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On the topology and geometry of Kac–Moody groups

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Dipl.-Math. Andreas Mars
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Referent: PD dr. Ralf Gramlich
1. Korreferent: Prof. Dr. Nils Scheithauer
2. Korreferent: Prof. Dr. Linus Kramer

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Deutsche Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit topologischen und geometrischen Fragestellungen innerhalb der Theorie der Kac–Moody-Gruppen. Diese sind natürliche Verallgemeinerungen von Chevalley-Gruppen über kommutativen Ringen mit Eins. Im Laufe des Promotionsprojektes war die Beantwortung folgender Fragestellungen von zentraler Bedeutung.

- Sei $G_{\mathcal{D}}(\mathbb{F})$ eine Kac–Moody-Gruppe, definiert über einem topologischen Körper $\mathbb{F}$, welcher ein k_ω-Körper[1] ist. Macht die Kac–Peterson-Topologie auf $G_{\mathcal{D}}(\mathbb{F})$ die Gruppe zu einer Hausdorffschen topologischen Gruppe? Ist diese Gruppe k_ω?

Diese Frage ergab sich natürlicherweise aus der Arbeit [GGH10]. Dort wurde gezeigt, dass die Kac–Peterson-Topologie reelle und komplexe Kac–Moody-Gruppen zu k_ω-Gruppen macht. Dieses Resultat wird in Kapitel 3 verallgemeinert.

- Sei $G_{\mathcal{D}}(R)$ eine Kac–Moody-Gruppe über einem Integritätsbereich R. Ist es möglich, die Isomorphismen zwischen zwei Kac–Moody-Gruppen $G_{\mathcal{D}}(R)$ und $G'_{\mathcal{D}'}(R)$ zu klassifizieren? Falls $G_{\mathcal{D}}(R) \cong G'_{\mathcal{D}'}(R)$, sind dann auch die zugehörigen Wurzeldaten $\mathcal{D}$ und $\mathcal{D}'$ isomorph? Wie verhalten sich die Automorphismen von $G_{\mathcal{D}}(R)$ im Vergleich zu denen von $G_{\mathcal{D}}(\mathbb{F})$, wobei $\mathbb{F}$ der Quotientenkörper von R ist?

In [Cap09] wurden die Isomorphismen zwischen zwei Kac–Moody-Gruppen über Körpern bestimmt. Der Beweis benutzt die Wirkung auf dem zugehörigen Zwillingsgebäude. Ich verwende, dass Kac–Moody-Gruppen über Integritätsbereichen auf den Gebäuden der Kac–Moody-Gruppen über den Quotientenkörpern wirken und bestimme die Isomorphismen mit Hilfe eines lokal-zu-global-Arguments.

- Ist das natürliche Zwillingsgebäude einer Kac–Moody-Gruppe $G_{\mathcal{D}}(\mathbb{F})$ (ausgestattet mit der Kac–Peterson-Topologie) über einem k_ω-Körper $\mathbb{F}$ ein topologisches Zwillingsgebäude im Sinne von [Har06, Definition 3.1.1]? Falls ja, wie sieht die topologische Bahnenstruktur spezieller Untergruppen von $G_{\mathcal{D}}(\mathbb{F})$ auf dem Gebäude aus?

[1] Zur Erinnerung: Ein Hausdorffscher topologischer Raum ist ein k_ω-Raum, wenn er direkter Limes einer aufsteigenden abzählbaren Folge von kompakten Teilmengen ist. Ein k_ω-Körper ist ein topologischer Körper, dessen zu Grunde liegender topologischer Raum ein k_ω-Raum ist.

Im sphärischen Fall wurde in [BS87] ein Zusammenhang zwischen Lie-Gruppen und sphärischen topologischen Gebäuden nachgewiesen. Die Arbeit [Har06] verallgemeinert die Resultate, welche wiederum hier in noch allgemeinerem Kontext bewiesen werden.

Diese Fragen werden in der vorliegenden Arbeit diskutiert und gelöst, einige weiterführende Fragestellungen werden formuliert und mögliche Verallgemeinerungen der präsentierten Resultate skizziert.

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CHAPTER 1

Introduction

The topic of this thesis are topological Kac–Moody groups, their topological and geometrical properties and related problems. We shall first give a short historical overview, then sketch the main results of this thesis.

There is a well-established theory of complex semisimple Lie groups and their corresponding complex Lie algebras. Moreover, exponentiation and differentiation, respectively, provide a close connection between these two concepts. As generalisation of complex finite-dimensional semisimple Lie algebras, Kac–Moody algebras over fields of characteristic zero were first introduced independently of each other by both V. Kac [Kac68] and R. Moody [Moo68] in the late 1960's. As is well known, a complex finite-dimensional semisimple Lie algebra may be reconstructed from its Cartan matrix, a certain positive definite integral matrix. It was observed omitting the assumption *positive definite* still yields a Lie algebra using the same method of construction, however, the resulting Lie algebra will no longer be finite-dimensional in general.

Soon after in [KP83b], [KP85] and [Kac85], Kac–Moody groups were defined. Firstly, the groups were obtained using integration of the adjoint representation of a Kac–Moody algebra on itself in a fashion similar to the one used when (re)constructing adjoint Lie groups or algebraic groups from their Lie algebras. In this sense, Kac–Moody groups associated to Kac–Moody algebras behave like Lie groups associated to Lie algebras. The main difference compared to finite-dimensional Lie groups is that the underlying root system is no longer assumed to be spherical, or equivalently, finite.

Another well-known generalisation of (linear) Lie groups are algebraic groups over arbitrary fields and Chevalley groups over arbitrary commutative rings. From this point of view, it seems natural to ask if one can also generalise Kac–Moody groups to other fields, or maybe even commutative rings. Trying to generalise the original definition in the straightforward manner hits upon a serious obstacle: Integration of the adjoint representation on the Kac–Moody algebra only works in the case where the underlying field is of characteristic zero, otherwise the denominator in the exponential series may become zero. As solution to this problem, J. Tits gave a functorial description of Kac–Moody groups over the category of commutative unital rings in [Tit87] (see also Section 2.5 of this thesis). For fields of characteristic zero these two approaches coincide (cf. [Tit87]), hence the functorial definition generalises the construction sketched above.

Many results about Kac–Moody groups have been obtained treating them as abstract groups, see [Tit87] or [KP83a]. For example, generators and relations have been computed in [KP85] and [DMGH09]. Moreover, P.-E. Caprace and B. Mühlherr gave a solution to the isomorphism problem for split Kac–Moody groups in [CM06], [CM05] and [Cap09] in analogy to the spherical case of Chevalley groups defined over fields. This result has been extended recently to quasi-split groups in characteristic zero in [Hai10].

In a series of papers (amongst others [KP83b], [KP85] and later [Kit08], [Kum02]) topology entered the picture. In [KP83b], the authors introduce the weak and strong Zariski topology on split Kac–Moody groups defined over algebraically closed fields of characteristic zero and prove a Peter–Weyl-type Theorem, but as in the case of algebraic groups, this topology does not turn the Kac–Moody group into a topological group. By contrast, consider the Kac–Peterson topology introduced in [KP83b, Section 4G] for Kac–Moody groups over locally compact algebraically closed fields of characteristic zero. In [GGH10] it is shown that a Kac–Moody group defined over $\mathbb{R}$ or $\mathbb{C}$ and equipped with this topology is a k_ω-group. In many ways the Kac-Peterson topology is a natural generalisation of the Lie group topology on Lie groups defined over local fields.

After J. Tits introduced the theory of (twin) buildings, a number of efforts have been made to combine the geometry of buildings with the topology of Lie groups, cf. [Kra02], [BS87], [Har06], [RS90]. There is a well-established close connection between abstract Kac–Moody groups and twin buildings. A topological connection between finite-dimensional Lie groups and spherical topological twin buildings has been obtained in [BS87]. The case of compact Moufang n-gons was treated in much detail in [Kra94]. But to date, there has been no definitive proof of a similar close connection between topological Kac–Moody groups and topological twin buildings, except in the spherical case. However, this connection was partially shown and conjectured in [Har06].

1.1 Structure of this thesis and main results

We now briefly outline the topics and results of the respective chapters of this thesis.

Chapter 2

In Chapter 2, we document the terminology and notation commonly used in the literature. Including examples and references to the literature, we also prove a number of basic results which seek to give the reader unfamiliar with the subject an insight into the objects of study. In addition, this chapter serves as a reference library for the thesis.

Chapter 3

We analyse the Kac–Peterson topology on split Kac–Moody groups defined over k_ω-fields. While the Kac–Peterson topology on a split Kac–Moody group defined over a local field naturally generalises the Lie group topology, to date (to the knowledge of the author) almost nothing is known for topological Kac–Moody group defined over topological fields in general. This chapter gives a description of the basic properties of the Kac–Peterson topology and compares the approach presented in Section 2.8 against the original one by Kac and Peterson, showing an equivalence between them.

While the Kac–Peterson topology of a non-spherical split Kac–Moody group over a locally compact field is no longer metrisable in general (cf. [FT77]), it satisfies the following property which was first shown for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ in [GGH10, Section 6].

Theorem (A, Theorem 3.1.11)**.** *Let $\mathbb{F}$ be a k_ω-field and let $G_{\mathcal{D}}(\mathbb{F})$ be a simply connected or adjoint split Kac–Moody group over $\mathbb{F}$. Then $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ is a k_ω-group.*

Consequently, if θ is a continuous flip (i.e. an involutory automorphism satisfying some additional conditions) of $G_{\mathcal{D}}(\mathbb{F})$, then $(G_\theta, \tau_{KP}) := (\mathrm{Stab}_{G_{\mathcal{D}}(\mathbb{F})}(\theta), \tau_{KP})$ is a k_ω-group.

The main difficulty in the proof of Theorem A is to show that the Kac–Peterson topology on $G_{\mathcal{D}}(\mathbb{F})$ is Hausdorff. This is obtained using the adjoint representation of $G_{\mathcal{D}}(\mathbb{F})$ on the universal enveloping algebra of the associated complex Kac–Moody algebra.

In Kac–Moody-theory, many results on non-spherical Kac–Moody groups use local-to-global arguments. This strategy aims at transferring results known in the spherical case to the non-spherical case, as is done in Chapter 4. For this method to be applicable, one needs to know that the local structure of a Kac–Moody group already contains enough information of the whole group. Using the results obtained in Chapter 3, we prove that the Kac–Peterson topology on two-spherical split Kac–Moody groups defined over fields admits a description using the fundamental subgroups of rank one and two only. The following Theorem is therefore a topological version of the Curtis–Tits Theorem.

Theorem (B, Theorem 3.1.12)**.** *Let $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ be a simply connected or adjoint split Kac–Moody group over some k_ω-field $\mathbb{F}$ of cardinality at least four. Assume that $\mathcal{D}$ is of two-spherical type. Let Φ^{re} be the set of real roots and let Π be a basis of simple roots for Φ^{re}. Construct an amalgam $\mathcal{A}$ as follows: For $\alpha, \beta \in \Pi$, set $G_\alpha := \varphi_\alpha(\mathrm{SL}_2(\mathbb{F}))$ and $G_{\alpha\beta} := \langle\, G_\alpha \cup G_\beta \,\rangle$. Moreover, let $\iota_{\alpha\beta} : G_\alpha \hookrightarrow G_{\alpha\beta}$ be the canonical inclusion morphisms.*

Then the group $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ is a universal enveloping group of the amalgam $\mathcal{A} = \{G_\alpha, G_{\alpha\beta}; \iota_{\alpha\beta}\}$ in the categories of

(i) abstract groups,

(ii) *Hausdorff topological groups and*

(iii) k_ω*-groups.*

The above result makes use of the fact that a split Kac–Moody group acts strongly transitively on its associated twin building.

Chapter 4

Let $n \geq 3$ and consider the group $\mathrm{SL}_n(\mathbb{Z})$ as a subgroup of $\mathrm{SL}_n(\mathbb{R})$. Then $\mathrm{SL}_n(\mathbb{Z})$ is discrete and of finite covolume in $\mathrm{SL}_n(\mathbb{R})$, i.e. a lattice. Applying deep theory (see [Mar91, Theorem VII.7.1]), one may conclude that $\mathrm{SL}_n(\mathbb{Z})$ is Mostow-rigid in $\mathrm{SL}_n(\mathbb{R})$. This implies in particular that every automorphism of $\mathrm{SL}_n(\mathbb{Z})$ uniquely lifts to a continuous automorphism of $\mathrm{SL}_n(\mathbb{R})$. In order to apply the theory developed by Margulis, one needs the fact that on a locally compact group there exists a Haar measure. By contrast, there is no Haar measure on a non-spherical Kac–Moody group, as it is not locally compact. The author is not aware of any generalisation of the Haar measure to which the theory applies.

To achieve a similar result, we use the algebraic methods developed by P.-E. Caprace and B. Mühlherr in [Cap09], [CM05], [CM06] (which were also used in [Mar07, Chapter 4]) as refined in [Hai10, Section 6]. We obtain the solution of the isomorphism problem for split Kac–Moody groups over a class of integral domains containing $\mathbb{Q}$, the so-called rank two-rigid rings, see Definition 4.1.5.

Theorem (C, Theorem 4.5.5)**.** *Let R be a rank two-rigid ring containing $\mathbb{Q}$ and let $\mathcal{D}$ and $\mathcal{D}'$ be two-spherical Kac–Moody root data without G_2-residues or direct factors of type A_1. Let $G_{\mathcal{D}}(R)$ and $G'_{\mathcal{D}'}(R)$ be the associated split Kac–Moody groups and let $\varphi : G_{\mathcal{D}}(R) \to G'_{\mathcal{D}'}(R)$ be an isomorphism. Denote by $A = (a_{ij})_{1\leq i,j\leq n}$ and $A' = (a'_{ij})_{1\leq i,j\leq n'}$ their respective generalised Cartan matrices. Then there exist*

(i) *a bijection $\pi\colon I \to I'$ such that $a_{ij} = a'_{\pi(i)\pi(j)}$,*

(ii) *an inner automorphism ν of $G'_{\mathcal{D}'}(R)$,*

(iii) *for all distinct $i, j \in I$ in the same connected component of the Dynkin diagram a diagonal-by-ring-by-sign automorphism γ_{ij} of the rank two group $X_{ij} = X(R)$ with $X \in \{A_1 \times A_1, A_2, B_2\}$ such that the diagram*

$$\begin{tikzcd} X_{ij} \arrow[r, "\gamma_{ij}"] \arrow[d, "\mathrm{id}"'] & X_{\pi(i)\pi(j)} \arrow[d, "\mathrm{id}"] \\ G_{\mathcal{D}}(R) \arrow[r, "\nu\circ\varphi"] & G'_{\mathcal{D}'}(R) \end{tikzcd}$$

commutes.

Chapter 5

The work [BS87] gives a description and classification of spherical topological buildings. In particular, the authors prove that the topological automorphism group of a spherical topological twin building is a Lie group. The description has been generalised to topological twin buildings in [Har06], conjecturing a similar correspondence between topological twin buildings and topological Kac–Moody groups. We give a proof of this conjecture for split Kac–Moody groups.

Theorem (D, Theorem 5.1.6)**.** *Let $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ be a simply connected or adjoint split Kac–Moody group over a k_ω-field $\mathbb{F}$. Then the canonical twin building $\Delta(G_{\mathcal{D}}(\mathbb{F})) = ((G_{\mathcal{D}}(\mathbb{F})/B_+, \delta_+), (G_{\mathcal{D}}(\mathbb{F})/B_-, \delta_-), \delta^*)$ associated to $G_{\mathcal{D}}(\mathbb{F})$, equipped with the quotient topology, is a topological twin building.*

This result is the starting point for the analysis conducted in the remainder of this chapter. Since quotient maps of topological groups are open, the canonical quotient map taking a topological split Kac–Moody group to its associated topological twin building is continuous and open. Therefore, we may combine the topological results from Chapter 3 with the geometry of twin building, obtaining a machinery which unifies and combines topological and geometrical properties.

For example, the following result links the topological orbit structure of a Borel subgroup to the codistance of chambers with respect to the chamber fixed by the Borel subgroup. In particular, we obtain a description of orbit closures with respect to Borel subgroups using the Weyl group and its Bruhat order only.

Theorem (E, Theorem 5.2.5)**.** *Let $G_{\mathcal{D}}(\mathbb{F})$ be a split Kac–Moody group over a non-discrete k_ω-field $\mathbb{F}$, let W be its Weyl group, $\leq$ the Bruhat order of W and consider the orbits of the action of B_- on Δ_+ by multiplication from the left. Then the following hold:*

(i) For $w \in W$ the closure relation

$$\overline{B_- w B_+} = \bigsqcup_{w' \geq w} B_- w' B_+$$

holds.

(ii) Let $w \in W$. The unique smallest open B_--invariant set containing the B_--orbit $B_- w B_+$ is

$$\bigsqcup_{w' \leq w} B_- w' B_+.$$

In particular, it consists of finitely many B_--orbits.

(iii) *The set* $\bigsqcup_{l(w)\le n} B_-wB_+$ *is open in* $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$.

In particular, $\left\{\bigsqcup_{l(w)\le n} B_-wB_+\right\}_{n\in\mathbb{N}}$ *is a filtration of* $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ *consisting of open sets.*

The very same question may be asked about the unitary form G_θ acting on one half of the twin building. The problem encountered is that G_θ in general does not fix a chamber, hence it does not preserve the codistance from a given chamber. Using the theory developed in [Hor10] and [GHM], we may introduce a filtration with respect to the θ-codistance of chambers that is preserved by the action of G_θ. Again, we may reduce the topological description of the G_θ-orbits to the Bruhat order of the Weyl group.

Theorem (F, Theorem 5.3.4)**.** *Assume that* $\mathbb{F}$ *is a non-discrete* k_ω*-field and let* $G_{\mathcal{D}}(\mathbb{F})$ *be a simply connected or adjoint split Kac–Moody group over* $\mathbb{F}$*. Let* θ *be a flip of* $G_{\mathcal{D}}(\mathbb{F})$ *and consider the filtration* $\{\Delta_w \mid w \in \mathrm{Cod}(\theta)\}$ *of* Δ_+ *with respect to the* θ*-codistance. Let* $X \subseteq \mathrm{Cod}(\theta)$*. Then the following hold:*

(i) *The union*

$$\Delta_X := \bigcup_{w\in X} \Delta_w$$

is open if and only if X *is a lower set in* $\mathrm{Cod}(\theta)$ *with respect to the order induced by the Bruhat order on* W*.*

(ii) *The set* Δ_X *is closed if and only if* X *is an upper set in* $\mathrm{Cod}(\theta)$ *with respect to the order induced by the Bruhat order on* W*.*

(iii) *Let* $w \in \mathrm{Cod}(\theta)$*. Then the closure relation*

$$\overline{\Delta_w} = \Delta_{\ge w} := \bigcup_{w'\ge w} \Delta_{w'}$$

holds.

(iv) *Let* $w \in \mathrm{Cod}(\theta)$*. Then the smallest open* G_θ*-invariant set containing* Δ_w *coincides with*

$$\Delta_{\le w} = \bigcup_{w'\le w} \Delta_{w'}.$$

Chapter 6

In this chapter, we present results concerning central extensions of two-spherical split Kac–Moody groups $G_{\mathcal{D}}(R)$, where R is a ring which is assumed to have nice units for $\mathcal{D}$.

