) + A(u(t))u(t) = F(t, u(t)) + f(t), & \text{for } t \in \mathbb{R}, \\ u(t) = u(t+T), & \text{for } t \in \mathbb{R}. \end{cases} \tag{8.1}$$

We denote by A_0 the linearization of A at zero, i. e., $A_0 := A(0)$. In the sequel, we assume that $A_0 \in \mathcal{PS}(\mathrm{X})$ with spectral angle $\phi_{A_0} < \pi/2$ is densely defined. For $\theta \in (0,1)$ and $p \in [1, \infty)$, let us recall from (2.3) the associated trace space $\mathrm{D}_{A_0}(\theta, p)$ defined by

$$\mathrm{D}_{A_0}(\theta, p) := \left\{ x \in \mathrm{X} : [x]_{\theta,p} := \left(\int_0^\infty \left\| t^{1-\theta} A_0 \mathrm{e}^{-tA_0} x \right\|_{\mathrm{X}}^p \frac{\mathrm{d}t}{t} \right)^{1/p} < \infty \right\}.$$

We also invoke the appropriate *data* and *maximal regularity space* given by

$$\begin{aligned} &\mathbb{E}_{0,\theta} := \mathrm{L}^p(0, T; \mathrm{D}_{A_0}(\theta, p)) \text{ as well as} \\ &\mathbb{E}_{1,\theta} := \left\{ u \in \mathrm{W}^{1,p}(0, T; \mathrm{D}_{A_0}(\theta, p)) : A_0 u \in \mathbb{E}_{0,\theta} \text{ and } u(0) = u(T) \right\}. \end{aligned} \tag{8.2}$$

Moreover, E_1 denotes the domain of the realization of A_0 on $\mathrm{D}_{A_0}(\theta, p)$, so

$$\mathrm{E}_1 := \{ u \in \mathrm{D}(A_0) : A_0 u \in \mathrm{D}_{A_0}(\theta, p) \}. \tag{8.3}$$

In that respect, E_γ represents the emerging trace space in this situation, i. e.,

$$\mathrm{E}_\gamma := (\mathrm{D}_{A_0}(\theta, p), \mathrm{E}_1)_{1-1/p,p}, \text{ for } p \in (1, \infty), \tag{8.4}$$

while for $p = 1$, we consider $\mathrm{E}_\gamma := \mathrm{D}_{A_0}(\theta, 1)$.

We now formulate the main assumption of this section. Similarly as in Chapter 7, we suppose that A and F fulfill suitable Lipschitz estimates, and we also impose conditions on the linearized operator A_0.

Assumption 8.1.1. *(a) The operators $A\colon \mathrm{E}_\gamma \to \mathcal{L}(\mathrm{E}_1, \mathrm{D}_{A_0}(\theta, p))$ constitute a family of closed linear operators. Moreover, there exists R_0 such that for all $R \in (0, R_0)$, there is $L(R) > 0$ with*

$$\|(A(u_1(\cdot)) - A(u_2(\cdot)))v(\cdot)\|_{\mathbb{E}_{0,\theta}} \le L(R) \cdot \|u_1 - u_2\|_{\mathbb{E}_{1,\theta}} \cdot \|v\|_{\mathbb{E}_{1,\theta}}$$

for all u_1, $u_2 \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, R)$ and $v \in \mathbb{E}_{1,\theta}$.

(b) There exists R_0 so that for all $R \in (0, R_0)$, the map $F\colon \mathrm{E}_\gamma \to \mathrm{D}_{A_0}(\theta, p)$ satisfies $F(\cdot, u(\cdot)) \in \mathbb{E}_{0,\theta}$ for all $u \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, R)$, and for some $C(R) > 0$, the estimate

$$\|F(\cdot, u_1(\cdot)) - F(\cdot, u_2(\cdot))\|_{\mathbb{E}_{0,\theta}} \leq C(R) \cdot \|u_1 - u_2\|_{\mathbb{E}_{1,\theta}}$$

is valid for all u_1, $u_2 \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, R)$.

(c) We have $A_0 \in \mathcal{S}(\mathrm{X})$ with spectral angle $\phi_{A_0} < \pi/2$ as well as $0 \in \rho(A_0)$.

Under the above assumption, we obtain the following existence and uniqueness result for the time periodic quasilinear abstract Cauchy problem (8.1).

Theorem 8.1.2. *Let $\theta \in (0, 1)$ and $p \in [1, \infty)$, and suppose that Assumption 8.1.1 holds true. Moreover, assume that for some $R \in (0, R_0]$, we have $C(R) < \delta_1$, with $\delta_1 > 0$ sufficiently small. Then there are $r \in (0, R)$ and $\delta_2 = \delta_2(r) > 0$ such that if $\| F(\cdot, 0)|_{(0,T)} \|_{\mathbb{E}_{0,\theta}} + \| f(\cdot)|_{(0,T)} \|_{\mathbb{E}_{0,\theta}} < \delta_2$, then there exists a T-periodic solution $u\colon \mathbb{R} \to \mathrm{D}_{A_0}(\theta, p)$ with $u|_{(0,T)} \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, r)$ to* (8.1). *In addition, u is unique in $\overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, r)$.*

Proof. The proof relies on the periodic version of the Da Prato-Grisvard theorem, Proposition 2.2.6, to handle the linearized system. We rewrite the present problem as a fixed point problem on the time period $(0, T)$ and then extend the solution periodically to the whole real line. Equivalently to finding a unique solution to (8.1), we can establish the existence of a unique fixed point of the linearized problem

$$\begin{cases} u'(t) + A_0 u(t) = A(0)v(t) - A(v(t))v(t) \\ \qquad\qquad\qquad + F(t, v(t)) + f(t), & \text{for } t \in (0, T), \\ u(0) = u(T). \end{cases} \tag{8.5}$$

Let us denote by Φ the solution map to (8.5), so

$$\Phi\colon \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, r) \to \mathbb{E}_{1,\theta}, \text{ with } \Phi(v) := u,$$

and $u \in \mathbb{E}_{1,\theta}$ represents the unique solution for $v \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, r)$.

The first step is to prove that Φ is a well-defined map. This is in fact a consequence of Assumption 8.1.1: Aspect (c) yields that the linearized operator lies within the scope of Proposition 2.2.6. In addition, it follows from (a) and (b) as well as the assumption on f that the right-hand side in (8.5) is contained in the data space for $R > 0$ sufficiently small. Therefore, the existence of a unique solution in the desired space is implied by Proposition 2.2.6.

The remainder of the proof is dedicated to showing that Φ is a self map and contraction, yielding the existence of a unique fixed point to (8.5) by the contraction mapping principle. We denote by $M > 0$ the infimum of all constants $C > 0$ satisfying the resulting maximal regularity type estimate (2.11) from Proposition 2.2.6. Let also $v \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, r)$ for some $r \le R$ with $R \in (0, R_0)$. By (2.11), Assumption 8.1.1(a) and (b) and the estimates of $F(\cdot, 0)$ and $f(\cdot)$ in the assertion, for $v \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, r)$, we conclude

$$\begin{aligned}\|\Phi(v)\|_{\mathbb{E}_{1,\theta}} &\le M \cdot \Big(\|(A(0) - A(v(\cdot)))v(\cdot)\|_{\mathbb{E}_{0,\theta}} + \|F(\cdot, v(\cdot))\|_{\mathbb{E}_{0,\theta}} + \|f(\cdot)\|_{\mathbb{E}_{0,\theta}}\Big) \\ &\le ML(R) \cdot \|v\|^2_{\mathbb{E}_{1,\theta}} + M \cdot \|F(\cdot, v(\cdot)) - F(\cdot, 0)\|_{\mathbb{E}_{0,\theta}} \\ &\quad + M \cdot \Big(\|F(\cdot, 0)\|_{\mathbb{E}_{0,\theta}} + \|f(\cdot)\|_{\mathbb{E}_{0,\theta}}\Big) \\ &\le ML(R) \cdot \|v\|^2_{\mathbb{E}_{1,\theta}} + MC(R) \cdot \|v\|_{\mathbb{E}_{1,\theta}} + M\delta_2 \\ &\le ML(R)r^2 + MC(R)r + M\delta_2.\end{aligned}$$

Therefore, if $C(R) < \delta_1$ with $\delta_1 := 1/4M$, $r \le \min\{1/4ML(R), R/2\}$, $\delta_2 := r/2M$ and $\|F(\cdot, 0)\|_{\mathbb{E}_{0,\theta}} + \|f(\cdot)\|_{\mathbb{E}_{0,\theta}} < \delta_2$, we get $\|\Phi(v)\|_{\mathbb{E}_{1,\theta}} \le r$, so Φ is a self map.

For v_1, $v_2 \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, r)$, we next observe that the difference $u := u_1 - u_2$, with $u_i := \Phi(v_i)$, solves

$$\begin{aligned}u' + A_0 u &= A(0)v_1 - A(v_1)v_1 + F(t, v_1) - A(0)v_2 + A(v_2)v_2 - F(t, v_2) \\ &= (A(0) - A(v_1))(v_1 - v_2) + (A(v_2) - A(v_1))v_2 \\ &\quad + F(t, v_1) - F(t, v_2),\end{aligned}$$

and $u_1 - u_2$ is also T-periodic. Thus, exploiting (2.11) as well as Assumption 8.1.1(a) and (b), we obtain

$$\begin{aligned}&\|\Phi(v_1) - \Phi(v_2)\|_{\mathbb{E}_{1,\theta}} \\ &\le M \cdot \Big(\|A(0) - A(v_1(\cdot))(v_1(\cdot) - v_2(\cdot))\|_{\mathbb{E}_{0,\theta}} \\ &\quad + \|(A(v_2(\cdot)) - A(v_1(\cdot)))v_2(\cdot)\|_{\mathbb{E}_{0,\theta}} + \|F(\cdot, v_1(\cdot)) - F(\cdot, v_2(\cdot))\|_{\mathbb{E}_{0,\theta}}\Big) \\ &\le M\Big(L(R) \cdot \Big(\|v_1\|_{\mathbb{E}_{1,\theta}} + \|v_2\|_{\mathbb{E}_{1,\theta}}\Big) + C(R)\Big) \cdot \|v_1 - v_2\|_{\mathbb{E}_{1,\theta}} \\ &\le M(2rL(R) + C(R)) \cdot \|v_1 - v_2\|_{\mathbb{E}_{1,\theta}}.\end{aligned}$$

Thanks to the above choice of δ_1, r and $\delta_2 = \delta_2(r)$, we get

$$\|\Phi(v_1) - \Phi(v_2)\|_{\mathbb{E}_{1,\theta}} \le \frac{3}{4} \cdot \|v_1 - v_2\|_{\mathbb{E}_{1,\theta}},$$

so Φ is also a contraction mapping. Consequently, there exists a unique fixed point $u \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, r)$ to (8.5). Finally, we extend the solution u to the whole real line thanks to $u(0) = u(T)$. □

In the sequel, we also comment on possible extensions of Theorem 8.1.2 as seen in Chapter 7. Theorem 8.1.2 corresponds to Corollary 7.1.4 from the previous section on the approach to time periodic solutions via the Arendt-Bu theorem. We remark that Theorem 8.1.2 can also be generalized to the situation of solutions close to an equilibrium solution. In other words, an analogue of Theorem 8.1.2 in the spirit of Theorem 7.1.2 can be shown. However, for simplicity of presentation, we do not explicitly state the respective result here.

The last part of this section is dedicated to the semilinear setting. In this case, we consider a densely defined operator $A \in \mathcal{PS}(\mathrm{X})$ with spectral angle $\phi_A < \pi/2$. The trace spaces $\mathrm{D}_A(\theta, p)$ and resulting spaces are defined accordingly. For a T-periodic force $f \colon \mathbb{R} \to \mathrm{D}_A(\theta, p)$ with $f|_{(0,T)} \in \mathbb{E}_{0,\theta}$, the time periodic semilinear abstract Cauchy problem reads as

$$\begin{cases} u'(t) + Au(t) = F(t,u) + f(t), & \text{for } t \in \mathbb{R}, \\ \qquad\qquad u(t) = u(t+T), & \text{for } t \in \mathbb{R}. \end{cases} \tag{8.6}$$

We also adjust the assumptions to the present context.

