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Characterizations of Synchronizability

in terms of road-colored graphs, Markov chains and quantum Markov processes

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P

PREFACE

> STORYTELLING IS THE MOST POWERFUL WAY
> TO PUT IDEAS INTO THE WORLD.
>
> — Robert McKee

Zusammenfassung

Diese Thesis widmet sich der Untersuchung der Eigenschaft der Synchronisierbarkeit straßengefärbter gerichteter Graphen, die auch als deterministische endliche Automaten verstanden werden können. In diesem Rahmen werden Markovketten, eine spezielle Klasse stochastischer Prozesse, betrachtet. Diese besitzen eine kanonische Darstellung als gerichtete Graphen und auch umgekehrt induziert ein gerichteter Graph, versehen mit einer Wahrscheinlichkeitsverteilung auf den ausgehenden Kanten der einzelnen Knoten, eine Markovkette. Eine solche Identifikation ist auch zwischen Markovketten und straßengefärbten gerichteten Graphen möglich. Insbesondere hinsichtlich der Eigenschaft der Synchronisierbarkeit, hat dieser Zusammenhang bereits zu interessanten Erkenntnissen in beiden Gebieten geführt (vgl. bspw. [GKL06], [Lan03], [Haa06], [Sis11], [YY11], [Yan13]), deren Vertiefung und Ausbau eines der Hauptziele dieser Arbeit ist. Darüber hinaus wurde im Bereich der Quantenwahrscheinlichkeitstheorie bereits gezeigt, dass der aus der Streutheorie motivierte Begriff der asymptotischen Vollständigkeit bestimmter Quantenmarkovprozesse eng mit der klassischen Idee der Synchronisierbarkeit verknüpft ist. Tatsächlich liefert die Synchronisierbarkeit eines straßengefärbten Graphen bei der Betrachtung kommutativer Systeme eine äquivalente Charakterisierung der asymptotischen Vollständigkeit des zugehörigen Quantenmarkovprozesses (vgl. bspw. [GKL06], [Lan03], [Haa06], [Sch12]). In diesem Rahmen befasst sich die vorliegende Thesis mit möglichen Verallgemeinerungen des klassischen Konzepts der Synchronisierbarkeit in einem quantenmechanischen Kontext.

In Kapitel 3 wird eine Modifikation von J.G. Propps und D.B. Wilsons Coupling-from-the-Past-Algorithmus eingeführt, die auf straßengefärbte Graphen anwendbar ist. Dadurch wird gezeigt, dass der ursprüngliche Algorithmus als Suche nach synchronisierenden Worten interpretiert werden kann [Sis11]. Außerdem werden zwei Konzepte für nichtkommutative Farben diskutiert, die beide eine Verallgemeinerung des klassischen Begriffs der monochromatischen Matrizen darstellen. Kapitel 4 präsentiert eine hinreichende und notwendige Bedingung für die Existenz einer synchronisierbaren Färbung für straßenfärbbare Graphen. Dies trägt zum besseren Verständnis eines bestimmten Graphenprodukts, das eine vorgegebene Färbung berücksichtigt, bei. Der sogenannte *dual extended transition operator* kann als eine Verallgemeinerung dieses Graphenprodukts angesehen werden. Es wurde bereits gezeigt, dass die Regularität dieses Operators äquivalent zur asymptotischen Vollständigkeit des zugehörigen Quantenmarkovprozesses ist [GKL06]. Diese Charakterisierung wird in ein allgemeineres Hilbertraumsetting übertragen und erweitert. In Kapitel 5 werden straßengefärbte Graphen mit einer abzählbar undendlichen Knotenmenge untersucht. Für eine bestimmte Klasse solcher Graphen, bzw. der zugehörigen Markovketten, wird eine hinreichende Bedingung formuliert, bei der die fast sichere Konvergenz der Übergangsfunktionen eine Charakterisierung der asymptotischen Vollständigkeit des entsprechenden Quantenmarkovprozesses liefert. Kapitel 6 zeigt zunächst, dass im klassischen Setting anstelle der fast sicheren Konvergenz der Übergangsfunktionen eines straßengefärbten Graphen auch deren fast gleichmäßige Konvergenz betrachtet werden kann. In der nichtkommutativen Mathematik gibt es einen analogen Begriff fast gleichmäßiger Konvergenz. Die Thesis schließt mit Betrachtungen zweier Fälle, bei denen die asymptotische Vollständigkeit des Quantenmarkovprozesses die fast gleichmäßige Konvergenz einer Folge von Operatoren impliziert.

Introduction

Generally, the publication of A. Kolmogorov, on the foundations of probability theory in 1933, is considered the birth of modern probability theory [Kol33]. Since that time, stochastic processes have become an integral part of mathematics. The crucial characteristic of Markov chains, a special kind of stochastic processes, is that they lose the memory of where they have been in the past so that where they go next depends solely on their current state. Together with the simplicity of this class of stochastic processes, their memory loss makes it possible to predict the Markov chain's behavior and to quantify it by assigning probabilities or expected values to it (compare, for example, [Nor98]). This allows Markov chains to efficiently model systems with sequential behavior and limited memory, making them valuable tools for prediction and decision-making. A Markov chain with values in a countable state space possesses a canonical representation as a directed graph. The vertices of the graph correspond to the states and its edges display the possible transitions between them.

There exists a close connection between Markov chains and the concept of deterministic automata, which can be understood as road-colored directed graphs. Here, directed graphs, whose vertices possess the same amount of outgoing edges, are considered. Fixing a correspondingly large set of labels or colors, the outgoing edges of each vertex can be labeled so that it possesses exactly one outgoing edge of each color. If a probability distribution is assigned to this set of colors and an initial distribution on the vertices is chosen, then the road-colored graph defines a Markov chain [Kit98]. Conversely, every Markov chain with finite state space can be represented by such a road-colored graph [Haa06], [YY11]. In the 1960s, the idea of synchronizing finite deterministic automata, and hence road-colored graphs with finitely many vertices, arose, efficiently making the automaton resistant against input errors [Hen64], [Čer64], [Boo67]. Intuitively, this means that in a city whose network of colored one-way roads conforms to the prescribed properties, no traveler can get lost. Regardless of which place within the city she finds herself, she can always find a way back home, following a single sequence of finitely many colored roads; the synchronizing word. Indeed, every aperiodic irreducible road-colorable graph can be colored in such a way that it possesses a synchronizing word [AGW77], [Tra09].

Simultaneously to these developments, the establishment of quantum probability theory, which was born in the early 20th century, gained more and more momentum (compare, for example, [Acc76], [AFL82], [Küm85], [Mey85], [Par92], [KM00]). A significant result was the introduction of a class of quantum Markov processes [Küm85]. For these processes, essential ideas of the classical scattering theory which describes the probability of particles undergoing an interaction and being deflected in different directions, aiming to predict the outcome of the interaction, have been transferred to the non-commutative setting, amounting to the property of asymptotic completeness of the process [KM00]. An equivalent characterization has been formulated in terms of universal preparability of the states of a quantum mechanical system [Haa06], [GHK17]. Here, a necessity that arises automatically when working with quantum mechanical systems is addressed: the desire to prepare the system in a specific state.

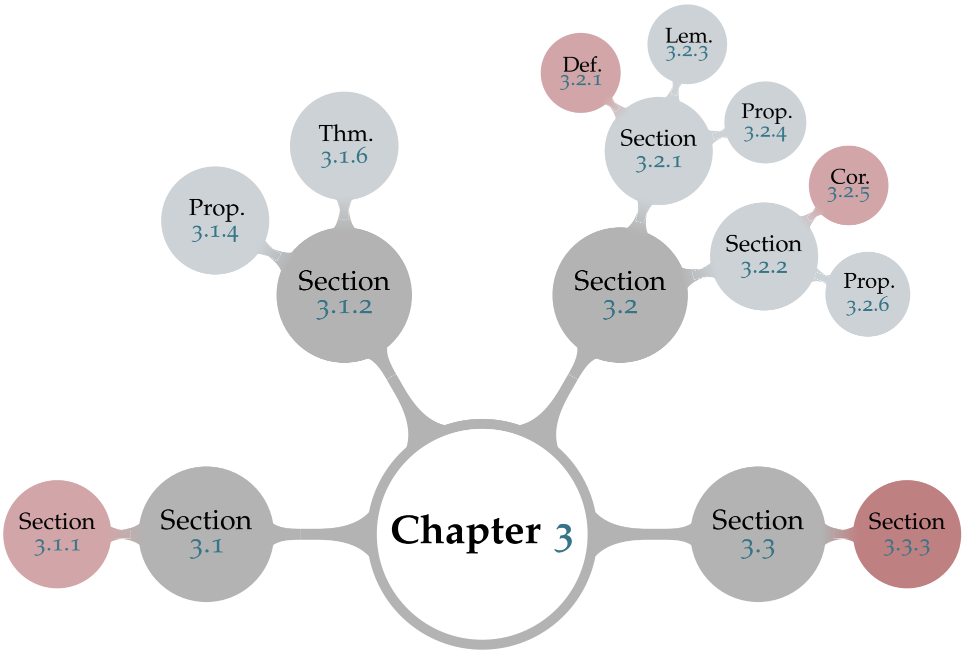
In the classical setting, on the other hand, to every road-colored graph a commutative quantum Markov process of the above kind can be assigned. In particular, when the graph possesses only finitely many vertices, it has been shown that its asymptotic completeness and the existence of a synchronizing word for the corresponding road-colored graph coincide [Lan03], [GKL06]. From this observation, the idea arose to leverage the characteristics of the classical theory of synchronizability in order to gain an understanding on the concept of asymptotic completeness in the quantum mechanical setting [Lan03], [Sis11], [Sch12]. The present thesis is dedicated to furthering the exploration of these aspects. Its primary objective is to deepen the understanding of synchronizability in classical mathematics. Following that, it aims to propose potential generalizations of these concepts within the realm of quantum mechanics.

Outline

In many aspects, writing about mathematics is like telling a story. In our case, it is the tale of synchronizable graphs and how they aid us in better comprehending asymptotic completeness. First and foremost, providing the reader with a sense of orientation is crucial. Chapter 1 delves into the classical concept of synchronizability of road-colored graphs. Furthermore, it introduces Markov chains, focusing on the correspondence between them and road-colorable graphs. The connection to quantum mechanics is established in Chapter 2. Here, the algebraic concepts that are essential for subsequent comprehension are presented, along with a brief introduction to the provided quantum probability theory. This ultimately leads to the introduction of a specific class of one-sided quantum Markov processes, for which a scattering theory has been established. In the commutative scenario, this setting precisely corresponds to the quest of finding synchronizing words in a road-colored graph.

In the following, we give a short summary of each chapter, together with an overview of this thesis' contribution to the research field. For each chapter, the new contents are illustrated graphically. Hereby, we differentiate between the expansion of already communicated ideas, the introduction of new concepts and results as well as new perspectives that lead to alternative proofs or reformulations of existing knowledge:

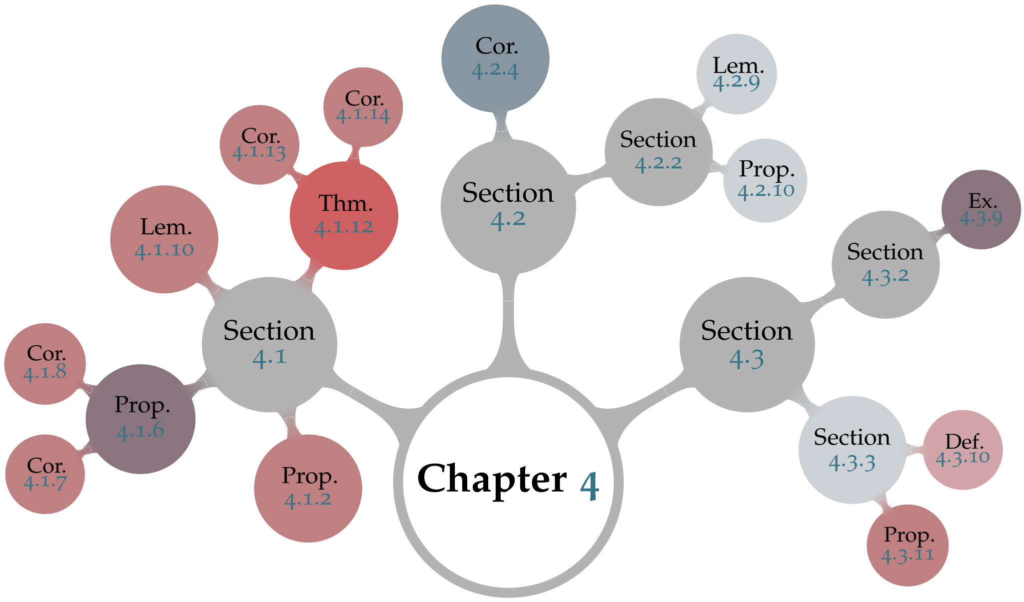
- new idea or concept
- new result
- new main result
- expansion of an existing idea or new example
- reformulation of existing knowledge
- new proof for existing knowledge



New content in Chapter 3

The main idea in Chapter 3 is that J.G. Propp's and D.B. Wilson's Coupling from the Past algorithm [PW96] does, in fact, nothing else but search for synchronizing words in an according representation of the underlying Markov chain as a road-colored graph. The idea was introduced in [Sis11] and also mentioned in [YY11]. This thesis implements it by introducing two new concepts – the Colored Coupling from the Past algorithm and the trivial coloring – allowing the transfer of the considerations on Propp's and Wilson's algorithm in [Sis11] to the setting of road-colored graphs. Section 3.2.2 concludes that the Colored Coupling from the Past algorithm, applied to trivially colored graphs, corresponds to the original algorithm. In this sense, Propp's and Wilson's algorithm can be understood as a special case of our Colored Coupling from the Past algorithm.

In Section 3.3 two possible concepts of non-commutative colors are presented. They correspond both to a generalization of the concept of monochromatic matrices, the first originating from an idea of B. Kümmerer and the second in [Sis11]. Subsequently, the question of whether these concepts coincide, is answered with a counterexample in Section 3.3.3.

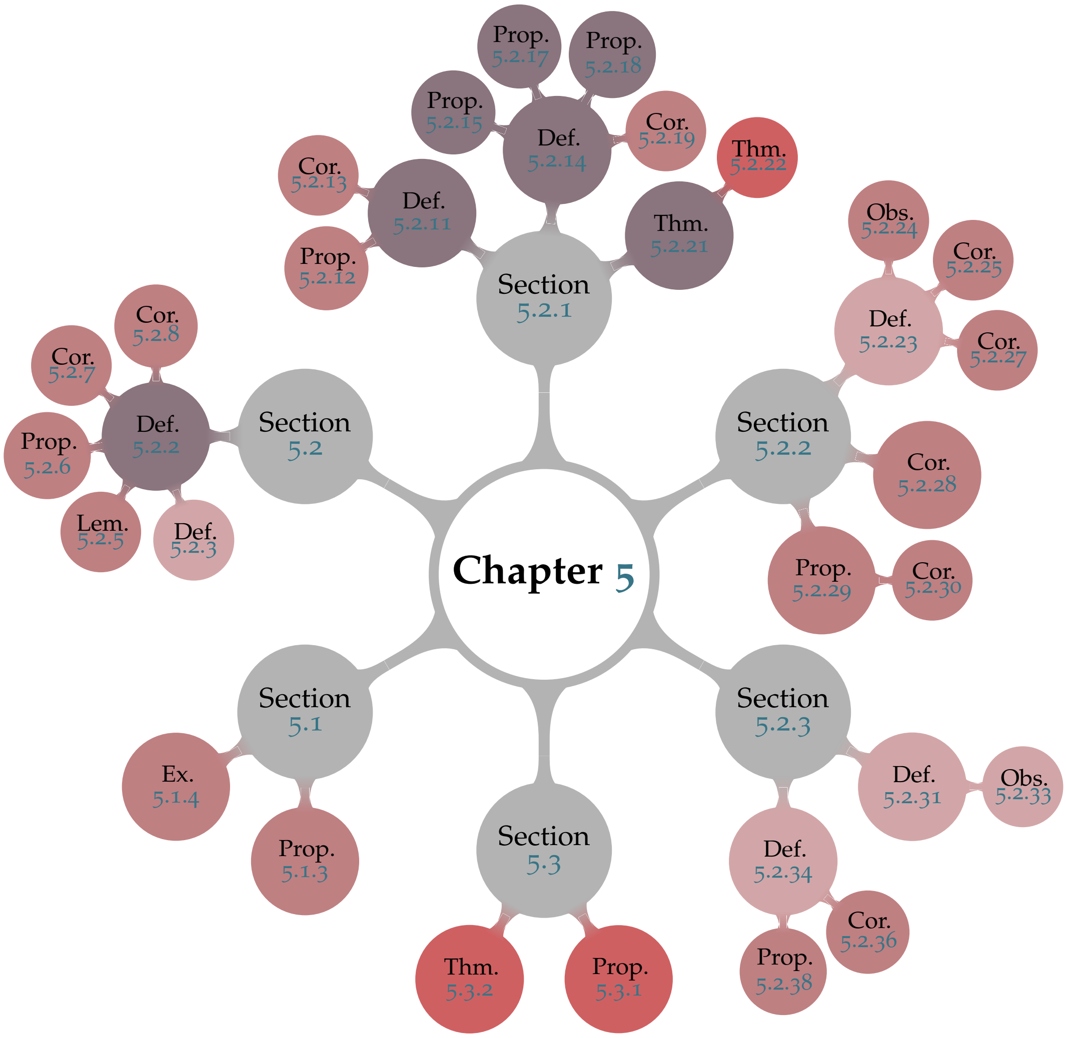


New content in Chapter 4

Since the proof of the Road-Coloring Conjecture [Tra09], it is known that every aperiodic and irreducible road-colorable graph can be colored in such a way that it possesses a synchronizing word. In the first section of Chapter 4, we present some minor results, leading to a necessary and sufficient condition for an arbitrary road-colorable graph to possess a synchronizing coloring.

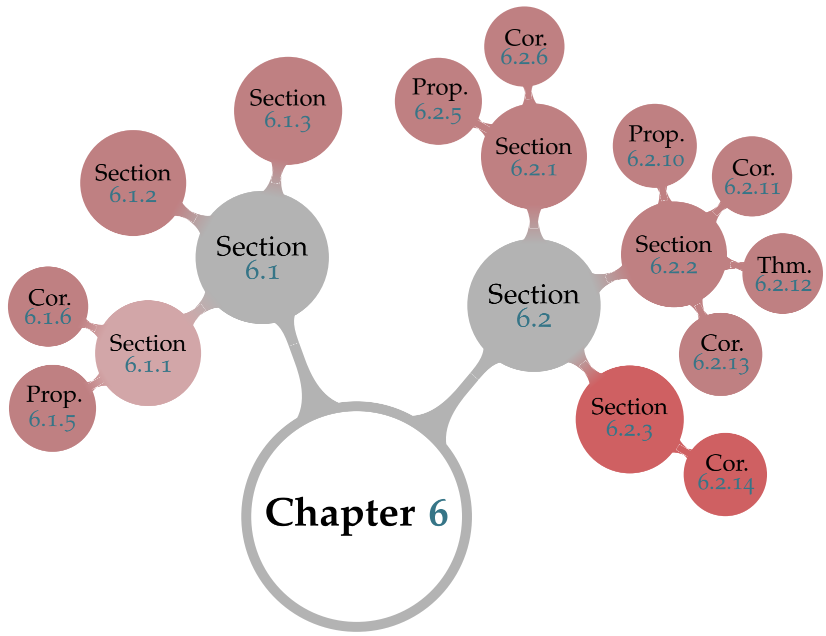
In Section 4.2, the label product of road-colored graphs is introduced, based on the elaborations in [Lan03] and [GKL06], where it has been shown that it can be used as a means to discern whether a given coloring is synchronizable. We show that this result is a direct consequence of our findings at the beginning of the chapter and reflect on the form of the label product in Section 4.2.2.

Section 4.3 presents the dual extended transition operator. With its introduction in [Goh04], it was shown that the regularity of the dual extended transition operator coincides with the asymptotic completeness of the transition (compare also [GKL06]). In particular, it can be understood as a possibility to algebraize the idea of the label product. The chapter closes with the introduction of a new perspective, transferring the concept of the dual extended transition operator into a Hilbert space setting, providing a new proof of the just-mentioned result.



New content in Chapter 5

In Section 5.1, we present the notion of synchronizability of road-colored graphs with infinitely many vertices, which was introduced in [Haa06]. It succeeds in transferring the equivalence between the existence of a synchronizing word and the asymptotic completeness of the corresponding transition to a countably infinite setting. Inspired by an example communicated in [GHK19], Section 5.2 develops the therein presented ideas further and introduces a class of infinite road-colored graphs, so-called integer random walks, for which a sufficient condition can be stated such that asymptotic completeness is equivalent to the almost sure convergence of the corresponding transition functions. In Section 5.3, it is shown that this condition is satisfied whenever the induced Markov chain is assumed to be positive recurrent.



New content in Chapter 6

Chapter 6 introduces the notion of almost uniform convergence of a sequence of measurable functions with values in the real or complex numbers. In the classical setting, Egorov's Theorem guarantees that the notions of almost sure and almost uniform convergence of a sequence of functions coincide [Ego11], [Sev10], [Rie22], [Rie28]. We show that this remains true if transition functions of road-colored graphs are considered. This is then illustrated for finite road-colored graphs and for integer random walks in Sections 6.1.2 and 6.1.3. This motivates the idea to characterize the asymptotic completeness of the transition corresponding to a road-colored graph by the almost uniform convergence of its transition functions.

There exist a non-commutative analog to almost uniform convergence and a corresponding counterpart of Egorov's Theorem, which are introduced in Section 6.2. This raises the idea to characterize asymptotic completeness of the transition corresponding to a one-sided quantum Markov process in terms of pointwise almost uniform convergence of a special sequence of operators towards the so-called Møller operator, arising from scattering theory. The approach is supported by the corresponding considerations of two marginal cases in Sections 6.2.1 and 6.2.2.

-

1 ROAD-COLORED GRAPHS AND MARKOV CHAINS

That's what storytellers do.
We restore order
with imagination.
— Walt Disney

The goal of this chapter is not the stringing together of definitions but rather to take the readers by the hand and lead them along a journey through the different mathematical worlds that one may encounter when examining the classical concept of synchronizability. We start with a short historical outline of the emergence of the concept of synchronizability in the literature. For a more detailed elaboration, we refer to [Vol08].

Dear Friend,
Some time ago I bought this old house, but found it to be haunted by two ghostly noises – a ribald Singing and a sardonic Laughter. [...] I have found that their behaviour is subject to certain laws, abscure but infallible, and that they can be affected by my playing the organ or burning incense.
In each minute, each noise is either sounding or silent – they show no degrees. What each will do during the ensuing minute depends, in the following exact way, on what has been happening during the preceding minute:
The Singing in the succeeding minute, will go on as it was during the preceding minute (sounding or silent) unless there was organ-playing with no Laughter, in which case it will change to the opposite (sounding to silent or vice versa).
As for the Laughter, if there was incense burning, then it will sound or not according as the Singing was sounding or not (so that the Laughter copies the Singing a minute later). If however there was no incense burning, the Laughter will do the opposite of what the Singing did.
At this minute of writing, the Laughter and Singing are both sounding. Please tell me what manipulation of incense and organ I should make to get the house quiet, and keep it so.

This letter, which was presented as an example by W.R. Ashby in 1956 [Ash95, Section 4/15], marks the very first occurrence of a synchronizing automaton that can be traced down in literature – it will be constructed in Example 1.1.21. Contemporaneously, E.F. Moore provided the framework for further research in this area [Moo56]. He considered deterministic finite automata and introduced the now-called Homing Problem, namely the question of how one can restore control over a finite automata, i.e., determine the terminal state if the initial state is unknown but outputs can be produced. He was the first to suggest reducing automata via input sequences, observing the output sequences, and then drawing conclusions about the automatas' structures and conditions. He called this procedure an experiment and estimated the lengths of the input sequences. The next to further develop Moore's ideas and to improve his bounds on the lengths of the input sequences was S. Ginsburg [Gin58]. F.C. Hennie [Hen64] was probably the first to introduce the notion of synchronization in 1964. Based on Moore's and Ginsburg's results, J. Černý introduced the idea of synchronized automata and suggested an upper bound for the length of a synchronizing word [Čer64]. But for the next eight years, Černý's results remained unknown to the English speaking world, probably due to the fact that he published his paper in Slovak. That is why synchronizing automata were rediscovered in 1967 by T.L. Booth [Boo67]. He explained the Synchronizing Problem on the example of a digital decoder in a digital information transmission system that might occasionally receive an improper sequence of input symbols. The existence of a synchronizing word would ensure the neutralization of these errors. In addition, Booth presented a test to check whether an automaton is synchronizing or not. It was only with the translation of P.H. Starke's book [Sta69] in 1972 that Černý became better known to the English-speaking public, for Starke seems to be the first who used Černý, as well as Moore and most of his followers, as a reference.

On the other hand, in 1977, in the context of symbolic dynamics, R.L. Adler, L.W. Goodwyn, and B. Weiss were the first to state the so-called *Road-Coloring Conjecture*, which was soon transferred to the context of automata theory. It was in 1995 that D. Lind and B. Marcus [LM95] shaped the concept of road-colored directed graphs and introduced the notion of synchronizing road-colorings so that the conjecture was formulated for directed graphs, stating that certain graphs might always allow a labeling of their edges in such a way that synchronizability is attained. It remained an open problem for a long time and was finally solved by A. Trahtman in 2009 [Tra09].

The notion of synchronizablity is mostly used in the context of automata theory. Nonetheless, as this thesis combines the worlds of synchronizing automata, Markov chains and dynamical systems, and since (road-colored) directed graphs can be used to represent each of these concepts, we choose to follow the notion of road-colorable graphs in [LM95]. In the first section, the necessary graph theoretical definitions are presented, as well as the Road-Coloring Theorem. Also, the connection to deterministic finite automata will be drawn. Section 1.2 provides the stochastical background and introduces Markov chains. The connection between Markov chains and road-colorable graphs will be pointed out in Section 1.2.3, showcasing how these two concepts can be linked and identified with one another.

1.1 ROAD-COLORED GRAPHS

We follow the definitions and graph theoretical concepts as presented in [LM95].

Definition 1.1.1. A *graph* (A, E) consists of a finite set A of vertices together with a finite set of directed edges E between them. Each edge $e \in E$ starts in a vertex $a \in A$ and ends in a vertex $b \in A$ (here, a and b can be the same vertex). We say that this edge has *initial vertex* a and *terminal vertex* b. It is denoted by e_{ab} and commonly visualized as $a \xrightarrow{e_{ab}} b$.
If edges have the same initial vertex and also all lead to the same vertex, they are called *multiple edges*, whereas an edge with an identical initial and terminal vertex is a *loop*.
A finite sequence of consecutive edges, i.e., the terminal vertex of an edge corresponds to the initial vertex of the following edge and so on, is called a *path*. A *cycle* is a path with identical initial and terminal vertex.
Analogously to an edge, a path $w \in E^m$ of length $m \in \mathbb{N}$, leading from vertex a to b, is visualized as $a \xrightarrow{w} b$. We say also that a is *connected* to b.

Usually, a distinction is made between directed and undirected graphs. However, since we will exclusively consider directed graphs throughout this thesis, the notion of the term *directed* will be omitted. In the context of automata theory and Markov chains, the elements of A are also called *states*.

Definition 1.1.2. Let (A, E) be a graph. For $a, b \in A$, let M_{ab} denote the number of edges in the graph with initial vertex a and terminal vertex b. Then the *adjacency matrix of* (A, E) is given by $M := (M_{ab})_{a,b \in A}$.

Conversely, let $n \in \mathbb{N}$, then for every $n \times n$-matrix with non-negative integer entries, the corresponding graph consists of n vertices $A \simeq \{1, \ldots, n\}$ with M_{ab} edges leading from vertex a to b. From now on, we will frequently identify a graph with its adjacency matrix and vice versa.

Analogously, for every $m \in \mathbb{N}$, the m-th power of the adjacency matrix displays the paths of length m. It is denoted by $M^m =: (M_{ab}^{(m)})_{a,b \in A}$, where $M_{ab}^{(m)}$ equals the number of paths of length m, starting in a and ending in b.

Definition 1.1.3. An (adjacency) matrix is called *irreducible*, if for every pair $a, b \in A$, there exists $m \in \mathbb{N}$, such that $M_{ab}^{(m)} > 0$. For the corresponding graph, this means that for every pair of vertices $a, b \in A$ there exists a path starting in a and terminating in b. A graph is called *irreducible* if its adjacency matrix is irreducible.

We remark that, in terms of graph theory, irreducible graphs are also called *strongly connected*.

Let $N = (N_{ij})_{i,j=1}^n$ be a non-negative $n \times n$-matrix. The *period of a state* $i \in \{1, \ldots, n\}$ is the greatest common divisor of all $m \in \mathbb{N}$ such that $(N^m)_{ii} > 0$. It can be shown that matrices have a cyclic structure in the sense that they possess a characteristic period, so that the vertices of the corresponding graph can be grouped into classes that move cyclically with *period* $p \in \mathbb{N}$.

Definition 1.1.4. The *period* of a non-negative matrix corresponds to the greatest common divisor of the periods of its states. Matrices with a period equal to one will be called *aperiodic*. An *aperiodic graph* is a graph with aperiodic adjacency matrix.

It is easy to see that in irreducible matrices, all vertices have the same period [LM95, Lem. 4.5.3].

PROPOSITION 1.1.5. Let N be an irreducible non-negative matrix, then all its states have the same period. In particular, the period of the matrix corresponds to the period of any of its states.

Aperiodicity of an irreducible graph means that there is no natural number $p \in \mathbb{N}$, which divides all cycle lengths in the graph. In other words, the greatest common divisor of all cycle lengths in the graph is equal to one. A very important characterization of aperiodic and irreducible matrices is the following [LM95, Thm. 4.5.8]:

PROPOSITION 1.1.6. For a non-negative matrix N, the following assertions are equivalent:

(a) N is aperiodic and irreducible.

(b) There exists $m \in \mathbb{N}$ such that $(N^m)_{ij} > 0$ for all i, j.

(c) There exists $m \in \mathbb{N}$ such that $(N^n)_{ij} > 0$ for all $n \geq m$ and all i, j.

In terms of graph theory this implies that in an aperiodic irreducible graph, there exists a natural number m such that for any pair of vertices a, b in A, we find a path of length m leading from a to b.

1.1.1 *Road-colorings and synchronizability*

For the concept of road-colored graphs, the following notions are crucial.

Definition 1.1.7. Consider a graph (A, E) and a finite set C. A *labeling* $l : E \to C$ assigns to each edge e of the graph a label $l(e) \in C$. A finite sequence of elements in C is called a *word*. The *labeled graph* is denoted as (A, C, l) and we speak of (A, E) as the *original* or *underlying graph*.
A labeling that establishes for each state $a \in A$ a one-to-one correspondence between C and the edges starting at a will be called a *transition function*. It is denoted by $\gamma : A \times C \to A$, where $\gamma(a; c) = b$ if and only if $a \xrightarrow{c} b$.
(C, γ) is then referred to as a *road-coloring* of the graph and C is called the *set of colors*. A graph is called *road-colorable* if it has a road-coloring. The resulting *road-colored graph* will be denoted as (A, C, γ).

Definition 1.1.8. Let (A, E) be a graph. The number of outgoing edges from one vertex $a \in A$ is called the *out-degree of a* and is denoted by d_a. If all vertices in a graph have the same out-degree, we say that the graph has *constant out-degree*.

The following observation is a direct consequence of this definition.

Observation 1.1.9. For a graph (A, E), the following assertions are equivalent:

(a) The graph is road-colorable.

(b) The graph has constant out-degree.

Before we finally come to the concept of synchronizability, we introduce some crucial notations on transition functions. For any word $w = (c_1, \dots, c_m) \in C^m$ of length $m \in \mathbb{N}$ we can inductively define the *m-step transition function*

$$\begin{aligned}
\gamma_m \;:\; A \times C^m \to A, \quad (a, w) \mapsto \gamma_m(a; w) \;&:=\; \gamma_m(a; c_1, \dots, c_m) \\
&:=\; \gamma(\gamma_{m-1}(a; c_1, \dots, c_{m-1}); c_m) \\
&\;\;\vdots \\
&:=\; \gamma(\gamma(\cdots \gamma(\gamma(a; c_1); c_2) \cdots ; c_{m-1}); c_m).
\end{aligned}$$

By a slight abuse of notation, we will also call every γ_m a *transition function*.

For a subset $B \subseteq A$ of vertices we denote by $\gamma(B; c)$ the set of vertices in A that can be reached when starting in B and following the c-colored edges, i.e.

$$\gamma(B; c) = \{a \in A \ : \ a = \gamma(b; c) \text{ for a } b \in B\}.$$

Example 1.1.10. The road-colored graph, illustrated in Figure 1.1, will be our classical toy example throughout the whole thesis:

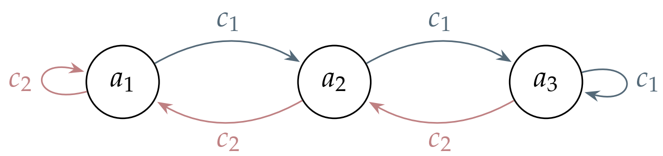


Figure 1.1: Our toy example

With $A = \{a_1, a_2, a_3\}$ and colors $C = \{c_1, c_2\}$. The transition function can be read off; clearly $\gamma(a_1; c_1) = a_2$, $\gamma(a_2; c_1) = a_3$, $\gamma(a_3; c_1) = a_3$, and analogously $\gamma(a_1; c_2) = a_1$, $\gamma(a_2; c_2) = a_1$, $\gamma(a_3; c_2) = a_2$.

Definition 1.1.11. Consider a road-colored graph (A, C, γ). A word $s \in C^m$ of length $m \in \mathbb{N}$ is called *synchronizing word*, if the induced path leads to a single vertex, independent of the chosen initial vertex, i.e., there exists $b \in A$, such that $\gamma_m(a; s) = b$ for every $a \in A$.
A road-colored graph is called *synchronizable* if it possesses a synchronizing word, whereas the corresponding road-coloring (C, γ) is called a *synchronizing coloring* (for the underlying road-colorable graph (A, E)).

LEMMA 1.1.12. Every word that contains a synchronizing word is synchronizing, too.

Proof. Let $s \in C^m$ be a synchronizing word, sending every $a \in A$ into the same vertex $b \in A$. It suffices to show the assertion for cs and $sc \in C^{m+1}$, with $c \in C$. For any $a \in A$ we have:

$$\begin{aligned}
\gamma_{m+1}(a; cs) &= \gamma_m(\underbrace{\gamma(a; c)}_{\in A}; s) = b, \\
\gamma_{m+1}(a; sc) &= \gamma(\underbrace{\gamma_m(a; s)}_{=b}; c) = \gamma(b; c).
\end{aligned}$$

In other words, the functions $\gamma_m(\cdot\,; cs)$ and $\gamma_m(\cdot\,; sc)$ on A are independent of the argument $a \in A$ and hence constant. $\square$

We conclude that, with the occurrence of a synchronizing word, the value of the transition function becomes independent of the initial vertex a.

Observation 1.1.13. Let (A, C, γ) be synchronizable and $s \in C^m$ a synchronizing word and let $l > m$, then for every word $w \in C^l$ which contains s, the map $\gamma_l(\cdot\,; w) : A \to A$ is constant.

This observation will play a key role in this thesis's further considerations.

Example 1.1.14. For the road-colored graph in Example 1.1.10, c_1c_1 and c_2c_2 are the shortest synchronizing words, and every other word that contains them is synchronizing too.

In some situations, for example, when proving assertions on the existence of a synchronizing word, the following concept will prove to be useful.

Definition 1.1.15. A road-colored graph (A, C, γ) is called *pairwise synchronizable*, if for any pair of vertices, there exists a word, mapping them to the same terminal vertex.

PROPOSITION 1.1.16. For a road-colored graph (A, C, γ), the following assertions are equivalent:

(a) (A, C, γ) is synchronizable.

(b) (A, C, γ) is pairwise synchronizable.

Proof. Clearly, if (A, C, γ) is synchronizable, assertion (b) follows directly. For the other direction, consider a subset $B \subsetneq A$ that contains at least two elements and let $s \in C^m$ be a synchronizing word for an arbitrary pair of vertices in B, then $|\gamma_m(B; s)| < |B|$. The assertion follows inductively. Since the graph consists of only finitely many vertices, it is synchronized when a subset containing only a single vertex is attained. □

For a road-colorable graph (A, E), the question of whether it possesses a synchronizing coloring (C, γ) arises naturally. The following assertion is due to [AGW77, Prop. 4].

PROPOSITION 1.1.17. Every irreducible road-colorable graph (A, E) that possesses a synchronizing coloring is aperiodic.

Proof. Let $a \in A$ and $c \in C$. By assumption there exist a natural number n and a word $w \in C^n$ that synchronizes a and $\gamma(a; c)$, i.e., $\gamma_n(a; w) = b = \gamma_n(\gamma(a; c); w)$. Moreover, due to irreducibility, we find $m \in \mathbb{N}$ and a word $v \in C^m$, which maps b into a. But then, vw and vwc are two cycles of coprime length. So, the vertex a has period one and since the graph is irreducible, all its vertices have the same period, i.e., (A, C, γ) is aperiodic. □

Further, a conjecture was raised in [AGW77, Section 4], suggesting that every aperiodic irreducible road-colorable graph might possess such a coloring. For a long time, this conjecture remained an open problem (compare, for example, [LM95]) until 2007, when A.N. Trahtman, who had dedicated himself to the research on synchronizing automata, provided a complete solution. The proof was officially published in 2009 [Tra09] and, remarkably, utilizes merely basic instruments of graph theory. For an extensive elaboration on the proof, compare, for example, [Kno13].

THEOREM 1.1.18 (ROAD-COLORING THEOREM). Every aperiodic irreducible road-colorable graph possesses a synchronizing coloring.

Taking the last two assertions together, we find that aperiodicity is a necessary and sufficient condition for an irreducible road-colorable graph to possess a synchronizing coloring.

COROLLARY 1.1.19. For an irreducible road-colorable graph, the following assertions are equivalent:

(a) The graph possesses a synchronizing coloring.

(b) The graph is aperiodic.

1.1.2 An insight into the framework of synchronizing automata

The goal of this section is to illustrate that the use of the notion of road-colorable graphs instead of deterministic finite automata is justified. The following definition shows that the triple (A, C, γ), describing a road-colored graph, can also be understood as a *deterministic finite automaton*. A nice overview can be found in [Vol08].

Definition 1.1.20. A *deterministic finite automaton* is given by a tuple $(Q, \Sigma, \delta, q_0, F)$, where Q denotes a state set, Σ stands for an input alphabet. The elements in Σ are called *letters* and every $w \in \Sigma^m$, with $m \in \mathbb{N}$ is a *word*. $\delta : Q \times \Sigma \to Q$ is a transition function, defining an action of the letters in Σ on Q. For every $m \in \mathbb{N}$ it extends uniquely to an action $\delta_m : Q \times \Sigma^m \to Q$ of words on the states in Q. Further, q_0 is the *initial or start state* and F denotes the set of *accepting states*, which help determining when a computation is successful.

The automaton is called *synchronizing*, if there exist $m \in \mathbb{N}$ and a word $w \in \Sigma^m$ whose action $\delta_m(\cdot, w)$ resets the automaton, i.e., it leaves the automaton in one particular state, no matter which state in Q it started at. Such a word is called *synchronizing or reset word*.

Example 1.1.21. Consider again the letter printed in the introduction of this chapter and set $A := \{SL, S\neg L, \neg SL, \neg S\neg L\}$, where S (respectively L) denotes that the singing (respectively laughter) is on and $\neg S$ (respectively $\neg L$) that it is turned off. Let further the set of colors $C = \{c_1, c_2, c_3, c_4\}$ be given by the actions

$$\{\text{organ \& incense}, \text{organ \& no incense}, \text{no organ \& incense}, \text{no organ \& no incense}\},$$

The possible transitions between the vertices (or states in the language of automata) are described in the text and lead to the road-colored graph (or deterministic finite automaton), depicted in Figure 1.2. We can thus easily answer the haunted house's

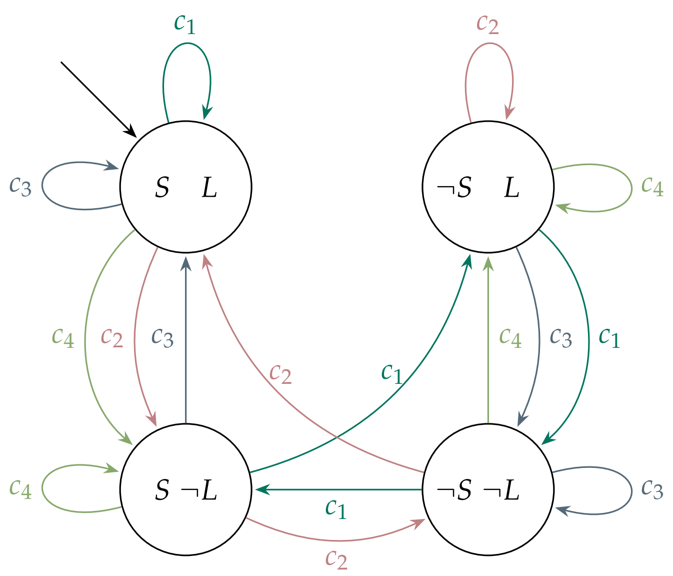


Figure 1.2: The road-colored graph, representing the haunted house

owner's question at the end of the letter:

$$s = \text{no organ \& no incense}\ \text{organ \& no incense}\ \text{no organ \& incense} = c_4 c_2 c_3.$$

In the letter, SL was indicated as the initial state. However, indeed, the synchronizing word s leads to $\neg S\neg L$ and hence silences all singing and laughter, no matter which situation may be encountered in the beginning.

1.1.3 *Infinite road-colored graphs*

A substantial part of this thesis will deal with infinite graphs, i.e., graphs with countable sets of vertices. The transfer of the concept of finite road-colored graphs to the infinite setting is quite natural. We follow the elaborations presented in the preprint [GHK19].

Definition 1.1.22. An *infinite road-colored graph* corresponds to the triple (A, C, γ), where the sets of vertices A and of colors C are countable, and $\gamma : A \times C \to A$ is a transition function.

The notions of irreducibility and aperiodicity for infinite graphs can be translated one-to-one from the finite setting. An infinite road-colored (A, C, γ) graph is *irreducible* if for every pair of vertices $a, b \in A$ there exist $m \in \mathbb{N}$ and a path $w \in C^m$ such that $\gamma_m(a; w) = b$. The period of a vertex $a \in A$ corresponds to the greatest common divisor of the lengths of all cycles containing a. Likewise, an irreducible infinite road-colored graph (A, C, γ) is called *aperiodic* if the greatest common divisor of all cycle lengths in the graph is equal to one.

The concept of synchronizability, on the other hand, cannot be easily applied to graphs with infinite vertex sets. The main difficulty when dealing with synchronizability is that a synchronizing word is a very finite concept. The question as to what could be the correct generalization of a synchronizing word has been addressed in the literature (compare, for example, [Haa06], [DMS11], [Doy+14], [ACS17]). This will be further elaborated on in Chapter 5, where we follow the concept of synchronizability introduced by F. Haag in [Haa06].

In the course of this thesis, when dealing with infinite graphs, we will always mention the term *infinite*.

1.2 MARKOV CHAINS

We will find that road-colored graphs can also be examined from the perspective of stochastics. Indeed, we will see that every (finite or infinite) road-colored graph gives rise to a Markov chain. The other way around, every Markov chain with finite state space can be represented by a road-colorable graph. In Chapter 2, it is demonstrated how the concept of Markov chains can be transferred to the setting of quantum mechanics, forming an essential cornerstone of quantum probability theory.

We start with the general introduction of Markov chains in classical probability theory and address the existence of invariant probability distributions for aperiodic irreducible Markov chains with finite, as well as countably infinite, state space. In Section 1.2.3, a correspondence between Markov chains with finite state space and road-colorable graphs is established.

1.2.1 Fundamentals of Markov chains

The basic knowledge of Markov chains that will be introduced in the following is based mainly on [Kle08] and [Shi96]. For extensive reflections on Markov chains (with a finite state space), we recommend taking a look in [Beh00], whereas [Sen06] provides further considerations on stochastic matrices.

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and A a countable set. By Σ_A we denote the power set of A. A *(classical) random variable with values in A* corresponds to a measurable function $X : (\Omega, \Sigma) \to (A, \Sigma_A)$. A family of such variables $(X_n)_{n\in\mathbb{N}_0} : (\Omega, \Sigma) \to (A, \Sigma_A)$ is called a *(classical) stochastic process with values in A*. In this context, A is also called the *state space* of the stochastic process and an element $a \in A$ is a *state*.

Consider now a random variable X with values in A and a subset $B \subseteq A$. By definition of a random variable, the subset $X^{-1}(B) = \{\omega \in \Omega \ : \ X(\omega) \in B\}$ lies in Σ, equivalently $X^{-1}(\{a\}) \in \Sigma$ for every $a \in A$. We follow the common praxis in stochastics to omit any notion of the underlying probability space and write $\mathbb{P}(X \in B)$ instead of $\mathbb{P}(\{\omega \in \Omega \ : \ X(\omega) \in B\}) = \mathbb{P}(X^{-1}(B))$ and respectively $\mathbb{P}(X = a)$.

