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Brownian Motion with Limited Occupation Times

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Abstract

In this thesis, we condition a Brownian motion on spending limited time outside an interval. More precisely, we bound the occupation time outside the interval by a deterministic constant. This is accomplished by conditioning on approximations of the event in question and passing to the weak limit.
We start our analysis with the case of an unbounded interval, where we describe the resulting process in terms of a path decomposition. In particular, we exactly determine the distributions of the total occupation time outside and the last entrance time into the interval. Additionally, we provide limiting theorems for the mentioned quantities as the starting point tends to ∞ or $-\infty$, respectively.
If the interval is bounded, we focus on starting points inside. In this setting, we prove that the resulting process does not leave the interval at all, but satisfies the very same SDE as a Brownian motion which is conditioned to stay inside the interval. This result is a very rare extreme example of entropic repulsion. On our way, we explicitly determine the exact asymptotic behavior of the probability that a Brownian motion spends limited time outside the interval during the first T time units, as $T \to \infty$.

Zusammenfassung

In dieser Arbeit bedingen wir eine Brownsche Bewegung darauf, nur begrenzt Zeit außerhalb eines Intervalls zu verbringen. Genauer beschränken wir die Aufenthaltszeit außerhalb des Intervalls durch eine deterministische Konstante. Dies wird durch Bedingen auf Approximationen des besagten Ereignisses und Übergang zum schwachen Grenzwert erreicht.
Wir starten unsere Analyse mit dem Fall eines unbeschränkten Intervalls, in dem wir den resultierenden Prozess durch eine Pfadzerlegung beschreiben. Insbesondere bestimmen wir die exakten Verteilungen der gesamten Aufenthaltszeit außerhalb des Intervalls und der letzten Eintrittszeit in dasselbe. Zudem formulieren wir Grenzwertsätze für die genannten Größen, wenn der Startpunkt gegen ∞ bzw. $-\infty$ divergiert.
Falls das Intervall beschränkt ist, fokussieren wir uns auf Startpunkte innerhalb. In diesem Setting beweisen wir, dass der resultierende Prozess das Intervall überhaupt nicht verlässt, sondern genau die gleiche stochastische Differentialgleichung löst wie eine Brownsche Bewegung, die darauf bedingt wird innerhalb des Intervalls zu bleiben. Dieses Resultat ist ein sehr ungewöhnliches Extrembeispiel entropischer Abstoßung. Auf dem Weg dorthin bestimmen wir die exakte Asymptotik der Wahrscheinlichkeit, dass eine Brownsche Bewegung innerhalb der ersten T Zeiteinheiten wenig Zeit außerhalb des Intervalls verbringt, für $T \to \infty$.

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Contents

1 Introduction

1.1 Motivation and Overview

Conditioning stochastic processes on avoiding certain sets is a classical problem in probability theory. In [Doo57], Doob developed his celebrated theory of h-transforms. This lead, in particular, to the result that a Brownian motion starting in $y > 0$ which is conditioned to avoid the negative half-line is nothing but a three-dimensional Bessel process starting in y. A more modern presentation of this result can be found in [Pit75]. Proceeding from Doob's work, similar problems have been considered for more complicated processes and more complicated or time-dependent sets to be avoided. Many examples are referred to in the introduction of [Bar20].

In this thesis, we proceed in a different direction: We condition a Brownian motion on spending *limited time outside* of intervals. More formally, our setting will be as follows: We consider a Brownian motion $B = (B_t)_{t\geq 0}$ with starting point $y \in \mathbb{R}$ as well as a non-trivial interval $I \subsetneq \mathbb{R}$ and define

$$\Gamma_T := \int_0^T \mathbb{1}_{\{B_t \notin I\}} \mathrm{d}t, \quad T \in [0, \infty],$$

to be the occupation time of B outside of I until time T. Given a maximum occupation time $s > 0$, our aim is to condition the process B on the event $\{\Gamma_\infty \leq s\}$, which has probability 0:

Lemma 1.1. *In the above setting, we have $\mathbb{P}(\Gamma_\infty \leq s) = 0$.*

Later on, we will in fact determine the exact asymptotic behavior of $\mathbb{P}(\Gamma_T \leq s)$ as $T \to \infty$ (cf. the proof of Proposition 2.9 as well as Theorem 3.2). For now, we restrict ourselves to the above result and provide an elementary proof based on the strong Markov property of B:

Proof. Since I^c contains an open interval of finite length, it suffices to prove that B spends an infinite amount of time in each such interval. By the scaling property and translation invariance of B, it suffices to consider the interval $(-1, 1)$. Setting $\tau^0 := 0$, we define the a.s. finite stopping times

$$\sigma^n := \inf\{t \geq \tau^{n-1} : B_t = 0\} \quad \text{and} \quad \tau^n := \inf\{t \geq \sigma^n : |B_t| = 1\}, \qquad n \in \mathbb{N}.$$

The strong Markov property implies that $(\{\tau^n - \sigma^n \geq 1\})_{n\in\mathbb{N}}$ is a family of independent events with

$$\mathbb{P}(\tau^n - \sigma^n \geq 1) = \mathbb{P}(\tau^1 - \sigma^1 \geq 1) > 0, \quad n \in \mathbb{N}.$$

Hence the second Borel-Cantelli lemma yields

$$\mathbb{P}\left(\int_0^\infty \mathbb{1}_{\{B_t\in(-1,1)\}}\mathrm{d}t = \infty\right) \geq \mathbb{P}(\tau^n - \sigma^n \geq 1 \text{ for infinitely many } n \in \mathbb{N}) = 1$$

proving the claim. □

Consequently, it is not possible to simply consider the conditional law $\mathbb{P}(B \in \cdot \,|\, \Gamma_\infty \leq s)$. We bypass this problem by approximating the event $\{\Gamma_\infty \leq s\}$ in perhaps the most natural way: Instead of restricting the occupation time on the whole infinite time axis, we restrict it on a large but finite time horizon $T \geq 0$ and let T tend to ∞. More formally, we consider the probability measures

$$\mathbb{P}(B \in \cdot \,|\, \Gamma_T \leq s), \quad T \in [0, \infty),$$

and show that, as $T \to \infty$, they converge weakly on the space $\mathcal{C}([0,\infty))$. Here and in what follows, this function space is, as usual, endowed with the topology of locally uniform convergence and the corresponding Borel σ-algebra.
Provided that a weak limit exists and assuming for a moment that I is closed to simplify the argument, it is straightforward to check that the total occupation time of a corresponding limiting process X outside I must be bounded by s: For every $T_0 \geq 0$, the event

$$C_{T_0} := \left\{ f \in \mathcal{C}([0,\infty)) : \int_0^{T_0} \mathbb{1}_{\{f_t \notin I\}}\mathrm{d}t \leq s \right\}$$

is closed in $\mathcal{C}([0,\infty))$ with $\mathbb{P}(B \in C_{T_0} \,|\, \Gamma_T \leq s) = 1$ for each $T \geq T_0$ so that the portmanteau theorem implies $\mathbb{P}(X \in C_{T_0}) = 1$. Taking $T_0 \to \infty$ yields the claim.
Naively, one might expect that the allowed s time units outside I are exhausted because the enforced condition already is a very severe restriction in comparison to the typical behavior of B. However, this is not the case: The limiting process will a.s. spend less than s time units outside I. While this is true for every choice of I, the behavior of the resulting process is still fundamentally different depending on whether I is bounded or not.
In Chapter 2, we deal with the case that I is unbounded. Under the additional assumption that B starts on the boundary point of I, this problem was already considered in [RY10] and [BB11]. The focus of our contribution lies on a thorough analysis of the resulting process and its properties in dependence on the starting point $y \in \mathbb{R}$, which can be chosen arbitrarily. In Theorem 2.1, we give an explicit description of the limiting process X^y in terms of a path decomposition. In particular, we determine the distribution of its last entrance time into I, which is a.s. finite. Theorem 2.2 provides the exact law of the occupation time of X^y outside I. In Theorem 2.4, we finally discuss how the distributions of the two quantities just mentioned behave asymptotically as $y \to \pm\infty$ by providing weak limit theorems.
Chapter 3 is concerned with the case that I is bounded. In this setting, we focus on starting points inside I. In Theorem 3.1, we show that the resulting process is exactly the diffusion obtained when conditioning B (by a similar limiting procedure) on not

leaving I at all. In particular, the limiting process does not only spend less than s time units outside I but a.s. absolutely no time. The key to this result is Theorem 3.2, where we determine the exact asymptotic behavior of $\mathbb{P}_y(\Gamma_T \leq s)$ as $T \to \infty$, including all terms of sub-exponential and polynomial order. In Corollary 3.3, we briefly discuss why a limiting process cannot exist in the usual sense for starting points outside $\overline{I}$.
The fact that the allowed time outside I is not exhausted completely is in line with related research, as we will discuss in the following section. However, our extreme results in the case of a bounded interval are utterly unusual.

1.2 Entropic Repulsion and Related Work

Our results can be seen seen as instances of a phenomenon called *entropic repulsion*. The term *entropy* was introduced by the physicist Clausius as the name of a thermodynamical state function. It is derived from the (Ancient) Greek word $\tau\rho o\pi\eta$, meaning *turning* or *transformation*, and is supposed to bear resemblance to the word *energy* (see p. 390 in [Cla65]). Naturally, the notion of entropy and, more specifically, of entropic repulsion found its way into the probabilistic language through mathematical physics and statistical mechanics: One of the first high-impact papers by probabilists explicitly mentioning entropic repulsion is [BDZ95], which deals with the lattice free field. In connection with Brownian motion, the notion of entropic repulsion first appeared in [BB10], where Benjamini and Berestycki characterize it as follows:

> Roughly speaking, entropic repulsion describes the fact that the easiest way to achieve a certain global constraint for a random process is to achieve much more than required. [BB10, p. 820]

Kolb and Savov gave a similar description of entropic repulsion:

> This phenomenon [...] usually refers to the fact that conditioning on an unlikely event often results in a process whose behavior appears to be even more unlikely than the one which the process is conditioned on. [KS16, p. 4085]

Despite the fact that the notion of entropic repulsion has not been used in the context of Brownian motion at that time, the result going back to [Doo57] mentioned at the beginning of this introduction can also be regarded as an instance of this phenomenon: A Brownian motion starting in $y > 0$ which is conditioned to stay non-negative a.s. has a strictly positive minimum. As already mentioned, the proof is based on Doob's theory of h-transforms, which he develops in the very same paper.

In the following subsections, we discuss a few more results related to this thesis. In all presented theorems, entropic repulsion is clearly visible. Nevertheless, with the exception of the rather exotic penalization discussed in Subsection 1.2.5, each of the limiting processes somehow makes use of the respective conditions and may, in a suitable sense, even come arbitrarily close to exhausting the condition with positive probability.
The setting will always be as follows: Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion. Further, for each $t \geq 0$, let $\mathcal{F}_t \subseteq \mathcal{B}(\mathcal{C}([0,\infty)))$ be the initial σ-algebra associated with $\mathcal{B}(\mathcal{C}([0,t]))$ under the canonical projection $\mathcal{C}([0,\infty)) \to \mathcal{C}([0,t])$.

1.2.1 Uniformly Bounded Local Time Everywhere

In [BB10], Benjamini and Berestycki condition B on having uniformly bounded local time, w.l.o.g. by the constant 1, at every point. Clearly, the resulting process $(X_t)_{t\geq 0}$ must be transient with $\limsup_{t\to\infty} \frac{|X_t|}{t} \geq 1$ a.s. We recall that the unconditioned Brownian motion B grows significantly slower: According to the law of the iterated logarithm (see, e.g., Theorem 5.1 in [MP10]), we have

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log(\log(t))}} = 1 \quad \text{a.s.}$$

Nevertheless, it turns out that $\lim_{t\to\infty} \frac{|X_t|}{t}$ exists and is equal to a constant $\gamma > 4.5$ a.s. This means that, even though the process is already forced to grow untypically fast, it still grows more than 4.5-times faster than required.
To formulate the precise result, let $(L_t^x)_{t\geq 0, x\in\mathbb{R}}$ be a jointly continuous version of the Brownian local time, i.e., a real-valued stochastic process such that

$$\mathbb{R} \times [0,\infty) \to \mathbb{R}, \quad (x,t) \mapsto L_t^x,$$

is a.s. continuous with

$$L_t^x = \lim_{\varepsilon\searrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{|B_s - x| \leq \varepsilon\}} \mathrm{d}s, \quad t \geq 0,\ x \in \mathbb{R}, \quad \text{a.s.}$$

For the existence of such a random field, alternative characterizations and further information, we refer to Chapter 6 of [MP10], Chapter VI of [RY99] and Chapter 29 of [Kal21]. Note that some other authors use a definition, which differs from the above by a factor of 2.
The event we would like to condition on is given by $\{\sup_{t\geq 0, x\in\mathbb{R}} L_t^x \leq 1\}$ and has probability 0. To approximate this event, let $\tau_a' := \inf\{t \geq 0 : |B_t| = a\}$ be the first exit time from $(-a,a)$ for each $a > 0$. Moreover, we define

$$\gamma := \frac{3}{1 - 2j_0^{-2}} \approx 4.5860\ldots,$$

where j_0 denotes the first non-negative zero of the Bessel function J_0 of the first kind and order 0. The announced result, which can be found in Remark 4 on Theorem 2 of [BB10], can be stated as follows:

Theorem 1.2. *As $a \to \infty$, the probability measures*

$$\mathbb{P}\left(B \in \cdot \;\middle|\; \sup_{t\in[0,\tau_a'],\, x\in\mathbb{R}} L_t^x \le 1\right) \tag{1.1}$$

converge weakly on the Skorokhod space $\mathcal{D}([0,\infty))$ to the law of a process $X = (X_t)_{t\ge 0}$ satisfying

$$\mathbb{P}\left(\lim_{t\to\infty} \frac{X_t}{t} = \gamma\right) = \mathbb{P}\left(\lim_{t\to\infty} \frac{X_t}{t} = -\gamma\right) = \frac{1}{2}.$$

Theorem 2 of [BB10] actually is a one-sided version of this result: If the first exit time τ_a' is replaced by the first hitting time $\tau_a := \inf\{t \ge 0 : B_t = a\}$ for each $a > 0$, the limiting process X escapes to ∞ with $\lim_{t\to\infty} \frac{X_t}{t} = \gamma$ a.s. The chosen approximation of the event of interest is closely connected with the proof, which we summarize in the following paragraph.
Apart from a few coupling arguments, the key ingredient of the proof is the Ray-Knight theorem: It connects the law of the local time at time τ_a, when considered as a process of the space variable x, with the laws of squared Bessel processes of dimensions 0 and 2 (see, e.g., Theorem VI.52.1 in [RW00]). In particular, it implies that $\{\sup_{t\le\tau_a, x\in\mathbb{R}} L_t^x \le 1\}$ is, for each $a > 0$, essentially equivalent to the condition that a suitable two-dimensional Brownian motion stays inside the unit ball $D \subseteq \mathbb{R}^2$ until time a. This is used at first to establish the existence of the limiting process X by proving that the probability measures in (1.1) form a Cauchy sequence w.r.t. a suitable topology of local convergence in total variation. Afterwards, it is used again to associate an eigenvalue problem for the Laplacian on the unit disc D with Dirichlet boundary condition on the one hand with the growth rate γ on the other hand.
We remark that Theorem 5 of [BB10] provides a similar result for the symmetric simple random walk. Further, the still unpublished preprint [Mot14] suggests that the growth rate γ in the above countinuous time setting highly depends on the chosen approximation of $\{\sup_{t\ge 0, x\in\mathbb{R}} L_t^x \le 1\}$.

1.2.2 Limited Local Time in 0

Multiple authors have considered a standard Brownian motion B with restricted local time $(L_t)_{t\ge 0} := (L_t^0)_{t\ge 0}$ in the starting point 0.

In [RVY06], Roynette, Valois and Yor develop multiple rather general penalization results w.r.t. several classes of functionals of B. In this subsection, we restrict ourselves to the presentation of a special case of their result concerning local time in 0: If $(L_t)_{t\ge 0}$ is forced to be bounded by 1, then the local time in 0 of the resulting process is uniformly distributed on $[0, 1]$. After its last zero, which is finite a.s., the absolute value of the limiting process behaves like a three-dimensional Bessel process (cf. the result going back to [Doo57] mentioned before) and hence diverges to ∞ a.s. The detailed result, which is a special case of Proposition 3.11 as well as Theorems 3.13 and 4.8 of [RVY06] with $h^+ = h^- = \mathbb{1}_{[0,1]}$, reads as follows:

Theorem 1.3. *(a) The process*

$$(M_t)_{t\geq 0} := \big(1 - L_t \wedge 1 + |B_t| \mathbb{1}_{\{L_t \leq 1\}}\big)_{t\geq 0}$$

is a martingale w.r.t. the filtration generated by B *and solves the SDE problem*

$$M_t = 1, \qquad \mathrm{d}M_t = \operatorname{sgn}(B_t)\mathbb{1}_{\{L_t\leq 1\}}\mathrm{d}B_t, \quad t \geq 0.$$

(b) There exists a probability measure $\mathbb{Q}$ *on* $\mathcal{C}([0,\infty))$ *satisfying*

$$\mathbb{Q}(A_t) = \lim_{T\to\infty} \mathbb{P}(B \in A_t \,|\, L_T \leq 1) = \mathbb{E}\left[\mathbb{1}_{A_t} M_t\right], \quad A_t \in \mathcal{F}_t,\ t \geq 0.$$

(c) Let $X = (X_t)_{t\geq 0} \sim \mathbb{Q}$ *be a corresponding limiting process. Further, let* $(L^X_t)_{t\geq 0}$ *be its local time process in* 0 *and*

$$g := \sup\{t \geq 0 : X_t = 0\} \in [0, \infty]$$

its last zero. Then the following assertions hold:

(i) The last zero g *is finite a.s.*

(ii) The total local time $L^X_\infty := \lim_{t\to\infty} L^X_t$ *is uniformly distributed on* $[0,1]$.

(iii) The processes $(X_{t\wedge g})_{t\geq 0}$ *and* $(X_{g+t})_{t\geq 0}$ *are independent.*

(iv) For each $l \in [0,1]$, *conditioned on* $\{L^X_\infty = l\}$, *the process* $(X_{t\wedge g})_{t\geq 0}$ *is a standard Brownian motion stopped at* $\inf\{t \geq 0 : L^X_t = l\}$.

(v) The process $(|X_{g+t}|)_{t\geq 0}$ *is a three-dimensional Bessel process starting in* 0.

(vi) The sign of $(X_{g+t})_{t\geq 0}$ *is uniformly distributed on* $\{-1, 1\}$.

Part (b), in particular, implies that $\mathbb{P}(B \in \cdot \mid L_T \leq 1)$ converges to $\mathbb{Q}$ weakly on $\mathcal{C}([0,\infty))$ as $T \to \infty$ (cf. Theorem 5 of [Whi70]).
Part (a) is a consequence of the so-called balayage formulas (see, e.g., Section VI.4 of [RY99]). Part (b) follows by direct calculations involving the Markov property and an application of Kolmogorov's existence theorem. Finally, the proof of part (c) is based on martingale theory, a progressive enlargement of the filtration and Itô's formula.

Remark 1.4. The existence of a weak limit and the construction of a corresponding limiting process is shown independently and with a different focus by Benjamini and Berestycki in Theorem 2 of [BB11], where a proof based on Itô's excursion theory (see, e.g., Section III.4.3 of [IW81]) is given. The key argument is that conditioning on the event $\{L_T \leq 1\}$ is asymptotically equivalent, as $T \to \infty$, to conditioning on B having exactly one excursion which is longer than T before the local time exceeds 1.

Applying the mentioned results of [RVY06] to $h^+ = 2\mathbb{1}_{[0,1]}$ and $h^- = 0$ instead, one obtains a biased version of Theorem 1.3, where the resulting process $(X_t)_{t\geq 0}$ a.s. escapes to $+\infty$. More precisely, it satisfies the assertions (i)-(iv) of Theorem 1.3 and, with g defined as above, the process $(X_{g+t})_{t\geq 0}$ is a three-dimensional Bessel process starting in 0.
Intermediate cases, in a suitable sense, are covered as well by the results of [RVY06]. Theorem 2 of [Deb09] provides a similar result for the symmetric simple random walk.

Since the local time is a pathwise non-decreasing process of time, we have

$$\{L_T \leq 1\} = \{L_t \leq 1 \text{ for all } t \in [0,T]\}, \quad T \geq 0.$$

Consequently, Theorem 1.3 implies that, if we condition B on $\{L_t \leq 1 \text{ for all } t \geq 0\}$, the resulting process X will be *transient*, i.e., satisfy $\lim_{t\to\infty} |X_t| = \infty$ a.s. If we instead condition B on $\{L_t \leq t \text{ for all } t \geq 0\}$, the resulting process X will trivially remain a Brownian motion and hence be *recurrent*, i.e., satisfy $\sup\{t \geq 0 : X_t = 0\} = \infty$ a.s. These observations naturally lead to the question how B behaves when being conditioned on

$$\{L_t \leq f(t) \text{ for all } t \geq 0\}$$

for a general function $f : [0,\infty) \to (0,\infty)$. In particular, it is of interest if a sharp transition between transience and recurrence occurs and, if so, at which growth rate of f. Replacing f by $t \mapsto \sup_{s\in[0,t]} f(s)$, one may assume that f is non-decreasing. Noting that the scaling property of B implies $L_t \overset{\mathrm{d}}{=} \sqrt{t}L_1$ for each $t \geq 0$, it is reasonable to only consider functions f growing slower than $\sqrt{\cdot}$ in the sense that $\frac{f}{\sqrt{\cdot}}$ is non-increasing.
The problem was first discussed by Benjamini and Berestycki: Under the above assumptions on f, Theorem 1 of [BB11] states that the family

$$(\mathbb{P}(B \in \cdot \mid L_t \leq f(t) \text{ for all } t \in [0,T]))_{T\geq 0} \tag{1.2}$$

of approximating probability measures on $\mathcal{C}([0,\infty))$ is tight and that

$$I(f) := \int_1^\infty \frac{f(t)}{t^{\frac{3}{2}}}\mathrm{d}t < \infty \tag{1.3}$$

implies that any subsequential weak limit is transient a.s. The authors also conjecture that this integral test is sharp in the sense that $I(f) = \infty$ implies that any subsequential weak limit is recurrent a.s.
In [KS16], Kolb and Savov prove that the approximating probability measures in (1.2) actually converge weakly and verify the above conjecture – both under mild technical assumptions on f, which are satisfied by functions in the critical regime of the integral test.
In the transient case, Theorem 1 of [KS16] characterizes the limiting process X explicitly. In particular, it is shown that $|X|$ a.s. eventually behaves like a three-dimensional Bessel process. Additionally, a description of the marginal laws of the (right-continuous inverse) local time of X in 0 is given.
In the recurrent case covered by Theorems 3 and 4 of [KS16], another kind of entropic repulsion phenomenon is discovered: Confirming a second conjecture of [BB11], it is proved that the local time process of X in 0 is, with high probability, asymptotically dominated by a deterministic function $\tilde{f} : [0,\infty) \to (0,\infty)$ with $\lim_{t\to\infty} \frac{\tilde{f}(t)}{f(t)} = 0$. Moreover, the set of all such functions is characterized analytically.
Instead of dealing with the local time itself, Kolb and Savov mainly work with the right-continuous inverse local time

$$(\tau_l)_{l\geq 0} := (\inf\{t > 0 : L_t > l\})_{l\geq 0},$$

which is a stable subordinator of index $\frac{1}{2}$ (see, e.g., p. 240 in [RY99]). The corresponding time-changed version of the conditioning events is given by

$$(O_l)_{l\geq 0} := (\{\tau_\lambda > f^{-1}(\lambda) : \lambda \in [0,l]\})_{l\geq 0},$$

where $f^{-1} : [0,\infty) \to [0,\infty)$ is a non-decreasing left-inverse of f. The first main step of the proof is to determine the exact asymptotic behavior of $\mathbb{P}(O_l)$ and, in the recurrent case, of $\int_0^l \mathbb{P}(O_\lambda)\mathrm{d}\lambda$ as $l \to \infty$. This is done with the help of the one large jump principle of $(\tau_l)_{l\geq 0}$, which can be seen as the time-changed equivalent to the one long excursion principle used in [BB11] (see Remark 1.4). Regarding the integral, a thorough case distinction w.r.t. to jump sizes of $(\tau_l)_{l\geq 0}$ is necessary. With the help of the asymptotics, the limiting behavior of the inverse local time can be analyzed in both cases. Then Itô's excursion theory is used to prove the claimed weak convergence as well as the explicit description of the limiting process in the transient case.
In [Bar20], the results of [KS16] are generalized to suitable classes of recurrent Markov processes. The integral test distinguishing between recurrence and transience of the resulting process becomes

$$I(f) := \int_1^\infty f(t)\mathrm{d}\nu(t) < \infty, \tag{1.4}$$

where ν denotes the Lévy measure of the inverse local time subordinator of the considered Markov process. In the Brownian case, this measure has a Lebesgue density given by

$$\mathbb{R}\setminus\{0\} \to \mathbb{R}, \quad t \mapsto \mathbb{1}_{(0,\infty)}(t)\frac{1}{\sqrt{2\pi}t^{\frac{3}{2}}},$$

so that the tests (1.3) and (1.4) are equivalent.

1.2.3 Limited Number of Down-Crossings

As already mentioned in the previous subsection, [RVY06] contains penalization results w.r.t. different classes of functionals of B. Besides functionals of local time, supremum and infimum, the authors also consider functionals associated with the number of down-crossings. For the sake of simplicity, we, once again, restrict ourselves to the presentation of the special case of their result which is most closely related to this thesis.
Let $a, b \in \mathbb{R}$ with $a < b$ and let $m \in \mathbb{N}_0$. If B is conditioned on having at most m down-crossings through the interval $[a,b]$, the number of down-crossings of the resulting process will be uniformly distributed on $\{0,\dots,m\}$. With probability $\frac{1}{2}$, the limiting process will up-cross the interval again after its last down-crossing. After the last (down- or up-)crossing, it will essentially behave like a three-dimensional Bessel process. In particular, the limiting process is transient.
For the sake of completeness, let us recall the formal definition of the number of down-crossings: Setting $\tau^0 := 0$, we first define the a.s. finite stopping times

$$\sigma^n := \inf\{t \geq \tau^{n-1} : B_t > b\} \quad \text{and} \quad \tau^n := \inf\{t \geq \sigma^n : B_t < a\}, \qquad n \in \mathbb{N}.$$

Then

$$D_T := \sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau^n \leq T\}}, \quad T \geq 0,$$

is the number of down-crossings of B through $[a,b]$ up to time T.
The announced result, which follows immediately from Theorems 3.21 and 4.11 of [RVY06] by applying them to

$$G : \mathbb{N}_0 \to \mathbb{R}, \quad n \mapsto \max\left\{\frac{n}{m+1}, 0\right\},$$

reads as follows:

Theorem 1.5. *(a) There exists a probability measure $\mathbb{Q}$ on $\mathcal{C}([0,\infty))$ satisfying*

$$\mathbb{Q}(A_t) = \lim_{T\to\infty} \mathbb{P}(B \in A_t \,|\, D_T \leq m), \quad A_t \in \mathcal{F}_t,\ t \geq 0.$$

(b) Let $X = (X_t)_{t\geq 0} \sim \mathbb{Q}$ be a corresponding limiting process. For each $t \geq 0$, let D_t^X be the number of down-crossings of the process X through $[a,b]$ until time t and let $D_\infty^X := \lim_{t\to\infty} D_t^X$ be the total number of down-crossings. Further, let

$$g := \inf\{t \geq 0 : D_t^X = D_t^\infty\} \quad \text{and} \quad \overline{g} := \inf\{t \geq g : X_t > b\}$$

(with the usual convention $\inf \emptyset := \infty$) be the times the last down-crossing and the subsequent up-crossing end, respectively. Then the following assertions hold:

- *(i) The number of down-crossings D_∞^X is uniformly distributed on $\{0,\dots,m\}$.*
- *(ii) We have $\mathbb{P}(\overline{g} < \infty) = \frac{1}{2}$.*
- *(iii) For each $n \in \{0,\dots,m\}$, conditioned on $\{D_\infty^X = n\}$, the process $(X_{t\wedge g})_{t\geq 0}$ is a standard Brownian motion stopped at the end of the n-th down-crossing.*
- *(iv) Conditioned on $\{\overline{g} = \infty\}$, the processes $(X_{t\wedge g})_{t\geq 0}$ and $(X_{g+t})_{t\geq 0}$ are independent. Moreover, $(2b-a-X_{g+t})_{t\geq 0}$ is a three-dimensional Bessel process started at $2(b-a)$ and conditioned to stay above $b-a$.*
- *(v) Conditioned on $\{\overline{g} < \infty\}$, the processes $(X_{t\wedge g})_{t\geq 0}$, $(X_{(g+t)\wedge\overline{g}})_{t\geq 0}$ and $(X_{\overline{g}+t})_{t\geq 0}$ are independent. Moreover, $(2b-a-X_{(g+t)\wedge\overline{g}})_{t\geq 0}$ is a three-dimensional Bessel process started at $2(b-a)$ and stopped when hitting $b-a$ for the first time. Finally, $(X_{\overline{g}+t}-a)_{t\geq 0}$ is a three-dimensional Bessel process started at $b-a$.*

Similar to Theorem 1.3, part (a) implies that $\mathbb{P}(B \in \cdot \,|\, D_T \leq m)$ converges to $\mathbb{Q}$ weakly on $\mathcal{C}([0,\infty))$ as $T \to \infty$. The proof is similar as well: At first, it is shown that there exists a martingale $(M_t)_{t\geq 0}$ w.r.t. the filtration generated by B satisfying

$$\lim_{T\to\infty} \mathbb{P}(B \in A_t \,|\, D_T \leq m) = \mathbb{E}\left[\mathbb{1}_{A_t} M_t\right], \quad A_t \in \mathcal{F}_t,\ t \geq 0.$$

Together with Kolmogorov's existence theorem, this guarantees the existence of $\mathbb{Q}$. The process $(M_t)_{t\geq 0}$ is, in fact, determined explicitly in Proposition 3.20 of [RVY06]. The

proof of part (b) is based on the optional stopping theorem, a progressive enlargement of the filtration and Itô's formula.
The structural similarity between the above result and Theorem 1.3 is not a mere coincidence. According to Theorem 1.2.15 of [NRY09], there exists a universal σ-finite measure $\mathbb{W}$ on $\mathcal{C}([0,\infty))$, which is described explicitly in Theorem 1.1.6 of [NRY09], such that the limiting laws $\mathbb{Q}$ of the two mentioned results are absolutely continuous w.r.t. $\mathbb{W}$. In fact, this abstract theorem holds for a rather large class of penalizations by functionals. Roughly speaking, the crucial assumption is the existence of a deterministic compact interval $J \subseteq \mathbb{R}$ such that those sample paths in $\mathcal{C}([0,\infty))$, which leave J eventually, do not get penalized anymore after leaving J for the last time. According to Theorem 1.2.14 of [NRY09], the assumptions in particular require the expected penalization (in our setting: the probability of the conditioning) to decay like a multiple of $\frac{1}{\sqrt{T}}$ as $T \to \infty$.

