

## RESEARCH ARTICLE

# Explicit improvements for $L^p$ -estimates related to elliptic systems

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**Abstract**

We give a simple argument to obtain  $L^p$ -boundedness for heat semigroups associated to uniformly strongly elliptic systems on  $\mathbb{R}^d$  by using Stein interpolation between Gaussian estimates and hypercontractivity. Our results give  $p$  explicitly in terms of ellipticity. It is optimal at the endpoint  $p = \infty$ . We also obtain  $L^p$ -estimates for the gradient of the semigroup, where  $p > 2$  depends on ellipticity but not on dimension.

**MSC 2020**

35J47, 47A60 (primary), 46B70 (secondary)

## 1 | INTRODUCTION

In dimension  $d \geq 3$ , we consider uniformly strongly elliptic systems on  $\mathbb{R}^d$  of  $N \geq 1$  equations in divergence form

$$(Lu)^\alpha = -(\operatorname{div}(A\nabla u))^\alpha = -\sum_{i,j=1}^d \sum_{\beta=1}^N \partial_i A_{i,j}^{\alpha,\beta} \partial_j u^\beta \quad (\alpha = 1, \dots, N)$$

with bounded, measurable and complex coefficients, see Section 2 for precise definitions. This gives rise to a contraction semigroup  $(e^{-tL})_{t>0}$  in  $(L^2)^N := L^2(\mathbb{R}^d; \mathbb{C}^N)$ . Surprisingly, little is known about explicit  $L^p$ -estimates when no further regularity on the coefficients is imposed. For systems with minimally smooth coefficients, we refer, for example, to [8]. More precisely, consider the set

$$\mathcal{J}(L) := \{p \in (1, \infty) : e^{-tL} \text{ is bounded in } L^p, \text{ uniformly for } t > 0\}.$$

By complex interpolation, it is an interval around 2, the endpoints of which are often denoted by  $p_{\pm}(L)$ . All of our results will be stable under taking adjoints. As  $p_-(L) = (p_+(L^*))'$ , we shall concentrate on the upper endpoint  $p_+(L)$ . It is known that  $p_+(L) > 2^*$ , where  $2^* := 2d/(d-2)$  is the Sobolev conjugate of 2, and that the improvement  $p_+(L) - 2^*$  can be arbitrarily small even when  $N = 1$  [12, section 2.2]. What seems to be missing though, are explicit lower bounds for  $p_+(L)$  in terms of the data of  $L$ , such as ellipticity constants and dimensions, in particular when the improvement is expected to be large or even covers  $p_+(L) = \infty$ . Indeed, all results for systems (that we are aware of) are perturbative from the general  $L^2$ -theory and provide small, nonquantifiable improvements [2, 3, 19]. In contrast, we proceed by interpolation from the  $L^\infty$ -theory for special systems described further below. Our results are new also for elliptic equations ( $N = 1$ ).

The number  $p_+(L)$  is related to the optimal ranges of various  $L^p$ -estimates for  $L$ , such as Riesz transforms, boundary value problems and functional calculus, see the introduction of [4] for a comprehensive account on the literature. Thus, improving lower bounds for  $p_+(L)$ , as we shall do here, leads to automatic improvements in all these topics.

All of our results are perturbative from the diagonal Laplacian system corresponding to  $A = \mathbf{1}_{(\mathbb{C}^N)^d} := (\delta_{\alpha,\beta} \delta_{i,j})_{i,j}^{\alpha,\beta}$  but not necessarily on a small scale. This is in the nature of things, because every uniformly strongly elliptic  $A$  is an  $L^\infty$ -perturbation of  $\mathbf{1}_{(\mathbb{C}^N)^d}$  of size smaller than 1 up to normalization:

$$d(A) := \min_{t \geq 0} \|\mathbf{1}_{(\mathbb{C}^N)^d} - tA\|_{L^\infty(\mathbb{R}^d; \mathcal{L}((\mathbb{C}^N)^d))} < 1.$$

The “distance”  $d(A)$  is a well-known measure of ellipticity [14]. It can be bounded from above and below in terms of the usual ellipticity constants and when  $A = A^*$ , there is an easy formula (Lemma 2.1). The dimensional constant

$$\delta(d) := \left(1 + \frac{(d-2)^2}{d-1}\right)^{-\frac{1}{2}}$$

will play an important role in this paper. Our main result is as follows.

**Theorem 1.1.** *The following three statements hold.*

(i) *If  $d(A) \geq \delta(d)$ , then (with  $2^*/0 := \infty$ )*

$$p_+(L) \geq \frac{2^*}{1 - \frac{\ln(d(A))}{\ln(\delta(d))}}.$$

(ii) *If  $d(A) < \delta(d)$ , then  $p_+(L) = \infty$ .*

(iii) *The result in (ii) is optimal in the sense that for each  $\varepsilon > 0$  there is some  $A_\varepsilon$  with  $d(A_\varepsilon) \leq \delta(d) + \varepsilon$  and  $p_+(L_\varepsilon) < \infty$ .*

The dimensional constants in Theorem 1.1 are quite large in small dimensions and we collect some values in Figure 1.

Part (ii) is proved in Section 3 by combining results of Koshelev [14], see also [15], with a characterization of Gaussian estimates and Hölder regularity of the kernel associated with  $e^{-tL}$  due to Auscher–Tchamitchian [5]. In fact, in Theorem 3.3 we shall not only prove that  $d(A) < \delta(d)$  implies  $p_+(L) = \infty$ , but that  $e^{-tL}$  has a Hölder regular integral kernel with Gaussian decay. As

$d$	$\delta(d)$	$-\ln(\delta(d))^{-1}$	$\varrho(A)$
3	0.8165	4.9326	0.1010
4	0.6547	2.3604	0.2087
5	0.5547	1.6968	0.2864
6	0.4880	1.3936	0.3441

**FIGURE 1** Approximate constants in Theorem 1.1 in small dimensions. The third column contains the ellipticity ratio  $\varrho(A) = \lambda(A)/\Lambda(A) \in (0, 1]$  that is sufficient for having  $d(A) = \delta(d)$  in the special case  $A = A^*$ , see Lemma 2.1.

$\delta(d) > 1/\sqrt{d}$ , this disproves the conjecture in [5, chapter 1, section 1.4.6] that the best possible perturbation result would be  $d(A) \in O(d^{-1})$ . The optimality statement in (iii) is almost classical, see Proposition 3.5.

Part (i) is proved in Section 4. The idea is to rewrite  $d(A) < 1$  as  $A = \tau(\mathbf{1}_{(\mathbb{C}^N)^d} - B)$ , where  $\|B\|_\infty = d(A)$  and  $\tau > 0$ . We embed  $A$  as  $A_1$  into an analytic family of elliptic matrices given by

$$A_z := \tau(\mathbf{1}_{(\mathbb{C}^N)^d} - zB),$$

where  $r \leq |z| \leq R$  with  $0 < r < 1 < R$ . Then, in the spirit of Stein interpolation, we estimate  $e^{-tL} = e^{-tL_1}$  by using the generic information  $p_+(L_z) \geq 2^*$  on the outer circle  $|z| = R$  and  $p_+(L_z) = \infty$  on the inner circle  $|z| = r$  provided  $r$  is small. This gives a lower bound for  $p_+(L_1)$  that becomes the larger, the closer  $z = 1$  is to the inner circle and the farther away it is from the outer one. Thus, the best bound is achieved when  $r, R$  are the largest possible and the optimal choice for  $r$  comes from (ii). We believe that this simple analytic perturbation argument is of independent interest and has multiple applications to other types of  $L^p$ -estimates for divergence form operators.