Assumption 8.1.3. *(a) There is R_0 so that for all $R \in (0, R_0)$ as well as $u \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, R)$, the right-hand side $F \colon \mathrm{E}_\gamma \to \mathrm{D}_A(\theta, p)$ has the property that $F(\cdot, u(\cdot)) \in \mathbb{E}_{0,\theta}$, and there is $C(R) > 0$ with*

$$\|F(\cdot, u_1(\cdot)) - F(\cdot, u_2(\cdot))\|_{\mathbb{E}_{0,\theta}} \le C(R) \cdot \|u_1 - u_2\|_{\mathbb{E}_{1,\theta}}$$

for all u_1, $u_2 \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, R)$.

(b) It is valid that $A \in \mathcal{S}(\mathrm{X})$ with spectral angle $\phi_A < \pi/2$ and $0 \in \rho(A)$.

The result below in the semilinear case follows directly from Theorem 8.1.2.

Corollary 8.1.4. *Consider $\theta \in (0,1)$ and $p \in [1, \infty)$, assume that Assumption 8.1.3 is valid, and suppose that for some $R \in (0, R_0]$, we have $C(R) < \delta_1$, with $\delta_1 > 0$ sufficiently small. Then there exist $r \in (0, R)$ and $\delta_2 = \delta_2(r) > 0$ such that if $\| F(\cdot, 0)|_{(0,T)} \|_{\mathbb{E}_{0,\theta}} + \| f(\cdot)|_{(0,T)} \|_{\mathbb{E}_{0,\theta}} < \delta_2$, there is a T-periodic solution $u \colon \mathbb{R} \to \mathrm{D}_A(\theta, p)$ to (8.6) with $u|_{(0,T)} \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, r)$. Additionally, u is unique in $\overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0, r)$.*

8.2. Application to Keller-Segel Systems

In this section, we apply the framework developed in Section 8.1 to quasilinear Keller-Segel systems. Before introducing these systems, we provide a brief bibliographic overview. The Keller-Segel model is a common model to describe so-called *chemotaxis*, i. e., the direct movements of cells and organisms in response to chemical gradients. It has first been considered by Keller and Segel [76]. Nowadays, there is vast literature on mathematical analysis of this model. We refer here for instance to the articles of Hillen and Painter [70], Kozono and Sugiyama [81] and Bellomo et al. [9], and to the references therein. In particular, we will investigate quasilinear Keller-Segel systems with nonlinear diffusion in the sequel. Global existence and blow-up results for these systems have been obtained by Cieślak and Stinner [26] or Bellomo et al. [9, Section 3.6].

More precisely, we will focus on a version of the quasilinear Keller-Segel system in which the classical cross diffusion term depends linearly on the cell density. This corresponds to the situation in [70, Section 2.5]. The specific case $m < 0$ in the nonlinear diffusion term has been considered in another article of Cieślak and Stinner [27, Section 2.5] in the context of *volume filling* models as introduced by Painter and Hillen [110].

Let now $\Omega \subset \mathbb{R}^d$, $d \geq 2$, denote a bounded domain with boundary of class C^2. Besides, we denote by $n\colon \mathbb{R}\times\Omega \to \mathbb{R}$ the *density of a cell population*, while $c\colon \mathbb{R}\times\Omega \to \mathbb{R}$ represents the *concentration of a chemoattractant*. In the sequel, we consider a *quasilinear Keller-Segel system* given by

$$\begin{cases} \partial_t n - \nabla\cdot((n+1)^m \nabla n) = -\nabla\cdot(n\nabla c), & \text{in } \mathbb{R}\times\Omega, \\ \partial_t c - \Delta c + c - n = 0, & \text{in } \mathbb{R}\times\Omega, \\ \partial_\nu n = \partial_\nu c = 0, & \text{on } \mathbb{R}\times\partial\Omega. \end{cases}$$

We are particularly interested in time periodic solutions to quasilinear Keller-Segel systems, so for a fixed time period $T \in (0,\infty)$, we consider a T-periodic function $f = (f_{\mathrm{n}}, f_{\mathrm{c}})$. The *time periodic quasilinear Keller-Segel system* reads as

$$\begin{cases} \partial_t n - \nabla\cdot((n+1)^m \nabla n) = -\nabla\cdot(n\nabla c) + f_{\mathrm{n}}, & \text{in } \mathbb{R}\times\Omega, \\ \partial_t c - \Delta c + c - n = f_{\mathrm{c}}, & \text{in } \mathbb{R}\times\Omega, \\ \partial_\nu n = \partial_\nu c = 0, & \text{on } \mathbb{R}\times\partial\Omega, \\ n(t) = n(t+T),\ \ c(t) = c(t+T), & \text{in } \mathbb{R}\times\Omega. \end{cases} \tag{8.7}$$

In order to show that (8.7) lies in the scope of Section 8.1, we first reformulate it as a time periodic quasilinear abstract Cauchy problem. One main obstacle is the necessary invertibility of the underlying operator matrix, and that term $-\nabla\cdot(n\nabla c)$ is of second order in c. This motivates the choice of the ground space as

$$\mathrm{X}_0 = \mathrm{L}^q_0(\Omega)\times \mathrm{W}^{1,q}(\Omega),$$

where $\mathrm{L}^q_0(\Omega)$ denotes the space of L^q-functions on Ω with spatial mean value zero as introduced in (1.1). Moreover, let us recall the Neumann Laplacian operator Δ_{N} with domain $\mathrm{D}(\Delta_{\mathrm{N}}) = \mathrm{W}^{2,q}_{\mathrm{N}}(\Omega)$ from (2.16). In addition, we require the L^q_0-realization of the Laplacian operator subject to Neumann boundary conditions $\Delta_{\mathrm{N,m}}$ as introduced in (2.17). The other operator used in the sequel is $(-\Delta_{\mathrm{N}}+1)_1$. It represents the translated negative Neumann Laplacian on $\mathrm{W}^{1,q}(\Omega)$, and its domain is given by

$$\mathrm{D}((-\Delta_{\mathrm{N}}+1)_1) = \mathrm{W}^{3,q}_{\mathrm{N}} \coloneqq \Big\{c\in \mathrm{W}^{3,q}(\Omega) : \partial_\nu c = 0,\ \text{ on } \partial\Omega\Big\}.$$

For n_0 sufficiently smooth, we also take into account the linearized operator

$$\nabla\cdot((n_0+1)^m\nabla),\ \text{ where } \mathrm{D}(\nabla\cdot((n_0+1)^m\nabla)) = \mathrm{W}^{2,q}_{\mathrm{N}}(\Omega)\cap \mathrm{L}^q_0(\Omega).$$

Hence, using $w=(n,c)$ to denote the principle variable, we set

$$A(w) \coloneqq \begin{pmatrix} -\nabla\cdot((n+1)^m\nabla) & 0 \\ -1 & (-\Delta_{\mathrm{N}}+1)_1 \end{pmatrix} \text{ and } F(w)\coloneqq \begin{pmatrix} -\nabla\cdot(n\nabla c) \\ 0\end{pmatrix}.$$

As a result, for $z=(z_1,z_2)^\top$ and $A_0\coloneqq A(0)$, we have

$$A_0 = \begin{pmatrix} -\Delta_{\mathrm{N,m}} & 0 \\ -1 & (-\Delta_{\mathrm{N}}+1)_1 \end{pmatrix},\quad A(w)z = \begin{pmatrix} -\nabla\cdot((n+1)^m\nabla z_1) \\ -z_1 + (-\Delta_{\mathrm{N}}+1)_1 z_2\end{pmatrix}. \tag{8.8}$$

The above objects allow us to rewrite the time periodic problem (8.7) as

$$\begin{cases} w'(t) + A(w(t))w(t) = F(w(t)) + f(t), & \text{for } t\in\mathbb{R}, \\ w(t) = w(t+T), & \text{for } t\in\mathbb{R}. \end{cases} \tag{8.9}$$

In a first step, we verify that A_0 satisfies Assumption 8.1.1(c).

Lemma 8.2.1. *The operator A_0 from* (8.8) *fulfills $A_0\in\mathcal{S}(\mathrm{X}_0)$ with spectral angle $\phi_{A_0}=0$ and $0\in\rho(A_0)$.*

Proof. Let us recall from Lemma 2.3.20 that $0\in\rho(-\Delta_{\mathrm{N,m}})$, while it holds that $0\in\rho(-\Delta_{\mathrm{N}}+1)$. This property carries over to the restriction of $-\Delta_{\mathrm{N}}+1$

to $\mathrm{W}^{1,q}(\Omega)$, so $0 \in \rho((-\Delta_\mathrm{N}+1)_1)$. From the upper triangular structure of A_0 as well as the inclusion $\mathrm{D}(-\Delta_{\mathrm{N,m}}) \subset \mathrm{W}^{1,q}(\Omega)$, we derive that $0 \in \rho(A_0)$.

It remains to show $A_0 \in \mathcal{S}(\mathrm{X}_0)$. To this end, we recall from Lemma 2.3.20 that $-\Delta_{\mathrm{N,m}} \in \mathcal{S}(\mathrm{L}^q_0(\Omega))$ with spectral angle zero. Lemma 2.3.19(b) yields that $-\Delta_\mathrm{N}+1 \in \mathcal{BIP}(\mathrm{L}^q(\Omega))$ with power angle $\theta_{-\Delta_\mathrm{N}+1} = 0$. Upon noting that the domain of the fractional power operator $(-\Delta_\mathrm{N}+1)^{1/2}$ takes the shape

$$\mathrm{D}((-\Delta_\mathrm{N}+1)^{1/2}) \cong \Big[\mathrm{L}^q(\Omega), \mathrm{W}^{2,q}_\mathrm{N}(\Omega)\Big]_{1/2} = \mathrm{H}^{1,q}(\Omega) = \mathrm{W}^{1,q}(\Omega)$$

by Lemma 2.3.4 as well as Lemma 1.3.6, we deduce from Lemma 2.3.9 the validity of $(-\Delta_\mathrm{N}+1)_1 \in \mathcal{BIP}(\mathrm{W}^{1,q}(\Omega))$ with power angle $\theta_{(-\Delta_\mathrm{N}+1)_1} = 0$. In view of (2.14), we infer that $(-\Delta_\mathrm{N}+1)_1 \in \mathcal{S}(\mathrm{W}^{1,q}(\Omega))$ with $\phi_{(-\Delta_\mathrm{N}+1)_1} = 0$. Thanks to the upper triangular structure of A_0 and $\mathrm{D}(-\Delta_{\mathrm{N,m}}) \subset \mathrm{W}^{1,q}(\Omega)$ as above, we deduce from there that $A_0 \in \mathcal{S}(\mathrm{X}_0)$ with spectral angle $\phi_{A_0} = 0$. □

Before addressing the estimates of the nonlinear terms in order to complement Assumption 8.1.1, we first discuss the shape of the trace spaces. In view of the triangular structure of A_0 from (8.8), we will also write

$$\mathrm{D}_{A_0}(\theta,p) = \mathrm{D}^1_{A_0}(\theta,p) \times \mathrm{D}^2_{A_0}(\theta,p) = \mathrm{D}_{-\Delta_{\mathrm{N,m}}}(\theta,p) \times \mathrm{D}_{(-\Delta_\mathrm{N}+1)_1}(\theta,p).$$

The lemma below on the trace space follows from Lemma 2.1.9 as well as the interpolation of the closed subspace $\mathrm{L}^q_0(\Omega)$ and the boundary conditions as made precise in Lemma 1.3.5 and Lemma 1.3.6, respectively. As before, the subscript $_\mathrm{N}$ indicates Neumann boundary conditions.

