# Competitive Analysis for Incremental Maximization

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# Zusammenfassung

Die Aufgabe in der inkrementellen Maximierung ist es, im Laufe der Zeit eine Lösung aufzubauen, die den Wert einer gegebenen monotonen Zielfunktion maximiert. Dabei fügt man der Lösung ein Element nach dem anderen hinzu, ohne zu wissen, wie viele Elemente letztendlich in der Lösung enthalten sein dürfen. Diese Information wird erst in dem Moment bekannt, in dem das letzte zulässige Element hinzugefügt wurde. Daher besteht das Ziel in der inkrementellen Maximierung darin, eine Reihenfolge anzugeben, in der die Elemente zu der Lösung hinzugefügt werden, sodass deren Wert zu jedem Zeitpunkt größtmöglich ist. In dieser Arbeit befassen wir uns mit der kompetitiven Analyse dieses Problems und beweisen obere und untere Schranken für den kompetitiven Faktor dieses Problems.

Anfangs analysieren wir einen Greedy Algorithmus, der in jedem Schritt das Element hinzufügt, das zu diesem Zeitpunkt den Wert der Lösung am meisten steigert. Es ist bekannt, dass dieser Algorithmus für einige Teilklassen des Problems einen beschränkten kompetitiven Faktor hat. Wir führen die neue Klasse mit $\gamma$-$\alpha$-augmentierbaren Zielfunktionen ein und zeigen, dass sie mehrere aus der Literatur bekannte Teilklassen vereint. Darüber hinaus zeigen wir eine obere Schranke von $\frac{\alpha-(1-c)\gamma}{\gamma} \cdot \frac{\mathrm{e}^{\alpha-(1-c)\gamma}}{\mathrm{e}^{\alpha-(1-c)\gamma}-1}$ für den kompetitiven Faktor des Greedy Algorithmus für diese Teilklasse, wobei $c \in [0, 1]$ die Krümmung der Zielfunktion ist. Für $c = 1$ präsentieren wir eine entsprechende untere Schranke.

Anschließend befassen wir uns mit dem Problem unter der Annahme, dass die Zielfunktion verantwortlich (engl.: *accountable*) ist. Verantwortlichkeit ist eine Eigenschaft, die sich als vorteilhaft für inkrementelle Maximierung erwiesen hat. Wir zeigen, dass der kompetitive Faktor in diesem Fall mit dem der Teilklasse mit separierbaren Instanzen übereinstimmt. Separierbarkeit ist eine neue Eigenschaft, die garantiert, dass das Problem eine einfache Struktur besitzt. Um dieses vereinfachte Problem zu analysieren, führen wir eine Kontinuisierungstechnik ein, mit der man untere Schranken an den kompetitiven Faktor zeigen kann. Wir verwenden diese Technik, um ein Indiz dafür zu geben, dass die obere Schranke von $\varphi + 1 \approx 2.618$ an den kompetitiven Faktor dem tatsächlichen kompetitiven Faktor entspricht, wobei $\varphi = \frac{1}{2}(1 + \sqrt{5})$ der goldene Schnitt ist. Desweiteren nutzen wir die Kontinuisierungstechnik, um eine verbesserte untere Schranke von $2.246$ zu beweisen.

Wir analysieren mehrere Skalierungsalgorithmen für die Teilklasse mit separierbaren Instanzen und bestimmen die exakten kopetitiven Faktoren der deterministischen Algorithmen CardinalityScaling, ValueScaling und DensityScaling. Wir präsentieren den randomisierten Algorithmus RandScaling und zeigen, dass er einen randomisierten kompetitiven Faktor von höchstens $1.772$ hat. Dieser oberen Schranke an den randomisierten kompetitiven Faktor der Teilklasse mit separierbaren Instanzen stellen wir eine untere Schranke von $1.357$ gegenüber, die wir mittels Yao's Prinzip erhalten.

Um Schranken an den kompetitiven Faktor von mehr und größeren Teilklassen zu finden, führen wir $\beta$-Verantwortlichkeit (engl.: *$\beta$-accountability*) als eine Relaxierung von Verantwortlichkeit ein. Für die von dieser Eigenschaft induzierte Subklasse zeigen wir eine obere Schranke von $\frac{1}{2\beta} + 1 + \sqrt{\frac{1}{4\beta^2} + 1}$ an den kompetitiven Faktor. Da jede subadditive Funktion auch $\frac{1}{2}$-verantwortlich ist, erhalten wir folglich mit einem Wert von $2 + \sqrt{2}$ die erste bekannte obere Schranke an den kompetitiven Faktor der Teilklasse mit subadditiven Zielfunktionen. Dieser oberen Schranke stellen wir eine untere Schranke von $\frac{1}{\beta} \cdot \left(1 + \frac{1}{\lceil \frac{1}{\beta} \rceil + 1}\right)$ entgegen, die für $\beta \to 0$ strikt ist.

Abschließend genaralisieren wir das Problem der inkrementellen Maximierung und nehmen an, dass anstelle einer unbekannten Kardinalitätsschranke eine unbekannte Knapsackschranke gegeben ist. Wir zeigen, dass der strikte kompetitive Faktor dieses Problems mit monotonen, fraktional subadditiven und $M$-beschränkten Zielfunktionen im Intervall $[\max\{\varphi + 1, M\}, \max\{3.293\sqrt{M}, 2M\}]$ liegt. Untere Schranken, die wir durch die zuvor eingeführte Kontinuisierungstechnik erhalten, übertragen sich auf den nicht-strikten kompetitiven Faktor im Problem mit Knapsackschranke. Also ist der nicht-strikte kompetitive Faktor dieses Problems wenigstens $2.246$. Wir komplementieren diese untere Schranke mit einer oberen Schranke von $\varphi + 1$.

# Abstract

In incremental maximization, we are tasked with building up a solution over time by adding elements from a groundset one by one. We want to maximize the monotone objective value of the assembled solution. However, the information how many elements may be added to the final solution is only revealed when the last feasible element is added. Thus, the goal is to give an ordering in which to add the elements such that the value of the solution in each step is maximized. In this thesis, we investigate this problem in the sense of competitive analysis and present upper and lower bounds on the competitive ratio for various subclasses of this problem.

We start by considering a simple greedy algorithm that always adds the element that yields the largest increase in the objective value. It is known to have a bounded competitive ratio for various problem classes. We introduce the new class with $\gamma$-$\alpha$-augmentable objective functions and show that it generalizes multiple subclasses from the literature. Furthermore, we prove that the greedy algorithm has a competitive ratio of at most $\frac{\alpha-(1-c)\gamma}{\gamma} \cdot \frac{e^{\alpha-(1-c)\gamma}}{e^{\alpha-(1-c)\gamma}-1}$ for this new subclass where $c \in [0,1]$ is the curvature of the objective. This bound is tight for $c = 1$.

Next, we consider the subclass of instances with accountable objectives. Accountability has proven to be a favorable property in incremental maximization. We show that the competitive ratio of this class is the same as the competitive ratio of the subclass of separable instances. Separability is a new property that guarantees the problem to have a simple structure. For its analysis, we introduce a continuization technique that can be used to show lower bounds on the competitive ratio of this subclass. We utilize it to give evidence that the known upper bound of $\varphi + 1 \approx 2.618$ on this competitive ratio might actually be tight. Here, $\varphi = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio. Furthermore, we use the continuization technique to show an improved lower bound of $2.246$ on the competitive ratio.

We analyze multiple scaling algorithms for separable problem instances and prove tight competitive ratios for the deterministic algorithms CardinalityScaling, ValueScaling, and DensityScaling. We introduce the randomized algorithm RandScaling and show that it has a randomized competitive ratio of at most $1.772$. This upper bound on the randomized competitive ratio of the subclass of separable instances is complemented with

a lower bound of $1.357$ by using Yao's principle.

In order to find bounds on the competitive ratio of more and larger problem classes, we introduce $\beta$-accountability, a relaxation of accountability. For the subclass with $\beta$-accountable objective functions, we show an upper bound on the competitive ratio of $\frac{1}{2\beta} + 1 + \sqrt{\frac{1}{4\beta^2} + 1}$. Since every subadditive function is $\frac{1}{2}$-accountable, we obtain the first upper bound on the competitive ratio of the subclass with subadditive objective functions with a value of $2 + \sqrt{2}$. We complement the upper bound on the competitive ratio of the subclass with $\beta$-accountable objectives with a lower bound of $\frac{1}{\beta} \cdot \left(1 + \frac{1}{\lceil \frac{1}{\beta} \rceil + 1}\right)$ which is tight in the limit $\beta \to 0$.

Lastly, we generalize the incremental maximization problem by considering an unknown knapsack constraint instead of an unknown cardinality constraint. We show that, for monotone, fractionally subadditive, and $M$-bounded objective functions, this problem has a strict competitive ratio in $[\max\{\varphi + 1, M\}, \max\{3.293\sqrt{M}, 2M\}]$. The lower bounds from the newly introduced continuization technique also yield lower bounds on the non-strict competitive ratio in the knapsack setting, i.e., the non-strict competitive ratio is at least $2.246$. We complement this with an upper bound of $\varphi + 1$.

# Contents

# 1. Introduction

Consider the development of expansive road or railroad networks, where it is initially unknown how large the network will be. In the beginning, only a small network can be constructed that may need to be adapted to larger demands over time. Similarly, corporate enterprises, such as car rental companies and delivery services, build depots designed to serve limited areas initially. However, as these businesses thrive, they may want to expand and build new depots to serve a larger area. With this in mind, it might make sense to build the first depot a bit off-center so that its service area overlaps less with that of later depots. In both cases, the challenge is to select an ordering in which to build the infrastructure - be it networks or depots - such that, at every point in time, it is as good as possible.

We aim to capture problems like the above with the incremental maximization problem (IncMax). In this problem, we are given a countable ground set $U$ containing the elements that can be added to the solution over time, together with a *monotone* objective function $f\colon 2^U \to \mathbb{R}_{\geq 0}$ that maps every subset of $U$ to some non-negative value. The function $f$ is called monotone if, for all $A \subseteq B \subseteq U$, we have $f(A) \leq f(B)$. The associated offline problem to IncMax is cardinality constrained maximization where it is known how many elements can be added to the solution. We denote the optimum for this problem by

$$\textsc{Opt}(C) := \sup\{f(S) \mid S \subseteq U, |S| \leq C\}.$$

We assume that $\textsc{Opt}(C) \in \mathbb{R}$ for all $C \in \mathbb{N}$, i.e., that it is not unbounded. Furthermore, we define the optimum solution $O(C) \subseteq U$ of cardinality $C$ to be a set with $|O(C)| \leq C$ and $f(O(C)) = \textsc{Opt}(C)$, where we break ties in an arbitrary but fixed manner.[1]

In contrast to cardinality constrained maximization, in the online IncMax problem, the cardinality constraint $C \in \mathbb{N}$ is not known in advance and only revealed when no

[1]Note that such a set $O(C)$ does not need to exist if the groundset is infinite. In this case, we define $O(C)$ to be a set that approximates the value $\textsc{Opt}(C)$ arbitrarily close and satisfies $|O(C)| \leq C$. Throughout this thesis, the sets $O(1), O(2), \dots$ are only used to define algorithms where they are added to the solution one element at the time. Thus, we only lose the arbitrarily small approximation error in the performance guarantee of the algorithm.

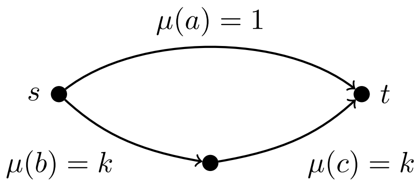


Figure 1.1.: Example of the incremental maximum $s$-$t$-flow problem where no incremental solution with competitive ratio $\rho < k$ exists.

more elements can be added to the solution. An *incremental solution* $X$ for the IncMax problem is an ordering $X = (e_1, e_2, \dots)$ of elements in $U$. The *solution* of $X$ for cardinality $C \in \mathbb{N}$ is given by $X(C) = \{e_1, \dots, e_C\}$. Since the cardinality constraint $C$ is not known in the IncMax problem, the solution of $X$ has to be good for every cardinality $C \in \mathbb{N}$. As it is usual in competitive analysis, we measure the quality of the incremental solution $X$ using the competitive ratio. We call $X$ *$\rho$-competitive* for $\rho \geq 1$ if, for all $C \in \mathbb{N}$, we have $\rho \cdot f(X(C)) \geq \textsc{Opt}(C)$. The *competitive ratio* of the solution $X$ is $\inf\{\rho \geq 1 \mid X \text{ is } \rho\text{-competitive}\}$. We call an algorithm Alg for a subclass $\mathcal{P}$ of IncMax *$\rho$-competitive* for $\rho \geq 1$ if the incremental solution of Alg for all instances in $\mathcal{P}$ is $\rho$-competitive. The *competitive ratio* of Alg is $\inf\{\rho \geq 1 \mid \textsc{Alg} \text{ is } \rho\text{-competitive}\}$. A solution/algorithm is called *competitive* if its competitive ratio is finite. The *competitive ratio* of the problem class $\mathcal{P}$ is

$$\inf\{\rho \geq 1 \mid \text{there exists a } \rho\text{-competitive algorithm for } \mathcal{P}\}.$$

As an example, we will consider the *incremental maximum $s$-$t$-flow problem*. Here, the ground set $U$ corresponds to the set of edges in a directed graph $G = (V, E)$ with two designated vertices $s, t \in V$. Each edge has a capacity $\mu(e) \in \mathbb{R}_{\geq 0}$. The value $f(S)$ of a subset $S \subseteq E$ is defined as the value of a maximum $s$-$t$-flow in $G_S = (V, S)$. Even in this special case, a competitive incremental solution may fail to exist. For illustration, consider the graph in Figure 1.1. Every incremental solution $X$ has to add edge $a$ first in order to be competitive. Otherwise, we would have $\textsc{Opt}(1) = 1$ and $f(X(1)) = 0$. On the other hand, every incremental solution that adds edge $a$ first cannot have a competitive ratio better than $k$, because for $C = 2$, we have for $\textsc{Opt}(2) = k$ and $f(X(2)) = 1$.

This shows that, in general, the competitive ratio of the IncMax problem is unbounded. Thus, throughout this thesis, we will investigate natural subclasses of the IncMax problem for which we can show that competitive solutions exist. An instance of IncMax is defined by giving a ground set $U$ and the monotone objective function $f \colon 2^U \to \mathbb{R}_{\geq 0}$. The subclasses of IncMax that we will consider in this thesis will be defined by giving properties that

| objective | weights | ALG | lower bound | upper bound |
|---|---|---|---|---|
| $\gamma$-$\alpha$-augmentable | $w(e)=1$ | GREEDY | $\frac{\alpha}{\gamma} \cdot \frac{e^{\alpha}}{e^{\alpha}-1}$ **Cor. 2.22**, [15, Thm. 2] | $\frac{\alpha-(1-c)\gamma}{\gamma} \cdot \frac{e^{\alpha-(1-c)\gamma}}{e^{\alpha-(1-c)\gamma}-1}$ **Thm. 2.17** |
| accountable | $w(e)=1$ | - | $2.246$ **Thm. 3.27**, [19, Thm. 4] | $2.618$ [5, Thm. 1] |
| | | random. | $1.357$ **Thm. 4.22** | $1.772$ **Thm. 4.21**, [19, Thm. 5] |
| subadditive | $w(e)=1$ | - | $2.246$ | $3.415$ **Thm. 5.9** |
| $\beta$-accountable | $w(e)=1$ | - | $\frac{1}{\beta} + \frac{1}{\beta\lceil\frac{1}{\beta}\rceil+\beta}$ **Thm. 5.5** | $\frac{1}{2\beta} + 1 + \sqrt{\frac{1}{4\beta^2} + 1}$ **Thm. 5.4** |
| frac. subadditive | - | - | $\max\{2.618, M\}$ **Thm. 6.16**, [17, Thm. 1.6] | $\max\{3.292\sqrt{M}, 2M\}$ **Thm. 6.8**, [20, Thm. 1] |

Figure 1.2.: Overview over bounds on the competitive ratio for incremental maximization under a knapsack constraint for monotone objectives in different settings. Here, $c \in [0,1]$ is the curvature, and $M := \max_{e_1,e_2 \in U} \frac{f(\{e_1\})}{f(\{e_2\})}$.

the objectives of the instances in the subclass should have. For example, the subclass of submodular objectives is the class containing all instances in INCMAX that have a submodular objective function.

In the following, we provide an overview of the structure of this work. For definitions of the objective properties mentioned, we refer to Definitions 1.2 to 1.5 later in this chapter. An overview over the most important results in this work and the literature can be found in Figure 1.2.

**overview.** First, in Chapter 2 we present the GREEDY algorithm for the INCMAX problem. This algorithm is known to have a bounded competitive ratio for the subclass of INCMAX where the objective is the weighted rank function in some independence system [40], for the subclass of objectives with a bounded submodularity ratio [14], as well as for the subclass of $\alpha$-augmentable objectives [5]. We propose the new subclass of INCMAX with $\gamma$-$\alpha$-augmentable objectives, and show that it encompasses all of the aforementioned problem classes. We give an upper bound of $\frac{\alpha-(1-c)\gamma}{\gamma} \cdot \frac{e^{\alpha-(1-c)\gamma}}{e^{\alpha-(1-c)\gamma}-1}$ on the competitive ratio of this class where $c \in [0,1]$ is the curvature of the objective function. Furthermore, we show that its competitive ratio generalizes the competitive ratios of the three other classes. For $c = 1$, we show that the competitive ratio is tight. The lower bound that we present can also be used to show that the known upper bound on the competitive ratio of the class with $\alpha$-augmentable objectives is tight, which closes a gap left in the analysis in [5].

In Chapter 3, we turn to the class $\text{IncMax}_{\text{acc}}$, the subclass of IncMax containing the instances with an accountable objective, and give an improved lower bound on its competitive ratio using a new continuization technique. In order to do this, we introduce the new subclass IncMaxSep that contains instances where the ground set is partitioned into subsets and the objective is the maximum over modular functions that each assign the same value for all elements within one subset and $0$ for all other elements. We show that IncMaxSep has the same competitive ratio as $\text{IncMax}_{\text{acc}}$. We introduce IncMaxCont, a continuous version of IncMaxSep, where we assume that we have one subset containing $c$ (fractional) elements for each $c > 0$ and that the solution may also add fractional elements. The smooth structure of this problem better lends itself to analysis. We show that lower bounds on the competitive ratio of this problem are also lower bounds on the competitive ratio of IncMaxSep. In order to investigate the competitive ratio of IncMaxCont, we introduce an optimal algorithm for this problem. We present strong evidence that its competitive ratio is not better than $\varphi + 1 \approx 2.618$ which would tightly match the known upper bound on the competitive ratio of $\text{IncMax}_{\text{acc}}$. Here, $\varphi = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio. Subsequently, we give a lower bound of $2.246$ on the competitive ratio of IncMaxCont that transfers back to the class $\text{IncMax}_{\text{acc}}$ and improves upon the best known lower bound of $2.18$ on the competitive ratio of this class.

We continue investigating the problem class IncMaxSep in Chapter 4 and present multiple deterministic algorithms to solve it. The algorithm CardinalityScaling was introduced by Bernstein et al. [5] and is currently the best known algorithm for the problem class $\text{IncMax}_{\text{acc}}$ with a known upper bound of $\varphi + 1$ on its competitive ratio. We show that this bound is actually tight. Afterwards, we present two new algorithms, ValueScaling and DensityScaling, that have a similar idea to CardinalityScaling. All of these algorithms add optimum solutions of increasing cardinalities where the cardinalities are scaled such that, from one cardinality to the next, the cardinality, value, or density increases or decreases at least by some fixed scaling factor. We show that CardinalityScaling and ValueScaling both have a tight competitive ratio of $\varphi + 1$ and that DensityScaling has a tight competitive ratio of $4$. In the remainder of the chapter, we consider randomized algorithms for this problem. We introduce the RandScaling algorithm that operates similar to the CardinalityScaling algorithm but chooses the first cardinality randomly. We show an upper bound of $1.772$ on its randomized competitive ratio which beats the lower bound of $2.246$ that we have shown in Chapter 3 on the deterministic competitive ratio of IncMaxSep. We complement this upper bound on the randomized competitive ratio with a lower bound of $1.357$ using Yao's principle.

In Chapter 5, we observe that $\text{IncMax}_{\text{acc}}$ does not contain all problem instances in IncMax with a bounded competitive ratio. For example, the subclass of IncMax of instances with subadditive objectives yields a bounded competitive ratio and contains instances that are not in $\textsc{IncMax}_{\text{acc}}$, and vice versa. We propose the new property of $\beta$-accountability, $\beta \in (0,1]$, a relaxation of accountability, and give upper and lower bound on the competitive ratio of the class induced by it. For $\beta \to 0$, these bounds are tight. Furthermore, we show that this new class subsumes the subclass of IncMax with subadditive objectives, the subclass with $\gamma$-$\alpha$-augmentable objectives, and the subclass of accountable objectives. With a value of $2+\sqrt{2}$, we obtain the first upper bound on the competitive ratio of the subclass of IncMax with subadditive objectives.

Lastly, in Chapter 6 we consider a generalization of the IncMax problem where, instead of an unknown cardinality constraint, we are given an unknown knapsack constraint. For this, every element has some weight and the goal is to define an ordering of the elements such that, for each capacity $C$, the largest prefix with a combined weight of at most this capacity has a value that is as large as possible. We assume that the objective function is fractionally subadditive and that the value of single elements in in $[1, M]$ for some fixed $M \geq 1$. We show upper and lower bounds on the strict and non-strict competitive ratio of this problem. On the one hand, the strict competitive ratio is in $[\max\{\varphi+1, M\}, \max\{3.293\sqrt{M}, 2M\}]$, i.e., it is linearly growing with $M$. On the other hand, the non-strict competitive ratio turns out to be in $[2.246, \varphi+1]$, i.e., it is constant.

We remark that the results in Chapter 2 are in large parts based on joint work Yann Disser [15]. Further, large parts of Chapter 3 as well as Section 4.2 were published together with Yann Disser, Max Klimm and Kevin Schewior [19]. Lastly, Chapter 6 appeared in parts as joint work with Yann Disser and Max Klimm [20]. For further information see the introduction of the respective chapters and sections.

## 1.1. Related Work

The problem IncMax was first proposed by Bernstein et al. [5]. They consider the natural Greedy algorithm for this problem and give an upper bound of $\alpha\frac{e^\alpha}{e^\alpha - 1}$ on its competitive ratio for the subclass with $\alpha$-augmentable objective functions, which is tight for $\alpha \in \{1,2\}$ and for $\alpha \to \infty$. Furthermore, they introduce *accountable* functions. For the subclass of IncMax induced by this property they show that every algorithm has a competitive ratio of at least $2.18$, and they give a $(\varphi+1)$-competitive algorithm where $\varphi = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$ is the golden ratio. A special case of incremental maximization was investigated by Zhu et al. [73] where the goal is to incrementally insert edges into a graph such that the number of internal nodes in the resulting graph is maximized. They provide an algorithm with a competitive ratio of at most $\frac{12}{7} \approx 1.714$.

**greedy algorithm.** The Greedy algorithm was first studied for the offline variant of IncMax, which is cardinality constrained maximization. Yet, it does not use any knowledge of the size of the cardinality constraint. Thus, it can also be used for incremental maximization and results regarding the competitive/approximation ratio of the Greedy algorithm for one of these problems also hold for the other. The Greedy algorithm is known to calculate the optimum solution for all weighted rank functions over matroids due to a result by Rado [63]. Edmonds [23] proved the inverse direction of this statement, i.e., if the Greedy algorithm produces an optimum solution for all weighted rank functions over some independence system, then the independence system is a matroid. The combination of these results is often called the Rado-Edmonds theorem. Jenkyns [40] generalized the upper bound and showed that the Greedy algorithm is $\frac{1}{q}$-competitive for weighted rank functions over an independence system with rank quotient $q$. This bound was shown to be tight by Korte and Hausmann [48]. The same result was later shown by Mestre [55] with a focus on highlighting the structure that yields a competitive ratio of $\frac{1}{q}$. A different relaxation of the matroid constraint was considered by Bouchet [9] who introduced symmetric matroids and showed that the Greedy algorithm produced optimum solutions.

Nemhauser et al. [58] considered the performance of the Greedy algorithm for submodular objectives under a cardinality constraint and gave tight bounds of $\frac{\mathrm{e}}{\mathrm{e}-1}$ on the competitive ratio. This was later shown to be best possible for any algorithm that runs in polynomial time, unless $P = NP$, due to a result by Feige [26]. For submodular objectives under the constraint that the solution lies in the intersection of $p \in \mathbb{N}$ matroids, Nemhauser et al. [59] gave an upper bound of $p + 1$ on the competitive ratio. This was later refined to $p + c$ by Conforti and Cornuejols [13], where $c \in [0, 1]$ is the curvature of the objective. Since modular functions have a curvature of $c = 0$, this result also generalizes the result by Jenkyns [40]. Conforti and Cornuejols also refined the the bound of $\frac{\mathrm{e}}{\mathrm{e}-1}$ by Nemhauser et al. [58] to $c\frac{\mathrm{e}^c}{\mathrm{e}^c-1}$. The analysis of the Greedy algorithm if only an approximation of the best element can be added was considered by Goundan and Schulz [34].

Das and Kempe [14] introduced the submodularity ratio as a generalization of submodularity and gave an upper bound of $\frac{\mathrm{e}^\gamma}{\mathrm{e}^\gamma-1}$ on the competitive ratio of the Greedy algorithm under a cardinality constraint for objectives with submodularity ratio at least $\gamma$. This bound was refined by Bian et al. [6] by using the curvature $c$ of the objective. They gave an upper bound of $c\frac{\mathrm{e}^{\gamma c}}{\mathrm{e}^{\gamma c}-1}$ and proved that this is tight. Bernstein et al. [5] introduced $\alpha$-augmentability as a different generalization of submodularity and gave an upper bound of $\alpha\frac{\mathrm{e}^\alpha}{\mathrm{e}^\alpha-1}$ on the competitive ratio of the Greedy algorithm in this setting under a cardinality constraint. They gave a tight lower bound for $\alpha \in \{1, 2\}$. Krause et al. [50] considered functions that are given by the minimum of two submodular functions and showed that, in general, no competitive solutions exist.

The *continuous greedy* algorithm was introduce by Vondrák [69] as a randomized algorithm that achieves an $\frac{e}{e-1}$-approximation for combinatorial auctions. Calinescu et al. [11] showed that the continuous greedy algorithm has a competitive ratio of $\frac{e}{e-1}$ for IncMax with a submodular objective functions under any matroid constraint. This was refined by Vondrák [70] who gave an upper bound of $c\frac{e^c}{e^c-1}$ where $c$ is the curvature of the objective function. Sviridenko et al. [66] show that the continuous greedy has competitive ratio at most $\frac{e}{e-c}$ for IncMax with a monotone submodular objective function, which was later shown by Yoshida [72] to also hold under a knapsack constraint. A generalization of the continuous greedy algorithm was shown to be $(e+o(1))$-competitive for non-monotone submodular objectives under a cardinality constraint by Feldman et al. [29]. Buchbinder et al. [10] improved this upper bound to $e-0.029$ and Ene and Nguyen [24] to $e-0.03$ each by adapting the algorithm further.

Other variants of the Greedy algorithm include a Greedy algorithm that is combined with a partial enumeration technique [65] and a Greedy algorithm that is called multiple times [28].

**IncMax under a knapsack constraint.** A generalization of IncMax is a variation of the problem where, instead of an unknown cardinality constraint, we are given an unknown knapsack constraint. The competitive ratio of this problem is unbounded but Megow and Mestre [54] gave an instance sensitive near-optimal solution when the objective is modular. Under the mild assumption that every item fits into the knapsack, Navarra and Pinotti [57] were able to show an upper bound of $2$ on the competitive ratio of the problem with a modular objective, which improves to $\frac{86}{49}$ if the items have unit densities. When the objective is modular and we allow to discard items that do not fit into the knapsack and pack others instead, Disser et al. [18] were able to show that the competitive ratio is exactly $2$ in the general case and $\varphi \approx 1.618$ in the unit density case. Kawase et al. [45] considered the problem with discarding and a submodular objective. They gave an $\frac{2e}{e-1}$-competitive randomized algorithm and a $\frac{21e}{2(e-1)}$-competitive deterministic algorithm. This deterministic bound was later improved by Klimm and Knaack [46]. They parameterized their upper bound with the curvature of the objective such that it generalizes the tight bound by for monotone objectives by Disser et al. [18] and gives a bound for general submodular objectives of $2.795$.

**robustness.** Incremental Maximization is closely related to robust maximization, where the goal is to give a solution for a problem that contains a good solution of every cardinality. Hassin and Rubinstein [38] proposed the problem of robust weighted matching where a matching in a weighted graph has to be given that contains a heavy matching of every cardinality. They showed that this problem has a tight competitive ratio of $\sqrt{2}$. Matuschke et al. [53] consider randomized algorithms for this problem and provide an $\ln(4)$-competitive randomized algorithm. A generalization of this problem is the problem of giving a robust solution in the intersection of two matroids, for which Fujita et al. [30] provide a $\sqrt{2}$-competitive algorithm. Kakimura and Makino [41] generalized the problem even further by considering robust solutions in independence systems. Here, they give a $\sqrt{\mu}$-competitive solution where $\mu$ is the exchangeability of the independence system. Hassin and Segev [39] consider robust paths and trees in a weighted graph and give a $\frac{1}{\alpha}$-competitive solution that contains $\frac{\alpha}{1-\alpha^2}$ edges. Robust solutions of knapsack problems were considered by Kakimura et al. [42] and Kobayashi and Takazawa [47]. Anari et al. [2] investigated robustness problems with submodular objective function under multiple combinatorial constraints in online and offline settings, and Orlin et al. [61] provided approximation guarantees for a setting where an adversary may remove elements and the objective is monotone and submodular.

**related problems.** A related problem to IncMax is the problem of incremental maximization with a sum-objective, where the goal is not to maximize the value of the solution in every time step but rather the sum of the solution of all previous time steps. Kalinowski et al. [43] gave a 1.5-competitive algorithm for the special case of incremental maximum flow. For a subclass of problems containing matchings and matroid intersections, Goemans and Unda [32] showed an upper bound of $(9 + \sqrt{21})/15$ on the competitive ratio. In his PhD thesis, Unda Surawski [68] considered incremental minimization and maximization problems with a sum-objective and gave multiple algorithms for various problem settings, including an $\frac{\mathrm{e}+1}{\mathrm{e}}$-competitive algorithm for monotone submodular objectives and an $\frac{2\mathrm{e}-1}{\mathrm{e}}$-competitive algorithm for maximum weighted matchings.

Incremental maximization was also studied in settings where in each time step the environment changes. Hartline and Sharp consider this setting with a sum-objective for bipartite matching, maximum flow, and knapsack [36, 37]. In his PhD thesis, Sharp [64] considers even more problems, as well as the incremental minimization problem. Thielen et al. [67] investigate a form of the knapsack problem where in each step the capacity increases and new items are released and can be added to the solution.

The incremental minimization problem was investigated for various problem classes including $k$-center [33], $k$-median [56, 62, 12], and facility location [62]. A generalized approach for incremental minimization was given by Lin et al. [52]. They also note that results for the minimum latency problem, which was studied by Blum et al. [8] and Goemans and Kleinberg [31], yield results for the incremental $k$-minimum spanning tree problem. Incremental minimization with a sum-objective was investigated by Engel et al. [25] for minimum spanning trees and by Baxter et al. [4] for shortest paths.

## 1.2. Notations and Preliminaries

We denote by $\mathbb{N} = \{1, 2, 3, \dots\}$ the set of all *natural numbers*, by $\mathbb{Q}$ the set of all *rational numbers*, by $\mathbb{R}$ the set of all *real numbers*, and by $\mathbb{C}$ the set of all *complex numbers*. For a set of numbers $\mathbb{S} \in \{\mathbb{N}, \mathbb{Q}, \mathbb{R}\}$ and some real number $r \in \mathbb{R}$, we define $\mathbb{S}_{\geq r} := \{s \in \mathbb{S} \mid s \geq r\}$, and analogously $\mathbb{S}_{>r}$. For $k \in \mathbb{N}$, let $[k] := \{1, 2, \dots, k\}$, and let $[\infty] := \mathbb{N}$. We denote the *golden ratio* by $\varphi = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$.

**competitive analysis.** Now, we define some variations of the competitive ratio, namely the non-strict, the strict, and the randomized competitive ratio. The definition of the strict competitive ratio will coincide with the definition of the competitive ratio in the introduction. Whenever we talk about the strict competitive ratio and no other competitive ratios in the same context, we simply write "competitive ratio" throughout this thesis.

Let $\alpha \geq 0$. For $\rho \geq 1$, we call an incremental solution $X$ *non-strictly $\rho$-competitive with additive constant $\alpha$* if, for all $C \in \mathbb{N}$, we have $\rho \cdot f(X(C)) \geq \textsc{Opt}(C) - \alpha$. The *non-strict competitive ratio with additive constant $\alpha$* of the solution $X$ is

$$\inf\{\rho \geq 1 \mid X \text{ is non-strictly } \rho\text{-competitive with additive constant } \alpha\}.$$

For $\rho \geq 1$, we call an algorithm $\textsc{Alg}$ for a problem class $\mathcal{P}$ *non-strictly $\rho$-competitive with additive constant $\alpha$* if the incremental solution of $\textsc{Alg}$ is non-strictly $\rho$-competitive with additive constant $\alpha$ for all instances in $\mathcal{P}$. The *non-strict competitive ratio with additive constant $\alpha$* of the algorithm $\textsc{Alg}$ is

$$\inf\{\rho \geq 1 \mid \textsc{Alg} \text{ is non-strictly } \rho\text{-competitive with additive constant } \alpha\}.$$

The *non-strict competitive ratio with additive constant $\alpha$* of a problem class $\mathcal{P}$ is

$$\inf\{\rho \geq 1 \mid \text{there exists a non-strictly } \rho\text{-competitive algorithm with additive constant } \alpha \text{ for } \mathcal{P}\}.$$

If the additive constant is $\alpha = 0$, we call the competitive ratio "strict" instead of "non-strict" and omit the addition "with additive constant $\alpha$".

A *randomized algorithm* for $\textsc{IncMax}$ is one that picks a deterministic algorithm randomly according to some distribution over the class of deterministic algorithms and returns the incremental solution of this deterministic algorithm. Since the algorithm is chosen randomly, the incremental solution of the randomized algorithm $X = (e_1, e_2, \dots)$ is a random variable. Similarly, the solution of $X$ for cardinality $C \in \mathbb{N}$, $X(C) = \{e_1, \dots, e_C\}$ is a random variable. The *randomized competitive ratio* of $X$ is

$$\sup_{C \in \mathbb{N}} \frac{\textsc{Opt}(C)}{\mathbb{E}[f(X(C))]}.$$

Similar to the definition of the (non-strict) competitive ratio, the *randomized competitive ratio* of a problem class is the infimum over the randomized competitive ratios of all randomized algorithms for this problem class.

**Remark 1.1.** *As it is usual in competitive analysis, we do not only consider algorithms that have a polynomial running time, but also those with worse running times. Yet, all algorithms presented in this work can run in polynomial time if we are given an algorithm that returns the set $O(C)$ for a given $C \in \mathbb{N}$ in polynomial time.*

**objective properties.** Throughout this thesis, we consider multiple subclasses of IncMax that are induced by different properties of the objective. In the following, we define some of these properties. Figure 1.3 gives an overview how the different properties relate to each other. For the definitions of $\gamma$-$\alpha$-augmentability and $\beta$-accountability, we refer to Definitions 2.3 and 5.1.

**Definition 1.2.** *A function $f \colon 2^U \to \mathbb{R}_{\geq 0}$ is called*

- modular *if, for all $A \subseteq U$ and $e \in U$, we have $f(A \cup \{e\}) = f(A) + f(\{e\})$,*
- submodular *if, for all $A, B \subseteq U$, we have $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$,*
- subadditive *if, for all $A, B \subseteq U$, we have $f(A \cup B) \leq f(A) + f(B)$,*
- fractionally subadditive *if, for all $A, B_1, \dots, B_k \subseteq U$ and all $\alpha_1, \dots, \alpha_k \in \mathbb{R}_{\geq 0}$ with $\sum_{i \in [k] : e \in B_i} \alpha_i \geq 1$ for all $e \in A$, we have*

$$f(A) \leq \sum_{i=1}^{k} \alpha_i f(B_i),$$

- accountable *if, for all finite $S \subseteq U$, there exists $e \in S$ with $f(S \setminus \{e\}) \geq (1 - 1/|S|) f(S)$,*

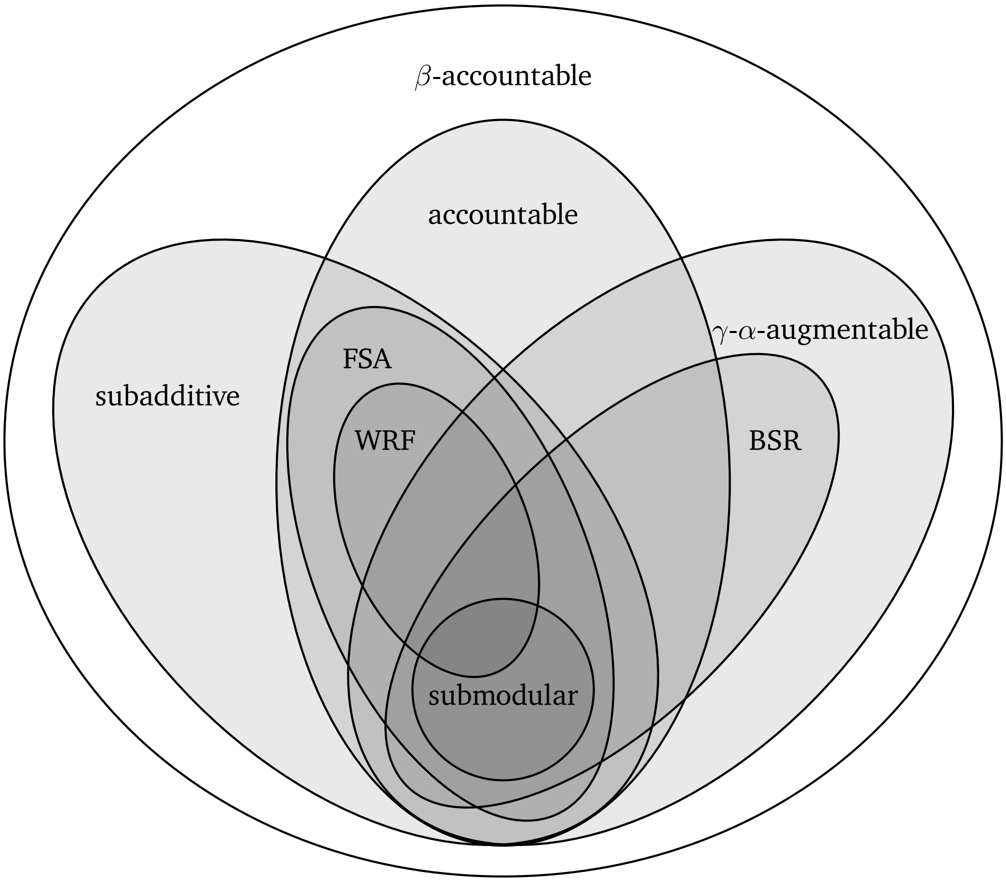


Figure 1.3.: Relation of the different objective properties used throughout this paper. We use the following abbreviations: FSA - fractionally subadditive, WRF - weighted rank function, BSR - bounded submodularity ratio. For the definitions of $\gamma$-$\alpha$-augmentability and $\beta$-accountability, we refer to Definitions 2.3 and 5.1. The parameter $\beta \in (0, 1]$ has to be chosen as $\beta = \min\{\frac{1}{2}, \frac{\gamma}{\alpha}\}$.

- $\alpha$-augmentable *for* $\alpha \geq 1$ *if, for all* $A, B \subseteq U$ *with* $B \setminus A \neq \emptyset$*, there exists* $b \in B \setminus A$ *with*
$$f(A \cup \{b\}) - f(A) \geq \frac{f(A \cup B) - \alpha f(A)}{|B|}.$$

Note that every modular function is submodular, and every submodular function is fractionally subadditive [51] and 1-augmentable [5]. Furthermore, every fractionally subadditive function is subadditive [51].

**Definition 1.3** ([14])**.** *The* submodularity ratio *of* $f\colon 2^U \to \mathbb{R}_{\geq 0}$ *is (using* $\frac{0}{0} := 1$*)*

$$\gamma(f) := \inf_{A \in 2^U, B \subseteq U \setminus A} \frac{\sum_{b \in B}(f(A \cup \{b\}) - f(A))}{f(A \cup B) - f(A)} \in [0, 1].$$

By definition, submodular functions have submodularity ratio 1.

**Definition 1.4** ([6])**.** *The* curvature *of* $f\colon 2^U \to \mathbb{R}_{\geq 0}$ *is*

$$1 - \inf\left\{ \frac{f(A \cup B \cup \{e\}) - f(A \cup B)}{f(A \cup \{e\}) - f(A)} \middle| A, B \subseteq U, e \in U \setminus (A \cup B) \right\}.$$

The curvature is a measure how close a function is to being modular. If the curvature is 0 it is modular, and if it is 1 there are items that completely lose their value when added to large sets.

**Definition 1.5.** *An* independence system *is a tuple* $(U, \mathcal{I})$*, where* $\mathcal{I} \subseteq 2^U$ *is closed under taking subsets and* $\emptyset \in \mathcal{I}$*. For a given weight function* $w\colon U \to \mathbb{R}_{\geq 0}$*, the* weighted rank function *of* $(U, \mathcal{I})$ *is given by* $f(X) = \max\{\sum_{x \in Y} w(x) | Y \in \mathcal{I} \cap 2^X\}$*. The set* $\mathcal{B}(X)$ *of all* bases *of some set* $X \subseteq U$ *is defined to be the set of inclusion-wise maximal subsets of* $\mathcal{I} \cap 2^X$*, i.e.,* $\mathcal{B}(X) := \{B \in \mathcal{I} \cap 2^X | \forall x \in X \setminus B\colon B \cup \{x\} \notin \mathcal{I}\}$*. The* rank quotient *of an independence system* $(U, \mathcal{I})$ *is* $q(U, \mathcal{I}) := \min_{X \subseteq U} \min_{B, B' \in \mathcal{B}(X)} |B|/|B'|$*, where we set* $\frac{0}{0} := 1$.

The weighted rank function of an independence system can be represented as the maximum over modular functions and is therefore fractionally subadditive [1].

**graphs.** We denote a *directed graph* by $G = (V, E)$ where $V$ is the *vertex set* and $E \subseteq V \times V$ the *edge set*. For a vertex $v \in V$, we define $\delta^+(v) := (\{v\} \times V) \cap E$ to be the set of *outgoing edges* from $v$, and $\delta^-(v) := (V \times \{v\}) \cap E$ to be the set of *incoming edges* to $v$. A *flow* in $G$ with respect to a capacity function $\mu\colon E \to \mathbb{R}_{\geq 0}$ is a function $\vartheta\colon E \to \mathbb{R}_{\geq 0}$ that satisfies

$$\begin{aligned} \vartheta(e) &\leq \mu(e) & \forall e \in E \quad & \text{(capacity constraint)}, \\ \mathrm{ex}_\vartheta(v) &= 0 & \forall v \in V \quad & \text{(flow conservation)}, \end{aligned}$$

where the *excess* of a vertex $v \in V$ is defined as

$$\mathrm{ex}_\vartheta(v) := \sum_{e \in \delta^-(v)} \vartheta(e) - \sum_{e \in \delta^+(v)} \vartheta(e),$$

and the excess of a set $V' \subseteq V$ is $\mathrm{ex}_\vartheta(V') = \sum_{v \in V'} \mathrm{ex}_\vartheta(v)$. Let $G = (V, E)$ be a directed graph with a designated *source* $s \in V$ and a set of *sinks* $T \subseteq V \setminus \{s\}$. An *$s$-$T$-flow* in $G$ with respect to a capacity function $\mu\colon E \to \mathbb{R}_{\geq 0}$ is a function $\vartheta\colon E \to \mathbb{R}_{\geq 0}$ that satisfies

$$\begin{aligned}
\vartheta(e) &\leq \mu(e) && \forall e \in E && \text{(capacity constraint)},\\
\mathrm{ex}_\vartheta(v) &= 0 && \forall v \in V \setminus (\{s\} \cup T) && \text{(flow conservation)},\\
\mathrm{ex}_\vartheta(t) &\geq 0 && \forall t \in T && (T \text{ are sinks}).
\end{aligned}$$

A *maximum $s$-$T$-flow* $\vartheta^*\colon E \to \mathbb{R}_{\geq 0}$ is an $s$-$T$-flow that maximizes the excess of $T$, i.e., $\mathrm{ex}_{\vartheta^*}(T) = \max\{\mathrm{ex}_\vartheta(T) \mid \vartheta \text{ is an } s\text{-}T\text{-flow in } G\}$. If $T$ contains only one vertex $T = \{t\}$, we may also write $s$-$t$-flow instead of $s$-$T$-flow.

**a useful estimate.** We will often encounter sequences $(c_1, c_2, \dots)$ where every value in the sequence is at least as large as the previous value multiplied by some $\delta > 0$. The following estimate will be useful in this context.

**Lemma 1.6.** *Let $\delta > 1$ and $(c_1, \dots, c_k)$ with $c_{i+1} \geq \delta c_i$ for all $i \in [k-1]$. Then $\sum_{i=1}^k c_i < \frac{\delta}{\delta-1} c_k$. In particular, if $\delta = \varphi + 1$, we have $\sum_{i=1}^k c_i < \varphi c_k$.*

*Proof.* The fact that, for all $i \in [k-1]$, we have $c_{i+1} \geq \delta c_i$ yields $c_k \geq \delta^{k-i} c_i$ for all $i \in [k]$. We obtain

$$\sum_{i=1}^k c_i \leq c_k \sum_{i=1}^k \frac{1}{\delta^{k-i}} < c_k \sum_{i=1}^\infty \frac{1}{\delta^{k-i}} = \frac{1}{1 - \frac{1}{\delta}} c_k = \frac{\delta}{\delta - 1} c_k.$$

If $\delta = \varphi + 1$, we have

$$\frac{\delta}{\delta - 1} = \frac{\varphi + 1}{\varphi} = \frac{\varphi^2}{\varphi} = \varphi.$$

□

# 2. The Greedy Algorithm

Probably, the most natural approach to define an incremental solution for the IncMax problem is to add the elements of $U$ one by one, and, in each step, to choose the element that yields the largest increase in the objective value.[1] The algorithm that does this is the Greedy algorithm. For $i \in \mathbb{N}$, it iteratively chooses elements

$$e_i \in \arg\max_{e \in U \setminus \{e_1,\dots,e_{i-1}\}} f(\{e_1 \dots, e_{i-1}, e\}).$$

If the choice is not unique, it chooses an element from the set in an arbitrary but fixed way. The incremental solution of the Greedy algorithm is $X^{\mathrm{G}} := (e_1, e_2, \dots)$. Note that we have $X^{\mathrm{G}}(i) = \{e_1, \dots, e_i\}$ for all $i \in \mathbb{N}$. While this algorithm is widely used in practical applications, the competitive ratio of the Greedy algorithm can be arbitrarily bad. To see this, consider the following example.

**Example 2.1.** *Let $k \in \mathbb{N}$, and let $U_1, U_2$ be disjoint sets with $|U_1| = |U_2| = k$. We define $U := U_1 \cup U_2$ and, with $\varepsilon \in (0, \frac{1}{k}]$, the objective function*

$$f(S \subseteq U) := \begin{cases} 0, & \text{if } S = \emptyset, \\ \max\{1 + |S \cap U_1|\varepsilon, |S \cap U_2|\}, & \text{else.} \end{cases}$$

*We set the (unknown) capacity constraint to be $C = k$. The first element added by the Greedy algorithm is one from the set $U_1$ because it increases the objective value by $1 + \varepsilon$, while the elements from set $U_2$ increase the objective value only by $1$. In the following $k - 1$ iterations, the Greedy algorithm adds the remaining elements from $U_1$ because they each increase the objective value by $\varepsilon$, while the elements from $U_2$ do not increase the objective value. After $k$ steps, the value of the greedy solution is $1 + k\varepsilon \leq 2$, while the optimum solution is the set $U_2$ with a value of $k$. As we can choose $k \in \mathbb{N}$ arbitrarily large, the competitive ratio of the Greedy algorithm can be arbitrarily bad.*

[1] If the groundset is infinitely large, such an element might not exists. In this case, we can instead add an arbitrarily good approximation. The competitive ratio of the Greedy algorithm is negligibly worse. Thus, throughout the chapter we assume that there always exists an element maximizing the marginal value when it is added.

A natural question in this context is, for which objective functions $f$ the Greedy algorithm gives a good solution, i.e., one with a bounded competitive ratio. We are interested in characterizing these objective functions.

A well-known class of functions for which the Greedy algorithm has a bounded competitive ratio of (exactly) $\frac{e}{e-1}$ are the monotone, submodular functions [58]. This class includes, for example, the maximum coverage problem, but fails to capture many other greedily approximable settings. See Figure 2.1 along with the following.

Das and Kempe [14] introduced the class of functions with bounded submodularity ratio (cf. Definition 1.3) as a generalization of submodular functions. This class was further generalized by Bian et al. [6] to the class of functions with bounded weak submodularity ratio.

**Definition 2.2** ([6]). *The* weak submodularity ratio *of* $f\colon 2^U \to \mathbb{R}_{\geq 0}$ *is (using* $\frac{0}{0} := 1$*)*

$$\gamma(f) := \inf_{A \in \{X^G(0), X^G(1), \dots\}, B \subseteq U \setminus A} \frac{\sum_{b \in B}(f(A \cup \{b\}) - f(A))}{f(A \cup B) - f(A)} \in [0, 1].$$

Das and Kempe [14] showed an upper bound of $\frac{e^\gamma}{e^\gamma - 1}$ on the competitive ratio of the Greedy algorithm for the set of all monotone functions with submodularity ratio at least $\gamma > 0$. Bian et al. [6] extended this to a tight bound that is additionally parameterized by the curvature $c$ of the objective. This tight bound is $c\frac{e^{c\gamma}}{ec\gamma - 1}$. Since submodular functions have submodularity ratio 1 and curvature $c \in [0, 1]$, this bound generalizes the submodular bound. Crucially, it is easy to verify that these results carry over to the set $\tilde{\mathcal{F}}_\gamma$ of all monotone functions with *weak* submodularity ratio at least $\gamma > 0$.[2]

Bernstein et al. [5] proposed another generalization of submodularity, $\alpha$-augmentability (cf. Definition 1.2). They showed that the Greedy algorithm has a competitive ratio of at most $\alpha \cdot \frac{e^\alpha}{e^\alpha - 1}$ on the set $\mathcal{F}_\alpha$ of monotone, $\alpha$-augmentable functions, for $\alpha \geq 1$, and that this bound is tight for $\alpha \in \{1, 2\}$ and in the limit $\alpha \to \infty$. Since submodular functions are 1-augmentable, this bound again generalizes the submodular bound. The class of $\alpha$-augmentable problems captures the objective of the maximum (weighted) $\alpha$-dimensional matching problem, which is not submodular. In this chapter, we introduce a natural $\alpha$-commodity flow variant that is $\alpha$-augmentable, and we prove a tight lower bound on the competitive ratio for all $\alpha \geq 1$.

Another, well-known setting, besides submodularity, where the Greedy algorithm has a bounded competitive ratio, are weighted rank functions of independence systems of bounded rank quotient [49] (cf. Definition 1.5). Jenkyns [40] and Korte and Hausmann [48] showed that the Greedy algorithm has a competitive ratio of exactly $1/q$ on

[2] Here and throughout we use the notation $\tilde{\mathcal{F}}$ as opposed to $\mathcal{F}$ to refer to a function class based on a *weak* definition.

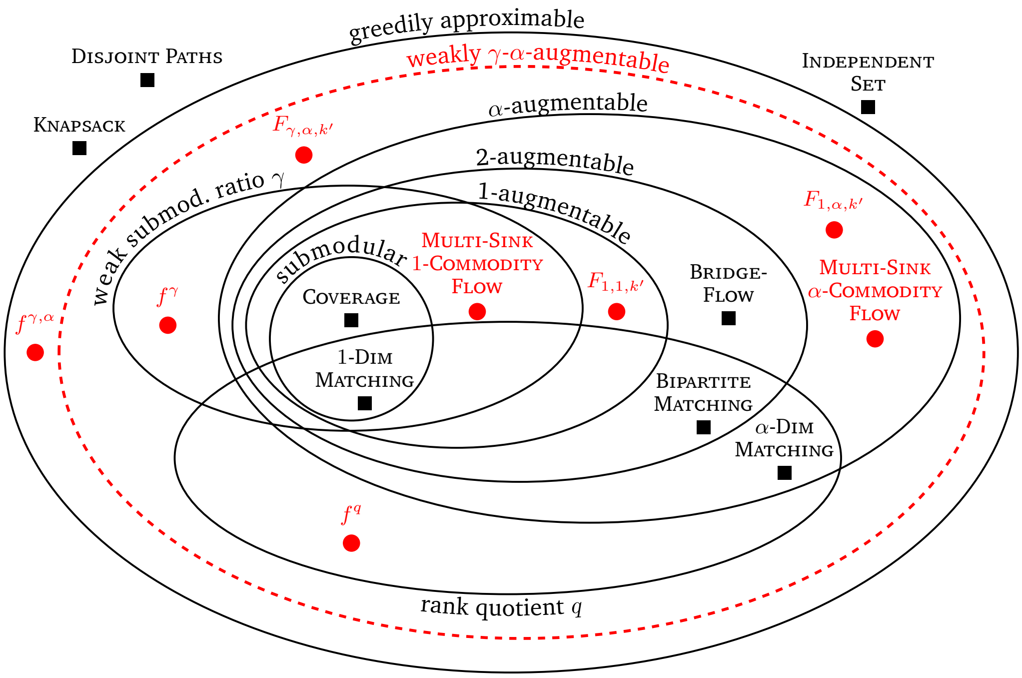


Figure 2.1.: Relation of the different problem classes (nodes) and objective properties (ellipses). Anything that is contained within one ellipse has the property the ellipse stands for. Newly introduced classes and problems are marked in red with dashed lines and round nodes. The parameter $k'$ is chosen sufficiently large, depending on $\gamma$ and $\alpha$.

the set $\mathcal{F}_q$ of all weighted rank functions of independence systems with rank quotient at least $q > 0$.[3]

The goal of this chapter is to unify and to generalize the above classes of functions for which the Greedy algorithm has a bounded approximation ratio.

In Section 2.1 we introduce a natural $\alpha$-augmentable variant of multi-commodity flow. Besides the $\alpha$-dimensional matching problem, to our knowledge, this problem is the only other natural $\alpha$-augmentable problem to date. We will construct a family of instances of this problem that yield a tight lower bound for the competitive ratio of the Greedy algorithm on the class of monotone and $\alpha$-augmentable problems for $\alpha \in \mathbb{N}$. This closes

[3] Note that we abuse notation, since, e.g., $\mathcal{F}_\alpha \neq \mathcal{F}_q$ for $\alpha = q = 1$. However, the set of functions we are referring to will always be clear by the naming of the indices.

the gap left [5] for $\alpha \in \mathbb{N}_{\geq 3}$.

In Section 2.2 we will observe that each of the classes $\tilde{\mathcal{F}}_\gamma$, $\mathcal{F}_\alpha$, and $\mathcal{F}_q$ captures greedily approximable objectives that are not contained in either of the other two classes (cf. Figure 2.1). This motivates the definition of the following property.

**Definition 2.3.** *The function* $f\colon 2^U \to \mathbb{R}_{\geq 0}$ *is* $\gamma$-$\alpha$-augmentable *for* $\gamma \in (0,1]$ *and* $\alpha \geq \gamma$ *if, for all* $A, B \subseteq U$ *with* $B \setminus A \neq \emptyset$*, there exists* $b \in B$ *with*

$$f(A \cup \{b\}) - f(A) \geq \frac{\gamma f(A \cup B) - \alpha f(A)}{|B|}.$$

*We call* $f$ weakly $\gamma$-$\alpha$-augmentable *if this only holds for all* $A \in \{X^G(0), X^G(1), \dots\}$.

In order to capture as many functions as possible, we will consider the weak variant of this definition, which enforces its defining property only for "greedy sets". However, any upper bound on the approximation ratio immediately carries over to the same bound in the stronger definition. Also note that $\gamma$-$\alpha$-augmentability only requires $\alpha \geq \gamma$, unlike $\alpha$-augmentability where $\alpha \geq 1$. This is in line with the definitions of $\alpha$-augmentability where $\gamma = 1$ and of the submodularity ratio where $\alpha = \gamma$. We let $\tilde{\mathcal{F}}_{\gamma,\alpha}$ denote the set of all weakly $\gamma$-$\alpha$-augmentable functions.

Finally, in Section 2.3 we show that the function class $\tilde{\mathcal{F}}_{\gamma,\alpha}$ contains the other three classes $\tilde{\mathcal{F}}_\gamma$, $\mathcal{F}_\alpha$, and $\mathcal{F}_q$, as well as additional functions (cf. Figure 2.1). We show that the competitive ratio of the GREEDY algorithm for INCMAX problems with $\gamma$-$\alpha$-augmentable objective with curvature $c$ (cf. Definition 1.4) is at most $\frac{\alpha-(1-c)\gamma}{\gamma} \cdot \frac{e^{\alpha-(1-c)\gamma}}{e^{\alpha-(1-c)\gamma}-1}$ which recovers the known upper bounds for the class of monotone, $\alpha$-augmentable functions, and for the class of monotone functions of bounded (weak) submodularity ratio with curvature $c$. For curvature $c = 1$, we show that this bound is tight. For $\gamma = 1$, the tight lower bound is obtained with an $\alpha$-augmentable function. This means that, in particular, we are able to close the gap left in [5] by showing a tight lower bound for monotone, $\alpha$-augmentable objectives for all $\alpha \geq 1$. Lastly, we recover the upper bound for the competitive ratio of the GREEDY algorithm on the class of weighted rank functions of some independence system by showing that, for $\gamma$-$\alpha$-augmentable weighted rank functions, the competitive ratio is $\frac{\alpha}{\gamma}$.

An extended abstract with most of the results in this chapter was published in [15] and a full version will soon appear in [16]. A new result in this thesis is the introduction of the dependency on the curvature in Theorem 2.17.

## 2.1. The Multi-Sink $\alpha$-Commodity Flow problem

In this section, we introduce a natural $\alpha$-commodity flow problem that models, e.g., production processes where the output is limited by availability of all components. The objective of this problem is (exactly) $\alpha$-augmentable, but, for $\alpha \in \mathbb{N} \setminus \{1\}$, does not have a bounded (weak) submodularity ratio and cannot be expressed as a weighted rank function over an independence system. We will show that this problem also gives a tight lower bound for the competitive ratio of the Greedy algorithm on the class of monotone, $\alpha$-augmentable functions, for $\alpha \in \mathbb{N}$. We will extend this lower bound to all $\alpha \geq 1$ in Section 2.3.1, and thus close a gap left in [5].

We extend the notion of $s$-$T$-flows to multi-commodity flows, where each commodity has an independent capacity function.

**Definition 2.4.** *Let $\alpha \in \mathbb{N}$ and $G = (V, E)$ be a graph with $s \in V$ and $T \subseteq V$. Furthermore, let $\boldsymbol{\mu} = (\mu_i \colon E \to \mathbb{R}_{\geq 0})_{i \in [\alpha]}$ be capacity functions. A* multicommodity-flow *in $G$ w.r.t. $\boldsymbol{\mu}$ is a tuple $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_\alpha)$, where $\vartheta_i$ is an $s$-$T$-flow in $G$ with respect to capacity function $\mu_i$. The* minimum-excess *of a sink vertex $t \in T$ in $\boldsymbol{\vartheta}$ is*

$$\operatorname{minex}_{\boldsymbol{\vartheta}}(t) := \min_{i \in [\alpha]} \operatorname{ex}_{\vartheta_i}(t).$$

For convenience, we define $\mu(u, v) := \mu((u, v))$ and $\vartheta(u, v) := \vartheta((u, v))$ For $T' \subseteq T$, we let $\operatorname{minex}_{\boldsymbol{\vartheta}}(T') := \sum_{t \in T'} \operatorname{minex}_{\boldsymbol{\vartheta}}(t)$ in the following.

An instance of the problem Multi-Sink $\alpha$-Commodity Flow, for $\alpha \in \mathbb{N}$, is given by a tuple $(G, s, T, \boldsymbol{\mu})$, where $G = (V, E)$ is a directed graph, $s \in V$ is a source vertex, $T \subseteq V$ is a set of sink vertices, and $\boldsymbol{\mu} = (\mu_i \colon E \to \mathbb{R}_{\geq 0})_{i \in [\alpha]}$ are capacity functions. The problem is to find a subset of sinks $S \subseteq T$ with $|S| \leq k$ that maximizes the objective function

$$f(S) = \max_{\boldsymbol{\vartheta} \in \mathcal{M}_{G,\mu}} \operatorname{minex}_{\boldsymbol{\vartheta}}(S),$$

where $\mathcal{M}_{G,\mu}$ denotes the set of all multicommodity-flows in $G$ w.r.t. capacities $\boldsymbol{\mu}$.

**Example 2.5.** *For a prototypical application of Multi-Sink $\alpha$-Commodity Flow, consider a factory where $k \in \mathbb{N}$ machines are to be built in a set $T$ of potential locations. Each machine produces the same item and needs a number $\alpha \in \mathbb{N}$ of different resources. The output of a machine is limited by the resource it has available the least. All resources are delivered to the machines along different routes within the factory, e.g., some liquids might be transported via pipes, other resources might be transported on a conveyor belt or on pallets. The objective is to determine in which $k$ locations the machines should be constructed in order to maximize overall production.*

**Theorem 2.6.** *For every $\alpha \in \mathbb{N}$, the objective of* Multi-Sink *$\alpha$*-Commodity Flow *is monotone and $\alpha$-augmentable.*

*Proof.* Let $A \subseteq T$ and $t \in T \setminus A$. To prove monotonicity, fix some multicommodity-flow $\boldsymbol{\vartheta}$ with $\mathrm{minex}_{\boldsymbol{\vartheta}}(A) = f(A)$. By definition, $\mathrm{minex}_{\boldsymbol{\vartheta}}(A) \leq \mathrm{minex}_{\boldsymbol{\vartheta}}(A \cup \{t\}) \leq f(A \cup \{t\})$ holds and thus $f$ is monotone.

To show $\alpha$-augmentability, let $(G, s, T, \mu)$ be an instance of Multi-Sink $\alpha$-Commodity Flow. Let $A, B \subseteq T$ such that $B' := B \setminus A \neq \emptyset$. We show that there exists $b \in B'$ with

$$f(A \cup \{b\}) - f(A) \geq \frac{f(A \cup B') - \alpha f(A)}{|B'|}.$$

This suffices because, with

$$\frac{f(A \cup B') - \alpha f(A)}{|B'|} = \frac{f(A \cup B) - \alpha f(A)}{|B'|} \geq \frac{f(A \cup B) - \alpha f(A)}{|B|},$$

$\alpha$-augmentability of $f$ follows.

Let $\boldsymbol{\vartheta}^{A \cup B'} = (\vartheta_1^{A \cup B'}, \ldots, \vartheta_\alpha^{A \cup B'})$ be a multicommodity-flow in $G$ that maximizes the minimum-excess $\mathrm{minex}_{\boldsymbol{\vartheta}^{A \cup B'}}(A \cup B')$, i.e., $\mathrm{minex}_{\boldsymbol{\vartheta}^{A \cup B'}}(A \cup B') = f(A \cup B')$, such that $\vartheta_i^{A \cup B'}$ is a maximum $s$-$(A \cup B')$-flow w.r.t. capacity $\mu_i$ for all $i \in [\alpha]$. Such a multicommodity-flow can, for example, be obtained by augmenting a flow that maximizes $\mathrm{minex}_{\boldsymbol{\vartheta}^{A \cup B'}}(A \cup B')$ with the Edmonds-Karp algorithm (cf. [49]). Furthermore, we let $\boldsymbol{\vartheta}^A = (\vartheta_1^A, \ldots, \vartheta_\alpha^A)$ be a multicommodity-flow in $G$ with $\mathrm{minex}_{\boldsymbol{\vartheta}^A}(A) = f(A)$, as well as $\mathrm{ex}_{\vartheta_i^A}(A) = f(A)$ and $\mathrm{ex}_{\vartheta_i^A}(T \setminus A) = 0$ for all $i \in [\alpha]$, i.e., $\boldsymbol{\vartheta}^A$ maximizes the minimum-excess of the set $A$ while the values of all flows $\vartheta_i^A$ are as small as possible. This multicommodity-flow can be obtained by reducing the flows of a multicommodity-flow that maximizes $\mathrm{minex}_{\boldsymbol{\vartheta}^A}(A \cup B')$ along paths of a path decomposition of the flow (cf. [49]). We define the function $g \colon A \to [\alpha]$, such that, for all $x \in A$, no flow $\tilde{\vartheta}$ w.r.t. capacity $\mu_{g(x)}$ exists with $\mathrm{ex}_{\tilde{\vartheta}}(x') \geq \mathrm{ex}_{\vartheta^A_{g(x)}}(x')$ for all $x' \in A \setminus \{x\}$ and with $\mathrm{ex}_{\tilde{\vartheta}}(x) > \mathrm{ex}_{\vartheta^A_{g(x)}}(x)$. This means that the flow $\vartheta^A_{g(x)}$ is one of the flows limiting the value of $\mathrm{minex}_{\boldsymbol{\vartheta}^A}(x)$. Let $g^{-1}(i) = \{x \in A \mid g(x) = i\}$ for all $i \in [\alpha]$ be the preimage of $g$. Obviously

$$\bigcup_{i=1}^{\alpha} g^{-1}(i) = A. \tag{2.1}$$

We add a super sink $t$ to $G$ and, with $\tilde{V} := V \cup \{t\}$ and $\tilde{E} := E \cup \{(v, t) \mid v \in (A \cup B')\}$, let $\tilde{G} = (\tilde{V}, \tilde{E})$ denote the resulting graph. Furthermore, we define the capacity functions

$\tilde{\mu}_i\colon \tilde{E} \to \mathbb{R}_{\geq 0}$ for all $i \in [\alpha]$ such that, for $(u,v) \in \tilde{E}$,

$$\tilde{\mu}_i(u,v) := \begin{cases} \mu_i(u,v), & \text{if } (u,v) \in E, \\ \max\{\mathrm{ex}_{\vartheta_i^A}(u), \mathrm{ex}_{\vartheta_i^{A\cup B'}}(u)\}, & \text{if } (u,v) \in A \times \{t\}, \\ \mathrm{ex}_{\vartheta_i^{A\cup B'}}(u), & \text{if } (u,v) \in B' \times \{t\}. \end{cases}$$

Now we extend the flow $\boldsymbol{\vartheta}^{A\cup B'}$ to a flow $\tilde{\boldsymbol{\vartheta}}^{A\cup B'}$ in $\tilde{G}$, such that, for all $i \in [\alpha]$ and $(u,v) \in \tilde{E}$,

$$\tilde{\vartheta}_i^{A\cup B'}(u,v) := \begin{cases} \vartheta_i^{A\cup B'}(u,v), & \text{if } (u,v) \in E, \\ \mathrm{ex}_{\vartheta_i^{A\cup B'}}(u), & \text{else}, \end{cases}$$

holds, and analogously, we extend the flow $\boldsymbol{\vartheta}^A$ to a flow $\tilde{\boldsymbol{\vartheta}}^A$ in $\tilde{G}$. With this definition, $\tilde{\vartheta}_i^{A\cup B'}$ is a maximum $s$-$t$-flow w.r.t. capacity $\tilde{\mu}_i$, because $\vartheta_i^{A\cup B'}$ is a maximum $s$-$(A\cup B')$-flow w.r.t. capacity $\mu_i$.