Let $E, F \in \Sigma$ be two events, then the probability that the event F happens under the assumption that E has already occurred is given by

$$\mathbb{P}(F|E) := \begin{cases} \frac{\mathbb{P}(E\cap F)}{\mathbb{P}(E)}, & \text{if } \mathbb{P}(E) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

We call $\mathbb{P}(\cdot\,|E)$ the *conditional probability, given E*. Clearly, if $\mathbb{P}(E) > 0$, then $\mathbb{P}(\cdot\,|E)$ denotes a probability measure on (Ω, Σ) [Kle08, Thm. 8.4]. The crucial property of memory loss described in the above introduction is formulated in terms of conditional probabilities.

Definition 1.2.1. A stochastic process $(X_n)_{n\in\mathbb{N}_0}$ with values in A is called a *Markov chain*, if the *Markov property* holds, i.e.:

$$\mathbb{P}(X_{n+1} = a_{n+1} \mid X_0 = a_0, \ldots, X_n = a_n) = \mathbb{P}(X_{n+1} = a_{n+1} \mid X_n = a_n),$$

for every choice of states $a_0, \ldots, a_{n+1} \in A$, provided that the probability measure $\mathbb{P}(\cdot \mid X_0 = a_0, \ldots, X_n = a_n)$ exists.
The Markov chain is *homogeneous*, if for all $a_n, a_{n+1} \in A$, the value

$$\mathbb{P}(X_{n+1} = a_{n+1} \mid X_n = a_n)$$

is independent of the choice of $n \in \mathbb{N}_0$. We call $t_{a_n a_{n+1}} := \mathbb{P}(X_{n+1} = a_{n+1} \mid X_n = a_n)$ the *transition probability (from a_n to a_{n+1})*.

From the definition, it follows that for every homogeneous Markov chain, we have

$$\mathbb{P}(X_0 = a_0, \ldots, X_n = a_n) = \mathbb{P}(X_0 = a_0) \cdot \prod_{m=0}^{n-1} t_{a_m a_{m+1}},$$

for any choice of $a_0, \ldots, a_n \in A$.

The transition probabilities define a *stochastic matrix*, i.e., a matrix with non-negative entries and row-sums equal to one. Indeed, $t_{ab} \geq 0$ for all $a, b \in A$ and a short calculations shows that $\sum_{b\in A} t_{ab} = 1$ for every $a \in A$.

Conversely, due to Kolmogorov's extension theorem (compare, for example, [Kle08, Thm. 14.36]), it can be shown that a homogeneous Markov chain is solely defined by a stochastic matrix and a probability distribution μ on A.

THEOREM 1.2.2. Let $(X_n)_{n\in\mathbb{N}_0}$ be a stochastic process with values in A, $\mu = (\mu_a)_{a\in A}$ a probability distribution on A and $T = (t_{ab})_{a,b\in A}$ a stochastic matrix. The following assertions are equivalent:

(a) $(X_n)_{n\in\mathbb{N}_0}$ is a homogeneous Markov chain with $\mu_a = \mathbb{P}(X_0 = a)$ and transition probabilities t_{ab}.

(b) For every choice of $n \in \mathbb{N}_0$ and $a_0, \ldots, a_n \in A$ holds

$$\mathbb{P}(X_0 = a_0, \ldots, X_n = a_n) = \mu_{a_0} \cdot \prod_{m=0}^{n-1} t_{a_m a_{m+1}}.$$

Definition 1.2.3. The stochastic matrix $T := (t_{ab})_{a,b\in A}$ is called the *transition matrix* of the Markov chain and $\mu = (\mu_a)_{a\in A}$ is its *initial probability distribution* on A.
We denote the corresponding homogeneous Markov chain by $(A; \mu, T)$.

From now on, whenever we refer to a Markov chain $(A; \mu, T)$, it is to be understood in the sense of Theorem 1.2.2. In particular, we will exclusively consider homogeneous Markov chains $(A; \mu, T)$ and refer to them simply as Markov chains.

1.2.2 *Aperiodic irreducible Markov chains*

We have already encountered the terms aperiodicity and irreducibility for graphs. In the setting of finite road-colored graphs, they guarantee the existence of a synchronizing coloring (Theorem 1.1.18). In fact, these terms are also part of the basic vocabulary of Markov chains. For the remainder of this section, let $(A; \mu, T)$ be a Markov chain with countable (finite or infinite) state space A. It is uniquely determined by its transition matrix. The powers of stochastic matrices with a countably infinite index set are well-defined by the natural extension of the rule of matrix multiplication and are again stochastic matrices [Sen06, Section 4.1]. In this setting, the notions of irreducibility (Definition 1.1.3) and aperiodicity (Definition 1.1.4) can be transferred to the transition matrices of Markov chains with countable state space.

Definition 1.2.4. A Markov chain $(A; \mu, T)$ is called *aperiodic*, if the transition matrix is aperiodic.
Likewise, $(A; \mu, T)$ is called *irreducible*, if T is irreducible. For $m \in \mathbb{N}$, $T^m =: (t_{ab}^{(m)})_{a,b\in A}$ is the *m-step transition matrix*.

Clearly, T^m describes the probabilities of the transitions in the Markov chain $(A; \mu, T)$ that are possible in exactly m steps.

Definition 1.2.5. An initial probability distribution μ on A is said to be *stationary* or *invariant*, if $\mu T = \mu$.

The notions *stationary* and *invariant* will be used interchangeably throughout this thesis. A stationary (probability) distribution can be understood as a left-eigenvector with eigenvalue equal to one of the transition matrix. Its existence is due to the *Perron-Frobenius Theorem* (compare, for example, [Sen06]).

When working with Markov chains, their asymptotic behavior is of great interest. For aperiodic irreducible Markov chains, it is already very well understood. For Markov chains with finite state space A, it can be described by the following limit theorem.

THEOREM 1.2.6. An aperiodic irreducible Markov chain with finite state space A has a unique stationary probability distribution π on A and for every $a, b \in A$ holds

$$t_{ab}^{(m)} \xrightarrow[m\to\infty]{} \pi_b.$$

There are different formulations of this theorem, but they all amount to the fact that for increasing m, the rows of the transition matrix tend all to the stationary distribution (no matter which initial distribution was chosen). The proof can be found in most literature on Markov chains (compare, for example, [Sen06, Thm. 4.1 and 4.2], [Kle08, Section 17.6]).

If the Markov chain $(A; \mu, T)$ has a countably infinite state space A, irreducibility and aperiodicity are necessary, but in contrast to the finite case, no longer sufficient conditions for the existence of a unique invariant probability distribution. In this case, it is mandatory to classify the states of a Markov chain in terms of the asymptotic properties of the m-step transition probabilities (compare, for example, [Shi96, Section VIII §3.]). This leads to the notion of *positive recurrence*. We forego a precise definition in terms of conditional probabilities and content ourselves with an intuitive interpretation, as it suffices for our purposes.

A state $a \in A$ is *recurrent* if the underlying Markov chain returns almost surely infinitely often to this state when it has started in it. The state is further called *positive recurrent* if the expected time of the first return into a, when starting in a, is finite. The Markov chain is *positive recurrent*, if every state in A is positive recurrent.

It is noteworthy that positive recurrence is a class property. In particular, when the Markov chain is irreducible and at least one state is positive recurrent, this implies that every state and hence also the Markov chain is positive recurrent. This leads to the following limit theorem [Sen06, Thm. 5.5].

THEOREM 1.2.7. An aperiodic irreducible and positive recurrent Markov chain with countably infinite state space A has a unique stationary probability distribution π on A and for every $a, b \in A$ holds

$$t_{ab}^{(m)} \xrightarrow[m\to\infty]{} \pi_b.$$

In particular, irreducible positive recurrent Markov chains can be characterized by the existence of an invariant probability distribution [Nor98, Thm. 1.7.7].

LEMMA 1.2.8. For an irreducible Markov chain, the following assertions are equivalent:

(a) The Markov chain is positive recurrent.

(b) The Markov chain possesses an invariant probability distribution.

Definition 1.2.9. A Markov chain $(A; \mu, T)$ with invariant initial probability distribution μ is called *stationary*.

1.2.3 *Markov chains and road-colored graphs*

We have seen that Markov chains are discrete stochastic processes with values in a countable state space A. A transition from one state to another occurs with a probability that is independent of the previous history. A Markov chain can thus be considered as a directed graph with vertices A and edges corresponding to the allowed transitions between the states. The edges are then labeled with the corresponding positive transition probabilities. Thus, every Markov chain possesses a canonical representation as a directed graph. The following will show how the road-coloring fits into the picture.

Let (A, C, γ) be a (finite or infinite) road-colored graph. By definition, every vertex in A possesses exactly one outgoing edge of each color. Just like the adjacency matrix for the whole graph (compare Definition 1.1.2), we can now define adjacency matrices corresponding to the individual colors.

Definition 1.2.10. The adjacency matrix with respect to a color $c \in C$ is given by $M_c := ((M_c)_{ab})_{a,b \in A}$, where

$$(M_c)_{ab} := \begin{cases} 1, & \gamma(a; c) = b, \\ 0, & \text{otherwise.} \end{cases}$$

M_c is called the *monochromatic matrix (with respect to $c \in C$)*.

COROLLARY 1.2.11. The adjacency matrix of (A, C, γ) corresponds to $M = \sum_{c \in C} M_c$.

If we equip the sets of vertices A and colors C with probability distributions μ and ν, respectively, we obtain a Markov chain.

COROLLARY 1.2.12. A road-colored graph (A, C, γ) together with probability distributions μ on A and ν on C induces a Markov chain with initial distribution μ and transition matrix $T = \sum_{c \in C} \nu_c \cdot M_c$, where

$$t_{ab} = \sum_{\substack{c \in C : \\ \gamma(a;c)=b}} \nu_c,$$

for $a, b \in A$.

Proof. This is a direct consequence of Theorem 1.2.2 since T is a stochastic matrix. □

Definition 1.2.13. Let (A, C, γ) be a road-colored graph together with probability distributions μ on A and ν on C and monochromatic matrices M_c, for $c \in C$. Then we call $T_c := \nu_c \cdot M_c$ the *monochromatic transition matrix (with respect to c)* of the graph. Likewise, $T = \sum_{c \in C} \nu_c \cdot M_c$ is the *transition matrix of the road-colored graph*.

Example 1.2.14. If we equip our toy Example 1.1.10 with the probability distribution $\nu = \left(\frac{1}{3}, \frac{2}{3}\right)$, we obtain the following graph and transition matrix:

$$T = \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

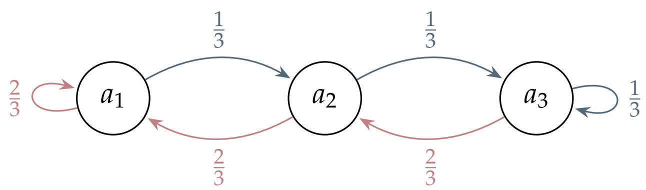


Figure 1.3: Transition probabilities for our toy example

The m-step transition matrix is then given by the m-th power of T, displaying the probabilities to reach a state $b \in A$ in m steps, when having started in $a \in A$:

$$t_{ab}^{(m)} = \sum_{\substack{w \in C^m: \\ \gamma_m(a;w)=b}} \nu^m(w).$$

Consider now the probability distribution μ on the set of vertices A, then – thinking of it as a row vector – for a subset $B \subseteq A$ the vector $\mu T^m(B)$ gives the probability to end up in B after m time steps, with respect to the initial distribution μ:

$$\begin{aligned}\mu T^m(B) &= \mu \otimes \nu^m(\{(a,w) \in A \times C^m \ : \ \gamma_m(a;\, w) \in B\}) \\ &= \mu \otimes \nu^m(\gamma_m^{-1}(B)).\end{aligned}$$

In particular, if μ corresponds to a point measure with respect to a state $a \in A$ then $\mu T^m(B)$ denotes the probability to end up in B after m steps when having started in a.

In this context, it is essential to note the following connection between invariance of the probability measure and the transition function being *measure-preserving*, i.e., for every measurable subset $B \subseteq A$ holds $\mu(B) = \mu \otimes \nu(\gamma^{-1}(B))$, where $\gamma^{-1}(B)$ corresponds to the subset of pairs $(a, c) \in A \times C$, such that $\gamma(a;\, c) \in B$.

LEMMA 1.2.15. Let $(A, C, \gamma;\, \mu, \nu)$ be a road-colored graph and T the induced transition matrix. Then the following assertions are equivalent:

(a) μ is invariant with respect to T.

(b) The transition function γ is measure-preserving.

Moreover, if γ is measure-preserving, so is γ_m for every $m \in \mathbb{N}$.

Proof. The application of T turns μ into the probability distribution μT. In other words, for every subset $B \subseteq A$ holds

$$\mu T(B) = \mu \otimes \nu(\{(a,c) \in A \times C \ : \ \gamma(a;\, c) \in B\}) = \mu \otimes \nu(\gamma^{-1}(B)).$$

In particular, $\mu T(B) = \mu(B)$ if μ is an invariant measure and on the other hand $\mu \otimes \nu(\gamma^{-1}(B)) = \mu(B)$, if γ is measure-preserving.
By an m-fold application, one can conclude that γ_m is measure-preserving too. □

So, every road-colored (finite or infinite) graph (A, C, γ), whenever equipped with probability distributions μ on A and ν on C, induces a Markov chain $(A, C, \gamma;\, \mu, T)$. Moreover, if the underlying graph is aperiodic and irreducible, the same holds for the Markov chain. And the transition function γ is measure-preserving if and only if the Markov chain is stationary.

But what about the other way around? Clearly, for a Markov chain to be represented by a road-colored graph, the following must hold:

PROPOSITION 1.2.16. Let $(A; \mu, T)$ be a Markov chain. A road-colored graph (A, C, γ) with probability distributions μ on A and ν on C and monochromatic transition matrices M_c for $c \in C$ represents the Markov chain if and only if $T = \sum_{c \in C} \nu_c M_c$.

Every homogeneous Markov chain can be associated with a graph, where the transition probabilities induce the edges. Nevertheless, the question remains of whether it can be represented by a road-colored (or road-colorable) graph. For Markov chains with finite state space, this is indeed the case. In order to show this, it is necessary to introduce additional edges so that a road-colorable graph can be crafted which still accurately represents the Markov chain (compare, for example, [YY11, Lem. 3.1]). In Section 3.2, we will provide a proof of this assertion by introducing, in a way, the most general road-colorable graph that still represents the Markov chain.

In Chapter 5, we will then turn again to the setting of countable state spaces and introduce a class of Markov chains, which possess a canonical representation as infinite road-colored graphs. This will lead to some fruitful considerations on the topic of sychronizability.

So from now on, we identify road-colored graphs that are equipped with probability distributions on the vertices and colors with their induced Markov chain, i.e., we denote by $(A, C, \gamma; \mu, \nu)$ a road-colored graph with Markov chain defined through μ and ν. Whenever a Markov chain $(A; \mu, T)$ can be represented by a road-colored graph (A, C, γ), we identify the Markov chain and the graph representation and denote them by $(A, C, \gamma; \mu, T)$.

2 ONE-SIDED QUANTUM MARKOV PROCESSES

Story is a yearning
meeting an obstacle.
— Robert Olen Butler

On December 14, 1900, M. Planck postulated the hypothesis that energy occurs in the smallest packets, later called quanta, as a necessity to describe the black-body radiation observed at that time. Meanwhile, this day is also considered the birth of quantum mechanics. The theoretical foundations of quantum mechanics were independently formulated in the 1920s by W. Heisenberg [Hei26] and E. Schrödinger [Sch26]. Only later did they realize that their results amounted to the same thing. In 1932, J. von Neumann laid the mathematical groundwork for quantum probability theory [Neu32]. Since in quantum mechanics, observables do not necessarily commute with each other, classical probability theory turned out to be insufficient to even describe rudimentary quantum mechanical experiments, as can be reviewed in [KM98]. The framework of operator – and especially von Neumann – algebras turned out to provide the necessary tools to describe infinite systems and capture the nature of quantum mechanics. It describes quantum mechanical systems by the algebra of all bounded operators on some Hilbert space. Actually, quantum mechanics also needs the theory of unbounded self-adjoint operators. However, since every measuring device has a finite scale and ultimately only functions with bounded range are considered, the functional calculus for unbounded self-adjoint operators allows them to be bound by an appropriate function. Therefore, it suffices to consider bounded operators, whereas classical probability systems can be represented by commutative von Neumann algebras. As especially quantum statistical mechanics, quantum field theory, and quantum optics began to gain more and more importance in the seventies, the necessity of a common mathematical framework providing a stochastic description of the physical systems arose. The development of quantum probability theory and with it the theory of quantum Markov processes gained more and more momentum (compare, for example, [Acc76], [AFL82], [Küm85], [Küm88], [Mey85], [Par92]).

After a familiarization with the graph or automata theoretical and stochastical basics in the preceding chapter, the concepts of quantum probability and scattering theory that are relevant for the understanding of this thesis are provided. The structure of this section is inspired by [GKL06], [Lan03] and [Sch12]. Having provided the necessary operator algebraic basics, fundamental quantum probabilistic notions will be introduced. Finally, the connection to scattering theory is made, which is also where we will re-encounter the idea of synchronizing automata. For the fundamental concepts in functional analysis and operator algebras, compare, for example, [Sak71], [Tak02], [Bla06], and [Con07].

2.1 VON NEUMANN ALGEBRAS AND OTHER OPERATOR ALGEBRAIC BASICS

In Physics, a *classical system* is given by a probability space $(\Omega, \Sigma, \mathbb{P})$, where Ω often corresponds to the so-called *phase space*, the set of conceivable states of the system. In other situations, it can describe energy states, angular momentum, or other properties describing the system. Whereas the σ-algebra Σ denotes the permissible events in the system, the probability measure $\mathbb{P}$ describes our knowledge about it. A random variable $X : \Omega \to \Omega_0$ with values in Ω_0, where (Ω_0, Σ_0) is another classical system, describes the influence of Ω onto Ω_0. A real-valued random variable $X : \Omega \to \mathbb{R}$ is called an *observable*. It describes a measurement or observation. In other words, if the system is in the state $\omega \in \Omega$, then a measurement of X yields the value $X(\omega) \in \mathbb{R}$.

In order to introduce quantum mechanical systems, a series of rudimentary algebraic definitions is necessary. We start with some basic notation and vocabulary in the abstract setting. Denote by $\mathcal{A}$ an algebra over $\mathbb{C}$, by a, b elements in $\mathcal{A}$ and by α, β elements in $\mathbb{C}$. An *involution* $^* : \mathcal{A} \to \mathcal{A}$ is an involutive ($(a^*)^* = a$), conjugate linear ($(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$) and anti-multiplicative ($(ab)^* = b^* a^*$) map on $\mathcal{A}$. It turns $\mathcal{A}$ into a **-algebra*. If the algebra $\mathcal{A}$ is equipped with a norm $\|\cdot\|_{\mathcal{A}}$, such that $\|ab\|_{\mathcal{A}} \leq \|a\|_{\mathcal{A}} \|b\|_{\mathcal{A}}$ for all $a, b \in \mathcal{A}$, it is called a *Banach algebra*. Every Banach algebra with involution is called a *Banach *-algebra*, if the involution is isometric, i.e., $\|a\|_{\mathcal{A}} = \|a^*\|_{\mathcal{A}}$ for every a. If further $\|a^* a\|_{\mathcal{A}} = \|a\|_{\mathcal{A}}^2$, it is called an *abstract C*-algebra*.

We move now towards algebras of operators on Hilbert spaces and remain in this setting. All Hilbert spaces that will be met in this thesis are considered over the complex field $\mathbb{C}$. They are denoted by $\mathcal{H}$ or $\mathcal{K}$, where the corresponding scalar product $\langle \cdot, \cdot \rangle$ is linear in the first component and conjugate-linear in the second. Elements in a Hilbert space are symbolized by ξ, η. The algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$, bounded linear operators by x, y and the *operator norm* by $\|\cdot\|$. Originally, the operator norm is denoted by $\|\cdot\|_{\mathrm{op}}$, but as all other norms that we use in the course of this thesis are marked accordingly, we omit the index for the operator norm. The adjoint of a linear operator x on a Hilbert space $\mathcal{H}$ is commonly denoted by x^* and uniquely determined by $\langle x^*\xi, \eta\rangle = \langle \xi, x\eta\rangle$ for $\xi, \eta \in \mathcal{H}$. The operator x is called *self-adjoint* if $x = x^*$ and *positive* if it is of the form $x = y^*y$ for an $y \in \mathcal{B}(\mathcal{H})$. We write $x \geq 0$, if x is positive. A self-adjoint operator x is called a *projection* if $x = x^* = x^2$ and by u we denote a *unitary* linear operator on a Hilbert space, i.e., $uu^* = \mathbb{1} = u^*u$, where $\mathbb{1}$ denotes the identity operation, also called *identity*, in $\mathcal{B}(\mathcal{H})$.

If $\mathcal{H}$ corresponds to the finite-dimensional Hilbert space $\mathbb{C}^n$, the bounded linear operators $\mathcal{B}(\mathbb{C}^n)$ can be identified with the algebra of complex-valued $n \times n$-matrices $\mathrm{M}_n(\mathbb{C})$, subsequently denoted by M_n.

A *quantum mechanical system* is characterized by a complex Hilbert space $\mathcal{H}$ with a scalar product $\langle \cdot, \cdot \rangle$ such that the states are given by one-dimensional subspaces, represented by their generating unit vectors. The *observables* correspond to self-adjoint operators on the Hilbert space. If a quantum mechanical system is in the state $\xi \in \mathcal{H}$, then a measurement of the observable $x : \mathcal{H} \to \mathcal{H}$ yields the expected value $\langle x\xi, \xi\rangle \in \mathbb{R}$.

2.1.1 *Von Neumann algebras and linear functionals*

The introduction of von Neumann algebras and hence the concepts of non-commutative probability theory require some fundamental knowledge on topologies. We will present only the most necessary and refer the reader to the aforementioned literature for more detail. The *norm topology* on $\mathcal{B}(\mathcal{H})$ is induced by the operator norm $\|\cdot\|$. The *weak operator topology* (wop) corresponds to the locally convex topology of pointwise weak convergence and is defined by the seminorms $\{p_{\xi\eta} : \xi, \eta \in \mathcal{H}\}$, where $p_{\xi\eta}(x) = |\langle x\xi, \eta\rangle|$ for $x \in \mathcal{B}(\mathcal{H})$. The *strong operator topology* (stop) is the locally convex topology of pointwise norm convergence and is induced by the family of seminorms $\{p_\xi : \xi \in \mathcal{H}\}$, where $p_\xi(x) = \|x\xi\|$ for $x \in \mathcal{B}(\mathcal{H})$.

A *-subalgebra of bounded linear operators $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called an *operator algebra*. We assume that every operator algebra contains the identity operation $\mathbb{1} \in \mathcal{B}(\mathcal{H})$. Every operator algebra that is closed with respect to the operator norm is a *concrete C*-algebra*. In general, it is simply referred to as a *C*-algebra*.

The door opener to the theory of von Neumann algebras and hence quantum probability theory is *von Neumann's Bicommutant Theorem*, which has its origin in [Neu30]. The proof may be reviewed, for example, in [Sak71, Thm. 1.20.3.].

For a *-subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, we define the *commutant of* $\mathcal{A}$ by

$$\mathcal{A}' := \{b \in \mathcal{B}(\mathcal{H}) \ : \ ab = ba \text{ for all } a \in \mathcal{A}\}.$$

$\mathcal{A}'$ contains the identity $\mathbb{1} \in \mathcal{B}(\mathcal{H})$ and is a strongly closed operator algebra of $\mathcal{B}(\mathcal{H})$. The *bicommutant* $(\mathcal{A}')'$ is denoted by $\mathcal{A}''$. Obviously, we have $\mathcal{A} \subseteq \mathcal{A}''$ and $\mathcal{A}' = (\mathcal{A}'')'$.

THEOREM 2.1.1 (Bicommutant Theorem). Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be an operator algebra. The following assertions are equivalent:

(a) $\mathcal{A}$ is closed in the strong operator topology.

(b) $\mathcal{A}$ is closed in the weak operator topology.

(c) $\mathcal{A} = \mathcal{A}''$.

If an operator algebra fulfills one and hence all of these conditions, it is called a *von Neumann algebra*. We note that every von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is the dual of a uniquely defined Banach space, i.e., $\mathcal{A} = (\mathcal{A}_*)^*$, where the Banach space $\mathcal{A}_*$ is called the *predual of* $\mathcal{A}$.

The predual of a von Neumann algebra $\mathcal{A}$ can be isometrically identified with a subset of the dual $\mathcal{A}^*$ and any functional $\varphi \in \mathcal{A}_*$ is called a *normal functional*. This gives rise to a weak-*-topology on $\mathcal{A}$, namely the locally convex topology that is induced by the seminorms $\{p_\varphi \ : \ \varphi \in \mathcal{A}_*\}$ with $p_\varphi(a) = |\varphi(a)|$ for $a \in \mathcal{A}$. As $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, this topology is called the *σ-weak operator topology* (σ-wop).

A linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is *positive*, if $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$ and *faithful* if in addition, $\varphi(a^*a) = 0$ implies $a = 0$. The operator norm induces a *dual norm* via $\|\varphi\| = \sup_{\|a\| \leq 1} |\varphi(a)|$. Positive linear functionals $\varphi \geq 0$ are characterized by the property $\|\varphi\| = \varphi(\mathbb{1})$. A positive linear functional is called a *state*, if it is of norm one, i.e., $\|\varphi\| = \varphi(\mathbb{1}) = 1$. The states on $\mathcal{A}$ form a convex set, denoted by $\mathcal{S}(\mathcal{A})$ and its extremal points are called *pure states*. A linear functional φ on $\mathcal{A}$ is said to be *tracial* if $\varphi(a^*a) = \varphi(aa^*)$ for all $a \in \mathcal{A}$ and a *tracial state*, if φ is also a state.

For a Hilbert space $\mathcal{H}$ and a vector $\xi \in \mathcal{H}$, the equation $\omega_\xi(x) := \langle x\xi, \xi \rangle$ for $x \in \mathcal{B}(\mathcal{H})$ defines a positive linear functional on $\mathcal{B}(\mathcal{H})$ of norm $\|\xi\|^2$. So, if $\|\xi\| = 1$, then ω_ξ is a state. Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then the restriction of ω_ξ to $\mathcal{A}$ defines a positive linear functional on $\mathcal{A}$. The states on $\mathcal{A}$ arising in this way from unit vectors in $\mathcal{H}$ are called *vector states of* $\mathcal{A}$.

Example 2.1.2. The pure states on M_2 correspond to vector states $\omega_\xi(\cdot) = \langle \cdot\, \xi, \xi \rangle$, where the unit vector ξ is, up to a phase, given by $\left(\alpha, \quad \sqrt{1-\alpha^2}e^{i\delta}\right)^t$, with $\alpha \in [0,1]$ and $\delta \in [0, 2\pi]$.

Let φ be a faithful normal state on $\mathcal{A}$. The *φ-norm* is defined as $\|a\|_\varphi := \varphi(a^*a)^{1/2}$ and analogously the *φ-scalar product* as $\langle a, b \rangle_\varphi := \varphi(b^*a)$, where $a, b \in \mathcal{A}$. On bounded subsets of $\mathcal{A}$, the topology induced by $\|\cdot\|_\varphi$ corresponds to the strong operator topology [Tak02, Prop. III.5.3]. This fact will turn out to be useful later.

Let $\mathcal{H}$ be *separable*, i.e., it admits a countable orthonormal basis $(\xi_i)_{i\in I} \subseteq \mathcal{H}$, then the *trace* of a positive operator $x \in \mathcal{B}(\mathcal{H})$ is defined as $\mathrm{tr}(x) := \sum_{i\in I} \langle x\xi_i, \xi_i \rangle$. We note that the trace takes values in $[0, \infty]$ and is independent of the chosen orthonormal basis.

Example 2.1.3. The following examples will be relevant in the course of this thesis.

1. On a commutative von Neumann algebra, every state is tracial.

2. A trace on the matrix algebra M_n is given by $\mathrm{tr}(a) := \sum_{i=1}^n a_{ii}$, for a matrix $a = (a_{ij})_{i,j=1}^n \in \mathrm{M}_n$, and $\tau := \frac{1}{n}\mathrm{tr}$ is a tracial state on M_n.

3. Every finite-dimensional von Neumann algebra $\mathcal{A}$ can be identified with
$$\mathcal{A} \simeq \mathrm{M}_{n_1} \oplus \cdots \oplus \mathrm{M}_{n_k},$$
for $n_1, \ldots, n_k \in \mathbb{N}$. In particular, every finite-dimensional von Neumann algebra has a tracial state.

The *trace class operators* $\mathcal{T}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ consist of those $x \in \mathcal{B}(\mathcal{H})$ with $\mathrm{tr}((x^*x)^{1/2}) < \infty$. As the following proposition shows, the predual of the von Neumann algebra $\mathcal{B}(\mathcal{H})$ is isometrically isomorphic to $\mathcal{T}(\mathcal{H})$ [Sak71, Thm. 1.15.3.].

PROPOSITION 2.1.4. Let $\mathcal{H}$ be separable. For a linear functional $\varphi \in \mathcal{B}(\mathcal{H})^*$ the following assertions are equivalent:

(a) φ is normal, i.e., $\varphi \in \mathcal{B}(\mathcal{H})_*$.

(b) There exists a trace class operator $\Phi \in \mathcal{T}(\mathcal{H})$ with $\Phi \geq 0$ and $\operatorname{tr}(\Phi) = 1$ such that $\varphi(x) = \operatorname{tr}(\Phi x)$ for every $x \in \mathcal{B}(\mathcal{H})$.

(c) There exists a sequence $(\xi_i)_{i\in I}$ of elements in $\mathcal{H}$ with $\sum_{i\in I} \|\xi_i\|^2 < \infty$, such that $\varphi(x) = \sum_{i\in I} \langle x\xi_i, \xi_i\rangle$ for every $x \in \mathcal{B}(\mathcal{H})$.

Throughout this thesis, we will often consider finite-dimensional von Neumann algebras. For the following result, we refer the reader to [Tak02, Ex. III.2.5].

PROPOSITION 2.1.5. For a von Neumann algebra $\mathcal{A}$, the following assertions are equivalent.

(a) Every state on $\mathcal{A}$ is normal.

(b) $\mathcal{A}$ is finite-dimensional.

Example 2.1.6. We present two standard examples that will be referred to repeatedly.

1. In the finite-dimensional matrix algebra M_n every operator is of trace class and every state φ on M_n is normal. There exists hence a trace class operator $\Phi \in \mathrm{M}_n$ with $\Phi \geq 0$ and $\operatorname{tr}(\Phi) = 1$ such that $\varphi(\cdot) = \operatorname{tr}(\Phi\,\cdot)$. Φ is called a *density matrix*. Then $\operatorname{rk}(\Phi)$, the rank of Φ, indicates the required minimal number of pure states ω_k such that $\varphi = \sum_{k=1}^{\operatorname{rk}(\Phi)} \lambda_k\omega_k$, where $\lambda_k \in (0,1]$. In particular, $\operatorname{rk}(\Phi) = 1$ if φ is pure and $\operatorname{rk}(\Phi) = n$ if φ is faithful. With respect to the φ-orthonormal basis, the density matrix is diagonal.

2. For a probability space $(\Omega, \Sigma, \mathbb{P})$, the algebra of the essentially bounded functions $L^\infty(\Omega, \Sigma, \mathbb{P})$ is canonically represented on the Hilbert space $L^2(\Omega, \Sigma, \mathbb{P})$, where an essentially bounded function f acts on a square integrable function g by pointwise multiplication $f(g) : \omega \mapsto f(\omega) \cdot g(\omega)$, for almost every $\omega \in \Omega$. So, $L^\infty(\Omega, \Sigma, \mathbb{P})$, together with pointwise multiplication, is a commutative von Neumann algebra. Its predual is isometrically isomorphic to the integrable functions $L^1(\Omega, \Sigma, \mathbb{P})$, so any $h \in L^1(\Omega, \Sigma, \mathbb{P})$ can be identified with the functional $\varphi_h(f) := \int_\Omega f \cdot h\, d\mathbb{P}$ for all $f \in L^\infty(\Omega, \Sigma, \mathbb{P})$.

From now on, if not mentioned otherwise, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is assumed to be a von Neumann algebra.

2.1.2 *Tensor products*

This section will introduce the von Neumann tensor product, which will be relevant for our non-commutative considerations. Since many physical phenomena involve multiple particles, like entangled states or subsystems interacting together, the mathematical model for quantum mechanics must allow the investigation of systems composed of multiple individual quantum subsystems. Tensor products offer a possibility to combine two vector spaces in order to create a new vector space. Moreover, they allow the modeling of entangled states, such as Bell states, and have thus proven to be the correct choice for describing composed systems.

The *Hilbert space tensor product* of two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is denoted by $\mathcal{H} \otimes \mathcal{K}$. For operators $x \in \mathcal{B}(\mathcal{H})$ and $y \in \mathcal{B}(\mathcal{K})$, the *tensor product operator* $x \otimes y \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ corresponds to the continuous linear extension of $(x \otimes y)(\xi \otimes \eta) := x\xi \otimes y\eta$ for all $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$.

Let now $\mathcal{A}$ and $\mathcal{C}$ be two von Neumann algebras. Their *von Neumann tensor product* $\mathcal{A} \otimes \mathcal{C}$ is defined as the stop-closure of the subalgebra of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ which is generated by $a \otimes c$ with $a \in \mathcal{A}$ and $c \in \mathcal{C}$. The tensor product $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ is isomorphic to $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$. This identification will be used throughout the thesis.

Example 2.1.7. Building on Example 2.1.6, we now consider the following:

1. The tensor product $\mathcal{A} \otimes \mathrm{M}_n$ of an arbitrary von Neumann algebra $\mathcal{A}$ with the algebra M_n can be identified with $\mathrm{M}_n(\mathcal{A})$, via $a \otimes y \simeq (y_{ij}a)_{i,j=1}^n$, where $a \in \mathcal{A}$ and $y = (y_{ij})_{i,j=1}^n \in \mathrm{M}_n$.
   In particular, if $\mathcal{A} = \mathrm{M}_m$, then we have $\mathrm{M}_m \otimes \mathrm{M}_n \simeq \mathrm{M}_n(\mathrm{M}_m) \simeq \mathrm{M}_{n \cdot m}$.

2. Consider two probability spaces $(\Omega_1, \Sigma_1, \mathbb{P}_1)$ and $(\Omega_2, \Sigma_2, \mathbb{P}_2)$. Then the Hilbert space tensor product $L^2(\Omega_1, \Sigma_1, \mathbb{P}_1) \otimes L^2(\Omega_2, \Sigma_2, \mathbb{P}_2)$ is isometrically isomorphic to $L^2(\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$, where $\mathbb{P}_1 \otimes \mathbb{P}_2$ is the product measure of $\mathbb{P}_1$ and $\mathbb{P}_2$. Then, the von Neumann tensor product $L^\infty(\Omega_1, \Sigma_1, \mathbb{P}_1) \otimes L^\infty(\Omega_2, \Sigma_2, \mathbb{P}_2)$ is isomorphic to the von Neumann algebra $L^\infty(\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$.

The *n-fold tensor product* $\mathcal{A} \otimes \bigotimes_{k=1}^n \mathcal{C}$ can be defined inductively. Naturally, the question arises whether it is also possible to speak about infinite tensor products of von Neumann algebras. The answer is yes, but the construction needs some preparation.

2.1.3 The GNS-construction

The Gelfand-Naimark-Segal representation serves as a crucial method for bridging quantum probability theory and the extensive theory of Hilbert spaces, which will be of great import at several points in this thesis. Among others, it allows to construct infinite von Neumann tensor products, as we will see in the next section.

We state the general assertion for C*-algebras. So, for the present, let $\mathcal{A}$ be a C*-algebra with identity $\mathbb{1}$ and φ a state on $\mathcal{A}$. A *representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$* is given by a *-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. A vector $\xi \in \mathcal{H}$ is called *cyclyc with respect to the representation* π, if $\pi(\mathcal{A})\xi$ is dense in $\mathcal{H}$. For finite-dimensional $\mathcal{A}$ follows equality $\pi(\mathcal{A})\xi = \mathcal{H}$. The following theorem guarantees the existence of a representation for a C*-algebra with respect to a state φ on it. In particular, it holds also for a von Neumann algebra with normal state, the case that will mainly be considered throughout this thesis.

THEOREM 2.1.8 (GNS – Gelfand Naimark Segal construction). Let $\mathcal{A}$ be a C*-algebra with identity and φ a state on it. There exist a Hilbert space $\mathcal{H}_\varphi$ and a representation $\pi_\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\varphi)$ with cyclic vector ξ_φ such that for all $a \in \mathcal{A}$ one has

$$\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle_{\mathcal{H}_\varphi}.$$

The triple $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ is called *GNS-triple* or *GNS-representation* and is unique up to *unitary equivalence*, i.e., for every other representation $(\pi_1, \mathcal{H}_1, \xi_1)$ there exists a unitary $u : \mathcal{H}_1 \to \mathcal{H}_\varphi$, such that $\pi_1(a) = u^*\pi_\varphi(a)u$ for all $a \in \mathcal{A}$. In particular, $u\xi_1 = \xi_\varphi$.

The GNS-construction is a standard tool in operator algebras. We refrain from presenting the proof and refer the reader to the standard literature (compare, for example, [Con07, Thm. VIII. §5.14]).

Example 2.1.9. Let $\mathcal{A} = \mathrm{M}_n$ and τ the normalized trace on M_n. By φ we denote a faithful state on M_n. A *system of matrix units in M_n* corresponds to a set $\{e_{ij} \ : \ 1 \leq i, j \leq n\}$ such that for all $1 \leq i, j, k, l \leq n$:

1. $e_{ij}^* = e_{ji}$,
2. $e_{ij} \cdot e_{kl} = \delta_{jk} \cdot e_{il}$,
3. $\sum_{i=1}^n e_{ii} = 1$,

where δ_{jk} is the *Kronecker delta*, which is equal to one if and only if $j = k$ and otherwise it is zero. With respect to the τ-scalar product, a system of matrix units forms an orthonormal basis of $\mathcal{H}_\tau$. This, in turn, induces an orthonormal basis on the GNS-Hilbert space $\mathcal{H}_\varphi$. Indeed, from the preceding example follows that with respect to the φ-scalar product

$$f_{ij} := \frac{1}{\varphi(e_{ij})^{1/2}} \cdot e_{ij}, \quad \text{for } 1 \leq i, j \leq n,$$

defines an orthonormal basis of $\mathcal{H}_\varphi$. It is also called a *φ-orthonormal basis*.
Let Φ be the density matrix corresponding to φ. The GNS-Hilbert space with respect to φ can be identified with $\mathbb{C}^{n \cdot \mathrm{rk}(\Phi)}$ (compare Example 2.1.6).

2.1.4 *Infinite von Neumann tensor products*

This section will roughly outline the construction of an infinite tensor product of matrix algebras $\mathcal{A} := \mathrm{M}_n$. This case is quite illustrative and gives a good impression of infinite tensor products. We follow the construction presented in *von Neumann Algebra*-lectures in 2014 and 2016 by B. Kümmerer. A slightly different approach can be found, for example, in [Emc72, Section 3.1.e] or [Bla06, Section III.3.1].

Let $n \in \mathbb{N}$. For any $k \in \mathbb{N}$ we can define $\mathrm{M}_{[0,k]} := \underbrace{\mathrm{M}_n \otimes \cdots \otimes \mathrm{M}_n}_{k+1 \text{ times}} \simeq \mathrm{M}_{n^{k+1}}$ and consider the injective embedding of $\mathrm{M}_{[0,k]}$ into $\mathrm{M}_{[0,k+1]}$ via

$$\begin{array}{cccl} i_k & : & \mathrm{M}_{[0,k]} & \to & \mathrm{M}_{[0,k]} \otimes \mathrm{M}_n = \mathrm{M}_{[0,k+1]} \simeq \mathrm{M}_{n^{k+2}} \\ & & x & \mapsto & x \otimes \mathbb{1}_n. \end{array}$$

We set $\mathcal{A}^0_\infty := \bigcup_{k \in \mathbb{N}} \mathrm{M}_{[0,k]}$. Then, for any $x \in \mathcal{A}^0_\infty$, there exists a natural number $k \in \mathbb{N}$ such that $x \in \mathrm{M}_{[0,k]}$. Moreover we have

$$\|x\|_{\mathrm{M}_{[0,k]}} = \|x \otimes \mathbb{1}\|_{\mathrm{M}_{[0,k+1]}} = \|i_k(x)\|_{\mathrm{M}_{[0,k+1]}}.$$

Consequently, $\|x\|_\infty := \|x\|_{\mathrm{M}_{[0,k]}}$ is a well-defined C*-norm on $\mathcal{A}^0_\infty$. The completion of $\mathcal{A}^0_\infty$ with respect to this norm is a C*-algebra. We denote it by $\mathcal{A}_\infty$.

In order to obtain a von Neumann algebra, we consider a state φ on M_n and define a state on $\mathrm{M}_{[0,k]}$ as follows:

$$\varphi_{[0,k]} := \underbrace{\varphi \otimes \cdots \otimes \varphi}_{k+1 \text{ times}}.$$

It is evident that $\varphi_{[0,k+1]} = \varphi_{[0,k]} \otimes \varphi$, resulting in $\varphi_{[0,k+1]} \circ i_k = \varphi_{[0,k]}$. Consequently, we can define $\varphi_\infty(x) := \varphi_{[0,k]}(x)$, for $x \in \mathrm{M}_{[0,k]}$. This establishes a state on $\mathcal{A}^0_\infty$ which can be uniquely extended to a state on the C*-algebra $\mathcal{A}_\infty$. We denote this extension by φ^+. With respect to this state we can consider the GNS-representation $\pi_{\varphi^+}(\mathcal{A}_\infty) \subseteq \mathcal{B}(\mathcal{H}_{\varphi^+})$. From the Bicommutant Theorem 2.1.1 follows that the closure in the strong operator topology of $\pi_{\varphi^+}(\mathcal{A}_\infty)$, or equivalently the bicommutant $\pi_{\varphi^+}(\mathcal{A}_\infty)''$, is a von Neumann algebra in $\mathcal{B}(\mathcal{H}_{\varphi^+})$.

Definition 2.1.10. $(\mathcal{A}^+, \varphi^+) := (\pi_{\varphi_\infty}(\mathcal{A}_\infty)'', \varphi^+) =: \bigotimes_{i\in\mathbb{N}}(\mathrm{M}_n, \varphi)$ is called the *infinite von Neumann tensor product* of (M_n, φ).

Clearly, the constructed von Neumann algebra depends on the chosen state φ. We remark that the above construction is also possible for general von Neumann algebras $\mathcal{A}_i$ with normal state φ_i, for $i \in I$, i.e., the infinite von Neumann tensor product $\bigotimes_{i\in I}(\mathcal{A}_i, \varphi_i)$ exists (compare, for example, [Bla06, Section III.3.1]).

2.1.5 *Completely positive operators*

The inception of the theory of completely positive operators can be attributed to W.F. Stinespring in 1955 [Sti55]. Subsequently, this theory has played a pivotal role in advancing the field of operator algebras.

Let $\mathcal{A}, \mathcal{B}$ be two von Neumann algebras. A linear operator $T : \mathcal{A} \to \mathcal{B}$ is called *positive*, if $T(a) \geq 0$ for all $a \geq 0$. For $n \in \mathbb{N}$, the operator is said to be *n-positive*, if the map

$$\begin{aligned} \mathcal{A} \otimes \mathrm{M}_n &\to \mathcal{B} \otimes \mathrm{M}_n \\ a \otimes y &\mapsto T(a) \otimes y \end{aligned}$$

is positive. And T is *completely positive*, if it is n-positive for every $n \in \mathbb{N}$.

Example 2.1.11. Every positive linear functional on a von Neumann algebra is completely positive [Sti55, Thm. 3].

> PROPOSITION 2.1.12. Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras and $T : \mathcal{A} \to \mathcal{B}$ a positive operator. If $\mathcal{A}$ or $\mathcal{B}$ is commutative, then T is also completely positive.

For the proof, we refer to [Tak02, Cor. IV.3.5, Prop. IV.3.9].

A positive operator $T : \mathcal{A} \to \mathcal{B}$ is called *normal* if for every bounded increasing net of positive elements $(a_i)_{i\in I} \subseteq \mathcal{A}$ follows $T(\sup_{i\in I} a_i) = \sup_{i\in I} T(a_i)$. It can be shown that this is the case if and only if T is continuous with respect to the σ-weak topologies on $\mathcal{A}$ and $\mathcal{B}$ [Bla06, Prop. III.2.2.2].

Large parts of the theory of normal completely positive operators on von Neumann algebras rely on the so-called *Stinespring representation*, a representation theorem that was introduced by W.F. Stinespring [Sti55, Thm. 1], [Bla06, Thm. III.2.2.4]. At this point, we refrain from presenting it. Instead, we introduce the *Kraus representation*, which goes back to K. Kraus [Kra71] and M.D. Choi [Cho75] and can be derived from the Stinespring representation.

THEOREM 2.1.13. Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$ be von Neuamm algebras and $T : \mathcal{A} \to \mathcal{B}$ a linear operator. Then, the following assertions are equivalent:

(a) T is a normal completely positive operator.

(b) There exists a family $(x_i)_{i \in I} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$, such that for every $a \in \mathcal{A}$ one has

$$T(a) = \sum_{i \in I} x_i^* a x_i,$$

where the sum converges in the stop-topology.

For the proof of this assertion, we refer the reader to [Küm86, Thm. 1.1.3]. In the case of assertion (b), we speak of a *Kraus decomposition of* T and call the corresponding operators $(x_i)_{i \in I}$ the *Kraus operators*.