Writing Theorem 1.1(i) as  $1/2^* - 1/p_+(L) \geq \ln(d(A))/2^* \ln(\delta(d)) =: \varepsilon(d, d(A))$ , we see that  $\varepsilon(d, d(A)) \rightarrow 0$  as  $d \rightarrow \infty$ . Inspired by Stein’s result [18] on dimensionless bounds for the Riesz transform, we ask whether an improvement can be given independently of  $d$ . To this end, it will be advantageous to consider

$$\mathcal{N}(L) := \left\{ p \in (1, \infty) : \sqrt{t} \nabla e^{-tL} \text{ is bounded in } L^p, \text{ uniformly for } t > 0 \right\},$$

instead of  $\mathcal{J}(L)$ . It is again an interval around 2. The left and right endpoints of  $\mathcal{N}(L)$  are denoted by  $q_\pm(L)$  and it is a fact that  $q_-(L) = p_-(L)$  and  $p_+(L) \geq q_+(L)^*$ , the Sobolev conjugate of  $q_+(L)$  [2, section 3.4]. It follows that the improvement  $q_+(L) - 2$  can be arbitrarily small. In the next result, proved in Section 6, we improve  $q_+(L)$  in terms of  $d(A)$  alone. Writing the conclusion as  $1/2 - 1/q_+(L) \geq \varepsilon(d(A))$  gives the dimensionless improvement  $1/2^* - 1/p_+(L) \geq \varepsilon(d(A))$ .

**Theorem 1.2.** *It holds*

$$q_+(L) \geq \begin{cases} \frac{2}{1 + \frac{\sigma-1}{\sigma^2} \ln(d(A))} & \text{if } \frac{1}{4(\sigma-1)^2} \leq d(A), \\ \frac{1}{2\sqrt{d(A)}} + 1 & \text{if } d(A) \leq \frac{1}{4(\sigma-1)^2}, \end{cases}$$

where  $\sigma \approx 5.69061$  is the unique real solution to

$$\ln(2\sigma - 2) = \frac{\sigma(\sigma - 2)}{2(\sigma - 1)}.$$

For curiosity, let us mention that the first bound in Theorem 1.2 produces a larger improvement for  $p_+(L)$  compared to Theorem 1.1(i) in dimension  $d \geq 922100$ .

It would be interesting to know to what extent our results can be extended to more general domains and boundary conditions. In case of Theorem 1.1, we provide an extension to bounded  $C^1$ -domains with Dirichlet boundary conditions in Section 5.

**Implicit constants.** We write  $X \lesssim_a Y$ , if  $X \leq cY$  for some  $c = c(a) > 0$ .

## 2 | UNIFORMLY STRONGLY ELLIPTIC SYSTEMS

Let  $d \geq 3, N \geq 1$  and  $A : \mathbb{R}^d \rightarrow \mathcal{L}((\mathbb{C}^N)^d)$  be measurable. We assume that  $A$  is “uniformly strongly elliptic”, that is,

$$\lambda(A) := \operatorname{ess\,inf}_{x \in \mathbb{R}^d} \min_{|\xi|=1} \operatorname{Re}(A(x)\xi \cdot \bar{\xi}) > 0 \quad \& \quad \Lambda(A) := \|A\|_{L^\infty(\mathbb{R}^d; \mathcal{L}((\mathbb{C}^N)^d))} < \infty.$$

Let  $L = -\operatorname{div}(A\nabla \cdot)$  be realized as an  $m$ -accretive operator in  $(L^2)^N$  via the sesquilinear form

$$a(u, v) := \int_{\mathbb{R}^d} A\nabla u \cdot \bar{\nabla} v \, dx = \sum_{i,j=1}^d \sum_{\alpha,\beta=1}^N \int_{\mathbb{R}^d} A_{i,j}^{\alpha,\beta} \partial_j u^\beta \overline{\partial_i v^\alpha} \, dx \quad (u, v \in (W^{1,2})^N)$$

and let  $(e^{-tL})_{t \geq 0}$  be the associated contraction semigroup, see [13, chapter 6]. We use the “distance function”,

$$d(A) := \min_{t \geq 0} \| \mathbf{1}_{(\mathbb{C}^N)^d} - tA \|_{L^\infty(\mathbb{R}^d; \mathcal{L}((\mathbb{C}^N)^d))},$$

to measure ellipticity. By compactness, the minimum is attained in some  $t^* \geq 0$ . Let us verify that  $t^* > 0$  and that  $d(A)$  is an appropriate quantity to measure ellipticity.

**Lemma 2.1.** *If  $\varrho(A) := \lambda(A)/\Lambda(A)$  denotes the ellipticity quotient of  $A$ , then*

$$\frac{1 - \varrho(A)}{1 + \varrho(A)} \leq d(A) \leq \sqrt{1 - \varrho(A)^2}.$$

Furthermore, if  $A = A^*$ , then the first inequality becomes an equality.

*Proof.* We have for all  $t \geq 0$ , each normalized  $\xi \in (\mathbb{C}^N)^d$  and almost every  $x \in \mathbb{R}^d$  that

$$|\xi - tA(x)\xi|^2 = 1 - 2t \operatorname{Re}(A(x)\xi | \xi) + t^2 |A(x)\xi|^2 \leq 1 - 2t\lambda(A) + t^2\Lambda(A)^2.$$

We choose  $t := \lambda(A)/\Lambda(A)^2$  to get the upper bound for  $d(A)$ .

Now, fix  $t^* > 0$  such that  $\| \mathbf{1}_{(\mathbb{C}^N)^d} - t^*A \|_\infty = d(A)$ . Then

$$\operatorname{Re}(t^*A(x)\xi | \xi) = 1 - \operatorname{Re}((\mathbf{1}_{(\mathbb{C}^N)^d} - t^*A(x))\xi | \xi) \geq 1 - d(A).$$

Hence,  $t^*\lambda(A) \geq 1 - d(A) > 0$  and by the triangle inequality  $t^*\Lambda(A) \leq 1 + d(A)$ . Rearranging gives the lower bound for  $d(A)$ .

For the second claim we take  $t := 2/(\Lambda(A)+\lambda(A))$ . Then  $\mathbf{1}_{(\mathbb{C}^N)^d} - tA(x)$  is a self-adjoint matrix for almost every  $x \in \mathbb{R}^d$  with eigenvalues contained in

$$[1 - t\Lambda(A), 1 - t\lambda(A)] = \left[ -\frac{\Lambda(A)-\lambda(A)}{\Lambda(A)+\lambda(A)}, \frac{\Lambda(A)-\lambda(A)}{\Lambda(A)+\lambda(A)} \right] = \left[ -\frac{1-\varrho(A)}{1+\varrho(A)}, \frac{1-\varrho(A)}{1+\varrho(A)} \right],$$

and thus

$$d(A) \leq \frac{1-\varrho(A)}{1+\varrho(A)}$$

by the spectral radius formula. □

The next smoothing of the coefficients lemma will be important in Sections 3 and 5 to absorb terms, which are a priori not finite for nonsmooth coefficients. We include the simple proof for convenience. To this end, we let  $\eta \in C_c^\infty(B(0, 1))$  be nonnegative with  $\int_{\mathbb{R}^d} \eta \, dx = 1$  and put  $\eta_n(x) := n^d \eta(nx)$  for  $n \in \mathbb{N}$  and  $x \in \mathbb{R}^d$ . We define the smoothed coefficients  $A_n := A * \eta_n$ .