For $i \in [\alpha]$, let $\tilde{\vartheta}_i$ be a maximum $s$-$t$-flow w.r.t. capacity $\tilde{\mu}_i$ in $\tilde{G}$ obtained from $\tilde{\vartheta}_i^A$ by using the Edmonds-Karp algorithm. Then its value is exactly

$$\mathrm{ex}_{\vartheta_i^{A\cup B'}}(A\cup B')$$

because $\vartheta_i^{A\cup B'}$ is a maximum $s$-$(A\cup B')$-flow. We project $\tilde{\vartheta}_i$ onto a flow in $G$, i.e., we set $\vartheta_i := \tilde{\vartheta}_i|_E$ for $i \in [\alpha]$ and define $\boldsymbol{\vartheta} := (\vartheta_1, \ldots, \vartheta_\alpha)$. For all $x \in A$, by definition of $\boldsymbol{\vartheta}$, we have $\mathrm{ex}_{\vartheta_i}(x) \geq \mathrm{ex}_{\vartheta_i^A}(x)$, and thus, by definition of $g$,

$$\mathrm{ex}_{\vartheta_{g(x)}}(x) = \mathrm{ex}_{\vartheta_{g(x)}^A}(x). \tag{2.2}$$

Because $\tilde{\vartheta}_i$ is a maximum $s$-$t$-flow in $\tilde{G}$ w.r.t. capacity $\tilde{\mu}_i$, $\vartheta_i$ is a maximum $s$-$(A\cup B')$-flow w.r.t. capacity $\mu_i$ in $G$. Since $\vartheta_i^{A\cup B'}$ is also a maximum $s$-$(A\cup B')$-flow w.r.t. capacity $\mu_i$, we have

$$\mathrm{ex}_{\vartheta_i}(A\cup B') = \mathrm{ex}_{\vartheta_i^{A\cup B'}}(A\cup B'). \tag{2.3}$$

For all $x \in A$, we know that the excess of $x$ in $\vartheta_i$ is as large as the flow $\tilde{\vartheta}_i(x,t)$, i.e.,

$$\begin{aligned} \mathrm{ex}_{\vartheta_i}(x) &= \tilde{\vartheta}_i(x,t) \leq \tilde{\mu}_i(x,t) \\ &= \max\{\mathrm{ex}_{\vartheta_i^A}(x), \mathrm{ex}_{\vartheta_i^{A\cup B'}}(x)\} \\ &\leq \mathrm{ex}_{\vartheta_i^A}(x) + \mathrm{ex}_{\vartheta_i^{A\cup B'}}(x). \end{aligned} \tag{2.4}$$

By maximality of $\boldsymbol{\vartheta}^A$ and because $\mathrm{ex}_{\vartheta_i}(x) \geq \mathrm{ex}_{\vartheta_i^A}(x)$ for all $x \in A$, we have

$$\mathrm{minex}_{\boldsymbol{\vartheta}}(A) = \mathrm{minex}_{\boldsymbol{\vartheta}^A}(A) = f(A). \tag{2.5}$$

Since $A \cap B' = \emptyset$, we obtain

$$\begin{aligned}
& \mathrm{ex}_{\vartheta_i^{A\cup B'}}(B') - \mathrm{ex}_{\vartheta_i}(B') \\
= \; & \mathrm{ex}_{\vartheta_i^{A\cup B'}}(A \cup B') - \mathrm{ex}_{\vartheta_i}(A \cup B') - \mathrm{ex}_{\vartheta_i^{A\cup B'}}(A) + \mathrm{ex}_{\vartheta_i}(A) \\
\overset{(2.3)}{=} \; & \mathrm{ex}_{\vartheta_i}(A) - \mathrm{ex}_{\vartheta_i^{A\cup B'}}(A) \\
= \; & \sum_{x \in A \setminus g^{-1}(i)} \big( \mathrm{ex}_{\vartheta_i}(x) - \mathrm{ex}_{\vartheta_i^{A\cup B'}}(x) \big) + \sum_{x \in g^{-1}(i)} \big( \mathrm{ex}_{\vartheta_i}(x) - \mathrm{ex}_{\vartheta_i^{A\cup B'}}(x) \big) \\
\overset{(2.4),(2.2)}{\leq} \; & \sum_{x \in A \setminus g^{-1}(i)} \mathrm{ex}_{\vartheta_i^{A}}(x) + \sum_{x \in g^{-1}(i)} \big( \mathrm{ex}_{\vartheta_i^{A}}(x) - \mathrm{ex}_{\vartheta_i^{A\cup B'}}(x) \big) \\
= \; & f(A) - \sum_{x \in g^{-1}(i)} \mathrm{ex}_{\vartheta_i^{A\cup B'}}(x), \qquad (2.6)
\end{aligned}$$

where we used minimality of $\boldsymbol{\vartheta}^A$. Using this we can compute

$$\begin{aligned}
\mathrm{minex}_{\boldsymbol{\vartheta}^{A\cup B'}}(B') \; & = \sum_{b \in B'} \min_{i \in [\alpha]} \big\{ \mathrm{ex}_{\vartheta_i^{A\cup B'}}(b) \big\} \\
& = \sum_{b \in B'} \min_{i \in [\alpha]} \big\{ \mathrm{ex}_{\vartheta_i}(b) + \big( \mathrm{ex}_{\vartheta_i^{A\cup B'}}(b) - \mathrm{ex}_{\vartheta_i}(b) \big) \big\} \\
& \leq \sum_{b \in B'} \left( \min_{i \in [\alpha]} \big\{ \mathrm{ex}_{\vartheta_i}(b) \big\} + \sum_{i=1}^{\alpha} \big( \mathrm{ex}_{\vartheta_i^{A\cup B'}}(b) - \mathrm{ex}_{\vartheta_i}(b) \big) \right) \\
& = \mathrm{minex}_{\boldsymbol{\vartheta}}(B') + \sum_{i=1}^{\alpha} \big( \mathrm{ex}_{\vartheta_i^{A\cup B'}}(B') - \mathrm{ex}_{\vartheta_i}(B') \big) \\
& \overset{(2.6)}{\leq} \mathrm{minex}_{\boldsymbol{\vartheta}}(B') + \sum_{i=1}^{\alpha} \left( f(A) - \sum_{x \in g^{-1}(i)} \mathrm{ex}_{\vartheta_i^{A\cup B'}}(x) \right) \\
& \overset{(2.1)}{=} \mathrm{minex}_{\boldsymbol{\vartheta}}(B') + \alpha f(A) - \sum_{x \in A} \mathrm{ex}_{\vartheta_{g(x)}^{A\cup B'}}(x). \qquad (2.7)
\end{aligned}$$

Finally, because $A \cap B' = \emptyset$, we get

$$\begin{aligned} f(A \cup B') &= \operatorname{minex}_{\boldsymbol{\vartheta}^{A \cup B'}}(A) + \operatorname{minex}_{\boldsymbol{\vartheta}^{A \cup B'}}(B') \\ &= \sum_{x \in A} \big(\min_{i \in [\alpha]} \operatorname{ex}_{\vartheta_i^{A \cup B'}}(x)\big) + \operatorname{minex}_{\boldsymbol{\vartheta}^{A \cup B'}}(B') \\ &\overset{(2.7)}{\leq} \sum_{x \in A} \operatorname{ex}_{\vartheta_{g(x)}^{A \cup B'}}(x) + \operatorname{minex}_{\boldsymbol{\vartheta}}(B') + \alpha f(A) - \sum_{x \in A} \operatorname{ex}_{\vartheta_{g(x)}^{A \cup B'}}(x) \\ &= \sum_{b \in B'} \operatorname{minex}_{\boldsymbol{\vartheta}}(b) + \alpha f(A), \end{aligned}$$

which is equivalent to

$$\sum_{b \in B'} \operatorname{minex}_{\boldsymbol{\vartheta}}(b) \geq f(A \cup B') - \alpha f(A). \tag{2.8}$$

Now, we show that $f(A \cup \{b\}) - f(A) \geq \operatorname{minex}_{\boldsymbol{\vartheta}}(b)$ for all $b \in B'$, which will complete the proof, because then

$$\begin{aligned} |B'| \big(\max_{b \in B'} f(A \cup \{b\}) - f(A)\big) &\geq \sum_{b \in B'} (f(A \cup \{b\}) - f(A)) \\ &\geq \sum_{b \in B'} \operatorname{minex}_{\boldsymbol{\vartheta}}(b) \\ &\overset{(2.8)}{\geq} f(A \cup B') - \alpha f(A). \end{aligned}$$

In order to show that $f(A \cup \{b\}) - f(A) \geq \operatorname{minex}_{\boldsymbol{\vartheta}}(b)$ holds for all $b \in B'$, let $b \in B'$. Since $A \cap B' = \emptyset$, we have

$$\operatorname{minex}_{\boldsymbol{\vartheta}}(A \cup \{b\}) = \operatorname{minex}_{\boldsymbol{\vartheta}}(A) + \operatorname{minex}_{\boldsymbol{\vartheta}}(b) \overset{(2.5)}{=} f(A) + \operatorname{minex}_{\boldsymbol{\vartheta}}(b).$$

Furthermore, we have $f(A \cup \{b\}) \geq \operatorname{minex}_{\boldsymbol{\vartheta}}(A \cup \{b\})$ because $\boldsymbol{\vartheta}$ is a multicommodity-flow in $G$. Combining these two insights yields $f(A \cup \{b\}) - f(A) \geq \operatorname{minex}_{\boldsymbol{\vartheta}}(b)$. Thus, we can conclude that $f$ is $\alpha$-augmentable. □

**Proposition 2.7.** *For every $\gamma, q \in (0,1)$, and $\alpha \in \mathbb{N}_{\geq 2}$, there exists an instance of* Multi-Sink *$\alpha$-*Commodity Flow *where the objective is not in $\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_q$.*

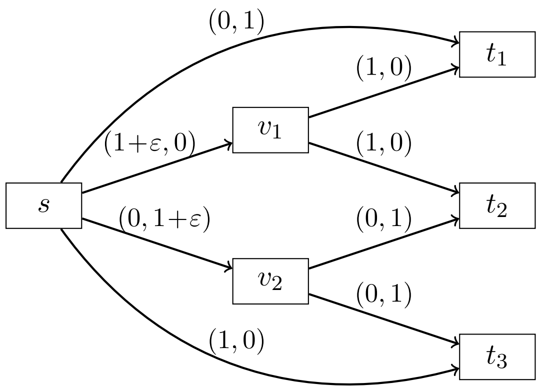


Figure 2.2.: An instance of MULTI-SINK $\alpha$-COMMODITY FLOW for $\alpha = 2$ where the objective has an arbitrarily small (weak) submodularity ratio and cannot be modeled as weighted rank function of some independence system.

*Proof.* We will define such an instance of MULTI-SINK $\alpha$-COMMODITY FLOW (cf. Figure 2.2). Let

$$\begin{aligned} T &:= \{t_1, t_2, t_3\}, \\ V &:= \{s, v_1, v_2\} \cup T, \\ E &:= \{(s, v_1), (s, v_2), (s, t_1), (s, t_3), (v_1, t_1), (v_1, t_2), (v_2, t_2), (v_2, t_3)\}, \\ G &:= (V, E), \end{aligned}$$

and, with $0 < \varepsilon < \frac{\gamma}{2}$, let

$$\mu\colon E \to \mathbb{R}^{\alpha}_{\geq 0}, \mu(e) = \begin{cases} (1+\varepsilon, 0, 0, \dots, 0), & \text{if } e = (s, v_1), \\ (0, 1+\varepsilon, 1+\varepsilon, \dots, 1+\varepsilon), & \text{if } e = (s, v_2), \\ (1, 0, 0, \dots, 0), & \text{if } e \in \{(s, t_3), (v_1, t_1), (v_1, t_2)\}, \\ (0, 1, 1, \dots, 1), & \text{else.} \end{cases}$$

With proper tie breaking (or by adding small extra capacities), the GREEDY algorithm picks the sink $t_2$ in the first iteration. Adding any other sink to this increases the objective value by $\varepsilon$, i.e., for all $t \in T$, we have $\sum_{t \in T} (f(X^{\mathrm{G}}(1) \cup \{t\}) - f(X^{\mathrm{G}}(1))) = 2\varepsilon$. But since $f(X^{\mathrm{G}}(1) \cup \{t_1, t_3\}) - f(X^{\mathrm{G}}(1)) = 1$, the weak submodularity ratio of this problem is $\frac{2\varepsilon}{1} < \gamma$.

If $f$ could be modeled as the weighted rank function of some independence system, the corresponding weight function would have to satisfy $w(t_1) = w(t_2) = w(t_3) = 1$ because each sink alone has a minimum-excess of 1. Yet, we have $f(\{t_1, t_2\}) = 1 + \varepsilon$. This cannot be possible if $f$ is the weighted rank function of some independence system, as, in this case, we would have $f(\{t_1, t_2\}) \in \{0, 1, 2\}$, depending on which sets are independent. □

We will now construct a family of instances of the Multi-Sink $\alpha$-Commodity Flow problem to show a tight lower bound on the competitive ratio of the Greedy algorithm for the class of monotone, $\alpha$-augmentable objectives for $\alpha \in \mathbb{N}$.

For $\alpha = 2$, the Multi-Sink $\alpha$-Commodity Flow problem is equivalent to the BridgeFlow problem that was used in [5] to show the tight lower bound for $\alpha = 2$. We generalize the tight lower bound construction for BridgeFlow to arbitrary $\alpha \in \mathbb{N}$.

For $k \in \mathbb{N}$, $k \geq 2$ we let $x := \frac{k}{k-1}$. Now, we define the graphs $G_k = (V_k, E_k)$ (cf. Figure 2.3) via

$$\begin{aligned}
V_k &:= \{s, v_1, \dots, v_{\alpha k}, t_1, \dots, t_{2\alpha k}\},\\
E^1_{k,i} &:= \{(s, t_{(\alpha+i-1)k+1}), \dots, (s, t_{(\alpha+i)k})\},\\
E^\infty_{k,i} &:= \{(s, t_{\alpha k+1}), \dots, (s, t_{2\alpha k})\} \setminus E^1_{k,i},\\
E_{k,i,j} &:= \{(s, v_j), (v_j, t_j)\} \forall j \in [\alpha k],\\
E'_{k,i,j} &:= \{(v_j, t_{(\alpha+i-1)k+1}), \dots, (v_j, t_{(\alpha+i)k})\} \forall j \in [\alpha k],\\
\mathcal{E}_{k,i} &:= E^1_{k,i} \cup E^\infty_{k,i} \cup \bigcup_{j=1}^{\alpha k} E_{k,i,j} \cup \bigcup_{j=1}^{\alpha k} E'_{k,i,j},\\
E_k &:= \bigcup_{i=1}^{\alpha} \mathcal{E}_{k,i},
\end{aligned}$$

capacity functions $\mu^k = (\mu^k_1, \dots, \mu^k_\alpha)$ with $\mu^k_i \colon E^k \to \mathbb{R}_{\geq 0}$ for $i \in [\alpha]$ and

$$\mu^k_i(e) = \begin{cases} 1, & \text{if } e \in E^1_{k,i},\\ \infty, & \text{if } e \in E^\infty_{k,i},\\ x^{\alpha k - j + 1}, & \text{if } e \in E_{k,i,j} \text{ for some } j \in [\alpha k],\\ \frac{1}{k} x^{\alpha k - j + 1}, & \text{if } e \in E'_{k,i,j} \text{ for some } j \in [\alpha k],\\ 0, & \text{else.} \end{cases}$$

Note that only the edges in $\mathcal{E}_{k,i}$ allow a flow of commodity $i$. We define $s$ to be the source vertex and $T := \{t_1, \dots, t_{2\alpha k}\}$ to be the set of sink vertices.

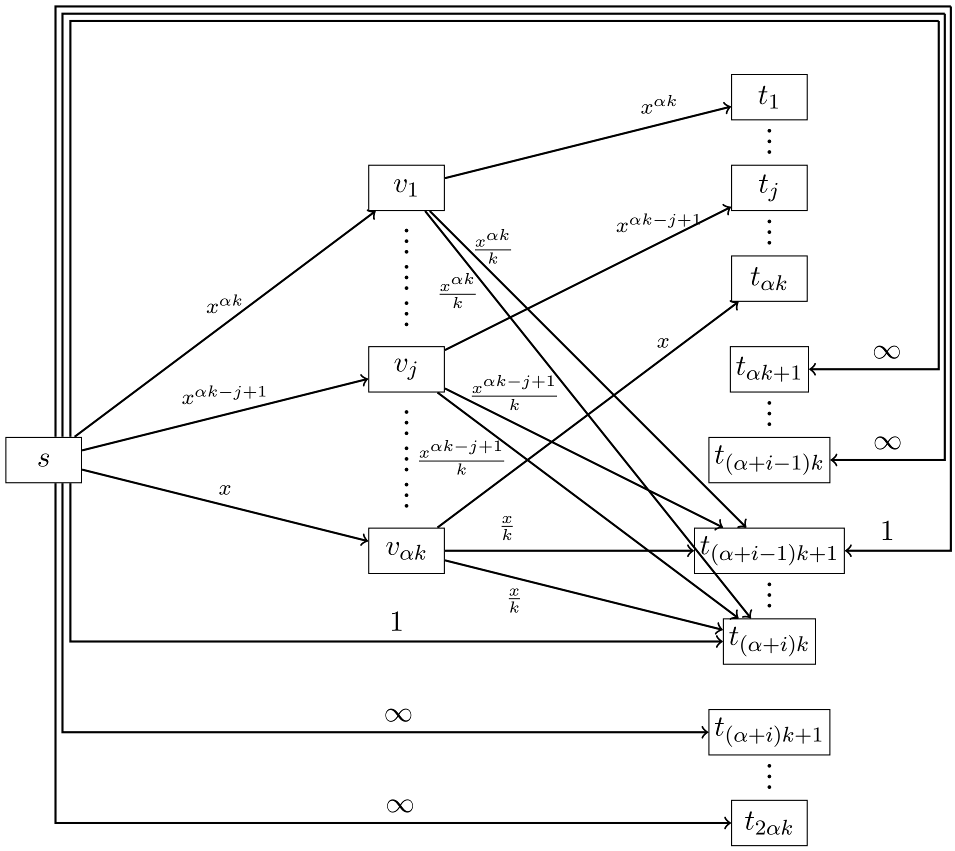


Figure 2.3.: The graph $G_k$ only with edges from $\mathcal{E}_{k,i}$ and capacities $\mu_i^k$.

In the next proof we will need the following observation: Using $x = \frac{k}{k-1}$ and with $i \in \mathbb{N}$ the equation

$$\begin{aligned} 1 + \frac{1}{k}\sum_{j=1}^{i} x^j &= 1 + \frac{1}{k}\left(\frac{x^{i+1}-1}{x-1} - 1\right) = 1 + \frac{1}{k}\left(\frac{\left(\frac{k}{k-1}\right)^{i+1}-1}{\left(\frac{k}{k-1}\right)-1} - 1\right) \\ &= 1 + \frac{1}{k}\left((k-1)\left(\left(\frac{k}{k-1}\right)^{i+1} - 1\right) - 1\right) \\ &= 1 + \left(\frac{k}{k-1}\right)^{i} - \frac{1}{k}((k-1)+1) = \left(\frac{k}{k-1}\right)^{i} = x^i \end{aligned} \tag{2.9}$$

holds.

We will now show in which order the Greedy algorithm picks the vertices from the set $T$. We assume that the tie-breaking works out in our favor. This can be achieved by introducing small offsets to the capacities. For better readability we omit this here.

**Lemma 2.8.** *Let $\alpha, k \in \mathbb{N}$ and $\ell \in [\alpha k]$. In iteration $\ell$, the Greedy algorithm picks sink vertex $t_\ell$. A multicommodity-flow $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_\alpha)$ that maximizes the minimum-excess of the vertices $\{t_1, \ldots, t_\ell\}$ fully saturates all edges in $E_{k,i,j}$ for all $i \in [\alpha]$ and $j \in [\ell]$.*

*Proof.* We will prove the statement by induction. In iteration $\ell = 1$, the gain of picking vertex $t_j$ with $j \in [\alpha k]$ is $x^{\alpha k - j + 1}$, because for all $i \in [\alpha]$ we can have a flow of value $x^{\alpha k - j+1}$ of commodity $i$ from $s$ via $v_j$ and the edges in $E_{k,i,j}$ to $t_j$ and no more flows to $t_j$ are possible, since the only incoming edge to $t_j$, which allows a flow of commodity $i$, is the edge $(v_j, t_j) \in E_{k,i,j}$. For $j \in \{\alpha k + 1, \ldots, 2\alpha k\}$, the gain of picking vertex $t_j$ is the minimum of all commodities flowing to $t_j$ and there is only one commodity which does not allow an unbounded flow to $t_j$, because for $i \in [\alpha] \setminus \left\{\left\lceil \frac{j-\alpha}{k} \right\rceil\right\}$ there is an edge from $s_i$ to $t_j$ in $E'_{k,i,j}$ with infinite capacity for commodity $i$. The maximum flow of the commodity with a finite flow to $t_j$ is

$$1 + \frac{1}{k}\sum_{j=1}^{\alpha k} x^j \overset{(2.9)}{=} x^{\alpha k},$$

and, thus, with proper tie-breaking, the Greedy algorithm chooses vertex $t_1$. For $i \in [\alpha]$, the only incoming path that allows a flow of commodity $i$ from $s$ to $t_1$ is along the edges in $E_{k,i,1}$, so they have to be fully saturated by a multicommodity-flow with maximum minimum-excess.

Now suppose the statement is true for some $\ell \in [\alpha k - 1]$, i.e., the Greedy algorithm has picked edges $t_1, \ldots, t_\ell$ and a multicommodity-flow with maximum minimum-excess of the vertices $\{t_1, \ldots, t_\ell\}$ fully saturates all edges in $E_{k,i,j}$ for all $i \in [\alpha]$ and $j \in [\ell]$. Then the gain of picking vertex $t_j$ for $j \in \{\ell+1, \ldots, \alpha k\}$ is still $x^{\alpha k - j + 1}$, because all $s$-$t_j$-paths for $i \in [\alpha]$ do not carry flow that contributes to the maximum minimum-excess. The gain of picking vertex $t_j$ for $j \in \{\alpha k + 1, \ldots, 2\alpha k\}$ is still the minimum of all commodities flowing to $t_j$, and again there is only one commodity which does not allow an unbounded flow to $t_j$. Because all incoming flow at vertices $v_1, \ldots, v_\ell$ already saturates all incoming edges, there is no flow of this commodity via a vertex in $\{v_1, \ldots, v_\ell\}$ to $t_j$ possible without reducing the minimum-excess of another sink vertex by the same amount. Thus, the maximal flow of this commodity to $t_j$ is

$$1 + \frac{1}{k} \sum_{j=1}^{\alpha k - \ell} x^j \overset{(2.9)}{=} x^{\alpha k - \ell},$$

so, with proper tie-breaking, the GREEDY algorithm picks vertex $t_{\ell+1}$ next. For $i \in [\alpha]$ and $j \in [\ell]$, the only incoming path that allows a flow of commodity $i$ from $s$ to $t_j$ is along the edges in $E_{k,i,j}$, so they have to be fully saturated by a multicommodity-flow with maximum minimum-excess. □

This enables us to calculate the competitive ratio of the GREEDY algorithm for this instance of MULTI-SINK $\alpha$-COMMODITY FLOW, which gives us a lower bound for the competitive ratio of the GREEDY algorithm for the MULTI-SINK $\alpha$-COMMODITY FLOW problem.

**Proposition 2.9.** *For $\alpha \in \mathbb{N}$, the GREEDY algorithm has a competitive ratio of at least $\alpha \frac{e^\alpha}{e^\alpha - 1}$ for MULTI-SINK $\alpha$-COMMODITY FLOW.*

*Proof.* We will consider the (unknown) cardinality constraint $C = \alpha k$. By Lemma 2.8, the GREEDY algorithm picks the sinks $t_1, \ldots, t_{\alpha k}$ in the first $\alpha k$ iterations and the objective increases by $x^{\alpha k - j + 1}$ when sink vertex $t_j$ is picked and thus the minimum-excess of the greedy solution is

$$f(X^{\mathrm{G}}(k)) = \sum_{j=1}^{\alpha k} x^j \overset{(2.9)}{=} k(x^{\alpha k} - 1).$$

We compare this to the solution that picks the vertices $t_{\alpha k+1}, \ldots, t_{2\alpha k}$ (which is, in fact, an optimum solution for cardinality $\alpha k$). Increasing the flow to one of these vertices does not reduce the flow to the others, so the minimum-excess of any of these vertices is

$$1 + \frac{1}{k} \sum_{j=1}^{\alpha k} x^j \overset{(2.9)}{=} x^{\alpha k},$$

and their combined minimum-excess thus is $(\alpha k)x^{\alpha k}$. Using this and $x = \frac{k}{k-1}$, we calculate the ratio between this solution and the greedy solution to get

$$\frac{\alpha k x^{\alpha k}}{k(x^{\alpha k}-1)} = \alpha \frac{x^{\alpha k}}{x^{\alpha k}-1} = \alpha \frac{\left(\frac{k}{k-1}\right)^{\alpha k}}{\left(\frac{k}{k-1}\right)^{\alpha k}-1} = \alpha \frac{\left(\left(\frac{k}{k-1}\right)^{k}\right)^{\alpha}}{\left(\left(\frac{k}{k-1}\right)^{k}\right)^{\alpha}-1}.$$

Using the identity $\lim_{k\to\infty}(k/(k-1))^k = \mathrm{e}$, we obtain the limit

$$\lim_{k\to\infty} \alpha \frac{\left(\left(\frac{k}{k-1}\right)^{k}\right)^{\alpha}}{\left(\left(\frac{k}{k-1}\right)^{k}\right)^{\alpha}-1} = \alpha \frac{\mathrm{e}^{\alpha}}{\mathrm{e}^{\alpha}-1}.$$

□

As Multi-Sink $\alpha$-Commodity Flow is monotone and $\alpha$-augmentable, we obtain a lower bound for the competitive ratio of the Greedy algorithm on $\mathcal{F}_\alpha$ for $\alpha \in \mathbb{N}$ that tightly matches the upper bound of [5].

**Corollary 2.10.** *For $\alpha \in \mathbb{N}$, the competitive ratio of the Greedy algorithm for IncMax with $\alpha$-augmentable objectives is exactly $\alpha \cdot \frac{\mathrm{e}^{\alpha}}{\mathrm{e}^{\alpha}-1}$.*

In particular, it follows that the objective of Multi-Sink $\alpha$-Commodity Flow is not $\alpha'$-augmentable for any $\alpha' < \alpha$. We will generalize the lower bound to all $\alpha \geq 1$ in Section 2.3.1.

## 2.2. Separating Function Classes

In this section we will show that the function classes $\tilde{\mathcal{F}}_\gamma$, $\mathcal{F}_\alpha$, and $\mathcal{F}_q$ each contain functions that are not in either of the other two function classes, i.e., we show the following.

**Theorem 2.11.** *For every $\gamma, q \in (0,1)$ and $\alpha \in \mathbb{N}$, $\alpha \geq 2$, it holds that*

$$\tilde{\mathcal{F}}_\gamma \nsubseteq (\mathcal{F}_\alpha \cup \mathcal{F}_q) \quad \text{and} \quad \mathcal{F}_\alpha \nsubseteq (\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_q) \quad \text{and} \quad \mathcal{F}_q \nsubseteq (\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_\alpha).$$

Note that we only show this for $\alpha \in \mathbb{N}_{\geq 2}$. The case $\alpha \geq 1$ will be addressed in Section 2.3.1.

We start with the second part, i.e., we separate $\mathcal{F}_\alpha$ for $\alpha \in \mathbb{N}_{\geq 2}$. This follows immediately from Theorem 2.6 and Proposition 2.7.

**Proposition 2.12.** *For every $\gamma, q \in (0,1)$, and $\alpha \in \mathbb{N}_{\geq 2}$, it holds that $\mathcal{F}_\alpha \nsubseteq (\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_q)$.*

*Proof.* Let $\gamma, q \in (0,1)$, and $\alpha \in \mathbb{N}_{\geq 2}$. By Theorem 2.6, every objective of an instance of MULTI-SINK $\alpha$-COMMODITY FLOW is $\alpha$-augmentable. By Proposition 2.7, there exists an instance of MULTI-SINK $\alpha$-COMMODITY FLOW, where the objective is not in $\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_q$. Combining this yields the desired result. □

We proceed to show the first and third part of Theorem 2.11 (for all $\alpha \geq 1$).

**Proposition 2.13.** *For every $\gamma, q \in (0,1)$, $\alpha \geq 1$, it holds that $\tilde{\mathcal{F}}_\gamma \not\subseteq (\mathcal{F}_\alpha \cup \mathcal{F}_q)$.*

*Proof.* Consider the set $U = \{a, b\}$ and the objective function

$$f^\gamma \colon 2^U \to \mathbb{R}_{\geq 0}, f^\gamma(S) = \begin{cases} |S|, & \text{if } |S| \leq 1, \\ \frac{2}{\gamma}, & \text{else.} \end{cases}$$

If $f^\gamma$ could be modeled as the weighted rank function of an independence system $(U, \mathcal{I})$, then we would have $U \in \mathcal{I}$ because $f(U) > f(S)$ for all $S \subsetneq U$. Then $\mathcal{I} = 2^U$ and $f^\gamma$ would be linear which is not true. Thus $f^\gamma$ cannot be modeled as the weighted rank function of an independence system, and $f^\gamma \notin \mathcal{F}_q$.

Furthermore, $f^\gamma \notin \mathcal{F}_\alpha$. To see this, consider $A = \emptyset$ and $B = \{a, b\}$. Then we have $f^\gamma(A \cup \{y\}) - f^\gamma(A) = 1$ for all $y \in B$, and we have $\frac{f^\gamma(A \cup B) - \alpha f^\gamma(A)}{|B|} = \frac{1}{\gamma}$. Since $\gamma < 1$, the problem is not $\alpha$-augmentable.

It remains to show that $f^\gamma \in \tilde{\mathcal{F}}_\gamma$. Let $A, B \subseteq U$ with $A \cap B = \emptyset$. For $B = \emptyset$, the ratio in the definition of the weak submodularity ratio is $\frac{0}{0} = 1$. Thus, assume $|B| \geq 1$. If $A = \emptyset$, we have

$$\frac{\sum_{b \in B} f^\gamma(A \cup \{b\}) - f^\gamma(A)}{f^\gamma(A \cup B) - f^\gamma(A)} = \frac{|B|}{f^\gamma(B)} \in \{1, \gamma\}.$$

Otherwise, if $|A| = 1$, then $|B| = 1$ and the ratio in the definition of the (weak) submodularity ratio is 1. In both cases, the ratio is at least $\gamma$, thus the (weak) submodularity ratio of this problem is $\gamma$, and $f^\gamma \in \tilde{\mathcal{F}}_\gamma$. □

**Proposition 2.14.** *For every $\gamma, q \in (0,1)$, $\alpha \geq 1$, it holds that $\mathcal{F}_q \not\subseteq (\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_\alpha)$.*

*Proof.* We fix $i, j \in \mathbb{N}$ with $q \leq \frac{i}{j} < 1$ and $\alpha \geq 1$. Let

$$\begin{aligned} U_1 &:= \{a_1, \dots, a_{\lceil \alpha \rceil j}\}, \\ U_2 &:= \{b_1, \dots, b_{\lceil \alpha \rceil j}\}, \\ U_3 &:= \{c\} \\ U &:= U_1 \cup U_2 \cup U_3, \\ \mathcal{I} &:= 2^{U_1} \cup 2^{U_2} \cup \{S \subset U \mid |S| \leq \lceil \alpha \rceil i\}. \end{aligned}$$

We consider the independence system $(U, \mathcal{I})$ and the weight function $w\colon U \to \mathbb{R}_{\geq 0}$ defined by

$$w(e) = \begin{cases} 1, & e \in U_1, \\ \lceil\alpha\rceil(j-i)+1, & \text{else.} \end{cases}$$

The weighted rank function $f$ is given by $f^q\colon U \to \mathbb{R}_{\geq 0}$,

$$f^q(S) = \max\{w(S') \mid S' \subseteq S, S' \in \mathcal{I}\}.$$

Obviously we have $q(U, \mathcal{I}) = \frac{i}{j} < q$, i.e., $f^q \in \mathcal{F}_q$.

For $A = U_1$, $B = U_2$ and $b \in B$, we calculate

$$\begin{aligned} f^q(A) &= \lceil\alpha\rceil j, \\ f^q(A \cup \{b\}) &= \max\{\lceil\alpha\rceil j, (\lceil\alpha\rceil i - 1) + (\lceil\alpha\rceil(j-i)+1)\} = \lceil\alpha\rceil j, \\ f^q(A \cup B) &= \lceil\alpha\rceil j(\lceil\alpha\rceil(j-i)+1). \end{aligned}$$

Suppose, $f^q$ was $\alpha$-augmentable. Then

$$f^q(A \cup \{b\}) - f^q(A) \geq \frac{f^q(A \cup B) - \alpha f^q(A)}{|B|},$$

i.e.,

$$\lceil\alpha\rceil j - \lceil\alpha\rceil j \geq \frac{\lceil\alpha\rceil j(\lceil\alpha\rceil(j-i)+1) - \alpha\lceil\alpha\rceil j}{\lceil\alpha\rceil j},$$

which is equivalent to

$$\alpha \geq \lceil\alpha\rceil(j-i)+1.$$

Since $j > i$, this is a contradiction, i.e., $f^q \notin \mathcal{F}_\alpha$.

Now, with $A = \{c, b_1, \ldots, b_{\lceil\alpha\rceil i - 1}\}$ and $B = U_2 \setminus A = \{b_{\lceil\alpha\rceil i}, \ldots, b_{\lceil\alpha\rceil j}\}$, we have

$$\frac{\sum_{b \in B} f^q(A \cup \{b\}) - f^q(A)}{f^q(A \cup B) - f^q(A)} = 0.$$

Thus, and because the set $A$ is the greedy solution $X^{\mathrm{G}}(\lceil\alpha\rceil i)$ (if we break ties in our favor or change the weights of some elements slightly), the weak submodularity ratio of this problem is $\gamma(f^q) = 0$, i.e., $f^q \notin \tilde{\mathcal{F}}_\gamma$. □

## 2.3. $\gamma$-$\alpha$-Augmentability

In this section, we argue that the function class $\tilde{\mathcal{F}}_{\gamma,\alpha}$ of monotone, weakly $\gamma$-$\alpha$-augmentable functions unifies the classes $\tilde{\mathcal{F}}_{\gamma}$, $\mathcal{F}_{\alpha}$, and $\mathcal{F}_{q}$. We show that the competitive ratio of the GREEDY algorithm on the function class $\tilde{\mathcal{F}}_{\gamma,\alpha}$ is at most $\frac{\alpha-(1-c)\gamma}{\gamma} \cdot \frac{e^{\alpha-(1-c)\gamma}}{e^{\alpha-(1-c)\gamma}-1}$ where $c \in [0,1]$ is the curvature of the objective. Later, in Section 2.3.1, we will introduce a critical family of functions that are monotone and weakly $\gamma$-$\alpha$-augmentable. These functions give a tight lower bound for the competitive ratio of the GREEDY algorithm for curvature $c = 1$, separate the class $\tilde{\mathcal{F}}_{\gamma,\alpha}$ from the classes $\tilde{\mathcal{F}}_{\gamma}$, $\mathcal{F}_{\alpha}$, and $\mathcal{F}_{q}$, and give a tight lower bound for the competitive ratio of the GREEDY algorithm on the class of monotone and $\alpha$-augmentable functions, $\mathcal{F}_{\alpha}$, for all $\alpha \geq 1$. Finally, in Section 2.3.2, we show an improved upper bound of $\frac{\alpha}{\gamma}$ for the competitive ratio of the GREEDY algorithm when the objective is a (weakly) $\gamma$-$\alpha$-augmentable weighted rank function of some independence system.

We start by proving the following simple lemma.

**Lemma 2.15.** *Let $(U, \mathcal{I})$ be an independence system with weight function $w\colon U \to \mathbb{R}_{\geq 0}$ and weighted rank function $f$. Furthermore, let $i \in \mathbb{N}$ and $x \in U \setminus X^{\mathrm{G}}(i)$ with $w(x) > 0$. Then, the following are equivalent:*

*(i)* $X^{\mathrm{G}}(i) \cup \{x\} \in \mathcal{I}$

*(ii)* $f(X^{\mathrm{G}}(i) \cup \{x\}) - f(X^{\mathrm{G}}(i)) = w(x)$

*(iii)* $f(X^{\mathrm{G}}(i) \cup \{x\}) - f(X^{\mathrm{G}}(i)) > 0$

*Proof.* "*(i)* ⇒ *(ii)*": Because $f$ is a weighted rank function and $X^{\mathrm{G}}(i) \cup \{x\} \in \mathcal{I}$, we have

$$f(X^{\mathrm{G}}(i) \cup \{x\}) - f(X^{\mathrm{G}}(i)) = \sum_{x' \in X^{\mathrm{G}}(i) \cup \{x\}} w(x') - \sum_{x' \in X^{\mathrm{G}}(i)} w(x') = w(x).$$

"*(ii)* ⇒ *(iii)*": This follows immediately from the fact that $w(x) > 0$.

"*(iii)* ⇒ *(i)*": Let $x \in U \setminus X^{\mathrm{G}}(i)$ with $f(X^{\mathrm{G}}(i) \cup \{x\}) - f(X^{\mathrm{G}}(i)) > 0$. Suppose there is some $s' \in X^{\mathrm{G}}(i)$ with $w(x) > w(s')$. This means that $x$ was considered by the GREEDY algorithm before and not added to the solution, i.e., $\{s \in X^{\mathrm{G}}(i) \mid w(s) \geq w(x)\} \cup \{x\} \notin \mathcal{I}$. The fact that $f(X^{\mathrm{G}}(i) \cup \{x\}) - f(X^{\mathrm{G}}(i)) > 0$ implies that there is a non-empty set $S \subseteq X^{\mathrm{G}}(i)$ with $X^{\mathrm{G}}(i) \setminus S \cup \{x\} \in \mathcal{I}$ and $w(S) < w(x)$. The last inequality implies that, for $s \in S$, we have $w(s) < w(x)$, which means that $\{s \in X^{\mathrm{G}}(i) \mid w(s) \geq w(x)\} \subseteq X^{\mathrm{G}}(i) \setminus S$. But then $\{s \in X^{\mathrm{G}}(i) \mid w(s) \geq w(x)\} \cup \{x\} \in \mathcal{I}$, which is a contradiction. Thus, we have $w(x) \leq w(s)$ for all $s \in X^{\mathrm{G}}(i)$. If $X^{\mathrm{G}}(i) \cup \{x\} \notin \mathcal{I}$ would hold, then the equality $f(X^{\mathrm{G}}(i) \cup \{x\}) - f(X^{\mathrm{G}}(i)) = 0$ would hold because every element in $X^{\mathrm{G}}(i)$ has a greater weight than $x$ and because $X^{\mathrm{G}}(i) \in \mathcal{I}$. Thus, *(i)* holds. □

We are now ready to prove that the class of monotone, weakly $\gamma$-$\alpha$-augmentable functions unifies the three function classes $\tilde{\mathcal{F}}_\gamma$, $\mathcal{F}_\alpha$, and $\mathcal{F}_q$.

**Theorem 2.16.** *For every $\gamma, q \in (0,1]$, and every $\alpha \geq 1$, it holds that*

$$\mathcal{F}_\alpha \subseteq \tilde{\mathcal{F}}_{1,\alpha} \qquad \text{and} \qquad \tilde{\mathcal{F}}_\gamma \subseteq \tilde{\mathcal{F}}_{\gamma,\gamma} \qquad \text{and} \qquad \mathcal{F}_q \subseteq \tilde{\mathcal{F}}_{\gamma,\gamma/q}.$$

*Proof.* If $f \in \mathcal{F}_\alpha$, then, for all $A, B \subseteq U$ and in particular $A \in \{X^{\mathrm{G}}(0), X^{\mathrm{G}}(1), \dots\}$, there exists $b \in B$ with

$$f(A \cup \{b\}) - f(A) \geq \frac{1 \cdot f(A \cup B) - \alpha f(A)}{|B|},$$

which means that $f \in \tilde{\mathcal{F}}_{1,\alpha}$.

For the second part of the proof, let $f \in \tilde{\mathcal{F}}_\gamma$, $A \in \{X^{\mathrm{G}}(0), X^{\mathrm{G}}(1), \dots\}$ and $B \subseteq U$ with $B' := B \setminus A \neq \emptyset$. Furthermore, let $b^* \in \arg\max_{b \in B} f(A \cup \{b\})$. Then, by definition of the submodularity ratio $\gamma(f)$, we have

$$\begin{aligned} |B|(f(A \cup \{b^*\}) - f(A)) &\geq \sum_{b \in B} (f(A \cup \{b\}) - f(A)) \\ &= \sum_{b \in B'} (f(A \cup \{b\}) - f(A)) \\ &\geq \gamma(f) f(A \cup B') - \gamma(f) f(A) \\ &= \gamma(f) f(A \cup B) - \gamma(f) f(A). \end{aligned}$$

Since $\gamma(f) \geq \gamma$, this means that $f$ is weakly $\gamma$-$\gamma$-augmentable, i.e., $f \in \tilde{\mathcal{F}}_{\gamma,\gamma}$.

For the last part of the proof, let $f \in \mathcal{F}_q$ be the weighted rank function of an independence system $(U, \mathcal{I})$, and let $w \colon U \to \mathbb{R}_{\geq 0}$ be the associated weight function. Furthermore, let $A \in \{X^{\mathrm{G}}(0), X^{\mathrm{G}}(1), \dots\}$ and $B \subseteq U$ with $B \setminus A \neq \emptyset$. We prove that, for every $\gamma \in (0,1]$, there exists $b \in B$ with

$$f(A \cup \{b\}) - f(A) \geq \frac{\gamma f(A \cup B) - \frac{\gamma}{q(U,\mathcal{I})} f(A)}{|B|}.$$

If $f(A \cup B) - \frac{1}{q(U,\mathcal{I})} f(A) < 0$, the inequality holds by monotonicity of $f$. Thus, assume from now on that

$$f(A \cup B) - \frac{1}{q(U,\mathcal{I})} f(A) \geq 0. \tag{2.10}$$

Let $A' \subseteq A$ and $B' \subseteq B$ with $A' \cup B' \in \mathcal{I}$ and $f(A \cup B) = w(A' \cup B')$. Furthermore, let $b^* := \arg\max_{b \in B'} f(A \cup \{b\})$. We define

$$\tilde{B} := \begin{cases} \{b \in B' \mid w(b) > w(b^*)\}, & \text{if } f(A \cup \{b^*\}) > f(A), \\ B', & \text{if } f(A \cup \{b^*\}) = f(A). \end{cases}$$

Note that, for all $b \in \tilde{B}$, we have $f(A \cup \{b\}) - f(A) = 0$. We define the independence system $(\tilde{U}, \tilde{\mathcal{I}})$ with

$$\begin{aligned} \tilde{U} &:= A \cup \tilde{B}, \\ \tilde{\mathcal{I}} &:= 2^A \cup 2^{A' \cup \tilde{B}}. \end{aligned}$$

We have $\tilde{U} \subseteq U$ and $\tilde{\mathcal{I}} \subseteq \mathcal{I}$ and thus, by Lemma 2.15, $q(\tilde{U}, \tilde{\mathcal{I}}) \geq q(U, \mathcal{I})$. The greedy solution for the maximization problem on the independence system $(\tilde{U}, \tilde{\mathcal{I}})$ of cardinality $i := |A|$ is $A$ because all elements in $\tilde{U}$ are also in $U$. The next element $e_{i+1}$ added by the Greedy would be from the set $\tilde{B}$, i.e., $f(A \cup \{e_{i+1}\}) - f(A) = 0$. Then, as shown in [40, 48], we have

$$f(A) \geq q(\tilde{U}, \tilde{\mathcal{I}}) f(\tilde{U}) \geq q(U, \mathcal{I}) f(A' \cup \tilde{B}) = q(U, \mathcal{I}) w(A' \cup \tilde{B}). \tag{2.11}$$

If $f(A \cup \{b^*\}) > f(A)$, Lemma 2.15 yields $f(A \cup \{b^*\}) - f(A) = w(b^*)$, and otherwise, if $f(A \cup \{b^*\}) = f(A)$, by definition of $\tilde{B}$, we have $|B' \setminus \tilde{B}| = 0$. Using this and the definition of $\tilde{B}$, we get

$$\begin{aligned} |B|(f(A \cup \{b^*\}) - f(A)) &\geq |B' \setminus \tilde{B}| w(b^*) \\ &\geq w(B' \setminus \tilde{B}) \\ &\overset{(2.11)}{\geq} w(B' \setminus \tilde{B}) + w(A' \cup \tilde{B}) - \frac{1}{q(U, \mathcal{I})} f(A) \\ &= w(A' \cup B') - \frac{1}{q(U, \mathcal{I})} f(A) \\ &= f(A \cup B) - \frac{1}{q(U, \mathcal{I})} f(A) \\ &\overset{(2.10), 1 \geq \gamma}{\geq} \gamma f(A \cup B) - \frac{\gamma}{q(U, \mathcal{I})} f(A). \end{aligned}$$

Since $q(U, \mathcal{I}) \geq q$, this yields weak $\gamma$-$\frac{\gamma}{q}$-augmentability, i.e., $f \in \tilde{\mathcal{F}}_{\gamma, \gamma/q}$. ☐

Having shown that $\tilde{\mathcal{F}}_{\gamma,\alpha}$ subsumes the other three classes of functions, we now prove an upper bound for the competitive ratio of the Greedy algorithm on this class. Observe that this upper bound trivially carries over to $\mathcal{F}_{\gamma,\alpha}$, the class of monotone, $\gamma$-$\alpha$-augmentable functions.

**Theorem 2.17.** *For $\gamma \in (0, 1]$, $\alpha \geq \gamma$, and $c \in [0, 1]$, the competitive ratio of the Greedy algorithm for IncMax with weakly $\gamma$-$\alpha$-augmentable objectives with $f(\emptyset) = 0$ and curvature $c$ is at most*

$$\rho = \frac{\alpha - (1-c)\gamma}{\gamma} \cdot \frac{\mathrm{e}^{\alpha - (1-c)\gamma}}{\mathrm{e}^{\alpha - (1-c)\gamma} - 1}.$$

*Proof.* Let $C \in \mathbb{N}$ be the cardinality constraint. First, we consider the case that there exists some $i \in [C]$, with $f(X^{\mathsf{G}}(i-1)) = f(X^{\mathsf{G}}(i))$. Then, we have

$$\max_{e \in U} f(X^{\mathsf{G}}(i-1) \cup \{e\}) - f(X^{\mathsf{G}}(i-1)) = 0, \tag{2.12}$$

which implies that the curvature is $c = 1$. By $\gamma$-$\alpha$-augmentability of $f$, for all $e \in O(C)$, we obtain

$$\begin{aligned} 0 &\overset{(2.12)}{=} |O(C)|\big(f(X^{\mathsf{G}}(i-1) \cup \{e\}) - f(X^{\mathsf{G}}(i-1))\big) \\ &\geq \gamma f(X^{\mathsf{G}}(i-1) \cup O(C)) - \alpha f(X^{\mathsf{G}}(i-1)) \\ &\geq \gamma f(O(C)) - \alpha f(X^{\mathsf{G}}(i-1)). \end{aligned}$$

This and monotonicity of $f$ yields

$$\begin{aligned} \textsc{Opt}(C) = f(O(C)) \leq \frac{\alpha}{\gamma} f(X^{\mathsf{G}}(i-1)) &\leq \frac{\alpha}{\gamma} f(X^{\mathsf{G}}(C)) \\ &< \frac{\alpha}{\gamma} \cdot \frac{\mathrm{e}^{\alpha}}{\mathrm{e}^{\alpha} - 1} f(X^{\mathsf{G}}(C)) \overset{c=1}{=} \rho f(X^{\mathsf{G}}(C)). \end{aligned}$$

Now consider the case that, for all $i \in [C]$, we have $f(X^{\mathsf{G}}(i-1)) < f(X^{\mathsf{G}}(i))$. Let $A, B \subseteq U$ be two disjoint sets and $B = \{b_1, \dots, b_{|B|}\}$. By definition of the curvature we have

$$\begin{aligned} f(A \cup B) &= f(A) + \sum_{j=1}^{|B|} f(A \cup \{b_1, \dots, b_j\}) - f(A \cup \{b_1, \dots, b_{j-1}\}) \\ &\geq f(A) + \sum_{j=1}^{|B|} (1-c)\big(f(\{b_1, \dots, b_j\}) - f(\{b_1, \dots, b_{j-1}\})\big) \\ &= f(A) + (1-c) f(B) - (1-c) f(\emptyset) \\ &= f(A) + (1-c) f(B). \end{aligned} \tag{2.13}$$

For ease of notation, we define the gain of the Greedy algorithm in iteration $i$ to be $\delta_i := f(X^{\mathsf{G}}(i)) - f(X^{\mathsf{G}}(i-1))$ for all $i \in [C]$. By $\gamma$-$\alpha$-augmentability, for all $i \in [C]$, we have

$$\begin{aligned}
\delta_i &= \max_{e \in U} f(X^{\mathrm{G}}(i-1) \cup \{e\}) - f(X^{\mathrm{G}}(i-1)) \\
&\geq \max_{e \in O(C)} f(X^{\mathrm{G}}(i-1) \cup \{e\}) - f(X^{\mathrm{G}}(i-1)) \\
&\geq \frac{\gamma f(X^{\mathrm{G}}(i-1) \cup O(C)) - \alpha f(X^{\mathrm{G}}(i-1))}{|O(C)|} \\
&\overset{(2.13)}{\geq} \frac{\gamma}{|O(C)|}\big(f(O(C)) + (1-c) f(X^{\mathrm{G}}(i-1))\big) - \frac{\alpha}{|O(C)|} f(X^{\mathrm{G}}(i-1)) \\
&= \frac{\gamma}{C} f(O(C)) - \frac{\alpha - (1-c)\gamma}{C} f(X^{\mathrm{G}}(i-1)). \qquad (2.14)
\end{aligned}$$

We prove by induction that, for all $\ell \in \{0, \ldots, C\}$, we have

$$f(O(C)) - \frac{\alpha - (1-c)\gamma}{\gamma} f(X^{\mathrm{G}}(\ell)) \leq f(O(C)) \left(1 - \frac{\alpha - (1-c)\gamma}{C}\right)^{\ell}. \qquad (2.15)$$

For $\ell = 0$ the inequality holds because

$$\frac{\alpha - (1-c)\gamma}{\gamma} \geq \frac{\gamma - (1-c)\gamma}{\gamma} = c\gamma \geq 0.$$

Now suppose that (2.15) holds for some $\ell \in \{0, \ldots, C-1\}$. Then, for $\ell + 1$, we have

$$\begin{aligned}
& f(O(C)) - \frac{\alpha - (1-c)\gamma}{\gamma} f(X^{\mathrm{G}}(\ell+1)) \\
= \;& f(O(C)) - \frac{\alpha - (1-c)\gamma}{\gamma} f(X^{\mathrm{G}}(\ell)) - \frac{\alpha - (1-c)\gamma}{\gamma} \delta_{\ell+1} \\
\overset{(2.14)}{\leq} \;& f(O(C)) - \frac{\alpha - (1-c)\gamma}{\gamma} f(X^{\mathrm{G}}(\ell)) \\
& - \frac{\alpha - (1-c)\gamma}{\gamma} \left(\frac{\gamma}{C} f(O(C)) - \frac{\alpha - (1-c)\gamma}{C} f(X^{\mathrm{G}}(\ell))\right) \\
= \;& \left(f(O(C)) - \frac{\alpha - (1-c)\gamma}{\gamma} f(X^{\mathrm{G}}(\ell))\right)\left(1 - \frac{\alpha - (1-c)\gamma}{C}\right) \\
\overset{(2.15)}{\leq} \;& f(O(C)) \left(1 - \frac{\alpha - (1-c)\gamma}{C}\right)^{\ell+1},
\end{aligned}$$

and (2.15) continues to hold.

Because of $1 + x \leq \mathrm{e}^x$ for $x \in \mathbb{R}$, we have

$$f(O(C)) - \frac{\alpha - (1-c)\gamma}{\gamma} f(X^{\mathrm{G}}(\ell)) \overset{(2.15)}{\leq} f(O(C))\left(1 - \frac{\alpha - (1-c)\gamma}{C}\right)^{\ell} \leq \mathrm{e}^{-\frac{\alpha-(1-c)\gamma}{C}\ell} f(O(C)).$$

Rearranging this for $\ell = C$ yields

$$f(X^{\mathrm{G}}(C)) \geq \frac{\gamma}{\alpha - (1-c)\gamma} \cdot \frac{\mathrm{e}^{\alpha-(1-c)\gamma} - 1}{\mathrm{e}^{\alpha-(1-c)\gamma}} f(O(C)) = \frac{1}{\rho}\textsc{Opt}(C).$$

□

Since every function $f\colon 2^U \to \mathbb{R}_{\geq 0}$ with weak submodularity ratio $\gamma \in (0,1]$ is weakly $\gamma$-$\gamma$-augmentable, we obtain an upper bound of $c\frac{\mathrm{e}^{c\gamma}}{\mathrm{e}^{c\gamma}-1}$ for objectives with submodularity ratio $\gamma \in (0,1]$ and curvature $c \in [0,1]$. This recovers the bound shown in [6].

**Remark 2.18.** *Note that in the proof of Theorem 2.17, the requirement $f(\emptyset) = 0$ was only needed to show* (2.13). *If $f(\emptyset) > 0$, we can only make the estimate $f(A \cup B) \geq f(A)$ by monotonicity. In this case, the upper bound on the competitive ratio we obtain is*

$$\frac{\alpha}{\gamma} \cdot \frac{\mathrm{e}^{\alpha}}{\mathrm{e}^{\alpha} - 1},$$

*which is exactly the bound shown in [15].*

### 2.3.1. A Critical Function

To show that the lower bound in Theorem 2.17 for the class of problems with monotone and weakly $\gamma$-$\alpha$-augmentable objectives is tight for curvature $c = 1$ and to separate this class from $\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_\alpha \cup \mathcal{F}_q$, we introduce a function that is inspired by a lower bound construction in [6] for the submodularity ratio.

We fix $\gamma \in (0,1]$ and $\alpha \geq \gamma$. Let $k \in \mathbb{N}$ with $k > \alpha$, and let $U_1 = \{a_1, \dots, a_k\}$ and $U_2 = \{b_1, \dots, b_k\}$ be disjoint sets. We set $U = U_1 \cup U_2$, define $\xi_i := \frac{1}{k}(\frac{k-\alpha}{k})^{i-1}$ and let $h(x) := \frac{\gamma^{-1}-1}{k-1}x^2 + \frac{k-\gamma^{-1}}{k-1}x$. For our purpose, the important facts about $h$ are $h(0) = 0$, $h(1) = 1$, $h(k) = \frac{k}{\gamma}$ and that $h$ is convex and non-decreasing on $[0,k]$. With this in mind, we define the function $F_{\gamma,\alpha,k}\colon 2^U \to \mathbb{R}_{\geq 0}$ by

$$F_{\gamma,\alpha,k}(S) = \max_{S' \subseteq S}\left\{\frac{h(|\{b_1\} \cap S'| \cdot |U_2 \cap S'|)}{k}\left(1 - \alpha \sum_{\substack{i \in [k]:\\ a_i \in U_1 \cap S'}} \xi_i\right) + \sum_{\substack{i \in [k]:\\ a_i \in U_1 \cap S'}} \xi_i\right\}$$

If $h(|\{b_1\} \cap S| \cdot |U_2 \cap S|) > \frac{k}{\alpha}$, we have

$$F_{\gamma,\alpha,k}(S) = \frac{h(|\{b_1\} \cap S| \cdot |U_2 \cap S|)}{k},$$

and otherwise, if $h(|\{b_1\} \cap S| \cdot |U_2 \cap S|) \leq \frac{k}{\alpha}$, we have

$$F_{\gamma,\alpha,k}(S) = \frac{h(|\{b_1\} \cap S| \cdot |U_2 \cap S|)}{k} \left(1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap S}} \xi_i\right) + \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap S}} \xi_i.$$

We observe that, for $S \subseteq U_2$, convexity of $h$, $h(0) = 0$, $h(k) = k/\gamma$ and $|S| \leq |U_2| = k$ imply that

$$h(|\{b_1\} \cap S| \cdot |S|) \leq \frac{|\{b_1\} \cap S| \cdot |S|}{\gamma}, \tag{2.16}$$

and, for $\ell \in \{0, \ldots, k\}$, we have

$$\sum_{i=1}^{\ell} \xi_i = \sum_{i=1}^{\ell} \frac{1}{k} \left(\frac{k-\alpha}{k}\right)^{i-1} = \frac{1}{k} \cdot \frac{1 - \left(\frac{k-\alpha}{k}\right)^{\ell}}{1 - \frac{k-\alpha}{k}} = \frac{1 - \left(\frac{k-\alpha}{k}\right)^{\ell}}{\alpha}. \tag{2.17}$$

We show that our modification of the function introduced in [6] retains the same structure in regard to greedy solutions.

**Lemma 2.19.** *For $i \in [k]$, the* Greedy *algorithm picks the element $a_i$ in iteration $i$, and, for $i \in [2k] \setminus [k]$, the* Greedy *algorithm picks the element $b_{i-k}$ in iteration $i$.*

*Proof.* First, we consider the case $i \in [k]$. Suppose that in iteration $i$, the initial solution is $\{a_1, \ldots, a_{i-1}\}$, where $\{a_1, \ldots, a_0\} = \emptyset$, with objective value $\sum_{\ell=1}^{i-1} \xi_\ell$. Adding an element from $\{b_2, \ldots, b_k\}$ does not increase the objective value because, for all $b \in \{b_2, \ldots, b_k\}$, we have $\{b_1\} \cap \{b\} = \emptyset$. For $j \in \{i, \ldots, k\}$, adding $a_j$ increases the objective value by $\xi_j = \frac{1}{k}(\frac{k-\alpha}{k})^{j-1}$. Since $k > \alpha$, we have $\xi_i \geq \xi_j$ for $j \geq i$. Adding the element $b_1$ to the solution $\{a_1, \ldots, a_{i-1}\}$ increases the objective value by

$$\frac{1}{k}\left(1 - \alpha \sum_{\ell=1}^{i-1} \xi_\ell\right) \overset{(2.17)}{=} \frac{1}{k}\left(1 - \alpha \frac{1 - \left(\frac{k-\alpha}{k}\right)^{i-1}}{\alpha}\right) = \frac{1}{k}\left(\frac{k-\alpha}{k}\right)^{i-1}.$$

Thus, with proper tie breaking, the Greedy algorithm picks the element $a_i$ in iteration $i$ for $i \in [k]$.

Now, we consider the case that $i \in \{k+1, \ldots, 2k\}$. For $i = k+1$, adding an element from $\{b_2, \ldots, b_k\}$ does not increase the objective value, while adding $b_1$ increases it by $\frac{1}{k}(\frac{k-\alpha}{k})^k$.

Thus, in iteration $k+1$, the element $b_1$ is added to the solution. For $i \geq k+2$, adding any element from $U_2 \setminus X^{\mathrm{G}}(i-1)$ to the greedy solution $X^{\mathrm{G}}(i-1)$ increases the function value by the same amount. Therefore, with proper tie breaking, the GREEDY algorithm picks the element $b_{i-k}$ in iteration $i$ for $i \in \{k+1, \ldots, 2k\}$. □

With this, we can show that $F_{\gamma,\alpha,k}$ is weakly $\gamma$-$\alpha$-augmentable.

**Lemma 2.20.** *Let $\gamma \in (0,1]$, $\alpha \geq \gamma$, and $k \in \mathbb{N}_{>\alpha}$. Then $F_{\gamma,\alpha,k} \in \tilde{\mathcal{F}}_{\gamma,\alpha}$.*

*Proof.* The monotonicity of $F_{\gamma,\alpha,k}$ immediately follows from the maximum in the definition. In order to prove weak $\gamma$-$\alpha$-augmentability, let $A \in \{X^{\mathrm{G}}(0), \ldots, X^{\mathrm{G}}(2k)\}$ and $B \subseteq U$ with $B' := B \setminus A \neq \emptyset$. For better readability, we will write $F := F_{\gamma,\alpha,k}$.

First, consider the case that $A \subseteq U_1$. Then $F(A) = \sum_{i \in [k] : a_i \in A} \xi_i$ because $h(0) = 0$. Thus and because $h(1) = 1$, for all $y \in B'$, we have

$$F(A \cup \{y\}) - F(A) = \begin{cases} \xi_i, & \text{if } y = a_i \in (U_1 \cap B'), \\ \frac{1}{k}(1 - \alpha \sum_{i \in [k] : a_i \in A} \xi_i), & \text{if } y = b_1, \\ 0, & \text{else.} \end{cases}$$

This yields

$$\begin{aligned} & |B'|\big(\max_{y \in B'} F(A \cup \{y\}) - F(A)\big) \\ \geq & \left( \sum_{y \in U_1 \cap B'} \big(F(A \cup \{y\}) - F(A)\big) \right) + |U_2 \cap B'| \max_{y \in U_2 \cap B'} F(A \cup \{y\}) - F(A) \\ = & \left( \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap B'}} \xi_i \right) + |\{b_1\} \cap B'| \cdot |U_2 \cap B'| \frac{1}{k} \left( 1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in A}} \xi_i \right). \end{aligned} \tag{2.18}$$

If $h(|\{b_1\} \cap B'| \cdot |U_2 \cap B'|) \leq \frac{k}{\alpha}$, we use the fact that $F(A) = \sum_{i \in [k]: a_i \in A} \xi_i$ to calculate

$$\begin{aligned}
&\gamma F(A \cup B) - \alpha F(A) \\
\overset{U_2 \cap A = \emptyset}{=} \quad & \gamma \left( \frac{h(|\{b_1\} \cap B'| \cdot |U_2 \cap B'|)}{k} \left( 1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in A \cup (U_1 \cap B')}} \xi_i \right) + \sum_{\substack{i \in [k]: \\ a_i \in A \cup (U_1 \cap B')}} \xi_i \right) \\
& - \alpha \sum_{\substack{i \in [k]: \\ a_i \in A}} \xi_i \\
= \quad & \left[ \frac{\gamma}{k} h(|\{b_1\} \cap B'| \cdot |U_2 \cap B'|) \left( 1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in A}} \xi_i \right) \right] \\
& + \left[ \gamma \left( 1 - \frac{\alpha}{k} h(|\{b_1\} \cap B'| \cdot |U_2 \cap B'|) \right) \left( \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap B'}} \xi_i \right) \right] + \left[ (\gamma - \alpha) \sum_{\substack{i \in [k]: \\ a_i \in A}} \xi_i \right] \\
\leq \quad & \left[ \frac{1}{k} |\{b_1\} \cap B'| \cdot |U_2 \cap B'| \left( 1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in A}} \xi_i \right) \right] + \left[ \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap B'}} \xi_i \right] + [0]. \qquad (2.19)
\end{aligned}$$

The first part of the last inequality follows from (2.16). The second part of the inequality follows from the fact that $\gamma \in (0, 1]$ and, for $x \geq 0$, we have $\frac{\alpha}{k} h(x) \geq 0$. The last part follows from the fact that $\gamma \leq \alpha$. Combining equations (2.18) and (2.19) together with the fact that $B' \subseteq B$ yields weak $\gamma$-$\alpha$-augmentability.

Otherwise, if $h(|\{b_1\} \cap B'| \cdot |U_2 \cap B'|) > \frac{k}{\alpha}$, we have

$$\begin{aligned}
\gamma F(A \cup B) - \alpha F(A) \quad = \quad & \gamma \frac{h(|\{b_1\} \cap B'| \cdot |U_2 \cap B'|)}{k} - \alpha \sum_{\substack{i \in [k]: \\ a_i \in A}} \xi_i \\
\overset{(2.16)}{\leq} \quad & \frac{1}{k} |\{b_1\} \cap B'| \cdot |U_2 \cap B'| - \alpha \sum_{\substack{i \in [k]: \\ a_i \in A}} \xi_i \\
\overset{|U_2| = k}{\leq} \quad & \frac{1}{k} |\{b_1\} \cap B'| \cdot |U_2 \cap B'| \left( 1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in A}} \xi_i \right) \\
\overset{(2.18)}{\leq} \quad & |B'| \big( \max_{y \in B'} F(A \cup \{y\}) - F(A) \big),
\end{aligned}$$

which yields $\gamma$-$\alpha$-augmentability also in this case.

Now, consider the case that $A \not\subseteq U_1$. Then, by Lemma 2.19, we have $A = U_1 \cup \{b_1, \ldots, b_i\}$ for some $i \in [k]$. We have $i < k$ because $B' \neq \emptyset$. The fact that $h$ is convex and non-decreasing on $[0, k]$ yields

$$\frac{h(i + |B'|) - h(i)}{|B'|} \overset{|B'|+i \leq |U_2|}{\leq} \frac{h(|U_2|) - h(i)}{|U_2| - i}. \tag{2.20}$$

With

$$H(i) := (k - i)\frac{h(i + 1) - h(i)}{h(k) - h(i)},$$

we have

$$H'(i) = (k - 1)\frac{2 - 3\gamma + \gamma^2}{(k - 1 + i - \gamma i)^2} \geq 0,$$

which yields

$$H(i) \geq H(0) = k\frac{1 - 0}{\frac{k}{\gamma} - 0} = \gamma. \tag{2.21}$$

Combining this with (2.20), we obtain

$$\frac{|B'|\big(h(i + 1) - h(i)\big)}{h(i + |B'|) - h(i)} \overset{(2.20)}{\geq} \frac{(|U_2| - i)\big(h(i + 1) - h(i)\big)}{h(|U_2|) - h(i)} \overset{|U_2|=k}{=} H(i) \overset{(2.21)}{\geq} \gamma. \tag{2.22}$$

Recall that $h$ is increasing for positive values. In the following let $b \in B'$.

If $\frac{k}{\alpha} < h(i) \leq h(i + 1) \leq h(i + |B'|)$, then we have

$$\begin{aligned} |B|\big(F(A \cup \{b\}) - F(A)\big) &= |B'|\frac{h(i + 1) - h(i)}{k} \\ &\overset{(2.22)}{\geq} \frac{\gamma\big(h(i + |B'|) - h(i)\big)}{k} \\ &\geq \gamma\frac{h(i + |B'|)}{k} - \alpha\frac{h(i)}{k} \\ &= \gamma F(A \cup B) - \alpha F(A), \end{aligned}$$

i.e., $F$ is weakly $\gamma$-$\alpha$-augmentable.

Now, consider the case that $\frac{k}{\alpha} \geq h(i)$. We will start by showing that we have

$$|B|\big(F(A \cup \{b\}) - F(A)\big) \geq \left(\gamma\frac{h(i + |B'|)}{k} - \alpha\frac{h(i)}{k}\right)\left(1 - \alpha\sum_{j=1}^{k}\xi_j\right). \tag{2.23}$$

If $h(i) \leq h(i+1) \leq \frac{k}{\alpha}$, we have

$$\begin{aligned}
|B|\big(F(A\cup\{b\})-F(A)\big) &\geq |B'|\frac{h(i+1)-h(i)}{k}\left(1-\alpha\sum_{j=1}^{k}\xi_j\right)\\
&\overset{(2.22)}{\geq} \frac{\gamma\big(h(i+|B'|)-h(i)\big)}{k}\left(1-\alpha\sum_{j=1}^{k}\xi_j\right)\\
&\overset{\gamma\leq\alpha}{\geq} \left(\gamma\frac{h(i+|B'|)}{k}-\alpha\frac{h(i)}{k}\right)\left(1-\alpha\sum_{j=1}^{k}\xi_j\right).
\end{aligned}$$

Otherwise, if $h(i) \leq \frac{k}{\alpha} < h(i+1)$, then

$$\frac{\alpha}{k}h(i+1) > \frac{\alpha}{k}\cdot\frac{k}{\alpha} = 1, \tag{2.24}$$

which implies that

$$F(A\cup\{b\}) = \frac{h(i+1)}{k} \overset{(2.24)}{>} \frac{h(i+1)}{k}\left(1-\alpha\sum_{j=1}^{k}\xi_j\right)+\sum_{j=1}^{k}\xi_j. \tag{2.25}$$

Thus,

$$\begin{aligned}
|B|\big(F(A\cup\{b\})-F(A)\big) &\overset{(2.25)}{\geq} |B'|\frac{h(i+1)-h(i)}{k}\left(1-\alpha\sum_{j=1}^{k}\xi_j\right)\\
&\overset{(2.22)}{\geq} \frac{\gamma\big(h(i+|B'|)-h(i)\big)}{k}\left(1-\alpha\sum_{j=1}^{k}\xi_j\right)\\
&\geq \frac{\gamma h(i+|B'|)-\alpha h(i)}{k}\left(1-\alpha\sum_{j=1}^{k}\xi_j\right).
\end{aligned}$$

Now that we have established (2.23), we will show that

$$\left(\gamma\frac{h(i+|B'|)}{k}-\alpha\frac{h(i)}{k}\right)\left(1-\alpha\sum_{j=1}^{k}\xi_j\right) \geq \gamma F(A\cup B)-\alpha F(A). \tag{2.26}$$

Combining (2.23) and (2.26) yields $\gamma$-$\alpha$-augmentability.