Lemma 8.2.2. *Let $\theta \in (0,1)$, $p \in [1,\infty)$ and $q \in (1,\infty)$.*

(a) If $\theta < 1/2q$, then we have

$$\mathrm{D}_{A_0}(\theta,p) = \mathrm{D}^1_{A_0}(\theta,p) \times \mathrm{D}^2_{A_0}(\theta,p) = \mathrm{B}^{2\theta}_{qp}(\Omega) \cap \mathrm{L}^q_0(\Omega) \times \mathrm{B}^{2\theta+1}_{qp}(\Omega).$$

(b) For $1/2q < \theta < 1/2 + 1/2q$, it holds that

$$\mathrm{D}_{A_0}(\theta,p) = \mathrm{D}^1_{A_0}(\theta,p) \times \mathrm{D}^2_{A_0}(\theta,p) = \mathrm{B}^{2\theta}_{qp}(\Omega) \cap \mathrm{L}^q_0(\Omega) \times \mathrm{B}^{2\theta+1}_{qp,\mathrm{N}}(\Omega).$$

(c) In the case $1/2 + 1/2q < \theta < 1$, it is valid that

$$\mathrm{D}_{A_0}(\theta,p) = \mathrm{D}^1_{A_0}(\theta,p) \times \mathrm{D}^2_{A_0}(\theta,p) = \mathrm{B}^{2\theta}_{qp,\mathrm{N}}(\Omega) \cap \mathrm{L}^q_0(\Omega) \times \mathrm{B}^{2\theta+1}_{qp,\mathrm{N}}(\Omega).$$

In all the above cases, we have equality of the spaces with equivalent norms.

The last preparatory result with regard to the Lipschitz estimates concerns embeddings of the maximal regularity space $\mathbb{E}_{1,\theta}$ as introduced in (8.2). Similarly as for $\mathrm{D}_{A_0}(\theta, p)$, we write $\mathbb{E}_{1,\theta} = \mathbb{E}^1_{1,\theta} \times \mathbb{E}^2_{1,\theta}$. Since all nonlinear terms concern the first component, we are mainly interested in embeddings of $\mathbb{E}^1_{1,\theta}$.

Lemma 8.2.3. *Let $\theta \in (0,1)$, $p \in (2,\infty)$ and $q \in (1,\infty)$ with $\theta < 1/2 + 1/2q$ as well as $\theta > d/2q$.*

(a) It follows that $\mathbb{E}^1_{1,\theta} \hookrightarrow \mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta+1}_{qp}(\Omega))$.

(b) We have $\mathbb{E}^1_{1,\theta} \hookrightarrow \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))$, and the second component admits the embedding $\mathbb{E}^2_{1,\theta} \hookrightarrow \mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta+1}_{qp}(\Omega)) \hookrightarrow \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))$.

(c) Moreover, if it additionally holds that

$$\frac{1}{p} + \frac{d-1}{2q} < \frac{1}{2}, \tag{8.10}$$

then $\mathbb{E}^1_{1,\theta} \hookrightarrow \mathrm{L}^\infty(0,T;\mathrm{W}^{2,\infty}(\Omega))$.

Proof. First, we observe that $\mathrm{L}^q_0(\Omega)$ is in particular a UMD space as a closed subspace of the UMD space $\mathrm{L}^q(\Omega)$, see Lemma 2.1.19(g). Moreover, we recall from Lemma 2.3.20 that $-\Delta_{\mathrm{N,m}} \in \mathcal{S}(\mathrm{L}^q_0(\Omega))$ with spectral angle $\phi_{-\Delta_{\mathrm{N,m}}} = 0$ and $0 \in \rho(-\Delta_{\mathrm{N,m}})$. We also observe that $-\Delta_{\mathrm{N,m}}$ and the time derivative commute, and Lemma 1.3.4 implies

$$\left[\mathrm{B}^{2\theta}_{qp}(\Omega), \mathrm{B}^{2\theta+2}_{qp}(\Omega)\right]_\alpha = \mathrm{B}^{2\theta+2\alpha}_{qp}(\Omega).$$

The mixed derivative theorem in real interpolation spaces given in Proposition 2.4.10 and the shape of the trace spaces from Lemma 8.2.2 then yield

$$\mathbb{E}^1_{1,\theta} \hookrightarrow \mathrm{H}^{1-\alpha,p}(0,T;\mathrm{B}^{2\theta+2\alpha}_{qp}(\Omega)). \tag{8.11}$$

From (1.8), we deduce $\mathbb{E}^1_{1,\theta} \hookrightarrow \mathrm{H}^{1-\alpha,p}(0,T;\mathrm{B}^{2\theta+2\alpha}_{qp}(\Omega)) \hookrightarrow \mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta+1}_{qp}(\Omega))$ provided $1 - \alpha - 1/p > 0$ as well as $\alpha \geq 1/2$. We can find such $\alpha \in (0,1)$ if $p > 2$, so the assertion of (a) is already implied.

The assumptions on θ especially yield that $2\theta + 1 - d/q > 0$, so the first part of (b) follows from (a) together with the embedding in (1.8). For the second part of (b), we first use Lemma 8.2.2 and the definition of $\mathbb{E}^2_{1,\theta}$ to get

$$\mathbb{E}^2_{1,\theta} \hookrightarrow \mathrm{W}^{1,p}(0,T;\mathrm{B}^{2\theta+1}_{qp}(\Omega)) \hookrightarrow \mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta+1}_{qp}(\Omega)),$$

where the second embedding is implied by the embedding of $\mathrm{W}^{1,p}$ into L^∞ in one dimension. Thus, the second part of (b) can be shown precisely as the first part from there. With regard to (c), we also rely on (8.11) and need that

$$\mathrm{H}^{1-\alpha,p}(0,T;\mathrm{B}^{2\theta+2\alpha}_{qp}(\Omega)) \hookrightarrow \mathrm{L}^\infty(0,T;\mathrm{W}^{2,\infty}(\Omega)).$$

By (1.8), this requires that $1-\alpha-1/p > 0$ and $2\theta+2\alpha > 2+d/q$. Such $\alpha \in (0,1)$ can exist provided $1 + d/2q - \theta < 1 - 1/p$, so $1/p + d/2q < \theta$. Taking into consideration $\theta \in (d/2q, 1/2 + 1/2q)$, we find that such α and θ exist in the situation that p and q satisfy (8.10), showing the claim. □

We are equipped with all the tools to attack the Lipschitz estimates of the nonlinear terms. First, we address the operator matrix A from (8.8). For this purpose, let us recall that E_1 denotes the domain of the realization of A_0 in $\mathrm{D}_{A_0}(\theta,p)$ as made precise in (8.3), while E_γ represents the resulting trace space as introduced in (8.4).

Lemma 8.2.4. *Let $\theta \in (0,1)$, $p \in (2,\infty)$, $q \in (1,\infty)$ fulfill $\theta \in (d/2q, 1/2+1/2q)$ as well as* (8.10). *Then*

(a) the family of operators $A\colon \mathrm{E}_\gamma \to \mathcal{L}(\mathrm{E}_1, \mathrm{D}_{A_0}(\theta,p))$ is a family of closed linear operators, and

(b) there exists $R_0 > 0$ such that for all $R \in (0,R_0)$, there is $L(R) > 0$ with

$$\|(A(w_1(\cdot)) - A(w_2(\cdot)))z(\cdot)\|_{\mathbb{E}_{0,\theta}} \le L(R) \cdot \|w_1 - w_2\|_{\mathbb{E}_{1,\theta}} \cdot \|z\|_{\mathbb{E}_{1,\theta}}$$

for all w_1, $w_2 \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$ and $z \in \mathbb{E}_{1,\theta}$.

Hence, Assumption 8.1.1(a) is satisfied.

Proof. From Lemma 8.2.2 and the present ranges of θ, p and q, we first recall

$$\mathrm{D}_{A_0}(\theta,p) = \mathrm{D}^1_{A_0}(\theta,p) \times \mathrm{D}^2_{A_0}(\theta,p) = \mathrm{B}^{2\theta}_{qp}(\Omega) \cap \mathrm{L}^q_0(\Omega) \times \mathrm{B}^{2\theta+1}_{qp,\mathrm{N}}(\Omega).$$

For $w = (\tilde{n},\tilde{c}) \in \mathrm{E}_\gamma$ and $z \in \mathrm{E}_1$, we thus have to verify that the first component of $A(w)z$ has spatial mean value zero, while the second component must satisfy Neumann boundary conditions. From $z = (n,c)$, it follows that $n \in \mathrm{B}^{2\theta+2}_{qp}(\Omega)$ with $\partial_\nu n = 0$ on $\partial\Omega$, and $c \in \mathrm{B}^{2\theta+3}_{qp}(\Omega)$ with $\partial_\nu c = \partial_\nu \Delta c = 0$ on $\partial\Omega$. By virtue of the divergence theorem, the first component of the operator matrix $A(w)z$ from (8.8) satisfies

$$\int_\Omega \nabla\cdot((\tilde{n}+1)^m \nabla n)\,\mathrm{d}x = \int_{\partial\Omega} (\tilde{n}+1)^m \partial_\nu n \,\mathrm{dS} = 0.$$

The second component of $A(w)z$ reads as $-(n+(\Delta-1)c)$ and thus fulfills a Neumann boundary condition by the above. Hence, $A\colon \mathrm{E}_\gamma \to \mathcal{L}(\mathrm{E}_1, \mathrm{D}_{A_0}(\theta,p))$ is a family of closed linear operators, and (a) is shown.

Concerning the Lipschitz estimate, we start by showing that the diffusion term does not degenerate for $R_0 > 0$ sufficiently small. Thanks to Lemma 8.2.3(b), we deduce the existence of $C_1 > 0$ such that

$$\|w\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} \le C_1 \cdot \|w\|_{\mathbb{E}_{1,\theta}} \le C_1 R$$

for $w = (n,c) \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$. For $R_0 \coloneqq 1/2C_1$ and $R \in (0,R_0)$, we then get

$$\frac{1}{2} \le n+1 \le \frac{3}{2}. \tag{8.12}$$

Furthermore, Lemma 1.3.4 yields $\mathrm{B}^{2\theta}_{qp}(\Omega) = (\mathrm{L}^q(\Omega), \mathrm{W}^{2,q}(\Omega))_{\theta,p}$, which in turn results in $\mathrm{W}^{2,q}(\Omega) \hookrightarrow \mathrm{B}^{2\theta}_{qp}(\Omega)$. We clearly have $\mathrm{W}^{2,\infty}(\Omega) \hookrightarrow \mathrm{W}^{2,q}(\Omega)$ by the boundedness of the domain $\Omega \subset \mathbb{R}^d$. From a direct computation, $n+1>0$ thanks to (8.12) and $n \in \mathrm{L}^\infty(0,T;\mathrm{W}^{2,\infty}(\Omega))$ by Lemma 8.2.3(c), we conclude $m(n+1)^{m-1} \in \mathrm{L}^\infty(0,T;\mathrm{W}^{2,\infty}(\Omega))$. In particular, this leads to

$$\|m(n+1)^{m-1}\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))} < \infty. \tag{8.13}$$