In the following sections, we will transfer the classical theory of Markov chains to the non-commutative setting of quantum mechanics. In particular, completely positive identity-preserving operators will play a significant role. The following example illustrates that they are a natural choice for the non-commutative version of the classical notion of a transition matrix.

Example 2.1.14. A classical transition matrix $T \in \mathrm{M}_n$ can also be understood as a linear map on $\mathbb{C}^n$, the algebra of complex-valued functions on n points. As a stochastic matrix, T consists only of non-negative entries and has row-sums equal to one. Consequently, it maps positive functions to positive functions and, in particular, $T\mathbb{1} = \mathbb{1}$, so it is identity-preserving and by Proposition 2.1.12 also completely positive.

2.2 ESSENTIALS OF QUANTUM PROBABILITY THEORY

In the classical framework, stochastic processes correspond to a family of random variables on a probability space. We will now introduce the quantum mechanical concept of probability spaces and random variables. Subsequently, we provide a definition for quantum Markov processes. Nice surveys can be found in [Küm88], [Küm06] and [Hel02].

2.2.1 *Quantum probability spaces and dynamical systems*

Definition 2.2.1. A *quantum probability space* corresponds to a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a von Neumann algebra and φ is a faithful normal state on it.

Throughout this thesis, we will often restrict ourselves to probability spaces with finite-dimensional von Neumann algebras. This will be mentioned accordingly.

We have already introduced $L^\infty(\Omega, \Sigma, \mathbb{P})$ as an example of a commutative von Neumann algebra (Examples 2.1.6 and 2.1.7). Indeed, it turns out that every commutative quantum probability space can be identified with the essentially bounded functions on a suitable measure space.

PROPOSITION 2.2.2. For a quantum probability space $(\mathcal{A}, \varphi)$ the following assertions are equivalent:

(a) $(\mathcal{A}, \varphi)$ is commutative.

(b) There exists some localizable measure space $(\Omega, \Sigma, \mathbb{P})$, such that $\mathcal{A} \simeq L^\infty(\Omega, \Sigma, \mathbb{P})$.

A proof may be found in [Sak71, Prop. 1.18.1.]. We note that, in the setting of the above proposition, the state φ on $\mathcal{A}$ is defined uniquely by the measure $\mathbb{P}$ via $\varphi := \int_\Omega \cdot \, d\mathbb{P}$. So, a commutative quantum probability space $(\mathcal{A}, \varphi)$ is commonly identified with $(L^\infty(\Omega, \Sigma, \mathbb{P}), \int_\Omega \cdot \, d\mathbb{P})$. For our purposes, it suffices to conclude that a classical probability space $(\Omega, \Sigma, \mathbb{P})$ induces the commutative quantum probability space $(L^\infty(\Omega, \Sigma, \mathbb{P}), \int_\Omega \cdot \, d\mathbb{P})$.

Equipped with the φ-scalar product $\langle \cdot, \cdot \rangle_\varphi$, a quantum probability space becomes a pre-Hilbert space. So the completion with respect to the φ-norm turns $(\mathcal{A}, \varphi)$ into a Hilbert space. In the commutative case, this implies that the $\|\cdot\|_\varphi$-closure of a commutative von Neumann algebra can be identified with the Hilbert space $L^2(\Omega, \Sigma, \mathbb{P})$. This will turn out to be very useful in Chapter 5.

We introduce the concept of a dynamical system, a fundamental mathematical model used to describe how a system changes over time.

Definition 2.2.3. A *morphism* T from one quantum probability space $(\mathcal{A}, \varphi)$ into another quantum probability space $(\mathcal{B}, \psi)$ is a completely positive identity-preserving operator $T : \mathcal{A} \to \mathcal{B}$ that leaves the states invariant, i.e., $\psi \circ T = \varphi$. Abbreviating, we write $T : (\mathcal{A}, \varphi) \to (\mathcal{B}, \psi)$. If $(\mathcal{B}, \psi) = (\mathcal{A}, \varphi)$, we call T a *morphism of* $(\mathcal{A}, \varphi)$. If additionally, T is a *-automorphism, we call it an *automorphism of* $(\mathcal{A}, \varphi)$.
We denote a *dynamical system* by $(\mathcal{A}, \varphi, T)$, where $(\mathcal{A}, \varphi)$ is a quantum probability space and T is a morphism of $(\mathcal{A}, \varphi)$. The dynamical system is called *reversible*, if T is an automorphism of $(\mathcal{A}, \varphi)$.

The following assertion is based on [Tak02, Thm. II.2.6, Prop. III.4.8] and can be reviewed, for example, in [Hel02, Thm. A.1.1].

PROPOSITION 2.2.4. A morphism of a quantum probability space $(\mathcal{A}, \varphi)$ is normal.

Occasionally, in lieu of a quantum probability space $(\mathcal{A}, \varphi)$, we will consider its GNS-representation. In this case, a morphism of $(\mathcal{A}, \varphi)$ can be canonically extended to a contraction on the GNS-Hilbert space.

2.2.2 *Conditional expectations*

In classical probability theory the notion of conditional probabilities is a central concept when defining Markov chains. When leaving the discrete setting, we speak about classical Markov processes. A possibility to formulate the Markov property for them is in terms of conditional expectations with respect to sub-σ-algebras of Σ (compare, for example, [Kle08]). We will not give a complete introduction to classical Markov processes since concrete time-continuous constructions will not be relevant for our considerations. Instead, we are content with a short definition of classical conditional expectations with respect to sub-σ-algebras.

Definition 2.2.5. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, $X \in L^1(\Omega, \Sigma, \mathbb{P})$ a random variable and $\Sigma_0 \subseteq \Sigma$ a sub-σ-algebra of Σ. A *conditional expectation of X given Σ_0* is a Σ_0-measurable random variable, normally denoted by $\mathbb{E}(X|\Sigma_0)$, satisfying

$$\int_{A_0} X \, d\mathbb{P} = \int_{A_0} \mathbb{E}(X|\Sigma_0) \, d\mathbb{P}$$

for all $A_0 \in \Sigma_0$.

Further information on classical conditional expectations and Markov processes can be found in the usual literature (for example, [Shi96], [Kle08]). The relevant fact for quantum probability is that classical conditional expectations possess a non-commutative analog that will prove to be central, too, for the definition of quantum Markov processes. We start with the definition of conditional expectations on C*-algebras (compare [Bla06, Section II.6.10]).

Let $\mathcal{A}$ be a C*-algebra and and $\mathcal{A}_0 \subseteq \mathcal{A}$ a C*-subalgebra. A completely positive contraction $P : \mathcal{A} \to \mathcal{A}_0$ is called a *conditional expectation from* $\mathcal{A}$ *to* $\mathcal{A}_0$, if $P(a) = a$ for $a \in \mathcal{A}_0$ and the *module property* holds, i.e., $P(axb) = aP(x)b$ for all $a, b \in \mathcal{A}_0$ and $x \in \mathcal{A}$. This can be transferred to the quantum mechanical setting.

Definition 2.2.6. Let $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ be two quantum probability spaces and let $P : (\mathcal{A}, \varphi) \to (\mathcal{B}, \psi)$ be a morphism such that there exists a *-homomorphism i from $(\mathcal{B}, \psi)$ into $(\mathcal{A}, \varphi)$, satisfying $P \circ i = \mathrm{id}_{\mathcal{B}}$. Then, $i \circ P$ is a conditional expectation in the above sense and P is called *conditional expectation from* $(\mathcal{A}, \varphi)$ *to* $(\mathcal{B}, \psi)$.
Typically, we will identify $i(\mathcal{B})$ with a von Neumann subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ as well as $\psi = \varphi \circ i$ with φ. Then, $P_0 := i \circ P$ is a morphism of $(\mathcal{A}, \varphi)$, satisfying $P_0(\mathcal{A}) = \mathcal{A}_0$. We refer to P_0 as the *conditional expectation with respect to* φ, or just *conditional expectation* when the association with the state φ is self-evident. We write $P_0 : (\mathcal{A}, \varphi) \to \mathcal{A}_0$.

It can be easily verified that P_0 has the module property. We note that on the GNS-Hilbert space, with respect to the faithful normal state φ, the conditional expectation is an orthogonal projection. This coincides with the classical understanding of conditional expectations.

Unlike in classical theory, the presence of a conditional expectation is not assured in the non-commutative setting, and even if it does exist, its uniqueness is not guaranteed. In the early 1970s, M. Takesaki emerged as a pioneering figure in this field by introducing a condition that ensures the existence of a conditional expectation on quantum probability spaces (compare, for example, [Tak71, Thm. 7.1], [Tak72]). It states that, for a von Neumann algebra with a faithful normal state φ, a conditional expectation exists if and only if $\mathcal{A}_0$ is invariant under the so-called modular automorphism group associated with the given faithful normal state. Moreover, the condition that φ is an invariant state for P_0 quarantees the uniqueness of a conditional expectation – provided it exists. At this point, we will not delve deeper into this matter, as it suffices for our purposes to be aware of these facts. Indeed, the automorphism group is trivial for a von Neumann algebra with tracial normal state.

Example 2.2.7. In classical probability theory, we consider the commutative von Neumann algebra $\mathcal{A} := L^\infty(\Omega, \Sigma, \mathbb{P})$, where the probability measure $\mathbb{P}$ induces a state $\varphi(\cdot) = \int_\Omega \cdot \, d\mathbb{P}$ (compare Proposition 2.2.2). Then a von Neumann subalgebra of $\mathcal{A}$ is of the form $\mathcal{A}_0 = L^\infty(\Omega, \Sigma_0, \mathbb{P}_0)$, where Σ_0 is a σ-subalgebra of Σ and $\mathbb{P}_0 = \mathbb{P}|_{\Sigma_0}$. Let now P_0 be a conditional expectation with respect to φ in the sense of Definition 2.2.6, then P_0 leaves φ invariant and has the module property. So, due to commutativity, for all $f \in \mathcal{A}$ and $g \in \mathcal{A}_0$ holds

$$\begin{aligned} \varphi(f \cdot g) &= \varphi \circ P_0(f \cdot g) \\ &= \varphi(P_0(g \cdot f)) \\ &= \varphi(g \cdot P_0(f)). \end{aligned}$$

Let now A_0 be any set in Σ_0, then $g = \chi_{A_0}$ lies in $\mathcal{A}_0 = L^\infty(\Omega, \Sigma_0, \mathbb{P}_0)$ and we obtain thus

$$\int_{A_0} f \, d\mathbb{P} = \int_{A_0} P_0(f) \, d\mathbb{P},$$

for all $A_0 \in \Sigma_0$ and $f \in \mathcal{A} = L^\infty(\Omega, \Sigma, \mathbb{P})$. This coincides with the definition of classical conditional expectations (compare Definition 2.2.5).

Consider von Neumann algebras $\mathcal{A}$ and $\mathcal{C}$ and a normal state ψ on $\mathcal{C}$. Then

$$\begin{aligned} P_\psi : \mathcal{A} \otimes \mathcal{C} &\to \mathcal{A} \otimes \mathbb{1} \\ a \otimes c &\mapsto \psi(c) \cdot a \otimes \mathbb{1} \end{aligned}$$

extends to a conditional expectation on the von Neumann tensor product of $\mathcal{A}$ and $\mathcal{C}$. Clearly, for any state φ on $\mathcal{A}$, we have $\varphi \otimes \psi \circ P_\psi = \varphi \otimes \psi$.

Definition 2.2.8. Let $\mathcal{A}, \mathcal{C}$ be von Neumann algebras, ψ a normal state on $\mathcal{C}$ and P_ψ defined as above, then P_ψ is called a *conditional expectation of tensor type*.

2.2.3 *Random variables and quantum processes*

We follow the definition of a quantum random variable, motivated by the algebraization of the original concept in classical stochastics and introduced in [Acc76]. We present the quantum mechanical point of view presented in [Küm88]. Let $(\Omega, \Sigma, \mathbb{P})$ denote a classical probability space. A random variable $X : \Omega \to \Omega_0$ is a measurable function with values in a measurable state space (Ω_0, Σ_0). By $\mathbb{P}_0 := \mathbb{P} \circ X$, a probability measure on (Ω_0, Σ_0) is induced. The probability space $(\Omega, \Sigma, \mathbb{P})$ describes the world that influences the system of interest, namely $(\Omega_0, \Sigma_0, \mathbb{P}_0)$. This system is then observed on a scale by *observables*, i.e., measurable functions $f : \Omega_0 \to \mathbb{R}$.

$$\underset{\text{The world}}{(\Omega, \Sigma, \mathbb{P})} \quad \underset{\text{influences}}{\xrightarrow{X}} \quad \underset{\text{a system,}}{(\Omega_0, \Sigma_0, \mathbb{P}_0)} \quad \underset{\substack{\text{which is}\\ \text{observed}}}{\xrightarrow{f}} \quad \underset{\substack{\text{on a}\\ \text{scale.}}}{\mathbb{R}}$$

Reformulating this algebraically, the random variable X gives rise to an injective *-homomorphism on von Neumann algebras

$$\begin{aligned} i_X : L^\infty(\Omega_0, \Sigma_0, \mathbb{P}_0) &\to L^\infty(\Omega, \Sigma, \mathbb{P}) \\ f \quad &\mapsto \quad f \circ X. \end{aligned}$$

The probability measures induce the normal states φ on $L^\infty(\Omega, \Sigma, \mathbb{P})$ and $\varphi_0 := \varphi \circ i_X$ on $L^\infty(\Omega_0, \Sigma_0, \mathbb{P}_0)$ via $\varphi(f) = \int_\Omega f \, d\mathbb{P}$ for $f \in L^\infty(\Omega, \Sigma, \mathbb{P})$. This leads to the following generalization.

Definition 2.2.9. Let $(\mathcal{A}, \varphi)$ be a quantum probability space and $\mathcal{A}_0$ a von Neumann algebra. A *random variable on* $(\mathcal{A}, \varphi)$ *with values in* $\mathcal{A}_0$ is an injective identity-preserving *-homomorphism $i : \mathcal{A}_0 \to (\mathcal{A}, \varphi)$. The faithful state φ induces a faithful state on $\mathcal{A}_0$ by $\varphi_0 := \varphi \circ i$, the *distribution of* i.
A *quantum process* is a family $(i_t)_{t \in \mathbb{T}}$ of random variables i_t on a quantum probability space $(\mathcal{A}, \varphi)$ with values in the von Neumann algebra $\mathcal{A}_0$, where $\mathbb{T} \in \{\mathbb{Z}, \mathbb{N}_0\}$.

If for a random variable $i : (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$, there exists a left-inverse P from $(\mathcal{A}, \varphi)$ to $(\mathcal{A}_0, \varphi_0)$, i.e., $P \circ i = \mathrm{id}_{\mathcal{A}_0}$, then $i \circ P$ defines a conditional expectation from $(\mathcal{A}, \varphi)$ onto the subalgebra $i(\mathcal{A}_0) \subseteq \mathcal{A}$. It follows that $i(\mathcal{A}_0)$ is a von Neumann algebra and i is called *random variable with conditional expectation*.

2.3 A CLASS OF ONE-SIDED QUANTUM MARKOV PROCESSES

It is now possible to define (stationary) quantum Markov processes. The corresponding Markov property is then formulated in terms of conditional expectations. The non-commutative analog of a classical transition matrix corresponds to a completely positive operator, the so-called transition operator. It turned out to be one of the biggest challenges in constructing quantum Markov processes to assure compatibility with the non-commutative analog of stationarity. Also, in contrast to classical Markov theory, where a Markov chain is uniquely determined by its transition matrix, no canonical procedure allows the construction of a quantum Markov process from a transition operator. We refrain from introducing the whole theoretical background. For detailed elaborations on the definition of quantum Markov processes and the challenges faced, we recommend looking at [Küm06]. Instead, we present a class of quantum Markov processes pioneered by B. Kümmerer in 1985 [Küm85], which we will deal with throughout this thesis.

After the introduction of dilations (compare, for example, [Sz-+70]), unitary dilations on Hilbert spaces have been studied extensively. It turned out to be quite fruitful to understand Markov processes in terms of dilations. In the context of dilation theory for completely positive operators on von Neumann algebras, B. Kümmerer presented a class of quantum Markov processes, so-called *Markov dilations* [Küm85], subsequently blossoming into a fertile theory with numerous applications in physics (compare, for example, [Wel+00], [GKL06]). Some years later, in 2000, together with H. Maassen, he showed that these processes can be viewed as a coupling between a quantum probability space (A, φ) and a non-commutative Bernoulli shift, a so-called *coupling representation of a Markov process* [KM00]. In their paper, they extended some of the ideas of the scattering theory for unitary dilations, that P. Lax and R. Philips established in 1967 [LP67], to the framework of operator algebras and developed thus a scattering theory for quantum Markov dilations.

In the following, let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ be quantum probability spaces. We follow closely the construction in [Lan03] and [GKL06].

Definition 2.3.1. A *transition* is a random variable J on $(\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$ with values in $\mathcal{A}$, such that $\varphi = (\varphi \otimes \psi) \circ J$. We denote this by $J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$.

The faithful state ψ on $\mathcal{C}$ gives rise to the conditional expectation of tensor type $P_\psi : \mathcal{A} \otimes \mathcal{C} \to \mathcal{A}$. The algebraization of the idea of a classical transition matrix (compare Example 2.1.14) leads now to a special class of completely positive operators.

Definition 2.3.2. The *transition operator (associated to the transition J)* is a completely positive identity-preserving operator $T_\psi : \mathcal{A} \to \mathcal{A}$, defined as $T_\psi := P_\psi \circ J$.

Since for all $a \in \mathcal{A}$ holds

$$\varphi \circ T_\psi(a) = \varphi \circ P_\psi(J(a)) = \varphi \otimes \psi(J(a)) = \varphi(a),$$

it follows that φ is invariant under T_ψ.

We define the following infinite von Neumann tensor products:

$$\begin{aligned}(\mathcal{C}^+,\psi^+) &:= \bigotimes_{\mathbb{N}_0}(\mathcal{C},\psi)\\ (\mathcal{A}^+,\varphi^+) &:= (\mathcal{A},\varphi)\otimes(\mathcal{C}^+,\psi^+)\simeq(\mathcal{A}\otimes\mathcal{C}^+,\varphi\otimes\psi^+).\end{aligned}$$

The tensor right shift S^+ on $(\mathcal{C}^+,\psi^+)$, i.e., $S^+(c_1\otimes c_2\otimes\cdots)=\mathbb{1}_\mathcal{C}\otimes c_1\otimes c_2\otimes\cdots$, is obviously a homomorphism of $(\mathcal{C}^+,\psi^+)$ and can be trivially extended to a tensor right shift on $\mathcal{A}^+$. The transition J leads to an injective *-homomorphism $J^+:\mathcal{A}^+\to\mathcal{A}^+$, where

$$J^+(a\otimes c^+)=\underbrace{J(a)}_{\in\mathcal{A}\otimes\mathcal{C}}\otimes\underbrace{c^+}_{\in S^+(\mathcal{C}^+)}\in(\mathcal{A}\otimes\mathcal{C})\otimes\bigotimes_{k=1}^{\infty}\mathcal{C},$$

for $a\otimes c^+\in\mathcal{A}\otimes\mathcal{C}^+$. It can easily be verified that $\varphi^+\circ J^+=\varphi^+$. The faithful state ψ^+ on $\mathcal{C}^+$ gives rise to a conditional expectation of tensor type $P_{\psi^+}:\mathcal{A}^+\to\mathcal{A}$, which is a left-inverse of the random variable $i^+:\mathcal{A}\to(\mathcal{A}^+,\varphi^+)$, $a\to a\otimes\mathbb{1}_{\mathcal{C}^+}$. It can easily be checked that $P_{\psi^+}\circ(J^+)^n\circ i^+=T^n_\psi$ and that the transition operators $(T^n_\psi)_{n\in\mathbb{N}_0}$ form a semigroup on $\mathcal{A}$. This is the non-commutative analog of the classical (discrete) Chapman-Kolmogorov equation (compare, for example, [Kle08, Section 14.4]).

The above construction is indeed a *one-sided quantum Markov process*, it is also called a *one-sided coupling representation of a Markov process*. In particular, such a Markov process is always *stationary* in a quantum mechanical sense. We content ourselves with pointing out these facts and refrain from elaborating on them further. For the necessary theoretical background and verification, we refer to [Küm06]. We denote such a stationary one-sided quantum Markov process by

$$(\mathcal{A}^+,\varphi^+,J^+;\mathcal{A},i^+).$$

In the following, when referring to a one-sided quantum Markov process, we will always mean one of the above form.

This construction inspires the interpretation of J as some *quantum mechanical automaton*. Its observables are characterized by the algebra $\mathcal{A}$. In each step, it takes another quantum mechanical system with observables $\mathcal{C}$ as an input and interacts with this input. Thus, J mathematically characterizes the interaction of the automaton and the input system [Sch12, Section 4.4]. The situation is illustrated in Figure 2.1.

$$\begin{tikzcd}
(\mathcal{A},\varphi) \arrow[dr, "J"] & \otimes & (\mathcal{C},\psi) \arrow[drr, "S^+"] & \otimes & (\mathcal{C},\psi) \arrow[drr, "S^+"] & \otimes & (\mathcal{C},\psi) \arrow[drr, "S^+"] & \otimes & \cdots \\
(\mathcal{A},\varphi) & \overset{\frown}{\otimes} & (\mathcal{C},\psi) & \otimes & (\mathcal{C},\psi) & \otimes & (\mathcal{C},\psi) & \otimes & \cdots
\end{tikzcd}$$

Figure 2.1: Construction of one-sided quantum Markov processes

In the beginning of this section, we remarked that this class of quantum Markov processes originates from dilation theory. Indeed, in the dilation setting, the diagram, depicted in Figure 2.2, commutes for every $n \geq 0$.

$$\begin{tikzcd}
(\mathcal{A},\varphi) \arrow[r, "T_\psi^n"] \arrow[d, "i^+"'] & (\mathcal{A},\varphi) \\
(\mathcal{A}^+,\varphi^+) \arrow[r, "(J^+)^n"'] & (\mathcal{A}^+,\varphi^+) \arrow[u, "P_{\psi^+}"']
\end{tikzcd}$$

Figure 2.2: One-sided quantum Markov process from the perspective of dilations

Inspired by Definition 2.3.3, we can now transfer the notion of irreducibility of a classical transition matrix to the non-commutative setting.

Definition 2.3.3. A completely positive identity-preserving operator $T : \mathcal{A} \to \mathcal{A}$ is called *irreducible*, if for a projection $p \in \mathcal{A}$, $T(p) \leq p$ implies either $p = 0$ or $p = \mathbb{1}$.
A transition $J : \mathcal{A} \to \mathcal{A}^+$ is called *irreducible*, if for a projection $p \in \mathcal{A}$, $J(p) \leq p \otimes \mathbb{1}_{\mathcal{C}^+}$ implies $p = 0$ or $p = \mathbb{1}_{\mathcal{A}}$.

2.3.1 *Asymptotic completeness*

One of the main objectives of this thesis is to contribute some understanding of the concept of synchronizing words for road-colored graphs. In the classical finite setting, the existence of a synchronizing word can also be characterized in terms of scattering theory [Lan03], [GKL06], which is a branch of physics that describes the probability of particles undergoing an interaction and being deflected in different directions. Essentially, it aims to predict the outcome of such interactions by examining the relation between an undisturbed evolution and a perturbation of it. In 1967, scattering theory for unitary dilations on Hilbert spaces was established by P. Lax and R. Phillips and has since been an important and fruitful area of research [LP67]. B. Kümmerer and H. Maassen extended some of these ideas to the framework of operator algebras and developed a scattering theory for two-sided quantum Markov dilations [KM00], where instead of a transition J an automorphism of $(\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$ is considered. Subsequently, this theory was transferred to the setting of one-sided quantum Markov processes [Lan03, Section 2.2], [GKL06].

Let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ be quantum probability spaces and $(\mathcal{A}^+, \varphi^+, J^+; \mathcal{A}, i^+)$ a one-sided quantum Markov process. In order to apply the scattering theoretical results in [KM00] it is necessary to trivially extend $(\mathcal{A}^+, \varphi^+, J^+; \mathcal{A}, i^+)$ to a two-sided process. By $(\mathcal{C}^-, \psi^-)$, we denote the left-sided infinite von Neumann tensor product $\bigotimes_{-\mathbb{N}}(\mathcal{C}, \psi)$ and set

$$\begin{aligned} (\hat{\mathcal{C}}, \hat{\psi}) &:= (\mathcal{C}^-, \psi^-) \otimes (\mathcal{C}^+, \psi^+) = \bigotimes_{\mathbb{Z}} (\mathcal{C}, \psi), \\ (\hat{\mathcal{A}}, \hat{\varphi}) &:= (\mathcal{C}^-, \psi^-) \otimes (\mathcal{A}^+, \varphi^+). \end{aligned}$$

Let $\hat{S}$ denote the trivial extension of the tensor right shift on $\hat{\mathcal{C}}$ to an automorphism of $(\hat{\mathcal{A}}, \hat{\varphi})$ and $\hat{J}$ the trivial extension of J^+, respectively J, onto $\hat{\mathcal{A}}$, where

$$\begin{aligned} \hat{J}(\mathbb{1}_{\mathcal{C}^-} \otimes x) &= \mathbb{1}_{\mathcal{C}^-} \otimes J^+(x), \quad \text{for } x \in \mathcal{A}^+, \\ \hat{J}(c^- \otimes \mathbb{1}_{\mathcal{A}^+}) &= c^- \otimes \mathbb{1}_{\mathcal{A}^+}, \quad \text{for } c^- \in \mathcal{C}^-. \end{aligned}$$

We consider further the canonical embeddings:

$$\begin{aligned} i &: \mathcal{A}^+ \to \hat{\mathcal{A}}, \quad x \mapsto \mathbb{1}_{\mathcal{C}^-} \otimes x, \\ \hat{\imath} &: \mathcal{A} \to \hat{\mathcal{A}}, \quad a \mapsto a \otimes \mathbb{1}_{\hat{\mathcal{C}}}. \end{aligned}$$

By P_φ we denote the conditional expectation of tensor type from $(\mathcal{A}^+, \varphi^+)$ to $(\mathcal{C}^+, \psi^+)$, where $\mathcal{C}^+$ is identified with $\mathbb{1} \otimes \mathcal{C}^+ \subset \mathcal{A}^+$.

Figuratively speaking, scattering theory is interested in the question, under what conditions every element of $\mathcal{A}^+$ eventually ends up in $\mathcal{C}^+$. This property is called asymptotic completeness and turned out to be – when considering classical Markov chains with finite state space – equivalent to the existence of a synchronizing word in the corresponding road-colored graph [GKL06], [Lan03]. Initially, it was formulated for two-sided coupling representations [KM00].

Definition 2.3.4. The transition $J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$ is called *asymptotically complete*, if for all $a \in \mathcal{A}$ holds

$$\left\|(J^+)^n i^+(a) - P_\varphi (J^+)^n i^+(a)\right\|_{\varphi^+} \underset{n\to\infty}{\longrightarrow} 0.$$

This leads to the following characterization of asymptotically complete transitions [GKL06, Lem. 1.5], [Lan03, Thm. 2.2.2].

THEOREM 2.3.5. For the one-sided quantum Markov process $(\mathcal{A}^+, \varphi^+, J^+; \mathcal{A}, i^+)$, the following assertions are equivalent:

(a) The transition J is asymptotically complete.

(b) For all $a \in \mathcal{A}$ holds $\left\|P_\varphi (J^+)^n i^+(a)\right\|_{\varphi^+} \underset{n\to\infty}{\longrightarrow} \|a\|_\varphi$

(c) For all $a \in \mathcal{A}$ exists $\|\cdot\|_{\hat{\varphi}} - \lim_{n\to\infty} \hat{S}^{-n}\hat{J}^n\hat{\imath}(a)$ and lies in $\mathcal{C}^- \otimes \mathbb{1}_{\mathcal{A}^+}$.

(d) For all $x \in \mathcal{A}^+$ exists $\|\cdot\|_{\hat{\varphi}} - \lim_{n\to\infty} \hat{S}^{-n}\hat{J}^n i(x)$ and lies in $\hat{\mathcal{C}}$.

In the original result on two-sided quantum Markov processes, the $\|\cdot\|_{\hat{\varphi}}$-limit in the matching formulation of assertion (d) corresponds to the *Møller operator* Φ_- from scattering theory [KM00, Lem. 3.1, Thm. 3.1]. In the one-sided case, we present an analog definition.

Definition 2.3.6. Let $J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C} \otimes \varphi \otimes \psi)$ be an asymptotically complete transition, then we set

$$\begin{aligned} \Phi_J \;:\; \quad \mathcal{A}^+ &\to \quad \hat{\mathcal{C}} \\ x &\mapsto \quad \|\cdot\|_{\hat{\varphi}} - \lim_{n\to\infty} \hat{S}^{-n}\hat{J}^n i(x). \end{aligned}$$

In this one-sided case, we and call the injective *-homomorphism Φ_J *Møller operator* as well. When applying the Møller operator to $a \in \mathcal{A}$, we consider $\Phi_J(a \otimes \mathbb{1}_{\mathcal{C}^+})$.

At the end of this section, we introduce an important special case. To this aim, let T be an automorphism of $(\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$. It can be shown that $J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$, $a \mapsto T(a \otimes \mathbb{1}_\mathcal{C})$ is a transition with conditional expectation [GKL06], [Lan03, Prop. 2.2.5]. The assignment of the one-sided process to the corresponding two-sided process is quite canonical and respects asymptotic completeness in the sense that J is asymptotically complete if and only if the automorphism T is asymptotically complete in the two-sided setting [Lan03, Prop. 2.2.6].

The following Proposition shows that the just described special case is broadly applicable [Küm93, Thm. 2.3.1].

Proposition 2.3.7. For finite-dimensional quantum probability spaces $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$, with $\mathcal{A} = \mathrm{M}_n$, there exists for every transition $J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$ an automorphism T of $(\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$, such that $J(a) = T(a \otimes \mathbb{1})$ for all $a \in \mathcal{A}$.

2.3.2 *Asymptotic completeness and road-colored graphs*

For this section, we find ourselves in the classical setting introduced in Chapter 1. We proceed by transferring the general classical situation to the language of scattering theory, following the construction in [GHK19]. In the case of Markov chains with finite state space, we can then further present the already announced equivalence between the asymptotic completeness of the transition, corresponding to the road-colored graph, and the existence of a synchronizing word [Lan03].

In the following, we consider an aperiodic irreducible (finite or countably infinite) road-colored graph $(A, C, \gamma; \mu, \nu)$ with invariant probability distribution μ and hence measure-preserving transition function γ (compare Lemma 1.2.15).

(A, μ) and (C, ν) induce quantum probability spaces $(\mathcal{A}, \varphi) = (L^\infty(A, \mu), \varphi)$ and $(\mathcal{C}, \psi) = (L^\infty(C, \nu), \psi)$, where $\varphi(\cdot) = \int_A \cdot \, d\mu$ and $\psi(\cdot) = \int_C \cdot \, d\nu$, respectively. Application of the GNS-construction leads to the Hilbert spaces $\mathcal{H}_\varphi = L^2(A, \mu)$ and $\mathcal{K}_\psi = L^2(C, \nu)$, whereas the measure-preserving transition functions γ_n induce isometries $J_n : L^2(A, \mu) \to L^2(A \times C^n, \mu \otimes \nu^n)$ by $J_n(f) := f \circ \gamma_n$, for $n \in \mathbb{N}$.

Let $b \in A$ and χ_b denote the corresponding characteristic function, then it follows that

$$J(\chi_b) = \chi_b \circ \gamma_n = \chi_{\{(a,w) \in A \times C^n \,:\, \gamma_n(a;w) = b\}} = \chi_{\gamma^{-1}(b)}.$$

Equivalently, $J_n(\chi_b) = \chi_{\gamma_n^{-1}(b)}$ for $n \in \mathbb{N}$.

We will frequently identify the L^2-spaces on product spaces with the tensor product of the L^2-spaces, i.e., for example, $L^2(A \times C^n, \mu \otimes \nu^n)$ with $L^2(A, \mu) \otimes L^2(C^n, \nu^n)$. In particular, we can then understand $\mathbb{1} \otimes L^2(C^n, \nu^n)$ as a subspace of $L^2(A \times C^n, \mu \otimes \nu^n)$. Similarly it will be convenient, to identify a function $g \in L^2(C^n, \nu^n)$ with the function $\mathbb{1} \otimes g \in L^2(A, \mu) \otimes L^2(C^n, \nu^n)$.

We are now in the framework of quantum probability and scattering theory and can translate the concepts that were introduced in the preceding sections to the language of road-colored graphs. In order to construct the commutative one-sided quantum Markov process, we need to consider infinite von Neumann tensor products, which give rise to the set of *infinite color sequences* and the corresponding L^2-spaces:

$$\begin{array}{rcccl} (\mathcal{C}^+,\psi^+) & \rightsquigarrow & (C^+,\nu^+) := (\times_{\mathbb{N}_0} C, \otimes_{\mathbb{N}_0}\nu) & \rightsquigarrow & L^2(C^+,\nu^+), \\ (\mathcal{C}^-,\psi^-) & \rightsquigarrow & (C^-,\nu^-) := (\times_{-\mathbb{N}} C, \otimes_{-\mathbb{N}}\nu) & \rightsquigarrow & L^2(C^-,\nu^-), \\ (\hat{\mathcal{C}},\hat{\psi}) & \rightsquigarrow & (\hat{C},\hat{\nu}) := (C^- \times C^+, \nu^- \otimes \nu^+) & \rightsquigarrow & L^2(\hat{C},\hat{\nu}), \end{array}$$

and, respectively,

$$\begin{array}{rcccl} (\mathcal{A}^+,\varphi^+) & \rightsquigarrow & (A^+,\mu^+) := (A \times \mathcal{C}^+, \mu \otimes \nu^+) & \rightsquigarrow & L^2(A,\mu) \otimes L^2(C^+,\nu^+), \\ (\hat{\mathcal{A}},\hat{\varphi}) & \rightsquigarrow & (\hat{A},\hat{\mu}) := (A \times \hat{C}, \mu \otimes \hat{\nu}) & \rightsquigarrow & L^2(A,\mu) \otimes L^2(\hat{C},\hat{\nu}), \end{array}$$

where ν^- and ν^+ correspond to the probability measures, such that $\psi^-(\cdot) = \int_{C^-} \cdot\, d\nu^-$ and $\psi^+(\cdot) = \int_{C^+} \cdot\, d\nu^+$. We denote the one-sided infinite color sequences in C^+ and C^- by $c^+ := (c_0, c_1, \dots)$ and $c^- := (\dots, c_{-2}, c_{-1})$. For convenience, we sometimes write $(c^-, a, c^+) \in C^- \times A \times C^+$ instead of $(a, \hat{c}) \in \hat{A}$.

Lemma 2.3.8. Let $(A, C, \gamma; \mu, \nu)$ be a road-colored graph with invariant probability distribution μ and let γ_n^- denote the trivial extension of γ_n to a map on $A \times C^-$:

$$\begin{array}{rrcl} \gamma_n^- : & A \times C^- & \to & A \\ & (a, c^-) & \mapsto & \gamma_n(a; c_{-n} \dots, c_{-1}), \end{array}$$

for $n \in \mathbb{N}$. Then, for the corresponding one-sided quantum Markov process, for all $f \in L^2(A,\mu)$ and all $n \in \mathbb{N}$, we have ν^--almost everywhere

$$\hat{S}^{-n} \hat{J}^n \hat{\imath}(f) = f \circ \gamma_n^-.$$

Proof. The transition functions γ_n can be extended to

$$\begin{array}{rrcl} \hat{\gamma}_n : & \hat{A} & \to & A \\ & (c^-, a, c^+) & \mapsto & \gamma_n(a; c_0 \dots, c_{n-1}). \end{array}$$

Then, for every $f \in L^2(A,\mu)$ follows $\hat{\mu}$-almost everywhere $f \circ \hat{\gamma}_n = \hat{J}^n \hat{\imath}(f)$. Moreover, we can consider the trivial extension of the right shift on $\hat{C}$ to $\hat{A}$:

$$\begin{array}{rccc} \sigma^{-1} : & \hat{A} & \to & \hat{A} \\ & (\dots, c_{-1}, a, c_0, c_1, \dots) & \mapsto & (\dots, c_{-2}, a, c_{-1}, c_0, \dots). \end{array}$$

For $n \in \mathbb{N}$ this leads to:

$$\sigma^{-n}(\dots, c_{-n}, \dots, c_{-1}, a, c_0, c_1, \dots) = (\dots, c_{-(n+1)}, a, c_{-n}, \dots, c_0, \dots).$$

Hence, for every $f \in L^2(A,\mu)$ the following functions are equal $\hat{\mu}$-almost everywhere, i.e. $S^{-n}\hat{J}_n\hat{\imath}(f) = f\hat{\gamma}_n\sigma^{-n}$. In particular, for $\hat{\mu}$-almost every $(a,\hat{c}) \in \hat{A}$ this means:

$$\begin{aligned} S^{-n}\hat{J}_n\hat{\imath}(f)(a,\hat{c}) &= f\gamma_n^+\sigma^{-n}(a,\hat{c}) \\ &= f\left(\gamma_n^+(\dots, c_{-(n+1)}, a, c_{-n}, \dots, c_0, \dots)\right) \\ &= f\left(\gamma_n^+(a;\, c_{-n}, \dots, c_0, \dots)\right) \\ &= f\left(\gamma_n(a;\, c_{-n}, \dots, c_{-1})\right) \\ &= f\left(\gamma_n^-(a;\, c^-)\right). \end{aligned}$$

So, for every $f \in \mathcal{A}$ we have $S^{-n}\hat{J}_n\hat{\imath}(f) = f\gamma_n^+\sigma^{-n} = f \circ \gamma_n^-$ $\hat{\mu}$-almost everywhere. □

A similar argument, as presented in the above proof, can be found in [Lan03, Section 4.3]. Lemma 2.3.8 illustrates that in terms of scattering theory it is quite convenient to identify a transition function γ_n with its trivial extension to a map from $A \times C^-$ to A. From now on, we will do so often and simply write $\gamma_n(a;\, c^-) = \gamma_n(a;\, c_{-n}, \dots, c_{-1})$.

For every square-integrable function $f \in L^2(A,\mu)$ the transitions induce a family of functions $(f_n)_{n\in\mathbb{N}_0}$ in $L^2(A,\mu) \otimes L^2(C^-,\nu^-)$. For $n \in \mathbb{N}$ and $(a,c^-) \in A \times C^-$ we set

$$\begin{aligned} f_0(a,c^-) &:= \mathrm{id}_{L^2(A)} \circ f(a) &&= f(a), \\ f_n(a,c^-) &:= J_n \circ f(a,c^-) &&= f(\gamma_n(a;\, c^-)). \end{aligned}$$

It follows that $\|f\|_{L^2(A)} = \|J_n f\|_{L^2(A\times C^-)} = \|f_n\|_{L^2(A\times C^-)}$, since the J_n are isometries.

We can now define the conditional expectation $P_n : L^2(A \times C^n, \mu \otimes \nu^n) \to \mathbb{1} \otimes L^2(C^n, \nu^n)$, for $n \in \mathbb{N}$. We note that it is an orthogonal projection (compare Section 2.2.2). Let

$h \in L^2(A \times C^n, \mu \otimes \nu^n)$, then it is given by

$$P_n h(a, w) = \sum_{b \in A} \mu(b) \cdot h(b, w),$$

for all $(a, w) \in A \times C^n$. In particular, $P_n h$ can be thought of as an element of $L^2(C^n, \nu^n)$, as its value no longer depends on a. Analogously to Definition 2.3.4, the transition $J : L^2(A, \mu) \to L^2(A \times C^-, \mu \otimes \nu^-)$ is asymptotically complete if for all $f \in L^2(A, \mu)$:

$$\|P_n J_n f - J_n f\|_{L^2(A \times C^n)} \xrightarrow[n \to \infty]{} 0.$$

The adaption of Theorem 2.3.5 to the present setting leads thus to the following characterization of asymptotic completeness [GKL06, Section 2], [GHK19].

PROPOSITION 2.3.9. For an infinite road-colored graph $(A, C, \gamma; \mu, \nu)$ with invariant probability measure μ and corresponding transition J, the following assertions are equivalent:

(a) The transition $J : L^2(A, \mu) \to L^2(A \times C^-, \mu \otimes \nu^-)$ is asymptotically complete.

(b) For all $f \in L^2(A, \mu)$, the limit $\|\cdot\|_{L^2(A \times C^-)} - \lim_{n \to \infty} f_n$, where the f_n are defined as above, exists in $L^2(A \times C^-, \mu \otimes \nu^-)$ and does not depend on the first variable $a \in A$, i.e., $\|\cdot\|_{L^2(A \times C^-)} - \lim_{n \to \infty} f_n = \mathbb{1} \otimes g$ for a function $g \in L^2(C^-, \nu^-)$.

We take into account that $\|\cdot\|_{\varphi^-}$-convergence in the quantum probability spaces corresponds to stop-convergence when considering the GNS-Hilbert spaces. As these correspond to L^2-spaces, convergence in the strong operator topology is given by $\|\cdot\|_{L^2}$-convergence. So, this proposition follows directly from Theorem 2.3.5.

2.3.3 *Asymptotic completeness and synchronizability*

For this section, we restrict ourselves to the consideration of finite road-colored graphs. Let $(A, C, \gamma; \mu, \nu)$ be a finite road-colored graph with invariant probability measure μ.

Now, we have finally reached the point where the concept of synchronizability comes back into play. In the past, the connection between the existence of a synchronizing word for the road-colored graph and asymptotic completeness of the corresponding transition function $J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$ has been extensively studied [Lan03], [Haa06], [GKL06]. It turned out that not only the synchronizability of the finite road-colored graph is a possible characterization of asymptotic completeness, but also the

ν^--almost sure convergence of the transition functions $(\gamma_n)_{n\in\mathbb{N}}$ towards a function that does no longer depend on the chosen initial vertex. The results go back to [GKL06, Prop. 2.3], [Lan03, Thm. 4.3.5 and 4.3.8]. We summarize this in the following theorem.

THEOREM 2.3.10. For a road-colored graph $(A, C, \gamma; \mu, \nu)$ with finite state space A and invariant probability distribution μ on it, the following assertions are equivalent:

(a) The transition J is asymptotically complete.

(b) The road-colored graph (A, C, γ) possesses a synchronizing word.

(c) The sequence of transition functions $(\gamma_n)_{n\in\mathbb{N}}$ converges ν^--almost surely and the limit is independent of the initial vertex $a \in A$.

In this case, we identify the ν^--almost sure limit $\lim_{n\to\infty} \gamma_n : A \times C^- \to A$ with a map $\gamma_\infty : C^- \to A$ and write $\lim_{n\to\infty} \gamma_n(\cdot\,; c^-) =: \gamma_\infty(c^-)$.

With the ulterior motive of better understanding the concept of synchronizability, this characterization motivates the idea of generalizing the almost sure convergence of the transition functions. We will deal with this idea several times throughout this thesis. Of course, the question immediately arises to what extent Theorem 2.3.10 can be applied to the setting of infinite road-colored graphs. At this point, we can already conclude one direction.

COROLLARY 2.3.11. Let $(A, C, \gamma; \mu, \nu)$ be an infinite road-colored graph with invariant probability measure μ such that the transition functions $(\gamma_n)_{n\in\mathbb{N}} : A \times C^- \to A$ converge ν^--almost surely towards a function $\gamma_\infty : C^- \to A$ that does no longer depend on the first component. Then, the corresponding transition J is asymptotically complete.

Proof. Let $f \in L^2(A, \mu)$. The proof is a direct consequence of Proposition 2.3.9 and the definition of $f_n := f(\gamma_n)$. Then, ν^--almost sure convergence of $(\gamma_n)_{n\in\mathbb{N}}$ induces ν^--almost sure convergence of $(f_n)_{n\in\mathbb{N}}$ towards $f(\gamma_\infty)$, which, in turn, implies the $\|\cdot\|_{L^2(A\times C^-)}$-convergence of the functions $(f_n)_{n\in\mathbb{N}}$ towards $f(\gamma_\infty)$. $\square$

In Chapter 5, we will demonstrate that the converse holds true for a special class of infinite road-colored graphs and, hence, also their corresponding Markov chains. To this aim we will introduce a concept of synchronizability for infinite graphs that goes back to [Haa06]. The objective of Chapter 6 will then be to extend the concept of almost sure convergence of the transition functions into the non-commutative setting.

3 SYNCHRONIZABILITY AND COUPLING FROM THE PAST

There are probably a number of ways to tell your story right...

— Anne Lamott

This chapter starts with the presentation of the Coupling from the Past algorithm that was introduced in 1996 by J.G. Propp and D.B. Wilson [PW96] in order to generate the single invariant probability distribution of an aperiodic irreducible Markov chain. Instead of long-time approximations, as it was standard until then (compare, for example, [Met+53], [Has70], [AGT92], [DS98]), it returns exact draws from the target distribution. In 2011, N. Sissouno provided a detailed elaboration on Propp's and Wilson's proof of the fact that for an aperiodic irreducible Markov chain, the Coupling from the Past algorithm terminates almost surely in finite time, and the outputted random sample is distributed according to the invariant distribution of the Markov chain [PW96, Thm. 1], [Sis11, Thm. 3.10]. He showed further that from the right viewpoint, this procedure could be understood in terms of road-colored graphs representing the aperiodic irreducible Markov chain [Sis11, Section 3.2]. This idea was also mentioned in [YY11]. In the present chapter, we adopt it by reformulating the elaborations in [Sis11] on the Coupling from the Past algorithm fully in terms of road-colored graphs. Through this approach, it becomes possible to express the previously mentioned assertion directly in the language of road-colored graphs and thereby avoid the use of further abstract concepts. In Section 3.1.2, we introduce a modification of the algorithm that can be applied to road-colored graphs and subsequently demonstrate that, when approached from the correct perspective, the modified algorithm essentially accomplishes to identify synchronizing words. By means of a special coloring, these results can then be transferred back to the original algorithm.