**Lemma 2.2.** *Let  $O \subseteq \mathbb{R}^d$  be open and bounded,  $u \in W^{1,2}(O)^N$  be a weak solution to  $Lu = 0$  in  $O$  and  $u_n \in W^{1,2}(O)^N$  be the unique weak solution to*

$$-\operatorname{div}(A_n \nabla u_n) = 0 \quad \text{in } O \quad \& \quad u - u_n \in W_0^{1,2}(O)^N.$$

*Then the following assertions are satisfied.*

- (i) *For all  $n \in \mathbb{N}$ , it holds  $\lambda(A_n) \geq \lambda(A)$ ,  $\Lambda(A_n) \leq \Lambda(A)$ ,  $d(A_n) \leq d(A)$  and  $u_n$  is smooth in  $O$ .*
- (ii) *Along a subsequence  $u_n \rightarrow u$  in  $L^2(O)^N$  and a.e. on  $O$ .*

*Proof.* As  $A_n$  is smooth, so is  $u_n$  by elliptic regularity theory, for example, [10, section 6.3.1, Theorem 3] adapted to systems. The rest of (i) follows from

$$A_n(x)\xi \cdot \bar{\zeta} = \int_{\mathbb{R}^d} \eta_n(y)A(x-y)\xi \cdot \bar{\zeta} \, dy \quad (\xi, \zeta \in (\mathbb{C}^N)^d).$$

For instance, to prove that  $d(A_n) \leq d(A)$ , we let  $t > 0$  be such that  $\|\mathbf{1}_{(\mathbb{C}^N)^d} - tA\|_\infty = d(A)$  and use  $\int_{\mathbb{R}^d} \eta_n(y) \, dy = 1$  twice in order to get for all  $\xi \in (\mathbb{C}^N)^d$  that

$$|(\mathbf{1}_{(\mathbb{C}^N)^d} - tA_n(x))\xi| = \left| \int_{\mathbb{R}^d} \eta_n(y)(\xi - tA(x-y)\xi) \, dy \right| \leq d(A) \int_{\mathbb{R}^d} \eta_n(y) \, dy = d(A).$$

To prove (ii), let us show in a first step that  $(v_n)_n := (u_n - u)_n \subseteq W_0^{1,2}(O)^N$  is bounded. Indeed, as

$$-\operatorname{div}(A_n \nabla v_n) = \operatorname{div}(A_n \nabla u) \quad \text{in } O,$$

this follows from the Lax–Milgram lemma and (i). Thus, we can find a subsequence  $(v_k)_k$ , and some  $v \in W_0^{1,2}(O)^N$  such that  $v_k \rightarrow v$  weakly in  $W_0^{1,2}(O)^N$ . By compactness, we can additionally assume  $v_k \rightarrow v$  strongly in  $L^2(O)^N$  and a.e. on  $O$ . Put  $w := v + u$ . In particular,  $u_k \rightarrow w$  in  $L^2(O)^N$  and a.e. on  $O$ , and  $\nabla u_k \rightarrow \nabla w$  weakly in  $L^2(O)^{dN}$ . We claim that  $w = u$ . To this end, we fix some

$\varphi \in C_c^\infty(O)^N$ . Then

$$0 = \int_O A_k \nabla u_k \cdot \overline{\nabla \varphi} \, dx = \int_O \nabla u_k \cdot \overline{A_k^* \nabla \varphi} \, dx \longrightarrow \int_O \nabla w \cdot \overline{A^* \nabla \varphi} \, dx = \int_O A \nabla w \cdot \overline{\varphi} \, dx,$$

using also strong  $L^2$ -convergence  $A_k^* \nabla \varphi \rightarrow A^* \nabla \varphi$ , which follows from dominated convergence. This proves that  $w$  solves

$$Lw = 0 \quad \text{in } O \quad \& \quad w - u \in W_0^{1,2}(O)^N,$$

hence  $w = u$ . □

*Remark 2.3.* If  $A$  would be only defined on  $O$ , then we can extend it to  $\mathbb{R}^d$  without changing the “distance”: Simply let  $t^* > 0$  be such that  $\|\mathbf{1}_{(\mathbb{C}^N)^d} - t^* A\|_{L^\infty(O)} = d(A)$  and extend  $A$  to  $\mathbb{R}^d$  by  $(t^*)^{-1} \mathbf{1}_{(\mathbb{C}^N)^d}$ . Hence, we can always assume that  $A$  is defined on  $\mathbb{R}^d$ .

### 3 | NEW THOUGHTS ON OLD RESULTS OF KOSHELEV

In a series of articles, culminating in the monograph [14], Koshelev studied qualitative (Hölder) regularity of weak solutions to elliptic systems. In this section, we explain how they lead us to an optimal perturbation result for Gaussian estimates for heat semigroups, when reinterpreted appropriately as quantitative statements.

**Definition 3.1.** Let  $O \subseteq \mathbb{R}^d$  be open. We call a function  $u \in W^{1,2}(O)^N$   **$L$ -harmonic in  $O$** , if we have for all  $\varphi \in C_c^\infty(O)^N$  that

$$\int_O A \nabla u \cdot \overline{\nabla \varphi} \, dx = 0.$$

The appropriate setting to study regularity of  $L$ -harmonic functions turns out to be the following “weighted Morrey spaces”  $H_\alpha(O)^N$ ,  $\alpha \in \mathbb{R}$ , which are defined as the spaces of all  $u \in W^{1,2}(O)^N$  modulo  $\mathbb{C}^N$  for which the norm

$$\|u\|_{H_\alpha(O)} := \sup_{x_0 \in O} \|u\|_{H_{\alpha,x_0}(O)}, \quad \text{where} \quad \|u\|_{H_{\alpha,x_0}(O)} := \left( \int_O |\nabla u|^2 |x - x_0|^\alpha \, dx \right)^{\frac{1}{2}},$$

is finite. For  $\alpha > d - 2$  sufficiently close to  $d - 2$  and  $\varepsilon > 0$  small enough we have

$$c(\alpha, d, \varepsilon) := \left( 1 + \frac{\alpha(d-2)}{d-1} + \varepsilon \right)^{\frac{1}{2}} \left( 1 - \frac{\alpha(\alpha - (d-2))}{2(d-1)} - \varepsilon \right)^{-1} > 0. \tag{3.1}$$

This quantity will play an important role. In fact,  $c(\alpha, d, \varepsilon) \rightarrow \delta(d)^{-1}$  in the limit as  $\alpha \rightarrow d - 2$  and  $\varepsilon \rightarrow 0$ . From now on we shall assume  $d(A) < \delta(d)$ .

We begin by looking at  $L$ -harmonic functions on the unit ball  $B$ . Let  $t > 0$ . Guided by the perturbation principle in Lemma 2.1, it begins with writing the equation  $Lu = 0$  in  $B$  in the weak

sense as

$$-\Delta u = -\operatorname{div}(F) \quad \text{with} \quad F := (\mathbf{1}_{(\mathbb{C}^N)^d} - tA)\nabla u.$$

Due to technical reasons, we replace  $A$  by  $A_n$  and  $u$  by  $u_n$  as defined in Lemma 2.2 and call the term on the right-hand side  $-\operatorname{div}(F_n)$ . In addition, we choose  $t > 0$  such that  $d(A_n) = \|\mathbf{1}_{(\mathbb{C}^N)^d} - tA_n\|_\infty$ .