Central extensions of algebraic groups have been studied extensively for example in [Mat69], [Ste62], [Ste68]. More general results for split Kac–Moody groups defined over fields are shown in [MR90] as well as [Cap07] together with [DMT09].

Theorem (G, Theorem 6.2.2)**.** *Assume that $\mathcal{D}$ is centred and the Weyl group of $\mathcal{D}$ is two-spherical and has no direct factors of type A_1. Let R be a ring having nice units for $\mathcal{D}$. Then the universal central extension of $G_{\mathcal{D}}(R)$ is the Steinberg group $\mathrm{St}_A(R)$.*

Under some stronger assumptions on the underlying ring R, we also compute the kernel of this universal central extension.

CHAPTER 2

Definitions and Basics

In this chapter, we briefly outline the definitions and fundamental properties of the objects under consideration. References containing more detailed information will be provided at the beginning of each section.

Throughout this thesis, **all rings are assumed to be commutative and unital**.

2.1 RGD systems

Our main references for this section are [AB08] and [CR09], see also [Tit92].

Let G be a group and let $\{U_\alpha\}_{\alpha\in\Phi}$ be a family of subgroups of G, indexed by some root system Φ of type (W, S), and let T be a subgroup of G. The triple $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ is called an **RGD system** of type (W, S) if it satisfies:

(RGD0) For all roots $\alpha \in \Phi$, we have $U_\alpha \neq \{1\}$.

(RGD1) For every prenilpotent pair $\{\alpha, \beta\} \subseteq \Phi$ of distinct roots, the commutator relation $[U_\alpha, U_\beta] \subseteq \langle\, U_\gamma \mid \gamma \in]\alpha, \beta[\, \rangle$ holds.

(RGD2) For every fundamental reflection $s \in S$ and every $u \in U_{\alpha_s} \setminus \{1\}$, there exist $u', u'' \in U_{-\alpha_s}$ such that $\mu(u) := u'uu''$ conjugates U_β to $U_{s_\alpha(\beta)}$ for each $\beta \in \Phi$.

(RGD3) For all $s \in S$ it holds that $U_{-\alpha_s} \nsubseteq U_+ := \langle\, U_\alpha \mid \alpha \in \Phi^+ \,\rangle$.

(RGD4) $G = T.\langle\, U_\alpha \mid \alpha \in \Phi \,\rangle$.

(RGD5) The group T normalises every U_α.

The groups U_α are called the **root groups** and $(\{U_\alpha\}_{\alpha\in\Phi}, T)$ is referred to as **root group datum** of G. See [CR09], [AB08, Definition 7.82 and Section 8.6.1] for details.

Define $G_\alpha := \langle U_{\pm\alpha} \rangle$. A root group datum $(\{U_\alpha\}_{\alpha\in\Phi}, T)$ is called **$\mathbb{F}$-locally split** if T is abelian and there is a field $\mathbb{F}$ such that $G_\alpha \cong \mathrm{(P)SL}_2(\mathbb{F})$ and $\{U_\alpha, U_{-\alpha}\}$ is isomorphic to the canonical root group datum of rank one. The RGD system is called **centred** if G is generated by its root subgroups.

Example 2.1.1. Let $n \geq 2$ and consider the Coxeter system (W, S) of type A_{n-1}. In this case, $W \cong S_n$, the symmetric group on n elements and S is the standard generating set consisting of neighbour transpositions. For each pair $1 \leq i, j \leq n$ with $i \neq j$ the root system of type A_{n-1} has a root $\alpha_{i,j}$.

Take

- $\mathbb{F}$ to be a field,
- $G := \mathrm{GL}_n(\mathbb{F})$ to be the invertible $n \times n$-matrices over $\mathbb{F}$,
- $T \leq G$ to be the diagonal matrices,
- and $U_{\alpha_{i,j}}$ to be the one-parameter subgroup which has arbitrary entries in the (i, j)-coordinate, ones on the diagonal and zeroes elsewhere.

Then $(G, \{U_{\alpha_{i,j}}\}, T)$ is an $\mathbb{F}$-locally split root group datum of type (W, S) (see [AB08, Example 7.133]).

The same is true for G replaced with $G' := \mathrm{SL}_n(\mathbb{F})$, T replaced with $T' := T \cap \mathrm{SL}_n(\mathbb{F})$ and $U_{\alpha_{i,j}}$ as above.

2.2 (Twin) BN-pairs

For this section, our main references are [AB08], [Tit74], [Bou02].

Let G be a group and let B, N be subgroups of G. The pair (B, N) is called a **BN-pair** for G if $G = \langle B, N \rangle$, the intersection $T := B \cap N$ is normal in N, and the quotient group $W := N/T$ admits a set of generating involutions S such that

(BN1) for all $w \in W$ and $s \in S$ we have that $wBs \subseteq BwsB \cup BwB$, and

(BN2) $sBs \nsubseteq B$ for all $s \in S$.

If (B, N) is a BN-pair, then the group W is called the **Weyl group**, the quadruple (G, B, N, S) is often called **Tits system**.

Note that the pair (W, S) is a Coxeter system. If G is a group with BN-pair, then G admits a **Bruhat decomposition** $G = \bigsqcup_{w\in W} BwB$, cf. [AB08, Theorems 6.17 and 6.56].

If (G, B_+, N, S) and (G, B_-, N, S) are two Tits systems with the property $B_+ \cap N = B_- \cap N$, then the triple (B_+, B_-, N) is called a **twin BN-pair** if the following additional conditions are satisfied:

(TBN1) for $\varepsilon \in \{+, -\}$ and all $w \in W$, $s \in S$ such that $l(ws) < l(w)$, we have $B_\varepsilon s B_\varepsilon w B_{-\varepsilon} = B_\varepsilon sw B_{-\varepsilon}$, and

(TBN2) for all $s \in S$ we have $B_+ s \cap B_- = \emptyset$.

The quintuple (G, B_+, B_-, N, S) is called **twin Tits system**. It is called **saturated** if the equality $B_+ \cap B_- = T$ holds. A group G with a twin Tits system admits a **Birkhoff decomposition** $G = \bigsqcup_{w \in W} B_\varepsilon w B_{-\varepsilon}$ (disjoint union).

Conjugates of the **fundamental** or **standard Borel subgroups** B_+ and B_- are called **Borel subgroups** of G. The intersection $T := B_+ \cap B_-$ is called the **fundamental maximal torus** or **standard maximal torus** of G; each of its conjugates is called a **maximal torus**. A **fundamental (or standard) parabolic subgroup** P_ε of G is a subgroup containing a fundamental Borel group B_ε. Any conjugate of a fundamental parabolic subgroup is simply called **parabolic subgroup**.

Proposition 2.2.1 ([AB08, Theorem 8.80])**.** *Let G be a group with a root group datum $(\{U_\alpha\}_{\alpha \in \Phi}, T)$. Let $\mu_\alpha \colon U_\alpha \to U_{-\alpha} U_\alpha U_{-\alpha}$ be the map provided by (RGD2). Then the groups*

$$\begin{aligned} N &:= T.\langle\, \mu_\alpha(u) \mid u \in U_\alpha \setminus \{1\}, \alpha \in \Pi \,\rangle \\ B_+ &:= T.U_+ \\ B_- &:= T.U_- \end{aligned}$$

make (B_+, B_-, N) a twin BN-pair for the group G. □

Example 2.2.2. Consider the group $G = \mathrm{SL}_n(\mathbb{F})$ from Example 2.1.1 again. Define

- $U_+ := \langle\, U_\alpha \mid \alpha \in \Phi^+ \,\rangle$, the strict upper triangular matrices,
- $B_+ := T.U_+$, the upper triangular matrices,
- $U_- := \langle\, U_\alpha \mid \alpha \in \Phi^- \,\rangle$, the strict lower triangular matrices,
- $B_- := T.U_-$, the lower triangular matrices,
- N to be the subgroup of invertible monomial matrices (which is exactly the normaliser of T).

Then (B_+, B_-, N) is a saturated twin BN-pair for $\mathrm{SL}_n(\mathbb{F})$.

The same construction applies for the group $\mathrm{GL}_n(\mathbb{F})$.

2.3 (Twin) buildings

The reader is referred to [AB08], [Ron09] and [Wei03] for a detailed treatment of buildings and twin buildings.

Definition 2.3.1. Let (W, S) be a Coxeter system. A **building** of type (W, S) is a pair (Δ, δ) consisting of a set Δ (whose elements are called **chambers**) together with a distance function $\delta\colon \Delta \times \Delta \to W$ satisfying the following axioms, where $\delta(x, y) = w$:

(Bu1) $w = 1$ if and only if $x = y$,

(Bu2) if $z \in \Delta$ such that $\delta(y, z) = s \in S$, then $\delta(x, z) \in \{ws, w\}$. If additionally $l(ws) > l(w)$, then $\delta(x, z) = ws$ holds.

(Bu3) If $s \in S$, there exists $z \in \Delta$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$. □

Example 2.3.2. Let G be a group with BN-pair. Then we obtain a building as follows. Let $\Delta := G/B$ and define $\delta\colon \Delta \times \Delta \to W$ via $\delta(gB, hB) = w$ if and only if $Bg^{-1}hB = BwB$. Note that this is well-defined due to the Bruhat decomposition.

For a group G with BN-pair, the chamber $c := B \in G/B$ is called **fundamental chamber**.

The **rank** of a building of type (W, S) is defined to be $|S|$, its **dimension** is $|S| - 1$. A building is called **thick** or **thin**, if for all $s \in S$ and all chambers $c \in \Delta$ the **s-panel** $P_s(c) := \{d \in \Delta \mid \delta(c, d) \in \langle\, s \,\rangle\}$ has at least three elements or exactly two elements, respectively. A building is called **spherical** if (W, S) is spherical, or equivalently, W is finite. If Δ is spherical, then $c, d \in \Delta$ are called **opposite** if $\delta(c, d) = w_0$, the longest element of (W, S).

For a given chamber c and a spherical residue R there exists a unique chamber $d \in R$ such that $l(\delta(c, d)) = \min\{l(\delta(c, x)) \mid x \in R\}$, cf. [AB08, Proposition 5.34]. The chamber d is called the **projection** of c onto R and is denoted by $\mathrm{proj}_R(c)$.

Example 2.3.3. Let (W, S) be any Coxeter system. Then $\Delta := W$ and $\delta\colon \Delta \times \Delta \to W, (x, y) \mapsto x^{-1}y$ yields a thin building of type (W, S). Moreover, any thin building of type (W, S) is isometric to this building, cf. [AB08, Exercise 4.12].

Let Δ be a building of type (W, S). A subset of Δ which is isometric to the thin building described in Example 2.3.3 is called an **apartment** of Δ.

Example 2.3.4. Let $n \geq 2$ and let (W, S) be a Coxeter system of type A_{n-1}. As in Example 2.1.1, then $W \cong S_n$ and S is the generating set of transpositions. Let $\mathbb{F}$ be a field and let V be an n-dimensional $\mathbb{F}$-vector space.

Define $\Delta := \{(V_1, \dots, V_{n-1}) \mid \{0\} \subsetneq V_1 \subsetneq \dots \subsetneq V_{n-1} \subsetneq V\}$ to be the set of all maximal flags of non-trivial proper subspaces of V. Given i with $1 \leq i \leq n - 1$, we define

$\delta\colon \Delta \times \Delta \to S$ via $\delta((V_1, \ldots, V_{n-1}), (V'_1, \ldots, V'_{n-1})) := s_i$ if and only if for all $j \neq i$ the equality $V_j = V'_j$ holds. This defines an equivalence relation $\sim_i$ (for each i) on the set of maximal flags. Then by [AB08, Definition 4.25] this gives rise to a building of type (W, S).

Definition 2.3.5. A **twin building** of type (W, S) is a triple $((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$, where (Δ_+, δ_+) and (Δ_-, δ_-) are two buildings of the same type (W, S), and the **codistance** function $\delta^*\colon (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \to W$ satisfies the following conditions, where $\varepsilon \in \{+.-\}$, $x \in \Delta_\varepsilon$, $y \in \Delta_{-\varepsilon}$ and $\delta^*(x, y) = w$:

(Tw1) $\delta^*(y, x) = w^{-1}$,

(Tw2) if $z \in \Delta_{-\varepsilon}$ such that $\delta_{-\varepsilon}(y, z) = s \in S$, and $l(ws) < l(w)$, then $\delta^*(x, z) = ws$, and

(Tw3) if $s \in S$, then there exists $z \in \Delta_{-\varepsilon}$ such that $\delta_{-\varepsilon}(y, z) = s$ and $\delta^*(x, z) = ws$. □

Remark 2.3.6. Let G be a group with a twin BN-pair (B_+, B_-, N) and let (Δ_+, δ_+) and (Δ_-, δ_-) be the two buildings obtained from G with respect to B_+ and B_-, cf. Example 2.3.2. Define a map $\delta^*\colon (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \to W$ via $\delta^*(gB_\varepsilon, hB_{-\varepsilon}) = w$ if and only if $B_\varepsilon g^{-1} h B_{-\varepsilon} = B_\varepsilon w B_{-\varepsilon}$ in the Birkhoff decomposition of G.
Then $((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ is a twin building.

If G is a group with twin BN-pair, then by Remark 2.3.6 the chambers of the associated twin building are precisely the conjugates of the standard Borel subgroups B_+, B_-. Then the chambers $c_+ := B_+ \in G/B_+$ and $c_- := B_- \in G/B_-$ are called **fundamental chambers**. We shall use this description in the later chapters.

Definition 2.3.7. Two chambers $c \in \Delta_\varepsilon$ and $d \in \Delta_{-\varepsilon}$ are called **opposite** if $\delta^*(c, d) = 1$. Two residues are **opposite**, if they have the same type and contain opposite chambers. A **twin apartment** of Δ is a pair (Σ_+, Σ_-) such that both of Σ_+ and Σ_- is an apartment in its respective building, and additionally every chamber in $\Sigma_+ \cup \Sigma_-$ is opposite to precisely one other chamber of $\Sigma_+ \cup \Sigma_-$.
Given a spherical residue $R \subseteq \Delta_\varepsilon$ and a chamber $c \in \Delta_{-\varepsilon}$, there exists a unique chamber $d \in R$ such that $\delta^*(c, d)$ is of maximal length in the set $\delta^*(c, R)$, cf. [AB08, Lemma 5.149]. Then $d := \mathrm{proj}_R(c)$ is called the **projection** of c onto R. □

Let $\leq$ be the Bruhat order of the Weyl group W. In order to fix notation to be used in later chapters, for a given twin building $\Delta = (\Delta_+, \Delta_-, \delta^*)$ and a chamber $c_\varepsilon \in \Delta_\varepsilon$ we write

$$E_w(c_\varepsilon) := \{d \in \Delta_\varepsilon \mid \delta_\varepsilon(c_\varepsilon, d) = w \in W\},$$
$$E_{\leq w}(c_\varepsilon) := \{d \in \Delta_\varepsilon \mid \delta_\varepsilon(c_\varepsilon, d) \leq w \in W\},$$

and similarly,

$$E^*_w(c_\varepsilon) := \{d \in \Delta_{-\varepsilon} \mid \delta^*(c_\varepsilon, d) = w \in W\},$$
$$E^*_{\leq w}(c_\varepsilon) := \{d \in \Delta_{-\varepsilon} \mid \delta^*(c_\varepsilon, d) \leq w \in W\}.$$

The set $E_w(c_\varepsilon)$ is called a **Schubert cell**, the set $E^*_w(c_\varepsilon)$ is called **co-Schubert cell** of c_ε. Moreover, we set

$$\Delta^{\pm}_w := \{(c,d) \in \Delta_\varepsilon \times \Delta_{-\varepsilon} \mid \delta^*(c,d) = w\}.$$

2.4 Topological twin buildings

The following definition is taken from [Har06]. See also [Kra02].

Definition 2.4.1. Let $\Delta = (\Delta_+, \Delta_-, \delta^*)$ be a thick twin building of type (W, S). Let τ be a topology on the set $\Delta_+ \cup \Delta_-$. Then (Δ, τ) is called a **topological twin building** if the following axioms are satisfied.

(TTB1) The topology τ is Hausdorff.

(TTB2) Let $s \in S$. Then the restriction of the projection map

$$\begin{aligned} p_s \colon \Delta^{\pm}_1 &\to \Delta_\varepsilon \cup \Delta_{-\varepsilon}, \\ (c,d) &\mapsto \mathrm{proj}_{P_s(c)}(d) \end{aligned}$$

is continuous. Moreover, the set $\Delta^{\pm}_1 = \{(c,d) \in \Delta_\varepsilon \times \Delta_{-\varepsilon} \mid \delta^*(c,d) = 1\}$ of opposite chambers is open in $\Delta_\varepsilon \times \Delta_{-\varepsilon}$ with respect to the product topology.

(TTB3) There exist chambers c_+, c_- such that the two halves of Δ are the direct limits of the spaces $E_{\leq w}(c_\varepsilon)$ centred at c_+ and c_-, respectively, i.e.

$$\Delta_\varepsilon = \lim_{\to} E_{\leq w}(c_\varepsilon). \qquad \square$$

Remark 2.4.2. We do not require that panels are compact. This axiom is not necessary for the results presented here, in [Har06, Definition 3.1.1] this is required. In general, compactness of panels is used in [Har06] (see for example Section 3.2 of loc. cit.) to conclude that every continuous surjective map is a quotient map. We show in Lemma 5.1.9 that for twin buildings associated to split Kac–Moody groups, the panels are compact if the underlying Kac–Moody group is defined over a locally compact field.

Example 2.4.3. Let $\mathbb{F}$ be a Hausdorff topological field. Then the geometry of one-dimensional subspaces of a two-dimensional vector space over $\mathbb{F}$ is a (twin) building of rank one (cf. Example 2.3.4). Equipping it with the quotient topology induced from $\mathbb{F}^2$, we obtain a topological (twin) building. It is homeomorphic to the projective line $\mathbb{P}^1(\mathbb{F})$, hence compact if $\mathbb{F}$ is locally compact.

2.5 Kac–Moody groups

The main references for this section are [Tit87] and [Rém02]. See also [Tit92], [KP85] and [Kac90].

We first turn our attention to Kac–Moody Lie algebras.