Leaving the commutative context, the chapter concludes with some reflections on possible quantum mechanical concepts for the classical notions of colors and synchronizing words by presenting and juxtaposing two concepts that arose in the past years.

3.1 COUPLING FROM THE PAST

Let A be a finite state space and $(A; \mu, T)$ an aperiodic irreducible Markov chain with values in A. By $\pi = (\pi_a)_{a \in A}$ we denote the unique invariant probability distribution of the Markov chain (compare Theorem 1.2.6).

As a starting point, we present the algorithm, with its representation being based on the approach in [Sis11]. In the next step, we introduce a variation of the algorithm, which takes into account Markov chains possessing a representation as a road-colored graph.

The fundamental principle of the Coupling from the Past algorithm lies in stepping back into the distant past and considering multiple copies of the Markov chain simultaneously, such that every state $a \in A$ is an initial state of one copy. These Markov chains are then run forward in time until the resulting paths coalesce into a common state in the present.

Coupling from the Past Algorithm (CFTP)

Step 1. Start in the present (time $t_0 = 0$).

Step 2. For $m \geq 1$:
Go one step backward into the past (time $t_m = -m$). For each $a \in A$, choose one successor randomly, according to the transition probabilities.

Step 3. Check whether the $|A|$ paths, exactly one starting from each of the $|A|$ states at time $t_m = -m$, coalesce in the present t_0 at the latest. If this is the case, then proceed to Step 4.
Otherwise, go back to step 2.

step 4. Put out the unique terminal state of all $|A|$ paths at time t_0 as a random sample.

Throughout this chapter, we make use of the fact that a Markov chain possesses a canonical representation as a directed graph. In this sense, the Coupling from the Past algorithm can be applied to a directed graph with transition probabilities assigned to its edges. The algorithm is applied to our toy example in the following example. We subsequently discuss why the backward consideration is crucial for obtaining an exact random sample.

$$T = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

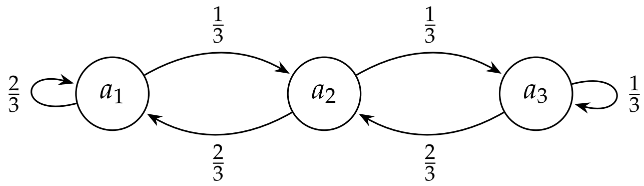


Figure 3.1: The underlying Markov chain of our toy example

Example 3.1.1. Consider the underlying Markov chain $(A; \mu, T)$ of our toy example. Figure 3.1 illustrates its transition matrix and canonical graph representation. Figures 3.2 and 3.3 present two possible runs of the Coupling from the Past algorithm.

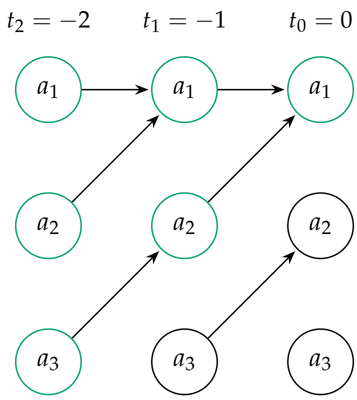


Figure 3.2: Run 1 of the CFTP algorithm

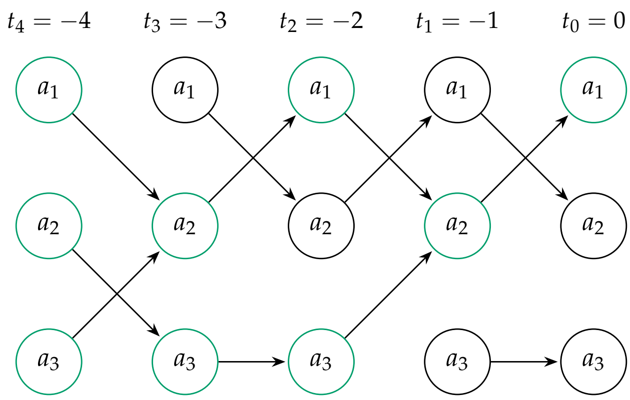


Figure 3.3: Run 2 of the CFTP algorithm

The outputted sample would be biased if the Markov chain were run forward instead of backward. To illustrate this, we consider a slightly different Markov chain, in which some states have only a single predecessor.

Example 3.1.2. Consider again our toy example. We could change the edge $a_1 \xrightarrow{e_{11}} a_1$ into $a_1 \xrightarrow{e_{12}} a_2$. The result would still be an aperiodic irreducible Markov chain, but now state a_2 has only a_1 as a predecessor (compare Figure 3.4). Clearly, in a forward

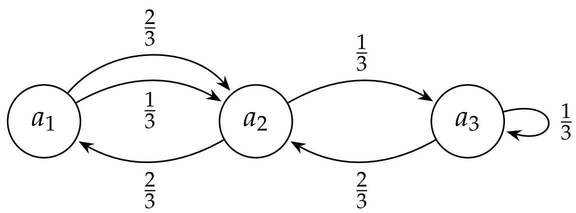


Figure 3.4: A modification of our toy example

algorithm, it would never be possible for any choice of paths to coalesce in a_2, as they would all have to go through a_1 first. That is to say, the coalescence would already have happened in a_1, leading to a termination of the algorithm without ever having a_2 as an output. In contrast, the backward algorithm allows a_2 as an output, as it terminates only if three paths are obtained, having a_1, a_2, and a_3 as initial states.

3.1.1 A modification of the algorithm

We made use of the fact that a Markov chain can be identified with its canonical representation as a directed graph. Therefore, the Coupling from the Past algorithm can also be applied to the uncolored directed graph. We introduce an adaption of the algorithm, which is applicable to road-colored graphs. The termination of the algorithm then corresponds to the occurrence of a synchronizing word. To be concrete, we will show that the application of the algorithm to a road-colored graph produces the desired random sample if and only if the coloring is synchronizing.

We consider a road-colored graph (A, C, γ) with finitely many vertices in A. Equipped with probability distributions μ on A and ν on C, it induces a Markov chain $(A, C, \gamma;\, \mu, \nu)$ with values in A. In the following, we present the modification of the Coupling from the Past algorithm. The difference compared to the original algorithm is depicted in blue. It enables us to approach the subject entirely in terms of road-colored graphs and synchronizability.

COLORED COUPLING FROM THE PAST ALGORITHM (CCFTP)

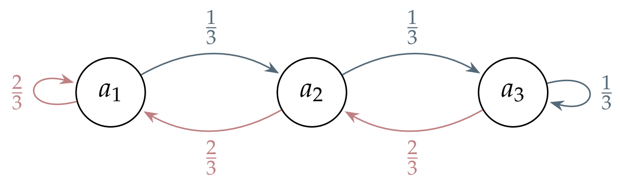
STEP 1. Start in the present (time $t_0 = 0$).

STEP 2. For $m \geq 1$:
Go one step backward into the past (time $t_m = -m$). Choose one color randomly, according to the probability distribution ν on C and pick for each $a \in A$ the successor with respect to this color.

STEP 3. Check whether the $|A|$ paths, exactly one starting from each of the $|A|$ states at time $t_m = -m$, coalesce in the present t_0 at the latest. If this is the case, then proceed to STEP 4.
Otherwise, go back to STEP 2.

STEP 4. Put out the unique terminal state of all $|A|$ paths at time t_0 as a random sample.

Example 3.1.3. We assign a road-coloring to Example 3.1.1, turning it in our toy example (compare Example 1.1.10), and illustrate the difference between the two algorithms.



The first run of the original algorithm, presented in Example 3.1.1, Figure 3.2, can be realized by a choice of colors as well:

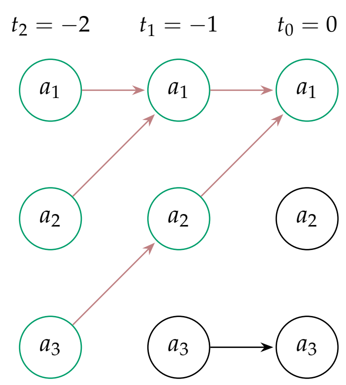


Figure 3.5: Run 1 of the CFTP is also possible in the CCFTP

The second run of the CFTP algorithm (compare Figure 3.3), is not possible in terms of the CCFTP algorithm. Indeed, as Figure 3.6 illustrates, not a single of the chosen transitions is allowed since a color must be chosen in each step.

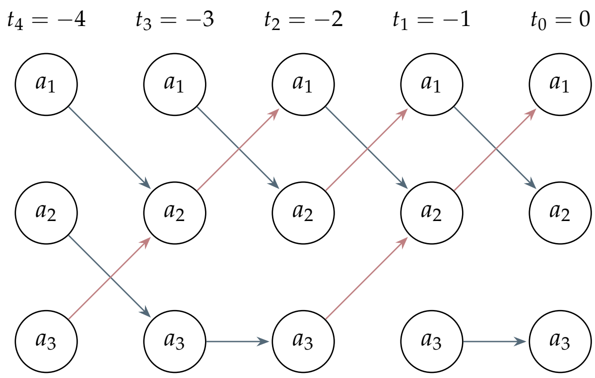


Figure 3.6: Run 2 of the CFTP is not possible in the CCFTP

3.1.2 *Synchronizability and Coupling from the Past*

Let $(A, C, \gamma; \mu, \nu)$ be an aperiodic irreducible road-colored graph. By π we denote the unique invariant probability distribution of the induced Markov chain. As agreed to in Section 2.3.2, the transition functions γ_n can also be understood as maps from $A \times C^-$ to A. From Theorem 2.3.10 we know that the road-colored graph is synchronizable if and only if the sequence of transition functions $(\gamma_n)_{n\in\mathbb{N}}$ converges ν^--almost surely towards a function $\gamma_\infty : C^- \to A$, that does no longer depend on the choice of initial vertex $a \in A$. In particular, for $(a, c^-) \in A \times C^-$, the limit $\lim_{n\to\infty} \gamma_n(a; c^-) = \gamma_\infty(c^-)$ exists if and only if c^- contains a synchronizing word.

Clearly, instead of choosing in each step of the CCFTP algorithm a color, prior to starting the algorithm, we can also choose an element $c^- \in C^-$ and then, at time t_m pick the corresponding color c_{-m}. The following proposition shows that the algorithm terminates if and only if a synchronizing word occurs in c^-.

Proposition 3.1.4. Let $(A, C, \gamma; \mu, \nu)$ be a road-colored graph and $c^- \in C^-$. Then, the following assertions are equivalent:

(a) There occurs a synchronizing word in c^-.

(b) The run of the CCFTP algorithm, corresponding to c^-, terminates in finite time.

Proof. Clearly, every choice of $c^- \in C^-$ induces a run of the algorithm. So, let $c^- \in C^-$ and $m \in \mathbb{N}$.

$(a) \Rightarrow (b)$. Assume that $(c_{-m}, \ldots, c_{-1})$ corresponds to the first occurrence of a synchronizing word when running m towards infinity. Then clearly $\gamma_m(a; c^-) = \gamma_m(b; c^-)$ for any pair of vertices $a, b \in A$. The corresponding paths, starting in any of the vertices in A coalesce, and the algorithm terminates.

$(b) \Rightarrow (a)$. The termination of the algorithm after m steps (i.e., at time $t_m = -m$) is possible if and only if all the paths coalesce for the first time when running backward, i.e., $\gamma_m(a; c^-) = \gamma_m(b; c^-)$ for every pair of vertices a, b in A for the first time. In other words, $(c_{-m}, \ldots, c_{-1})$ is the first synchronizing word. In particular, there exists no $l < m$ such that $(c_{-l}, \ldots, c_{-1})$ is synchronizing. Otherwise, the CCFTP algorithm would already have terminated at time $-l > -m$.

□

Definition 3.1.5. Let $(A, C, \gamma; \mu, \nu)$ be a road-colored graph. We define its *set of synchronizing color sequences*:

$$\begin{aligned} C_\infty :=& \{c^- \in C^- \ : \ c^- \text{ contains a synchronizing word for } A\} \\ =& \{c^- \in C^- \ : \ \lim_{n \to \infty} \gamma_n(\cdot\,; c^-) = \gamma_\infty(c^-) \text{ exists}\} \end{aligned}$$

And for $b \in A$ we set $C_b := \{c^- \in C_\infty \ : \ \gamma_\infty(c^-) = b\}$.

Obviously, the sets C_b form a partition of the synchronizing color sequences:

$$C_\infty = \bigcup_{b \in A} C_b.$$

An adaption of [Sis11, Thm. 3.10] to our setting shows that the CCFTP does, indeed, what we promised.

THEOREM 3.1.6. Consider an aperiodic irreducible Markov chain that can be represented by a synchronizable road-colored graph $(A, C, \gamma; \mu, \nu)$ and its unique invariant probability distribution π. For the Colored Coupling from the Past algorithm, the following holds:

1. The algorithm terminates ν^--almost surely in finite time, i.e., $\nu^-(C_\infty) = 1$.

2. The random sample produced by the algorithm is distributed according to the invariant distribution π, i.e., for all $b \in A$ holds $\nu^-(C_b) = \pi_b$.

Proof. 1. By assumption, there exists a synchronizing word $s := (c_{-m}, \dots, c_{-1})$, so clearly

$$\nu^-(C_\infty) \geq \nu^m(s) = \prod_{k=1}^{m} \nu(c_{-k}) > 0.$$

We can easily conclude that $\nu^-(C_\infty) = 1$, since in every infinite sequence of colors, every finite word occurs ν^--almost surely infinitely often. The proof is standard and due to continuity from above of ν^-.

2. We fix $b \in A$ and consider the subset of color sequences, where the first synchronizing word occurs at time $t_m = -m$ and maps A into b:

$$\begin{aligned} C_b &= \{c^- \in C_\infty \; : \; \gamma_\infty(c^-) = b\} \\ &= \{c^- \in C_\infty \; : \; \text{there exists } m \in \mathbb{N} \text{ s.t. } \gamma_m(A; c^-) = b\} \\ &=: \bigcup_{m \in \mathbb{N}} C_{m,b}, \end{aligned}$$

where $C_{m,b} := \{c^- \in C_b \; : \; (c_{-m}, \dots, c_{-1}) \text{ synchronizes } A \text{ into } b\}$. The sets $C_{m,b}$ are measurable and form an increasing sequence of subsets. Indeed, with Lemma 1.1.12 follows that for all $m \in \mathbb{N}$ holds

$$C_{m,b} \subseteq C_{m+1,b} \subseteq C_b.$$

So by continuity from below, C_b is measurable and for every $a \in A$ holds

$$\left.\begin{aligned} \nu^-(C_b) &= \lim_{m\to\infty} \nu^-(C_{m,b}) \\ &= \lim_{m\to\infty} \nu^-(\{c^- \in C_b \; : \; (c_{-m}, \dots, c_{-1}) \text{ synchronizes } A \text{ into } b\}) \\ &\leq \lim_{m\to\infty} \nu^-(\{c^- \in C^- \; : \; \gamma_m(a; c^-) = b\}) \\ &= \lim_{m\to\infty} t_{ab}^m \\ &\underset{\text{Thm.1.2.6}}{=} \pi_b. \end{aligned}\right\} (\star)$$

On the other hand, since C_∞ corresponds to a disjoint union of the sets C_b and due to assertion 1., the following holds:

$$\left.\begin{aligned} 1 &= \nu^-(C_\infty) \\ &= \sum_{b\in A} \nu^-(C_b) \\ &\underset{(\star)}{\leq} \sum_{b\in A} \pi_b \\ &= 1. \end{aligned}\right\} (\star\star)$$

And consequently, $\nu^-(C_b) = \pi_b$ for every $b \in A$.

□

We note that the first assertion in Theorem 3.1.6 is a direct consequence of the well-known fact that every finite word occurs almost surely infinitely often in an infinite sequence of letters. As for the second statement, we observe that the last step in equation $(\star)$ is possible since the graph was chosen to be aperiodic and irreducible. Moreover, when the probability distribution μ on A is chosen to be the unique invariant probability distribution of the Markov chain, this leads to the following assertion.

LEMMA 3.1.7. Consider $(A, C, \gamma; \mu, \nu)$ and suppose that μ is the unique invariant probability distribution on A. Suppose that the transition functions $(\gamma_n)_{n\in\mathbb{N}}$ converge ν^--almost surely towards a function γ_∞. Then γ_∞ is measure-preserving.

Proof. For convenience, we include a short proof. We refer to the notation introduced in the proof of the second assertion of Theorem 3.1.6 and make use of the fact that μ is invariant if and only if the transition functions γ_n are measure-preserving (compare Lemma 1.2.15). Consequently, for every $b \in A$ holds

$$\begin{aligned} \nu^-(\gamma_\infty^{-1}(b)) &= \nu^-(C_b) \\ &\leq \lim_{m\to\infty} \mu \otimes \nu^-(\{(a, c^-) \ : \ \gamma_m(a; c^-) = b\}) \\ &= \lim_{m\to\infty} \mu \otimes \nu^-(\gamma_m^{-1}(b)) \\ &\underset{\gamma_m \text{ meas. pres.}}{=} \lim_{m\to\infty} \mu(b) \\ &= \mu_b. \end{aligned}$$

The equality $\nu^-(\gamma_\infty^{-1}(b)) = \mu_b$ follows then with the same argument $(\star\star)$ as in the proof of Theorem 3.1.6. □

3.2 A TRIVIAL COLORING FOR MARKOV CHAINS

We have seen that the random sample produced by the Colored Coupling from the Past algorithm is distributed according to the invariant distribution whenever the underlying road-coloring possesses a synchronizing word. Naturally, the question arises of how this might help to conclude the same for the original CFTP algorithm. We reformulate the arguments in [Sis11, Chapter 3], leading to the introduction of a road-colored representation of a Markov chain, which corresponds, in a way, to the most general road-coloring that is possible.

3.2.1 *The trivial coloring*

Let $(A; \mu, T)$ be a Markov chain. Its canonical graph representation, where the vertices correspond to A, and the directed edges display the transitions according to T, does not necessarily need to be road-colorable. Indeed, different vertices $a \neq b$ in A may have different out-degrees $d_a \neq d_b$. We solve this problem by multiplying edges in order to obtain a road-colorable graph. The now-following construction is not the standard way to prove the assertion that the Markov chain can be represented by a road-colored graph. In general, the edges are multiplied until all vertices have the same out-degree and a coloring can be chosen with respect to the resulting transitions [YY11, Lem. 3.1]. Our approach generates a much larger graph, which, broadly speaking, contains all possible colorings simultaneously. In this way, there are no restrictions on the possible transitions when considering multiple copies of the Markov chain simultaneously, as would be the case when specifying a particular coloring. This allows us to understand the CFTP algorithm as a special case of the CCFTP algorithm and offers a new perspective on uncolored graphs.

Although the statements in this section become quickly clear in a descriptive manner, the construction is quite technical and requires the introduction of a somewhat extensive notation. Unfortunately, this cannot be avoided. It is advisable to focus on the concept when reading and consider the technical notation as necessary.

For better readability, we set $A := \{1, \ldots, n\}$. The out-degree of vertex $i \in A$ is denoted by d_i. To the outgoing edges of i we attribute an order by assigning to each $1 \leq j \leq d_i$ the corresponding terminal vertex j_i. We call such a tuple (i, j_i) *admissible* and denote the set of admissible tuples with initial vertex i by $A_i \subseteq \{i\} \times A$, then clearly $|A_i| = d_i$. The n-tuples in $\bigtimes_{i=1}^{n} A_i$ are called *admissible n-tuples*. They are of the form $((1, j_1), (2, j_2), \ldots, (n, j_n))$ and denote all the possibilities to move one step along the graph (one transition in the Markov chain) when starting simultaneously in all vertices of A. It follows $|\bigtimes_{i=1}^n A_i| = \prod_{i=1}^n d_i := d$. Every element in $\bigtimes_{i=1}^n A_i$ induces an adjacency matrix, where the non-null entries correspond exactly to the entries $(1, j_1), (2, j_2), \ldots, (n, j_n)$. Such an adjacency matrix can now also be understood as a monochromatic matrix M_c, assigning to all edges corresponding to the admissible tuples $((1, j_1), (2, j_2), \ldots, (n, j_n))$ the same color c. On the other hand, every possibility to assign one color to edges in the graph corresponds to a unique element in $\bigtimes_{i=1}^n A_i$. We find hence a set of d monochromatic matrices $M_D := \{M_1, \ldots, M_d\}$, to which can be assigned d colors $D := \{c_1, \ldots, c_d\}$, and a bijection between $\bigtimes_{i=1}^n A_i$ and M_D, respectively D.

So, if the graph representing $(A; \mu, T)$ were road-colorable, every coloring (C, γ) would correspond to a subset of monochromatic matrices of M_D. In this sense, the colors C of the coloring can be understood as a subset of D.

Definition 3.2.1. Consider the canonical graph representation of the Markov chain. One way to turn it into a road-colorable graph is to copy every outgoing edge of a vertex $i \in A$ exactly $\frac{d}{d_i} - 1$ times. Then, each of the d_i outgoing edges of i is turned into exactly $\frac{d}{d_i} = \prod_{i' \neq i} d_{i'}$ multiple edges. The thus obtained road-colorable graph has constant out-degree d and we call it *trivially colorable*. Then $\delta : A \times D \to A$ with $\delta(i; c_k) = j_i$, such that $(M_{c_k})_{i, j_i} = 1$ is a transition function and we call (D, δ) the *trivial coloring*.

Obviously, up to permutation, the trivial coloring is the only possible coloring of a trivially colorable graph. We define a distribution $\nu = (\nu_k)_{k=1}^d$ on D by

$$\nu_k := \prod_{\substack{i, j_i \,:\\ \delta(i; c_k) = j_i}} t_{i, j_i},$$

for $1 \leq k \leq d$. Before we show that this is a probability distribution on D, [Sis11, Prop. 3.3], we present an example.

Example 3.2.2. We carry this out by way of our standard example and start with the Markov chain from Example 3.1.1. There are $2^3 = 8$ possibilities to go one step along the graph, starting simultaneously in each of the three vertices. The trivially colorable graph is hence obtained by adding three copies of each edge, so that every vertex has out-degree equal to eight. We illustrate this in Figure 3.7.

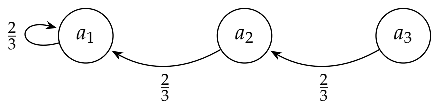
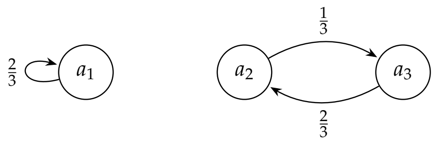
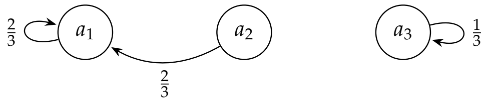
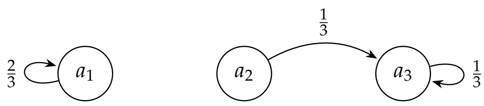
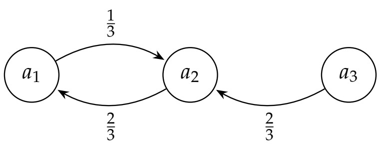
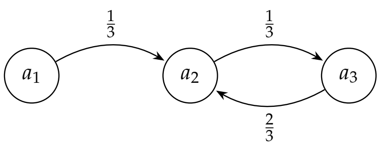
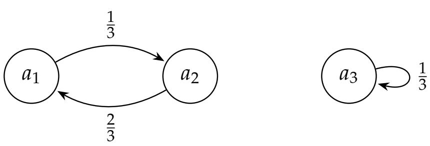
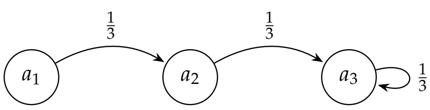
subgraph corresponding to a color	monochromatic matrix M_k	probability distribution on the colors ν_k
	$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\nu_1 = \frac{2 \cdot 2 \cdot 2}{3^3} = \frac{8}{27}$
	$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\nu_2 = \frac{2 \cdot 1 \cdot 2}{3^3} = \frac{4}{27}$
	$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\nu_3 = \frac{2 \cdot 2 \cdot 1}{3^3} = \frac{4}{27}$
	$M_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\nu_4 = \frac{2 \cdot 1 \cdot 1}{3^3} = \frac{2}{27}$
	$M_5 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\nu_5 = \frac{1 \cdot 2 \cdot 2}{3^3} = \frac{4}{27}$
	$M_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\nu_6 = \frac{1 \cdot 1 \cdot 2}{3^3} = \frac{2}{27}$
	$M_7 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\nu_7 = \frac{1 \cdot 2 \cdot 1}{3^3} = \frac{2}{27}$
	$M_8 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\nu_8 = \frac{1 \cdot 1 \cdot 1}{3^3} = \frac{1}{27}$

Figure 3.7: The trivial coloring for our toy example (compare Example 3.2.2)

LEMMA 3.2.3. In the just described setting, ν is a probability distribution on D.

Proof. We verify that the entries of ν sum up to one since there is a bijection between D and the admissible n-tuples $\times_{i=1}^{n} A_i$.

$$\begin{aligned}
\sum_{k=1}^{d} \nu_k &= \sum_{k=1}^{d} \prod_{\substack{i,j_i\,: \\ \delta(i;c_k)=j_i}} t_{i,j_i} \\
&= \sum_{\substack{(i,j_i)\in \\ \times_{i=1}^{n} A_i}} \prod_{i=1}^{n} t_{i,j_i} \\
&= \sum_{\substack{(i,j_i)\in \\ \times_{i=1}^{n} A_i}} t_{1,j_1} \cdot t_{2,j_2} \cdot \ldots \cdot t_{(n-1),j_{n-1}} \cdot t_{n,j_n} \\
&= \sum_{\substack{j_1\,: \\ (1,j_1)\in A_1}} t_{1,j_1} \cdot \underbrace{\left(\cdots \underbrace{\left(\sum_{\substack{j_{n-1}\,: \\ (n-1,j_{n-1})\in A_{(n-1)}}} t_{n-1,j_{n-1}} \cdot \underbrace{\left(\sum_{\substack{j_n\,: \\ (n,j_n)\in A_n}} t_{n,j_n} \right)}_{=1} \right)}_{=1} \cdots \right)}_{=1} \\
&= 1
\end{aligned}$$

So, ν defines a probability distribution on D. $\square$

Last but not least, we verify that the trivially colored graph $(A, D, \delta; \mu, \nu)$ does indeed represent the Markov chain $(A; \mu, T)$ [Sis11, Prop. 3.27].

PROPOSITION 3.2.4. Every Markov chain $(A; \mu, T)$ possesses a representation as a road-colored graph.

Proof. We consider the corresponding trivially colored graph $(A, D, \delta; \mu, \nu)$ and show that $T = \sum_{c_k \in D} \nu_k \cdot M_k$, where M_k is the monochromatic matrix corresponding to color $c_k \in D$. We check this componentwise. By definition $(M_k)_{i,j} = 0$, if $t_{i,j} = 0$. So, for each admissible tuple, it needs to be verified that $\sum_{c_k \in D} \nu_k (M_k)_{i,j_i} = t_{i,j_i}$. Without loss of generality, we consider the admissible tuple $(1, j_1) \in A_1$ and proceed as in the proof of Lemma 3.2.3:

$$
\begin{aligned}
\sum_{c_k \in D} \nu_k (M_k)_{1,j_1} &= \sum_{\substack{c_k : \\ \delta(1;c_k)=j_1}} \nu_k \\
&= \sum_{\substack{c_k : \\ \delta(1;c_k)=j_1}} \prod_{\substack{i,j_i : \\ \delta(i;c_k)=j_i}} t_{i,j_i} \\
&= t_{1,j_1} \cdot \sum_{\substack{c_k : \\ i \neq 1 \\ \delta(1;c_k)=j_1}} \prod_{\substack{i,j_i : \\ \delta(i;c_k)=j_i}} t_{i,j_i} \\
&= t_{1,j_1} \cdot \sum_{\substack{(i,j_i) \in \\ \times_{i=2}^n A_i}} \prod_{i=2}^{n} t_{i,j_i} \\
&= t_{1,j_1} \cdot \underbrace{\sum_{\substack{(i,j_i) \in \\ \times_{i=2}^n A_i}} t_{2,j_2} \cdot \ldots \cdot t_{n-1,j_{n-1}} \cdot t_{n,j_n}}_{=1 \text{ (compare Lem. 3.2.3)}} \\
&= t_{1,j_1}.
\end{aligned}
$$

□

It is crucial to become aware of the fact that the trivially colored graph representation of a Markov chain $(A; \mu, T)$ takes a special role compared to all other possible colorings. The difference becomes apparent when starting in all n vertices simultaneously and comparing the possible outcomes after one step along the colored edges (i.e., proceeding precisely as the CCFTP algorithm):

$(A; \mu, T)$: If the underlying graph is uncolored, in each step, one can choose between the transitions corresponding to all d admissible n-tuples.

$(A, C, \gamma; \mu, T)$: If the underlying graph is colored (with $C \neq D$), one can choose only between the transitions, corresponding to the choice of a same-colored edge for all vertices. There are, hence, only $|C| < d$ allowed transitions.

$(A, D, \delta; \mu, T)$: If the underlying graph is trivially colored, by definition, there corresponds to every admissible n-tuple a color. So, to every transition that is possible in the uncolored setting can be assigned a color from the trivial coloring.

Apart from the trivial coloring, every coloring imposes a constraint on the possible transitions. In comparison, the trivial coloring allows us to obtain a road-colored graph, which can still behave like the uncolored canonical graph representation of the Markov chain. In this sense, the naming *trivial coloring* is quite intuitive. As a coloring itself, it is not very interesting, aside from the fact that it is a coloring.

3.2.2 *Back to the original algorithm*

We have seen that every Markov chain $(A; \mu, T)$ can be represented by a trivially colored graph. The application of the CFTP algorithm to an uncolored graph can hence be understood as applying the CCFTP algorithm to the corresponding trivially colored graph. In this sense, the CFTP algorithm can be seen as a special case of the CCFTP algorithm.

COROLLARY 3.2.5. Each step in the Coupling from the Past algorithm corresponds to the choice of a color in the trivial coloring.

It remains to be shown that the the algorithm, applied to $(A, D, \delta; \mu, T)$, terminates ν^--almost surely. From Theorem 3.1.6, we know this would be the case if the trivial coloring (D, δ) were synchronizing. Indeed, this is the case for aperiodic, irreducible Markov chains [Sis11, Lem. 3.7].

PROPOSITION 3.2.6. Every trivially colored aperiodic irreducible graph $(A, D, \delta; \mu, \nu)$ possesses a synchronizing word.

Proof. The assertion is a direct consequence of the graph being aperiodic and irreducible. From Proposition 1.1.6 follows the existence of $m \in \mathbb{N}$, such that the m-th power T^m of the transition matrix T has only positive entries. In particular, for fixed $b \in A$ and for every $a \in A$, we have $t_{ab}^m > 0$. Every vertex can hence be mapped to b in m steps. By construction of the trivial coloring (D, δ), when staring in all vertices simultaneously and choosing the first transition in the corresponding paths, we can also pick the corresponding color in D. Proceeding iteratively, we obtain thus a synchronizing word of length m. □

We conclude the elaborations concerning the Coupling from the Past algorithm, which does, in fact, nothing different than finding synchronizing words in the trivially colored graph, representing the underlying aperiodic irreducible Markov chain. As almost every color sequence contains a synchronizing word, the simulation terminates indeed almost surely (compare Theorem 3.1.6).

COROLLARY 3.2.7. For every aperiodic irreducible Markov chain, the following holds:

1. The CFTP algorithm terminates almost surely in finite time.
2. The random sample is distributed according to the unique invariant distribution of the Markov chain.

3.3 REFLECTIONS ON THE CONCEPT OF NON-COMMUTATIVE COLORS

Monochromatic matrices turned out to be a handy tool in examining synchronizing words in the classical setting. Indeed, every monochromatic matrix represents a classical color, and any product thereof illustrates the outcome when starting in all vertices simultaneously and following the corresponding word of colors. So, the following is an obvious observation.

COROLLARY 3.3.1. For a road-colored graph (A, C, γ) with corresponding monochromatic matrices M_k, each M_k representing a color $c_k \in C$, the following assertions are equivalent:

(a) The word $(c_{i_1}, \ldots, c_{i_n})$ is synchronizing.

(b) The matrix product $M_{i_n} \cdot \ldots \cdot M_{i_1}$ is of rank one.

We present two promising concepts of non-commutative colors, both building on the correspondence between monochromatic matrices (respecting the allowed transitions) and colors in the classical setting. The first concept generalizes the idea of a monochromatic transition matrix, whereas the second aims to transfer Corollary 3.3.1 to the non-commutative setting. This will lead to a common issue in quantum probability theory: two different perceptions of the same classical phenomenon lead to two different non-commutative concepts.

3.3.1 *A proposition for non-commutative colors*

When algebraizing the classical setting, an interesting observation was communicated by B. Kümmerer. Let $(A, C, \gamma; \mu, \nu)$ be a road-colored graph representing a Markov chain. Suppose that A consists of n vertices and C of d colors. The commutative algebras of complex-valued functions on n and d points can be identified with $\mathcal{A} \simeq \mathbb{C}^n$ and $\mathcal{C} \simeq \mathbb{C}^d$, respectively. Via GNS-construction with respect to the faithful tracial states induced by μ and ν, we obtain commutative quantum probability spaces $(\mathcal{A}, \varphi)$ and

$(\mathcal{C},\psi)$, which can be identified with the (commutative) subalgebras of diagonal matrices in M_n and M_d, respectively, i.e. $\mathcal{A} \simeq \mathit{Diag}(\mathrm{M}_n)$ and $\mathcal{C} \simeq \mathit{Diag}(\mathrm{M}_d)$. In particular, we identify φ and ψ with their canonical extensions to M_n and M_d, given by the density matrices

$$\Phi = \begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n \end{pmatrix}, \quad \text{and} \quad \Psi = \begin{pmatrix} \nu_1 & & & \\ & \nu_2 & & \\ & & \ddots & \\ & & & \nu_d \end{pmatrix}.$$

It turns out that the transition operator $T_\psi : \mathit{Diag}(\mathrm{M}_n) \to \mathit{Diag}(\mathrm{M}_n)$ can be trivially extended to a completely positive operator on M_n by

$$\begin{aligned} \tilde{T}_\psi \ : \quad \mathrm{M}_n &\to \mathrm{M}_n \\ x &\mapsto \sum_{k=1}^{d} \nu_k \cdot M_{\varphi,k} x M_{\varphi,k}^* \end{aligned}$$

where $M_{\varphi,k}$ corresponds to the representation of the *monochromatic matrix to the color $c_k \in C$, with respect to the φ-orthonormal basis*, as introduced in Example 2.1.9, i.e. with respect to the canonical system of matrix units $(e_{ij})_{i,j=1}^n$ of M_n, it is of the form:

$$\begin{aligned} (M_{\varphi,k})_{ij} &= \begin{cases} \left(\frac{\varphi(e_{ii})}{\varphi(e_{jj})}\right)^{1/2}, & \text{if } \gamma(a_i; c_k) = a_j, \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} \left(\frac{\mu_i}{\mu_j}\right)^{1/2}, & \text{if } \gamma(a_i; c_k) = a_j, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

For the computation of $M_{\varphi,k}$ compare [GKL06, Section 5], where it was introduced as *normalized adjacency matrix for the color c_k*. In particular, for $\varphi = \tau$, or equivalently if μ corresponds to the uniform probability distribution on A, we obtain

$$M_{\tau,k} = M_k$$

for all colors $c_k \in C$. In this sense, every monochromatic matrix, and hence a classical color, induces a completely positive operator on M_n mapping $x \in \mathrm{M}_n$ to $\nu_k \cdot M_k x M_k^*$.

Example 3.3.2. We present this procedure for our toy example (compare Example 1.1.10). Consider $A = \{a_1, a_2, a_3\}$ with uniform distribution $\mu = \frac{1}{3}(1,1,1)$ and colors

$C = \{c_1, c_2\}$ with probability distribution $\nu = (\nu_1, \nu_2)$. We identify $\mathcal{A} \simeq Diag(\mathrm{M}_3)$ with $\mathbb{C}^3$ and $\mathcal{C} \simeq Diag(\mathrm{M}_2)$ with $\mathbb{C}^2$. Let $(e_i)_{i=1}^3$ and $(f_k)_{k=1}^2$ denote the canonical orthonormal bases for $\mathbb{C}^3$ and $\mathbb{C}^2$. This leads to the following transition (compare [GKL06, Section 5]):

$$\begin{aligned} J \; : \quad \mathbb{C}^3 &\to \mathbb{C}^3 \otimes \mathbb{C}^2 \\ e_j &\mapsto \sum_{\substack{(i,k)\,:\\ \gamma(a_i;c_k)=a_j}} e_i \otimes f_k. \end{aligned}$$

We denote by ψ the state induced by ν and obtain

$$\begin{aligned} T_\psi \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} &= P_\psi \circ J \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\ &= P_\psi \left(\begin{pmatrix} a_2 \\ a_3 \\ a_3 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &= \nu_1 \cdot \begin{pmatrix} a_2 \\ a_3 \\ a_3 \end{pmatrix} + \nu_2 \cdot \begin{pmatrix} a_1 \\ a_1 \\ a_2 \end{pmatrix} \\ &= \begin{pmatrix} \nu_1 a_2 + \nu_2 a_1 \\ \nu_1 a_3 + \nu_2 a_1 \\ \nu_1 a_3 + \nu_2 a_2 \end{pmatrix} \\ &= \begin{pmatrix} \nu_2 & \nu_1 & 0 \\ \nu_2 & 0 & \nu_1 \\ 0 & \nu_2 & \nu_1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\ &= T \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \end{aligned}$$

where T is the classical transition matrix.

Let now denote M_1 and M_2 the monochromatic transition matrices corresponding to c_1 and c_2. Application of $\tilde{T}_\psi$ to an element in $Diag(\mathrm{M}_3)$ leads to:

$$\begin{aligned}\tilde{T}_\psi \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} &= \nu_1 \cdot \begin{pmatrix} a_2 & & \\ & a_3 & a_3 \\ & a_3 & a_3 \end{pmatrix} + \nu_2 \cdot \begin{pmatrix} a_1 & a_1 & \\ a_1 & a_1 & \\ & & a_2 \end{pmatrix} \\ &= \begin{pmatrix} \nu_a a_2 + \nu_2 a_1 & \nu_s a_2 & \\ \nu_2 a_1 & \nu_1 a_3 + \nu_2 a_1 & \nu_1 a_3 \\ & \nu_1 a_3 & \nu_1 a_3 + \nu_2 a_2 \end{pmatrix}.\end{aligned}$$

So, on the diagonal, $\tilde{T}_\psi$ coincides with the transition operator T_ψ and can hence be understood as an extension of the classical transition matrix T to the non-commutative setting.

In particular, the classical transition matrix is given by a convex combination of the monochromatic matrices, which, in turn, induce Kraus operators and hence corresponding completely positive operators. Let us leave the commutative context and consider any quantum probability space $(\mathcal{A}, \varphi)$. As a faithful state, ψ is a convex combination of pure states $\omega_k \in \mathcal{S}(\mathcal{C})$. The above observation motivates the idea of associating the pure states on $\mathcal{C}$ with all possible monochromatic matrices corresponding to the trivial coloring. The resulting operators $T_{\omega_k} := P_{\omega_k} \circ J$ can thus be understood as non-commutative colors. We remark that these operators are no transition operators associated to a transition in the sense of Definition 2.3.2, as pure states are not faithful.

3.3.2 *Synchronizing convex decomposition*

Another approach to transfer the concept of synchronizing words to the quantum mechanical setting, which emerged from the study of monochromatic matrices, was introduced in [Sis11, Chapter 4].

Let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ be probability spaces.

Definition 3.3.3. A completely positive identity-preserving operator $T : \mathcal{A} \to \mathcal{A}$ is said to be *synchronizable* if it allows a convex decomposition

$$T = \sum_{i \in I} \lambda_i T_i,$$

where the $T_i : \mathcal{A} \to \mathcal{A}$ are identity-preserving and completely positive operators, such that there exist a state $\varphi \in \mathcal{S}(\mathcal{A})$, a natural number n and $(i_{-n}, \ldots, i_{-1}) \in I^n$ with $\eta \circ T_{i_{-n}} \circ \cdots \circ T_{i_{-1}} = \varphi$ for all $\eta \in \mathcal{S}(\mathcal{A})$. In this case, $(i_{-n}, \ldots, i_{-1})$ is called a *(non-commutative) synchronizing word* and $\sum_{i \in I} \lambda_i T_i$ is said to be a *synchronizing convex decomposition (of T)*.

Applied to the transition operator T_ψ, corresponding to a transition J, this definition generalizes the classical intuition that a transition matrix corresponds to a convex combination of monochromatic transition matrices and a synchronizing word occurs if the corresponding product of monochromatic (transition) matrices is of rank one (compare Corollary 3.3.1). In this setting, non-commutative colors correspond to the completely positive operators T_i, forming the convex decomposition.

3.3.3 *Non-commutative colors are not what they seem*

We have presented two possible concepts of non-commutative colors. One is to understand operators of the form T_ω, where ω is a pure state, as the natural extension of monochromatic transition matrices to the general setting. The other concept introduces the completely positive operators that form a synchronizing convex decomposition as a possible generalization of monochromatic transition matrices. It is only natural to pose the question of whether these two concepts of monochromatic transition matrices coincide. This is motivated further by the following considerations.

Let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ be quantum probability spaces with finite-dimensional von Neumann algebras $\mathcal{A}$, $\mathcal{C}$ and let $J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$ be a transition. Since every state on $\mathcal{C}$ can be represented as a convex combination of the extremal states in $\mathcal{S}(\mathcal{C})$, i.e. the pure states on $\mathcal{C}$, we find pure states $\omega_1, \ldots, \omega_k$ on $\mathcal{C}$ and $\lambda_k > 0$ with $\sum_{k=1}^d \lambda_k = 1$, such that

$$\psi = \sum_{k=1}^{d} \lambda_k \omega_k.$$

LEMMA 3.3.4. Let $\psi = \sum_{k=1}^d \lambda_k \omega_k$ be a convex decomposition of ψ into pure states ω_k on $\mathcal{C}$, then $T_\psi = \sum_{k=1}^d \lambda_k T_{\omega_k}$, where $T_{\omega_k} = P_{\omega_k} \circ J$ and P_{ω_k} corresponds to the conditional expectations of tensor type with respect to ω_k.

Proof. A short calculation shows that indeed for all $a \in \mathcal{A}$ and $c \in \mathcal{C}$

$$P_\psi(a \otimes c) = \psi(c) \cdot a \otimes \mathbb{1} = \sum_k \lambda_k \omega_k(c) \cdot a \otimes \mathbb{1} = \sum_k \lambda_k P_{\omega_k}(a \otimes c).$$

And hence, $T_\psi = P_\psi \circ J = \sum_k \lambda_k P_{\omega_k} \circ J = \sum_k \lambda_k T_{\omega_k}$. □

So clearly, every transition operator T_ψ, corresponding to J, is given by a convex decomposition of operators T_{ω_k}. It seems only reasonable to ask whether the two concepts coincide and if $\sum_k \lambda_k T_{\omega_k}$ can form a synchronizing convex decomposition of the transition operator T_ψ. Unfortunately, these two approaches are incompatible, as the following example will show. It originated within the framework of a joint workshop, together with members of the C*-AG [Die+18].