If $h(i) \leq h(i+|B'|) \leq \frac{k}{\alpha}$, we have

$$
\begin{aligned}
& \left(\gamma \frac{h(i+|B'|)}{k} - \alpha \frac{h(i)}{k}\right)\left(1 - \alpha \sum_{j=1}^{k} \xi_j\right) \\
\overset{\gamma \leq \alpha}{\geq} \quad & \gamma \left[\frac{h(i+|B'|)}{k}\left(1 - \alpha \sum_{j=1}^{k} \xi_j\right) + \sum_{j=1}^{k} \xi_j\right] - \alpha \left[\frac{h(i)}{k}\left(1 - \alpha \sum_{j=1}^{k} \xi_j\right) + \sum_{j=1}^{k} \xi_j\right] \\
= \quad & \gamma F(A \cup B) - \alpha F(A).
\end{aligned}
$$

Otherwise, if $h(i) \leq \frac{k}{\alpha} \leq h(i+|B'|)$, we have

$$
\begin{aligned}
& \left(\gamma \frac{h(i+|B'|)}{k} - \alpha \frac{h(i)}{k}\right)\left(1 - \alpha \sum_{j=1}^{k} \xi_j\right) \\
= \quad & \gamma \frac{h(i+|B'|)}{k} - \alpha \frac{h(i)}{k}\left(1 - \alpha \sum_{j=1}^{k} \xi_j\right) - \frac{\gamma}{k} h(i+|B'|) \alpha \sum_{j=1}^{k} \xi_j \\
\overset{h(i+|B'|) \leq h(k) = \frac{k}{\gamma}}{\geq} \quad & \gamma \frac{h(i+|B'|)}{k} - \alpha \left[\frac{h(i)}{k}\left(1 - \alpha \sum_{j=1}^{k} \xi_j\right) + \sum_{j=1}^{k} \xi_j\right] \\
= \quad & \gamma F(A \cup B) - \alpha F(A).
\end{aligned}
$$

This establishes (2.26) and, thus, completes the proof. □

It is straightforward to bound the competitive ratio of the Greedy algorithm for $F_{\gamma,\alpha,k}$.

**Proposition 2.21.** *Let $\gamma \in (0,1]$, $\alpha \geq \gamma$ and $k \in \mathbb{N}_{>\alpha}$. Then, the competitive ratio of the Greedy algorithm for the instance with objective $F_{\gamma,\alpha,k}$ is at least*

$$
\frac{\alpha}{\gamma} \cdot \frac{\mathrm{e}^{\alpha}}{\mathrm{e}^{\alpha} - 1}.
$$

*Proof.* We set the cardinality constraint to be $C = k$. We compare the objective values of the greedy solution $X^{\mathrm{G}}(k)$ of cardinality $k$ and the solution $U_2$, which also has cardinality $k$. By Lemma 2.19, we have $X^{\mathrm{G}}(k) = U_1$, and thus

$$
F(X^{\mathrm{G}}(k)) = F(U_1) = \sum_{i=1}^{k} \xi_i \overset{(2.17)}{=} \frac{1 - \left(\frac{k-\alpha}{k}\right)^k}{\alpha}
$$

and

$$F(U_2) = \frac{h(k)}{k} = \frac{\frac{k}{\gamma}}{k} = \frac{1}{\gamma}.$$

Thus, the GREEDY algorithm has a competitive ratio of at least

$$\frac{\textsc{Opt}(k)}{F(X^{\mathrm{G}}(k))} \geq \frac{F(U_2)}{F(X^{\mathrm{G}}(k))} = \frac{\alpha}{\gamma} \cdot \frac{1}{1 - \left(\frac{k-\alpha}{k}\right)^k}.$$

The lower bound follows, since

$$\lim_{k \to \infty} \frac{1}{1 - \left(\frac{k-\alpha}{k}\right)^k} = \frac{1}{1 - \mathrm{e}^{-\alpha}} = \frac{\mathrm{e}^{\alpha}}{\mathrm{e}^{\alpha} - 1}.$$

□

As $F_{\gamma,\alpha,k} \in \tilde{\mathcal{F}}_{\gamma,\alpha}$ by Lemma 2.20, this immediately yields the following result.

**Corollary 2.22.** *Let $\gamma \in (0, 1]$ and $\alpha \geq \gamma$. Then, the competitive ratio of the GREEDY algorithm for INCMAX with weakly $\gamma$-$\alpha$-augmentable objectives is at least*

$$\frac{\alpha}{\gamma} \cdot \frac{\mathrm{e}^{\alpha}}{\mathrm{e}^{\alpha} - 1}.$$

By combining Theorem 2.17 for curvature $c = 1$ with Corollary 2.22, we obtain a tight bound on the competitive ratio for monotone, weakly $\gamma$-$\alpha$-augmentable functions.

**Theorem 2.23.** *Let $\gamma \in (0, 1]$ and $\alpha \geq \gamma$. Then, the competitive ratio of the GREEDY algorithm for INCMAX with weakly $\gamma$-$\alpha$-augmentable objectives is exactly*

$$\frac{\alpha}{\gamma} \cdot \frac{\mathrm{e}^{\alpha}}{\mathrm{e}^{\alpha} - 1}.$$

It even turns out that, for $\gamma = 1$, the function $F_{\gamma,\alpha,k}$ is $\alpha$-augmentable. This allows to carry the lower bound over to the class $\mathcal{F}_{\alpha}$.

**Proposition 2.24.** *Let $\alpha \geq 1$ and $k \in \mathbb{N}_{\geq \alpha}$. Then $F_{1,\alpha,k} \in \mathcal{F}_{\alpha}$.*

*Proof.* By Lemma 2.20, $F_{1,\alpha,k}$ is monotone. Thus, it suffices to prove that the function is $\alpha$-augmentable. For better readability, we write $F := F_{1,\alpha,k}$. Observe that, since $\gamma = 1$, we have $h(x) = x$ for all $x \in \mathbb{R}$. Let $A, B \subseteq U$ and $B' := B \setminus A$.

**Case 1:** $|\{b_1\} \cap (A \cup B)| \cdot |U_2 \cap (A \cup B)| \leq \frac{k}{\alpha}$.
Then, for $b \in B'$, we have

$$
\begin{aligned}
&F(A \cup \{b\}) - F(A) \\
= &\begin{cases} \left(1 - \frac{|\{b_1\}\cap A| \cdot |U_2 \cap A|}{k}\alpha\right)\xi_i, & \text{if } b = a_i \in U_1 \cap B', \\ \frac{|\{b_1\}\cap(A\cup\{b\})| \cdot |U_2 \cap (A \cup \{b\})| - |\{b_1\}\cap A| \cdot |U_2\cap A|}{k} & \\ \quad \cdot \left(1 - \alpha \sum_{i \in [k]: a_i \in U_1 \cap A} \xi_i\right), & \text{if } b \in U_2 \cap B'. \end{cases}
\end{aligned}
\tag{2.27}
$$

This yields

$$
\begin{aligned}
&F(A \cup B) - \alpha F(A) \\
= &\left(\frac{|\{b_1\} \cap (A \cup B')| \cdot |U_2 \cap (A \cup B')|}{k}\left(1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap (A \cup B')}} \xi_i\right) + \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap (A \cup B')}} \xi_i\right) \\
&-\alpha\left(\frac{|\{b_1\} \cap A| \cdot |U_2 \cap A|}{k}\left(1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i\right) + \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i\right) \\
= &\left[\frac{|\{b_1\} \cap (A \cup B')| \cdot |U_2 \cap (A \cup B')| - \alpha|\{b_1\} \cap A| \cdot |U_2 \cap A|}{k}\left(1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i\right)\right] \\
&+\left[\left(1 - \frac{|\{b_1\} \cap (A \cup B')| \cdot |U_2 \cap (A \cup B')|}{k}\alpha\right) \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap B'}} \xi_i\right] + \left[(1-\alpha) \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i\right] \\
\leq &\left[|U_2 \cap B'| \max_{b \in U_2 \cap B'}\left\{\frac{|\{b_1\} \cap (A \cup \{b\})| \cdot |U_2 \cap (A \cup \{b\})| - |\{b_1\} \cap A| \cdot |U_2 \cap A|}{k}\right.\right. \\
&\left.\left.\left(1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i\right)\right\}\right] + \left[\left(1 - \frac{|\{b_1\} \cap A| \cdot |U_2 \cap A|}{k}\alpha\right) \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap B'}} \xi_i\right] + [0] \\
\overset{(2.27)}{=} &|U_2 \cap B'|\left(\max_{b \in U_2 \cap B'} F(A \cup \{b\}) - F(A)\right) + \sum_{b \in U_1 \cap B'} (F(A \cup \{b\}) - F(A)) \\
\leq &|B|\left(\max_{b \in B} F(A \cup \{b\}) - F(A)\right).
\end{aligned}
$$

This establishes $\alpha$-augmentability if $|\{b_1\} \cap (A \cup B)| \cdot |U_2 \cap (A \cup B)| \leq \frac{k}{\alpha}$.

**Case 2:** $|\{b_1\} \cap A| \cdot |U_2 \cap A| \leq \frac{k}{\alpha} < |\{b_1\} \cap (A \cup B)| \cdot |U_2 \cap (A \cup B)|$**.**
Let $b \in U_2 \cap B'$. If $|\{b_1\} \cap (A \cup \{b\})| \cdot |U_2 \cap (A \cup \{b\})| \leq \frac{k}{\alpha}$, then

$$\begin{aligned} & F(A \cup \{b\}) - F(A) \\ \overset{(2.27)}{=} & \frac{|\{b_1\} \cap (A \cup \{b\})| \cdot |U_2 \cap (A \cup \{b\})| - |\{b_1\} \cap A| \cdot |U_2 \cap A|}{k} \left(1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right). \end{aligned}$$

Otherwise, if

$$|\{b_1\} \cap (A \cup \{b\})| \cdot |U_2 \cap (A \cup \{b\})| > \frac{k}{\alpha}, \tag{2.28}$$

then

$$|U_2 \cap (A \cup \{b\})| = |\{b_1\} \cap (A \cup \{b\})| \cdot |U_2 \cap (A \cup \{b\})|, \tag{2.29}$$

which yields

$$\begin{aligned} & F(A \cup \{b\}) - F(A) \\ = & \frac{|U_2 \cap (A \cup \{b\})|}{k} - \frac{|\{b_1\} \cap A| \cdot |U_2 \cap A|}{k} \left(1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right) - \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \\ \overset{(2.28),(2.29)}{\geq} & \frac{|\{b_1\} \cap (A \cup \{b\})| \cdot |U_2 \cap (A \cup \{b\})| - |\{b_1\} \cap A| \cdot |U_2 \cap A|}{k} \left(1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right). \end{aligned}$$

By combining the two cases, for all $b \in U_2 \cap B'$, we obtain

$$\begin{aligned} & F(A \cup \{b\}) - F(A) \\ \geq & \frac{|\{b_1\} \cap (A \cup \{b\})| \cdot |U_2 \cap (A \cup \{b\})| - |\{b_1\} \cap A| \cdot |U_2 \cap A|}{k} \cdot \big(1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \big). \end{aligned} \tag{2.30}$$

Since we consider $|\{b_1\} \cap A| \cdot |U_2 \cap A| \leq \frac{k}{\alpha} < |\{b_1\} \cap (A \cup B)| \cdot |U_2 \cap (A \cup B)|$, we have

$$B' \cap U_2 \neq \emptyset \quad \text{and} \quad b_1 \in A \cup B = A \cup B'. \tag{2.31}$$

This yields

$$
\begin{aligned}
& F(A \cup B) - \alpha F(A) \\
= \; & \frac{|U_2 \cap (A \cup B')|}{k} - \alpha \left( \frac{|\{b_1\} \cap A| \cdot |U_2 \cap A|}{k} \left( 1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right) + \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right) \\
= \; & \left[ \frac{|U_2 \cap (A \cup B')|}{k} - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right] - \left[ \alpha \frac{|\{b_1\} \cap A| \cdot |U_2 \cap A|}{k} \left( 1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right) \right] \\
\overset{\alpha \geq 1}{\leq} \; & \left[ \frac{|U_2 \cap (A \cup B')|}{k} - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right] - \left[ \frac{|\{b_1\} \cap A| \cdot |U_2 \cap A|}{k} \left( 1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right) \right] \\
= \; & \left[ \frac{|U_2 \cap (A \cup B')|}{k} - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right] \\
& - \left[ (|U_2 \cap B'| - (|U_2 \cap B'| - 1)) \frac{|\{b_1\} \cap A| \cdot |U_2 \cap A|}{k} \left( 1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right) \right] \\
\leq \; & \left[ \frac{|U_2 \cap (A \cup B')|}{k} \left( 1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right) \right] \\
& - \left[ \frac{|U_2 \cap B'| \cdot |\{b_1\} \cap A| \cdot |U_2 \cap A| - (|U_2 \cap B'| - 1)|U_2 \cap A|}{k} \left( 1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right) \right] \\
= \; & |U_2 \cap B'| \frac{|U_2 \cap A| + 1 - |\{b_1\} \cap A| \cdot |U_2 \cap A|}{k} \left( 1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right) \\
\overset{(2.31)}{=} \; & |U_2 \cap B'| \max_{b \in U_2 \cap B'} \left\{ \frac{|\{b_1\} \cap (A \cup \{b\})| \cdot |U_2 \cap (A \cup \{b\})| - |\{b_1\} \cap A| \cdot |U_2 \cap A|}{k} \right. \\
& \qquad \left. \cdot \left( 1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in U_1 \cap A}} \xi_i \right) \right\} \\
\overset{(2.30)}{\leq} \; & |U_2 \cap B'| \left( \max_{b \in U_2 \cap B'} F(A \cup \{b\}) - F(A) \right).
\end{aligned}
$$

Thus, the function $F$ is $\alpha$-augmentable.

**Case 3:** $\frac{k}{\alpha} < |\{b_1\} \cap A| \cdot |U_2 \cap A|$.

For $b \in B'$, we have

$$F(A \cup \{b\}) - F(A) = \begin{cases} 0 & \text{if } b = a_i \in U_1 \cap B', \\ \frac{|U_2 \cap (A \cup \{b\})| - |U_2 \cap A|}{k} & \text{if } b \in U_2 \cap B', \end{cases} \quad (2.32)$$

which yields

$$\begin{aligned} F(A \cup B) - \alpha F(A) &= \frac{|U_2 \cap (A \cup B')|}{k} - \alpha \frac{|U_2 \cap A|}{k} \\ &\overset{\alpha \geq 1}{\leq} \frac{|U_2 \cap (A \cup B')| - |U_2 \cap A|}{k} \\ &= \frac{|U_2 \cap B'|}{k} \\ &= |U_2 \cap B'| \max_{b \in U_2 \cap B'} \frac{|U_2 \cap (A \cup \{b\})| - |U_2 \cap A|}{k} \\ &\leq |B| \max_{b \in U_2 \cap B'} \frac{|U_2 \cap (A \cup \{b\})| - |U_2 \cap A|}{k} \\ &\overset{(2.32)}{=} |B| \big( \max_{b \in B} F(A \cup \{b\}) - F(A) \big). \end{aligned}$$

This establishes $\alpha$-augmentability. □

Combining this with Proposition 2.21 extends the lower bound of Corollary 2.10 to all $\alpha \geq 1$.

**Theorem 2.25.** *Let $\alpha \geq 1$. Then, the competitive ratio of the* GREEDY *algorithm for* INCMAX *with $\alpha$-augmentable objectives is exactly $\alpha \cdot \frac{e^\alpha}{e^\alpha - 1}$.*

Now, we will use the function $F_{\gamma,\alpha,k}$ in order to separate the class $\tilde{\mathcal{F}}_{\gamma,\alpha}$ from the class $\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_\alpha \cup \mathcal{F}_q$.

**Lemma 2.26.** *For every $\gamma' \in (0,1)$, $\alpha' \geq \gamma'$, $\alpha \geq 1$ and $k \in \mathbb{N}_{>\alpha'}$, it holds that $F_{\gamma',\alpha',k} \notin \mathcal{F}_\alpha$. For every $\gamma, \gamma', q \in (0,1]$ and $\alpha' \geq \gamma'$, there exists $k' \in \mathbb{N}_{>\alpha}$ such that $F_{\gamma',\alpha',k'} \not\in \tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_q$.*

*Proof.* For the first part, let $\gamma' \in (0,1)$, $\alpha' \geq \gamma'$ and $k \in \mathbb{N}_{>\alpha'}$. Furthermore, let $A = \emptyset$ and $B = U_2$. For all $b \in B$, we have

$$F_{\gamma',\alpha',k}(A \cup \{b\}) - F_{\gamma',\alpha',k}(A) = F_{\gamma',\alpha',k}(\{b\}) \leq F_{\gamma',\alpha',k}(\{b_1\}) = \frac{1}{k}$$

and, for all $\alpha \geq 1$, we have

$$\frac{F_{\gamma',\alpha',k}(A \cup B) - \alpha F_{\gamma',\alpha',k}(A)}{|B|} = \frac{F_{\gamma',\alpha',k}(U_2)}{k} = \frac{1}{k\gamma'} > \frac{1}{k}$$

because $\gamma' < 1$. Thus, $F_{\gamma',\alpha',k}$ is not $\alpha$-augmentable for any $\alpha \geq 1$.

For the second part, let $\gamma' \in (0,1]$, $\alpha' \geq \gamma'$ and $k \in \mathbb{N}_{>\alpha'}$. Furthermore, let $A = U_1 = X^{\mathrm{G}}(k)$ and $B = U_2$. For all $b \in B$, we have

$$F_{\gamma',\alpha',k}(A \cup \{b\}) - F_{\gamma',\alpha',k}(A) = \begin{cases} \frac{1}{k}\big(1 - \alpha' \sum_{i=1}^{k} \xi_i\big) = \frac{1}{k}\big(\frac{k-\alpha'}{k}\big)^k & \text{if } b = b_1, \\ 0, & \text{else,} \end{cases}$$

and

$$F_{\gamma',\alpha',k}(A \cup B) - F_{\gamma',\alpha',k}(A) = F_{\gamma',\alpha',k}(U_2) - F_{\gamma',\alpha',k}(U_1) = \frac{1}{\gamma'} - \frac{1}{\alpha'}\big(1 - (\frac{k-\alpha'}{k})^k\big).$$

For $k \to \infty$, the (weak) submodularity ratio gets arbitrarily close to 0 because

$$\lim_{k\to\infty} \frac{\sum_{b\in B}\big(F_{\gamma',\alpha',k}(A \cup \{b\}) - F_{\gamma',\alpha',k}(A)\big)}{F_{\gamma',\alpha',k}(A \cup B) - F_{\gamma',\alpha',k}(A)} = \lim_{k\to\infty} \frac{\frac{1}{k}\big(\frac{k-\alpha'}{k}\big)^k}{\frac{1}{\gamma'} - \frac{1}{\alpha'}\big(1 - \big(\frac{k-\alpha'}{k}\big)^k\big)} = 0,$$

i.e., for $k = k'$ large enough, $F_{\gamma',\alpha',k'} \notin \tilde{\mathcal{F}}_\gamma$. It remains to show that $F_{\gamma',\alpha',k'} \not\subseteq \mathcal{F}_q$. If $F_{\gamma',\alpha',k'} \in \mathcal{F}_q$ would hold, then there would be some independence system with weight function $w$ such that $F_{\gamma',\alpha',k'}$ was the associated weighted rank function. The fact that $F_{\gamma',\alpha',k'}(\{b_2\}) = 0$ implies that $b_2$ must have weight 0 or $\{b_2\}$ is not independent, and the fact that $F_{\gamma',\alpha',k'}(\{b_1, b_2\}) - F_{\gamma',\alpha',k'}(\{b_1\}) = \frac{h(2)-h(1)}{k} > 0$ implies that $b_2$ must have a weight greater 0 and that $\{b_2\}$ has to be independent, which contradict each other. Thus, $F_{\gamma',\alpha',k'}$ cannot be modeled as the weighted rank function of an independence system, i.e., $F_{\gamma',\alpha',k'} \notin \mathcal{F}_q$. □

This lemma immediately yields the following.

**Theorem 2.27.** *Let $\gamma, q \in (0,1]$, $\gamma' \in (0,1)$, $\alpha \geq 1$, and $\alpha' \geq \gamma'$. Then*

$$\tilde{\mathcal{F}}_{\gamma',\alpha'} \not\subseteq \tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_\alpha \cup \mathcal{F}_q.$$

Finally, we can extend Proposition 2.12 to all $\alpha \geq 1$ by combining the fact that, for $\alpha \geq 1$, we have $\{F_{1,\alpha,k} \mid k \in \mathbb{N}, k > \alpha\} \subseteq \mathcal{F}_\alpha$ by Proposition 2.24 and the fact that, for every $\gamma, q \in (0,1]$, we have $\{F_{1,\alpha,k} \mid k \in \mathbb{N}, k > \alpha\} \not\subseteq \tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_q$ by Lemma 2.26.

**Corollary 2.28.** *For every* $\gamma, q \in (0,1]$, $\alpha \geq 1$, *it holds that* $\mathcal{F}_\alpha \nsubseteq (\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_q)$.

Combining Propositions 2.13 and 2.14 with Corollary 2.28 yields the following.

**Theorem 2.29.** *For every* $\gamma, q \in (0,1)$ *and* $\alpha \geq 1$, *it holds that*

$$\tilde{\mathcal{F}}_\gamma \nsubseteq (\mathcal{F}_\alpha \cup \mathcal{F}_q) \quad \text{and} \quad \mathcal{F}_\alpha \nsubseteq (\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_q) \quad \text{and} \quad \mathcal{F}_q \nsubseteq (\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_\alpha).$$

### 2.3.2. $\gamma$-$\alpha$-Augmentability on Independence Systems

To tightly capture the class $\mathcal{F}_q$ of weighted rank functions on independence systems, we show a stronger bound for the competitive ratio of the Greedy algorithm on the class of monotone, (weakly) $\gamma$-$\alpha$-augmentable weighted rank functions. In particular, it was already shown in [5] that the objective function of $\alpha$-Dimensional Matching is (exactly) $\alpha$-augmentable, while the Greedy algorithm yields a competitive ratio of $\alpha$, which beats the upper bound of $\alpha \cdot \frac{\mathrm{e}^\alpha}{\mathrm{e}^\alpha - 1}$ for this case. We show that this can be explained by the fact that $\alpha$-Dimensional Matching can be represented via a weighted rank function over an independence system. We denote the set of all weighted rank functions on some independence system by $\mathcal{F}_{\mathrm{IS}} := \bigcup_{q \in (0,1]} \mathcal{F}_q$.

**Proposition 2.30.** *The competitive ratio of the Greedy algorithm for IncMax with objectives in* $\tilde{\mathcal{F}}_{\gamma,\alpha} \cap \mathcal{F}_{\mathrm{IS}}$ *is at most* $\frac{\alpha}{\gamma}$, *for every* $\gamma \in (0,1]$ *and* $\alpha \geq \gamma$.

*Proof.* Let $f \in \tilde{\mathcal{F}}_{\gamma,\alpha} \cap \mathcal{F}_{\mathrm{IS}}$, and let $w \colon U \to \mathbb{R}_{\geq 0}$ be the weight function that induces $f$. We use induction over $C$ to show that, for all $C \in \mathbb{N}$, we have

$$f(X^{\mathrm{G}}(C)) \geq \frac{\gamma}{\alpha} \textsc{Opt}(C). \tag{2.33}$$

For $C = 0$, the statement holds obviously.

Now suppose, the statement holds for some $C \in \mathbb{N}$. If $f(X^{\mathrm{G}}(C)) \geq \frac{\gamma}{\alpha} f(O(C+1))$, then, by monotonicity of $f$, $f(X^{\mathrm{G}}(C+1)) \geq f(X^{\mathrm{G}}(C)) \geq \frac{\gamma}{\alpha} f(O(k+1))$. On the other hand, if $f(X^{\mathrm{G}}(C)) < \frac{\gamma}{\alpha} f(O(C+1))$, then the weak $\gamma$-$\alpha$-augmentability of $f$ guarantees the existence of $e \in O(C+1)$ with

$$\begin{aligned} f(X^{\mathrm{G}}(C) \cup \{e\}) - f(X^{\mathrm{G}}(C)) \quad &\geq \quad \frac{\gamma f(X^{\mathrm{G}}(C) \cup O(C+1)) - \alpha f(X^{\mathrm{G}}(C))}{|O(C+1)|} \\ &\geq \quad \frac{\gamma f(O(C+1)) - \alpha f(X^{\mathrm{G}}(C))}{C+1} > 0. \end{aligned}$$

By Lemma 2.15, this is equivalent to $f(X^{\mathrm{G}}(C) \cup \{e\}) = f(X^{\mathrm{G}}(C)) + w(e)$. We conclude

$$\begin{aligned} f(X^{\mathrm{G}}(C+1)) &\geq f(X^{\mathrm{G}}(C) \cup \{e\}) \\ &= f(X^{\mathrm{G}}(C)) + w(e) \\ &\overset{(2.33)}{\geq} \frac{\gamma}{\alpha} f(O(C)) + w(e) \\ &\geq \frac{\gamma}{\alpha} f(O(C+1) \setminus \{e\}) + w(e) \\ &\overset{\alpha \geq \gamma}{\geq} \frac{\gamma}{\alpha} f(O(C+1) \setminus \{e\}) + \frac{\gamma}{\alpha} w(e) \\ &\geq \frac{\gamma}{\alpha} f(O(C+1)), \end{aligned}$$

i.e., the GREEDY algorithm has a competitive ratio of at most $\frac{\alpha}{\gamma}$. □

The tight lower bound follows directly from the well-known tight bound of $1/q$ for $\mathcal{F}_q$.

**Proposition 2.31.** *The competitive ratio of the GREEDY algorithm for INCMAX with objectives in $\tilde{\mathcal{F}}_{\gamma,\alpha} \cap \mathcal{F}_{\mathrm{IS}}$ is at least $\frac{\alpha}{\gamma}$, for every $\gamma \in (0,1]$ and $\alpha \geq \gamma$.*

*Proof.* Let $\gamma \in (0,1]$, $\alpha \geq \gamma$ and $q \in [\frac{\gamma}{\alpha}, 1] \cap \mathbb{Q}$. In [40] it was shown that the competitive ratio of the GREEDY algorithm on the set $\mathcal{F}_q$ is exactly $1/q$. By definition of $\mathcal{F}_{\mathrm{IS}}$, we have $\mathcal{F}_q \subseteq \mathcal{F}_{\mathrm{IS}}$, and, by Theorem 2.16, $\mathcal{F}_q \subseteq \tilde{\mathcal{F}}_{\gamma,\gamma/q} \subseteq \tilde{\mathcal{F}}_{\gamma,\alpha}$ holds, where we use the fact that $\frac{\gamma}{q} \leq \frac{\gamma}{\gamma/\alpha} = \alpha$. Thus, we can conclude that the competitive ratio of the GREEDY algorithm on the class $\tilde{\mathcal{F}}_{\gamma,\alpha} \cap \mathcal{F}_{\mathrm{IS}}$ is at least $1/q$, and since $q$ can be chosen arbitrarily close to $\frac{\gamma}{\alpha}$, the statement follows. □

Combining Propositions 2.30 and 2.31 yields the following.

**Theorem 2.32.** *The competitive ratio of the GREEDY algorithm for INCMAX with objectives in $\tilde{\mathcal{F}}_{\gamma,\alpha} \cap \mathcal{F}_{\mathrm{IS}}$ is exactly $\frac{\alpha}{\gamma}$, for every $\gamma \in (0,1]$ and $\alpha \geq \gamma$.*

It can be shown that the lower bound of Proposition 2.31 already holds for the class of $\gamma$-$\alpha$-augmentable functions, i.e., for the non-weak subclass of $\tilde{\mathcal{F}}_{\gamma,\alpha}$. It follows that the tight bound carries over to this, in some sense more natural, class of functions. Since every $\alpha$-augmentable function is 1-$\alpha$-augmentable, and vice-versa, we additionally obtain the following. Note that this tightly captures the performance of the GREEDY algorithm for the $\alpha$-DIMENSIONAL MATCHING problem, which can be represented as the maximization of an $\alpha$-augmentable weighted rank function over an independence system [5].

**Corollary 2.33.** *Let $\alpha \geq 1$. Then, the competitive ratio of the GREEDY algorithm for INCMAX with objectives in $\mathcal{F}_\alpha \cap \mathcal{F}_{\mathrm{IS}}$ is exactly $\alpha$.*

# 3. Incremental Maximization via Continuization

In the previous chapter, we analyzed the performance of the GREEDY algorithm for the INCMAX problem. Now, we also consider other algorithms and work towards finding the competitive ratio of the INCMAX problem. As we have seen in the instance in Figure 1.1, the competitive ratio of the INCMAX problem itself is unbounded. Thus, we can only hope to find meaningful subclasses with a bounded competitive ratio. The problem with the instance in Figure 1.1 is that there exists a set with a very large value, whose proper subsets have rather small values. Thus, the solution we obtain while assembling this set is not competitive compared to other sets. In order to avoid this, Bernstein et al. [5] introduced accountability which guarantees that in every set $S \subseteq U$ there exists an element that can be removed without decreasing the value of the set too much (cf. Definition 1.2). The authors in [5] showed that the competitive ratio of INCMAX$_{\text{acc}}$, the subclass of INCMAX with instances that have an accountable objective, lies in $[2.18, \varphi + 1]$, where $\varphi \approx 1.618$ is the golden ratio. We give an intuition why accountability is a desirable property for the objective of INCMAX.

**Lemma 3.1.** *A function $f\colon 2^U \to \mathbb{R}_{\geq 0}$ is accountable if and only if, for every finite $S \subseteq U$, there exists an ordering $(e_1, \ldots, e_{|S|})$ of $S$ with $f(\{e_1, \ldots, e_i\}) \geq \frac{i}{|S|} f(S)$ for all $i \in [|S|]$.*

*Proof.* "⇐": Let $S \subseteq U$ be finite. There exists an ordering $(e_1, \ldots, e_{|S|})$ of $S$ with $f(\{e_1, \ldots, e_i\}) \geq \frac{i}{k} f(S)$ for all $i \in [|S|]$. In particular, we have

$$f(S \setminus \{e_{|S|}\}) = f(\{e_1, \ldots, e_{|S|-1}\}) \geq \frac{|S| - 1}{|S|} f(S) = f(S) - \frac{f(S)}{|S|},$$

i.e., $f$ is accountable.

"⇒": Let $S \subseteq U$ be finite, and let $f$ be accountable and $S_{|S|} := S$. We define $S_{|S|-1}, \ldots, S_1$ recursively. Suppose, for $i \in [|S| - 1]$, the set $S_{i+1}$ is defined and $|S_{i+1}| = i + 1$. By accountability of $f$, there exists $e \in S_{i+1}$ with

$$f(S_{i+1} \setminus \{e\}) \geq f(S_{i+1}) - \frac{f(S_{i+1})}{|S_{i+1}|} = \frac{i}{i+1} f(S_{i+1}).$$

Let $S_i := S_{i+1} \setminus \{e\}$. We define the ordering $(e_1, \ldots, e_{|S|})$ to be the unique ordering of $S$ such that $S_i = \{e_1, \ldots, e_i\}$ for all $i \in [|S|]$. Then, for $i \in [|S| - 1]$, we have

$$f(\{e_1, \ldots, e_i\}) = f(S_i) \geq \frac{i}{i+1} f(S_{i+1}) \geq \cdots \geq \frac{i}{i+1} \cdot \ldots \cdot \frac{|S|-1}{|S|} f(S_{|S|}) = \frac{i}{|S|} f(S). \quad \square$$

Lemma 3.1 immediately yields that the optimum value for larger cardinality cannot grow too fast with increasing cardinality.

**Corollary 3.2.** *Let $C, C' \in \mathbb{N}$ with $C \leq C' \leq |U|$. Then, for every instance in* INCMAX$_{\text{acc}}$*,*

$$\text{OPT}(C) \geq \frac{C}{C'} \text{OPT}(C').$$

*Proof.* Let $(e_1, \ldots, e_{C'})$ be the ordering of the optimum solution $O(C')$ given by Lemma 3.1. Then

$$\text{OPT}(C) \geq f(\{e_1, \ldots, e_C\}) \overset{\text{Lem. 3.1}}{\geq} \frac{C}{C'} f(O(C')) = \frac{C}{C'} \text{OPT}(C'). \quad \square$$

In this chapter, we dive deeper into the analysis of the competitive ratio of INCMAX$_{\text{acc}}$. We are going to introduce a continuization technique to reduce the problem to a continuous one and use this continuous problem to show improved lower bounds for the (non-strict) competitive ratio of INCMAX$_{\text{acc}}$. We give an overview over the contents of this chapter.

In Section 3.1, we introduce the problem class INCMAXSEP - a subclass of INCMAX where the instances have a simpler structure and are easier to analyze. The elements of such an instance are partitioned into (countably many) subsets, where the objective within one of the subsets is a simple weight function where each element has the same weight. The objective value of a set of elements from different subsets is simply the maximum over the value of the weight functions on the subsets. We show that this objective is monotone and accountable and that INCMAXSEP has the same competitive ratio as INCMAX$_{\text{acc}}$.

Subsequently, in Section 3.2, we define the INCMAXCONT problem, a continuization of the INCMAXSEP problem, where we assume that there exists one such subset every size $c > 0$. Note that we have to assume that there also exist fractional elements in order to have sets of non-integral sizes. The smooth structure of this problem is more beneficial to analysis. We show that the strict competitive ratio of the INCMAXCONT problem gives a lower bound on the (non-strict) competitive ratio of the INCMAXSEP problem, i.e., in order to find lower bounds for the (non-strict) competitive ratio of INCMAX$_{\text{acc}}$, we can instead find a lower bound for the strict competitive ratio of INCMAXCONT.

In order to do this, we introduce the continuous algorithm GREEDYSCALING$(c_1, \rho)$ in Section 3.2.1. This algorithm adds a sequence of subsets to the solution, starting with the subset of size $c_1 > 0$ and proceeding with a sequence of subsets such that, in every step, the size of the next subset is as large as possible without violating $\rho$-competitiveness. We show that this algorithm is optimal for the correct choice of $c_1 > 0$ and $\rho \geq 1$. Furthermore, we show that, for any reasonably small choice of $c_1$ and with $\rho = \varphi + 1$, the algorithm is always $(\varphi + 1)$-competitive. We conclude the analysis of GreedyScaling$(c_1, \rho)$ by showing that, if we restrict the algorithm to choose $c_1$ from some fixed countable set, then the algorithm cannot be better than $(\varphi + 1)$-competitive. While this gives a lower bound if we restrict the starting value, it does not transfer to the problem class IncMaxCont because, for this, we would have to find one problem instance that shows this for all starting values $c_1 > 0$, and not only a countable set.

In Section 3.2.2, we extrapolate the techniques used before to show a lower bound of $2.246$ on the competitive ratio of the problem class IncMaxCont. This yields a lower bound of $2.246$ on the (non-strict) competitive ratio of IncMax$_{\text{acc}}$ which improves upon the lower bound of $2.18$ from [5].

An extended abstract with most of the results in this chapter appeared in [19]. A new result in this thesis is Theorem 3.20.

## 3.1. Separability of Accountable Incremental Maximization

As a first step to bound the competitive ratio of IncMax$_{\text{acc}}$, we introduce IncMaxSep, a class of instances of IncMax with a relatively simple structure. We show that the (non-strict) competitive ratios of IncMax$_{\text{acc}}$ and IncMaxSep coincide. Thus, we can analyze IncMaxSep to obtain bounds on the (non-strict) competitive ratio of IncMax$_{\text{acc}}$.

**Definition 3.3.** *An instance of IncMax with objective $f\colon 2^U \to \mathbb{R}_{\geq 0}$ is called* separable *if there exist a partition $U = U_1 \cup U_2 \cup \ldots$ of $U$ and values $d_i > 0$ such that, for all $S \subseteq U$,*

$$f(S) = \max_{i \in \mathbb{N}} \{|S \cap U_i| \cdot d_i\}.$$

*We refer to $d_i$ as the* density *of set $U_i$ and to $v_i := |U_i| \cdot d_i$ as the* value *of set $U_i$. The restriction of IncMax to separable instances will be denoted by IncMaxSep.*

It turns out that the class IncMaxSep is a subclass of IncMax$_{\text{acc}}$.

**Lemma 3.4.** *The objective of every instance in IncMaxSep is monotone and accountable.*

*Proof.* Let $f$ be the objective function of some instance in IncMaxSep. The function $f$ is the maximum over modular functions with non-negative values for single elements and therefore monotone.

To show that $f$ is accountable, let $S \subseteq U$. Let $i \in \mathbb{N}$ such that $f(S) = |S \cap U_i| \cdot d_i$. If $S \setminus U_i \neq \emptyset$, we consider some element $e \in S \setminus U_i$. We have

$$f(S \setminus \{e\}) = |S \cap U_i| \cdot d_i = f(S) \geq f(S) - \frac{f(S)}{|S|},$$

i.e., $f$ is accountable. Otherwise, if $S \setminus U_i = \emptyset$, we let $e \in S$ be chosen arbitrarily and obtain

$$\begin{aligned} f(S \setminus \{e\}) &\geq |(S \setminus \{e\}) \cap U_i| \cdot d_i = (|S \cap U_i| - 1) \cdot d_i \\ &= \frac{|S \cap U_i| - 1}{|S \cap U_i|} f(S) = \frac{|S| - 1}{|S|} f(S) = f(S) - \frac{f(S)}{|S|}, \end{aligned}$$

i.e., also in this case, $f$ is accountable. □

Lemma 3.4 implies that lower bound on the (non-strict) competitive ratio of IncMaxSep are also lower bound on the (non-strict) competitive ratio of IncMax$_{\text{acc}}$. We will show in the remainder of this chapter that the competitive ratios of the two problem classes coincide. In order to do this, we show that we can restrict ourselves to instances in IncMaxSep with a couple of nice properties.

**Lemma 3.5.** *Any instance of IncMaxSep can be transformed into one with the same (non-strict) competitive ratio, that satisfies the following properties.*
*(i) There is exactly one set of every cardinality, i.e.,* $|U_i| = i$.
*(ii) Densities are decreasing, i.e.,* $1 \geq d_1 \geq d_2 \geq \dots$.
*(iii) Values are increasing, i.e.,* $v_1 \leq v_2 \leq \dots$.

*Proof.* We show this by transforming a given instance that does not satisfy *(i)-(iii)* into one that does, without changing the optimum value for any cardinality, and without changing the value of the best incremental solution. Thus the (non-strict) competitive ratio of the two instances coincide.

If there are two sets $U_i, U_j$ with $|U_i| = |U_j|$, it only makes sense to consider the one with higher density, as every incremental solution adding elements from the set of smaller density can be improved by adding elements from the other set instead, i.e., we can remove the set with smaller density. If there is $i \in \mathbb{N}_{\geq 2}$ such that there is no set with $i$ elements, we can add a new set $U_i$ with $i$ elements to the instance with value $v_i := v_{i-1}$. Then, every incremental solution that adds elements from the newly introduced set can be improved by adding elements from set $U_{i-1}$ instead. Thus, we neither change the value of the optimum solution of a given cardinality, nor the value of the best incremental solution for any cardinality. If there is no set $U_1$ with $1$ element, we can introduce it with

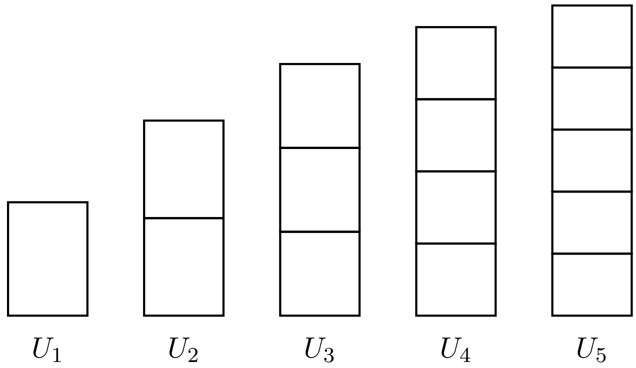


Figure 3.1.: Illustration of an instance of IncMaxSep with $N = 5$ sets. Each set $U_i$ consists of $i$ elements. The height of the elements represents their value. As in Lemma 3.5, the values of the single elements decreases the larger $i$ is, while the value of the whole set $U_i$ increases.

density $d_2$. Then, every incremental solution that adds this one element can instead also add one element from $U_2$. With these changes we obtain an instance that satisfies *(i)*.

The property that $1 \geq d_1$ can be made without loss of generality by rescaling the objective $f$. If there was $i \in \mathbb{N}$ with $d_i < d_{i+1}$, every incremental solution to the problem instance that adds elements from the set $U_i$ could be improved by adding elements from the set $U_{i+1}$ instead. Since $|U_{i+1}| \geq |U_i|$, this is possible. Thus, we can change the density $d_i$ to be equal to $d_{i+1}$ without changing the (non-strict) competitive ratio of the instance. With this, we obtain an instance satisfying *(i)* and *(ii)*.

We can assume that *(iii)* holds because, if there was $i \in \mathbb{N}$ with $v_i > v_{i+1}$, an incremental solution that adds elements from $U_{i+1}$ can be improved by adding elements from $U_i$ instead. This would mean that we could set $v_{i+1}$ to be equal to $v_i$ without changing the (non-strict) competitive ratio. □

In the following, we assume that every instance satisfies the properties from Lemma 3.5.

**Definition 3.6.** *We say that an incremental solution for IncMaxSep can be* represented by a sequence $(c_1, c_2, \dots)$ *with* $c_i \in \mathbb{N}$ *for all* $i \in \mathbb{N}$ *if it first adds all elements from the set* $U_{c_1}$*, then all elements from the set* $U_{c_2}$*, and so on.*

An incremental solution of IncMaxSep can only improve if it is modified in a way such that it can be represented by a sequence $(c_1, c_2, \dots)$. If not all elements of one set $U_{c_i}$ are added to the solution at some point in time, the incremental solution does not degrade if elements from a smaller set are added instead because the density of the smaller set is at least as large as the density of the larger set. Adding all elements of one set consecutively is better because the value of the incremental solution increases faster this way.

**Lemma 3.7** ([5, Observation 2]). *For every instance of* IncMaxSep, *there is an incremental solution achieving the best-possible competitive ratio that can be represented by a sequence* $(c_1, c_2, \dots)$. *We can assume that* $v_{c_i} < v_{c_{i+1}}$ *and thus, since the values* $(v_i)_{i\in\mathbb{N}}$ *are non-decreasing,* $c_i < c_{i+1}$ *for all* $i \in \mathbb{N}$.

From now on, we restrict ourselves to the analysis of incremental solutions that can be represented by a sequence $(c_1, c_2, \dots)$ with $c_i < c_{i+1}$ for all $i \in \mathbb{N}$.

**Proposition 3.8.** *Let* $\rho \geq 1$ *and* $\alpha \geq 0$. *If there exists an algorithm with (non-strict) competitive ratio* $\rho$ *with additive constant* $\alpha$ *for* IncMaxSep, *then there also exists an algorithm with (non-strict) competitive ratio* $\rho$ *with additive constant* $\alpha$ *for* $\text{IncMax}_{\text{acc}}$.

*Proof.* Let $f\colon U \to \mathbb{R}_{\geq 0}$ be the monotone and accountable objective of some instance of $\text{IncMax}_{\text{acc}}$. We construct an instance of IncMaxSep such that, for $\rho \geq 1$, every incremental solution that is non-strictly $\rho$-competitive with additive constant $\alpha \geq 0$ induces an incremental solution for the initial instance of $\text{IncMax}_{\text{acc}}$ that is non-strictly $\rho$-competitive with additive constant $\alpha$.

By $O(i)$ we refer to the optimum solution of cardinality $i$ for the instance of $\text{IncMax}_{\text{acc}}$, and by $\textsc{Opt}(i)$ to the value $f(O(i))$.

First, we define the instance of IncMaxSep. Let $n := |U|$ and let $U_1, \dots, U_n$ be disjoint sets such that, for $i \in [n]$, $|U_i| = i$. For $i \in [n]$, let $d_i := \textsc{Opt}(i)/i$, i.e., we have $v_i = \textsc{Opt}(i)$. Let the objective of this problem be denoted by $f_{\text{sep}}$. Note that this new instance contains significantly more elements than the instance of $\text{IncMax}_{\text{acc}}$.

Let $X$ be an incremental solution of the separable problem instance that is non-strictly $\rho$-competitive with additive constant $\alpha \geq 0$ that can be represented by a sequence $(c_1, \dots, c_n)$. We define an incremental solution $\tilde{X}$ to the $\text{IncMax}_{\text{acc}}$ problem as follows. First, we add all elements from the set $O(c_1)$, then all elements from the set $O(c_2)$ and so on, until we added all optimum solutions $O(c_1), \dots, O(c_n)$. For all $i \in [n]$, the elements of the set $O(c_i)$ are added in the order given by Lemma 3.1. We show that this incremental solution is non-strictly $\rho$-competitive with additive constant $\alpha$, as well. For this, fix a cardinality $C \in [n]$. Let $i \in [n]$ be such that the last element added to the solution $\tilde{X}(C)$ is from the set $O(c_i)$. Note that, for cardinality $C$, $\tilde{X}$ has added $O(c_{i-1})$, the optimum solution of cardinality $c_{i-1}$, completely and $C - \sum_{j=1}^{i-1} c_j$ elements from the set $O(c_i)$. By Lemma 3.1 and monotonicity of $f$, the value of the solution $\tilde{X}(C)$ is

$$f(\tilde{X}(C)) = \max\left\{\textsc{Opt}(c_{i-1}), \frac{C - \sum_{j=1}^{i-1} c_j}{c_i}\textsc{Opt}(c_i)\right\}. \tag{3.1}$$

Similar to $\tilde{X}$, the solution $X(C)$ of the separable problem instance contains all elements from the set $U_{c_{i-1}}$ and $C - \sum_{j=1}^{i-1} c_j$ elements from the set $U_{c_i}$. Thus, the value of the solution $X(C)$ is

$$f_{\text{sep}}(X(C)) = \max\left\{v_{c_{i-1}}, \left(C - \sum_{j=1}^{i-1} c_j\right) d_{c_i}\right\} = \max\left\{v_{c_{i-1}}, \frac{C - \sum_{j=1}^{i-1} c_j}{c_i} v_{c_i}\right\}.$$

Combining this with the fact that $v_{c_j} = \textsc{Opt}(c_j)$ for all $j \in [n]$ as well as with (3.1), we obtain

$$f(\tilde{X}(C)) = f_{\textsc{sep}}(X(C)) \geq \frac{1}{\rho} v(C) - \alpha = \frac{1}{\rho}\textsc{Opt}(C) - \alpha. \qquad \square$$

By Lemma 3.4, IncMaxSep is a subclass of $\text{IncMax}_{\text{acc}}$, which yields that the competitive ratio of IncMaxSep is smaller or equal to that of $\text{IncMax}_{\text{acc}}$. Furthermore, by Proposition 3.8, the competitive ratio of IncMaxSep cannot be smaller than that of $\text{IncMax}_{\text{acc}}$. Combining these results immediately yields the following.

**Theorem 3.9.** *The (non-strict) competitive ratios of $\text{IncMax}_{\text{acc}}$ and IncMaxSep coincide.*

## 3.2. Continuization of Accountable Incremental Maximization

In order to find lower bounds on the (non-strict) competitive ratio of IncMaxSep, we transform the problem into a continuous one, the IncMaxCont problem. We will only consider the strict competitive ratio of this problem because it will turn out that lower bounds for the strict competitive ratio of the IncMaxCont problem are also lower bounds on the non-strict competitive ratio of the IncMaxSep problem.

Before we state the definition of the IncMaxCont problem, we give an intuition what the problem is about. In the IncMaxSep problem, we restricted ourselves to instances where the ground set $U$ is partitioned into disjoint subsets $U_1, U_2, \dots$ with $|U_i| = i$ for all $i \in \mathbb{N}$. Within one such subset, the objective function $f$ is modular and every element has the same value. We have seen that, without loss of generality, an incremental solution for such a problem can be represented by a sequence $(c_1, c_2, \dots)$ with $c_i \in \mathbb{N}$, i.e., the incremental solution starts by adding the elements in $U_{c_1}$ one by one, then the elements in $U_{c_2}$, and so on. In the continuized version of this problem we assume that, instead of the sets $U_1, U_2, \dots$, we are given a family of disjoint sets $(U_c)_{c \in \mathbb{R}_{>0}}$ with $|U_c| = c$ for all $c > 0$. Note that sets may now contain fractional elements. Further we assume that we can add fractional items to the solution. Within one such subset, the objective will still be modular in the sense that the value of $S \subseteq U_c$ is given by $|S| \cdot d(c)$, where $d\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

maps the index of the subset $U_c$ to the value of one element in this subset. An incremental solution for this problem will also be a sequence $(c_1, c_2, \dots)$ with the idea that first the subset $U_{c_1}$ is added to the solution, then $U_{c_2}$, and so on. Since $|U_c|$ is not necessarily an integer, instead of the cardinality of a set, we will talk about its size.

**Definition 3.10.** *In the* IncMaxCont *problem, we are given a* density function $d\colon \mathbb{R}_{\geq 0} \to (0,1]$ *and a* value function $v(c) := cd(c)$*. As for the discrete problem, we denote an* incremental solution $X$ *for* IncMaxCont *by a sequence* $X = (c_1, c_2, \dots)$*. For a given size* $c \geq 0$*, we denote the* solution *of this size by* $X(c)$*. With* $k \in \mathbb{N}$ *such that* $\sum_{i=1}^{k-1} c_i < c \leq \sum_{i=1}^{k} c_i$*, the value of* $X(c)$ *is defined as*

$$f(X(c)) := \max\left\{ \max_{i\in[k-1]} v(c_i), \left(c - \sum_{i=1}^{k-1} c_i\right) d(d_k) \right\}.$$

*An incremental solution* $X$ *is* $\rho$-competitive *if* $\rho \cdot f(X(c)) \geq$ Opt$(c)$ *for all* $c > 0$*. The* competitive ratio *of* $X$ *is defined as* $\inf\{\rho \geq 1 \mid X$ *is* $\rho$*-competitive*$\}$.

As we did for the discrete version of the problem, without loss of generality, we assume that the density function $d$ is non-increasing and the value function $v$ is non-decreasing. These assumptions imply that $d$ is continuous: If this was not the case and $d$ was not continuous for some size $c'$, i.e., $\lim_{c \nearrow c'} d(c) > \lim_{c \searrow c'} d(c)$, then $\lim_{c \nearrow c'} v(c) > \lim_{c \searrow c'} v(c)$ by definition of $v$, i.e., $v$ would not be increasing in $c$. So $d$ is continuous, and, by definition of $v$, also $v$ is continuous. Furthermore, without loss of generality, we assume that $d(0) = 1$.

**Remark 3.11.** *As the function* $v$ *is increasing and* $d$ *is decreasing, we have* Opt$(c) = v(c)$*. Thus, an incremental solution* $X$ *is* $\rho$-competitive *if* $\rho \cdot f(X(c)) \geq v(c)$ *for all* $c > 0$*.*

For a fixed size $c \geq 0$, we define $p(c) := \max\{c' \geq 0 \mid v(c') \leq \rho v(c)\}$. This value gives the size up to which a solution with value $v(c)$ is $\rho$-competitive. Throughout our analysis, we assume that $p(c)$ is defined for every $c \geq 0$, i.e., that $\lim_{c\to\infty} v(c) = \infty$. Otherwise, any algorithm can terminate when the value of its solution is at least $\frac{1}{\rho} \sup_{c \in \mathbb{R}_{\geq 0}} v(c)$.

**Proposition 3.12.** *For every additive constant* $\alpha \geq 0$*, the non-strict competitive ratio of* IncMaxSep *is greater or equal to the strict competitive ratio of* IncMaxCont*.*

*Proof.* Let an instance of the IncMaxCont problem with value function $v\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and density function $d\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be given, and let $\rho \geq 1$, $\alpha \geq 0$, $\varepsilon > 0$. We will construct an instance of the IncMaxSep problem such that every incremental solution for this problem instance with non-strict competitive ratio $\rho$ with additive constant $\alpha$ yields an incremental solution for the IncMaxCont problem with competitive ratio $\rho + \varepsilon$. Let $\varepsilon' > 0$ be small enough such that

$$\rho < (\rho + \varepsilon)(1 - \varepsilon'), \tag{3.2}$$

$$\frac{1 - 2\varepsilon'}{\rho} \geq \frac{1}{\rho + \varepsilon}, \tag{3.3}$$

$$\varepsilon' < 1 - \frac{1}{\rho}. \tag{3.4}$$

Furthermore, let $c_{\text{min}} \geq 0$ be the largest value with $d((\rho + 1)c_{\text{min}}) = 1 - \varepsilon'$.

Let $k \in \mathbb{N}$ be large enough such that, for all $C \geq c_{\text{min}}$,

$$\frac{\varepsilon'}{\rho} \lfloor k c_{\text{min}} \rfloor \geq \alpha, \tag{3.5}$$

$$\frac{\rho + \varepsilon}{\rho} v\left(C - \frac{1}{k}\right) - \frac{(\rho + \varepsilon)\alpha}{k} \geq v(C), \tag{3.6}$$

$$\rho k \cdot c_{\text{min}} < \lfloor k(\rho + 1) c_{\text{min}} \rfloor (1 - \varepsilon') - \alpha, \tag{3.7}$$

$$\frac{1 - \varepsilon'}{\rho} - \frac{\alpha}{\lfloor k \rho c_{\text{min}} \rfloor} \geq \frac{1}{\rho + \varepsilon}, \tag{3.8}$$

$$\tag{3.9}$$

where the last two are possible because of (3.2). We define the IncMaxSep problem as follows. Let $U_1, U_2, \dots$ be disjoint sets of elements with $|U_i| = i$. For $i \in \mathbb{N}$, let

$$d_i := d\left(\frac{i}{k}\right) \tag{3.10}$$

be the density of set $U_i$, i.e., the value of $U_i$ is

$$v_i = i \cdot d_i = i \cdot d\left(\frac{i}{k}\right) = k \cdot v\left(\frac{i}{k}\right) \tag{3.11}$$

Let $X$ be an incremental solution that is non-strictly $\rho$-competitive with additive constant $\alpha$ for this instance of IncMaxSep and can be represented by the sequence $(c_1, c_2, \dots)$. Without loss of generality, we can assume that $c_i < c_{i+1}$ for all $i \in \mathbb{N}$. Furthermore, we can assume that

$$d_{c_2} < \frac{1}{\rho}. \tag{3.12}$$

If this was not the case, we could simply consider the incremental solution represented by $(c_2, c_3, \dots)$, which would also be non-strictly $\rho$-competitive with additive constant $\alpha$.

We will define an incremental solution $\tilde{X} = (\tilde{c_1}, \tilde{c_2}, \dots)$ for the instance of IncMaxCont depending on the values $c_1$ and $c_{\text{min}}$.

**Case 1:** $\frac{c_1}{k} \geq c_{\text{min}}$.

In this case, we define $\tilde{X} = \left(\frac{c_1}{k}, \frac{c_2}{k}, \dots\right)$. The fact that $X$ is non-strictly $\rho$-competitive with additive constant $\alpha$, together with the fact that $c_1 \geq kc_{\text{min}}$ yields

$$\begin{aligned}
\lfloor kc_{\text{min}} \rfloor d_{c_1} &= f(X(\lfloor kc_{\text{min}} \rfloor)) \geq \frac{1}{\rho} v_{\lfloor kc_{\text{min}} \rfloor} - \alpha \\
&= \frac{1}{\rho} \lfloor kc_{\text{min}} \rfloor d_{\lfloor kc_{\text{min}} \rfloor} - \alpha \overset{(3.10)}{=} \frac{1}{\rho} \lfloor kc_{\text{min}} \rfloor d\left(\frac{\lfloor kc_{\text{min}} \rfloor}{k}\right) - \alpha \\
&\geq \frac{1}{\rho} \lfloor kc_{\text{min}} \rfloor d(c_{\text{min}}) - \alpha \geq \frac{1-\varepsilon'}{\rho} \lfloor kc_{\text{min}} \rfloor - \alpha \\
&\overset{(3.5)}{\geq} \frac{1-\varepsilon'}{\rho} \lfloor kc_{\text{min}} \rfloor - \frac{\varepsilon'}{\rho} \lfloor kc_{\text{min}} \rfloor = \frac{1-2\varepsilon'}{\rho} \lfloor kc_{\text{min}} \rfloor \\
&\overset{(3.3)}{\geq} \frac{1}{\rho+\varepsilon} \lfloor kc_{\text{min}} \rfloor,
\end{aligned}$$

i.e., we have

$$d(\tilde{c_1}) = d\left(\frac{c_1}{k}\right) \overset{(3.10)}{=} d_{c_1} \geq \frac{1}{\rho+\varepsilon}.$$

**Case 2:** $\frac{c_1}{k} < c_{\text{min}}$.

In this case, we define $\tilde{X} = \left(\frac{c_2}{k}, \frac{c_3}{k}, \dots\right)$, i.e., we skip the first size $\frac{c_1}{k}$. The solution $X(\lfloor k(\rho+1)c_{\text{min}} \rfloor)$ contains the set $U_{c_1}$ completely because we have $c_1 < kc_{\text{min}} \leq \lfloor k(\rho+1)c_{\text{min}} \rfloor$. Furthermore, because

$$d_{\lfloor k(\rho+1)c_{\text{min}} \rfloor} = d\left(\frac{\lfloor k(\rho+1)c_{\text{min}} \rfloor}{k}\right) \geq d((\rho+1)c_{\text{min}}) = 1-\varepsilon' \overset{(3.4)}{>} \frac{1}{\rho} \overset{(3.12)}{>} d_{c_2},$$

we have $c_2 > \lfloor k(\rho+1)c_{\text{min}} \rfloor$. Thus, $X(\lfloor k(\rho+1)c_{\text{min}} \rfloor)$ contains exactly $\lfloor k(\rho+1)c_{\text{min}} \rfloor - c_1$ elements from the set $U_{c_2}$ and, except for the set $U_{c_1}$, nothing else. Thus,

$$f(X(\lfloor k(\rho+1)c_{\text{min}} \rfloor)) = \max\{v_{c_1}, (\lfloor k(\rho+1)c_{\text{min}} \rfloor - c_1)d_{c_2}\}. \tag{3.13}$$

We have

$$\begin{aligned}
\lfloor k(\rho+1)c_{\text{min}} \rfloor (1-\varepsilon') &= \lfloor k(\rho+1)c_{\text{min}} \rfloor d((\rho+1)c_{\text{min}}) \\
&\leq \lfloor k(\rho+1)c_{\text{min}} \rfloor d\left(\frac{\lfloor k(\rho+1)c_{\text{min}} \rfloor}{k}\right) \\
&= k \cdot v\left(\frac{\lfloor k(\rho+1)c_{\text{min}} \rfloor}{k}\right) \\
&\overset{(3.11)}{=} v_{\lfloor k(\rho+1)c_{\text{min}} \rfloor}
\end{aligned} \tag{3.14}$$

and thus

$$\begin{aligned}\rho v_{c_1} &\overset{(3.11)}{=} \rho k \cdot v\left(\frac{c_1}{k}\right) \le \rho k \cdot v(c_{\min}) \le \rho k \cdot c_{\min} \\ &\overset{(3.7)}{<} \lfloor k(\rho+1)c_{\min}\rfloor(1-\varepsilon') - \alpha \overset{(3.14)}{\le} v_{\lfloor k(\rho+1)c_{\min}\rfloor} - \alpha. \end{aligned} \tag{3.15}$$

Since $X$ has a non-strict competitive ratio of $\rho$ with additive constant $\alpha$, (3.13) and (3.15) yield

$$f(X(\lfloor k(\rho+1)c_{\min}\rfloor)) = (\lfloor k(\rho+1)c_{\min}\rfloor - c_1)d_{c_2}.$$

Combined with the fact that $X$ is non-strictly $\rho$-competitive with additive constant $\alpha$, this implies

$$\begin{aligned}(\lfloor k(\rho+1)c_{\min}\rfloor - c_1)d_{c_2} &\ge \frac{1}{\rho}v_{\lfloor k(\rho+1)c_{\min}\rfloor} - \alpha \\ &\overset{(3.14)}{\ge} \frac{1}{\rho}\lfloor k(\rho+1)c_{\min}\rfloor(1-\varepsilon') - \alpha \\ &\ge \frac{1}{\rho}(\lfloor k(\rho+1)c_{\min}\rfloor - c_1)(1-\varepsilon') - \alpha, \end{aligned} \tag{3.16}$$

i.e.,

$$\begin{aligned}d(\tilde{c_1}) &= d\left(\frac{c_2}{k}\right) \overset{(3.10)}{=} d_{c_2} \\ &\overset{(3.16)}{\ge} \frac{1-\varepsilon'}{\rho} - \frac{\alpha}{\lfloor k(\rho+1)c_{\min}\rfloor - c_1} \\ &\ge \frac{1-\varepsilon'}{\rho} - \frac{\alpha}{\lfloor k\rho c_{\min}\rfloor + \lfloor kc_{\min}\rfloor - c_1} \\ &\overset{c_1\in\mathbb{N},\, c_1<kc_{\min}}{\ge} \frac{1-\varepsilon'}{\rho} - \frac{\alpha}{\lfloor k\rho c_{\min}\rfloor} \overset{(3.8)}{\ge} \frac{1}{\rho+\varepsilon}. \end{aligned}$$

At the beginning of this case we have already established that $c_2 > \lfloor k(\rho+1)c_{\min}\rfloor \ge kc_{\min}$. As $\tilde{c_1} = \frac{c_2}{k}$, we have $\tilde{c_1} \ge c_{\min}$.

In both cases, we have defined $\tilde{X}$ such that

$$d(\tilde{c_1}) \ge \frac{1}{\rho+\varepsilon} \tag{3.17}$$

and

$$\tilde{c_1} \ge c_{\min}. \tag{3.18}$$

To show that $\tilde{X}$ has a strict competitive ratio of $\rho + \varepsilon$, fix some $C \geq 0$. If $C \leq \tilde{c_1}$, we have

$$f(\tilde{X}(C)) = C \cdot d(\tilde{c_1}) \overset{(3.17)}{\geq} \frac{1}{\rho + \varepsilon} C \geq \frac{1}{\rho + \varepsilon} v(C).$$

Otherwise, if $C > \tilde{c_1}$,

$$\begin{aligned}
(\rho + \varepsilon) f_{\mathrm{c}}(\tilde{X}(C)) &\geq (\rho + \varepsilon) f_{\mathrm{c}}\left(\tilde{X}\left(\frac{\lfloor kC \rfloor}{k}\right)\right) \\
&\overset{(3.11)}{=} (\rho + \varepsilon) \frac{1}{k} f(X(\lfloor kC \rfloor)) \\
&\geq \frac{(\rho + \varepsilon)}{\rho} \frac{1}{k} v_{\lfloor kC \rfloor} - \frac{(\rho + \varepsilon)\alpha}{k} \\
&\overset{(3.11)}{=} \frac{(\rho + \varepsilon)}{\rho} v\left(\frac{\lfloor kC \rfloor}{k}\right) - \frac{(\rho + \varepsilon)\alpha}{k} \\
&\geq \frac{(\rho + \varepsilon)}{\rho} v\left(\frac{kC - 1}{k}\right) - \frac{(\rho + \varepsilon)\alpha}{k} \\
&= \frac{(\rho + \varepsilon)}{\rho} v\left(C - \frac{1}{k}\right) - \frac{(\rho + \varepsilon)\alpha}{k} \\
&\overset{(3.6)}{\geq} v(C),
\end{aligned}$$

where for the last inequality we use the fact that, by (3.18), we have $C > c_{\text{min}}$. □

Proposition 3.12 implies that, instead of devising a lower bound for the (non-strict) competitive ratio of the IncMaxSep problem, we can construct a lower bound for the strict competitive ratio of the IncMaxCont problem.

Note that it is not clear whether the (non-)strict competitive ratio of IncMaxSep and the strict competitive ratio of IncMaxCont coincide. This is due to the fact that an incremental solution to the IncMaxCont problem may add fractional elements while an incremental solution to the IncMaxSep problem may only add an integral number of items. There are even discrete instances where every continuization of the instance has a competitive ratio smaller than the initial instance.

**Observation 3.13.** *There exists an instance of IncMaxSep that has a strict competitive ratio that is strictly larger than that of every instance of IncMaxCont that monotonically interpolates the IncMaxSep instance, i.e., with $v(i) = v_i$ for all $i \in \mathbb{N}$.*

*Proof.* Consider the instance of IncMaxSep with $N = 16$ sets and

$$\begin{aligned} d_1 &= 1, \\ d_3 = d_4 &= \frac{17}{40}, \\ d_{12} = d_{13} = d_{14} = d_{15} = d_{16} &= \frac{16473}{107200}. \end{aligned}$$

For $i \in \{2,5,6,7,8,9,10,11\}$, we choose $d_i$ such that $i \cdot d_i = v_i = v_{i-1} = (i-1)d_{i-1}$. We show that every incremental solution represented by a sequence $(c_1, c_2, \dots)$ has a competitive ratio of at least $1.446$ for this problem instance. If $c_1 \geq 2$, then $d_{c_1} \leq \frac{1}{2}$, i.e., for cardinality $1$, the solution has value $d_{c_1} \leq \frac{1}{2}$ which implies that the incremental solution has a competitive ratio of at least $2$. Thus assume that $c_1 = 1$. If $c_2 \geq 5$, we can, without loss of generality, assume that $c_2 \geq 12$. Otherwise we can improve the incremental solution by choosing $c_2 = 4$ instead. Then, the value of the solution for cardinality $4$ is $\max\{1, 3 \cdot d_{c_2}\} = \max\{1, 3 \cdot \frac{16473}{107200}\} = 1$, while the optimum solution has value $4d_4 = \frac{17}{10}$, i.e., the competitive ratio of the incremental solution is at least $1.7$. Without loss of generality, we can assume that $v_{c_2} > v_{c_1}$, i.e., that $c_2 \geq 3$. It remains to consider the case that $c_2 \in \{3,4\}$. We can assume that $d_{c_3} < d_{c_2}$ because otherwise, we could improve the incremental solution by removing $c_2$. Thus, and because $v_{c_3} > v_{c_2}$, we have $c_3 \geq 12$, i.e., $d_{c_3} = \frac{16473}{107200}$. For cardinality $4c_2$, the value of the solution is

$$\max\{\frac{17}{40}c_2, (4c_2 - 1 - c_2)\frac{16473}{107200}\} = \frac{17}{40}c_2$$

and the optimum solution has value at least $4c_2 \cdot \frac{16473}{107200}$. Thus, the competitive ratio is $4c_2 \cdot \frac{16473}{107200} / (\frac{17}{40}c_2) = \frac{969}{670} > 1.446$.

Now, we consider an instance of IncMaxCont with $d(i) = d_i$ for all $i \in [16]$. Let $\rho = \frac{57}{40} = 1.425$. We show that the incremental solution $(c_1, c_2, c_3) = (\frac{1}{\rho}, 4, 12 - \frac{1}{\rho})$ is $\rho$-competitive. Note that $c_1 \geq \frac{1}{\rho}$. As we will see in Lemma 3.14, it suffices to show that

$$d(c_i) \geq \frac{v(c_{i-1})}{p(c_{i-1}) - \sum_{j=1}^{i-1} c_j} \tag{3.19}$$

for $i \in \{2,3\}$. We have $p(c_1) \leq \frac{\rho v(c_1)}{d_3} = \frac{57}{17} v(c_1)$ and thus

$$d(c_2) = d(4) = d_4 = \frac{17}{40} = \frac{1}{\frac{57}{17} - 1} \overset{d(c_1) \geq d_1 = 1}{\geq} \frac{1}{\frac{57}{17} - \frac{1}{d(c_1)}} \overset{v(c_1) = c_1 d(c_1)}{=} \frac{v(c_1)}{p(c_1) - c_1}.$$

We have $p(c_2) = p(4) = \frac{\rho v(4)}{d_{12}} = \frac{\frac{57}{40} \cdot 4 \cdot \frac{17}{40}}{\frac{16473}{107200}} = \frac{268}{17}$ and thus

$$d(c_3) \overset{c_3<12}{\geq} d_{12} = \frac{16473}{107200} = \frac{4 \cdot \frac{17}{40}}{\frac{268}{17} - \frac{40}{57} - 4} = \frac{v(c_2)}{p(c_2) - c_1 - c_2}.$$

Therefore, (3.19) holds for $i \in \{2, 3\}$ and thus, the incremental solution $(c_1, c_2, c_3)$ is $\rho$-competitive. $\square$

Note that, even though this shows that there are instances where the continuous problem is easier than the discrete one, this does not rule out that the competitive ratios of IncMaxSep and IncMaxCont coincide. This is due to the fact that the instance in the proof is not a worst-case instance.

### 3.2.1. Optimal Continuous Online Algorithm

In this section, we present an optimal algorithm to solve the IncMaxCont problem and analyze it. By giving a lower bound on the competitive ratio of such an optimal algorithm, one can derive a lower bound for the competitive ratio of the IncMaxCont problem. To get an idea what the algorithm does, consider the following lemma. It gives a characterization what it means for an incremental solution $(c_1, c_2, \dots)$ to be $\rho$-competitive, depending on $(c_1, c_2, \dots)$, $v$ and $d$.