Definition 2.5.1 ([Kac90, Sections 0.3 and 3.3], [Rém02, Definition 7.3.1], [KP83a, Section 1], [Kac85, Chapter 2], [KP85, Section 2], [KP83b, Section 1])**.** A **generalised Cartan matrix** is a matrix $A = (a_{ij})_{1 \leq i,j \leq n} \in \mathbb{Z}^{n \times n}$ satisfying $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$ and $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Let A be a generalised Cartan matrix. Following the construction in [Kac90, Section 1.2], let $\mathfrak{g}_A = \mathfrak{g}(A)$ be the associated complex Lie algebra and $\mathfrak{g} := \mathfrak{g}'_A$ its derived Kac–Moody algebra. By [Kac90, Theorem 9.11], $\mathfrak{g}$ admits a standard generating set $\{e_i, f_i, \alpha_i^\vee \mid i \in \{1, \ldots, n\}\}$ with the following defining relations, also known as **Serre's relations**:

$$\begin{aligned}
[e_i, f_j] &= \delta_{ij}\alpha_i^\vee, \\
[h, h'] &= 0, \\
[h, e_i] &= \alpha_i(h)e_i, \\
[h, f_i] &= -\alpha_i(h)f_i, \\
(\mathrm{ad}_{e_i})^{1-a_{ij}}(e_j) &= 0 \quad (i \neq j), \\
(\mathrm{ad}_{f_i})^{1-a_{ij}}(f_j) &= 0 \quad (i \neq j),
\end{aligned}$$

where $i, j \in \{1, \ldots, n\}$, $h, h' \in \mathfrak{h}$ ($\mathfrak{h}$ denotes the Cartan subalgebra of $\mathfrak{g}$). □

The algebra $\mathfrak{g}_A$ admits a root space decomposition $\mathfrak{g}_A = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, which in turn induces a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, cf. [Kac90, Section 1.3].

The set of roots Φ decomposes into the set Φ^{re} of real roots and the set Φ^{im} of imaginary roots. If α is a real root, then the root space $\mathfrak{g}_\alpha$ is integrable in the sense of [KP85, Section 2].

Let W be the Weyl group of the root system Φ. A **basis** of the real roots is a minimal set $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ (where n is the size of the generalised Cartan matrix) such that $\Phi^{re} = W.\Pi$. Note that the set of real roots is independent of the chosen basis, as by [Kac90, Proposition 5.9] any root basis is conjugate to the standard basis Π_0 or its opposite $-\Pi_0$ under W.

If $\{e_i, f_i, \alpha_i^\vee \mid i = 1, \ldots, n\}$ is the set of standard generators for $\mathfrak{g}$, then it follows from Serre's relations that the e_i and f_i are ad-locally nilpotent, i.e. for all $v \in \mathfrak{g}$ there exists $m \in \mathbb{N}$ (which may depend on the vector v chosen) we have that $\mathrm{ad}_{e_i}^m.v = \mathrm{ad}_{f_i}^m.v = 0$, cf. [Kac90, Sections 3.4 and 3.5]. Fix an index $i \in \{1, \ldots, n\}$, then the span of $\{e_i, f_i, \alpha_i^\vee\}$ is isomorphic to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of traceless 2×2-matrices and called a **fundamental rank one**

subalgebra. As $\mathfrak{h}$ acts diagonally, the elements of the fundamental $\mathfrak{sl}_2$-subalgebras of $\mathfrak{g}$ can be written as Lie algebra combinations of ad-locally nilpotent elements.

Let $I = \{1, \ldots, n\}$ and let $A = (a_{ij})_{1\leq i,j\leq n}$ be a generalised Cartan matrix. A quintuple $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ is called a **Kac–Moody root datum** if Λ is a free $\mathbb{Z}$-module, each c_i is an element of Λ and every h_i is in the $\mathbb{Z}$-dual of Λ, which we will denote by $\Lambda^\vee$, such that the following relation holds for all $i, j \in I$:

$$h_i(c_j) = a_{ij}.$$

Moreover, the root datum $\mathcal{D}$ is called **simply connected** if the set $\{h_i \mid i \in I\}$ is a $\mathbb{Z}$-basis of $\Lambda^\vee$. The root datum $\mathcal{D}$ is called **adjoint** if the set $\{c_i \mid i \in I\}$ is a $\mathbb{Z}$-basis of Λ.

Given a generalised Cartan matrix $A \in \mathbb{Z}^{n\times n}$, we obtain a simply connected Kac–Moody root datum as follows. Let $I := \{1, \ldots, n\}$, $\Lambda := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$, $c_i := \sum_{j\in I} a_{ji}\alpha_j$ and h_j be chosen such that $h_j(\alpha_i) = \delta_{ij}$. Then by construction $\{h_i \mid i \in I\}$ is a $\mathbb{Z}$-basis of $\Lambda^\vee$ and hence $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ is a simply connected Kac–Moody root datum.

Throughout this thesis, unless stated otherwise, r and s are elements of a ring R and u, v denote units of R.

Definition 2.5.2. Given a Kac–Moody root datum $\mathcal{D}$, a **basis** for a **Tits functor** $\mathcal{G}$ of type $\mathcal{D}$ is a triple $\mathcal{F} = (\mathcal{G}, \{\varphi_i\}_{i\in I}, \eta)$, where φ_i are maps $\mathrm{SL}_2(R) \to \mathcal{G}(R)$ and η is a natural transformation $\mathrm{Hom}(\Lambda, -^\times) \to \mathcal{G}$ which satisfy the following axioms:

(KMG1) If $\mathbb{F}$ is a field, then the group $\mathcal{G}(\mathbb{F})$ is generated by the images of the φ_i and $\eta(\mathbb{F})$.

(KMG2) For all rings R the homomorphism $\eta(R)\colon \mathrm{Hom}(\Lambda, R^\times) \to \mathcal{G}(R)$ is injective.

(KMG3) Given a ring R, $i \in I$ and $u \in R^\times$, it holds that $\varphi_i \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = \eta(\lambda \mapsto u^{h_i(\lambda)})$.

(KMG4) If R is a ring, $\mathbb{F}$ is a field and $\iota\colon R \to \mathbb{F}$ is an injection, then $\mathcal{G}(\iota)\colon \mathcal{G}(R) \to \mathcal{G}(\mathbb{F})$ is injective as well.

(KMG5) If $\mathfrak{g}_A$ is the Kac-Moody algebra of type A over the field of complex numbers, then there exists a homomorphism $\mathrm{Ad}\colon \mathcal{G}(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{g}_A)$ such that $\ker(\mathrm{Ad}) \subseteq \eta(\mathrm{Hom}(\Lambda, \mathbb{C}^\times))$ and for a given $z \in \mathbb{C}$:

$$\mathrm{Ad}\left(\varphi_i \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) = \exp(\mathrm{ad}_{ze_i}),$$
$$\mathrm{Ad}\left(\varphi_i \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right) = \exp(\mathrm{ad}_{-zf_i});$$

and for every homomorphism $\gamma \in \mathrm{Hom}(\Lambda, \mathbb{C}^\times)$ it holds that

$$\mathrm{Ad}(\eta(\gamma))(e_i) = \gamma(c_i)e_i, \quad \mathrm{Ad}(\eta(\gamma))(f_i) = \gamma(-c_i)f_i.$$

The value $\mathcal{G}(R) = G_{\mathcal{D}}(R)$ of a Tits functor with Kac–Moody root datum $\mathcal{D}$ at a ring R is called **split Kac–Moody group** of type $\mathcal{D}$ over R, see [Tit87, Section 3.6], [Rém02, Chapter 8] for details. $\square$

Throughout this thesis, all Kac–Moody groups are assumed to be split.

The main result of [Tit87] states that any functor defined on the category of fields satisfying the above axioms must coincide with $\mathcal{G}$ (under some non-degeneracy assumptions on the images of the maps φ_i; see [Tit87, Theorem 1] for a precise statement).

The image $\eta(\mathrm{Hom}(\Lambda, R^\times))$ then is the **standard maximal torus**. If $\mathbb{F}$ is a field, then $T_{\mathbb{F}} := \eta(\mathrm{Hom}(\Lambda, \mathbb{F}^\times)) \leq G_{\mathcal{D}}(\mathbb{F})$ is called **standard maximal $\mathbb{F}$-torus**. Its conjugates are called **maximal $\mathbb{F}$-tori**.

The Kac–Moody root datum $\mathcal{D}$ is called **centred** if the following assumption is satisfied, which clearly is stronger than (KMG1).

(KMG1') If $\mathbb{F}$ is a field, then the group $\mathcal{G}(\mathbb{F})$ is generated by the images of the φ_i.

It follows that if $\mathcal{D}$ is centred, then $G_{\mathcal{D}}(R) = \langle\, U_\alpha \mid \alpha \in \Phi \,\rangle$.

Example 2.5.3. Let $\mathcal{D}$ be a simply connected centred Kac–Moody root datum of type A_{n-1}. Then the corresponding Kac–Moody group coincides with the group $\mathrm{SL}_n(R)$, cf. Example 2.1.1. If $\mathcal{D}$ is adjoint, then the Kac–Moody group obtained is $\mathrm{PSL}_n(R)$.

Definition 2.5.4. The **Steinberg group** $\mathrm{St}_A(R)$ of type A over a ring R (note that the definition depends only on the matrix A, not on the whole root datum $\mathcal{D}$) is the group generated by symbols $x_\alpha(r)$ for all real roots α in the root system of A and $r \in R$ with the defining relations:

(A) x_α is additive, i.e. $x_\alpha(r)x_\alpha(s) = x_\alpha(r+s)$,

(B) if $\{\alpha, \beta\}$ a prenilpotent pair of roots, then $[x_\alpha(r), x_\beta(s)] = \prod_{i,j>0} x_{i\alpha+j\beta}(c_{\alpha\beta ij} r^i s^j)$.

The structure constants $c_{\alpha\beta ij}$ in relation (B) are integers which depend on i, j, α and β, but not on r and s. By [Tit87, Remark 3.7 (f)] they are computable once the roots in the Kac–Moody algebra have been explicitly chosen. $\square$

By [Mor87, Theorem 1] the number $c_{\alpha\beta 11}$ depends on the α-string through β, i.e. the roots of the form $\beta - p\alpha, \ldots, \beta, \ldots, \beta + q\alpha$. Assume that $\alpha + \beta \in \Phi$ is a root and let $p, q \in \mathbb{N}_0$ be maximal such that $\beta - p\alpha, \ldots, \beta, \ldots, \beta + q\alpha$ are roots. Then $c_{\alpha\beta 11} = \pm(p+1)$.

In fact, given a prenilpotent pair of roots $\{\alpha, \beta\}$ and assuming that $\mathbb{N}\alpha + \mathbb{N}\beta$ does not contain an imaginary root, then by [Mor87, Theorem 2] the structure constants with respect to the standard Chevalley basis are given as follows, where $Q_{\alpha\beta} := (\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Phi$ and we assume without loss of generality that β is not shorter than α:

- If $Q_{\alpha\beta} = \emptyset$, then $x_\alpha(r)$ and $x_\beta(s)$ commute.
- If $Q_{\alpha\beta} = \{\alpha + \beta\}$, then $[x_\alpha(r), x_\beta(s)] = x_{\alpha+\beta}(\pm mrs)$, where m is the largest natural number such that $\beta - m\alpha$ is a root.

- If $Q_{\alpha\beta} = \{\alpha+\beta, 2\alpha+\beta\}$, then $[x_\alpha(r), x_\beta(s)] = x_{\alpha+\beta}(\pm rs)x_{2\alpha+\beta}(\pm r^2 s)$.
- If $Q_{\alpha\beta} = \{\alpha+\beta, 2\alpha+\beta, \alpha+2\beta\}$, then
$$[x_\alpha(r), x_\beta(s)] = x_{\alpha+\beta}(\pm 2rs)x_{2\alpha+\beta}(\pm 3r^2 s)x_{\alpha+2\beta}(\pm 3rs^2).$$
- If $Q_{\alpha\beta} = \{\alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$, then
$$[x_\alpha(r), x_\beta(s)] = x_{\alpha+\beta}(\pm rs)x_{2\alpha+\beta}(\pm r^2 s)x_{3\alpha+\beta}(\pm rs^2)x_{3\alpha+2\beta}(\pm 2r^3 s^2).$$

Example 2.5.5 ([Tit87, Section 3.5])**.** This example shows that the constant m as above may take any value a priori. Let R be the field of complex numbers, let $n \in \mathbb{N}$ and define the generalised Cartan matrix $A := \begin{pmatrix} 2 & -1 \\ -n & 2 \end{pmatrix}$. Denote by α_0, α_1 the simple roots and define $\alpha := \alpha_0 + \alpha_1 = s_{\alpha_0}(\alpha_1)$. We put $\Psi := \{\alpha_0, -\alpha_1, \alpha\}$. Then Ψ is a prenilpotent pair of roots, as $s_{\alpha_1}(\Psi) = \{\alpha_1, \alpha_0 + (n-1)\alpha_1, \alpha_0 + n\alpha_1\}$ and $s_{\alpha_1}s_{\alpha_0}(\Psi) = s_{\alpha_1}(-\Psi) = -s_{\alpha_1}(\Psi)$.

If e_α, f_α and h_α denote the corresponding generators of the rank one subalgebras $\mathfrak{g}(\alpha)$ of the Kac–Moody algebra, then we obtain $[e_{\alpha_0}, e_{\alpha_1}] = e_\alpha$. It follows that
$$[f_{\alpha_1}, e_\alpha] = -[f_{\alpha_1}, [e_{\alpha_1}, e_{\alpha_0}]] = -[h_{\alpha_1}, e_{\alpha_0}] = ne_{\alpha_0}.$$
Hence we conclude that in the group U_Ψ the commutator relation $[x_{-\alpha_1}(r_1), x_\alpha(r_2)] = x_{\alpha_0}(nr_1r_2)$ holds.

Using matrix calculations in the group $\mathrm{SL}_2(R)$, we see that with respect to $s_\alpha(u) := x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u)$ the identity $s_\alpha(u)x_\alpha(r)s_\alpha(u)^{-1} = x_\alpha(-u^{-2}r)$ holds. In general, when conjugating $x_\alpha(r)$ with $s_\beta(u)$, a different sign in the argument may occur, depending on the roots α and β.

Remark 2.5.6 ([Mat69, Lemme 5.1 (c)])**.** The sign $\varepsilon = \varepsilon_{\alpha,\beta}$ satisfies the following identities:
$$\begin{aligned}
\varepsilon_{\alpha,\alpha} = \varepsilon_{\alpha,-\alpha} &= -1, \\
\varepsilon_{\alpha,\beta} &= \varepsilon_{\alpha,-\beta}, \\
\varepsilon_{\alpha,\beta} &= 1 \quad \text{if} \quad \alpha \pm \beta \neq 0, \alpha \pm \beta \notin \Phi, \\
\varepsilon_{\alpha,\beta}\varepsilon_{\beta,\alpha} &= -1 \quad \text{if} \quad \alpha(\beta^*)\beta(\alpha^*) = -1, \\
\varepsilon_{\alpha,\beta} &= -1 \quad \text{if} \quad \alpha(\beta^*) = 1, \alpha \pm \beta \in \Phi.
\end{aligned}$$

Proposition 2.5.7. *Let A be a generalised Cartan matrix and let $\mathcal{D}$ be centred. Then the associated simply connected split Kac–Moody group $G_{\mathcal{D}^{sc}}(R)$ over R is a quotient of the Steinberg group $\mathrm{St}_A(R)$ with the additional relations*

(B') $s_\alpha(u)x_\beta(r)s_\alpha(-u) = x_{s_\beta(\alpha)}(\varepsilon u^{-\alpha(\beta^*)}r)$ *(where $s_\beta(\alpha)$ denotes the image of α under the reflection associated to the root β and ε is a sign as in Remark 2.5.6),*

(C) $h_\alpha(u)h_\alpha(v) = h_\alpha(uv)$,

where $h_\alpha(u) = s_\alpha(u)s_\alpha(-1)$, $u \in R^\times$ *and* $s_\alpha(u) = x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u)$.

Proof. Using the defining relations, one deduces that in the group G given by relations (A), (B), (B') and (C) the identity

$$h_{\alpha_i}(v)x_{\alpha_j}(r)h_{\alpha_i}(v)^{-1} = x_{\alpha_j}(v^{h_{\alpha_i}(\alpha_j)}r)$$

holds. Hence we may choose $c_i := \sum_{j\in I} a_{ji}\alpha_j$ and define the root lattice $\Lambda := \bigoplus_{i\in I}\mathbb{Z}\alpha_i$. We also define elements h_j in the coroot lattice such that $h_j(\alpha_i) = \delta_{ij}$. Then by construction the relation $h_j(c_i) = a_{ji}$ holds.

In particular, the set $\{h_j\}$ is a $\mathbb{Z}$-basis for $\Lambda^\vee$, the $\mathbb{Z}$-dual of Λ, and hence G is a simply connected Kac–Moody group of type A. We therefore conclude that $G = G_{\mathcal{D}^{sc}}(R)$. $\square$

Note that the special case of relation (B') for $\alpha = \beta$ is

$$s_\alpha(u)x_\alpha(r)s_\alpha(-u) = x_{-\alpha}(-u^{-2}r), \tag{2.1}$$

as can be obtained by matrix calculations in $\mathrm{SL}_2(R)$ as described above.

Using this approach, we see immediately that a simply connected centred split Kac–Moody group of a given type A over R is a quotient of the Steinberg group of the same type over R.

However, here is an alternative description of a Kac–Moody group with generators and relations.

Theorem 2.5.8 ([Tit87, Section 3.6], [Rém02, Definition 8.3.3])**.** *Let R be a ring and let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a Kac–Moody root datum. Define an action of* $\mathrm{Hom}(\Lambda, R^\times)$ *on* $\mathrm{St}_A(R)$ *via*

(i) $tx_{\alpha_i}(r)t^{-1} := x_{\alpha_i}(t(\alpha_i)r)$.

Then the associated split Kac–Moody group $G_{\mathcal{D}}(R)$ is the quotient of the semidirect product $\mathrm{St}_A(R) \rtimes \mathrm{Hom}(\Lambda, R^\times)$ *modulo the normal closure of the following elements, where* $t \in \mathrm{Hom}(\Lambda, R^\times), r \in R, u \in R^\times, \alpha_i \in \Pi, s_{\alpha_i}(u) := x_{\alpha_i}(u)x_{-\alpha_i}(-u^{-1})x_{\alpha_i}(u)$ *and* $s_{\alpha_i} := s_{\alpha_i}(1)$*:*

(ii) $s_{\alpha_i}ts_{\alpha_i}^{-1}s_{\alpha_i}(t)^{-1}$, *where $s_{\alpha_i}(t)$ denotes the image of t under the action of $s_{\alpha_i} \in W$,*

(iii) $s_{\alpha_i}s_{\alpha_i}(u)^{-1}u^{h_{\alpha_i}}$, *where $u^{h_{\alpha_i}}$ denotes the element $\lambda \mapsto u^{h_{\alpha_i}(\lambda)}$ of* $\mathrm{Hom}(\Lambda, R^\times)$ *and $h_{\alpha_i} = h_i$ is the associated element of the underlying Kac–Moody root datum $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$,*

(iv) $s_{\alpha_i}x_\alpha(r)s_{\alpha_i}^{-1}x_{s_{\alpha_i}(\alpha)}(\varepsilon r)^{-1}$, *where ε is as in Remark 2.5.6.*

In particular, if $\mathcal{D}$ is simply connected, then relations (i), (ii), (iii) and (iv) above are equivalent to relations (A), (B), (B') and (C) of Definition 2.5.4 and Proposition 2.5.7. $\square$

Note that the elements in Theorem 2.5.8 (iii) are slightly different compared to [Tit87, Section 3.6], see [Rém02, Remarque 8.3.3]. Also note that by (iii) the group $N = W \ltimes T$ decomposes as semidirect product, cf. [Kum02, Corollary 6.1.8].