Similarly as above, the first component of $A(w_1)z - A(w_2)z$ has spatial average zero. Together with the shapes of the trace spaces as revealed in Lemma 8.2.2, the Banach algebra structure of the underlying Besov spaces as stated in Lemma 1.3.8 as well as the mean value theorem joint with (8.13), this yields

$$\begin{aligned}
&\|A(w_1(\cdot))z(\cdot) - A(w_2(\cdot))z(\cdot)\|_{\mathbb{E}_{0,\theta}} \\
&\le \|\nabla\cdot((n_1+1)^m\nabla n) - \nabla\cdot((n_2+1)^m\nabla n)\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))} \\
&\le \Big(\|m(n_1+1)^{m-1}(\nabla n_1 - \nabla n_2)\nabla n\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))} \\
&\quad + \|(m(n_1+1)^{m-1} - m(n_2+1)^{m-1})\nabla n_2\cdot\nabla n\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))} \\
&\quad + \|((n_1+1)^m - (n_2+1)^m)\Delta n\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\Big) \\
&\le C_2\cdot\Big(\|\nabla n_1 - \nabla n_2\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\cdot\|\nabla n\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))} \\
&\quad + \|n_1 - n_2\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\cdot\|\nabla n_2\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\cdot\|\nabla n\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))} \\
&\quad + \|n_1 - n_2\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\cdot\|\Delta n\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\Big) \\
&\le C_3\Big(\|n_1 - n_2\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta+1}_{qp}(\Omega))}\cdot\|n\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta+1}_{qp}(\Omega))} \\
&\quad + \|n_1 - n_2\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\cdot\|n_2\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta+1}_{qp}(\Omega))}\cdot\|n\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta+1}_{qp}(\Omega))} \\
&\quad + \|n_1 - n_2\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\cdot\|n\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta+2}_{qp}(\Omega))}\Big)
\end{aligned}$$

for w_1, $w_2 \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$, $w_i = (n_i, c_i)$, and $z = (n,c) \in \mathbb{E}_{1,\theta}$. Invoking the embedding $\mathbb{E}^1_{1,\theta} \hookrightarrow \mathrm{L}^p(0,T;\mathrm{B}^{2\theta+2}_{qp}(\Omega))$, which is a direct consequence of the shape of the maximal regularity space $\mathbb{E}^1_{1,\theta}$, and recalling the embedding of the maximal regularity space from Lemma 8.2.3(a), we conclude

$$\begin{aligned}\|A(w_1(\cdot))z(\cdot) - A(w_2(\cdot))z(\cdot)\|_{\mathbb{E}_{0,\theta}} &\le C_4 \cdot \Big(\|n_1 - n_2\|_{\mathbb{E}^1_{1,\theta}} \cdot \|n\|_{\mathbb{E}^1_{1,\theta}} \\ &\quad + \|n_1 - n_2\|_{\mathbb{E}^1_{1,\theta}} \cdot \|n_2\|_{\mathbb{E}^1_{1,\theta}} \cdot \|n\|_{\mathbb{E}^1_{1,\theta}} \\ &\quad + \|n_1 - n_2\|_{\mathbb{E}^1_{1,\theta}} \cdot \|n\|_{\mathbb{E}^1_{1,\theta}}\Big) \\ &\le C_5(R+1) \cdot \|w_1 - w_2\|_{\mathbb{E}_{1,\theta}} \cdot \|z\|_{\mathbb{E}_{1,\theta}}.\end{aligned}$$

This completes the proof of the lemma. □

With regard to Assumption 8.1.1, we still require an estimate of the term on the right-hand side. This is the topic of the lemma below.

Lemma 8.2.5. *Let $\theta \in (0,1)$, $p \in [1,\infty)$, $q \in (1,\infty)$ with $\theta \in (d/2q, 1/2 + 1/2q)$, or $\theta \in [d/2q, 1/2 + 1/2q)$ in the case $p = 1$. Then for all $R > 0$, it holds that*

(a) $F\colon \mathrm{E}_\gamma \to \mathrm{D}_{A_0}(\theta,p)$ fulfills $F(w(\cdot)) \in \mathbb{E}_{0,\theta}$ for all $w \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$, and

(b) there exists $C_F > 0$ such that for all w_1, $w_2 \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$, we have

$$\|F(w_1(\cdot)) - F(w_2(\cdot))\|_{\mathbb{E}_{0,\theta}} \le C_F R \cdot \|w_1 - w_2\|_{\mathbb{E}_{1,\theta}}.$$

Proof. In order to verify (a), we first deduce from the divergence theorem and the Neumann boundary conditions that

$$\int_\Omega \nabla \cdot (n\nabla c)\,\mathrm{d}x = 0,$$

yielding $\nabla \cdot (n\nabla c) \in \mathrm{L}^q_0(\Omega)$ for $w = (n,c) \in \mathbb{E}_{1,\theta}$. Moreover, we invoke

$$\begin{aligned}&\mathbb{E}^1_{1,\theta} \hookrightarrow \mathrm{L}^p(0,T;\mathrm{B}^{2\theta+2}_{qp}(\Omega)),\ \ \mathbb{E}^2_{1,\theta} \hookrightarrow \mathrm{L}^p(0,T;\mathrm{B}^{2\theta+2}_{qp}(\Omega)) \\ &\mathbb{E}^1_{1,\theta} \hookrightarrow \mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega)) \text{ and } \mathbb{E}^2_{1,\theta} \hookrightarrow \mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta+1}_{qp}(\Omega)),\end{aligned}$$

following directly from the shape of the maximal regularity space as well as the elementary embedding $\mathrm{W}^{1,p} \hookrightarrow \mathrm{L}^\infty$ in one dimension. Additionally recalling the trace spaces from Lemma 8.2.2, and making use of the Leibniz rule as well as the Banach algebra structure of the underlying Besov spaces as seen in Lemma 1.3.8, we find that

$$\begin{aligned}\|F(w(\cdot))\|_{\mathbb{E}_{0,\theta}} &\le \|-\nabla\cdot(n\nabla c)\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\\ &\le \|\nabla n\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\cdot\|\nabla c\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\\ &\quad+\|n\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\cdot\|\Delta c\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\\ &\le C_1\cdot\Big(\|n\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta+1}_{qp}(\Omega))}\cdot\|c\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta+1}_{qp}(\Omega))}\\ &\quad+\|n\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\cdot\|c\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta+2}_{qp}(\Omega))}\Big)\\ &\le C_2\cdot\|n\|_{\mathbb{E}^1_{1,\theta}}\cdot\|c\|_{\mathbb{E}^2_{1,\theta}}\end{aligned}$$

for $w=(n,c)\in\mathbb{E}_{1,\theta}$. In particular, (a) is implied for any $R_0>0$.

Next, we focus on the Lipschitz estimate. Similarly as above, we have

$$\begin{aligned}&\|F(w_1(\cdot))-F(w_2(\cdot))\|_{\mathbb{E}_{0,\theta}}\\ &\le\|\nabla n_1\cdot(\nabla c_1-\nabla c_2)\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}+\|(\nabla n_1-\nabla n_2)\cdot\nabla c_2\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\\ &\quad+\|n_1(\Delta c_1-\Delta c_2)\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}+\|(n_1-n_2)\Delta c_2\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}\\ &\le C_3\cdot\Big(\|n_1\|_{\mathbb{E}^1_{1,\theta}}\cdot\|c_1-c_2\|_{\mathbb{E}^2_{1,\theta}}+\|n_1-n_2\|_{\mathbb{E}^1_{1,\theta}}\cdot\|c_2\|_{\mathbb{E}^2_{1,\theta}}\Big)\\ &\le C_4R\cdot\|w_1-w_2\|_{\mathbb{E}_{1,\theta}}\end{aligned}$$

for w_1, $w_2\in\overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$, as desired. □

We are now in the position to state this section's main theorem on the existence of a unique T-periodic strong solution to (8.7) in a neighborhood of zero for sufficiently small T-periodic external force f.

Theorem 8.2.6. *Let $T>0$, $\theta\in(0,1)$, $p\in(2,\infty)$ and $q\in(1,\infty)$ such that $\theta\in({}^{d}/_{2q},{}^{1}/_{2}+{}^{1}/_{2q})$ and* (8.10)*. Moreover, let $f=(f_{\mathrm{n}},f_{\mathrm{c}})\colon\mathbb{R}\to\mathrm{D}_{A_0}(\theta,p)$ be T-periodic. Then there exist $R>0$ and $\delta=\delta(R)>0$ sufficiently small so that if $\|\,f|_{(0,T)}\|_{\mathbb{E}_{0,\theta}}<\delta$, there is a T-periodic solution w to* (8.9)*, or, equivalently, to* (8.7)*, and it is valid that $w|_{(0,T)}\in\overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$. Furthermore, w is unique in $\overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$.*

Proof. We prove this theorem by applying the general framework from Section 8.1. Concatenating Lemma 8.2.1, Lemma 8.2.4 and Lemma 8.2.5, we conclude the existence of $R_0>0$ such that Assumption 8.1.1 is satisfied. In particular, thanks to the explicit shape of the Lipschitz constant $C(R)=C_FR>0$ of F as unveiled in Lemma 8.2.5, for any $\delta_1>0$, we find $R>0$ with $C(R)<\delta_1$. The proof follows from Theorem 8.1.2 upon observing $F(0)=0$. □

The last part of this section is dedicated to the discussion of possible adjustments and extensions of the above Theorem 8.2.6. In fact, as n and c represent densities and concentrations, it is reasonable to demand n, $c \geq 0$. In the following, we thus consider such solutions, where we drop the assumption on n having mean value zero. As shown below, this non-negativity assumption already implies that $f_{\mathrm{n}} = 0$, and the spatial average of the resulting solution is constant in time.

Lemma 8.2.7. *Consider a non-negative solution $w = (n, c)$ to (8.7) with T-periodic forcing term $f = (f_{\mathrm{n}}, f_{\mathrm{c}}) \colon \mathbb{R} \to \mathrm{D}_{A_0}(\theta, p)$. Then $f_{\mathrm{n}} \equiv 0$, and*

$$V(t) \coloneqq \frac{1}{|\Omega|} \int_\Omega n(t, x) \, \mathrm{d}x$$

is constant in time, so $V(t) \equiv V$ for all $t \in \mathbb{R}$.

Proof. From the non-negativity of (n, c), we already deduce that both components of the solution are non-negative at time zero, so $n(0, \cdot)$, $c(0, \cdot) \geq 0$. The comparison principle yields that f_{n} and f_{c} are also necessarily non-negative in order to ensure non-negativity of the solution. Integrating the resulting first equation in (8.7), using the divergence theorem, and invoking the Neumann boundary conditions of n, we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega n(t, x) \, \mathrm{d}x = \int_\Omega f_{\mathrm{n}}(t, x) \, \mathrm{d}x =: f_{\mathrm{n,avg}}(t),$$

where $f_{\mathrm{n,avg}}$ denotes the spatial average of f_{n}. In particular,

$$V(t) = V(0) + \int_0^s f_{\mathrm{n,avg}}(s) \, \mathrm{d}s$$

by an integration of the preceding identity. The T-periodicity of n leads to

$$\int_0^T f_{\mathrm{n,avg}}(s) \, \mathrm{d}s = 0,$$

that means $f_{\mathrm{n}} \equiv 0$ due to $f_{\mathrm{n}} \geq 0$, so $V(t) = V(0) =: V$ for all $t \in \mathbb{R}$. $\square$

For a non-negative solution (N, C) to (8.7) with $f_{\mathrm{n}} = 0$ and V resulting from Lemma 8.2.7, we set $(n, c) \coloneqq (N - V, C - V)$. Then (n, c) solves

$$\text{(8.14)} \quad \begin{cases} \partial_t n - \nabla \cdot ((n + V + 1)^m \nabla n) = -\nabla \cdot ((n + V) \nabla c), & \text{in } \mathbb{R} \times \Omega, \\ \partial_t c - \Delta c + c - n = f_{\mathrm{c}}, & \text{in } \mathbb{R} \times \Omega, \\ \partial_\nu n = \partial_\nu c = 0, & \text{on } \mathbb{R} \times \partial\Omega, \\ n(t) = n(t + T), \;\; c(t) = c(t + T), & \text{in } \mathbb{R} \times \Omega. \end{cases}$$

Arguing as in the proof of Theorem 8.2.6, and noting that the Lipschitz constant emerging from an analogue of Lemma 8.2.5 depends directly on V and decreases to zero for $R \to 0$ and $V \to 0$, we derive the following result on the existence of a unique time periodic strong solution to (8.14).

Corollary 8.2.8. *Consider $T > 0$, $\theta \in (0,1)$, $p \in (2,\infty)$ and $q \in (1,\infty)$ satisfying $\theta \in (d/2q, 1/2 + 1/2q)$ and (8.10), and let $f = (0, f_c)\colon \mathbb{R} \to \mathrm{D}_{A_0}(\theta,p)$ be T-periodic. Then there are $R > 0$, $\delta = \delta(R) > 0$ and $V_0 > 0$ sufficiently small such that if $\| f|_{(0,T)} \|_{\mathbb{E}_{0,\theta}} < \delta$ and $V < V_0$, there exists a T-periodic solution $w\colon \mathbb{R} \to \mathrm{D}_{A_0}(\theta,p)$ with $w|_{(0,T)} \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$. In addition, w is unique in $\overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$.*

In particular, $(N,C) := (n+V, c+V)$ is a non-negative T-periodic strong solution to (8.7) with $f_n = 0$.