1.2.4 Limited Maximum Excursion Length

In [RVY09], Roynette, Valois and Yor condition B on having uniformly bounded excursion lengths. In other words, they force every completed excursion of B (i.e., every excursion returning to 0) to last for at most 1 time unit. Then the supremum of the lengths of all completed excursions of the resulting process will have the same distribution as the square of a uniform distribution on $[0,1]$. Further, the limiting process will a.s. have a finite last zero after which the absolute value of the limiting process behaves like a three-dimensional Bessel process.
Before we formulate the result, we introduce some notation. Let

$$g_t := \sup\{s \in [0,t] : B_s = 0\} \quad \text{and} \quad d_t := \inf\{s \geq t : B_s = 0\}, \qquad t \geq 0,$$

be the last zero before and the first zero after time t, respectively. Than the length of the excursion around t is given by $d_t - g_t$ for each $t \geq 0$. We define

$$\Sigma_T := \sup\{d_t - g_t : t \geq 0,\, d_t \leq T\}, \quad T \geq 0, \tag{1.5}$$

to be the supremum of the lengths of all excursions completed before time T. Further, for each $t \geq 0$, let $E_t := t - g_t$ be the duration the ongoing excursion at time t already lasts. Theorem 2.1 of [RVY09] then states:

Theorem 1.6. *(a) There exists a probability measure $\mathbb{Q}$ on $\mathcal{C}([0,\infty))$ satisfying*

$$\mathbb{Q}(A_t) = \lim_{T\to\infty} \mathbb{P}(B \in A_t \,|\, \Sigma_T \leq 1), \quad A_t \in \mathcal{F}_t,\ t \geq 0.$$

(b) Let $X = (X_t)_{t\geq 0} \sim \mathbb{Q}$ be a corresponding limiting process and let Σ_∞^X be the supremum of the lengths of all its completed excursions (which is defined similar to (1.5)). Further, let L_∞^X be the total local time of X in 0 and let

$$g := \sup\{t \geq 0 : X_t = 0\} \in [0,\infty]$$

be its last zero. Then the following assertions hold:

(i) *The last zero* g *is finite a.s. and satisfies* $\mathbb{P}(g < t) = \mathbb{E}[E_{t\wedge\sigma}]$ *for each* $t \geq 0$ *with* $\sigma := \inf\{t \geq 0 : E_t = 1\}$.

(ii) *The random variable* $\sqrt{\Sigma_\infty^X}$ *is uniformly distributed on* $[0, 1]$.

(iii) *The processes* $(X_{t\wedge g})_{t\geq 0}$ *and* $(X_{g+t})_{t\geq 0}$ *are independent.*

(iv) *The process* $(|X_{g+t}|)_{t\geq 0}$ *is a three-dimensional Bessel process starting in* 0.

(v) *The sign of* $(X_{g+t})_{t\geq 0}$ *is uniformly distributed on* $\{-1, 1\}$.

(vi) *The random variable* $\sqrt{\frac{2}{\pi}} L_\infty^X$ *is exponentially distributed with parameter* 1.

Similar to Theorems 1.3 and 1.5, part (a) implies that $\mathbb{P}(B \in \cdot \,|\, \Sigma_T \leq 1)$ converges to $\mathbb{Q}$ weakly on $\mathcal{C}([0,\infty))$ as $T \to \infty$. The general structure of the proof is similar as well: Using a generalization of Itô's formula, it is shown that some explicitly constructed process $(M_t)_{t\geq 0}$ is a (local) martingale w.r.t. the filtration generated by B. The existence of $\mathbb{Q}$ is then guaranteed by proving

$$\lim_{T\to\infty} \mathbb{P}(B \in A_t \,|\, \Sigma_T \leq 1) = \mathbb{E}\left[\mathbb{1}_{A_t} M_t\right], \quad A_t \in \mathcal{F}_t,\ t \geq 0.$$

To this end, the asymptotic behavior of $\mathbb{P}(\Sigma_T \leq 1)$ as $T \to \infty$ is determined. The first two assertions of part (b) are proved with the help of Itô's excursion theory. The proof of the remaining assertions is based on a progressive enlargement of the filtration and Itô's formula.
Theorem 3 of [Deb09] provides a similar result for the symmetric simple random walk.

1.2.5 Limited Number of Passages of the Integrated Process

As a final example, we discuss a result of [Pro15], where Profeta penalizes B by functionals of the integrated Brownian motion

$$Z := (Z_t)_{t\geq 0} := \left(\int_0^t B_s \mathrm{d}s\right)_{t\geq 0}.$$

Letting $z > 0$ and setting $\tau_z^0 := 0$, we define

$$\tau_z^n := \inf\{t > \tau_z^{n-1} : Z_t = z\}, \quad n \in \mathbb{N}.$$

On the one hand, the stopping time τ_z^n is a.s. finite (and well-defined) for each $n \in \mathbb{N}$ since Z is recurrent. On the other hand, the sequence $(\tau_z^n)_{n\in\mathbb{N}}$ is a.s. strictly increasing due to $B_{\tau_z^n} \neq 0$ a.s. for all $n \in \mathbb{N}$. Formally, both properties of $(\tau_z^n)_{n\in\mathbb{N}}$ follow immediately from the density formulas developed in Section 3 of [McK63]. Consequently, it is justified for each $n \in \mathbb{N}$ to call τ_z^n the time of the n-th passage of Z through z.
Theorem 3.3(ii) of [Pro15] now shows that the process obtained by conditioning B such that Z passes through z at most $m \in \mathbb{N}_0$ times does not depend on m. This can be seen as a very rare extreme example of entropic repulsion. Further properties of the resulting process are discussed in Theorem 2.4 and Corollary 2.8 of [Pro15]. The results may be summarized as follows:

Theorem 1.7. *There exist a harmonic function $h_z : \mathbb{R} \times (-\infty, z) \to \mathbb{R}$ and a probability measure $\mathbb{Q}_z$ on $\mathcal{C}([0,\infty))$ with*

$$\begin{aligned}\mathbb{Q}_z(A_t) &= \lim_{T\to\infty} \mathbb{P}(B \in A_t \,|\, \tau_z^{m+1} > T) \\ &= \mathbb{E}\left[\mathbb{1}_{A_t} \frac{h_z(B_t, Z_t)}{h_z(0,0)} \mathbb{1}_{\{\tau_z^1 > t\}}\right], \quad A \in \mathcal{F}_t,\ t \geq 0,\ m \in \mathbb{N}_0.\end{aligned}$$

A corresponding limiting process $(X_t^z)_{t\geq 0} \sim \mathbb{Q}_z$ satisfies

$$\mathbb{P}\left(\int_0^\infty X_s^z \mathrm{d}s = -\infty\right) = 1$$

and

$$\mathbb{P}\left(\int_0^t X_s^z \mathrm{d}s < a \textit{ for all } t \geq 0\right) = \frac{h(0, z-a)}{h(0,0)}, \quad a \in (0, z].$$

Similar to theorems from previous subsections, the first statement implies weak convergence of the conditional laws to $\mathbb{Q}_z$. The function h_z is given explicitly by a double integral.

The main ingredients of the proof are the Markov property of (B, Z) as well as the asymptotic behavior of $\mathbb{P}(\tau_z^{m+1} > T)$ as $T \to \infty$ for $m \in \mathbb{N}_0$. The latter is obtained by analytic calculations involving integral transforms, which Profeta himself describes as "purely computational and rather technical" [Pro15, p. 158].

At least for $z \searrow 0$, occupation times of the limiting process above curves have been studied. In particular, Theorem 6.1 of [GJW99] shows that, for any $k > 0$, the expected time spend above $t \mapsto -kt^\alpha$ is finite if, and only if, $\alpha < \frac{9}{10}$ holds.

2 Limited Time Outside an Unbounded Interval

2.1 Overview and Main Results

In this chapter, we allow a Brownian motion with an arbitrary starting point to spend limited time in the negative half-line. More precisely, we fix $y \in \mathbb{R}$ and let $B = (B_t)_{t \geq 0}$ be a Brownian motion starting in y. For each $T \geq 0$, let

$$\Gamma_T := \int_0^T \mathbb{1}_{\{B_s < 0\}} \mathrm{d}s$$

be the time B spends below 0 until time T. In Theorem 2.1, we will show that

$$\mathbb{P}(B \in \cdot \mid \Gamma_T \leq 1)$$

convergences weakly in $\mathcal{C}([0,\infty))$ as $T \to \infty$. The limiting process can informally be described as follows: Up to a random time $g \geq 0$, the distribution of which we will determine explicitly, the limiting process is a Brownian bridge starting in y, ending in 0 and conditioned to have limited occupation time. Afterwards, it is a three-dimensional Bessel process. For $y > 0$, the event $\{g = 0\}$ occurs with positive probability. If it occurs, the Bessel process starts immediately so that the limiting process stays positive all the time. A rigorous construction is given below.

Even though we allow the limiting process to spend a total of 1 time unit below 0, its actual total occupation time Γ below 0 is a.s. strictly smaller than 1. In Theorem 2.2, we will explicitly determine the distribution of Γ. As discussed in the introductory chapter, the phenomenon that the condition is not exhausted completely can be seen as an instance of entropic repulsion and occurs frequently when conditioning stochastic processes on negligible events.

Our Theorems 2.1 and 2.2 generalize Theorem 4 of [BB11], which covers the special case $y = 0$. This result in turn can be seen as a special case of Theorem 3.1 of [RY10] (applied to $h := \frac{1}{2}\mathbb{1}_{[0,1]}$), which discusses more general restrictions of the occupation time of a standard Brownian motion in the negative half-line. However, the two results were proved independently with completely different methods. The main upshot of the present work is the complete understanding of the limiting process for general $y \in \mathbb{R}$. In particular, we describe the distributions of the last zero $g = g^y$ and the occupation time $\Gamma = \Gamma^y$ below 0 of the limiting process as explicit functions of y.

Without relying on the explicit distributions, Proposition 2.3 will provide an identity connecting the distributions of g and Γ as well as the first entrance time of the limiting process to the negative half-line.
Finally, we will take a closer look at the behavior of the distributions of g^y and Γ^y as functions of y. As $y \to -\infty$, the transformed occupation time $y^2(1-\Gamma^y)$ is approximately exponentially distributed while the weak limit of $y^2(1-g^y)$ has some other explicit distribution (see part (b) of Theorem 2.4). In particular, Γ^y and g^y both converge weakly to 1 as $y \to -\infty$. For $y > 0$, we have to take into account that the limiting process may stay positive permanently. Conditioned on spending time below 0 at all, the distribution of Γ^y is independent of y for $y \geq 0$ while the conditional distribution of $\frac{g^y}{y^2}$ converges weakly to an inverse chi-squared distribution as $y \to \infty$ (see part (a) of Theorem 2.4). In particular, conditional on the existence of a zero, the last zero diverges weakly to ∞ as $y \to \infty$.
In comparison with the overview given in Section 1.1, the above assumptions on the considered interval and the maximum allowed occupation time are made w.l.o.g.: In view of the scaling property of Brownian motion, it is straightforward to replace the single time unit the process is allowed to spend outside $[0, \infty)$ by any other amount $s > 0$ of time. Since Brownian motion is homogeneous in space and symmetric, similar results hold when limiting the time to be spent outside another unbounded interval.

In order to make the informal description of the limiting process given above rigorous, we introduce an auxiliary notation: Let

$$q(t,u) := q^y(t,u) := \mathbb{P}\left(\int_0^t \mathbb{1}_{\{b'_s<0\}} \mathrm{d}s \leq u\right), \quad t,u \geq 0, \tag{2.1}$$

be the probability that a Brownian bridge $(b'_s)_{s\in[0,t]}$ of length t with $b'_0 = y$ and $b'_t = 0$ spends at most u time units below 0. Explicit formulas for q are given in (2.4), (2.7) and (2.8).
Now the limiting process X can be constructed as follows:

1. Depending on the sign of y, let $g = g^y$ be a non-negative random variable with

$$\mathbb{P}(g \leq x) = \begin{cases} \mathbb{1}_{\{x\leq 1\}}\dfrac{\sqrt{x}}{2} + \mathbb{1}_{\{x>1\}}\left(1 - \dfrac{1}{2\sqrt{x}}\right), & x \geq 0,\ y = 0, \\[2ex] \dfrac{\int_0^x q^y(t,1)\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t}{2\int_0^1 \frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t}, & x \geq 0,\ y < 0, \\[2ex] \dfrac{2\sqrt{2\pi}y + \int_0^x q^y(t,1)\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t}{2\sqrt{2\pi}y + 4}, & x \geq 0,\ y > 0. \end{cases} \tag{2.2}$$

   We will see in Corollary 2.8 that the law of g is well-defined with $\mathbb{P}(g < \infty) = 1$ in all three cases. A plot of the distribution function of g for different values of y can be found in Figure 2.1 at the end of this section.

2. Let $b = (b_t)_{t\in[0,1]}$ be a process which, if restricted to $\{g = x\}$ for $x \geq 0$, is a standard Brownian bridge conditioned on

$$\int_0^x \mathbb{1}_{\left\{\sqrt{x}b_{\frac{s}{x}}+y-\frac{s}{x}y<0\right\}}\mathrm{d}s \leq 1.$$

3. Let $Z = (Z_t)_{t\geq 0}$ be a three-dimensional Bessel process starting in 0, independent of (g, b).

4. For $y > 0$, let $Y = (Y_t)_{t\geq 0}$ be a three-dimensional Bessel process starting in y, independent of (g, b, Z).

5. We define the process $X = (X_t)_{t\geq 0}$ by

$$X := \left(\mathbb{1}_{\{g>0,\,t<g\}}\left(\sqrt{g}b_{\frac{t}{g}} + y - \frac{t}{g}y\right) + \mathbb{1}_{\{g>0,\,t\geq g\}}Z_{t-g} + \mathbb{1}_{\{g=0,y>0\}}Y_t\right)_{t\geq 0}.$$

Theorem 2.1. *As $T \to \infty$, the probability measures $\mathbb{P}(B \in \cdot \mid \Gamma_T \leq 1)$ converge weakly on $\mathcal{C}([0,\infty))$ to the law of X.*

Let

$$\Gamma := \Gamma^y := \int_0^\infty \mathbb{1}_{\{X_s<0\}}\mathrm{d}s = \int_0^g \mathbb{1}_{\{X_s<0\}}\mathrm{d}s$$

be the total time X spends below 0. By construction of X, we have $\Gamma \leq 1$. The distribution function of Γ has the following explicit form:

Theorem 2.2. *We have*

$$\mathbb{P}(\Gamma \leq u) = \begin{cases} \dfrac{\int_0^u \frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t}{\int_0^1 \frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t}, & u \in [0,1],\ y \leq 0, \\[2ex] \dfrac{\sqrt{2\pi}y + 2\sqrt{u}}{\sqrt{2\pi}y + 2}, & u \in [0,1],\ y \geq 0. \end{cases} \tag{2.3}$$

A plot of this distribution function for different values of y can be found in Figure 2.1 at the end of this section. Noting

$$q(t,u) = 1, \quad u \geq t \geq 0, \tag{2.4}$$

a comparison of (2.2) and (2.3) yields

$$2\mathbb{P}(g \leq u) = \mathbb{P}(\Gamma \leq u), \quad u \in [0,1],\ y \leq 0. \tag{2.5}$$

This identity is not a mere coincidence. Let

$$\tau := \inf\{t \geq 0 : X_t \leq 0\}$$

be the first entrance time of the limiting process to the negative half-line. Noting $g > 0$ and $\tau = 0$ a.s. for $y \leq 0$, Proposition 2.3 below provides a generalization of (2.5), which is valid for all $y \in \mathbb{R}$. We will prove this result, which is a consequence of the arcsine laws and the strong Markov property, without relying on the explicit formulas (2.2) and (2.3).

Proposition 2.3. *We have*

$$2\mathbb{P}(g \in (0,u]) = \mathbb{P}(\tau + \Gamma \leq u), \quad u \in [0,1].$$

Finally, let us discuss the behavior of the distributions of g^y and Γ^y, respectively, as $y \to \pm\infty$. According to (2.2) and (2.3), we have

$$\mathbb{P}(g^y = 0) = \mathbb{P}(\Gamma^y = 0) = \frac{\sqrt{2\pi}y}{\sqrt{2\pi}y + 2} > 0, \quad y > 0.$$

In particular, the random variables g^y and Γ^y both converge weakly to 0 as $y \to \infty$. Note that the event $\{g^y = 0\} = \{\Gamma^y = 0\}$ (with equality up to a set of probability 0) corresponds to the situation where the limiting process stays positive all the time. Formula (2.3) implies

$$\mathbb{P}(\Gamma^y \leq u \,|\, \Gamma^y > 0) = \sqrt{u} = \mathbb{P}(\Gamma^0 \leq u), \quad u \in [0,1],\ y \geq 0.$$

Consequently, conditioned on spending time below 0 at all, the distribution of the occupation time Γ^y below 0 is given by the square of a uniform distribution on $[0,1]$ for each $y \geq 0$. In particular, this conditional distribution is independent of the starting point $y \geq 0$. The reason is as follows: After the limiting process X^y (starting in $y \geq 0$) hits 0 for the first time, it behaves in distribution like the limiting process X^0 (starting in 0).
The following theorem covers the behavior of g^y as $y \to \infty$ conditioned on the existence of a zero as well as the behavior of Γ^y and g^y as $y \to -\infty$:

Theorem 2.4. *(a) As $y \to \infty$, the conditional distribution $\mathbb{P}\big(\frac{g^y}{y^2} \in \cdot \,|\, g^y > 0\big)$ converges weakly to an inverse chi-squared distribution with Lebesgue density*

$$\mathbb{R} \to [0,\infty), \quad s \mapsto \mathbb{1}_{\{s>0\}} \frac{1}{\sqrt{2\pi s^3}}\, \mathrm{e}^{-\frac{1}{2s}}\,. \tag{2.6}$$

In particular, g^y conditional on $\{g^y > 0\}$ diverges in distribution to ∞ as $y \to \infty$.

(b) As $y \to -\infty$, the random variable $y^2(1-\Gamma^y)$ converges weakly to an exponential distribution with parameter $\frac{1}{2}$ while $y^2(1-g^y)$ converges weakly to a random variable g' with distribution function given by

$$\mathbb{P}(g' \leq u) = \begin{cases} \displaystyle\int_0^\infty \frac{2z}{\sqrt{2\pi(2z-u)}}\, \mathrm{e}^{-z}\, \mathrm{d}z, & u \leq 0, \\[2ex] 1 - \dfrac{1}{2}\, \mathrm{e}^{-\frac{u}{2}}, & u \geq 0. \end{cases}$$

In particular, Γ^y and g^y both converge in distribution to 1 as $y \to -\infty$.

Let us recall that $\inf\{t \geq 0 : B_t^1 = 0\}$, the first zero of a Brownian motion starting in 1, precisely has the Lebesgue density given in (2.6) (see, e.g., Remark 2.8.3 in [KS91]). By the scaling property, $\frac{1}{y^2} \inf\{t \geq 0 : B_t^y = 0\}$ has the same distribution for each $y > 0$. Consequently, we can expect a result similar to part (a) to hold for the *first* zero of X^y. On the other hand, the distribution of the time between the first zero and the last zero of X^y – conditional on their existence – does not depend on y as a consequence of the strong Markov property. As $y \to \infty$, the influence of this portion becomes negligible compared to the amount of time needed to reach zero in the first place.

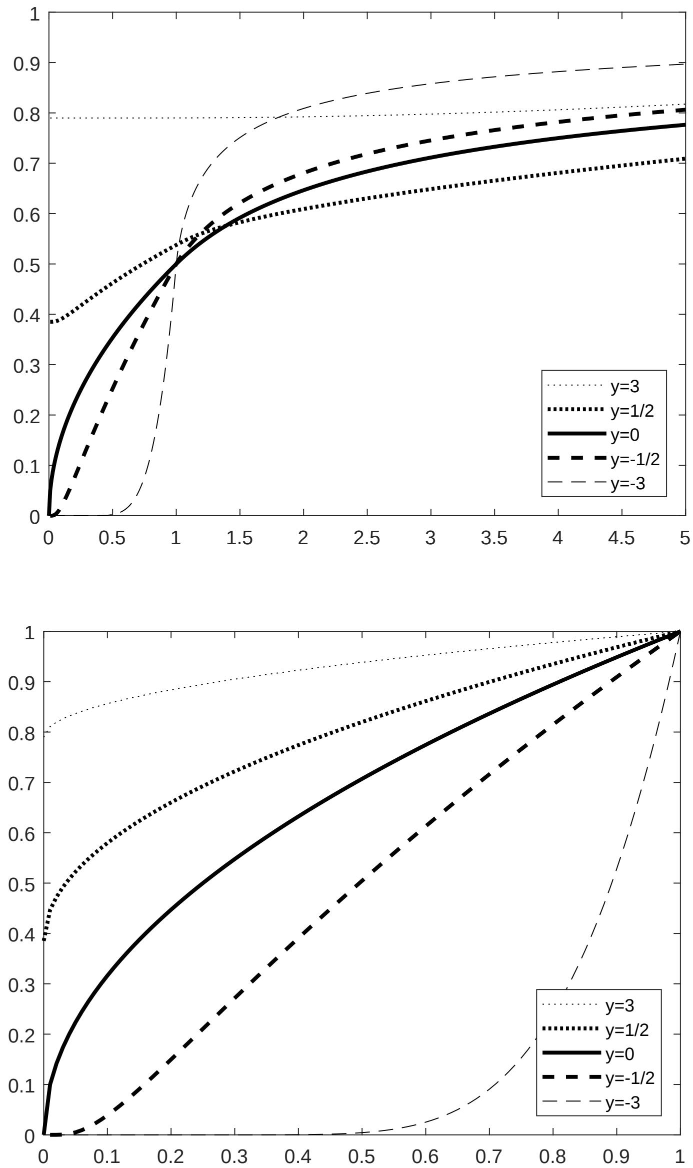

Figure 2.1: Distribution functions of g^y and of Γ^y, respectively, for several values of y

While it certainly is possible to use an abstract martingale approach as in [RY10] to establish Theorem 2.1, we follow a more illustrative path decomposition ansatz as in [BB11]. In contrast to [BB11], we cannot rely on symmetry arguments and it is non-trivial to check that g is well-defined (i.e., a.s. finite).

The outline of the rest of this chapter is as follows: Section 2.2 is concerned with the distribution of g and, in particular, the well-definedness of g. The subsequent two sections are devoted to the proof of Theorem 2.1, which is based on a path decomposition (see Proposition 2.15) around

$$g_T := \max\{s \in [0,T] : B_s = 0\},$$

the last zero of B before time T. The key step will be Proposition 2.9, which states that the conditional distribution of g_T under $\{\Gamma_T \leq 1\}$ converges to the law of g, the last zero of the limiting process, in total variation as $T \to \infty$. In Section 2.5, we will prove the remaining results. For the sake of completeness, we finally collect and prove a few required auxiliary results, such as the mentioned path decomposition, in Section 2.6. Most parts of the first five sections of this chapter were published in [AS22].

2.2 Formulas for q and Finiteness of g

Before we start proving Theorem 2.1, we take a closer look at the distribution of g given in (2.2). More precisely, we provide explicit formulas for q (and hence for the distribution of g) and check that g is a.s. finite. This is crucial for the limiting process X to be well-defined. We start with an auxiliary result, which follows straight from formula (2.15) in [BO99]:

Lemma 2.5. *Given $y \neq 0$ as well as $T > 0$ and $z \in \mathbb{R}$, let $(b_t^z)_{t\in[0,T]}$ be a Brownian bridge of length T with $b_0^z = y$ and $b_T^z = z$. We define*

$$\tau^z := \min\{s \in [0,T] : b_s^z = 0\}$$

(with the convention $\min\emptyset := T$). Then we have

$$\mathbb{P}(\tau^z \in \mathrm{d}t) = \frac{|y|\sqrt{T}}{\sqrt{2\pi t^3(T-t)}}\, \mathrm{e}^{\frac{(y-z)^2}{2T} - \frac{z^2}{2(T-t)} - \frac{y^2}{2t}}\, \mathrm{d}t, \quad t \in (0,T).$$

This is a proper probability density integrating to 1 if and only if $zy \leq 0$ holds.

Proof. Let us consider a Brownian bridge $(\bar{b}_t^z)_{t\in[0,T]}$ of length T with $\bar{b}_T^z = (y-z)\,\mathrm{sgn}(y)$ and $\bar{b}_0^z = 0$. Using a symmetry argument in the case $y > 0$, we get

$$\tau^z \overset{\mathrm{d}}{=} \min\{s \in [0,T] : \bar{b}_s^z = |y|\}.$$

The claim now follows immediately from formula (2.15) of [BO99]. Note that this formula is valid for all $\eta \in \mathbb{R}$ and not only for $\eta < \beta$ (compare Theorem 2.1 in [BO99]). □

Combining this lemma with Lévy's result that the occupation time below 0 of a standard Brownian bridge is uniformly distributed (see [Lév40]), we compute the distribution function $q = q^y$ (see (2.1)) of the occupation time of a Brownian bridge with drift starting in y.

Lemma 2.6. *Given $t > 0$ and $u \in [0, t)$, we have*

$$q(t,u) = \begin{cases} \dfrac{u}{t}, & y = 0, \\ \displaystyle\int_0^u \frac{\sqrt{t}(u-x)|y|}{\sqrt{2\pi x^3(t-x)^3}} \, \mathrm{e}^{\frac{y^2}{2t}-\frac{y^2}{2x}} \, \mathrm{d}x, & y < 0, \\ \displaystyle\int_0^{t-u} \frac{\sqrt{t}uy}{\sqrt{2\pi x^3(t-x)^3}} \, \mathrm{e}^{\frac{y^2}{2t}-\frac{y^2}{2x}} \, \mathrm{d}x + \int_{t-u}^t \frac{\sqrt{t}y}{\sqrt{2\pi x^3(t-x)}} \, \mathrm{e}^{\frac{y^2}{2t}-\frac{y^2}{2x}} \, \mathrm{d}x, & y > 0. \end{cases} \tag{2.7}$$

After a linear time change, formula (E-4) of [Pec], which is proved using Girsanov's theorem, provides the alternative representation

$$q(t,u) = \begin{cases} -2\left(\dfrac{t-u}{t}\left(1-\dfrac{y^2}{t}\right)-1\right)\Phi\left(\dfrac{y\sqrt{t-u}}{\sqrt{tu}}\right) \\ \quad +\dfrac{\sqrt{2u(t-u)}y}{\sqrt{\pi t^3}}\,\mathrm{e}^{\frac{y^2}{2t}-\frac{y^2}{2u}}, & y \leq 0, \\ 1+2\left(\dfrac{u}{t}\left(1-\dfrac{y^2}{t}\right)-1\right)\Phi\left(-\dfrac{y\sqrt{u}}{\sqrt{t(t-u)}}\right) \\ \quad +\dfrac{\sqrt{2u(t-u)}y}{\sqrt{\pi t^3}}\,\mathrm{e}^{\frac{y^2}{2t}-\frac{y^2}{2(t-u)}}, & y \geq 0, \end{cases} \tag{2.8}$$

for all $t > 0$ and $u \in [0, t)$, where $\Phi : \mathbb{R} \to \mathbb{R}$ denotes the distribution function of the standard normal distribution. We remark this formula just for the sake of completeness and will not rely on it in what follows.

Proof of Lemma 2.6. The formula for $y = 0$ is a classical result by Lévy (see [Lév40]). Now let $y \neq 0$ and let $(b'_s)_{s\in[0,t]}$ be a Brownian bridge of length t with $b'_0 = y$ and $b'_t = 0$. Furthermore, let $\tau' := \min\{s \in [0, t] : b'_s = 0\}$ be the first zero of b', which is a.s. strictly smaller than t. Then

$$\left(\hat{b}_s\right)_{s\in[0,1]} := \left(\frac{1}{\sqrt{t-\tau'}} b'_{\tau'+s(t-\tau')}\right)_{s\in[0,1]}$$

is a standard Brownian bridge independent of τ' (see Corollary 2.17). By Lévy's result, $\int_0^1 \mathbb{1}_{\{\hat{b}_s<0\}}\mathrm{d}s$ is uniformly distributed on $[0, 1]$. Now let $y < 0$. We observe $b'_s < 0$

for all $s < \tau'$. Together with Lemma 2.5, we obtain

$$\begin{aligned} q(t,u) &= \mathbb{P}\left(\int_{\tau'}^{t} \mathbb{1}_{\{b'_s<0\}} \mathrm{d}s \le u-\tau'\right) \\ &= \mathbb{P}\left(\int_0^1 \mathbb{1}_{\{\hat{b}_s<0\}} \mathrm{d}s \le \frac{u-\tau'}{t-\tau'}\right) \\ &= \int_0^u \mathbb{P}\left(\int_0^1 \mathbb{1}_{\{\hat{b}_s<0\}} \mathrm{d}s \le \frac{u-x}{t-x}\right) \mathbb{P}(\tau' \in \mathrm{d}x) \\ &= \int_0^u \frac{u-x}{t-x} \cdot \frac{\sqrt{t}|y|}{\sqrt{2\pi x^3(t-x)}} \,\mathrm{e}^{\frac{y^2}{2t}-\frac{y^2}{2x}} \,\mathrm{d}x. \end{aligned}$$

Given $y > 0$, we similarly observe $b'_s > 0$ for all $s < \tau'$ and obtain

$$\begin{aligned} q(t,u) &= \mathbb{P}\left(\int_{\tau'}^{t} \mathbb{1}_{\{b'_s<0\}} \mathrm{d}s \le u\right) \\ &= \mathbb{P}\left(\int_0^1 \mathbb{1}_{\{\hat{b}_s<0\}} \mathrm{d}s \le \frac{u}{t-\tau'}\right) \\ &= \int_0^t \mathbb{P}\left(\int_0^1 \mathbb{1}_{\{\hat{b}_s<0\}} \mathrm{d}s \le \frac{u}{t-x}\right) \mathbb{P}(\tau' \in \mathrm{d}x) \\ &= \int_0^t \left(\frac{u}{t-x} \wedge 1\right) \frac{\sqrt{t}y}{\sqrt{2\pi x^3(t-x)}} \,\mathrm{e}^{\frac{y^2}{2t}-\frac{y^2}{2x}} \,\mathrm{d}x \end{aligned}$$

proving the claim. □

Next we prove that g is a.s. finite. This will essentially follow from the subsequent lemma as we will see in Corollary 2.8 below. The more general formulation of the lemma will help us to determine the distribution of Γ (see Theorem 2.2). Besides, it is rather difficult to verify the formulas directly for a single $u > 0$.