Let $\mathcal{A} = \mathcal{C} = \mathrm{M}_2$ and an automorphism on $\mathcal{A} \otimes \mathcal{C} = \mathrm{M}_2 \otimes \mathrm{M}_2$ be given by the unitary

$$u = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & c & s & 0 \\ \hline 0 & -s & c & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) =: \left(\begin{array}{c|c} u_{11} & u_{12} \\ \hline u_{21} & u_{22} \end{array}\right) \in \mathrm{M}_2 \otimes \mathrm{M}_2,$$

with $c := \cos\vartheta$ and $s := \sin\vartheta$, for $\vartheta \in [0, 2\pi]$. Then, a transition $J : \mathrm{M}_2 \to \mathrm{M}_2 \otimes \mathrm{M}_2$ is given by

$$\begin{aligned} J(a) &= u^*(a \otimes \mathbb{1})u \\ &= \left(\begin{array}{c|c} u_{11}^* a u_{11} + u_{21}^* a u_{21} & u_{11}^* a u_{12} + u_{21}^* a u_{22} \\ \hline u_{12}^* a u_{11} + u_{22}^* a u_{21} & u_{12}^* a u_{12} + u_{22}^* a u_{22} \end{array}\right) \\ &:= \left(\begin{array}{c|c} T_{11}(a) & T_{12}(a) \\ \hline T_{21}(a) & T_{22}(a) \end{array}\right). \end{aligned}$$

Every pure state ω on M_2 is of the form $\omega(\cdot) = \langle \cdot\, \xi_\omega, \xi_\omega \rangle$ (compare Example 2.1.2), with

$$\xi_\omega = \begin{pmatrix} \alpha \\ \sqrt{1-\alpha^2} e^{i\delta}. \end{pmatrix},$$

where $\alpha \in [0,1]$ and $\delta \in [0,2\pi]$. Let now w be any pure state, with corresponding ξ_ω and let $(f_{ij})_{i,j=1}^2$ denote the canonical matrix units in M_2. Let $a \in \mathcal{A} = M_2$. Then, the corresponding operator T_ω on M_2 is given by

$$\begin{aligned} T_\omega(a) &= P_\omega \circ J(a) \\ &= P_\omega \left(\sum_{i,j=1}^{2} T_{ij}(a) \otimes f_{ij} \right) \\ &= \sum_{i,j=1}^{2} P_\omega \left(T_{ij}(a) \otimes f_{ij} \right) \\ &= \sum_{i,j=1}^{2} \omega(f_{ij}) \cdot T_{ij}(a) \\ &= \sum_{i,j=1}^{2} \left\langle f_{ij}\xi_\omega, \xi_\omega \right\rangle \cdot T_{ij}(a). \end{aligned}$$

A short calculation shows that

$$\omega(f_{11}) = \alpha^2, \quad \omega(f_{12}) = \alpha\sqrt{1-\alpha^2}e^{i\delta}, \quad \omega(f_{21}) = \alpha\sqrt{1-\alpha^2}e^{-i\delta}, \quad \omega(f_{22}) = 1-\alpha^2.$$

Since M_2 and $\mathbb{C}^4$ can be identified, every operator on M_2, and hence also the operator T_ω, possesses a representation as a 4×4-matrix in $M_4 \simeq M_2 \otimes M_2 \simeq \mathcal{B}(M_2)$. Thus, T_ω can be identified with the matrix (the calculation is simple but tedious, compare [Die+18]).

$$T_\omega = \begin{pmatrix} \alpha^2 + c^2(1-\alpha^2) & -s\alpha\sqrt{1-\alpha^2}e^{i\varphi} & -s\alpha\sqrt{1-\alpha^2}e^{-i\varphi} & s^2(1-\alpha^2) \\ cs\alpha\sqrt{1-\alpha^2}e^{-i\varphi} & c & 0 & -cs\alpha\sqrt{1-\alpha^2}e^{-i\varphi} \\ cs\alpha\sqrt{1-\alpha^2}e^{i\varphi} & 0 & c & -cs\alpha\sqrt{1-\alpha^2}e^{i\varphi} \\ \alpha^2 s^2 & s\alpha\sqrt{1-\alpha^2}e^{i\varphi} & s\alpha\sqrt{1-\alpha^2}e^{-i\varphi} & c^2\alpha^2 + (1-\alpha^2) \end{pmatrix}$$

A synchronizing word in terms of the synchronizing convex decomposition corresponds to finitely many operators T_ω, such that the product of their representations as 4×4-matrices is of rank one. We compute the determinant for such an operator and find that it is given by

$$\det(T_\omega) = c^4 + 4c^2s^2\alpha^2(1-\alpha^2).$$

In other words, whenever $c \neq 0$ follows $\det(T_\omega) > 0$ and the matrix representing the transition operator is of full rank. But then, no such operatos' product will ever be of rank one.

The convex decomposition of the state ψ on $\mathcal{C}$ into pure states can thus never lead to a synchronizing convex decomposition of the transition operator T_ψ with respect to operators T_ω. Of course, this does not mean that either of these concepts has to be wrong.

In the following chapter, we will focus on a third possibility to translate the idea of colors into the non-commutative setting. It also arises from the observations in Section 3.3.1. As we have seen, the monochromatic matrices, in their representation with respect to the φ-orthonormal basis, induce Kraus operators of a completely positive operator $\tilde{T}_\psi$ on M_n extending the transition operator T_ψ. A first choice could be to further consider this operator as a generalization of the transition matrix to the non-commutative setting. However, here we quickly encounter the problem that $\tilde{T}_\psi$ is not identity-preserving on M_n. In Chapter 4, we will introduce another concept that will turn out to be strongly linked to the idea in Section 3.3.1 and lead to a completely positive identity-preserving operator that can be understood as a natural extension of the transition J.

4

SYNCHRONIZABILITY AND REGULARITY

THE PURPOSE OF A STORYTELLER
IS NOT TO TELL YOU HOW TO THINK,
BUT TO GIVE YOU QUESTIONS TO THINK UPON.

— Brandon Sanderson

In the study of synchronizability of road-colorable graphs, irreducibility is typically assumed as a default condition. Obviously, the existence of a synchronizing word requires the underlying uncolored graph to possess paths that map every vertex to the same target state, indicating the need for a certain level of connectivity. In the first section of this chapter, we provide a necessary and sufficient condition for a road-colorable graph to possess a synchronizing coloring, thus showing that the irreducibility of the entire graph is no necessary requirement. However, the concept cannot be waived completely.

Moving on to Section 4.2, we will present the so-called label product. It was introduced in [LM95] as a special graph product with respect to the given road-coloring (or labeling, in the language of automata, thence the naming). By leveraging the insights of Section 4.1, it can be shown that the label product offers an alternative characterization of the existence of a synchronizing word in a road-colored graph [GKL06], [Lan03].

Section 4.3 focuses on the non-commutative concept of the dual extended transition operator. It was established by [Goh04] in the context of extensions of completely positive identity-preserving operators on a von Neumann algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ to all of $\mathcal{B}(\mathcal{H})$ and can be understood as a possible answer to the reflections on non-commutative colors in Section 3.3 as well as a generalization of the concept of the label product. Indeed, it provides an alternative characterization of the asymptotic completeness of a transition corresponding to a one-sided quantum Markov process [Goh04], [GKL06]. We establish a suitable framework and provide an alternative proof of this crucial equivalence within a Hilbert space setting, thereby creating a new perspective on the dual extended transition operator and asymptotic completeness through the lens of Hilbert space theory.

4.1 A CHARACTERIZATION OF SYNCHRONIZABLE GRAPHS

We introduce the notion of absorbing vertices of a graph and show that they play a special role when examining the question of whether a road-colored graph is synchronizable. This idea finds a generalization in the concept of recurrent and transient communicating classes of vertices, which will give us the possibility to formulate necessary and sufficient conditions concerning the questions of whether a given coloring is synchronizing and if such a coloring exists at all. For the whole section, we consider exclusively finite directed graphs and rely on [Sen06] for the basic definitions.

Definition 4.1.1. Let (A, E) be a road-colorable graph. A vertex $a_0 \in A$ is called *absorbing*, if all its outgoing edges form a loop.

In other words, the a_0-th row of the corresponding adjacency matrix has only a single non-null entry, namely the diagonal entry.

PROPOSITION 4.1.2. Suppose that a road-colorable graph (A, E) possesses an absorbing state, such that every other vertex is connected to a_0. Then, every road-coloring of the graph is synchronizable.

Proof. Let (C, γ) be any coloring of (A, E). It suffices to show pairwise synchronizability (compare Proposition 1.1.16). By assumption, for every vertex $a_i \in A$ there exist $n_i \in \mathbb{N}$ and a path $w_i \in C^{n_i}$, mapping a_i into a_0. Let now a_i, a_j be any pair of vertices in A. Then, $\gamma_{n_i}(a_i; w_i) = a_0$ and $\gamma_{n_i}(a_j; w_i) =: a_k$. If $k = 0$, we are finished, else we find $n_k \in \mathbb{N}$ and a word $w_k \in C^{n_k}$, mapping a_k into a_0. As a_0 is absorbing, it is mapped to itself by w_k. So, $w_k w_i$ synchronizes a_i and a_j into a_0. $\square$

Clearly, in the setting of Proposition 4.1.2, there can exist only one absorbing state $a_0 \in A$. Otherwise, it would not be possible for all other vertices in $A \setminus \{a_0\}$ to be connected to it.

We will shortly see that a generalization of this idea to a subset of states will lead to a necessary and sufficient condition for a road-colorable graph to be synchronizable, as well as for a given coloring to be synchronizing. The following definitions are standard when examining stochastic matrices and Markov chains.

Definition 4.1.3. Let (A, E) be a directed graph. A vertex $a \in A$ is called *transient*, if there exists $b \in A$ such that $a \longrightarrow b$, but $b \not\longrightarrow a$. Otherwise, a is called *recurrent*, i.e., $a \longrightarrow b$ implies $b \longrightarrow a$.

Two states $a, b \in A$ *communicate*, if they are connected to each other, i.e., if $a \longrightarrow b$ and $b \longrightarrow a$. A subset of states in A forms a *communicating class* if all its states communicate with each other.

Obviously, transience and recurrence are class properties, i.e., if one state in a communicating class is recurrent, then the remaining states in the class are recurrent, too. The same holds for transience. Consequently, all recurrent vertices can be subdivided into *recurrent communicating classes*, whereas the transient vertices may be divided into *transient communicating classes*. Often, the term *communicating* is omitted. In literature, recurrent and transient states are sometimes called *essential* and *inessential*, respectively [Sen06].

In place of the vertices in a graph, we can now also consider its communicating classes. In this sense, notions such as incoming and outgoing edges or cycles can be transferred to communicating classes.

We state a few basic observations that can easily be verified.

Observation 4.1.4. The following assertions hold for every directed graph with finite vertex set:

1. Every vertex in a graph is either recurrent or transient.
2. A recurrent class possesses no outgoing edges, i.e., edges that start in the recurrent class and lead out of it.
3. Transient classes cannot form a cycle. In particular, it is not possible to leave a transient class and then return back to it.

Clearly, the property of a vertex to be recurrent or transient is independent of the fact whether the graph is road-colored or not. The next assertion follows directly [Sen06, Lem. 1.1].

PROPOSITION 4.1.5. Let (A, E) be a directed graph. If every vertex possesses at least one outgoing edge, then every transient class possesses at least one outgoing edge.

Proof. Let $A_0 \subseteq A$ be a transient class. By definition, there exist $a \in A_0$ and $b \in A \setminus A_0$, such that $a \longrightarrow b$ but $b \not\longrightarrow a$. Due to the assumption, we can exclude the case where a transient class consists of only a single vertex with no outgoing edges. So, the assertion follows. □

We now show some further simple properties of communicating classes in road-colorable graphs.

PROPOSITION 4.1.6. Every road-colorable graph (A, E) possesses at least one recurrent class.

Proof. This is a direct consequence of Proposition 4.1.5. Suppose that A consists only of transient communicating classes $A_0, \ldots, A_m$. Since there are only finitely many vertices and since every transient class possesses at least one outgoing edge, some of them must form a cycle. However, this contradicts the third assertion in Observation 4.1.4. □

So, in a road-colored graph, every transient class possesses an outgoing edge and is connected to a recurrent class. But what happens if synchronizability comes into play?

COROLLARY 4.1.7. Let (A, C, γ) be a synchronizable road-colored graph. Every synchronizing word leads into a recurrent class.

Proof. By Proposition 4.1.6, there exists at least one recurrent class $A_0 \subseteq A$. As A_0 possesses no outgoing edges, we have $\gamma_n(A_0; w) \subseteq A_0$ for all $n \in \mathbb{N}$ and $w \in C^n$. □

COROLLARY 4.1.8. Every synchronizable road-colorable graph (A, C, γ) possesses exactly one recurrent class.

Proof. Suppose that there exist at least two recurrent classes A_0 and A_1. Then every pair of vertices in $A_0 \times A_1$ cannot be synchronizable by Corollary 4.1.7. □

A necessary condition for a road-colorable graph to possess a synchronizing coloring is, hence, that it possesses exactly one recurrent communicating class. The following concept of regularity originates from stochastic matrices and Markov chains and can be transferred to directed graphs.

Definition 4.1.9. A matrix is said to be *regular* if its recurrent states form a single communicating class, which is aperiodic. A *regular Markov chain* is a Markov chain with a regular transition matrix.
A directed graph is called *regular in the matrix-sense*, if all its recurrent vertices form a single aperiodic communicating class.

We remark that the notion *regular graph* is already defined in graph theory and has nothing to do with our setting. So, in order to avoid confusion, we introduced the somewhat cumbersome term of regularity in the matrix-sense.

Clearly, an aperiodic irreducible graph is a graph that is regular in the matrix-sense, such that the aperiodic recurrent communicating class consists of all vertices.

Lemma 4.1.10. Let (A, C, γ) be a road-colored graph and $A_0 \subseteq A$ a recurrent communicating class. Then, the subgraph consisting of the vertices in A_0 and the (colored) outgoing edges of these vertices is irreducible and road-colored.

Proof. Since the vertices in A_0 form a recurrent communicating class, there are no edges leading to a vertex that is not in A_0. So, the vertices in the subgraph corresponding to A_0 have constant out-degree $|C|$. The irreducibility follows directly since all vertices in A_0 communicate. □

Definition 4.1.11. Let (A, C, γ) be a graph that is regular in the matrix-sense and $A_0 \subseteq A$ the corresponding aperiodic recurrent class. The corresponding irreducible road-colored subgraph is denoted by (A_0, C, γ_0), where $\gamma_0 := \gamma|_{A_0 \times C}$.

In the following, we present a necessary and sufficient condition for a road-colorable graph to possess a synchronizing coloring.

Theorem 4.1.12. For a road-colorable graph (A, E), the following assertions are equivalent:

(a) The graph possesses a synchronizing coloring.

(b) The graph is regular in the matrix-sense.

Proof. $(a) \Rightarrow (b)$. Suppose the graph possesses a synchronizing coloring (C, γ). From Corollary 4.1.8 follows the existence of a unique recurrent communicating class $A_0 \subseteq A$. With Lemma 4.1.10 follows, that the subgraph (A_0, C, γ_0) is irreducible and road-colored. Since the road-coloring was assumed to be synchronizable, (A_0, C, γ_0) is synchronizable too by Corollary 4.1.7. But for a synchronizing irreducible road-colored graph follows aperiodicity by Proposition 1.1.17. The original graph (A, C, γ) is thus regular in the matrix-sense.

$(b) \Rightarrow (a)$. Suppose the graph is regular in the matrix sense. Then, the aperiodic irreducible subgraph (A_0, C, γ_0) possesses a synchronizing coloring due to the Road-Coloring Theorem 1.1.18. By definition, every vertex in $A \setminus A_0$ lies in a transient class with at least one outgoing edge (compare Proposition 4.1.5). As A is finite and since transient classes cannot form cycles, every vertex in $A \setminus A_0$ is connected to A_0.

Proceeding analogously to Proposition 4.1.2, it can be shown that, independent of the chosen coloring, every pair of vertices in $(A \setminus A_0) \times (A \setminus A_0)$ can be mapped to a pair of vertices in (A_0, A_0), which is synchronizable since (A_0, C, γ_0) possesses a synchronizing coloring. Consequently, (A, C, γ) is pairwise synchronizable when choosing the synchronizing coloring (C, γ_0) on the recurrent subgraph and extending it arbitrarily to the whole graph.

□

COROLLARY 4.1.13. Let (A, C, γ) be regular in the matrix-sense and (A_0, C, γ_0) be the subgraph corresponding to the single aperiodic recurrent class. Then, the following assertions are equivalent.

(a) The graph (A, C, γ) is synchronizable.

(b) The subgraph (A_0, C, γ_0) is synchronizable.

Let us now turn to the setting of Markov chains. Indeed, since every graph representing a regular Markov chain is regular in the matrix-sense, the above results can be transferred to the setting of regular Markov chains. We obtain the following assertion as a direct consequence of Proposition 3.2.4 and Theorem 4.1.12.

COROLLARY 4.1.14. Any regular Markov chain can be represented by a synchronizable road-colored graph.

This section started with the introduction of absorbing states. It can now be concluded that graphs, which are regular in the matrix-sense, are indeed a fitting generalization of a graph with an absorbing state to a graph with an absorbing subset, respecting synchronizability. Theorem 4.1.12 can thus be understood as the corresponding generalization of Proposition 4.1.2.

Further, the considerations on the (Colored) Coupling from the Past algorithm can easily be transferred to the regular case since the following holds [Sen06, Thm. 4.7].

PROPOSITION 4.1.15. A regular Markov chain $(A; \mu, T)$ with finite state space has a unique invariant probability distribution π on A. In particular, $\pi = \pi|_{A_0} \oplus 0$, where $A_0 \subseteq A$ denotes the aperiodic recurrent class in A and $\pi|_{A_0}$ corresponds to the unique invariant probability distribution of the aperiodic irreducible Markov chain $(A_0, C, \gamma_0; \mu|_{A_0}, T|_{A_0})$.

Regularity in a matrix-sense provides a necessary and sufficient condition for a road-colorable graph to possess a synchronizing coloring. In particular, any extension of a synchronizable road-colored aperiodic irreducible graph to a graph that is regular in the matrix-sense is synchronizable. At first glance, regularity seems to provide no further information on the question of whether a given coloring is synchronizable. In the following section, we will see that the concept is indeed quite helpful when determining if a given road-coloring is synchronizable, i.e., the corresponding transition is asymptotically complete.

4.2 SYNCHRONIZABILITY AND THE LABEL PRODUCT

This section answers the question as to how the regularity of graphs helps in the quest to determine whether a given coloring is synchronizable. For an aperiodic irreducible road-colored graph (A, C, γ), we will present the so-called label product, a special graph product where the property of it being regular in the matrix-sense already implies synchronizability of the underlying graph (A, C, γ). It was introduced in [LM95, Def. 3.4.5]. We will later see that it can be understood as a special coupling of the corresponding Markov chain to itself.

Definition 4.2.1. Let (A, C, γ) be an aperiodic irreducible road-colored graph. We define its *label product* as the road-colored graph with vertices $A \times A$, colors C and transition function

$$\begin{aligned} \gamma^{\text{prod}}: \quad (A \times A) \times C &\to A \times A \\ ((a_1, a_2), c) &\mapsto (\gamma(a_1; c), \gamma(a_2; c)). \end{aligned}$$

The label product is denoted by $(A \times A, C, \gamma^{\text{prod}})$.

From the definition, a particular form of the label product directly emerges, whose significance becomes apparent when considering the results from the last section.

COROLLARY 4.2.2. Let (A, C, γ) be an aperiodic irreducible road-colorable graph. The set of vertices $A_0 := \{(a, a) \; : \; a \in A\} \subset A \times A$ forms a recurrent aperiodic class in the label product. In particular, $(A_0, C, \gamma_0^{\text{prod}})$, with $\gamma_0^{\text{prod}} := \gamma^{\text{prod}}|_{A_0 \times C}$, is a road-colored subgraph and can be identified with the original graph (A, C, γ).

Proof. Clearly, $\gamma^{\text{prod}}((a, a); c) = (b, b)$ if and only if $\gamma(a; c) = b$. So, $(A_0, C, \gamma_0^{\text{prod}})$ can be identified with (A, C, γ) and since the original graph is aperiodic and irreducible, A_0 forms an aperiodic recurrent class. $\square$

Example 4.2.3. Consider the toy example (Example 1.1.10) and set $a_{ij} := (a_i, a_j)$ for better readability. The form of the original graph is independent of the chosen coloring. In contrast, the label product depends on it – by definition (compare Figures 4.1 and 4.2).

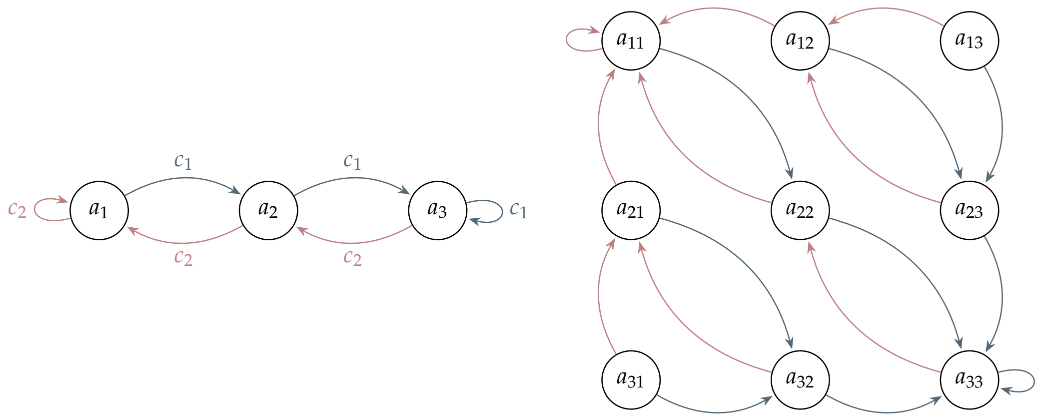


Figure 4.1: A synchronizable coloring and the corresponding label product

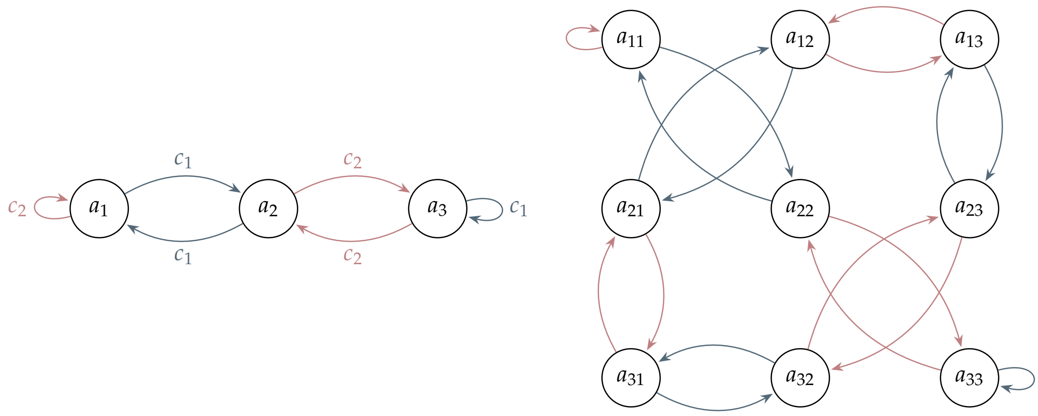


Figure 4.2: A non-synchronizable coloring and the corresponding label product

An important result on label products was introduced and shown in [GKL06, Lem. 5.2], [Lan03, Thm. 4.2.8]. It provides an answer to the question of whether a given coloring is synchronizable. From Theorem 4.1.12, our main theorem on the synchronizability of road-colorable graphs, we obtain this answer as a corollary. So, the result itself may not be new, but the perspective on it is.

COROLLARY 4.2.4. For an aperiodic irreducible road-colored graph $(A, C, \gamma; \mu, \nu)$, the following assertions are equivalent:

(a) The corresponding transition $J : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ is asymptotically complete.

(b) The original graph (A, C, γ) is synchronizable.

(c) The label product $(A \times A, C, \gamma^{\text{prod}})$ is synchronizable.

(d) The label product is regular in the matrix-sense.

Proof. $(a) \Leftrightarrow (b)$ follows from Theorem 2.3.10, $(b) \Leftrightarrow (c)$ from Corollary 4.1.13 and $(c) \Leftrightarrow (d)$ from Theorem 4.1.12. □

Considering Example 4.2.3 again, we can directly see that the label product illustrated in Figure 4.2 possesses two recurrent classes $\{a_{11}, a_{22}, a_{33}\}$ and $\{a_{12}, a_{13}, a_{21}, a_{23}, a_{31}, a_{32}\}$. So neither the label product nor the underlying original graph can be synchronizable. However, the vertices of the label product in Figure 4.1 form, apart from the aperiodic recurrent class $\{a_{11}, a_{22}, a_{33}\}$, four transient classes, namely $\{a_{13}\}$, $\{a_{12}, a_{23}\}$, $\{a_{21}, a_{32}\}$, and $\{a_{31}\}$.

4.2.1 *Pairwise synchronizability and couplings of Markov chains*

This section is to be understood as an interpretation of the label product from a probability theoretical viewpoint. In classical probability theory, a method exists to specify a common probability space for two distributions so that the original distributions correspond to the marginal distributions. This leads to the notion of *couplings*. We outline some basics on the coupling method, referring to [Kle08, Section 18.2], [Beh00, Section II.13] for more detail.

Definition 4.2.5. Let $(\Omega_1, \Sigma_1, \mathbb{P}_1)$ and $(\Omega_2, \Sigma_2, \mathbb{P}_2)$ be two probability spaces. Then, every probability measure $\mathbb{P}$ on $(\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2)$ such that $\mathbb{P}(\cdot \times \Omega_2) = \mathbb{P}_1(\cdot)$ and, respectively, $\mathbb{P}(\Omega_1 \times \cdot) = \mathbb{P}_2(\cdot)$ is called a *coupling of* $\mathbb{P}_1$ *and* $\mathbb{P}_2$.

There are different kinds of couplings. Of special interest is the so-called *coupled Markov chain* where two Markov chains, defined on the same probability space with identical transition matrix T, are coupled. Letting them start at the same time in different initial states, the following behavior of the resulting walks can be observed [Beh00, Section II.13]: Each of the walks, considered by itself, corresponds to the ordinary walk given by the transition matrix T. From the moment on, when the two walks meet for the first time, i.e., they occupy the same state at the same time, they stay together for the future.

Consider now two Markov chains with identical state space and transition matrix T. It can be easily verified that the transition matrix of the coupled Markov chain can be identified with $T \otimes T$ [Kle08, Ex. 18.11].

Such a coupling can also be expressed in terms of graph-theory [LM95, Section 3.4].

Definition 4.2.6. Let (A_1, E_1) and (A_2, E_2) be two directed graphs. Their *graph product* corresponds to the graph $(A_1 \times A_2, E_1 \times E_2)$.

In other words, the graph product of the canonical representations as directed graphs of two Markov chains with identical state spaces and transition matrices coincides with the canonical representation as a directed graph of the coupled Markov chain. In this sense, the label product (compare Definition 4.2.1) can be understood as the canonical extension of this concept when considering representations as road-colored graphs.

If the aperiodic irreducible Markov chains $(A; \mu_1, T)$ and $(A; \mu_2, T)$ are represented by a road-colored graph (A, C, γ), then the label product corresponds to the coupled Markov chain, respecting the coloring of the underlying graphs. It displays the possible transitions when starting in two vertices simultaneously. The coupled Markov chain with respect to the road-coloring is then represented by the label product $(A \times A, C, \gamma^{\text{prod}}; \mu_1 \otimes \mu_2, \nu)$. Suppose now that the label product is regular in the matrix-sense with unique aperiodic recurrent class $A_0 \subset A \times A$. Then every transient vertex in $(A \times A) \setminus A_0$, i.e., every pair of two different vertices in the original graph, is connected to a pair of the form $(a, a) \in A_0$. In this sense, regularity in the matrix-sense of the label product can be understood as pairwise synchronizability.

COROLLARY 4.2.7. For an aperiodic irreducible road-colored graph (A, C, γ), the following assertions are equivalent:

(a) The original graph is pairwise synchronizable.

(b) The label product is regular in the matrix-sense.

Following this line of argumentation, the results from Chapter 3 can also be interpreted in this light. Defining the n-fold label product and the n-fold couplings recursively, one step in the (Colored) Coupling from the Past algorithm in Section 3.1.1 can be understood as one transition in the n-fold label product of (A, C, γ). Due to the general form of Theorem 4.1.12, the above results can be directly transferred to an n-fold label product. The algorithm corresponds then to a search of a path, mapping an n-tuple of pairwise different vertices into the unique recurrent aperiodic class. It terminates whenever such a path is found.

4.2.2 *Reflections on the form of the label product*

Before we delve into the non-commutative setting in the next section, we take a closer look at the precise form of the label product, starting from the monochromatic (transition) matrices of the underlying original graph. This will provide some understanding of how the shape of the label product depends on the choice of coloring.

Let $(A, C, \gamma; \mu_i, \nu)_{i=1,2}$ be an aperiodic irreducible road-colored graph with probability distribution $\nu = (\nu_c)_{c \in C}$ on C, representing two aperiodic irreducible Markov chains with identical state space and transition probabilities. By $\{M_c\}_{c \in C}$, we denote the monochromatic matrices corresponding to the coloring (compare Definition 1.2.13).

Definition 4.2.8. In the just described setting, we denote by M_c^{prod} the *monochromatic matrices of the label product* and by $T_c^{\text{prod}} := \nu_c M_c^{\text{prod}}$ the corresponding *monochromatic transition matrices*.

We set $n := |A|$. Since $\mathrm{M}_n \otimes \mathrm{M}_n \simeq \mathrm{M}_n(\mathrm{M}_n) \simeq \mathrm{M}_{n^2}$ (compare Example 2.1.7), every element in the tensor product $x \otimes y \in \mathrm{M}_n \otimes \mathrm{M}_n$ can be identified with an $n \times n$-block matrix $\left(y_{ij} \cdot x\right)_{i,j=1}^n \in \mathrm{M}_{n^2}$. The following elaborations are to be understood in this sense.

Lemma 4.2.9. Let $(A, C, \gamma; \mu_i, \nu)_{i=1,2}$ be defined as above. The monochromatic matrices M_c^{prod} of the label product $(A \times A, C, \gamma^{\text{prod}}; \mu_1 \otimes \mu_2, \nu)$ can be identified with $M_c \otimes M_c$, for every color $c \in C$. The monochromatic transition matrices then correspond to $T_c^{\text{prod}} = \sqrt{\nu_c} M_c \otimes \sqrt{\nu_c} M_c$.

Proof. Suppose that the original graph has $n := |A|$ vertices. Then, the label product has $|A \times A| = n^2$ vertices. We fix $(a_1, a_2), (b_1, b_2) \in A \times A$ and show that for every color $c \in C$ the $n^2 \times n^2$-matrices M_c^{prod} and $M_c \otimes M_c$ match entrywise. On the one hand, $M_c^{\text{prod}} = \sum_{\substack{(a_1,a_2),\\(b_1,b_2)\in A\times A}} \left(M_c^{\text{prod}}\right)_{(a_1,a_2)(b_1,b_2)}$, where

$$\begin{aligned}
\left(M_c^{\text{prod}}\right)_{(a_1,a_2)(b_1,b_2)} &= \begin{cases} 1, & \gamma^{\text{prod}}((a_1,a_2);\, c) = (b_1,b_2) \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & (M_c)_{a_1b_1} = 1 = (M_c)_{a_2b_2} \\ 0, & \text{otherwise} \end{cases} \\
&= (M_c)_{a_1b_1} \cdot (M_c)_{a_2b_2} \\
&= \left((M_c)_{a_2b_2} \cdot M_c\right)_{a_1b_1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
M_c \otimes M_c &= \left((M_c)_{a_2b_2} \cdot M_c\right)_{a_2,b_2\in A} \\
&= \left((M_c)_{a_2b_2} \cdot \left((M_c)_{a_1b_1}\right)_{a_1,b_1\in A}\right)_{a_2,b_2\in A} \\
&= \Bigg(\bigg(\underbrace{\left((M_c)_{a_2b_2} \cdot M_c\right)_{a_1b_1}}_{=\left(M_c^{\text{prod}}\right)_{(a_1,a_2)(b_1,b_2)}}\bigg)_{a_1,b_1\in A}\Bigg)_{a_2,b_2\in A}.
\end{aligned}$$

We can thus identify $M_c^{\text{prod}} \in \mathrm{M}_{n^2}$ and $M_c \otimes M_c \in \mathrm{M}_n \otimes \mathrm{M}_n$.

By construction, ν is a probability distribution on the colors C. It does not change when starting in two vertices simultaneously. So, the transition matrix of the coupled Markov chain is given by

$$T^{\text{prod}} = \sum_{c\in C} \nu_c \cdot M_c^{\text{prod}}.$$

□

The following assertion is a direct consequence.

Corollary 4.2.10. For a color $c \in C$, let T_c denote the monochromatic transition matrices of the original road-colored graph. The transition matrix T^{prod} of the label product can then be identified with

$$T^{\text{prod}} = \sum_{c\in C} \frac{1}{\nu_c} T_c \otimes T_c.$$

Example 4.2.11. Consider the label product corresponding to our toy example (compare Figure 4.1 in Example 4.2.3). Then, the monochromatic transition matrices of the label product correspond to

$$
T_1^{\text{prod}} = \frac{1}{3} \cdot
\begin{array}{rc}
 & \begin{array}{ccccccccc} a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} & a_{31} & a_{32} & a_{33} \end{array} \\
\begin{array}{r} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{21} \\ a_{22} \\ a_{23} \end{array} &
\left(\begin{array}{ccc|ccc|ccc}
 & & & & 1 & & & & \\
 & & & & & 1 & & & \\
 & & & & & 1 & & & \\
\hline
 & & & & & & & 1 & \\
 & & & & & & & & 1 \\
 & & & & & & & & 1 \\
\hline
 & & & & & & & 1 & \\
 & & & & & & & & 1 \\
 & & & & & & & & 1
\end{array}\right)
\end{array}
$$

$$
= \frac{1}{3} \cdot \begin{pmatrix} & 1 & \\ & & 1 \\ & & 1 \end{pmatrix} \otimes \begin{pmatrix} & 1 & \\ & & 1 \\ & & 1 \end{pmatrix} = \frac{1}{3} \cdot T_1 \otimes T_1,
$$

$$
T_2^{\text{prod}} = \frac{2}{3} \cdot
\begin{array}{rc}
 & \begin{array}{ccccccccc} a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} & a_{31} & a_{32} & a_{33} \end{array} \\
\begin{array}{r} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{21} \\ a_{22} \\ a_{23} \end{array} &
\left(\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
1 & & & & & & & & \\
 & 1 & & & & & & & \\
\hline
1 & & & & & & & & \\
1 & & & & & & & & \\
 & 1 & & & & & & & \\
\hline
 & & & 1 & & & & & \\
 & & & 1 & & & & & \\
 & & & & 1 & & & &
\end{array}\right)
\end{array}
$$

$$
= \frac{2}{3} \cdot \begin{pmatrix} 1 & & \\ 1 & & \\ & 1 & \end{pmatrix} \otimes \begin{pmatrix} 1 & & \\ 1 & & \\ & 1 & \end{pmatrix} = \frac{2}{3} \cdot T_2 \otimes T_2.
$$

4.3 ASYMPTOTIC COMPLETENESS AND REGULARITY

As we have seen, in the classical setting, the label product of a road-colored graph is a useful tool for the determination of whether the corresponding transition is asymptotically complete. In this section, we will present the non-commutative concept of a dual extended transition operator. It was first introduced by R. Gohm [Goh04], who showed that it provides a new characterization of asymptotic completeness, which can indeed be understood as a non-commutative version of the assertion that the label product is regular in the matrix-sense [Goh04, Thm. 2.7.4], [GKL06].

Definition 4.3.1. Let $\mathcal{A}$ be a von Neumann algebra. A completely positive and identity-preserving operator $S : \mathcal{A} \to \mathcal{A}$ is called *regular*, if it is σ-wop continuous and if there exists a normal state φ, such that for all $a \in \mathcal{A}$ holds $\sigma\text{-wop-}\lim_{n\to\infty} S^n(a) = \varphi(x)\mathbb{1}$.

It has been shown that the regularity of the transition operator is a direct consequence of the asymptotic completeness of the transition [GKL06, Prop. 1.6].

> PROPOSITION 4.3.2. Let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ be quantum probability spaces and let $J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$ be an asymptotically complete transition. Then, the corresponding transition operator T_ψ is regular. In particular, T_ψ and J are irreducible.

This assertion can be understood as a non-commutative version of Corollary 4.1.8. However, for the other direction, the regularity of the transition operator is not sufficient to show the asymptotic completeness of the corresponding transition. This coincides with the intuition gained from the classical case. Here, the transition operator corresponds to the classical transition matrix. The question of whether a synchronizing word exists and the transition is thus asymptotically complete depends solely on the chosen coloring. However, the coloring is not recognizable from the transition matrix of the original graph. All information is contained in the transition function γ and thus in the corresponding transition. Only the label product displays the coloring in the corresponding transition matrix. This motivates the idea to consider an extension of the transition, respectively, the transition operator.

4.3.1 The dual extended transition operator

At the end of Chapter 3, we announced a third possibility to translate the idea of colors into the non-commutative setting: This section presents the dual extended transition operator and discusses its application in the context of asymptotic completeness of a transition corresponding to a one-sided quantum Markov process. It is further illustrated that the corresponding Kraus operators can indeed be understood as non-commutative colors. We follow the elaborations and construction as presented by R. Gohm, B. Kümmerer and T. Lang in [GKL06].

Let $\mathcal{A}$ and $\mathcal{C}$ be finite-dimensional von Neumann algebras and consider quantum probability spaces $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$. By $J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$ is denoted a transition with corresponding transition operator $T_\psi : (\mathcal{A}, \varphi) \to (\mathcal{A}, \varphi)$. The GNS-construction of $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ leads to Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ with cyclic vectors $\xi_\varphi \in \mathcal{H}$ and $\eta_\psi \in \mathcal{K}$, respectively, such that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{C} \subseteq \mathcal{B}(\mathcal{K})$ with

$$\begin{aligned} \varphi(a) &= \langle a\xi_\varphi, \xi_\varphi \rangle_{\mathcal{H}} \quad \text{for all } a \in \mathcal{A} \\ \psi(c) &= \langle c\eta_\psi, \eta_\psi \rangle_{\mathcal{K}} \quad \text{for all } c \in \mathcal{C}. \end{aligned}$$

Lemma 4.3.3. In the just described setting, the transition J induces an isometry on the Hilbert spaces

$$\begin{aligned} v : \quad \mathcal{H} &\to \mathcal{H} \otimes \mathcal{K} \\ a\xi_\varphi &\mapsto J(a)\xi_\varphi \otimes \eta_\psi, \quad \text{for all } a \in \mathcal{A}. \end{aligned}$$

Proof. This can be verified in a few lines. For all $a, b \in \mathcal{A}$ holds

$$\begin{aligned} \langle J(a)\xi_\varphi \otimes \eta_\psi, J(b)\xi_\varphi \otimes \eta_\psi \rangle_{\mathcal{H}\otimes\mathcal{K}} &= \langle \xi_\varphi \otimes \eta_\psi, J(a^*b)\xi_\varphi \otimes \eta_\psi \rangle_{\mathcal{H}\otimes\mathcal{K}} \\ &= \varphi \otimes \psi(J(a^*b)) \\ &= \varphi(a^*b) \\ &= \langle a\xi_\varphi, b\xi_\varphi \rangle_{\mathcal{H}}, \end{aligned}$$

where we used that J is a *-homomorphism in the second step and $\varphi = (\varphi \otimes \psi) \circ J$ in the second to last step. Consequently, v is well-defined. Indeed, if $a\xi_\varphi = b\xi_\varphi$, then follows with the above:

$$0 = \|a\xi_\varphi - b\xi_\varphi\|_{\mathcal{H}} = \|J(a)\xi_\varphi \otimes \eta_\psi - J(b)\xi_\varphi \otimes \eta_\psi\|_{\mathcal{H}\otimes\mathcal{K}}$$

and hence $J(a)\xi_\varphi \otimes \eta_\psi = J(b)\xi_\varphi \otimes \eta_\psi$.

By definition we have $\langle va\xi_\varphi, vb\xi_\varphi \rangle_{\mathcal{H}\otimes\mathcal{K}} = \langle J(a)\xi_\varphi \otimes \eta_\psi, J(b)\xi_\varphi \otimes \eta_\psi \rangle_{\mathcal{H}\otimes\mathcal{K}}$, so with the above follows also that v is an isometry. □

In this setting, we introduce the following definition.

Definition 4.3.4. We define the *dual extended transition operator associated to the transition* $J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$ as

$$\begin{array}{rccl} Z' : & \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & x & \mapsto & v^*(x \otimes \mathbb{1}_{\mathcal{K}})v. \end{array}$$

Let now $(\eta_k)_k \subset \mathcal{K}$ be an orthonormal basis of $\mathcal{K}$, then there are linear operators $(a_k)_k \in \mathcal{B}(\mathcal{H})$, such that for all $\xi \in \mathcal{H}$ holds $v\xi = \sum_k a_k\xi \otimes \eta_k$. This leads to a Kraus decomposition of the dual extended transition operator with respect to the chosen basis:

$$Z'(x) = \sum_k a_k^* x a_k, \quad \text{for all } x \in \mathcal{B}(\mathcal{H}).$$

Z' is completely positive (compare Theorem 2.1.13) and it is identity-preserving, since v is an isometry.

Example 4.3.5. Consider an automorphism T of $\mathcal{A} \otimes \mathcal{C}$, given by $T(x) = u^*(x)u$ for all $x \in \mathcal{A} \otimes \mathcal{C}$, where u is a unitary in $\mathcal{A} \otimes \mathcal{C}$ and let $i_0 : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ be the canonical embedding $a \mapsto a \otimes \mathbb{1}$. Then $J = T \circ i_0$ is a transition (compare the comment at the end of Section 2.3.1). In particular, for every $a \in \mathcal{A}$ holds

$$J(a) = T \circ i_0(a) = T(a \otimes \mathbb{1}) = u^*(a \otimes \mathbb{1})u.$$

We present this in more detail for matrix algebras with complex entries, i.e., $\mathcal{A} = \mathrm{M}_n$ and $\mathcal{C} = \mathrm{M}_d$ with normal faithful states φ and ψ, respectively. The GNS-construction leads to the Hilbert spaces $\mathcal{H}_\varphi \simeq \mathrm{M}_n$ and $\mathcal{H}_\psi \simeq \mathrm{M}_d$. Then any unitary $u \in M_n \otimes M_d \simeq \mathrm{M}_d(\mathrm{M}_n)$ is a block matrix of the form:

$$u = \begin{pmatrix} u_{11} & \cdots & u_{1d} \\ \vdots & & \vdots \\ u_{d1} & \cdots & u_{dd} \end{pmatrix}$$

where $u_{ij} \in \mathrm{M}_n$. Then, the transition $J : \mathrm{M}_n \to \mathrm{M}_n \otimes \mathrm{M}_d$ is given by

$$J(a) = u^* \begin{pmatrix} a & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & a \end{pmatrix} u =: \begin{pmatrix} T_{11}(a) & \cdots & T_{1d}(a) \\ \vdots & & \vdots \\ T_{d1}(a) & \cdots & T_{dd}(a) \end{pmatrix}$$

The T_{ij} are then operators on M_n. We obtain a ψ-orthonormal basis $(f_{ij})_{i,j=1}^d$ of M_n from a system of matrix units $(e_{ij})_{i,j=1}^d$ by $f_{ij} := \frac{1}{\psi(e_{ij})^{1/2}} e_{ij}$ (compare Example 2.1.9). The Kraus operators of the dual extended transition operator Z' are thus given by

$$a_{ij} := \frac{1}{\psi(e_{ij})^{1/2}} T_{ij}, \quad \text{for } i,j \in \{1,\dots,d\}.$$

So, $Z'(x) = \sum_{i,j=1}^d a_{ij}^* x a_{ij} = \sum_{i,j=1}^d \frac{1}{\psi(e_{ij})} T_{ij}^* x T_{ij}$ for $x \in \mathcal{B}(\mathcal{H}_\varphi) \simeq M_n \otimes M_n$.

We turn again to the classical framework and consider an aperiodic irreducible road-colored graph $(A, C, \gamma; \mu, \nu)$. We follow the elaborations in [GKL06, Section 5] and find, analogously to the construction in Section 3.3.1, that the dual extended transition operator Z' is of the form

$$\begin{aligned} Z' : \quad \mathrm{M}_n &\to \mathrm{M}_n \\ x &\mapsto \sum_{k=1}^d \nu_k \cdot M_{\varphi,k}^* x M_{\varphi,k} \end{aligned}$$

where $M_{\varphi,k}$ corresponds to the representation of the *monochromatic matrix to the color $c_k \in C$, with respect to the φ-orthonormal basis*. It is then further shown that Z' is regular in the sense of Definition 4.3.4 if and only if the label product of $(A, C, \gamma; \mu, \nu)$ is regular in the matrix-sense [GKL06, Thm. 5.1], [Lan03, Section 4.5]. So, with respect to the tracial state τ on M_n, the monochromatic matrices of the underlying road-colored graph form the Kraus operators of the dual extended transition operator Z'. This motivates the idea to understand the Kraus operators of Z' as *non-commutative colors* and the dual extended transition operator as a non-commutative version of the label product.

4.3.2 *Regularity of the dual extended transition operator*

This section introduces the repeatedly announced main result of [GKL06], namely that the regularity of the dual extended transition operator associated with the transition of a one-sided quantum Markov process is an equivalent characterization of asymptotic completeness of said transition. The following results on operators on $\mathcal{B}(\mathcal{H})$ will prove to be handy. They were formulated and shown in [GKL06, Prop. 3.2, 3.3]. We state them without proof.

PROPOSITION 4.3.6. Let $S : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a completely positive identity-preserving operator and $\xi \in \mathcal{H}$ a unit vector, such that the induced vector state on $\mathcal{B}(\mathcal{H})$, denoted by $\omega_\xi(x) := \langle x\xi, \xi\rangle$ is invariant, i.e., $\omega_\xi \circ S = \omega_\xi$. The following assertions are equivalent:

(a) S is regular with invariant state ω_ξ.

(b) The fixed space of S is one-dimensional, i.e., it consists of multiples of $\mathbb{1}$.

(c) Let p_ξ denote orthogonal projection onto $\mathbb{C}\xi$, then stop-$\lim_{n\to\infty} S^n(\mathbb{1} - p_\xi) = 0$.

This reminds of the classical setting. If the transition matrix of a Markov chain is regular in the sense of Definition 4.1.9, then there exists a unique invariant probability measure that "lives" only on the recurrent part (compare Proposition 4.1.15).

If the underlying Hilbert space $\mathcal{H}$ is finite-dimensional, there is a useful further characterization of regularity.

PROPOSITION 4.3.7. Consider the same setting as in Proposition 4.3.6 under the assumption that the Hilbert space $\mathcal{H}$ is finite-dimensional and a Kraus decomposition of S is given by $S(x) = \sum_{i=1}^{k} a_i^* x a_i$, for some $a_i \in \mathcal{B}(\mathcal{H})$. Then the following assertions are equivalent:

(a) S is regular with invariant state φ_ξ.

(b) The vector ξ is cyclic for $\{a_i^* \,:\, 1 \le i \le k\}$, i.e., the Hilbert space $\mathcal{H}$ is spanned by the vectors $\{a_{i_1}^* \cdots a_{i_n}^* \xi \,:\, n \in \mathbb{N}, 1 \le i_1, \ldots, i_n \le k\}$.