Remark 2.5.9. At this point, one should remark the following about the standard torus $T = \eta(R)$. By the construction in Definition 2.5.2 (KMG3), the torus T is identified with the images of diagonal matrices in $\mathrm{SL}_2(R)$ via $\varphi_i \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = \eta(\lambda \mapsto u^{h_i(\lambda)})$. Hence T coincides with the group $\mathrm{Hom}(\Lambda, R^\times)$. The latter group is isomorphic to $(R^\times)^n$, where n is the rank of Λ. This identification is in fact $G_{\mathcal{D}}(R)$-equivariant. Hence we may consider $T \cong \mathrm{Hom}(\Lambda, R^\times) \cong (R^\times)^n$ as appropriate.

Proposition 2.5.10. *Let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a centred Kac–Moody root datum and consider the simply connected root datum $\mathcal{D}^{sc}$ associated to the matrix A. Then the group $G_{\mathcal{D}}(R)$ is a quotient of $G_{\mathcal{D}^{sc}}(R)$ such that the kernel of the canonical morphism $G_{\mathcal{D}^{sc}}(R) \to G_{\mathcal{D}}(R)$ is central.*

Moreover, the adjoint group $G_{\mathcal{D}^{ad}}(R)$ is obtained as split Kac–Moody group as in Definition 2.5.2 with respect to the root datum $\mathcal{D}^{ad} = (I, A, \Lambda', \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$, where $\Lambda' = \bigoplus_{i\in I} \mathbb{Z}.c_i$.

Proof. Let $\mathcal{D}^{sc} = (I, A, \Lambda^{sc}, \{c_i^{sc}\}_{i\in I}, \{h_i^{sc}\}_{i\in I})$ be the simply connected root datum with respect to A. Since $\mathrm{Hom}(\Lambda^{sc}, R^\times) = \langle h_i^{sc} \rangle$, by $\mathbb{Z}$-linearity the unique morphism $h_i^{sc} \mapsto h_i$ extends to a morphism $\mathrm{Hom}(\Lambda^{sc}, R^\times) \to \mathrm{Hom}(\Lambda, R^\times)$. This induces a morphism $\varphi\colon \mathrm{St}_A(R) \rtimes \mathrm{Hom}(\Lambda^{sc}, R^\times) \to \mathrm{St}_A(R) \rtimes \mathrm{Hom}(\Lambda, R^\times)$ and hence a homomorphism $\varphi\colon G_{\mathcal{D}^{sc}}(R) \to G_{\mathcal{D}}(R)$. Now for a simple root α_j and $t = \sum_{i\in I} z_i h_i^{sc} \in \ker(\varphi)$ it follows that:

$$\begin{aligned} t x_{\alpha_j}(r) t^{-1} &= x_{\alpha_j}(t(\alpha_j) r) \\ &= x_{\alpha_j}(\sum_{i\in I} z_i h_i^{sc}(\alpha_j) r) \\ &= x_{\alpha_j}(\sum_{i\in I} z_i h_i(\alpha_j) r) \\ &= x_{\alpha_j}(\varphi(t)(\alpha_j) r) \\ &= x_{\alpha_j}(r). \end{aligned}$$

Hence $\ker(\varphi)$ is central. $\square$

In particular, Proposition 2.5.10 implies that the class of centred Kac–Moody root data with respect to a fixed generalised Cartan matrix A has an *initial* and a *terminal* object, namely the simply connected and the adjoint root datum, respectively. The adjoint Kac–Moody group is obtained by taking the quotient of the simply connected Kac–Moody group modulo its centre.

Proposition 2.5.11 ([Rém02, Proposition 8.4.1], [Cap09, Lemma 1.4])**.** *Let $\mathbb{F}$ be a field and let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a Kac–Moody root datum. Let $G_{\mathcal{D}}(\mathbb{F}) := \mathcal{G}(\mathbb{F})$ be the split Kac–Moody group of type $\mathcal{D}$ over $\mathbb{F}$.*

Then $G_{\mathcal{D}}(\mathbb{F})$ admits an RGD system as follows. Let $M(A)$ be the associated Coxeter matrix of type (W, S) and choose a set of simple roots $\Pi = \{\alpha_i \mid i \in I\}$ such that the reflection associated to α_i is s_i. Given $i \in I$, let U_{α_i} and $U_{-\alpha_i}$ be the image of strict upper or lower triangular matrices of $\mathrm{SL}_2(R)$ under the map φ_i. Define $T := \bigcap_{\alpha\in\Phi^{re}} N_{G_{\mathcal{D}}(\mathbb{F})}(U_\alpha)$. Then $(G_{\mathcal{D}}(\mathbb{F}), \{U_\alpha\}_{\alpha\in\Phi^{re}}, T)$ is an RGD system and $W \cong N_{G_{\mathcal{D}}(\mathbb{F})}(T)/T$. □

By Proposition 2.5.11 it follows that Kac–Moody groups defined over fields are examples of groups with an RGD system. Note that if $G_{\mathcal{D}}(R)$ is a split Kac–Moody group defined over a ring R which is not a field, then the statement of Proposition 2.5.11 becomes false.

2.6 The adjoint representation of a Kac–Moody group

Many of the results obtained for Kac–Moody groups use the adjoint representation on the Kac–Moody algebra. However, in characteristic $p > 0$ this tool is not as useful, because there is no exponential series which links the group to the algebra. In order to solve this problem, B. Rémy has shown in [Rém02, Chapter 9] that there is an adjoint representation on the universal enveloping algebra of the Kac–Moody algebra which may also be used in positive characteristic.

Recall that the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the associative algebra $\left(\bigotimes_{n\in\mathbb{N}} \mathfrak{g}^n\right)/I$, where $I := \langle\, [x,y] - xy + yx \,\rangle$. Let R be a ring, let $\mathfrak{g}$ be the complex Kac–Moody algebra associated to the generalised Cartan matrix A and let $\mathcal{U} := \mathcal{U}(\mathfrak{g})$ be its universal enveloping algebra, cf. [Rém02, Section 7.3.1]. For each $u \in \mathcal{U}$, we let $u^{[n]} := (n!)^{-1}u^n$ and we define $\binom{u}{n} := (n!)^{-1}u(u-1)\cdots(u-n+1)$.

Let $Q := \sum_{\alpha\in\Pi} \mathbb{Z}\alpha$ be the root lattice (or free abelian group) generated by the simple roots. Then, as in [Rém02, Section 7.3.1], $\mathcal{U}$ and $\mathfrak{g}$ admit an abstract Q-grading by declaring e_i and f_i to be of degree α_i and $-\alpha_i$, respectively, and extending linearly to $\mathcal{U}$. The elements of $\mathfrak{h}$ have degree 0. As example, the element $e_i f_j \alpha_k^\vee$ has degree $\alpha_i - \alpha_j$.

With this notation, we set $\mathcal{U}_0$ to be the subring of $\mathcal{U}$ generated by the degree 0-elements of the form $\binom{h}{n}$, where $h \in \mathfrak{h}, n \in \mathbb{N}$. Moreover, we define $\mathcal{U}_{\alpha_i}$ and $\mathcal{U}_{-\alpha_i}$ to be the subrings $\sum_{n\in\mathbb{N}} \mathbb{Z}e_i^{[n]}$ and $\sum_{n\in\mathbb{N}} \mathbb{Z}f_i^{[n]}$, respectively.

Definition 2.6.1 ([Rém02, Section 7.4])**.** Let $\mathcal{U}_{\mathbb{Z}}$ be the subring of $\mathcal{U}$ generated by $\mathcal{U}_0$ and $\{\mathcal{U}_{\pm\alpha} \mid \alpha \in \Pi\}$. Then $\mathcal{U}_{\mathbb{Z}}$ is a $\mathbb{Z}$-form of $\mathcal{U}$, i.e. the natural map $\mathcal{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \to \mathcal{U}$ is a bijection (cf. [Tit87, Section 4], [Rém02, Proposition 7.4.3]). This construction allows to replace the field $\mathbb{C}$ with some ring R and we will write $\mathcal{U}_R := \mathcal{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$. Since $\mathcal{U}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module, it follows that $\mathcal{U}_R$ is a free R-module. □

In order to shorten notation, for arbitrary real roots β we denote $x_\beta(r) := \varphi_\beta \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ and $x_{-\beta}(r) := \varphi_\beta \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$, then we have that $x_\beta(R) = U_\beta$ and $x_{-\beta}(R) = U_{-\beta}$. We let

$\mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_R)$ be the automorphism group of $\mathcal{U}_R$ which preserves the above Q-grading.

Proposition 2.6.2 ([Rém02, Proposition 9.5.2]). *Let $G_{\mathcal{D}}(R)$ be a split Kac–Moody group over a ring R and let T denote its standard maximal torus. Then there exists a morphism of groups*

$$\mathrm{Ad}\colon G_{\mathcal{D}}(R) \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_R)$$

which is characterised by the following axioms, where α_i is a real root, $r \in R$ and $h \in T$:

(i) $\mathrm{Ad}(x_{\alpha_i}(r)) = \exp(\mathrm{ad}_{e_i} \otimes r) = \sum_{n=0}^{\infty} \frac{(\mathrm{ad}_{e_i})^n}{n!} \otimes r^n$,

(ii) $\mathrm{Ad}(T)$ *fixes* $\mathcal{U}_0$,

(iii) $\mathrm{Ad}(h)(e_i \otimes r) = h^*(\alpha_i^\vee)(e_i \otimes r)$. □

In general, the adjoint representation Ad is not faithful. We shall see now that its kernel coincides with the centre of $G_{\mathcal{D}}(R)$, justifying the name *adjoint* representation.

Proposition 2.6.3. *Let $\mathcal{D}$ be non-spherical and let $G_{\mathcal{D}^{ad}}(R)$ be the adjoint split Kac–Moody group type of type $\mathcal{D}$ over R.*

Then $\mathrm{Ad}\colon G_{\mathcal{D}^{ad}}(R) \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_R)$ *is injective.*

Proof. It suffices to show this for $G_{\mathcal{D}^{ad}}(\mathbb{Z})$, as $\mathcal{G} = G_{\mathcal{D}}$ is a $\mathbb{Z}$-functor and Ad is a natural transformation, cf. [Rém02, Theorem 9.5.3]. It then follows that $\ker(\mathrm{Ad})$ is a functor and coincides with the trivial functor $R \mapsto \{1\}$ mapping a ring to the trivial subgroup of the adjoint Kac–Moody group.

By [Rém02, Proposition 9.6.2], $\mathrm{Ad}\colon G_{\mathcal{D}^{ad}}(\mathbb{Q}) \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathbb{Q}})$ is injective. Hence $\mathrm{Ad}\colon G_{\mathcal{D}^{ad}}(\mathbb{Z}) \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathbb{Q}})$ is injective as well.

Let $1 \neq x \in G_{\mathcal{D}^{ad}}(\mathbb{Z})$. Since $\mathrm{Ad}(x) \neq 1$, there exists a vector $v \in \mathcal{U}_{\mathbb{Q}}$ such that $\mathrm{Ad}(x).v \neq v$. Write $v = \sum_{\alpha \in Q} v_\alpha \otimes q_\alpha$, where $k \in \mathbb{N}$, $q_\alpha \in \mathbb{Q}$ and the sum has only finitely many non-zero terms. Put n to be the least common multiple of the denominators of the q_α. Then we have that $nv \in \mathcal{U}_{\mathbb{Z}}$, and moreover, that

$$\mathrm{Ad}(x).(nv) = n(\mathrm{Ad}(x).v) \neq nv.$$

Hence $1 \neq \mathrm{Ad}(x) \in \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathbb{Z}})$. □

The fact that $\ker(\mathrm{Ad}) = Z(G_{\mathcal{D}}(R))$ is now an easy consequence.

Corollary 2.6.4. *Let R be a ring. Then* $\ker(\mathrm{Ad}) = Z(G_{\mathcal{D}}(R))$, *the centre of* $G_{\mathcal{D}}(R)$.

Proof. By Proposition 2.6.3, $\ker(\mathrm{Ad}) \subseteq Z(G_{\mathcal{D}}(R)$. Conversely, let $x \in Z(G_{\mathcal{D}}(R)) \subseteq T$. Then $x(\alpha) = 1$ for all roots α, hence by Proposition 2.6.2 (iii) we conclude that $\mathrm{Ad}(x)(e_\alpha \otimes r) = e_\alpha \otimes r$ and $\mathrm{Ad}(x).(f_\alpha \otimes r) = f_\alpha \otimes r$ for all e_α and f_α, hence $\mathrm{Ad}(x) = 1$. The claim follows. □

Corollary 2.6.5. *Let $\mathcal{D}$ be a Kac–Moody root datum. Then the image of* $\mathrm{Ad}\colon G_{\mathcal{D}}(R) \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_R)$ *is isomorphic to $G_{\mathcal{D}^{ad}}(R)$, the adjoint Kac–Moody group of type $\mathcal{D}$.* □

2.7 Group actions on buildings

References for this section are provided by [AB08] and [Ron09].

A group G **acts** on a twin building $\Delta = ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta^*)$ if it acts on both halves simultaneously, preserving the distances and the codistance. The action is called **strongly transitive** if it is transitive on the set $\Delta_1^{\pm}$ of pairs of opposite chambers, or equivalently, if G acts transitively on ordered pairs (c, Σ), where Σ is an apartment and $c \in \Sigma$ is a chamber. If G is a group acting strongly transitively on a thick twin building, then by [AB08, Corollary 6.79] G admits a twin BN-pair.

Conversely, as in Remark 2.3.6, to every group G with a twin BN-pair there is associated a canonical twin building $\Delta = (G/B_+, G/B_-, \delta^*)$ on which G acts. Applying Proposition 2.2.1, we see that to every group G with an RGD system (hence in particular to Kac–Moody groups), there exists a twin building, called the **twin building associated to** G.

Example 2.7.1. As in Example 2.3.4, let Δ be the spherical building of type A_{n-1} associated to an n-dimensional vector space V over a field $\mathbb{F}$. Recall that Δ is obtained by taking the flag complex of maximal flags of proper non-trivial subspaces of V. Then the group $\mathrm{SL}_n(\mathbb{F})$ acts strongly transitively on the building Δ, as it is transitive on the set of all frames and the stabiliser of a frame $\mathcal{F}$ is transitive on all maximal flags obtained by using elements of $\mathcal{F}$.

Lemma 2.7.2. *Let $G_{\mathcal{D}}(\mathbb{F})$ be a split Kac–Moody group and let $\Delta = \Delta(G_{\mathcal{D}}(\mathbb{F}))$ be its associated twin building. Let $c \in \Delta_+$ be a chamber and let B_ε be the Borel subgroups associated to the fundamental chambers c_ε of Δ_ε. Assume that $\delta^*(c, c_-) = w$. Then $B_-.c = B_-wB_+$.*

In particular, $B_-.c$ is exactly a double coset of the Birkhoff decomposition and every such double coset arises as a B_--orbit.

Proof. By [AB08, Lemma 6.70], B_- acts transitively on the chambers at codistance w from c_+. Using Remark 2.3.6, we see that the set of chambers at codistance w from c_+ is exactly B_-wB_+. □

We also note that the sets $\{x \in \Delta_+ \mid \delta^*(x, c_-) = w \in W\}$ yield a partition of Δ_+ with respect to the Weyl group.

Definition 2.7.3. Let $c \in \Delta_\varepsilon$ be a chamber and let Σ be a twin apartment of Δ containing c. Then the map $\rho = \rho_{c,\Sigma} \colon \Delta \to \Sigma$ which fixes c pointwise and maps every apartment containing c isometrically onto Σ is called the **retraction** (onto Σ) centred at c. □

Note that $\rho = \rho_{c,\Sigma}$ does not depend on the apartment system chosen, cf. [AB08, Proposition 4.39]. Since every two chambers are contained in a common apartment, it is easily verified that ρ preserves distances from c. Moreover, ρ is distance-decreasing, i.e. $\delta(\rho(d), \rho(e)) \leq \delta(d, e)$ for any two chambers d, e.

Lemma 2.7.4. *Let $c \in \Delta_\varepsilon$, $d \in \Delta_{-\varepsilon}$ be chambers and let $g \in G$, where G is a group acting strongly transitively on Δ. Assume that $\delta^*(d,c) = w$ and $\delta_{-\varepsilon}(g.d, d) = v$.*

Then $\delta^*(g.d, c) = v'w$, where v' is a subexpression of v.

Proof. Let $\rho = \rho_{c,\Sigma}$ be the retraction map for some twin apartment Σ containing c. Then $\delta^*(\rho(d), \rho(c)) = \delta^*(\rho(d), c) = w = \delta^*(d, c)$ as ρ preserves distances from c and fixes c. Since ρ is (Weyl-)distance-decreasing, it follows that $\delta_{-\varepsilon}(\rho(g.d), \rho(d)) \leq v$. We obtain

$$\delta^*(g.d, c) = \delta_{-\varepsilon}(\rho(g.d), \rho(d))\delta^*(\rho(d), c) = \delta_{-\varepsilon}(\rho(g.d), \rho(d))w,$$

and hence $\delta^*(g.d, c) \in \{v_0 w \mid v_0 \leq v\}$. $\square$

2.8 Topology

We shall be working in the category of k_ω-spaces. A survey on the topic of k_ω-spaces can be found in [FT77], see also [GGH10] for a treatment of k_ω-groups.

Definition 2.8.1. Let X be a Hausdorff topological space. Then X is said to be a **k_ω-space** if there exists a countable ascending sequence $K_1 \subseteq K_2 \subseteq \ldots \subseteq X$ of compact sets such that $X = \bigcup_{n\in\mathbb{N}} K_n$ and $U \subseteq X$ is open if and only if $U \cap K_n$ is open in K_n for each n in the induced topology.

A **k_ω-group**, **k_ω-ring** or **k_ω-field** is a topological group, ring or field, respectively, whose underlying topological space is a k_ω-space. $\square$

Note that a k_ω-space is in particular a k-space, cf. [Eng77, Section 3.3].

Example 2.8.2. The category of k_ω-spaces contains

- every compact Hausdorff space,
- the ring of adèles $\mathbb{A}_\mathbb{F}$ of a global field $\mathbb{F}$,
- all countable spaces with the discrete topology,
- more generally, any σ-compact locally compact Hausdorff space.

The last point includes the real and complex numbers $\mathbb{R}$, $\mathbb{C}$ with their natural topology induced by the absolute value, the p-adic fields $\mathbb{Q}_p$ (and finite extensions of them) with the topology induced by the p-adic valuation, the fields $\mathbb{F}_q((t))$ of formal Laurent series with finite residue field with the topology induced from the t-adic valuation. Note that a k_ω-space is not necessarily locally compact, cf. [FT77, Page 116].

Even countable Hausdorff spaces may fail to be a k_ω-space. An example of a countable space which is not k_ω is provided by the rationals $\mathbb{Q}$ with their order topology.

Proposition 2.8.3 ([GGH10, Proposition 4.2])**.** *The following hold:*

(i) Closed subspaces of k_ω-spaces are k_ω.

(ii) Finite products of k_ω-spaces are k_ω.

(iii) Hausdorff quotients of k_ω-spaces are k_ω.