Lemma 2.7. *We have*

$$\int_u^\infty q(t,u) \frac{1}{\sqrt{t}} \,\mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t = \int_0^u \frac{1}{\sqrt{t}} \,\mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t, \quad u \ge 0,\ y < 0$$

and

$$\int_0^u \frac{1}{\sqrt{t}} \,\mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t + \int_u^\infty q(t,u) \frac{1}{\sqrt{t}} \,\mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t = 4\sqrt{u}, \quad u \ge 0,\ y > 0.$$

Corollary 2.8. *The random variable g, as defined in* (2.2)*, is a.s. finite.*

Proof of Corollary 2.8. For $y = 0$, the claim is clear from the definition. Applying (2.2) and (2.4) as well as Lemma 2.7 with $u = 1$, we obtain

$$\begin{aligned} \mathbb{P}(g < \infty) &= \frac{\int_0^\infty q(t,1) \frac{1}{\sqrt{t}} \,\mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t}{2\int_0^1 \frac{1}{\sqrt{t}} \,\mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t} \\ &= \frac{\int_0^1 \frac{1}{\sqrt{t}} \,\mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t + \int_1^\infty q(t,1) \frac{1}{\sqrt{t}} \,\mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t}{2\int_0^1 \frac{1}{\sqrt{t}} \,\mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t} = 1, \quad y < 0. \end{aligned}$$

Similarly, we get

$$\mathbb{P}(g<\infty)=\frac{2\sqrt{2\pi}y+\int_0^1\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t+\int_1^\infty q(t,1)\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t}{2\sqrt{2\pi}y+4}=1,\quad y>0,$$

proving the claim. □

Proof of Lemma 2.7. Noting $q(t,0)=0$ for all $t>0$, both formulas hold for $u=0$. Hence it suffices to show that the derivatives of the left- and the right-hand side coincide in both cases.
First we consider $y<0$. Given $t>0$, an application of (2.7) and of Leibniz's rule yields

$$\frac{\mathrm{d}}{\mathrm{d}u}q(t,u)\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}=\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u\frac{(u-x)|y|}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x=\int_0^u\frac{|y|}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x$$

for all $u\in(0,t)$. Now fix $u_0>0$. Noting

$$\begin{aligned}\int_{u_0}^\infty\sup_{u\in(0,u_0)}\left|\frac{\mathrm{d}}{\mathrm{d}u}q(t,u)\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\right|\mathrm{d}t&=\int_{u_0}^\infty\sup_{u\in(0,u_0)}\left|\int_0^u\frac{|y|}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\right|\mathrm{d}t\\&=\int_{u_0}^\infty\int_0^{u_0}\frac{|y|}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\mathrm{d}t\\&=\int_0^{u_0}\frac{2|y|}{\sqrt{2\pi x^3(u_0-x)}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\\&<\infty,\end{aligned}$$

we can differentiate w.r.t. $u\in(0,u_0)$ under the integral $\int_{u_0}^\infty$. Together with Leibniz's rule applied to the integral $\int_u^{u_0}$ and with Tonelli's theorem, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}\int_u^\infty q(t,u)\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t&=\int_u^\infty\frac{\mathrm{d}}{\mathrm{d}u}q(t,u)\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t-q(u,u)\frac{1}{\sqrt{u}}\,\mathrm{e}^{-\frac{y^2}{2u}}\\&=\int_u^\infty\int_0^u\frac{|y|}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\mathrm{d}t-\frac{1}{\sqrt{u}}\,\mathrm{e}^{-\frac{y^2}{2u}}\\&=\int_0^u\frac{2|y|}{\sqrt{2\pi x^3(u-x)}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x-\frac{1}{\sqrt{u}}\,\mathrm{e}^{-\frac{y^2}{2u}}\\&=\int_0^u\frac{|y|\sqrt{u}}{\sqrt{2\pi x^3(u-x)}}\,\mathrm{e}^{\frac{y^2}{2u}-\frac{y^2}{2x}}\,\mathrm{d}x\cdot\frac{2}{\sqrt{u}}\,\mathrm{e}^{-\frac{y^2}{2u}}-\frac{1}{\sqrt{u}}\,\mathrm{e}^{-\frac{y^2}{2u}}\end{aligned}$$

for all $u\in(0,u_0)$ and consequently for all $u>0$. Given a Brownian bridge $(b'_{s,u})_{s\in[0,u]}$ of length u with $b'_{0,u}=y$ and $b'_{u,u}=0$, Lemma 2.5 implies

$$1=\mathbb{P}(\min\{s\in[0,u]:b'_{s,u}=0\}\in(0,u))=\int_0^u\frac{|y|\sqrt{u}}{\sqrt{2\pi x^3(u-x)}}\,\mathrm{e}^{\frac{y^2}{2u}-\frac{y^2}{2x}}\,\mathrm{d}x$$

so that we can deduce

$$\frac{\mathrm{d}}{\mathrm{d}u}\int_u^\infty q(t,u)\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t = 1\cdot\frac{2}{\sqrt{u}}\,\mathrm{e}^{-\frac{y^2}{2u}} - \frac{1}{\sqrt{u}}\,\mathrm{e}^{-\frac{y^2}{2u}} = \frac{1}{\sqrt{u}}\,\mathrm{e}^{-\frac{y^2}{2u}} = \frac{\mathrm{d}}{\mathrm{d}u}\int_0^u \frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t$$

for all $u > 0$, as claimed.
Now we consider $y > 0$. Given $t > 0$, an application of (2.7) and of Leibniz's rule yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}u}q(t,u)\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\\
&= \frac{\mathrm{d}}{\mathrm{d}u}\int_0^{t-u}\frac{uy}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}u}\int_{t-u}^{t}\frac{y}{\sqrt{2\pi x^3(t-x)}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\\
&= \int_0^{t-u}\frac{y}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x - \frac{uy}{\sqrt{2\pi(t-u)^3u^3}}\,\mathrm{e}^{-\frac{y^2}{2(t-u)}} + \frac{y}{\sqrt{2\pi(t-u)^3u}}\,\mathrm{e}^{-\frac{y^2}{2(t-u)}}\\
&= \int_0^{t-u}\frac{y}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x
\end{aligned}\tag{2.9}$$

for all $u \in (0,t)$. Now fix $u_0, \varepsilon > 0$. We get

$$\begin{aligned}
\int_{u_0}^\infty \sup_{u\in(\varepsilon,u_0)}\left|\frac{\mathrm{d}}{\mathrm{d}u}q(t,u)\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\right|\mathrm{d}t &= \int_{u_0}^\infty \sup_{u\in(\varepsilon,u_0)}\left|\int_0^{t-u}\frac{y}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\right|\mathrm{d}t\\
&= \int_{u_0}^\infty\int_0^{t-\varepsilon}\frac{y}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\mathrm{d}t\\
&= \int_0^\infty\int_{u_0\vee(x+\varepsilon)}^\infty\frac{y}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}t\mathrm{d}x\\
&= \int_0^\infty\frac{2y}{\sqrt{2\pi x^3((u_0-x)\vee\varepsilon)}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\\
&< \infty
\end{aligned}$$

so that we can differentiate w.r.t. $u \in (\varepsilon, u_0)$ under the integral $\int_{u_0}^\infty$. Similar to the previous case, the integral $\int_u^{u_0}$ can be differentiated with the help of Leibniz's rule. Together with $q(u,u) = 1$ and (2.9) in the second step, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}u}\left(\int_0^u\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t + \int_u^\infty q(t,u)\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t\right)\\
&= \frac{1}{\sqrt{u}}\,\mathrm{e}^{-\frac{y^2}{2u}} + \int_u^\infty\frac{\mathrm{d}}{\mathrm{d}u}q(t,u)\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t - q(u,u)\frac{1}{\sqrt{u}}\,\mathrm{e}^{-\frac{y^2}{2u}}\\
&= \int_u^\infty\int_0^{t-u}\frac{y}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

for all $u \in (\varepsilon, u_0)$ and consequently for all $u > 0$. Substituting $x = \frac{y^2}{z^2}$, we get

$$\begin{aligned}
\int_u^\infty \int_0^{t-u} \frac{y}{\sqrt{2\pi x^3 (t-x)^3}} \,\mathrm{e}^{-\frac{y^2}{2x}} \,\mathrm{d}x\mathrm{d}t &= \int_0^\infty \int_{x+u}^\infty \frac{y}{\sqrt{2\pi x^3 (t-x)^3}} \,\mathrm{e}^{-\frac{y^2}{2x}} \,\mathrm{d}t\mathrm{d}x \\
&= \int_0^\infty \frac{2y}{\sqrt{2\pi x^3 u}} \,\mathrm{e}^{-\frac{y^2}{2x}} \,\mathrm{d}x \\
&= \int_0^\infty \frac{2yz^3}{\sqrt{2\pi y^6 u}} \,\mathrm{e}^{-\frac{z^2}{2}} \frac{2y^2}{z^3} \mathrm{d}z \\
&= \frac{2}{\sqrt{u}} \int_0^\infty \frac{2}{\sqrt{2\pi}} \,\mathrm{e}^{-\frac{z^2}{2}} \,\mathrm{d}z \\
&= \frac{2}{\sqrt{u}}
\end{aligned}$$

proving

$$\frac{\mathrm{d}}{\mathrm{d}u} \left(\int_0^u \frac{1}{\sqrt{t}} \,\mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t + \int_u^\infty q(t,u) \frac{1}{\sqrt{t}} \,\mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t \right) = \frac{\mathrm{d}}{\mathrm{d}u} 4\sqrt{u}, \quad u > 0.$$

This completes the proof. □

2.3 Convergence of the Distribution of the Last Zero

Recall that

$$g_T := \max\{s \in [0,T] : B_s = 0\}$$

denotes the last zero of B before time $T > 0$ with the convention $\max \emptyset := 0$. As already mentioned at the end of Section 2.1, the following result is the key part of the proof of Theorem 2.1:

Proposition 2.9. *As $T \to \infty$, the probability measures $\mathbb{P}(g_T \in \cdot \mid \Gamma_T \leq 1)$ converge to the law of g in total variation.*

To prove this result, we compute the asymptotics of the density

$$\frac{\mathbb{P}(g_T \in \mathrm{d}x, \Gamma_T \leq 1)}{\mathrm{d}x}, \quad x > 0,$$

and of the conditioning probability $\mathbb{P}(\Gamma_T \leq 1)$ as $T \to \infty$. The latter has essentially been computed before in Section 2.8 of [RY10]. Here we give a different argument. Regarding the latter, the main idea is to condition on g_T. To avoid a case distinction and obtain the rather compact formula (2.2), we additionally condition on B_T. After considering the mass in 0, corresponding to the case that the process has no zero, the proposition follows from Scheffé's lemma.

Proof of Proposition 2.9.
Step 1: We start by explicitly computing the density $\frac{\mathbb{P}(g_T\in \mathrm{d}x,\Gamma_T\leq 1)}{\mathrm{d}x}$ and its asymptotics for $x>0$. Let $x_0\geq 0$ and $T>x_0+1$. Moreover, let $z>0$ and let $b^z=(b^z_t)_{t\in[0,T]}$ be a Brownian bridge of length T starting in y and ending in z. Once again with the convention $\max\emptyset := 0$, we define $\gamma^z_T := \max\{s\in[0,T] : b^z_s = 0\}$. Conditioned on $\{\gamma^z_T>0\}$ (i.e., on the existence of a zero of b^z), the process

$$\left(\hat{b}^z_s\right)_{s\in[0,1]} := \left(\frac{1}{\sqrt{\gamma^z_T}}\left(b^z_{s\gamma^z_T}-y+sy\right)\right)_{s\in[0,1]}$$

is a standard Brownian bridge independent of γ^z_T (see Proposition 2.16). Consequently, the process $(\sqrt{t}\hat{b}^z_{\frac{s}{t}}+y-\frac{s}{t}y)_{s\in[0,t]}$ is a Brownian bridge of length t starting in y and ending in 0 for each $t>0$. Using $T>x_0+1$ in the first step and recalling $B_0=y$, we get

$$\begin{aligned}
&\mathbb{P}(g_T\in(0,x_0],\Gamma_T\leq 1)\\
&=\mathbb{P}\big(g_T\in(0,x_0],\Gamma_{g_T}\leq 1,B_T>0\big)\\
&=\int_0^\infty \mathbb{P}\left(\gamma^z_T\in(0,x_0],\int_0^{\gamma^z_T}\mathbb{1}_{\{b^z_s<0\}}\mathrm{d}s\leq 1\right)\mathbb{P}(B_T\in\mathrm{d}z)\\
&=\int_0^\infty \mathbb{P}\left(\gamma^z_T\in(0,x_0],\gamma^z_T\int_0^1\mathbb{1}_{\left\{b^z_{s\gamma^z_T}<0\right\}}\mathrm{d}s\leq 1\right)\mathbb{P}(B_T\in\mathrm{d}z)\\
&=\int_0^\infty \mathbb{P}\left(\gamma^z_T\in(0,x_0],\gamma^z_T\int_0^1\mathbb{1}_{\left\{\sqrt{\gamma^z_T}\hat{b}^z_s+y-sy<0\right\}}\mathrm{d}s\leq 1\right)\mathbb{P}(B_T\in\mathrm{d}z)\\
&=\int_0^\infty\int_{(0,x_0]}\mathbb{P}\left(t\int_0^1\mathbb{1}_{\left\{\sqrt{t}\hat{b}^z_s+y-sy<0\right\}}\mathrm{d}s\leq 1\right)\mathbb{P}(\gamma^z_T\in\mathrm{d}t)\mathbb{P}(B_T\in\mathrm{d}z)\\
&=\int_0^\infty\int_{(0,x_0]}\mathbb{P}\left(\int_0^t\mathbb{1}_{\left\{\sqrt{t}\hat{b}^z_{\frac{s}{t}}+y-\frac{s}{t}y<0\right\}}\mathrm{d}s\leq 1\right)\mathbb{P}(\gamma^z_T\in\mathrm{d}t)\mathbb{P}(B_T\in\mathrm{d}z)\\
&=\int_0^\infty\int_{(0,x_0]}q(t,1)\mathbb{P}(\gamma^z_T\in\mathrm{d}t)\mathbb{P}(B_T\in\mathrm{d}z).
\end{aligned}\tag{2.10}$$

Let $\left(\bar{b}^z_s\right)_{s\in[0,T]}$ be a Brownian bridge of length T starting in z and ending in y. According to Lemma 2.5, we have

$$\mathbb{P}(\min\{s\in[0,T]:\bar{b}^z_s=0\}\in\mathrm{d}t)=\frac{z\sqrt{T}}{\sqrt{2\pi t^3(T-t)}}\,\mathrm{e}^{\frac{(z-y)^2}{2T}-\frac{y^2}{2(T-t)}-\frac{z^2}{2t}}\,\mathrm{d}t,\quad t\in(0,T),$$

and hence

$$\begin{aligned}
\mathbb{P}(\gamma^z_T\in\mathrm{d}t)&=\mathbb{P}(T-\min\{s\in[0,T]:\bar{b}^z_s=0\}\in\mathrm{d}t)\\
&=\frac{z\sqrt{T}}{\sqrt{2\pi(T-t)^3t}}\,\mathrm{e}^{\frac{(z-y)^2}{2T}-\frac{y^2}{2t}-\frac{z^2}{2(T-t)}}\,\mathrm{d}t,\quad t\in(0,T).
\end{aligned}$$

Combining this with (2.10), we obtain

$$\begin{aligned}
&\mathbb{P}(g_T \in (0,x_0], \Gamma_T \leq 1)\\
&= \int_0^\infty \int_0^{x_0} q(t,1) \frac{z\sqrt{T}}{\sqrt{2\pi(T-t)^3 t}} \, \mathrm{e}^{\frac{(z-y)^2}{2T} - \frac{y^2}{2t} - \frac{z^2}{2(T-t)}} \, \mathrm{d}t \frac{1}{\sqrt{2\pi T}} \, \mathrm{e}^{-\frac{(z-y)^2}{2T}} \, \mathrm{d}z\\
&= \frac{1}{2\pi} \int_0^{x_0} q(t,1) \, \mathrm{e}^{-\frac{y^2}{2t}} \int_0^\infty \frac{z}{\sqrt{t(T-t)^3}} \, \mathrm{e}^{-\frac{z^2}{2(T-t)}} \, \mathrm{d}z\mathrm{d}t\\
&= \frac{1}{2\pi} \int_0^{x_0} q(t,1) \, \mathrm{e}^{-\frac{y^2}{2t}} \frac{1}{\sqrt{t(T-t)}} \mathrm{d}t\\
&= \frac{1}{2\pi} \int_0^{x_0} q(t,1) \frac{1}{\sqrt{t\left(1-\frac{t}{T}\right)}} \, \mathrm{e}^{-\frac{y^2}{2t}} \, \mathrm{d}t \cdot \frac{1}{\sqrt{T}}.
\end{aligned}$$

Since $x_0 > 0$ has been chosen arbitrarily, we can deduce

$$\begin{aligned}
\frac{\mathbb{P}(g_T \in \mathrm{d}x, \Gamma_T \leq 1)}{\mathrm{d}x} &= \frac{1}{2\pi} q(x,1) \frac{1}{\sqrt{x\left(1-\frac{x}{T}\right)}} \, \mathrm{e}^{-\frac{y^2}{2x}} \cdot \frac{1}{\sqrt{T}}\\
&\sim \frac{1}{2\pi} q(x,1) \frac{1}{\sqrt{x}} \, \mathrm{e}^{-\frac{y^2}{2x}} \cdot \frac{1}{\sqrt{T}}, \quad T \to \infty,\ x > 0.
\end{aligned} \tag{2.11}$$

Step 2: Next we compute the asymptotics of $\mathbb{P}(\Gamma_T \leq 1)$ and, combining it with (2.11), prove

$$\frac{\mathbb{P}(g_T \in \mathrm{d}x \,|\, \Gamma_T \leq 1)}{\mathrm{d}x} \to \frac{\mathbb{P}(g \in \mathrm{d}x)}{\mathrm{d}x}, \quad T \to \infty,\ x > 0.$$

To this end, let $T > 1$.
First we consider $y = 0$. Using the scaling property of B, Lévy's arcsine law and the definition of the derivative, we get

$$\begin{aligned}
\mathbb{P}(\Gamma_T \leq 1) &= \mathbb{P}\left(\int_0^1 \mathbb{1}_{\{B_s<0\}} \mathrm{d}s \leq \frac{1}{T}\right)\\
&= \frac{2}{\pi} \arcsin\left(\frac{1}{\sqrt{T}}\right) \sim \frac{2}{\pi} \arcsin'(0) \frac{1}{\sqrt{T}} = \frac{2}{\pi} \cdot \frac{1}{\sqrt{T}}, \quad T \to \infty.
\end{aligned}$$

Combining this with (2.11) in the first step, using (2.4) and (2.7) in the second step as well as (2.2) in the third, we deduce

$$\begin{aligned}
\frac{\mathbb{P}(g_T \in \mathrm{d}x \,|\, \Gamma_T \leq 1)}{\mathrm{d}x} \to \frac{1}{4} q(x,1) \frac{1}{\sqrt{x}} &= \mathbb{1}_{\{x \leq 1\}} \frac{1}{4\sqrt{x}} + \mathbb{1}_{\{x>1\}} \frac{1}{4\sqrt{x^3}}\\
&= \frac{\mathbb{P}(g \in \mathrm{d}x)}{\mathrm{d}x}, \quad T \to \infty,\ x > 0,\ y = 0.
\end{aligned}$$

Now let $y < 0$ and let $B^0 = (B^0_t)_{t\geq 0}$ be a Brownian motion starting in 0. Using the scaling property and the symmetry of B^0, we obtain

$$\begin{aligned}\mathbb{P}(\Gamma_T \leq 1) &= \mathbb{P}\left(\int_0^T \mathbb{1}_{\{B^0_s<-y\}}\mathrm{d}s \leq 1\right)\\ &= \mathbb{P}\left(\int_0^1 \mathbb{1}_{\left\{B^0_s<-\frac{y}{\sqrt{T}}\right\}}\mathrm{d}s \leq \frac{1}{T}\right)\\ &= \mathbb{P}\left(\int_0^1 \mathbb{1}_{\left\{B^0_s>-\frac{y}{\sqrt{T}}\right\}}\mathrm{d}s \geq 1-\frac{1}{T}\right)\\ &= 1-\mathbb{P}\left(\int_0^1 \mathbb{1}_{\left\{B^0_s>-\frac{y}{\sqrt{T}}\right\}}\mathrm{d}s \leq 1-\frac{1}{T}\right).\end{aligned}$$

Applying formula (12) of [Tak98] and substituting $z = \frac{t}{T}$, we deduce

$$\mathbb{P}(\Gamma_T \leq 1) = \frac{1}{\pi}\int_0^{\frac{1}{T}} \frac{1}{\sqrt{z(1-z)}}\,\mathrm{e}^{-\frac{y^2}{2zT}}\,\mathrm{d}z = \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{t(1-\frac{t}{T})}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t\cdot\frac{1}{\sqrt{T}}.$$

The dominated convergence theorem yields

$$\mathbb{P}(\Gamma_T \leq 1) \sim \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t\cdot\frac{1}{\sqrt{T}}, \quad T\to\infty.$$

Combining this with (2.11) and (2.2), we obtain

$$\frac{\mathbb{P}(g_T\in \mathrm{d}x\,|\,\Gamma_T\leq 1)}{\mathrm{d}x} \to \frac{q(x,1)\frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}}{2\int_0^1 \frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t} = \frac{\mathbb{P}(g\in\mathrm{d}x)}{\mathrm{d}x}, \quad T\to\infty,\ x>0,\ y<0.$$

Finally, we consider $y > 0$. Since B^0 is symmetric, we have

$$\begin{aligned}\mathbb{P}(\Gamma_T \leq 1) &= \mathbb{P}\left(\int_0^T \mathbb{1}_{\{B^0_s<-y\}}\mathrm{d}s \leq 1\right)\\ &= \mathbb{P}\left(\int_0^1 \mathbb{1}_{\left\{B^0_s<-\frac{y}{\sqrt{T}}\right\}}\mathrm{d}s \leq \frac{1}{T}\right) = \mathbb{P}\left(\int_0^1 \mathbb{1}_{\left\{B^0_s>\frac{y}{\sqrt{T}}\right\}}\mathrm{d}s \leq \frac{1}{T}\right).\end{aligned}$$

Using formula (12) of [Tak98] and the density of the arcsine distribution, we deduce

$$\begin{aligned}\mathbb{P}(\Gamma_T \leq 1) &= 1-\frac{1}{\pi}\int_0^{1-\frac{1}{T}} \frac{1}{\sqrt{z(1-z)}}\,\mathrm{e}^{-\frac{y^2}{2zT}}\,\mathrm{d}z\\ &= \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{z(1-z)}}\mathrm{d}z - \frac{1}{\pi}\int_0^{1-\frac{1}{T}} \frac{1}{\sqrt{z(1-z)}}\,\mathrm{e}^{-\frac{y^2}{2zT}}\,\mathrm{d}z\\ &= \frac{1}{\pi}\left(\int_0^{1-\frac{1}{T}} \frac{1}{\sqrt{z(1-z)}}\left(1-\mathrm{e}^{-\frac{y^2}{2zT}}\right)\mathrm{d}z + \int_{1-\frac{1}{T}}^1 \frac{1}{\sqrt{z(1-z)}}\mathrm{d}z\right).\end{aligned} \tag{2.12}$$

Recalling $\arcsin'(0) = 1$, the definition of the derivative yields

$$\sqrt{T}\int_{1-\frac{1}{T}}^{1} \frac{1}{\sqrt{z(1-z)}}\mathrm{d}z = \sqrt{T}\int_{0}^{\frac{1}{T}} \frac{1}{\sqrt{(1-z)z}}\mathrm{d}z$$
$$= 2\sqrt{T}\arcsin\left(\frac{1}{\sqrt{T}}\right) \to 2, \quad T \to \infty. \tag{2.13}$$

To obtain the asymptotics of the first term in (2.12), we split the integral at $\frac{1}{2}$. Noting

$$1 - \mathrm{e}^{-\frac{y^2}{2zT}} \leq 1 - \mathrm{e}^{-\frac{y^2}{T}} \leq \frac{y^2}{T}, \quad z \in \left[\frac{1}{2}, 1\right],$$

the dominated convergence theorem implies

$$\sqrt{T}\int_{\frac{1}{2}}^{1-\frac{1}{T}} \frac{1}{\sqrt{z(1-z)}}\left(1 - \mathrm{e}^{-\frac{y^2}{2zT}}\right)\mathrm{d}z \to 0, \quad T \to \infty. \tag{2.14}$$

Using the substitution $t = \frac{y^2}{x^2}$ and integration by parts, we get

$$\int_0^\infty \frac{1}{\sqrt{t}}\left(1 - \mathrm{e}^{-\frac{y^2}{2t}}\right)\mathrm{d}t = \int_0^\infty \frac{2y}{x^2}\left(1 - \mathrm{e}^{-\frac{x^2}{2}}\right)\mathrm{d}x = \sqrt{2\pi}y.$$

In particular, we have

$$\int_0^\infty \sup_{T\geq 1}\left|\frac{1}{\sqrt{t(1-\frac{t}{T})}}\left(1 - \mathrm{e}^{-\frac{y^2}{2t}}\right)\mathbb{1}_{\left\{t\leq\frac{T}{2}\right\}}\right|\mathrm{d}t \leq \int_0^\infty \sup_{T\geq 1}\left|\frac{\sqrt{2}}{\sqrt{t}}\left(1 - \mathrm{e}^{-\frac{y^2}{2t}}\right)\mathbb{1}_{\left\{t\leq\frac{T}{2}\right\}}\right|\mathrm{d}t$$
$$= \int_0^\infty \frac{\sqrt{2}}{\sqrt{t}}\left(1 - \mathrm{e}^{-\frac{y^2}{2t}}\right)\mathrm{d}t < \infty$$

so that we can apply the dominated convergence theorem: Together with the substitution $z = \frac{t}{T}$, we get

$$\sqrt{T}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{z(1-z)}}\left(1 - \mathrm{e}^{-\frac{y^2}{2zT}}\right)\mathrm{d}z = \int_0^{\frac{T}{2}} \frac{1}{\sqrt{t(1-\frac{t}{T})}}\left(1 - \mathrm{e}^{-\frac{y^2}{2t}}\right)\mathrm{d}t$$
$$\to \int_0^\infty \frac{1}{\sqrt{t}}\left(1 - \mathrm{e}^{-\frac{y^2}{2t}}\right)\mathrm{d}t$$
$$= \sqrt{2\pi}y, \quad T \to \infty. \tag{2.15}$$

In view of the three limits (2.13), (2.14) and (2.15), equation (2.12) implies

$$\mathbb{P}(\Gamma_T \leq 1) \sim \frac{1}{\pi}(\sqrt{2\pi}y + 2)\frac{1}{\sqrt{T}}, \quad T \to \infty. \tag{2.16}$$

Combining this with (2.11) and (2.2), we can deduce

$$\frac{\mathbb{P}(g_T \in \mathrm{d}x \,|\, \Gamma_T \le 1)}{\mathrm{d}x} \to \frac{q(x,1)\frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}}{2\sqrt{2\pi}y+4} = \frac{\mathbb{P}(g \in \mathrm{d}x)}{\mathrm{d}x}, \quad T \to \infty,\ x>0,\ y>0.$$

Step 3: Based on the results of step 2 and taking account of the mass in 0 in the case $y > 0$, we finally prove the claimed convergence in total variation. As a consequence of $T > 1$, the reflection principle and the dominated convergence theorem, we have

$$\begin{aligned}
\mathbb{P}(g_T = 0, \Gamma_T \le 1) &= \mathbb{P}(B_t > 0 \text{ for all } t \in [0,T]) \\
&= \mathbb{P}(B_t^0 < y \text{ for all } t \in [0,T]) \\
&= \mathbb{P}(|B_T^0| < y) \\
&= \mathbb{1}_{\{y>0\}} \frac{2}{\sqrt{2\pi}} \int_0^y \mathrm{e}^{-\frac{z^2}{2T}}\,\mathrm{d}z \cdot \frac{1}{\sqrt{T}} \\
&\sim \mathbb{1}_{\{y>0\}} \frac{2}{\sqrt{2\pi}} y \cdot \frac{1}{\sqrt{T}}, \quad T \to \infty.
\end{aligned}$$

Combining this with (2.16) and using (2.2), we obtain

$$\mathbb{P}(g_T = 0 \,|\, \Gamma_T \le 1) \to \mathbb{1}_{\{y>0\}} \frac{\sqrt{2\pi}y}{\sqrt{2\pi}y+2} = \mathbb{P}(g=0), \quad T \to \infty. \tag{2.17}$$

Recalling $\mathbb{P}(g < \infty) = 1$, as shown in Corollary 2.8, we deduce

$$\begin{aligned}
\int_{(0,\infty)} \frac{\mathbb{P}(g_T \in \mathrm{d}x \,|\, \Gamma_T \le 1)}{\mathrm{d}x}\mathrm{d}x &= \mathbb{P}(g_T > 0 \,|\, \Gamma_T \le 1) \\
&\to \mathbb{P}(g > 0) = \int_{(0,\infty)} \frac{\mathbb{P}(g \in \mathrm{d}x)}{\mathrm{d}x}\mathrm{d}x, \quad T \to \infty.
\end{aligned}$$

Since we have already proved

$$\frac{\mathbb{P}(g_T \in \mathrm{d}x \,|\, \Gamma_T \le 1)}{\mathrm{d}x} \to \frac{\mathbb{P}(g \in \mathrm{d}x)}{\mathrm{d}x}, \quad T \to \infty,\ x > 0,$$

Scheffé's lemma implies that the restriction of $\mathbb{P}(g_T \in \cdot \,|\, \Gamma_T \le 1)$ to $((0,\infty), \mathcal{B}((0,\infty)))$ converges in total variation to the corresponding restriction of the law of g as $T \to \infty$. Combining this with (2.17) yields the claim. $\square$

2.4 Proof of the Weak Convergence

To prove the weak convergence claimed in Theorem 2.1, we will (additionally) condition on g_T, the last zero before T. In view of Proposition 2.9, the following result guarantees that it essentially suffices to show the weak convergence of the conditioned process.