The following theorem provides the connection between the asymptotic completeness of a transition and the regularity of the corresponding dual extended transition operator. For the original proof of the theorem, we refer the reader to [GKL06, Thm. 4.3]. We will present an alternative approach in Section 4.3.3.

THEOREM 4.3.8. Let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ be quantum probability spaces with finite-dimensional von Neumann algebras $\mathcal{A}$ and $\mathcal{C}$. The following assertions are equivalent:

(a) The transition $J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$ is asymptotically complete.

(b) The dual extended transition operator associated to J is regular.

Example 4.3.9. We present a concrete calculation for the setting described in Examples 2.1.9 and 4.3.5, with $\mathcal{A} = \mathcal{C} = \mathrm{M}_2$ and $\varphi = \tau = \psi$ the normalized trace on M_2. It was introduced during a workshop with members of the C*-AG [Die+18]. Let an automorphism of $\mathcal{A} \otimes \mathcal{C} = \mathrm{M}_2 \otimes \mathrm{M}_2$ be given by the unitary

$$u = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & c & s & 0 \\ \hline 0 & -s & c & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) =: \left(\begin{array}{c|c} u_{11} & u_{12} \\ \hline u_{21} & u_{22} \end{array}\right) \in \mathrm{M}_2 \otimes \mathrm{M}_2,$$

with $u_{ij} \in \mathcal{A} = \mathrm{M}_2$ and $c := \cos\vartheta$, $s := \sin\vartheta$. The transition J from M_2 to $\mathrm{M}_2 \otimes \mathrm{M}_2$ is then given by

$$\begin{aligned} J(a) &= u^*(a \otimes \mathbb{1})u \\ &= \left(\begin{array}{c|c} u_{11}^* a u_{11} + u_{21}^* a u_{21} & u_{11}^* a u_{12} + u_{21}^* a u_{22} \\ \hline u_{12}^* a u_{11} + u_{22}^* a u_{21} & u_{12}^* a u_{12} + u_{22}^* a u_{22} \end{array}\right) \\ &:= \left(\begin{array}{c|c} T_{11}(a) & T_{12}(a) \\ \hline T_{21}(a) & T_{22}(a) \end{array}\right). \end{aligned}$$

Then, the Kraus operators of the dual extended transition operator correspond to $a_{ij} := \frac{1}{\sqrt{2}} T_{ij}$ and hence, for $x \in \mathcal{B}(\mathcal{H}) \simeq \mathrm{M}_2 \otimes \mathrm{M}_2$, we obtain:

$$v = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}, \qquad Z'(x) = v^* x v = \sum_{ij=1}^{2} \frac{1}{2} T_{ij}^* x T_{ij}.$$

We use Proposition 4.3.7 and show that $\mathbb{1}$ is cyclic for $\{a_{ij}^* \,:\, 1 \leq i, j \leq 2\}$, if $s \neq 0$ and $c \neq 0$. To this aim, we determine the adjoints $a_{ij}^* : \mathrm{M}_2 \to \mathrm{M}_2$:

$$\begin{aligned} a_{11}^*(\,\cdot\,) &= \frac{1}{2}\left(u_{11} \cdot u_{11}^* + u_{21} \cdot u_{21}^*\right), \quad & a_{21}^*(\,\cdot\,) &= \frac{1}{2}\left(u_{12} \cdot u_{11}^* + u_{22} \cdot u_{21}^*\right), \\ a_{21}^*(\,\cdot\,) &= \frac{1}{2}\left(u_{12} \cdot u_{11}^* + u_{22} \cdot u_{21}^*\right), \quad & a_{22}^*(\,\cdot\,) &= \frac{1}{2}\left(u_{12} \cdot u_{12}^* + u_{22} \cdot u_{22}^*\right). \end{aligned}$$

We can easily see that a single application of the a_{ij}^* to $\mathbb{1}$ leads to two dimensions:

$$\left.\begin{array}{ll} a_{11}^*(\mathbb{1}) = \frac{1}{2}\begin{pmatrix} 1+s^2 & 0 \\ 0 & c^2 \end{pmatrix}, & a_{12}^*(\mathbb{1}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\[2ex] a_{21}^*(\mathbb{1}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & a_{22}^*(\mathbb{1}) = \frac{1}{2}\begin{pmatrix} c^2 & 0 \\ 0 & 1+s^2 \end{pmatrix}. \end{array}\right\} \text{2 dimensions}$$

With $a_{12}^*(a_{11}^*(\mathbb{1}))$ and $a_{21}^*(a_{11}^*(\mathbb{1}))$, we obtain the remaining two dimensions, i.e., elements on the off-diagonal:

$$\left.\begin{array}{l} a_{12}^*\begin{pmatrix} 1+s^2 & 0 \\ 0 & c^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & s(c^2-1+s^2) \\ 0 & 0 \end{pmatrix}, \\[2ex] a_{21}^*\begin{pmatrix} 1+s^2 & 0 \\ 0 & c^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ s(c^2-1+s^2) & 0 \end{pmatrix}. \end{array}\right\} \text{2 dimensions}$$

Consequently, due to Proposition 4.3.7, the dual extended transition operator Z' is regular and with Theorem 4.3.8 follows asymptotic completeness of the transition J.

4.3.3 *A Hilbert space formulation*

We will now transfer the concept of the dual extended transition operator into a Hilbert space setting and show that the isometry, induced by a transition J, is asymptotically complete in a fitting sense if and only if the dual extended transition operator Z' is regular. This result not only implies the assertion in Theorem 4.3.8, but also gives rise to the possibility of a further theory of the dual extended transition operator on Hilbert spaces. It corresponds to a joint work during a workshop of the C*-AG [Die+16].

Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces. For later purposes we choose $\mathcal{L}$ do be infinite-dimensional. For an isometry $v : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ we define

$$Z'(x) := v^*(x \otimes \mathbb{1}_{\mathcal{K}})v,$$

for all $x \in \mathcal{B}(\mathcal{H})$. Let $w : \mathcal{K} \otimes \mathcal{L} \to \mathcal{L}$ be an isometry. We set

$$v_+ := (\mathbb{1}_{\mathcal{H}} \otimes w)(v \otimes \mathbb{1}_{\mathcal{L}}) \in \mathcal{B}(\mathcal{H} \otimes \mathcal{L}).$$

Since $w^*w = \mathbb{1}_{\mathcal{L}}$, the following holds for all $x \in \mathcal{B}(\mathcal{H})$:

$$\begin{aligned} v_+^*\,(x\otimes \mathbb{1}_{\mathcal{L}})\,v_+ &= (v^*\otimes \mathbb{1}_{\mathcal{L}})\,(\mathbb{1}_{\mathcal{H}}\otimes w^*)\,(x\otimes \mathbb{1}_{\mathcal{L}})\,(\mathbb{1}_{\mathcal{H}}\otimes w)\,(v\otimes \mathbb{1}_{\mathcal{L}}) \\ &= (v^*\otimes \mathbb{1}_{\mathcal{L}})\,(x\otimes w^*w)\,(v\otimes \mathbb{1}_{\mathcal{L}}) \\ &= \Big(v^*\,(x\otimes \mathbb{1}_{\mathcal{K}})\,v\Big)\otimes \mathbb{1}_{\mathcal{L}} \\ &= Z'\,(x)\otimes \mathbb{1}_{\mathcal{L}}\,. \end{aligned}$$

Iteratively, for $n \in \mathbb{N}$, this leads to $(v_+^*)^n\,(x\otimes \mathbb{1}_{\mathcal{L}})\,v_+^n = (Z')^n\,(x)\otimes \mathbb{1}_{\mathcal{L}}$.

The definition of asymptotic completeness needs to be adapted to this setting.

Definition 4.3.10. Let $\xi_0 \in \mathcal{H}$ and p_0 the corresponding orthogonal projection onto $\mathbb{C}\xi_0$. The isometry v is called ξ_0*-asymptotically complete*, if

$$\lim_{n\to\infty} \|v_+^n\xi_+ - (p_0\otimes \mathbb{1}_{\mathcal{L}})\,v_+^n\xi_+\|_{\mathcal{H}\otimes\mathcal{L}} = 0$$

holds for every $\xi_+ \in \mathcal{H}\otimes\mathcal{L}$.

The following proposition can now be proved.

PROPOSITION 4.3.11. For $\xi_0 \in \mathcal{H}$, the following assertions are equivalent:

(a) The isometry v is ξ_0-asymptotically complete.

(b) For all $x \in \mathcal{B}(\mathcal{H})$ holds $\lim_{n\to\infty}(Z')^n\,(x) = \langle x\xi_0, \xi_0\rangle \cdot \mathbb{1}_{\mathcal{H}}$.

Proof. In a first step we show that for arbitrary $\xi, \xi' \in \mathcal{H}$ and $\eta, \eta' \in \mathcal{L}$ holds:

$$\begin{aligned} \big\langle (Z')^n\,(p_0)\,\xi, \xi'\big\rangle_{\mathcal{H}}\,\langle \eta, \eta'\rangle_{\mathcal{L}} &= \big\langle\big((Z')^n\,(p_0)\otimes \mathbb{1}_{\mathcal{L}}\big)\,(\xi\otimes\eta)\,, \xi'\otimes\eta'\big\rangle_{\mathcal{H}\otimes\mathcal{L}} \\ &= \big\langle (v_+^*)^n\,(p_0\otimes \mathbb{1}_{\mathcal{L}})\,v_+^n\,(\xi\otimes\eta)\,, \xi'\otimes\eta'\big\rangle_{\mathcal{H}\otimes\mathcal{L}} \\ &= \big\langle (p_0\otimes \mathbb{1}_{\mathcal{L}})\,v_+^n\,(\xi\otimes\eta)\,, v_+^n\,(\xi'\otimes\eta')\big\rangle_{\mathcal{H}\otimes\mathcal{L}} \\ &= \big\langle (p_0\otimes \mathbb{1}_{\mathcal{L}})\,v_+^n\,(\xi\otimes\eta)\,, (p_0\otimes \mathbb{1}_{\mathcal{L}})\,v_+^n\,(\xi'\otimes\eta')\big\rangle_{\mathcal{H}\otimes\mathcal{L}}\,. \end{aligned}$$

$(a) \Rightarrow (b)$. Suppose that the isometry ν is ξ_0-asymptotically complete. Setting $\xi = \xi'$ and $\eta = \eta'$ such that $\|\eta\| = 1$, then with the above calculation follows:

$$\big\langle (Z')^n\,(p_0)\,\xi, \xi\big\rangle = \|(p_0\otimes \mathbb{1}_{\mathcal{L}})\,v_+^n\,(\xi\otimes\eta)\|^2 \xrightarrow[n\to\infty]{} \|\xi\otimes\eta\|^2 = \|\xi\|^2 = \langle \mathbb{1}_{\mathcal{H}}\xi, \xi\rangle\,.$$

Together with Proposition 4.3.6 (c), this implies assertion (b).

$(b) \Rightarrow (a)$. Setting $\xi = \xi'$ and $\eta = \eta'$, the above calculation implies

$$\|(p_0 \otimes \mathbb{1}_{\mathcal{L}})\, v_+^n\, (\xi \otimes \eta)\|^2 = \left\langle (Z')^n\, (p_0)\, \xi, \xi \right\rangle \langle \eta, \eta \rangle \underset{n\to\infty}{\longrightarrow} \|\xi \otimes \eta\|^2 .$$

So, for all $\xi \otimes \eta \in \mathcal{L}$ we have $\lim_{n\to\infty} \|(p_0 \otimes \mathbb{1}_{\mathcal{L}})\, v_+^n\, (\xi \otimes \eta)\| = \|\xi \otimes \eta\|$. Since p_0 is an orthogonal projection, this is equivalent to the ξ_0-asymptotic completeness of the isometry v.

□

This construction corresponds to a slightly more general setting than we have previously considered within the scope of our one-sided quantum Markov processes (compare Section 2.3). Let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ be quantum probability spaces with finite-dimensional $\mathcal{A}$ and $\mathcal{C}$ and let $(\mathcal{A}^+, \varphi^+, J^+; \mathcal{A}, i^+)$ be a one-sided quantum Markov process. The GNS-construction of $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ leads to Hilbert spaces with cyclic vectors $(\mathcal{H}, \xi_\varphi)$ and $(\mathcal{K}, \eta_\psi)$. Let $(\mathcal{K}^+, \eta^+) := \bigotimes_{\mathbb{N}_0} (\mathcal{K}, \eta_\psi)$ denote the *infinite Hilbert space tensor product of $\mathcal{K}$ with respect to η_ψ*, i.e., the completion of the set of linear combinations of elementary tensors of the form $\eta_0 \otimes \eta_1 \otimes \cdots \otimes \eta_n \otimes \eta_\psi \otimes \eta_\psi \otimes \cdots$ for an $n \in \mathbb{N}$ (compare, for example, [Bla06, Section III.3.1]). In particular, $(\mathcal{K}^+, \eta^+)$ arises from the GNS-construction of $(\mathcal{C}^+, \psi^+)$. A transition $J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$ induces an isometry $v : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$, whereas $J^+ : (\mathcal{A}^+, \varphi^+) \to (\mathcal{A}^+, \varphi^+)$ induces an isometry v^+ on $\mathcal{H} \otimes \mathcal{K}^+$:

$$\begin{array}{rcl} v_+ : \mathcal{H} \otimes \mathcal{K}^+ & \to & \mathcal{H} \otimes \mathcal{K}^+ \\ \xi \otimes \eta_0 \otimes \cdots \otimes \eta_n \otimes \eta_\psi \otimes \eta_\psi \otimes \cdots & \mapsto & \underbrace{v\,(\xi)}_{\in \mathcal{H} \otimes \mathcal{K}} \otimes \eta_0 \otimes \cdots \otimes \eta_n \otimes \eta_\psi \otimes \eta_\psi \otimes \cdots . \end{array}$$

So, Theorem 4.3.8 is a direct consequence of Proposition 4.3.11, when setting $\mathcal{L} := \mathcal{K}^+$.

5 SYNCHRONIZABILITY OF INTEGER RANDOM WALKS

THERE IS NO GREATER AGONY
THAN BEARING AN UNTOLD STORY INSIDE YOU.

— Maya Angelou

Until now, we have considered Markov chains with finite state space, and hence road-colored graphs with finitely many vertices. Theorem 2.3.10 introduced the idea to understand the existence of a synchronizing word as an almost sure convergence of the transition functions towards a function depending solely on the chosen color sequence. In the finite setting, this is a direct consequence of the facts that asymptotic completeness of the transition is equivalent to the existence of a synchronizing word and that almost every color sequence possesses every finite (synchronizing) word infinitely often. This chapter generalizes the assertions of Theorem 2.3.10 to a class of infinite road-colored graphs relying on the notation and elaborations introduced in Section 2.3.2. In the literature, there is no obvious answer to the question of what is the best approach to transfer the concept of a synchronizing word for finitely many vertices to an infinite setting. In automata theory, generalizations to various infinite systems have been proposed over the years. For example, in [DMS11], where infinite synchronizing words for finite probabilistic automata are introduced, or [Doy+14], where two classes of infinite state spaces that can be synchronized in finite time are presented. We follow the notion of synchronizability for road-colored graphs with countable sets of vertices and colors, introduced by F. Haag in [Haa06]. Here, synchronizability means that every finite subset of vertices in the countable vertex set A possesses a synchronizing word. This definition proves to be the right generalization for our considerations as it is still equivalent to asymptotic completeness of the corresponding transition [Haa06, Thm. 3.2.9].

Based on an example in the preprint [GHK19], Section 5.2 introduces a sufficient condition for a special class of infinite road-colored graphs so that asymptotic completeness and almost sure convergence of the transition functions are equivalent. This condition, in turn, is satisfied whenever positive recurrence is given, as is shown in Section 5.3.

5.1 SYNCHRONIZABILITY AND INFINITE GRAPHS

Let (A, C, γ) be an infinite road-colored graph, as introduced in Definition 1.1.22. We define synchronizability as in [Haa06].

Definition 5.1.1. An infinite road-colored graph (A, C, γ) is *synchronizable* if every finite subset $B \subset A$ possesses a synchronizing word. It is *pairwise synchronizable* if for any two vertices $a, b \in A$ exists a synchronizing word.

It is crucial to note that the vertex into which a finite subset $B \subset A$ is mapped does not need to be an element in B.

COROLLARY 5.1.2. Let (A, C, γ) be an infinite road-colored graph. The following assertions are equivalent:

(a) (A, C, γ) is synchronizing.

(b) (A, C, γ) is pairwise synchronizing.

Proof. The proof corresponds to the same inductive argument as its analog for finite graphs in Proposition 1.1.16. □

In Lemma 1.1.12, it has been shown that in a finite graph, a synchronizing word s remains synchronizing if we add one more color $c \in C$ to it. Regardless of whether it is added to the end (sc) or in front of the word (cs). When dealing with infinite graphs, the situation presents itself somewhat differently. If we consider a finite set $B \subset A$, with synchronizing word s, The word sc is still synchronizing for B. But for cs this may no longer be true, since $\gamma(B; c)$ produces a new finite set, which does not need to be synchronizable by s. More generally, different finite subsets of A may have different synchronizing words. This difficulty needs to be dealt with. These problems are also addressed in [ACS17], where a part of the reflections is devoted to possible generalizations of the notion of synchronizability to infinite road-colored graphs. It was concluded that the notion of synchronizability in terms of Definition 5.1.1 was too weak for the algebraic considerations in [ACS17]. Nonetheless, in the following, we will see that it is the right tool for our scattering theoretical considerations.

Just like in the finite setting, irreducibility and synchronizability imply aperiodicity (compare Proposition 1.1.17). The proof of this assertion can be directly transferred to infinite road-colored graphs.

Let us reflect shortly on the two standard questions that arise when considering synchronizability: Does a synchronizing road-coloring exist? Given a road-coloring, how can it be determined whether it is synchronizing?

In the finite case, the question concerning the existence of a synchronizing coloring has been answered in Theorem 4.1.12. For every finite road-colorable graph that is regular in the matrix-sense, there exists a synchronizing coloring. To date, neither this assertion nor the Road-Coloring Theorem 1.1.18, could be transferred to infinite graphs. The first assertion that will be proved in this chapter presents a simple method to construct a synchronizing coloring whenever the graph is irreducible and has at least one loop.

PROPOSITION 5.1.3. Let (A, E) be an aperiodic, irreducible graph with at most countably many vertices in A. If there exists a vertex $a \in A$ with a loop, then the graph possesses a synchronizing coloring.

Proof. Let $a_0 \in A$ be a vertex with loop. Thanks to the assumed irreducibility, we can find a spanning subgraph consisting of only one tree with root a_0. The construction of such a spanning subgraph is quite simple. Fix a_0 as the root. For an arbitrary $a_1 \in A$ fix a path with initial vertex a_1 and terminal vertex a_0. Next, choose a vertex $a_2 \in A$, which is not yet contained in the tree (i.e., the path leading from a_1 to a_0) and consider again a path leading from a_2 to a_0. If it intersects the first path, change its course from the time of intersection such that the two paths coalesce. Otherwise, take the path as a new branch of the subgraph. Proceeding inductively, this provides a spanning subgraph consisting of only one tree with root a_0.
We obtain a synchronizing coloring when all the edges in the tree as well as the edge forming the loop at a_0 are colored in the same color c. The remaining edges can be colored arbitrarily. Then, every finite subset of vertices can be synchronized by a word of the form c^n, with n large enough.

□

If we equip an infinite road-colored graph (A, C, γ) with probability measures μ on A and ν on C, then we can identify $(A, C, \gamma; \mu, \nu)$ with the corresponding Markov chain (compare Section 1.2.3). In the finite case, a converse assertion is also true. Indeed, to every Markov chain with finite state space, it is possible to construct the corresponding trivially colorable graph (compare Section 3.2). This is no longer possible for Markov chains with infinite state space, as Lemma 3.2.3 does not hold in this setting. We will circumvent this problem by considering exclusively Markov chains, which allow, by construction, a representation as a road-colorable graph.

Example 5.1.4. Let $(A; \mu, T)$ be a Markov chain with countably infinite state space. If the states can be indexed as $A = \{a_n \ : \ n \in \mathbb{N}_0\}$ such that a_0 possesses a loop and the transition probabilities are $t_{a_n a_{n-1}} > \frac{1}{2}$ for all $n > 0$, and $t_{a_0 a_0} > \frac{1}{2}$ then every road-colored graph that represents the Markov chain is synchronizable.
Let $a_m, a_n \in A$ with $m \neq n$. By assumption, we have

$$\begin{aligned} \nu(\{c \in C \ : \ \gamma(a_m; c) = a_{m-1}\}) &= t_{a_m a_{m-1}} > \frac{1}{2}, \\ \nu(\{c \in C \ : \ \gamma(a_n; c) = a_{n-1}\}) &= t_{a_n a_{n-1}} > \frac{1}{2}, \\ \nu(\{c \in C \ : \ \gamma(a_0; c) = a_0\}) &= t_{a_0 a_0} > \frac{1}{2}. \end{aligned}$$

Consequently, these sets have pairwise nonempty intersection. In particular,

$$\begin{aligned} \{c \in C \ : \ \gamma(a_m; c) = a_{m-1}\} \cap \{c \in C \ : \ \gamma(a_n; c) = a_{n-1}\} \neq \emptyset, \\ \{c \in C \ : \ \gamma(a_m; c) = a_{m-1}\} \cap \{c \in C \ : \ \gamma(a_0; c) = a_0\} \neq \emptyset. \end{aligned}$$

So, for every $a_m \in A$, we find a word $w \in C^m$ of length m, such that

$$\gamma_m(a_m; w) = a_0 = \gamma_m(a_0; w).$$

Every pair of the form (a_m, a_0) is hence synchronizable. Analogously, we find for a pair (a_m, a_n), where $m < n$ without loss of generality, a word $v \in C^m$ of length m, such that a_m and a_n are both "moved towards" a_0, i.e., $\gamma_m(a_m; v) = a_0$ and $\gamma_m(a_n; v) = a_{n-m}$. In other words, (a_m, a_n) can be mapped to the synchronizable pair of vertices (a_0, a_{n-m}) and is hence synchronizable itself.

In [Haa06, Thm. 3.2.9] it has been shown that asymptotic completeness in the classical infinite setting is still equivalent to the existence of a synchronizing word in the sense of Definition 5.1.1. A further elaboration of this is worked out in [GHK19]. This shows that the notion of synchronizability for infinite graphs is the correct generalization for the scattering theoretical considerations in this thesis.

THEOREM 5.1.5. For an infinite road-colored graph $(A, C, \gamma; \mu, \nu)$ with invariant probability distribution μ and corresponding transition J, the following assertions are equivalent:

(a) The transition J is asymptotically complete.

(b) The infinite road-colored graph (A, C, γ) is irreducible and synchronizing.

In the following section, we will present a class of synchronizable infinite road-colored graphs and formulate a sufficient condition for the corresponding transition functions to converge ν^--almost surely.

5.2 INTEGER RANDOM WALKS

In [GHK19], it is shown that for the simple one-sided random walk presented in Example 5.2.1, the corresponding transition functions converge ν^--almost surely towards a function that does no longer depend on the chosen initial vertex whenever the probability distribution on the colors is chosen such that $\lambda > \frac{1}{2}$.

Example 5.2.1. The simple random walk is a generalization of our toy example to the infinite case. It is clearly synchronizable.

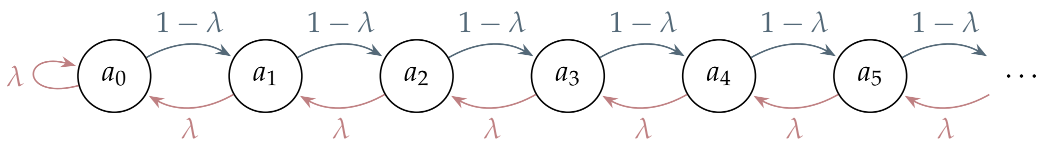


Figure 5.1: The birth-and-death chain presented in [GHK19]

This chapter is dedicated to expanding the above example to a more general class of infinite road-colored graphs representing a Markov chain.

Definition 5.2.2. We call a Markov chain $(A; \mu, T)$ an *integer random walk* if there exist a finite subset of integers $L \subset \mathbb{Z}$ and a probability distribution $\nu = (\nu_k)_{k \in L}$ on L, and if the vertices can be indexed as $A = \{a_n \; : \; n \in \mathbb{N}_0\}$ such that the transition matrix corresponds to

$$T = \sum_{\substack{k \in L: \\ -k > 0}} \nu_k \cdot \begin{pmatrix} 1 & 0 & \cdots & \cdots \\ 1 & 0 & \ddots & \\ 0 & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}^k + \nu_0 \cdot \begin{pmatrix} 1 & 0 & \cdots & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix} + \sum_{\substack{k \in L: \\ k > 0}} \nu_k \cdot \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}^k$$

where $\nu_0 := 0$, if $0 \notin L$.

In the first instance, we notice that the canonical graph representing the Markov chain need not be road-colorable, as vertex a_0 possesses exactly $|L \cap \mathbb{N}_0|$ outgoing edges, whereas every vertex a_n with $n > \max\{l \in L\}$ has out-degree $|L|$.

By the following procedure, the canonical graph representation can be turned into a road-colored graph. Consider the canonical representation of the integer random walk with $L \subset \mathbb{Z}$ and a probability distribution ν on L as a directed graph. Set $d := |L|$. Every vertex has out-degree smaller or equal to d. If a vertex a_n has less than d outgoing edges, then, by construction, $t_{n0} > 0$. The graph possesses hence an edge $a_n \longrightarrow a_0$. Copying this edge, i.e., adding further edges with initial vertex a_n and terminal vertex a_0 until a_n has out-degree d, leads to a road-colorable graph.

We denote the colors by $c_k := k \in L$ and set $C := \{c_1, \ldots, c_d\} = L \subset \mathbb{Z}$. Then a transition function $\gamma : A \times C \to A$ is defined by

$$\gamma(a_n; c_k) := \begin{cases} a_{n+c_k}, & \text{if } n + c_k > 0, \\ a_0, & \text{if } n + c_k \leq 0. \end{cases}$$

We obtain thus a road-coloring (C, γ) and ν is a probability distribution on the colors.

Definition 5.2.3. Let $(A; \mu, T)$ be an integer random walk as introduced in Definition 5.2.2. We call the just constructed road-colored graph the *canonical representation (of the integer random walk) as a road-colored graph.*

Example 5.2.4. Set $L = \{-3, -1, 2\}$ and let $\nu = (\nu_{-3}, \nu_{-1}, \nu_2)$ be a probability distribution on it. Consider the integer random walk defined by the (one-sided infinite) transition matrix

$$T = \begin{pmatrix} (\nu_{-3} + \nu_{-1}) & 0 & \nu_2 & 0 & 0 & \cdots & \cdots \\ (\nu_{-3} + \nu_{-1}) & 0 & 0 & \nu_2 & 0 & 0 & \cdots \\ \nu_{-3} & \nu_{-1} & 0 & 0 & \nu_2 & 0 & \ddots \\ \nu_{-3} & 0 & \nu_{-1} & 0 & 0 & \nu_2 & \ddots \\ 0 & \nu_{-3} & 0 & \nu_{-1} & 0 & 0 & \ddots \\ \vdots & 0 & \nu_{-3} & 0 & \nu_{-1} & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

This Markov chain is then represented by the road-colored graph in Figure 5.2.

Consider the canonical road-colored graph $(A, C, \gamma; \mu, \nu)$ representing an integer random walk $(A; \mu, T)$. We want to emphasize the fact that the colors, which usually correspond to an abstract set, form a subset of $\mathbb{Z}$ whenever integer random walks are

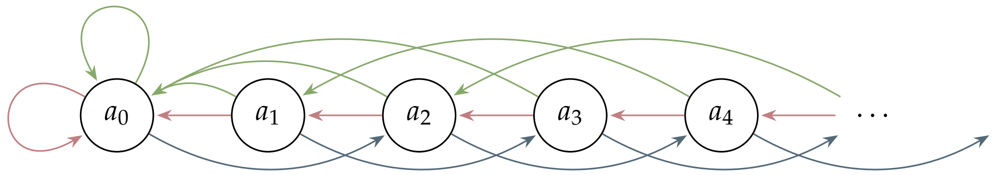


Figure 5.2: The integer random walk as defined in Example 5.2.4

considered. We will take advantage of this particular property in the following and speak of *negative colors*, whenever $c < 0$ and *non-negative* or *positive colors*, if $c \geq 0$ or $c > 0$, respectively. Let $n, m \in \mathbb{N}_0$, then the transition probabilities t_{nm} are given by:

1. For $n \geq 0$ and $m > 0$, we have

$$t_{nm} = \begin{cases} \nu_k, & \text{if } a_m = \gamma(a_n; c_k) = a_{n+c_k} \text{ for a } c_k \in C, \\ 0, & \text{otherwise.} \end{cases}$$

2. For $n \geq 0$ and $m = 0$, we have $t_{n0} = \sum_{\substack{c_k \in C\,: \\ \gamma(a_n; c_k) = a_0}} \nu_k$.

The transition probabilities correspond, hence, to $t_{nm} := \nu(\{c \in C : \gamma(a_n; c) = a_m\})$.

In the following, we identify the Markov chain with this canonical representation as a road-colored graph. So, when talking about an *integer random walk*, we address the corresponding canonical infinite road-colored graph $(A, C, \gamma; \mu, \nu)$.

Let us now state some simple observations.

LEMMA 5.2.5. An integer random walk $(A, C, \gamma; \mu, \nu)$ is synchronizable if and only if $C \cap -\mathbb{N} \neq \emptyset$, i.e., if C contains negative colors.

Proof. The proof arises from careful observation of the graph. Let $c < 0$, then

$$\gamma(a_m; c) = \begin{cases} a_{m-c}, & \text{if } m > c, \\ a_0, & \text{if } m \leq c. \end{cases}$$

So, starting in different initial vertices and applying enough negative colors will lead any vertex in a_0. However, the application of positive colors (i.e., $c > 0$) moves any two

different vertices $a_m \neq a_n$ into $a_{m+c} \neq a_{n+c}$. We formalize this:
Clearly, if $c \in C$ with $c < 0$, there exists $l \in \mathbb{N}$ such that $l \cdot c \geq \max\{m, n\}$ and $\gamma_l(a_m; c^l) = a_0 = \gamma_l(a_n; c^l)$, i.e., the pair (a_n, a_m) is synchronized.
Conversely, if $C \subset \mathbb{N}$, then by definition, $\gamma(a_m; c) = a_{m+c}$ for every $c \in C$ and $m \in \mathbb{N}_0$. So for every word $(c_{-l}, \ldots, c_{-1}) \in C^l$ and for all $m, n \in \mathbb{N}_0$ follows

$$
\begin{aligned}
\gamma_l(a_m; c_{-l}, \ldots, c_{-1}) &= a_{m+\sum_{k=1}^{l} c_{-k}}, \\
\gamma_l(a_n; c_{-l}, \ldots, c_{-1}) &= a_{n+\sum_{k=1}^{l} c_{-k}}.
\end{aligned}
$$

So it is not possible to synchronize two different vertices $a_m \neq a_n$. □

PROPOSITION 5.2.6. An integer random walk $(A, C, \gamma; \mu, \nu)$ is aperiodic if and only if $C \cap \mathbb{N}_0 \neq \emptyset$, i.e., if C contains non-negative colors.

Proof. Suppose that C contains non-negative colors. Then, the graph possesses at least one loop in a_0 and is hence aperiodic. For the converse, suppose that C consists only of positive colors. Then, by construction, the graph contains no cycle and cannot be aperiodic. □

COROLLARY 5.2.7. If an integer random walk $(A, C, \gamma; \mu, \nu)$ is irreducible, then $C \cap \mathbb{N} \neq \emptyset$ and $C \cap -\mathbb{N} \neq \emptyset$, i.e., C contains positive as well as negative colors.

Proof. This is obvious, as every vertex $a_m \in A$ must be connected to all other vertices $a_n \in A$. □

Combining these assertions, we can conclude that every irreducible integer random walk is automatically synchronizable.

COROLLARY 5.2.8. Every irreducible integer random walk $(A, C, \gamma; \mu, \nu)$ is aperiodic and synchronizable.

Example 5.2.9. The simple random walk introduced in Example 5.2.4 is an irreducible integer random walk.

5.2.1 Asymptotic completeness and almost sure convergence

In preparation for the asymptotic considerations, we now turn to one-sided infinite color sequences. As before, to spare the reader further elaborate notation, we write γ for the transition function on $A \times C$ as well as for its extension to a function from

$A \times C^-$ to A. Together with the results of the preceding sections, Theorem 5.1.5 leads to the following assertion regarding asymptotic completeness of transitions corresponding to integer random walks.

COROLLARY 5.2.10. For an integer random walk $(A, C, \gamma; \mu, \nu)$ with invariant probability distribution μ and corresponding transition J, the following assertions are equivalent:

(a) The transition J is asymptotically complete.

(b) The underlying road-colored graph (A, C, γ) is irreducible.

The following definition will be important for our considerations. We remark that it makes only sense since the colors C form is no abstract set but a subset of $\mathbb{Z}$.

Definition 5.2.11. Let (A, C, γ) be the road-colored graph representing an integer random walk. For a color sequence $c^- \in C^-$, and $n \in \mathbb{N}$, the *n-step-length of c^-* is a function $S_n : C^- \to \mathbb{Z}$, where

$$S_n(c^-) := \sum_{k=1}^{n} c_{-k}.$$

Integer random walks form a very special class of Markov chains. The crucial property is that – as long as we are far away from a_0 – the graph behaves identically in all vertices. Figuratively speaking, n-step length $S_n(c^-)$ of a color sequence c^- indicates how "far away" an initial vertex a_m can be moved when the last n steps of the color sequence are executed. This is formalized in the next assertions.

PROPOSITION 5.2.12. Let (A, C, γ) be the road-colored graph representing an integer random walk and let $c^- \in C^-$ and $n \in \mathbb{N}$. If $\gamma_{l+1}(a_0; c_{-n}, \ldots, c_{-(n-l)}) \neq a_0$ for every $0 \leq l < n$, then $\gamma_n(a_0; c^-) = a_{S_n(c^-)}$.

Before we come to the proof, we briefly interpret the assertion. If a color sequence c^- is such that the corresponding walk, when starting in a_0, does not return to a_0 before time $n \in \mathbb{N}$, then the n-step length $S_n(c^-)$ indicates the position of the walk at time n.

Proof. Fix $n \in \mathbb{N}$. We prove the assertion iteratively for $1 \leq l < n$.
Let $l = 0$, then $\gamma(a_0; c_{-n}) \neq a_0$ if and only if $c_{-n} > 0$ and by definition this implies $\gamma(a_0; c_{-n}) = a_{c_{-n}}$.

Let $l = 1$ and suppose that $\gamma_2(a_0; c_{-n}, c_{-(n-1)}) \neq a_0$. Then, the following holds

$$\begin{aligned} a_0 &\neq \gamma_2(a_0; c_{-n}, c_{-(n-1)}) \\ &= \gamma(\underbrace{\gamma(a_0; c_{-n})}_{=a_{c_{-n}} \neq a_0}; c_{-(n-1)}) \\ &= \gamma(a_{c_{-n}}; c_{-(n-1)}). \end{aligned}$$

But this implies that $c_{-n} + c_{-(n-1)} > 0$. So, by construction of the coloring (compare Definition 5.2.3), follows $\gamma(a_{c_{-n}}; c_{-(n-1)}) = a_{c_{-n}+c_{-(n-1)}}$.
We proceed iteratively and obtain thus $\gamma_l(a_0, c_{-n}, \ldots, c_{-(n-l+1)}) = a_{\sum_{k=l-1}^{n} c_k}$ for every $0 \leq l < n$ and hence

$$\gamma_n(a_0; c^-) = \gamma_n(a_0; c_{-n}, \ldots, c_{-1}) = a_{\sum_{k=1}^{n} c_k} = a_{S_n(c^-)}.$$

□

Proposition 5.2.12 can be transferred to an arbitrary initial vertex a_m, as long as the considered walk does not land in a_0 before time $n \in \mathbb{N}$.

COROLLARY 5.2.13. Let (A, C, γ) be the road-colored graph representing an integer random walk and let $c^- \in C^-$ and $m, n \in \mathbb{N}$. If $\gamma_{l+1}(a_m; c_{-n}, \ldots, c_{-(n-l)}) \neq a_0$ for every $0 \leq l < n$, then $\gamma_n(a_m; c^-) = a_{m+S_n(c^-)}$.

Definition 5.2.14. Let (A, C, γ) be the road-colored graph representing an integer random walk and let $M \in \mathbb{N}$. We introduce the following disjoint subsets of C^-:

$$\begin{aligned} C_M &:= \{c^- \in C^- : \max_{n \in \mathbb{N}} \{S_n(c^-)\} = M\}, \\ C_0 &:= \{c^- \in C^- : \max_{n \in \mathbb{N}} \{S_n(c^-)\} \leq 0\}. \end{aligned}$$

The elements in C_M and C_0 are called *color sequences of maximum distance M*, respectively *color sequence of no distance*.

The following propositions describe some properties of the color sequences of maximum and of no distance. This will lead to the insight that the naming – which might seem somewhat arbitrary – is indeed quite intuitive.

PROPOSITION 5.2.15. Let (A, C, γ) be the road-colored graph representing an integer random walk. Consider a color sequence of no distance, i.e., $c^- \in C_0$. Then for all $n \in \mathbb{N}$, follows $\gamma_n(a_0; c^-) = a_0$.

Proof. We prove the assertion by induction.
Let $n = 1$. By assumption, $S_1(c^-) = c_{-1} \leq 0$ and hence, $\gamma(a_0; c^-) = \gamma(a_0; c_{-1}) = a_0$.
Suppose that the assertion holds for $n \in \mathbb{N}$.
For $c_{-(n+1)} \leq 0$, this implies: $\gamma_{n+1}(a_0; c^-) = \gamma_n(\underbrace{\gamma(a_0; c_{-(n+1)})}_{=a_0}; c^-) = 0$.
For $c_{-(n+1)} > 0$, we find with the assumption that $c^- \in C_0$:

$$0 \geq S_{n+1}(c^-) = \underbrace{c_{-(n+1)}}_{>0} + S_n(c^-) \quad \Longleftrightarrow \quad S_n(c^-) \leq -c_{-(n+1)}.$$

In particular, this implies:

$$\gamma_{n+1}(a_0; c^-) = \gamma_n(\underbrace{\gamma(a_0; c_{-(n+1)})}_{=a_{c_{-(n+1)}}}; \underbrace{c_{-n}, \ldots, c_{-1}}_{S_n(c^-) \leq -c_{-(n+1)}}) = a_0.$$

□

Consequently, for a color sequence that is of no distance, whenever starting in a_0, it is not possible to leave a_0. So, $\gamma_n(a_0; c^-)$ remains equal to a_0 at all times $n \in \mathbb{N}$.

The goal of the following considerations is to show that every color sequence of maximum distance $M \in \mathbb{N}$ consists of a color sequence of no distance to which only a finite part is added.

Definition 5.2.16. Let (A, C, γ) be the road-colored graph representing an integer random walk and let $c^- \in C_M$. We define the *time of maximum distance (of c^-)* as the first time, at which the n-step length of c^- reaches the maximum distance M (when n tends to infinity) and denote it as

$$n_M(c^-) := \min\{n \in \mathbb{N} \ : \ S_n(c^-) = M\}.$$

Whenever the color sequence is fixed, and there is no risk of confusion, we omit the argument and denote the time of maximum distance simply by n_M.

In the following, we will refer to different points of time that will be passed when n tends to infinity. In order to avoid confusion, it might be helpful to remember from which perspective we are looking at the point in time.

PROPOSITION 5.2.17. Let (A, C, γ) be the road-colored graph representing an integer random walk. Consider further a color sequence of maximum distance M with time of maximum distance $n_M = n_M(c^-)$, i.e., $c^- = (\dots, c_{-(n_M+1)}, c_{-n_M}, \dots, c_{-1}) \in C_M$. Then, for all times $l < n_M$, holds

$$\gamma_{l+1}(a_0, c_{-n_M}, \dots, c_{-(n_M-l)}) \neq a_0.$$

Proof. Assume there where $1 \leq l < n_M$ such that $\gamma_{l+1}(a_0; c_{-n_M}, \dots, c_{-(n_M-l)}) = a_0$ This would imply $\sum_{k=n_M-l}^{n_M} c_{-k} \leq 0$ and hence

$$M = S_{n_M}(c^-) = \underbrace{\sum_{k=n_M-l}^{n_M} c_{-k}}_{\leq 0} + S_{n_M-l-1}(c^-).$$

But then $S_{n_M-l-1}(c^-) \geq M$ which contradicts either the assertion that $c^- \in C_M$, or that n_M corresponds to the first time that the maximum distance M is achieved. □

PROPOSITION 5.2.18. Let (A, C, γ) be the road-colored graph representing an integer random walk. Consider further a color sequence of maximum distance M, i.e., $c^- = (\dots, c_{-(n_M+1)}, c_{-n_M}, \dots, c_{-1}) \in C_M$. Then, $\gamma_{n_M}(a_0; c^-) = a_M$ and the remainder of the sequence is of no distance, i.e., $(\dots, c_{-(n_M+1)}) \in C_0$.

Proof. Fix a color sequence of maximum distance $c^- \in C_M$.
From Proposition 5.2.17 follows that $\gamma_{l+1}(a_0; c_{-n_M}, \dots, c_{-(n_M-l)}) \neq a_0$ holds for every $l < n_M$. So, by Proposition 5.2.12 follows $\gamma_{n_M}(a_0, c^-) = a_{S_{n_M}(c^-)} = a_M$.
The second assertion follows directly since for all times $n > n_M$ holds

$$M \geq S_n(c^-) = \sum_{k=n_M+1}^{n} c_{-k} + \underbrace{S_{n_M}(c^-)}_{=M}.$$

Consequently, $\sum_{k=n_M+1}^{n} c_{-k} \leq 0$ for all $n > n_M$ and hence, by definition, the remainder $(\dots, c_{-(n_M+1)})$ of the original color sequence, is itself a color sequence of no distance, i.e., lies in C_0. □

Consequently, when starting in a_0, the application of a color sequence of maximum distance M will always lead into a vertex in $\{a_0, \dots, a_M\}$. In other words, the color sequences in C_0 and C_M can be described with respect to the maximum distance that can be achieved when starting in a_0. In this sense, the naming turns out to be quite intuitive. This is summarized in the following corollary.

COROLLARY 5.2.19. Let (A, C, γ) be the road-colored graph representing an integer random walk. The color sequences of maximum distance $M \in \mathbb{N}$ and those of no distance can be identified with

$$\begin{aligned} C_0 &= \{c^- \in C^- \ : \ \gamma_n(a_0; c^-) = a_0 \text{ for all } n \in \mathbb{N}\}, \\ C_M &= \{c^- \in C^- \ : \ M = \max_{n\in\mathbb{N}}\{m : \gamma_n(a_0; c^-) = a_m\}\}. \end{aligned}$$

After having understood these crucial properties of integer random walks, the following demonstration will show that they allow the ν^--almost sure convergence of the sequence of transition functions $(\gamma_n)_{n\in\mathbb{N}}$ towards a function $\gamma_\infty : C^- \to A$ that does no longer depend on the choice of an initial vertex.

Definition 5.2.20. Let (A, C, γ, μ, ν) be an integer random walk. We define the following subset of C^-:

$$C_\nu := \left\{c^- \ : \ \lim_{n\to\infty} S_n(c^-) = -\infty\right\}.$$

THEOREM 5.2.21. Let (A, C, γ, μ, ν) be an irreducible integer random walk. If $\nu^-(C_\nu) = 1$, then for every $a \in A$ and ν^--almost every $c^- \in C^-$ the limit

$$\lim_{n\to\infty} \gamma_n(a; c^-)$$

exists and is independent of the initial vertex $a \in A$. We denote it by $\gamma_\infty(c^-)$.
Moreover, up to ν^--null sets, C_0 and $\{C_M\}_{M\in\mathbb{N}}$ form a disjoint partition of C^- and γ_∞ is ν^--almost surely given by

$$\gamma_\infty(c^-) = \begin{cases} a_0, & c^- \in C_0 \cap C_\nu, \\ a_M, & c^- \in C_M \cap C_\nu. \end{cases}$$

Proof. We proceed in four steps. In the first instance, we show that the infinite color sequences can ν^--almost surely be classified according to their maximum distances. If the initial vertex corresponds to a_0, the ν^--almost sure convergence of the transition functions can easily be concluded. In the last two steps, the desired convergence for an arbitrary initial vertex is shown for color sequences with no distance and with maximum distance, respectively.

STEP 1. Classification of C^-.

By assumption the n-step lengths S_n of the color sequences in C^- tend ν^--almost surely towards $-\infty$. For ν^--almost every $c^- \in C^-$, there can thus be determined a maximum $M := \max_{n\in\mathbb{N}}\{0, S_n(c^-)\}$. By Corollary 5.2.19, such a c^- belongs either to C_0 or to C_M for an $M \in \mathbb{N}$, i.e., $\nu^-(C_0 \cup \bigcup_{M\in\mathbb{N}} C_M) = 1$. In particular, since by assumption $\nu^-(C_\nu) = 1$, we have also

$$\nu^-\Big((C_0 \cap C_\nu) \cup \bigcup_{M\in\mathbb{N}} (C_M \cap C_\nu)\Big)\nu^-\Big(\Big(C_0 \cup \bigcup_{M\in\mathbb{N}} C_M\Big) \cap C_\nu\Big) = 1.$$

STEP 2. The ν^--almost sure convergence of $(\gamma_n(a_0; c^-))_{n\in\mathbb{N}}$.