Finally, we briefly address classical *semilinear* Keller-Segel systems. Similar results as Theorem 8.2.6 and Corollary 8.2.8 can be deduced in the situation of time periodically forced semilinear Keller-Segel problems of the shape

$$\begin{cases} \partial_t n - \Delta n = -\nabla \cdot (n \nabla c) + f_n, & \text{in } \mathbb{R} \times \Omega, \\ \partial_t c - \Delta c + c - n = f_c, & \text{in } \mathbb{R} \times \Omega, \\ \partial_\nu n = \partial_\nu c = 0, & \text{on } \mathbb{R} \times \partial\Omega, \\ n(t) = n(t+T), \;\; c(t) = c(t+T), & \text{in } \mathbb{R} \times \Omega. \end{cases} \tag{8.15}$$

The strategy to obtain a result in this context is to use the semilinear framework as presented in Assumption 8.1.3 as well as Corollary 8.1.4. Indeed, equation (8.15) can be rewritten as a time periodic semilinear abstract Cauchy problem in an analogous way as the quasilinear Keller-Segel system. Moreover, Lemma 8.2.1 remains valid and yields Assumption 8.1.3(b), whereas part (a) of the aforementioned assumption follows from Lemma 8.2.5. Hence, we get the existence of a unique T-periodic strong solution in the same space as before provided $\theta \in (0,1)$, $p \in [1,\infty)$ and $q \in (1,\infty)$ fulfill $\theta \in (d/2q, 1/2 + 1/2q)$, or $\theta \in [d/2q, 1/2 + 1/2q)$ in the case $p = 1$, and if the T-periodic external forcing terms are sufficiently small.

8.3. Application to a Nernst-Planck-Poisson type System

The second application of the general framework developed in Section 8.1 concerns a Nernst-Planck-Poisson type system of equations from electrochemistry. For more background information on this system, we refer for instance to the works of Rubinstein [121] or Newman and Thomas-Alyea [109]. With regard to mathematical analysis, we mention the articles of Bothe, Fischer and Saal [13] on the local existence of solutions to electrokinetic flows as well as global existence in two dimenions, Constantin and Ignatova [30] on global existence and stability for large data for ionic electrodiffusion in fluids, or Prüss, Simonett and Wilke [116, Section 5.2], where so-called *critical spaces* of these equations are investigated.

By $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, we denote a bounded domain with C^2-boundary. The model variables are the *concentrations of oppositely charged ions* $u\colon \mathbb{R}\times\Omega \to \mathbb{R}$ and $v\colon \mathbb{R}\times\Omega \to \mathbb{R}$ as well as the *induced electrical potential* $w\colon \mathbb{R}\times\Omega \to \mathbb{R}$. For constants μ_{u}, $\mu_{\mathrm{v}} > 0$, the resulting system reads as

$$\begin{cases} \partial_t u - \mu_{\mathrm{u}}\Delta u = \nabla\cdot(u\nabla w), & \text{in } \mathbb{R}\times\Omega, \\ \partial_t v - \mu_{\mathrm{v}}\Delta v = -\nabla\cdot(v\nabla w), & \text{in } \mathbb{R}\times\Omega, \\ \partial_t w - \Delta w = u - v, & \text{in } \mathbb{R}\times\Omega, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & \text{on } \mathbb{R}\times\partial\Omega. \end{cases}$$

In the following, for convenience of notation, we restrict ourselves to the situation that $\mu_{\mathrm{u}} = \mu_{\mathrm{v}} = 1$. For a given time period $T > 0$, we investigate the situation of time periodic solutions provided the above system of equations is subject to a time periodic force $f = (f_{\mathrm{u}}, f_{\mathrm{v}}, f_{\mathrm{w}})$. The resulting *time periodic Nernst-Planck-Poisson type system* is given by

$$\text{(8.16)} \quad \begin{cases} \partial_t u - \mu_{\mathrm{u}}\Delta u = \nabla\cdot(u\nabla w) + f_{\mathrm{u}}, & \text{in } \mathbb{R}\times\Omega, \\ \partial_t v - \mu_{\mathrm{v}}\Delta v = -\nabla\cdot(v\nabla w) + f_{\mathrm{v}}, & \text{in } \mathbb{R}\times\Omega, \\ \partial_t w - \Delta w - u + v = f_{\mathrm{w}}, & \text{in } \mathbb{R}\times\Omega, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & \text{on } \mathbb{R}\times\partial\Omega, \\ u(t) = u(t+T), \ \ v(t) = v(t+T), \ \ w(t) = w(t+T), & \text{in } \mathbb{R}\times\Omega. \end{cases}$$

Next, we reformulate (8.16) as a time periodic semilinear abstract Cauchy problem. The task is again to ensure invertibility of the underlying operator matrix. Another important aspect is that the nonlinear terms appearing in the equations of u and v are of second order in w. In order to bypass this difficulty, we choose a ground space with different orders of regularity in u and v on the one hand and w on the other hand. In fact, we set

$$\mathrm{X}_0 = \mathrm{L}^q_0(\Omega) \times \mathrm{L}^q_0(\Omega) \times \mathrm{W}^{2,q}_{\mathrm{N}}(\Omega) \cap \mathrm{L}^q_0(\Omega).$$

The subscript $_\mathrm{N}$ indicates Neumann boundary conditions. We also recall from (2.17) that the third component of the ground space is precisely the domain of the L^q_0-realization of the Laplacian operator subject to Neumann boundary conditions. In other words, we choose the third component one order higher in the scale generated by $-\Delta_{\mathrm{N,m}}$ on the Banach space $\mathrm{L}^q_0(\Omega)$. The emerging operator on $\mathrm{W}^{2,q}_{\mathrm{N}}(\Omega) \cap \mathrm{L}^q_0(\Omega)$ is denoted by $(-\Delta_{\mathrm{N,m}})_2$. Its domain is

$$\mathrm{D}((-\Delta_{\mathrm{N,m}})_2) = \Big\{ w \in \mathrm{W}^{4,q}(\Omega) \cap \mathrm{L}^q_0(\Omega) : \partial_\nu w = \partial_\nu \Delta w = 0, \text{ on } \partial\Omega \Big\}.$$

Denoting by $z = (u, v, w)$ the principal variable and additionally recalling the Neumann Laplacian operator $\Delta_{\mathrm{N,m}}$ on $\mathrm{L}^q_0(\Omega)$ from (2.17), we introduce the resulting operator matrix A and right-hand side $F(z)$ taking the shape

$$(8.17) \quad A \coloneqq \begin{pmatrix} -\Delta_{\mathrm{N,m}} & 0 & 0 \\ 0 & -\Delta_{\mathrm{N,m}} & 0 \\ -1 & 1 & (-\Delta_{\mathrm{N,m}})_2 \end{pmatrix} \quad \text{and} \quad F(z) \coloneqq \begin{pmatrix} \nabla \cdot (u \nabla w) \\ -\nabla \cdot (v \nabla w) \\ 0 \end{pmatrix}.$$

For a given T-periodic forcing term $f = (f_\mathrm{u}, f_\mathrm{v}, f_\mathrm{w})$, we can thus rewrite the time periodic Nernst-Planck-Poisson type system (8.16) as a time periodic semilinear abstract Cauchy problem of the form

$$(8.18) \qquad \begin{cases} z'(t) + Az(t) = F(z(t)) + f(t), & \text{for } t \in \mathbb{R}, \\ z(t) = z(t+T), & \text{for } t \in \mathbb{R}. \end{cases}$$

As we will attack (8.18) by applying Corollary 8.1.4, we need to verify that Assumption 8.1.3 is satisfied. For this, we begin with aspect (b) on the sectoriality and invertibility of the linear operator A.

Lemma 8.3.1. *The operator A from* (8.17) *satisfies $A \in \mathcal{S}(\mathrm{X}_0)$ with spectral angle $\phi_A = 0$ and $0 \in \rho(A)$. Thus, A lies in the scope of Assumption 8.1.3(b).*

Proof. First, we recall from Lemma 2.3.20 that $0 \in \rho(-\Delta_{\mathrm{N,m}})$. This also carries over to the restriction of $-\Delta_{\mathrm{N,m}}$ to $\mathrm{D}(-\Delta_{\mathrm{N,m}})$, so $0 \in \rho((-\Delta_{\mathrm{N,m}})_2)$. Hence, the triangular structure of A and u, $v \in \mathrm{D}(-\Delta_{\mathrm{N,m}})$, implying that the term $-u + v$ is contained in the ground space of $(-\Delta_{\mathrm{N,m}})_2$, yield $0 \in \rho(A)$.

From Lemma 2.3.20, we also deduce $-\Delta_{\mathrm{N,m}} \in \mathcal{BIP}(\mathrm{L}^q_0(\Omega)) \subset \mathcal{S}(\mathrm{L}^q_0(\Omega))$ with $\theta_{-\Delta_{\mathrm{N,m}}} = \phi_{-\Delta_{\mathrm{N,m}}} = 0$. We infer from Lemma 2.3.9 on the Banach scales that $(-\Delta_{\mathrm{N,m}})_2 \in \mathcal{BIP}(\mathrm{W}^{2,q}_{\mathrm{N}}(\Omega) \cap \mathrm{L}^q_0(\Omega))$ with $\theta_{(-\Delta_{\mathrm{N,m}})_2} = 0$, so $(-\Delta_{\mathrm{N,m}})_2$ is especially sectorial with spectral angle zero. The upper triangular structure of A together with the above observation leads to $A \in \mathcal{S}(\mathrm{X}_0)$ with $\phi_A = 0$. □

Next, we unveil the shape of the spaces in the present setting. Because of the triangular structure, we also write

$$\begin{aligned}\mathrm{D}_A(\theta,p) &= \mathrm{D}_A^1(\theta,p)\times\mathrm{D}_A^2(\theta,p)\times\mathrm{D}_A^3(\theta,p)\\ &= \mathrm{D}_{-\Delta_{\mathrm{N,m}}}(\theta,p)\times\mathrm{D}_{-\Delta_{\mathrm{N,m}}}(\theta,p)\times\mathrm{D}_{(-\Delta_{\mathrm{N,m}})^2}(\theta,p).\end{aligned}$$

The following lemma discusses the precise shape of the trace spaces and can be obtained in the same way as Lemma 8.2.2. Again, $_\mathrm{N}$ represents Neumann boundary conditions.

Lemma 8.3.2. *Consider $\theta\in(0,1)$, $p\in[1,\infty)$ and $q\in(1,\infty)$.*

(a) For $0<\theta<1/2+1/2q$, we have

$$\begin{aligned}\mathrm{D}_A(\theta,p) &= \mathrm{D}_A^1(\theta,p)\times\mathrm{D}_A^2(\theta,p)\times\mathrm{D}_A^3(\theta,p)\\ &= \mathrm{B}_{qp}^{2\theta}(\Omega)\cap\mathrm{L}_0^q(\Omega)\times\mathrm{B}_{qp}^{2\theta}(\Omega)\cap\mathrm{L}_0^q(\Omega)\times\mathrm{B}_{qp,\mathrm{N}}^{2\theta+2}(\Omega)\cap\mathrm{L}_0^q(\Omega).\end{aligned}$$

(b) In the case $1/2+1/2q<\theta<1$, we have

$$\begin{aligned}\mathrm{D}_A(\theta,p) &= \mathrm{D}_A^1(\theta,p)\times\mathrm{D}_A^2(\theta,p)\times\mathrm{D}_A^3(\theta,p)\\ &= \mathrm{B}_{qp,\mathrm{N}}^{2\theta}(\Omega)\cap\mathrm{L}_0^q(\Omega)\times\mathrm{B}_{qp,\mathrm{N}}^{2\theta}(\Omega)\cap\mathrm{L}_0^q(\Omega)\times\mathrm{D}_A^3(\theta,p),\quad \textit{with}\\ \mathrm{D}_A^3(\theta,p) &= \left\{w\in\mathrm{B}_{qp}^{2\theta+2}(\Omega)\cap\mathrm{L}_0^q(\Omega) : \partial_\nu w=\partial_\nu\Delta w=0,\quad \textit{on } \partial\Omega\right\}.\end{aligned}$$

In both cases, the spaces coincide with equivalent norms.