**Lemma 3.14.** *Let $X = (c_1, c_2, \dots)$ be an incremental solution for an instance of the IncMaxCont problem. The following are equivalent:*

*(i) $X$ is $\rho$-competitive.*

*(ii) We have $d(c_1) \geq \frac{1}{\rho}$ and, for all $i \in \mathbb{N}$, $d(c_{i+1}) \geq \frac{v(c_i)}{p(c_i) - \sum_{j=1}^{i} c_j}$.*

*(iii) We have $d(c_1) \geq \frac{1}{\rho}$, and, for all $i \in \mathbb{N}$, $p(c_i) > \sum_{j=1}^{i} c_j$ and $d(c_{i+1}) \geq \frac{v(c_i)}{p(c_i) - \sum_{j=1}^{i} c_j}$.*

*Proof.* $(i) \Rightarrow (iii)$: Since $X$ is $\rho$-competitive, for all $C \geq 0$, we have $f(X(C)) \geq v(C)/\rho$. If $d(c_1) < \frac{1}{\rho}$ was true, $X$ would not be $\rho$-competitive for all sizes $C \geq 0$ with $d(C) > \rho d(c_1)$, which exist because $d(0) = 1$. Thus, we have $d(c_1) \geq \frac{1}{\rho}$. Now suppose $p(c_i) < \sum_{j=1}^{i} c_j$. By definition of $p$ and monotonicity of $v$, we know that

$$\rho v(c_i) = v(p(c_i)) < v\left(\sum_{j=1}^{i} c_j\right),$$

which means that $X$ is not $\rho$-competitive for size $\sum_{j=1}^{i} c_i$. This is a contradiction and thus we have $p(c_i) \geq \sum_{j=1}^{i} c_j$. Suppose $p(c_i) = \sum_{j=1}^{i} c_j$. Let $x \in \left(0, \frac{v(c_i)}{d(c_{i+1})}\right)$. We have

$f(X(p(c_i)+x)) = v(c_i)$ but, by definition of $p$ and monotonicity of $v$, we know

$$\rho v(c_i) = v(p(c_i)) < v\left(\left(\sum_{j=1}^{i} c_j\right) + x\right),$$

and thus we have $p(c_i) > \sum_{j=1}^{i} c_j$. Assume that

$$d(c_{i+1}) < \frac{v(c_i)}{p(c_i) - \sum_{j=1}^{i} c_j}.$$

We established $p(c_i) > \sum_{j=1}^{i} c_j$ and because $\big(p(c_i) - \sum_{j=1}^{i} c_j\big)d(c_{i+1}) < v(c_i)$, we have $f(X(p(c_i))) = v(c_i) = v(p(c_i))/\rho$. Furthermore, for the same reason there is $\varepsilon > 0$ with $\big(p(c_i) + \varepsilon - \sum_{j=1}^{i} c_j\big)d(c_{i+1}) < v(c_i)$. This implies that

$$f(X(p(c_i)+\varepsilon)) = \max\left\{\left(p(c_i) + \varepsilon - \sum_{j=1}^{i} c_j\right)d(c_{i+1}), v(c_i)\right\} = v(c_i).$$

Yet, by definition of $p$, $v(p(c_i)+\varepsilon) > \rho v(c_i)$ holds and thus $X$ is not $\rho$-competitive. This is a contradiction, i.e., $(iii)$ must hold.

$(iii) \Rightarrow (i)$: Suppose $(iii)$ holds but $X$ was not $\rho$-competitive. Then there exist $C \geq 0$ and $\varepsilon > 0$ such that $X$ is $\rho$-competitive for all sizes in $[0, C]$ and not $\rho$-competitive for all sizes in $(C, C+\varepsilon]$ because $v(c)$ and $f(X(c))$ are both continuous in $c$. Let $i \in \mathbb{N}$ and $0 < x \leq c_i$ such that $C = \big(\sum_{j=1}^{i-1} c_j\big) + x$. If $i = 1$, we have

$$X(C) = xd(c_1) \geq \frac{1}{\rho}x \geq \frac{1}{\rho}xd(x) = \frac{1}{\rho}v(x).$$

This is a contradiction to the fact that $X$ is not $\rho$-competitive for size $C$ and therefore we have $i \geq 2$. Assume that $x = c_i$. Then, $f(X(C)) = v(c_i)$ holds. For all $0 < x' \leq \min\{\varepsilon, \frac{v(c_i)}{d(c_{i+1})}\}$, we have $f(X(C+x')) = v(c_i)$ and, by definition of $C$, $\varepsilon$ and $x'$, $X$ is not $\rho$-competitive for size $C + x'$, i.e., $v(C+x') > \rho v(c_i)$. Since this holds for arbitrarily small $x' > 0$ and because $\rho$-competitiveness for size $C$ of $X$ implies $v(C) \leq \rho v(c_i)$, we know that $p(c_i) = C = \sum_{j=1}^{i} c_j$, which is a contradiction to $(iii)$ and thus $x \neq c_i$, i.e., $x < c_i$. Let $x' \in (x, \min\{c_i, x+\varepsilon\})$ be chosen arbitrarily and let $C' := \big(\sum_{j=1}^{i-1} c_j\big) + x'$. Now suppose that

$$xd(c_i) < v(c_{i-1}). \tag{3.20}$$

Then we have $f(X(C)) = v(c_{i-1})$ and since $X$ is $\rho$-competitive for size $C$, we have $C \leq p(c_{i-1})$. But since $C' > p(c_{i-1})$ for any $x' > x$, i.e., for any $C' > C$, we have

$C = p(c_{i-1})$. Non-negativity of $v$ and $(iii)$ imply $p(c_{i-1}) - \sum_{j=1}^{i-1} c_j \geq 0$ and thus

$$x \overset{(3.20)}{<} \frac{v(c_{i-1})}{d(c_i)} \overset{(iii)}{\leq} p(c_{i-1}) - \sum_{j=1}^{i-1} c_j$$

or, equivalently, $C = \left(\sum_{j=1}^{i-1} c_j\right) + x < p(c_{i-1})$. This is a contradiction and therefore (3.20) does not hold, i.e., we have $xd(c_i) \geq v(c_{i-1})$. Thus, we have $f(X(C)) = xd(c_i)$. Since $X$ is $\rho$-competitive for size $C$, we have

$$v(C) \leq \rho \cdot xd(c_i). \tag{3.21}$$

This implies

$$d(C) = \frac{v(C)}{C} \overset{(3.21)}{\leq} \rho \frac{x}{C} d(c_i) \leq \rho d(c_i). \tag{3.22}$$

We can conclude

$$\begin{aligned} v(C') &= C'd(C') \overset{d \text{ non-inc.}}{\leq} C'd(C) \\ &= Cd(s) + (C' - C)d(C) \\ &= v(C) + (x' - x)d(C) \\ &\overset{(3.20),(3.22)}{\leq} \rho \cdot xd(c_i) + (x' - x)\rho d(c_i) \\ &= \rho \cdot x'd(c_i), \end{aligned}$$

which is a contradiction to the fact that $X$ is not $\rho$-competitive for size $C'$ and therefore $(i)$ must hold.

$(iii) \Rightarrow (ii)$: This follows immediately.

$(ii) \Rightarrow (iii)$: Suppose $(ii)$ holds. We have to show that $p(c_i) > \sum_{j=1}^{i} c_j$ for all $i \in \mathbb{N}$. The rest of $(iii)$ follows immediately from $(ii)$. We will prove that this is the case by induction on $i$. For $i = 1$ we have $p(c_1) > c_1$ by definition of $p$, continuity of $v$ and the fact that $c_1 > 0$. Now suppose

$$p(c_i) > \sum_{j=1}^{i} c_j \tag{3.23}$$

holds for some $i \in \mathbb{N}$. If $p(c_i) \geq \sum_{j=1}^{i+1} c_j$, then (3.23) holds for $i+1$ because $p(c_{i+1}) > p(c_i)$. So suppose

$$p(c_i) < \sum_{j=1}^{i+1} c_j. \tag{3.24}$$

In that case, we have

$$
\begin{aligned}
v\left(\sum_{j=1}^{i+1} c_j\right) = \left(\sum_{j=1}^{i+1} c_j\right) \cdot d\left(\sum_{j=1}^{i+1} c_j\right) &\overset{(3.24)}{\leq} \left(\sum_{j=1}^{i+1} c_j\right) \cdot d(p(c_i)) \\
= \quad & p(c_i)d(p(c_i)) + \left(\left(\sum_{j=1}^{i+1} c_j\right) - p(c_i)\right) d(p(c_i)) \\
= \quad & v(p(c_i)) + \left(\left(\sum_{j=1}^{i+1} c_j\right) - p(c_i)\right) d(p(c_i)) \\
\overset{\text{def of } p}{=} \quad & \rho v(c_i) + \left(\left(\sum_{j=1}^{i+1} c_j\right) - p(c_i)\right) \frac{\rho v(c_i)}{p(c_i)} \\
\overset{(3.23)}{<} \quad & \rho \cdot \left(p(c_i) - \sum_{j=1}^{i} c_j\right) \frac{v(c_i)}{p(c_i) - \sum_{j=1}^{i} c_j} + \left(\left(\sum_{j=1}^{i+1} c_j\right) - p(c_i)\right) \frac{\rho v(c_i)}{p(c_i) - \sum_{j=1}^{i} c_j} \\
= \quad & \rho \cdot c_{i+1} \frac{v(c_i)}{p(c_i) - \sum_{j=1}^{i} c_j} \overset{(ii)}{\leq} \rho \cdot c_{i+1} d(c_{i+1}) = \rho v(c_{i+1}) = v(p(c_{i+1})).
\end{aligned}
$$

Since $v$ is increasing and continuous, this implies that $p(c_{i+1}) > \sum_{j=1}^{i+1} c_j$. □

The intuition behind the fraction

$$\frac{v(c_i)}{p(c_i) - \sum_{j=1}^{i} c_j}$$

is the following: The value of the partial incremental solution $(c_1, \ldots, c_{i-1}, c_i)$ is $v(c_i)$ and this value is $\rho$-competitive up to size $p(c_i)$. The total size required for this partial incremental solution is $\sum_{j=1}^{i} c_j$. Thus, in order to stay competitive, the size of the optimum solution added next, namely $c_{i+1}$, needs to be chosen such that $\big(p(c_i) - \sum_{j=1}^{i} c_j\big) d(c_{i+1}) \geq v(c_i)$, i.e., the density $d(c_{i+1})$ has to be large enough such that the value of the solution of size $p(c_i)$ is $\big(p(c_i) - \sum_{j=1}^{i} c_j\big) d(c_{i+1})$.

We use this fraction to define an algorithm for solving the IncMaxCont Problem. For the algorithm, we assume that $v$ is strictly increasing and $d$ is strictly decreasing to make the definition of our algorithm unique. Every instance of IncMaxCont can be transformed to satisfy this with an arbitrarily small loss by simply "tilting" constant parts of $d$ and $v$ slightly. The algorithm GreedyScaling$(c_1, \rho)$ starts by adding the optimum solution of

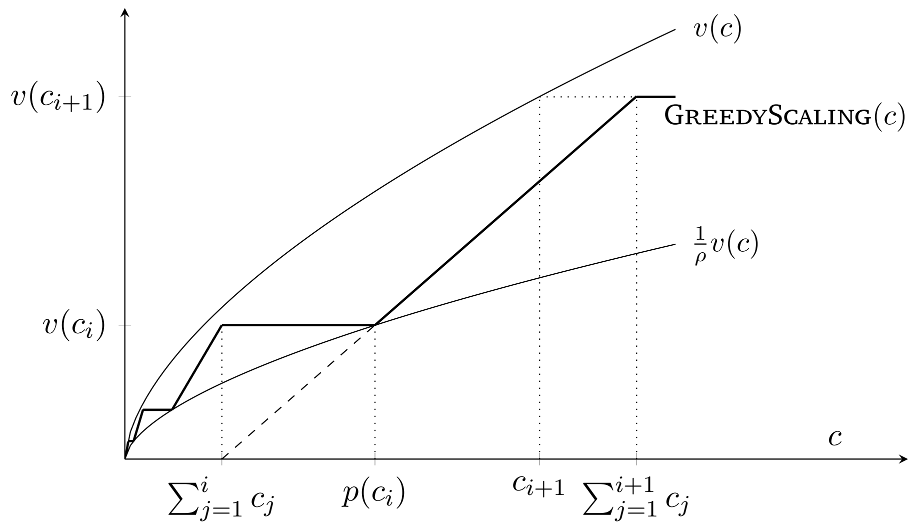


Figure 3.2.: Illustration of GreedyScaling$(c_1, \rho)$. Between size $\sum_{j=1}^{i} c_j$ and size $\sum_{j=1}^{i+1} c_j$, the algorithm adds the optimum solution of size $c_{i+1}$. This size is chosen in a way such that the value of the partially added optimum solution of size $c_{i+1}$ has value $v(c_i)$ exactly at size $p(c_i)$, i.e., when the previously added optimum solution of size $c_i$ loses $\rho$-competitiveness.

size $c_1 > 0$ and iteratively chooses the size $c_{i+1}$ such that

$$d(c_{i+1}) = \frac{v(c_i)}{p(c_i) - \sum_{j=1}^{i} c_j}, \tag{3.25}$$

i.e., as large as possible while still satisfying the inequality in Lemma 3.14. The incremental solution is given by $(c_1, c_2, \dots)$. An illustration of the algorithm can be found in Figure 3.2.

Using the definition of the algorithm in (3.25) and Lemma 3.14, we are able to prove the following.

**Proposition 3.15.** *The algorithm* GreedyScaling$(c_1, \rho)$ *is $\rho$-competitive if and only if $d(c_1) \geq \frac{1}{\rho}$ and $c_i < c_{i+1}$ for all $i \in \mathbb{N}$.*

*Proof.* Let $X = (c_1, c_2, \dots)$ denote the solution of GreedyScaling$(c_1, \rho)$.

"$\Leftarrow$": If $c_i < c_{i+1}$ for all $i \in \mathbb{N}$ and $d(c_1) \geq \frac{1}{\rho}$, we can simply apply Lemma 3.14 and obtain that $X$ is $\rho$-competitive.

"$\Rightarrow$": If $d(c_1) < \frac{1}{\rho}$, Lemma 3.14 yields that $X$ is not $\rho$-competitive. Now, suppose that $c_{k+1} \leq c_k$ for some $k \in \mathbb{N}$. If $c_i \leq 0$ for some $i \in \mathbb{N}$, then $X$ is not valid and thus not $\rho$-competitive. Thus, assume that $c_i > 0$ for all $i \in \mathbb{N}$. We will now iteratively show that, for all $i \in \{k, k+1, \dots\}$, we have $c_{i+1} < c_i$. For this, suppose that $c_{i+1} \leq c_i$. Then

$$\begin{aligned} d(c_{i+2}) &= \frac{v(c_{i+1})}{p(c_{i+1}) - \sum_{j=1}^{i+1} c_j} = \frac{1}{\frac{\rho}{d(p(c_{i+1}))} - \frac{1}{v(c_{i+1})}\sum_{j=1}^{i+1} c_j} \\ &\overset{c_{i+1}\leq c_i}{\geq} \frac{1}{\frac{\rho}{d(p(c_i))} - \frac{1}{v(c_i)}\sum_{j=1}^{i+1} c_j} > \frac{1}{\frac{\rho}{d(p(c_i))} - \frac{1}{v(c_i)}\sum_{j=1}^{i} c_j} \\ &= \frac{v(c_i)}{p(c_i) - \sum_{j=1}^{i} c_j} = d(c_{i+1}). \end{aligned}$$

Because $d$ is non-increasing, we have $c_{i+2} < c_{i+1}$. By an iterative argument it follows that, for all $i \in \{k, k+1, \dots\}$, we have $c_{i+1} < c_i$. This implies that the value of $X$ is smaller or equal to $v(c_k)$ for all sizes. Yet, for large sizes $C \in \mathbb{N}$, we have $v(C) > \rho v(c_k)$ as $\lim_{c\to\infty} v(c) = \infty$. □

To show that GreedyScaling$(c_1, \rho)$ with the correct choice of $c_1$ and $\rho$ computes the best-possible incremental solution, we need the following lemma. It states that if it is possible to find a $\rho$-competitive incremental solution for the problem, then, for any $k \in \mathbb{N}$, there exists an incremental solution such that we cannot reduce any of the sizes in any prefix with a combined size of up to $k$ without losing $\rho$-competitiveness. The idea behind its proof is the following. We start with some $\rho$-competitive incremental solution and iteratively reduce the sizes until we converge to some incremental solution that satisfies the sought property. To avoid running into some sequence starting with 0's, we start our search with an incremental solution with a minimal number of chosen sizes up to a combined size of $k$.

**Lemma 3.16.** *Let $v$ be strictly increasing, let $k \in \mathbb{N}$, and let $\rho \geq 1$ such that there exists a $\rho$-competitive incremental solution. Then there exists a $\rho$-competitive incremental solution $(c_1^*, c_2^*, \dots)$ such that, with $n := \min\{\ell \in \mathbb{N} \mid \sum_{i=1}^{\ell} c_i^* \geq k\}$, there exists no other $\rho$-competitive incremental solution $(c_1', c_2', \dots)$ with $c_i' \leq c_i^*$ for all $i \in [n-1]$, $c_i' = c_i^*$ for all $i \in \mathbb{N}_{\geq n}$ and $c_i' < c_i^*$ for at least one $i \in [n-1]$. Furthermore, there is some $C > 0$ that is independent from $k$ such that $c_1^* > C$.*

*Proof.* We fix a $\rho$-competitive incremental solution $(c_1, c_2, \dots)$ with $c_1 < c_2 < \dots$. We define $n := \min\{\ell \in \mathbb{N} \mid \sum_{i=1}^{\ell} c_i \geq k\}$. For $i \in \mathbb{N}$, let $\mathcal{S}_i$ be the set of all $\rho$-competitive incremental solutions $(c_1', c_2', \dots)$ with $c_{n+j} = c_{i+j}'$ for all $j \in \mathbb{N} \cup \{0\}$. Furthermore, let

$m := \min\{i \in \mathbb{N} \mid \mathcal{S}_i \neq \emptyset\}$. This value exists since $(c_1, c_2, \dots) \in \mathcal{S}_n$. For every incremental solution $(c'_1, c'_2, \dots) \in \mathcal{S}_m$, we have

$$0 < c'_1 < c'_2 < \dots < c'_m = c_n \tag{3.26}$$

because otherwise it would be possible to skip a size, which is a contradiction to the minimality of $m$. For every $(c'_1, c'_2, \dots) \in \mathcal{S}_m$, we define the set

$$\mathcal{S}_m((c'_1, c'_2, \dots)) := \{(c''_1, c''_2, \dots) \in \mathcal{S}_m \mid c''_i \leq c'_i \; \forall i \in \mathbb{N}\}.$$

To prove the lemma, it suffices to show that there exists some incremental solution $(c^*_1, c^*_2, \dots)$ with

$$\mathcal{S}_m((c^*_1, c^*_2, \dots)) = \{(c^*_1, c^*_2, \dots)\}. \tag{3.27}$$

It is easy to see that, for every $(c''_1, c''_2, \dots) \in \mathcal{S}_m((c'_1, c'_2, \dots))$, we have

$$\mathcal{S}_m((c''_1, c''_2, \dots)) \subseteq \mathcal{S}_m((c'_1, c'_2, \dots)). \tag{3.28}$$

For $(c'_1, c'_2, \dots) \in \mathcal{S}_m$, we define

$$s_m((c'_1, c'_2, \dots)) := \inf\left\{ \sum_{i=1}^{m} c''_i \;\middle|\; (c''_1, c''_2, \dots) \in \mathcal{S}_m((c'_1, c'_2, \dots)) \right\}.$$

Since (3.26) holds, this value is larger than 0 and smaller than $\sum_{j=1}^{n} c_j$, and therefore exists.

We fix some incremental solution $(c^1_1, c^1_2, \dots) \in \mathcal{S}_m$ and recursively define a sequence of incremental solutions such that, for all $i, j \in \mathbb{N}$, we have $(c^{j+1}_1, c^{j+1}_2, \dots) \in \mathcal{S}_m((c^j_1, c^j_2, \dots))$, and such that

$$\left( \sum_{i=1}^{m} c^{j+1}_i \right) - s_m((c^j_1, c^j_2, \dots)) \leq \left( \frac{1}{2} \right)^j. \tag{3.29}$$

This sequence exists because the infimum can be approximated arbitrarily close.

*Claim 1:* The limit $(c^*_1, c^*_2, \dots) := \lim_{j\to\infty} (c^j_1, c^j_2, \dots)$ exists.

*Proof of Claim 1:* The sequence $\big(s_m((c^j_1, c^j_2, \dots))\big)_{j\in\mathbb{N}}$ is increasing because of (3.28), and the sequence $\big(\sum_{i=1}^{m} c^j_i\big)_{j\in\mathbb{N}}$ is decreasing because $(c^{j+1}_1, c^{j+1}_2, \dots) \in \mathcal{S}_m((c^j_1, c^j_2, \dots))$. Furthermore,

$$s_m((c^j_1, c^j_2, \dots)) \leq \sum_{i=1} c^j_i$$

and (3.29) imply

$$\lim_{j\to\infty} s_m((c^j_1, c^j_2, \dots)) = \lim_{j\to\infty} \sum_{i=1} c^j_i.$$

Because $c_i^{j+1} \leq c_i^j$ for all $i, j \in \mathbb{N}$, the sequence $\big((c_1^j, c_2^j, \dots)\big)_{j\in\mathbb{N}}$ converges to some incremental solution $(c_1^*, c_2^*, \dots)$ with

$$\sum_{i=1}^{m} c_i^* = \lim_{j\to\infty} \sum_{i=1}^{m} c_i^j = \lim_{j\to\infty} s_m((c_1^j, c_2^j, \dots)). \tag{3.30}$$

*Claim 2:* We have $(c_1^*, c_2^*, \dots) \in \mathcal{S}_m$.

*Proof of Claim 2:* For every $j \in \mathbb{N}$, we have $c_{m+\ell}^j = c_{n+\ell}$ for all $\ell \in \mathbb{N} \cup \{0\}$ and therefore also $c_{m+\ell}^* = c_{n+\ell}$ for all $\ell \in \mathbb{N} \cup \{0\}$. So, it remains to prove that the incremental solution $(c_1^*, c_2^*, \dots)$ is $\rho$-competitive. For all $j \in \mathbb{N}$, we have $c_1^j \geq \frac{1}{\rho}$ and thus $c_1^* = \lim_{j\to\infty} c_1^j \geq \frac{1}{\rho}$. Next, we show that

$$d(c_{i+1}^*) \geq \frac{v(c_i^*)}{p(c_i^*) - \sum_{\ell=1}^{i} c_\ell^*}$$

holds for all $i \in \mathbb{N}$. The function $p$ is continuous by continuity and strict monotonicity of $v$. Continuity of $v$ and $p$ imply that $v(c_i')/(p(c_i') - \sum_{\ell=1}^{i} c_\ell')$ is continuous in $c_1', \dots, c_i'$. By Lemma 3.14, we have

$$d(c_{i+1}^j) \geq \frac{v(c_i^j)}{p(c_i^j) - \sum_{\ell=1}^{i} c_\ell^j}$$

for all $j \in \mathbb{N}$ because $(c_1^j, c_2^j, \dots)$ is $\rho$-competitive. Both sides of this inequality are continuous in $c_1^j, c_2^j, \dots$, and we have $\lim_{j\to\infty} c_i^j = c_i^*$. Those two facts imply that

$$d(c_{i+1}^*) \geq \frac{v(c_i^*)}{p(c_i^*) - \sum_{\ell=1}^{i} c_\ell^*}$$

holds. To prove $\rho$-competitiveness of $(c_1^*, c_2^*, \dots)$, by Lemma 3.14, it remains to show that $0 < c_1^* < c_2^* < \dots$ holds. We know that this holds for all incremental solutions $(c_1^j, c_2^j, \dots)$, $j \in \mathbb{N}$. Therefore, we have $0 \leq c_1^* \leq \dots \leq c_m^* < c_{m+1}^* < \dots$. Bit if $c_i^* = c_{i+1}^*$ for some $i \in [m-1]$, then we could remove $c_{i+1}$ and would still be left with a $\rho$-competitive solution. This would be a contradiction to the minimality of $m$. Therefore, we have $0 < c_1^* < c_2^* < \dots$, which concludes the proof of Claim 2.

We established $(c_1^*, c_2^*, \dots) \in \mathcal{S}_m$ and therefore $(c_1^*, c_2^*, \dots) \in \mathcal{S}_m((c_1^j, c_2^j, \dots))$, which implies that $\mathcal{S}_m((c_1^*, c_2^*, \dots)) \subseteq \mathcal{S}_m((c_1^j, c_2^j, \dots))$. Combined with the fact that we have $(c_1^*, c_2^*, \dots) \in \mathcal{S}_m((c_1^*, c_2^*, \dots))$, this implies

$$\sum_{i=1}^{m} c_i^* \geq s_m((c_1^*, c_2^*, \dots)) \geq \lim_{j\to\infty} s_m((c_1^j, c_2^j, \dots)) \overset{(3.30)}{=} \sum_{i=1}^{m} c_i^*,$$

i.e., $s_m((c_1^*, c_2^*, \dots)) = \sum_{i=1}^m c_i^*$. Thus, (3.27) holds.

It remains to show that there is some $C > 0$ that is independent from $k$ such that $c_1^* > C$. Suppose the contrary, i.e., that $c_1^*$ is not bounded from $0$ for varying values of $k$. Let $\varepsilon > 0$ be small enough such that $\frac{\rho}{1-\varepsilon} - 1 \leq \rho$. Furthermore, let $k \in \mathbb{N}$ such that $d(p(c_1^*)) \geq 1 - \varepsilon$, which is possible because $d(0) = 1$, $d$ is continuous, and $c_1^*$ is not bounded from $0$. By minimality of $m$ and Lemma 3.14, we have

$$\frac{1}{\rho} > d(c_2^*) \geq \frac{v(c_1^*)}{p(c_1^*) - c_1^*} = \frac{1}{\frac{\rho}{d(p(c_1^*))} - \frac{1}{d(c_1^*)}} \geq \frac{1}{\frac{\rho}{1-\varepsilon} - 1} \geq \frac{1}{\rho}$$

which is a contradiction. Thus, $c_1^*$ is bounded from $0$. □

Using this lemma, we can show that GreedyScaling$(c_1, \rho)$ for the correct choice of $c_1$ and $\rho$ can achieve every possible competitive ratio.

**Lemma 3.17.** *Let $v$ be strictly increasing and $d$ be strictly decreasing, and let $\rho \geq 1$ such that there exists a $\rho$-competitive incremental solution. Then, there exists a starting value $c_1^* \in \left[d^{-1}\left(\frac{\rho-1}{\rho}\right), d^{-1}\left(\frac{1}{\rho}\right)\right]$ such that GreedyScaling$(c_1^*, \rho)$ is $\rho$-competitive.*

*Proof.* By Lemma 3.16, for every $k \in \mathbb{N}$, there exists a $\rho$-competitive incremental solution $(c_1^k, c_2^k, \dots)$ such that, with $n(k) := \min\{\ell \in \mathbb{N} \mid \sum_{i=1}^{\ell} c_i \geq k\}$, there exists no other $\rho$-competitive incremental solution $(c'_1, c'_2, \dots)$ with $c'_i \leq c_i^k$ for all $i \in [n(k) - 1]$, $c'_i = c_i^k$ for all $i \in \mathbb{N}_{\geq n(k)}$ and $c'_i < c_i^k$ for at least one $i \in [n(k) - 1]$. Furthermore, there is $C > 0$ such that $c_1^k \geq C$ for all $k \in \mathbb{N}$. Without loss of generality, we can assume that $v(c_{i+1}^k) \geq v(c_i^k)$ for all $i \in \mathbb{N}$. Because $(c_1^k, c_2^k, \dots)$ is $\rho$-competitive, by Lemma 3.14, we know that, for all $i \in \mathbb{N}$, we have

$$d(c_{i+1}^k) \geq \frac{v(c_i^k)}{p(c_i^k) - \sum_{j=1}^i c_j^k}. \tag{3.31}$$

Suppose there was $i' \in [n(k) - 1]$ such that (3.31) does not hold with equality. By continuity of $v$, $d$ and $p$, we can find $c'_{i'} < c_{i'}^k$ such that

$$d(c_{i'+1}^k) > \frac{v(c'_{i'})}{p(c'_{i'}) - c'_{i'} - \sum_{j=1}^{i'-1} c_j} > \frac{v(c_{i'}^k)}{p(c_{i'}^k) - \sum_{j=1}^{i'} c_j^k}. \tag{3.32}$$

The incremental solution $(c'_1, c'_2, \dots) := (c_1^k, \dots, c_{i'-1}^k, c'_{i'}, c_{i'+1}^k, \dots)$ satisfies

$$d(c'_{i+1}) \geq \frac{v(c'_i)}{p(c'_i) - \sum_{j=1}^i c'_j}$$

for all $i \in \mathbb{N}$. For $i \in [i'-1]$, this follows immediately from (3.31), for $i = i'$, this follows from (3.32), and, for $i \in \{i'+1, i'+2, \dots\}$, this is due to (3.31) and the fact that $c'_{i'} < c^k_{i'}$. We have $c'_i \leq c^k_i$ for all $i \in [n(k)-1]$, $c'_i = c^k_i$ for all $i \in \{n(k), n(k)+1, \dots\}$ and $c'_{i'} < c^k_{i'}$, which is a contradiction to our initial choice of $(c^k_1, c^k_2, \dots)$. Thus, for all $i \in [n(k)-1]$, (3.31) holds with equality.

The sequence $\left(c^k_1\right)_{k\in\mathbb{N}}$ is bounded since $C \leq c^k_1 \leq d^{-1}(\frac{1}{\rho})$. Therefore, by the Bolzano-Weierstrass theorem, it contains a converging subsequence $\left(c^{k_\ell}_1\right)_{\ell\in\mathbb{N}}$ with $k_\ell \in \mathbb{N}$ and $k_{\ell+1} > k_\ell$ for all $\ell \in \mathbb{N}$. We define the limit $c^*_i := \lim_{\ell\to\infty} c^{k_\ell}_i$ for all $i \in \mathbb{N}$. We have $v(c^*_1) \geq v(d^{-1}(C)) > 0$ and $v(c^*_{i+1}) \geq v(c^*_i)$ by continuity of $v$ and because this holds for all sequences $\left(c^k_i\right)_{i\in\mathbb{N}}$. By Lemma 3.14, we have $d(c^k_1) \geq \frac{1}{\rho}$ and

$$d(c^k_{i+1}) \geq \frac{v(c^k_i)}{p(c^k_i) - \sum_{j=1}^i c^k_j}$$

for all $i, k \in \mathbb{N}$. Continuity of $d$, $v$ and $p$ yields $d(c^*_1) \geq \frac{1}{\rho}$ and

$$d(c^*_{i+1}) \geq \frac{v(c^*_i)}{p(c^*_i) - \sum_{j=1}^i c^*_j}.$$

Thus, the incremental solution $\left(c^*_1, c^*_2, \dots\right)$ is $\rho$-competitive. It remains to show that

$$c^*_{i+1} = d^{-1}\left(\frac{v(c^*_i)}{p(c^*_i) - \sum_{j=1}^i c^*_j}\right).$$

Note that $n(k)$ increases in $k$. This implies that for every $i \in \mathbb{N}$, there exists some $K \in \mathbb{N}$ such that

$$d(c^k_{i+1}) = \frac{v(c^k_i)}{p(c^k_i) - \sum_{j=1}^i c^k_j}$$

for all $k \in \mathbb{N}$ with $k \geq K$. Since $c^*_i = \lim_{\ell\to\infty} c^{k_\ell}_i$, the desired equality follows, i.e., the incremental solution $(c^*_1, c^*_2, \dots)$ is produced by the algorithm GreedyScaling$(c^*_1, \rho)$. □

This result immediately yields the following.

**Theorem 3.18.** *For every instance of IncMaxCont, there exists a starting value $c_1$ such that the algorithm GreedyScaling$(c_1, \rho^*)$ achieves the best-possible competitive ratio $\rho^* \geq 1$.*

For all starting values $c_1$ that are reasonably small, we are able to show an upper bound of $\varphi + 1$ on the competitive ratio of GreedyScaling$(c_1, \varphi + 1)$, where $\varphi$ is the golden ratio.

**Theorem 3.19.** *GreedyScaling*$(c_1, \varphi + 1)$ *is* $(\varphi + 1)$*-competitive if and only if* $d(c_1) \geq \frac{1}{\varphi+1}$.

*Proof.* "⇒": By Lemma 3.14, $d(c_1) \geq \frac{1}{\varphi+1}$ holds because the algorithm is $(\varphi+1)$-competitive.

"⇐": Let $(c_1, c_2, \dots)$ be the incremental solution produced by GreedyScaling$(c_1, \varphi+1)$. To show $(\varphi + 1)$-competitiveness, by Proposition 3.15, it suffices to show that $c_i \leq c_{i+1}$.

*Claim:* We have $c_{i+1} \geq (\varphi + 1)c_i$ for all $i \in \mathbb{N}$.

We have $p(c_i) = \max\{c \geq 0 \mid v(c) \leq (\varphi + 1)v(c_i)\}$. This implies $v(p(c_i)) = (\varphi + 1)v(c_i)$ by continuity of $v$, and thus

$$p(c_i) = \frac{v(p(c_i))}{d(p(c_i))} = \frac{(\varphi + 1)v(c_i)}{d(p(c_i))} \geq \frac{(\varphi + 1)v(c_i)}{d(c_i)} = (\varphi + 1)c_i, \tag{3.33}$$

where the inequality holds because $v(p(c_i)) = (\varphi + 1)v(c_i) > v(c_i)$, $v$ is non-decreasing, $d$ is non-increasing.

*Proof of claim:* We will prove the claim by induction. For $i = 1$, we have

$$\begin{aligned} d(c_2) &= \frac{v(c_1)}{p(c_1) - c_1} \overset{(3.33)}{\leq} \frac{v(c_1)}{p(c_1) - \frac{1}{\varphi+1}p(c_1)} = \frac{\varphi + 1}{\varphi} \cdot \frac{v(c_1)}{p(c_1)} \\ &= \frac{1}{\varphi} \cdot \frac{v(p(c_1))}{p(c_1)} = \frac{1}{\varphi} d(p(c_1)) < d(p(c_1)), \end{aligned}$$

Together with (3.33), this yields $c_2 \geq p(c_1) \geq (\varphi + 1)c_1$.

Let $i \in \mathbb{N}$ and suppose the claim holds for all $j \in [i]$. Then,

$$\begin{aligned} d(c_{i+2}) &= \frac{v(c_{i+1})}{p(c_{i+1}) - \sum_{j=1}^{i+1} c_j} \overset{\text{Lem. 1.6}}{<} \frac{v(c_{i+1})}{p(c_{i+1}) - \varphi c_{i+1}} \overset{(3.33)}{\leq} \frac{v(c_{i+1})}{p(c_{i+1}) - \frac{\varphi}{\varphi+1}p(c_{i+1})} \\ &= (\varphi + 1)\frac{v(c_{i+1})}{p(c_{i+1})} = \frac{v(p(c_{i+1}))}{p(c_{i+1})} = d(p(c_{i+1})), \end{aligned}$$

which implies $c_{i+1} > p(c_i)$ because $d$ is decreasing. Together with (3.33), this yields the claim. □

If we assume that the value function is concave, we are able to show an even better upper bound on the competitive ratio for GreedyScaling$(c_1, \rho)$.

**Theorem 3.20.** *Consider an instance of IncMaxCont with concave value function, and let* $\rho = 2.508$. *Then, GreedyScaling*$(c_1, \rho)$ *is* $\rho$*-competitive if and only if* $d(c_1) \geq \frac{1}{\rho}$.

*Proof.* "⇒": By Lemma 3.14, $d(c_1) \geq \frac{1}{\rho}$ holds because the algorithm is $\rho$-competitive.

"$\Leftarrow$": Let $(c_1, c_2, \dots)$ be the incremental solution produced by GreedyScaling$(c_1, \rho)$. To show $\rho$-competitiveness, by Proposition 3.15, it suffices to show that $c_i \leq c_{i+1}$. Let $\delta = 3.287$. We will show that, for all $i \in \mathbb{N}$, we have

$$c_{i+1} \geq \delta c_i, \tag{3.34}$$

which immediately yields the desired result. For all $i \in \mathbb{N}$, because $p(c_i) > c_i$, we have

$$p(c_i) = \frac{v(p(c_i))}{d(p(c_i))} = \frac{\rho v(c_i)}{d(p(c_i))} \geq \frac{\rho v(c_i)}{d(c_i)} = \rho c_i. \tag{3.35}$$

We show (3.34) by induction. For $i = 1$, we have

$$d(c_2) \overset{(3.25)}{=} \frac{v(c_1)}{p(c_1) - c_1} = \frac{v(c_1)}{\frac{\rho v(c_1)}{d(p(c_1))} - c_1} = \frac{d(p(c_1))}{\rho - \frac{d(p(c_1))}{d(c_1)}} \overset{p(c_1) \geq c_1}{\leq} \frac{d(p(c_1))}{\rho - 1} \overset{\rho \geq 2}{\leq} d(p(c_1)),$$

i.e., $c_2 \geq p(c_1)$. Thus,

$$c_2 = \frac{v(c_2)}{d(c_2)} \overset{c_2 \geq p(c_1)}{\geq} \frac{v(p(c_1))}{d(c_2)} \overset{(3.25)}{=} \frac{\rho v(c_1)}{\frac{v(c_1)}{p(c_1) - c_1}} = \rho(p(c_1) - c_1) \overset{(3.35)}{\geq} \rho(\rho - 1)c_1 > 3.77 c_1 > \delta c_1.$$

Now fix $i \in \mathbb{N}$ and suppose that, for all $j \in [i-1]$, (3.34) holds. For all $j \in [i]$, we have

$$\begin{aligned} d(c_{j+1}) &\overset{(3.25)}{=} \frac{v(c_j)}{p(c_j) - \sum_{\ell=1}^{j} c_\ell} \overset{\text{Lem. 1.6}}{<} \frac{v(c_j)}{p(c_j) - \frac{\delta}{\delta - 1} c_j} \overset{(3.35)}{\leq} \frac{1}{1 - \frac{\delta}{\delta - 1} \cdot \frac{1}{\rho}} \cdot \frac{v(c_j)}{p(c_j)} \\ &< 2.343 \frac{v(c_j)}{p(c_j)} < \frac{\rho v(c_j)}{p(c_j)} = \frac{v(p(c_j))}{p(c_j)} = d(p(c_j)), \end{aligned}$$

i.e., $c_{j+1} > p(c_j)$ because $d$ is non-increasing. Furthermore, for all $j \in [i]$,

$$c_{j+1} = \frac{v(c_{j+1})}{d(c_{j+1})} \overset{(3.25)}{=} \frac{v(c_{j+1})}{v(c_j)} \left( p(c_j) - \sum_{\ell=1}^{j} c_\ell \right). \tag{3.36}$$

Since we assume that $v$ is concave, we have

$$\rho v(c_i) = v(p(c_i)) \leq v(c_i) + \frac{v(c_i) - v(p(c_{i-1}))}{c_i - p(c_{i-1})} (p(c_i) - c_i),$$

which is equivalent to

$$p(c_i) - c_i \geq \frac{(\rho - 1) v(c_i)(c_i - p(c_{i-1}))}{v(c_i) - \rho v(c_{i-1})} = \frac{(\rho - 1)(c_i - p(c_{i-1}))}{1 - \rho \frac{v(c_{i-1})}{v(c_i)}}.$$

This yields

$$\begin{aligned}
p(c_i) \quad &\geq \quad \left(\frac{(\rho-1)\Big(1-\frac{p(c_{i-1})}{c_i}\Big)}{1-\rho\frac{v(c_{i-1})}{v(c_i)}}+1\right)c_i \\
&= \quad \left((\rho-1)\frac{\frac{v(c_i)}{v(c_{i-1})}-\frac{v(c_i)}{v(c_{i-1})}\cdot\frac{p(c_{i-1})}{c_i}}{\frac{v(c_i)}{v(c_{i-1})}-\rho}+1\right)c_i \\
&\overset{(3.36)}{=} \quad \left((\rho-1)\frac{\frac{v(c_i)}{v(c_{i-1})}-\frac{p(c_{i-1})}{p(c_{i-1})-\sum_{\ell=1}^{i-1}c_\ell}}{\frac{v(c_i)}{v(c_{i-1})}-\rho}+1\right)c_i \\
&= \quad \left((\rho-1)\frac{\frac{v(c_i)}{v(c_{i-1})}-\frac{1}{1-\frac{1}{p(c_{i-1})}\sum_{\ell=1}^{i-1}c_\ell}}{\frac{v(c_i)}{v(c_{i-1})}-\rho}+1\right)c_i.
\end{aligned}$$

By combining this with

$$\sum_{\ell=1}^{i-1}c_\ell \overset{\text{Lem. 1.6}}{<} \frac{\delta}{\delta-1}c_{i-1} \overset{(3.35)}{\leq} \frac{\delta}{\delta-1}\cdot\frac{1}{\rho}p(c_{i-1}),$$

we obtain

$$\frac{p(c_i)}{c_i} > (\rho-1)\frac{\frac{v(c_i)}{v(c_{i-1})}-\frac{1}{1-\frac{\delta}{\delta-1}\cdot\frac{1}{\rho}}}{\frac{v(c_i)}{v(c_{i-1})}-\rho}+1 = (\rho-1)\frac{\frac{v(c_i)}{v(c_{i-1})}-\frac{\rho(\delta-1)}{\delta\rho-\rho-\delta}}{\frac{v(c_i)}{v(c_{i-1})}-\rho}+1. \tag{3.37}$$

Furthermore, by Lemma 1.6 and the fact that $d$ is non-increasing, we have

$$\sum_{\ell=1}^{i}c_\ell < \frac{\delta}{\delta-1}c_{i-1}+c_i \leq \left(\frac{\delta}{\delta-1}\cdot\frac{v(c_{i-1})}{v(c_i)}+1\right)c_i. \tag{3.38}$$

Similar to the case that $i=1$, for $j\in[i]$, we have

$$\begin{aligned}
d(c_{j+1}) \quad &\overset{(3.25)}{=} \quad \frac{v(c_j)}{p(c_j)-\sum_{\ell=1}^{j}c_\ell} \overset{\text{Lem. 1.6}}{<} \frac{v(c_j)}{p(c_j)-\frac{\delta}{\delta-1}c_j} \\
&= \quad \frac{d(p(c_j))}{\rho-\frac{\delta}{\delta-1}\frac{d(p(c_j))}{d(c_j)}} \overset{p(c_j)\geq c_j}{\leq} \frac{d(p(c_j))}{\rho-\frac{\delta}{\delta-1}} < d(p(c_j))
\end{aligned}$$

Thus, $c_{j+1} \geq p(c_j)$, which yields

$$v(c_{j+1}) \geq v(p(c_j)) = \rho v(c_j). \tag{3.39}$$

We can conclude that

$$\begin{aligned}
\frac{c_{i+1}}{c_i} &\overset{(3.36)}{=} \frac{1}{c_i}\frac{v(c_{i+1})}{v(c_i)}\left(p(c_i) - \sum_{\ell=1}^{i} c_\ell\right) \\
&\overset{(3.39)}{\geq} \rho\left(\frac{p(c_i)}{c_i} - \frac{1}{c_i}\sum_{\ell=1}^{i} c_\ell\right) \\
&\overset{(3.37),(3.38)}{\geq} \rho\left(\left((\rho-1)\frac{\frac{v(c_i)}{v(c_{i-1})} - \frac{\rho(\delta-1)}{\delta\rho-\rho-\delta}}{\frac{v(c_i)}{v(c_{i-1})} - \rho} + 1\right) - \left(\frac{\delta}{\delta-1}\cdot\frac{v(c_{i-1})}{v(c_i)} + 1\right)\right) \\
&= \rho\left((\rho-1)\frac{\frac{v(c_i)}{v(c_{i-1})} - \frac{\rho(\delta-1)}{\delta\rho-\rho-\delta}}{\frac{v(c_i)}{v(c_{i-1})} - \rho} - \frac{\delta}{\delta-1}\cdot\frac{v(c_{i-1})}{v(c_i)}\right).
\end{aligned}$$

Since $\delta = 3.287$ and $\rho = 2.508$ are fixed, the only variable in this expression is the ratio $\frac{v(c_i)}{v(c_{i-1})}$. By (3.39), we have $\frac{v(c_i)}{v(c_{i-1})} \geq \rho$. Analyzing the function

$$\rho\left((\rho-1)\frac{x - \frac{\rho(\delta-1)}{\delta\rho-\rho-\delta}}{x-\rho} - \frac{\delta}{\delta-1}\cdot\frac{1}{x}\right)$$

for $x \geq \rho$ yields that it has a global minimum at $x \approx 4.285$ with a function value larger than $3.2871 > \delta$, i.e., we have $c_{i+1} \geq \delta c_i$. □

Since GreedyScaling$(c_1, \rho)$ with the correct starting value $c_1$ is the best-possible algorithm for a fixed instance, we can give a lower bound of $\rho > 1$ for the IncMaxCont problem by finding an instance that is a lower bound for GreedyScaling$(c_1, \rho)$ for all starting values $c_1 > 0$ that satisfy $d(c_1) \leq 1/\rho$. Theorem 3.20 implies that a lower bound instance where the value function is concave cannot give a lower bound larger than $2.508$. Since the best upper bound for the class IncMax$_{\text{acc}}$ is $\varphi + 1 \approx 2.618$, we aim to find better lower bounds than $2.508$. Thus, in the following, we will derive lower bound instances that are not convex, but rather some sort of step function, where we alternate between intervals of constant value function and constant density function (cf. Figure 3.4 later in the chapter).

We start by showing that, for every countable set $S \subseteq \mathbb{R}_{>0}$ of starting values, there is an instance where GreedyScaling$(c_1, \rho)$ cannot have a competitive ratio of better than $\varphi + 1$ for every $c_1 \in S$. In order to do this, we need the following lemma.

**Lemma 3.21.** *For $\alpha, \beta, \rho, \varepsilon \in \mathbb{R}_{\geq 0}$ with $\beta > 0$, consider the recursively defined sequence $(t_n)_{n\in\mathbb{N}}$ with*

$$t_0 = \beta, \qquad t_{n+1} = \frac{1}{\frac{\rho}{t_n(1-\varepsilon)} - \left(\sum_{j=0}^{n} \frac{(\rho+\varepsilon)^{j-n}}{t_j}\right) - \frac{\alpha}{(\rho+\varepsilon)^n}} \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

*If $1 < \rho < \varphi + 1$, then there exists $\varepsilon' > 0$ such that, for all $\varepsilon \in (0, \varepsilon']$, there is $\ell \in \mathbb{N}$ with $t_\ell < 0$.*

*Proof.* Rearranging terms, for all $n \in \mathbb{N}$, we obtain

$$t_0 = \beta, \qquad \frac{(\rho+\varepsilon)^{n+1}}{t_{n+1}} = \frac{\rho(\rho+\varepsilon)^{n+1}}{t_n(1-\varepsilon)} - \left(\sum_{j=0}^{n} \frac{(\rho+\varepsilon)^{j+1}}{t_j}\right) - \alpha(\rho+\varepsilon).$$

We substitute $a_n = 1/t_n$ for all $n \in \mathbb{N}$ and obtain the recursively defined sequence $(a_n)_{n\in\mathbb{N}}$ with $a_0 = 1/\beta$ and, for all $n \in \mathbb{N}$,

$$a_{n+1}(\rho+\varepsilon)^{n+1} = a_n \frac{\rho}{1-\varepsilon}(\rho+\varepsilon)^{n+1} - \left(\sum_{j=0}^{n} a_j(\rho+\varepsilon)^{j+1}\right) - \alpha(\rho+\varepsilon), \tag{3.40}$$

which implies

$$a_n(\rho+\varepsilon)^n = a_{n-1}\frac{\rho}{1-\varepsilon}(\rho+\varepsilon)^n - \left(\sum_{j=0}^{n-1} a_j(\rho+\varepsilon)^{j+1}\right) - \alpha(\rho+\varepsilon). \tag{3.41}$$

Subtracting (3.41) from (3.40), for all $n \in \mathbb{N}$, we obtain

$$a_{n+1}(\rho+\varepsilon)^{n+1} - a_n(\rho+\varepsilon)^n = a_n\frac{\rho}{1-\varepsilon}(\rho+\varepsilon)^{n+1} - a_{n-1}\frac{\rho}{1-\varepsilon}(\rho+\varepsilon)^n - a_n(\rho+\varepsilon)^{n+1},$$

which yields

$$a_{n+1} = a_n\left(\frac{1}{\rho+\varepsilon} + \frac{\rho}{1-\varepsilon} - 1\right) - a_{n-1}\frac{\rho}{(1-\varepsilon)(\rho+\varepsilon)}.$$

Together with the start values $a_0 = 1/\beta$ and

$$a_1 = \frac{1}{t_1} = \frac{\rho}{\beta(1-\varepsilon)} - \frac{1}{\beta} - \alpha$$

this yields a uniquely defined linear homogeneous recurrence relation with characteristic polynomial

$$0 = x^2 - \left(\frac{1}{\rho+\varepsilon} + \frac{\rho}{1-\varepsilon} - 1\right)x + \frac{\rho}{(1-\varepsilon)(\rho+\varepsilon)}.$$

Let $D(\rho,\varepsilon) = \left(\frac{1}{2(\rho+\varepsilon)} + \frac{\rho}{2(1-\varepsilon)} - \frac{1}{2}\right)^2 - \frac{\rho}{(1-\varepsilon)(\rho+\varepsilon)}$. The roots of the characteristic polynomial are then

$$\begin{aligned} x &= \frac{1}{2(\rho+\varepsilon)} + \frac{\rho}{2(1-\varepsilon)} - \frac{1}{2} - \sqrt{D(\rho,\varepsilon)}, \\ y &= \frac{1}{2(\rho+\varepsilon)} + \frac{\rho}{2(1-\varepsilon)} - \frac{1}{2} + \sqrt{D(\rho,\varepsilon)}. \end{aligned} \tag{3.42}$$

We claim that if $\rho < \varphi + 1$, then there is $\varepsilon > 0$ such that $D(\rho,\varepsilon) < 0$. To see this claim, consider the function

$$D(\rho, 0) = \left(\frac{1}{2\rho} + \frac{\rho}{2} - \frac{1}{2}\right)^2 - 1.$$

The function $h(\rho) = \frac{1}{2\rho} + \frac{\rho}{2} - \frac{1}{2}$ has the derivative $h'(\rho) = -\frac{1}{2\rho^2} + \frac{1}{2} > 0$ for $\rho > 1$. Thus, $h$ is strictly increasing for $\rho \in (1,\infty)$, and, hence, $D(\rho,0)$ is also strictly increasing for $\rho \in (1,\infty)$. Thus, $D(\rho,0)$ has at most one root $\rho_0 \in (1,\infty)$. This root satisfies

$$\frac{1}{2\rho_0} + \frac{\rho_0}{2} - \frac{1}{2} = 1.$$

Rearranging terms yields $1 + \rho_0^2 = 3\rho_0$. The only solution $\rho_0 > 1$ to this equation is $\varphi + 1$. We have shown that $D(\rho,0) < 0$ for all $\rho < \varphi + 1$. Since $D(\rho,\varepsilon)$ is continuous in $\varepsilon$, there is $\varepsilon' > 0$ such that also $D(\rho,\varepsilon) < 0$ for all $\varepsilon \in (0,\varepsilon']$. For $\rho \in (1, \varphi+1)$ and $\varepsilon$ chosen small enough, we have that the roots of the characteristic polynomial (3.42) are distinct and complex valued. We then obtain that the sequence $(a_n)_{n\in\mathbb{N}}$ has the closed-form expression

$$a_n = \lambda x^n + \mu y^n \quad \text{for all } n \in \mathbb{N} \cup \{0\}, \tag{3.43}$$

where the constants $\lambda, \mu \in \mathbb{C}$ are chosen in such a way that the equations for the starting values

$$a_0 = \frac{1}{\beta} = \lambda + \mu \qquad \text{and} \qquad a_1 = \frac{\rho}{\beta(1-\varepsilon)} - \frac{1}{\beta} - \alpha = \lambda x + \mu y \tag{3.44}$$

are satisfied. Note that by (3.42), $x$ and $y$ are complex conjugate, and hence, by (3.44), also $\lambda$ and $\mu$ are complex conjugate. We can, thus, reformulate (3.43) as

$$a_n = \lambda x^n + \bar{\lambda}\bar{x}^n = \lambda x^n + \overline{\lambda x}^n = 2\Re(\lambda x^n) \tag{3.45}$$

for all $n \in \mathbb{N}$, where for the second equation we used that conjugation is distributive with multiplication and for the third equation we used that for a complex number $z \in \mathbb{C}$ its real part can be computed as $\Re(z) = \frac{z+\bar{z}}{2}$. We will show that $\Re(\lambda x^\ell)$ is negative for some

$\ell \in \mathbb{N}$. The idea behind this part of the proof is visualized in Figure 3.3. Going to polar coordinates, we obtain

$$\lambda = r_\lambda \exp(\mathrm{i}\varphi_\lambda) \qquad \text{and} \qquad x = r_x \exp(\mathrm{i}\varphi_x)$$

for some $r_\lambda, r_x \in \mathbb{R}_{\geq 0}$ and some $\varphi_\lambda, \varphi_x \in [0, 2\pi)$. By exchanging the roles of $x$ and $y$, it is without loss of generality to assume that $\varphi_x \in [0, \pi]$. We obtain

$$a_n \overset{(3.45)}{=} 2\Re(\lambda x^n) = 2\Re\big(r_\lambda r_x^n \exp(\mathrm{i}(\varphi_\lambda + n\varphi_x))\big) \qquad \text{for all } n \in \mathbb{N}.$$

Let $k = \lceil \pi/\varphi_x \rceil$. We claim that $a_0, \dots, a_k$ are not strictly increasing. To see this, note that $a_0 = 1/\beta$, and thus,

$$1 = \operatorname{sgn}(a_0) = \operatorname{sgn}\big(2\Re\big(r_\lambda \exp(\mathrm{i}\varphi_\lambda)\big)\big) = \operatorname{sgn}\big(2\Re\big(\exp(\mathrm{i}\varphi_\lambda)\big)\big).$$

On the other hand, we have

$$-1 = \operatorname{sgn}\big(2\Re\big(\exp(\mathrm{i}\varphi_\lambda + \pi)\big)\big).$$

Since $\varphi_x \leq \pi$, this implies that either $\operatorname{sgn}(a_k) = -1$ or $\operatorname{sgn}(a_{k-1}) = -1$ (or both). In any case, this implies that there is $\ell \in \mathbb{N}$ with $a_\ell < 0$. Since $t_n = 1/a_n$ for all $n \in \mathbb{N}$, this further implies that $t_\ell < 0$. □

The following lemma shows that, given points $((x_0, v_0), \dots, (x_k, v_k)) \in (\mathbb{R}_{>0} \times \mathbb{R}_{>0})^{k+1}$ with $v_i < v_{i+1} < \frac{x_{i+1}}{x_i} v_i$ for all $i \in \{0, \dots, k-1\}$, we can construct an instance of IncMaxCont with $v(x_i) = v_i$ for all $i \in \{0, \dots, k-1\}$ simply by linearly interpolating between these points.

**Lemma 3.22.** *Let an instance of IncMaxCont with value function $\bar{v}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and density function $\bar{d}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be given. Let $k \in \mathbb{N}$ and $((x_0, v_0), \dots, (x_k, v_k)) \in (\mathbb{R}_{>0} \times \mathbb{R}_{>0})^{k+1}$ with $\bar{v}(x_0) = v_0$ and $v_i < v_{i+1} < \frac{x_{i+1}}{x_i} v_i$ for all $i \in \{0, \dots, k-1\}$. Then there exist an instance of IncMaxCont with value function $v\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and density function $d\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $v(x) = \bar{v}(x)$ for all $x \in [0, x_0]$ and $v(x_i) = v_i$ for all $i \in \{0, \dots, k\}$.*

*Proof.* For all $c \in [0, x_0]$, we define

$$v(x) := \bar{v}(x) \tag{3.46}$$

and, for all $i \in \{0, \dots, k-1\}$ and $x \in (x_i, x_{i+1}]$,

$$v(x) := v_i + \frac{x - x_i}{x_{i+1} - x_i}\big(v_{i+1} - v_i\big). \tag{3.47}$$

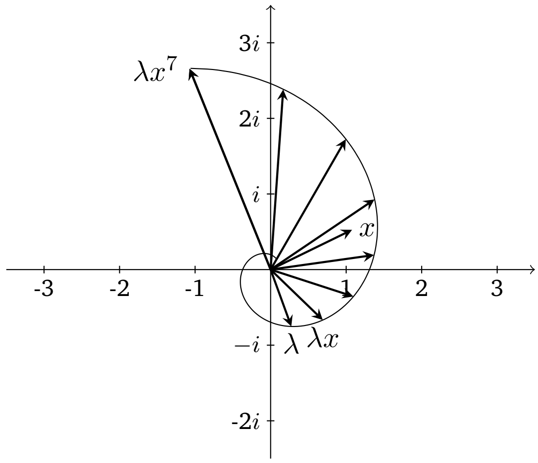


Figure 3.3.: Multiplying $\lambda$ repeatedly by $x \in (\mathbb{C} \setminus \mathbb{R})$ is equivalent to a rotation around the origin that, at some point, reaches the half-plane corresponding to negative real parts.

Note that $x_{i+1} > x_i$ because $\frac{x_{i+1}}{x_i} v_i > v_{i+1}$ and $v_i < v_{i+1}$. Furthermore, for $x > x_k$, we define

$$v(x) := v_{k-1} + \frac{x - x_{k-1}}{x_k - x_{k-1}}(v_k - v_{k-1}). \tag{3.48}$$

We set $d(0) := 1$ and $d(x) := \frac{v(x)}{x}$ for all $x > 0$.

We have to show that
$(i)$ $v$ is strictly increasing,
$(ii)$ $d$ is strictly decreasing,
$(iii)$ $d(0) = 1$,
$(iv)$ $v(x) = xd(x)$ for all $x \in \mathbb{R}_{\geq 0}$,
$(v)$ $v(x) = \bar{v}(x)$ for all $x \in [0, x_0]$, and
$(vi)$ $v(x_i) = v_i$ for all $i \in \{0, \ldots, k\}$.

On the interval $[0, x_0]$, $(i)$ holds because of (3.46) and because $\bar{v}$ is strictly increasing. For $x > x_0$, we have

$$v(x) \overset{(3.47),(3.48)}{=} v_i + \frac{x - x_i}{x_{i+1} - x_i}\big(v_{i+1} - v_i\big)$$

for some $i \in \{0, \dots, k-1\}$, and thus

$$v'(x) \overset{(3.47),(3.48)}{=} \frac{v_{i+1} - v_i}{x_{i+1} - x_i} \overset{v_{i+1}>v_i, x_{i+1}>x_i}{>} 0,$$

i.e., $(i)$ holds for $x > x_0$.

On the interval $[0, x_0]$, $(ii)$ holds because $\bar{d}$ is strictly decreasing and because $d(x) = \bar{d}(x)$ by definition of $d$ and by (3.46). For $x > x_0$, we have

$$d(x) \overset{(3.47),(3.48)}{=} \frac{v_i}{x} + \frac{1 - \frac{x_i}{x}}{x_{i+1} - x_i}(v_{i+1} - v_i)$$

for some $i \in \{0, \dots, k-1\}$, and thus

$$d'(x) \overset{(3.47),(3.48)}{=} -\frac{1}{x^2}\left(v_i - x_i \frac{v_{i+1} - v_i}{x_{i+1} - x_i}\right) \overset{v_{i+1}<\frac{x_{i+1}}{x_i}v_i}{<} -\frac{1}{x^2}\left(v_i - \frac{x_{i+1}v_i - x_i v_i}{x_{i+1} - x_i}\right) = 0,$$

i.e., $(ii)$ holds for $x > x_0$.

By definition of $d$, we have $d(0) = 1$, i.e., $(iii)$ holds.

We have $v(0) = \bar{v}(0) = 0 \cdot \bar{d}(0) = 0 \cdot d(0)$ and $v(x) = xd(x)$ by definition of $d$, i.e., $(iv)$ holds.

By (3.46), $(v)$ holds.

We have $v(x_0) \overset{(3.46)}{=} \bar{v}(x_0) = v_0$ and, for $i \in [k]$, we have

$$v(x_i) \overset{(3.47)}{=} v_{i-1} + \frac{x_i - x_{i-1}}{x_i - x_{i-1}}(v_i - v_{i-1}) = v_i,$$

i.e., $(vi)$ holds. □

The following calculations are needed to show that Lemma 3.22 can be applied to a sequence of points in the proof of Proposition 3.24.

**Lemma 3.23.** *Let* $1 < \rho < \varphi + 1$, $x_0, v_0, z > 0$,

$$t_0 = \frac{v_0}{x_0}, \qquad t_{n+1} = \frac{1}{\frac{\rho}{t_n(1-\varepsilon)} - \left(\sum_{j=0}^{n} \frac{(\rho+\varepsilon)^{j-n}}{t_j}\right) - \frac{z}{(\rho+\varepsilon)^n v_0}} \quad \text{for all } n \in \mathbb{N} \cup \{0\}$$

*and let* $\varepsilon \in (0, 1)$ *be small enough. By Lemma 3.21, there exists* $\ell' \in \mathbb{N}$ *with* $t_{\ell'} < 0$. *Let* $\ell \in \{0, \dots, \ell' - 1\}$ *be the smallest index such that* $\frac{1}{t_\ell} > \frac{1}{t_{\ell+1}}$. *Then,*

*(i)* $(\rho + \varepsilon)^n v_0 < \rho(\rho + \varepsilon)^n v_0$ *for all* $n \in \{0, \dots, \ell\}$,

*(ii)* $\rho(\rho + \varepsilon)^n v_0 < (\rho + \varepsilon)^{n+1} v_0$ *for all* $n \in \{0, \dots, \ell - 1\}$,

*(iii)* $\rho(\rho+\varepsilon)^n v_0 < \frac{\frac{\rho(\rho+\varepsilon)^n v_0}{(1-\varepsilon)t_n}}{\frac{(\rho+\varepsilon)^n v_0}{t_n}}(\rho+\varepsilon)^n v_0$ *for all* $n \in \{0,\ldots,\ell\}$*, and*

*(iv)* $(\rho+\varepsilon)^{n+1} v_0 < \frac{\frac{(\rho+\varepsilon)^{n+1} v_0}{t_{n+1}}}{\frac{\rho(\rho+\varepsilon)^n v_0}{(1-\varepsilon)t_n}}\rho(\rho+\varepsilon)^n v_0$ *for all* $n \in \{0,\ldots,\ell-1\}$.

*Proof.* Inequalities $(i)$ and $(ii)$ hold because $\rho > 1$, $\varepsilon > 0$, and $v_0 > 0$.

Let $n \in \{0,\ldots,\ell\}$. Inequality $(iii)$ is equivalent to $\rho < \frac{\rho}{1-\varepsilon}$, which holds because $\varepsilon \in (0,1)$.

Let $n \in \{0,\ldots,\ell-1\}$. Inequality $(iv)$ is equivalent to

$$\frac{1}{t_n} < (1-\varepsilon)\frac{1}{t_{n+1}} \quad \text{for all } n \in \{0,\ldots,\ell-1\}. \tag{3.49}$$

For fixed $1 < \rho < \varphi + 1$, we define the ratio $r(n,\varepsilon) := \frac{t_n}{t_{n+1}}$ for all $n \in \mathbb{N} \cup \{0\}$. Note that $r(0,\varepsilon) = \frac{\rho}{1-\varepsilon} - 1 - \frac{z}{v_0}t_0$ and, for $n \in \mathbb{N}$,

$$\begin{aligned}
r(n,\varepsilon) &= \frac{t_n}{t_{n+1}} = \left(\frac{\rho}{t_n(1-\varepsilon)} - \left(\sum_{j=0}^{n}\frac{(\rho+\varepsilon)^{j-n}}{t_j}\right) - \frac{z}{(\rho+\varepsilon)^n v_0}\right)t_n \\
&= \Bigg(\frac{\rho}{t_n(1-\varepsilon)} - \frac{1}{t_n} - \frac{\rho}{t_{n-1}(\rho+\varepsilon)(1-\varepsilon)} \\
&\quad + \frac{1}{\rho+\varepsilon}\left(\frac{\rho}{t_{n-1}(1-\varepsilon)} - \left(\sum_{j=0}^{n-1}\frac{(\rho+\varepsilon)^{j-n+1}}{t_j}\right) - \frac{z}{(\rho+\varepsilon)^{n-1}v_0}\right)\Bigg)t_n \\
&= \left(\frac{\rho}{t_n(1-\varepsilon)} - \frac{1}{t_n} - \frac{\rho}{t_{n-1}(\rho+\varepsilon)(1-\varepsilon)} + \frac{1}{\rho+\varepsilon}\cdot\frac{1}{t_n}\right)t_n \\
&= \frac{\rho}{1-\varepsilon} - 1 - \frac{\rho}{(\rho+\varepsilon)(1-\varepsilon)}\cdot\frac{t_n}{t_{n-1}} + \frac{1}{\rho+\varepsilon} \\
&= \frac{\rho}{1-\varepsilon} - 1 + \frac{1}{\rho+\varepsilon} - \frac{\rho}{(\rho+\varepsilon)(1-\varepsilon)}\cdot\frac{1}{r(n-1,\varepsilon)}.
\end{aligned} \tag{3.50}$$

*Claim:* For all $n \in \{0,\ldots,\ell-1\}$, we have

$$r(n,\varepsilon) > \frac{r(n,0)}{1-\varepsilon}. \tag{3.51}$$

*Proof of claim:* We prove the claim by induction. For $n = 0$, we have

$$\frac{r(0,\varepsilon)}{r(0,0)} = \frac{\frac{\rho}{1-\varepsilon} - 1 - \frac{z}{v_0}t_0}{\rho - 1 - \frac{z}{v_0}t_0} = \frac{\rho - (1-\varepsilon)\big(1 + \frac{z}{v_0}t_0\big)}{\rho - \big(1 + \frac{z}{v_0}t_0\big)} \cdot \frac{1}{1-\varepsilon} \overset{\varepsilon>0}{>} \frac{1}{1-\varepsilon},$$

i.e., (3.51) holds for $n = 0$.

If $\rho < 2$, without loss of generality, we can assume that $\varepsilon < 1 - \frac{\rho}{2}$ because $\varepsilon$ is small enough. This yields

$$t_0 = r(0,\varepsilon)t_1 = \Big(\frac{\rho}{1-\varepsilon} - 1 - \frac{z}{v_0}t_0\Big)t_1 < \Big(\frac{\rho}{1-\varepsilon} - 1\Big)t_1 \overset{\varepsilon<1-\frac{\rho}{2}}{<} t_1,$$

i.e., $\ell = 0$ and we are done with the proof. Thus, assume from now on that $\rho \geq 2 > \varphi$. Suppose (3.51) holds for some $n \in \{0, \ldots, \ell - 2\}$. Then

$$\frac{\rho}{\rho+\varepsilon} \cdot \frac{1}{r(n,\varepsilon)} < \frac{1}{r(n,\varepsilon)} \overset{(3.51)}{<} \frac{1-\varepsilon}{r(n,0)} < \frac{1}{r(n,0)} \tag{3.52}$$

and

$$\begin{aligned} \varepsilon\rho(\rho+\varepsilon) + (1-\varepsilon)\rho &= \varepsilon\rho^2 + \varepsilon^2\rho + \rho - \varepsilon\rho \\ &= (\rho+\varepsilon) + (\rho^2 - \rho - 1)\varepsilon + \varepsilon^2\rho \\ &\overset{\rho\geq 2}{\geq} \rho + \varepsilon + \varepsilon^2\rho > \rho + \varepsilon, \end{aligned}$$

which is equivalent to

$$\varepsilon + \frac{1-\varepsilon}{\rho+\varepsilon} > \frac{1}{\rho}. \tag{3.53}$$

This yields

$$\begin{aligned} \frac{r(n+1,\varepsilon)}{r(n+1,0)} &\overset{(3.50)}{=} \frac{\frac{\rho}{1-\varepsilon} - 1 + \frac{1}{\rho+\varepsilon} - \frac{\rho}{(\rho+\varepsilon)(1-\varepsilon)} \cdot \frac{1}{r(n,\varepsilon)}}{\rho - 1 + \frac{1}{\rho} - \frac{1}{r(n,0)}} \\ &= \frac{\rho - 1 + \varepsilon + \frac{1-\varepsilon}{\rho+\varepsilon} - \frac{\rho}{\rho+\varepsilon} \cdot \frac{1}{r(n,\varepsilon)}}{\rho - 1 + \frac{1}{\rho} - \frac{1}{r(n,0)}} \cdot \frac{1}{1-\varepsilon} \\ &\overset{(3.52),(3.53)}{>} \frac{\rho - 1 + \frac{1}{\rho} - \frac{1}{r(n,0)}}{\rho - 1 + \frac{1}{\rho} - \frac{1}{r(n,0)}} \cdot \frac{1}{1-\varepsilon} \\ &= \frac{1}{1-\varepsilon}, \end{aligned}$$

which proves the claim.