Proposition 2.10. *On a measurable space $(\Omega, \mathcal{F})$, let $\mu_\infty, \mu_1, \mu_2, \dots$ be probability measures such that $(\mu_n)_{n\in\mathbb{N}}$ converges to μ_∞ in total variation. Given a metric space E, let $\nu_\infty, \nu_1, \nu_2, \dots : \Omega \times \mathcal{B}(E) \to [0,1]$ be Markov kernels from $(\Omega, \mathcal{F})$ to $(E, \mathcal{B}(E))$ satisfying $\nu_n(t,\,\cdot\,) \Rightarrow \nu_\infty(t,\,\cdot\,)$ as $n \to \infty$ for μ_∞-almost every $t \in \Omega$. Then we have*

$$\int_\Omega \nu_n(t,\,\cdot\,)\mu_n(\mathrm{d}t) \Rightarrow \int_\Omega \nu_\infty(t,\,\cdot\,)\mu_\infty(\mathrm{d}t), \quad n \to \infty.$$

Proof. Let $F \subseteq E$ be closed. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}\int_\Omega \nu_n(t,F)\mu_n(\mathrm{d}t) &\leq \int_\Omega \nu_n(t,F)\,|\mu_n - \mu_\infty|(\mathrm{d}t) + \int_\Omega \nu_n(t,F)\mu_\infty(\mathrm{d}t)\\ &\leq |\mu_\infty - \mu_n|(\Omega) + \int_\Omega \nu_n(t,F)\mu_\infty(\mathrm{d}t),\end{aligned}$$

where $|\mu_\infty - \mu_n|$ denotes the variation of the signed measure $\mu_\infty - \mu_n$. Applying Fatou's lemma and the portmanteau theorem, we deduce

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega \nu_n(t,F)\mu_n(\mathrm{d}t) &\leq \lim_{n\to\infty} |\mu_\infty - \mu_n|(\Omega) + \limsup_{n\to\infty} \int_\Omega \nu_n(t,F)\mu_\infty(\mathrm{d}t)\\ &\leq 0 + \int_\Omega \limsup_{n\to\infty} \nu_n(t,F)\mu_\infty(\mathrm{d}t) \leq \int_\Omega \nu_\infty(t,F)\mu_\infty(\mathrm{d}t)\end{aligned}$$

so that the portmanteau theorem yields the claim. □

Around g_T, we can decompose $(B_t)_{t\in[0,T]}$ into a scaled Brownian bridge with drift and, up to sign, a scaled Brownian meander (see Definition 2.14 and Proposition 2.15). Proposition 2.11 below, which can be considered mathematical folklore, shows that a scaled Brownian meander of infinite length is nothing but a three-dimensional Bessel process starting in 0. Recall that Z is such a process.

Proposition 2.11. *Let $t_0 > 0$ and $x \in [0, t_0]$. Moreover, let $(B_s^+)_{s\in[0,1]}$ be a Brownian meander. Then we have*

$$\lim_{T\to\infty} \mathbb{P}\left(\left(\sqrt{T-x}B^+_{\frac{t}{T-x}}\right)_{t\in[0,t_0-x]} \in A\right) = \mathbb{P}((Z_t)_{t\in[0,t_0-x]} \in A), \quad A \in \mathcal{B}(\mathcal{C}([0,t_0-x])),$$

and consequently

$$\left(\sqrt{T-x}B^+_{\frac{t}{T-x}}\right)_{t\in[0,t_0-x]} \Rightarrow (Z_t)_{t\in[0,t_0-x]}, \quad T \to \infty.$$

Proof. Let $f : \mathcal{C}([0, t_0 - x]) \to \mathbb{R}$ be a Borel-measurable bounded function, e.g., $f = \mathbb{1}_A$ for some $A \in \mathcal{B}(\mathcal{C}([0, t_0 - x]))$, and let $T > t_0$. Section 4 of [Imh84] provides the change of measure formula

$$\mathbb{P}\left((B_s^+)_{s\in[0,1]} \in \cdot\right) = \mathbb{E}\left[\frac{1}{Z_1}\sqrt{\frac{\pi}{2}}\mathbb{1}_{(Z_s)_{s\in[0,1]}\in\cdot}\right]. \tag{2.18}$$

Together with the scaling property of Z, we get

$$\begin{aligned}\mathbb{E}\left[f\left(\left(\sqrt{T-x}B^+_{\frac{t}{T-x}}\right)_{t\in[0,t_0-x]}\right)\right] &= \mathbb{E}\left[\frac{1}{Z_1}\sqrt{\frac{\pi}{2}}f\left(\left(\sqrt{T-x}Z_{\frac{t}{T-x}}\right)_{t\in[0,t_0-x]}\right)\right]\\ &= \mathbb{E}\left[\frac{\sqrt{T-x}}{Z_{T-x}}\sqrt{\frac{\pi}{2}}f\left((Z_t)_{t\in[0,t_0-x]}\right)\right].\end{aligned}$$

Given a three-dimensional Brownian motion $(W_t)_{t\geq 0}$ with $Z = \|W\|_2$, the process

$$Z' := (Z'_t)_{t\geq 0} := (\|W_{t_0-x+t} - W_{t_0-x}\|_2)_{t\geq 0}$$

is a three-dimensional Bessel process independent of $(Z_t)_{t\in[0,t_0-x]}$. The triangle inequality implies $Z_{T-x} \leq Z'_{T-t_0} + Z_{t_0-x}$. Combining this with the scaling property of Z' and the dominated convergence theorem, we obtain

$$\begin{aligned}&\mathbb{E}\left[\frac{\sqrt{T-x}}{Z_{T-x}}\sqrt{\frac{\pi}{2}}f\left((Z_t)_{t\in[0,t_0-x]}\right)\right]\\ &\geq \mathbb{E}\left[\frac{\sqrt{T-x}}{Z'_{T-t_0}+Z_{t_0-x}}\sqrt{\frac{\pi}{2}}f\left((Z_t)_{t\in[0,t_0-x]}\right)\right]\\ &= \mathbb{E}\left[\frac{\sqrt{T-x}}{\sqrt{T-t_0}Z'_1+Z_{t_0-x}}\sqrt{\frac{\pi}{2}}f\left((Z_t)_{t\in[0,t_0-x]}\right)\right]\\ &= \frac{\sqrt{T-x}}{\sqrt{T-t_0}}\mathbb{E}\left[\frac{1}{1+\frac{Z_{t_0-x}}{\sqrt{T-t_0}Z'_1}}\cdot\frac{1}{Z'_1}\sqrt{\frac{\pi}{2}}f\left((Z_t)_{t\in[0,t_0-x]}\right)\right]\\ &\to \mathbb{E}\left[\frac{1}{Z'_1}\sqrt{\frac{\pi}{2}}f\left((Z_t)_{t\in[0,t_0-x]}\right)\right], \quad T\to\infty.\end{aligned}$$

On the other hand, the triangle inequality also implies $Z_{T-x} \geq Z'_{T-t_0} - Z_{t_0-x}$. Together with $\lim_{T\to\infty} Z'_{T-t_0} = \infty$ a.s., a similar computation yields

$$\begin{aligned}&\limsup_{T\to\infty}\mathbb{E}\left[\mathbb{1}_{\{2Z_{t_0-x}\leq Z'_{T-t_0}\}}\frac{\sqrt{T-x}}{Z_{T-x}}\sqrt{\frac{\pi}{2}}f\left((Z_t)_{t\in[0,t_0-x]}\right)\right]\\ &\leq \mathbb{E}\left[\frac{1}{Z'_1}\sqrt{\frac{\pi}{2}}f\left((Z_t)_{t\in[0,t_0-x]}\right)\right].\end{aligned}$$

Using $Z_{T-x} \leq Z'_{T-t_0} + Z_{t_0-x}$ again, we get

$$
\begin{aligned}
0 &\leq \mathbb{E}\left[\mathbb{1}_{\{2Z_{t_0-x} > Z'_{T-t_0}\}} \frac{\sqrt{T-x}}{Z_{T-x}} \sqrt{\frac{\pi}{2}} f\left((Z_t)_{t\in[0,t_0-x]}\right)\right] \\
&\leq \sqrt{\frac{\pi}{2}} \|f\|_\infty \mathbb{E}\left[\mathbb{1}_{\{3Z_{t_0-x} > Z_{T-x}\}} \frac{\sqrt{T-x}}{Z_{T-x}}\right] \\
&= \sqrt{\frac{\pi}{2}} \|f\|_\infty \mathbb{E}\left[\mathbb{1}_{\left\{3Z_{\frac{t_0-x}{T-x}} > Z_1\right\}} \frac{1}{Z_1}\right] \\
&\to \sqrt{\frac{\pi}{2}} \|f\|_\infty \mathbb{E}\left[\mathbb{1}_{\{0 > Z_1\}} \frac{1}{Z_1}\right] \\
&= 0, \quad T \to \infty.
\end{aligned}
$$

As a consequence of (2.18), we have $\mathbb{E}\left[\frac{1}{Z'_1}\sqrt{\frac{\pi}{2}}\right] = 1$. Using this fact as well as the independence of Z' and $(Z_t)_{t\in[0,t_0-x]}$ in the final step, we can deduce

$$
\begin{aligned}
\mathbb{E}\left[f\left(\left(\sqrt{T-x}B^+_{\frac{t}{T-x}}\right)_{t\in[0,t_0-x]}\right)\right] &= \mathbb{E}\left[\frac{\sqrt{T-x}}{Z_{T-x}}\sqrt{\frac{\pi}{2}} f\left((Z_t)_{t\in[0,t_0-x]}\right)\right] \\
&\to \mathbb{E}\left[\frac{1}{Z'_1}\sqrt{\frac{\pi}{2}} f\left((Z_t)_{t\in[0,t_0-x]}\right)\right] \\
&= \mathbb{E}\left[f\left((Z_t)_{t\in[0,t_0-x]}\right)\right], \quad T \to \infty,
\end{aligned}
$$

proving the claim. □

Finally, we are ready to prove Theorem 2.1. In view of the topological structure of $\mathcal{C}([0,\infty))$, it suffices to prove weak convergence in $\mathcal{C}([0,t_0])$ for each $t_0 > 0$ (see Theorem 5 of [Whi70]). If there exists a zero before time $T > t_0$, we use the mentioned path decomposition and the auxiliary result we just proved. If there is no zero before T, we simply use the fact that Brownian motion conditioned to stay positive is nothing but a three-dimensional Bessel process.

Proof of Theorem 2.1. Let $t_0 > 0$ and $T > t_0$. Conditioned on $\{g_T > 0\}$, the process

$$
(b'_s)_{s\in[0,1]} := \left(\frac{1}{\sqrt{g_T}}\left(B_{sg_T} - y + sy\right)\right)_{s\in[0,1]}
$$

is a standard Brownian bridge while

$$
(B^+_s)_{s\in[0,1]} := \left(\frac{1}{\sqrt{T-g_T}}\left|B_{g_T+(T-g_T)s}\right|\right)_{s\in[0,1]}
$$

is a Brownian meander. Moreover, the two processes, g_T and the sign of B_T are mutually independent (see Proposition 2.15). Rewriting the definitions, we get

$$
\begin{aligned}
&(B_t)_{t\in[0,T]} \\
&= \left(\mathbb{1}_{\{t<g_T\}}\left(\sqrt{g_T}b'_{\frac{t}{g_T}} + y - \frac{t}{g_T}y\right) + \mathbb{1}_{\{t\geq g_T\}}\operatorname{sgn}(B_T)\sqrt{T-g_T}B^+_{\frac{t-g_T}{T-g_T}}\right)_{t\in[0,T]}.
\end{aligned}
$$

Now let $x > 0$ and $T \geq x+1$. Using this assumption in the first step and the mentioned independence in the third, we obtain

$$\begin{aligned}
&\mathbb{P}\left((B_t)_{t\in[0,t_0]} \in \cdot \,\middle|\, \Gamma_T \leq 1, g_T = x\right) \\
&= \mathbb{P}\left((B_t)_{t\in[0,t_0]} \in \cdot \,\middle|\, \Gamma_x \leq 1, B_T > 0, g_T = x\right) \\
&= \mathbb{P}\left(\left(\mathbb{1}_{\{t<x\}}\left(\sqrt{x}b'_{\frac{t}{x}} + y - \frac{t}{x}y\right) + \mathbb{1}_{\{t\geq x\}}\sqrt{T-x}B^+_{\frac{t-x}{T-x}}\right)_{t\in[0,t_0]} \in \cdot \,\middle|\, \dots\right. \\
&\qquad\qquad \left.\dots \,\middle|\, \int_0^x \mathbb{1}_{\left\{\sqrt{x}b'_{\frac{s}{x}} + y - \frac{s}{x}y < 0\right\}} \mathrm{d}s \leq 1, B_T > 0, g_T = x\right) \\
&= \mathbb{P}\left(\left(\mathbb{1}_{\{t<x\}}\left(\sqrt{x}b'_{\frac{t}{x}} + y - \frac{t}{x}y\right) + \mathbb{1}_{\{t\geq x\}}\sqrt{T-x}B^+_{\frac{t-x}{T-x}}\right)_{t\in[0,t_0]} \in \cdot \,\middle|\, \dots\right. \\
&\qquad\qquad \left.\dots \,\middle|\, \int_0^x \mathbb{1}_{\left\{\sqrt{x}b'_{\frac{s}{x}} + y - \frac{s}{x}y < 0\right\}} \mathrm{d}s \leq 1\right).
\end{aligned} \tag{2.19}$$

On the one hand, we have

$$\begin{aligned}
&\mathbb{P}\left(\left(\sqrt{x}b'_{\frac{t}{x}} + y - \frac{t}{x}y\right)_{t\in[0,x]} \in \cdot \,\middle|\, \int_0^x \mathbb{1}_{\left\{\sqrt{x}b'_{\frac{s}{x}} + y - \frac{s}{x}y < 0\right\}} \mathrm{d}s \leq 1\right) \\
&= \mathbb{P}\left(\left(\sqrt{g}b_{\frac{t}{g}} + y - \frac{t}{g}y\right)_{t\in[0,x]} \in \cdot \,\middle|\, g = x\right).
\end{aligned} \tag{2.20}$$

On the other hand, Proposition 2.11 implies

$$\mathbb{P}\left(\left(\sqrt{T-x}B^+_{\frac{t-x}{T-x}}\right)_{t\in[x,t_0]} \in \cdot\right) \Rightarrow \mathbb{P}\left((Z_{t-x})_{t\in[x,t_0]} \in \cdot\right), \quad T \to \infty. \tag{2.21}$$

Note that this convergence and the following considerations are trivial for $x > t_0$. The law in (2.19) is supported on

$$\mathcal{C}_x := \{f \in \mathcal{C}([0,t_0]) : f(x) = 0\} \in \mathcal{B}(\mathcal{C}([0,t_0])),$$

which can be identified with the product space

$$\{f \in \mathcal{C}([0,x]) : f(x) = 0\} \times \{f \in \mathcal{C}([x,t_0]) : f(x) = 0\}.$$

Since the individual spaces are separable, the Borel σ-algebra on $\mathcal{C}_x$ is precisely the product σ-algebra. Further, the law in (2.19) is nothing but the product measure of the left-hand sides of (2.20) and (2.21) as a consequence of the independence of the path decomposition. The product measure of the right-hand sides of (2.20) and (2.21) is nothing but $\mathbb{P}((X_t)_{t\in[0,t_0]} \in \cdot \,|\, g = x)$ by (the independence assumptions in the) construction of X. Combining (2.19) with (2.20) and (2.21), Theorem 2.8 in [Bil99] consequently yields

$$\mathbb{P}\left((B_t)_{t\in[0,t_0]} \in \cdot \,\middle|\, \Gamma_T \leq 1, g_T = x\right) \Rightarrow \mathbb{P}((X_t)_{t\in[0,t_0]} \in \cdot \,|\, g = x), \quad T \to \infty,$$

on $\mathcal{C}_x$ and hence also on $\mathcal{C}([0,t_0])$.

For $y > 0$, we additionally have (see, e.g., Example 3 in [Pin85])

$$\begin{aligned}
\mathbb{P}\left((B_t)_{t\in[0,t_0]} \in \cdot \,\middle|\, \Gamma_T \le 1, g_T = 0\right) &= \mathbb{P}\left((B_t)_{t\in[0,t_0]} \in \cdot \,\middle|\, B_t > 0 \text{ for all } t \in [0,T]\right) \\
&\Rightarrow \mathbb{P}\left((Y_t)_{t\in[0,t_0]} \in \cdot\right) \\
&= \mathbb{P}\big((X_t)_{t\in[0,t_0]} \in \cdot \,\big|\, g = 0\big), \quad T \to \infty.
\end{aligned}$$

In view of Proposition 2.9 and of $\mathbb{P}(g = 0) = 0$ for $y \le 0$, Proposition 2.10 yields

$$\begin{aligned}
&\mathbb{P}\left((B_t)_{t\in[0,t_0]} \in \cdot \,\middle|\, \Gamma_T \le 1\right) \\
= &\int_{[0,\infty)} \mathbb{P}\left((B_t)_{t\in[0,t_0]} \in \cdot \,\middle|\, \Gamma_T \le 1, g_T = x\right) \mathbb{P}(g_T \in \mathrm{d}x \,|\, \Gamma_T \le 1) \\
\Rightarrow &\int_{[0,\infty)} \mathbb{P}\left((X_t)_{t\in[0,t_0]} \in \cdot \,\middle|\, g = x\right) \mathbb{P}(g \in \mathrm{d}x) \\
= &\ \mathbb{P}\left((X_t)_{t\in[0,t_0]} \in \cdot\right), \quad T \to \infty.
\end{aligned}$$

Since $t_0 > 0$ has been chosen arbitrarily, Theorem 5 of [Whi70] yields the claimed weak convergence in $\mathcal{C}([0,\infty))$. $\square$

2.5 Proofs of the Remaining Results

To prove Theorem 2.2, we condition on g, the last zero of the limiting process, and perform an explicit calculation using Lemma 2.7.

Proof of Theorem 2.2. Let $u \in [0,1]$. By construction of X, we have

$$\Gamma = \int_0^g \mathbb{1}_{\{X_s<0\}} \mathrm{d}s = \int_0^g \mathbb{1}_{\left\{\sqrt{g} b_{\frac{s}{g}} + y - \frac{s}{g} y < 0\right\}} \mathrm{d}s.$$

Now let $(b'_s)_{s\in[0,1]}$ be a standard Brownian bridge. Using $u \le 1$ and the definition of b in the second step, we obtain

$$\begin{aligned}
\mathbb{P}(\Gamma \le u) &= \mathbb{P}(g \le u) + \int_u^\infty \mathbb{P}\left(\int_0^g \mathbb{1}_{\left\{\sqrt{g} b_{\frac{s}{g}} + y - \frac{s}{g} y < 0\right\}} \mathrm{d}s \le u \,\middle|\, g = x\right) \mathbb{P}(g \in \mathrm{d}x) \\
&= \mathbb{P}(g \le u) + \int_u^\infty \frac{\mathbb{P}\Big(\int_0^x \mathbb{1}_{\left\{\sqrt{x} b'_{\frac{s}{x}} + y - \frac{s}{x} y < 0\right\}} \mathrm{d}s \le u\Big)}{\mathbb{P}\Big(\int_0^x \mathbb{1}_{\left\{\sqrt{x} b'_{\frac{s}{x}} + y - \frac{s}{x} y < 0\right\}} \mathrm{d}s \le 1\Big)} \mathbb{P}(g \in \mathrm{d}x) \\
&= \mathbb{P}(g \le u) + \int_u^\infty \frac{q(x,u)}{q(x,1)} \mathbb{P}(g \in \mathrm{d}x).
\end{aligned}$$

For $y < 0$, we apply (2.2) and (2.4) as well as Lemma 2.7 in the final step to deduce

$$\begin{aligned}\mathbb{P}(\Gamma \le u) &= \frac{\int_0^u q(x,1)\frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x}{2\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x} + \frac{\int_u^\infty q(x,u)\frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x}{2\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x} \\ &= \frac{\int_0^u \frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x}{2\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x} + \frac{\int_u^\infty q(x,u)\frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x}{2\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x} = 2\frac{\int_0^u \frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x}{2\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x}.\end{aligned}$$

For $y > 0$, we similarly get

$$\begin{aligned}\mathbb{P}(\Gamma \le u) &= \frac{2\sqrt{2\pi}y + \int_0^u q(x,1)\frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x}{2\sqrt{2\pi}y + 4} + \frac{\int_u^\infty q(x,u)\frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x}{2\sqrt{2\pi}y + 4} \\ &= \frac{2\sqrt{2\pi}y + \int_0^u \frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x + \int_u^\infty q(x,u)\frac{1}{\sqrt{x}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x}{2\sqrt{2\pi}y + 4} \\ &= \frac{2\sqrt{2\pi}y + 4\sqrt{u}}{2\sqrt{2\pi}y + 4}.\end{aligned}$$

For $y = 0$, we can proceed as in the proof of Theorem 4 in [BB11]: Applying (2.2) as well as (2.4) and (2.7), we obtain

$$\begin{aligned}\mathbb{P}(\Gamma \le u) &= \frac{\sqrt{u}}{2} + \frac{\int_u^1 \frac{q(x,u)}{q(x,1)\sqrt{x}}\mathrm{d}x}{4} + \frac{\int_1^\infty \frac{q(x,u)}{q(x,1)\sqrt{x^3}}\mathrm{d}x}{4} \\ &= \frac{\sqrt{u}}{2} + \frac{\int_u^\infty \frac{u}{\sqrt{x^3}}\mathrm{d}x}{4} = \sqrt{u} = \frac{\int_0^u \frac{1}{\sqrt{x}}\mathrm{d}x}{\int_0^1 \frac{1}{\sqrt{x}}\mathrm{d}x}\end{aligned}$$

showing the claim. $\square$

As already mentioned, we prove Proposition 2.3 without relying on the explicit distributions of g, τ and Γ but on the arcsine laws and the strong Markov property. Instead of working with the explicit definition of g given in (2.2), we use the characterization of g as the conditioned limiting law of g_t, proved in Propositon 2.9.

Proof of Proposition 2.3. Let $u \in [0,1]$ and $T > 2$. Further, let $\bar{\tau} := \inf\{t \ge 0 : B_t = 0\}$ be the first zero of B. By the strong Markov property, $(\bar{B}_t)_{t\ge0} := (B_{\bar{\tau}+t})_{t\ge0}$ is a standard Brownian motion independent of $\bar{\tau}$. For each $t \ge 0$, we define

$$\bar{g}_t := \sup\{s \in [0,t] : \bar{B}_s = 0\} \quad \text{and} \quad \bar{\Gamma}_t := \int_0^t \mathbb{1}_{\{\bar{B}_s<0\}}\mathrm{d}s.$$

On $\{\bar{\tau} \le T\}$, we have

$$g_T = \bar{\tau} + \bar{g}_{T-\bar{\tau}} \qquad \text{and} \qquad \Gamma_T = \mathbb{1}_{\{y\le0\}}\bar{\tau} + \bar{\Gamma}_{T-\bar{\tau}} = \begin{cases}\bar{\tau} + \bar{\Gamma}_{T-\bar{\tau}}, & y \le 0,\\ \bar{\Gamma}_{T-\bar{\tau}}, & y \ge 0.\end{cases} \tag{2.22}$$

Using the symmetry of $\bar{B}$ in the first step and the arcsine laws in the second, we obtain

$$2\mathbb{P}(\bar{g}_{T-t} \leq u-t, \bar{B}_{T-t} > 0) = \mathbb{P}(\bar{g}_{T-t} \leq u-t) = \mathbb{P}(\bar{\Gamma}_{T-t} \leq u-t), \quad t \in [0,u].$$

Using $T-u > 1$ in the first step, inserting the first part of (2.22) in the third step and recalling that $\bar{\tau}$ is independent of both $\bar{g}$ and $\bar{\Gamma}$, we can deduce

$$\begin{aligned}
2\mathbb{P}(g_T \in (0,u], \Gamma_T \leq 1) &= 2\mathbb{P}(g_T \in (0,u], B_T > 0) \\
&= 2\mathbb{P}(g_T \leq u, B_T > 0, \bar{\tau} \leq u) \\
&= 2\mathbb{P}(\bar{g}_{T-\bar{\tau}} \leq u-\bar{\tau}, \bar{B}_{T-\bar{\tau}} > 0, \bar{\tau} \leq u) \\
&= \int_{[0,u]} 2\mathbb{P}(\bar{g}_{T-t} \leq u-t, \bar{B}_{T-t} > 0)\mathbb{P}(\bar{\tau} \in \mathrm{d}t) \\
&= \int_{[0,u]} \mathbb{P}(\bar{\Gamma}_{T-t} \leq u-t)\mathbb{P}(\bar{\tau} \in \mathrm{d}t) \\
&= \mathbb{P}(\bar{\Gamma}_{T-\bar{\tau}} \leq u-\bar{\tau}, \bar{\tau} \leq u).
\end{aligned}$$

Noting $\{\Gamma_T \leq u\} \subseteq \{\bar{\tau} \leq u\}$ for $y \leq 0$, the second part of (2.22) implies

$$\begin{aligned}
2\mathbb{P}(g_T \in (0,u], \Gamma_T \leq 1) = \mathbb{P}(\bar{\Gamma}_{T-\bar{\tau}} \leq u-\bar{\tau}, \bar{\tau} \leq u) &= \mathbb{P}\left(\mathbb{1}_{\{y>0\}}\bar{\tau} + \Gamma_T \leq u, \bar{\tau} \leq u\right) \\
= \mathbb{P}\left(\mathbb{1}_{\{y>0\}}\bar{\tau} + \Gamma_T \leq u\right) &= \mathbb{P}\left(\mathbb{1}_{\{y>0\}}\bar{\tau} + \Gamma_T \leq u, \Gamma_T \leq 1\right)
\end{aligned}$$

proving

$$2\mathbb{P}(g_T \in (0,u] \,|\, \Gamma_T \leq 1) = \mathbb{P}\left(\mathbb{1}_{\{y>0\}}\bar{\tau} + \Gamma_T \leq u \,\middle|\, \Gamma_T \leq 1\right). \tag{2.23}$$

As a consequence of Proposition 2.9, the left-hand side converges to $2\mathbb{P}(g \in (0,u])$ as $T \to \infty$. Now assume that the distribution function of $\tau + \Gamma$ is continuous in u. Then, in view of the very same proposition, a path decomposition and conditioning argument similar to that in the proof of Theorem 2.1 implies that the right-hand side of (2.23) converges to

$$\mathbb{P}\big(\mathbb{1}_{\{y>0\}} \inf\{t \geq 0 : X_t = 0\} + \Gamma \leq u\big) = \mathbb{P}(\tau + \Gamma \leq u)$$

as $T \to \infty$. For general $u \in [0,1]$, the claimed equality now is a consequence of the right-continuity of distribution functions. $\square$

Theorem 2.4 follows from the formulas for the distribution functions of g^y and Γ^y by direct computations.

Proof of Theorem 2.4. Part (a): Let $u > 0$ and $y > \frac{2}{\sqrt{u}}$. Equations (2.2) and (2.7) imply

$$\begin{aligned}
&\mathbb{P}\left(\frac{g^y}{y^2} > u \,\middle|\, g^y > 0\right) \\
&= \frac{1}{4}\int_{uy^2}^{\infty} q^y(t,1)\frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t \\
&= \frac{1}{4}\int_{uy^2}^{\infty} \int_0^{t-1} \frac{y}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x + \int_{t-1}^{t} \frac{y}{\sqrt{2\pi x^3(t-x)}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.24}$$

Regarding the first double integral, we use Tonelli's theorem and two linear substitutions to obtain

$$\begin{aligned}
&\int_{uy^2}^{\infty}\int_0^{t-1} \frac{y}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^{\infty}\int_{uy^2\vee(x+1)}^{\infty} \frac{y}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}t\mathrm{d}x \\
&= \int_0^{uy^2-1} \frac{2y}{\sqrt{2\pi x^3(uy^2-x)}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x + \int_{uy^2-1}^{\infty} \frac{2y}{\sqrt{2\pi x^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x \\
&= \int_0^1 \frac{2y(uy^2-1)}{\sqrt{2\pi(uy^2-1)^3z^3(uy^2-(uy^2-1)z)}}\,\mathrm{e}^{-\frac{y^2}{2(uy^2-1)z}}\,\mathrm{d}z \\
&\quad + \int_u^{\infty} \frac{2y^3}{\sqrt{2\pi(sy^2-1)^3}}\,\mathrm{e}^{-\frac{y^2}{2(sy^2-1)}}\,\mathrm{d}s \\
&= \frac{1}{y}\int_0^1 \frac{2\big(u-\frac{1}{y^2}\big)}{\sqrt{2\pi\big(u-\frac{1}{y^2}\big)^3z^3\big(u-\big(u-\frac{1}{y^2}\big)z\big)}}\,\mathrm{e}^{-\frac{1}{2\left(u-\frac{1}{y^2}\right)z}}\,\mathrm{d}z \\
&\quad + \int_u^{\infty} \frac{2}{\sqrt{2\pi\big(s-\frac{1}{y^2}\big)^3}}\,\mathrm{e}^{-\frac{1}{2\left(s-\frac{1}{y^2}\right)}}\,\mathrm{d}s.
\end{aligned}$$

Noting that the initial assumption on y implies $\frac{3}{4}u \le s - \frac{1}{y^2} \le s$ for all $s \ge u$, the dominated convergence theorem yields

$$\int_{uy^2}^{\infty}\int_0^{t-1} \frac{y}{\sqrt{2\pi x^3(t-x)^3}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\mathrm{d}t \to 0 + \int_u^{\infty} \frac{2}{\sqrt{2\pi s^3}}\,\mathrm{e}^{-\frac{1}{2s}}\,\mathrm{d}s, \quad y\to\infty.$$

Regarding the second double integral in (2.24), two linear substitutions lead to

$$\begin{aligned}
&\int_{uy^2}^{\infty}\int_{t-1}^{t} \frac{y}{\sqrt{2\pi x^3(t-x)}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\mathrm{d}t \\
&= \int_u^{\infty}\int_{sy^2-1}^{sy^2} \frac{y^3}{\sqrt{2\pi x^3(sy^2-x)}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\mathrm{d}s \\
&= \int_u^{\infty}\int_0^1 \frac{y^3}{\sqrt{2\pi(sy^2-1+z)^3(1-z)}}\,\mathrm{e}^{-\frac{y^2}{2(sy^2-1+z)}}\,\mathrm{d}z\mathrm{d}s \\
&= \int_u^{\infty}\int_0^1 \frac{1}{\sqrt{2\pi(s-\frac{1}{y^2}+\frac{z}{y^2})^3(1-z)}}\,\mathrm{e}^{-\frac{1}{2\left(s-\frac{1}{y^2}+\frac{z}{y^2}\right)}}\,\mathrm{d}z\mathrm{d}s.
\end{aligned}$$

As above, we can apply the dominated convergence theorem to deduce

$$\begin{aligned}
\int_{uy^2}^{\infty}\int_{t-1}^{t} \frac{y}{\sqrt{2\pi x^3(t-x)}}\,\mathrm{e}^{-\frac{y^2}{2x}}\,\mathrm{d}x\mathrm{d}t &\to \int_u^{\infty}\int_0^1 \frac{1}{\sqrt{2\pi s^3(1-z)}}\,\mathrm{e}^{-\frac{1}{2s}}\,\mathrm{d}z\mathrm{d}s \\
&= \int_u^{\infty} \frac{2}{\sqrt{2\pi s^3}}\,\mathrm{e}^{-\frac{1}{2s}}\,\mathrm{d}s, \quad y\to\infty.
\end{aligned}$$

Combining the two limits with (2.24), we get

$$\mathbb{P}\left(\frac{g^y}{y^2} > u \,\middle|\, g^y > 0\right) \to \int_u^\infty \frac{1}{\sqrt{2\pi s^3}}\,\mathrm{e}^{-\frac{1}{2s}}\,\mathrm{d}s, \quad y \to \infty,$$

proving the first claim. The second claim follows immediately.