Let $c^- \in C_0 \cap C_\nu$. In Proposition 5.2.15 we have shown $\gamma_n(a_0; c^-) = a_0$ for every $n \in \mathbb{N}$. The almost sure convergence in the case of $c^- \in C_M \cap C_\nu$, is a direct consequence of Proposition 5.2.18. Indeed, for every $n > n_M$ holds

$$\gamma_n(a_0; c^-) = \gamma_{n_M}(\gamma_{n-n_M}(a_0; \underbrace{(\dots, c_{-(n_M+1)})}_{\in C_0}); c_{-n_M}, \dots, c_{-1}) = \gamma_{n_M}(a_0; c^-) = a_M.$$

So, the limit $\lim_{n\to\infty} \gamma_n(a_0; c^-)$ corresponds ν^--almost surely to a_0 or a_M, depending on whether $c^- \in C_0 \cap C_\nu$ or $c^- \in C_M \cap C_\nu$, respectively.

STEP 3. For $c^- \in C_0 \cap C_\nu$ holds $\lim_{n\to\infty} \gamma_n(a_m; c^-) = a_0$ for all $a_m \in A$.

Let $c^- \in C_0 \cap C_\nu$ and $m \in \mathbb{N}$. Since the n-step lengths are not bounded from below, there exists a time $l_m \in \mathbb{N}$, such that for every $n \geq l_m$ holds $S_n(c^-) \leq -m$. Let now $n \geq l_m$ and set $u_m := \max_{u\leq n}\{u \;:\; \sum_{k=u+1}^{n} c_{-k} \leq -m\}$. Then we have $\gamma_{n-u_m}(a_m; c_{-n}, \dots, c_{-(u_m+1)}) = a_0$ for the first time, i.e., when starting in a_m at time $-n$, then $u_m + 1$ corresponds to the first time that the path induced by c^- goes through a_0. In particular this implies that $m + \sum_{k=u_m+1}^{n} c_{-k} \leq 0$ and $m + \sum_{k=u+1}^{n} c_{-k} > 0$ for all $n \geq u > u_m$. Consequently, for all times $n > l_m$ holds

$$\begin{aligned}
\gamma_n(a_m; c^-) &= \gamma_n(a_m; c_{-n}, \dots, c_{-u_(m+1)}, c_{-u_m}, \dots, c_{-1}) \\
&= \gamma_{u_m}(\underbrace{\gamma_{n-u_m}(a_m; c_{-n}, \dots, c_{-(u_m+1)})}_{=a_0}; c_{-u_m}, \dots, c_{-1}) \\
&= \gamma_{u_m}(a_0; c^-) \\
&= a_0.
\end{aligned}$$

To be exact, as u_m depends on the chosen $n > l_m$, it should have been denoted as $u_m(n)$. The argument was omitted in favor of better readability.

STEP 4. For $c^- \in C_M \cap C_\nu$ holds $\lim_{n\to\infty} \gamma_n(a_m; c^-) = a_M$ for all $a_m \in A$.

Let $c^- \in C_M \cap C_\nu$, for $M \in \mathbb{N}$ with corresponding time of maximum distance n_M. Then by Proposition 5.2.18 it follows that the remainder of c^- up to time n_M, i.e., $(\ldots, c_{-(n_M+1)})$ lies in C_0. So, for any $m \in \mathbb{N}$, STEP 3 can be applied to this remainder. We find $l_m > n_M$ as above and for every $n > l_m$ there exists u_m as above. Consequently, for all times $n > l_m$ holds

$$\begin{aligned}
\gamma_n(a_m; c^-) &= \gamma_n(a_m; c_{-n}, \ldots, c_{u_m}, \ldots, c_{-n_M}, \ldots, c_{-1}) \\
&= \gamma_{u_m}(\underbrace{\gamma_{n-u_m}(a_m; c_{-n}, \ldots, c_{-(u_m+1)})}_{=a_0}; c_{u_m}, \ldots, c_{-n_M}, \ldots, c_{-1}) \\
&= \gamma_{n_M}(\gamma_{u_m-n_M}(a_0; \underbrace{\ldots, c_{-(n_M+1)}}_{\in C_0}); c_{-n_M}, \ldots, c_{-1}) \\
&= \gamma_{n_M}(a_0; c_{-n_M}, \ldots, c_{-1}) \\
&= a_M.
\end{aligned}$$

Thus, for all $a_m \in A$ and ν^--almost every infinite color sequence c^- follows

$$\gamma_\infty(c^-) := \lim_{n\to\infty} \gamma_n(a_m; c^-) = \begin{cases} a_0, & c^- \in C_0 \cap C_\nu, \\ a_M, & c^- \in C_M \cap C_\nu. \end{cases}$$

□

We have thus provided a sufficient condition for the transfer of Theorem 2.3.10 to the setting of integer random walks.

THEOREM 5.2.22. Let $(A, C, \gamma; \mu, \nu)$ be an integer random walk with invariant probability distribution μ and let J be the corresponding transition. Suppose that $\nu^-(C_\nu) = 1$. Then, the following assertions are equivalent:

(a) The transition J is asymptotically complete.

(b) The underlying road-colored graph (A, C, γ) is irreducible and synchronizable.

(c) The underlying road-colored graph (A, C, γ) is irreducible.

(d) The sequence of transition functions $(\gamma_n)_{n\in\mathbb{N}}$ converges ν^--almost surely towards a function γ_∞ that does no longer depend on the chosen initial vertex.

Proof. The equivalence $(a) \Leftrightarrow (b)$ is due to Theorem 2.3.10, $(b) \Rightarrow (c)$ follows from Corollary 2.3.11. Theorem 5.2.21 implies $(c) \Rightarrow (d)$ and $(d) \Rightarrow (a)$ was shown in Corollary 2.3.11. □

Clearly, integer random walks form a quite special class of graphs. In particular, they are characterized by the property that it can be guaranteed that a designated vertex a_0 is reached at a certain time. Then, for the determination of the limit, only the last finitely many entries of the color sequence c^- are relevant. The question remains as to which infinite cases allow a similar characterization of asymptotic completeness or synchronizability in terms of an almost sure convergence of the corresponding transition functions.

In the following sections, we will present two further generalizations of the class of integer random walks.

5.2.2 *Finite transient additions*

When asking for the existence of a synchronizing coloring for a finite road-colorable graph, we have seen in Chapter 4 that the assumption of aperiodicity and irreducibility can be weakened to regularity (compare Theorem 4.1.12). We will now deal with the question of whether the results in Section 5.2.1 remain true for regular infinite graphs whose aperiodic recurrent class corresponds to an integer random walk. We will find that this is still the case if the set of transient vertices is finite.

Definition 5.2.23. Consider an infinite road-colored graph $(A \cup T, C, \gamma; \mu, \nu)$ that is regular in the matrix-sense, where A is countable and T a finite set. Suppose that the aperiodic recurrent class is given by $(A, C, \gamma|_A; \mu|_A, \nu)$ and corresponds to an irreducible integer random walk. We call such a graph an *integer random walk with finite transient addition*.
The transient part corresponds to the road-colored subgraph $(T, C, \gamma|_T; \mu|_T, \nu)$ and is called *finite transient addition (to an integer random walk)*.

The double assignment of the letter T is accepted in favor of an intuitive naming of the set of transient vertices. In fact, the transition matrix T will not appear in this section so in the following, T always stands for the set of transient nodes.

We state some simple observations.

Observation 5.2.24. Let $(A \cup T, C, \gamma; \mu, \nu)$ be an integer random walk with finite transient addition, then $T \cap A = \emptyset$ and there is no transition from A into T.

COROLLARY 5.2.25. Let $(A \cup T, C, \gamma; \mu, \nu)$ be an integer random walk with finite transient addition. The vertices in T form one ore more transient communicating classes and there exists $n_T \in \mathbb{N}$ and a word $w_T \in C^{n_T}$ such that $\gamma_{n_T}(T; w_T) \subset A$.

Proof. For every transient vertex $t \in T$, there exists $n_t \in \mathbb{N}$ and a word $w_t \in C^{n_t}$ that maps it into the recurrent class, that is to say, into A. As T is finite, and there is no transition from A (back) into T, a finite combination of such words maps T into A. □

Definition 5.2.26. Let $(A \cup T, C, \gamma; \mu, \nu)$ be an integer random walk with finite transient addition. We define the *set of T-target points* as

$$A_T := \{a_m \in A \ : \ \gamma(t; c) = a_m \text{ for a } t \in T \text{ and } c \in C\} \subset A.$$

COROLLARY 5.2.27. The set of T-target points A_T of an integer random walk with finite transient addition $(A \cup T, C, \gamma; \mu, \nu)$ is finite.

We remark that our notation in Definition 5.2.23 is not quite exact. Whereas the subgraph $(A, C, \gamma|_A)$ actually corresponds to a road-colored graph in the sense of Definition 1.1.7, the triple $(T, C, \gamma|_T)$ is to be understood as a subgraph of $(A \cup T, C, \gamma)$, since $\gamma|_T : T \times C \to T \cup A_T$. So, strictly speaking, $\gamma|_A$ is a transition function and $\gamma|_T$ is not quite. But as this does not affect our considerations, it suffices to bear this in mind when referring to $(T, C, \gamma|_T)$ as a subgraph.

COROLLARY 5.2.28. Let $(A \cup T, C, \gamma; \mu, \nu)$ be an integer random walk with finite transient addition, then the following assertions are equivalent:

(a) The graph $(A \cup T, C, \gamma)$ is synchronizable.

(b) The underlying integer random walk $(A, C, \gamma|_A)$ is synchronizable.

Proof. (a) $\Rightarrow$ (b) is obvious. (a) $\Rightarrow$ (b) is a direct consequence of Corollary 5.2.25. Indeed, every pair of vertices in $A \cup T$ can be mapped to a pair of vertices in A, which is synchronizable by assumption. □

PROPOSITION 5.2.29. Let $(A \cup T, C, \gamma; \mu, \nu)$ be an integer random walk with finite transient addition. If $\nu^-(C_\nu) = 1$, then for every $a \in A \cup T$ and ν^--almost every $c^- \in C^-$ the limit $\lim_{n\to\infty} \gamma_n(a; c^-)$ exists and corresponds to $\gamma_\infty(c^-)$, where γ_∞ is defined as in Theorem 5.2.21.

Proof. We note that for every $a_m \in A$, the assertion follows directly from Theorem 5.2.21. It must hence only be proved for initial vertices $t \in T$. Fixing $c^- \in C_0 \cap C_\nu$, we show that for every $t \in T$ follows $\lim_{n\to\infty} \gamma_n(t; c^-) = a_0$. Then, thanks to Proposition 5.2.18, follows directly $\lim_{n\to\infty} \gamma_n(t; c^-) = a_M$ for every $c^- \in C_M \cap C_\nu$. The ν^--almost sure convergence follows then, since ν^--almost every color sequence is contained in $(C_0 \cap C_\nu) \cup \bigcup_{M\in\mathbb{N}}(C_M \cap C_\nu)$.

Let $c^- \in C_0 \cap C_\nu$. The idea behind this approach is to show that for every large enough $n \in \mathbb{N}$, the vertex $t \in T$ is – somewhere along the way – sent into a_0. But since c^- is a color sequence of no distance, it is not possible to leave a_0 ever again.
From Corollary 5.2.28, we know that T is synchronizable. So, we can define a set containing every word that synchronizes T into the vertex a_0.

$$W_0 := \{w_0 \ : \ \text{there is } n_0 \in \mathbb{N} \text{ and } w_0 \in C^{n_0} \text{ such that } \gamma_{n_0}(T; w_0) = a_0\}.$$

Since c^- was chosen to be in C_0 as well as C_ν, for every $m \in \mathbb{N}_0$ there exists a time l_m, such that $S_n(c^-) \leq -m$ for all $n \geq l_m$ (compare the third step in the proof of Theorem 5.2.21). We can define a common such time for the finitely many T-target points in A_T and set

$$l_T := \max\{l_m \ : \ a_m \in A_T\}.$$

Then, for all $n \geq l_T$ follows $S_n(c^-) \leq -\max\{m \ : \ a_m \in A_T\}$.

Every finite word occurs ν^--almost surely infinitely often in c^-. So, when letting n tend to infinity, we find words from W_0 that synchronize T into a_0 after having passed time l_T. Consider the first occurrence of such a word after the passage of l_T:

$$N(c^-) := \min_{n>l_T}\{n \in \mathbb{N} \ : \ \text{there exists } n_0 \in \mathbb{N} \text{ such that } (c_{-(n+n_0)}, \dots, c_{-(n+1)}) \in W_0\}.$$

In the following, the argument c^- will be omitted for better readability. The color sequence is then of the form

$$c^- = (\dots, \underbrace{c_{-(N+n_0)}, \dots, c_{-(N+1)}}_{\in W_0}, c_{-N}, \dots, c_{-l_T}, \dots, c_{-1}).$$

N exists ν^--almost surely and for every $n \geq N + n_0$ and $t \in T$, follows

$$\gamma_n(t; c^-) = \gamma_{N+n_0}(\gamma_{n-(N+n_0)}(t; \dots, c_{-(N+n_0+1)}); c_{-(N+n_0)}, \dots, c_{-(N+1)}, c_{-N}, \dots, c_{-1}).$$

Clearly, the value of $\gamma_{n-(N+n_0)}(t;\ldots,c_{-(N+n_0+1)})$ is crucial. There exist two cases:

1. CASE. Suppose that $\gamma_{n-(N+n_0)}(t,\ldots,c_{-(N+n_0+1)}) \in T$.
The first possibility is that we are still in T at time $N+n_0+1$, then, as the synchronizing word from W_0 starts in $c_{-(N+n_0)}$, we are sent directly into a_0. And since c^- was chosen to be a color sequence of no distance, a_0 cannot be left:

$$\begin{aligned}\gamma_n(t;c^-) &= \gamma_N(\underbrace{\gamma_{n_0}(\underbrace{\gamma_{n-(N+n_0)}(t;\ldots,c_{-(N+n_0+1)})}_{\in T};\underbrace{\ldots,c_{-(N+1)}}_{\in W_0})}_{=a_0};\ldots,c_{-1}) \\ &= \gamma_N(a_0;\underbrace{c_{-N},\ldots,c_{-1}}_{\in C_0}) \\ &= a_0.\end{aligned}$$

2. CASE. Suppose that $\gamma_{n-(N+n_0)}(t,\ldots,c_{-(N+n_0+1)}) = a_m \in A$ for an $m \in \mathbb{N}$.
The second possibility is that we are no longer in T at time $N+n_0+1$, so between c_{-n} and $c_{-(N+n_0+1)}$ there must have occurred a transition from T into the set of T-target points A_T. As T is transient, once entered A, it is not possible to come back into T (compare Corollary 5.2.24). So, such a transition occurs exactly once. We can thus determine a unique T-target point $a_{m_T} \in A_T$. Let n_{m_T} denote the corresponding moment when the transition occurs. We show now that, since $n \geq l_T$, the T-target point a_{m_T} is sent into a_0. And as c^- was chosen to be a color sequence of no distance, a_0 cannot be left after that.
Let n_{m_T} be such that $\gamma_{n-n_{m_T}+1}(t;c_{-n},\ldots,c_{-n_{m_T}}) = a_{m_T}$. By assumption we have $n > n_{m_T} \geq N+n_0+1 > l_T$. Analogously to the third step in the proof of Theorem 5.2.21, we find for the T-target point in $a_{m_T} \in A_T$ a time u_{m_T} such that u_{m_T+1} corresponds to the first time that a_{m_T} is mapped into a_0 by c^-, when starting at time $n_{m_T}+1$. In other words, c^- is of the form:

$$c^- = (\underbrace{\ldots,c_{-n},\ldots,c_{-n_{m_T}}}_{t\longrightarrow a_{m_T}},\underbrace{\underbrace{c_{-(n_{m_T}-1)},\ldots,c_{-(u_{m_T}+1)}}_{\text{sum }\leq -m_T\text{ for the first time}}}_{a_{m_T}\longrightarrow a_0},c_{-u_{m_T}},\ldots,c_{-1}).$$

Consequently, since c^- was chosen to be of no distance, it follows that

$$\gamma_n(t;c^-) = \gamma_{u_{m_T}}(a_0;c^-) = a_0$$

Then, $\gamma_n(t;c^-) = a_M$ for every $c^- \in C_M \cap C_\nu$ follows with Proposition 5.2.18. $\square$

Of course, the question arises whether Proposition 5.2.29 remains true if T, the set of vertices that form the transient addition, is no longer finite but countable. The crucial part of the above proof is that it must be guaranteed that during the observed time span, a transition from T into A occurs. Moreover, this transition must happen before a time u_m (i.e., during the application of $(c_{-n}, \ldots, c_{-(u_m+1)})$) such that the corresponding T-target point can still be mapped into a_0 in time. In other words, it must be guaranteed that every vertex in T can be mapped into a_0 in finite time. A possibility to generalize Proposition 5.2.29 is the following.

COROLLARY 5.2.30. Let $(A \cup T, C, \gamma; \mu, \nu)$ be a road-colored graph that is regular in the matrix-sense. Suppose that the aperiodic recurrent class corresponds to A and $(A, C, \gamma|_A; \mu|_A, \nu)$ is an integer random walk, and T is a countable set of vertices such that A_T remains finite. Suppose further that there exists a finite synchronizing word for the set T and that $\nu^-(C_\nu) = 1$. Then, for every $a \in A \cup T$ and ν^--almost every $c^- \in C^-$ the limit

$$\lim_{n\to\infty} \gamma_n(a; c^-)$$

exists and corresponds to $\gamma_\infty(c^-)$, where γ_∞ is defined as in Theorem 5.2.21.

5.2.3 *Quotient graphs*

This section deals with another generalization of the almost sure convergence of the transition functions for integer random walks. To this aim, we introduce the concept of *quotient graphs*.

In the following, let (A, C, γ) be a road-colored graph with countable A and finitely many colors C and let $\{A_m\}_{m\in\mathbb{N}_0}$ be a *partition of* A. We consider the *equivalence relation, induced by the partition*.

$$a \sim b \quad :\Leftrightarrow \quad a, b \in A_m \text{ for an } m \geq 0.$$

In the following, the representative corresponding to the elements in A_m is denoted as $[A_m]$ and we set $[A] := \{[A_m] : m \in \mathbb{N}_0\}$.

Definition 5.2.31. The *quotient graph* of (A, C, γ) consists of vertices $[A]$ and edges, labeled with the colors in C. There is a c-colored edge from $[A_n] \xrightarrow{c} [A_m]$, if in the original graph, there is a vertex in A_n, which has a c-colored edge, leading to a vertex in A_m.

As the following example illustrates, the quotient graph is not necessarily road-colored, since there may possibly lead several edges of the same color $c \in C$ from vertices in A_n into vertices in different sets $A_{m_1}, A_{m_2}, \dots$

Example 5.2.32. As the underlying original graph, consider the simple random walk (compare Example 5.2.1) with the following partition of $A = \{a_m : m \in \mathbb{N}_0\}$:

$$A_0 := \{a_0\}, \quad A_1 := \{a_1, a_3\}, \quad A_2 := \{a_2, a_4\}, \quad A_m := \{a_{m+2}\}, \text{ for } m \geq 3.$$

The quotient graph corresponds then to

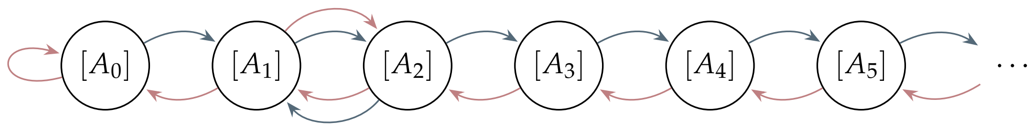


Figure 5.3: Quotient graphs of road-colorable graphs are not necessarily road-colorable

Observation 5.2.33. A quotient graph of (A, C, γ) with respect to a partition $\{A_m\}_{m\in\mathbb{N}_0}$ of A is road-colored if and only if for every $A_m \in \{A_m\}_{m\in\mathbb{N}_0}$ and $c \in C$ there exists a unique $A_{m_c} \in \{A_m\}_{m\in\mathbb{N}_0}$, such that $\gamma(A_m; c) \subseteq A_{m_c}$.

Definition 5.2.34. If for a graph (A, C, γ) there exists a partition $\{A_m\}_{m\in\mathbb{N}_0}$ of A, such that the induced quotient graph is an integer random walk, then we call the underlying original graph an *integer random walk of second order (with respect to* $\{A_m\}_m$*)* and denote the corresponding quotient graph by $([A], C, [\gamma]; \{A_m\}_m)$, where $[\gamma] : [A] \times C \to [A]$ is defined in the obvious way.
Without loss of generality the color sequences can then be classified according to their *maximum distance* from $[A_0]$ into sets $C_{[0]}$ and $C_{[M]}$ (analogously to Corollary 5.2.19), where to each $c^- \in C_{[M]}$ corresponds a *time of maximum distance* n_M (compare Definition 5.2.16).

Example 5.2.35. Consider the graph depicted in Figure 5.4. With respect to the partition given by $A_0 = \{a_0\}, A_1 = \{a_{1.1}, a_{1.2}\}, A_2 = \{a_2\}, A_3 = \{a_{3.1}, a_{3.2}\}, A_4 = \{a_{4.1}, a_{4.2}\}$, and $A_m = \{a_m\}$, for $m \geq 5$, it is an integer random walk of second order with colors $(-1, 0, 1)$. The corresponding quotient graph is illustrated in Figure 5.5.

The assertions in Section 5.2.1 can hence be transferred one-to-one to the setting of integer random walks of second order by simply replacing the vertices $a_m \in A$ with $[A_m] \in [A]$.

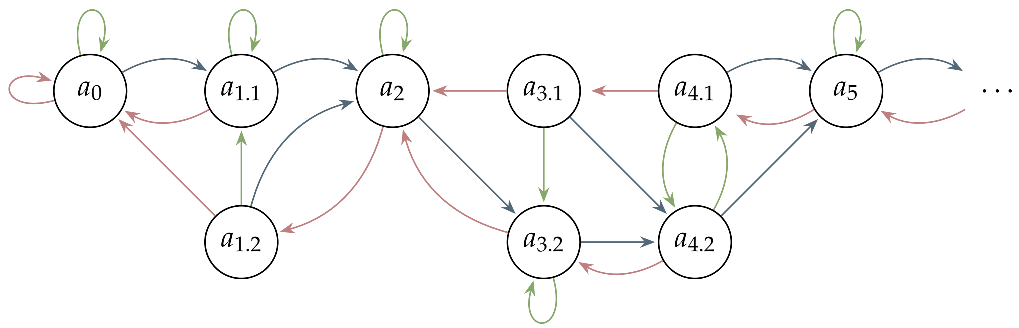


Figure 5.4: Example of an integer random walk of second order

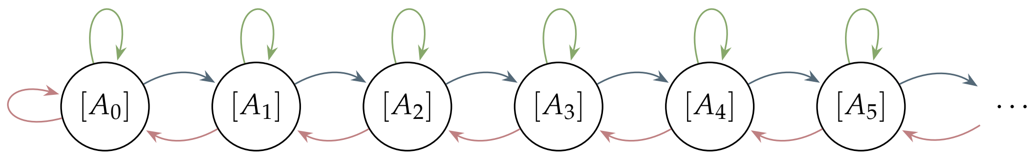


Figure 5.5: The quotient graph of the integer random walk of second order in Figure 5.4

COROLLARY 5.2.36. Let $([A], C, [\gamma]; [\mu], \nu; \{A_m\}_m)$ be a quotient graph that is an irreducible integer random walk, where $[\mu] := (\mu(A_m))_m$. If $\nu^-(C_\nu) = 1$, then for every $[A_m] \in [A]$ and ν^--almost every $c^- \in C^-$ the limit

$$\lim_{n\to\infty} [\gamma]_n([A_m]; c^-)$$

exists and corresponds to

$$[\gamma]_\infty(c^-) := \begin{cases} [A_0], & \text{for } c^- \in C_{[0]} \cap C_\nu, \\ [A_M], & \text{for } c^- \in C_{[M]} \cap C_\nu. \end{cases}$$

The obvious question that now arises is whether the almost sure convergence of the transition functions of the quotient graph also implies the same for the underlying integer random walk of second order. On careful consideration of the above corollary, it becomes clear that this is not always the case. Indeed, for arbitrary $a \in A$ it can only be concluded that $\lim_{n\to\infty} \gamma_n(a; c^-) \in A_0$ or A_M, respectively. Also, synchronizability of the quotient graph does not automatically imply synchronizability of the original underlying graph in the sense of Definition 5.1.1, as the following example shows.

Example 5.2.37. The following graph illustrated in Figure 5.6 is an integer random walk of second order with respect to the partitioning $A_m := \{a_{m.1}, a_{m.2}\}$ for $m \in \mathbb{N}_0$ of

A. Its quotient graph corresponds to the graph depicted in Figure 5.5. It is, hence, an integer random walk and, in particular, synchronizable, whereas this is not true for the original graph. Indeed, no pair of vertices of the form $(a_{n.1}, a_{m.2})$, with $m, n \geq 0$ is synchronizable.

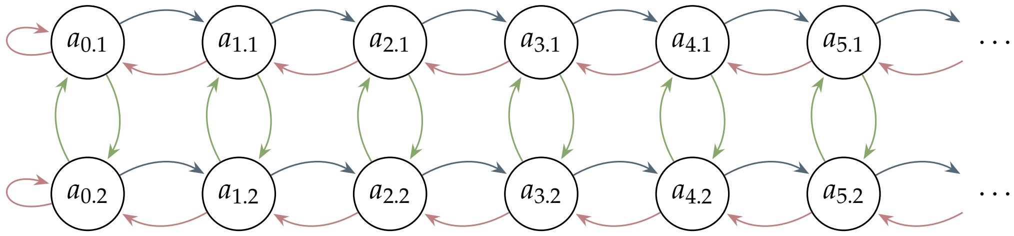


Figure 5.6: Synchronizability of the quotient graph does not guarantee the same for the integer random walk of second order

It is thus necessary to exclude the possibility that the limit value oscillates between the vertices in one of the subsets forming the partition. However, both problems can be easily circumvented by assuming that the set A_0 contains only a single element.

PROPOSITION 5.2.38. Let $(A, C, \gamma;\ \mu, \nu;\ \{A_m\}_m)$ be an integer random walk of second order with respect to a partition $\{A_m\}_{m\in\mathbb{N}_0}$ of A, where $A_0 =: \{a_0\}$ is a one-element subset of A. If the quotient graph $([A], C, [\gamma]; [\mu], \nu;\ \{A_m\}_m)$, with $[\mu] := (\mu(A_m))_m$, is an irreducible integer random walk and if $\nu^-(C_\nu) = 1$, then for all $a \in A$ and ν^--almost every $c^- \in C^-$ the limit $\lim_{n\to\infty} \gamma_n(a;\ c^-)$ exists and corresponds to

$$\gamma_\infty(c^-) := \begin{cases} a_0, & \text{if } c^- \in C_{[0]} \cap C_\nu, \\ \gamma_{n_M}(a_0;\ c^-), & \text{if } c^- \in C_{[M]} \cap C_\nu. \end{cases}$$

Proof. By definition, the quotient graph is an integer random walk, so irreducibility and synchronizability induce the ν^--almost sure convergence of its transition functions $([\gamma]_n)_{n\in\mathbb{N}}$. Let now $a \in A$, then there exists $m \in \mathbb{N}_0$, such that $a \in A_m$, i.e., a is represented by $[A_m]$. By Theorem 5.2.21 and Corollary 5.2.36 follows for $c^- \in C_{[0]} \cap C_\nu$:

$$\gamma_n(a;\ c^-) \in [\gamma]_n([A_m];\ c^-) \underset{n\to\infty}{\longrightarrow} [\gamma]_N([A_0];\ c^-) = [A_0] = a_0$$

and respectively for $c^- \in C_{[M]} \cap C_\nu$:

$$\gamma_n(a;\ c^-) \in [\gamma]_n([A_m];\ c^-) \underset{n\to\infty}{\longrightarrow} [\gamma]_{n_M}([A_0];\ c^-) = \gamma_{n_M}(a_0;\ c^-).$$

Synchronizability of the underlying graph follows by the same argument. Any word that synchronizes a subset of the quotient graph into $A_0 = \{a_0\}$ is also synchronizing for the corresponding vertices in the original graph. □

Nevertheless, it is reasonable to assume that it should be possible to infer almost certain convergence also for the case when the original graph (A, C, γ) is synchronizable without having to reduce A_0 to a one-element set. We briefly present one of the difficulties that have to be dealt with in this setting. Suppose that a finite synchronizable word exists for each subset A_m of the respective partition. In fact, such a word almost certainly appears in every color sequence c^-. In this case, however, it would also have to be guaranteed that at the time of the appearance of the synchronizing word, the initial vertex has been mapped into the corresponding set, which is then synchronized.

5.3 POSITIVE RECURRENT INTEGER RANDOM WALKS

By construction, the one-sided quantum Markov processes considered in this thesis are stationary. Consequently, the classical Markov chains that are considered when dealing with scattering theoretical questions, in particular, with the topic of asymptotic completeness, are assumed to possess an invariant probability distribution. For irreducible Markov chains with countable state space, this amounts to the property of positive recurrence (compare Lemma 1.2.8). In this context, one thought arises immediately: How does the sufficient condition $\nu^-(C_\nu) = 1$ relate to positive recurrence? This is motivated by the observation that in Example 5.2.1, positive-recurrence is given if and only if $\lambda > \frac{1}{2}$ [Kle08, Ex. 17.32]. In this case, it follows obviously that $\nu^-(C_\nu) = 1$. We show that this is also the case for general integer random walks. The idea for the following proof arose in a discussion with F. Aurzada.

> PROPOSITION 5.3.1. Let $(A, C, \gamma; \mu, \nu)$ be an irreducible positive recurrent integer random walk. Then, $\nu^-(C_\nu) = 1$.

Proof. By definition, positive recurrence means that for any choice of an initial vertex, the expected time of the first return to the initial vertex is finite. In particular, it implies recurrence, i.e., for ν^--almost every color sequence, the corresponding walk returns infinitely often to the chosen initial vertex.

We remember that, $C_\nu = \{c^- \in C^- : \lim_{n\to\infty} S_n(c^-) = -\infty\}$. From a stochastic perspective, $(S_n)_{n\in\mathbb{N}}$ is a random walk with independent identically distributed random variables $S_1, S_2, \ldots$ In particular, $1 = \nu^-(C_\nu) = \nu^-(\lim_{n\to\infty} S_n = -\infty)$ if and only if

$\mathbb{E}(\lim_{n\to\infty} S_n) = -\infty$, where $\mathbb{E}$ denotes the expected value. We will show this in the following.

Let $c^- \in C^-$, $n \in \mathbb{N}$ and fix a_0 as initial vertex.
We are particularly interested in the first step after returning to a_0 and define thus the following set:

$$R_n(c^-) := \{0 \leq r < n \ : \ \gamma_{n-r}(a_0; c_{-n}, \ldots c_{-r+1}) = a_0\}.$$

In other words, two consecutive elements in $R_n(c^-)$ mark the beginning and the end of a path in c^- that has a_0 as initial and terminal vertex but does not return to a_0 in-between. In the following, the size of the set $R_n(c^-)$ is considered. To guarantee better readability, we write $|R_n|$ instead of $|R_n(c^-)|$. Setting $r_0 := n$, we can also write $R_n(c^-) := \{r_1, \ldots, r_{|R_n|}\}$, where for every $1 \leq i < |R_n|$ holds $r_{i+1} < r_i$ and

$$\gamma_{r_i - r_{i+1}}(a_0; c_{-r_i}, \ldots, c_{-r_{i+1}+1}) = a_0$$

for the first time. That is, for all $r_{i+1} \leq l < r_i$ holds $\gamma_{r_i - l}(a_0; c_{-r_i}, \ldots, c_{-l+1}) \neq a_0$.
Next, for every time $r_i \in R_n(c^-)$ we consider the corresponding element in the color sequence c_{-r_i}. As the colors C are a subset of $\mathbb{Z}$, we differentiate between three cases:

$$c_{-r_i} > 0, \quad c_{-r_i} = 0 \quad \text{and} \quad c_{-r_i} < 0.$$

The set $R_n(c^-)$ corresponds hence to a disjoint union of the following three sets:

$$\begin{aligned}
R_{n>}(c^-) &:= \{r_i \in R_n(c^-) \ : \ c_{-r_i} > 0\}, \\
R_{n=}(c^-) &:= \{r_i \in R_n(c^-) \ : \ c_{-r_i} = 0\}, \\
R_{n<}(c^-) &:= \{r_i \in R_n(c^-) \ : \ c_{-r_i} < 0\}.
\end{aligned}$$

Now, the n-step length of c^- can be decomposed with respect to $R_n(c^-)$ and $n := r_0$:

$$\begin{aligned}
S_n(c^-) &= \underbrace{\sum_{k=r_1+1}^{r_0} c_{-k} + \sum_{k=r_2+1}^{r_1} c_{-k} + \cdots + \sum_{k=r_{|R_n|}+1}^{r_{|R_n|-1}} c_{-k}}_{\text{each sum corresponds to a path with initial and terminal vertex } a_0} + \underbrace{\sum_{k=1}^{r_{|R_n|}} c_{-k}}_{(\star)} \\
&= \sum_{i=0}^{r_{|R_n|-1}} \left(\sum_{k=r_{i+1}+1}^{r_i} c_{-k} \right) + \sum_{k=1}^{r_{|R_n|}} c_{-k},
\end{aligned}$$

where $(\star)$ corresponds to a path, which, when starting in a_0 at time $r_{|R_n|}$, does not return to a_0. For the terms in the first sum, we differentiate between the above introduced three cases:

$r_i \in R_{n>}(c^-)$. In this case, $\gamma(a_0; c_{-r_i}) = a_{c_{-r_i}} \neq a_0$. In particular follows $r_i > r_{i+1} + 1$ and $\sum_{k=r_{i+1}+1}^{r_i} c_{-k} \leq 0$, since every step leading away from a_0 has to be reversed on the way back to a_0.

$r_i \in R_{n=}(c^-)$. In this case, $\gamma(a_0; c_{-r_i}) = a_0$. In particular follows $r_i = r_{i+1} + 1$ and since $c_{-r_i} = 0$ it follows that $\sum_{k=r_{i+1}+1}^{r_i} c_{-k} = 0$.

$r_i \in R_{n<}(c^-)$. In this case, $\gamma(a_0; c_{-r_i}) = a_0$. In particular follows $r_i = r_{i+1} + 1$ and since $c_{-r_i} < 0$ it follows that $\sum_{k=r_{i+1}+1}^{r_i} c_{-k} = c_{-r_i} < 0$.

It is thus possible to bound the value $S_n(c^-)$ from above:

$$S_n(c^-) \leq \sum_{r_i \in R_{n<}} c_{-r_i} + \sum_{k=1}^{r_{|R_n|}} c_{-k} \leq -|R_{n<}| + \sum_{k=1}^{r_{|R_n|}} c_{-k}.$$

To the time of first return in the initial vertex and to the number of visits in the initial vertex correspond two independent random variables. We obtain thus an upper bound for the expected value of $\lim_{n\to\infty} S_n$:

$$\begin{aligned}
\mathbb{E}(\lim_{n\to\infty} S_n) \quad &\leq \quad (-1) \cdot \underbrace{\mathbb{E}(\text{number of visits in the initial vertex } a_0)}_{=\infty,\text{ by recurrence}} \cdot \underbrace{\nu(\{c \in C \;:\; c < 0\})}_{>0,\text{ by Cor. 5.2.7}} \\
&\quad \cdot \underbrace{\mathbb{E}(\text{time of first return})}_{<\infty,\text{ by positive recurrence}} \cdot \underbrace{\max\{c \in C\}}_{<\infty} \\
&= \quad -\infty.
\end{aligned}$$

□

It is thus possible to replace the sufficient condition $\nu^-(C_\nu) = 1$ in the preceding sections with the assumption that the considered integer random walks are positive recurrent. In particular, Theorem 5.2.22 can be reformulated accordingly.

THEOREM 5.3.2. Let $(A, C, \gamma; \mu, \nu)$ be a positive-recurrent integer random walk with invariant probability distribution μ and let J be the corresponding transition. Then, the following assertions are equivalent:

(a) The transition J is asymptotically complete.

(b) The underlying road-colored graph (A, C, γ) is irreducible and synchronizable.

(c) The underlying road-colored graph (A, C, γ) is irreducible.

(d) The sequence of transition functions $(\gamma_n)_{n\in\mathbb{N}}$ converges ν^--almost surely towards a function γ_∞ that does no longer depend on the chosen initial vertex.

6 SYNCHRONIZABILITY AND ALMOST UNIFORM CONVERGENCE

True stories can't be told forward, only backward.
We invent them from the vantage point of an ever-changing
present and tell ourselves how they unfolded.

— Siri Hustvedt

After primarily focusing on gaining a better understanding of synchronizability in the classical setting, we now turn our attention to one-sided quantum Markov processes. The motivation for these non-commutative considerations stemmed from an idea proposed by B. Kümmerer, to extend the classically observed property of almost sure convergence of the n-step transition functions in a suitable way into the non-commutative framework. In a first instance, one quickly runs into the problem that ν^--almost sure convergence is formulated with respect to the base space $A \times C^-$ and has no algebraic analog. In the early 1900s, the mathematicians D.F. Egorov and C. Severini, independent of one another, established a condition for the uniform convergence of a pointwise convergent sequence of measurable functions [Ego11], [Sev10]. From the theorem it follows that almost sure convergence of a sequence of measurable functions is equivalent to its almost uniform convergence, which can be expressed algebraically without any reference to the base space. Section 6.1 transfers this to a sequence of transition functions. Further, their almost uniform convergence is illustrated for aperiodic irreducible finite road-colored graphs and positive recurrent integer random walks.

In Section 6.2, we present the non-commutative counterpart to the notion of almost uniform convergence. The first to actually apply this concept within the framework of von Neumann algebras were C. Lance [Lan76] and B. Kümmerer [Küm78]. Utilizing non-commutative martingale theory, we demonstrate that the asymptotic completeness of the transition corresponding to a one-sided quantum Markov process implies the almost uniform convergence of the sequences $(P_{\hat{\varphi}}\hat{S}^{-n}\hat{J}^n i(x))_{n\in\mathbb{N}}$ and $(P_{\varphi}\hat{S}^{-n}\hat{J}^n i(x))_{n\in\mathbb{N}}$ for every choice of $x \in \mathcal{A}^+$. This opens the door to a new interpretation of synchronizability in terms of scattering theory in the non-commutative setting.

6.1 IN THE CLASSICAL SETTING

Let $\mathbb{K}$ correspond to $\mathbb{C}$ or $\mathbb{R}$. We present the notion of almost uniform convergence of a sequence of measurable $\mathbb{K}$-valued functions on a classical probability space $(\Omega, \Sigma, \mathbb{P})$, which is, due to the so-called Egorov's Theorem, equivalent to the $\mathbb{P}$-almost sure convergence of the sequence. The necessary theoretical background is based on [Els18, Section VI. §3].

Definition 6.1.1. A sequence of measurable functions $(f_n)_{n\in\mathbb{N}}$, with $f_n : \Omega \to \mathbb{K}$ *converges almost uniformly* towards a measurable function $f : \Omega \to \mathbb{K}$, if for all $\varepsilon > 0$ exists $S_\varepsilon \in \Sigma$, such that $\mathbb{P}(\Omega \setminus S_\varepsilon) < \varepsilon$ and $(f_n)_{n\in\mathbb{N}}$ converges on S_ε uniformly towards f, i.e., for every $\delta > 0$ exists an $N \in \mathbb{N}$, such that for all $n \geq N$ and all $\omega \in S_\varepsilon$ holds $|f_n(\omega) - f(\omega)| < \delta$.

In other words, almost uniform convergence corresponds to uniform convergence in the complement of appropriate sets of arbitrarily small measure. It is easy to verify that almost uniform convergence of a sequence of measurable functions implies their $\mathbb{P}$-almost sure convergence [Els18, Lem. VI.3.4].

LEMMA 6.1.2. Let $(f_n)_{n\in\mathbb{N}}$ be an almost uniformly convergent sequence of measurable functions on Ω. Then there exists a measurable function $f : \Omega \to \mathbb{K}$, such that $(f_n)_{n\in\mathbb{N}}$ converges $\mathbb{P}$-almost surely towards f.

The following result was first published by C. Severini in 1910 [Sev10]. However, it initially went unnoticed and was then independently introduced by D.F. Egorov in 1911 [Ego11]. F. Riesz proved it later in a more modern measure theoretical setting [Rie22], [Rie28]. It is now known as Egorov's Theorem, or sometimes also Egorov-Severini Theorem. The result was originally formulated for finite measure spaces, so it is, in particular, true for the probability space $(\Omega, \Sigma, \mathbb{P})$. Strictly speaking, the following assertion corresponds to a special case of the original theorem, which can, for example, be reviewed in [Els18, Thm. VI.3.5].

THEOREM 6.1.3 (Egorov's Theorem). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n : \Omega \to \mathbb{K}$, which converges $\mathbb{P}$-almost everywhere to a measurable function $f : \Omega \to \mathbb{K}$. Then it converges almost uniformly towards f.

6.1.1 *Almost uniform convergence for discrete metric spaces*

The above assertions can also be shown for sequences of functions with values in a separable metric space. For our considerations, it is sufficient to consider a discrete metric space (X, d), with discrete metric $d : X \times X \to \{0, 1\}$, where

$$d(x, y) := \begin{cases} 0, & \text{for } x = y, \\ 1, & \text{for } x \neq y. \end{cases}$$

Since d takes only the values 0 and 1, one easily realizes that the definition of almost uniform convergence in this context can be formulated as follows:

Definition 6.1.4. A sequence of measurable X-valued functions $(f_n)_{n\in\mathbb{N}}$ on Ω *converges almost uniformly* towards a measurable function $f : \Omega \to X$, if for all $\varepsilon > 0$ exists $S_\varepsilon \in \Sigma$, such that $\mathbb{P}(\Omega \setminus S_\varepsilon) < \varepsilon$ and $(f_n)_{n\in\mathbb{N}}$ converges on S_ε uniformly towards f. In this particular case, this means that there exists an $N \in \mathbb{N}$, such that for all $n \geq N$ and all $\omega \in S_\varepsilon$ holds $d(f_n(\omega), f(\omega)) = 0$, i.e. $f_n(\omega) = f(\omega)$.

It is easy to prove the equivalence of almost uniform convergence and $\mathbb{P}$-almost sure convergence of a sequence of measurable X-valued functions. For convenience, we include a proof.

PROPOSITION 6.1.5. Let f and f_n, for all $n \in \mathbb{N}$, be measurable X-valued functions on Ω. Then, the following assertions are equivalent:

(a) The sequence $(f_n)_{n\in\mathbb{N}}$ converges $\mathbb{P}$-almost surely towards f.

(b) The sequence $(f_n)_{n\in\mathbb{N}}$ converges almost uniformly towards f.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of X-valued measurable functions on Ω.

$(a) \Rightarrow (b)$. Suppose that $(f_n)_{n\in\mathbb{N}}$ converges $\mathbb{P}$-almost surely towards a measurable function $f : \Omega \to X$. Since the functions f_n and f take values in the discrete set X, there exists a subset $D \subseteq \Omega$, such that $\mathbb{P}(D) = 1$ and for every $\omega \in D$, the values $f_n(\omega)$ become eventually constant. There exists hence a natural number N_ω such that for all $n \geq N_\omega$ holds $f_n(\omega) = f_{N_\omega}(\omega)$.
Let now $\varepsilon > 0$, and set $D_n := \{\omega \in D \ : \ n \geq N_\omega\} \subseteq D$. Then, $D_n \subseteq D_{n+1}$, $\bigcup_{n\in\mathbb{N}} D_n = D$ and by continuity from below of the probability measure $\mathbb{P}$ we can choose $N \in \mathbb{N}$, such that the subset D_N has measure $\mathbb{P}(D_N) \geq 1 - \varepsilon$. For every $\omega \in D_N$ and all $n \geq N$ holds $f_n(\omega) = f_N(\omega)$, i.e., $d(f_n(\omega), f_N(\omega)) = 0$.

$(b) \Rightarrow (a)$. Suppose that $(f_n)_{n\in\mathbb{N}}$ converges almost uniformly towards a measurable function $f : \Omega \to X$. Then, for every $k \in \mathbb{N}$ exists a set $S_k \in \Sigma$ such that

$\mathbb{P}(S_k) \geq 1 - \frac{1}{k}$ and $(f_n)_{n\in\mathbb{N}}$ converges uniformly towards f on S_k.
For $m \in \mathbb{N}$, we set

$$D_m := \bigcup_{k=1}^{m} S_k,$$

then clearly $D_m \subseteq D_{m+1}$ and $(f_n)_{n\in\mathbb{N}}$ is still uniformly convergent on D_m. In particular, $\mathbb{P}(\bigcup_{m\in\mathbb{N}} D_m) = \lim_{m\to\infty} \mathbb{P}(D_m) = 1$.
Let $\omega \in \bigcup_{m\in\mathbb{N}} D_m$, then exists $m \in \mathbb{N}$, such that $\omega \in D_m$. Since $(f_n)_{n\in\mathbb{N}}$ converges uniformly on D_m, there exists a natural number N, such that for all $n \geq N$ holds $d(f_n(\omega), f(\omega)) = 0$. In other words $\lim_{n\to\infty} f_n(\omega) = f(\omega)$.