Temporarily, fix  $x_0 \in \frac{1}{4}B$ . To derive optimal Morrey estimates for the solutions  $u_n$ , Koshelev considers two variational integrals

$$X'(u_n, v_n) := \int_{\frac{1}{4}B} \nabla u_n \cdot \nabla v_n \, dx,$$

$$Y'_{-\alpha}(v_n) := \int_{\frac{1}{4}B} |\nabla v_n|^2 |x - x_0|^\alpha \, dx,$$

where  $\alpha > d - 2$  is as above and  $v_n$  is an ingeniously chosen test function for the equation on  $\frac{1}{4}B$  that they constructs from  $u_n$  using spherical harmonics [14, eq. (2.3.2)]. The precise formula for  $v_n$  is not needed here – it suffices to use the estimates below “off-the-shelf”. In fact, this specific  $v_n$  dates back to Giaquinta and Nečas [16]. Koshelev goes on by proving in [14, Corollary 2.3.1] the bounds

$$X'(u_n, v_n) \geq \left(1 - \frac{\alpha(\alpha - (d - 2))}{2(d - 1)} - \varepsilon\right) \|u_n\|_{H_{-\alpha, x_0}(\frac{1}{4}B)}^2 - C(\alpha, d, \varepsilon) \|\nabla u_n\|_{L^2(\frac{1}{4}B)}^2,$$

$$Y'_{-\alpha}(v_n) \leq \left(1 + \frac{(d - 2)\alpha}{d - 1} + \varepsilon\right) \|u_n\|_{H_{-\alpha, x_0}(\frac{1}{4}B)}^2 + C(\alpha, d, \varepsilon) \|\nabla u_n\|_{L^2(\frac{1}{4}B)}^2.$$

As  $v_n$  is a test function for the equation for  $u_n$  in  $\frac{1}{4}B$ , we have

$$X'(u_n, v_n) = \int_{\frac{1}{4}B} (\mathbf{1}_{(\mathbb{C}^N)^d} - tA_n)\nabla u_n \cdot \nabla v_n \, dx$$

and the Cauchy–Schwarz inequality along with the bound  $d(A_n) \leq d(A)$  in Lemma 2.2(i) yields

$$|X'(u_n, v_n)| \leq d(A) \|u_n\|_{H_{-\alpha, x_0}(\frac{1}{4}B)} Y'_{-\alpha}(v_n)^{\frac{1}{2}}.$$

Combining the previous three estimates and recalling the definition of  $c(\alpha, d, \varepsilon)$  leads to

$$\|u_n\|_{H_{-\alpha, x_0}(\frac{1}{4}B)}^2 \leq d(A) \|u_n\|_{H_{-\alpha, x_0}(\frac{1}{4}B)} \left[ c(\alpha, d, \varepsilon) \|u_n\|_{H_{-\alpha, x_0}(\frac{1}{4}B)} \right. \\ \left. + C(\alpha, d, \varepsilon) \|\nabla u_n\|_{L^2(\frac{1}{4}B)} \right] + C(\alpha, d, \varepsilon) \|\nabla u_n\|_{L^2(\frac{1}{4}B)}^2,$$

where  $C(\alpha, d, \varepsilon)$  varies from line to line. The smoothing of the coefficients guarantees that the first summand on the right-hand side is finite and this is the very reason why we have to include this

argument. By Young’s inequality, it follows that

$$\|u_n\|_{H_{-\alpha, x_0}(\frac{1}{4}B)}^2 \leq (c(\alpha, d, \varepsilon) + \varepsilon) d(A) \|u_n\|_{H_{-\alpha, x_0}(\frac{1}{4}B)}^2 + C(\alpha, d, \varepsilon) \|\nabla u_n\|_{L^2(\frac{1}{4}B)}^2.$$

As  $d(A) < \delta(d)$ , we can fix  $\varepsilon$  small and  $\alpha$  close to  $d - 2$  depending only on dimension and  $d(A)$  such that the first term on the right can be absorbed. This is the key point and the result is

$$\|u_n\|_{H_{-\alpha, x_0}(\frac{1}{4}B)} \lesssim_{d, d(A)} \|\nabla u_n\|_{L^2(\frac{1}{4}B)}.$$

After taking the supremum over all  $x_0 \in \frac{1}{4}B$ , we arrive at

$$\|u_n\|_{H_{-\alpha}(\frac{1}{4}B)} \lesssim_{d, d(A)} \|\nabla u_n\|_{L^2(\frac{1}{4}B)}.$$

Then, Koshelev proves in [14, Theorem 2.1.1] that the left-hand side controls the Hölder seminorm of order  $\mu = (\alpha - d + 2)/2$  on  $\frac{1}{4}B$ . Applying Caccioppoli’s inequality on the right-hand side eventually leads to

$$[u_n]_{\frac{1}{4}B}^{(\mu)} := \sup_{x, y \in \frac{1}{4}B, x \neq y} \frac{|u_n(x) - u_n(y)|}{|x - y|^\mu} \lesssim_{d, d(A)} \|u_n\|_{L^2(B)}.$$

Finally, we invoke Lemma 2.2(ii) in order to deduce

$$[u]_{\frac{1}{4}B}^{(\mu)} \leq \limsup_{n \rightarrow \infty} [u_n]_{\frac{1}{4}B}^{(\mu)} \lesssim_{d, d(A)} \|u\|_{L^2(B)}.$$

This estimate holds for any  $L$ -harmonic function on the unit ball. As we have

$$d(A) = d(A^*) \quad \& \quad d(A) = d(A(x_0 + r \cdot))$$

for each  $x_0 \in \mathbb{R}^d$  and  $r > 0$ , a scaling argument shows that the outcome of revisiting Koshelev’s results is the following proposition.

**Proposition 3.2.** *Suppose  $d(A) < \delta(d)$ . There are  $\mu \in (0, 1]$  and  $C > 0$ , both depending only on  $d$  and  $d(A)$ , such that we have for all balls  $B = B(x, r) \subseteq \mathbb{R}^d$  and every  $L$ - or  $L^*$ -harmonic  $u$  in  $B$  that*

$$r^\mu [u]_{\frac{1}{4}B}^{(\mu)} \leq Cr^{-\frac{d}{2}} \|u\|_{L^2(B)}.$$

The quantitative Hölder estimate in Proposition 3.2 appeared much later in a different context. Namely, Auscher and Tchamitchian [3, 5] called it property (H) and proved that it implies that  $e^{-tL}$  has a kernel with pointwise Gaussian bounds. If we combine their Theorem 10 in [5, chapter 1, section 1.4.1] with Proposition 3.2, then we obtain the following result for Gaussian estimates. It can be seen as a perturbation result from the Laplacian.

**Theorem 3.3.** *Suppose  $d(A) < \delta(d)$ . The kernel of  $(e^{-tL})_{t>0}$  is represented by a Hölder regular function  $(K_t)_{t>0}$ , which admits pointwise Gaussian estimates: There are  $c, a > 0$  and  $\mu \in (0, 1)$  such*



that

$$|K_t(x, y)| \leq ct^{-\frac{d}{2}} e^{-a \frac{|x-y|^2}{t}},$$

$$|K_t(x, y) - K_t(x', y')| \leq ct^{-\frac{d}{2} - \frac{\mu}{2}} (|x - x'| + |y - y'|)^\mu$$

for all  $t > 0$  and  $x, x', y, y' \in \mathbb{R}^d$ . The constants  $c, a, \mu$  depend only on  $d, d(A), \lambda(A)$  and  $\Lambda(A)$ .

From Young’s inequality for convolutions, we obtain:

**Corollary 3.4.** *If  $d(A) < \delta(d)$ , then  $p_+(L) = \infty$ .*

We shall see next that the “radius”  $r = \delta(d)$  is optimal for the conclusion in Corollary 3.4 and hence also for the one in Theorem 3.3. Again this is implicit in Koshelev’s work and relies on a counterexample due to De Giorgi.