(iv) Countable disjoint unions of k_ω-spaces are k_ω. □

The following definition is motivated by V. Kac and D. Peterson ([KP83a, Remark (iii)] or [KP83b, Section 4G]), who first introduced a group topology on complex and real split Kac–Moody groups. Let $\mathbb{F}$ be a topological field. We equip the group $\mathrm{SL}_2(\mathbb{F})$ with the subspace topology coming from the product topology on $\mathbb{F}^{2\times 2}$, the 2×2-matrices over $\mathbb{F}$, which turns $\mathrm{SL}_2(\mathbb{F})$ into a topological group. Moreover, we let $\eta\colon (\mathbb{F}^\times)^n \to T$ be the embedding of the standard torus provided by Definition 2.5.2, where $(\mathbb{F}^\times)^n$ carries the topology induced from $\mathbb{F}$.

Definition 2.8.4 (Kac–Peterson topology)**.** Let $G_{\mathcal{D}}(\mathbb{F})$ be a split Kac–Moody group over a topological field $\mathbb{F}$. Then the **Kac–Peterson topology** on $G_{\mathcal{D}}(\mathbb{F})$ is defined to be the final group topology for the maps η and φ_α, where $\alpha \in \Phi^{re}$. In other words, the Kac–Peterson topology on $G_{\mathcal{D}}(\mathbb{F})$ is the finest topology such that each of the group morphisms η and φ_α is continuous and $G_{\mathcal{D}}(\mathbb{F})$ is a (not necessarily Hausdorff) topological group. We denote the Kac–Peterson topology by τ_{KP}. □

Note that the Kac–Peterson topology is well-defined, since the images of η and φ_α generate $G_{\mathcal{D}}(\mathbb{F})$, see also [GGH10, Section 1 or Proposition 5.8].

Remark 2.8.5. (i) We will see in Proposition 3.1.8 and Theorem 3.1.11 that the Kac–Peterson topology on any split Kac–Moody group over a k_ω-field is indeed Hausdorff, however, this fact requires some work.

(ii) The above description is different from the original definition Kac and Peterson gave (see Definition 3.2.1). We will see in Proposition 3.2.4 that the two definitions are equivalent.

Example 2.8.6. Assume that $\mathbb{F}$ is a local field and that $\mathcal{D}^{sc}$ is a simply connected spherical root datum. Then $G_{\mathcal{D}^{sc}}(\mathbb{F})$ has the algebraic structure of a finite-dimensional simply connected semisimple Lie group over $\mathbb{F}$. Moreover, the Kac–Peterson topology on $G_{\mathcal{D}^{sc}}(\mathbb{F})$ coincides with the Lie group topology coming from $\mathbb{F}$, see Remark 3.1.13.

2.9 Flips

Let $\mathbb{F}$ be a field. In the category of algebraic $\mathbb{F}$-groups, there is a deep theory concerning algebraic involutions defined over $\mathbb{F}$, see for example [RS90], [HW93], [Hel88], [Hel91], [Hel97], [HS01] or [Ric82]. Recently, this theory has been extended to abstract involutions on algebraic and Kac–Moody groups in [Hor10] and [GHM].

Definition 2.9.1. Let $G_{\mathcal{D}}(\mathbb{F})$ be a split Kac–Moody group over a topological field $\mathbb{F}$. Let θ be a continuous involution of $G_{\mathcal{D}}(\mathbb{F})$ mapping a positive Borel subgroups to an opposite (necessarily negative) one. Then by [GHM, Proposition 3.1] θ induces an involution of the Weyl group. We call θ a **flip of** $G_{\mathcal{D}}(\mathbb{F})$ if the induced involution $W \to W$ is the identity. □

Note that a flip as in Definition 2.9.1 is a proper flip in terminology of [GHM].

In [GHM], a slightly more general definition is used. It is required in loc. cit. that a quasi-flip θ maps a positive Borel subgroup to a negative one, which need not be opposite. They also allow that the induced map on W permutes the generating set S non-trivially.

Note that a flip θ is a homeomorphism, as the order of θ is two (in particular finite), hence $\theta^{-1} = \theta$ is continuous.

Given a flip θ, we may define the **Lang map** (also known as twist map)

$$\begin{array}{rcl} \tau_\theta \colon G_{\mathcal{D}}(\mathbb{F}) & \to & G_{\mathcal{D}}(\mathbb{F}) \\ g & \mapsto & g^{-1}\theta(g). \end{array}$$

This map is continuous and open. Moreover, it will play an important role in (the proof of) Theorem 5.3.4. See also [KW92, Chapter 5], [GHM, Chapter 6], [HW93, Sections 1, 2 and 6], [Hor10, Section 2.7] for results using the map τ_θ.

Example 2.9.2. Let ω be the Chevalley involution on a split Kac–Moody group $G_{\mathcal{D}}(\mathbb{F})$ and let θ denote the composition of ω with some continuous field automorphism σ of order one or two. Then θ induces on each rank one-subgroup of $G_{\mathcal{D}}(\mathbb{F})$ (which are isomorphic to $\mathrm{SL}_2(\mathbb{F})$) the contragredient automorphism of that group, composed with the field automorphism σ. Similarly, θ induces the field involution σ on the torus $T \cong (\mathbb{F}^\times)^n$.

Since the rank one-subgroups together with the torus generate $G_{\mathcal{D}}(\mathbb{F})$, the involution θ is unique with this property. It follows that θ is continuous. Hence θ is a flip of $G_{\mathcal{D}}(\mathbb{F})$. We call θ **(σ-twisted) Chevalley involution**.

This construction in particular applies if $\mathbb{F}$ is an extension of order two of some other field $\mathbb{K}$ such that the non-trivial field automorphism of $\mathbb{F}$ which fixes $\mathbb{K}$ is continuous.

Let $\varphi_\alpha \colon \mathrm{SL}_2(\mathbb{F}) \to G_\alpha = \mathrm{Im}(\varphi_\alpha)$ be the embedding provided by Definition 2.5.2. Moreover, setting $G_\theta^\alpha := \mathrm{Stab}_{G_\alpha}(\theta)$, by restriction and co-restriction we obtain isomorphisms $\psi_\alpha = \varphi_\alpha|_{\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{F})}(\theta)}^{G_\theta^\alpha} \colon \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{F})}(\theta) \to G_\theta^\alpha$.

Definition 2.9.3 (Unitary form)**.** Let $G_{\mathcal{D}}(\mathbb{F})$ be a split Kac–Moody group over a topological field $\mathbb{F}$ and let θ be a flip of $G_{\mathcal{D}}(\mathbb{F})$.

Then the fixed point subgroup $G_\theta := \mathrm{Stab}_{G_{\mathcal{D}}(\mathbb{F})}(\theta)$ is called **unitary form** of $G_{\mathcal{D}}(\mathbb{F})$ with respect to θ. The subspace topology on G_θ induced from $G_{\mathcal{D}}(\mathbb{F})$ is also called the **Kac–Peterson topology** and denoted by τ_{KP}. □

Example 2.9.4. Assume that $G = \mathrm{SL}_n(\mathbb{C})$, $\sigma\colon \mathbb{C} \to \mathbb{C}$ is complex conjugation and θ is the σ-twisted Chevalley involution of G. Then θ maps a matrix to its transpose-inverse and applies σ to the entries. Hence $G_\theta = \mathrm{SU}_n(\mathbb{C})$, which carries its natural compact Lie group topology, cf. Remark 3.1.13.

If θ is the Chevalley involution without twist, then θ takes a matrix to its transpose-inverse, and hence $G_\theta = \mathrm{SO}_n(\mathbb{C})$.

Assume that $G_{\mathcal{D}}(\mathbb{F})$ is defined over a field $\mathbb{F}$. Then by [GHM, Proposition 3.1], a flip θ induces a flip on the building Δ of $G_{\mathcal{D}}(\mathbb{F})$. Hence we may require $\delta_\varepsilon(c, d) = \delta_{-\varepsilon}(\theta(c), \theta(d))$ and $\delta^*(c, d) = \delta^*(\theta(c), \theta(d))$ for all chambers c and d, cf. [GHM, Definition 3.5, Proposition 3.1]. This is in fact equivalent to the conditions given in Definition 2.9.1.

Finally, for a chamber $c \in \Delta_\varepsilon$, we define its **θ-codistance** to be

$$\delta^\theta(c) := \delta^*(c, \theta(c)),$$

we call c a **Phan chamber** if $\delta^\theta(c) = 1_W$, the collection of all Phan chambers of Δ_ε is called **flip-flop system** and is denoted by Δ_1 or Δ_θ.

2.10 Amalgams

Detailed information about the concept of amalgams can be found in [BGHS03], [Gra09], [IS02] and [BS04].

Definition 2.10.1. Let $\Gamma = (V, E)$ be a Dynkin diagram, where V denotes the set of vertices and E the set of edges. An **amalgam** over Γ is a family of groups $\mathcal{A} = \{G_i, G_{ij} \mid i, j \in V\}$ together with a family $\iota = \{\iota_{ij} \mid i, j \in I\}$ of injective homomorphisms $\iota_{ij}\colon G_i \hookrightarrow G_{ij}$.

An **enveloping group** of an amalgam $\mathcal{A}$ is a group G together with a family $\varphi = \{\varphi_i, \varphi_{ij} \mid i, j \in V\}$ of homomorphisms $\varphi_i\colon G_i \to G$ and $\varphi_{ij}\colon G_{ij} \to G$ such that $\varphi_{ij} \circ \iota_{ij} = \varphi_i$ for all $i, j \in V$ and G is generated by the images of the maps φ_i and φ_{ij}. □

Moreover, an eveloping group $(\overline{G}, \overline{\varphi})$ is called **universal** if for any enveloping group (G, φ) there is a unique quotient map $\pi\colon \overline{G} \to G$ satisfying $\varphi = \pi \circ \overline{\varphi}$.

Proposition 2.10.2. *Let $\mathcal{A} = \{G_i\}$ be an amalgam and consider the group with generators and relations*

$$U(\mathcal{A}) := \langle\, u_x, x \in \mathcal{A} \mid u_x u_y = u_z \text{ if } x, y, z \in G_i \text{ for some } i \text{ and } xy = z \,\rangle.$$

Moreover, define $\nu\colon \mathcal{A} \to U(\mathcal{A}) : x \mapsto u_x$.

Then $(U(\mathcal{A}), \nu)$ is the universal enveloping group of $\mathcal{A}$. □

We call the amalgam $\mathcal{A}$ **non-collapsing** if $U(\mathcal{A}) \neq \{1\}$.

2.11 Central extensions

The main references for this section are [Mil71] and [HO89], see also [Ste68], [Ste62], [Mat69].

Definition 2.11.1. Let G be a group. A **central extension** of G is a pair (E, ψ) such that $\psi\colon E \to G$ is a surjective homomorphism with $\ker(\psi) \subseteq Z(E)$.

A central extension (E, ψ) of G is called **universal** if for every central extension (X, φ) of G there exists a unique morphism μ such that the diagram

$$\begin{tikzcd}
E \arrow[dr, "\psi"] \arrow[dd, dashed, "\exists!\mu"'] & \\
& G \\
X \arrow[ur, "\forall\varphi"'] &
\end{tikzcd}$$

commutes. □

Clearly, if a universal central extension exists, then it is unique up to isomorphism. Observe also that if (E, ψ) is a universal central extension and (X, φ) is a central extension of G, then (E, μ) is a central extension of $\mathrm{Im}(\mu)$.

Example 2.11.2. Let G be a group and let A be an abelian group such that $G \ltimes A$ is a semidirect product. Then the short exact sequence

$$1 \to A \to G \ltimes A \to G \to 1$$

defines a central extension of G. It is called a **split extension**, as the above sequence splits.

Recall that a group G is called **perfect** if it coincides with its commutator subgroup, i.e. $G = G' := [G, G]$. Assuming the group E of a central extension (E, ψ) to be perfect rules out the split case as above.

Theorem 2.11.3 ([Mil71, Theorem 5.7], [HO89, 1.4.11])**.** *Let G be a group. Then there exists a universal central extension of G if and only if G is perfect.* □

There is an alternative description of the universal central extension which we shall use in Chapter 6. A **section** of a central extension (E, ψ) of a group G is a homomorphism $\hat{\psi}\colon G \to E$ such that $\psi \circ \hat{\psi}$ is the identity on G.

Theorem 2.11.4 ([Mil71, Theorem 5.3], [HO89, 1.4.10])**.** *Let (E, ψ) be a central extension of a perfect group G. Then (E, ψ) is a universal central extension of G if and only if E is perfect and every central extension of E admits a section.* □

The existence of a section is in fact equivalent to the condition that the central extension splits as described in Example 2.11.2. Indeed, the existence of a section implies that the groups C and E of the sequence

$$1 \to C \to \tilde{E} \to E \to 1$$

intersect trivially. Moreover, $\tilde{E}$ is generated by C and E and finally, C is normal in $\tilde{E}$ being the kernel of a homomorphism.

Before closing this chapter, we briefly comment on cocycles.

Definition 2.11.5. Let $A = \{A_n\}_{n\in\mathbb{Z}}$ be a family of G-modules together with a family $\{d_n\}_{n\in\mathbb{Z}}$ of morphisms $d_n \colon A_n \to A_{n+1}$ such that for all $n \in \mathbb{Z}$ the condition $d_{n+1} \circ d_n = 0$ holds. Then the sequence

$$\dots \xrightarrow{d_{n-1}} A_{n-1} \xrightarrow{d_n} A_n \xrightarrow{d_{n+1}} A_{n+1} \to \dots$$

is a **cochain complex of G-modules**. □

Given a cochain complex A, we may define the groups $H^i(G, A) := \ker(d_{i+1})/\mathrm{Im}(d_i)$. In this context, $H^i(G, A)$ is called **i-th cohomology group**. An element of $H^2(G, A)$ is called **2-cocycle**.

If A is a trivial G-module, then by [HS71, Theorem 10.3] the group $H^2(G, A)$ maps bijectively onto the equivalence classes of central extensions $1 \to A \to E \to G \to 1$. In particular, the trivial element of $H^2(G, A)$ corresponds to the split extension $1 \to A \to G \times A \to G \to 1$.

If G is an algebraic group defined over a field $\mathbb{F}$, then the elements of $H^2(G, A)$ are called **Steinberg cocycles** in [Mat69, Section 5].

2.12 Open problem

Describe the lattice of groups H satisfying $B_+ \leq H \leq G_{\mathcal{D}}(R)$. For fields, it is known that this lattice consists exactly of the parabolic subgroups. Certainly, these will turn up in the general case as well, but there will be more intermediate groups. Let us illustrate this with a small example. Consider the group $\mathrm{SL}_3(R)$ over a ring R and assume that R has two proper non-trivial ideals $\mathfrak{a}$ and $\mathfrak{b}$ satisfying $\mathfrak{a}^2 \subseteq \mathfrak{b}$. Then one verifies that

$$\left\{ \begin{pmatrix} * & * & * \\ a & * & * \\ b & a & * \end{pmatrix} \mid a \in \mathfrak{a}, b \in \mathfrak{b} \right\}$$

is a subgroup of $\mathrm{SL}_3(R)$, which obviously contains B_+, but certainly is not parabolic.

Some partial results are known, i.e. if $\mathcal{D}$ is spherical and R is semilocal (with some additional assumptions), cf. for example [Vav84, Theorem 1] or [Vav87, Theorem 1]. The author is not aware of any general results, in particular in the non-spherical case.

Knowledge of the overgroups of B_+ would allow to adapt the strategy used in [Rém02, Proposition 9.6.2] to determine the kernel ker(Ad) of the adjoint representation.

CHAPTER 3

The Kac–Peterson topology

Let $G_{\mathcal{D}}(\mathbb{F})$ be a split Kac–Moody group defined over a k_ω-field $\mathbb{F}$ as introduced in Section 2.5. In this chapter, we prove that the Kac–Peterson topology τ_{KP} turns $G_{\mathcal{D}}(\mathbb{F})$ into a k_ω-group. Moreover, given a flip θ of $G_{\mathcal{D}}(\mathbb{F})$, we also show that the unitary form G_θ is a k_ω-group.

In Section 3.2, we show that the original approach to the Kac–Peterson topology given by V. Kac and D. Peterson in [KP83b] in the special case of Kac–Moody groups defined over a locally compact field of characteristic zero is equivalent to the approach presented here.

3.1 Properties of the Kac–Peterson topology

We start off with some observations regarding the Kac–Peterson topology. Before, let us fix notation.

Definition 3.1.1. Let $\{(X_i, \mathcal{O}_i)\}_{i\in I}$ be any family of topological spaces and let $X \subseteq \prod_{i\in I} X_i$. Then we denote by $\mathcal{O}_{pw}$ the topology on X induced from the product topology on $\prod_{i\in I} X_i$, also known as the topology of pointwise convergence.

The above construction in particular applies to matrix groups, e.g. $\mathrm{SL}_n(\mathbb{F})$, considered as subgroup of the space $\mathbb{F}^{n\times n} \cong \mathbb{F}^{n^2}$.

Let $(i, \overline{\alpha}) = (i, \alpha_1, \ldots, \alpha_k) \in \{0,1\} \times (\Phi^{re})^k$. Composing the map

$$\eta^i \times \varphi_{\alpha_1} \times \ldots \times \varphi_{\alpha_k} \colon ((\mathbb{F}^\times)^{in} \times (\mathrm{SL}_2(\mathbb{F}))^k, \mathcal{O}_{pw}) \to T^i \times G_{\alpha_1} \times \ldots \times G_{\alpha_k}$$

with the multiplication map of $G_{\mathcal{D}}(\mathbb{F})$, we obtain a map $p_{(i,\overline{\alpha})} \colon ((\mathbb{F}^\times)^{in} \times (\mathrm{SL}_2(\mathbb{F})^k, \mathcal{O}_{pw})) \to T^i.G_{\alpha_1} \ldots G_{\alpha_k}$. Then we define $\tau_{(i,\overline{\alpha})}$ to be the quotient topology on $G_{(i,\overline{\alpha})} := \mathrm{Im}(p_{(i,\overline{\alpha})}) = T^i.G_{\alpha_1} \ldots G_{\alpha_k}$ with respect to the map $p_{(i,\overline{\alpha})}$. Note that in general $G_{(i,\overline{\alpha})}$ is not a subgroup. $\square$

Lemma 3.1.2. *Let $\mathbb{F}$ be a k_ω-field. Then the topological group $(\mathrm{SL}_n(\mathbb{F}), \mathcal{O}_{pw})$ and each of its closed subgroups is a k_ω-group. In particular, the group of units $\mathbb{F}^\times$ is a k_ω-group.*

Proof. Since $(\mathrm{SL}_n(\mathbb{F}), \mathcal{O}_{pw})$ is a closed subset of $(\mathbb{F}^{n\times n}, \mathcal{O}_{pw})$ defined by the equation $\det(A) = 1$, the result follows from (i) and (ii) of Proposition 2.8.3.

The group $\mathbb{F}^\times$ is homeomorphic to the set $(\{(x, y) \in \mathbb{F}^2 \mid xy = 1\}, \mathcal{O}_{pw})$, which is closed in $\mathbb{F}^2$. Hence Proposition 2.8.3 implies that $\mathbb{F}^\times$ is a k_ω-group. □

The next result shows that in fact the torus and the *fundamental* rank one subgroups suffice to describe the Kac–Peterson topology.