Note that $\ell$ depends on $\varepsilon$. Thus, we write $\ell(\varepsilon)$ from now on. By definition of $\ell(0)$, for all $n \in \{0, \ldots, \ell - 1\}$, we have

$$r(n,0) \geq 1 \tag{3.54}$$

and $r(\ell, 0) < 1$. Note that, by definition, $(t_n)_{n\in\mathbb{N}\cup\{0\}}$ is continuous in $\varepsilon$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, if $\varepsilon > 0$ is small enough (which we assumed), we have $r(\ell, \varepsilon) < 1$ and, for all $n \in \{0, \dots, \ell - 1\}$,

$$r(n, \varepsilon) \overset{(3.51)}{>} \frac{r(n, 0)}{1 - \varepsilon} \overset{(3.54)}{\geq} \frac{1}{1 - \varepsilon}.$$

This immediately implies (3.49) and thus completes the proof. □

With these preparations, we are now ready to define an instance of IncMaxCont where GreedyScaling$(c_1, \rho)$ cannot be better than $(\varphi + 1)$-competitive for one given starting value $c_1$.

**Proposition 3.24.** *Let an instance of IncMaxCont with value function $\bar{v}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be given. Let $\rho \in (1, \varphi + 1)$ and $0 < c_1 < C$. Then there exists an instance of IncMaxCont with value function $v\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $v(c) = \bar{v}(c)$ for all $c \in [0, C]$ and GreedyScaling$(c_1, \rho)$ is not $\rho$-competitive for this instance. Furthermore, there is some $\overline{C} > 0$ such that $v(c)$ can be altered for all $c \geq \overline{C}$ without losing the fact that GreedyScaling$(c_1, \rho)$ is not $\rho$-competitive for this instance.*

*Proof.* Let $\bar{d}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the density function with $\bar{v}(c) = c\bar{d}(c)$ for all $c \geq 0$.

If GreedyScaling$(c_1, \rho)$ is not $\rho$-competitive for the instance given by $\bar{v}$, we can simply choose $v = \bar{v}$ and $\overline{C}$ to be some value for which GreedyScaling$(c_1, \rho)$ is not $\rho$-competitive. Suppose GreedyScaling$(c_1, \rho)$ is $\rho$-competitive for the instance given by $\bar{v}$, and let $(\bar{c}_1, \bar{c}_2, \dots)$ with $\bar{c}_1 = c_1$ be the incremental solution produced by GreedyScaling$(c_1, \rho)$ on this instance. We will define the function $v$ such that GreedyScaling$(c_1, \rho)$ is not $\rho$-competitive on the instance given by $v$. In order to do this, we will give a sequence of points together with values such that GreedyScaling$(c_1, \rho)$ is forced to choose these points and Lemma 3.21 will show that this incremental solution is not a valid one because the sizes in the incremental solution are not increasing. Let $k \in \mathbb{N}_{\geq 2}$ such that $\bar{c}_{k-1} \leq C < \bar{c}_k$, let $v_k := \bar{v}(\bar{c}_k)$, and let $z := \sum_{j=1}^{k-1} \bar{c}_j$.

We will modify the value function $\bar{v}$ for $c > \bar{c}_k$, but we have to ensure that $\bar{c}_k \geq p(\bar{c}_{k-1}$ holds. If this is not the case and $\bar{c}_k < p(\bar{c}_{k-1})$, we change the instance in such a way that, for the incremental solution $(c'_1, c'_2, \dots)$ with $c'_1 = c_1$ of GreedyScaling$(c_1, \rho)$, we have $c'_i = \bar{c}_i$ for all $i \in [k - 1]$ and $c'_k \geq p(c'_{k-1})$. This can be achieved by leaving $\bar{v}$ unchanged on the interval $[0, \max\{p(\bar{c}_{k-2}), \bar{c}_k\}]$ and linearly interpolating from there on such that $\bar{v}\big(\frac{\rho}{\rho-1} z\big) = \rho\bar{v}(\bar{c}_{k-1})$. To show that this is possible, we first observe that, because $\bar{c}_k < p(\bar{c}_{k-1})$, we have

$$\bar{d}(p(\bar{c}_{k-1})) < \bar{d}(\bar{c}_k) \overset{(3.25)}{=} \frac{\bar{v}(\bar{c}_{k-1})}{p(\bar{c}_{k-1}) - z} = \frac{1}{\frac{\rho}{d(p(\bar{c}_{k-1}))} - \frac{z}{\bar{v}(\bar{c}_{k-1})}}.$$

This yields

$$\rho - 1 < \bar{d}(p(\bar{c}_{k-1}))\frac{z}{v(\bar{c}_{k-1})}. \tag{3.55}$$

By Lemma 3.22, the above change of $\bar{v}$ is possible because

$$\rho\bar{v}(\bar{c}_{k-1}) > \max\{\rho\bar{v}(\bar{c}_{k-2}), \bar{v}(\bar{c}_k)\} = \bar{v}(\max\{p(\bar{c}_{k-2}), \bar{c}_k\})$$

and

$$\frac{\rho\bar{v}(\bar{c}_{k-1})}{\frac{\rho}{\rho-1}z} = (\rho-1)\frac{\bar{v}(\bar{c}_{k-1})}{z} \overset{(3.55)}{<} \bar{d}(p(\bar{c}_{k-1}) \leq \bar{d}(\max\{p(\bar{c}_{k-2}), \bar{c}_k\}).$$

With this altered value function, we have $p(c'_{k-1}) = \frac{\rho}{\rho-1}z$ and thus

$$\bar{d}(c'_k) \overset{(3.25)}{=} \frac{\bar{v}(c'_{k-1})}{p(c'_{k-1}) - z} = \frac{\bar{v}(c'_{k-1})}{\frac{\rho}{\rho-1}z - z} = \frac{\bar{v}(c'_{k-1})}{\frac{1}{\rho-1}z} = \rho\frac{\bar{v}(c'_{k-1})}{p(c'_{k-1})} = d(p(c'_{k-1})),$$

i.e., we have $c'_k = d(p(c'_{k-1}))$. We are now ready to modify the instance for $c > c'_k$. For ease of notation, we redefine $\bar{c}_k := c'_k$.

We consider the recursively defined sequence

$$t_0 = \bar{d}(\bar{c}_k), \qquad t_{n+1} = \frac{1}{\frac{\rho}{t_n(1-\varepsilon)} - \left(\sum_{j=0}^{n}\frac{(\rho+\varepsilon)^{j-n}}{t_j}\right) - \frac{z}{(\rho+\varepsilon)^n v_k}} \quad \text{for all } n \in \mathbb{N}\cup\{0\}.$$

Since $\rho < \varphi + 1$, by Lemma 3.21, there exists $\varepsilon' > 0$ such that, for all $\varepsilon \in (0, \varepsilon']$, there is $\ell' \in \mathbb{N}$ with $t_{\ell'} < 0$. Let $\varepsilon \in (0, \varepsilon']$ be small enough. Since $t_{\ell'} < 0$, we have

$$\frac{\rho}{t_{\ell'-1}(1-\varepsilon)} - \left(\sum_{j=0}^{\ell'-1}\frac{(\rho+\varepsilon)^{j-\ell'+1}}{t_j}\right) - \frac{z}{(\rho+\varepsilon)^{\ell'-1}v_k} < 0,$$

i.e., we can define $\ell \in \{0, \dots, \ell'-1\}$ to be the smallest index such that

$$\frac{1}{t_\ell} > \frac{\rho}{t_\ell(1-\varepsilon)} - \left(\sum_{j=0}^{\ell}\frac{(\rho+\varepsilon)^{j-\ell}}{t_j}\right) - \frac{z}{(\rho+\varepsilon)^\ell v_k} = \frac{1}{t_{\ell+1}}. \tag{3.56}$$

For $n \in \{0, \dots, \ell\}$, let

$$x_{2n} := \frac{(\rho+\varepsilon)^n v_k}{t_n}, \qquad v_{2n} := (\rho+\varepsilon)^n v_k \tag{3.57}$$

and

$$x_{2n+1} := \frac{\rho(\rho+\varepsilon)^n v_k}{(1-\varepsilon)t_n}, \qquad v_{2n+1} := \rho(\rho+\varepsilon)^n v_k. \tag{3.58}$$

For $c \in [0, x_0]$, we let $v(c) = \bar{v}(c)$ and for $c > x_0$, we let $v$ be the function with $v(x_n) = v_n$ for all $n \in \{0, \dots, 2\ell\}$ that linearly interpolates between these points. By Lemmas 3.22 and 3.23, this is a valid value function for an instance of IncMaxCont. It remains to show that GreedyScaling$(c_1, \rho)$ is not $\rho$-competitive for the instance given by $v$.

Let $(c_1, c_2, \dots)$ be the incremental solution generated by GreedyScaling$(c_1, \rho)$ on the instance given by $v$. Note that $c_i = \bar{c}_i$ for all $i \in [k-1]$ since $v(c) = \bar{v}(c)$ for all $c \in [0, \max\{p(\bar{c}_{k-2}), \bar{c}_k\}]$, i.e., $z = \sum_{j=1}^{k-1} c_j$.

*Claim:* For all $n \in \{0, \dots, \ell+1\}$, we have

$$d(c_{k+n}) = t_n. \tag{3.59}$$

*Proof of Claim:* We prove this by induction. For $n = 0$, we have

$$d(c_k) \overset{(3.25)}{=} \bar{d}(\bar{c}_k) = t_0.$$

Suppose, for some $n \in \{0, \dots, \ell\}$, the claim holds for all $i \in \{0, \dots, n\}$. Because $v(c) = cd(d)$ for all $c \in \mathbb{R}_{\geq 0}$, we have, for all $j \in \{0, \dots, n\}$,

$$d(x_{2j}) = \frac{v_{2j}}{c_{2j}} \overset{(3.57)}{=} t_j \overset{(3.59)}{=} d(c_{k+j}),$$

i.e.,

$$c_{k+j} = x_{2j} \tag{3.60}$$

because $d$ is strictly decreasing. Furthermore, we have

$$v(x_{2j+1}) \overset{(3.58)}{=} \rho(\rho+\varepsilon)^j v_k \overset{(3.57)}{=} \rho v(x_{2j}) \overset{(3.60)}{=} \rho v(c_{k+j}),$$

i.e.,

$$p(c_{k+j}) = x_{2j+1}. \tag{3.61}$$

because $v$ is strictly increasing. This yields

$$\begin{aligned}
d(c_{k+n+1}) &\overset{(3.25)}{=} \frac{v(c_{k+n})}{p(c_{k+n}) - \sum_{j=1}^{k+n} c_j} \\
&\overset{(3.60),(3.61)}{=} \frac{v(x_{2n})}{x_{2n+1} - z - \sum_{j=0}^{n} x_{2j}} \\
&\overset{(3.57),(3.58)}{=} \frac{(\rho+\varepsilon)^n v_k}{\frac{\rho(\rho+\varepsilon)^n v_k}{(1-\varepsilon)t_n} - z - \sum_{j=0}^{n} \frac{(\rho+\varepsilon)^j v_k}{t_j}} \\
&= \frac{1}{\frac{\rho}{(1-\varepsilon)t_n} - \frac{z}{(\rho+\varepsilon)^n v_k} - \sum_{j=0}^{n} \frac{(\rho+\varepsilon)^{j-n}}{t_j}} \\
&= t_{n+1},
\end{aligned}$$

which proves the claim.

This implies

$$\frac{1}{d(c_{k+\ell+1})} \overset{(3.59)}{=} \frac{1}{t_{\ell+1}} \overset{(3.56)}{<} \frac{1}{t_\ell} \overset{(3.59)}{=} \frac{1}{d(c_{k+\ell})},$$

i.e., either $d(c_{k+\ell+1}) > d(c_{k+\ell})$ and thus $c_{k+\ell+1} < c_{k+\ell}$ because $d$ is strictly decreasing, or $d(c_{k+\ell+1}) < 0$. In both cases, by Proposition 3.15, this implies that GreedyScaling$(c_1, \rho)$ is not $\rho$-competitive for the instance given by $v$.

Note that the values $v(c)$ for $c > x_{2\ell+1}$ are never used throughout this proof. Thus, we can set $\overline{C} = x_{2\ell+1} + 1$. □

We can use Proposition 3.24 iteratively in order to exclude any countable set of starting values. Proposition 3.24 states that we can modify the value function of an instance of IncMaxCont in such a way that one starting value gets excluded, $v(c)$ for small values of $c$, i.e., for $c \in [0, C]$, remains unchanged, and $v(c)$ for large values of $c$, i.e., for $c \geq \overline{C}$, can be changed without losing the fact that the starting value is excluded. Thus, after we modified the instance to exclude one starting value, we can modify the part for $c \geq \overline{C}$ in order to exclude one more starting value. We can repeat this countably often to exclude a countable set of starting values.

**Corollary 3.25.** *For every countable set $S \subset \mathbb{R}_{>0}$ of starting values, there exists an instance of IncMaxCont such that GreedyScaling$(c_1, \rho)$ is not $\rho$-competitive for any $c_1 \in S$ and $\rho < \varphi + 1$.*

### 3.2.2. General Lower Bound

Now we want to employ the techniques we used to prove Lemma 3.21 in order to give a lower bound on the competitive ratio of IncMaxCont. Let $\rho^* \approx 2.246$ be the unique real root $\rho \geq 1$ of the polynomial $-4\rho^6 + 24\rho^4 - \rho^3 - 30\rho^2 + 31\rho - 4$. We will show that there is no algorithm that is better than $\rho^*$-competitive for IncMaxCont. As before, we need to show that a recursively defined sequence becomes negative at some point.

**Lemma 3.26.** *For $\rho \in \mathbb{R}_{\geq 0}$ and $\varepsilon > 0$, consider the recursively defined sequence $(t_n)_{n\in\mathbb{N}}$ with*

$$t_0 = 1, \qquad t_1 = \frac{1-\varepsilon}{\rho}, \qquad t_n = \frac{1-\varepsilon}{\frac{\rho}{t_{n-1}} - \frac{1}{t_{n-2}} - \frac{1}{\rho}\sum_{j=0}^{n-3}\frac{(\rho+\varepsilon)^{j+2-n}}{t_j}} \quad \text{for all } n \in \mathbb{N}_{\geq 2}.$$

*If $1 < \rho < \rho^*$, then there exists $\varepsilon' > 0$ such that, for all $\varepsilon \in [0, \varepsilon']$, there is $\ell \in \mathbb{N}$ with $t_\ell < 0$.*

*Proof.* Let $1 < \rho < \rho^*$. Rearranging terms, for all $n \in \mathbb{N}_{\geq 2}$, we obtain

$$\frac{1-\varepsilon}{t_n} = \frac{\rho}{t_{n-1}} - \frac{1}{t_{n-2}} - \frac{1}{\rho}\left(\sum_{j=0}^{n-3}\frac{(\rho+\varepsilon)^{j+2-n}}{t_j}\right).$$

We substitute $a_n = 1/t_n$ for all $n \in \mathbb{N} \cup \{0\}$ and obtain the recursively defined sequence $(a_n)_{n\in\mathbb{N}}$ with $a_0 = 1$ and, for all $n \in \mathbb{N}_{\geq 2}$,

$$(1-\varepsilon)a_n = \rho a_{n-1} - a_{n-2} - \frac{1}{\rho}\left(\sum_{j=0}^{n-3}(\rho+\varepsilon)^{j+2-n}a_j\right), \tag{3.62}$$

which implies

$$(1-\varepsilon)(\rho+\varepsilon)a_{n+1} = \rho(\rho+\varepsilon)a_n - (\rho+\varepsilon)a_{n-1} - \frac{1}{\rho}\left(\sum_{j=0}^{n-2}(\rho+\varepsilon)^{j+2-n}a_j\right). \tag{3.63}$$

Subtracting (3.62) from (3.63), for all $n \in \mathbb{N}_{\geq 2}$, we obtain

$$(1-\varepsilon)(\rho+\varepsilon)a_{n+1} - (1-\varepsilon)a_n = \left(\rho(\rho+\varepsilon)a_n - (\rho+\varepsilon)a_{n-1} - \frac{1}{\rho}a_{n-2}\right) - \big(\rho a_{n-1} - a_{n-2}\big),$$

which yields

$$(1-\varepsilon)(\rho+\varepsilon)a_{n+1} = (\rho^2 + 1 + \rho\varepsilon - \varepsilon)a_n - (2\rho+\varepsilon)a_{n-1} + \left(1 - \frac{1}{\rho}\right)a_{n-2}.$$

Together with the start values

$$a_0 = 1, \qquad a_1 = \frac{\rho}{1-\varepsilon}, \quad \text{and} \quad a_2 = \frac{\rho a_1 - a_0}{1-\varepsilon} = \frac{\rho^2 - 1 + \varepsilon}{(1-\varepsilon)^2}, \tag{3.64}$$

this yields a uniquely defined linear homogeneous recurrence relation with characteristic polynomial

$$0 = x^3 - \frac{\rho^2 + 1 + \rho\varepsilon - \varepsilon}{(1-\varepsilon)(\rho+\varepsilon)}x^2 + \frac{2\rho + \varepsilon}{(1-\varepsilon)(\rho+\varepsilon)}x - \frac{1-\frac{1}{\rho}}{(1-\varepsilon)(\rho+\varepsilon)}. \tag{3.65}$$

Using

$$\begin{aligned} a &= -\frac{\rho^2 + 1 + \rho\varepsilon - \varepsilon}{(1-\varepsilon)(\rho+\varepsilon)}, \\ b &= \frac{2\rho + \varepsilon}{(1-\varepsilon)(\rho+\varepsilon)}, \\ c &= -\frac{1-\frac{1}{\rho}}{(1-\varepsilon)(\rho+\varepsilon)}, \end{aligned}$$

the discriminant of this polynomial is

$$D(\rho,\varepsilon) = \left(\frac{a^3}{27} - \frac{ab}{6} + \frac{c}{2}\right)^2 + \left(\frac{b}{3} - \frac{a^2}{9}\right)^3.$$

In particular, we have

$$D(\rho,0) = \frac{-4\rho^6 + 24\rho^4 - \rho^3 - 30\rho^2 + 31\rho - 4}{108\rho^5} > 0$$

because $1 < \rho < \rho^*$. Note that $a, b, c$ are all continuous in $\varepsilon$ and, thus, so is $D(\rho,\varepsilon)$. Therefore, there is $\varepsilon' > 0$ such that, for all $\varepsilon \in [0, \varepsilon']$, we have $D(\rho,\varepsilon) > 0$. The fact that for the discriminant of the polynomial we have $D(\rho,\varepsilon) > 0$ implies that (3.65) has one real root $r_1$ and two complex conjugate roots $r_2$ and $r_3 = \overline{r_2}$. We want to express the recurrence relation in terms of the roots, i.e., we want to find $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ such that

$$a_n = \lambda_1 r_1^n + \lambda_2 r_2^n + \lambda_3 r_3^n. \tag{3.66}$$

We will now show that $\lambda_3 = \overline{\lambda_2}$. Using the starting values (3.64) together with (3.66), we obtain

$$\begin{aligned} 1 &= \lambda_1 + \lambda_2 + \lambda_3, \\ \frac{\rho}{1-\varepsilon} &= \lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3, \\ \frac{\rho^2 - 1 + \varepsilon}{(1-\varepsilon)^2} &= \lambda_1 r_1^2 + \lambda_2 r_2^2 + \lambda_3 r_3^2. \end{aligned}$$

This implies

$$0 = \mathfrak{Im}(\lambda_1) + \mathfrak{Im}(\lambda_2) + \mathfrak{Im}(\lambda_3), \tag{3.67}$$

$$0 = \mathfrak{Im}(\lambda_1 r_1) + \mathfrak{Im}(\lambda_2 r_2) + \mathfrak{Im}(\lambda_3 r_3), \tag{3.68}$$

$$0 = \mathfrak{Im}(\lambda_1 r_1^2) + \mathfrak{Im}(\lambda_2 r_2^2) + \mathfrak{Im}(\lambda_3 r_3^2). \tag{3.69}$$

We obtain

$$\begin{aligned}
& \mathfrak{Re}(r_1 - r_2)\mathfrak{Im}(\lambda_2 + \lambda_3) + \mathfrak{Im}(r_1 - r_2)\mathfrak{Re}(\lambda_2 - \lambda_3) \\
= \;& \mathfrak{Re}(r_1 - r_2)\mathfrak{Im}(\lambda_2) + \mathfrak{Im}(r_1 - r_2)\mathfrak{Re}(\lambda_2) \\
& + \mathfrak{Re}(r_1 - r_2)\mathfrak{Im}(\lambda_3) - \mathfrak{Im}(r_1 - r_2)\mathfrak{Re}(\lambda_3) \\
\overset{r_1 \in \mathbb{R}, r_3 = \overline{r_2}}{=} \;& \mathfrak{Re}(r_1 - r_2)\mathfrak{Im}(\lambda_2) + \mathfrak{Im}(r_1 - r_2)\mathfrak{Re}(\lambda_2) \\
& + \mathfrak{Re}(r_1 - r_3)\mathfrak{Im}(\lambda_3) + \mathfrak{Im}(r_1 - r_3)\mathfrak{Re}(\lambda_3) \\
= \;& \mathfrak{Im}\big((r_1 - r_2)\lambda_2 + (r_1 - r_3)\lambda_3\big) \\
\overset{r_1 \in \mathbb{R}}{=} \;& r_1\mathfrak{Im}(\lambda_2) - \mathfrak{Im}(\lambda_2 r_2) + r_1\mathfrak{Im}(\lambda_3) - \mathfrak{Im}(\lambda_3 r_3) \\
\overset{(3.67),(3.68)}{=} \;& -r_1\mathfrak{Im}(\lambda_1) + \mathfrak{Im}(\lambda_1 r_1) \\
\overset{r_1 \in \mathbb{R}}{=} \;& 0
\end{aligned} \tag{3.70}$$

and

$$\begin{aligned}
& \mathfrak{Re}(r_1^2 - r_2^2)\mathfrak{Im}(\lambda_2 + \lambda_3) + \mathfrak{Im}(r_1^2 - r_2^2)\mathfrak{Re}(\lambda_2 - \lambda_3) \\
= \;& \mathfrak{Re}(r_1^2 - r_2^2)\mathfrak{Im}(\lambda_2) + \mathfrak{Im}(r_1^2 - r_2^2)\mathfrak{Re}(\lambda_2) \\
& + \mathfrak{Re}(r_1^2 - r_2^2)\mathfrak{Im}(\lambda_3) - \mathfrak{Im}(r_1^2 - r_2^2)\mathfrak{Re}(\lambda_3) \\
\overset{r_1 \in \mathbb{R}, r_3 = \overline{r_2}}{=} \;& \mathfrak{Re}(r_1^2 - r_2^2)\mathfrak{Im}(\lambda_2) + \mathfrak{Im}(r_1^2 - r_2^2)\mathfrak{Re}(\lambda_2) \\
& + \mathfrak{Re}(r_1^2 - r_3^2)\mathfrak{Im}(\lambda_3) + \mathfrak{Im}(r_1^2 - r_3^2)\mathfrak{Re}(\lambda_3) \\
= \;& \mathfrak{Im}\big((r_1^2 - r_2^2)\lambda_2 + (r_1^2 - r_3^2)\lambda_3\big) \\
\overset{r_1 \in \mathbb{R}}{=} \;& r_1^2\mathfrak{Im}(\lambda_2) - \mathfrak{Im}(\lambda_2 r_2^2) + r_1^2\mathfrak{Im}(\lambda_3) - \mathfrak{Im}(\lambda_3 r_3^2) \\
\overset{(3.67),(3.69)}{=} \;& -r_1^2\mathfrak{Im}(\lambda_1) + \mathfrak{Im}(\lambda_1 r_1^2) \\
\overset{r_1 \in \mathbb{R}}{=} \;& 0.
\end{aligned} \tag{3.71}$$

Consider the real matrix

$$M := \begin{pmatrix} \mathfrak{Re}(r_1 - r_2) & \mathfrak{Im}(r_1 - r_2) \\ \mathfrak{Re}(r_1^2 - r_2^2) & \mathfrak{Im}(r_1^2 - r_2^2) \end{pmatrix} \overset{r_1 \in \mathbb{R}}{=} \begin{pmatrix} r_1 - \mathfrak{Re}(r_2) & -\mathfrak{Im}(r_2) \\ r_1^2 - \mathfrak{Re}(r_2^2) & -\mathfrak{Im}(r_2^2) \end{pmatrix}.$$

This matrix does not have rank 0 because $-\mathfrak{Im}(r_2) \neq 0$. Suppose $M$ has rank 1. Then there exists $x \in \mathbb{R}$ with

$$x \cdot (-\mathfrak{Im}(r_2)) = -\mathfrak{Im}(r_2^2) = -2\mathfrak{Re}(r_2)\mathfrak{Im}(r_2),$$

which yields

$$x = 2\mathfrak{Re}(r_2), \tag{3.72}$$

as well as

$$\begin{aligned} x(r_1 - \mathfrak{Re}(r_2)) &= r_1^2 - \mathfrak{Re}(r_2^2) = r_1^2 - \mathfrak{Re}(r_2)^2 + \mathfrak{Im}(r_2)^2 \\ &= (r_1 - \mathfrak{Re}(r_2))(r_1 + \mathfrak{Re}(r_2)) + \mathfrak{Im}(r_2)^2, \end{aligned}$$

which yields

$$x = r_1 + \mathfrak{Re}(r_2) + \frac{\mathfrak{Im}(r_2)^2}{r_1 - \mathfrak{Re}(r_2)}. \tag{3.73}$$

Combining (3.72) and (3.73) yields

$$\mathfrak{Re}(r_2) - r_1 = \frac{\mathfrak{Im}(r_2)^2}{r_1 - \mathfrak{Re}(r_2)}.$$

This is equivalent to $-(r_1 - \mathfrak{Re}(r_2))^2 = \mathfrak{Im}(r_2)^2$, which can only hold if both sides are $0$, because the left side is non-positive and the right is non-negative. Yet, we have $r_2 \notin \mathbb{R}$, which is a contradiction. Thus, $M$ has rank $2$, i.e., the system of equations

$$M \cdot (\mathfrak{Im}(\lambda_2 + \lambda_3), \mathfrak{Re}(\lambda_2 - \lambda_3))^T = (0,0)^T$$

given by (3.70) and (3.71) only has the solution $(0,0)^T$. This yields $\mathfrak{Im}(\lambda_2) = -\mathfrak{Im}(\lambda_3)$ and $\mathfrak{Re}(\lambda_2) = \mathfrak{Re}(\lambda_3)$, i.e., we have $\lambda_3 = \overline{\lambda_2}$, and, by (3.67), we have $\lambda_1 \in \mathbb{R}$. Thus, the recurrence relation can be written as

$$a_n = \lambda_1 r_1^n + \lambda_2 r_2^n + \overline{\lambda_2 r_2^n} = \lambda_1 r_1^n + 2\mathfrak{Re}(\lambda_2 r_2^n).$$

We have $|r_1| < 1$ and $|r_2| > 1$. There exists $\ell \in \mathbb{N}$ such that $\mathfrak{Re}(\lambda_2 r_2^\ell) < -\frac{\lambda_1}{2}$ because $r_2 \notin \mathbb{R}$ and $|r_2| > 1$ (cf. Figure 3.3). Thus, $a_\ell = \lambda_1 r_1^\ell + 2\mathfrak{Re}(\lambda_2 r_2^\ell) < \lambda_1 r_1^\ell - \lambda_1 < 0$ where the last inequality follows from the fact that $|r_1| < 1$. □

With this lemma, we are ready to construct our lower bound on the competitive ratio of IncMaxCont, which, via Theorem 3.9 and Proposition 3.12, gives a lower bound on the (non-strict) competitive ratio of $\text{IncMax}_{\text{acc}}$. Recall that $\rho^* \approx 2.246$ is the unique solution $\rho \geq 1$ to the equation $-4\rho^6 + 24\rho^4 - \rho^3 - 30\rho^2 + 31\rho - 4$.

**Theorem 3.27.** *The competitive ratio of* IncMaxCont *is at least* $2.246$.

*Proof.* Let $\rho < \rho^*$. By Lemma 3.26, there is $\varepsilon' > 0$ such that, for all $\varepsilon \in (0, \varepsilon']$, the recursively defined sequence $(t_n)_{n \in \mathbb{N}}$ with

$$t_0 = 1, \qquad t_1 = \frac{1-\varepsilon}{\rho}, \qquad t_n = \frac{1-\varepsilon}{\frac{\rho}{t_{n-1}} - \frac{1}{t_{n-2}} - \frac{1}{\rho}\sum_{j=0}^{n-3}\frac{(\rho+\varepsilon)^{j+2-n}}{t_j}} \quad \text{for all } n \in \mathbb{N}_{\geq 2}$$

becomes negative at some point. Thus, for $\varepsilon > 0$, we can define $\ell(\varepsilon) \in \mathbb{N}$ to be the smallest value such that $\frac{1}{t_{\ell(\varepsilon)}} \geq \frac{1}{t_{\ell(\varepsilon)+1}}$. Note that this is the case when either $t_{\ell(\varepsilon)+1} < 0$ or $t_{\ell(\varepsilon)+1} \geq t_{\ell(\varepsilon)}$. This implies that $t_0 > t_1 > \cdots > t_{\ell(\varepsilon)}$. Let $v_i = (\rho+\varepsilon)^i$, and let $\varepsilon \in (0, \varepsilon']$ be small enough such that $\ell := \ell(\varepsilon) = \ell(0)$ and such that, for all $n \in \{0, \ldots, \ell - 1\}$, we have

$$\frac{1}{t_{n+1}} + \frac{1}{v_{n+1}} \sum_{j=0}^{n-1} \frac{v_j}{t_j} > \frac{1}{t_n} + \frac{1}{\rho v_n} \sum_{j=0}^{n-1} \frac{v_j}{t_j}, \tag{3.74}$$

which is possible because $v_{n+1} = (\rho+\varepsilon)v_n$, i.e., the inequality holds for $\varepsilon = 0$.

We consider the instance of IncMaxCont with the value function that linearly interpolates between the points $v(0) = 0$,

$$v\left(\frac{v_n}{t_n}\right) = v\left(\frac{v_n}{t_{n+1}}\right) = v_n \quad \text{for all } n \in \{0, 1, \ldots, \ell - 1\},$$

and $v(\frac{v_\ell}{t_\ell}) = v_\ell$. This means that, for $0 \leq c \leq 1$, we have $d(c) = 1$, for $\frac{v_n}{t_n} \leq c \leq \frac{v_n}{t_{n+1}}$, we have $v(c) = v_n$, and, for $\frac{v_n}{t_{n+1}} \leq c \leq \frac{v_{n+1}}{t_{n+1}}$, we have $d(c) = t_{n+1}$ (cf. Figure 3.4).

Suppose there was a $\rho$-competitive incremental solution $(c_0, \ldots, c_k)$ for this problem instance. We will show that these capacities have to satisfy $d(c_n) = t_n$, which is not possible because the sequence $t_0, \ldots, t_{\ell+1}$ is not decreasing.

Without loss of generality, we can assume that, for all $n \in \{0, \ldots, k\}$, we have

$$d(c_n) \in \{t_0, t_1, \ldots, t_\ell\}. \tag{3.75}$$

If this was not the case, we have $\frac{v_i}{t_i} < c_n < \frac{v_i}{t_{i+1}}$ for some $i \in \{0, \ldots, k-1\}$. Then, we can improve the incremental solution by setting $c_n = \frac{v_i}{t_i}$ because $v(c_n) = v_i = v\big(\frac{v_i}{t_i}\big)$ and $\frac{v_i}{t_i} < c_n$, i.e., the modified incremental solution obtains the same value faster and can start adding the next optimum solution earlier. Furthermore, we can assume that

$$d(c_n) > d(c_{n+1}) \tag{3.76}$$

for all $n \in \{0, \ldots, k-1\}$. Otherwise we can improve the incremental solution by removing the smaller of $c_n$ and $c_{n+1}$. This also implies that $k \leq \ell$.

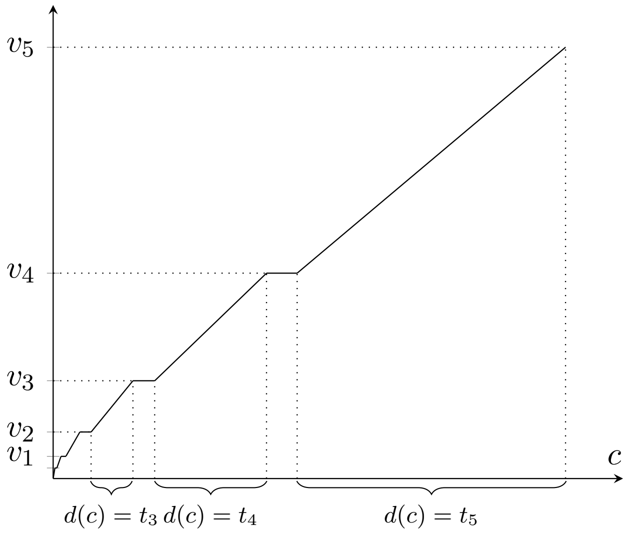


Figure 3.4.: Lower bound construction from the proof of Theorem 3.27 for $\rho = 2.1$. Here, we have $\ell = 5$.

We will now show by induction, that, for $n \in \{0, \ldots, k\}$, we have

$$d(c_n) > t_{n+1}, \tag{3.77}$$

and, for $n \in \{0, \ldots, k-1\}$, we have

$$c_n \geq \frac{1}{\rho} \cdot \frac{v_n}{t_n}. \tag{3.78}$$

For $n = 0$, by Lemma 3.14, we have $d(c_0) \geq \frac{1}{\rho} > \frac{1-\varepsilon}{\rho} = t_1$, i.e., (3.77) holds. This implies $d(c_0) = t_0$ because $t_0 > \cdots > t_\ell$. If $c_0 < \frac{1}{\rho} \cdot \frac{v_0}{t_0} = \frac{1}{\rho}$, then the solution achieved by only adding the optimum solution of size $c_0$ is $\rho$-competitive up to size $p(c_0) = \rho c_0$. By Lemma 3.14, we have

$$d(c_1) \geq \frac{v(c_0)}{p(c_0) - \sum_{j=0}^{0} c_j} = \frac{c_0 t_0}{\rho c_0 - c_0} = \frac{1}{\rho - 1} > \frac{1-\varepsilon}{\rho} = t_1,$$

i.e., using (3.75), we obtain $d(c_1) = t_0$, which is a contradiction to the fact that $d(c_0) = t_0$. Thus, $c_0 \geq \frac{1}{\rho} \cdot \frac{v_0}{t_0}$, i.e., also (3.78) holds.

Now, assume that, for some $n \in \{0, \dots, k-1\}$, (3.77) and (3.78) hold for all natural numbers at most $n$. As $c_n \geq \frac{1}{\rho} \cdot \frac{v_n}{t_n}$, we have

$$p(c_n) = \frac{\rho v(c_n)}{t_{n+1}}. \tag{3.79}$$

Note that, for all $i \in \{0, \dots, k\}$, we have $d(c_i) \in \{t_0, \dots, t_\ell\}$ and $d(c_{i+1}) < d(c_i)$. Because $t_0 > \dots > t_\ell$ and because, for all $i \in \{0, \dots, n\}$, (3.77) holds, we have $d(c_i) = t_i$ for all $i \in \{0, \dots, n\}$. Thus, $c_n \leq \frac{v_n}{t_n}$ and therefore

$$v(c_n) \leq v\left(\frac{v_n}{t_n}\right) = v_n. \tag{3.80}$$

By Lemma 3.14, $c_{n+1}$ has to satisfy

$$\begin{aligned}
d(c_{n+1}) &\geq \frac{v(c_n)}{p(c_n) - \sum_{j=0}^{n} c_j} \overset{(3.79)}{=} \frac{v(c_n)}{\frac{\rho v(c_n)}{t_{n+1}} - \sum_{j=0}^{n} c_j} \\
&\overset{(3.78)}{\geq} \frac{v(c_n)}{\frac{\rho v(c_n)}{t_{n+1}} - c_n - \frac{1}{\rho}\sum_{j=0}^{n-1} \frac{v_j}{t_j}} = \frac{1}{\frac{\rho}{t_{n+1}} - \frac{1}{t_n} - \frac{1}{\rho v(c_n)}\sum_{j=0}^{n-1} \frac{(\rho+\varepsilon)^j}{t_j}} \\
&\overset{(3.80)}{\geq} \frac{1}{\frac{\rho}{t_{n+1}} - \frac{1}{t_n} - \frac{1}{\rho v_n}\sum_{j=0}^{n-1} \frac{(\rho+\varepsilon)^j}{t_j}} = \frac{1}{\frac{\rho}{t_{n+1}} - \frac{1}{t_n} - \frac{1}{\rho}\sum_{j=0}^{n-1} \frac{(\rho+\varepsilon)^{j-n}}{t_j}} \\
&= \frac{1}{1-\varepsilon} t_{n+2} > t_{n+2},
\end{aligned}$$

i.e., (3.77) holds for $n+1$. As $d(c_{n+1}) < d(c_n) = t_n$ by (3.76) and because $t_0 > \dots > t_\ell$, we have $d(c_{n+1}) = t_{n+1}$. Now, assume that $n \in \{0, \dots, k-2\}$. We will show that (3.78) holds for $n+1$. For the sake of contradiction, suppose that (3.78) does not hold for $n+1$, i.e., we have $c_{n+1} < \frac{1}{\rho} \cdot \frac{v_{n+1}}{t_{n+1}}$. Then,

$$v(c_{n+1}) = c_{n+1} t_{n+1} < \frac{1}{\rho} v_{n+1}, \tag{3.81}$$

i.e., the prefix $(c_0, \dots, c_{n+1})$ of the incremental solution is $\rho$-competitive up to size $p(c_{n+1}) = \frac{\rho v(c_{n+1})}{t_{n+1}} = \rho c_{n+1}$. By Lemma 3.14, the next size in the incremental solution has to satisfy

$$\begin{aligned}
d(c_{n+2}) &\geq \frac{v(c_{n+1})}{p(c_{n+1}) - \sum_{j=0}^{n+1} c_j} = \frac{v(c_{n+1})}{\rho c_{n+1} - \sum_{j=0}^{n+1} c_j} \\
&\overset{(3.78)}{\geq} \frac{v(c_{n+1})}{\rho c_{n+1} - c_{n+1} - c_n - \frac{1}{\rho}\sum_{j=0}^{n-1} \frac{v_j}{t_j}} \geq \frac{v(c_{n+1})}{\rho c_{n+1} - c_{n+1} - \frac{1}{\rho}\sum_{j=0}^{n-1} \frac{v_j}{t_j}} \\
&= \frac{1}{\frac{\rho}{t_{n+1}} - \frac{1}{t_{n+1}} - \frac{1}{\rho v(c_{n+1})}\sum_{j=0}^{n-1} \frac{v_j}{t_j}} \overset{(3.81)}{>} \frac{1}{\frac{\rho}{t_{n+1}} - \frac{1}{t_{n+1}} - \frac{1}{v_{n+1}}\sum_{j=0}^{n-1} \frac{v_j}{t_j}} \\
&\overset{(3.74)}{\geq} \frac{1}{\frac{\rho}{t_{n+1}} - \frac{1}{t_n} - \frac{1}{\rho v_n}\sum_{j=0}^{n-1} \frac{v_j}{t_j}} = \frac{1}{\frac{\rho}{t_{n+1}} - \frac{1}{t_n} - \frac{1}{\rho}\sum_{j=0}^{n-1} \frac{(\rho+\varepsilon)^{j-n}}{t_j}} \\
&= \frac{1}{1-\varepsilon} t_{n+2} > t_{n+2}.
\end{aligned}$$

As $t_0 > \cdots > t_\ell$, we have $d(c_{n+2}) \notin \{t_{n+2}, \ldots, t_\ell\}$. But, for all $i \in \{0, \ldots, n+1\}$, we also have $d(c_i) = t_i$ and thus $d(c_{n+2}) \notin \{t_0, \ldots, t_{n+1}\}$. This is a contradiction to (3.75) and therefore (3.78) holds for $n + 1$.

We have established that (3.77) holds for all $n \in \{0, \ldots, k\}$. Together with the fact that $d(c_n) \in \{t_0, \ldots, t_\ell\}$ for $n \in \{0, \ldots, k\}$, $d(c_0) > \cdots > d(c_k)$ and $t_0 > \cdots > t_\ell$, we obtain $d(c_n) = t_n$ for all $n \in \{0, \ldots, k\}$. If $k < \ell$, the solution obtains a value of $v(c_k)$ for size $\frac{v_\ell}{t_\ell}$. Yet, the optimum solution for this size has value

$$v\Big(\frac{v_\ell}{t_\ell}\Big) = v_\ell = (\rho+\varepsilon)^{\ell-k} v_k \geq (\rho+\varepsilon)^{\ell-k} v(c_k) > \rho v(c_k).$$

Thus, $(c_0, \ldots, c_k)$ would not be $\rho$-competitive. Therefore, we have $k = \ell$. By a similar argument, we find that $c_k = c_\ell \geq \frac{1}{\rho} \cdot \frac{v_\ell}{t_\ell}$. By (3.77), we know that $d(c_\ell) > t_{\ell+1}$. If $t_{\ell+1} \geq t_\ell$, we know that $d(c_\ell) \neq t_\ell$. But we also have $d(c_\ell) \notin \{t_0, \ldots, t_{\ell-1}\}$ as $d(c_\ell) < d(c_{\ell-1}) = t_{\ell-1}$ and $t_0 < \cdots < t_{\ell-1}$. This is a contradiction to the assumption that $d(c_\ell) \in \{t_0, \ldots, t_\ell\}$. Therefore, $t_{\ell+1} < t_\ell$. By definition of $\ell$, we have $\frac{1}{t_{\ell+1}} < \frac{1}{t_\ell}$, which implies that

$$0 > t_{\ell+1} = \frac{1-\varepsilon}{\frac{\rho}{t_\ell} - \frac{1}{t_{\ell-1}} - \frac{1}{\rho}\sum_{j=0}^{\ell-2} \frac{(\rho+\varepsilon)^{j+1-\ell}}{t_j}} > \frac{1}{\frac{\rho}{t_\ell} - \frac{1}{t_{\ell-1}} - \frac{1}{\rho}\sum_{j=0}^{\ell-2} \frac{(\rho+\varepsilon)^{j+1-\ell}}{t_j}}.$$

This is equivalent to

$$\frac{\rho}{t_\ell} - \frac{1}{t_{\ell-1}} - \frac{1}{\rho}\sum_{j=0}^{\ell-2} \frac{(\rho+\varepsilon)^{j+1-\ell}}{t_j} < 0. \tag{3.82}$$

We have $p(c_{\ell-1}) = \frac{\rho v(c_{\ell-1})}{t_\ell}$ and thus

$$\begin{aligned}
\frac{1}{v_{\ell-1}}\left(p(c_{\ell-1}) - \sum_{j=0}^{\ell-1} c_j\right) &= \frac{1}{v_{\ell-1}}\left(\frac{\rho v(c_{\ell-1})}{t_\ell} - c_{\ell-1} - \sum_{j=0}^{\ell-2} c_j\right) \\
&\overset{(3.78)}{\leq} \frac{1}{v_{\ell-1}}\left(\frac{\rho v(c_{\ell-1})}{t_\ell} - c_{\ell-1} - \frac{1}{\rho}\sum_{j=0}^{\ell-2} \frac{v_j}{t_j}\right) \\
&= \frac{c_{\ell-1}}{v_{\ell-1}}\left(\rho\frac{t_{\ell-1}}{t_\ell} - 1\right) - \frac{1}{\rho}\sum_{j=0}^{\ell-2} \frac{(\rho+\varepsilon)^{j+1-\ell}}{t_j}.
\end{aligned}$$

As $c_{\ell-1} \leq \frac{v_{\ell-1}}{t_{\ell-1}}$, we obtain

$$\begin{aligned}
\frac{1}{v_{\ell-1}}\left(p(c_{\ell-1}) - \sum_{j=0}^{\ell-1} c_j\right) &\leq \frac{1}{t_{\ell-1}}\left(\rho\frac{t_{\ell-1}}{t_\ell} - 1\right) - \frac{1}{\rho}\sum_{j=0}^{\ell-2} \frac{(\rho+\varepsilon)^{j+1-\ell}}{t_j} \\
&= \frac{\rho}{t_\ell} - \frac{1}{t_{\ell-1}} - \frac{1}{\rho}\sum_{j=0}^{\ell-2} \frac{(\rho+\varepsilon)^{j+1-\ell}}{t_j} \\
&\overset{(3.82)}{<} 0.
\end{aligned}$$

Since $v_{\ell-1} = (\rho+\varepsilon)^{\ell-1} > 0$, we find that $p(c_{\ell-1}) - \sum_{j=0}^{j-1} c_j < 0$, which is a contradiction to Lemma 3.14 *(iii)* and the fact that $(c_0, \ldots, c_k)$ is $\rho$-competitive. Thus, a $\rho$-competitive incremental solution cannot exist. □

As the strict competitive ratio of IncMaxCont is smaller or equal to the (non-strict) competitive ratio of IncMax$_{\text{acc}}$ we immediately obtain the following.

**Theorem 3.28.** *The (non-strict) competitive ratio of IncMax$_{\text{acc}}$ is at least* 2.246.

# 4. Scaling Algorithms for Separable Incremental Maximization

Now that we have derived an improved lower bound on the competitive ratio of IncMax$_{\text{acc}}$, the subclass of IncMax of instances with monotone and accountable objectives, we will turn to investigate multiple algorithms for this problem class. More precisely, we will consider algorithms for the problem class IncMaxSep. By Proposition 3.8, these algorithms induce algorithms for IncMax$_{\text{acc}}$ with the same competitive ratios. Because of the simple structure of the problems in IncMaxSep, incremental solutions for problems in this class can be represented as a sequence of cardinalities $(c_1, c_2, \dots)$ (cf. Lemma 3.7). Even though, we presented evidence that the upper bound of $\varphi+1$ on the competitive ratio of IncMax$_{\text{acc}}$ might be tight, analyzing different algorithms for this problem class might lead to a better understanding of the problem and help in finding better lower bounds on the competitive ratio. Furthermore, it enables us to improve the upper bound by randomizing one of the algorithms in Section 4.2.

CardinalityScaling is the best known algorithm for the incremental maximization problem with monotone and accountable objective and was introduced in [5]. It adds the optimum solutions for increasing cardinalities $c_1 < c_2 < \dots$, where the cardinalities increase by a factor $\delta = \varphi + 1$, which we call the scaling parameter. In the context of separable incremental maximization problems, this corresponds to the solution represented by the cardinalities $(c_1, c_2, \dots)$. We will further analyze the competitive ratio of this algorithm and introduce similar algorithms that all follow the same idea of scaling up the cardinalities in the sequence representation.

In Section 4.1, we will investigate three deterministic algorithms for the IncMaxSep problem, CardinalityScaling, ValueScaling, and DensityScaling. Each of these algorithms scales the cardinalities in its sequence representation $(c_1, c_2, \dots)$ such that either the cardinalities $c_i$, the values $v_i$, or the densities $d_i$ are at least scaled by some fixed scaling factor $\delta$. We will show that CardinalityScaling and ValueScaling each have a tight competitive ratio of $\varphi + 1 \approx 2.618$, and that DensityScaling has a tight competitive ratio of $4$.

In Section 4.2, we combine the idea of CardinalityScaling with a randomization approach from [44] and present the algorithm RandScaling. This algorithm chooses the first cardinality in the sequence representation randomly and, similar to the CardinalityScaling algorithm, iteratively scales the next cardinalities by a fixed scaling parameter $\delta$. We show that this algorithm has a randomized competitive ratio of at most $1.772$. We complement this upper bound on the randomized competitive ratio of IncMaxSep with a lower bound of $1.357$ by employing Yao's principle.

Recall that the groundset of an instance of IncMaxSep is partitioned into disjoint sets $U_1, U_2, \dots$. Since $v_1 \leq v_2 \leq \dots$ and $d_1 \geq d_2 \geq \dots$, the optimum solution of cardinality $c \in \mathbb{N}$ is given by the set $U_c$. Thus, $\textsc{Opt}(c) = v_c$ for all $c \in \mathbb{N}$.

An extended abstract with the results in Section 4.2 appeared in [19].

## 4.1. Deterministic Scaling Algorithms

### 4.1.1. Cardinality Scaling

We start off with the CardinalityScaling algorithm that was introduced in [5] for the $\textsc{IncMax}_{\text{acc}}$ problem. For this problem, the algorithm is the best known algorithm with a competitive ratio of $\varphi + 1$. In this section, we will adapt this algorithm for the IncMaxSep problem and show that the competitive ratio of $\varphi + 1$ is tight.

The CardinalityScaling algorithm operates as follows. It fixes a scaling parameter $\delta > 1$ and calculates $c_1 = 1$, and $c_{i+1} = \lceil \delta c_i \rceil$ for all $i \in \mathbb{N}$. The incremental solution of CardinalityScaling is the one represented by the sequence $(c_1, c_2, \dots)$.[1]

An upper bound on the competitive ratio of this algorithm was given in [5].

**Theorem 4.1** ([5, Theorem 3]). *CardinalityScaling with scaling factor $\delta = \varphi + 1$ is $(\varphi + 1)$-competitive for IncMaxSep.*

We will now derive lower bounds on the competitive ratio of CardinalityScaling to show that its competitive ratio is exactly $\varphi + 1$. In the analysis, the following estimate is very useful.

The fact that $c_{i+1} = \lceil \delta c_i \rceil$ yields $c_i \geq \frac{1}{\delta}(c_{i+1} - 1)$ for all $i \in \mathbb{N}$. Using this iteratively, for $i \in [k]$, we obtain

$$c_i \geq \frac{1}{\delta^{k-i}} c_k - \sum_{j=1}^{k-i} \frac{1}{\delta^j} > \frac{1}{\delta^{k-i}} c_k - \sum_{j=1}^{\infty} \frac{1}{\delta^j} = \frac{1}{\delta^{k-i}} c_k - \frac{1}{\delta - 1}.$$

[1]Note that, with this definition, the algorithm does not terminate on finite instances. To avoid this, it suffices to stop calculating the cardinalities $c_i$ when they become larger than the number of elements in the instance. This will also be the case for the other algorithms presented in this chapter.

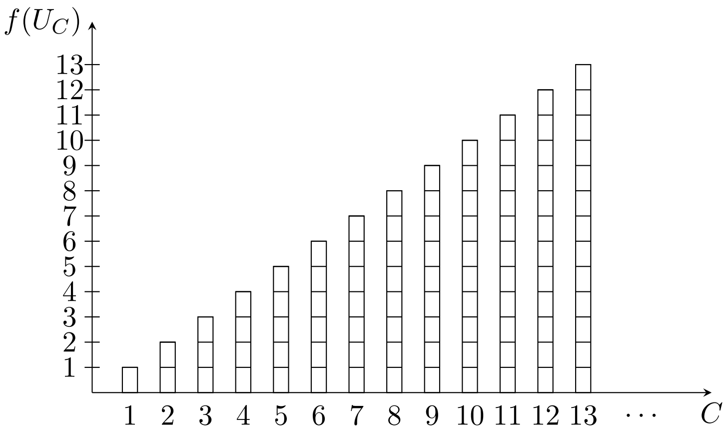


Figure 4.1.: Visualization of the lower bound instance in the proof of Proposition 4.2.

This yields

$$\begin{aligned}\sum_{i=1}^{k} c_i &> \sum_{i=1}^{k}\left(\frac{1}{\delta^{k-i}}c_k - \frac{1}{\delta-1}\right)\\ &= \frac{1-\frac{1}{\delta^{k+1}}}{1-\frac{1}{\delta}}c_k - \frac{k}{\delta-1}\\ &= \left(\frac{\delta}{\delta-1} - \frac{1}{\delta^k(\delta-1)} - \frac{1}{\delta-1}\frac{k}{c_k}\right)c_k \end{aligned} \tag{4.1}$$

We will now give a lower bound for small scaling parameters $\delta \leq \varphi + 1$.

**Proposition 4.2.** *For all $\delta > 1$, the competitive ratio of CardinalityScaling with scaling factor $\delta$ for IncMaxSep is at least $1 + \frac{\delta}{\delta-1}$ for all $\delta > 1$.*

*Proof.* We define the following lower bound instance of IncMaxSep (cf. Figure 4.1). Let $U_1, U_2, \ldots$ be disjoint sets such that $|U_c| = c$ for all $c \in \mathbb{N}$. We define the groundset $U := \bigcup_{c\in\mathbb{N}} U_c$ and the densities $d_c = 1$ for all $c \in \mathbb{N}$.

We denote the incremental solution of CardinalityScaling by $X$ and the sequence that represents $X$ by $(c_1, c_2, \ldots)$. Let $k \in \mathbb{N}$ be some large integer. We consider the cardinality constraint $C = \left(\sum_{i=1}^{k} c_i\right) + c_k$. The solution $X(C)$ contains the sets $U_{c_1}, \ldots, U_{c_k}$ and $c_k$ elements from the set $U_{c_{k+1}}$. This and $d_{c_k} = d_{c_{k+1}} = 1$ imply $f(X(C)) = c_k$. Furthermore,

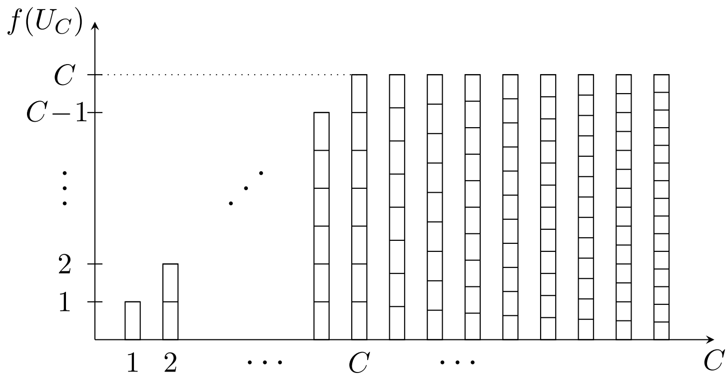


Figure 4.2.: Visualization of the lower bound instance in the proof of Proposition 4.3.

$\textsc{Opt}(C) = v_C = Cd_C = C$. Combining all of the above gives us

$$\frac{\textsc{Opt}(C)}{X(C)} = \frac{C}{c_k} = \frac{\left(\sum_{i=1}^{k} c_i\right) + c_k}{c_k} \overset{(4.1)}{>} \frac{\delta}{\delta - 1} - \frac{1}{\delta^k(\delta - 1)} - \frac{1}{\delta - 1}\frac{k}{c_k} + 1.$$

The lower bound of $1 + \frac{\delta}{\delta-1}$ follows in the limit for $k \to \infty$ because $c_k$ grows exponentially fast in $k$. □

We complement this lower bound for small scaling parameters with one for large scaling parameters $\delta \geq \varphi + 1$.

**Proposition 4.3.** *For all $\delta \geq \varphi + 1$, the competitive ratio of* CardinalityScaling *with scaling factor $\delta$ for* IncMaxSep *is at least $\frac{\delta}{2(\delta-1)} + \sqrt{\frac{\delta^2}{4(\delta-1)^2} + \delta}$ for all $\delta \geq \varphi + 1$.*

*Proof.* Let $k \in \mathbb{N}$ be some large integer, $\rho = \frac{\delta}{2(\delta-1)} + \sqrt{\frac{\delta^2}{4(\delta-1)^2} + \delta}$ and $C = \lfloor \rho c_k \rfloor$ be the cardinality constraint. We define the following lower bound instance of IncMaxSep (cf. Figure 4.2). Let $U_1, U_2, \ldots$ be disjoint sets such that $|U_c| = c$ for all $c \in \mathbb{N}$. We define the groundset $U := \bigcup_{c \in \mathbb{N}} U_c$ and the densities

$$d_c := \begin{cases} 1, & \text{if } c \in [C] \\ \frac{C}{c}, & \text{else.} \end{cases}$$

Then the values of the sets are

$$v_c = \begin{cases} c, & \text{if } c \in [C] \\ C, & \text{else.} \end{cases}$$

We denote the incremental solution of CardinalityScaling by $X$ and the sequence that represents $X$ by $(c_1, c_2, \dots)$. Due to the fact that $c_{i+1} \geq \delta c_i$ for all $i \in \mathbb{N}$, we have

$$\sum_{i=1}^{k} c_i \overset{\text{Lem. 1.6}}{<} \frac{\delta}{\delta - 1} c_k = \left( \frac{\delta}{2(\delta-1)} + \sqrt{\frac{\delta^2}{4(\delta-1)^2}} \right) c_k < \rho c_k.$$

Because $\sum_{i=1}^{k} c_i \in \mathbb{N}$ and $C = \lfloor \rho c_k \rfloor$, we obtain

$$\sum_{i=1}^{k} c_i \leq C. \tag{4.2}$$

By the fact that $\delta \geq \varphi + 1$, we have $\frac{\delta-1}{\delta-2} \leq \delta$. This yields

$$\delta \leq \delta^2 \frac{\delta - 2}{\delta - 1} = \delta^2 \left(1 - \frac{1}{\delta - 1}\right) = \delta^2 - \frac{\delta^2}{\delta - 1},$$

and therefore

$$\frac{\delta^2}{4(\delta-1)^2} + \delta \leq \frac{\delta^2}{4(\delta-1)^2} + \delta^2 - \frac{\delta^2}{\delta-1} = \left(\delta - \frac{\delta}{2(\delta-1)}\right)^2.$$

This yields

$$\rho = \frac{\delta}{2(\delta-1)} + \sqrt{\frac{\delta^2}{4(\delta-1)^2} + \delta} \leq \frac{\delta}{2(\delta-1)} + \left(\delta - \frac{\delta}{2(\delta-1)}\right) = \delta,$$

which implies

$$C = \lfloor \rho c_k \rfloor \leq \rho c_k \leq \delta c_k \leq \lceil \delta c_k \rceil = c_{k+1}. \tag{4.3}$$

The solution $X(C)$ contains the sets $U_{c_1}, \dots, U_{c_k}$ because of (4.2) and exactly $C - \sum_{i=1}^{k} c_i$ elements from the set $U_{k+1}$ because of (4.3). Thus,

$$f(X(C)) = \max\left\{ v_{c_k}, \left(C - \sum_{i=1}^{k} c_i\right) d_{c_{k+1}} \right\} = \max\left\{ c_k, \left(C - \sum_{i=1}^{k} c_i\right) \frac{C}{c_{k+1}} \right\}. \tag{4.4}$$

We have

$$
\begin{aligned}
\left(C - \sum_{i=1}^{k} c_i\right)\frac{C}{c_{k+1}} &\overset{\text{Def. } C, c_{k+1}}{\leq} \left(\rho c_k - \sum_{i=1}^{k} c_i\right)\frac{\rho}{\delta} \\
&\overset{(4.1)}{\leq} \left(\rho - \frac{\delta}{\delta-1} + \frac{1}{\delta^k(\delta-1)} + \frac{k}{c_k(\delta-1)}\right)\frac{\rho}{\delta} c_k \\
&\overset{\text{Def. } \rho}{=} \left(\frac{1}{\delta}\left(\frac{\delta}{2(\delta-1)} + \sqrt{\frac{\delta^2}{4(\delta-1)^2} + \delta}\right)^2 - \frac{1}{\delta-1}\left(\frac{\delta}{2(\delta-1)} + \sqrt{\frac{\delta^2}{4(\delta-1)^2} + \delta}\right)\right. \\
&\qquad \left. + \frac{\rho}{\delta^{k+1}(\delta-1)} + \frac{\rho k}{c_k \delta(\delta-1)}\right) c_k \\
&= \left(\left(\frac{\delta}{4(\delta-1)^2} + \frac{1}{\delta-1}\sqrt{\frac{\delta^2}{4(\delta-1)^2} + \delta} + \frac{\delta}{4(\delta-1)^2} + 1\right)\right. \\
&\qquad \left. - \left(\frac{\delta}{2(\delta-1)^2} + \frac{1}{\delta-1}\sqrt{\frac{\delta^2}{4(\delta-1)^2} + \delta}\right) + \frac{\rho}{\delta^{k+1}(\delta-1)} + \frac{\rho k}{c_k \delta(\delta-1)}\right) c_k \\
&= \left(1 + \frac{\rho}{\delta^{k+1}(\delta-1)} + \frac{\rho k}{c_k \delta(\delta-1)}\right) c_k.
\end{aligned}
$$

Together with (4.4), this yields

$$
f(X(C)) \leq \left(1 + \frac{\rho}{\delta^{k+1}(\delta-1)} + \frac{\rho k}{c_k \delta(\delta-1)}\right) c_k
$$

and thus

$$
\frac{\textsc{Opt}(C)}{f(X(C))} = \frac{C}{f(X(C))} \geq \frac{\rho c_k - 1}{\left(1 + \frac{\rho}{\delta^{k+1}(\delta-1)} + \frac{\rho k}{c_k \delta(\delta-1)}\right) c_k}.
$$

In the limit $k \to \infty$, the terms $\frac{\rho}{\delta^{k+1}(\delta-1)}$ and $\frac{\rho k}{c_k \delta(\delta-1)}$ vanish and we are left with

$$
\frac{\textsc{Opt}(C)}{f(X(C))} \geq \rho - \frac{1}{c_k} - \varepsilon
$$

for arbitrarily small $\varepsilon > 0$. The fraction $\frac{1}{c_k}$ vanishes as well for $k \to \infty$ and we are done. □

By taking the maximum of the two lower bounds in Propositions 4.2 and 4.3, we obtain the following lower bound on the competitive ratio of CardinalityScaling.

**Theorem 4.4.** *For all $\delta > 1$, the competitive ratio of* CardinalityScaling *with scaling factor $\delta$ for* IncMaxSep *is at least $\varphi + 1$.*

Combining the above results yields the following.

**Corollary 4.5.** *The competitive ratio of* CardinalityScaling *with scaling factor $\delta = \varphi + 1$ for* IncMaxSep *is exactly $\varphi + 1$.*

By Proposition 3.8, an incremental solution $(c_1, c_2, \dots)$ for IncMaxSep can be transformed into one for $\text{IncMax}_{\text{acc}}$ by adding $O_{c_1}$, then $O_{c_2}$, and so on in the accountable instance. Thus, by interpreting the solution returned by CardinalityScaling in this way, we obtain the following result.

**Corollary 4.6.** *The competitive ratio of* CardinalityScaling *with scaling factor $\delta = \varphi + 1$ for* $\text{IncMax}_{\text{acc}}$ *is exactly $\varphi + 1$.*

### 4.1.2. Value Scaling

Now, we present an algorithm that has a similar idea to the CardinalityScaling algorithm, but instead of scaling the cardinalities by a scaling parameter, the ValueScaling algorithm increases the cardinalities such that the corresponding values of the optimum solutions are scaled by a scaling parameter.

The ValueScaling algorithm operates as follows. It fixes a scaling parameter $\delta > 1$ and calculates $c_1 = 1$, and $c_{i+1} = \min\{c \in \mathbb{N} \mid v_c \geq \delta v_{c_i}\}$ for all $i \in \mathbb{N}$. The incremental solution of ValueScaling is the one represented by the sequence $(c_1, c_2, \dots)$.

Note that, for all $i \in \mathbb{N}$, we have

$$c_{i+1} = \frac{v_{c_{i+1}}}{d_{c_{i+1}}} \geq \frac{\delta v_{c_i}}{d_{c_i}} = \delta c_i. \tag{4.5}$$

We show that ValueScaling is as good as CardinalityScaling.

**Theorem 4.7.** *ValueScaling with scaling factor $\delta = \varphi + 1$ is $(\varphi+1)$-competitive for IncMaxSep.*

*Proof.* Fix an instance of $\text{IncMax}_{\text{acc}}$ with objective $f$, and let $X$ denote the incremental solution of ValueScaling. Furthermore, let $(c_1, c_2, \dots)$ denote the sequence representing $X$. First, suppose that, for some $k \in \mathbb{N}$, we have $C \in \left[\sum_{i=1}^{k} c_i, (\varphi+1)c_k\right]$. Then $X(C)$ contains all sets $U_{c_1}, \dots, U_{c_k}$, i.e.,

$$f(X(C)) \geq v_{c_k} = c_k d_{c_k} \overset{C \geq c_k}{\geq} c_k d_C \geq \frac{1}{\varphi+1} C d_C = \frac{1}{\varphi+1} v_C.$$

Now suppose that, for some $k \in \mathbb{N}$, we have $C \in \left[(\varphi+1)c_k, \sum_{i=1}^{k+1} c_i\right]$. This implies $C \geq c_{k+1} = \lceil(\varphi+1)c_k\rceil$ because $C \in \mathbb{N}$ and $C \geq (\varphi+1)c_k$. By Lemma 1.6 and (4.5), we have $\sum_{i=1}^{k} c_i \leq \varphi c_k < C$, i.e., the solution $X(C)$ contains all elements from the sets $U_{c_1}, \dots, U_{c_k}$. Thus and because $C \leq \sum_{i=1}^{k+1} c_i$, $X(C)$ contains exactly $C - \sum_{i=1}^{k} c_i$ elements from the set $U_{c_{k+1}}$. Therefore,

$$\begin{aligned} f(X(C)) &\geq \left(C - \sum_{i=1}^{k} c_i\right) d_{c_{k+1}} \overset{\text{Lem. 1.6}}{\geq} (C - \varphi c_k) d_{c_{k+1}} \overset{C \geq c_{k+1}}{\geq} (C - \varphi c_k) d_C \\ &= \frac{C - \varphi c_k}{C} v_C \geq \left(1 - \frac{\varphi c_k}{(\varphi+1)c_k}\right) v_C = \frac{1}{\varphi+1} v_C. \end{aligned}$$
□

We show that this upper bound is indeed tight. We do this again by giving two lower bounds, starting with the lower bound for small scaling parameters $\delta \leq \varphi + 1$.

**Proposition 4.8.** *For all $\delta > 1$, the competitive ratio of* ValueScaling *with scaling factor $\delta$ for* IncMaxSep *is at least $1 + \frac{\delta}{\delta-1}$.*

*Proof.* We will show this lower bound with the same construction as in the proof of Proposition 4.2. Let $U_1, U_2, \dots$ be disjoint sets such that $|U_c| = c$ for all $c \in \mathbb{N}$. We define the groundset $U := \bigcup_{c \in \mathbb{N}} U_c$ and the densities $d_c = 1$ for all $c \in \mathbb{N}$.

Let $X$ denote the incremental solution of ValueScaling and let it be represented by the sequence $(c_1, c_2, \dots)$. Note that $v_c = c$ for all $c \in \mathbb{N}$. This yields that $c_1 = 1$ and

$$c_{i+1} = \min\{c \in \mathbb{N} \mid v_c \geq \delta v_{c_i}\} = \lceil \delta c_i \rceil$$

for all $i \in \mathbb{N}$. Thus, $X$ is identical to the incremental solution of CardinalityScaling for this instance. This implies that the competitive ratios of the solution of ValueScaling cannot be better than the competitive ratio of the solution of CardinalityScaling. Thus, also ValueScaling has a competitive ratio of at least $1 + \frac{\delta}{\delta-1}$ for this instance. □

We complement this with a lower bound for large scaling parameters $\delta \leq \varphi + 1$.

**Proposition 4.9.** *For all $\delta > 1$, the competitive ratio of* ValueScaling *with scaling factor $\delta$ for* IncMaxSep *is at least $\delta$.*

*Proof.* We will prove this statement via contradiction. For this suppose, there was a scaling parameter $\delta > 1$ such that ValueScaling was $\rho$-competitive for $\rho < \delta$. Let $C = \lceil \frac{\rho+\delta}{2} \rceil$. We define a lower bound instance of the IncMaxSep problem (cf. Figure 4.3). Let $U_1, U_2, \dots$

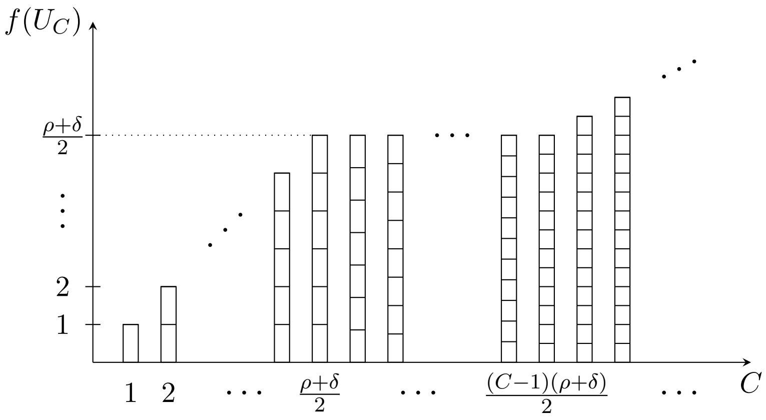


Figure 4.3.: Visualization of the lower bound instance in the proof of Proposition 4.9.

be disjoint sets such that $|U_c| = c$ for all $c \in \mathbb{N}$. We define the groundset $U := \bigcup_{c\in\mathbb{N}} U_c$ and the densities

$$d_c = \begin{cases} 1, & \text{if } c \leq \frac{\rho+\delta}{2}, \\ \frac{\rho+\delta}{2c}, & \text{if } \frac{\rho+\delta}{2} < c \leq \frac{(C-1)(\rho+\delta)}{2}, \\ \frac{1}{C-1}, & \text{else.} \end{cases}$$

Then

$$v_c = \begin{cases} c, & \text{if } c \leq \frac{\rho+\delta}{2}, \\ \frac{\rho+\delta}{2}, & \text{if } \frac{\rho+\delta}{2} < c \leq \frac{(C-1)(\rho+\delta)}{2}, \\ \frac{c}{C-1}, & \text{else.} \end{cases}$$

We consider the cardinality constraint $C$. Let $X$ denote the incremental solution of ValueScaling that is represented by the sequence $(c_1, c_2, \dots)$. We have $c_1 = 1$ and $v_{c_1} = 1$. Since $v_c \leq \frac{\rho+\delta}{2} < \delta = \delta v_{c_1}$ for all $c \in \left[\left\lfloor\frac{(C-1)(\rho+\delta)}{2}\right\rfloor\right]$, we have $c_2 > \frac{(C-1)(\rho+\delta)}{2}$. Thus, $d_{c_2} = \frac{1}{C-1}$. The solution $X(C)$ contains the set $U_1$ and $C-1$ elements from the set $U_{c_2}$. Thus,

$$f(X(C)) = \max\{v_1, (C-1)d_{c_2}\} = \max\left\{1, (C-1)\frac{1}{C-1}\right\} = 1.$$

Yet, we have $v_C = \frac{\rho+\delta}{2} > \rho = \rho f(X(C))$, i.e., ValueScaling is not $\rho$-competitive. □

By taking the maximum of the two lower bounds in Propositions 4.8 and 4.9, we obtain the following lower bound on the competitive ratio of ValueScaling.