Let  $c > 0$  and  $D \geq (c^2+1)/(d-2)c$ . Define for  $x \in \mathbb{R}^d \setminus \{0\}$  the elliptic system with coefficients

$$(A_{DG}(x))_{i,j}^{\alpha,\beta} := \left( c\delta_{ij} + D \frac{x_i x_j}{|x|^2} \right) \left( c\delta_{\alpha\beta} + D \frac{x_\alpha x_\beta}{|x|^2} \right) \quad (i, j, \alpha, \beta = 1, \dots, d).$$

Then  $u(x) := x/|x|^b$  with

$$b = \frac{d}{2} - \left( \frac{d^2}{4} - \frac{d(d-1)cD + (d-1)D^2}{1 + (c+D)^2} \right)^{\frac{1}{2}} \in [1, d/2)$$

solves the elliptic system  $-\operatorname{div}(A_{DG} \nabla u) = 0$  in the weak sense in  $B(0, 1)$ , see [14, section 2.5] or [11]. Note that  $b = 1$  if and only if  $D = (c^2+1)/(d-2)c$ . Koshelev continues in [14, section 2.5] by showing that for this choice of  $D$  they can pick  $c = c(d) > 0$  such that  $d(A_{DG}) = \delta(d)$ . As  $d(A_{DG})$  depends continuously on  $D$  and  $c$ , we can pick these parameters for any given  $\varepsilon > 0$  in such a way that

$$b > 1 \quad \& \quad d(A_{DG}) < \delta(d) + \varepsilon. \tag{3.2}$$

Now we use a localization argument from [5, chapter 1, section 1.3] to prove:

**Proposition 3.5.** *For any  $\varepsilon > 0$  there is  $A$  such that  $d(A) < \delta(d) + \varepsilon$  and  $p_+(L) < \infty$ .*

*Proof.* We pick  $A = A_{DG}$  as in (3.2) and set  $u(x) := x/|x|^b$ . Let  $\phi \in C_c^\infty$  be such that  $\mathbf{1}_{B(0,1/2)} \leq \phi \leq \mathbf{1}_{B(0,1)}$ . Then  $v := \phi u \in D(L)$  can be seen as follows: using the equation for  $u$ , we deduce

$$Lv = -\operatorname{div}(A(\nabla\phi \otimes u)) - \operatorname{div}(\phi A \nabla u) = -\operatorname{div}(A(\nabla\phi \otimes u)) - \nabla\phi \odot (A \nabla u).$$

Our notation should be interpreted as what comes out from the product rule. The only thing that matters is that  $\nabla\phi$  vanishes near the origin and hence  $Lv \in (C_c^\infty)^N$ . If we had  $p_+(L) = \infty$ , then according to [2, Proposition 5.3] we would get  $v \in (L^q)^N$  for every  $q \in (2^*, \infty)$ . However,  $|v(x)| = |x|^{1-b}$  in a neighborhood of 0 cannot belong to  $L^q$  for  $q \geq d/(b-1)$ . □

### 4 | THE INTERPOLATION ARGUMENT

We come to the proof of our main result, Theorem 1.1, for the case  $d(A) \geq \delta(d)$ . We will use basic properties of semigroups and vector-valued holomorphic functions. For further background, we refer to [1].

We begin with a Stein-type interpolation principle tailored to our needs (taking care of implicit constants in particular). We write

$$S_\delta := \{z \in \mathbb{C} : -\delta < \operatorname{Re}(z) < 1 + \delta\} \quad \& \quad S := S_0.$$

For a matrix  $A = A_z$  depending on a parameter  $z$ , we let  $a_z$  be the sesquilinear form corresponding to  $L_z := -\operatorname{div}(A_z \nabla \cdot)$ .

**Proposition 4.1.** *Let  $\delta > 0$ . Suppose that  $\{A_z\}_{z \in S_\delta} \subseteq L^\infty(\mathbb{R}^d; \mathcal{L}((\mathbb{C}^N)^d))$  are uniformly strongly elliptic matrices such that:*

- (i) *there are  $0 < \lambda \leq \Lambda$  with  $\lambda(A_z) \geq \lambda$  and  $\Lambda(A_z) \leq \Lambda$  for all  $z \in \bar{S}$ ;*
- (ii) *we have  $c := \sup_{t \in \mathbb{R}} d(A_{it}) < \delta(d)$ ;*
- (iii) *for all  $u, v \in (W^{1,2})^N$  the map  $z \mapsto a_z(u, v)$  is holomorphic in  $S_\delta$ .*

Then for  $\theta \in [0, 1]$ , we have  $p_+(L_\theta) \geq 2^*/\theta$ .

*Proof.* Fix  $t > 0$  and  $\theta \in [0, 1]$ . Let us define  $\Phi(z) := e^{-tL_z}$  for  $z \in S_\delta$ . As an  $\mathcal{L}((L^2)^N)$ -valued map,  $\Phi$  is bounded by 1, holomorphic on  $S_\delta$  and in particular continuous on  $\bar{S}$ . This follows from (iii), see [20]. Now, we estimate  $\Phi$  on the boundary of  $S$ . Let  $f \in (L^2)^N$ .

**(A) Estimate on the left boundary.** Let  $z \in \bar{S}$  with  $\operatorname{Re}(z) = 0$ . Due to (ii) and Theorem 3.3 we have Gaussian estimates for the kernel  $(K_{t,z})_{t>0}$  of  $(e^{-tL_z})_{t>0}$  at our disposal. By (i), implicit constants depend only on  $d, c, \lambda$  and  $\Lambda$ . Young’s inequality for convolutions yields

$$\|e^{-tL_z} f\|_\infty \lesssim_{d,c,\lambda,\Lambda} t^{-\frac{d}{4}} \|f\|_2.$$

**(B) Estimate on the right boundary.** Let  $z \in \bar{S}$  with  $\operatorname{Re}(z) = 1$ . By holomorphy of the semigroup it follows that  $e^{-tL_z} f \in (W^{1,2})^N$ . Thus, by a Sobolev embedding, (i) and Hölder’s inequality, we get

$$\begin{aligned} \|e^{-tL_z} f\|_{2^*}^2 &\lesssim_d \|\nabla e^{-tL_z} f\|_2^2 \\ &\lesssim_\lambda |a_z(e^{-tL_z} f, e^{-tL_z} f)| \\ &= |(L_z e^{-tL_z} f | e^{-tL_z} f)_2| \\ &\leq \|L_z e^{-tL_z} f\|_2 \|e^{-tL_z} f\|_2. \end{aligned}$$

The semigroup is contractive on  $(L^2)^N$  in the sector  $\{z \in \mathbb{C} : |\arg(z)| < \varphi\}$ , where  $\tan(\varphi) = \lambda/\Lambda$ . In particular,  $\|e^{-tL_z} f\|_2 \leq \|f\|_2$  and by Cauchy’s formula for the complex derivative

$$\|L_z e^{-tL_z} f\|_2 = \left\| \frac{1}{2\pi i} \int_{\partial B(t,r)} \frac{e^{-wL_z} f}{(w-t)^2} dw \right\|_2 \lesssim_{\lambda,\Lambda} t^{-1} \|f\|_2$$

with  $r = t \sin(\varphi)$ . Altogether, this gives

$$\| e^{-tL_z} f \|_{2^*} \lesssim_{d,c,\lambda,\Lambda} t^{-\frac{1}{2}} \| f \|_2.$$

Combining (A) and (B), Stein’s interpolation theorem [17, Theorem 1] implies that

$$\| e^{-tL_\theta} f \|_{\frac{2^*}{\theta}} \lesssim_{d,c,\lambda,\Lambda,\theta} t^{\frac{d}{2 \cdot 2^*/\theta} - \frac{d}{4}} \| f \|_2.$$

In the language of  $L^p - L^q$ -estimates, this means that  $(e^{-tL_\theta})_{t>0}$  is  $L^2 - L^{2^*/\theta}$ -bounded. A general principle for these estimates [2, Proposition 3.2] implies  $\| e^{-tL_\theta} f \|_q \lesssim_{d,c,\lambda,\Lambda,q} \| f \|_q$  for all  $q \in (2, 2^*/\theta)$  as claimed. □

*Proof of Theorem 1.1 (i).* Let  $A$  be elliptic such that  $d(A) \in [\delta(d), 1)$ . Fix  $\varepsilon \in (0, 1 - d(A))$  and  $t^* > 0$  such that  $\| \mathbf{1}_{(\mathbb{C}^N)^d} - t^* A \|_\infty = d(A)$ . We abbreviate  $B := \mathbf{1}_{(\mathbb{C}^N)^d} - t^* A$ , which means  $A = (t^*)^{-1}(\mathbf{1}_{(\mathbb{C}^N)^d} - B)$ . As sketched in the introduction, we perturb  $B$  by multiplication with complex numbers from a suitable annulus. However, it will be convenient to parameterize these numbers via an analytic function  $F$  defined on  $S$ .