Lemma 3.1.3 (cf. [GGH10, Lemma 6.2])**.** *Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a basis of simple roots for Φ with respect to the standard torus T. Then the Kac–Peterson topology on $G_{\mathcal{D}}(\mathbb{F})$ is the final group topology for the maps η and φ_{α_i}, where $i = 1, \dots, n$.*

Proof. Clearly, if every map φ_α, for $\alpha \in \Phi^{re}$, is continuous, then the maps φ_{α_i}, for $\alpha_i \in \Pi$, are continuous.

Let $\alpha \in \Phi^{re} = W.\Pi$ be a real root. Then there exist $w \in W$ and $\alpha_i \in \Pi$ such that $\alpha = w.\alpha_i$. Let $\tilde{w}$ be the canonical representative of w in $G_{\mathcal{D}}(\mathbb{F})$. Then by the above each rank one group $G_\alpha = G_{w.\alpha_i}$ coincides with $\tilde{w}G_{\alpha_i}\tilde{w}^{-1}$, a conjugate of the fundamental rank one group G_{α_i}.

Now let τ be a group topology on $G_{\mathcal{D}}(\mathbb{F})$ for which η and each map φ_{α_i}, where α_i is simple, is continuous. Then by continuity of conjugation in $G_{\mathcal{D}}(\mathbb{F})$ all other maps $\varphi_\alpha = c_{\tilde{w}} \circ \varphi_{\alpha_i}$ (where $c_{\tilde{w}}$ denotes conjugation with $\tilde{w}$), are continuous with respect to τ, from which the claim follows. □

In positive characteristic, the exponential series $\sum_{n=0}^{\infty} \frac{\mathrm{ad}_{e_i}^n}{n!}$ obtained by integrating the adjoint action of $G_{\mathcal{D}}(\mathbb{F})$ on $\mathfrak{g}$ is no longer well-defined, as the denominator becomes zero when $n \cdot 1_{\mathbb{F}} = 0$ for some $n \in \mathbb{Z} \setminus \{0\}$. To circumvent this problem, we shall use the adjoint action of $G_{\mathcal{D}}(\mathbb{F})$ on the algebra $\mathcal{U}_{\mathbb{F}}$ introduced in Section 2.6.

Remark 3.1.4. Assume that V is a direct summand of $\mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathbb{F}})$. Then the topology of pointwise convergence $\mathcal{O}_{pw}$ is the coarsest topology on $\mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathbb{F}})$ making the projection $\mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathbb{F}}) \to (\mathrm{GL}(V), \mathcal{O}_{pw})$ continuous. Equivalently, the topology of pointwise convergence is the initial topology with respect to projections onto the direct factors.

Lemma 3.1.5. *Let $v \in \mathcal{U}_{\mathbb{F}} = \mathcal{U}_{\mathbb{Z}} \otimes \mathbb{F}$, let α be a real root and consider the associated rank one group G_α of $G_{\mathcal{D}}(\mathbb{F})$. Then v is contained in an $\mathrm{Ad}\,|_{T.G_\alpha}$-invariant finite-dimensional sub-$\mathbb{F}$-vector space V of $\mathcal{U}_{\mathbb{F}}$. Moreover, for all $w \in V$ the orbit map $(T.G_\alpha, \tau_{(1,\alpha)}) \to (V, \mathcal{O}_{pw}|_V), tg \mapsto \mathrm{Ad}(tg).w$ on that submodule is continuous.*

Proof. The Bruhat decomposition of the rank one subgroup G_α of $G_{\mathcal{D}}(\mathbb{F})$ implies that the product map $U_\alpha \times U_{-\alpha} \times U_\alpha \times U_{-\alpha} \to G_{\alpha,\mathbb{Q}}$ is surjective, as is deduced from the calculations in

[Ste68, Lemma 24]. Since every $u_\alpha(q) \in U_\alpha \subseteq G_{\alpha,\mathbb{Q}}$ acts via $\mathrm{Ad}(u_\alpha(q)) = \sum_{n=0}^{\infty}(\frac{(\mathrm{ad}_{e_i})^n}{n!} \otimes q^n)$, we obtain that the space

$$V := \sum_{k,l,m,n\in\mathbb{N}} \left\langle \left(\frac{(\mathrm{ad}_{e_\alpha})^k}{k!} \otimes 1\right)\left(\frac{(\mathrm{ad}_{f_\alpha})^l}{l!} \otimes 1\right)\left(\frac{(\mathrm{ad}_{e_\alpha})^m}{m!} \otimes 1\right)\left(\frac{(\mathrm{ad}_{f_\alpha})^n}{n!} \otimes 1\right).v \right\rangle_{\mathbb{F}}$$

contains $\langle\, G_\alpha.v \,\rangle_{\mathbb{F}}$. Since T acts via scalars, V in fact even contains $\langle\, T.G_\alpha.v \,\rangle_{\mathbb{F}}$. Moreover, by construction V is $\mathrm{Ad}\,|_{T.G_\alpha}$-invariant. From the local nilpotency of ad_{e_i} and ad_{f_i}, we may conclude that the above sum is finite and hence V is finite-dimensional.

For the second claim, note that the subspace topology on V coincides with the product topology induced from the field $\mathbb{F}$. We therefore conclude that the map

$$\mathrm{Ad}\,|_{T.G_\alpha}^{\mathrm{GL}(V)} \colon (T.G_\alpha, \tau_{(1,\alpha)}) \to (\mathrm{GL}(V), \mathcal{O}_{pw})$$

is continuous. Since the evaluation map $\mathrm{eval}_w \colon \mathrm{GL}(V) \to V$, $A \mapsto A.w$ is continuous, it follows that the orbit map

$$\mathrm{eval}_w \circ \mathrm{Ad}\,|_{G_\alpha}^{\mathrm{GL}(V)}$$

is continuous as well. □

Proposition 3.1.6. *Let $(i, \overline{\alpha}) = (i, \alpha_1, \ldots, \alpha_k) \in \{0,1\} \times (\Phi^{re})^k$ and let $v \in \mathcal{U}_{\mathbb{F}}$. Then v is contained in a $\mathrm{Ad}\,|_{T^i.G_{\alpha_1}\cdots G_{\alpha_k}}$-invariant finite-dimensional sub-$\mathbb{F}$-vector space V of $\mathcal{U}_{\mathbb{F}}$. Moreover, for each $w \in V$ the orbit map*

$$(T^i.G_{\alpha_1} \ldots G_{\alpha_k}, \tau_{(i,\alpha_1,\ldots,\alpha_k)}) \to (V, \mathcal{O}_{pw}|_V) \colon g \mapsto \mathrm{Ad}(g).w$$

is continuous.

Proof. We argue by induction on k, the length of $\overline{\alpha}$. The case $k = 1$ is shown in Lemma 3.1.5.

By induction, assume that $\mathrm{Ad}(G_{\alpha_2} \cdots G_{\alpha_k}).v$ is contained in an $\mathrm{Ad}\,|_{T^i.G_{\alpha_2}\cdots G_{\alpha_k}}$-invariant finite-dimensional sub-$\mathbb{F}$-vector space V_0 of $\mathcal{U}_{\mathbb{F}}$. Let $B = \{b_1, \ldots, b_n\}$ be a basis of V_0.

By Lemma 3.1.5, for each b_i there exists a $\mathrm{Ad}\,|_{G_{\alpha_1}}$-invariant finite-dimensional sub-$\mathbb{F}$-vector space V_i of $\mathcal{U}_{\mathbb{F}}$. Hence the sub-$\mathbb{F}$-vector space $V := \sum_{i=1}^n V_i$ is $\mathrm{Ad}\,|_{T^i.G_{\alpha_1}\cdots G_{\alpha_k}}$-invariant, finite-dimensional, and contains $V_0 = \langle\, b_1, \ldots, b_n \,\rangle_{\mathbb{F}}$.

The second claim is shown as in Lemma 3.1.5. □

Proposition 3.1.7. *Assume that $\mathcal{D}$ is simply connected or adjoint. Then for all $(i, \overline{\alpha})$ the space $(G_{(i,\overline{\alpha})}, \tau_{(i,\overline{\alpha})})$ is Hausdorff and hence k_ω.*

Proof. Corollary 2.6.4 shows that $\ker(\mathrm{Ad}) = Z(G_{\mathcal{D}}(\mathbb{F}))$.

Let $(i, \overline{\alpha}) \in \{0,1\} \times (\Phi^{re})^k$ and let $Z_{(i,\overline{\alpha})} := Z(G_{\mathcal{D}}(\mathbb{F})) \cap G_{(i,\overline{\alpha})}$ be the intersection of the centre of $G_{\mathcal{D}}(\mathbb{F})$ with the spaces $G_{(i,\overline{\alpha})}$. Then the map $\mathrm{Ad}\,|_{G_{(i,\overline{\alpha})}}$ separates the points of the quotient $G_{(i,\overline{\alpha})}/Z_{(i,\overline{\alpha})}$. Let $g \neq h \in G_{(i,\overline{\alpha})}/Z_{(i,\overline{\alpha})}$, then it follows that $\mathrm{Ad}(g) \neq \mathrm{Ad}(h)$. In particular, there exists a vector $v \in \mathcal{U}_{\mathbb{F}}$ such that $\mathrm{Ad}(g).v \neq \mathrm{Ad}(h).v$.

The orbit map

$$\varphi_v\colon (G_{(i,\overline{\alpha})}, \tau_{(i,\overline{\alpha})}) \to (V, \mathcal{O}_{pw}|_V), g \mapsto \mathrm{Ad}(g).v$$

is continuous by Proposition 3.1.6. As $(\mathcal{U}_{\mathbb{F}}, \mathcal{O}_{pw})$ is a topological vector space over a k_ω-field carrying the product topology, it is Hausdorff. Thus there exist disjoint open neighbourhoods of $\varphi_v(g) = \mathrm{Ad}(g).v$ and $\varphi_v(h) = \mathrm{Ad}(h).v$, whose preimages under the continuous map φ_v are disjoint open neighbourhoods in $G_{(i,\overline{\alpha})}/Z_{(i,\overline{\alpha})}$ containing g and h, respectively.

Hence $G_{(i,\overline{\alpha})}/Z_{(i,\overline{\alpha})}$ is Hausdorff. By continuity of φ_v, we conclude that $G_{(i,\overline{\alpha})}$ is homeomorphic to $G_{(i,\overline{\alpha})}/Z_{(i,\overline{\alpha})} \times Z_{(i,\overline{\alpha})}$. Since $Z(G_{\mathcal{D}}(\mathbb{F}))$ is contained in the standard torus and hence Hausdorff, so is $Z_{(i,\overline{\alpha})}$. It follows that $(G_{(i,\overline{\alpha})}, \tau_{(i,\overline{\alpha})})$ is Hausdorff. Now Proposition 2.8.3 implies that $(G_{(i,\overline{\alpha})}, \tau_{(i,\overline{\alpha})})$ is k_ω. ☐

Define a partial order on $\bigcup_{k\in\mathbb{N}}\{0,1\} \times (\Phi^{re})^k$ by $(i, \overline{\alpha}) \leq (j, \overline{\beta})$ if and only if $i \leq j$ and $\overline{\alpha}$ is a subsequence of $\overline{\beta}$. Note that this order in fact makes $\bigcup_{k\in\mathbb{N}}\{0,1\} \times (\Phi^{re})^k$ a directed set. Recall that a **cofinal sequence** ω in a directed set $(X, \leq)$ is a sequence $\omega = (x_1, x_2, \ldots)$ such that for every element $x \in X$ there is $n \in \mathbb{N}$ such that $x \leq x_n$.

We are going to apply the next result a couple of times later on.

Proposition 3.1.8. *Assume that $\mathcal{D}$ is simply connected or adjoint. Then $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ is the direct limit topology for the directed system $\{(G_{(i,\overline{\alpha})}, \tau_{(i,\overline{\alpha})}) \mid (i, \overline{\alpha}) \in \{0,1\} \times (\Phi^{re})^k\}$.*

In particular, $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ is Hausdorff.

Proof. We introduce a filtration of the set of real roots via $\Phi_i := \{\alpha \in \Phi^{re} \mid |\mathrm{ht}(\alpha)| = i\}$ using their height. Note that for every $i \in \mathbb{N}$ there are only finitely many real roots of height i.

Let a_i be the number of elements in Φ_i and number them $\alpha_{i1}, \ldots, \alpha_{ia_i}$. Then

$$(1, \alpha_{11}, \ldots, \alpha_{1a_1}, \alpha_{21}, \ldots, \alpha_{2a_2}, \alpha_{11}, \ldots, \alpha_{1a_1}, \alpha_{21}, \ldots, \alpha_{2a_2}, \alpha_{31}, \ldots, \alpha_{3a_3}, \alpha_{11}, \ldots)$$

is a cofinal sequence in $\mathcal{W}$, as every finite sequence occurs as a subsequence.

Consider the associated sequence $(p_{(1,\alpha_{11})}, p_{(1,\alpha_{11},\alpha_{12})}, \ldots)$ of product maps (which are quotient maps by construction). Let τ_0 be the direct limit topology on $G_{\mathcal{D}}(\mathbb{F})$ with respect to the sets $(G_{(i,\overline{\alpha})}, \tau_{(i,\overline{\alpha})})$. Then for each $(i_1, \overline{\alpha}) \in \{0,1\} \times (\Phi^{re})^{k_1}$, $(i_2, \overline{\beta}) \in \{0,1\} \times (\Phi^{re})^{k_2}$ the concatenation map

$$(G_{(i_1,\overline{\alpha})}, \tau_{(i_1,\overline{\alpha})}) \times (G_{(i_2,\overline{\beta})}, \tau_{(i_2,\overline{\beta})}) \to (G_{(\max\{i_1,i_2\},\overline{\alpha}(\overline{\beta})^{-1})}, \tau_{(\max\{i_1,i_2\},\overline{\alpha}(\overline{\beta})^{-1})}, (x,y) \mapsto xy^{-1}$$

is continuous, and hence

$$(G_{(i_1,\overline{\alpha})}, \tau_{(i_1,\overline{\alpha})}) \times (G_{(i_2,\overline{\beta})}, \tau_{(i_2,\overline{\beta})}) \to (G_{\mathcal{D}}(\mathbb{F}), \tau_0), (x,y) \mapsto xy^{-1}$$

is continuous. Since in the category of k_ω-spaces, direct limits and direct product commute, it follows that the maps $(G_{\mathcal{D}}(\mathbb{F}), \tau_0) \times (G_{\mathcal{D}}(\mathbb{F}), \tau_0) \to (G_{\mathcal{D}}(\mathbb{F}), \tau_0), (x,y) \mapsto xy^{-1}$ is continuous and hence $(G_{\mathcal{D}}(\mathbb{F}), \tau_0)$ is a topological group. Since moreover each space $(G_{(i,\overline{\alpha})}, \tau_{(i,\overline{\alpha})})$ is k_ω by Proposition 3.1.7, by [GGH10, Proposition 5.8] it follows that $(G_{\mathcal{D}}(\mathbb{F}), \tau_0)$ is Hausdorff and k_ω.

Let τ_1 be a group topology on $G_{\mathcal{D}}(\mathbb{F})$ for which the maps η and φ_{α_i} are continuous. Then for every $(i, \overline{\alpha})$ also the product map $p_{(i,\overline{\alpha})}$ is continuous. In particular, we obtain that τ_1 is coarser than τ_0. This characterises τ_0 as the finest group topology on $G_{\mathcal{D}}(\mathbb{F})$ for which the maps η and φ_α are continuous. Hence $\tau_0 = \tau_{KP}$. The claim follows. □

Proposition 3.1.8 also implies the following, which we note for future reference.

Corollary 3.1.9. *The quotient topology $\tau_{(i,\overline{\alpha})}$ on $G_{(i,\overline{\alpha})}$ with respect to the quotient map $p_{(i,\overline{\alpha})}$ coincides with the subspace topology induced from the Kac–Moody group $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$.*

In particular, the product maps $p_{(i,\overline{\alpha})}$ are continuous with respect to the Kac–Peterson topology on $G_{\mathcal{D}}(\mathbb{F})$. □

The map $\eta\colon (\mathrm{Hom}(\Lambda, \mathbb{F}^\times), \mathcal{O}_{pw}) \to (G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ is injective and the kernel of each map $\varphi_\alpha\colon (\mathrm{SL}_2(\mathbb{F}), \mathcal{O}_{pw}) \to (G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ is finite, hence discrete. It follows that the images of η and each φ_α are Hausdorff. Since by Proposition 2.8.3 Hausdorff quotients of k_ω-groups are k_ω again, Lemma 3.1.2 and Corollary 3.1.9 imply that the torus $(T, \tau_{(1,\emptyset)}) = (\mathrm{Im}(\eta), \tau_{(1,\emptyset)})$ and the fundamental rank one subgroups $(G_\alpha, \tau_{(0,\alpha)}) = (\mathrm{Im}(\varphi_\alpha), \tau_{(0,\alpha)})$ of $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ are k_ω.

Corollary 3.1.10. *The adjoint representation* $\mathrm{Ad}\colon (G_{\mathcal{D}}(\mathbb{F}), \tau_{KP}) \to (\mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathbb{F}}), \mathcal{O}_{pw})$ *given by Proposition 2.6.2 is continuous.* □

Here is one of the announced main results.

Theorem 3.1.11. *Let $\mathbb{F}$ be a k_ω-field and let $G_{\mathcal{D}}(\mathbb{F})$ be a simply connected or adjoint split Kac–Moody group over $\mathbb{F}$. Then $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ is a k_ω-group.*

Proof. By Proposition 3.1.8, the group $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ is Hausdorff. The result then follows from [GGH10, Proposition 5.8]. □

We may therefore deduce the following topological variation of the Curtis–Tits Theorem.

Theorem 3.1.12 (Curtis–Tits Theorem)**.** *Let $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ be a two-spherical simply connected or adjoint split Kac–Moody group over some k_ω-field $\mathbb{F}$ of cardinality at least four. Let Φ^{re} be the set of real roots and let Π be a basis of simple roots for Φ^{re}. Construct an amalgam $\mathcal{A}$ as follows: For $\alpha, \beta \in \Pi$, set $G_\alpha := \varphi_\alpha(\mathrm{SL}_2(\mathbb{F}))$ and $G_{\alpha\beta} := \langle\, G_\alpha \cup G_\beta \,\rangle$. Moreover, let $\iota_{\alpha\beta}\colon G_\alpha \hookrightarrow G_{\alpha\beta}$ be the canonical inclusion morphisms.*

Then the group $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ is a universal enveloping group of the amalgam $\mathcal{A} = \{G_\alpha, G_{\alpha\beta}; \iota_{\alpha\beta}\}$ in the categories of

(i) abstract groups,

(ii) Hausdorff topological groups and

(iii) k_ω-groups.

Proof. (i) This is the main result of [AM97].

(ii) It follows from Lemma 3.1.3 together with (i) that the Kac–Peterson topology on $G_{\mathcal{D}}(\mathbb{F})$ is characterised as the final group topology with respect to the maps φ_α for simple roots $\alpha \in \Pi$ only. The claim is now an immediate consequence of part (i) together with Theorem 3.1.11.