**Theorem 4.10.** *For all $\delta > 1$, the competitive ratio of* ValueScaling *with scaling factor $\delta$ for* IncMaxSep *is at least $\varphi + 1$.*

Combining the above results yields the following.

**Corollary 4.11.** *The competitive ratio of* ValueScaling *with scaling factor $\delta = \varphi + 1$ for* IncMaxSep *is exactly $\varphi + 1$.*

As reasoned before Corollary 4.6, we can interpret the solution returned by ValueScaling to work for accountable problem instances and obtain the following result.

**Corollary 4.12.** *The competitive ratio of* ValueScaling *with scaling factor $\delta = \varphi + 1$ for* $\text{IncMax}_{\text{acc}}$ *is exactly $\varphi + 1$.*

### 4.1.3. Density Scaling

In this section, we present the DensityScaling algorithm. Similar to the two algorithms in the two previous sections, the idea is to increase the cardinalities in such a way that some value associated with the sets $U_i$ is scaled by some scaling parameter. This time we scale the density $d_i$ of the sets.

The DensityScaling algorithm operates as follows. It fixes a scaling parameter $\delta \in (0, 1)$ and calculates $c_1 = 1$ and $c_{i+1} = \max\{c \in \mathbb{N} \mid d_c \geq \delta d_{c_i}\}$ for all $i \in \mathbb{N}$. Note that this time we are not scaling up some value, but are scaling it down to be smaller in the next step. The incremental solution of DensityScaling is the one represented by $(c_1, c_2, \dots)$.

**Remark 4.13.** *In order to be able to show bounds on the competitive ratio of* DensityScaling, *we have to assume that $\lim_{c\to\infty} d_c = 0$. Otherwise, we might have the problem that $c_{i+1}$ does not exist for some $i \in \mathbb{N}$. This would be the case for the lower bound construction in the proof of Proposition 4.2, where $d_c = 1$ for all $c \in \mathbb{N}$. Here,* DensityScaling *would terminate after adding one element and would therefore have an unbounded competitive ratio.*

Note that, because $v_1 \leq v_2 \leq \dots$, we have

$$d_{\lfloor \frac{c_i}{\delta} \rfloor} = \frac{v_{\lfloor \frac{c_i}{\delta} \rfloor}}{\lfloor \frac{c_i}{\delta} \rfloor} \geq \frac{v_{c_i}}{\lfloor \frac{c_i}{\delta} \rfloor} \geq \delta \frac{v_{c_i}}{c_i} = \delta d_{c_i},$$

which yields

$$c_{i+1} \geq \left\lfloor \frac{c_i}{\delta} \right\rfloor. \tag{4.6}$$

In particular, if $\delta = \frac{1}{k}$ for some integer $k \in \mathbb{N}$, then $c_{i+1} \geq k c_i$.

We give an upper bound on the competitive ratio of DensityScaling.

**Theorem 4.14.** *DensityScaling with scaling factor $\delta = \frac{1}{2}$ is 4-competitive for IncMaxSep.*

*Proof.* Let $C \in \mathbb{N}$ be the cardinality constraint. Let $k \in \mathbb{N}$ such that $C \in \left[\sum_{i=1}^{k} c_i, \sum_{i=1}^{k+1} c_i\right)$, and let $X$ denote the incremental solution of DensityScaling that is represented by the sequence $(c_1, c_2, \dots)$.

First, consider the case that $C \le 4c_k$. Recall that $C \ge \sum_{i=1}^{k} c_i$. Thus, the solution $X(C)$ contains all elements from the sets $U_{c_1}, \dots, U_{c_k}$, i.e.,

$$f(X(C)) \ge v_{c_k} = c_k d_{c_k} \ge c_k d_{4c_k} = \frac{1}{4}(4c_k) d_{4c_k} = \frac{1}{4} v_{4c_k} \ge \frac{1}{4} v_C.$$

Now, consider the case that $C \ge 4c_k$. As $C < \sum_{i=1}^{k+1} c_i$, $X(C)$ does not contain the whole set $U_{c_{k+1}}$. Yet, it contains all the sets $U_{c_1}, \dots, U_{c_k}$. Thus, $X(C)$ contains exactly $C - \sum_{i=1}^{k} c_i$ elements from the set $U_{c_{k+1}}$, i.e., we have

$$\begin{aligned} f(X(C)) &\ge \left(C - \sum_{i=1}^{k} c_i\right) d_{c_{k+1}} \overset{\text{Lem. 1.6, (4.6)}}{\ge} (C - 2c_k) d_{c_{k+1}} \\ &\overset{\text{Def. } c_{k+1}}{\ge} (C - 2c_k)\delta d_{c_k} \overset{C \ge c_k}{\ge} (C - 2c_k)\delta d_C \\ &= \delta \frac{C - 2c_k}{C} v_C \overset{C \ge 4c_k}{\ge} \delta\left(1 - \frac{2c_k}{4c_k}\right) v_C = \frac{1}{4} v_C. \end{aligned}$$

□

We complement this upper bound with a tight lower bound. We start with a bound for small scaling parameters $\delta \le \frac{1}{2}$.

**Proposition 4.15.** *For all $\delta \in \left(0, \frac{1}{2}\right]$, the competitive ratio of DensityScaling with scaling parameter $\delta$ for IncMaxSep is at least $\frac{1}{\delta(1-\delta)} \ge 4$.*

*Proof.* Let $\varepsilon > 0$ be arbitrarily small. We define $t_1 := 1$, $t_{i+1} := \left\lfloor \frac{1}{\delta} t_i \right\rfloor$ for $i \in \mathbb{N}$. Since $\delta \le \frac{1}{2}$, we have

$$t_{i+1} = \left\lfloor \frac{1}{\delta} t_i \right\rfloor \ge \lfloor 2t_i \rfloor = 2t_i > t_i,$$

i.e., $(t_1, t_2, \dots)$ is strictly increasing.

Let $k \in \mathbb{N}$ be large enough such that, with $C := \left(\sum_{i=1}^{k} t_i\right) + \left\lfloor \frac{1}{\delta} t_k \right\rfloor + 1$, we have

$$\left(1 + \delta \frac{1}{t_k}\right) \frac{C+1}{C} \le 1 + \varepsilon \tag{4.7}$$

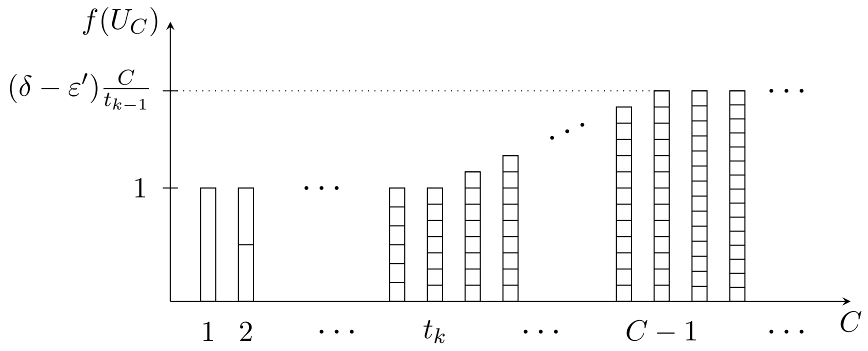


Figure 4.4.: Visualization of the lower bound instance in the proof of Proposition 4.15.

and

$$\left(\frac{1-\delta^{k+1}}{1-\delta}+\frac{1}{\delta}\right)\left(\frac{1}{\delta}t_{k-1}-1\right) \geq (1-\varepsilon)\left(\frac{1}{1-\delta}+\frac{1}{\delta}\right)\frac{1}{\delta}t_{k-1}. \tag{4.8}$$

Let $\varepsilon' > 0$ such that

$$\varepsilon' < \varepsilon \tag{4.9}$$

and

$$(\delta-\varepsilon')\frac{1}{t_{k-1}} \geq \delta\frac{1}{t_k}, \tag{4.10}$$

which is possible because $t_k > t_{k-1}$.

With these preparations, we are ready to define the lower bound instance of IncMaxSep (cf. Figure 4.4). Let $U_1, U_2, \dots$ be disjoint sets such that $|U_c| = c$ for all $c \in \mathbb{N}$. We define the groundset $U := \bigcup_{c\in\mathbb{N}} U_c$ and, the densities

$$d_c = \begin{cases} \frac{1}{c}, & \text{if } c \leq t_k, \\ (\delta-\varepsilon')\frac{1}{t_{k-1}}, & \text{if } t_k+1 \leq c \leq C-1, \\ (\delta-\varepsilon')\frac{1}{t_{k-1}}\cdot\frac{C}{c}, & \text{else.} \end{cases}$$

The densities are non-increasing. For $c \in \mathbb{N}\setminus\{t_k\}$, $d_c \geq d_{c+1}$ follows immediately from the definition. For $c = t_k$, we have

$$d_{t_k} = \frac{1}{t_k} = \frac{1}{\left\lfloor\frac{1}{\delta}t_{k-1}\right\rfloor} \geq \delta\frac{1}{t_{k-1}} > (\delta-\varepsilon')\frac{1}{t_{k-1}} = d_{t_k+1}.$$

Thus,

$$v_i = \begin{cases} 1, & \text{if } c \le t_k, \\ (\delta - \varepsilon')\frac{c}{t_{k-1}}, & \text{if } t_k + 1 \le c \le C - 1, \\ (\delta - \varepsilon')\frac{C}{t_{k-1}}, & \text{else.} \end{cases}$$

The values are non-decreasing. For $c \in \mathbb{N} \setminus \{t_k\}$, $v_c \le v_{c+1}$ can be seen immediately. For $c = t_k$, we have

$$v_{t_k} = 1 < \delta \overset{(4.10)}{\le} (\delta - \varepsilon')\frac{t_k}{t_{k-1}} < (\delta - \varepsilon')\frac{t_k + 1}{t_{k-1}} = v_{t_k+1}.$$

Note that $t_{i+1} = \left\lfloor \frac{1}{\delta} t_i \right\rfloor \le \frac{1}{\delta} c_i$ holds for all $i \in [k-1]$, which yields

$$\sum_{i=1}^{k} t_i \ge \sum_{i=1}^{k} \delta^{k-i} t_k = \frac{1 - \delta^{k+1}}{1 - \delta} t_k. \tag{4.11}$$

This yields

$$\begin{aligned} C &= \left(\sum_{i=1}^{k} t_i\right) + \left\lfloor \frac{1}{\delta} t_k \right\rfloor + 1 \overset{(4.11)}{\ge} \frac{1 - \delta^{k+1}}{1 - \delta} t_k + \frac{1}{\delta} t_k \\ &= \left(\frac{1 - \delta^{k+1}}{1 - \delta} + \frac{1}{\delta}\right) \left\lfloor \frac{1}{\delta} t_{k-1} \right\rfloor \ge \left(\frac{1 - \delta^{k+1}}{1 - \delta} + \frac{1}{\delta}\right) \left(\frac{1}{\delta} t_{k-1} - 1\right) \\ &\overset{(4.8)}{\ge} (1 - \varepsilon)\left(\frac{1}{1 - \delta} + \frac{1}{\delta}\right) \frac{1}{\delta} t_{k-1} = \frac{1 - \varepsilon}{\delta} \frac{1}{\delta(1 - \delta)} t_{k-1}. \end{aligned} \tag{4.12}$$

Let $X$ denote the incremental solution of DensityScaling with scaling parameter $\delta$, and let $(c_1, c_2, \dots)$ be its representation.

*Claim:* For $i \in [k]$, we have

$$c_i = t_i. \tag{4.13}$$

*Proof of Claim:* We show this by induction. For $i = 1$, we have $c_1 = 1 = t_1$ by definition. Now assume that, for some $i \in [k-1]$, $c_i = t_i$ holds. Then

$$d_{t_{i+1}} = \frac{1}{t_{i+1}} = \frac{1}{\left\lfloor \frac{1}{\delta} t_i \right\rfloor} \ge \delta \frac{1}{t_i} = \delta d_{t_i} = \delta d_{c_i},$$

which implies that $c_{i+1} \ge t_{i+1}$. If $i \le k - 2$, we have

$$d_{t_{i+1}+1} = \frac{1}{t_{i+1} + 1} = \frac{1}{\left\lfloor \frac{1}{\delta} t_i \right\rfloor + 1} < \delta \frac{1}{t_i} = \delta d_{t_i} = \delta d_{c_i}.$$

Otherwise, if $i = k-1$, then

$$d_{t_{i+1}+1} = d_{t_k+1} = (\delta - \varepsilon')\frac{1}{t_{k-1}} < \delta\frac{1}{t_{k-1}} = \delta d_{t_{k-1}} = \delta d_{t_i} = \delta d_{c_i}.$$

In both cases, we have $d_{t_{i+1}+1} < \delta d_{c_i}$, which yields $c_{i+1} < t_{i+1}+1$. Together with the fact that $c_{i+1} \geq t_{i+1}$, we have $c_{i+1} = t_{i+1}$, which establishes the claim.

By definition of $\varepsilon'$, we have

$$d_C = (\delta - \varepsilon')\frac{1}{t_{k-1}} \overset{(4.10)}{\geq} \delta\frac{1}{t_k} = \delta d_{t_k} \overset{(4.13)}{=} \delta d_{c_k},$$

i.e., $c_{k+1} \geq C$. This and the definition of $c_{k+1}$ imply

$$d_{c_{k+1}} = \frac{v_{c_{k+1}}}{c_{k+1}} = \frac{c_{k+1}+1}{c_{k+1}}\frac{v_{c_{k+1}+1}}{c_{k+1}+1} = \frac{c_{k+1}+1}{c_{k+1}}d_{c_{k+1}+1} < \frac{c_{k+1}+1}{c_{k+1}}\delta d_{c_k}. \tag{4.14}$$

The solution $X(C)$ contains all the sets $U_{c_1}, \dots, U_{c_k}$ because

$$C = \left(\sum_{i=1}^{k} t_i\right) + \left\lfloor \frac{1}{\delta} t_k \right\rfloor + 1 > \sum_{i=1}^{k} t_i \overset{(4.13)}{=} \sum_{i=1}^{k} c_i.$$

Furthermore, $X(C)$ contains $\lfloor \frac{1}{\delta} t_k \rfloor + 1$ elements from the set $U_{c_{k+1}}$. We have

$$\begin{aligned}\left(\left\lfloor \frac{1}{\delta} t_k \right\rfloor + 1\right) d_{c_{k+1}} &\overset{(4.14)}{<} \left(\left\lfloor \frac{1}{\delta} t_k \right\rfloor + 1\right)\frac{c_{k+1}+1}{c_{k+1}}\delta d_{c_k} = \left(\left\lfloor \frac{1}{\delta} t_k \right\rfloor + 1\right)\frac{c_{k+1}+1}{c_{k+1}}\delta\frac{1}{t_k} \\ &\leq \left(1 + \delta\frac{1}{t_k}\right)\frac{c_{k+1}+1}{c_{k+1}} \overset{c_{k+1}\geq C}{\leq} \left(1 + \delta\frac{1}{t_k}\right)\frac{C+1}{C} \overset{(4.7)}{\leq} 1 + \varepsilon.\end{aligned}$$

Thus, and because $v_i = 1$ for all $i \in [c_k]$, we have $f(X(C)) \leq 1 + \varepsilon$. Yet, we have $\textsc{Opt}(C) = v_C = (\delta - \varepsilon')\frac{C}{t_{k-1}}$. Thus, the competitive ratio of DensityScaling is at least

$$\frac{\textsc{Opt}(C)}{f(X(C))} \geq \frac{(\delta - \varepsilon')C}{(1+\varepsilon)t_{k-1}} \overset{(4.9),(4.12)}{\geq} \frac{(\delta-\varepsilon)(1-\varepsilon)}{(1+\varepsilon)\delta} \cdot \frac{1}{\delta(1-\delta)}.$$

The lower bound follows in the limit $\varepsilon \to 0$. □

With this, we can now show a lower bound on the competitive ratio of DensityScaling for all scaling parameters.

**Theorem 4.16.** *For all $\delta \in (0,1)$, the competitive ratio of DensityScaling with scaling parameter $\delta$ for IncMaxSep is at least $4$.*

*Proof.* For $\delta \in \left(0, \frac{1}{2}\right]$ this immediately follows from Proposition 4.15. Thus, suppose that $\delta \in \left(\frac{1}{2}, 1\right)$. Consider an instance of IncMaxSep with $d_1 = 1$, $d_2 = \frac{1}{2}$, and arbitrary $d_i \in \left(0, \frac{1}{2}\right]$ for all $i \in \mathbb{N}_{\geq 3}$ such that $d_2 \geq d_3 \geq \dots$ and $\lim_{c \to \infty} v_c = \infty$. DensityScaling chooses $c_1 = 1$ and

$$c_2 = \max\{c \in \mathbb{N} \mid d_c \geq \delta d_{c_1}\} = \max\{1\} = 1.$$

Here, we use the fact that $d_i \leq \frac{1}{2}$ for all $i \in \mathbb{N}_{\geq 2}$ and $\delta > \frac{1}{2}$. This continues, i.e., we have $c_i = 1$ for all $i \in \mathbb{N}$. Thus, DensityScaling only adds one element and stops after that. Since $\lim_{c \to \infty} v_c = \infty$, DensityScaling cannot be competitive. □

Combining the above results yields the following.

**Corollary 4.17.** *The competitive ratio of DensityScaling with scaling factor $\delta = \frac{1}{2}$ for IncMaxSep is exactly* $4$.

As reasoned before Corollaries 4.6 and 4.12, we can interpret the solution returned by DensityScaling to work for accountable problem instances and obtain the following result.

**Corollary 4.18.** *The competitive ratio of DensityScaling with scaling factor $\delta = \frac{1}{2}$ for IncMax$_{acc}$ is exactly* $4$.

## 4.2. Randomized Algorithms

We turn to analyzing randomized algorithms to solve the IncMaxSep problem. In contrast to deterministic algorithms, we do not compare the value obtained by the algorithm to an optimum solution, but rather the expected value obtained by the algorithm. This enables us to find an algorithm with randomized competitive ratio smaller than the lower bound of $2.24$ on the competitive ratio of deterministic algorithms we have seen in Theorem 3.28.

### 4.2.1. A Randomized Scaling Algorithm

As we have stated before, the best known deterministic algorithm is the CardinalityScaling algorithm [5]. In the analysis, it turns out that, on average, a scaling algorithm performs better than the actual competitive ratio, which is only tight for few cardinalities. By randomizing the initial cardinality $c_1$, we manage to average out the worst-case cardinalities in the analysis.

We describe the randomized algorithm RandScaling for IncMaxSep. In its analysis, we need the unique maximum of the function

$$
\begin{aligned}
g(x) \quad = \quad & z - \frac{1-x^{-3}}{2(x-1)\log(x)} - \frac{2x^{2+z}}{\left(\sqrt{\left(\frac{x^3-1}{x-1}x^z-1\right)^2+4x^{5+2z}} - \frac{x^3-1}{x-1}x^z+1\right)\log(x)} \\
& + \frac{1-\sqrt{\left(\frac{x^3-1}{x-1}x^z-1\right)^2+4x^{5+2z}}}{2\log(x)x^{3+z}} - (1-z)\frac{1-x^{-3}}{x-1} - \left(\frac{1-x^{-3}}{x-1} - \frac{1}{x^{3+z}}\right) \\
& \cdot\left(\log_x\left(\sqrt{\left(\frac{x^3-1}{x-1}x^z-1\right)^2+4x^{5+2z}} - \frac{x^3-1}{x-1}x^z+1\right) - \log_x(2) - 3\right) \\
& + \frac{2}{\log(x)} - \left(1+\frac{1}{x^{3+z}}\right)\left(\log_x(x^{3+z}+1)+\log_x(x-1)-\log_x(x^4-1)\right)
\end{aligned}
$$

with $z = \log_x\left(\frac{x^4-1}{x-1}-1\right)-3$. Let $\delta \approx 5.165$ be the unique maximum of $g$ with $g(\delta) \approx 0.564$. The upper bound on the randomized competitive ratio that we show later in the chapter will be $1/g(\delta) < 1.772$. The algorithm RandScaling starts by choosing $\varepsilon \in (0,1)$ uniformly at random. For all $i \in \mathbb{N}$, it calculates $\tilde{c}_i := \delta^{i-1+\varepsilon}$ and $c_i := \lfloor \tilde{c}_i \rfloor$ and returns the incremental solution represented by $(c_1, c_2, c_3, \dots)$. This approach is similar to a randomized algorithm to solve the CowPath problem in [44], which also calculates such a sequence with a different choice of $\delta \in \mathbb{R}$ in order to explore a star graph.

For better readability, we define

$$
\tilde{t}_i := \sum_{j=1}^{i} \tilde{c}_j = \delta^\varepsilon \frac{\delta^i - 1}{\delta - 1} \qquad \text{and} \qquad t_i := \sum_{j=1}^{i} c_j,
$$

as well as $\tilde{c}_0 = c_0 = \tilde{t}_0 = t_0 = 0$. Note that, for all $i \in \mathbb{N}$, we have

$$
t_i \leq \tilde{t}_i = \delta^\varepsilon \frac{\delta^i - 1}{\delta - 1} \overset{\delta > 2}{<} \delta^{i+\varepsilon} - \delta^\varepsilon \leq \delta^{i+\varepsilon} - 1 = \tilde{c}_{i+1} - 1 \leq c_{i+1}. \tag{4.15}
$$

We denote the incremental solution of RandScaling by $X$.

In order to find an upper bound on the randomized competitive ratio of RandScaling, we need the following lemma. It gives an estimate on the expected value of the solution for a fixed cardinality $C \in \mathbb{N}$ of RandScaling depending on the interval in which $C$ falls.

**Lemma 4.19.** *Let $C \in \mathbb{N}$.*

*(i) For $i \in \mathbb{N}$ with $\mathbb{P}[C \in (t_{i-1}, t_i]] > 0$, we have*

$$
\mathbb{E}\big[f(X(C)) \mid C \in (t_{i-1}, t_i]\big] \geq \mathbb{E}\left[\max\left\{\frac{c_{i-1}}{C}, \frac{C - t_{i-1}}{\max\{C, c_i\}}\right\} \,\middle|\, C \in (t_{i-1}, t_i]\right] \cdot v_C.
$$

*(ii) For $i \in \mathbb{N}$ with $\mathbb{P}[C \in (\tilde{c}_i, \tilde{t}_i - 1]] > 0$, we have*

$$\mathbb{E}\big[f(X(C)) \mid C \in (\tilde{c}_i, \tilde{t}_i - 1]\big] \geq \mathbb{E}\left[1 - \frac{\tilde{t}_{i-1}}{C} \;\middle|\; C \in (\tilde{c}_i, \tilde{t}_i - 1]\right] \cdot v_C.$$

*(iii) For $i \in \mathbb{N}$ with $\mathbb{P}[C \in (\tilde{t}_{i-1} - 1, \tilde{c}_i]] > 0$, we have*

$$\mathbb{E}\big[f(X(C)) \mid C \in (\tilde{t}_{i-1} - 1, \tilde{c}_i]\big] \geq \mathbb{E}\left[\max\left\{\frac{\tilde{c}_{i-1} - 1}{C}, \frac{C - \tilde{t}_{i-1}}{\tilde{c}_i}\right\} \;\middle|\; C \in (\tilde{t}_{i-1} - 1, \tilde{c}_i]\right] \cdot v_C.$$

*Proof.* We prove *(i)*. For $C \in (t_{i-1}, c_i]$ with $i \in \mathbb{N}$, the solution $X(C)$ contains the sets $U_1, \ldots, U_{c_{i-1}}$ and some elements from the set $U_{c_i}$. Thus,

$$\begin{aligned} f(X(C)) &= \max\{v_{c_{i-1}}, (C - t_{i-1})d_{c_i}\} \\ &= \max\left\{c_{i-1}d_{c_{i-1}}, (C - t_{i-1})\frac{v_{c_i}}{c_i}\right\} \\ &\overset{d \text{ dec.}, v \text{ inc.}}{\geq} \max\left\{c_{i-1}d_C, (C - t_{i-1})\frac{v_C}{c_i}\right\} \\ &= \max\left\{\frac{c_{i-1}}{C}, \frac{C - t_{i-1}}{c_i}\right\} \cdot v_C. \end{aligned} \tag{4.16}$$

Now, assume $C \in (c_i, t_i]$ for some $i \in \mathbb{N}$. By (4.15), we have $t_{i-1} \leq c_i$, i.e., the solution $X(C)$ contains the sets $U_1, \ldots, U_{c_{i-1}}$ and some elements from the set $U_{c_i}$. Thus, we have

$$\begin{aligned} f(X(C)) &= \max\{v_{c_{i-1}}, (C - t_{i-1})d_{c_i}\} \\ &= \max\{c_{i-1}d_{c_{i-1}}, (C - t_{i-1})d_{c_i}\} \\ &\overset{d \text{ dec.}}{\geq} \max\{c_{i-1}d_C, (C - t_{i-1})d_C\} \\ &= \max\left\{\frac{c_{i-1}}{C}, \frac{C - t_{i-1}}{C}\right\} \cdot v_C. \end{aligned} \tag{4.17}$$

Combining (4.16) and (4.17), for $C \in (c_{i-1}, c_i]$, we obtain

$$f(X(C)) \geq \max\left\{\frac{c_{i-1}}{C}, \frac{C - t_{i-1}}{\max\{C, c_i\}}\right\} \cdot v_C.$$

By monotonicity of the conditional expectation, *(i)* follows.

We prove *(ii)*. If $C \in (\tilde{c}_i, \tilde{t}_i - 1]$ for some $i \in \mathbb{N}$, we have

$$C > \tilde{c}_i \geq c_i \overset{(4.15)}{\geq} t_{i-1},$$

i.e., the solution $X(C)$ contains the sets $U_1, \dots, U_{c_{i-1}}$ and either some, or all elements from the set $U_{c_i}$. Thus,

$$\begin{aligned} f(X(C)) &\geq \min\{C - t_{i-1}, c_i\} d_{c_i} \geq \min\{C - \tilde{t}_{i-1}, c_i\} d_{c_i} \\ &\overset{C \leq \tilde{t}_i - 1}{=} (C - \tilde{t}_{i-1}) d_{c_i} = \left(1 - \frac{\tilde{t}_{i-1}}{C}\right) \frac{d_{c_i}}{d_C} \cdot v_C \\ &\overset{d \text{ dec.}}{\geq} \left(1 - \frac{\tilde{t}_{i-1}}{C}\right) \cdot v_C. \end{aligned}$$

By monotonicity of the conditional expectation, *(ii)* follows.

Lastly, we prove *(iii)*. If $C \in (\tilde{t}_{i-1} - 1, \tilde{c}_i]$ for some $i \in \mathbb{N}$, we have $C > \tilde{t}_{i-1} - 1 \geq t_{i-1} - 1$. Together with the fact that $C$ and $t_{i-1}$ are both integral, this implies $C \geq t_{i-1}$, i.e., the solution $X(C)$ contains the sets $U_1, \dots, U_{c_{i-1}}$ and some elements from the set $U_{c_i}$. Furthermore, $C \leq \tilde{c}_i$ and $C \in \mathbb{N}$ imply that $C \leq c_i = \lfloor \tilde{c}_i \rfloor$. Together this yields

$$\begin{aligned} f(X(C)) &= \max\{v_{c_{i-1}}, (C - t_{i-1}) d_{c_i}\} \\ &= \max\left\{c_{i-1} d_{c_{i-1}}, (C - t_{i-1}) \frac{v_{c_i}}{c_i}\right\} \\ &\overset{d \text{ dec.}, v \text{ inc.}}{\geq} \max\left\{c_{i-1} d_C, (C - t_{i-1}) \frac{v_C}{c_i}\right\} \\ &= \max\left\{\frac{c_{i-1}}{C}, \frac{C - t_{i-1}}{c_i}\right\} \cdot v_C \\ &\geq \max\left\{\frac{\tilde{c}_{i-1} - 1}{C}, \frac{C - \tilde{t}_{i-1}}{c_i}\right\} \cdot v_C. \end{aligned}$$

By monotonicity of the conditional expectation, *(iii)* follows. □

In the analysis of the algorithm, additionally, the following estimate is needed.

**Lemma 4.20.** *Let $k \in \mathbb{N}$ and $r \in (0,1]$ such that $\delta^{k+r} \geq \sum_{i=0}^{3} \delta^i$. Then*

$$\begin{aligned} g(\delta) \leq I(k,r) := & \int_{\min\{1,\mu(k)\}}^{1} 1 - \frac{\tilde{t}_{k-1}}{\delta^{k+r}} \, d\varepsilon \quad + \int_{\min\{1,\nu(k)\}}^{\min\{1,\mu(k)\}} \frac{\tilde{c}_k - 1}{\delta^{k+r}} \, d\varepsilon \\ & + \int_{r}^{\min\{1,\nu(k)\}} \frac{\delta^{k+r} - \tilde{t}_k}{\tilde{c}_{k+1}} \, d\varepsilon \quad + \int_{\max\{0,\mu(k+1)\}}^{r} 1 - \frac{\tilde{t}_k}{\delta^{k+r}} \, d\varepsilon \\ & + \int_{\max\{0,\nu(k+1)\}}^{\max\{0,\mu(k+1)\}} \frac{\tilde{c}_{k+1} - 1}{\delta^{k+r}} \, d\varepsilon + \int_{0}^{\max\{0,\nu(k+1)\}} \frac{\delta^{k+r} - \tilde{t}_{k+1}}{\tilde{c}_{k+1}} \, d\varepsilon \end{aligned}$$

*where*

$$\mu(i) = \log_\delta(\delta^{k+r}+1) + \log_\delta(\delta-1) - \log_\delta(\delta^i - 1),$$
$$\nu(i) = \log_\delta\left(\sqrt{\left(\delta^{k+r}\frac{1-\delta^{-i}}{\delta-1}-1\right)^2 + 4\delta^{2k+2r-1}} - \delta^{k+r}\frac{1-\delta^{-i}}{\delta-1}+1\right) - \log_\delta(2) - i + 1.$$

*Proof.* Note that, because $\delta^{k+r} \geq \sum_{i=0}^{3}\delta^i$, we have $k \geq 3$.

For the analysis of the expression $I(k,r)$, we need to distinguish between different cases, depending on the value of $\mu(i)$ and $\nu(i)$ for $i \in \{k, k+1\}$. For this, the following observations are useful. For $m \in \{0,1\}$, we have

$$\begin{aligned}
\mu(i) \leq m \quad &\Leftrightarrow \quad (\delta^{k+r}+1)\frac{\delta-1}{\delta^i-1} \leq \delta^m \\
&\Leftrightarrow \quad \delta^{k+r} \leq \delta^m\frac{\delta^i-1}{\delta-1} - 1 \\
&\Leftrightarrow \quad r \leq \log_\delta\left(\delta^m\frac{\delta^i-1}{\delta-1}-1\right) - 1
\end{aligned}$$

and

$$\begin{aligned}
&\nu(i) \leq m \\
\Leftrightarrow\quad & \sqrt{\left(\delta^{k+r}\frac{1-\delta^{-i}}{\delta-1}-1\right)^2 + 4\delta^{2k+2r-1}} - \delta^{k+r}\frac{1-\delta^{-i}}{\delta-1}+1 \leq 2\delta^{i+m-1} \\
\Leftrightarrow\quad & \left(\delta^{k+r}\frac{1-\delta^{-i}}{\delta-1}-1\right)^2 + 4\delta^{2k+2r-1} \leq \left(2\delta^{i+m-1} + \delta^{k+r}\frac{1-\delta^{-i}}{\delta-1}-1\right)^2 \\
\Leftrightarrow\quad & 4\delta^{2k+2r-1} \leq 4\delta^{2(i+m-1)} + 4\delta^{i+m-1}\left(\delta^{k+r}\frac{1-\delta^{-i}}{\delta-1}-1\right) \\
\Leftrightarrow\quad & (\delta^r)^2 - \delta^{m-k}\frac{\delta^i-1}{\delta-1}\delta^r - \delta^{2(i+m-k)-1} + \delta^{i+m-2k} \leq 0 \\
\overset{r\geq 0}{\Leftrightarrow}\quad & \delta^r \leq \delta^{m-k}\frac{\delta^i-1}{2(\delta-1)} + \sqrt{\left(\delta^{m-k}\frac{\delta^i-1}{2(\delta-1)}\right)^2 + \delta^{2(i+m-k)-1} - \delta^{i+m-2k}}.
\end{aligned}$$

We define

$$b_\mu(k,i,m) := \log_\delta\left(\delta^m\frac{\delta^i-1}{\delta-1}-1\right) - 1$$

and

$$b_\nu(k,i,m) := \log_\delta\left(\delta^{m-k}\frac{\delta^i-1}{2(\delta-1)} + \sqrt{\left(\delta^{m-k}\frac{\delta^i-1}{2(\delta-1)}\right)^2 + \delta^{2(i+m-k)-1} - \delta^{i+m-2k}}\right)$$

to obtain

$$\mu(i) \le m \quad \Leftrightarrow \quad r \le b_\mu(k,i,m), \tag{4.18}$$

$$\nu(i) \le m \quad \Leftrightarrow \quad r \le b_\nu(k,i,m). \tag{4.19}$$

Depending on $r$ and $k$, only 3 or 4 of the integrals in $I(k,r)$ are non-zero. We distinguish between the different possibilities.

**Case 1:** $\nu(k+1) > 0$. Then

$$\begin{aligned}
&\nu(k) + \log_\delta(2) + k - 1 \\
= \quad & \log_\delta\left(\sqrt{\left(\delta^{k+r}\frac{1-\delta^{-k}}{\delta-1} - 1\right)^2 + 4\delta^{2k+2r-1}} - \delta^{k+r}\frac{1-\delta^{-k}}{\delta-1} + 1\right) \\
\ge \quad & \log_\delta\left(\sqrt{\left(\delta^{k+r}\frac{1-\delta^{-(k+1)}}{\delta-1} - 1\right)^2 + 4\delta^{2k+2r-1}} - \delta^{k+r}\frac{1-\delta^{-(k+1)}}{\delta-1} + 1\right) \\
= \quad & \nu(k+1) + \log_\delta(2) + k,
\end{aligned}$$

i.e., we have $\nu(k) \ge \nu(k+1) + 1 > 1$. For the inequality, we used the fact that, for $a > b > 0$ and $x > 0$, we have $\sqrt{b^2+x} - b > \sqrt{a^2+x} - a$. We obtain

$$\begin{aligned}
&I(k,r) \\
= \quad & \int_r^1 \frac{\delta^{k+r} - \frac{\delta^k-1}{\delta-1}\delta^\varepsilon}{\delta^{k+\varepsilon}} \mathrm{d}\varepsilon \\
& + \int_{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)}^{r} 1 - \frac{\frac{\delta^k-1}{\delta-1}\delta^\varepsilon}{\delta^{k+r}} \mathrm{d}\varepsilon \\
& + \int_{\log_\delta\left(\sqrt{\left(\frac{\delta^k-\delta^{-1}}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-\delta^{-1}}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k}^{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)} \frac{\delta^{k+\varepsilon}-1}{\delta^{k+r}} \mathrm{d}\varepsilon \\
& + \int_0^{\log_\delta\left(\sqrt{\left(\frac{\delta^k-\delta^{-1}}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-\delta^{-1}}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k} \frac{\delta^{k+r} - \frac{\delta^{k+1}-1}{\delta-1}\delta^\varepsilon}{\delta^{k+1+\varepsilon}} \mathrm{d}\varepsilon \\
= \quad & \left[-\frac{\delta^{k+r}}{\log(\delta)\delta^{k+\varepsilon}} - \frac{1-\delta^{-k}}{\delta-1}\varepsilon\right]_r^1
\end{aligned}$$

$$
\begin{aligned}
& +\Big[\varepsilon - \frac{(\delta^k - 1)\delta^\varepsilon}{\log(\delta)(\delta-1)\delta^{k+r}}\Big]_{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)}^{r} \\
& +\Big[\frac{\delta^{k+\varepsilon}}{\log(\delta)\delta^{k+r}} - \frac{\varepsilon}{\delta^{k+r}}\Big]_{\log_\delta\Big(\sqrt{\big(\frac{\delta^k-\delta^{-1}}{\delta-1}\delta^r-1\big)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-\delta^{-1}}{\delta-1}\delta^r+1\Big)-\log_\delta(2)-k}^{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)} \\
& +\Big[-\frac{\delta^{k+r}}{\log(\delta)\delta^{k+1+\varepsilon}} - \frac{1-\delta^{-(k+1)}}{\delta-1}\varepsilon\Big]_0^{\log_\delta\Big(\sqrt{\big(\frac{\delta^k-\delta^{-1}}{\delta-1}\delta^r-1\big)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-\delta^{-1}}{\delta-1}\delta^r+1\Big)-\log_\delta(2)-k} \\
= \; & r + \frac{2+\delta^{-(k+r)}}{\log(\delta)} - \frac{\sqrt{\big(\frac{\delta^k-\delta^{-1}}{\delta-1}\delta^r-1\big)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-\delta^{-1}}{\delta-1}\delta^r+1}{2\log(\delta)\delta^{k+r}} - \left(1+\frac{1}{\delta^{k+r}}\right) \\
& \big(\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)\big) - \frac{1-\delta^{-k}}{\delta-1}\left(1-r+\frac{1}{\log(\delta)}\right) \\
& - \frac{2\delta^{k-1+r}}{\log(\delta)\Big(\sqrt{\big(\frac{\delta^k-\delta^{-1}}{\delta-1}\delta^r-1\big)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-\delta^{-1}}{\delta-1}\delta^r+1\Big)} - \Big(\frac{1-\delta^{-(k+1)}}{\delta-1} - \frac{1}{\delta^{k+r}}\Big) \\
& \left(\log_\delta\left(\sqrt{\left(\frac{\delta^k-\delta^{-1}}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-\delta^{-1}}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k\right) \\
=: \; & f_1(k,r).
\end{aligned}
$$

We have $\nu(k+1) > 0$, which, by (4.19), implies $r \geq b_\nu(k, k+1, 0)$. By analyzing $f_1(k,r)$ for $r \in [b_\nu(k,k+1,0), 1]$, one can see that it is minimized for small $k$, i.e., for $k = 3$. Analyzing $f_1(3,r)$ yields that it is minimized for small $r$, i.e., for $r = b_\nu(3,4,0)$. By combining the above, we obtain

$$I(k,r) = f_1(k,r) \geq f_1(3, b_\nu(3,4,0)) > 0.566 > g(\delta).$$

**Case 2:** $\nu(k+1) \leq 0$ and $\nu(k) > 1$. Then

$$
\begin{aligned}
& I(k,r) \\
= \; & \int_r^1 \frac{\delta^{k+r} - \frac{\delta^k-1}{\delta-1}\delta^\varepsilon}{\delta^{k+\varepsilon}} \mathrm{d}\varepsilon \\
& + \int_{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)}^{r} 1 - \frac{\frac{\delta^k-1}{\delta-1}\delta^\varepsilon}{\delta^{k+r}} \mathrm{d}\varepsilon \\
& + \int_0^{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)} \frac{\delta^{k+\varepsilon}-1}{\delta^{k+r}} \mathrm{d}\varepsilon
\end{aligned}
$$

$$
\begin{aligned}
&= \left[-\frac{\delta^{k+r}}{\log(\delta)\delta^{k+\varepsilon}} - \frac{1-\delta^{-k}}{\delta-1}\varepsilon\right]_r^1 \\
&\quad + \left[\varepsilon - \frac{(\delta^k-1)\delta^{\varepsilon}}{\log(\delta)(\delta-1)\delta^{k+r}}\right]_{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)}^r \\
&\quad + \left[\frac{\delta^{k+\varepsilon}}{\log(\delta)\delta^{k+r}} - \frac{\varepsilon}{\delta^{k+r}}\right]_0^{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)} \\
&= -\frac{\delta^r}{\log(\delta)\delta} - \frac{1-\delta^{-k}}{\delta-1}\left(1-r+\frac{1}{\log(\delta)}\right) + \frac{2+\delta^{-(k+r)}}{\log(\delta)} - \frac{1}{\log(\delta)\delta^r} \\
&\quad + r - \left(1+\frac{1}{\delta^{k+r}}\right)\left(\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)\right) \\
&=: f_2(k,r)
\end{aligned}
$$

We have $\nu(k+1) \leq 0$ and $\nu(k) > 1$, which, by (4.19), implies $r \leq b_\nu(k, k+1, 0)$ and $r \geq b_\nu(k,k,1)$. By analyzing $f_2(k,r)$ for $r \in [b_\nu(k,k,1), b_\nu(k,k+1,0)]$, one can see that it is minimized for small $k$, i.e., for $k=3$. Analyzing $f_2(3,r)$ yields that it is minimized for small $r$, i.e., for $r = b_\nu(3,3,1)$. By combining the above, we obtain

$$
I(k,r) = f_2(k,r) \geq f_2(3, b_\nu(3,3,1)) > 0.566 > g(\delta).
$$

**Case 3:** $\nu(k) \leq 1$ and $\mu(k+1) > 0$. Then

$$
\begin{aligned}
\mu(k) &= \mu(k+1) + \log_\delta(\delta^{k+1}-1) - \log_\delta(\delta^k-1) \\
&= \mu(k+1) + 1 + \log_\delta\left(\frac{\delta^k-\delta^{-1}}{\delta^k-1}\right) \\
&\geq \mu(k+1)+1 \\
&> 1. 
\end{aligned} \tag{4.20}
$$

We obtain

$$
\begin{aligned}
&I(k,r) \\
&= \int_{\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k+1}^{1} \frac{\delta^{k-1+\varepsilon}-1}{\delta^{k+r}}\mathrm{d}\varepsilon \\
&\quad + \int_r^{\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k+1} \frac{\delta^{k+r}-\frac{\delta^k-1}{\delta-1}\delta^{\varepsilon}}{\delta^{k+\varepsilon}}\mathrm{d}\varepsilon \\
&\quad + \int_{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)}^{r} 1 - \frac{\frac{\delta^k-1}{\delta-1}\delta^{\varepsilon}}{\delta^{k+r}}\mathrm{d}\varepsilon
\end{aligned}
$$

$$
\begin{aligned}
& + \int_0^{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)} \frac{\delta^{k+\varepsilon}-1}{\delta^{k+r}} \mathrm{d}\varepsilon \\
= \quad & \left[\frac{\delta^{k-1+\varepsilon}}{\log(\delta)\delta^{k+r}} - \frac{\varepsilon}{\delta^{k+r}}\right]_{\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k+1}^{1} \\
& + \left[-\frac{\delta^{k+r}}{\log(\delta)\delta^{k+\varepsilon}} - \frac{1-\delta^{-k}}{\delta-1}\varepsilon\right]_r^{\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k+1} \\
& + \left[\varepsilon - \frac{(\delta^k-1)\delta^\varepsilon}{\log(\delta)(\delta-1)\delta^{k+r}}\right]_{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)}^{r} \\
& + \left[\frac{\delta^{k+\varepsilon}}{\log(\delta)\delta^{k+r}} - \frac{\varepsilon}{\delta^{k+r}}\right]_0^{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)} \\
= \quad & r - \frac{1-\delta^{-k}}{2(\delta-1)\log(\delta)} - \frac{2\delta^{k-1+r}}{\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)\log(\delta)} \\
& + \frac{1-\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}}{2\log(\delta)\delta^{k+r}} - (1-r)\frac{1-\delta^{-k}}{\delta-1} - \left(\frac{1-\delta^{-k}}{\delta-1} - \frac{1}{\delta^{k+r}}\right) \\
& \cdot\left(\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}} - \frac{\delta^k-1}{\delta-1}\delta^r+1\right) - \log_\delta(2)-k\right) \\
& + \frac{2}{\log(\delta)} - \left(1+\frac{1}{\delta^{k+r}}\right)\left(\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)\right) \\
=: \quad & f_3(k,r).
\end{aligned}
$$

We have $\nu(k) \leq 1$ and $\mu(k+1) > 0$, which, by (4.18) and (4.19), implies $r \leq b_\nu(k,k,1)$ and $r \geq b_\mu(k,k+1,0)$. By analyzing $f_3(k,r)$ for $r \in [b_\mu(k,k+1,0), b_\nu(k,k,1)]$, one can see that it is minimized for small $k$, i.e., for $k=3$. Analyzing $f_3(3,r)$ yields that it is minimized for small $r$, i.e., for $r = b_\mu(3,4,0)$. By combining the above, we obtain

$$
I(k,r) = f_3(k,r) \geq f_3(3, b_\mu(3,4,0)) = g(\delta).
$$

**Case 4:** $\mu(k+1) \leq 0$ and $\mu(k) > 1$. Then

$$
\begin{aligned}
& I(k,r) \\
= \quad & \int_{\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k+1}^{1} \frac{\delta^{k-1+\varepsilon}-1}{\delta^{k+r}} \mathrm{d}\varepsilon
\end{aligned}
$$

$$
\begin{aligned}
&+\int_r^{\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k+1} \frac{\delta^{k+r}-\frac{\delta^k-1}{\delta-1}\delta^\varepsilon}{\delta^{k+\varepsilon}}\mathrm{d}\varepsilon \\
&+\int_0^r 1-\frac{\frac{\delta^k-1}{\delta-1}\delta^\varepsilon}{\delta^{k+r}}\mathrm{d}\varepsilon \\
= \quad & \left[\frac{\delta^{k-1+\varepsilon}}{\log(\delta)\delta^{k+r}}-\frac{\varepsilon}{\delta^{k+r}}\right]^1_{\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k+1} \\
&+\left[-\frac{\delta^{k+r}}{\log(\delta)\delta^{k+\varepsilon}}-\frac{1-\delta^{-k}}{\delta-1}\varepsilon\right]_r^{\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k+1} \\
&+\left[\varepsilon-\frac{(\delta^k-1)\delta^\varepsilon}{\log(\delta)(\delta-1)\delta^{k+r}}\right]_0^r \\
= \quad & \frac{1}{\log(\delta)\delta^r}-\frac{1}{\delta^{k+r}}-\frac{1}{2\log(\delta)\delta^{k+r}}+\frac{1}{\log(\delta)}+\frac{1-\delta^{-k}}{\delta-1}r \\
&-\frac{2\delta^{k-1+r}}{\log(\delta)\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)}-\left(\frac{1-\delta^{-k}}{\delta-1}-\frac{1}{\delta^{k+r}}\right) \\
&\left(\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k+1\right) \\
&-\frac{\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}}{2\log(\delta)\delta^{k+r}}+r-\left(\frac{1}{2}-\frac{1}{\delta^r}\right)\frac{1-\delta^{-k}}{\log(\delta)(\delta-1)} \\
=: \quad & f_4(k,r).
\end{aligned}
$$

We have $\mu(k+1)\le 0$ and $\mu(k)>1$, which, by (4.18), implies $r\le b_\mu(k,k+1,0)$ and $r\ge b_\mu(k,k,1)$. Furthermore, $0\ge\mu(k+1)=\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^{k+1}-1)$ is equivalent to

$$
\delta^{k+r}+1\le\frac{\delta^{k+1}-1}{\delta-1}=\sum_{i=0}^k\delta^i.
$$

Since $\delta^{k+r}\ge\sum_{i=0}^3\delta^i$, we have $\sum_{i=0}^k\delta^i\ge 1+\sum_{i=0}^3\delta^i$ which implies $k\ge 4$. By analyzing $f_4(k,r)$ for $r\in[b_\mu(k,k,1),b_\mu(k,k+1,0)]$, one can see that it is minimized for small $k$, i.e., for $k=4$. Analyzing $f_4(4,r)$ yields

$$
I(k,r)=f_4(k,r)\ge f_4(4,r)>0.566>g(\delta).
$$

**Case 5:** $\mu(k) \leq 1$. Then

$$
\begin{aligned}
& I(k,r) \\
= \; & \int_{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^k-1)}^{1} 1 - \frac{\frac{\delta^{k-1}-1}{\delta-1}\delta^\varepsilon}{\delta^{k+r}} \mathrm{d}\varepsilon \\
& + \int_{\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k+1}^{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^k-1)} \frac{\delta^{k-1+\varepsilon}-1}{\delta^{k+r}} \mathrm{d}\varepsilon \\
& + \int_r^{\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k+1} \frac{\delta^{k+r}-\frac{\delta^k-1}{\delta-1}\delta^\varepsilon}{\delta^{k+\varepsilon}} \mathrm{d}\varepsilon \\
& + \int_0^r 1 - \frac{\frac{\delta^k-1}{\delta-1}\delta^\varepsilon}{\delta^{k+r}} \mathrm{d}\varepsilon \\
= \; & \left[\varepsilon - \frac{(\delta^{k-1}-1)\delta^\varepsilon}{\log(\delta)(\delta-1)\delta^{k+r}}\right]_{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^k-1)}^{1} \\
& + \left[\frac{\delta^{k-1+\varepsilon}}{\log(\delta)\delta^{k+r}} - \frac{\varepsilon}{\delta^{k+r}}\right]_{\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k+1}^{\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^k-1)} \\
& + \left[-\frac{\delta^{k+r}}{\log(\delta)\delta^{k+\varepsilon}} - \frac{1-\delta^{-k}}{\delta-1}\varepsilon\right]_r^{\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k+1} \\
& + \left[\varepsilon - \frac{(\delta^k-1)\delta^\varepsilon}{\log(\delta)(\delta-1)\delta^{k+r}}\right]_0^r \\
= \; & 1 + r + \frac{2}{\log(\delta)} - \left(1+\frac{1}{\delta^{k+r}}\right)\left(\log_\delta(\delta^{k+r}+1)+\log_\delta(\delta-1)-\log_\delta(\delta^k-1)\right) \\
& + \frac{\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)-\log_\delta(2)-k+1}{\delta^{k+r}} \\
& - \frac{2\delta^{k+r}}{\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)\log(\delta)\delta} + \frac{3}{2\log(\delta)\delta^{k+r}} \\
& - \frac{1-\delta^{-k}}{\delta-1}\left(\log_\delta\left(\sqrt{\left(\frac{\delta^k-1}{\delta-1}\delta^r-1\right)^2+4\delta^{2k+2r-1}}-\frac{\delta^k-1}{\delta-1}\delta^r+1\right)\right.
\end{aligned}
$$

$$- \log_\delta(2) + \frac{1}{2\log(\delta)} - k + 1 - r \Bigg) - \frac{\sqrt{\left(\frac{\delta^{k}-1}{\delta-1}\delta^r - 1\right)^2 + 4\delta^{2k+2r-1}}}{2\log(\delta)\delta^{k+r}}$$
$$=: \quad f_5(k,r).$$

We have $\mu(k) \leq 1$, which, by (4.18) implies $r \leq b_\mu(k,k,1)$. Because (4.20) does not hold and (4.20) followed from the fact that $\mu(k+1) > 0$, we can conclude that $\mu(k+1) \leq 0$. Thus, by the same argumentation as in Case 4, we have $k \geq 4$. By analyzing $f_5(k,r)$ for $r \in [0, b_\mu(k,k,1)]$, one can see that it is minimized for small $k$, i.e., for $k = 4$. Analyzing $f_5(4,r)$ yields that it is minimized for large $r$, i.e., for $r = b_\mu(4,4,1)$. By combining the above, we obtain

$$I(k,r) = f_5(k,r) \geq f_5(4, b_\mu(4,4,1)) > 0.566 > g(\delta). \qquad \square$$

With these lemmas, we are ready to prove an upper bound of $1/g(\delta)$ on the randomized competitive ratio of RandScaling.

**Theorem 4.21.** *The randomized competitive ratio of RandScaling for IncMaxSep is at most* $1/g(\delta) < 1.772$.

*Proof.* Let $X$ denote the incremental solution obtained by RandScaling, and let $(c_1, c_2, \dots)$ be its representation. Let $C \in \mathbb{N}$. First, assume that $C < \sum_{i=0}^{3} \delta^i$. Then

$$C < \sum_{i=0}^{3} \delta^i < \sum_{i=0}^{3} \delta^{i+\varepsilon} = t_4.$$

For better readability, let $\mathcal{I} := \{i \in [4] \mid \mathbb{P}[C \in (t_{i-1}, t_i]] > 0\}$ denote the set of indices such that the value $C$ can fall into the interval $(t_{i-1}, t_i]$. Let

$$S_4 := \{(\lfloor \delta^x \rfloor, \lfloor \delta^{1+x} \rfloor, \lfloor \delta^{2+x} \rfloor, \lfloor \delta^{3+x} \rfloor) \mid x \in (0,1)\}$$

denote the finite set of all possible realizations of the sequence $(c_1, c_2, c_3, c_4)$. We denote $\sigma \in S_4$ as $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. By Lemma 4.19 *(i)*, for $i \in \mathcal{I}$, we have

$$\begin{aligned}
& \mathbb{E}\big[f(X(C)) \mid C \in (t_{i-1}, t_i]\big] \\
\geq \; & \mathbb{E}\left[\max\left\{\frac{c_{i-1}}{C}, \frac{C - t_{i-1}}{\max\{C, c_i\}}\right\} \,\middle|\, C \in (t_{i-1}, t_i]\right] \cdot v_C \\
= \; & \sum_{\sigma \in S_4} \max\left\{\frac{\sigma_{i-1}}{C}, \frac{C - \sigma_1 - \dots - \sigma_{i-1}}{\max\{C, \sigma_i\}}\right\} \\
& \cdot \mathbb{P}\big[(c_1, c_2, c_3, c_4) = \sigma \big| C \in (t_{i-1}, t_i]\big] \cdot v_C
\end{aligned} \tag{4.21}$$

This yields (with $\sigma_0 = 0$)

$$\begin{aligned}
&\frac{\mathbb{E}\big[f(X(C))\big]}{v_C} \\
=\;& \frac{1}{v_C}\sum_{i\in\mathcal{I}} \mathbb{E}\big[f(X(C)) \mid C \in (t_{i-1}, t_i]\big] \cdot \mathbb{P}\big[C \in (t_{i-1}, t_i]\big] \\
\overset{(4.21)}{\geq}\;& \sum_{i\in\mathcal{I}}\sum_{\sigma\in S_4} \max\left\{\frac{\sigma_{i-1}}{C}, \frac{C-\sigma_1-\cdots-\sigma_{i-1}}{\max\{C,\sigma_i\}}\right\} \\
& \qquad \cdot \mathbb{P}\big[\big((c_1,c_2,c_3,c_4)=\sigma\big) \cap \big(C \in (t_{i-1}, t_i]\big)\big] \\
=\;& \sum_{i\in\mathcal{I}}\sum_{\sigma\in S_4} \max\left\{\frac{\sigma_{i-1}}{C}, \frac{C-\sigma_1-\cdots-\sigma_{i-1}}{\max\{C,\sigma_i\}}\right\} \\
& \qquad \cdot \mathbb{P}\big[(c_1,c_2,c_3,c_4)=\sigma\big] \cdot \mathbb{1}\big[C \in (\sigma_{i-1}, \sigma_i]\big] \\
=\;& \sum_{\sigma\in S_4} \max\left\{\frac{\sigma_{i(C,\sigma)-1}}{C}, \frac{C-\sigma_1-\cdots-\sigma_{i(C,\sigma)-1}}{\max\{C,\sigma_i\}}\right\} \cdot \mathbb{P}\big[(c_1,c_2,c_3,c_4)=\sigma\big],
\end{aligned}$$

where we write $i(C,\sigma)$ for the unique value $i \in [4]$ with $\sum_{j=1}^{i-1} \sigma_j < C \leq \sum_{j=1}^{i}$. For $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in S_4$, we have

$$\begin{aligned}
\mathbb{P}\big[(c_1,c_2,c_3,c_4)=\sigma\big] &= \mathbb{P}\Big[\bigcap_{i\in[4]} \big(\delta^{i-1+\varepsilon} \in [\sigma_i, \sigma_i+1)\big)\Big] \\
&= \mathbb{P}\Big[\bigcap_{i\in[4]} \big(\varepsilon \in [\log_\delta(\sigma_i) - i + 1, \log_\delta(\sigma_i+1) - i + 1)\big)\Big] \\
&= \mathbb{P}\Big[\varepsilon \in \Big[\max_{i\in[4]}(\log_\delta(\sigma_i) - i + 1), \min_{i\in[4]}(\log_\delta(\sigma_i+1) - i + 1)\Big)\Big].
\end{aligned}$$

This gives us an explicit formula to calculate an upper bound for the randomized competitive ratio. A visualization of the inverse of these upper bounds can be found in Figure 4.5, while the calculated values can be found in the appendix in Figure A.1. We can see that

$$\mathbb{E}\big[f(X(C))\big] > g(\delta) \cdot v_C.$$

Now, suppose $C \geq \sum_{i=0}^{3} \delta^i$ and let $k \in \mathbb{N}$ and $r \in (0,1]$ be uniquely defined such that

$$C = \delta^{k+r} \tag{4.22}$$

We have

$$\tilde{c}_k = \delta^{k-1+\varepsilon} \overset{\varepsilon\leq 1}{\leq} \delta^k \overset{(4.22)}{<} C \overset{(4.22)}{\leq} \delta^{k+1} \overset{\varepsilon\geq 0}{\leq} \delta^{k+1+\varepsilon} = \tilde{c}_{k+2}$$

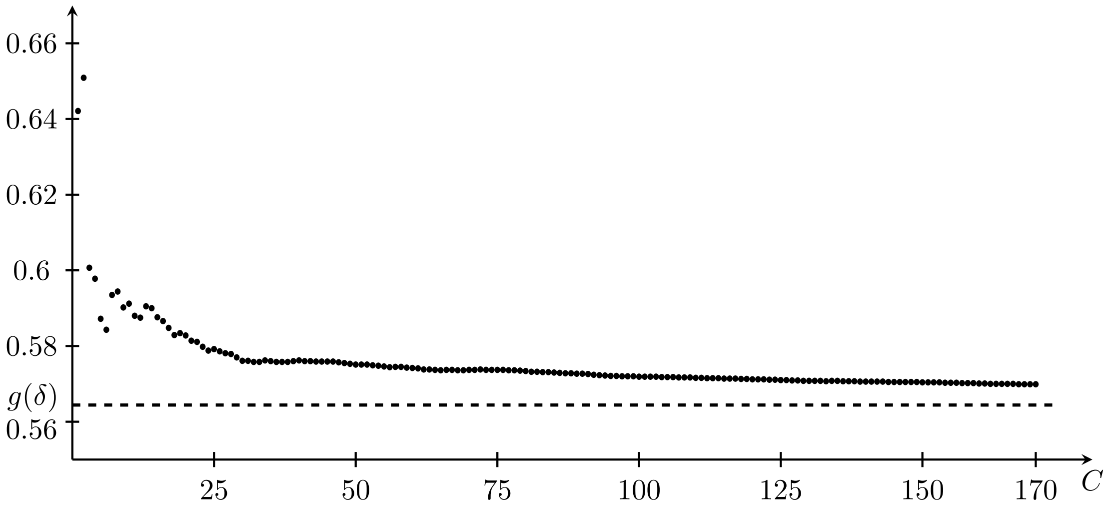


Figure 4.5.: Lower bound on $\mathbb{E}[f(X(C))]/v_C$, the inverse of the competitive ratio of RandScaling, for cardinalities $C \in \left[\lfloor \sum_{i=0}^{3} \delta^i \rfloor\right]$. The dashed horizontal line in the right graphic represents the value $g(\delta) \approx 0.5644$.

and, for all $i \in \mathbb{N}$,

$$\tilde{t}_i - 1 \overset{(4.15)}{<} \tilde{c}_{i+1} < \tilde{t}_{i+1} - 1$$

where the second inequality follows from the fact that

$$(\tilde{t}_{i+1} - 1) - \tilde{c}_{i+1} = \tilde{t}_i - 1 \geq \tilde{c}_0 - 1 = \delta^{\varepsilon} - 1 > 0.$$

This means that there are $4$ intervals, in which $C$ can fall:

$$I_1 = (\tilde{c}_k, \tilde{t}_k - 1], \quad I_2 = (\tilde{t}_k - 1, \tilde{c}_{k+1}], \quad I_3 = (\tilde{c}_{k+1}, \tilde{t}_{k+1} - 1], \quad I_4 = (\tilde{t}_{k+1} - 1, \tilde{c}_{k+2}].$$

We will calculate for which values of $\varepsilon$ the value $C$ falls into which interval. For $i \in \mathbb{N}$, we have

$$\begin{aligned} C \leq \tilde{c}_i \quad &\Leftrightarrow \quad \delta^{k+r} \leq \delta^{i-1+\varepsilon} \\ &\Leftrightarrow \quad \varepsilon \geq k - i + 1 + r \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} C \leq \tilde{t}_i - 1 \quad &\Leftrightarrow \quad \delta^{k+r} \leq \delta^{\varepsilon} \frac{\delta^i - 1}{\delta - 1} - 1 \\ &\Leftrightarrow \quad \varepsilon \geq \log_\delta(\delta^{k+r} + 1) + \log_\delta(\delta - 1) - \log_\delta(\delta^i - 1) =: \mu(i). \end{aligned} \tag{4.24}$$

The expected value of $X(C)$ is

$$
\begin{aligned}
& \mathbb{E}[f(X(C))] \\
= & \sum_{i=1}^{5} \mathbb{E}[f(X(C)) \mid C \in I_i] \cdot \mathbb{P}[C \in I_i] \\
\overset{\text{Lem. 4.19 } (ii),(iii)}{\geq} & \Bigg( \mathbb{E}\left[1 - \frac{\tilde{t}_{k-1}}{C} \,\middle|\, C \in (\tilde{c}_k, \tilde{t}_k - 1]\right] \cdot \mathbb{P}[C \in I_1] \\
& + \mathbb{E}\left[\max\left\{\frac{\tilde{c}_k - 1}{C}, \frac{C - \tilde{t}_k}{\tilde{c}_{k+1}}\right\} \,\middle|\, C \in (\tilde{t}_k - 1, \tilde{c}_{k+1}]\right] \cdot \mathbb{P}[C \in I_2] \\
& + \mathbb{E}\left[1 - \frac{\tilde{t}_k}{C} \,\middle|\, C \in (\tilde{c}_{k+1}, \tilde{t}_{k+1} - 1]\right] \cdot \mathbb{P}[C \in I_3] \\
& + \mathbb{E}\left[\max\left\{\frac{\tilde{c}_{k+1} - 1}{C}, \frac{C - \tilde{t}_{k+1}}{\tilde{c}_{k+2}}\right\} \,\middle|\, C \in (\tilde{t}_{k+1} - 1, \tilde{c}_{k+2}]\right] \cdot \mathbb{P}[C \in I_4] \Bigg) \cdot v_C \\
\overset{(4.23),(4.24)}{=} & \Bigg( \int_{\min\{1,\mu(k)\}}^{1} 1 - \frac{\tilde{t}_{k-1}}{C} \, d\varepsilon \\
& + \int_{r}^{\min\{1,\mu(k)\}} \max\left\{\frac{\tilde{c}_k - 1}{C}, \frac{C - \tilde{t}_k}{\tilde{c}_{k+1}}\right\} d\varepsilon \\
& + \int_{\max\{0,\mu(k+1)\}}^{r} 1 - \frac{\tilde{t}_k}{C} \, d\varepsilon \\
& + \int_{0}^{\max\{0,\mu(k+1)\}} \max\left\{\frac{\tilde{c}_{k+1} - 1}{C}, \frac{C - \tilde{t}_{k+1}}{\tilde{c}_{k+2}}\right\} d\varepsilon \Bigg) \cdot v_C
\end{aligned}
$$

Furthermore, for $i \in \mathbb{N}$ we have

$$
\begin{aligned}
& \frac{\tilde{c}_i - 1}{C} \geq \frac{C - \tilde{t}_i}{\tilde{c}_{i+1}} \\
\Leftrightarrow \quad & \delta^{i+\varepsilon}(\delta^{i-1+\varepsilon} - 1) \geq \delta^{k+r}\left(\delta^{k+r} - \delta^{\varepsilon}\frac{\delta^i - 1}{\delta - 1}\right) \\
\Leftrightarrow \quad & \delta^{2i-1}(\delta^{\varepsilon})^2 + \left(\delta^{k+r}\frac{\delta^i - 1}{\delta - 1} - \delta^i\right)\delta^{\varepsilon} - \delta^{2k+2r} \geq 0 \\
\overset{\delta^{\varepsilon} \geq 0}{\Leftrightarrow} \quad & \delta^{\varepsilon} \geq -\frac{1}{2\delta^{i-1}}\left(\delta^{k+r}\frac{1 - \delta^{-i}}{\delta - 1} - 1\right) \\
& \quad + \sqrt{\frac{1}{4\delta^{2i-2}}\left(\delta^{k+r}\frac{1 - \delta^{-i}}{\delta - 1} - 1\right)^2 + \delta^{2k-2i+2r+1}}
\end{aligned}
$$

$$\Leftrightarrow \quad \varepsilon \geq \log_\delta\left(\sqrt{\left(\delta^{k+r}\frac{1-\delta^{-i}}{\delta-1}-1\right)^2+4\delta^{2k+2r-1}}-\delta^{k+r}\frac{1-\delta^{-i}}{\delta-1}+1\right) \\ -\log_\delta(2)-i+1=:\nu(i),$$

i.e., instead of one integral over a maximum, we can evaluate two separate integrals, which yields

$$\begin{aligned}\mathbb{E}[f(X(C))] \geq &\left(\int_{\min\{1,\mu(k)\}}^{1} 1-\frac{\tilde{t}_{k-1}}{C}\,d\varepsilon \quad + \int_{\min\{1,\nu(k)\}}^{\min\{1,\mu(k)\}} \frac{\tilde{c}_k-1}{C}\,d\varepsilon\right.\\ &+\int_{r}^{\min\{1,\nu(k)\}} \frac{C-\tilde{t}_k}{\tilde{c}_{k+1}}\,d\varepsilon \quad + \int_{\max\{0,\mu(k+1)\}}^{r} 1-\frac{\tilde{t}_k}{C}\,d\varepsilon\\ &\left.+\int_{\max\{0,\nu(k+1)\}}^{\max\{0,\mu(k+1)\}} \frac{\tilde{c}_{k+1}-1}{C}\,d\varepsilon + \int_{0}^{\max\{0,\nu(k+1)\}} \frac{C-\tilde{t}_{k+1}}{\tilde{c}_{k+2}}\,d\varepsilon\right)\cdot v_C\\ \overset{\text{Lem. 4.20}}{\geq}\ & g(\delta)\cdot v_C. \end{aligned}$$

□

### 4.2.2. Randomized Lower Bound

Now, we want to complement the upper bound on the randomized competitive ratio of IncMaxSep of $1.772$ in Theorem 4.21 with a lower bound. To do this, we employ Yao's principle [71]. In [19], the authors used Yao's principle to show a lower bound of $1.447$ on the randomized competitive ratio. Yet, the way Yao's principle was used by the authors is flawed and thus the lower bound might not hold. We fix the error made in the analysis and present a new lower bound of $1.357$.