As for the trace spaces, we use $\mathbb{E}_{1,\theta}=\mathbb{E}_{1,\theta}^1\times\mathbb{E}_{1,\theta}^2\times\mathbb{E}_{1,\theta}^3$ to denote the three components of the maximal regularity space. The embeddings below follow directly from the definition of the maximal regularity space together with the shape of the trace spaces as revealed in Lemma 8.3.2 as well as the embedding $\mathrm{W}^{1,p}\hookrightarrow\mathrm{L}^\infty$ in one spatial dimension.

Lemma 8.3.3. *Let $\theta\in(0,1)$, $p\in[1,\infty)$ and $q\in(1,\infty)$ with $\theta\in(0,1/2+1/2q)$.*

(a) Then we obtain the embeddings $\mathbb{E}_{1,\theta}^1$, $\mathbb{E}_{1,\theta}^2\hookrightarrow\mathrm{L}^p(0,T;\mathrm{B}_{qp}^{2\theta+2}(\Omega))$ as well as $\mathbb{E}_{1,\theta}^3\hookrightarrow\mathrm{L}^p(0,T;\mathrm{B}_{qp}^{2\theta+4}(\Omega))$, and

(b) it is valid that $\mathbb{E}_{1,\theta}^1$, $\mathbb{E}_{1,\theta}^2\hookrightarrow\mathrm{W}^{1,p}(0,T;\mathrm{B}_{qp}^{2\theta}(\Omega))\hookrightarrow\mathrm{L}^\infty(0,T;\mathrm{B}_{qp}^{2\theta}(\Omega))$ and $\mathbb{E}_{1,\theta}^3\hookrightarrow\mathrm{W}^{1,p}(0,T;\mathrm{B}_{qp}^{2\theta+2}(\Omega))\hookrightarrow\mathrm{L}^\infty(0,T;\mathrm{B}_{qp}^{2\theta+2}(\Omega))$.

The previous lemmas provide a good toolbox for the Lipschitz estimates of the term $F(z)$.

Lemma 8.3.4. *Let* $\theta \in (0,1)$*,* $p \in [1,\infty)$ *as well as* $q \in (1,\infty)$ *be such that* $\theta \in ({}^{d}/_{2q}, {}^{1}/_{2} + {}^{1}/_{2q})$*, or* $\theta \in [{}^{d}/_{2q}, {}^{1}/_{2} + {}^{1}/_{2q})$ *if* $p = 1$*. Then for all* $R > 0$*,*

(a) the map $F\colon \mathrm{E}_\gamma \to \mathrm{D}_A(\theta, p)$ *satisfies* $F(z(\cdot)) \in \mathbb{E}_{0,\theta}$ *if* $z \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$*, and*

(b) there is $C_F > 0$ *so that*

$$\|F(z_1(\cdot)) - F(z_2(\cdot))\|_{\mathbb{E}_{0,\theta}} \le C_F R \cdot \|z_1 - z_2\|_{\mathbb{E}_{1,\theta}}$$

holds for all z_1*,* $z_2 \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$*, so Assumption 8.1.3(a) is fulfilled.*

Proof. The divergence theorem and boundary conditions first imply

$$\int_\Omega \nabla \cdot (u \nabla w)\,\mathrm{d}x = \int_\Omega \nabla \cdot (v \nabla w)\,\mathrm{d}x = 0,$$

so $\nabla \cdot (u\nabla w)$, $\nabla \cdot (v \nabla w) \in \mathrm{L}^q_0(\Omega)$ is valid for $z = (u,v,w) \in \mathbb{E}_{1,\theta}$. On the other hand, for such z, we conclude from the Leibniz rule, the Banach algebra structure of the present Besov spaces as asserted in Lemma 1.3.8 as well as the embeddings from Lemma 8.3.3 that

$$\begin{aligned}
\|\nabla \cdot (u \nabla w)\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))} &\le \|\nabla u\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))} \cdot \|\nabla w\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))} \\
&\quad + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))} \cdot \|\Delta w\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))} \\
&\le C_1 \cdot \Big(\|u\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta+1}_{qp}(\Omega))} \cdot \|w\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta+1}_{qp}(\Omega))} \\
&\quad + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))} \cdot \|w\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta+2}_{qp}(\Omega))} \Big) \\
&\le C_2 \cdot \|u\|_{\mathbb{E}^1_{1,\theta}} \cdot \|w\|_{\mathbb{E}^3_{1,\theta}}.
\end{aligned}$$

Likewise, we estimate $\| - \nabla \cdot (v \nabla w)\|_{\mathrm{L}^p(0,T;\mathrm{B}^{2\theta}_{qp}(\Omega))}$. Thus, for every $R > 0$, we conclude $F(z(\cdot)) \in \mathbb{E}_{1,\theta}$ for $z \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$. In the same way as in the proof of Lemma 8.2.5, we then also obtain the estimate

$$\|F(z_1(\cdot)) - F(z_2(\cdot))\|_{\overline{\mathbb{B}}_{\mathbb{E}_{0,\theta}}} \le C_3 R \cdot \|z_1 - z_2\|_{\overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}}$$

for z_1, $z_2 \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$, completing the proof. □

The theorem below asserts the existence of a unique time periodic strong solution to the Nernst-Planck-Poisson type system (8.16) subject to T-periodic forces. It is a direct consequence of Corollary 8.1.4 in view of Lemma 8.3.1 as well as Lemma 8.3.4, yielding the validity of Assumption 8.1.3, and $F(0) = 0$.

Theorem 8.3.5. *Let* $T > 0$*,* $\theta \in (0,1)$*,* $p \in [1,\infty)$ *and* $q \in (1,\infty)$ *satisfy* $\theta \in (d/2q, 1/2 + 1/2q)$*. In addition, consider a* T*-periodic external forcing term* $f = (f_\mathrm{u}, f_\mathrm{v}, f_\mathrm{w})\colon \mathbb{R} \to \mathrm{D}_A(\theta, p)$ *. Then there are* $R > 0$ *as well as* $\delta = \delta(R) > 0$ *sufficiently small such that if* $\| f|_{(0,T)} \|_{\mathbb{E}_{0,\theta}} < \delta$*, there exists a* T*-periodic solution* $z = (u, v, w)$ *to* (8.18)*, or, equivalently, to* (8.16)*, and we have* $z|_{(0,T)} \in \overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$*. Besides,* z *is unique in* $\overline{\mathbb{B}}_{\mathbb{E}_{1,\theta}}(0,R)$*.*

We find $\theta \in (0,1)$ and $q \in (1,\infty)$ satisfying the assumptions from Theorem 8.3.5 if $q > d/2$ as well as $q > d-1$.

APPENDIX A

Appendix: Properties of the Inoue-Wakimoto Transform

We discuss here several properties of the Inoue-Wakimoto transform as introduced in Section 4.2, and we also refer to this section for the notation. The properties are mainly collected from Section 6 of the article of Geissert, Götze and Hieber [48] on the fluid-structure interaction of Newtonian and generalized Newtonian fluids with a rigid body. Let us observe that the properties of the transform from there also carry over to the present situation even though sea ice is not incompressible.

In the sequel, we use the subscript $i \in \{1, 2\}$ to denote the dependence of objects on the rigid body velocities (ℓ_i, ω_i). In particular, X_i and Y_i represent the diffeomorphisms associated to (ℓ_i, ω_i) and deduced therefrom by the procedure as described in Remark 4.4.4.

The following lemma collects properties and estimates of the transform. For a thorough proof, we refer to [48, Section 6.1].

Lemma A.1.1. *Let* (ℓ_1, ω_1), $(\ell_2, \omega_2) \in \mathrm{W}^{1,p}(0, T)^3$.

(a) It holds that X_i, $Y_i \in \mathrm{C}^1\left(0, T; \mathrm{C}^\infty(\mathbb{R}^2)^2\right)$, *and we get the estimates*

$$\|\partial^\alpha X_i\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} + \|\partial^\alpha Y_i\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \le C$$

as well as

$$\begin{aligned} &\|\partial^\beta (X_1 - X_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} + \|\partial^\beta (Y_1 - Y_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \\ &\le CT \cdot \Big(\|\ell_1 - \ell_2\|_{\mathrm{L}^\infty(0,T)} + \|\omega_1 - \omega_2\|_{\mathrm{L}^\infty(0,T)} \Big) \end{aligned}$$

for all multi-indices α and β with $1 \leq |\alpha| \leq 3$ and $0 \leq |\beta| \leq 3$. In the above, the constants only depend on $K_i := \|\ell_i\|_{\mathrm{L}^\infty(0,T)} + \|\omega_i\|_{\mathrm{L}^\infty(0,T)}$ and not directly on ℓ_i or ω_i. For $i, k, m \in \{1, 2\}$, we especially have

$$\|\partial_k \partial_i Y_m\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \leq CT(R + C_T^*).$$

(b) The original body velocities (ξ_i, Ω_i) derived from (ℓ_i, ω_i) as described in Remark 4.4.4(a) and (b) satisfy

$$\|\xi_1 - \xi_2\|_{\mathrm{L}^\infty(0,T)} \leq C \cdot \left(\|\ell_1 - \ell_2\|_{\mathrm{L}^\infty(0,T)} + \|\omega_1 - \omega_2\|_{\mathrm{L}^\infty(0,T)}\right) \text{ and}$$
$$\|\Omega_1 - \Omega_2\|_{\mathrm{L}^\infty(0,T)} \leq C \cdot \|\omega_1 - \omega_2\|_{\mathrm{L}^\infty(0,T)}.$$

For the matrix Q deduced from Remark 4.4.4(a), and for K_i as introduced in (a), we get

$$\|Q_1 - Q_2\|_{\mathrm{L}^\infty(0,T)} \leq CT \cdot \|M_1 - M_2\|_{\mathrm{L}^\infty(0,T)} \leq CT \cdot \|\omega_1 - \omega_2\|_{\mathrm{L}^\infty(0,T)}$$

and

$$\|Q_i\|_{\mathrm{L}^\infty(0,T)} + \|Q_i^\top\|_{\mathrm{L}^\infty(0,T)} \leq C(1 + TK_i \mathrm{e}^{TK_i}).$$

(c) For all multi-indices β with $0 \leq |\beta| \leq 3$, the term b_i from (4.14) and related to ξ_i and Ω_i admits the estimates

$$\|\partial^\beta b_i\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \leq C \text{ and}$$
$$\|\partial^\beta (b_1 - b_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \leq C \cdot \left(\|\xi_1 - \xi_2\|_{\mathrm{L}^\infty(0,T)} + \|\Omega_1 - \Omega_2\|_{\mathrm{L}^\infty(0,T)}\right).$$

(d) For all multi-indices β with $0 \leq |\beta| \leq 3$, the right-hand side $b^{(Y_i)}$ from (4.16) and emerging from ξ_i and Ω_i satisfies

$$\left\|\partial^\beta (b^{(Y_1)} - b^{(Y_2)})\right\|_{\mathrm{L}^\infty(0,T;\mathrm{C}(\mathbb{R}^2))} \leq C \cdot \|\partial^\beta (b_1 - b_2)\|_{\mathrm{L}^\infty(0,T;\mathrm{C}^1(\mathbb{R}^2))}.$$

In the next step, we estimate the contravariant tensor g^{ij} from (4.21).