Part (b): Similar to part (a), it suffices to prove the weak convergence of $y^2(1-\Gamma^y)$ and $y^2(1-g^y)$. Let $u \geq 0$ and $y < -\sqrt{u}$. Substituting $t = \frac{y^2}{y^2+2z}$ or equivalently $z = -\frac{y^2(t-1)}{2t}$ and applying the dominated convergence theorem, we obtain

$$\begin{aligned}\int_0^{1-\frac{u}{y^2}} \frac{y^2}{2\sqrt{t}}\,\mathrm{e}^{\frac{y^2}{2}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t &= \int_{\frac{u}{2}\left(1-\frac{u}{y^2}\right)^{-1}}^\infty \frac{y^2\sqrt{y^2+2z}}{2|y|}\,\mathrm{e}^{-z}\,\frac{2y^2}{(y^2+2z)^2}\mathrm{d}z \\ &= \int_0^\infty \mathbb{1}_{\left\{\frac{u}{2} \leq \left(1-\frac{u}{y^2}\right)z\right\}} \left(\frac{y^2}{y^2+2z}\right)^{\frac{3}{2}} \mathrm{e}^{-z}\,\mathrm{d}z \\ &\to \int_0^\infty \mathbb{1}_{\left\{\frac{u}{2} \leq z\right\}}\,\mathrm{e}^{-z}\,\mathrm{d}z = \mathrm{e}^{-\frac{u}{2}}, \quad y \to -\infty. \end{aligned} \tag{2.25}$$

Together with (2.3), we deduce

$$\begin{aligned}\mathbb{P}(y^2(1-\Gamma^y) \geq u) = \mathbb{P}\left(\Gamma^y \leq 1 - \frac{u}{y^2}\right) &= \frac{\int_0^{1-\frac{u}{y^2}} \frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t}{\int_0^1 \frac{1}{\sqrt{t}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t} \\ &= \frac{\int_0^{1-\frac{u}{y^2}} \frac{y^2}{2\sqrt{t}}\,\mathrm{e}^{\frac{y^2}{2}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t}{\int_0^1 \frac{y^2}{2\sqrt{t}}\,\mathrm{e}^{\frac{y^2}{2}}\,\mathrm{e}^{-\frac{y^2}{2t}}\,\mathrm{d}t} \to \mathrm{e}^{-\frac{u}{2}}, \quad y \to -\infty,\end{aligned}$$

proving the claimed convergence of $y^2(1-\Gamma^y)$.
Recalling $\mathbb{P}(g^y < \infty) = 1$ and using (2.5), we deduce

$$\begin{aligned}\mathbb{P}(y^2(1-g^y) \leq u) = 1 - \mathbb{P}\left(g^y < 1 - \frac{u}{y^2}\right) &= 1 - \frac{1}{2}\mathbb{P}\left(\Gamma^y < 1 - \frac{u}{y^2}\right) \\ &\to 1 - \frac{1}{2}\,\mathrm{e}^{-\frac{u}{2}} = \mathbb{P}(g' \leq u), \quad y \to -\infty.\end{aligned}$$

Now let $u \leq 0$ and $y < 0$. Using (2.7), substituting $x = \frac{y^2}{y^2+2z}$ as above and applying the dominated convergence theorem with majorant $\frac{4z}{\sqrt{2\pi(2z-u)}} \mathrm{e}^{-z}$, we obtain

$$\begin{aligned}
&\int_{1-\frac{u}{y^2}}^{\infty} q^y(t,1) \frac{y^2}{2\sqrt{t}} \mathrm{e}^{\frac{y^2}{2}} \mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t \\
&= \int_{1-\frac{u}{y^2}}^{\infty} \int_0^1 \frac{(1-x)|y|y^2}{2\sqrt{2\pi x^3(t-x)^3}} \mathrm{e}^{\frac{y^2}{2}} \mathrm{e}^{-\frac{y^2}{2x}} \,\mathrm{d}x\mathrm{d}t \\
&= \int_0^1 \frac{(1-x)|y|y^2}{\sqrt{2\pi x^3\left(1-\frac{u}{y^2}-x\right)}} \mathrm{e}^{\frac{y^2}{2}} \mathrm{e}^{-\frac{y^2}{2x}} \,\mathrm{d}x \\
&= \int_0^{\infty} \frac{2z|y|y^2\sqrt{(y^2+2z)^3}}{(y^2+2z)\sqrt{2\pi y^6\left(1-\frac{u}{y^2}-\frac{y^2}{y^2+2z}\right)}} \mathrm{e}^{-z} \frac{2y^2}{(y^2+2z)^2} \mathrm{d}z \\
&= \int_0^{\infty} \frac{4z}{\sqrt{2\pi\left(2z-u\frac{y^2+2z}{y^2}\right)}} \mathrm{e}^{-z} \frac{y^2}{y^2+2z} \mathrm{d}z \\
&\to \int_0^{\infty} \frac{4z}{\sqrt{2\pi(2z-u)}} \mathrm{e}^{-z} \,\mathrm{d}z, \quad y \to -\infty.
\end{aligned}$$

Recalling $\mathbb{P}(g^y < \infty) = 1$ again, applying (2.2) and combining the above convergence with (2.25), we deduce

$$\begin{aligned}
\mathbb{P}(y^2(1-g^y) \leq u) &= \mathbb{P}\left(g^y > 1 - \frac{u}{y^2}\right) \\
&= \frac{\int_{1-\frac{u}{y^2}}^{\infty} q^y(t,1) \frac{1}{\sqrt{t}} \mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t}{2\int_0^1 \frac{1}{\sqrt{t}} \mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t} \\
&= \frac{\int_{1-\frac{u}{y^2}}^{\infty} q^y(t,1) \frac{y^2}{2\sqrt{t}} \mathrm{e}^{\frac{y^2}{2}} \mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t}{2\int_0^1 \frac{y^2}{2\sqrt{t}} \mathrm{e}^{\frac{y^2}{2}} \mathrm{e}^{-\frac{y^2}{2t}} \,\mathrm{d}t} \\
&\to \frac{\int_0^{\infty} \frac{4z}{\sqrt{2\pi(2z-u)}} \mathrm{e}^{-z} \,\mathrm{d}z}{2} \\
&= \mathbb{P}(g' \leq u), \quad y \to -\infty,
\end{aligned}$$

proving $y^2(1-g^y) \Rightarrow g'$. □

2.6 Auxiliary Path Decomposition Results

For the sake of completeness and for the reader's convenience, we collect and prove a few path decomposition results concerning Brownian motions and bridges in this section. In preparation of the proofs, we start with a random version of the scaling property of Brownian motion.

Proposition 2.12. *Let $(W_t)_{t\geq 0}$ be a standard Brownian motion independent of some σ-algebra $\mathcal{G}$. Moreover, let σ be a $\mathcal{G}$-measurable random variable taking values in $[0,\infty)$. Then $W^\sigma := \big(\frac{1}{\sqrt{\sigma}} W_{\sigma t}\big)_{t\geq 0}$ is a standard Brownian motion independent of $\mathcal{G}$.*

Proof. By the classical scaling property, $W^s := (\frac{1}{\sqrt{s}} W_{st})_{t\geq 0}$ is a Brownian motion independent of $\mathcal{G}$ for each $s > 0$. We deduce

$$\begin{aligned}
\mathbb{P}(\{W^\sigma \in F\} \cap G) &= \int_{[0,\infty)} \mathbb{P}(\{W^\sigma \in F\} \cap G \,|\, \sigma = s)\mathbb{P}(\sigma \in \mathrm{d}s) \\
&= \int_{[0,\infty)} \mathbb{P}(\{W^s \in F\} \cap G \,|\, \sigma = s)\mathbb{P}(\sigma \in \mathrm{d}s) \\
&= \int_{[0,\infty)} \mathbb{P}(W^s \in F)\mathbb{P}(G \,|\, \sigma = s)\mathbb{P}(\sigma \in \mathrm{d}s) \\
&= \int_{[0,\infty)} \mathbb{P}(W \in F)\mathbb{P}(G \,|\, \sigma = s)\mathbb{P}(\sigma \in \mathrm{d}s) \\
&= \mathbb{P}(W \in F)\mathbb{P}(G), \quad F \in \mathcal{B}(\mathcal{C}[0,\infty)),\ G \in \mathcal{G},
\end{aligned}$$

proving the claim. $\square$

Our next auxiliary result states that a Brownian motion with a suitable non-linear time change becomes a Brownian bridge.

Lemma 2.13. *Let $(W_t)_{t\geq 0}$ be a standard Brownian motion. Then $\big(sW_{\frac{1-s}{s}}\big)_{s\in(0,1]}$ is a standard Brownian bridge.*

Proof. The process $\big(sW_{\frac{1-s}{s}}\big)_{s\in(0,1]}$ is a centered Gaussian process whose covariance function is given by

$$\mathbb{E}\left[sW_{\frac{1-s}{s}} \cdot tW_{\frac{1-t}{t}}\right] = st\left(\frac{1-s}{s} \wedge \frac{1-t}{t}\right) = (s \wedge t) - st, \quad s,t \in (0,1].$$

Consequently, it must be a standard Brownian bridge. $\square$

Before we state the main path decomposition result, let us recall the definition of a Brownian meander.

Definition 2.14. Let $(W_t)_{t\geq 0}$ be a standard Brownian motion and let

$$\gamma_1 := \max\{s \in [0,1] : W_s = 0\}$$

be its last zero before time 1. A stochastic process $B^+ = (B^+_t)_{t\in[0,1]}$ is called *(standard) Brownian meander* if it satisfies

$$B^+ \stackrel{\mathrm{d}}{=} \left(\frac{1}{\sqrt{1-\gamma_1}} |W_{\gamma_1 + (1-\gamma_1)s}|\right)_{s\in[0,1]}.$$

Further, let us recall that $(B_t)_{t\geq 0}$ is a Brownian motion with starting point $y \in \mathbb{R}$ and that

$$g_T := \max\{s \in [0,T] : B_s = 0\}$$

is the last zero of B before time $T > 0$ with the convention $\max \emptyset := 0$. The next proposition shows that, conditioned on $\{g_T > 0\}$ or equivalently on the existence of a zero before time T, we can decompose $(B_t)_{t\in[0,T]}$ around g_T into a Brownian bridge, a Brownian meander and the sign of the endpoint such that the four parts of the decomposition are independent. In the special case $y = 0$, this result can, for instance, be found in Section 7 of [YY13]. Regarding the general case, we start by generalizing (the proof of) Theorem 7.1.1 in [YY13] concerning the bridge part. Afterwards, we deduce the remaining claims from the special case $y = 0$ by considering the Brownian motion started in the first zero. Formally, the statement reads as follows:

Proposition 2.15. *Let $T > 0$. Conditioned on $\{g_T > 0\}$, the process*

$$(b'_s)_{s\in[0,1]} := \left(\frac{1}{\sqrt{g_T}}\left(B_{sg_T} - y + sy\right)\right)_{s\in[0,1]}$$

is a standard Brownian bridge while

$$(B^+_s)_{s\in[0,1]} := \left(\frac{1}{\sqrt{T-g_T}}\left|B_{g_T+(T-g_T)s}\right|\right)_{s\in[0,1]}$$

is a Brownian meander. Moreover, these two processes, g_T and $\operatorname{sgn}(B_T)$ are mutually independent.

Proof. By time inversion, $(W_t)_{t\geq 0} := \left(t\left(B_{\frac{1}{t}} - y\right)\right)_{t\geq 0}$ is a standard Brownian motion. Moreover,

$$\sigma_T := \inf\left\{t \geq \frac{1}{T} : W_t = -yt\right\} = \inf\left\{t \geq \frac{1}{T} : B_{\frac{1}{t}} = 0\right\} = \frac{1}{g_T}$$

is a stopping time with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$ generated by $(W_t)_{t\geq 0}$. From now on, we condition on $\{g_T > 0\}$ or equivalently on $\{\sigma_T < \infty\}$. By the strong Markov property, $(W_{\sigma_T+t} - W_{\sigma_T})_{t\geq 0}$ is a standard Brownian motion independent of $\mathcal{G}_{\sigma_T}$. According to Proposition 2.12, the process

$$\left(\bar{W}_t\right)_{t\geq 0} := \left(\frac{1}{\sqrt{\sigma_T}}\left(W_{\sigma_T+\sigma_T t} - W_{\sigma_T}\right)\right)_{t\geq 0}$$

is a standard Brownian motion independent of $\mathcal{G}_{\sigma_T}$ as well. We obtain

$$
\begin{aligned}
(b'_s)_{s\in(0,1]} &= \left(\frac{1}{\sqrt{g_T}}\left(sg_T W_{\frac{1}{sg_T}} + sy\right)\right)_{s\in(0,1]} \\
&= \left(s\sqrt{g_T}\left(W_{\frac{1}{sg_T}} + \frac{y}{g_T}\right)\right)_{s\in(0,1]} \\
&= \left(\frac{s}{\sqrt{\sigma_T}}\left(W_{\frac{\sigma_T}{s}} - W_{\sigma_T}\right)\right)_{s\in(0,1]} \\
&= \left(\frac{s}{\sqrt{\sigma_T}}\left(W_{\sigma_T+\sigma_T\frac{1-s}{s}} - W_{\sigma_T}\right)\right)_{s\in(0,1]} \\
&= \left(s\bar{W}_{\frac{1-s}{s}}\right)_{s\in(0,1]}.
\end{aligned}
$$

According to Lemma 2.13, this process is a standard Brownian bridge. Moreover, it is independent of $\mathcal{G}_{\sigma_T}$. By continuity, both properties extend from $(b'_s)_{s\in(0,1]}$ to $(b'_s)_{s\in[0,1]}$. Since $(W_t)_{t\geq 0}$ (with its natural filtration $(\mathcal{G}_t)_{t\geq 0}$) is constructed from $(B_t)_{t\geq 0}$ by time inversion and $\sigma_T = \frac{1}{g_T}$ holds, the triple $((B^+_s)_{s\in[0,1]}, g_T, \operatorname{sgn}(B_T))$ is $\mathcal{G}_{\sigma_T}$-measurable and hence independent of $(b'_s)_{s\in[0,1]}$.
Now consider the $(B_t)_{t\geq 0}$-stopping time

$$\tau_T := \min\{s \in [0,T] : B_s = 0\}$$

(with the convention $\min\emptyset := T$). Since we condition on $\{g_T > 0\}$, which is equivalent to the existence of a zero, we have $\tau_T < T$ and $B_{\tau_T} = 0$. By the strong Markov property, $(B_{\tau_T+t})_{t\geq 0}$ is a (standard) Brownian motion independent of τ_T. According to Proposition 2.12, the process

$$\left(\bar{B}_t\right)_{t\geq 0} := \left(\frac{1}{\sqrt{T-\tau_T}} B_{\tau_T+(T-\tau_T)t}\right)_{t\geq 0}$$

is a Brownian motion independent of τ_T as well. We define

$$\bar{g} := \max\{s \in [0,1] : \bar{B}_s = 0\} = \frac{g_T - \tau_T}{T-\tau_T}.$$

Then

$$
\begin{aligned}
(B^+_s)_{s\in[0,1]} &= \left(\frac{1}{\sqrt{T-g_T}}\left|B_{g_T+(T-g_T)s)}\right|\right)_{s\in[0,1]} \\
&= \left(\frac{1}{\sqrt{1-\bar{g}}}\frac{1}{\sqrt{T-\tau_T}}\left|B_{\tau_T+(T-\tau_T)(\bar{g}+(1-\bar{g})s)}\right|\right)_{s\in[0,1]} \\
&= \left(\frac{1}{\sqrt{1-\bar{g}}}\left|\bar{B}_{\bar{g}+(1-\bar{g})s}\right|\right)_{s\in[0,1]}
\end{aligned}
$$

is a Brownian meander. Now the path decomposition result for starting point $y = 0$ (see Lemma 7.6.1 in [YY13]) implies that $(B_s^+)_{s\in[0,1]}$ and $\bar{g}$ and $\operatorname{sgn}(\bar{B}_1)$ are independent. On the other hand, $((B_s^+)_{s\in[0,1]}, \operatorname{sgn}(\bar{B}_1))$ is independent of τ_T. We deduce the independence of $(B_s^+)_{s\in[0,1]}$ and $g_T = \tau_T + \bar{g}(T - \tau_T)$ and $\operatorname{sgn}(B_T) = \operatorname{sgn}(\bar{B}_1)$. □

If the underlying process already is a Brownian bridge (with drift), we use another kind of time inversion and the strong Markov property to show that the process up to the last zero is again a Brownian bridge.

Proposition 2.16. *Given $T > 0$ and $z \in \mathbb{R}$, let $(b'_t)_{t\in[0,T]}$ be a Brownian bridge of length T with $b'_0 = y$ and $b'_T = z$. We define*

$$\gamma_T := \max\{s \in [0, T] : b'_s = 0\}$$

with the convention $\max \emptyset := 0$. Conditioned on $\{\gamma_T > 0\}$, the process

$$\left(\frac{1}{\sqrt{\gamma_T}}(b'_{s\gamma_T} - y + sy)\right)_{s\in[0,1]}$$

is a standard Brownian bridge independent of γ_T.

Proof. The centered Gaussian process

$$(W_t)_{t\geq 0} := \left(\frac{t+1}{\sqrt{T}}\left(b'_{\frac{T}{t+1}} - \frac{1}{t+1}(z-y) - y\right)\right)_{t\geq 0}$$

satisfies

$$\begin{aligned}\mathbb{E}[W_t W_s] &= \frac{(t+1)(s+1)}{T}\operatorname{cov}\left(b'_{\frac{T}{t+1}}, b'_{\frac{T}{s+1}}\right)\\ &= \frac{(t+1)(s+1)}{T}\left(\left(\frac{T}{t+1} \wedge \frac{T}{s+1}\right) - \frac{T}{(t+1)(s+1)}\right) = s \wedge t, \quad s, t \geq 0.\end{aligned}$$

Consequently, $(W_t)_{t\geq 0}$ is a standard Brownian motion. Moreover,

$$\sigma_T := \inf\left\{t > 0 : W_t = -\frac{1}{\sqrt{T}}(z-y) - \frac{t+1}{\sqrt{T}}y\right\} = \inf\left\{t > 0 : b'_{\frac{T}{t+1}} = 0\right\}$$

is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by $(W_t)_{t\geq 0}$. From now on, we condition on $\{\gamma_T > 0\}$ or equivalently on $\{\sigma_T < \infty\}$. By the strong Markov property, $(W_{\sigma_T+t} - W_{\sigma_T})_{t\geq 0}$ is a Brownian motion independent of $\mathcal{F}_{\sigma_T}$. According to Proposition 2.12,

$$(\bar{W}_t)_{t\geq 0} := \left(\frac{1}{\sqrt{\sigma_T + 1}}\left(W_{\sigma_T+(\sigma_T+1)t} - W_{\sigma_T}\right)\right)_{t\geq 0}$$

is a Brownian motion independent of $\mathcal{F}_{\sigma_T}$ as well. Noting $\gamma_T = \frac{T}{\sigma_T+1}$ and

$$(b'_s)_{s\in(0,1]} = \left(\frac{s}{\sqrt{T}}W_{\frac{T-s}{s}} + y + \frac{s}{T}(z-y)\right)_{s\in(0,1]},$$

we get

$$\begin{aligned}
&\left(\frac{1}{\sqrt{\gamma_T}}\left(b'_{s\gamma_T} - y + sy\right)\right)_{s\in(0,1]} \\
&= \left(\frac{1}{\sqrt{\gamma_T}}\left(\frac{s\gamma_T}{\sqrt{T}}W_{\frac{T-s\gamma_T}{s\gamma_T}} + \frac{s\gamma_T}{T}(z-y) + sy\right)\right)_{s\in(0,1]} \\
&= \left(\frac{s}{\sqrt{\sigma_T+1}}\left(W_{\frac{\sigma_T+1-s}{s}} + \frac{1}{\sqrt{T}}(z-y) + \frac{\sigma_T+1}{\sqrt{T}}y\right)\right)_{s\in(0,1]} \\
&= \left(\frac{s}{\sqrt{\sigma_T+1}}\left(W_{\sigma_T+(\sigma_T+1)\frac{1-s}{s}} - W_{\sigma_T}\right)\right)_{s\in(0,1]} \\
&= \left(s\bar{W}_{\frac{1-s}{s}}\right)_{s\in(0,1]}.
\end{aligned}$$

According to Lemma 2.13, this process is a standard Brownian bridge. Moreover, it is independent of $\mathcal{F}_{\sigma_T}$ and hence of γ_T. Continuity of $(b'_s)_{s\in[0,1]}$ in 0 finally yields the claim. □

Reversing time and passing to a special case, we obtain the following result:

Corollary 2.17. *Let $(b'_s)_{s\in[0,t]}$ be a Brownian bridge of length t with $b'_0 = y$ and $b'_t = 0$. We define*

$$\tau := \min\{s \in [0,t] : b'_s = 0\}.$$

Then

$$\left(\frac{1}{\sqrt{t-\tau}}b'_{\tau+s(t-\tau)}\right)_{s\in[0,1]}$$

is a standard Brownian bridge independent of τ.

3 Limited Time Outside a Bounded Interval

3.1 Overview and Main Results

In this chapter, we will encounter a very rare extreme example of entropic repulsion: If Brownian motion is forced to spend *only limited time* outside a bounded interval, then the resulting process does not spend *any time at all* outside the interval. In other words, the resulting process does not make any use of the possibility to leave the interval, which is somewhat surprising.
In order to explain this result in precise terms, let us first introduce some basic notation. Fix $s > 0$, let $B = (B_t)_{t\geq 0}$ be a Brownian motion starting in $y \in (-1, 1)$ and let

$$\Gamma_T := \int_0^T \mathbb{1}_{\{|B_t|\geq 1\}} \mathrm{d}t, \quad T \geq 0,$$

be the time B spends outside the interval $(-1, 1)$ until time T. By the scaling property and translation invariance of Brownian motion, one can immediately transfer our results to the case of an arbitrary bounded interval.
Our main result can be stated as follows:

Theorem 3.1. *As $T \to \infty$, the probability measures*

$$\mathbb{P}_y(B \in \cdot \mid \Gamma_T \leq s) \quad \textit{and} \quad \mathbb{P}_y(B \in \cdot \mid \Gamma_T = 0)$$

converge weakly to the same limit on $\mathcal{C}([0, \infty))$. The limiting process $(X_t)_{t\geq 0}$ satisfies the SDE

$$X_0 = y, \quad \mathrm{d}X_t = \mathrm{d}W_t - \frac{\pi}{2} \tan\left(\frac{\pi X_t}{2}\right) \mathrm{d}t, \quad t \geq 0, \tag{3.1}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion.

This result shows that a Brownian motion which is forced to spend *less than s time units outside a bounded interval* will end up *not leaving the interval at all*, as announced above.
We note that one does not recover the case of limited occupation time in the negative half-line discussed in Chapter 2 by exhausting $(0, \infty)$ by bounded intervals: On the one hand, a Brownian motion which starts in $\tilde{y} > 0$ and is conditioned on spending at most one time unit outside $(0, \infty)$ will spend some time in $(-\infty, 0]$ with positive probability. On the other hand, a Brownian motion which starts in $\tilde{y} > 0$ and is conditioned on

spending at most one time unit outside an arbitrary interval of the form $(0,a)$ with $a > \tilde{y}$ will not spend any time in $(-\infty,0] \subseteq \mathbb{R} \setminus (0,a)$.
In addition to the above theorem, our methodology allows us to exhibit the exact asymptotic behavior of $\mathbb{P}_y(\Gamma_T \leq s)$, as $T \to \infty$, explicitly:

Theorem 3.2. *As $T \to \infty$, we have*

$$\begin{aligned}&\mathbb{P}_y(\Gamma_T \leq s)\\ &\sim \frac{\cos\left(\frac{\pi y}{2}\right) 2^{\frac{19}{6}}}{\sqrt{3}\pi^{\frac{13}{6}} s^{\frac{1}{6}} T^{\frac{1}{3}}} \exp\left(-\frac{\pi^2}{8}T + \frac{3}{2^{\frac{7}{3}}}\pi^{\frac{4}{3}} s^{\frac{1}{3}} T^{\frac{2}{3}} - \frac{3}{2^{\frac{5}{3}}}\pi^{\frac{2}{3}} s^{\frac{2}{3}} T^{\frac{1}{3}} + \frac{\pi^2+12}{24}s\right), \quad y \in (-1,1),\end{aligned}$$

as well as

$$\lim_{T\to\infty} \frac{\mathbb{P}_x(\Gamma_T \leq s)}{\mathbb{P}_y(\Gamma_T \leq s)} = 0, \quad x \in \mathbb{R} \setminus (-1,1),\ y \in (-1,1). \tag{3.2}$$

This asymptotic behavior is of interest in its own right due to the explicitness of the unusual polynomial and subexponential terms as well as the discontinuity w.r.t. the starting point implied by (3.2). It should be compared to the probability that B never leaves $(-1,1)$: The classical formula for first exit probabilities (see, e.g., Example 5a in [DS53]) yields

$$\begin{aligned}\mathbb{P}_y(\Gamma_T = 0) &= \mathbb{P}_y(|B_t| < 1 \text{ for all } t \in [0,T])\\ &= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\left(\frac{(2k+1)\pi y}{2}\right) \mathrm{e}^{-\frac{(2k+1)^2\pi^2}{8}T}\\ &\sim \frac{4}{\pi}\cos\left(\frac{\pi y}{2}\right)\mathrm{e}^{-\frac{\pi^2}{8}T}, \quad y \in (-1,1),\ T \to \infty. \end{aligned}\tag{3.3}$$

If we do not start inside $[-1,1]$ as assumed above, we cannot expect a limiting process to exist on the Wiener space $\mathcal{C}([0,\infty))$ or the Skorokhod space $\mathcal{D}([0,\infty))$ since such a process would have to jump into $[-1,1]$ immediately:

Corollary 3.3. *Fix $y \in \mathbb{R} \setminus [-1,1]$ and set $\tau := \inf\{t \geq 0 : |B_t| = 1\}$. Then we have*

$$\lim_{T\to\infty} \mathbb{P}_y(\tau \geq \varepsilon \mid \Gamma_T \leq s) = 0, \quad \textit{for any } \varepsilon > 0.$$

For starting points on the boundary, i.e., for $y \in \{-1,1\}$, the existence of a limiting process remains an open problem.

Let us outline the general strategy behind the proof of these results: Roughly speaking, using an analysis of the Laplace transform of $T \mapsto \mathbb{P}_y(\Gamma_T \leq s)\,\mathrm{e}^{\frac{\pi^2}{8}T}$ and an application of Tauberian arguments, we are able to prove Theorem 3.2. However, the required monotonicity in the Tauberian theorems seems to be difficult to establish for arbitrary starting points $y \in (-1,1)$. We bypass this problem by applying the Tauberian argument to a specific initial distribution ν supported on $(-1,1)$ for which monotonicity of a suitable function can be established. Then it remains to prove the convergence of the quotient $\frac{\mathbb{P}_y(\Gamma_T \leq s)}{\mathbb{P}_\nu(\Gamma_T \leq s)}$ for $y \in (-1,1)$. Theorem 3.2 together with tightness of the family of conditional laws will imply our main result.

Although our way of approximating the event $\{\Gamma_\infty \leq s\}$ seems to be the most natural one, we note that the resulting process depends on the chosen limiting procedure. Even for $s = 0$, i.e., when conditioning B on not spending any time outside $(-1, 1)$, it was shown in [Kni69] that different approximations lead to different processes. More precisely, given $y \in (-1, 1)$ and letting $(\sigma_l)_{l\geq 0}$ be the right-continuous inverse local time of B in 0, the laws $\mathbb{P}_y(B \in \cdot \mid \Gamma_{\sigma_l} \leq s)$ converge weakly on $\mathcal{C}([0, \infty))$, as $l \to \infty$, to the law of a process $(X'_t)_{t\geq 0}$ satisfying the SDE

$$X'_0 = y, \quad \mathrm{d}X'_t = \mathrm{d}W_t - \frac{\operatorname{sgn}(X'_t)}{1 - |X'_t|}\mathrm{d}t, \quad t \geq 0,$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion.
Further, while Theorem 3.2 provides the precise asymptotic behavior of $T \mapsto \mathbb{P}_y(\Gamma_T \leq s)$ for $y \in (-1, 1)$, a complementing result for $y \in \mathbb{R} \setminus (-1, 1)$ remains an open question. Here we conjecture, e.g.,

$$\begin{aligned}\mathbb{P}_y(\Gamma_T \leq s) &\sim \frac{2^{\frac{17}{6}} s^{\frac{1}{6}}}{\sqrt{3}\pi^{\frac{11}{6}} T^{\frac{2}{3}}} \exp\left(-\frac{\pi^2}{8}T + \frac{3}{2^{\frac{7}{3}}}\pi^{\frac{4}{3}} s^{\frac{1}{3}} T^{\frac{2}{3}} - \frac{3}{2^{\frac{5}{3}}}\pi^{\frac{2}{3}} s^{\frac{2}{3}} T^{\frac{1}{3}} + \frac{\pi^2 + 12}{24}s\right) \\ &\sim \left(\frac{\pi s}{2T}\right)^{\frac{1}{3}} \mathbb{P}_0(\Gamma_T \leq s), \quad y \in \{-1, 1\},\ T \to \infty,\end{aligned}$$

which would, in particular, be in accordance with (3.2). We are, in fact, able to prove the corresponding asymptotics of the Laplace transform. However, the asymptotic inversion requires a priori knowledge of monotonicity-type properties of $T \mapsto \mathbb{P}_y(\Gamma_T \leq s)\, \mathrm{e}^{\frac{\pi^2}{8}T}$, which, unfortunately, we cannot ensure for *any* deterministic starting point $y \in \mathbb{R}$.