The idea behind Yao's principle is to exchange the randomness in the algorithm for randomness in the (unknown) cardinality constraint. This can be done as follows. Fix some problem instance in IncMaxSep. Let $\mathcal{A}$ denote the set of deterministic algorithms for this instance. Let Alg$^*$ be some randomized algorithm[2], and let $C^*$ be a random cardinality constraint that is drawn from the probability distribution $p\colon \mathbb{N} \to [0,1]$. Then,

[2]In order to show a lower bound on the randomized competitive ratio, it makes sense to consider the randomized algorithm to be (near-)optimal.

we can derive a lower bound on the randomized competitive ratio of $\textsc{Alg}^*$ as follows:

$$\begin{aligned}
\sup_{C\in\mathbb{N}} \frac{\textsc{Opt}(C)}{\mathbb{E}[\textsc{Alg}^*(C)]} &= \sup_{C\in\mathbb{N}} \frac{1}{\mathbb{E}\left[\frac{\textsc{Alg}^*(C)}{\textsc{Opt}(C)}\right]} &&= \frac{1}{\inf_{C\in\mathbb{N}} \mathbb{E}\left[\frac{\textsc{Alg}^*(C)}{\textsc{Opt}(C)}\right]} \\
&\geq \frac{1}{\sum_{C\in\mathbb{N}} p(C)\mathbb{E}\left[\frac{\textsc{Alg}^*(C)}{\textsc{Opt}(C)}\right]} &&= \frac{1}{\mathbb{E}\left[\sum_{C\in\mathbb{N}} p(C)\frac{\textsc{Alg}^*(C)}{\textsc{Opt}(C)}\right]} \\
&\geq \frac{1}{\sup_{\textsc{Alg}\in\mathcal{A}} \sum_{C\in\mathbb{N}} p(C)\frac{\textsc{Alg}(C)}{\textsc{Opt}(C)}} &&= \inf_{\textsc{Alg}\in\mathcal{A}} \frac{1}{\sum_{C\in\mathbb{N}} p(C)\frac{\textsc{Alg}(C)}{\textsc{Opt}(C)}} \\
&= \inf_{\textsc{Alg}\in\mathcal{A}} \frac{1}{\mathbb{E}\left[\frac{\textsc{Alg}(C^*)}{\textsc{Opt}(C^*)}\right]}.
\end{aligned}$$

Thus, we obtain a lower bound on the competitive ratio of $\textsc{Alg}^*$ by analyzing the expected value of the ratio $\frac{\textsc{Alg}(C^*)}{\textsc{Opt}(C^*)}$ for all deterministic algorithms.

**Theorem 4.22.** *The randomized competitive ratio of* IncMaxSep *is at least* $1.357$.

*Proof.* We fix $N \in \mathbb{N}$, and let $U_1, \ldots, U_N$ be disjoint sets with $|U_c| = c$ for all $c \in [N]$. We define the groundset $U := \bigcup_{c\in[N]} U_c$, and let $d_1, \ldots, d_N$ denote the densities. The densities will be parameters to determine the instance; we denote the resulting instance by $I(d_1, \ldots, d_N)$. Note that, given a probability distribution $p_1, \ldots, p_N$ over the set of possible cardinality constraints $[N]$ in addition, Yao's principle yields

$$\inf_{\textsc{Alg}\in\mathcal{A}} \frac{1}{\sum_{i=1}^{N} p_i \cdot \frac{\textsc{Alg}(I(d_1,\ldots,d_N),i)}{i\cdot d_i}}$$

as a lower bound on the randomized competitive ratio of the problem. Here, $\textsc{Alg}(I, i)$ denotes the value of the first $i$ elements in the solution of $\textsc{Alg}$ on instance $I$. By Lemma 3.7, we may assume that

$$\mathcal{A} := \left\{ \textsc{Alg}_{c_1,\ldots,c_\ell} \;\middle|\; 1 \leq c_1 < \cdots < c_\ell \leq N, \sum_{i=1}^{\ell} c_i \leq N \right\},$$

where $\textsc{Alg}_{c_1,\ldots,c_\ell}$ is the algorithm that first includes all elements of $U_{c_1}$ into the incremental solution, then all elements of $U_{c_2}$, and so on. Once it has added the $c_\ell$ elements of $U_{c_\ell}$, it adds some arbitrary elements from then onwards.

We can formulate the problem of maximizing the lower bound on the randomized competitive ratio as an optimization problem:

$$
\begin{aligned}
\max \quad & \rho \\
\text{s.t.} \quad & \rho \le \frac{1}{\sum_{i=1}^{N} p_i \cdot \frac{\textsc{Alg}(I(d_1,\ldots,d_N),i)}{i \cdot d_i}} \qquad \forall \textsc{Alg} \in \mathcal{A}, \\
& \sum_{i=1}^{N} p_i = 1, \\
& d_1, \ldots, d_N \ge 0, \\
& p_1, \ldots, p_N \ge 0.
\end{aligned}
$$

Every feasible solution of this optimization problem yields a lower bound on the randomized competitive ratio of IncMaxSep.

Note that the expression $\textsc{Alg}_{c_1,\ldots,c_\ell}(I(d_1,\ldots,d_N),i)$ can also be written as a function of $c_1,\ldots,c_\ell$, $d_1,\ldots,d_N$, and $i$ by taking the maximum over all sets from which $\textsc{Alg}_{c_1,\ldots,c_\ell}$ selects elements, i.e.,

$$
\textsc{Alg}_{c_1,\ldots,c_\ell}(I(d_1,\ldots,d_N),i) = \max_{1 \le j \le \ell} \left\{ \min \left\{ i - \sum_{1 \le j' < j} c_{j'}, c_j \right\} \cdot d_{c_j} \right\}.
$$

A feasible solution to the above optimization problem with $N = 16$ is given by

$$
\begin{aligned}
&(\rho; d_1, \ldots, d_{16}; p_1, \ldots, p_{16}) \\
=&(1.357; \\
&1, 0.725, 0.593, 0.524, 0.48, 0.4284, 0.42431, 0.3713, 0.3666, 0.3666, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \\
&0.1382, 0.0646, 0.0937, 0.1602, 0.0596, 0, 0.1491, 0, 0, 0.1853, 0, 0, 0, 0, 0.0253, 0.124),
\end{aligned}
$$

with objective value 1.357. □

# 5. Incremental Maximization beyond Accountability

One major problem why the competitive ratio of IncMax is unbounded is because, for two sets $A \subsetneq B \subseteq U$, the value of $A$ can be very small compared to the value of the set $B$ (cf. Figure 1.1). In the previous chapters, we avoided this by restricting ourselves to the problem class $\text{IncMax}_{\text{acc}}$ of instances with an accountable objective. Unfortunately, this class does not contain all problem instances that admit a competitive solution. This is already the case for the relatively simple instance with $U = \{a, b, c\}$ and objective $f\colon 2^U \to \mathbb{R}_{\geq 0}$, such that, for all $S \subseteq U$,

$$f(S) = \begin{cases} 0, & \text{if } S = \emptyset, \\ 2, & \text{if } S = U, \\ 1, & \text{else.} \end{cases}$$

This objective is not accountable because, for all $e \in U$, we have

$$f(S \setminus \{e\}) = 1 < \frac{2}{3} \cdot 2 = \left(1 - \frac{1}{|U|}\right) f(U).$$

Yet, one can immediately see that any ordering of the elements in $U$ is a $1$-competitive incremental solution.

We recall that the problem with instances with unbounded competitive ratio was that the value of larger sets was not bounded by the value of its subsets. Subadditivity does exactly this. For two sets $A, B \subseteq U$, it bounds the value of their union $f(A \cup B) \leq f(A) + f(B)$. For example, the objective we considered above is subadditive. As we will see later in this chapter, the subclass of IncMax with subadditive objectives indeed admits a bounded competitive ratio. Yet, there are also problem instances where the objective is accountable, but not subadditive. As an example consider $U = \{a, b\}$ and the function $f\colon 2^U \to \mathbb{R}_{\geq 0}$

such that, for all $S \subseteq U$,

$$f(S) = \begin{cases} 0, & \text{if } S = \emptyset, \\ 2, & \text{if } S = \{a\}, \\ 1, & \text{if } S = \{b\}, \\ 4, & \text{if } S = U. \end{cases}$$

This function is not subadditive because $f(\{a\}) + f(\{b\}) = 3 < 4 = f(\{a, b\})$, but accountable because the ordering $(a, b)$ is one where each prefix of length $i \in \{1, 2\}$ has value at least $\frac{i}{2} f(U)$.

This gives rise to the question whether there exists a natural problem class that encompasses both, the instances with accountable objectives, as well as the instances with subadditive objectives. In order to answer this question, in this chapter, we introduce and investigate a relaxed version of accountability that also relaxes subadditivity.

## 5.1. $\beta$-Accountability

Lemma 3.1 shows that accountability of a function $f : 2^U \to \mathbb{R}_{\geq 0}$ implies that, for every finite set $S \subseteq U$ with $k := |S|$, there exists an ordering $(e_1, \dots, e_k)$ of the elements in $S$ such that, for all $i \in [k]$,

$$f(\{e_1, \dots, e_i\}) \geq \frac{i}{k} f(S).$$

In order to define a relaxed version of accountability, we relax this property.

**Definition 5.1.** *For $\beta \in (0, 1]$, a function $f : 2^U \to \mathbb{R}_{\geq 0}$ is called* $\beta$-accountable *if, for every $S \subseteq U$ with $k := |U|$, there exists an ordering $(e_1, \dots, e_k)$ of the elements in $S$ such that, for all $i \in [k]$,*

$$f(\{e_1, \dots, e_i\}) \geq \beta \frac{i}{k} f(S).$$

Similar to accountability, $\beta$-accountability implies that the optimum value for large cardinalities cannot grow too fast. As we did with accountability, we denote the subclass of IncMax with $\beta$-accountable objective functions by $\text{IncMax}_{\beta\text{-acc}}$.

**Lemma 5.2.** *Let $f$ be monotone and $\beta$-accountable for some $\beta \in (0, 1]$. Then, for $C, C' \in \mathbb{N}$ with $C < C'$, we have*

$$\text{Opt}(C) \leq \text{Opt}(C') \leq \frac{1}{\beta} \frac{C'}{C} \text{Opt}(C).$$

*Proof.* The fact that $\textsc{Opt}(C) \leq \textsc{Opt}(C')$ follows immediately from monotonicity.

By $\beta$-accountability, there is an ordering $(e_1, \dots, e_{C'})$ of $O(C')$, the optimum solution of cardinality $C'$, such that, for all $i \in [C']$, we have $f(\{e_1, \dots, e_k\}) \geq \beta \frac{i}{C'} \textsc{Opt}(C')$. Thus, $\textsc{Opt}(C) \geq f(\{e_1, \dots, e_C\}) \geq \beta \frac{C}{C'} \textsc{Opt}(C')$. □

The best known algorithm to solve the IncMax problem with an accountable objective is the CardinalityScaling algorithm that was introduced in [5], and that we investigated in Section 4.1.1. Since $\beta$-accountability is closely related to accountability, we introduce a modified version of CardinalityScaling to find an incremental solution for the IncMax problem with a $\beta$-accountable objective.

The algorithm $\textsc{CardinalityScaling}_\beta$ uses the scaling parameter

$$\delta = \frac{1}{2\beta} + 1 + \sqrt{\frac{1}{4\beta^2} + 1}$$

and chooses

$$c_1 \in \arg\max_{c \in \mathbb{N}} \frac{\textsc{Opt}(c)}{c}$$

in an arbitrary, but fixed way. Then, for $i \in \mathbb{N}$, it chooses

$$c_{i+1} \in \arg\max_{c \in \mathbb{N}_{\geq \delta c_i}} \frac{\textsc{Opt}(c)}{c}$$

also in an arbitrary, but fixed way. $\textsc{CardinalityScaling}_\beta$ operates in phases, and in phase $i \in \mathbb{N}$, it adds $O(c_i)$, the optimum solution of cardinality $c_i$, in the order given by Definition 5.1.

The following observation follows immediately from the definition of $c_i$ and Lemma 1.6.

**Observation 5.3.** *For all $i \in \mathbb{N}$, we have*

*(i)* $c_{i+1} \geq \delta c_i$,

*(ii)* $\sum_{j=1}^{i} c_j \leq \frac{\delta}{\delta - 1} c_i$,

*(iii)* $\frac{\textsc{Opt}(c_i)}{c_i} \geq \frac{\textsc{Opt}(c)}{c}$ *for all* $c \in \mathbb{N}_{\geq \delta c_i}$.

We are now ready to prove an upper bound on the competitive ratio of $\textsc{CardinalityScaling}_\beta$.

**Theorem 5.4.** *The algorithm $\textsc{CardinalityScaling}_\beta$ is $\delta$-competitive for $\textsc{IncMax}_{\beta\text{-acc}}$.*

*Proof.* Let $X$ denote the incremental solution of $\textsc{CardinalityScaling}_\beta$. Let $C \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $C \in \left(\sum_{j=1}^{i-1} c_j, \sum_{j=1}^{i} c_j\right]$. If $i = 1$, we let $x = 0$, otherwise, let

$$x = \frac{1}{\beta} \cdot \frac{\textsc{Opt}(c_{i-1})}{\textsc{Opt}(c_i)} c_i + \sum_{j=1}^{i-1} c_j, \tag{5.1}$$

i.e., we have

$$\textsc{Opt}(c_{i-1}) = \beta \frac{x - \sum_{j=1}^{i-1} c_j}{c_i} \textsc{Opt}(c_i).$$

The value $x$ is chosen such that, if $C \geq x$, then the value of the partially added optimum solution $O(c_i)$ in the solution $X(C)$ is at least as large as the value of the completely contained optimum solution $O(c_{i-1})$.

**Case 1:** $C < \lceil \delta c_{i-1} \rceil$**.**
As $X(C)$ contains the optimum solution $O(c_{i-1})$, by monotonicity, we have

$$\frac{\textsc{Opt}(C)}{f(X(C))} \leq \frac{\textsc{Opt}(C)}{\textsc{Opt}(c_{i-1})} \leq \frac{\textsc{Opt}(\lceil \delta c_{i-1} \rceil - 1)}{\textsc{Opt}(c_{i-1})} \overset{\text{Obs. 5.3 } (iii)}{\leq} \frac{\lceil \delta c_{i-1} \rceil - 1}{c_{i-1}} \leq \delta.$$

**Case 2:** $\lceil \delta c_{i-1} \rceil \leq C < x$**.**
Note that $C < x$ implies that $x > 0$, i.e., (5.1) holds. The solution $X(C)$ contains the optimum solution $O(c_{i-1})$. Thus,

$$\begin{aligned}
\frac{\textsc{Opt}(C)}{\textsc{Alg}(C)} &\leq \frac{\textsc{Opt}(C)}{\textsc{Opt}(c_{i-1})} \overset{\text{Obs. 5.3 } (iii)}{\leq} \frac{\textsc{Opt}(c_i)}{\textsc{Opt}(c_{i-1})} \cdot \frac{C}{c_i} \\
&\leq \frac{\textsc{Opt}(c_i)}{\textsc{Opt}(c_{i-1})} \cdot \frac{x}{c_i} \overset{(5.1)}{=} \frac{1}{\beta} + \frac{\textsc{Opt}(c_i)}{\textsc{Opt}(c_{i-1})} \cdot \frac{1}{c_i} \sum_{j=1}^{i-1} c_j \\
&\overset{\text{Obs. 5.3 } (iii)}{\leq} \frac{1}{\beta} + \frac{c_i}{c_{i-1}} \cdot \frac{1}{c_i} \sum_{j=1}^{i-1} c_j \overset{\text{Obs. 5.3 } (ii)}{\leq} \frac{1}{\beta} + \frac{\delta}{\delta - 1} = \delta,
\end{aligned}$$

where the last equality follows from the definition of $\delta$.

**Case 3:** $C \geq \lceil \delta c_{i-1} \rceil$, $C \geq x$**.**
If $i = 1$, by definition of $c_1$, we have

$$\frac{\textsc{Opt}(C)}{\textsc{Alg}(C)} \leq \frac{\textsc{Opt}(C)}{\beta \frac{C}{c_1} \textsc{Opt}(c_1)} \leq \frac{1}{\beta} < \delta.$$

Now, assume that $i \geq 2$, i.e., (5.1) holds. The solution $X(C)$ contains $C - \sum_{j=1}^{i-1} c_j$

elements from the optimum solution $O(c_i)$. Thus,

$$
\begin{aligned}
\frac{\textsc{Opt}(C)}{\textsc{Alg}(C)} &\leq \frac{\textsc{Opt}(C)}{\beta \frac{C-\sum_{j=1}^{i-1} c_j}{c_i} \textsc{Opt}(c_i)} \overset{\text{Obs. 5.3 } (iii)}{\leq} \frac{1}{\beta} \cdot \frac{C}{C - \sum_{j=1}^{i-1} c_j} \\
&\overset{C \geq x}{\leq} \frac{1}{\beta} \cdot \frac{x}{x - \sum_{j=1}^{i-1} c_j} \overset{(5.1)}{=} \frac{1}{\beta} \left( 1 + \frac{\sum_{j=1}^{i-1} c_j}{\frac{1}{\beta} \cdot \frac{\textsc{Opt}(c_{i-1})}{\textsc{Opt}(c_i)} c_i} \right) \\
&= \frac{1}{\beta} + \frac{\textsc{Opt}(c_i)}{\textsc{Opt}(c_{i-1})} \cdot \frac{1}{c_i} \sum_{j=1}^{i-1} c_j \overset{\text{Obs. 5.3 } (iii)}{\leq} \frac{1}{\beta} + \frac{c_i}{c_{i-1}} \cdot \frac{1}{c_i} \sum_{j=1}^{i-1} c_j \\
&\overset{\text{Obs. 5.3 } (ii)}{\leq} \frac{1}{\beta} + \frac{\delta}{\delta - 1} = \delta.
\end{aligned}
$$

□

We complement this upper bound with a lower bound, that, in particular, shows that for $\beta \to 0$, we cannot be better than $\frac{1}{\beta}$-competitive.

**Theorem 5.5.** *For all $\beta \in (0, 1]$, the competitive ratio of* $\textsc{IncMax}_{\beta\text{-acc}}$ *is at least*

$$\frac{1}{\beta} \cdot \left( 1 + \frac{1}{\lceil \frac{1}{\beta} \rceil + 1} \right).$$

*Proof.* Let $k := \lceil \frac{1}{\beta} \rceil + 2$ and $d := \frac{k-1}{k} \beta$. We will define an instance where no incremental solution can have a competitive ratio better than $\frac{1}{d}$. Let $U = \{e_1, \dots, e_{k+1}\}$ be the groundset and $f : 2^U \to \mathbb{R}_{\geq 0}$ be the objective such that, for $S \subseteq U$,

$$f(S) = \begin{cases} \frac{kd}{\beta} = k - 1, & \text{if } \{e_2, \dots, e_{k+1}\} \subseteq S, \\ \max\bigl\{ |\{e_1\} \cap S|, |\{e_2, \dots, e_{k+1}\} \cap S| \cdot d \bigr\}, & \text{else.} \end{cases}$$

This objective is monotone because of the maximum in the definition and because $\frac{kd}{\beta} \geq kd$. We show that $f$ is $\beta$-accountable. For this, let $S \subseteq U$. We have to show that there is an ordering $(e_{i_1}, \dots, e_{i_{|S|}})$ of $S$ such that $f(\{e_{i_1}, \dots, e_{i_j}\}) \geq \beta \frac{j}{|S|} f(S)$ for all $j \in [|S|]$. If we have $e_1 \in S$ and $f(S) = 1$, we can simply choose $e_1$ to be the first element in the ordering and obtain

$$f(\{e_{i_1}, \dots, e_{i_j}\}) = 1 = f(S) \geq \beta \frac{j}{|S|} f(S)$$

for all $j \in [|S|]$. Otherwise, if $e_1 \notin S$ or $f(S) > 1$, then, with $S' := S \cap \{e_2, \dots, e_{k+1}\}$, either $f(S) = \frac{kd}{\beta} = \frac{|S'|d}{\beta}$, or $f(S) = |S'|d \leq \frac{|S'|d}{\beta}$. In our ordering, we can put the elements from $S'$ in the beginning and, for $j \in [|S'|]$, obtain

$$f(\{e_{i_1}, \ldots, e_{i_j}\}) \geq j \cdot d = \beta \frac{j}{|S'|} \frac{|S'|d}{\beta} \geq \beta \frac{j}{|S'|} f(S) \geq \beta \frac{j}{|S|} f(S).$$

If $S' = S$, we are done. Otherwise, $|S| = |S'| + 1$ holds and, for $j = |S|$, we have $f(\{e_{i_1}, \ldots, e_{i_j}\}) = f(S) \geq \beta \frac{j}{|S|} f(S)$.

Let $X$ be an incremental solution for this instance. We consider two cases. First, assume that $e_1$ is not the last element in the ordering $X$. We have

$$(k-1)d = \frac{\left(\lceil \frac{1}{\beta} \rceil + 1\right)^2}{\lceil \frac{1}{\beta} \rceil + 2} \beta > \frac{\lceil \frac{1}{\beta} \rceil^2 + 2\lceil \frac{1}{\beta} \rceil}{\lceil \frac{1}{\beta} \rceil + 2} \beta = \left\lceil \frac{1}{\beta} \right\rceil \beta \geq 1.$$

The solution $X(k)$ contains $e_1$ and $k-1$ elements from $\{e_2, \ldots, e_{k+1}\}$. Thus,

$$f(X_1(k)) = \max\{1, (k-1)d\} = (k-1)d.$$

The optimum solution of cardinality $k$ is the set $\{e_2, \ldots, e_{k+1}\}$ with a value of $k-1$. Thus, in this case, the competitive ratio of $X$ is at least $\frac{1}{d}$.

Now, consider the other case, i.e., $e_1$ is the last element in the ordering $X$. Then the solution $X(1)$ contains exactly one element from the set $\{e_2, \ldots, e_{k+1}\}$ and has therefore value $f(X(1)) = d$. The optimum solution of cardinality $1$ is $\{e_1\}$ with a value of $1$. Thus, also in this case, the competitive ratio of $X$ is at least $\frac{1}{d}$. □

In Figure 5.1, we can see a plot of the upper bound from Theorem 5.4 and the lower bound from Theorem 5.5 in black, as well as a plot of their difference in red. On the one hand, one can see that both bounds diverge for small $\beta$. This seems plausible because for $\beta \to 0$, we have (almost) no guarantee that the value of large sets is bounded by the value of smaller sets. On the other hand, one can see in Figure 5.1 that the difference between upper and lower bound is almost $0$ for $\beta \to 0$. Thus, in the limit $\beta \to 0$, CardinalityScaling$_\beta$ performs optimally.

## 5.2. Comparison to Other Properties

We have seen that the class of $\beta$-accountable objectives yields a bounded competitive ratio for the IncMax problem. We turn to comparing the new property of $\beta$-accountability to other objective properties to see whether the upper bound results from Theorem 5.4 yield upper bounds for objectives with these other properties.

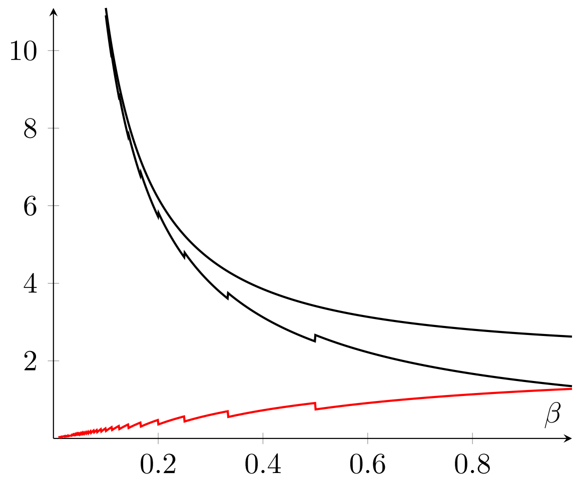


Figure 5.1.: Plot of the upper bound from Theorem 5.4, the lower bound from Theorem 5.5, and their difference (red).

We will show that every function that is subadditive, accountable, or $\gamma$-$\alpha$-augmentable is also $\beta$-accountable for $\beta = \frac{1}{2}$, $\beta = 1$, or $\beta = \frac{\gamma}{\alpha}$ respectively. Since the classes with these properties contain functions that are submodular, weighted rank functions of independence systems, fractionally subadditive, $\alpha$-augmentable, or have a bounded submodularity ratio (cf. Figure 1.3), we also obtain upper bounds on the competitive ratio for IncMax problems with these objectives.

**accountability.** We start by comparing $\beta$-accountability to accountability. Since $\beta$-accountability is simply a relaxation of accountability with relaxation parameter $\beta$, the sets of accountable and 1-accountable functions coincide. Theorem 5.4 gives an upper bound of $\frac{1}{2}(3+\sqrt{5})$ on the competitive ratio of the IncMax$_{\text{acc}}$ problem, which recovers the best known upper bound of $\varphi + 1$. This is not surprising because, for $\beta = 1$, CardinalityScaling$_\beta$ behaves exactly like CardinalityScaling.

**Proposition 5.6.** *Every fractionally subadditive function is accountable.*

*Proof.* Let $f\colon 2^U \to \mathbb{R}_{\geq 0}$ be fractionally subadditive. Then, the function is also an XOS-function as shown by [27]. Thus, there are $k \in \mathbb{N}$ and values $v_{e,i} \in \mathbb{R}$ for all $e \in U$ and $i \in [k]$ such that, for all $S \subseteq U$,

$$f(S) = \max_{i\in[k]} \sum_{e\in S} v_{e,i}.$$

To show that $f$ is accountable, fix a finite set $S \subseteq U$. Let $i^* \in [k]$ such that $f(S) = \sum_{e \in S} v_{e,i^*}$. Let $e^* = \arg\min_{e \in S} v_{e,i^*}$, i.e., we have $v_{e^*,i^*} \leq \frac{1}{|S|} f(S)$. Then

$$f(S \setminus \{e^*\}) = \max_{i \in [k]} \sum_{e \in S \setminus \{e^*\}} v_{e,i} \geq \sum_{e \in S \setminus \{e^*\}} v_{e,i^*} = f(S) - v_{e^*,i^*} \geq \left(1 - \frac{1}{|S|}\right) f(S).$$ □

Combined with Theorem 5.4, we obtain an upper bound on the competitive ratio of IncMax with fractionally subadditive.

**Corollary 5.7.** *CardinalityScaling$_1$ has a competitive ratio of $\varphi + 1$ for IncMax with a fractionally subadditive objective.*

**subadditivity.** Now, we compare $\beta$-accountability to subadditivity.

**Proposition 5.8.** *Every monotone, subadditive function is $\frac{1}{2}$-accountable.*

*Proof.* Let $f \colon 2^U \to \mathbb{R}_{\geq 0}$ be monotone and subadditive, $S \subseteq U$ be finite, and $k := |S|$. We define $\ell := \lceil \log_2 k \rceil$, i.e., we have $2^{\ell-1} < k \leq 2^\ell$. Furthermore, we define $S_\ell := S$ and iteratively, for $j \in \{\ell - 1, \ldots, 0\}$, $S_j \subseteq S_{j+1}$ with $|S_j| = \lceil \frac{k}{2^{\ell-j}} \rceil$ and $f(S_j) \geq \frac{1}{2} f(S_{j+1})$. This is possible as we will see now. We have $2\lceil \frac{k}{2^{\ell-j}} \rceil = \lceil 2 \lceil \frac{k}{2^{\ell-j}} \rceil \rceil \geq \lceil \frac{k}{2^{\ell-j-1}} \rceil$, i.e., we can choose $A, B \subseteq S_j$ with $|A| = |B| = \lceil \frac{k}{2^{\ell-j}} \rceil$ and $A \cup B = S_{j+1}$. By subadditivity, we have $f(A) \geq \frac{1}{2} f(S_{j+1})$ or $f(B) \geq \frac{1}{2} f(S_{j+1})$. Thus, we can choose $S_j \in \{A, B\}$ with the desired properties.

Consider any order $(x_1, \ldots, x_k)$ of $S$ with $\{e_1, \ldots, e_{|S_j|}\} = S_j$ for all $j \in \{0, \ldots, \ell\}$. Let $i \in [k]$ and $j \in \{0, \ldots, \ell\}$ such that $\lceil \frac{k}{2^{\ell-j}} \rceil \leq i < \lceil \frac{k}{2^{\ell-j-1}} \rceil$. This implies that $S_j \subseteq \{e_1, \ldots, e_i\}$ and, because $i \in \mathbb{N}$, that $i < \frac{k}{2^{\ell-j-1}}$. Together with monotonicity of $f$, we obtain

$$f(\{e_1, \ldots, e_i\}) \geq f(S_j) \geq \left(\frac{1}{2}\right)^{\ell-j} f(S_\ell) = \frac{1}{2^{\ell-j}} f(S) \geq \frac{1}{2} \frac{i}{k} f(S),$$

which yields $\frac{1}{2}$-accountability. □

We combine the results from Theorem 5.4 and Proposition 5.8 to obtain an upper bound on the competitive ratio of IncMax with a subadditive objective, which, to our knowledge, is the first for this problem setting.

**Theorem 5.9.** *CardinalityScaling$_{1/2}$ is $(2 + \sqrt{2} < 3.415)$-competitive for IncMax with a subadditive objective.*

We complement this upper bound on the competitive ratio of the subclass of IncMax with subadditive objective functions with a lower bound. In order to do this, we show that the objective function of every instance in IncMaxSep (cf. Section 3.1) is subadditive.

**Proposition 5.10.** *The objective function of every instance in IncMaxSep is subadditive.*

*Proof.* Let an instance from IncMaxSep with objective function $f\colon 2^U \to \mathbb{R}_{\geq 0}$ be given. Furthermore, let $U = U_1 \cup U_2 \cup \dots$ be a partition of $U$, and let $d_1, d_2, \dots > 0$ be the densities such that, for all $S \subseteq U$,

$$f(S) = \max_{i \in \mathbb{N}} |S \cap U_i| \cdot d_i.$$

In order to show subadditivity, we fix two sets $A, B \subseteq U$. Let $i^* \in \mathbb{N}$ be the index such that $f(A \cup B) = |(A \cup B) \cap U_{i^*}| \cdot d_{i^*}$. Then

$$\begin{aligned} f(A \cup B) &= |(A \cup B) \cap U_{i^*}| \cdot d_{i^*} \\ &= |A \cap U_{i^*}| \cdot d_{i^*} + |(B \setminus A) \cap U_{i^*}| \cdot d_{i^*} \\ &\leq |A \cap U_{i^*}| \cdot d_{i^*} + |B \cap U_{i^*}| \cdot d_{i^*} \\ &\leq \big(\max_{i \in \mathbb{N}} |A \cap U_i| \cdot d_i\big) + \big(\max_{i \in \mathbb{N}} |B \cap U_i| \cdot d_i\big) \\ &= f(A) + f(B). \end{aligned}$$ □

This result yields that the (non-strict) competitive ratio of IncMax with subadditive objectives is at least that of IncMaxSep. Therefore, the lower bound from Theorem 3.27 also holds here.

**Corollary 5.11.** *The (non-strict) competitive ratio of IncMax with a subadditive objective function is at least* $2.246$.

**$\gamma$-$\alpha$-augmentability.** We turn to comparing $\beta$-accountability to $\gamma$-$\alpha$-augmentability that we introduced to bound the competitive ratio of the Greedy algorithm (cf. Definition 2.3).

**Proposition 5.12.** *For all $\gamma \in (0,1]$ and $\alpha \geq 1$, every monotone, $\gamma$-$\alpha$-augmentable function is $\frac{\gamma}{\alpha}$-accountable.*

*Proof.* For $\gamma \in (0,1]$ and $\alpha \geq 1$, let $f\colon 2^U \to \mathbb{R}_{\geq 0}$ be monotone and $\gamma$-$\alpha$-augmentable. Let $S \subseteq U$ be finite and $k := |S|$. By $\gamma$-$\alpha$-augmentability, there exists an ordering $(e_1, \dots, e_k)$ of the elements in $S$ such that, for all $i \in \{0, \dots, k-1\}$,

$$f(\{e_1, \dots, e_{i+1}\}) - f(\{e_1, \dots, e_i\}) \geq \frac{\gamma f(S) - \alpha f(\{e_1, \dots, e_i\})}{k - i}.$$

For $i \in \{0, \ldots, k\}$, let $S_i := \{e_1, \ldots, e_i\}$. Then, for $i \in \{0, \ldots, k-1\}$, this yields

$$f(S_{i+1}) - f(S_i) \geq \alpha \frac{\frac{\gamma}{\alpha} f(S) - f(S_i)}{k-i} \overset{\alpha \geq 1}{\geq} \frac{\frac{\gamma}{\alpha} f(S) - f(S_i)}{k-i}. \tag{5.2}$$

To show $\frac{\gamma}{\alpha}$-accountability of $f$, we prove by induction that, for $i \in [k]$, we have

$$f(S_i) \geq \frac{\gamma}{\alpha} \frac{i}{k} f(S). \tag{5.3}$$

For i=1, (5.2) yields

$$f(S_1) \geq \frac{1}{k}\left(\frac{\gamma}{\alpha} f(S) - f(\emptyset)\right) + f(\emptyset) = \frac{\gamma}{\alpha}\frac{1}{k} f(S) + \left(1 - \frac{1}{k}\right) f(\emptyset) \geq \frac{\gamma}{\alpha}\frac{1}{k} f(S),$$

where the last inequality follows from non-negativity of $f$.

Now, suppose that (5.3) holds for some $i \in [k-1]$. Then

$$\begin{aligned} f(S_{i+1}) &\overset{(5.2)}{\geq} f(S_i) + \frac{\frac{\gamma}{\alpha} f(S) - f(S_i)}{k-i} \\ &= \frac{\gamma}{\alpha}\frac{1}{k-i} f(S) + \left(1 - \frac{1}{k-i}\right) f(S_i) \\ &\overset{(5.3)}{\geq} \frac{\gamma}{\alpha}\frac{1}{k-i} f(S) + \left(\frac{k-i-1}{k-i}\right)\frac{\gamma}{\alpha}\frac{i}{k} f(S) \\ &= \frac{\gamma}{\alpha}\frac{ki + k - i^2 - i}{(k-i)k} f(S) \\ &= \frac{\gamma}{\alpha}\frac{(k-i)(i+1)}{(k-i)k} f(S) \\ &= \frac{\gamma}{\alpha}\frac{i+1}{k} f(S), \end{aligned}$$

which concludes the induction. □

Note that we have to require that $f$ is $\gamma$-$\alpha$-augmentable, and not weakly $\gamma$-$\alpha$-augmentable, in order to have the estimate in (5.2) for all sets $S_0, \ldots, S_k$.

**Proposition 5.13.** *For $\gamma \in (0, 1]$, $\alpha \geq 1$, and $\varepsilon > 0$, there exists a monotone, $\gamma$-$\alpha$-augmentable function that is not $\left(\frac{\gamma}{\alpha} + \varepsilon\right)$-accountable.*

*Proof.* Let $n \in \mathbb{N}$ be large enough that $n \geq \alpha$ and

$$\frac{n-1}{n} > \frac{\frac{\gamma}{\alpha}}{\frac{\gamma}{\alpha} + \varepsilon}. \tag{5.4}$$

Consider some set $U$ with $|U| = n$ and the function $f\colon 2^U \to \mathbb{R}_{\geq 0}$ with

$$f(S) = \begin{cases} 0, & \text{if } |S| = 0, \\ \frac{\gamma}{\alpha}, & \text{if } 1 \leq |S| \leq n-1, \\ 1, & \text{if } |S| = n. \end{cases}$$

Obviously, this function is monotone. We show that it is $\gamma$-$\alpha$-augmentable. Let $A, B \subseteq U$ with $B \setminus A \neq \emptyset$. If $A = \emptyset$ and $B = U$, then, for all $b \in B$, we have $f(A \cup \{b\}) - f(A) = \frac{\gamma}{\alpha} - 0 \geq \frac{\gamma - 0}{n} = \frac{\gamma f(A \cup B) - \alpha f(A)}{|B|}$. If $A = \emptyset$ and $|B| \leq n-1$, then, for all $b \in B$, we have

$$f(A \cup \{b\}) - f(A) = \frac{\gamma}{\alpha} - 0 \geq \frac{\frac{\gamma}{\alpha} - 0}{|B|} = \frac{f(A \cup B) - f(A)}{|B|} \geq \frac{\gamma f(A \cup B) - \alpha f(A)}{|B|}.$$

Now suppose that $A \neq \emptyset$. Because $B \setminus A \neq \emptyset$, we have $1 \leq |A| \leq n-1$, which yields

$$f(A) = \frac{\gamma}{\alpha}. \tag{5.5}$$

For all $b \in B$, we have

$$f(A \cup \{b\}) - f(A) \geq 0 = \frac{\gamma - \alpha\frac{\gamma}{\alpha}}{|B|} \overset{(5.5)}{\geq} \frac{\gamma f(A \cup B) - \alpha f(A)}{|B|},$$

i.e., $f$ is $\gamma$-$\alpha$-augmentable.

Now we will show that $f$ is not $\left(\frac{\gamma}{\alpha} + \varepsilon\right)$-accountable. For the sake of contradiction suppose that $f$ is $\left(\frac{\gamma}{\alpha} + \varepsilon\right)$-accountable. Then there exists an ordering $(e_1, \ldots, e_n)$ of $U$ such that, for all $i \in [n]$, we have

$$f(\{e_1, \ldots, e_i\}) \geq \left(\frac{\gamma}{\alpha} + \varepsilon\right)\frac{i}{n} f(U) = \left(\frac{\gamma}{\alpha} + \varepsilon\right)\frac{i}{n}.$$

Yet, we have

$$f(\{e_1, \ldots, e_{n-1}\}) = \frac{\gamma}{\alpha} \overset{(5.4)}{<} \left(\frac{\gamma}{\alpha} + \varepsilon\right)\frac{n-1}{n},$$

which gives the desired contradiction. □

We combine the results from Proposition 5.12 and Theorem 5.4 to obtain an upper bound on the competitive ratio of IncMax with a monotone and $\gamma$-$\alpha$-augmentable objective.

**Theorem 5.14.** *For $\gamma \in (0,1]$ and $\alpha \geq 1$,* CardinalityScaling$_{\gamma/\alpha}$ *has a competitive ratio of at most*

$$\frac{\alpha}{2\gamma} + 1 + \sqrt{\frac{\alpha^2}{4\gamma^2} + 1}$$

*for* IncMax *with a $\gamma$-$\alpha$-augmentable objective.*

We compare this upper bound on the competitive ratio of IncMax with a $\gamma$-$\alpha$-augmentable objective to the upper bound

$$\frac{\alpha}{\gamma} \cdot \frac{e^{\alpha}}{e^{\alpha} - 1}$$

from Theorem 2.17. Let the ratio between the two upper bounds be denoted by

$$r(\gamma, \alpha) := \frac{\frac{\alpha}{2\gamma} + 1 + \sqrt{\frac{\alpha^2}{4\gamma^2} + 1}}{\frac{\alpha}{\gamma} \cdot \frac{e^{\alpha}}{e^{\alpha}-1}} = \frac{e^{\alpha} - 1}{e^{\alpha}} \cdot \left( \frac{1}{2} + \frac{\gamma}{\alpha} + \sqrt{\frac{1}{4} + \frac{\gamma^2}{\alpha^2}} \right).$$

We have $\lim_{\gamma \to 0} r(\gamma, \alpha) = \frac{e^{\alpha}-1}{e^{\alpha}} < 1$, i.e., for small values of $\gamma$, CardinalityScaling$_{\gamma/\alpha}$ performs better than Greedy. Since $\gamma \in (0,1]$ and $\alpha \geq 1$, we have $r(\gamma, \alpha) = 1$ if and only if $\gamma = \alpha \frac{e^{\alpha}}{e^{2\alpha}-1}$. This value lies in the interval $\left(0, \frac{e}{e^2-1}\right] \subseteq (0, 0.426)$, and for $\alpha \to \infty$, approaches $0$. Thus, for large $\alpha \geq 1$, Greedy performs better than CardinalityScaling$_{\gamma/\alpha}$ for almost all values of $\gamma \in (0,1]$. This is probably due to the fact that $\gamma$-$\alpha$-augmentability is a property that relaxes an inequality that is a core estimate in the analysis of the Greedy algorithm for monotone and submodular functions.

# 6. Incremental Maximization under a Knapsack Constraint

A natural generalization of the IncMax problem is one where, instead of an unknown cardinality constraint, we are given an unknown knapsack constraint. This problem lends itself better to model real life problems such as, e.g., infrastructure projects because we are able to model different construction times or costs in this model.

In the IncMaxKnap problem, we are given a countable ground set $U$ of elements. Each element $e \in U$ has a *weight* $w(e)$ that models the time or money that has to be spent to realize the element. In the following, for a set $S \subseteq U$, we write $w(S) := \sum_{e \in S} w(e)$. As in the IncMax problem, we are given a monotone objective $f\colon 2^U \to \mathbb{R}_{\geq 0}$ and the optimum for a capacity $C \in \mathbb{R}_{\geq 0}$ is defined as

$$\textsc{Opt}(C) := \sup\{f(S) \mid S \subseteq U, w(S) \leq C\}.$$

We denote a set $S \subseteq U$ for which the optimum is attained by $O(C)$[1], where we break ties in an arbitrary but fixed manner in order to obtain a unique set $O(C)$.

Again, we assume that we do not know the capacity constraint $C$. Thus, also for this problem, an *incremental solution* $X$ is given by an ordering $X = (e_1, e_2, \dots)$ of the elements of the ground set $U$. As before, for a capacity $C$, we denote by $X(C)$ the elements of the largest prefix of weight at most $C$, i.e.,

$$X(C) = \{e_1, e_2, \dots, e_k\}$$

with $k \in \mathbb{N}$ such that $\sum_{i=1}^{k} w(e_{\pi(i)}) \leq C$ and either $k = |U|$ or $\sum_{i=1}^{k+1} w(e_{\pi(i)}) > C$. All definitions regarding competitiveness are analogous to the IncMax problem. For example, we say that $X$ is *$\rho$-competitive* if, for all $C \in \mathbb{R}_{\geq 0}$, we have $\textsc{Opt}(C) \leq \rho f(X(C))$.

As an example, let us consider a modified version of the incremental maximum $s$-$t$-flow problem that we already considered in Chapter 1. Now, every edge additionally has a weight $w(e) \in \mathbb{R}_{\geq 0}$ and the combined weight of a solution may not exceed the (unknown)

[1] In the case that such a set does not exists, we use an arbitrarily close approximation as we did for the IncMax problem.

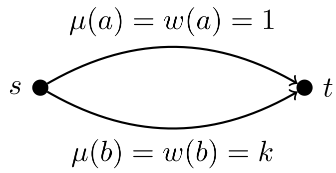


Figure 6.1.: Two examples of the incremental maximum $s$-$t$-flow problem where no $\rho$-competitive incremental solution with $\rho < k$ exists.

capacity constraint $C$. Consider the graph in Figure 6.1. Every competitive incremental solution has to put edge $a$ first in order to be competitive for $C = 1$, and every incremental solution that puts edge $a$ first is not better than $k$-competitive for $C = k$.

A closer inspection of the examples in Figure 1.1 (which is an instance of this problem with weights $w(a) = w(b) = w(c) = 1$) and in Figure 6.1 reveals that there are (at least) two effects that prevent the existence of competitive incremental solutions. The first is the *complementarity* of elements. In the graph in Figure 1.1, edges $b$ and $c$ are complementary in the sense that both edges together support an $s$-$t$-flow of $k$ while a single one of these edges alone cannot support any $s$-$t$-flow. For the graph in Figure 6.1, no two edges are complementary since the total $s$-$t$-flow supported by a subset of edges is here simply equal to the sum of the capacities of the edges. In this example, the non-existence of a competitive incremental solution is caused by the fact that the edges are too heterogeneous. More specifically, we have $f(\{a\}) = 1$, but $f(\{b\}) = k$, i.e., there are two singleton sets whose values differ by a factor of $k$.

As we will show, these are essentially the only two effects that prevent the existence of competitive incremental solutions. More specifically, we make two assumptions that exclude the two effects shown in Figure 1.1 and Figure 6.1. First, to avoid complementarities between elements, we assume that $f$ is fractionally subadditive (cf. Definition 1.2). Second, to avoid that there exist singleton sets that differ too much in their values, we assume that there is a constant $M \in \mathbb{R}_{\geq 0}$, $M \geq 1$ such that $f(\{e\}) \in [1, M]$ for all $e \in U$. We call such valuations *$M$-bounded*.

Summarizing the discussion, this chapter considers the IncMaxKnap with the following assumptions.

- $f$ is monotone, i.e., $f(A) \geq f(B)$ for $A \supseteq B$
- $f$ is $M$-bounded, i.e., $f(e) \in [1, M]$ for all $e \in U$
- $f$ is fractionally subadditive

Before giving an overview over the chapter, we illustrate the applicability of this framework to different settings.

**Example 6.1** (Submodular objective)**.** *It was shown by Lehmann et al. [51] that every monotone submodular function is also fractionally subadditive.*

*As a consequence our framework captures, e.g., the* Maximum Coverage *problem, where we are given a weighted family of sets $U \subseteq 2^E$ over a universe $E$. Every element of $E$ has a value $v\colon E \to \mathbb{R}_{\geq 0}$ associated with it, and $f(S) = v\left(\bigcup_{X \in S} X\right)$ for all $S \subseteq U$ where we write $v(X) := \sum_{x \in X} v(x)$ for a set $X \subseteq U$. In this context, the $M$-boundedness condition demands that $v(X) \in [1, M]$ for all $X \in U$. Further examples include maximization versions of clustering and location problems.*

**Example 6.2** (XOS objective)**.** *An objective function $f\colon 2^U \to \mathbb{R}$ is called* XOS *if it can be written as the pointwise maximum of modular functions, i.e., there are $k \in \mathbb{N}$ and values $v_{e,i} \in \mathbb{R}$ for all $e \in U$ and $i \in [k]$ such that*

$$f(S) = \max\left\{ \sum_{e \in S} v_{e,i} \;\middle|\; i \in [k] \right\} \quad \text{for all } S \subseteq U.$$

*As shown by Feige [27], the set of fractionally subadditive functions and the set of XOS functions coincide. XOS functions are a popular way to encode the valuations of buyers in combinatorial auctions (cf. [21, 22, 51, 60]).*

**Example 6.3** (Weighted rank function of an independence system)**.** *As shown by Amanatidis et al. [1], the weighted rank function of an independence system is fractionally subadditive. Thus, this setting includes problems like weighted $d$-dimensional matching, weighted set packing, or weighted maximum independent set.*

**Example 6.4** (Potential-based $s$-$t$-flows)**.** *Consider a variant of the incremental maximum $s$-$t$-flow problem on parallel edges in a directed graph $G = (\{s,t\}, E)$, as in Figure 6.1. Every edge $e$ has a capacity $\mu(e) \in \mathbb{R}_{\geq 0}$. In addition, we are given a continuous and strictly increasing* potential-loss function *$\psi : \mathbb{R} \to \mathbb{R}$ with $\lim_{x\to\infty} \psi(x) = \infty$ that describes the physical properties of the network. Every edge $e \in E$ has a* resistance *$\beta(e) \in \mathbb{R}_{\geq 0}$. A flow $\vartheta\colon E \to \mathbb{R}_{\geq 0}$ is called a* potential-based flow *if there are vertex potentials $p_s, p_t \in \mathbb{R}_{\geq 0}$ such that*

$$p_s - p_t = \beta(e)\psi(\vartheta(e)) \quad \text{for all } e \in E.$$

*The potentials correspond to physical properties at the nodes such as pressures or voltages; different choices of $\psi$ allow to model, e.g., gas flows, water flows, and electrical flows, see*

*Birkhoff and Diaz [7] and Groß et al. [35]. In our incremental framework, $w\colon U \to \mathbb{R}_{\geq 0}$ are interpreted as construction costs/times of pipes or cables and the objective is to maximize the potential-based $s$-$t$-flow, i.e., the objective $f\colon 2^U \to \mathbb{R}_{\geq 0}$ maps $S \subseteq U$ to the value*

$$f(S) = \max\left\{\sum_{e \in T} \psi^{-1}\left(\frac{p}{\beta(e)}\right) \;\middle|\; T \subseteq S, p \in \mathbb{R}_{\geq 0} \text{ with } \psi^{-1}\left(\frac{p}{\beta(e)}\right) \leq \mu(e) \text{ for all } e \in T\right\},$$

*where $p := p_s - p_t$. Note that we allow turning off the edges in $S \setminus T$ in order to make $f$ monotone. The $M$-boundedness condition corresponds to the assumption that $\mu(e) \in [1, M]$. As we will show in Proposition 6.21, this objective is fractionally subadditive.*

In Section 6.1 we analyze the strict competitive ratio of the IncMaxKnap problem. We introduce the algorithm DoubleScaling that combines the ideas of the CardinalityScaling and the ValueScaling by adding optimum solutions for different capacities one after the other such that the capacities and values of the added sets are scaled by at least some constant each. The order in which the elements from one set are added is chosen based on a primal-dual LP formulation that relies on fractional subadditivity. For the definition of the algorithm, we need access to two oracles. On the one hand, we need oracle access to the optimum solution of a given capacity; on the other hand, we need access to an XOS oracle. More information on this can be found in Remark 6.5. The algorithm DoubleScaling yields an upper bound of $\max\{3.293\sqrt{M}, 2M\}$ on the competitive ratio. We complement this with a problem instance that yields a lower bound of $\varphi + 1$.

In order to eliminate the dependence of the competitive ratios on the value of $M$, in Section 6.2, we analyze the non-strict competitive ratio of the IncMaxKnap problem. We show that a modified version of the CardinalityScaling algorithm achieves a non-strict competitive ratio of $\varphi + 1$ with additive constant $2M$. We complement this upper bound with a lower bound and show that every lower bound on the strict competitive ratio of IncMaxCont (cf. Section 3.2) is also a lower bound for the IncMaxKnap problem.

As an additional motivation, in Section 6.3 we show that our framework captures potential-based $s$-$t$-flows as described in Example 6.4. In this context, a 1-bounded objective corresponds to unit capacities. In comparison, we also show that the classical incremental maximum $s$-$t$-flow problem with capacities in $[1, M]$ admits a $2M$-competitive incremental solution, and that this is best-possible for the unit capacity case.

An extended abstract with the results in Section 6.1.1 appeared in [20] and a full version with the results in Section 6.1 will appear in [17].

## 6.1. Strict Analysis

### 6.1.1. A Capacity and Value Scaling Algorithm

In the following, we fix an instance from IncMaxKnap with ground set $U$, an $M$-bounded and fractionally subadditive objective $f\colon 2^U \to \mathbb{R}_{\geq 0}$, and weights $w\colon U \to \mathbb{R}_{\geq 0}$. We present an algorithm that combines the ideas of the CardinalityScaling algorithm from [5] and the ValueScaling algorithm that both were presented in Section 4.1. On a high level, the idea is to consider optimum solutions of increasing capacities, and to add all elements in these optimum solutions one at a time. By carefully choosing the order in which we add elements of a single optimum solution, we ensure that elements contributing the most to the objective are added first. In this way, we can guarantee that either the optimum solution we have assembled most recently, or the optimum solution we are currently assembling provides sufficient value to stay competitive. While CardinalityScaling and ValueScaling only scale either the capacity, or the value of the optimum solution, our algorithm DoubleScaling simultaneously scales both of them. In addition, we use a more sophisticated order in which we assemble the optimum solutions that is based on a primal-dual LP formulation.

We now describe DoubleScaling in detail. Let $\lambda \approx 3.2924$ be the unique real root of the equation

$$0 = \lambda^7 - 2\lambda^6 - 3\lambda^5 - 3\lambda^4 - 3\lambda^3 - 2\lambda^2 - \lambda - 1.$$

This yields

$$\left(\frac{1}{\lambda} + \frac{1}{\lambda^2}\right)\frac{\lambda^3}{\lambda^2+1} = \frac{\lambda^2}{\lambda+1} - 1 - \frac{\lambda^2+1}{\lambda^3}. \tag{6.1}$$

Furthermore, let $\delta := \frac{\lambda^3}{\lambda^2+1} \approx 3.0143$ and

$$\rho := \max\{\lambda\sqrt{M}, 2M\}. \tag{6.2}$$

Algorithm DoubleScaling operates in phases of increasing capacities $c_1, \dots, c_N \in \mathbb{R}_{\geq 0}$ with

$$c_1 := \min_{e \in U} w(e),$$
$$c_i := \min\{C \geq \delta c_{i-1} \mid \textsc{Opt}(C) \geq \rho\textsc{Opt}(c_{i-1})\} \quad \text{for all } i \in \mathbb{N},$$

where we set $\min \emptyset = w(U)$. We define $N \in \mathbb{N}$ to be the minimal index such that $c_N = w(U)$. In phase $i \in [N]$, DoubleScaling adds the elements of the set $O(c_i)$, the optimum solution for capacity $c_i$, one at a time. We may assume that previously added elements are added again (without any benefit), since this only hurts the algorithm.

To specify the order in which the elements of $O(c_i)$ are added, consider the following linear program parameterized by $X \subseteq U$ (cf. [27]):

$$\begin{aligned} \min \sum_{B \subseteq U} \alpha_B f(B) & & \quad (\text{LP}_X)\\ \text{s.t.} \sum_{B \subseteq U : e \in B} \alpha_B \geq 1, & \quad \text{for all } e \in X,\\ \alpha_B \geq 0, & \quad \text{for all } B \subseteq U, \end{aligned}$$

and its dual

$$\begin{aligned} \max \sum_{e \in X} \gamma_e & \\ \text{s.t.} \sum_{e \in B} \gamma_e \leq f(B), & \quad \text{for all } B \subseteq U,\\ \gamma_e \geq 0, & \quad \text{for all } e \in X. \end{aligned}$$

Fractional subadditivity of $f$ yields $f(X) \leq \sum_{B \subseteq U} \alpha_B f(B)$ for all $\alpha \in \mathbb{R}^{2^U}$ feasible for (LP$_X$). The solution $\alpha^* \in \mathbb{R}^{2^U}$ with $\alpha^*_X = 1$ and $\alpha^*_B = 0$ for $X \neq B \subseteq U$ is feasible and satisfies $f(X) = \sum_{B \subseteq U} \alpha^*_B f(B)$. Together this implies that $\alpha^*$ is an optimum solution to (LP$_X$). By strong duality, there exists an optimum dual solution $\gamma^*(X) \in \mathbb{R}^U$ with

$$f(X) = \sum_{e \in X} \gamma^*(X)_e. \tag{6.3}$$

In phase 1, the algorithm DoubleScaling adds the single element in $O(c_1)$. In phase 2, DoubleScaling adds an element $e \in O(c_2)$ first that maximizes $\gamma^*(O(c_2))_e$ and the other elements in an arbitrary order. In phase $i \in \{3, 4, \dots, N\}$, DoubleScaling adds the elements of $O(c_i)$ in an order $(e_1, \dots, e_{|O(c_i)|})$ such that, for all $j \in [|O(c_i)| - 1]$,

$$\frac{\gamma^*(O(c_i))_{e_j}}{w(e_j)} \geq \frac{\gamma^*(O(c_i))_{e_{j+1}}}{w(e_{j+1})}. \tag{6.4}$$

The reason why we do not use (6.4) in phase 2 is because so early on we want to increase the objective value as fast as possible which is not necessarily guaranteed by choosing the order of elements in $O(c_2)$ according to (6.4).

In the following, we denote the incremental solution of the DoubleScaling algorithm by $X^{\mathrm{A}}$. Fix some $0 \leq C \leq C' \leq w(U)$. Let $k := |O(C')|$, and let $(e_1, \dots, e_k)$ be an ordering of all elements in $O(C')$ such that, for all $j \in [k-1]$, (6.4) holds. With

$j := \max\{j \in [k] \mid w(\{e_1, \ldots, e_j\}) \leq C\}$, we define $O(C', C) := \{e_1, \ldots, e_j\}$ to be the largest prefix of the optimum solution $O(C')$ with capacity at most $C$.

Roughly, we show that this algorithm is competitive as follows: In the first phase DoubleScaling obviously performs optimally. In all other phases, the optimum solution added in the previous phase is large enough to be competitive until partially added optimum solution of the next phase has a larger value. From this point until the end of the phase, the partially added optimum solution of the next phase is competitive.

**Remark 6.5.** *In the construction of our algorithm, we assume to have oracle access to an optimum solution $O(C)$ of a given capacity $C \in \mathbb{R}_{\geq 0}$. Finding such an optimum solution may not be possible in polynomial time. Badanidiyuru et al. [3], give a $(2+\varepsilon)$-approximation algorithm that uses only a polynomial number of demand oracle queries. Furthermore, they show that no algorithm with a polynomial number of demand oracle queries can have an approximation ratio of less than $2$, unless $\mathsf{P} = \mathsf{NP}$. Our algorithm DoubleScaling can use an $\alpha$-approximation oracle instead of an oracle for the optimum solution, for a loss of factor $\alpha$ in its competitive ratio. Furthermore, we assume to have access to an XOS oracle. For a given set $X \subseteq U$ and $x \in X$, an XOS oracle gives the value of $x$ within the set $X$, which corresponds to the solution of the dual of $(\mathrm{LP}_X)$. Instead of an XOS oracle, our algorithm can use an $\beta$-approximation oracle for a loss of factor $\beta$ in its competitive ratio.*

For all $X \subseteq U$, the dual variables $\gamma^*(X)$ are a feasible solution for the dual of $(\mathrm{LP}_X)$. Thus, for all $Y \subseteq U$, we have

$$\sum_{e \in Y} \gamma^*(X)_e \leq f(Y). \tag{6.5}$$

i.e., $\gamma^*(X)$ associates a contribution to the overall objective to each element $e \in U$, and this association is consistent for all sets $Y \subseteq U$.

The following lemma establishes that the order in which we add the elements of each optimum solution are decreasing in density, in an approximate sense.

**Lemma 6.6.** *Let $0 \leq C \leq C' \leq w(U)$. Then*

$$\textsc{Opt}(C') \leq \frac{C'}{C}\big(f(O(C', C)) + M\big).$$

*Proof.* If $O(C') = O(C', C)$, the statement holds trivially. Suppose $|O(C')| > |O(C', C)|$. Let $j := |O(C', C)|$, and let $O(C') = \{e_1, \ldots, e_{|O(C')|}\}$ such that (6.4) holds. Note that, by definition, $O(C', C) = \{e_1, \ldots, e_j\}$ and

$$w(\{e_1, \ldots, e_j\}) \leq C < w(\{e_1, \ldots, e_{j+1}\}). \tag{6.6}$$

We have

$$
\begin{aligned}
\textsc{Opt}(C') &\overset{(6.3)}{=} \sum_{i=1}^{|O(C')|} w(e_i)\frac{\gamma^*(O(C'))_{e_i}}{w(e_i)} \\
&\overset{(6.4)}{\leq} \left(\sum_{i=1}^{j+1} \gamma^*(O(C'))_{e_i}\right) + \frac{\sum_{i=1}^{j+1} w(e_i)}{w(\{e_1,\dots,e_{j+1}\})} \sum_{i=j+2}^{|O(C')|} w(e_i)\frac{\gamma^*(O(C'))_{e_{j+1}}}{w(e_{j+1})} \\
&= \left(\sum_{i=1}^{j+1} \gamma^*(O(C'))_{e_i}\right) + \frac{\sum_{i=1}^{j+1} w(e_i)\frac{\gamma^*(O(C'))_{e_{j+1}}}{w(e_{j+1})}}{w(\{e_1,\dots,e_{j+1}\})} \sum_{i=j+2}^{|O(C')|} w(e_i) \\
&\overset{(6.4)}{\leq} \left(\sum_{i=1}^{j+1} \gamma^*(O(C'))_{e_i}\right) + \frac{\sum_{i=1}^{j+1} \gamma^*(O(C'))_{e_i}}{w(\{e_1,\dots,e_{j+1}\})} \sum_{i=j+2}^{|O(C')|} w(e_i) \\
&\overset{(6.6)}{<} \left(\sum_{i=1}^{j+1} \gamma^*(O(C'))_{e_i}\right) + \frac{\sum_{i=1}^{j+1} \gamma^*(O(C'))_{e_i}}{C}(C'-C) \\
&= \frac{C'}{C}\left[\left(\sum_{i=1}^{j} \gamma^*(O(C'))_{e_i}\right) + \gamma^*(O(C'))_{e_{j+1}}\right] \\
&\overset{(6.5)}{\leq} \frac{C'}{C}\big(f(\{e_1,\dots,e_j\}) + f(\{e_{j+1}\})\big) \\
&\leq \frac{C'}{C}\big(f(O(C',C)) + M\big).
\end{aligned}
$$

□

Since every set $S \subseteq U$ with $w(S) \leq C$ satisfies $f(S) \leq \textsc{Opt}(C)$, and since we have $w(O(C',C)) \leq C$, we immediately obtain the following.

**Corollary 6.7.** *Let $C, C' \in \mathbb{R}_{\geq 0}$ with $C \leq C' \leq w(U)$. Then*

$$\textsc{Opt}(C') \leq \frac{C'}{C}\big(\textsc{Opt}(C) + M\big).$$

With this, we are now ready to show an upper bound on the competitive ratio of DoubleScaling.

**Theorem 6.8.** *With $\rho = \max\{\lambda\sqrt{M}, 2M\} \approx \max\{3.2924\sqrt{M}, 2M\}$, DoubleScaling is $\rho$-competitive.*

*Proof.* We have to show that, for all capacities $C \in \mathbb{R}_{\geq 0}$, we have $\textsc{Opt}(C) \leq \rho f(X^{\mathrm{A}}(C))$. We do this by analyzing the different phases of the algorithm. Observe that, for all

$i \in \{2, \ldots, N-1\}$, by $M$-boundedness, we have

$$\textsc{Opt}(c_i) \geq \rho\textsc{Opt}(c_{i-1}) \geq \rho^{i-1}\textsc{Opt}(c_1) \geq \rho^{i-1} \geq (\lambda\sqrt{M})^{i-1}, \tag{6.7}$$

where for the first inequality, we use the definition of the algorithm DoubleScaling, and for the last inequality we use the definition of $\rho$ in (6.2).

In phase 1, we have $C \in (0, c_1]$. Since $c_1$ is the minimum weight of all elements and we start by adding $O(c_1)$, i.e., the optimum solution of capacity $c_1$, the value of $X^{\mathsf{A}}(C)$ is optimal.

Consider phase 2, and suppose $C \in (c_1, c_2)$. If $c_2 > \delta c_1$ holds, then $c_2$ is the smallest value such that $\textsc{Opt}(c_2) \geq \rho\textsc{Opt}(c_1)$, i.e., by monotonicity of $f$, we have

$$f(X^{\mathsf{A}}(C)) \geq f(X^{\mathsf{A}}(c_1)) = \textsc{Opt}(c_1) > \frac{1}{\rho}\textsc{Opt}(C).$$

Now assume $c_2 \leq \delta c_1$. If $C \in (c_1, 3c_1)$, i.e., any solution of capacity $C$ cannot contain more than two elements, or if $C \in (c_1, c_2)$ and $O(c_2)$ contains at most 2 elements, by fractional subadditivity and $M$-boundedness of $f$, we have $\textsc{Opt}(C) \leq |O(c_2)|M \leq 2M$ and thus,

$$f(X^{\mathsf{A}}(C)) \geq \textsc{Opt}(c_1) \geq 1 \geq \frac{1}{2M}\textsc{Opt}(C) \geq \frac{1}{\rho}\textsc{Opt}(C).$$

Now suppose that $C \in [3c_1, c_2)$ and that the set $O(c_2)$ contains at least 3 elements. The solution $X^{\mathsf{A}}(c_1 + c_2)$ contains all elements from the set $O(c_1) \cup O(c_2)$, the solution $X^{\mathsf{A}}(c_2) = X^{\mathsf{A}}((c_1 + c_2) - c_1)$ contains at least all but one elements from $O(c_2)$, and the solution $X^{\mathsf{A}}(c_2 - c_1)$ contains at least all but 2 elements from $O(c_2)$ because the weight of any element is at least $c_1$. Since

$$C \geq 3c_1 > (\delta - 1)c_1 \geq c_2 - c_1,$$

$X^{\mathsf{A}}(C)$ contains at least all but 2 elements from $O(c_2)$. Recall that in phase 2, the algorithm adds the element $e \in O(c_2)$ that maximizes $\gamma^*(O(c_2))_e$ first. Therefore, and because $|O(c_2)| \geq 3$, we have $f(X^{\mathsf{A}}(C)) \geq \frac{1}{3}f(O(c_2)) \geq \frac{1}{\rho}\textsc{Opt}(C)$.

Consider phase 2 and suppose $C \in [c_2, c_1 + c_2]$. We have

$$\textsc{Opt}(c_1 + c_2) \leq \textsc{Opt}(c_2) + M \tag{6.8}$$

because $f$ is subadditive and because $c_1$ is the minimum weight of all elements. Furthermore, we have

$$f\big(X^{\mathsf{A}}(c_2)\big) \geq \textsc{Opt}(c_2) - M \geq \rho - M \geq M \tag{6.9}$$

where the first inequality follows from subadditivity of $f$ and the fact that the solution $X^{\mathrm{A}}(c_2)$ contains at least all but one element from $O(c_2)$. We obtain

$$f\big(X^{\mathrm{A}}(c_2)\big) \overset{(6.9)}{\geq} \textsc{Opt}(c_2) - M \overset{(6.8)}{\geq} \textsc{Opt}(c_1 + c_2) - 2M \overset{(6.9)}{\geq} \textsc{Opt}(c_1 + c_2) - 2f(X^{\mathrm{A}}(c_2)),$$

i.e., by monotonicity,

$$\textsc{Opt}(C) \leq \textsc{Opt}(c_1 + c_2) \leq 3f(X^{\mathrm{A}}(c_2)) \leq \rho f(X^{\mathrm{A}}(c_2)) \leq \rho f(X^{\mathrm{A}}(C)).$$

Now consider phase $i \in \{3, \dots, N\}$ and $C \in \big(\sum_{j=1}^{i-1} c_j, \sum_{j=1}^{i} c_j\big]$. Note that, for $1 \leq j \leq i \leq N-1$, we have $c_i \geq \delta^{i-j} c_j$ and hence

$$\sum_{j=1}^{i-1} \frac{c_j}{c_i} \overset{\text{Lem. 1.6}}{<} \frac{\delta}{\delta - 1} - 1 = \frac{1}{\delta - 1} < 1.$$

This yields $\sum_{j=1}^{i-1} c_j \leq c_i \leq \sum_{j=1}^{i} c_j$. If $i = N$ and $\sum_{j=1}^{N-1} c_j \geq c_N = w(U)$, we have nothing left to show because $C \geq w(E)$. Thus, suppose that we have $\sum_{j=1}^{N-1} c_j \leq c_N$. Furthermore, if $i = N$ and $\rho\textsc{Opt}(c_{N-1}) > \textsc{Opt}(c_N) = \textsc{Opt}(w(U))$, we again have nothing to show as the solution $X^{\mathrm{A}}\big(\sum_{j=1}^{i-1} c_j\big) \subseteq X^{\mathrm{A}}(C)$ contains the set $O(c_{N-1})$ and has value at least $\textsc{Opt}(c_{N-1})$. Thus, assume that $\textsc{Opt}(c_N) \geq \rho\textsc{Opt}(c_{N-1})$. This implies that (6.7) also holds for $i = N$.

**Case 1:** $C \in \big(\sum_{j=1}^{i-1} c_j, c_i\big)$.

We show that in this case, the value of the optimum solution $O(c_{i-1})$, which is already added by the algorithm, is large enough to guarantee competitiveness. If $c_i > \delta c_{i-1}$ holds, then $c_i$ is the smallest integer such that $\textsc{Opt}(c_i) \geq \rho\textsc{Opt}(c_{i-1})$, i.e., using monotonicity of $f$, we obtain

$$f(X^{\mathrm{A}}(C)) \geq f\big(X^{\mathrm{A}}\big(\textstyle\sum_{j=1}^{i-1} c_j\big)\big) \geq \textsc{Opt}(c_{i-1}) > \frac{1}{\rho}\textsc{Opt}(C).$$

For the case that $c_i = \delta c_{i-1}$. Note that $c_i < \delta c_{i-1}$ is only possible if $i = N$.

**Case 1.1:** $i = 3$.

Let $c := \big(\frac{1}{\lambda\sqrt{M}} + \frac{1}{\lambda^2}\big)\delta c_2$. We have

$$\begin{aligned} c &\leq \left(\frac{1}{\lambda} + \frac{1}{\lambda^2}\right)\delta c_2 \overset{(6.1)}{=} \left(\frac{\lambda^2}{\lambda + 1} - 1 - \frac{1}{\delta}\right) c_2 \\ &\leq \frac{\lambda^2}{\frac{\lambda}{\sqrt{M}} + 1} c_2 - c_2 - c_1 = \frac{\lambda^2 M}{\lambda\sqrt{M} + M} c_2 - c_2 - c_1. \end{aligned} \tag{6.10}$$

We show that $X^{\mathrm{A}}(c_1+c_2)$ is competitive up to capacity $c_1+c_2+c$, and that $X^{\mathrm{A}}(c_1+c_2+c)$ is competitive up to capacity $c_3$. We have

$$\begin{aligned}
\textsc{Opt}(c_1+c_2+c) &\overset{\text{Cor. 6.7}}{\leq} \frac{c_1+c_2+c}{c_2}\left(\textsc{Opt}(c_2)+M\right) \\
&\overset{(6.10)}{\leq} \frac{c_1+c_2+\left(\frac{\lambda^2 M}{\lambda\sqrt{M}+M}c_2-c_2-c_1\right)}{c_2}\left(\textsc{Opt}(c_2)+M\right) \\
&= \frac{\lambda^2 M}{\lambda\sqrt{M}+M}\left(1+\frac{M}{\textsc{Opt}(c_2)}\right)\textsc{Opt}(c_2) \\
&\overset{(6.7)}{\leq} \frac{\lambda^2 M}{\lambda\sqrt{M}+M}\left(1+\frac{M}{\lambda\sqrt{M}}\right)\textsc{Opt}(c_2) \\
&= \lambda\sqrt{M}\frac{\lambda\sqrt{M}}{\lambda\sqrt{M}+M}\left(\frac{\lambda\sqrt{M}+M}{\lambda\sqrt{M}}\right)\textsc{Opt}(c_2) \\
&\leq \rho\textsc{Opt}(c_2) \leq \rho f(X^{\mathrm{A}}(c_1+c_2)),
\end{aligned}$$

where the last inequality follows from the fact that the algorithm starts by packing $O(c_1)$ and $O(c_2)$ before any other elements and needs capacity $c_1+c_2$ to assemble both sets, i.e., $O(c_2) \subseteq X^{\mathrm{A}}(c_1+c_2)$.