Lemma A.1.2. *For all multi-indices $0 \leq |\alpha| \leq 1$, the contravariant metric tensors associated to X, X_1, X_2 and Y, Y_1, Y_2 fulfill the estimates*

$$\|\partial^\alpha g^{ij}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \leq C \text{ and}$$
$$\|\partial^\alpha ((g_1)^{ij} - (g_2)^{ij})\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F}_0))} \leq CT \cdot \|(\ell_1 - \ell_2, \omega_1 - \omega_2)\|_{\mathrm{L}^\infty(0,T)}.$$

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List of notations

Numbers, Sets and Elementary Spaces

$\mathbb{N}, \mathbb{R}, \mathbb{C}, \mathbb{N}_0$	natural, real and complex numbers as well as natural numbers with zero ....
$\mathbb{R}_+, \mathbb{C}_+$	positive real numbers and complex numbers with positive real part, i. e., $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ ....
Σ_θ	sector of angle θ in the complex plane ....
$\operatorname{Re} z, \operatorname{Im} z$	real and imaginary part of a complex number z ....
$\mathbb{B}_{\mathrm{X}}(x_0, r)$	open ball of radius $r > 0$ and center $x_0 \in \mathrm{X}$ in a Banach space X ....
$\overline{\mathbb{B}}_{\mathrm{X}}(x_0, r)$	closed ball of radius $r > 0$ and center $x_0 \in \mathrm{X}$ ....
$\mathcal{L}(\mathrm{X}, \mathrm{Y}), \mathcal{L}(\mathrm{X})$	space of bounded linear operators between two Banach spaces X and Y and on a Banach space X ....
X'	dual space of a Banach space X ....
A'	adjoint of a densely defined operator $A\colon \mathrm{D}(A) \subset \mathrm{X} \to \mathrm{Y}$
$x^\perp$	orthogonal vector $(-x_2, x_1)^\top$ for $x = (x_1, x_2)^\top \in \mathbb{R}^2$ ....
$\mathrm{SO}(d)$	space of special orthogonal matrices on $\mathbb{R}^{d\times d}$ ....
$\operatorname{Cof} A$	cofactor matrix of A ....

Derivatives

$\frac{\mathrm{D}u}{\mathrm{D}t}$	material derivative ....
∂_j	j-th partial derivative ....
D_j	shorthand for $-\mathrm{i}\partial_j$ ....
∂^α	shorthand for $\partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$, where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multiindex ....
D^α	shorthand for $\mathrm{D}_1^{\alpha_1} \dots \mathrm{D}_d^{\alpha_d}$, where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multiindex ....
div_H	horizontal divergence ....
∇_H	horizontal gradient ....

Interpolation Theory

(X, Y)	interpolation couple of Banach spaces X and Y ....
$\mathrm{X} + \mathrm{Y}$	algebraic sum of Banach spaces X and Y ....
$(\mathrm{X}, \mathrm{Y})_{\theta,p}$	real interpolation space of X and Y for $\theta \in (0,1)$ and $p \in [1, \infty]$ ....
$[\mathrm{X}, \mathrm{Y}]_\theta$	complex interpolation space of X and Y for $\theta \in (0,1)$ ..

Function Spaces

C^m	m-times continuously differentiable functions ....
C_b^m	m-times continuously differentiable, bounded functions with bounded derivatives ....
$\mathrm{C}^{m,\alpha}$	m-times continuously differentiable functions with α-Hölder-continuous derivatives up to order m ....
BUC	bounded and uniformly continuous functions ....
BUC^α	bounded and uniformly α-Hölder continuous functions .

$\mathrm{C}_0(\mathbb{R}_+)$	continuous functions on $\mathbb{R}_+$ such that $\lim_{x\to\infty} f(x) = 0$ ....
L^p	Lebesgue spaces for $p \in [1,\infty]$ ....
L^p_0	Lebesgue spaces with average zero ....
$f_{\mathrm{avg}}, f_{\mathrm{m}}$	average and mean value zero part $f_{\mathrm{m}} = f - f_{\mathrm{avg}}$ of a function $f \in \mathrm{L}^q(\Omega)$ for Ω bounded ....
$\mathrm{W}^{m,p}, \mathrm{H}^m$	Sobolev space for $m \in \mathbb{N}_0$ and $p \in [1,\infty]$ as well as for $m \in \mathbb{N}_0$ and $p = 2$ ....
$\mathrm{S}(\mathbb{R}^d)$	Schwartz functions ....
$\mathrm{S}'(\mathbb{R}^d)$	tempered distributions ....
$\mathrm{H}^{s,p}$	Bessel potential space for $p \in (1,\infty)$ and $s \in \mathbb{R}$ ....
B^s_{pq}	Besov space for $p \in (1,\infty)$, $q \in [1,\infty]$ and $s \in \mathbb{R}$ ....
$\mathrm{W}^{s,p}$	Sobolev-Slobodeckij space for $p \in (1,\infty)$ and $s \geq 0$ ....
F^s_{pq}	Triebel-Lizorkin space for $p \in (1,\infty)$, $q \in [1,\infty]$, $s \in \mathbb{R}$.
γ, ∂_ν	trace and normal derivative ....
D, N, per	subscripts indicating Dirichlet, Neumann and periodic boundary conditions ....
$\mathrm{W}^{1,p}_0, \mathrm{H}^1_0$	functions in $\mathrm{W}^{1,p}$ and H^1 with trace zero ....
$\mathrm{L}^p_\mu(J;\mathrm{X})$	weighted Lebesgue space on an interval $J \subset \mathbb{R}_+$ and with values in X ....
$\mathrm{W}^{1,p}_\mu(J;\mathrm{X})$	weighted Sobolev space on an interval $J \subset \mathbb{R}_+$ and with values in X ....
L^q_σ	solenoidal L^q-functions ....
$\mathrm{L}^q_{\overline{\sigma}}$	hydrostatic solenoidal L^q-functions ....

Operators and Semigroups

$\mathrm{D}(A), \mathrm{N}(A), \mathrm{R}(A)$	domain, kernel and range of an operator A ....
X_A	domain of an operator A equipped with the graph norm

$\sigma(A), \rho(A)$	spectrum and resolvent set of an operator A
$R(\lambda, A)$	resolvent of an operator A for $\lambda \in \rho(A)$
$s(A)$	spectral bound of an operator A
e^{At}	semigroup generated by an operator A
$\omega_{\mathrm{sg}}(A)$	growth bound of a semigroup with generator A
$\mathcal{PS}(\mathrm{X})$	class of pseudo-sectorial operators on a Banach space X
$\mathcal{S}(\mathrm{X})$	class of sectorial operators on a Banach space X
ϕ_A	spectral angle of a (pseudo-)sectorial operator A
$\mathrm{D}_A(\theta, p)$	time trace spaces defined for $\theta \in (0,1)$ and $p \in [1, \infty)$

Maximal Regularity and $\mathcal{R}$-Sectoriality

$\mathcal{MR}_p(J; \mathrm{X})$	class of operators with maximal $\mathrm{L}^p(J)$-regularity
$\mathcal{MR}_p(\mathrm{X})$	shorthand for $\mathcal{MR}_p(\mathbb{R}_+; X)$
${}_0\mathcal{MR}_p(\mathrm{X})$	operators with maximal regularity of $\mathrm{L}^p(\mathbb{R}_+)$-type
$\mathcal{R}(\mathcal{T})$	$\mathcal{R}$-bound of an operator family $\mathcal{T}$
Hu	Hilbert transform of u
$\mathcal{RS}(\mathrm{X})$	class of $\mathcal{R}$-sectorial operators on a Banach space X
$\phi_A^{\mathcal{R}}$	$\mathcal{R}$-angle of an $\mathcal{R}$-sectorial operator A
$\mathcal{MR}_{p,\mu}(\mathrm{X})$	class of operators with maximal $\mathrm{L}^p_\mu(\mathbb{R}_+)$-regularity
$\mathcal{MR}_{\mathrm{per},p}(\mathrm{X})$	class of operators with maximal periodic L^p-regularity

Bounded Imaginary Powers and Banach Scales

$\mathcal{BIP}(\mathrm{X})$	class of operators with bounded imaginary powers on a Banach space X

θ_A	power angle of an operator A with bounded imaginary powers . . . .
X_{A^α}	fractional power spaces defined as $\mathrm{D}(A^\alpha)$ endowed with the graph norm . . . .
$[(\mathrm{X}_\alpha, A_\alpha) : \alpha \in I]$	Banach scale for an index set $I \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}_+, \mathbb{R}\}$ and a family of Banach spaces X_α as well as linear isomorphisms $A_\alpha \colon \mathrm{X}_{\alpha+1} \to \mathrm{X}_\alpha$ for $\alpha \in I$ . . . .
$[(\mathrm{X}_j, A_j) : j \in \mathbb{N}]$	power scale generated by (X, A) . . . .
$(\mathrm{X}_\alpha, A_\alpha)$	interpolation-extrapolation scale generated by (X_0, A) .

Bounded $\mathcal{H}^\infty$-Calculus

$\mathrm{H}^\infty(\Sigma_\theta)$	holomorphic and bounded functions on Σ_θ, $\theta \in (0, \pi]$ . .
$\mathcal{H}^\infty(\mathrm{X})$	class of operators with a bounded $\mathcal{H}^\infty$-calculus on a Banach space X . . . .
ϕ_A^∞	$\mathcal{H}^\infty$-angle of an operator A with a bounded $\mathcal{H}^\infty$-calculus

Concrete Operators

Δ	Laplacian operator . . . .
Δ_{D}	Dirichlet Laplacian operator . . . .
Δ_{N}	Neumann Laplacian operator . . . .
Δ_{H}	horizontal Laplacian operator; also used for the L^q-realization on $(0,1) \times (0,1)$ subject to periodic boundary conditions . . . .
$\mathfrak{E}, \mathfrak{R}$	extension and restriction operator . . . .
B_p	derivative operator on $\mathrm{L}^p(\mathbb{R}; \mathrm{X})$ . . . .
$B_{\mathbb{R}_+,p}$	derivative operator on $\mathrm{L}^p(\mathbb{R}_+; \mathrm{X})$ . . . .

$B_{J,p}$	derivative operator on $\mathrm{L}^p(J;\mathrm{X})$ for $J=(0,T)$ ....
$-B_{\mathbb{R}_+,p}$	negative derivative operator on $\mathrm{L}^p(\mathbb{R}_+;\mathrm{X})$ ....
$-B_{J,p}$	negative derivative operator on $\mathrm{L}^p(J;\mathrm{X})$ for $J=(0,T)$.

General Differential Operators

$\mathcal{A}(x,\mathrm{D})$	second-order differential operator ....
$\mathcal{A}_\#(x,\mathrm{D})$	principal part of a second-order differential operator ...
$\mathcal{B}(x,\mathrm{D})$	boundary differential operator associated to a second-order differential operator on domains with boundary ..
$\mathcal{B}_\#(x,\mathrm{D})$	principal part of a boundary differential operator ......
A_B	L^q-realization of $(\mathcal{A},\mathcal{B})$ ....

Quasilinear Parabolic Evolution Equations

X_0	ground space ....
X_1	regularity space with $\mathrm{X}_1 \hookrightarrow \mathrm{X}_0$ densely ....
$\mathrm{X}_{\gamma,\mu}, \mathrm{X}_\gamma$	trace space defined by $(\mathrm{X}_0,\mathrm{X}_1)_{\mu-1/p,p}$ or $(\mathrm{X}_0,\mathrm{X}_1)_{1-1/p,p}$ ..
J	time interval $J=(0,T)$, where $0<T\leq\infty$ ....
V_μ, V	open subset of $\mathrm{X}_{\gamma,\mu}$ or X_γ ....
A	quasilinear operator $A\colon V_\mu \to \mathcal{L}(\mathrm{X}_1,\mathrm{X}_0)$ ....
F	nonlinear term $F\colon V_\mu \to \mathrm{X}_0$ ....
u_0	initial data contained in V_μ ....
$\mathbb{E}_{0,\mu}$	data space $\mathrm{L}^p_\mu(J;\mathrm{X}_0)$ ....
$\mathbb{E}_{1,\mu}$	maximal regularity space $\mathrm{W}^{1,p}_\mu(J;\mathrm{X}_0)\cap\mathrm{L}^p_\mu(J;\mathrm{X}_1)$ ......
$\mathbb{E}_0, \mathbb{E}_1$	shorthand for $\mathbb{E}_{0,1}$ and $\mathbb{E}_{1,1}$, respectively ....

A_0	linearization of the operator A at zero ....