The structural outline of the rest of this chapter is as follows. In Section 3.2, we prove Theorem 3.2 following the strategy mentioned above: After some preliminaries, Subsection 3.2.1 contains a thorough analysis of the Laplace transform and its asymptotic behavior. In Subsection 3.2.2, this is used to determine the exact asymptotic behavior of the $T \mapsto \mathbb{P}_\nu(\Gamma_T \leq s)$ for a special initial distribution ν. In Subsection 3.2.3, we finally study the precise dependence of $T \mapsto \mathbb{P}_y(\Gamma_T \leq s)$ on the starting point y. Section 3.3 finishes the proof of Theorem 3.1 and includes the proof of Corollary 3.3.
Most parts of this chapter are contained in the preprint [AKS23].

3.2 Proof of the Asymptotics

Let us now start with the proof of Theorem 3.2. To this end, let $y \in [-1, 1]$. Recalling the behavior of $\mathbb{P}_y(\Gamma_T = 0)$ cited in (3.3) for $y \in (-1, 1)$ and observing $\mathbb{P}_y(\Gamma_T = 0) = 0$ for $y \in \{-1, 1\}$, we may focus on the analysis of

$$R := R_{y,s} : [0, \infty) \to [0, 1], \quad T \mapsto \mathbb{P}_y(\Gamma_T \in (0, s]).$$

Let

$$H := \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\}$$

be the complex right half-plane. The key tool in deriving the asymptotics of R is Ingham's Tauberian theorem. It connects the asymptotic behavior of the Laplace transform of R at 0 with that of R at ∞. However, we will not be able to apply Lemma 3.4 directly to R due to the required regularity conditions. The version of Inghams's Tauberian theorem we will work with reads as follows (cf. Theorem 1′ in [Ing41]):

Lemma 3.4. *Let $S : [0,\infty) \to [0,\infty)$ be a non-decreasing function such that*

$$\widehat{S}(\lambda) := \lambda \int_0^\infty S(T)\,\mathrm{e}^{-\lambda T}\,\mathrm{d}T \in \mathbb{C}, \quad \lambda \in H, \tag{3.4}$$

converges. Let $D \subseteq \mathbb{C}\setminus\{0\}$ be a domain containing all positive real numbers. Furthermore, let $U, V : D \to \mathbb{C}$ be holomorphic with $U(\lambda), V(\lambda) \in \mathbb{R}_{>0}$ for sufficiently small $\lambda \in \mathbb{R}_{>0}$ and with

$$\widehat{S}(\lambda) \sim U(\lambda)\,\mathrm{e}^{V(\lambda)}, \quad \text{for real } \lambda \searrow 0,$$

as well as

$$\widehat{S}(\lambda) = O\left(U(|\lambda|)\,\mathrm{e}^{V(|\lambda|)}\right), \quad \lambda \to 0,\ \lambda \in H. \tag{3.5}$$

Denoting by $d : D \to [0,\infty]$ the distance function to D^c, assume that there is some $k > 1$ with

$$\lambda V'(\lambda) \searrow -\infty, \quad \frac{\sqrt{V''(\lambda)}}{|V'(\lambda)|} = o\left(\frac{d(\lambda)}{\lambda}\right), \quad \lambda^k V'(\lambda) \nearrow 0, \qquad \text{for real } \lambda \searrow 0. \tag{3.6}$$

Further, assume

$$\sup_{\substack{z\in\mathbb{C}\\|z|<d(\lambda)}} |V''(\lambda+z)| = O(V''(\lambda)), \qquad \text{for real } \lambda \searrow 0, \tag{3.7}$$

and

$$\sup_{\substack{z\in\mathbb{C}\\|z|<d(\lambda)}} |U(\lambda+z)| = O(U(\lambda)), \qquad \text{for real } \lambda \searrow 0.$$

Then we have

$$S(T) \sim \frac{U(h(T))\,\mathrm{e}^{Th(T)+V(h(T))}}{h(T)\sqrt{2\pi V''(h(T))}}, \quad T \to \infty,$$

where h is the inverse of $-V'|_{(0,\varepsilon)}$ for sufficiently small $\varepsilon > 0$.

Remark 3.5. (a) In (3.6), the notations $\searrow$ and $\nearrow$ denote monotone convergence. In particular, $\lambda V'(\lambda) \searrow -\infty$ for real $\lambda \searrow 0$ implies $\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda V'(\lambda)) \geq 0$ and $V'(\lambda) < 0$ for all sufficiently small $\lambda \in \mathbb{R}_{>0}$. Thus there exists an $\varepsilon > 0$ with $V''(\lambda) \geq -\frac{V'(\lambda)}{\lambda} > 0$ for all $\lambda \in (0,\varepsilon)$ so that $-V'|_{(0,\varepsilon)}$ is strictly decreasing. Noting $\lim_{\lambda\searrow 0} -V'(\lambda) = \infty$, it must have an inverse $h : (T_0,\infty) \to (0,\varepsilon)$ for a suitable $T_0 > 0$.

(b) In [Ing41], the function S is assumed to satisfy $S(0) = 0$ because the result itself (but not the proof) is presented in terms of the Laplace-Stieltjes transform: Assuming in our formulation additionally that $S(0) = 0$, an integration by parts shows that $\widehat{S}$ is nothing but the Laplace-Stieltjes transform of S (see the beginning of the proof or, e.g., Proposition I.13.1 in [Kor04]). From the proof (as well as from the nature of the result), it is evident that the assumptions on S may be relaxed in the above way if we confine ourselves to defining $\widehat{S}$ by (3.4).

(c) For future use, we remark that the assumption that S is globally non-decreasing can be relaxed: It suffices to assume that S is non-decreasing for sufficiently large arguments. Once again, this is apparent from the proof (as well as from the nature of the result).

(d) In [Ing41], assumption (3.5) is replaced by a weaker condition which does not require uniformity on all of H but only on the complex wedge $\{\lambda \in H : |\lambda| \leq c\,\mathrm{Re}(\lambda)\}$ for each $c > 0$. The assumption that D contains the whole positive axis is relaxed in [Ing41] as well.

3.2.1 Analysis of the Laplace Transform

Throughout this subsection, we consider $y \in [-1, 1]$. Our first step is to compute the Laplace transform of R. For the purpose of readability, let us define

$$v : (0, \infty) \to \mathbb{C}, \quad \lambda \mapsto \sqrt{2\lambda}\tanh(\sqrt{2\lambda}),$$

and

$$u := u_y : (0, \infty) \to \mathbb{C}, \quad \lambda \mapsto \frac{\cosh(y\sqrt{2\lambda})}{\cosh(\sqrt{2\lambda})}.$$

These functions are related to the functions U and V to which we will apply Ingham's Tauberian theorem (Lemma 3.4). The following result essentially is formula 3.1.4.4 in [BS02]:

Lemma 3.6. *For almost every $T > 0$, we have*

$$R(T) = \frac{2}{\sqrt{2\pi}} \int_0^s \int_0^\infty \rho_{r,x}(T)\mathrm{d}x\mathrm{d}r,$$

where $(\rho_{r,x} : (0, \infty) \to \mathbb{R})_{r,x>0}$ is a family of functions with Laplace transforms given by

$$(\mathcal{L}(\rho_{r,x}))(\lambda) = u(\lambda)\,\mathrm{e}^{-v(\lambda)x - \lambda r - \frac{x^2}{2r}} \left(\frac{1}{\sqrt{r}} + \frac{v(\lambda)x}{2\lambda\sqrt{r^3}} \right), \quad \lambda, r, x > 0.$$

Proof. By symmetry, it suffices to prove the claim for $y \in [0, 1]$. For the time being, let $y = 1$. We start by observing

$$F_T(s) := R_{1,s}(T) = \mathbb{P}_0 \left(\int_0^T \mathbb{1}_{(-2,0)^c}(B_t)\mathrm{d}t \leq s \right), \quad s, T \geq 0.$$

Now let $\eta, \lambda > 0$. Kac's theory developed in [Kac49] and [Kac51] shows the existence and uniqueness of a function $\psi_{\eta,\lambda} \in \mathcal{C}_0(\mathbb{R},\mathbb{R})$ (vanishing at infinity) which is $\mathcal{C}^1$ on $\mathbb{R}\setminus\{0\}$ as well as $\mathcal{C}^2$ on $\mathbb{R}\setminus\{-2,0\}$ and solves the ordinary differential equation (ODE)

$$\psi''_{\eta,\lambda}(x) = 2\left(\lambda + \eta \mathbb{1}_{(-2,0)^c}(x)\right)\psi_{\eta,\lambda}(x), \quad x \in \mathbb{R}\setminus\{-2,0\},$$

subject to $\psi'_{\eta,\lambda}(0-) - \psi'_{\eta,\lambda}(0+) = 2$. Moreover, this function satisfies

$$\int_{-\infty}^{\infty} \psi_{\eta,\lambda}(x)\mathrm{d}x = \int_0^{\infty}\int_0^{\infty} \mathrm{e}^{-\eta r - \lambda T}\, F_T(\mathrm{d}r)\mathrm{d}T = \int_0^{\infty} \mathrm{e}^{-\lambda T}\int_0^{\infty} \mathrm{e}^{-\eta r}\, F_T(\mathrm{d}r)\mathrm{d}T. \tag{3.8}$$

Setting $c_1 := \sqrt{2(\lambda+\eta)}$ and $c_2 := \sqrt{2\lambda}$, it is straightforward to check that a solution (and hence the only one) of the above ODE problem is given by

$$\psi_{\eta,\lambda}(x) := \frac{2}{(c_2+c_1)^2\,\mathrm{e}^{4c_2} - (c_2-c_1)^2}\cdot\begin{cases} 2c_2\,\mathrm{e}^{2(c_2+c_1)}\,\mathrm{e}^{c_1 x}, & x < -2,\\ (c_2-c_1)\,\mathrm{e}^{-c_2 x} + (c_2+c_1)\,\mathrm{e}^{4c_2}\,\mathrm{e}^{c_2 x}, & x \in [-2,0],\\ \left((c_2-c_1)+(c_2+c_1)\,\mathrm{e}^{4c_2}\right)\mathrm{e}^{-c_1 x}, & x > 0.\end{cases}$$

Piecewise integration yields

$$\begin{aligned}
&\int_{-\infty}^{\infty}\psi_{\eta,\lambda}(x)\mathrm{d}x\\
&= 2\frac{2c_2\,\mathrm{e}^{2(c_2+c_1)}\,\frac{\mathrm{e}^{-2c_1}}{c_1}}{(c_2+c_1)^2\,\mathrm{e}^{4c_2} - (c_2-c_1)^2} + 2\frac{(c_2-c_1)\frac{\mathrm{e}^{2c_2}-1}{c_2} + (c_2+c_1)\,\mathrm{e}^{4c_2}\,\frac{1-\mathrm{e}^{-2c_2}}{c_2}}{(c_2+c_1)^2\,\mathrm{e}^{4c_2} - (c_2-c_1)^2}\\
&\qquad + 2\frac{\left((c_2-c_1)+(c_2+c_1)\,\mathrm{e}^{4c_2}\right)\frac{1}{c_1}}{(c_2+c_1)^2\,\mathrm{e}^{4c_2} - (c_2-c_1)^2}\\
&= \frac{2}{c_1c_2}\cdot\frac{\left((c_2+c_1)\,\mathrm{e}^{2c_2} + (c_2-c_1)\right)^2}{\left((c_2+c_1)\,\mathrm{e}^{2c_2}\right)^2 - (c_2-c_1)^2}\\
&= \frac{2}{c_1c_2}\cdot\frac{(c_2+c_1)\,\mathrm{e}^{2c_2} + (c_2-c_1)}{(c_2+c_1)\,\mathrm{e}^{2c_2} - (c_2-c_1)}\\
&= \frac{2}{c_2(\mathrm{e}^{2c_2}-1) + c_1(\mathrm{e}^{2c_2}+1)}\left(\frac{\mathrm{e}^{2c_2}+1}{c_1} + \frac{\mathrm{e}^{2c_2}-1}{c_2}\right)\\
&= \frac{2}{c_2\frac{\mathrm{e}^{2c_2}-1}{\mathrm{e}^{2c_2}+1} + c_1}\left(\frac{1}{c_1} + \frac{\frac{\mathrm{e}^{2c_2}-1}{\mathrm{e}^{2c_2}+1}}{c_2}\right)\\
&= 2\int_0^{\infty}\exp\left(-\left(c_2\frac{\mathrm{e}^{2c_2}-1}{\mathrm{e}^{2c_2}+1} + c_1\right)x\right)\left(\frac{1}{c_1} + \frac{\frac{\mathrm{e}^{2c_2}-1}{\mathrm{e}^{2c_2}+1}}{c_2}\right)\mathrm{d}x\\
&= 2\int_0^{\infty}\mathrm{e}^{-\sqrt{2\lambda}\tanh(\sqrt{2\lambda})x}\,\mathrm{e}^{-\sqrt{2(\lambda+\eta)}x}\left(\frac{1}{\sqrt{2(\lambda+\eta)}} + \frac{\tanh(\sqrt{2\lambda})}{\sqrt{2\lambda}}\right)\mathrm{d}x.
\end{aligned}$$

Using, e.g., formulas 4.5.27 and 4.5.28 of [EMOT54], the integrand can be written involving a Laplace transform as follows:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \psi_{\eta,\lambda}(x)\mathrm{d}x \\
&= 2\int_0^\infty \mathrm{e}^{-\sqrt{2\lambda}\tanh(\sqrt{2\lambda})x} \int_0^\infty \mathrm{e}^{-(\eta+\lambda)r} \left(\frac{1}{\sqrt{2\pi r}} + \frac{\tanh(\sqrt{2\lambda})x}{\sqrt{2\lambda}\sqrt{2\pi r^3}} \right) \mathrm{e}^{-\frac{x^2}{2r}}\,\mathrm{d}r\mathrm{d}x \\
&= \frac{2}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-\eta r} \int_0^\infty 1 \cdot \mathrm{e}^{-v(\lambda)x-\lambda r-\frac{x^2}{2r}} \left(\frac{1}{\sqrt{r}} + \frac{v(\lambda)x}{2\lambda\sqrt{r^3}} \right) \mathrm{d}x\mathrm{d}r \\
&= \frac{2}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-\eta r} \int_0^\infty \int_0^\infty \mathrm{e}^{-\lambda T} \rho_{r,x}(T)\mathrm{d}T\mathrm{d}x\mathrm{d}r \\
&= \int_0^\infty \mathrm{e}^{-\lambda T} \int_0^\infty \mathrm{e}^{-\eta r} \frac{2}{\sqrt{2\pi}} \int_0^\infty \rho_{r,x}(T)\mathrm{d}x\mathrm{d}r\mathrm{d}T,
\end{aligned}$$

observing $u_1(\lambda) = 1$ and using the definition of $\rho_{r,x}$ in the third step. In view of equation (3.8), the uniqueness of the Laplace(-Stieltjes) transform implies that F_T has a Lebesgue density which is given by $\frac{F_T(\mathrm{d}r)}{\mathrm{d}r} = \frac{2}{\sqrt{2\pi}} \int_0^\infty \rho_T(r,x)\mathrm{d}x$ for almost all $T, r > 0$. Noting $F_T(0) = 0$, integration yields the claim for $y = 1$.
Now let $y \in [0, 1)$ and $\tau := \inf\{t \geq 0 : |B_t| = 1\}$ (with $B_0 = y$). Observing $R_{-1,s} = R_{1,s}$, we get

$$R_{y,s}(T) = \int_0^T R_{1,s}(T-t)\mathbb{P}(\tau \in \mathrm{d}t), \quad s, T \geq 0.$$

The Laplace(-Stieltjes) transform of τ is given by $\mathbb{E}\,\mathrm{e}^{-\lambda\tau} = u_y(\lambda)$ for each $\lambda > 0$ (see, e.g., Example 5a in [DS53]). Recalling $u_1(\lambda) = 1$ for all $\lambda > 0$, the claim now follows straight from the fact that the Laplace transform of a convolution is nothing but the product of the individual Laplace(-Stieltjes) transforms. □

The next lemma shows that one can exchange the order of integration to obtain the Laplace transform of R. The resulting formula can be extended to the half-plane

$$H^{\leftarrow} := \left\{ \lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\frac{\pi^2}{8} \right\} = H - \frac{\pi^2}{8} :$$

Lemma 3.7. *The functions u and v as well as $\lambda \mapsto \frac{v(\lambda)}{2\lambda}$ are well-defined and holomorphic on $H^{\leftarrow}$. (The singularity of the latter in 0 is removable.) Moreover, the Laplace transform of R is well-defined on $H^{\leftarrow}$ and satisfies*

$$(\mathcal{L}(R))(\lambda) = \frac{2u(\lambda)}{\sqrt{2\pi}} \int_0^s \int_0^\infty \mathrm{e}^{-v(\lambda)x-\lambda r-\frac{x^2}{2r}} \left(\frac{1}{\sqrt{r}} + \frac{v(\lambda)x}{2\lambda\sqrt{r^3}} \right) \mathrm{d}x\mathrm{d}r, \quad \lambda \in H^{\leftarrow}. \tag{3.9}$$

Proof. The functions $z \mapsto z\tanh(z)$ and $z \mapsto \frac{\tanh(z)}{z}$ as well as $z \mapsto \frac{\cosh(yz)}{\cosh(z)}$ are well-defined (after removing the singularity of the second in 0), holomorphic and even on

$$\left\{ z \in \mathbb{C} : \frac{z^2}{2} \in H^{\leftarrow} \right\} \subseteq \mathbb{C} \setminus \left\{ \frac{(2n+1)\pi}{2} i : n \in \mathbb{Z} \right\}.$$

Consequently, the three functions mentioned in the lemma are well-defined and holomorphic as well.
For all $r, x > 0$, this implies that the Laplace transform of the function $\rho_{r,x}$ introduced in Lemma 3.6 has a (unique) holomorphic extension to $H^{\leftarrow}$. Now let $K \subseteq H^{\leftarrow}$ be compact. By continuity, $\bigcup_{\lambda\in K}\{u(\lambda), v(\lambda), \frac{v(\lambda)}{2\lambda}, \lambda\}$ is bounded by some $m_K > 0$. Letting $Z_r \sim \mathcal{N}(m_K r, r)$ be normally distributed with mean $m_K r$ and variance $r > 0$, we obtain

$$\begin{aligned}
&\int_0^s \int_0^\infty \sup_{\lambda\in K} \left| \frac{u(\lambda)}{\sqrt{2\pi}} \,\mathrm{e}^{-v(\lambda)x-\lambda r-\frac{x^2}{2r}} \cdot 1 \cdot \left(\frac{1}{\sqrt{r}} + \frac{v(\lambda)x}{2\lambda\sqrt{r^3}}\right)\right| \mathrm{d}x\mathrm{d}r \\
&\le \int_0^s \int_0^\infty \frac{m_K}{\sqrt{2\pi}} \,\mathrm{e}^{m_K x + m_K s - \frac{x^2}{2r}} \,\mathrm{e}^{\frac{m_K^2 (s-r)}{2}} \left(\frac{1}{\sqrt{r}} + \frac{m_K x}{\sqrt{r^3}}\right) \mathrm{d}x\mathrm{d}r \\
&= m_K \,\mathrm{e}^{m_K s + \frac{m_K^2 s}{2}} \int_0^s \int_0^\infty \left(1 + \frac{m_K x}{r}\right) \cdot \frac{1}{\sqrt{2\pi r}} \,\mathrm{e}^{-\frac{(x-m_K r)^2}{2r}} \,\mathrm{d}x\mathrm{d}r \\
&= m_K \,\mathrm{e}^{m_K s + \frac{m_K^2 s}{2}} \int_0^s \mathbb{E}\left[\mathbb{1}_{\{Z_r \ge 0\}} \left(1 + \frac{m_K Z_r}{r}\right)\right] \mathrm{d}r \\
&\le m_K \,\mathrm{e}^{m_K s + \frac{m_K^2 s}{2}} \int_0^s 1 + \frac{m_K}{r}(m_K r + \mathbb{E}|Z_r - m_K r|)\mathrm{d}r \\
&= m_K \,\mathrm{e}^{m_K s \frac{m_K^2 s}{2}} \int_0^s 1 + m_K^2 + m_K \sqrt{\frac{2}{\pi r}} \mathrm{d}r \\
&< \infty.
\end{aligned}$$

The dominated convergence theorem thus yields the continuity of the right-hand side of (3.9) as a function of $\lambda \in H^{\leftarrow}$. Now let γ be a closed path in $H^{\leftarrow}$. Bounding the path integral by the length of γ multiplied with the supremum of the integrand on $K := \operatorname{trace}(\gamma)$, the above computation implies

$$\int_0^s \int_0^\infty \int_\gamma \left| u(\lambda) \,\mathrm{e}^{-v(\lambda)x-\lambda r-\frac{x^2}{2r}} \left(\frac{1}{\sqrt{r}} + \frac{v(\lambda)x}{2\lambda\sqrt{r^3}}\right)\right| \mathrm{d}\lambda\mathrm{d}x\mathrm{d}r < \infty,$$

which allows us to apply Fubini's theorem. Since $\mathcal{L}(\rho_{r,x})$ is holomorphic, Cauchy's integral theorem implies

$$\begin{aligned}
&\int_\gamma \int_0^s \int_0^\infty u(\lambda) \,\mathrm{e}^{-v(\lambda)x-\lambda r-\frac{x^2}{2r}} \left(\frac{1}{\sqrt{r}} + \frac{v(\lambda)x}{2\lambda\sqrt{r^3}}\right) \mathrm{d}x\mathrm{d}r\mathrm{d}\lambda \\
&= \int_0^s \int_0^\infty \int_\gamma (\mathcal{L}(\rho_{r,x}))(\lambda)\mathrm{d}\lambda\mathrm{d}x\mathrm{d}r = 0.
\end{aligned}$$

By Morera's theorem,

$$H^{\leftarrow} \to \mathbb{C}, \quad \lambda \mapsto \frac{2u(\lambda)}{\sqrt{2\pi}} \int_0^s \int_0^\infty \mathrm{e}^{-v(\lambda)x-\lambda r-\frac{x^2}{2r}} \left(\frac{1}{\sqrt{r}} + \frac{v(\lambda)x}{2\lambda\sqrt{r^3}}\right) \mathrm{d}x\mathrm{d}r,$$

is holomorphic. By Lemma 3.6 and Tonelli's theorem, it coincides with $\mathcal{L}(R)$ on $(0, \infty)$ and hence, by the identity theorem for holomorphic function, on all of $H^{\leftarrow}$. □

Introducing

$$w : H^{\leftarrow} \to \mathbb{C}, \quad \lambda \mapsto \frac{(v(\lambda))^2}{2} - \lambda,$$

we can rewrite the Laplace transform of R in the following way, which is more suitable for our further analysis:

Lemma 3.8. *For each $\lambda \in H^{\leftarrow} \setminus \{0\}$, we have*

$$(\mathcal{L}(R))(\lambda) = \frac{u(\lambda)}{\sqrt{2\pi}\lambda} \left(\sqrt{2\pi} - 2\,\mathrm{e}^{w(\lambda)s} \int_0^\infty \mathrm{e}^{-\frac{(x+v(\lambda)\sqrt{s})^2}{2}}\,\mathrm{d}x + (v(\lambda)-1) \int_0^s \frac{1}{\sqrt{r}}\,\mathrm{e}^{-\lambda r}\,\mathrm{d}r \right).$$

Proof. Let $\lambda \in H^{\leftarrow}$. Then we have

$$\begin{aligned}
&\int_0^s \mathrm{e}^{w(\lambda)r} \int_0^\infty \mathrm{e}^{-\frac{(x+v(\lambda)\sqrt{r})^2}{2}} \frac{x+v(\lambda)\sqrt{r}}{\sqrt{r}} \mathrm{d}x\mathrm{d}r \\
&= \int_0^s \mathrm{e}^{w(\lambda)r}\,\mathrm{e}^{-\frac{(v(\lambda)\sqrt{r})^2}{2}} \frac{1}{\sqrt{r}} \mathrm{d}r \\
&= \int_0^s \frac{1}{\sqrt{r}}\,\mathrm{e}^{-\lambda r}\,\mathrm{d}r. \qquad (3.10)
\end{aligned}$$

Noting that

$$\int_0^\infty \sup_{r\in[\varepsilon,s]} \left| \mathrm{e}^{-\frac{(x+v(\lambda)\sqrt{r})^2}{2}} \frac{x+v(\lambda)\sqrt{r}}{2\sqrt{r}} \right| \mathrm{d}x \le \int_0^\infty \mathrm{e}^{-\frac{x^2}{2}+x|v(\lambda)|\sqrt{s}+\frac{|v(\lambda)|^2 s}{2}} \frac{x+|v(\lambda)|\sqrt{s}}{2\sqrt{\varepsilon}} \mathrm{d}x$$

is finite for each $\varepsilon \in (0, s)$, we are allowed to differentiate w.r.t. r under the x-integral in the following computation. Integrating by parts w.r.t. r in the second step and using (3.10) in the third, we get

$$\begin{aligned}
&\int_0^s \int_0^\infty \mathrm{e}^{-v(\lambda)\sqrt{r}x-\lambda r-\frac{x^2}{2}}\,\mathrm{d}x\mathrm{d}r \\
&= \int_0^s \mathrm{e}^{w(\lambda)r} \int_0^\infty \mathrm{e}^{-\frac{(x+v(\lambda)\sqrt{r})^2}{2}}\,\mathrm{d}x\mathrm{d}r \\
&= \frac{\mathrm{e}^{w(\lambda)s}}{w(\lambda)} \int_0^\infty \mathrm{e}^{-\frac{(x+v(\lambda)\sqrt{s})^2}{2}}\,\mathrm{d}x - \frac{\sqrt{2\pi}}{2w(\lambda)} + \int_0^s \frac{\mathrm{e}^{w(\lambda)r}}{w(\lambda)} \int_0^\infty \mathrm{e}^{-\frac{(x+v(\lambda)\sqrt{r})^2}{2}} \frac{x+v(\lambda)\sqrt{r}}{2\sqrt{r}} \mathrm{d}x\mathrm{d}r \\
&= \frac{1}{w(\lambda)} \left(\mathrm{e}^{w(\lambda)s} \int_0^\infty \mathrm{e}^{-\frac{(x+v(\lambda)\sqrt{s})^2}{2}}\,\mathrm{d}x - \frac{\sqrt{2\pi}}{2} + \int_0^s \frac{1}{2\sqrt{r}}\,\mathrm{e}^{-\lambda r}\,\mathrm{d}r \right). \qquad (3.11)
\end{aligned}$$

Substituting x by $\sqrt{r}x$ in the first step and plugging in (3.10) and (3.11) in the third, we get

$$\begin{aligned}
&\int_0^s \int_0^\infty \mathrm{e}^{-v(\lambda)x-\lambda r-\frac{x^2}{2r}} \left(\frac{1}{\sqrt{r}} + \frac{v(\lambda)x}{2\lambda\sqrt{r^3}}\right) \mathrm{d}x\mathrm{d}r \\
&= \int_0^s \int_0^\infty \mathrm{e}^{-v(\lambda)x\sqrt{r}-\lambda r-\frac{x^2}{2}} \left(1 + \frac{v(\lambda)x}{2\lambda\sqrt{r}}\right) \mathrm{d}x\mathrm{d}r \\
&= \int_0^s \int_0^\infty \mathrm{e}^{-v(\lambda)x\sqrt{r}-\lambda r-\frac{x^2}{2}} \left(-\frac{w(\lambda)}{\lambda} + \frac{v(\lambda)}{2\lambda} \cdot \frac{x+v(\lambda)\sqrt{r}}{\sqrt{r}}\right) \mathrm{d}x\mathrm{d}r \\
&= -\frac{w(\lambda)}{\lambda} \cdot \frac{1}{w(\lambda)} \left(\mathrm{e}^{w(\lambda)s} \int_0^\infty \mathrm{e}^{-\frac{(x+v(\lambda)\sqrt{s})^2}{2}} \,\mathrm{d}x - \frac{\sqrt{2\pi}}{2} + \int_0^s \frac{1}{2\sqrt{r}} \,\mathrm{e}^{-\lambda r} \,\mathrm{d}r\right) \\
&\quad + \frac{v(\lambda)}{2\lambda} \int_0^s \frac{1}{\sqrt{r}} \,\mathrm{e}^{-\lambda r} \,\mathrm{d}r.
\end{aligned}$$

Inserting this into (3.9) and simplifying, we obtain the claimed formula. □

Let us now start with the asymptotic analysis of $\mathcal{L}(R)$ near its rightmost singularity, i.e., near $-\frac{\pi^2}{8}$. In view of Ingham's Tauberian theorem (Lemma 3.4), we translate $-\frac{\pi^2}{8}$ to the origin. To this end, we set $u^{\rightarrow} := u\big(\,\cdot\, - \frac{\pi^2}{8}\big)$ and similarly define $v^{\rightarrow}$ and $w^{\rightarrow}$.