Let  $0 < r < 1 < R$  to be chosen. We embed  $A$  into the analytic family

$$A_z := (t^*)^{-1}(\mathbf{1}_{(\mathbb{C}^N)^d} - F(z)B), \quad \text{where} \quad F(z) := r^{1-z} R^z = r e^{z \ln(R/r)}.$$

Note that  $F$  is entire, bounded by  $r^{-\delta} R^{1+\delta}$  in any strip  $S_\delta$ , and maps  $\bar{S}$  onto  $\{z \in \mathbb{C} : r \leq |z| \leq R\}$ . At this point, we choose

$$r := \frac{\delta(d)}{d(A) + \varepsilon} \quad \& \quad R := \frac{1}{d(A) + \varepsilon}.$$

Let us show that for  $z$  in a strip  $S_\delta$  with sufficiently small  $\delta > 0$  we can define the interpolating operators  $L_z = -\operatorname{div}(A_z \nabla \cdot)$ , where  $A_z$  is still elliptic by Lemma 2.1. Indeed, for  $\delta > 0$  sufficiently small our choice of  $R$  delivers

$$\sup_{z \in S_\delta} \| A_z \|_\infty \leq (t^*)^{-1} (1 + \delta(d)^{-\delta} R d(A)) \quad \& \quad \sup_{z \in S_\delta} d(A_z) \leq \delta(d)^{-\delta} R d(A) < 1.$$

This also proves (i) in Proposition 4.1. Part (iii) follows immediately and our choice of  $r$  yields

$$\sup_{t \in \mathbb{R}} d(A_{it}) \leq \sup_{t \in \mathbb{R}} \| F(it)B \|_\infty = r d(A) < \delta(d),$$

which is (ii). Now, pick  $\theta \in (0, 1)$  such that  $1 = r^{1-\theta} R^\theta$ . Then  $L = L_\theta$  and Proposition 4.1 implies that  $p_+(L) \geq 2^*/\theta$ . Finally, we notice that

$$\theta = 1 - \frac{\ln(R)}{\ln(R/r)} \longrightarrow 1 - \frac{\ln(d(A))}{\ln(\delta(d))} \quad \text{as } \varepsilon \searrow 0. \quad \square$$

*Remark 4.2.* In Proposition 4.1, we assume that  $z \mapsto a_z(u, v)$  is holomorphic in a larger strip for convenience to get continuity of  $z \mapsto e^{-tL_z}$  up to  $\bar{S}$ . If this holds true for any other reason, it is enough to suppose that  $z \mapsto a_z(u, v)$  is holomorphic in  $S$ .

*Remark 4.3.* The proof of Theorem 1.1 reveals that the same results hold for divergence form operators with form domain  $V$  on general open sets  $O \subseteq \mathbb{R}^d$ , provided that Theorem 3.3 holds true with implicit constants depending only on geometry, ellipticity and dimension, and that we have additionally

$$\|u\|_{L^{2^*}(O)} \lesssim \|\nabla u\|_{L^2(O)} \quad (u \in V),$$

which was used in (B) above.

## 5 | EXTENSION TO BOUNDED $C^1$ -DOMAINS

Let us extend Theorems 1.1 and 3.3 to bounded  $C^1$ -domains with Dirichlet boundary conditions. The divergence form operator  $L = -\operatorname{div}(A\nabla \cdot)$  with uniformly strongly elliptic  $A \in L^\infty(\Omega; \mathcal{L}((\mathbb{C}^N)^d))$  is now realized in  $L^2(\Omega)^N$  as the  $m$ -accretive operator associated to the form

$$a(u, v) := \int_{\Omega} A\nabla u \cdot \overline{\nabla v} \, dx \quad (u, v \in W_0^{1,2}(\Omega)^N).$$

We fix our geometric setup.

**Assumption 5.1.** Throughout this section,  $\Omega \subseteq \mathbb{R}^d$ ,  $d \geq 3$ , is a bounded domain with  $C^1$ -boundary. This means that there is some  $M > 0$  such that for each  $x_0 \in \partial\Omega$  there is an open neighborhood  $U$  of  $x_0$  and a  $C^1$ -diffeomorphism  $\phi : U \rightarrow B(0, 1)$ ,  $\phi(x) = (x', \psi(x') - x_d)$  such that  $\phi(U \cap \Omega) = B(0, 1) \cap \mathbb{R}_+^d$  and  $\|D\psi\|_\infty \leq M$ .

We can choose  $M$  arbitrarily small by choosing the neighborhoods small enough. This is exactly the reason, why we assume that the boundary is  $C^1$  and not just Lipschitz.

**Theorem 5.2.** *In the setting above suppose that  $d(A) < \delta(d)$ . Then the kernel of  $(e^{-tL})_{t>0}$  is represented by a measurable function  $(K_t)_{t>0}$  for which there are  $c, a > 0$  and  $\mu \in (0, 1)$  such that*

$$|K_t(x, y)| \leq ct^{-\frac{d}{2}} e^{-a\frac{|x-y|^2}{t}},$$

$$|K_t(x, y) - K_t(x', y')| \leq ct^{-\frac{d}{2} - \frac{\mu}{2}} (|x - x'| + |y - y'|)^\mu$$

for all  $t > 0$  and  $x, x', y, y' \in \Omega$ . The constants  $c$  and  $a$  depend only on  $d, d(A), \lambda(A), \Lambda(A)$  and geometry.

*Proof.* The proof is very similar to the one of Theorem 3.3. We abbreviate

$$\Omega(x, r) := \Omega \cap B(x, r).$$

We use again elliptic estimates for  $L$ -harmonic functions due to Koshelev, this time also in the half space after localization and transformation. The kernel estimates will then follow from [6, Theorem 12] provided we can check what they call property (D)<sup>†</sup> for  $L$  and  $L^*$ . By the easy argument in [5, p. 37] it suffices to show a property similar to Proposition 3.2 and formulated as follows:

<sup>†</sup>Note carefully that the bounded  $C^1$ -domain  $\Omega$  falls into class (II) in [6]. In this case, and as we consider Dirichlet boundary conditions, the properties (D) and  $(D)_{\text{loc}}$  in [6] coincide by definition.

There are  $C > 0$  and  $\gamma \in (0, 1)$  such that for all  $x_0 \in \overline{\Omega}$ ,  $r \leq \rho_0$  and  $u \in W_0^{1,2}(\Omega)^N$  with  $Lu = 0$  or  $L^*u = 0$  in  $\Omega(x_0, r)$  it holds

$$r^\mu [u]_{\Omega(x_0, \gamma r)}^{(\mu)} \leq Cr^{-\frac{d}{2}} \|u\|_{L^2(\Omega(x_0, r))}. \tag{5.1}$$

Here  $\rho_0 \leq 1$  is chosen small as explained in [6, p. 20].