Since DoubleScaling adds the elements from $O(c_3)$ after those from $O(c_1)$ and $O(c_2)$, we have $O(c_3, c) \subseteq X^{\mathrm{A}}(c_1+c_2+c)$, and thus

$$\begin{aligned}
f\left(X^{\mathrm{A}}(c_1+c_2+c)\right) &\geq f(O(c_3,c)) \\
&\overset{\text{Lem. 6.6}}{\geq} \frac{c}{c_3}\textsc{Opt}(c_3) - M \\
&\geq \left[\left(\frac{1}{\lambda\sqrt{M}}+\frac{1}{\lambda^2}\right) - \frac{M}{\textsc{Opt}(c_3)}\right]\textsc{Opt}(c_3) \\
&\overset{(6.7)}{\geq} \left(\frac{1}{\lambda\sqrt{M}}+\frac{1}{\lambda^2}-\frac{M}{\lambda^2 M}\right)\textsc{Opt}(c_3) \\
&= \frac{1}{\lambda\sqrt{M}}\textsc{Opt}(c_3) \\
&\geq \frac{1}{\rho}\textsc{Opt}(c_3),
\end{aligned}$$

where for the first inequality we use monotonicity of $f$, and for the third we use $c_3 \leq \delta c_2$. This, together with monotonicity of $f$, implies $\textsc{Opt}(C) \leq \rho f(X^{\mathrm{A}}(C))$ for all $C \in (c_1 + c_2, c_3]$.

**Case 1.2:** $i \geq 4$.
Recall that $C \in \left(\sum_{j=1}^{i-1} c_j, c_i\right)$. We have

$$\begin{aligned}
\textsc{Opt}(C) &\leq \textsc{Opt}(c_i) \overset{\text{Cor. 6.7}}{\leq} \frac{c_i}{c_{i-1}}(\textsc{Opt}(c_{i-1}) + M) \\
&\leq \delta\left(1 + \frac{M}{\textsc{Opt}(c_{i-1})}\right)\textsc{Opt}(c_{i-1}) \overset{(6.7)}{\leq} \delta\left(1 + \frac{M}{\lambda^2 M}\right)\textsc{Opt}(c_{i-1}) \\
&= \frac{\lambda^3}{\lambda^2+1}\left(1 + \frac{1}{\lambda^2}\right)\textsc{Opt}(c_{i-1}) = \lambda \textsc{Opt}(c_{i-1}) \\
&\leq \rho f(X^{\mathrm{A}}(C)),
\end{aligned}$$

where for the first inequality we use monotonicity of $f$, and for the third we use $c_i \leq \delta c_{i-1}$. Thus, also in this case, we find $\textsc{Opt}(C) \leq \rho f(X^{\mathrm{A}}(C))$ for all $C \in \left(\sum_{j=1}^{i-1} c_j, c_i\right)$.

**Case 2:** $C \in \left[c_i, \sum_{j=1}^{i} c_j\right]$.
Since $c_N = w(U)$, we can assume that $i < N$. Up to this budget, the algorithm had a capacity of $C - \sum_{j=1}^{i-1} c_j > C - c_i \geq 0$ to pack elements from $O(c_i)$, i.e., we have $O(c_i, C - \sum_{j=1}^{i-1} c_j) \subseteq X^{\mathrm{A}}(C)$. We show that the value of this set is large enough to guarantee competitiveness in this case. We have

$$\begin{aligned}
f\big(X^{\mathrm{A}}(C)\big) &\geq f\Big(O\big(c_i, C - \textstyle\sum_{j=1}^{i-1} c_j\big)\Big) \\
&\overset{\text{Lem. 6.6}}{\geq} \frac{C - \sum_{j=1}^{i-1} c_j}{c_i}\textsc{Opt}(c_i) - M \\
&\overset{\text{Cor. 6.7}}{\geq} \frac{C - \sum_{j=1}^{i-1} c_j}{c_i}\left(\frac{c_i}{C}\textsc{Opt}(C) - M\right) - M \\
&= \left(\frac{C - \sum_{j=1}^{i-1} c_j}{C} - \frac{C - \sum_{j=1}^{i-1} c_j}{c_i} \cdot \frac{M}{\textsc{Opt}(C)} - \frac{M}{\textsc{Opt}(C)}\right)\textsc{Opt}(C) \\
&\geq \left(1 - \sum_{j=1}^{i-1}\frac{c_j}{c_i} - 1 \cdot \frac{M}{\textsc{Opt}(C)} - \frac{M}{\textsc{Opt}(C)}\right)\textsc{Opt}(C) \\
&\overset{(6.7),\ \text{Lem. 1.6}}{\geq} \left(1 - \left(\frac{\delta}{\delta-1} - 1\right) - \frac{2M}{\rho^{i-1}}\right)\textsc{Opt}(C) \\
&\geq \left(1 - \left(\frac{\delta}{\delta-1} - 1\right) - \frac{2M}{\lambda^2 M}\right)\textsc{Opt}(C) \\
&\geq 0.319 \cdot \textsc{Opt}(C) \geq \frac{1}{\rho}\textsc{Opt}(C),
\end{aligned}$$

where for the first inequality we use monotonicity of $f$, and for the fourth we use $c_i \leq C \leq \sum_{j=1}^{i} c_j$. □

For 1-bounded objectives, Theorem 6.8 immediately yields the following.

**Corollary 6.9.** *If $M = 1$, DoubleScaling is 3.293-competitive.*

### 6.1.2. Lower Bound

In this section, we give a lower bound on the competitive ratio of IncMaxKnap with a monotone, $M$-bounded, and fractionally subadditive objective, and we show a lower bound for the special case with $M = 1$.

**Theorem 6.10.** *The competitive ratio of IncMaxKnap with monotone, $M$-bounded, and fractionally subadditive objectives is at least $M$.*

*Proof.* Consider the set $U = \{e_1, e_2\}$ with weights $w(e_i) = i$ for $i \in \{1, 2\}$ and the values $v(e_1) = 1$ and $v(e_2) = M$. We define the objective $f(S) := \sum_{e \in S} v(e)$ for all $S \subseteq U$. It is easy to see that $f$ is monotone, $M$-bounded and modular and thus fractionally subadditive.

In order to be competitive for capacity 1, an algorithm has to add element $e_1$ first. Thus, the solution of the algorithm for capacity 2 cannot contain element $e_2$, i.e., the value of the solution of capacity 2 given by the algorithm has value 1. The optimum solution of capacity 2 has value $M$, and thus the algorithm cannot be better than $M$-competitive. □

We will now give a stronger lower bound for $M \in [1, \varphi+1)$ where $\varphi = \frac{1}{2}(1+\sqrt{5}) \approx 2.618$ is the golden ratio. This lower bound construction will be similar to an instance from IncMaxSep (cf. Definition 3.3), as, for some $N \in \mathbb{N}$, the ground set $U$ will be partitioned into $N$ disjoint sets $U_1, \dots, U_N$ with $|U_i| = i$ for all $i \in [N]$.

We proceed to give the actual construction. Let $N \in \mathbb{N}$, consider a problem instance $I$ with $\sum_{i=1}^{N} i = \frac{1}{2}N(N+1)$ elements partitioned into sets $U_1, U_2, \dots, U_N$ such that $|U_i| = i$ for all $i \in [N]$. We define the groundset $U := \bigcup_{i \in [N]} U_i$ and the objective $f \colon 2^U \to \mathbb{R}_{\geq 0}$ such that, for all $S \subseteq U$,

$$f(S) = \max_{i \in [N]} |S \cap U_i| \quad \text{for all } S \subseteq U. \tag{6.11}$$

The elements' weights are defined as $w(e) = b + i!$ for all $e \in U_i$ with *base weight* $b = (N+2)!$.

We note that the problem instance $I$ is built in such a way that the elements in all sets $U_1, \dots, U_N$ have roughly the same relative weight because $b$ is very large. First, we show that, for a given capacity $C \in \mathbb{N}$, the number of elements that can be packed without exceeding this capacity can vary by at most 1, regardless of which elements are packed. Yet, the weights of elements in $U_i$ increase quickly enough with increasing $i$ such that, for capacity $C = i(b + i!)$ it is only possible to pack $i$ elements if all $i$ elements are from the set $U_1 \cup \cdots \cup U_i$.

**Proposition 6.11.** *Let $X = (e_1, \ldots, e_{N(N+1)/2})$ be an incremental solution for the instance defined above and $i \in \left[\frac{1}{2}N(N+1)\right]$. Consider capacity $C = i(b + i!)$. Then (with $U_j = \emptyset$ for $j > N$), we have*

$$|X(C)| = \begin{cases} i & \textit{if } \{e_1, \ldots, e_i\} \subseteq U_1 \cup \cdots \cup U_i, \\ i-1 & \textit{else}. \end{cases}$$

*Proof.* First we show that $|X(C)| \in \{i-1, i\}$. Assume $|X(C)| < i - 1$. Then, we have

$$\begin{aligned} C - w(X(C)) &\geq i(b+i!) - (i-2)(b+N!) \geq 2b - i \cdot N! \\ &\geq b + (N+2)! - N(N+1)N! = b + 2(N+1)! \\ &\geq \max_{e \in U} w(e), \end{aligned}$$

contradicting the fact that $X(C)$ is the maximum prefix of the incremental solution $X$ with capacity at most $C$. Assume $|X(C)| > i$. Then, we have

$$w(X(C)) > (i+1)b > ib + (i+1)! > i(b+i!) = C,$$

which contradicts $w(X(C)) \leq C$. We have established that $|X(C)| \in \{i-1, i\}$.

If $X(C)$ contains $e \in U_j$ with $j > i$, we have

$$\begin{aligned} C - w(e) &= i(b+i!) - (b+j!) = (i-1)b + i \cdot i! - j! \\ &< (i-1)b < (i-1) \min_{e \in U} w(e). \end{aligned}$$

So $X(C) \setminus e$ contains at most $i - 2$ elements and $|X(C)| \leq i - 1$, which yields that $X(C)$ contains $i - 1$ elements. Otherwise, if the elements in $\{e_1, \ldots, e_i\}$ are from the set $U_1 \cup \cdots \cup U_i$, each element has weight at most $b+i!$. Thus, we have $w(X(C)) \leq i(b+i!) = C$. Therefore, $|X(C)| \geq i$ which yields that $X(C)$ contains $i$ elements. □

We say an incremental solution to the problem instance $I$ given above *is represented by* a sequence $(c_1, \ldots, c_\ell)$ with $c_i < c_{i+1}$ and $c_\ell = N$ if the incremental solution adds all elements from $U_{c_1}$, then all elements from $U_{c_2}$, and so on until adding all elements from $U_{c_\ell}$. Only afterwards all remaining elements are added in an arbitrary order. Note that elements added after the last element of $U_N$ in any incremental solution do not influence the objective value for any capacity since when they are added the incremental solution has already reached the maximum value of $N$. First, we will observe that every incremental solution of problem instance $I$ can be transformed into a solution that can be represented by a sequence $(c_1, \ldots, c_\ell)$ without decreasing the objective value for any capacity.

**Lemma 6.12.** *For every incremental solution $X$ there is a sequence $(c_1, \ldots, c_\ell)$ with $c_i < c_{i+1}$ and $c_\ell = N$ representing an incremental solution with objective value at least $f(X(C))$ for all capacities $C \geq 0$.*

*Proof.* First, we show that there is an incremental solution $X'$ satisfying the following three properties.

*(i)* We have $f(X'(C)) \geq f(X(C))$ for all $C \geq 0$.

*(ii)* For all $i \in [N-1]$, if at least one element from the set $U_i$ is added before the last element from $U_N$ is added, then this is true for all elements in $U_i$.

*(iii)* For all $i \in [N]$, if the last element from the set $U_i$ is added before the last element from $U_N$ is added, then the objective value of the solution increases from $i-1$ to $i$ when this last element is added.

To show this, fix some $i \in \mathbb{N}$ such that at least one element from the set $U_i$ is added before the last element from $U_N$ is added in the incremental solution $X$. Let $j \in \mathbb{N}$, $j \leq i$ be the largest number such that, when the $j$-th element from the set $U_i$ is added, the value of the solution increases from $j-1$ to $j$. If this does not exist, we set $j = 0$. If $j = i$, $U_i$ is completely added before the last element from $U_N$ and when the last element from $U_i$ is added to the incremental solution its value increases by one to the value of $i$. Thus, suppose that $j < i$. All elements from $U_i$ that are added after the $j$-th element do not increase the value of the solution and can thus be moved to the end of the whole order $X$. Since now, there are only $j$ elements from the set $U_i$ added before the last element from the set $U_N$ is added, it makes sense to add the elements from the set $U_j$ instead of these $j$ elements, as they have a smaller weight (if they are not already added). We can then move the $j$ elements from $U_i$ to the end of the order. After all these changes the incremental solution obtains all values at least as fast as before, i.e., *(i)* holds. Furthermore, *(ii)* and *(iii)* also hold because of the changes we made to the incremental solution.

Now, we show that we can reorder the elements in the incremental solution $X'$ such that the elements that are added before the last element from $U_N$ is added are ordered by the index of the set they belong to. Consider any two sets $U_i$, $U_j$, $i < j$ that are added before the last element from set $U_N$ is added. Recall that, when the last element from $U_i$ is added, the value of the solution is $i$. This implies that at that point at most $i-1$ elements from the set $U_j$ are added. Thus, swapping the elements of $U_i$ and $U_j$ in the incremental solution $X'$ until all elements from $U_i$ are added before the elements from $U_j$, does not decrease the value of the incremental solution for any capacity. By doing this for all pairs $(i, j)$, we obtain an incremental solution that can be represented by a sequence $(c_1, \ldots, c_\ell)$. $\square$

Utilizing the properties of the weights we mentioned before, we can find a collection of conditions which are necessary and sufficient for a sequence $(c_1, \ldots, c_\ell)$ to represent a $\rho$-competitive incremental solution for the problem instance $I$. In the following, we denote by $\ell'$ the index with $\rho c_{\ell'} < N$ and $\rho c_{\ell'+1} \geq N$ and set $t_i := \sum_{j=1}^{i} c_j$. The index $\ell'$ is needed because all indices $i > \ell'$ satisfy $\rho c_i \geq N$, i.e., after an incremental solution has added the set $U_{\ell'+1}$, it is $\rho$-competitive for all capacities.

**Lemma 6.13.** *Let $(c_1, \ldots, c_\ell)$ with $c_i < c_{i+1}$ and $c_\ell = N$ be a sequence that represents an incremental solution for instance $I$. Then, the incremental solution is $\rho$-competitive if and only if the following three conditions are satisfied:*
*(i) $c_1 = 1$,*
*(iia) $t_i + c_i \leq \lfloor \rho c_i \rfloor$ for all $i \in [\ell']$ with $c_{i+1} \leq \lfloor \rho c_i \rfloor + 1$,*
*(iib) $t_i + c_i + 1 \leq \lfloor \rho c_i \rfloor$ for all $i \in [\ell']$ with $c_{i+1} > \lfloor \rho c_i \rfloor + 1$.*

*Proof.* We first show that for a $\rho$-competitive incremental solution $X$ that can be represented by some sequence conditions *(i)*, *(iia)* and *(iib)* have to be satisfied.

If $c_1 \neq 1$, the incremental solution is not competitive for capacity $C = b + 1$, i.e., *(i)* must hold.

Consider capacity $C = (\lfloor \rho c_i \rfloor + 1)(b + (\lfloor \rho c_i \rfloor + 1)!)$ for $i \in [\ell']\}$. The optimum solution of capacity $C$ is $O(C) = U_{\lfloor \rho c_i \rfloor + 1}$ and has value $\textsc{Opt}(C) = \lfloor \rho c_i \rfloor + 1$. Furthermore, $f(X(C)) \geq c_i + 1$, since $\frac{1}{\rho}(\lfloor \rho c_i \rfloor + 1) > c_i$. Thus, $X(C)$ contains at least $c_i + 1$ elements from $U_{c_{i+1}}$.

If $c_{i+1} \leq \lfloor \rho c_i \rfloor + 1$, $X(C)$ contains $\lfloor \rho c_i \rfloor + 1 - t_i$ elements from $U_{c_{i+1}}$ by Proposition 6.11. Thus, we have $c_i + 1 \leq \lfloor \rho c_i \rfloor + 1 - t_i$ which implies *(iia)*. If $c_{i+1} > \lfloor \rho c_i \rfloor + 1$, $X(C)$ contains $\lfloor \rho c_i \rfloor - t_i$ elements from $U_{c_{i+1}}$ by Proposition 6.11. Thus, we have $c_i + 1 \leq \lfloor \rho c_i \rfloor - t_i$ which implies *(iib)*.

We proceed to show that, conversely, an incremental solution that can be represented by some sequence satisfying conditions *(i)*, *(iia)* and *(iib)* is $\rho$-competitive. To this end, fix an arbitrary incremental solution $X$ with these properties. Since all elements have integer weight, it suffices to show $\rho$-competitiveness for all capacities $C \in \mathbb{N}$.

For capacities $C \in [b + 1]$, the incremental solution is $\rho$-competitive because $b + 1$ is the smallest weight of all elements and $c_1 = 1$ by *(i)*, i.e., the element of smallest weight is added first.

Let $i \in [\ell']$. We show that, for all capacities in

$$\{t_i(b + t_i!) + 1, \ldots, t_{i+1}(b + t_{i+1}!)\},$$

$X$ is $\rho$-competitive. For all capacities

$$C \in \{t_i(b + t_i!) + 1, \ldots, (\lfloor \rho c_i \rfloor + 1)(b + (\lfloor \rho c_i \rfloor + 1)!) - 1\}, \quad (6.12)$$

we have $\textsc{Opt}(C) \le \lfloor \rho c_i \rfloor \le \rho c_i \le f(X(C))$ because $X(t_i(b + t_i!)) \subseteq X(C)$ contains at least all elements from $U_{c_1} \cup \cdots \cup U_{c_i}$ by Proposition 6.11. Thus, $X$ is $\rho$-competitive for all values $C$ as in (6.12). Next, suppose that

$$C \in \{(\lfloor \rho c_i \rfloor + 1)(b + (\lfloor \rho c_i \rfloor + 1)!), \ldots, t_{i+1}(b + t_{i+1}!)\}.$$

Let $a^* = \textsc{Opt}(C) \in \{\lfloor \rho c_i \rfloor + 1, \ldots, t_{i+1}\}$. This implies $C \ge a^*(b + a^*!)$. We consider two cases.

**Case 1:** $c_{i+1} \le \lfloor \rho c_i \rfloor + 1$.
By *(iia)*, we have $t_i + c_i \le \lfloor \rho c_i \rfloor$ and thus

$$a^* - t_i \ge c_i + a^* - \lfloor \rho c_i \rfloor \ge \frac{1}{\rho}\lfloor \rho c_i \rfloor + \frac{1}{\rho}(a^* - \lfloor \rho c_i \rfloor) = \frac{1}{\rho}a^*. \quad (6.13)$$

By Proposition 6.11, $X(C)$ contains at least $a^* - t_i$ elements from the set $U_{c_{i+1}}$ since $c_{i+1} \le \lfloor \rho c_i \rfloor + 1 \le a^*$. This implies that, by (6.13), $f(X(C)) \ge a^* - t_i \ge \frac{1}{\rho}a^*$, i.e., $X$ is $\rho$-competitive for capacity $C$.

**Case 2:** $c_{i+1} > \lfloor \rho c_i \rfloor + 1$.
By *(iib)*, we have $t_i + c_i + 1 \le \lfloor \rho c_i \rfloor$ and thus

$$a^* - t_i - 1 \ge c_i + a^* - \lfloor \rho c_i \rfloor \ge \frac{1}{\rho}\lfloor \rho c_i \rfloor + \frac{1}{\rho}(a^* - \lfloor \rho c_i \rfloor) = \frac{1}{\rho}a^*. \quad (6.14)$$

By Proposition 6.11 $X(C)$ contains at least $a^* - t_i - 1$ elements from the set $U_{c_{i+1}}$. This means that, by (6.14), $f(X(C)) \ge a^* - t_i - 1 \ge \frac{1}{\rho}a^*$, i.e., $X$ is $\rho$-competitive for capacity $C$.

We conclude that, for every capacity $C \in [t_{\ell'+1}(b + t_{\ell'+1}!)]$, $X$ is $\rho$-competitive. For all capacities $C > t_{\ell'+1}(b + t_{\ell'+1}!)$, the value of $X(C)$ is at least $c_{\ell'+1}$, while the optimum solution has value at most $N$. By definition of $\ell'$ we have $\rho c_{\ell'+1} \ge N$. Therefore, $X$ is $\rho$-competitive. □

If $t_i + c_i = \lfloor \rho c_i \rfloor$, then $t_i + c_i + 1 > \lfloor \rho c_i \rfloor$, i.e., contraposition of condition *(iib)* from Lemma 6.13 yields the following.

**Corollary 6.14.** *If a sequence* $(c_1, \ldots, c_\ell)$ *represents a* $\rho$*-competitive incremental solution and* $t_i + c_i = \lfloor \rho c_i \rfloor$ *for some* $i \in [\ell']$*, then* $c_{i+1} \le \lfloor \rho c_i \rfloor + 1$.

In the following we show that, for $2 \le \rho \le \varphi + 1$ and given some sequence $(c_1, \ldots, c_i)$, every algorithm is forced to choose $c_{i+1} \le \lfloor \rho c_i \rfloor + 1$ to be $\rho$-competitive for capacity $\lfloor \rho c_i \rfloor + 1$.

**Proposition 6.15.** *Let $\rho \in [2, \varphi + 1]$, and let $(c_1, \ldots, c_\ell)$ with $c_i < c_{i+1}$ be a sequence that represents an incremental solution. If the incremental solution is $\rho$-competitive, then $c_{i+1} \leq \lfloor \rho c_i \rfloor + 1$ for all $i \in [\ell']$.*

*Proof.* By Corollary 6.14, it suffices to show that we have $t_i + c_i = \lfloor \rho c_i \rfloor$ for all $i \in [\ell']$. We prove this by induction. For $i = 1$, we have

$$t_1 + c_1 = 1 + 1 = 2 = \lfloor \rho \rfloor = \lfloor \rho c_1 \rfloor,$$

where we use the fact that $c_1 = 1$ by Lemma 6.13*(i)*.

Suppose that

$$t_i + c_i = \lfloor \rho c_i \rfloor \tag{6.15}$$

holds for some $i \in [\ell' - 1]$. By Lemma 6.13 *(iia)* and *(iib)*, we have $t_{i+1} + c_{i+1} \leq \lfloor \rho c_{i+1} \rfloor$. Thus, it remains to show that

$$t_{i+1} + c_{i+1} \geq \lfloor \rho c_{i+1} \rfloor. \tag{6.16}$$

To prove this, we first calculate for $\rho \in (2, \varphi + 1]$:

$$\begin{aligned} \frac{(3-\rho)(\rho-1)}{\rho-2} &= \frac{-(\rho-2)^2+1}{\rho-2} = \frac{1}{\rho-2} - (\rho-2) \\ &\geq \frac{1}{\varphi-1} - (\varphi-1) = \varphi - (\varphi-1) = 1, \end{aligned} \tag{6.17}$$

where for the inequality we use that $\rho \leq \varphi + 1$. In the case $\rho = 2$ we can calculate directly $\rho - 2 = 0 \leq 1 = (3-\rho)(\rho-1)$. We obtain

$$\begin{aligned} (3-\rho)\lfloor(\rho-1)c_i\rfloor + 1 &> (3-\rho)((\rho-1)c_i - 1) + 1 \\ &= (3-\rho)(\rho-1)c_i + \rho - 2 \\ &\overset{(6.17)}{\geq} (\rho-2)c_i + \rho - 2 \\ &= (\rho-2)(c_i+1). \end{aligned} \tag{6.18}$$

Utilizing this inequality, we have

$$\begin{aligned} \lfloor(\rho-2)(\lfloor\rho c_i\rfloor + 1)\rfloor &= \big\lfloor(\rho-2)(\lfloor(\rho-1)c_i\rfloor + c_i + 1)\big\rfloor \\ &= \big\lfloor\lfloor(\rho-1)c_i\rfloor + (\rho-3)\lfloor(\rho-1)c_i\rfloor + (\rho-2)(c_i+1)\big\rfloor \\ &= \lfloor(\rho-1)c_i\rfloor + \big\lfloor(\rho-3)\lfloor(\rho-1)c_i\rfloor + (\rho-2)(c_i+1)\big\rfloor \\ &\overset{(6.18)}{<} \lfloor(\rho-1)c_i\rfloor + 1, \end{aligned}$$

where for the third equation we use that $\lfloor(\rho-1)c_i\rfloor \in \mathbb{N}$. Because both sides of this inequality are in $\mathbb{N}$, we have

$$\lfloor(\rho-2)(\lfloor\rho c_i\rfloor+1)\rfloor \leq \lfloor(\rho-1)c_i\rfloor \tag{6.19}$$

This yields

$$\begin{aligned}
\lfloor\rho c_{i+1}\rfloor &= \lfloor(\rho-2)c_{i+1}\rfloor + 2c_{i+1} \\
&\overset{\text{Cor. 6.14}}{\leq} \lfloor(\rho-2)(\lfloor\rho c_i\rfloor+1)\rfloor + 2c_{i+1} \\
&\overset{(6.19)}{\leq} \lfloor(\rho-1)c_i\rfloor + 2c_{i+1} \\
&= \lfloor\rho c_i\rfloor - c_i + 2c_{i+1} \\
&\overset{(6.15)}{=} t_i + 2c_{i+1} \\
&= t_{i+1} + c_{i+1},
\end{aligned}$$

i.e., (6.16) holds. By induction $t_i + c_i = \lfloor\rho c_i\rfloor$ follows for all $i \in [\ell']$. □

**Theorem 6.16.** *For $\rho < \varphi + 1$, there is no $\rho$-competitive algorithm for problem instance $I$ with sufficiently large $N \in \mathbb{N}$.*

*Proof.* Suppose, for $\rho < 2$, there was a $\rho$-competitive incremental solution represented by the sequence $(c_1, \dots, c_\ell)$. Yet, Lemma 6.13 implies that

$$2 = t_1 + c_1 \leq \lfloor\rho c_1\rfloor = 1$$

which is a contradiction, i.e., for $\rho < 2$, there is no $\rho$-competitive incremental solution.

Next, suppose that for $\rho \in [2, \varphi+1)$ there was a $\rho$-competitive incremental solution. Let the number of disjoint sets $N \in \mathbb{N}$ in the instance $I$ be sufficiently large, and let $(c_1, \dots, c_\ell)$ with $c_i > c_{i+1}$ for all $i \in [\ell-1]$ be the sequence representing a $\rho$-competitive incremental solution. By Lemma 6.13 and Proposition 6.15, we know that the following conditions are satisfied:

*(i)* $c_1 = 1$,

*(ii)* $t_i + c_i \leq \lfloor\rho c_i\rfloor$ for all $i \in [\ell']$,

*(iii)* $c_{i+1} \leq \lfloor\rho c_i\rfloor + 1$ for all $i \in [\ell']$.

For $1 \leq j \leq i \leq \ell'$, from *(i)* it follows that

$$c_j \geq \frac{1}{\rho}\lfloor\rho c_j\rfloor \overset{(iii)}{\geq} \frac{1}{\rho}(c_{j+1}-1) \geq \frac{1}{\rho}\left[\frac{1}{\rho}\Big(c_{j+2}-1\Big)-1\right] \geq \cdots \geq \frac{1}{\rho^{i-j}}c_i - \sum_{k=1}^{i-j}\frac{1}{\rho^k}.$$

This implies

$$
\begin{aligned}
t_i &= \sum_{j=1}^{i} c_j \geq \sum_{j=1}^{i}\left(\frac{1}{\rho^{i-j}}c_i - \sum_{k=1}^{i-j}\frac{1}{\rho^k}\right)\\
&= \left(\sum_{j=0}^{i-1}\frac{1}{\rho^j}\right)c_i - \sum_{j=1}^{i}\sum_{k=1}^{i-j}\frac{1}{\rho^k}\\
&= \frac{1-\rho^{-i}}{1-\rho^{-1}}c_i - \sum_{j=1}^{i}\left(\frac{1-\rho^{j-i-1}}{1-\rho^{-1}} - 1\right)\\
&\geq \frac{1-\rho^{-i}}{1-\rho^{-1}}c_i - i\frac{1}{1-\rho^{-1}}\\
&= \frac{1}{1-\rho^{-1}}\big((1-\rho^{-i})c_i - i\big). \qquad (6.20)
\end{aligned}
$$

For $i \in \{2, \ldots, \ell'\}$, we obtain

$$
\begin{aligned}
\rho &\geq \frac{1}{c_i}\lfloor \rho c_i \rfloor \overset{(ii)}{\geq} \frac{1}{c_i}(t_i + c_i)\\
&\overset{(6.20)}{\geq} \frac{1}{c_i}\cdot\frac{1}{1-\rho^{-1}}\big((1-\rho^{-i})c_i - i\big) + 1\\
&= \frac{1}{1-\rho^{-1}}\left(1-\rho^{-i} - \frac{i}{c_i}\right) + 1. \qquad (6.21)
\end{aligned}
$$

Observe that $c_{j+1} > c_j$ for all $j \in [\ell - 1]$ implies $c_j \geq j$ for all $j \in [\ell]$. It follows that

$$
c_i \geq \frac{1}{\rho - 1}\big(\lfloor \rho c_i \rfloor - c_i\big) \overset{(ii)}{\geq} \frac{1}{\rho-1}t_i \overset{c_j \geq j}{\geq} \frac{1}{\rho-1}\cdot\frac{i(i+1)}{2},
$$

which implies that

$$
\frac{i}{c_i} \leq \frac{2(\rho-1)}{i+1}. \qquad (6.22)
$$

By definition of $\ell'$ and by Proposition 6.15, $\ell'$ increases when $N$ is increased sufficiently. Thus, for every $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$
\frac{\ell'}{c_{\ell'}} \overset{(6.22)}{\leq} \frac{2(\rho-1)}{\ell'+1} \leq \frac{\varepsilon}{2} \qquad (6.23)
$$

and

$$
\rho^{-\ell'} \leq \frac{\varepsilon}{2}. \qquad (6.24)
$$

Since we chose $N$ sufficiently large, we can assume that this holds. Let $\varepsilon = 1 - \frac{\rho-1}{\rho}\varphi$. Note that $\varepsilon > 0$ because $\rho < \varphi + 1$. We obtain

$$\begin{aligned}
\rho &\overset{(6.21)}{\geq} \frac{1}{1-\rho^{-1}}\left(1 - \rho^{-\ell'} - \frac{\ell'}{c_{\ell'}}\right) + 1 \\
&\overset{(6.23),(6.24)}{\geq} \frac{1}{1-\rho^{-1}}(1-\varepsilon) + 1 \\
&= \frac{\rho}{\rho-1}\left(1 - \left(1 - \frac{\rho-1}{\rho}\varphi\right)\right) + 1 \\
&= \varphi + 1,
\end{aligned}$$

This yields a contradiction to the fact that $\rho < \varphi + 1$. Thus, there is no $\rho$-competitive algorithm for $\rho < \varphi + 1$. □

This result immediately yields the desired lower bound.

**Corollary 6.17.** *The competitive ratio of IncMaxKnap with monotone, $1$-bounded, and fractionally subadditive objectives is at least $\varphi + 1$.*

## 6.2. Non-Strict Analysis

In the upper and lower bound in Theorems 6.8 and 6.10 it becomes apparent that the value of $M$ has a lot of impact on the competitive ratio because we are always forced to add the element of smallest weight first. In order to avoid this one can allow some slack in the analysis, i.e., use the non-strict competitive ratio instead of the strict version. In this section, we investigate the non-strict competitive ratio of IncMaxKnap.

### 6.2.1. An Adapted Capacity Scaling Algorithm

Recall that the CardinalityScaling algorithm presented in Section 4.1.1 adds optimum solutions for capacities $c_1, c_2, \dots$ one after the other. The capacities are chosen such that $c_1 = 1$ and $c_{i+1} = \lceil \delta c_i \rceil$ for all $i \in \mathbb{N}$ with a scaling parameter $\delta > 1$. We adapt this algorithm to the IncMaxKnap problem. The algorithm KnapsackScaling sets $c_1 = \min_{e \in U} w(e)$ and calculates $c_{i+1} = \delta c_i$ for scaling parameter $\delta = \varphi + 1$ where $\varphi$ is the golden ratio. Then, similar to the DoubleScaling algorithm, in phase $i \in \mathbb{N}$, it adds the elements of the set $O(c_i)$ in an order such that (6.4) holds. The non-strict competitive ratio of this algorithm is smaller than the strict competitive ratio of DoubleScaling in Theorem 6.8 and, furthermore, independent of $M$ (except for the additive constant).

**Theorem 6.18.** *The algorithm* KNAPSACKSCALING *is non-strictly* $(\varphi + 1)$*-competitive with additive constant* $\alpha = (\varphi + 1)M$.

*Proof.* Let $X^{\mathrm{A}}$ denote the incremental solution of KNAPSACKSCALING. As $X^{\mathrm{A}}$ adds the optimum solution of capacity $c_1 = \min_{e \in U} w(e)$ first, we have have $X^{\mathrm{A}}(C) = \textsc{Opt}(C)$ for all $C \in [0, c_1]$.

Now, suppose that $C \in (\sum_{j=1}^{i} c_j, c_{i+1}]$ for some $i \in \mathbb{N}$. Note that the set $X^{\mathrm{A}}(C)$ contains the set $O(c_i)$, i.e.,

$$f(X^{\mathrm{A}}(C)) \geq \textsc{Opt}(c_i). \tag{6.25}$$

By monotonicity of $f$, we have

$$\begin{aligned} \textsc{Opt}(C) &\leq \textsc{Opt}(c_{i+1}) \\ &\overset{\text{Cor. 6.7}}{\leq} \frac{c_{i+1}}{c_i}\big(\textsc{Opt}(c_i) + M\big) \\ &\overset{(6.25)}{\leq} \frac{c_{i+1}}{c_i}\big(X^{\mathrm{A}}(C) + M\big) \\ &= (\varphi + 1)\big(X^{\mathrm{A}}(C) + M\big), \end{aligned}$$

which yields a non-strict competitive ratio of $\varphi + 1$ with additive constant $\alpha = (\varphi + 1)M$.

Now, suppose that

$$C \in \left(c_i, \sum_{j=1}^{i} c_j\right] \tag{6.26}$$

for some $i \in \mathbb{N}_{\geq 2}$. Then

$$\sum_{j=1}^{i-1} c_j \overset{\text{Lem. 1.6}}{<} \left(\frac{\varphi + 1}{\varphi} - 1\right) c_i = \frac{1}{\varphi} c_i < c_i < C.$$

Thus, the solution $X^{\mathrm{A}}(C)$ contains all optimum sets for capacities $c_1, \ldots, c_{i-1}$ taking up a capacity of at most $\sum_{j=1}^{i-1} c_j$. Thus, $X^{\mathrm{A}}(C)$ also contains the prefix of $O(c_i)$ of capacity at most $C - \sum_{j=1}^{i-1} c_j$, namely $O\big(c_i, C - \sum_{j=1}^{i-1} c_j\big)$. Thus

$$
\begin{aligned}
f(X^{\mathrm{A}}(C)) \quad &\geq \quad f\left(O\left(c_i, C - \sum_{j=1}^{i-1} c_j\right)\right) \\
&\overset{\text{Lem. 6.6}}{\geq} \quad \frac{C - \sum_{j=1}^{i-1} c_j}{c_i} \textsc{Opt}(c_i) - M \\
&\overset{\text{Cor. 6.7}}{\geq} \quad \frac{C - \sum_{j=1}^{i-1} c_j}{c_i} \left(\frac{c_i}{C} \textsc{Opt}(C) - M\right) - M \\
&= \quad \left(1 - \frac{1}{C} \sum_{j=1}^{i-1} c_j\right) \textsc{Opt}(C) - \left(\frac{C - \sum_{j=1}^{i-1} c_j}{c_i} + 1\right) M \\
&\overset{(6.26)}{\geq} \quad \left(1 - \frac{1}{c_i} \sum_{j=1}^{i-1} c_j\right) \textsc{Opt}(C) - 2M \\
&\overset{\text{Lem. 1.6}}{\geq} \quad (2 - \varphi) \textsc{Opt}(C) - 2M \\
&= \quad \frac{1}{\varphi + 1} \textsc{Opt}(C) - 2M,
\end{aligned}
$$

which, yields a non-strict competitive ratio of $\varphi + 1$ with additive constant $\alpha = 2M$. □

### 6.2.2. Lower Bound

We take a look at lower bounds on the non-strict competitive ratio of the IncMaxKnap problem.

In Proposition 3.12, we have seen that, for every additive constant $\alpha \geq 0$, the non-strict competitive ratio of IncMaxSep is greater or equal to the strict competitive ratio of IncMaxCont. The objective of every instance of IncMaxSep is the maximum over modular functions over the ground set. Thus, it is an XOS-function, and therefore fractionally subadditive. Furthermore, it is monotone by Lemma 3.4. If it was $M$-bounded, every lower bound on the non-strict competitive ratio of IncMaxSep would transfer to the IncMaxKnap problem with monotone, $M$-bounded, and fractionally subadditive objective. Yet, this is not the case. In order to circumvent this problem, we reason in the following that the proof of Proposition 3.12 can be adapted to a modified version of separable problem instances that are $1$-bounded.

In the proof of Proposition 3.12, a lower bound instance of IncMaxCont is discretized arbitrarily finely to generate an instance of IncMaxSep. If we also do this, but additionally change the value of non-empty sets to always be at least $1$, we obtain a monotone, $1$-bounded, and fractionally subadditive objective. The analysis of this new problem instance is very similar to the analysis in the proof of Proposition 3.12. This is due to the fact that, by discretizing very finely, the density of small sets (in the proof, sets of cardinality at most $(\rho + 1)c_{\text{min}}$) is very close to $1$. Thus, it (almost) makes no difference that the values of a single element is increased to $1$. We obtain the following.

**Corollary 6.19.** *For every additive constant $\alpha \geq 0$, the (non-strict) competitive ratio of* IncMaxKnap *with monotone, $M$-bounded, and fractionally subadditive objectives is greater or equal to the strict competitive ratio of* IncMaxCont*.*

Combining this with Theorem 3.27 yields the following

**Corollary 6.20.** *For every additive constant $\alpha \geq 0$, the (non-strict) competitive ratio of* IncMaxKnap *with monotone, $M$-bounded, and fractionally subadditive objectives is at least* $2.246$

## 6.3. Application to $s$-$t$-Flows

In this section we show that the algorithms from the two previous sections can be used to find competitive incremental solutions for incremental maximum potential-based $s$-$t$-flow problems as in Example 6.4. Further, we show that the lower bound construction for the strict competitive ratio from Section 6.1.2 can be modeled by a potential-based $s$-$t$-flow. Lastly, we consider the incremental maximum $s$-$t$-flow problem without potentials and derive upper and lower bounds for this.

Formally, for the *incremental maximum potential-based $s$-$t$-flow problem* on parallel edges, we are given a directed graph $G = (V, E)$ consisting of two nodes $s$ and $t$ with a collection of edges between them. The goal is to determine an order in which to build the edges while maintaining a potential-based $s$-$t$-flow that is as large as possible. To this end, we are given a continuous and strictly increasing *potential-loss function* $\psi : \mathbb{R} \to \mathbb{R}$ with $\lim_{x\to\infty} \psi(x) = \infty$. Every edge $e$ has an edge *resistance* $\beta(e) > 0$ and a *capacity* $\mu(e)$. Vertex *potentials* $p_s, p_t \in \mathbb{R}$ induce an $s$-$t$-flow of $\vartheta(e) = \psi^{-1}(p/\beta(e))$ on edge $e$ where $p = p_t - p_s$. This $s$-$t$-flow is only feasible if $\vartheta(e) \leq \mu(e)$. The goal is to choose vertex potentials $p_s, p_t \in \mathbb{R}$ together with a subset of active edges that maximize the total induced $s$-$t$-flow. This yields the objective

$$f(S) = \max\left\{ \sum_{e\in T} \psi^{-1}\left(\frac{p}{\beta(e)}\right) \middle| T \subseteq S, p \in \mathbb{R}_{\geq 0} \text{ with } \psi^{-1}\left(\frac{p}{\beta(e)}\right) \leq \mu(e) \text{ for all } e \in T \right\} \tag{6.27}$$

for all $S \subseteq E$. The function $f$ is obviously monotone. We further obtain that $f$ scaled by $(\min_{e\in E}\mu(e))^{-1}$ is $M$-bounded for $M := \frac{\max_{e\in E}\mu(e)}{\min_{e\in E}\mu(e)}$ because $f(\{e\}) = \mu(e)$. We proceed to show that the objective is fractionally subadditive.

**Proposition 6.21.** *The function $f : 2^E \to \mathbb{R}_{\geq 0}$ defined in (6.27) is fractionally subadditive.*

*Proof.* For $e \in E$, let

$$p_e := \beta(e)\psi(\mu(e))$$

be the maximum potential difference between $s$ and $t$ such that the flow along $e$ induced by the potential difference $p_e$ is still feasible, i.e., does not violate the capacity constraint $\mu(e)$. For $e, e' \in E$, we define $\vartheta_{e'}(e)$ to be the flow value along $e$ induced by a potential difference of $p_{e'}$ between $s$ and $t$ if this flow is feasible, and $0$ otherwise. For $S \subseteq E$, we have

$$\begin{aligned} f(S) &= \max\left\{\sum_{e\in T}\psi^{-1}\left(\frac{p}{\beta(e)}\right)\middle| T\subseteq S, p\in\mathbb{R}_{\geq 0} \text{ with } \psi^{-1}\left(\frac{p}{\beta(e)}\right)\leq \mu(e) \text{for all} e\in T\right\} \\ &= \max\left\{\sum_{e\in S}\vartheta_{e'}(e)\middle| e'\in E\right\}, \end{aligned}$$

i.e., $f$ is an XOS-function and thus fractionally subadditive (see Example 6.2). □

Since the objective is monotone and fractionally subadditive, we obtain the following corollaries.

**Corollary 6.22.** *The strict competitive ratio of the incremental maximum potential-based $s$-$t$-flow problem on parallel edges is at most*

$$\rho \in \left[\max\{\varphi+1, M\}, \max\{3.293\sqrt{M}, 2M\}\right],$$

*where* $M = \frac{\max_{e\in E}\mu(e)}{\min_{e\in E}\mu(e)}$.

Theorem 6.18 and Proposition 6.21 yield the following.

**Corollary 6.23.** *The non-strict competitive ratio with additive constant $(\varphi+1)\frac{\max_{e\in E}\mu(e)}{\min_{e\in E}\mu(e)}$ of the incremental maximum potential-based $s$-$t$-flow problem on parallel edges is at most $\varphi+1$.*

It is possible to define a problem instance of the incremental maximum potential-based $s$-$t$-flow problem on parallel edges which reflects the construction in Section 6.1.2. Thus, the lower bound on the competitive ratio translates also to this special case.

**Proposition 6.24.** *The strict competitive ratio of the incremental maximum potential-based $s$-$t$-flow problem on parallel edges is at least $\varphi + 1$.*

*Proof.* Let $N \in \mathbb{N}$. For $i = 1, \dots, N$ define $E_i$ to be a set of $i$ parallel edges from $s$ to $t$ with unit capacities. For $e \in E_i$, define its resistance to be $\beta(e) := \varepsilon^i$ for some $0 < \varepsilon < 1$. Let the potential loss function $\psi$ be continuous and strictly increasing with $\psi(0) = 0$. Let $p_i := \varepsilon^i \psi(1)$ be the potential difference between $s$ and $t$ inducing a flow of $1$ on all edges $e \in E_i$. Then, the maximum potential-based $s$-$t$-flow on a subset $S \subseteq E := \bigcup_{i=1}^N E_i$ is given by

$$\begin{aligned} f'(S) &= \max\left\{ \sum_{e \in T} \psi^{-1}\left(\frac{p}{\beta(e)}\right) \middle| T \subseteq S, p \in \mathbb{R}_{\geq 0} \text{ with } \psi^{-1}\left(\frac{p}{\beta(e)}\right) \leq u(e) \text{ for all } e \in T \right\} \\ &= \max\left\{ \sum_{j=1}^{i} \sum_{e \in E_j \cap S} \psi^{-1}\left(\frac{p_i}{\beta(e_j)}\right) \middle| i \in [N] \right\} \\ &= \max_{i \in [N]} |S \cap E_i| + \sum_{j=1}^{i-1} |S \cap E_j| \psi^{-1}(\varepsilon^{i-j}\psi(1)). \end{aligned}$$

For all edges $e \in E_i$, the weights that represent the construction cost of the edges is defined as in the problem instance in Section 6.1.2 to be $w(e_i) = b + i!$ for $b = (N+2)!$. Note that the problem instance that we just defined is similar to the construction in Section 6.1.2, with the only difference being the sum in the objective $f'$. This sum can be chosen arbitrarily small by choosing $\varepsilon$ small enough.

Assume there is a $\rho$-competitive incremental solution $X$ with $\rho < \varphi + 1$ for this problem. Let $\varepsilon' > 0$ with $\rho + \varepsilon' < \varphi + 1$. Consider the problem instance with objective $f \colon 2^U \to \mathbb{R}_{\geq 0}$ such that, for all $S \subseteq U$, we have

$$f(S) = \max_{i \in [N]} |S \cap E_i|.$$

This instance is the same as the one defined in Section 6.1.2. By Theorem 6.16 the solution $X$ cannot be better than $(\rho + \varepsilon')$-competitive for the instance given by $f$, i.e., there exists $C \in \mathbb{N}$ with

$$(\rho + \varepsilon') f(X(C)) < \textsc{Opt}_f(C), \tag{6.28}$$

where we write $\textsc{Opt}_f(c)$ for the optimum of capacity $c$ in the instance given by $f$. Analogously, we write $\textsc{Opt}_{f'}(c)$ for the optimum of capacity $c$ in the instance given by $f'$. As $X$ is $\rho$-competitive for the instance with objective function $f'$, we have $f'(X(C)) \geq 1$ and thus also

$$1 \leq f(X(C)) \overset{(6.28)}{<} \frac{1}{\rho + \varepsilon'} \textsc{Opt}_f(C) \tag{6.29}$$

Choose $\varepsilon$ small enough such that $\rho N^2 \psi^{-1}(\varepsilon\psi(1)) < \varepsilon'$. This yields

$$f'(S) - f(S) = \sum_{j=1}^{i-1} |S \cap E_j| \psi^{-1}(\varepsilon^{i-j}\psi(1)) \le n^2 \psi^{-1}(\varepsilon\psi(1)) < \frac{\varepsilon'}{\rho} \tag{6.30}$$

for all $S \subseteq U$. Then, we have

$$\begin{aligned} \rho f'(X(C)) &\overset{(6.30)}{<} \rho\left(f(X(C)) + \frac{\varepsilon'}{\rho}\right) \overset{(6.28)}{<} \rho\left(\frac{\textsc{Opt}_f(C)}{\rho+\varepsilon'} + \frac{\varepsilon'}{\rho}\right) \\ &= \textsc{Opt}_f(C) - \frac{\varepsilon'}{\rho+\varepsilon'}\textsc{Opt}_f(C) + \varepsilon' \overset{(6.29)}{<} \textsc{Opt}_f(C) \le \textsc{Opt}_{f'}(C). \end{aligned}$$

This is a contradiction to the fact that $X$ is a $\rho$-competitive incremental solution for the instance with objective $f'$. Therefore, a $\rho$-competitive algorithm for the incremental maximum potential-based $s$-$t$-flow problem cannot exist for $\rho < \varphi + 1$. □

We now return to the *incremental maximum $s$-$t$-flow problem* discussed in the beginning of the chapter. In this problem, we are given a directed graph $G = (V, E)$ with two designated vertices $s, t \in V$.

For $S \subseteq E$, the incremental maximum $s$-$t$-flow problem has the objective

$$f(S) = \max\{v \mid \text{there exists an } s\text{-}t\text{-flow of value } v \text{ in } G_S = (V, S)\}.$$

It is straightforward to verify that $f$ is modular (and, hence, also fractionally subadditive) for the case that $G$ has only the two vertices $s$ and $t$ and all edges go from $s$ to $t$. This problem can be solved by a greedy approach optimally. Thus, we will consider the problem on a general directed graph $G$. It is easy to see that the objective does not have to be fractionally subadditive in general. In fact, for the example of Figure 1.1, we have

$$f(\{b\}) = 0, \qquad f(\{c\}) = 0, \qquad f(\{b, c\}) = k.$$

This contradicts fractional subadditivity for the choices $A = \{b, c\}$, $B_1 = \{b\}$, $B_2 = \{c\}$, and $\alpha_1 = \alpha_2 = 1$.

We proceed to show that, despite the lack of (fractional) subadditivity, this problem admits a bounded competitive incremental solution. To solve this problem, we describe the algorithm Quickest-Increment that has been introduced by Kalinowski et al. [43] for a different incremental $s$-$t$-flow problem where the sum of the $s$-$t$-flow values for all integer capacities $C$ is to be maximized. The algorithm starts by adding the shortest path and then iteratively adds the smallest set of edges that increase the value of the maximum $s$-$t$-flow by at least 1. We denote by $r \in \mathbb{N}$ the number of iterations until QUICKEST-INCREMENT terminates. For $i \in \{0, 1, \dots, r\}$, let $\lambda_i$ be the size of the set added in iteration $i$, i.e., $\lambda_0$ is the length of the shortest $s$-$t$-path, $\lambda_1$ the size of the set added in iteration 1, and so on. We denote the incremental solution of the algorithm by $X^{\mathrm{A}}$.

With $v_{\max} \in \mathbb{R}_{\geq 0}$ defined as the maximum possible $s$-$t$-flow value in the underlying graph, for $j \in [\lfloor v_{\max} \rfloor]$, we denote by $k_j$ the minimum number of edges required to achieve an $s$-$t$-flow value of at least $j$. The values $\lambda_i$ and $k_j$ are related in the following way.

**Lemma 6.25** ([43, Lemma 4])**.** *When $w(e) = \mu(e) = 1$ for all $e \in E$, we have $\lambda_i \leq k_j/(j-i)$ for all $i, j \in \mathbb{N}$ with $0 \leq i < j \leq r$.*

Using this estimate, we can find a bound on the competitive ratio of QUICKEST-INCREMENT for the unit weight and unit capacity case.

**Theorem 6.26.** *For the incremental maximum $s$-$t$-flow problem with unit capacities and weights, the algorithm QUICKEST-INCREMENT is 2-competitive.*

*Proof.* Note that, since we consider the unit capacity case, we have $v_{\max} = r + 1$ because QUICKEST-INCREMENT increases the value of the solution by exactly 1 in each iteration.

Consider some capacity $C \in [|E|]$. If $C < k_1$, we have $f\big(O(C)\big) = 0$, i.e., every incremental solution is competitive. If $C \geq k_1$, let $j := f\big(O(C)\big)$. Note that we have $f\big(O(k_j)\big) = j = f\big(O(C)\big)$ and therefore $C \geq k_j$. By Lemma 6.25, we have

$$\begin{aligned} \sum_{i=0}^{\lceil j/2 \rceil - 1} \lambda_i &\leq \sum_{i=0}^{\lceil j/2 \rceil - 1} \frac{k_j}{j-i} = k_j \sum_{i=0}^{\lceil j/2 \rceil - 1} \frac{1}{j-i} \\ &\leq k_j \sum_{i=0}^{\lceil j/2 \rceil - 1} \frac{1}{j - \lceil \frac{j}{2} \rceil + 1} = k_j \left\lceil \frac{j}{2} \right\rceil \frac{1}{\lfloor \frac{j}{2} \rfloor + 1} \leq k_j \end{aligned} \tag{6.31}$$

This implies $f(X^{\mathrm{A}}(C)) \geq f(X^{\mathrm{A}}(k_j)) \overset{(6.31)}{\geq} \lceil \frac{j}{2} \rceil \geq \frac{1}{2} j = \frac{1}{2} f(O(C))$. ∎

Now, we turn to the case of unit capacities and rational weights. By rescaling the weights, we can assume that, without loss of generality, the weights are integral. To transform an instance with integral weights to one where all edges have unit weight, one can simply replace every edge $e \in E$ by a path of length $w(e)$ where every edge on the new path has unit weight. Then, Theorem 6.26 can be applied and we obtain the following.

**Corollary 6.27.** *The algorithm QUICKEST-INCREMENT is 2-competitive for the incremental maximum $s$-$t$-flow problem with unit capacities and $w(e) \in \mathbb{Q}_{\geq 0}$ for all $e \in E$.*

If we consider capacities that are in the interval $[1, M]$, one $s$-$t$-path can carry at most $M$ times as much $s$-$t$-flow as every other $s$-$t$-path. Combining this with the fact that the incremental solution of QUICKEST-INCREMENT for the instance with $\mu(e) = 1$ for all $e \in E$ is 2-competitive yields that adding the edges in the same order is always within a factor of $2M$ of the optimum solution.

**Corollary 6.28.** *The solution obtained by QUICKEST-INCREMENT when the capacities are all set to* $1$ *is* $2M$*-competitive for the incremental maximum* $s$-$t$*-flow problem with* $\mu(e) \in [1, M]$, $w(e) \in \mathbb{Q}_{\geq 0}$ *for all* $e \in E$.

As it turns out, the competitive ratio of QUICKEST-INCREMENT of 2 in the unit capacity case is optimal.

**Theorem 6.29.** *The competitive ratio of the incremental maximum* $s$-$t$*-flow problem with unit capacities and weights is at least* $2$.

*Proof.* Consider the directed graph $G = (V, E)$ with

$$\begin{aligned} V &:= \{s, t, u_1, u_2, u_3, v_1, v_2, v_3\}, \\ E &:= \{(s, u_1), (s, v_1), (u_1, u_2), (v_1, v_2), (u_2, u_3), (v_2, v_3), (u_3, t), (v_3, t), (u_1, v_3)\}, \end{aligned}$$

with unit capacities and unit weights (cf. Figure 6.2). Let $X$ be an arbitrary incremental solution that is $\rho$-competitive. If the first three elements added by $X$ are not the elements $(s, u_1)$, $(u_1, v_3)$, and $(v_3, t)$ (in any order) then the incremental solution is not competitive for capacity 3. Thus, any competitive solution starts by adding these three elements. This, however, implies that the first eight elements of $X$ cannot contain the elements of the upper and lower paths, i.e., we have

$$\{(s, u_1), (u_1, u_2), (u_2, u_3), (u_3, t)\} \cup \{(s, v_1), (v_1, v_2), (v_2, v_3), (v_3, t)\} \nsubseteq X(8).$$

This implies that $f(X(8)) = 1$. Since $\text{OPT}(8) = 2$, we obtain $\rho \geq 2$, as claimed. □

Furthermore, similar to the INCMAXKNAP problem with a fractionally subadditive, $M$-bounded objective function, no algorithm can have a competitive ratio better than $M$ when $\mu(e) \in [1, M]$ for all $e \in E$.

**Theorem 6.30.** *The competitive ratio of the incremental maximum* $s$-$t$*-flow problem with unit weights and* $\mu(e) \in [1, M]$ *for all* $e \in E$ *is at least* $M$.

*Proof.* Consider the directed graph $G = (V, E)$ with

$$\begin{aligned} V &:= \{s, t, v\}, \\ E &:= \{(s, t), (s, v), (v, t)\}, \end{aligned}$$

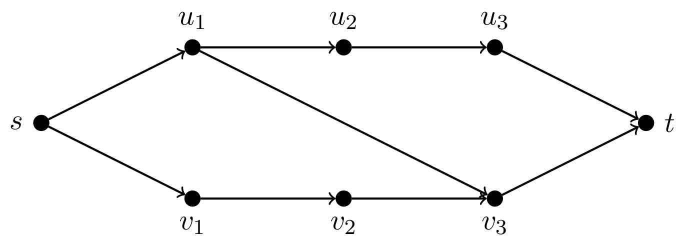


Figure 6.2.: A lower bound instance with best possible competitive ratio 2 for the incremental maximum $s$-$t$-flow problem

with unit weights and capacities $\mu((s,t)) = 1$, $\mu((s,v)) = \mu((v,t)) = M$ (cf. Figure 1.1 with $k = M$).

Let $X$ be an arbitrary incremental solution. If $X$ does not begin with element $(s,t)$, then it is not competitive for capacity $1$. This, however, implies that, if $X$ is competitive, we have $X(2) \neq \{(s,v),(v,t)\}$. Thus, we have $f(X(2)) = 1$ while $\textsc{Opt}(2) = M$. This implies $\rho \geq M$, as claimed. $\square$

# 7. Conclusion

The goal of this work was to investigate the competitive ratio of the IncMax problem. We have seen right from the start, in the instance in Figure 1.1, that the competitive ratio of IncMax is unbounded. Thus, we tried to find meaningful subclasses of IncMax that induce a bounded competitive ratio.

In Chapter 2, we analyzed the Greedy algorithm that iteratively adds the element that yields the largest increase in the objective value. For this, we introduced the class of $\gamma$-$\alpha$-augmentable problems and showed that it encompasses important classes of greedily approximable problems from the literature, e.g., problems with an $\alpha$-augmentable objective, objectives with a bounded submodularity ratio, or objectives that are weighted rank functions. We gave an upper bound on the competitive ratio of $\frac{\alpha-(1-c)\gamma}{\gamma} \cdot \frac{e^{\alpha-(1-c)\gamma}}{e^{\alpha-(1-c)\gamma}-1}$. This bound is tight for curvature $c = 1$ and recovers the known bounds for the subclass with $\alpha$-augmentable objectives, as well as for the subclass with objectives with bounded submodularity ratio and curvature $c$. Our tight lower bound also closed a gap left in the analysis of the class with $\alpha$-augmentable objectives left in [5].

Then, we turned to analyzing the competitive ratio of the class $\text{IncMax}_{\text{acc}}$ of instances with and accountable objectives. In Chapter 3, we reduced this problem to analyzing the competitive ratio of the well-structured subclass IncMaxSep. We introduced IncMaxCont as a continuization of IncMaxSep and showed that lower bounds on the strict competitive ratio of IncMaxCont are also a lower bound on the (non-strict) competitive ratio of IncMaxSep. We gave evidence that the upper bound of $\varphi+1$ on the competitive ratio might actually be tight. Using a similar technique, we gave an improved lower bound of $2.246$ on the competitive ratio of IncMaxCont. Subsequently, in Chapter 4, we took one step back and investigated the IncMaxSep problem again. We presented three deterministic algorithms, CardinalityScaling, ValueScaling, and DensityScaling, and gave tight bounds on their competitive ratio of $\varphi + 1$, $\varphi + 1$, and $4$, respectively. Lastly, we turned to randomized algorithms, where the RandScaling algorithm, a randomized version of CardinalityScaling, gave an upper bound of $1.772$ on the randomized competitive ratio of IncMaxSep. We complemented this with a lower bound of $1.357$ by using Yao's principle.

In Chapter 5, we introduced the new class of $\beta$-accountable functions that generalizes the classes of accountable, subadditive, and $\gamma$-$\alpha$-augmentable functions. We gave upper and lower bounds on the competitive ratio of the class with $\beta$-accountable objectives, that are tight for $\beta \to 0$. For $\beta = 1$, the upper bound exactly recovers the best upper bound of $\varphi + 1$ known for the class with accountable objectives. For $\beta = \frac{1}{2}$, we showed that we capture the class of subadditive objectives and obtain an upper bound of $2 + \sqrt{2}$ on its competitive ratio, which, to our knowledge, is the first one known.

Lastly, in Chapter 6, we considered IncMaxKnap, a variation of IncMax where we are not given an unknown cardinality constraint, but an unknown knapsack constraint. We started off by investigating the strict competitive ratio of this problem for objectives that are monotone, fractionally subadditive and $M$-bounded. We gave upper and lower bounds and could see that, for large $M$, the competitive ratio grows linearly in $M$. In order to avoid this, we turned to the non-strict competitive ratio and showed that an algorithm with constant non-strict competitive ratio $\varphi + 1$ and additive constant $(\varphi + 1)M$ exists. We also argued that every lower bound on the strict competitive ratio of IncMaxCont, e.g., the bound of $2.246$ from Section 3.2.2, is a lower bound on the non-strict competitive ratio for IncMax under a knapsack constraint with monotone, fractionally subadditive and $M$-bounded objective for all $M \geq 1$.

## 7.1. Future Work

While we were able to show a bounded competitive ratio for many natural problem classes, there are still relatively simple problems that admit a competitive solution, but that are not captured by any of our settings.

**Proposition 7.1.** *For all $\beta \in (0, 1]$, there exists an instance of IncMax that is not weakly $\beta$-accountable, and for which every ordering is an optimal incremental solution.*

*Proof.* Let $U = \mathbb{N}$ and consider the objective function $f\colon 2^U \to \mathbb{R}_{\geq 0}$ such that, for all $S \subseteq U$,

$$f(S) = |S|^2.$$

It is obvious that every ordering of the elements in $U$ is optimal because all elements are symmetrical with regard to $f$. Monotonicity of $f$ follows immediately from the fact that the function $x^2$ is non-decreasing for $x \geq 0$. It remains to show that $f$ is not $\beta$-accountable. For this, let $k = \lceil \frac{1}{\beta} \rceil + 1$ consider the set $S = [k] \subseteq U$. Let $(e_1, \dots, e_k)$ be any ordering of the elements in $S$. Then

$$f(\{e_1\}) = 1 < \beta k = \beta \frac{1}{k} k^2 = \beta \frac{1}{k} f(S),$$

i.e., $f$ is not $\beta$-accountable. □

One problem with the instance in Proposition 7.1 it that the objective value grows very fast with increasing set size. Thus, the objective is not $\beta$-accountable. Yet, the elements behave very symmetrically which is why a competitive incremental solution exists. We leave it as an open question to find a natural problem class that captures such problems where the elements are rather symmetrical, and to find a generalization of it that also encompasses $\beta$-accountable objectives.

We explored one generalization of the unknown cardinality constraint in Chapter 6 where we looked at the problem under an unknown knapsack constraint. A knapsack constraint requires that the value of our solution under some modular function does not exceed an unknown value. In future work, this can be generalized even further to not only consider modular functions, but any monotone function. A cardinality constraint can also be expressed as the requirement that the solution has to be independent in some uniform matroid. To generalize this, rather than considering an unknown uniform matroid constraint, one can consider some unknown matroid constraint or an unknown independence system constraint.

Many of our results are not tight, so, naturally, the open question arises to find tight bounds in these settings. Examples include the deterministic and randomized competitive ratio of $\textsc{IncMax}_{\text{acc}}$, the deterministic competitive ratio of the class $\textsc{IncMax}$ with $\beta$-accountable objectives, or the strict and non-strict competitive ratio of $\textsc{IncMax}$ under a knapsack constraint with monotone, fractionally subadditive and $M$-bounded objective.

We leave it open to find a natural generalization of weak $\gamma$-$\alpha$-augmentability that captures a larger set of greedily approximable objectives. The challenge is to find a meaningful generalization in terms of a natural definition that does not directly depend on the behavior of the $\textsc{Greedy}$ algorithm as it it the case with weak $\gamma$-$\alpha$-augmentability, but rather enforces some structural property of the objective function.

# A. Appendix

| C | bound | C | bound | C | bound | C | bound | C | bound |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.64208 | 35 | 0.57597 | 69 | 0.57364 | 103 | 0.57187 | 137 | 0.57071 |
| 2 | 0.65090 | 36 | 0.57576 | 70 | 0.57369 | 104 | 0.57184 | 138 | 0.57069 |
| 3 | 0.60073 | 37 | 0.57576 | 71 | 0.57370 | 105 | 0.57185 | 139 | 0.57064 |
| 4 | 0.59777 | 38 | 0.57582 | 72 | 0.57378 | 106 | 0.57179 | 140 | 0.57063 |
| 5 | 0.58725 | 39 | 0.57601 | 73 | 0.57374 | 107 | 0.57171 | 141 | 0.57063 |
| 6 | 0.58432 | 40 | 0.57618 | 74 | 0.57371 | 108 | 0.57171 | 142 | 0.57059 |
| 7 | 0.59353 | 41 | 0.57598 | 75 | 0.57369 | 109 | 0.57168 | 143 | 0.57057 |
| 8 | 0.59445 | 42 | 0.57600 | 76 | 0.57370 | 110 | 0.57162 | 144 | 0.57054 |
| 9 | 0.59025 | 43 | 0.57591 | 77 | 0.57364 | 111 | 0.57159 | 145 | 0.57050 |
| 10 | 0.59115 | 44 | 0.57588 | 78 | 0.57360 | 112 | 0.57153 | 146 | 0.57050 |
| 11 | 0.58804 | 45 | 0.57591 | 79 | 0.57354 | 113 | 0.57145 | 147 | 0.57049 |
| 12 | 0.58747 | 46 | 0.57593 | 80 | 0.57338 | 114 | 0.57146 | 148 | 0.57046 |
| 13 | 0.59052 | 47 | 0.57567 | 81 | 0.57325 | 115 | 0.57143 | 149 | 0.57045 |
| 14 | 0.59002 | 48 | 0.57545 | 82 | 0.57323 | 116 | 0.57139 | 150 | 0.57042 |
| 15 | 0.58765 | 49 | 0.57531 | 83 | 0.57313 | 117 | 0.57137 | 151 | 0.57038 |
| 16 | 0.58664 | 50 | 0.57511 | 84 | 0.57308 | 118 | 0.57130 | 152 | 0.57038 |
| 17 | 0.58482 | 51 | 0.57512 | 85 | 0.57302 | 119 | 0.57125 | 153 | 0.57036 |
| 18 | 0.58287 | 52 | 0.57510 | 86 | 0.57288 | 120 | 0.57125 | 154 | 0.57032 |
| 19 | 0.58344 | 53 | 0.57486 | 87 | 0.57280 | 121 | 0.57121 | 155 | 0.57030 |
| 20 | 0.58278 | 54 | 0.57479 | 88 | 0.57275 | 122 | 0.57116 | 156 | 0.57026 |
| 21 | 0.58143 | 55 | 0.57462 | 89 | 0.57268 | 123 | 0.57113 | 157 | 0.57021 |
| 22 | 0.58114 | 56 | 0.57444 | 90 | 0.57266 | 124 | 0.57106 | 158 | 0.57018 |
| 23 | 0.57983 | 57 | 0.57452 | 91 | 0.57259 | 125 | 0.57098 | 159 | 0.57016 |
| 24 | 0.57885 | 58 | 0.57446 | 92 | 0.57244 | 126 | 0.57096 | 160 | 0.57011 |
| 25 | 0.57921 | 59 | 0.57432 | 93 | 0.57232 | 127 | 0.57091 | 161 | 0.57007 |
| 26 | 0.57858 | 60 | 0.57422 | 94 | 0.57224 | 128 | 0.57087 | 162 | 0.57003 |
| 27 | 0.57814 | 61 | 0.57405 | 95 | 0.57214 | 129 | 0.57083 | 163 | 0.56999 |
| 28 | 0.57791 | 62 | 0.57380 | 96 | 0.57209 | 130 | 0.57078 | 164 | 0.56997 |
| 29 | 0.57698 | 63 | 0.57377 | 97 | 0.57201 | 131 | 0.57077 | 165 | 0.56997 |
| 30 | 0.57609 | 64 | 0.57369 | 98 | 0.57195 | 132 | 0.57076 | 166 | 0.56995 |
| 31 | 0.57609 | 65 | 0.57361 | 99 | 0.57195 | 133 | 0.57073 | 167 | 0.56995 |
| 32 | 0.57583 | 66 | 0.57369 | 100 | 0.57191 | 134 | 0.57075 | 168 | 0.56994 |
| 33 | 0.57581 | 67 | 0.57369 | 101 | 0.57185 | 135 | 0.57076 | 169 | 0.56990 |
| 34 | 0.57615 | 68 | 0.57359 | 102 | 0.57189 | 136 | 0.57073 | 170 | 0.56990 |

Figure A.1.: Evaluations for $C \in [170]$ of the lower bounds on the inverse of the randomized competitive ratio that was derived in (4.21) in the proof of Theorem 4.21.

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