Lemma 3.9. *We have*

$$v^{\rightarrow}(\lambda) = -\frac{\pi^2}{4\lambda} + \frac{3}{2} + \frac{3+\pi^2}{3\pi^2}\lambda + O(\lambda^2), \quad \lambda \to 0,\ \lambda \in H,$$

and

$$w^{\rightarrow}(\lambda) = \frac{\pi^4}{32\lambda^2} - \frac{3\pi^2}{8\lambda} + \frac{\pi^2+21}{24} + O(\lambda), \quad \lambda \to 0,\ \lambda \in H.$$

For $y \in (-1,1)$, we uniformly get

$$u^{\rightarrow}(\lambda) \sim \cos\left(y\sqrt{\frac{\pi^2}{4} - 2\lambda}\right) \frac{\pi}{2\lambda}, \quad \lambda \to 0,\ \lambda \in H.$$

Proof. Recalling

$$\tan(z) = -\left(z - \frac{\pi}{2}\right)^{-1} + \frac{1}{3}\left(z - \frac{\pi}{2}\right) + O\left(\left(z - \frac{\pi}{2}\right)^3\right), \quad z \to \frac{\pi}{2},$$

we obtain

$$-z\tan(z) = \frac{\pi}{2}\left(z - \frac{\pi}{2}\right)^{-1} + 1 - \frac{\pi}{6}\left(z - \frac{\pi}{2}\right) + O\left(\left(z - \frac{\pi}{2}\right)^2\right), \quad z \to \frac{\pi}{2}.$$

On the other hand, a Taylor expansion yields

$$\sqrt{\frac{\pi^2}{4} - 2\lambda} - \frac{\pi}{2} = -\frac{2}{\pi}\lambda - \frac{4}{\pi^3}\lambda^2 - \frac{16}{\pi^5}\lambda^3 + O(\lambda^4), \quad \lambda \to 0,\ \lambda \in H. \tag{3.12}$$

Using $zi\tanh(zi) = -z\tan(z)$ for $z \in \mathbb{C} \setminus \left\{\frac{(2n+1)\pi}{2} : n \in \mathbb{Z}\right\}$ in the second step and the above two expansions in the third, we get

$$\begin{aligned}
&v^{\rightarrow}(\lambda)\\
&= \sqrt{2\lambda - \frac{\pi^2}{4}}\tanh\left(\sqrt{2\lambda - \frac{\pi^2}{4}}\right)\\
&= -\sqrt{\frac{\pi^2}{4} - 2\lambda}\tan\left(\sqrt{\frac{\pi^2}{4} - 2\lambda}\right)\\
&= \frac{\pi}{2}\left(-\frac{2}{\pi}\lambda - \frac{4}{\pi^3}\lambda^2 - \frac{16}{\pi^5}\lambda^3 + O(\lambda^4)\right)^{-1} + 1 - \frac{\pi}{6}\left(-\frac{2}{\pi}\lambda + O(\lambda^2)\right) + O(\lambda^2)\\
&= -\frac{\pi^2}{4\lambda}\left(1 - \left(-\frac{2}{\pi^2}\lambda - \frac{8}{\pi^4}\lambda^2 + O(\lambda^3)\right)\right)^{-1} + 1 - \frac{\pi}{6}\left(-\frac{2}{\pi}\lambda + O(\lambda^2)\right) + O(\lambda^2)\\
&= -\frac{\pi^2}{4\lambda}\sum_{n=0}^{\infty}\left(-\frac{2}{\pi^2}\lambda - \frac{8}{\pi^4}\lambda^2 + O(\lambda^3)\right)^n + 1 + \frac{\pi^2}{3\pi^2}\lambda + O(\lambda^2)\\
&= -\frac{\pi^2}{4\lambda}\left(1 - \frac{2}{\pi^2}\lambda - \frac{8}{\pi^4}\lambda^2 + \frac{4}{\pi^4}\lambda^2 + O(\lambda^3)\right) + 1 + \frac{\pi^2}{3\pi^2}\lambda + O(\lambda^2)\\
&= -\frac{\pi^2}{4\lambda} + \frac{3}{2} + \frac{3+\pi^2}{3\pi^2}\lambda + O(\lambda^2), \quad \lambda \to 0,\ \lambda \in H.
\end{aligned}$$

We deduce

$$\begin{aligned}
w^{\rightarrow}(\lambda) = \frac{(v^{\rightarrow}(\lambda))^2}{2} - \lambda + \frac{\pi^2}{8} &= \frac{1}{2}\left(\frac{\pi^2}{16\lambda^2} - \frac{3\pi^2}{4\lambda} + \frac{9}{4} - \frac{3+\pi^2}{6}\right) + \frac{\pi^2}{8} + O(\lambda)\\
&= \frac{\pi^4}{32\lambda^2} - \frac{3\pi^2}{8\lambda} + \frac{\pi^2+21}{24} + O(\lambda), \quad \lambda \to 0,\ \lambda \in H.
\end{aligned}$$

Recalling $\cosh(iz) = \cos(z)$ for each $z \in \mathbb{C}$ and $\cos(\frac{\pi}{2} - z) \sim z$ as $z \to 0$ and using (3.12), we finally obtain

$$\begin{aligned}
u^{\rightarrow}(\lambda) &= \frac{\cosh\left(y\sqrt{2\lambda - \frac{\pi^2}{4}}\right)}{\cosh\left(\sqrt{2\lambda - \frac{\pi^2}{4}}\right)} = \frac{\cos\left(y\sqrt{\frac{\pi^2}{4} - 2\lambda}\right)}{\cos\left(\sqrt{\frac{\pi^2}{4} - 2\lambda}\right)}\\
&= \frac{\cos\left(y\sqrt{\frac{\pi^2}{4} - 2\lambda}\right)}{\cos\left(\frac{\pi}{2} - \frac{2}{\pi}\lambda + O(\lambda^2)\right)} \sim \frac{\cos\left(y\sqrt{\frac{\pi^2}{4} - 2\lambda}\right)}{\frac{2}{\pi}\lambda}, \quad \lambda \to 0,\ \lambda \in H,
\end{aligned}$$

uniformly in $y \in (-1,1)$. □

Let us now introduce the objects needed in Ingham's Tauberian theorem (Lemma 3.4): We define

$$S := S_y : [0,\infty) \to [0,\infty), \quad T \mapsto \mathrm{e}^{\frac{\pi^2}{8}T} R(T),$$

and

$$\widehat{S} := \widehat{S_y} : H \to \mathbb{C}, \quad \lambda \mapsto \lambda \int_0^\infty S(T)\,\mathrm{e}^{-\lambda T}\,\mathrm{d}T = \lambda \cdot (\mathcal{L}(R))\left(\lambda - \frac{\pi^2}{8}\right).$$

Moreover, we consider the complex wedge

$$D := \left\{\lambda \in H : |\arg(\lambda)| < \frac{\pi}{4}\right\},$$

where arg is a branch of the complex argument function taking values in $[-\pi, \pi]$, as well as the holomorphic function

$$V : D \to \mathbb{C}, \quad \lambda \mapsto \left(\frac{\pi^4}{32\lambda^2} - \frac{3\pi^2}{8\lambda} + \frac{\pi^2 + 21}{24}\right) s.$$

We observe $V(\lambda) \in \mathbb{R}_{>0}$ for sufficiently small $\lambda \in \mathbb{R}_{>0}$. More importantly, the asymptotic behavior of the transform $\widehat{S}$ has the required structure:

Lemma 3.10. *We have*

$$\widehat{S}(\lambda) \sim \frac{16}{\pi^2}\lambda u^{\to}(\lambda)\,\mathrm{e}^{V(\lambda)} \quad \textit{for real } \lambda \searrow 0,$$

as well as

$$\widehat{S}(\lambda) = O\left(|\lambda u^{\to}(\lambda)|\,\mathrm{e}^{V(|\lambda|)}\right), \quad \lambda \to 0,\ \lambda \in H,$$

both uniformly in $y \in [-1, 1]$.

Proof. Regarding the uniformity, we note that only $u^{\to}$ (implicitly) depends on y, while $v^{\to}$ and $w^{\to}$ do not. We start by observing

$$\left|\int_0^s \frac{1}{\sqrt{r}}\,\mathrm{e}^{-\left(\lambda - \frac{\pi^2}{8}\right)r}\,\mathrm{d}r\right| \le \mathrm{e}^{|\lambda| s}\int_0^s \frac{1}{\sqrt{r}}\,\mathrm{e}^{\frac{\pi^2}{8}r}\,\mathrm{d}r = O(1), \quad \lambda \to 0,\ \lambda \in H.$$

Together with Lemmas 3.8 and 3.9, we get

$$\begin{aligned}
&\widehat{S}(\lambda)\\
&= \frac{\lambda u^{\to}(\lambda)}{\sqrt{2\pi}\left(\lambda - \frac{\pi^2}{8}\right)}\\
&\qquad \cdot \left(\sqrt{2\pi} - 2\,\mathrm{e}^{w^{\to}(\lambda)s}\int_0^\infty \mathrm{e}^{-\frac{(x + v^{\to}(\lambda)\sqrt{s})^2}{2}}\,\mathrm{d}x + (v^{\to}(\lambda) - 1)\int_0^s \frac{1}{\sqrt{r}}\,\mathrm{e}^{-\left(\lambda - \frac{\pi^2}{8}\right)r}\,\mathrm{d}r\right)\\
&= \frac{16\lambda u^{\to}(\lambda)}{\pi^2\sqrt{2\pi}}(1 + o(1))\\
&\qquad \cdot \left(\mathrm{e}^{w^{\to}(\lambda)s}\int_0^\infty \mathrm{e}^{-\frac{(x + v^{\to}(\lambda)\sqrt{s})^2}{2}}\,\mathrm{d}x + O\left(\frac{1}{\lambda}\right)\right), \quad \lambda \to 0,\ \lambda \in H,
\end{aligned} \tag{3.13}$$

uniformly in $y \in [-1,1]$. Noting that Lemma 3.9 implies $\lim_{\lambda \searrow 0} v^{\rightarrow}(\lambda)\sqrt{s} = -\infty$, we deduce

$$\begin{aligned} \widehat{S}(\lambda) &= \frac{16}{\pi^2}\lambda u^{\rightarrow}(\lambda)(1+o(1))\left(\mathrm{e}^{w^{\rightarrow}(\lambda)s}\int_{v^{\rightarrow}(\lambda)\sqrt{s}}^{\infty}\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x + O\left(\frac{1}{\lambda}\right)\right) \\ &\sim \frac{16}{\pi^2}\lambda u^{\rightarrow}(\lambda)\,\mathrm{e}^{V(\lambda)}, \quad \text{for real } \lambda \searrow 0, \end{aligned}$$

uniformly in $y \in [-1,1]$. Since Lemma 3.9 also implies

$$\begin{aligned} \frac{(\operatorname{Re} v^{\rightarrow}(\lambda))^2}{2} &= \frac{1}{2}\left(-\frac{\pi^2}{4}\operatorname{Re}\frac{1}{\lambda} + \frac{3}{2} + O(|\lambda|)\right)^2 = \frac{\pi^4}{32}\left(\operatorname{Re}\frac{1}{\lambda}\right)^2 - \frac{3\pi^2}{8}\operatorname{Re}\frac{1}{\lambda} + O(1) \\ &\leq \frac{\pi^4}{32}\left|\frac{1}{\lambda}\right|^2 - \frac{3\pi^2}{8}\left|\frac{1}{\lambda}\right| + O(1) = \frac{V(|\lambda|)}{s} + O(1), \quad \lambda \to 0,\ \lambda \in H, \end{aligned}$$

we obtain

$$\begin{aligned} \left|\mathrm{e}^{w^{\rightarrow}(\lambda)s}\int_0^{\infty}\mathrm{e}^{-\frac{(x+v^{\rightarrow}(\lambda)\sqrt{s})^2}{2}}\,\mathrm{d}x\right| &\leq \int_0^{\infty}\left|\mathrm{e}^{-\frac{x^2}{2}-xv^{\rightarrow}(\lambda)\sqrt{s}-\left(\lambda-\frac{\pi^2}{8}\right)s}\right|\mathrm{d}x \\ &= \int_0^{\infty}\mathrm{e}^{-\frac{x^2}{2}-x\operatorname{Re} v^{\rightarrow}(\lambda)\sqrt{s}-\operatorname{Re}\lambda s+\frac{\pi^2}{8}s}\,\mathrm{d}x \\ &= \mathrm{e}^{\frac{(\operatorname{Re} v^{\rightarrow}(\lambda))^2}{2}s}\,\mathrm{e}^{-\operatorname{Re}\lambda s+\frac{\pi^2}{8}s}\int_0^{\infty}\mathrm{e}^{-\frac{(x+\operatorname{Re} v^{\rightarrow}(\lambda)\sqrt{s})^2}{2}}\,\mathrm{d}x \\ &= O\big(\mathrm{e}^{V(|\lambda|)}\big)O(1), \quad \lambda \to 0,\ \lambda \in H. \end{aligned}$$

Plugging this into (3.13), we deduce

$$\begin{aligned} \big|\widehat{S}(\lambda)\big| &= \frac{16|\lambda u^{\rightarrow}(\lambda)|}{\pi^2\sqrt{2\pi}}(1+o(1))\left|\mathrm{e}^{w^{\rightarrow}(\lambda)s}\int_0^{\infty}\mathrm{e}^{-\frac{(x+v^{\rightarrow}(\lambda)\sqrt{s})^2}{2}}\,\mathrm{d}x + O\left(\frac{1}{\lambda}\right)\right| \\ &= O\big(|\lambda u^{\rightarrow}(\lambda)|\,\mathrm{e}^{V(|\lambda|)}\big), \quad \lambda \to 0,\ \lambda \in H, \end{aligned}$$

uniformly in $y \in [-1,1]$. □

3.2.2 Inversion when Starting in the QSD

As already noted in the outline of the proof in Section 3.1, it seems unclear how to prove monotonicity of S_y, which is required to apply Ingham's Tauberian theorem (Lemma 3.4), for any deterministic starting point $y \in \mathbb{R}$. For a specific starting distribution ν, however, the monotonicity of the corresponding function is rather easy to check. This allows us to rigorously derive the asymptotic behavior of $\mathbb{P}_\nu(\Gamma_T \in (0,s])$ as $T \to \infty$ in a first step. Let

$$\tau := \inf\{t \geq 0 : |B_t| = 1\}$$

be the first exit time of B from the interval $(-1,1)$. We recall that there is a unique distribution ν supported on $(-1,1)$ which satisfies

$$\mathbb{P}_\nu(\tau \geq t, B_t \in \mathrm{d}y) = \mathrm{e}^{-\frac{\pi^2}{8}t}\,\mathrm{d}\nu(y), \quad y \in (-1,1),\ t \geq 0. \tag{3.14}$$

This distribution is called quasi-stationary distribution (qsd) in the literature. It has a Lebesgue density given by

$$\mathrm{d}\nu(y) = \frac{\pi}{4}\cos\left(\frac{\pi y}{2}\right)\mathrm{d}y, \quad y \in (-1,1). \tag{3.15}$$

Note that the dependence on y is the same as in Theorem 3.2. For more details and further results concerning quasi-stationary distributions of diffusions, we refer to [KS12], [CMS13] and [CV23]. The monotonicity of

$$S_\nu : [0,\infty) \to [0,\infty), \quad T \mapsto \mathrm{e}^{\frac{\pi^2}{8}T}\,\mathbb{P}_\nu(\Gamma_T \in (0,s]),$$

now follows straight from the strong Markov property and (3.14):

$$\begin{aligned}\mathbb{P}_\nu(\Gamma_{T+t} \in (0,s]) &\geq \mathbb{P}_\nu(\Gamma_{T+t} \in (0,s], \tau \geq t)\\ &= \mathbb{E}_\nu\left[\mathbb{1}_{\{\tau\geq t\}}\mathbb{P}_{B_t}(\Gamma_T \in (0,s])\right] = \mathrm{e}^{-\frac{\pi^2}{8}t}\,\mathbb{P}_\nu(\Gamma_T \in (0,s]), \quad T,t \geq 0.\end{aligned}$$

This is an observation by Kolb. Preparing the application of Inghams's Tauberian theorem (Lemma 3.4) to S_ν, we check that the function V satisfies the technical assumptions of the theorem:

Lemma 3.11. *The function V satisfies the conditions stated in* (3.6) *and* (3.7)*.*

Proof. The derivatives of V are given by

$$V'(\lambda) = \left(-\frac{\pi^4}{16\lambda^3} + \frac{3\pi^2}{8\lambda^2}\right)s \quad \text{and} \quad V''(\lambda) = \left(\frac{3\pi^4}{16\lambda^4} - \frac{3\pi^2}{4\lambda^3}\right)s, \qquad \lambda \in D.$$

In particular, we have $\lambda V'(\lambda) \searrow -\infty$ and $\lambda^4 V'(\lambda) \nearrow 0$ for real $\lambda \searrow 0$ (see part (a) of Remark 3.5 for the notation). Noting that the distance of λ to D^c is given by $d(\lambda) = \frac{\lambda}{\sqrt{2}}$ for $\lambda \in D \cap \mathbb{R}_{>0}$, we get

$$\frac{\sqrt{V''(\lambda)}}{|V'(\lambda)|} = o(1) = o\left(\frac{d(\lambda)}{\lambda}\right), \quad \text{for real } \lambda \searrow 0.$$

For every sufficiently small $\lambda \in \mathbb{R}_{>0}$ and each $z \in \mathbb{C}$ with $|z| < d(\lambda)$, we obtain

$$\frac{|V''(\lambda+z)|}{V''(\lambda)} = \frac{\lambda^4}{|\lambda+z|^4}\cdot\left|\frac{\frac{3\pi^4}{16} - \frac{3}{4}\pi^2(\lambda+z)}{\frac{3\pi^4}{16} - \frac{3}{4}\pi^2\lambda}\right| \leq \left(\frac{1}{1-\frac{1}{\sqrt{2}}}\right)^4 \cdot 2,$$

verifying the conditions stated in (3.7). ☐

As seen in part (a) of Remark 3.5, there are constants $\varepsilon, T_0 > 0$ such that $-V'|_{(0,\varepsilon)}$ has an inverse $h : (T_0,\infty) \to (0,\varepsilon)$. This function can be determined explicitly and behaves as follows:

Lemma 3.12. *The function h satisfies*

$$h(T) \sim \left(\frac{\pi^4 s}{2^4 T}\right)^{\frac{1}{3}} \quad \textit{and} \quad h(T)\sqrt{2\pi V''(h(T))} \sim \frac{\sqrt{3}\pi^{\frac{7}{6}}}{2^{\frac{1}{6}}} s^{\frac{1}{6}} T^{\frac{1}{3}}, \qquad T \to \infty,$$

as well as

$$Th(T) + V(h(T)) = \frac{3}{2^{\frac{7}{3}}}\pi^{\frac{4}{3}} s^{\frac{1}{3}} T^{\frac{2}{3}} - \frac{3}{2^{\frac{5}{3}}}\pi^{\frac{2}{3}} s^{\frac{2}{3}} T^{\frac{1}{3}} + \frac{\pi^2+12}{24}s + O\left(\frac{1}{T^{\frac{1}{3}}}\right), \qquad T \to \infty.$$

Proof. Recalling

$$V'(\lambda) = \left(-\frac{\pi^4}{2^4\lambda^3} + \frac{3\pi^2}{2^3\lambda^2}\right)s, \quad \lambda > 0,$$

the function h must satisfy

$$\frac{T}{s}(h(T))^3 = \frac{\pi^4}{2^4} - \frac{3\pi^2}{2^3}h(T), \quad T \in (T_0, \infty),$$

so that Cardano's formula yields

$$\begin{aligned} h(T) &= \left(\frac{\pi^4 s}{2^5 T} + \sqrt{\frac{\pi^8 s^2}{2^{10}T^2} + \frac{\pi^6 s^3}{2^9 T^3}}\right)^{\frac{1}{3}} + \left(\frac{\pi^4 s}{2^{10} T} - \sqrt{\frac{\pi^8 s^2}{2^{10}T^2} + \frac{\pi^6 s^3}{2^9 T^3}}\right)^{\frac{1}{3}} \\ &= \left(\frac{\pi^4 s}{2^5 T}\right)^{\frac{1}{3}} \left(\left(\sqrt{1+\frac{2s}{\pi^2 T}} + 1\right)^{\frac{1}{3}} - \left(\sqrt{1+\frac{2s}{\pi^2 T}} - 1\right)^{\frac{1}{3}}\right), \quad T \in (T_0, \infty). \end{aligned}$$

This implies $h(T) \sim \left(\frac{\pi^4 s}{2^4 T}\right)^{\frac{1}{3}}$ as $T \to \infty$. We deduce

$$\begin{aligned} h(T)\sqrt{2\pi V''(h(T))} &= \sqrt{2\pi\left(\frac{3\pi^4}{2^4(h(T))^2} - \frac{3\pi^2}{4h(T)}\right)s} \\ &\sim \sqrt{\frac{3\pi^5 s}{2^3}\left(\frac{2^4 T}{\pi^4 s}\right)^{\frac{2}{3}}} = \frac{\sqrt{3}\pi^{\frac{7}{6}}}{2^{\frac{1}{6}}} s^{\frac{1}{6}} T^{\frac{1}{3}}, \quad T \to \infty. \end{aligned}$$

Observing

$$\left(\sqrt{1+z} - 1\right)^{\frac{1}{3}} = z^{\frac{1}{3}} \cdot \frac{1}{\left(\sqrt{1+z}+1\right)^{\frac{1}{3}}} = z^{\frac{1}{3}}\left(\frac{1}{2^{\frac{1}{3}}} + O(z)\right), \quad z \to 0, \tag{3.16}$$

we obtain

$$\begin{aligned} Th(T) &= \left(\frac{\pi^4 s T^2}{2^5}\right)^{\frac{1}{3}} \left(\left(\sqrt{1+\frac{2s}{\pi^2 T}} + 1\right)^{\frac{1}{3}} - \left(\sqrt{1+\frac{2s}{\pi^2 T}} - 1\right)^{\frac{1}{3}}\right) \\ &= \left(\frac{\pi^4 s T^2}{2^5}\right)^{\frac{1}{3}} \left(\left(2^{\frac{1}{3}} + O\left(\frac{1}{T}\right)\right) - \left(\frac{2s}{\pi^2 T}\right)^{\frac{1}{3}}\left(\frac{1}{2^{\frac{1}{3}}} + O\left(\frac{1}{T}\right)\right)\right) \\ &= \frac{\pi^{\frac{4}{3}}}{2^{\frac{4}{3}}} s^{\frac{1}{3}} T^{\frac{2}{3}} - \frac{\pi^{\frac{2}{3}}}{2^{\frac{5}{3}}} s^{\frac{2}{3}} T^{\frac{1}{3}} + O\left(\frac{1}{T^{\frac{1}{3}}}\right), \quad T \to \infty. \end{aligned}$$

Similarly, (3.16) yields

$$\begin{aligned} H(z) &:= \frac{1}{\left(\sqrt{1+z}+1\right)^{\frac{1}{3}} - \left(\sqrt{1+z}-1\right)^{\frac{1}{3}}} \\ &= \frac{1}{2}\left(\left(\sqrt{1+z}+1\right)^{\frac{2}{3}} + z^{\frac{1}{3}} + \left(\sqrt{1+z}-1\right)^{\frac{2}{3}}\right) \\ &= \frac{1}{2}\left(\left(2^{\frac{2}{3}} + O(z)\right) + z^{\frac{1}{3}} + z^{\frac{2}{3}}\left(\frac{1}{2^{\frac{2}{3}}} + O(z)\right)\right) \\ &= \frac{1}{2^{\frac{1}{3}}} + \frac{1}{2} z^{\frac{1}{3}} + \frac{1}{2^{\frac{5}{3}}} z^{\frac{2}{3}} + O(z), \quad z \to 0, \end{aligned}$$

which implies

$$\begin{aligned} V(h(T)) &= \frac{\pi^4 s}{2^5}\left(\frac{2^5 T}{\pi^4 s}\right)^{\frac{2}{3}}\left(H\left(\frac{2s}{\pi^2 T}\right)\right)^2 - \frac{3\pi^2 s}{8}\left(\frac{2^5 T}{\pi^4 s}\right)^{\frac{1}{3}} H\left(\frac{2s}{\pi^2 T}\right) + \frac{\pi^2+21}{24}s \\ &= \frac{\pi^4 s}{2^5}\left(\frac{2^5 T}{\pi^4 s}\right)^{\frac{2}{3}}\left(\frac{1}{2^{\frac{2}{3}}} + \frac{1}{2^{\frac{1}{3}}}\left(\frac{2s}{\pi^2 T}\right)^{\frac{1}{3}} + \left(\frac{1}{2^2}+\frac{1}{2}\right)\left(\frac{2s}{\pi^2 T}\right)^{\frac{2}{3}} + O\left(\frac{1}{T}\right)\right) \\ &\qquad - \frac{3\pi^2 s}{2^3}\left(\frac{2^5 T}{\pi^4 s}\right)^{\frac{1}{3}}\left(\frac{1}{2^{\frac{1}{3}}} + \frac{1}{2}\left(\frac{2s}{\pi^2 T}\right)^{\frac{1}{3}} + O\left(\frac{1}{T^{\frac{2}{3}}}\right)\right) + \frac{\pi^2+21}{24}s \\ &= \frac{\pi^{\frac{4}{3}}}{2^{\frac{7}{3}}} s^{\frac{1}{3}} T^{\frac{2}{3}} - \frac{2\pi^{\frac{2}{3}}}{2^{\frac{5}{3}}} s^{\frac{2}{3}} T^{\frac{1}{3}} + \frac{\pi^2+12}{24}s + O\left(\frac{1}{T^{\frac{1}{3}}}\right), \quad T \to \infty. \end{aligned}$$

The claim follows by adding the expansions of $Th(T)$ and $V(h(T))$. □

For the sake of completeness, we record the following result, which is clear from a heuristic point of view. The proof is an easy coupling argument.

Lemma 3.13. *Let $x, y \in \mathbb{R}$ with $|y| \geq |x|$. Then we have*

$$\mathbb{P}_y\left(\Gamma_T \leq s\right) \leq \mathbb{P}_x\left(\Gamma_T \leq s\right), \quad T \geq 0.$$

Proof. By symmetry, we may w.l.o.g. assume $x, y \geq 0$. Let $(B^1_t)_{t\geq 0}$ and $(B^2_t)_{t\geq 0}$ be independent standard Brownian motions. Further, let $\tau_{x,y} := \inf\{t \geq 0 : B^1_t = -\frac{x+y}{2}\}$. By the strong Markov property, the processes $(W^x_t)_{t\geq 0}$ and $(W^y_t)_{t\geq 0}$ defined by

$$W^x_t := \begin{cases} x + B^1_t, & t \leq \tau_{x,y}, \\ -\frac{x+y}{2} + B^2_{t-\tau_{x,y}}, & t \geq \tau_{x,y}, \end{cases} \qquad t \geq 0,$$

and

$$W^y_t := \begin{cases} y + B^1_t, & t \leq \tau_{x,y}, \\ \frac{x+y}{2} - B^2_{t-\tau_{x,y}}, & t \geq \tau_{x,y}, \end{cases} \qquad t \geq 0,$$

are Brownian motions starting in x and y, respectively. For every $t \leq \tau_{x,y}$, we observe $W_t^y - W_t^x = y - x$ and $W_t^x \geq -\frac{y-x}{2}$ showing $|W_t^y| \geq |W_t^x|$. For each $t \geq \tau_{x,y}$, we have $|W_t^y| = |W_t^x|$. Consequently,

$$\int_0^T \mathbb{1}_{\{|W_t^x| \geq 1\}} \mathrm{d}t \leq \int_0^T \mathbb{1}_{\{|W_t^y| \geq 1\}} \mathrm{d}t, \quad T \geq 0,$$

holds proving the claim. □

Integrating in Lemma 3.10 w.r.t. ν, we can finally apply Ingham's Tauberian theorem (Lemma 3.4) to obtain the asymptotic behavior of S_ν:

Proposition 3.14. *We have*

$$S_\nu(T) \sim \frac{2^{\frac{7}{6}}}{\sqrt{3}\pi^{\frac{7}{6}} s^{\frac{1}{6}} T^{\frac{1}{3}}} \exp\left(\frac{3}{2^{\frac{7}{3}}} \pi^{\frac{4}{3}} s^{\frac{1}{3}} T^{\frac{2}{3}} - \frac{3}{2^{\frac{5}{3}}} \pi^{\frac{2}{3}} s^{\frac{2}{3}} T^{\frac{1}{3}} + \frac{\pi^2 + 12}{24} s \right), \quad T \to \infty.$$

Proof. Equation (3.15) implies $\int_{-1}^1 \frac{8}{\pi} \cos\left(\frac{\pi y}{2}\right) \mathrm{d}\nu(y) = 2$. Trivially, the constant function $U : D \to (0, \infty)$, $\lambda \mapsto 2$, satisfies all requirements of Lemma 3.4. We define

$$\widehat{S_\nu} : H \to \mathbb{C}, \quad \lambda \mapsto \lambda \int_0^\infty S_\nu(T) \,\mathrm{e}^{-\lambda T} \,\mathrm{d}T.$$

Now let $\lambda \in H$. Lemma 3.13 implies

$$\begin{aligned}
&\sup_{y \in [-1,1]} \int_0^\infty |S_y(T) \,\mathrm{e}^{-\lambda T}| \mathrm{d}t \\
&= \sup_{y \in [-1,1]} \int_0^\infty \mathbb{P}_y(\Gamma_T \in (0, s]) \,\mathrm{e}^{-\operatorname{Re}\left(\lambda - \frac{\pi^2}{8}\right) T} \,\mathrm{d}T \\
&\leq \int_0^\infty \mathbb{P}_0(\Gamma_T \leq s) \,\mathrm{e}^{-\operatorname{Re}\left(\lambda - \frac{\pi^2}{8}\right) T} \,\mathrm{d}T \\
&= \mathcal{L}(R_{0,s}) \left(\operatorname{Re}\left(\lambda - \frac{\pi^2}{8}\right)\right) + \int_0^\infty \mathbb{P}_0(\Gamma_T = 0) \,\mathrm{e}^{-\operatorname{Re}\left(\lambda - \pi^2/8\right) T} \,\mathrm{d}T,
\end{aligned}$$

which is finite as a consequence of Lemma 3.7 and equation (3.3). Since ν is a probability measure, Fubini's theorem is applicable and yields

$$\begin{aligned}
\widehat{S_\nu}(\lambda) &= \lambda \int_0^\infty \int_{-1}^1 S_y(T) \,\mathrm{e}^{-\lambda T} \,\mathrm{d}\nu(y) \mathrm{d}T \\
&= \lambda \int_{-1}^1 \int_0^\infty S_y(T) \,\mathrm{e}^{-\lambda T} \,\mathrm{d}T \mathrm{d}\nu(y) = \int_{-1}^1 \widehat{S_y}(\lambda) \mathrm{d}\nu(y).
\end{aligned}$$

Using the uniformity in Lemma 3.10 and in the last assertion of Lemma 3.9 in the first step as well as the dominated convergence theorem in the second, we obtain

$$\begin{aligned}
\int_{-1}^1 \widehat{S_y}(\lambda) \mathrm{d}\nu(y) &\sim \int_{-1}^1 \frac{8}{\pi} \cos\left(y \sqrt{\frac{\pi^2}{4} - 2\lambda} \right) \mathrm{e}^{V(\lambda)} \,\mathrm{d}\nu(y) \\
&\to \int_{-1}^1 \frac{8}{\pi} \cos\left(\frac{\pi y}{2}\right) \mathrm{e}^{V(\lambda)} \,\mathrm{d}\nu(y) = U(\lambda) \,\mathrm{e}^{V(\lambda)}, \quad \text{for real } \lambda \searrow 0.
\end{aligned}$$

Setting $\mathbb{C}_1 := \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$, continuity implies

$$\sup_{\lambda \in \mathbb{C}_1, y \in [-1,1]} \left| \cos\left(y \sqrt{\frac{\pi^2}{4} - 2\lambda} \right) \right| < \infty.$$

Again using the uniformity in Lemma 3.10 and in the last assertion of Lemma 3.9 in the first step, we obtain

$$\begin{aligned} \int_{-1}^{1} \widehat{S_y}(\lambda) \mathrm{d}\nu(y) &= O\left(\int_{-1}^{1} \frac{8}{\pi} \left| \cos\left(y \sqrt{\frac{\pi^2}{4} - 2\lambda} \right) \right| \mathrm{e}^{V(|\lambda|)} \, \mathrm{d}\nu(y) \right) \\ &= O\left(U(|\lambda|) \, \mathrm{e}^{V(|\lambda|)} \right), \quad \lambda \to 0, \ \lambda \in H. \end{aligned}$$

Recalling Lemma 3.11, Lemma 3.4 is applicable and implies

$$\widehat{S_\nu}(\lambda) \sim \frac{U(h(T)) \, \mathrm{e}^{Th(T)+V(h(T))}}{h(T)\sqrt{2\pi V''(h(T))}}, \quad T \to \infty.$$

The claim follows by inserting the asymptotics developed in Lemma 3.12. $\square$

3.2.3 Asymptotics for Deterministic Starting Points

In this subsection, we finally prove Theorem 3.2, i.e., we extend the precise asymptotics to the case of deterministic starting points. To this end, we discuss how $\frac{\mathbb{P}_y(\Gamma_T \in (0,s])}{\mathbb{P}_\nu(\Gamma_T \in (0,s])}$ behaves as $T \to \infty$. Apart from the analysis in Subsection 3.2.1, our argument relies on a rather elementary observation, which we have not seen used before. Moreover, the proof will explicitly highlight the role of the density of ν as noted in the introduction of the previous subsection.