(iii) Using part (ii), we may apply [GGH10, Corollary 5.10] and conclude that $G_{\mathcal{D}}(\mathbb{F})$ is indeed a universal enveloping group of $\mathcal{A}$ in the category of k_ω-groups. □

Remark 3.1.13. Let $\mathbb{F}$ be a non-discrete σ-compact locally compact field, let $\mathcal{D}$ be a spherical Kac–Moody root datum and let $G_{\mathcal{D}}(\mathbb{F})$ be the associated split Kac–Moody group. Then by [GGH10, Proposition 2.2] the Kac–Peterson topology and the Lie group topology on $G_{\mathcal{D}}(\mathbb{F})$ coincide.

3.2 An equivalent description of the Kac–Peterson topology

This section is devoted to establishing an equivalence between the definition of the Kac–Peterson topology as final group topology with respect to the torus and the subgroups of rank one given in Definition 2.8.4 and the one given in [KP83b, Section 4G] using parametrisations of the root groups.

First, we recall the definition from [KP83b]. Let $G_{\mathcal{D}}(\mathbb{F})$ be a split Kac–Moody group over a k_ω-field $\mathbb{F}$. For all real roots α, choose parametrisations $x_\alpha\colon \mathbb{F} \to U_\alpha$ of the root groups. Let $i \in \{0,1\}$ and choose k (not necessarily distinct nor simple) real roots $\beta_1, \dots, \beta_k$. Denote by $x_{\overline{\beta}}\colon \mathbb{F}^k \to G_{\mathcal{D}}(\mathbb{F}), (t_1, \dots, t_k) \mapsto x_{\beta_1}(t_1) \cdots x_{\beta_k}(t_k)$ the composition of the chosen parametrisations with the product map of $G_{\mathcal{D}}(\mathbb{F})$ and by $x_{(i,\overline{\beta})}$ the composition of the map $\eta^i \times x_{\overline{\beta}}$ with the product map of $G_{\mathcal{D}}(\mathbb{F})$. The image of $x_{(i,\overline{\beta})}$ is denoted by $U_{(i,\overline{\beta})} := \operatorname{Im}(x_{(i,\overline{\beta})}) \subseteq G_{\mathcal{D}}(\mathbb{F})$.

Definition 3.2.1. A subset $U \subseteq G_{\mathcal{D}}(\mathbb{F})$ is open if and only if for all $i \in \{0,1\}$, $k \in \mathbb{N}$ and all choices of real roots $\beta_1, \dots, \beta_k$ we have that $x^{-1}_{(i,\overline{\beta})}(U) \subseteq ((\mathbb{F}^\times)^{in} \times \mathbb{F}^k, \mathcal{O}_{pw})$ is open. We denote the topology on $G_{\mathcal{D}}(\mathbb{F})$ obtained in this way by τ.

Equivalently, τ is the final topology with respect to the maps $x_{(i,\overline{\beta})}$. □

We will establish the facts that each of τ and the Kac–Peterson topology τ_{KP} is finer than the other, hence they must coincide. The following result links Definition 3.2.1 to final group topologies.

Proposition 3.2.2. *The topology τ is the final group topology for the maps $x_{(i,\overline{\beta})}$, where $\overline{\beta}$ runs through the sequences of real roots.*

Proof. By definition, for every $k \in \mathbb{N}$ and every $(i, \overline{\beta}) = (i, \beta_1, \dots, \beta_k)$ the map $x_{(i,\overline{\beta})}\colon (\mathbb{F}^\times)^{in} \times \mathbb{F}^k \to (G_{\mathcal{D}}(\mathbb{F}), \tau)$ is continuous. Moreover, as $\operatorname{Im}(x_{(i,\overline{\beta})}) \subseteq G_{(i,\overline{\beta})}$, it follows from Proposition

3.1.7 that $\mathrm{Im}(x_{(i,\overline{\beta})})$ is Hausdorff and hence k_ω by Proposition 2.8.3. As in the proof of Proposition 3.1.8 we obtain that the final topology for the maps $x_{(i,\overline{\beta})}$ is indeed a final *group* topology.

Conversely, let τ_1 be a group topology on $G_{\mathcal{D}}(\mathbb{F})$ for which every map $x_{(i,\overline{\beta})}$ is continuous. Then in particular, for any open set $U \in \tau_1$ and $(i,\overline{\beta}) \in \{0,1\} \times (\Phi^{re})^k$, its preimage under the map $x_{(i,\overline{\beta})}$ is open in $(\mathbb{F}^\times)^{in} \times \mathbb{F}^k$, from which $U \in \tau$ follows. Hence τ is the finest group topology for which each map $x_{(i,\overline{\beta})}$ is continuous. $\square$

Lemma 3.2.3. *Let α be a real root. Then the map $\varphi_\alpha \colon (\mathrm{SL}_2(\mathbb{F}), \mathcal{O}_{pw}) \to (G_{\mathcal{D}}(\mathbb{F}), \tau)$ is continuous.*

Proof. As before, we note that the map $U_\alpha \times U_{-\alpha} \times U_\alpha \times U_{-\alpha} \to \mathrm{SL}_2(\mathbb{F})$ is surjective. Hence the following diagram commutes:

$$\begin{tikzcd}
U_\alpha \times U_{-\alpha} \times U_\alpha \times U_{-\alpha} \arrow[d, "p_{(0,\alpha,-\alpha,\alpha,-\alpha)}"'] \arrow[dr, "x_{(0,\alpha,-\alpha,\alpha,-\alpha)}"] & \\
\mathrm{SL}_2(\mathbb{F}) \arrow[r, "\varphi_\alpha"'] & G_{\mathcal{D}}(\mathbb{F})
\end{tikzcd}$$

where $p_{(0,\alpha,-\alpha,\alpha,-\alpha)} \colon U_\alpha \times U_{-\alpha} \times U_\alpha \times U_{-\alpha} \to (\mathrm{SL}_2(\mathbb{F}), \mathcal{O}_{pw})$ is the natural quotient map, which is continuous and open. It follows that φ_α is continuous. $\square$

Combining the results, we have shown the following. In particular, this description gives a criterion for openness of subsets of $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ using the root groups and the torus.

Proposition 3.2.4. *The Kac–Peterson topology τ_{KP} on $G_{\mathcal{D}}(\mathbb{F})$ coincides with τ.*

Proof. Lemma 3.2.3 shows that the Kac–Peterson topology is finer than τ. It remains to see that τ is finer than the Kac–Peterson topology, which follows from Proposition 3.1.8 together with Proposition A.3.1. $\square$

It is stated without proof in [KP83b, Section 4G] that if $G_{\mathcal{D}}(\mathbb{C})$ is equipped with the Kac–Peterson topology τ_{KP}, then it is a Hausdorff σ-compact topological group. This special case, however, follows from Proposition 3.2.4 and Theorem 3.1.11.

As a byproduct, we record the following alternative characterisation of the Kac–Peterson topology on a Kac–Moody group as direct limit topology.

Corollary 3.2.5. *Equip the sets $U_{(i,\overline{\beta})}$ with the quotient topology with respect to the maps $x_{(i,\overline{\beta})}$. Then the Kac–Peterson topology coincides with the direct limit topology given by*

$$\lim_{\rightarrow} U_{(i,\overline{\beta})},$$

where $(i,\overline{\beta}) \in \{0,1\} \times (\Phi^{re})^k$. $\square$

3.3 The induced topology on the unitary form

Let θ be a flip of $G_{\mathcal{D}}(\mathbb{F})$ and consider the unitary form $G_\theta = \mathrm{Stab}_{G_{\mathcal{D}}(\mathbb{F})}(\theta) \leq G_{\mathcal{D}}(\mathbb{F})$, cf. Definition 2.9.3.

Proposition 3.3.1. *Let $\mathbb{F}$ be a k_ω-field and let θ be a flip of a split Kac–Moody group $G_{\mathcal{D}}(\mathbb{F})$. Then the unitary form (G_θ, τ_{KP}) is a k_ω-group.*

Proof. Since θ is continuous, $(G_\theta, \tau_{KP}) = \tau_\theta^{-1}(\{1\})$ is closed in $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$. Hence the assertion follows from Theorem 3.1.11 and Proposition 2.8.3. □

We may also prove a characterisation of the Kac–Peterson topology on the unitary form G_θ similar to the one shown in Lemma 3.1.3. We will denote the induced flips on $G_\alpha \cong \mathrm{SL}_2(\mathbb{F})$ by θ_α and the induced involution on T by θ_T.

Proposition 3.3.2. *Let $\Pi = \{\alpha_i, \dots, \alpha_N\}$ be a basis of simple roots. Then the Kac–Peterson topology on the unitary form G_θ is the final group topology with respect to the maps $\eta_\theta \colon (\mathbb{F}^\times)^n_{\theta_T} \to G_\theta$ and $\psi_{\alpha_i} \colon \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{F})}(\theta_{\alpha_i}) \to G_\theta$.*

Proof. The arguments are similar to those given in [GGH10, Proposition 6.9]. The main difference is that in our situation the groups $G_\theta^{\alpha_i} = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{F})}(\theta_{\alpha_i})$ need not be compact.

Let τ_0 be the final group topology on G_θ with respect to the maps η_θ and ψ_{α_i}. By construction and continuity of conjugation, each of the maps η_θ and ψ_α (where α is a real root) is continuous, hence τ_0 is coarser than the Kac–Peterson topology on G_θ.

Since θ_{α_i} is a continuous involution on G_{α_i}, its fixed point group $G_\theta^{\alpha_i}$ is closed. Thus each group $G_\theta^{\alpha_i}$ is (homeomorphic to) a closed subgroup of $(\mathrm{SL}_2(\mathbb{F}), \mathcal{O}_{pw})$, hence is a k_ω-group by Lemma 3.1.2.

Thus for each simple root α_i there exists a k_ω-sequence $\{{G_\theta^{\alpha_i}}^{(n)}\}_{n\in\mathbb{N}}$ of compact sets such that

$$G_\theta^{\alpha_i} = \bigcup_{n\in\mathbb{N}} {G_\theta^{\alpha_i}}^{(n)}$$

and $G_\theta^{\alpha_i}$ carries the weak topology with respect to the family $\{{G_\theta^{\alpha_i}}^{(n)}\}_{n\in\mathbb{N}}$. Moreover, as T_{θ_T} is a k_ω-group, we also fix a k_ω-sequence $\{T_n\}_{n\in\mathbb{N}}$ of T_{θ_T}.

Given $n \in \mathbb{N}$, let $(i, \overline{\alpha}) = (i, \alpha_1, \dots, \alpha_n) \in \{0, 1\} \times \Pi^n$. Consider the product map

$$\psi^{(n)}_{(i,\overline{\alpha})} \colon (T_n)^i \times {G_\theta^{\alpha_1}}^{(n)} \times \dots \times {G_\theta^{\alpha_n}}^{(n)} \to (G_\theta, \tau_{KP}),$$

whose image is denoted by ${G_\theta^{(i,\overline{\alpha})}}^{(n)}$. Now every element in the family of maps $\{\psi^{(n)}_{(i,\overline{\alpha})}\}$ is continuous both with respect to τ_0 and the Kac–Peterson topology, cf. Proposition 3.1.8. By construction, $T_n \times {G_\theta^{\alpha_1}}^{(n)} \times \dots \times {G_\theta^{\alpha_n}}^{(n)}$ is a compact Hausdorff space, hence it satisfies the T_4-separation axiom. Consequently, every continuous surjective morphism is a quotient map. This observation implies that the family $\{\psi^{(n)}_{(i,\overline{\alpha})}\}$ from above consists of quotient maps (onto their respective images) with respect to both topologies. Further, this implies that the topologies agree on each of the sets $G_\theta^{(i,\overline{\alpha})^{(n)}}$.

Consider the set $\bigcup_{n\in\mathbb{N}}\{0,1\} \times (\Phi^{re})^n \times \mathbb{N}$ together with the partial order

$$(i, \overline{\alpha}, n) \leq (j, \overline{\beta}, m) \text{ if and only if } \overline{\alpha} \leq \overline{\beta} \text{ (as a subsequence) and } n \leq m, i \leq j.$$

It suffices to show that for each $(i, \overline{\alpha}, n) \in \mathcal{W}$ there exists a triple $(j, \overline{\beta}, m)$ such that

$$G_{(i,\overline{\alpha})} \cap G_\theta \subseteq G_\theta^{(j,\overline{\beta})^{(m)}},$$

then the result follows from the above conclusion. However, as G_θ is generated by the torus and the collection of its subgroups of rank one, and $\{G_\theta^{\alpha_i (n)}\}$ is a k_ω-sequence for $G_\theta^{\alpha_i}$, the direct limit of the sequence $G_\theta^{(i,\overline{\beta})^{(m)}}$ coincides with G_θ.

This implies that G_θ is the limit of the spaces associated to the cofinal sequence $(1, (\alpha_1), 1), (1, (\alpha_1, \alpha_2), 2), \ldots, (1, (\alpha_1, \ldots, \alpha_N, \alpha_1), N + 1), \ldots$ and so on. More precisely, denoting $\overline{\alpha_n} := (\alpha_1, \ldots, \alpha_N, \ldots, \alpha_{(n \mod N)})$, we have shown that $G_\theta = \lim_{\rightarrow} G_\theta^{(i,\overline{\alpha_n})^{(n)}}$. Hence the topologies agree on G_θ, which shows the claim. $\square$

3.4 Open problems

(i) The description of the Kac–Peterson topology as final group topology on the one hand and as direct limit topology on the other hand (Definition 2.8.4 and Proposition 3.1.8) uses the fact that the defining field is k_ω. More precisely, the argument involves that the sets $G_{(i,\overline{\alpha})}$ are k_ω in order to conclude that the final topology they induce is in fact a group topology. For this, we need the fact that $\lim(X_n \times Y_n) \cong \lim X_n \times \lim Y_n$ is true in the category of k_ω-spaces (cf. [GGH10, Proposition 4.7]). The author does not know whether the above description may be used in general, in particular whether the direct limit topology of the sets $G_{(i,\overline{\alpha})}$ defines a group topology on $G_{\mathcal{D}}(\mathbb{F})$.

Results in this direction could also be used to prove the results obtained in Chapter 5 in a more general setup.

(ii) Continuing (i) above in a slightly different direction, it is natural to ask about quasi-split groups. Clearly, first one has to think about well-definedness of the Kac–Peterson topology as described in Definition 2.8.4. To obtain a well-defined topology, one needs a *suitable* topology on the rank one groups and the torus. Using Galois descent (assuming that the Galois group acts continuously), this is certainly feasible.

Then, using the well-behavedness of k_ω-spaces or k_ω-groups, respectively, the main issue would be to prove an analogue of Proposition 3.1.8. Finally, one has to show that the adjoint representation of quasi-split groups behaves in a similar way regarding continuity and its kernel. The author would expect that most of the above results generalise to the quasi-split case, however, there are a number of subtle and/or technical details to be checked.

(iii) The author conjectures that the results presented in Section 3.1 generalise to the case of split Kac–Moody groups defined over k_ω-rings. The only problem occurs when dealing with continuity of the orbit map (Lemma 3.1.5 and Proposition 3.1.6), as the submodule constructed in the proofs need not have a complement a priori.

CHAPTER 4

Rigidity

Let $\mathcal{D}$ and $\mathcal{D}'$ be Kac–Moody root data and let $R \leq S$ be rings. By extension of scalars $\mathrm{SL}_2(R) \hookrightarrow \mathrm{SL}_2(S)$ and $\mathrm{Hom}(\Lambda, R^\times) \hookrightarrow \mathrm{Hom}(\Lambda, S^\times)$ we obtain an embedding $G_\mathcal{D}(R) \hookrightarrow G_\mathcal{D}(S)$. Under this embedding, we view $G_\mathcal{D}(R)$ as a subgroup of $G_\mathcal{D}(S)$.

We call $G_\mathcal{D}(R)$ **(topologically) Mostow-rigid** in $G_\mathcal{D}(S)$ if for every (continuous) isomorphism $\varphi\colon G_\mathcal{D}(R) \to G'_{\mathcal{D}'}(R)$ there exists a unique (continuous) isomorphism $\psi\colon G_\mathcal{D}(S) \to G'_{\mathcal{D}'}(S)$, whose restriction and corestriction coincides with φ.

In this chapter, we determine the automorphism group of $G_\mathcal{D}(R)$ in case that $\mathcal{D}$ is two-spherical and without residues of type G_2 or direct factors of type A_1 and R is a rank two rigid ring (Definition 4.1.5) containing $\mathbb{Q}$. This will be used to show Mostow-rigidity of $G_\mathcal{D}(R)$ in $G_\mathcal{D}(\mathbb{F})$, where $\mathbb{F}$ denotes the field of fractions of R.

4.1 Assumptions and preliminaries

We assume throughout this chapter that R is an integral domain. Moreover, we denote by $\mathbb{F}$ the field of fractions of the ring R. Then there is a canonical embedding $R \hookrightarrow \mathbb{F}$ which induces an embedding $G_\mathcal{D}(R) \hookrightarrow G_\mathcal{D}(\mathbb{F})$ (cf. also axiom (KMG4) of Definition 2.5.2). We therefore view $G_\mathcal{D}(R)$ as a subgroup of $G_\mathcal{D}(\mathbb{F})$ as above.

Definition 4.1.1. Let $T_\mathbb{F}$ be a maximal $\mathbb{F}$-torus of the group $G_\mathcal{D}(\mathbb{F})$. Then the group $T_R := T_\mathbb{F} \cap G_\mathcal{D}(R)$ is called **group of R-rational points** of $T_\mathbb{F}$.

We call $T_\mathbb{F}$ **defined over** R if the group of R-rational points T_R is (algebraically) isomorphic to $(R^\times)^n$, where $n = \mathrm{rk}(\mathcal{D})$.

Similarly, if $\Sigma \subseteq \Delta(G_\mathcal{D}(\mathbb{F}))$ is a twin apartment, then we call Σ **defined over** R if the torus $\mathrm{Fix}_{G_\mathcal{D}(\mathbb{F})}(\Sigma)$ is defined over R. $\square$

Remark 4.1.2. (i) Let $T_{\mathbb{F}}$ be the standard maximal torus of the group $G_{\mathcal{D}}(\mathbb{F})$. Then $T_R \cong (R^\times)^n$ follows by construction, hence $T_{\mathbb{F}}$ is defined over R, see also Remark 2.5.9.

(ii) We shall see in Corollary 4.3.4 that if $T_{\mathbb{F}}$ is any maximal $\mathbb{F}$-torus of $G_{\mathcal{D}}(\mathbb{F})$ defined over a rank two-rigid ring R, then its R-rational points T_R are conjugate to the R-rational points of the standard torus via an element of $G_{\mathcal{D}}(R)$. In particular, the R-rational points of any two maximal $\mathbb{F}$-tori defined over R are conjugate under an element of $G_{\mathcal{D}}(R)$.

Lemma 4.1.3. *Let $T_{\mathbb{F}} \leq G_{\mathcal{D}}(\mathbb{F})$ be a maximal $\mathbb{F}$-torus and set $N := N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$.*
Then $T_R = nT_Rn^{-1}$ for all $n \in N$.