□

Let now $(A, C, \gamma; \mu, \nu)$ be a road-colored graph with countable state space A. If we equip A with the discrete metric d, Proposition 6.1.5 can be applied to the sequence $(\gamma_n)_{n\in\mathbb{N}}$ of transition functions, together with Theorems 2.3.10 and 5.3.2, this leads to the following assertion.

COROLLARY 6.1.6. Let $(A, C, \gamma; \mu, \nu)$ be an aperiodic irreducible finite road-colored graph or a positive recurrent irreducible integer random walk with invariant probability distribution μ on A. Then, the following assertions are equivalent:

(a) The corresponding transition J is asymptotically complete.

(b) (A, C, γ) possesses a synchronizing word.

(c) The sequence of transition functions $(\gamma_n)_{n\in\mathbb{N}}$ converges ν^--almost surely towards a function γ_∞ that does no longer depend on the chosen initial vertex.

(d) The sequence of transition functions $(\gamma_n)_{n\in\mathbb{N}}$ converges almost uniformly towards a function γ_∞ that does no longer depend on the chosen initial vertex.

We want to remark that the almost sure convergence of the n-step transition functions can also be understood as a classical analog of the Møller operator. Indeed, we obtain an almost everywhere defined map on the base spaces

$$\begin{array}{cccc} \phi_J : & \hat{C} & \to & A \\ & (c^-, c^+) & \mapsto & \gamma_\infty(c^-). \end{array}$$

On the other hand, the Møller operator $\Phi_J : \mathcal{A}^+ \to \hat{\mathcal{C}}$ is defined for the corresponding transition (compare Definition 2.3.6). Let $f \in \mathcal{A}$, then for all $a \in A$ and ν^--almost every

$c^- \in C^-$ we obtain

$$\Phi_J(f)(a,\hat{c}) = \lim_{n\to\infty} f \circ \hat{\gamma}_n \sigma^{-n}(a,\hat{c}) = \lim_{n\to\infty} f \circ \gamma_n(a; c^-) = f \circ \gamma_\infty(c^-) = f \circ \phi_J(c^-),$$

where we adopted the notation that was introduced in Section 2.3.2 (in particular in the proof of Lemma 2.3.8). For further elaborations on this correspondence, we refer to [GKL06, Section 2], [Lan03, Section 4.3].

Consequently, the equivalence between almost sure and almost uniform convergence of measurable functions translates into the fact that for every $f \in \mathcal{A}$, the sequence $(f_n)_{n\in\mathbb{N}}$, where $f_n := f \circ \gamma_n$, converges almost uniformly towards $\Phi_J(f \otimes \mathbb{1}) = \mathbb{1} \otimes f(\gamma_\infty)$.

Of course, it would be quite revealing to understand the nature of the subsets S_ε on which the sequence of transitions $(\gamma_n)_{n\in\mathbb{N}}$, and hence also $(f_n)_{n\in\mathbb{N}}$ for every $f \in \mathcal{A}$, converges uniformly. In the following, we will see that this construct is quite intuitive for aperiodic irreducible road-colored graphs with finite state space.

6.1.2 *Almost uniform convergence and finite graphs*

Let $(A, C, \gamma; \mu, \nu)$ be an aperiodic irreducible road-colored graph with finite state space. Suppose that it is synchronizable. As before, we denote by C_∞ the subset of color sequences in C^- that contain a synchronizing word. From Theorem 3.1.6 we know that $\nu^-(C_\infty) = 1$ and hence also $\mu \otimes \nu^-(A \times C_\infty) = 1$.

Let $\varepsilon > 0$.
For $n \in \mathbb{N}$, we set

$$\begin{aligned} C_{n,\infty} &:= \{c^- \in C_\infty \,:\, \gamma_\infty(c^-) = \gamma_n(a; c^-) \text{ for all } a \in A\} \\ &= \{c^- \in C_\infty \,:\, (c_{-n}, \ldots, c_{-1}) \text{ synchronizes } A\}. \end{aligned}$$

Obviously $C_{n,\infty} \subseteq C_{(n+1),\infty}$ and $\bigcup_{n\in\mathbb{N}} C_{n,\infty} = C_\infty$, so by continuity from below of ν^-, there exists a natural number $n_\varepsilon \in \mathbb{N}$, such that $\nu^-(C_{n_\varepsilon,\infty}) \geq 1 - \varepsilon$. Moreover, for every $c^- \in C_{n_\varepsilon,\infty}$ follows that $\gamma_n(\cdot\,; c^-)$ is constant for $n \geq n_\varepsilon$ (i.e., independent of the chosen initial vertex $a \in A$).

Consequently, the sequence of transition functions $(\gamma_n)_{n\in\mathbb{N}}$ converges uniformly on the set $S_\varepsilon := A \times C_{n_\varepsilon,\infty}$ and $\mu \otimes \nu^-((A \times C^-) \setminus S_\varepsilon) < \varepsilon$.

6.1.3 Almost uniform convergence and integer random walks

The challenge in the examination of the synchronizability of road-colored graphs with countably infinite state space was to gain control over the sets of vertices that have been synchronized by a certain word. In particular, these vertices must remain synchronized when the word is prolonged, i.e., a larger section of the corresponding color sequence is considered.

Let $(A, C, \gamma; \mu, \nu)$ be a positive recurrent irreducible integer random walk with countably infinite state space $A = \{a_m \ : \ m \in \mathbb{N}\}$.

Let $\varepsilon > 0$ and choose $\delta > 0$, such that $1 - \varepsilon = (1 - \delta)^2$.
Since A is countable, there exists a smallest $m_\delta \in \mathbb{N}$ such that $\mu(\{a_1, \ldots, a_{m_\delta}\}) \geq 1 - \delta$. We set hence

$$A_\delta := \{a_1, \ldots, a_{m_\delta}\}.$$

Let now $c^- \in C^-$. From the third and fourth step in the proof of Theorem 5.2.21 we know that there exists ν^--almost surely a number $l_{m_\delta} \in \mathbb{N}$ such that for all $n \geq l_{m_\delta}$ and $a \in A_\delta$ holds

$$\gamma_n(a; c^-) = \gamma_\infty(c^-),$$

where γ_∞ is defined as in Theorem 5.2.21.

Clearly, l_{m_δ} depends on the choice of c^-, so $l_\delta(c^-) := l_{m_\delta}$ defines a function $l_\delta : C^- \to \mathbb{N}$. For $n \in \mathbb{N}$, we set

$$\begin{aligned} C_\infty &:= \{c^- \in C^- \ : \ \lim_{n\to\infty} \gamma_n(a; c^-) = \gamma_\infty(c^-)\}, \\ C_{n,\infty} &:= \{c^- \in C_\infty \ : \ n \geq l_\delta(c^-)\}. \end{aligned}$$

Then clearly $C_{n,\infty} \subseteq C_{(n+1),\infty}$ and $\bigcup_{n\in\mathbb{N}} C_{n,\infty} = C_\infty$. With Theorem 5.3.2 follows $\nu^-(C_\infty) = 1$, so by continuity from below of ν^-, there must exist a natural number $n_\delta \in \mathbb{N}$, such that $\nu^-(C_{n_\delta,\infty}) \geq 1 - \delta$. By definition, for all $(a, c^-) \in A_\delta \times C_{n_\delta,\infty}$ follows that $\gamma_n(a; c^-)$ is constant (and equal to $\gamma_\infty(c^-)$), for $n \geq n_\delta$.

Due to the choice of δ, the sequence of transition functions $(\gamma_n)_{n\in\mathbb{N}}$ converges hence uniformly on $S_\varepsilon := A_\delta \times C_{n_\delta,\infty}$ and $\mu \otimes \nu^-(S_\varepsilon) \geq (1 - \delta)(1 - \delta) = 1 - \varepsilon$.

6.2 IN THE NON-COMMUTATIVE SETTING

Motivated by the classical case, where the almost sure and hence almost uniform convergence of the transition functions of a road-colored graph with invariant initial distribution seems to be a promising candidate to characterize asymptotic completeness of the corresponding transition, we transfer the idea to the non-commutative setting and present the non-commutative version of almost uniform convergence. Closely following the elaborations of R. Jaijte [Jaj85, Chapter 1] and N. Dang-Ngoc [Dan79], we provide the necessary theoretical background for our considerations on asymptotic completeness. For the notation we refer the reader to Section 2.3.

Definition 6.2.1. Let $(\mathcal{A}, \varphi)$ be a quantum probability space. A sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ converges *almost uniformly* towards an element $a \in \mathcal{A}$, if for each $\varepsilon > 0$ there exists a projection $p \in \mathcal{A}$, such that:

$$\varphi(p) \geq 1 - \varepsilon \quad \text{and} \quad \lim_{n\to\infty} \|(a_n - a)p\| = 0.$$

Just like in the classical case, almost uniform convergence implies the convergence in the strong operator topology [Jaj85, Thm. 1.1.3].

PROPOSITION 6.2.2. Let $(\mathcal{A}, \varphi)$ be a quantum probability space. For a bounded, almost uniform convergent sequence of operators $(a_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ follows convergence in the strong operator topology as well.

In the classical setting, Egorov's theorem provides an equivalence between the two notions of convergence. We will now see it is, in general, not possible to preserve the equivalence. The generalization of Egorov's Theorem to von Neumann algebras is due to K. Saitô [Sai67, Thm. 1] (compare also [Jaj85, Thm. 1.3.2]).

THEOREM 6.2.3 (non-commutative Egorov's Theorem). Let $(\mathcal{A}, \varphi)$ be a quantum probability space and let $(a_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ converge to $a \in \mathcal{A}$ in the strong operator topology. Then, for any projection $q \in \mathcal{A}$ and each $\varepsilon > 0$, there exists a projection $p \leq q$ in $\mathcal{A}$ and a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$, such that:

$$\varphi(q - p) < \varepsilon \quad \text{and} \quad \lim_{n\to\infty} \|(a_{n_k} - a)p\| = 0.$$

In particular, almost uniform convergence implies convergence in the strong operator topology, which, in turn, only implies almost uniform convergence of a subsequence.

So, what does this mean for our setting?

Let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ be two quantum probability spaces with finite-dimensional von Neumann algebras $\mathcal{A}$ and $\mathcal{C}$. We assume the construction and notation introduced in Chapter 2.3.

Let $(\mathcal{A}^+, \varphi^+, J^+; \mathcal{A}, i^+)$ be a one-sided quantum Markov process as introduced in Section 2.3. By Theorem 2.3.5, it is asymptotically complete if and only if $(\hat{S}^{-n}\hat{J}^n i(x))_{n\in\mathbb{N}}$ converges in the $\|\cdot\|_{\hat{\varphi}}$-norm for all $x \in \mathcal{A}^+$. Let $(\mathcal{H}^+, \xi^+)$ denote the GNS-Hilbert space corresponding to $(\mathcal{A}^+, \varphi^+)$, then asymptotic completeness is given if and only if the corresponding sequence in $\mathcal{B}(\mathcal{H}^+)$ converges in the strong operator topology. The following is, hence, an immediate consequence of Proposition 6.2.2.

Corollary 6.2.4. Let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ be quantum probability spaces with finite-dimensional von Neumann algebras $\mathcal{A}$ and $\mathcal{C}$ and let $(\mathcal{A}^+, \varphi^+, J^+; \mathcal{A}, i^+)$ be a one-sided quantum Markov process. Suppose that the sequence $(\hat{S}^{-n}\hat{J}^n i(x))_{n\in\mathbb{N}}$ converges almost uniformly for every $x \in \mathcal{A}^+$. Then, the transition J is asymptotically complete.

On the other hand, when assuming asymptotic completeness, for $x \in \mathcal{A}^+$, the non-commutative Egorov's Theorem 6.2.3 offers almost uniform convergence only for a subsequence $(\hat{S}^{-n_k}\hat{J}^{n_k} i(x))_{k\in\mathbb{N}}$.

In the following we will show that it is, at least, possible to infer almost uniform convergence of the sequences $(P_{\hat{\psi}}\hat{S}^{-n}\hat{J}^n i(x))_{n\in\mathbb{N}}$ and $(P_{\varphi}\hat{S}^{-n}\hat{J}^n i(x))_{n\in\mathbb{N}}$.

6.2.1 *Almost uniform convergence on $\mathcal{A} \otimes \mathbb{1}$*

By $P_{\hat{\psi}}$ we denote the conditional expectation of tensor type from $\hat{\mathcal{A}} = \mathcal{A} \otimes \hat{\mathcal{C}}$ onto $\mathcal{A} \otimes \mathbb{1}_{\hat{\mathcal{C}}}$.

Proposition 6.2.5. Let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ be quantum probability spaces with finite-dimensional von Neumann algebras $\mathcal{A}$ and $\mathcal{C}$ and let $(\mathcal{A}^+, \varphi^+, J^+; \mathcal{A}, i^+)$ be a one-sided quantum Markov process. Suppose that the corresponding transition is asymptotically complete. Then, for every $x \in \mathcal{A}^+$, the sequence $(P_{\hat{\psi}}\hat{S}^{-n}\hat{J}^n i(x))_{n\in\mathbb{N}}$ converges in norm towards $P_{\hat{\psi}}\Phi_J(x)$, where $\Phi_J : \mathcal{A}^+ \to \mathbb{1}_{\mathcal{A}} \otimes \hat{\mathcal{C}}$ corresponds to the Møller operator.

Proof. Asymptotic completeness of J implies the $\|\cdot\|_{\hat{\varphi}}$-convergence of $(\hat{S}^{-n}\hat{J}^n i(x))_n$ towards $\Phi_J(x)$ for all $x \in \mathcal{A}^+$. Thus, an application of the conditional expectation $P_{\hat{\psi}}$

leads to

$$\begin{aligned}\underbrace{P_{\hat{\psi}}(\Phi_J(x))}_{\in \mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\hat{\mathcal{C}}}} &= P_{\hat{\psi}}\left(\|\cdot\|_{\hat{\varphi}} - \lim_{n\to\infty} \hat{S}^{-n}\hat{J}^n i(x)\right) \\ &= \|\cdot\|_{\hat{\varphi}} - \lim_{n\to\infty} \underbrace{P_{\hat{\psi}}\hat{S}^{-n}\hat{J}^n i(x)}_{\in \mathcal{A}\otimes \mathbb{1}_{\hat{\mathcal{C}}}} \\ &= \|\cdot\| - \lim_{n\to\infty} P_{\hat{\psi}}\hat{S}^{-n}\hat{J}^n i(x),\end{aligned}$$

for $x \in \mathcal{A}^+$, where the last equality is due to the fact that on finite-dimensional von Neumann algebras, stop-convergence and norm convergence coincide. □

COROLLARY 6.2.6. Let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ be quantum probability spaces with finite-dimensional von Neumann algebras $\mathcal{A}$ and $\mathcal{C}$ and let $(\mathcal{A}^+, \varphi^+, J^+; \mathcal{A}, i^+)$ be a one-sided quantum Markov process. Suppose that the corresponding transition is asymptotically complete. Then, for every $x \in \mathcal{A}^+$, the sequence $(P_{\hat{\psi}}\hat{S}^{-n}\hat{J}^n i(x))_{n\in\mathbb{N}}$ converges almost uniformly towards $\hat{\psi}(\Phi_J(x)) \cdot \mathbb{1}_{\hat{\mathcal{A}}}$.

Proof. This is a direct consequence of the proof of Proposition 6.2.5. Obviously, convergence in the norm topology implies almost uniform convergence. And since $\Phi_J(x) \in \mathbb{1}_{\mathcal{A}} \otimes \hat{\mathcal{C}}$ for all $x \in \mathcal{A}^+$, $P_{\hat{\psi}}(\Phi_J(x))$ is of the form $\hat{\psi}(\Phi_J(x)) \cdot \mathbb{1}_{\hat{\mathcal{A}}}$. □

6.2.2 *Almost uniform convergence on* $\mathbb{1} \otimes \hat{\mathcal{C}}$

In this section we will show that the asymptotic completeness of the corresponding transition J induces the almost uniform convergence of the sequence of operators $(P_\varphi \hat{S}^{-n}\hat{J}^n i(x))_{n\in\mathbb{N}}$ for all $x \in \mathcal{A}^+$, where P_φ denotes the conditional expectation of tensor type from $\hat{\mathcal{A}} = \mathcal{A} \otimes \hat{\mathcal{C}}$ onto $\mathbb{1}_{\mathcal{A}} \otimes \hat{\mathcal{C}}$. This needs some preparatory work. In the first step, we investigate the concrete form of the sequence $\hat{S}^{-n}\hat{J}^n i(x)$ for an $x \in \mathcal{A}^+$. In the second step, we give a short introduction to non-commutative martingale theory that goes back to N. Dang-Ngoc [Dan79]. By bringing this together, we then conclude the desired convergence.

By assumption, $\mathcal{A}$ and $\mathcal{C}$ are finite-dimensional von Neumann algebras. As every von Neumann algebra is a vector space and since finite-dimensional vector spaces possess a finite basis, we find bases $(a_i)_i$ of $\mathcal{A}$ and $(c_k)_k$ of $\mathcal{C}$. Then, with respect to the basis of $\mathcal{A}$ an element $x \in \mathcal{A}^+ = \mathcal{A} \otimes \mathcal{C}^+$ is a finite sum of the form $x = \sum_i a_i \otimes c_i^+$, for some elements $c_i^+ \in \mathcal{C}^+$. So, without loss of generality, consider $a \otimes c^+ \in \mathcal{A}^+$. For better readability we write $\mathbb{1}_{k]} := \otimes_{-\infty}^{k} \mathbb{1}_{\mathcal{C}}$ for $k < 0$, then, in particular, $i(a \otimes c^+) = \mathbb{1}_{-1]} \otimes a \otimes c^+$. With respect to the basis $(c_k)_k$ of C there exist $j_k : \mathcal{A} \to \mathcal{A}$, such that

$$\begin{aligned} \hat{J}i(a \otimes c^+) &= \mathbb{1}_{-1]} \otimes \sum_{k_1} j_{k_1}(a) \otimes c_{k_1} \otimes c^+, \\ &\vdots \\ \hat{J}^n i(a \otimes c^+) &= \mathbb{1}_{-1]} \otimes \sum_{k_1,\dots,k_n} j_{k_n} \circ \dots \circ j_{k_1}(a) \otimes c_{k_n} \otimes \dots \otimes c_{k_1} \otimes c^+, \end{aligned}$$

for all $n \in \mathbb{N}$. Application of the n-fold tensor shift now leads to

$$\hat{S}^{-n} \hat{J}^n i(a \otimes c^+) = \mathbb{1}_{-(n+1)]} \otimes c_{k_n} \otimes \dots \otimes c_{k_1} \otimes \sum_{k_1,\dots,k_n} j_{k_n} \circ \dots \circ j_{k_1}(a) \otimes c^+.$$

The goal of the following considerations is to show that the asymptotic completeness of J implies almost uniform convergence of sequences of the form $P_\varphi \hat{S}^{-n} \hat{J}^n i(x))_{n \in \mathbb{N}}$, for $x \in \mathcal{A}^+$. Here, the non-commutative martingale theory proves to be a useful tool [Dan79].

We introduce the results that are relevant for our considerations. For further theoretical background on this subject, we refer to [Dan79], [Jaj85, Section 3.2].

Definition 6.2.7. Let $(\mathcal{A}, \varphi)$ be a quantum probability space. An increasing sequence $(\mathcal{A}_n)_{n \in \mathbb{N}}$ of von Neumann subalgebras of $\mathcal{A}$, i.e., $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ for all $n \in \mathbb{N}$, with conditional expectations $P_n : \mathcal{A} \to \mathcal{A}_n$ with respect to φ, is called a *filtration*.
We say that a sequence $(x_n)_{n \in \mathbb{N}}$ of elements in $\mathcal{A}$ is a *martingale adapted to the filtration* $(\mathcal{A}_n)_{n \in \mathbb{N}}$ if the following conditions hold:

1. $x_n \in \mathcal{A}_n$ for all $n \in \mathbb{N}$.
2. $P_n(x_{n+1}) = x_n$ for all $n \in \mathbb{N}$.
3. $\sup_{n \in \mathbb{N}} \|x_n\| < \infty$.

Further, we denote by $\mathcal{A}_\infty$ the von Neumann subalgebra of $\mathcal{A}$ that is generated by the filtration $(\mathcal{A}_n)_{n \in \mathbb{N}}$, i.e.:

$$\mathcal{A}_\infty = \left(\bigcup_{n \in \mathbb{N}} \mathcal{A}_n \right)''.$$

N. Dang-Ngoc showed the following assertion on the almost uniform convergence of adapted martingales [Dan79, Thm. 9].

THEOREM 6.2.8. Let $(\mathcal{A},\varphi)$ be a quantum probability space and let $(x_n)_{n\in\mathbb{N}}$ be a martingale adapted to a filtration $(\mathcal{A}_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$. Then, there exists a unique $x \in \mathcal{A}_\infty$ such that $(x_n)_{n\in\mathbb{N}}$ converges to x ultrastrongly and almost uniformly. Furthermore, for every $n \in \mathbb{N}$ holds $x_n = P_n(x)$.

Example 6.2.9. Let $(\mathcal{A},\varphi)$ be a quantum probability space and let $(\mathcal{A}_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be a filtration. Then, for every $x \in \mathcal{A}$, the sequence $(P_n(x))_{n\in\mathbb{N}}$ is a martingale adapted to the filtration and converges almost uniformly towards $P_\infty(x)$, where P_∞ denotes the conditional expectation onto $\mathcal{A}_\infty$ [Dan79, Thm. 8].

We apply this to our setting, i.e., let $(\mathcal{A},\varphi)$ and $(\mathcal{C},\psi)$ be quantum probability spaces with finite-dimensional von Neumann algebras $\mathcal{A}$ and $\mathcal{C}$ and let $(\mathcal{A}^+,\varphi^+,J^+;\mathcal{A},i^+)$ be a one-sided quantum Markov process. The idea is to find a filtration with adapted martingale, such that the almost uniform convergence of the martingale induces the almost uniform convergence of $(P_\varphi \hat{S}^{-n}\hat{J}^n i(x))_{n\in\mathbb{N}}$ for all $x \in \mathcal{A}^+$.

First, we define a fitting filtration. For $n \in \mathbb{N}$ we set:

$$\hat{\mathcal{A}}_n := \mathbb{1}_{-(n+1)]} \otimes \underbrace{\mathcal{C} \otimes \cdots \otimes \mathcal{C}}_{n \text{ times}} \otimes \mathbb{1}_{\mathcal{A}} \otimes \mathcal{C}^+.$$

Clearly, $\hat{\mathcal{A}}_n \subset \hat{\mathcal{A}}_{n+1} \subset \hat{\mathcal{A}} = \mathcal{A} \otimes \hat{\mathcal{C}}$. We denote by P_n the corresponding conditional expectation of tensor type from $\hat{\mathcal{A}}$ onto $\hat{\mathcal{A}}_n$. Then the sequence of von Neumann subalgebras $(\hat{\mathcal{A}}_n)_{n\in\mathbb{N}}$ is a filtration. This leads to the following proposition.

PROPOSITION 6.2.10. Let $(\mathcal{A},\varphi)$ and $(\mathcal{C},\psi)$ be quantum probability spaces with finite-dimensional von Neumann algebras $\mathcal{A}$ and $\mathcal{C}$ and let $(\mathcal{A}^+,\varphi^+,J^+;\mathcal{A},i^+)$ be a one-sided quantum Markov process. For every $x \in \mathcal{A}^+$, the sequence of elements $(P_n\hat{S}^{-n}\hat{J}^n i(x))_{n\in\mathbb{N}}$ is a martingale adapted to the filtration $(\hat{\mathcal{A}}_n)_{n\in\mathbb{N}}$.

Proof. We verify the three conditions in Definition 6.2.7 for $a \otimes c^+ \in \mathcal{A}^+$ and $n \in \mathbb{N}$:

1. Clearly, $P_n\hat{S}^{-n}\hat{J}^n i(a \otimes c^+) \in \hat{\mathcal{A}}_n$.

2. Since for the conditional expectations holds

$$P_n P_{n+1} = P_n = P_{n+1} P_n,$$

the second condition follows if $P_n\hat{S}^{-(n+1)}\hat{J}^{n+1}i(a\otimes c^+) = P_n\hat{S}^{-n}\hat{J}^n i(a\otimes c^+)$.

$$\begin{aligned}
&P_n\hat{S}^{-(n+1)}\hat{J}^{n+1}i(a\otimes c^+)\\
&\quad= P_n\left(\sum_{k_1,\dots,k_{n+1}} \mathbb{1}_{-(n+2)]}\otimes c_{k_{n+1}}\otimes c_{k_n}\otimes\cdots\otimes c_{k_1}\otimes j_{k_{n+1}}\circ j_{k_n}\circ\cdots\circ j_{k_1}(a)\otimes c^+\right)\\
&\quad= \sum_{k_1,\dots,k_n}\underbrace{\sum_{k_{n+1}}\varphi\left(j_{k_{n+1}}\circ\cdots\circ j_{k_1}(a)\right)\psi(c_{k_{n+1}})}_{=:(\star)}\cdot\mathbb{1}_{-(n+1)]}\otimes c_{k_n}\otimes\cdots\otimes c_{k_1}\otimes\mathbb{1}_{\mathcal{A}}\otimes c^+\\
&\quad\underset{(\star)}{=} \sum_{k_1,\dots,k_n}\varphi\left(j_{k_n}\circ\cdots\circ j_{k_1}(a)\right)\cdot\mathbb{1}_{-(n+1)]}\otimes c_{k_n}\otimes\cdots\otimes c_{k_1}\otimes\mathbb{1}_{\mathcal{A}}\otimes c^+\\
&\quad= P_n\left(\sum_{k_1,\dots,k_n}\mathbb{1}_{-(n+1)]}\otimes c_{k_n}\otimes\cdots\otimes c_{k_1}\otimes j_{k_n}\circ\cdots\circ j_{k_1}(a)\otimes c^+\right)\\
&\quad= P_n\hat{S}^{-n}\hat{J}^n i(a\otimes c^+),
\end{aligned}$$

where $(\star)$ is due to the invariance $\varphi\otimes\psi\circ J = \varphi\otimes\psi$:

$$\begin{aligned}
(\star) &= \sum_{k_{n+1}}\varphi\left(j_{k_{n+1}}\circ j_{k_n}\circ\cdots\circ j_{k_1}(a)\right)\cdot\psi(c_{k_{n+1}})\\
&= \varphi\otimes\psi\left(\sum_{k_{n+1}} j_{k_{n+1}}\left(\underbrace{j_{k_n}\circ\cdots\circ j_{k_1}(a)}_{\in\mathcal{A}}\right)\otimes c_{k_{n+1}}\right)\\
&= \varphi\otimes\psi\left(J\left(j_{k_n}\circ\cdots\circ j_{k_1}(a)\right)\otimes\mathbb{1}_{\mathcal{C}}\right)\\
&= \varphi\otimes\psi\left(j_{k_n}\circ\cdots\circ j_{k_1}(a)\otimes\mathbb{1}_{\mathcal{C}}\right)\\
&= \varphi\left(j_{k_n}\circ\cdots\circ j_{k_1}(a)\right).
\end{aligned}$$

3. The third condition follows directly since all considered operators are bounded.

□

Corollary 6.2.11. Let $(\mathcal{A},\varphi)$ and $(\mathcal{C},\psi)$ be quantum probability spaces with finite-dimensional von Neumann algebras $\mathcal{A}$ and $\mathcal{C}$ and let $(\mathcal{A}^+,\varphi^+,J^+;\mathcal{A},i^+)$ be a one-sided quantum Markov process. Let $n\in\mathbb{N}$ and $x\in\mathcal{A}^+$, then for all $m\in\mathbb{N}$ holds $P_n\hat{S}^{-(n+m)}\hat{T}^{n+m}i(x) = P_n\hat{S}^{-n}\hat{T}^n i(x)$.

Proof. Follows inductively, from the second step in the proof of Proposition 6.2.10 □

Let us now turn to the von Neumann algebra generated by the filtration. One easily sees that

$$\hat{\mathcal{A}}_\infty = \left(\bigcup_{n \in \mathbb{N}} \hat{\mathcal{A}}_n \right)'' = \mathbb{1}_{\mathcal{A}} \otimes \hat{\mathcal{C}} \subset \hat{\mathcal{A}}.$$

This leads to the desired result.

THEOREM 6.2.12. Let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ be quantum probability spaces with finite-dimensional von Neumann algebras $\mathcal{A}$ and $\mathcal{C}$ and let $(\mathcal{A}^+, \varphi^+, J^+; \mathcal{A}, i^+)$ be a one-sided quantum Markov process. Suppose that the corresponding transition is asymptotically complete. Then, the sequence $(P_\varphi \hat{S}^{-n} \hat{J}^n i(x))_{n\in\mathbb{N}}$ converges almost uniformly towards $\Phi_J(x)$, for all $x \in \mathcal{A}^+$.

Proof. We fix $x \in \mathcal{A}^+$ and $n \in \mathbb{N}$.
By construction of the one-sided quantum Markov process, $\hat{S}^{-n}\hat{J}^n i(x)$ is an element in $\mathbb{1}_{-(n+1)]} \otimes \underbrace{\mathcal{C} \otimes \cdots \otimes \mathcal{C}}_{n \text{ times}} \otimes \mathcal{A} \otimes \mathcal{C}^+$ and hence $P_\varphi \hat{S}^{-n}\hat{J}^n i(x) \in \hat{\mathcal{A}}_n$. In particular, this implies

$$P_\varphi \hat{S}^{-n}\hat{J}^n i(x) = P_n \hat{S}^{-n}\hat{J}^n i(x).$$

So, due to Theorem 6.2.8 and Proposition 6.2.10, there exists a unique $\hat{x} \in \hat{\mathcal{A}}_\infty = \mathbb{1}_{\mathcal{A}} \otimes \hat{\mathcal{C}}$, such that $(P_\varphi \hat{S}^{-n}\hat{J}^n i(x))_n$ converges almost uniformly towards $\hat{x} = P_\varphi(\hat{x}) \in \mathbb{1}_{\mathcal{A}} \otimes \hat{\mathcal{C}}$.

Since the Møller operator Φ_J maps $\mathcal{A}^+$ onto $\mathbb{1}_{\mathcal{A}} \otimes \hat{\mathcal{C}}$, it follows also that $P_\varphi \Phi_J = \Phi_J$. By Theorem 2.3.5, asymptotic completeness of J implies hence that

$$\|\cdot\|_\varphi - \lim_{n\to\infty} P_\varphi \hat{S}^{-n}\hat{J}^n i(x) = P_\varphi \Phi_J(x) = \Phi_J(x).$$

With the non-commutative Egorov's Theorem 6.2.3 follows now that there exists a subsequence $(P_\varphi \hat{S}^{-n_k}\hat{J}^{n_k} i(x))_{k\in\mathbb{N}}$ which converges almost uniformly towards $\Phi_J(x)$. But since the the sequence converges also almost uniformly towards an $\hat{x} \in \hat{\mathcal{A}}_\infty$, it follows that $\hat{x} = \Phi_J(x)$.

□

COROLLARY 6.2.13. Let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ be quantum probability spaces with finite-dimensional von Neumann algebras $\mathcal{A}$ and $\mathcal{C}$ and let $(\mathcal{A}^+, \varphi^+, J^+; \mathcal{A}, i^+)$ be a one-sided quantum Markov process. Suppose that the corresponding transition is asymptotically complete. For all $n \in \mathbb{N}$ and all $x \in \mathcal{A}^+$ holds $P_n \Phi_J(x) = P_n \hat{S}^{-n}\hat{J}^n i(x)$.

Proof. The assertion is a direct consequence of Theorems 6.2.8 and 6.2.12. □

6.2.3 *Asymptotic completeness and almost uniform convergence*

Combining the results from the preceding sections, we conclude the considerations with the following assertion:

COROLLARY 6.2.14. Let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \psi)$ be quantum probability spaces with finite-dimensional von Neumann algebras $\mathcal{A}$ and $\mathcal{C}$ and let $(\mathcal{A}^+, \varphi^+, J^+; \mathcal{A}, i^+)$ be a one-sided quantum Markov process. Then, the following assertions hold:

1. If for all $x \in \mathcal{A}^+$ the sequence $(\hat{S}^{-n}\hat{J}^n i(x))_{n\in\mathbb{N}}$ converges almost uniformly towards $\Phi_J(x)$, then the transition $J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$ is asymptotically complete.

2. If the transition $J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$ is asymptotically complete, then for all $x \in \mathcal{A}^+$, the sequences $(P_{\hat{\psi}}\hat{S}^{-n}\hat{J}^n i(x))_{n\in\mathbb{N}}$ and $(P_{\varphi}\hat{S}^{-n}\hat{J}^n i(x))_{n\in\mathbb{N}}$ converge almost uniformly towards $\hat{\psi}(\Phi_J(x)) \cdot \mathbb{1}_{\hat{\mathcal{A}}}$ and $\Phi_J(x)$, respectively.

Proof. This is a direct result of Corollary 6.2.4, Proposition 6.2.5 and Theorem 6.2.12. □

In other words, the pointwise almost uniform convergence of $(\hat{S}^{-n}\hat{J}^n i)_{n\in\mathbb{N}}$ towards the Møller operator Φ_J implies asymptotic completeness of the transition J. On the other hand, given asymptotic completeness of J, the sequences $(P_{\hat{\psi}}\hat{S}^{-n}\hat{J}^n i)_{n\in\mathbb{N}}$ and $(P_{\varphi}\hat{S}^{-n}\hat{J}^n i)_{n\in\mathbb{N}}$ converge pointwise almost uniformly towards $P_{\hat{\psi}}\Phi_J$ and $P_{\varphi}\Phi_J$, respectively. In the non-commutative setting, the pointwise almost uniform convergence of $(\hat{S}^{-n}\hat{J}^n i)_{n\in\mathbb{N}}$ towards the Møller operator Φ_J seems hence to be a promising candidate to generalize the almost sure convergence of the transition functions, that can be observed in the classical finite setting and could thus lead to a further characterization of asymptotic completeness of a transition.

C

CONCLUSION AND OUTLOOK

A STORY IS NEVER COMPLETE.

— Jude Brigley

Just like a story, mathematics is never complete. At this point, a very fitting image by B. Kümmerer needs to be mentioned, who often compares mathematics with a sphere. Inside the sphere lies the already existing mathematical knowledge, while its surface corresponds to the open problems and questions. Consequently, each new mathematical result implies an enlargement of the sphere's surface. So it is not surprising that with this thesis, further open questions arise as well.

Taking advantage of the fact that a road-colored graph equipped with a probability distribution on the colors induces a Markov chain, we presented a modification of the Coupling of the Past algorithm, which can be applied to road-colored graphs, in Chapter 3. The question of whether a given road-colored graph possesses a synchronizing word also finds attention in current research on automata theory, as a brief insight shows. In [Tra12], [BP14], for instance, algorithms to find synchronizing words are presented, whereas probabilities for a deterministic finite automaton to possess a synchronizing word are determined in [Ber16], [CP23]. Of special interest is the estimation of the minimal length of a synchronizing word (compare, for example, [Ber14], [Nic19]). The perspective of the Colored Coupling to the Past algorithm might provide some interesting insights.

In Chapter 4, we addressed the idea of understanding the Kraus operators of the dual extended transition operator, which itself can be understood as a non-commutative generalization of the label product of a road-colored graph, as non-commutative colors. A better understanding of this connection would be desirable, particularly regarding whether it might lead to a possible concept of non-commutative synchronizing words and hence non-commutative synchronizability. Moreover, we introduced a generalization of the equivalence between the asymptotic completeness of a transition and the regularity of the corresponding dual transition operator to a Hilbert space setting, thus offering the opportunity to transfer further results in this setting.

Particular attention has been paid to the fact that the existence of a synchronizing word for a finite road-colored graph is equivalent to the almost sure convergence of the corresponding n-step transition functions. This has been generalized to special cases in the classical infinite and the non-commutative setting.

A special class of road-colored graphs with countably many vertices is introduced in Chapter 5. We showed that for these graphs, whenever positive recurrence is assumed, the existence of a synchronizing word, as introduced in [Haa06], implies the almost sure convergence of the corresponding n-step transition functions. Here, we took advantage of their particular structure, which allowed us to draw conclusions on the position of the walk corresponding to a chosen color sequence at certain points of time, independent of the initial state. In general, this represents a major challenge. To date, it has neither been possible to construct a counterexample nor to show more advanced statements for less specific cases. Although there are some very fruitful considerations in the preprint [GHK19] concerning the fact that a color sequence sends an arbitrary initial vertex with very high probability into a certain finite subset of vertices, which can then be synchronized in finitely many steps.

For a one-sided quantum Markov process $(\mathcal{A}^+, \varphi^+, J^+; \mathcal{A}, i^+)$ we proposed the pointwise almost uniform convergence of the sequence $(\hat{S}^{-n}\hat{J}^n i)_{n\in\mathbb{N}}$ towards the Møller operator Φ_J as a possible characterization of the asymptotic completeness of the transition. The approach presented in Chapter 6 opens the door to a a new possibility of generalizing the idea of synchronizability into the non-commutative setting.

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NOTATION

General notation

$\mathbb{N}$	natural numbers $\{1, 2, \ldots\}$
$\mathbb{N}_0$	natural numbers including zero
$\mathbb{Z}, \mathbb{R}, \mathbb{C}$	integers, real and complex numbers
$\mathbb{T}$	$\mathbb{Z}$ or $\mathbb{N}_0$
α, β, λ	elements in $\mathbb{C}$

Graph theory

A	vertex set
a, b	vertices in A
A^+	$A \times C^+$
$\hat{A}$	$A \times \hat{C}$
$a^+, \hat{a}$	elements in $A^+, \hat{A}$
(A, E)	directed graph with vertices in A and edges in E
(A, C, γ)	road-colored graph with vertices in A and coloring (C, γ)
$(A, C, \gamma; \mu, \nu)$	road-colored graph with probability distributions μ on A and ν on C (often, μ is invariant, i.e. γ is measure-preserving)
C	set of colors
c	color in C
C^-	one-sided (negative) infinite color sequences $\bigtimes_{-\mathbb{N}} C$
C^+	one-sided (non-negative) infinite color sequences $\bigtimes_{\mathbb{N}_0} C$
$\hat{C}$	two-sided infinite color sequences $C^- \times C^+$

$c^-, c^+, \hat{c}$	elements in $C^-, C^+, \hat{C}$
C_∞	color sequences in C^-, containing a synchronizing word (finite graphs)
C_b	color sequences in C^-, possessing a word that synchronizes into $b \in A$ (finite graphs)
(C, γ)	coloring with colors in C and transition function $\gamma : A \times C \to A$
d_a	out-degree of vertex $a \in A$
(D, δ)	trivial coloring
E	set of edges in a directed graph
e_{ab}	edge with initial vertex a and terminal vertex b
γ	transition function from $A \times C \to A$
	it is often identified with its trivial extension $\gamma^- : A \times C^- \to A$
γ_n	n-step transition function from $A \times C^n \to A$
	it is often identified with its trivial extension $\gamma_n^- : A \times C^- \to A$
$\gamma^-, \gamma^+, \hat{\gamma}$	trivial extensions of γ onto $A \times C^-, A^+$ and $\hat{A}$
$\gamma_n^-, \gamma_n^+, \hat{\gamma}_n$	trivial extensions of γ_n onto $A \times C^-, A^+$ and $\hat{A}$
γ_∞	ν^--almost sure limit of the transition functions $(\gamma_n)_n$
	usually identified with a map from $C^- \to A$
γ^{prod}	transition function of the label product
M	adjacency matrix of a directed graph
M_c	monochromatic matrix with respect to color $c \in C$
M_c^{prod}	monochromatic matrix of the label product
μ	probability distribution on A (often chosen to be invariant)
$\mu \otimes \nu$	product measure
$\mu^+, \hat{\mu}$	infinite product measure $\mu \otimes \nu^+$ and $\mu \otimes \hat{\nu}$
ν	probability distribution on C
$\nu^-, \nu^+, \hat{\nu}$	infinite product measure $\otimes_I \nu$, with $I = \{-\mathbb{N}, \mathbb{N}_0, \mathbb{Z}\}$
$(Q, \Sigma, \delta, q_0, F)$	deterministic finite automaton

Integer random walks

A_T	set of T-target points
C	colors, subset of $\mathbb{Z}$
C_0	color sequences in C^- of no distance
C_M	color sequences in C^- of maximum distance N
$n_M(c^-)$	time of maximum distance of c^-
S_n	n-step length, map from $C^- \to \mathbb{Z}$

Classical probability theory

A	state space
$(A;\, \mu, T)$	Markov chain with initial distribution μ and transition matrix T
$(A, C, \gamma;\, \mu, T)$	Markov chain, identified with a representation as road-colored graph
$\mathbb{E}$	expected value
$L^\infty(\Omega, \Sigma, \mathbb{P})$	essentially bounded functions on $(\Omega, \Sigma, \mathbb{P})$
$L^1(\Omega, \Sigma, \mathbb{P})$	integrable measurable functions on $(\Omega, \Sigma, \mathbb{P})$
$L^2(\Omega, \Sigma, \mathbb{P})$	square-integrable measurable functions on $(\Omega, \Sigma, \mathbb{P})$
$\mathbb{P}$	probability measure on Ω
$(\Omega, \Sigma, \mathbb{P})$	probability space
T	transition matrix
T_c	monochromatic transition matrix
T^{prod}	transition matrix of the label product

Hilbert spaces and operator algebras

$*$	involution
$\langle \cdot, \cdot \rangle$	scalar product, linear in the first component
$\lVert \cdot \rVert$	operator norm

$\langle \cdot, \cdot \rangle_\varphi$, $\|\cdot\|_\varphi$	φ-scalar product, φ-norm
$\mathbb{1}$	identity operation in $\mathcal{B}(\mathcal{H})$
$\otimes$	tensor product
$\oplus$	direct sum
$\mathcal{A}$	operator algebra over $\mathbb{C}$, usually a von Neumann algebra
a, b	elements in $\mathcal{A}$
$\mathcal{A}', \mathcal{A}''$	commutant, respectively, bicommutant of $\mathcal{A}$
$\mathcal{A}_*$	predual of $\mathcal{A}$
$\mathcal{B}(\mathcal{H})$	algebra of bounded operators on $\mathcal{H}$
$\mathcal{C}$	von Neumann algebra
c	element in $\mathcal{C}$
$\mathcal{H}, \mathcal{K}$	Hilbert spaces
$(\mathcal{H}^+, \xi^+), (\mathcal{K}^+, \eta^+)$	infinite Hilbert space tensor product of $(\mathcal{H}, \xi_\varphi)$, $(\mathcal{K}, \eta_\psi)$
id	identity on an operator algebra
M_n	complex $n \times n$-matrices $\mathrm{M}_n(\mathbb{C})$
ω	pure state
ω_ξ	vector state $\langle \cdot\, \xi, \xi \rangle$
φ	linear functional on $\mathcal{A}$, usually a faithful normal state
Φ, Ψ	density matrices
ψ	linear functional on $\mathcal{C}$, usually a faithful normal state
$(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$	GNS-triple
$\mathcal{S}(\mathcal{A}), \mathcal{S}(\mathcal{C})$	convex set of states on $\mathcal{A}$, respectively $\mathcal{C}$
stop	strong operator topology on $\mathcal{B}(\mathcal{H})$
$\mathcal{T}(\mathcal{H})$	trace class operators on $\mathcal{H}$
tr	trace
τ	tracial state on M_n
wop	weak operator topology on $\mathcal{B}(\mathcal{H})$
ξ, η	elements in $\mathcal{H}$, respectively $\mathcal{K}$
x, y	elements in $\mathcal{B}(\mathcal{H})$

Quantum probability theory

$(\mathcal{A}, \varphi)$	quantum probability space
$(\mathcal{A}, \varphi, T)$	dynamical system
$(\mathcal{A}^+, \varphi^+)$	$(\mathcal{A} \otimes \mathcal{C}^+, \varphi \otimes \psi^+)$
$(\hat{\mathcal{A}}, \hat{\varphi})$	$(\mathcal{A} \otimes \hat{\mathcal{C}}, \varphi \otimes \hat{\psi})$
$(\mathcal{A}^+, \varphi^+, J^+; \mathcal{A}, i^+)$	stationary one-sided quantum Markov process
$(\mathcal{C}, \psi)$	quantum probability space
$(\mathcal{C}^-, \psi^-)$	infinite von Neumann tensor product $\bigotimes_{-\mathbb{N}}(\mathcal{C}, \psi)$
$(\mathcal{C}^+, \psi^+)$	infinite von Neumann tensor product $\bigotimes_{\mathbb{N}_0}(\mathcal{C}, \psi)$
$(\hat{\mathcal{C}}, \hat{\psi})$	infinite von Neumann tensor product $\bigotimes_{\mathbb{Z}}(\mathcal{C}, \psi)$
i^+	canonical embedding of $\mathcal{A}$ into $\mathcal{A}^+$
i	canonical embedding of $\mathcal{A}^+$ into $\hat{\mathcal{A}}$
$\hat{\imath}$	canonical embedding of $\mathcal{A}$ into $\hat{\mathcal{A}}$
J	transition from (A, φ) into $(\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$
J^+	extension of J onto $\mathcal{A}^+$
$\hat{J}$	extension of J^+, respectively J, onto $\hat{\mathcal{A}}$
P	conditional expectation
P_φ, P_ψ	conditional expectations of tensor type
Φ_J	Møller operator
S	tensor right shift on $\mathcal{C}$
S^+	tensor right shift on $\mathcal{C}^+$
$\hat{S}$	trivial extension of the tensor right shift on $\hat{\mathcal{C}}$ to $\hat{\mathcal{A}}$
T	automorphism of (A, φ)
T_ψ	transition operator associated to a transition J

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