We start with an auxiliary result, which, again, is a consequence of our analysis in Subsection 3.2.1 and Ingham's Tauberian theorem (Lemma 3.4):

Lemma 3.15. *For any $T_0 > 0$, we have*

$$\lim_{T \to \infty} \frac{\int_{T-T_0}^{T} S_1(t) \mathrm{d}t}{\int_0^T S_1(t) \mathrm{d}t} = 0.$$

Proof. Clearly, the constant function $U : D \to (0, \infty)$, $\lambda \mapsto \frac{16}{\pi^2}$, satisfies the assumptions of Lemma 3.4. Integrating by parts, noting $u_1 = 1$ and applying Lemma 3.10, we get

$$\lambda \int_0^\infty \left(\int_0^T S_1(t) \mathrm{d}t \right) \mathrm{e}^{-\lambda T} \, \mathrm{d}\lambda = \frac{1}{\lambda} \widehat{S}_1(\lambda) \sim U(\lambda) \, \mathrm{e}^{V(\lambda)}, \quad \text{for real } \lambda \searrow 0,$$

and similarly

$$\lambda \int_0^\infty \left(\int_0^T S_1(t) \mathrm{d}t \right) \mathrm{e}^{-\lambda T} \, \mathrm{d}\lambda = O\left(U(|\lambda|) \, \mathrm{e}^{V(|\lambda|)} \right), \quad \lambda \to 0, \ \lambda \in H.$$

Since S_1 is non-negative, its primitive function is non-decreasing. Recalling Lemma 3.11, Lemma 3.4 is applicable and implies

$$\int_0^T S_1(t)\mathrm{d}t \sim \frac{U(h(T))\,\mathrm{e}^{Th(T)+V(h(T))}}{h(T)\sqrt{2\pi V''(h(T))}}, \quad T \to \infty.$$

Inserting the asymptotics developed in Lemma 3.12, we deduce

$$\int_0^T S_1(t)\mathrm{d}t \sim \frac{2^{\frac{25}{6}}}{\sqrt{3}\pi^{\frac{19}{6}} s^{\frac{1}{6}} T^{\frac{1}{3}}} \exp\left(\frac{3}{2^{\frac{7}{3}}}\pi^{\frac{4}{3}} s^{\frac{1}{3}} T^{\frac{2}{3}} - \frac{3}{2^{\frac{5}{3}}}\pi^{\frac{2}{3}} s^{\frac{2}{3}} T^{\frac{1}{3}} + \frac{\pi^2+12}{24}s\right), \quad T \to \infty.$$

Recalling $\lim_{T\to\infty} T^q - (T-T_0)^q = 0$ for all $q \in (0,1)$, we deduce

$$\lim_{T\to\infty} \frac{\int_0^{T-T_0} S_1(t)\mathrm{d}t}{\int_0^T S_1(t)\mathrm{d}t} = 1, \quad T_0 > 0,$$

proving the claim. ☐

We remark that the argument in the above proof to obtain the asymptotics of $\int_0^T S_y(t)\mathrm{d}t$ works for any starting point $y \in \mathbb{R}$.
Let us define

$$\sigma := \inf\left\{T > 0 : \int_0^T \mathbb{1}_{\{|W_t^1| \geq 1\}}\mathrm{d}t > s\right\},$$

where $(W_t^1)_{t\geq 0}$ denotes a Brownian motion with start in 1, independent of B. In the following argument by Kolb, who wishes to acknowledge a suggestion by Savov, we shall split a path with $\tau \leq T$ into a piece of length τ and an ingredient involving σ.

Lemma 3.16. *Given $\lambda_1 > \lambda_0 := \frac{\pi^2}{8} > 0$, let $e_0 \sim \operatorname{Exp}(\lambda_0)$ and $e_1 \sim \operatorname{Exp}(\lambda_1)$ be independent of σ. Then it holds*

$$\lim_{T\to\infty} \frac{\mathbb{P}(e_1+\sigma > T, e_1 \leq T)}{\mathbb{P}(e_0+\sigma > T, e_0 \leq T)} = \lim_{T\to\infty} \frac{\int_0^T \lambda_1 \mathrm{e}^{-\lambda_1 t}\,\mathbb{P}(\sigma > T-t)\mathrm{d}t}{\int_0^T \lambda_0 \mathrm{e}^{-\lambda_0 t}\,\mathbb{P}(\sigma > T-t)\mathrm{d}t} = 0.$$

Proof. Let $T \geq 0$. We start by observing

$$\begin{aligned}\mathbb{P}(e_0+\sigma > T, e_0 \leq T) &= \int_0^T \lambda_0 \mathrm{e}^{-\lambda_0 t}\,\mathbb{P}(\sigma > T-t)\mathrm{d}t \\ &= \lambda_0 \mathrm{e}^{-\lambda_0 T} \int_0^T S_1(T-t)\mathrm{d}t = \lambda_0 \mathrm{e}^{-\lambda_0 T} \int_0^T S_1(t)\mathrm{d}t.\end{aligned}$$

In the same way, we obtain

$$\mathbb{P}(e_1+\sigma > T, e_1 \leq T) = \lambda_1 \mathrm{e}^{-\lambda_0 T} \int_0^T \mathrm{e}^{-(\lambda_1-\lambda_0)t}\, S_1(T-t)\mathrm{d}t.$$

Now let $\varepsilon > 0$. Taking $T_\varepsilon > 0$ such that $\mathrm{e}^{-(\lambda_1-\lambda_0)t} \leq \varepsilon$ for all $t \geq T_\varepsilon$, we estimate

$$\begin{aligned}\int_0^T \mathrm{e}^{-(\lambda_1-\lambda_0)t}\, S_1(T-t)\mathrm{d}t &= \int_0^{T_\varepsilon} \mathrm{e}^{-(\lambda_1-\lambda_0)t}\, S_1(T-t)\mathrm{d}t + \int_{T_\varepsilon}^T \mathrm{e}^{-(\lambda_1-\lambda_0)t}\, S_1(T-t)\mathrm{d}t \\ &\leq \int_0^{T_\varepsilon} S_1(T-t)\mathrm{d}t + \varepsilon \int_{T_\varepsilon}^T S_1(T-t)\mathrm{d}t \\ &= \int_{T-T_\varepsilon}^T S_1(t)\mathrm{d}t + \varepsilon \int_0^{T-T_\varepsilon} S_1(t)\mathrm{d}t.\end{aligned}$$

We conclude

$$\begin{aligned}\frac{\mathbb{P}(e_1+\sigma > T, e_1 \leq T)}{\mathbb{P}(e_0+\sigma > T, e_0 \leq T)} &\leq \frac{\lambda_1 \left(\int_{T-T_\varepsilon}^T S_1(t)\mathrm{d}t + \varepsilon \int_0^{T-T_\varepsilon} S_1(t)\mathrm{d}t\right)}{\lambda_0 \int_0^T S_1(t)\mathrm{d}t} \\ &\leq \frac{\lambda_1 \int_{T-T_\varepsilon}^T S_1(t)\mathrm{d}t}{\lambda_0 \int_0^T S_1(t)\mathrm{d}t} + \frac{\lambda_1}{\lambda_0}\varepsilon, \quad T \geq T_\varepsilon.\end{aligned}$$

First letting $T \to \infty$, invoking Lemma 3.15, and then $\varepsilon \searrow 0$, we get the claim. $\square$

We are now ready to control the behavior of $\frac{\mathbb{P}_y(\Gamma_T \in (0,s])}{\mathbb{P}_\nu(\Gamma_T \in (0,s])}$ as announced at the beginning of this subsection. Most parts of the argument are due to Kolb.

Lemma 3.17. *Let $y \in (-1,1)$. We have*

$$\lim_{T\to\infty} \frac{\mathbb{P}_y(\Gamma_T \in (0,s])}{\mathbb{P}_\nu(\Gamma_T \in (0,s])} = \frac{4}{\pi} \cos\left(\frac{\pi y}{2}\right).$$

Proof. Defining

$$\lambda_n := \frac{(2n+1)^2\pi^2}{8} \quad \text{and} \quad \varphi_n(y) := \frac{4(-1)^n}{\pi(2n+1)} \cos\left(\frac{(2n+1)\pi y}{2}\right), \quad n \in \mathbb{N}_0,$$

a Lebesgue density $f_y : (0,\infty) \to (0,\infty)$ of τ under $\mathbb{P}_y$ is given by (see, e.g., Example 5a in [DS53] or equation (3.3) above)

$$f_y(t) = \sum_{n=0}^{\infty} \lambda_n\, \mathrm{e}^{-\lambda_n t}\, \varphi_n(y), \quad t > 0. \tag{3.17}$$

The dominated convergence theorem implies that f_y is continuous. Now let τ_1 and τ_{-1} be the first times B hits 1 and -1, respectively. Noting $\tau = \tau_{-1} \wedge \tau_1$, we observe

$$\mathbb{P}_y(\tau \in A) \leq \mathbb{P}_y(\tau_1 \in A) + \mathbb{P}_y(\tau_{-1} \in A), \quad A \in \mathcal{B}((0,\infty)).$$

Recalling that τ_1 and τ_{-1} follow inverse-chi-squared distributions (see, e.g., Remark 2.8.3 in [KS91]) and thus have bounded Lebesgue densities under $\mathbb{P}_y$, the continuous density f_y of τ must be bounded as well. Therefore,

$$g_y : (0,\infty) \to \mathbb{R}, \quad t \mapsto f_y(t) - \varphi_0(y)\lambda_0\, \mathrm{e}^{-\lambda_0 t}$$

is a bounded continuous function. Furthermore, equation (3.17) implies

$$|g_y(t)| = \mathrm{e}^{-\lambda_1 t} \left| \sum_{k=1}^{\infty} \varphi_k(y) \lambda_k \, \mathrm{e}^{-(\lambda_k - \lambda_1)t} \right| \le \mathrm{e}^{-\lambda_1 t} \sum_{k=1}^{\infty} |\varphi_k(y)| \lambda_k \, \mathrm{e}^{-(\lambda_k - \lambda_1)}, \quad t \ge 1.$$

Noting that the last series converges and recalling the boundedness of g_y, there exists a constant $K > 0$ with $|g_y(t)| \le K \, \mathrm{e}^{-\lambda_1 t}$ for all $t > 0$. Consequently, Lemma 3.16 yields

$$\frac{\left| \int_0^T g_y(t) \mathbb{P}(\sigma > T - t) \mathrm{d}t \right|}{\int_0^T \lambda_0 \, \mathrm{e}^{-\lambda_0 t} \, \mathbb{P}(\sigma > T - t) \mathrm{d}t} \le \frac{K \int_0^T \mathrm{e}^{-\lambda_1 t} \, \mathbb{P}(\sigma > T - t) \mathrm{d}t}{\lambda_0 \int_0^T \mathrm{e}^{-\lambda_0 t} \, \mathbb{P}(\sigma > T - t) \mathrm{d}t} \to 0, \quad T \to \infty.$$

We deduce

$$\begin{aligned} \mathbb{P}_y(\Gamma_T \in (0, s]) &= \mathbb{P}_y(\tau + \sigma > T, \tau \le T) \\ &= \int_0^T f_y(t) \mathbb{P}(\sigma > T - t) \mathrm{d}t \\ &= \varphi_0(y) \int_0^T \lambda_0 \, \mathrm{e}^{-\lambda_0 t} \, \mathbb{P}(\sigma > T - t) \mathrm{d}t + \int_0^T g_y(t) \mathbb{P}(\sigma > T - t) \mathrm{d}t \\ &\sim \varphi_0(y) \int_0^T \lambda_0 \, \mathrm{e}^{-\lambda_0 t} \, \mathbb{P}(\sigma > T - t) \mathrm{d}t, \quad T \to \infty. \end{aligned}$$

On the other hand, the characterization of the quasi-stationary distribution ν given in (3.14) implies

$$\mathbb{P}_\nu(\Gamma_T \in (0, s]) = \mathbb{P}_\nu(\tau + \sigma > T, \tau \le T) = \int_0^T \lambda_0 e^{-\lambda_0 t} \mathbb{P}(\sigma > T - t) \mathrm{d}t, \quad T \ge 0,$$

proving the claim. □

It essentially remains to combine Proposition 3.14 and Lemma 3.17:

Proof of Theorem 3.2. Let $y \in (-1, 1)$. Lemma 3.17 and Proposition 3.14 imply

$$\begin{aligned} &\mathbb{P}_y(\Gamma_T \in (0, s]) \\ &= \frac{\mathbb{P}_y(\Gamma_T \in (0, s])}{\mathbb{P}_\nu(\Gamma_T \in (0, s])} \cdot \mathrm{e}^{-\frac{\pi^2}{8} T} \, S_\nu(T) \\ &\sim \frac{4}{\pi} \cos\left(\frac{\pi y}{2}\right) \cdot \frac{2^{\frac{7}{6}}}{\sqrt{3} \pi^{\frac{7}{6}} s^{\frac{1}{6}} T^{\frac{1}{3}}} \exp\left(-\frac{\pi^2}{8} T + \frac{3}{2^{\frac{7}{3}}} \pi^{\frac{4}{3}} s^{\frac{1}{3}} T^{\frac{2}{3}} - \frac{3}{2^{\frac{5}{3}}} \pi^{\frac{2}{3}} s^{\frac{2}{3}} T^{\frac{1}{3}} + \frac{\pi^2 + 12}{24} s \right) \end{aligned}$$

as $T \to \infty$. Comparing this with equation (3.3), we see that

$$\mathbb{P}_y(\Gamma_T \le s) = \mathbb{P}_y(\Gamma_T = 0) + \mathbb{P}_y(\Gamma_T \in (0, s])$$

must have the same asymptotic behavior as $T \to \infty$ proving the first claim.
Now let $x \in \mathbb{R} \setminus (-1, 1)$. Using Lemma 3.13 and the case already settled, we get

$$\limsup_{T \to \infty} \frac{\mathbb{P}_x(\Gamma_T \le s)}{\mathbb{P}_0(\Gamma_T \le s)} \le \limsup_{T \to \infty} \frac{\mathbb{P}_y(\Gamma_T \le s)}{\mathbb{P}_0(\Gamma_T \le s)} = \cos\left(\frac{\pi y}{2}\right), \quad y \in (-1, 1),$$

which, taking $y \nearrow 1$, implies

$$\limsup_{T\to\infty} \frac{\mathbb{P}_x(\Gamma_T \le s)}{\mathbb{P}_0(\Gamma_T \le s)} = 0,$$

proving the second claim. $\square$

3.3 Proofs of the Remaining Results

Let us now prove Theorem 3.1. To this end, let $y \in (-1,1)$. It is a classical result (see, e.g., Example 1 in [Pin85]) that

$$\mathbb{P}_y(B \in \cdot \,|\, \Gamma_T = 0) = \mathbb{P}_y\left(B \in \cdot \,\big|\, |B_t| < 1 \text{ for all } t \in [0,T]\right)$$

converges weakly to a probability measure $\mathbb{Q}_y$ on $\mathcal{C}([0,\infty))$ as $T \to \infty$ and that the limiting process $(X_t)_{t\ge 0} \sim \mathbb{Q}_y$ satisfies the SDE (3.1). Regarding the weak convergence of $\mathbb{P}_y(B \in \cdot \,|\, \Gamma_T \le s)$ as $T \to \infty$, let us start by proving convergence of the finite dimensional distributions, based on a sketch by Kolb.

Lemma 3.18. *As $T \to \infty$, the probability measures $\mathbb{P}_y(B \in \cdot \,|\, \Gamma_T \le s)$ converge to $\mathbb{Q}_y$ in finite dimensional distributions.*

Proof. As before, let τ be the first exit time of B from $(-1,1)$. Further, let $t_1,\dots,t_d > 0$ with $t_1 < \ldots < t_d$ and let $C_1,\dots,C_d \in \mathcal{B}(\mathbb{R})$. We start by observing that the first part of Theorem 3.2 yields

$$\lim_{T\to\infty} \frac{\mathbb{P}_x(\Gamma_{T-t} \le s)}{\mathbb{P}_y(\Gamma_T \le s)} = \frac{\cos\left(\frac{\pi x}{2}\right)}{\cos\left(\frac{\pi y}{2}\right)} \mathrm{e}^{\frac{\pi^2}{8}t}, \quad x,y \in (-1,1),\ t \ge 0. \tag{3.18}$$

Together with the strong Markov property, this implies

$$\begin{aligned}
&\mathbb{P}_y(B_{t_1} \in C_1,\dots,B_{t_d} \in C_d, \tau \le t_d \,|\, \Gamma_T \le s) \\
&\le \mathbb{P}_y(\tau \le t_d \,|\, \Gamma_T \le s) \le \frac{\mathbb{P}_1(\Gamma_{T-t_d} \le s)}{\mathbb{P}_y(\Gamma_{T-t_d} \le s)} \cdot \frac{\mathbb{P}_y(\Gamma_{T-t_d} \le s)}{\mathbb{P}_y(\Gamma_T \le s)} \to 0, \quad T \to \infty.
\end{aligned} \tag{3.19}$$

Further, Lemma 3.13 and (3.18) guarantee the existence of a $T_0 \ge 0$ with

$$\frac{\mathbb{P}_{x_d}(\Gamma_{T-t_d} \le s)}{\mathbb{P}_y(\Gamma_T \le s)} \le \frac{\mathbb{P}_0(\Gamma_{T-t_d} \le s)}{\mathbb{P}_y(\Gamma_T \le s)} \le \frac{\mathrm{e}^{\frac{\pi^2}{8}t_d}}{\cos\left(\frac{\pi y}{2}\right)} + 1, \quad x_d \in (-1,1),\ T \ge T_0. \tag{3.20}$$

Now let

$$(p_t : (-1,1) \times (-1,1) \to [0,\infty))_{t>0}$$

be the family of (non-probability) transition densities of B absorbed in $\{-1,1\}$. We define $x_0 := y$ and $t_0 := 0$. Using (3.19), the Markov property as well as (3.18) together with the dominated convergence theorem, which is applicable as a consequence of (3.20), we obtain

$$
\begin{aligned}
&\lim_{T\to\infty} \mathbb{P}_y(B_{t_1} \in C_1, \ldots, B_{t_d} \in C_d \,|\, \Gamma_T \le s) \\
&= \lim_{T\to\infty} \mathbb{P}_y(B_{t_1} \in C_1, \ldots, B_{t_d} \in C_d, \tau > t_d \,|\, \Gamma_T \le s) \\
&= \lim_{T\to\infty} \int_{(-1,1)^d} \mathbb{1}_{C_1\times\cdots\times C_d}(x) \prod_{i=1}^{d} p_{t_i - t_{i-1}}(x_{i-1}, x_i) \frac{\mathbb{P}_{x_d}(\Gamma_{T-t_d} \le s)}{\mathbb{P}_y(\Gamma_T \le s)} \mathrm{d}x \\
&= \int_{(-1,1)^d} \mathbb{1}_{C_1\times\cdots\times C_d}(x) \prod_{i=1}^{d} p_{t_i - t_{i-1}}(x_{i-1}, x_i) \frac{\cos\left(\frac{\pi x_d}{2}\right)}{\cos\left(\frac{\pi y}{2}\right)} \mathrm{e}^{\frac{\pi^2}{8} t_d} \, \mathrm{d}x \\
&= \int_{(-1,1)^d} \mathbb{1}_{C_1\times\cdots\times C_d}(x) \prod_{i=1}^{d} \left(p_{t_i - t_{i-1}}(x_{i-1}, x_i) \frac{\cos\left(\frac{\pi x_i}{2}\right)}{\cos\left(\frac{\pi x_{i-1}}{2}\right)} \mathrm{e}^{\frac{\pi^2}{8}(t_i - t_{i-1})} \right) \mathrm{d}x,
\end{aligned}
$$

where $x_1, \ldots, x_d$ are the components of x. As pointed out, e.g., in the proof of Theorem 3.1 of [Kni69], the family $(\tilde{p}_t)_{t>0}$ of transition densities of the Markov process with law $\mathbb{Q}_y$ is precisely given by

$$
\tilde{p}_t(x_1, x_2) = p_t(x_1, x_2) \frac{\cos\left(\frac{\pi x_2}{2}\right)}{\cos\left(\frac{\pi x_1}{2}\right)} \mathrm{e}^{\frac{\pi^2}{8} t}, \quad x_1, x_2 \in (-1, 1),\ t > 0,
$$

completing the proof. □

To establish Theorem 3.1, it now suffices to prove tightness of the family of conditional laws. The proof, which is due to Kolb, relies on Kolmogorov's continuity theorem.

Lemma 3.19. *The family $(\mathbb{P}_y(B \in \cdot \mid \Gamma_T \le s))_{T\ge 0}$ of probability measures on $\mathcal{C}([0,\infty))$ is tight.*

Proof. Let $t_0 > 0$ and $t_1, t_2 \in [0, t_0]$ with $t_1 \le t_2$. Using the Markov property and Lemma 3.13, we get

$$
\begin{aligned}
\mathbb{E}_y\left[|B_{t_2} - B_{t_1}|^4 \,\middle|\, \Gamma_T \le s \right] &= \frac{1}{\mathbb{P}_y(\Gamma_T \le s)} \mathbb{E}_y\left[|B_{t_2} - B_{t_1}|^4 \mathbb{1}_{\{\Gamma_T \le s\}} \right] \\
&\le \frac{1}{\mathbb{P}_y(\Gamma_T \le s)} \mathbb{E}_y\left[|B_{t_2} - B_{t_1}|^4 \mathbb{1}_{\{\Gamma_T - \Gamma_{t_2} \le s\}} \right] \\
&= \frac{1}{\mathbb{P}_y(\Gamma_T \le s)} \mathbb{E}_y\left[|B_{t_2} - B_{t_1}|^4 \mathbb{P}_{B_{t_2}}(\Gamma_{T-t_2} \le s) \right] \\
&\le \frac{\mathbb{P}_0(\Gamma_{T-t_2} \le s)}{\mathbb{P}_y(\Gamma_T \le s)} \mathbb{E}_y\left[|B_{t_2} - B_{t_1}|^4 \right] \\
&\le \frac{\mathbb{P}_0(\Gamma_{T-t_0} \le s)}{\mathbb{P}_y(\Gamma_T \le s)} 3|t_2 - t_1|^2, \quad T \ge t_0.
\end{aligned}
$$

Since Theorem 3.2 implies $\limsup_{T\to\infty} \frac{\mathbb{P}_0(\Gamma_{T-t_0}\leq s)}{\mathbb{P}_y(\Gamma_T\leq s)} < \infty$ and the probabilities are continuous in T, there must be a constant $C > 0$ with

$$\mathbb{E}_y\left[|B_{t_2} - B_{t_1}|^4 \,\middle|\, \Gamma_T \leq s\right] \leq C|t_2 - t_1|^2, \quad t_1, t_2 \in [0, t_0],\ T \geq 0.$$

Now let $\varepsilon > 0$ and $\gamma \in (0, \frac{1}{2})$. As a consequence of the theorem of Kolmogorov-Chentsov (see, e.g., Theorem 3.4.16 in [Str93] for an applicable version), there exists a constant $K > 0$ (independent of T) with

$$\mathbb{P}_y\left(|X_{t_2} - X_{t_1}| \leq K|t_2 - t_1|^\gamma \text{ for all } t_1, t_2 \in [0, t_0] \,\middle|\, \Gamma_T \leq s\right) \geq 1 - \varepsilon, \quad T \geq 0.$$

Moreover, the set

$$\{f \in \mathcal{C}([0, t_0]) : |f_{t_2} - f_{t_1}| \leq K|t_2 - t_1|^\gamma \text{ for all } t_1, t_2 \in [0, t_0]\}$$

is compact in the space $\mathcal{C}([0, t_0])$. Hence we can apply Prohorov's theorem to conclude tightness on $\mathcal{C}([0, t_0])$. Corollary 5 in [Whi70] yields tightness on $\mathcal{C}([0, \infty))$. □

Finally, we deduce Corollary 3.3 from Theorem 3.2:

Proof of Corollary 3.3. W.l.o.g., let $\varepsilon \in (0, s)$ and $T \geq \varepsilon$. By symmetry, we may assume $y > 1$. Noting that the mapping $t \mapsto \{\Gamma_{T+t} \leq s + t\}$ is (weakly) monotonically increasing on $(-T, \infty)$ w.r.t. inclusion, we, on the one hand, obtain

$$\begin{aligned}\mathbb{P}_y(\tau \geq \varepsilon, \Gamma_T \leq s) = \mathbb{P}_y(\tau \in [\varepsilon, s], \Gamma_T \leq s) &= \int_\varepsilon^s \mathbb{P}_1(\Gamma_{T-t} \leq s - t)\mathbb{P}_y(\tau \in \mathrm{d}t)\\ &\leq \mathbb{P}_1(\Gamma_{T-\varepsilon} \leq s - \varepsilon)\mathbb{P}_y(\tau \in [\varepsilon, s]).\end{aligned}$$

Setting $\tau_0 := \inf\{t \geq 0 : B_t = 0\}$, we, on the other hand, get

$$\begin{aligned}\mathbb{P}_y(\Gamma_T \leq s) &\geq \mathbb{P}_y(\tau_0 \leq \varepsilon, \Gamma_T \leq s)\\ &\geq \int_0^\varepsilon \mathbb{P}_0(\Gamma_{T-t} \leq s - t)\mathbb{P}_y(\tau_0 \in \mathrm{d}t) \geq \mathbb{P}_0(\Gamma_{T-\varepsilon} \leq s - \varepsilon)\mathbb{P}_y(\tau_0 \in (0, \varepsilon)).\end{aligned}$$

Noting $\mathbb{P}_y(\tau_0 \in (0, \varepsilon)) > 0$, the second part of Theorem 3.2 yields the claim. □

4 Outlook

Let us finally give a brief overview of some open problems and possible directions of further research.
Similar to the results of [BB11] and [KS16] concerning limited local time in 0 discussed in Subsection 1.2.2, it is natural to ask what happens if the condition $\{\Gamma_T \leq s\}$, where $s > 0$ is constant, is replaced by

$$\{\Gamma_t \leq f(t) \text{ for all } t \in [0, T]\}, \quad T \geq 0,$$

where $f : [0, \infty) \to [0, \infty)$ belongs to a suitable class of non-decreasing functions.
Let us start with a discussion of the one-sided case: Up to a certain growth rate of f, the limiting process will eventually behave like a three-dimensional Bessel process and, in particular, be transient. For functions f growing faster, the resulting process should, apart from killing effects near the start, essentially behave like an unconditioned Brownian motion and, in particular, be recurrent.
In the two-sided case, the situation is more sophisticated. If f is growing sufficiently slowly, the resulting process should not leave the bounded interval at all, similar to the situation with constant f we considered. This may even be the case whenever the growth of f is strictly sub-linear. If f is growing fast enough, the limiting process will, in the long term, essentially behave like a Brownian motion. A priori, there could additionally be an intermediate regime where the resulting process leaves the interval but remains (pathwise) a.s. bounded.
A first step towards analyzing these rather difficult problems might be to condition on the significantly simpler event $\{\Gamma_T \leq f(T)\}$, similar to the approach in [BB11].

Another follow-up question consists in considering limited occupation times outside time-dependent intervals, i.e., in replacing the functional Γ_T by

$$\tilde{\Gamma}_T := \int_0^T \mathbb{1}_{\{B_t \leq g(t)\}} \mathrm{d}t \quad \text{or} \quad \tilde{\Gamma}_T := \int_0^T \mathbb{1}_{\{|B_t| \geq g(t)\}} \mathrm{d}t, \qquad T \geq 0,$$

respectively, for some function $g : [0, \infty) \to \mathbb{R}$, non-negative in the latter case. Again, the asymptotic growth of g will be an important ingredient. However, brief but extreme local fluctuations of g may have an influence as well.
Yet another possibility is to replace the considered process. In particular, one may study more general (one-dimensional) Markov processes conditioned on having limited occupation times, similar to the extensions of the results concerning limited local time in 0 given in [Bar20].

Furthermore, one could try to generalize the problems to multiple dimensions and condition a multi-dimensional Brownian motion on spending limited time outside cones or balls. Here it would be of particular interest how the additional spacial freedom influences the repulsive behavior. The possibilities are endless...

Notations

The following table lists some notations and abbreviations used throughout this thesis:

$\Rightarrow$	Weak convergence of probability measures or convergence in distribution of random variables
$\mathbb{1}_A$	The indicator function of a given set A
a.s.	Abbreviation of "almost surely"
$\mathcal{B}(E)$	The Borel σ-algebra on a given topological space E
$\mathcal{C}(I)$	The space of real-valued continuous functions on a given interval $I \subseteq \mathbb{R}$, endowed with the topology of locally uniform convergence
$\overset{\mathrm{d}}{=}$	Equality in distribution
$f\|_A$	The restriction of a given function f to a subset A of the domain of f
$\mathbb{E}[X]$	The expectation of a given random variable X
$\mathcal{L}(f)$	The Laplace transform of a given function $f : [0,\infty) \to \mathbb{C}$
$\mathbb{N}$	The set of natural numbers excluding 0
$\mathbb{N}_0$	The set of natural numbers including 0
$\mathbb{P}_y$	Probability if the considered process is started in a given point $y \in \mathbb{R}$
$\mathbb{P}_\nu$	Probability if the considered process is started in a given distribution ν
$\operatorname{Re} z$	The real part of a given complex number $z \in \mathbb{C}$
SDE	Abbreviation of "stochastic differential equation"
w.l.o.g.	Abbreviation of "without loss of generality"

Given a topological space E, a probability measure $\mathbb{Q} : \mathcal{B}(E) \to [0,1]$ is called a *probability measure on E*. Regarding the usage of $\mathbb{Q}$, we remark that the field of rational numbers does not appear (explicitly) in this thesis.

By $\sim$, we denote asymptotic equivalence of two functions, mapping from a common subset of $\mathbb{R}$ to $(0,\infty)$, at a specified point. For instance, given $f, g : [0,\infty) \to (0,\infty)$, we write $f(T) \sim g(T)$, as $T \to \infty$, if $\lim_{T\to\infty} \frac{f(T)}{g(T)} = 1$ holds.

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