Notation in the Context of the Primitive Equations

$\Omega = G \times (a, b)$	cylindrical domain with $G = (0, 1) \times (0, 1)$ ....
u, v, w	full, horizontal and vertical velocity, so $u = (v, w)$ ......
Γ_a, Γ_b	upper and lower part of the boundary ....
Γ_l	lateral part of the boundary ....
$\Gamma_\mathrm{D}, \Gamma_\mathrm{N}$	Dirichlet and Neumann part of the boundary ....
$\overline{v}$	vertical average of the horizontal velocity ....
$\tilde{v}$	fracturing part $\tilde{v} = v - \overline{v}$ of the horizontal velocity .....
$\mathbb{P}_\mathrm{H}$	two-dimensional Helmholtz projection ....
$\mathbb{P}$	hydrostatic Helmholtz projection $\mathbb{P}v = \mathbb{P}_\mathrm{H}\overline{v} + \tilde{v}$ ....
$A_{\mathrm{b.c.}}$	hydrostatic Stokes operator ....

Notation in the Context of Sea Ice

Ω	bounded domain in $\mathbb{R}^2$ with boundary of class C^2 ......
v_{ice}	horizontal sea ice velocity ....
h	mean ice thickness ....
a	ice compactness ....
u	principle variable (v_{ice}, h, a) ....
κ	parameter indicating the transition to open water ......
$h_\bullet$	threshold for thick ice ....
$\varepsilon = \varepsilon(v_{\mathrm{ice}})$	deformation tensor associated to the horizontal sea ice velocity ....
$p^*, c_\bullet$	positive constants appearing in the ice strength ....

$P = P(h, a)$	ice strength ....
e	ratio of major to minor axes on which the principal components of the stress lie ....
$\zeta = \zeta(\varepsilon, P)$	bulk viscosity ....
$\eta = \eta(\varepsilon, P)$	shear viscosity ....
σ	ice stress tensor ....
δ	regularization parameter ....
$\zeta_\delta, \eta_\delta$	regularized viscosities ....
σ_δ	regularized ice stress tensor ....
ρ_{ice}	sea ice density ....
m_{ice}	sea ice mass ....
c_{cor}	Coriolis parameter ....
g	gravity ....
H	sea surface dynamic height ....
$V_{\mathrm{atm}}, V_{\mathrm{ocn}}$	external surface wind and ocean velocity ....
$C_{\mathrm{atm}}, C_{\mathrm{ocn}}$	air and ocean drag coefficients ....
$\rho_{\mathrm{atm}}, \rho_{\mathrm{ocn}}$	densities for air and sea water ....
$R_{\mathrm{atm}}, R_{\mathrm{ocn}}$	rotation matrices acting on wind and current vectors ..
$\tau_{\mathrm{atm}}, \tau_{\mathrm{ocn}}(v_{\mathrm{ice}})$	atmospheric wind and ocean forces ....
τ_{ice}	shorthand for $\tau_{\mathrm{atm}} + \tau_{\mathrm{ocn}}(v_{\mathrm{ice}})$ ....
$S_{\mathrm{h}}, S_{\mathrm{a}}$	thermodynamic source terms ....
f_{gr}	ice growth rate function ....
$\mathbb{A}^{\mathrm{H}}$	Hibler's operator in differential form ....
$A_{\mathrm{D}}^{\mathrm{H}}$	L^q-realization of Hibler's operator subject to Dirichlet boundary conditions ....

$e^{A_D^H t}$	Hibler semigroup generated by the L^q-realization of the Hibler operator A_D^H on $L^q(\Omega)^2$ ....
u_*	equilibrium solution of the shape $(0, h_*, a_*)$, with $h_* > \kappa$ and $a_* \in (0,1)$ constant in time and space ....
f_{ice}, f_h, f_a	time periodic external forcing terms ....

Notation in the Context of the Interaction Problem of Sea Ice

$\mathcal{O}$	bounded domain in $\mathbb{R}^2$ with boundary of class C^2 ......
$\mathcal{S}(t)$	domain occupied by the rigid body at time t ....
$\mathcal{F}(t)$	domain $\mathcal{O} \setminus \mathcal{S}(t)$ filled with sea ice at time t ....
$\partial\mathcal{S}(t)$	interface of the rigid body with the sea ice at time t ...
$\mathcal{S}_0$	initial domain of the rigid body ....
$\mathcal{F}_0$	initial domain $\mathcal{O} \setminus \mathcal{S}_0$ of the sea ice ....
$\partial\mathcal{S}_0$	interface of the rigid body with the sea ice at time zero
$\overline{v}_{ice}$	horizontal sea ice velocity on the moving domain ......
$\overline{h}$	mean ice thickness on the moving domain ....
$\overline{a}$	ice compactness on the moving domain ....
$\overline{u}$	principle variable $(\overline{v}_{ice}, \overline{h}, \overline{a})$ for the sea ice part on the moving domain ....
$x_c(t)$	center of mass of the rigid body at time t ....
ξ	translational velocity of the rigid body ....
$\beta(t)$	rotation angle of the rigid body at time t ....
Ω	angular velocity of the rigid body ....
$v^{\mathcal{S}}$	velocity the rigid body ....
$Q(t)$	skew symmetric matrix associated to the rotation of the rigid body ....

$\rho_{\mathcal{S}}, m_{\mathcal{S}}$	density of the rigid body, assumed to satisfy $\rho_{\mathcal{S}} \equiv 1$, and resulting mass ....
J	inertia tensor of the rigid body, coinciding with the tensor J_0 at time zero ....
$\overline{F}, \overline{N}$	external force and torque ....
$\overline{\nu}$	unit outward normal to the boundary of $\mathcal{F}(t)$ ....
X, Y	diffeomorphism accounting for the transform to the fixed domain and its inverse ....
v_{ice}	horizontal sea ice velocity on the fixed domain ....
h	mean ice thickness on the fixed domain ....
a	ice compactness on the fixed domain ....
ℓ	translational velocity of the rigid body on the fixed domain ....
ω	angular velocity of the rigid body on the fixed domain .
z	principle variable $(v_{\mathrm{ice}}, h, a, \ell, \omega)$ on the fixed domain ...
F, N	transformed external force and torque ....
$\mathcal{T}_\delta$	transformed regularized stress tensor ....
I	transformed inertia tensor of the rigid body ....
ν	transformed unit outward normal to the boundary of $\mathcal{F}_0$
$\mathcal{N}(v_{\mathrm{ice}})$	transformed term associated to $(\overline{v}_{\mathrm{ice}} \cdot \nabla_{\mathrm{H}})\overline{v}_{\mathrm{ice}}$ ....
$\mathcal{M}(v_{\mathrm{ice}}, h)$	transformed term associated to $\mathrm{div}_{\mathrm{H}}\,(\overline{v}_{\mathrm{ice}}\overline{h})$ ....
$\mathcal{M}(v_{\mathrm{ice}}, a)$	transformed term associated to $\mathrm{div}_{\mathrm{H}}\,(\overline{v}_{\mathrm{ice}}\overline{a})$ ....
g^{ij}	metric contravariant tensor ....
$\mathcal{L}$	transformed (horizontal) Laplacian operator ....
$\mathcal{A}^{\mathrm{H}}$	transformed quasilinear Hibler operator ....
$\mathcal{B}$	transformed term related to $\mathrm{div}_{\mathrm{H}}\, {}^{P(\overline{h},\overline{a})}/_{2}\, \mathrm{Id}_2$ ....

Notation in the Context of the Coupled Model

G	sea ice domain given by $G = (0,1) \times (0,1)$ ....
v_{ice}	horizontal sea ice velocity ....
h	mean ice thickness ....
a	ice compactness ....
κ_1, κ_2	lower and upper bound of the mean ice thickness h ....
h_{atm}	sufficiently large fixed height of the atmosphere ....
h_{ocn}	fixed depth of the ocean ....
Ω_{atm}	domain of the atmosphere given by $G \times (\kappa_2, h_{\text{atm}})$ .....
$u_{\text{atm}}, v_{\text{atm}}, w_{\text{atm}}$	full, horizontal and vertical velocity of the atmospheric wind, i. e., $u_{\text{atm}} = (v_{\text{atm}}, w_{\text{atm}})$ ....
Ω_{ocn}	domain of the ocean taking the shape $G \times (-h_{\text{ocn}}, 0)$ ...
$u_{\text{ocn}}, v_{\text{ocn}}, w_{\text{ocn}}$	full, horizontal and vertical velocity of the ocean, i. e., $u_{\text{ocn}} = (v_{\text{ocn}}, w_{\text{ocn}})$ ....
v	principle variable associated to the coupled atmosphere-sea ice-ocean model, so $v = (v_{\text{atm}}, v_{\text{ocn}}, v_{\text{ice}}, h, a)$ ....
$\pi_{\text{atm}}, \pi_{\text{ocn}}$	pressures variables of the atmosphere and the ocean ...
Γ_u, Γ_b	upper and lower boundary, i. e., $\Gamma_u = G \times \{h_{\text{atm}}\}$ and $\Gamma_b = G \times \{-h_{\text{ocn}}\}$ ....
Γ_i, Γ_o	interfaces of the atmosphere and ocean and with the sea ice, so $\Gamma_i = G \times \{\kappa_2\}$ and $\Gamma_o = G \times \{0\}$ ....
$\Gamma_{l,\text{atm}}, \Gamma_{l,\text{ocn}}$	lateral boundaries given by $\Gamma_{l,\text{atm}} = \partial G \times \{\kappa_2, h_{\text{atm}}\}$ and $\Gamma_{l,\text{ocn}} = \partial G \times \{-h_{\text{ocn}}, 0\}$ ....
μ_{ocn}	viscosity parameter of the ocean ....
$f_{\text{atm}}, f_{\text{ocn}}$	external forcing terms in the context of the atmosphere and ocean equations ....
$\mathbb{P}_{\text{atm}}, \mathbb{P}_{\text{ocn}}$	hydrostatic Helmholtz projection for the atmosphere and the ocean ....

A^{atm}	hydrostatic Stokes operator for the atmosphere ....
$A^{\mathrm{ocn}}_{\mathrm{m}}$	maximal hydrostatic Stokes operator for the ocean .....
A^{ocn}_{0}	hydrostatic Stokes operator for the ocean with homogeneous boundary conditions ....
A^{H}	L^q-realization of Hibler's operator on G subject to periodic boundary conditions ....
L_0	hydrostatic Dirichlet operator ....
N_0	hydrostatic Dirichlet-to-Neumann operator ....

Notation in the Context of Quasilinear Keller-Segel Systems

Ω	bounded domain in $\mathbb{R}^d$, $d \geq 2$, with C^2-boundary ......
n	density of a cell population ....
c	concentration of a chemoattractant ....
$w = (n, c)$	principal variable ....
$f_{\mathrm{n}}, f_{\mathrm{c}}$	time periodic external forcing terms ....

Notation in the Context of the Nernst-Planck-Poisson type System

Ω	bounded domain in $\mathbb{R}^d$, $d \in \mathbb{N}$, with C^2-boundary ......
u, v	concentration of oppositely charged ions ....
w	induced electrical potential ....
$z = (u, v, w)$	principal variable ....

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Curriculum Vitæ

08/06/1997	Born in Darmstadt, Germany
until 06/16	**Highschool**, *Gymnasium Gernsheim*, Gernsheim, Germany, Abitur (1.0 / very good)
10/16 – 10/19	**Bachelor's studies (Mathematics)**, *TU Darmstadt*, Darmstadt, Germany, Bachelor of Science (1.25 / very good) Bachelor's thesis: *Calderón-Zygmund Decomposition and Singular Integral Operators*
09/18 – 02/19	**Exchange semester (Mathematics)**, *EPFL*, Lausanne, Switzerland
10/19 – 05/21	**Master's studies (Mathematics)**, *TU Darmstadt*, Darmstadt, Germany, Master of Science (1.05 / very good with distinction) Master's thesis: *Analysis of the Hibler Sea Ice Model*
since 10/21	**Research assistant**, *TU Darmstadt*, Darmstadt, Germany
03/12/24	**Submission of the doctor's thesis (Dissertation)** *Geophysical Flow Models: An Approach by Quasilinear Evolution Equations* at *TU Darmstadt*, Darmstadt, Germany
05/15/24	**Defense of the doctor's thesis**, overall assessment *summa cum laude*