In view of  $d(A) = d(A^*)$ , we stick to the case of  $L$ -harmonic functions. When  $x_0 \in \Omega$  and  $B(x_0, r) \subseteq \Omega$ , this estimate has already been obtained in Section 3 with  $\gamma = 1/4$ . By a case distinction (whether or not  $B(x_0, r/2)$  intersects  $\partial\Omega$ ) it suffices to treat in addition the case  $x_0 \in \partial\Omega$ .

So, let  $r \leq \rho_0$  and  $u \in W_0^{1,2}(\Omega)^N$  with  $Lu = 0$  in  $\Omega(x_0, r)$ . Let  $t > 0$  and pick  $\rho \simeq_M r$  small enough such that  $\phi^{-1}(B(0, \rho)) \subseteq B(x_0, r)$ . Put  $B_+ := \mathbb{R}_+^d \cap B$ , where  $B$  is again the unit ball, and write  $u_{\phi, \rho} := (u \circ \phi^{-1})(\rho \cdot)$ . A change of coordinates implies that  $u_{\phi, \rho} \in W^{1,2}(B_+)^N$  is a weak solution of

$$-\operatorname{div} \left( (D\phi)_{\phi, \rho} A_{\phi, \rho} (D\phi)_{\phi, \rho}^\top \nabla(u_{\phi, \rho}) \right) = 0 \quad \text{in } B_+$$

that vanishes on  $\partial\mathbb{R}_+^d \cap \partial B_+$ .

Next, we use a smoothing procedure as in Section 3. We let  $A_n$  and  $u_n$  be defined as in Lemma 2.2 with  $A$  replaced by  $(D\phi)_{\phi, \rho} A_{\phi, \rho} (D\phi)_{\phi, \rho}^\top$  (see also Remark 2.3). Then the same lemma assures that  $u_n \rightarrow u_{\phi, \rho}$  in  $L^2(B_+)^N$  and a.e. along a subsequence. In addition, we have  $u_n \in C^\infty(\overline{\frac{3}{4}B_+})^N$  by elliptic regularity [10, section 6.3.1, Theorem 5]. As in Section 3, we write  $-\operatorname{div}(A_n \nabla u_n) = 0$  as

$$-\Delta u_n = -\operatorname{div}(F_n) \quad \text{with} \quad F_n := (\mathbf{1}_{(\mathbb{C}^N)^d} - tA_n) \nabla u_n$$

and  $t > 0$  is chosen such that  $\|\mathbf{1}_{(\mathbb{C}^N)^d} - tA_n\|_\infty = d(A_n)$ . Note that

$$\|\mathbf{1}_{(\mathbb{C}^N)^d} - tA_n\|_\infty \leq d((D\phi)_{\phi, \rho} A_{\phi, \rho} (D\phi)_{\phi, \rho}^\top) \leq M^2 + (1 + M)^2 d(A), \tag{5.2}$$

where the first inequality is due to Lemma 2.2 (i) and the second one follows by definition of  $\phi$ . Koshelev proves in [14, (2.4.13)] for each  $x_0 \in \frac{1}{4}B_+$  that

$$\begin{aligned} \|u_n\|_{H_{-\alpha, x_0}(\frac{1}{4}B_+)} &\leq c(\alpha, d, \varepsilon) \|F_n\|_{L^2(\frac{1}{4}B_+)} |x - x_0|^{-\frac{\alpha}{2}} \\ &\quad + C(\alpha, d, \varepsilon) \left[ \|\nabla u_n\|_{L^2(\frac{1}{4}B_+)} + \|F_n\|_{L^2(\frac{1}{4}B_+)} \right], \end{aligned}$$

provided that the right-hand side is finite, and  $u_n$  and  $f_n$  are sufficiently smooth, which is the case thanks to our smoothing procedure. Here,  $c(\alpha, d, \varepsilon)$  is as in (3.1), where  $\alpha > d - 2$  and  $\varepsilon > 0$  are chosen such that  $c(\alpha, d, \varepsilon)$  is positive and finite. By definition of  $F_n$  and (5.2), we derive the estimate

$$\begin{aligned} \|u_n\|_{H_{-\alpha, x_0}(\frac{1}{4}B_+)} &\leq ((1 + M)^2 d(A) + M^2)(c(\alpha, d, \varepsilon) + \varepsilon) \|u_n\|_{H_{-\alpha, x_0}(\frac{1}{4}B_+)} \\ &\quad + C(\alpha, d, \varepsilon, M) \|\nabla u_n\|_{L^2(\frac{1}{4}B_+)}. \end{aligned}$$

As  $d(A) < \delta(d)$ , we can pick  $\varepsilon > 0$ ,  $\alpha > d - 2$  and  $M > 0$  depending only on  $d$  and  $d(A)$  such that the first term on the right can be absorbed in order to obtain

$$\|u_n\|_{H_{-\alpha}(\frac{1}{4}B_+)} \lesssim_{d,d(A)} \|\nabla u_n\|_{L^2(\frac{1}{4}B_+)}.$$

As in Section 3, we deduce

$$[u_{\phi,\rho}]_{\frac{1}{4}B_+}^{(\mu)} \lesssim_{d,d(A)} \|u_{\phi,\rho}\|_{L^2(B_+)},$$

where  $\mu := (\alpha - d + 2)/2$ .

Transforming back gives (5.1) for some  $\gamma = \gamma(M) \in (0, 1)$ . □

At this point, we are in the same situation as on  $\mathbb{R}^d$  and we can derive the following statement.

**Corollary 5.3.** *In the setup of this section, the following assertions hold true.*

(i) *If  $d(A) > \delta(d)$ , then*

$$p_+(L) \geq \frac{2^*}{1 - \frac{\ln(d(A))}{\ln(\delta(d))}}.$$

(ii) *If  $d(A) \leq \delta(d)$ , then  $p_+(L) = \infty$ .*

(iii) *Part (ii) is sharp in the sense that for all bounded  $C^1$ -domains  $\Omega \subseteq \mathbb{R}^d$  and for each  $\varepsilon > 0$  there is some  $A_\varepsilon$  with  $d(A_\varepsilon) \leq \delta(d) + \varepsilon$  and  $p_+(L_\varepsilon) < \infty$ .*

*Proof.* The estimates for  $p_+(L)$  follow as before, see also Remark 4.3. As for the sharpness of the radius  $d(A) = \delta(d)$  we can, after translation, assume  $0 \in \Omega$ . We take the same coefficients  $A = A_{DG}$  as in the proof of Proposition 3.5 and localize  $u$  to a ball contained in  $\Omega$ . As before, this produces some  $v \in D(L)$  with  $v \notin (L^q)^N$  for  $q$  large but  $Lv \in (C_c^\infty)^N$ . Arriving at a contradiction with  $p_+(L) = \infty$  requires a different (and in fact simpler) argument compared to the case  $\Omega = \mathbb{R}^d$ .

By ellipticity and Poincaré’s inequality, there is some  $\theta_2 > 0$  such that  $L - \theta_2$  is still  $m$ -accretive. Hence,  $L$  is invertible in  $(L^2)^N$  and the semigroup enjoys the exponential bound  $\|e^{-tL} f\|_2 \leq e^{-\theta_2 t} \|f\|_2$  for all  $t > 0$  and  $f \in (L^2)^N$ . By interpolation with the uniform bound on  $(L^p)^N$  for some  $p > q$ , we get  $\|e^{-tL} f\|_q \lesssim e^{-\theta_q t} \|f\|_q$  with some  $\theta_q > 0$ . But then the formula

$$L^{-1} f = \int_0^\infty e^{-tL} f \, dt,$$

valid in  $(L^q \cap L^2)^N$  by the exponential estimate, implies that  $L^{-1}$  maps  $(L^q \cap L^2)^N$  into itself, in contradiction with the properties of  $v$ . □

## 6 | DIMENSIONLESS IMPROVEMENTS

Here, we prove Theorem 1.2. For  $1 < p < \infty$  we denote by  $(\dot{W}^{1,p})^N$  the space of all  $\mathbb{C}^N$ -valued tempered distributions modulo  $\mathbb{C}^N$  for which the distributional gradient belongs to  $(L^p)^{dN}$ . We

endow this space with the norm  $\|\nabla \cdot\|_p$  and denote by  $(\dot{W}^{-1,p})^N$  the anti-dual space of  $(\dot{W}^{1,p'})^N$ . We define

$$-\operatorname{div} A \nabla : (\dot{W}^{1,p})^N \rightarrow (\dot{W}^{-1,p})^N, \quad \langle -\operatorname{div} A \nabla u \mid v \rangle := \int_{\mathbb{R}^d} A \nabla u \cdot \overline{\nabla v} \, dx. \tag{6.1}$$

By Hölder’s inequality, this is a bounded map.

We denote by  $c(p)$  the operator norm of the Riesz transform  $R := \nabla(-\Delta)^{-1/2}$  (defined via a Fourier multiplication operator with symbol  $-i \frac{\xi}{|\xi|} \otimes \mathbf{1}_{\mathbb{C}^N}$ ) from  $(L^p)^N$  to  $(L^p)^{dN}$ . By Plancherel’s theorem we have  $c(2) = 1$ .

**Proposition 6.1.** *Let  $p > 2$ . Then  $c(p) < 1/\sqrt{d(A)}$  implies  $q_+(L) \geq p$ .*

*Proof.* By the characterization of  $q_+(L)$  in [4, section 13.3], it suffices to prove that the map in (6.1) is invertible and that the inverse is compatible with the one for  $p = 2$ .

We borrow an idea from [2, Lemma 3.4]. Fix  $t^* > 0$  such that  $d(A) = \|\mathbf{1}_{(\mathbb{C}^N)^d} - t^* A\|_\infty$  and put  $B := \mathbf{1}_{(\mathbb{C}^N)^d} - t^* A$ . Then we can factorize

$$\begin{aligned} -\operatorname{div} A \nabla &= (t^*)^{-1}(-\Delta) + (t^*)^{-1} \operatorname{div} B \nabla \\ &= (t^*)^{-1}(-\Delta)^{\frac{1}{2}}(\mathbf{1} + R^* B R)(-\Delta)^{\frac{1}{2}}. \end{aligned}$$

As

$$\|R^* B R f\|_p \leq c(p)^2 d(A) \|f\|_p$$

for  $f \in (L^p \cap L^2)^N$ , it suffices that  $c(p) < 1/\sqrt{d(A)}$  to invert

$$(\mathbf{1} + R^* B R)^{-1} = \sum_{n=0}^\infty (-R^* B R)^n,$$

in  $(L^p)^N$ . The inverse is compatible because the same Neumann series converges also in  $(L^2)^N$  owing to  $c(2) = 1$ . Next, as  $(-\Delta)^{1/2} : (\dot{W}^{s,p})^N \rightarrow (\dot{W}^{s-1,p})^N$  is an isomorphism for  $s = 0, 1$  and all  $p \in (1, \infty)$ , it follows that also  $-\operatorname{div} A \nabla : (\dot{W}^{1,p})^N \rightarrow (\dot{W}^{-1,p})^N$  is invertible with compatible inverse. □

The constant  $c(p)$  has a long history and that  $c(p)$  is controlled from above by a dimensionless constant goes back to Stein [18]. Its exact value remains unknown to date. The best known estimates can be used to determine an improvement for  $q_+(L) - 2$  explicitly. Dragičević and Volberg have shown in [9, Corollary 0.2] that

$$c(p) \leq 2(p - 1) \quad (p \geq 2). \tag{6.2}$$

(Note that their short argument applies word-by-word to  $\mathbb{C}^N$ -valued functions.) However, this does not give  $c(2) = 1$ , suggesting that their bound can be improved by interpolation for  $p > 2$  not too far away from 2. In fact, this is the case we are most interested in and we include the proof of the following elementary lemma. A similar argument for the Ahlfors–Beurling transform is found in [7].

**Lemma 6.2.** *If  $\sigma \approx 5.69061$  is the unique real solution to*

$$\ln(2\sigma - 2) = \frac{\sigma(\sigma - 2)}{2(\sigma - 1)},$$

and  $2 \leq p \leq \sigma$ , then

$$c(p) \leq \left( e^{\frac{\sigma^2}{\sigma-1}} \right)^{\frac{1}{2} - \frac{1}{p}} \leq 2(p - 1). \tag{6.3}$$

*Proof.* Fix  $q \geq p$ . Riesz–Thorin interpolation and the fact that  $c(2) = 1$  yields

$$c(p) \leq c(q)^\theta \quad \text{with} \quad \frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}.$$

We insert this value of  $\theta$  and use (6.2) to get

$$c(p) \leq \left( (2(q - 1))^{\frac{2q}{q-2}} \right)^{\frac{1}{2} - \frac{1}{p}} = \left( e^{\frac{2q}{q-2} \ln(2q-2)} \right)^{\frac{1}{2} - \frac{1}{p}}.$$

As  $1/2 - 1/p > 0$ , we have to minimize the expression  $\frac{2q}{q-2} \ln(2q - 2)$  over  $q \geq p$ . A straightforward calculation shows that the minimum over  $q \geq 2$  is attained in  $q = \sigma$ . As  $p \leq \sigma$ , this must also be the global minimum over  $q \geq p$  and the first estimate in (6.3) follows. The second estimate in (6.3) follows simply because  $q = p$  cannot give a better bound. □

*Proof of Theorem 1.2.* Let

$$\Phi : [2, \infty) \rightarrow [1, \infty), \quad \Phi(p) := \begin{cases} \left( e^{\frac{\sigma^2}{\sigma-1}} \right)^{\frac{1}{2} - \frac{1}{p}} & (p \leq \sigma), \\ 2(p - 1) & (p > \sigma). \end{cases}$$

Note that  $\Phi$  is continuous, bijective and  $c(p) \leq \Phi(p)$  for all  $p \geq 2$ , see (6.2) and (6.3). Hence, Proposition 6.1 yields

$$q_+(L) \geq \sup \left\{ p > 2 : \Phi(p) < 1/\sqrt{d(A)} \right\} = \Phi^{-1}(1/\sqrt{d(A)}).$$

Inverting  $\Phi$  leads to

$$q_+(L) \geq \begin{cases} \frac{2}{1 + \frac{\sigma-1}{\sigma^2} \ln(d(A))} & \text{if } \frac{1}{4(\sigma-1)^2} \leq d(A), \\ \frac{1}{2\sqrt{d(A)}} + 1 & \text{if } d(A) \leq \frac{1}{4(\sigma-1)^2}, \end{cases}$$

which proves the theorem. □

**ACKNOWLEDGMENTS**

We would like to thank Salvatore Leonardi for helping us with the literature underlying Section 3 and Oliver Dragičević for helpful exchange about the constant  $c(p)$  in Section 6. We are



greatful to the anonymous referees for their valuable comments and their suggestions to expand on Koshelev's work in Section 3.

Open access funding enabled and organized by Projekt DEAL.

## JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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