Proof. Let $n \in N$. Then $n = s_{\alpha_1}(u_1)\cdots s_{\alpha_k}(u_k)$ for some $k \in \mathbb{N}$, $u_i \in \mathbb{F}^\times$ and simple roots α_i. Hence it suffices to show that the R-rational points T_R of $T_{\mathbb{F}}$ are invariant under the application of $s_\alpha(u)$, from which the result will follow by induction.
Let $t \in T_{\mathbb{F}}$, $\alpha \in \Pi$ and $u \in \mathbb{F}^\times$. By relation (iii) of Theorem 2.5.8 we conclude that

$$s_\alpha(u) = u^{h_\alpha} s_\alpha. \tag{4.1}$$

Since $T_{\mathbb{F}}$ is abelian, we obtain

$$\begin{aligned} s_\alpha(u)ts_\alpha(u)^{-1} &\overset{(4.1)}{=} u^{h_\alpha}s_\alpha t s_\alpha^{-1}u^{-h_\alpha} \\ &\overset{2.5.8(ii)}{=} u^{h_\alpha}s_\alpha(t)u^{-h_\alpha} \\ &= s_\alpha(t), \end{aligned}$$

where $s_\alpha(t)$ denotes the image of t under the reflection on $T_{\mathbb{F}}$ induced by s_α. Since $s_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1) \in G_{\mathcal{D}}(R)$ is of finite order, it follows that $t \in T_R$ if and only if $s_\alpha(t) \in T_R$, which implies the claim. $\square$

Lemma 4.1.4. *Assume that the ring R has at least three units. Let $T_{\mathbb{F}}$ and $T'_{\mathbb{F}}$ be two maximal $\mathbb{F}$-tori which are defined over R.*
If $T_R = T'_R$, then $T_{\mathbb{F}} = T'_{\mathbb{F}}$. In other words, a maximal $\mathbb{F}$-torus defined over R is uniquely determined by its R-rational points.

Proof. Since R has at least three units by assumption, it follows from [Cap09, Lemma 4.8 (iii)] that $T_R = T'_R$ fixes a unique twin apartment Σ of $\Delta(G_{\mathcal{D}}(\mathbb{F}))$. Since $|\mathbb{F}^\times| \geq |R^\times| \geq 3$, by loc. cit. every maximal $\mathbb{F}$-torus is the pointwise stabiliser of a unique twin apartment, and we conclude that $T_{\mathbb{F}} = \mathrm{Fix}_{G_{\mathcal{D}}(\mathbb{F})}(\Sigma) = T'_{\mathbb{F}}$. $\square$

Definition 4.1.5. Let R be a commutative unital ring. Then R is called **rank two-rigid** if

(i) R is an integral domain,

(ii) the group of units satisfies $|R^\times| \geq 3$ and there is a unit $u \in R^\times$ such that $u^2 - 1 \in R^\times$. □

Let $X \in \{A_2, B_2\}$ be an irreducible rank two group scheme. Then we call the pair $(R, \mathbb{K})$ a **(topological) rank two-rigid pair** if R is rank two-rigid and additionally $\mathbb{K}$ is a field containing R such that the group $X(R)$ is (topologically) Mostow-rigid in $X(\mathbb{K})$.

Let us briefly motivate this definition. A field of fractions $\mathbb{F}$ together with an embedding $R \hookrightarrow \mathbb{F}$ exists only if R is an integral domain, which is why we impose (i). We need (ii) to be able to prove Lemma 4.1.4 (which relates the R-rational points of a maximal $\mathbb{F}$-torus defined over R to a unique twin apartment of $\Delta(G_\mathcal{D}(\mathbb{F}))$) and Lemma 4.3.1. In the proof of the isomorphism theorem, we shall also assume that $\mathbb{Q} \subseteq R$ due to technical restrictions.

Note also that if R is an integral domain and $\mathbb{F}$ is its field of fractions, then every automorphism of R induces a unique automorphism of $\mathbb{F}$, cf. [Coh00, Theorem 1.6].

Remark 4.1.6. The strategy of Section 4.5 relies on the fact that the automorphisms of the fundamental subgroups of rank one and two determine the automorphisms of the ambient Kac–Moody group. Hence we need a description of the automorphism group of possible irreducible fundamental rank two subgroups. For the group $G_2(R)$, there is no general description known to the author, which is why residues of type G_2 are excluded in Definition 4.1.5.

Remark 4.1.7. Assume that R is rank two-rigid of characteristic not 2 and let $\mathbb{F}$ be the field of fractions of R. Let X be an irreducible rank two group scheme of type A_2 or B_2. Then the outer automorphisms of $X(R)$ and $X(\mathbb{F})$ coincide by [HO89, Theorem 3.2.29], [Ste68, Theorem 30] and [O'M68, Theorem C].

Moreover, by matrix calculations we see that the group $X(R)$ is self-normalising in $X(\mathbb{F})$ (up to the centre of $X(\mathbb{F})$), hence every inner automorphism of the group $X(R)$ uniquely lifts to an inner automorphism of the group $X(\mathbb{F})$. It follows that if R is rank two-rigid, then $(R, \mathbb{F})$ is a rank two-rigid pair.

The isomorphism problem for split Kac–Moody groups defined over fields of cardinality at least four has been solved in [Cap09] (see also [Cap05]). The author applied this machinery in [Mar07, Chapter 4] to solve the isomorphism problem for unitary forms of $G_\mathcal{D}(\mathbb{F}_{q^2})$ with respect to the twisted Chevalley involution. Recently, G. Hainke presented a refinement in [Hai10, Chapter 6], describing the isomorphisms of quasi-split Kac–Moody groups defined over fields of characteristic zero.

We shall apply the above strategy in our setting, determining the possible isomorphisms $G_\mathcal{D}(R) \stackrel{\cong}{\to} G'_{\mathcal{D}'}(R)$ if R is a rank two-rigid ring containing $\mathbb{Q}$ and $\mathcal{D}$ is two-spherical and without residues of type G_2 or direct factors of type A_1. As corollary of the solution to the isomorphism problem, we conclude that any automorphism of the group $G_\mathcal{D}(R)$ lifts to a unique automorphism of the group $G_\mathcal{D}(\mathbb{F})$ defined over the field of fractions of R. Moreover, we conclude that any isomorphism $G_\mathcal{D}(R) \stackrel{\cong}{\to} G'_{\mathcal{D}'}(R)$ induces an isomorphism of root data $\mathcal{D} \stackrel{\cong}{\to} \mathcal{D}'$.

Remark 4.1.8. As in the statement of Theorem 4.5.5, with the method of proof presented here it is in fact crucial to assume that W is two-spherical. Indeed, the strategy of [Cap09] used in order to solve the isomorphism problem for split Kac–Moody groups defined over fields is to show that rank one groups are mapped to rank one groups under an isomorphism. Secondly, the automorphisms of the fundamental rank one subgroups determine the automorphisms of the ambient Kac–Moody group.

For split Kac–Moody groups defined over rings, this cannot work, as for the group $\mathrm{SL}_2(R)$ there exist exceptional automorphisms which do not occur in higher rank, see Example 4.1.9. Hence we follow the strategy of showing that the automorphisms of the fundamental rank two subgroups determine the automorphisms of the Kac–Moody group, which is why we impose the condition of two-sphericity.

Example 4.1.9 (Reiner-type automorphism)**.** Let $\mathbb{F}$ be a field and let $R := \mathbb{F}[X]$ be the polynomial ring over $\mathbb{F}$ in one indeterminate. Consider the standard basis $B := \{1, X, X^2, X^3, \ldots\}$ for $\mathbb{F}[X]$ over $\mathbb{F}$ and choose a different $\mathbb{F}$-basis $B' = \{1, Y_1, Y_2, \ldots\}$. The group $\mathrm{SL}_2(R)$ is generated by the matrices $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & X^i \\ 0 & 1 \end{pmatrix}$ $(i \in \mathbb{N})$, $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $(a \in R^\times)$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Writing an element of $\mathrm{SL}_2(R)$ as product of the above generators, the map

$$\begin{pmatrix} 1 & X^i \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & Y_i \\ 0 & 1 \end{pmatrix}$$

extends to an automorphism of $\mathrm{SL}_2(R)$ by replacing every factor of the form $\begin{pmatrix} 1 & X^i \\ 0 & 1 \end{pmatrix}$ with the corresponding factor $\begin{pmatrix} 1 & Y_i \\ 0 & 1 \end{pmatrix}$. In general this automorphism (called **automorphism of Reiner type**) is not inner-by-diagonal-by-ring. See [Rei57] for details.

4.2 Conjugacy classes of tori

Let R be a rank two-rigid ring and let $T_{\mathbb{F}}$ be a maximal $\mathbb{F}$-torus of $G_{\mathcal{D}}(\mathbb{F})$ defined over R, with group of R-rational points T_R. In this section we present an answer to the following question: Under which conditions do elements $g \in G_{\mathcal{D}}(\mathbb{F})$ satisfy $gT_Rg^{-1} \subseteq G_{\mathcal{D}}(R)$?

For this, consider the following condition on $T_{\mathbb{F}}$:

(∗) For all $g \in G_{\mathcal{D}}(\mathbb{F})$ the following holds. The torus gT_Rg^{-1} is contained in $G_{\mathcal{D}}(R)$ if and only if there exists $n \in N = N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$ such that $gn \in G_{\mathcal{D}}(R)$.

Remark 4.2.1. Note that the if-part of condition $(*)$ is always satisfied.

Indeed, if there is $n \in N = N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$ such that $gn \in G_{\mathcal{D}}(R)$, then $G_{\mathcal{D}}(R) \supseteq g(nT_Rn^{-1})g^{-1} = gT_Rg^{-1}$ by Lemma 4.1.3.

The motivation for the rather technical condition $(*)$ is that using it we can show that maximal $\mathbb{F}$-tori defined over R have $G_{\mathcal{D}}(R)$-conjugate groups of R-rational points. In Section 4.3, we will show that any maximal $\mathbb{F}$-torus defined over R satisfies conditon $(*)$.

Proposition 4.2.2. *Let $\mathbb{F}$ be a field with at least three units. Let $T_{\mathbb{F}}$ and $T'_{\mathbb{F}}$ be two maximal $\mathbb{F}$-tori of $G_{\mathcal{D}}(\mathbb{F})$. Then there exist two opposite Borel subgroups B_1, B_2 such that*

(i) $T_{\mathbb{F}} \leq B_1$,

(ii) $u_1T_{\mathbb{F}}u_1^{-1} \leq B_1 \cap B_2$ *for some* $u_1 \in R_u(B_1)$ *and*

(iii) $u_2u_1T_{\mathbb{F}}u_1^{-1}u_2^{-1} = T'_{\mathbb{F}}$ *for some* $u_2 \in R_u(B_2)$.

Moreover, the element u_2u_1 is unique up to multiplication by an element of $N = N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$.

Proof. We denote by $\Sigma(T_{\mathbb{F}})$ and $\Sigma(T'_{\mathbb{F}})$ the unique twin apartments of $\Delta(G_{\mathcal{D}}(\mathbb{F}))$ which are fixed by $T_{\mathbb{F}}$ and $T'_{\mathbb{F}}$, respectively. Let $c \in \Sigma(T_{\mathbb{F}})$ be a chamber and let d be a chamber of $\Sigma(T'_{\mathbb{F}})$ opposite c (which exists by [AB08, Corollary 5.141 (ii)]). Finally, let $\Sigma(c, d)$ denote the (unique) twin apartment spanned by c and d.

Now $\Sigma(c, d)$ and $\Sigma(T_{\mathbb{F}})$, both containing c, are conjugate via a (unique) unipotent element u_1 of $B_{\mathbb{F}}(c) := \mathrm{Stab}_{G_{\mathcal{D}}(\mathbb{F})}(c)$ by [AB08, Corollary 7.67].

Since $\Sigma(c, d)$ and $\Sigma(T'_{\mathbb{F}})$ are two twin apartments containing d, by the same argument they are conjugate via a (unique) element $u_2 \in R_u(B_{\mathbb{F}}(d))$, see Figure 4.1. Note that $B_{\mathbb{F}}(c)$ and $B_{\mathbb{F}}(d)$ are opposite Borel subgroups, as c and d are opposite chambers by construction. In particular, we obtain $u_2u_1T_{\mathbb{F}}u_1^{-1}u_2^{-1} = T'_{\mathbb{F}}$.

The last claim follows from the fact that the transport in $G_{\mathcal{D}}(\mathbb{F})$ from $T_{\mathbb{F}}$ to $T'_{\mathbb{F}}$ is a coset of the normaliser $N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$. □

By Proposition 4.2.2, any two maximal $\mathbb{F}$-tori are conjugate under the product of two unipotent elements which are unique up to the normaliser of one of the tori. Since the R-rational points of an $\mathbb{F}$-torus $T_{\mathbb{F}}$ are invariant under the action of the normaliser $N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$ (see Lemma 4.1.3), we may focus our analysis on the two unipotent elements, which are uniquely determined by the two tori and the two Borel subgroups as in Proposition 4.2.2. In view of condition $(*)$ it therefore suffices to consider the action of unipotent elements on tori by conjugation.

Lemma 4.2.3. *Let $B_{\mathbb{F}}$ be a Borel subgroup and let $T_{\mathbb{F}} \leq B_{\mathbb{F}}$ be a maximal $\mathbb{F}$-torus. Let $u \in R_u(B_{\mathbb{F}}) = U_{\mathbb{F}}$ and let $t \in T_{\mathbb{F}}$.*

Then $utu^{-1} \in G_{\mathcal{D}}(R)$ implies that $t \in T_R$. In particular, the identity $uT_{\mathbb{F}}u^{-1} \cap G_{\mathcal{D}}(R) = uT_Ru^{-1} \cap G_{\mathcal{D}}(R)$ holds.

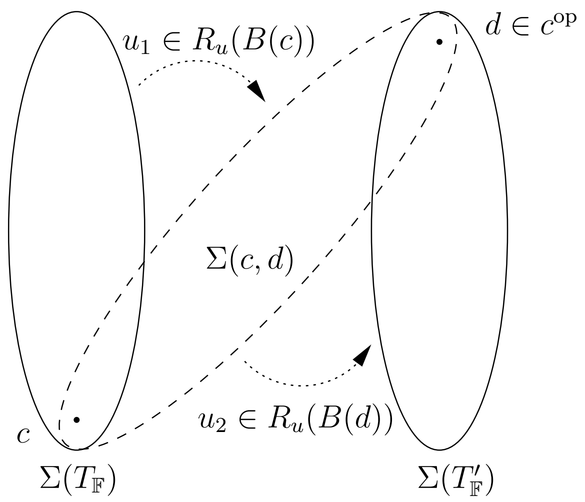

Figure 4.1.: The argument used in Proposition 4.2.2: Any two twin apartments are conjugate under two unipotent elements of opposite Borel subgroups.

Proof. Assume that $t \notin T_R$. Since the torus $T_{\mathbb{F}}$ normalises the unipotent radical $U_{\mathbb{F}}$, there exists $v \in U_{\mathbb{F}}$ such that the identity

$$utu^{-1} = t(t^{-1}ut)u^{-1} = tvu^{-1} \in T_{\mathbb{F}} \ltimes U_{\mathbb{F}}$$

holds. Moreover, since $T_{\mathbb{F}}$ and $U_{\mathbb{F}}$ intersect trivially, the elements $t \in T_{\mathbb{F}}$ and $vu^{-1} \in U_{\mathbb{F}}$ are uniquely determined by the product tvu^{-1}.

As the R-rational points of $B_{\mathbb{F}}$ are obtained via $B_R = T_R \ltimes U_R = (T_{\mathbb{F}} \cap G_{\mathcal{D}}(R)) \ltimes (U_{\mathbb{F}} \cap G_{\mathcal{D}}(R))$ and by assumption $t \notin T_R$, it follows that $utu^{-1} = tvu^{-1} \notin G_{\mathcal{D}}(R)$. □

Lemma 4.2.4. *Let $B_{\mathbb{F}}$ be a Borel subgroup, let $T_{\mathbb{F}} \leq B_{\mathbb{F}}$ be a maximal $\mathbb{F}$-torus and let $u \in R_u(B_{\mathbb{F}}) = U_{\mathbb{F}}$. Assume that $T_{\mathbb{F}}$ satisfies condition $(*)$.*

Then the maximal $\mathbb{F}$-torus $uT_{\mathbb{F}}u^{-1}$ satisfies condition $(*)$ if and only if $u \in U_R$.

Proof. Suppose that for some unipotent element $u_0 \in U_R \subseteq G_{\mathcal{D}}(R)$ the torus $u_0T_{\mathbb{F}}u_0^{-1}$ does not satisfy condition $(*)$. Then there exists $g \in G_{\mathcal{D}}(\mathbb{F})$ such that $g(u_0T_Ru_0^{-1})g^{-1} \subseteq G_{\mathcal{D}}(R)$ and $gn \notin G_{\mathcal{D}}(R)$ for all $n \in N_{G_{\mathcal{D}}(\mathbb{F})}(u_0T_{\mathbb{F}}u_0^{-1}) = u_0N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})u_0^{-1}$. We therefore conclude that $(u_0^{-1}gu_0)T_R(u_0^{-1}g^{-1}u_0) \subseteq G_{\mathcal{D}}(R)$, but $(u_0^{-1}gu_0)(u_0^{-1}nu_0) = u_0^{-1}gnu_0 \notin G_{\mathcal{D}}(R)$ for all $u_0^{-1}nu_0 \in N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$. Hence $T_{\mathbb{F}}$ does not satisfy $(*)$, a contradiction.

To show the only if-part, assume that $uT_{\mathbb{F}}u^{-1}$ satisfies $(*)$. Then it follows that

$$u^{-1}(uT_{\mathbb{F}}u^{-1} \cap G_{\mathcal{D}}(R))u \overset{4.2.3}{=} u^{-1}(uT_Ru^{-1} \cap G_{\mathcal{D}}(R))u \subseteq T_R \subseteq G_{\mathcal{D}}(R).$$

Hence by condition $(*)$ there exists $n \in N_{G_{\mathcal{D}}(\mathbb{F})}(uT_{\mathbb{F}}u^{-1})$ such that $u^{-1}n \in G_{\mathcal{D}}(R)$. As the group $N_{G_{\mathcal{D}}(\mathbb{F})}(uT_{\mathbb{F}}u^{-1})$ intersects $U_{\mathbb{F}}$ trivially, we conclude that $u^{-1} \in G_{\mathcal{D}}(R)$, from which $u \in G_{\mathcal{D}}(R)$ is immediate. □

Lemma 4.2.5. *Let $T_{\mathbb{F}}$ be a maximal $\mathbb{F}$-torus satisfying $(*)$ and let $B_{\mathbb{F},+}$, $B_{\mathbb{F},-}$ be opposite Borel subgroups. Let $u_0 \in R_u(B_{\mathbb{F},+})$ be such that*

$$u_0 T_{\mathbb{F}} u_0^{-1} \leq B_{\mathbb{F},+} \cap B_{\mathbb{F},-}.$$

If there exists $u \in R_u(B_{\mathbb{F},-})$ such that the torus $uu_0T_{\mathbb{F}}u_0^{-1}u^{-1}$ satisfies $(*)$, then the torus $u_0T_{\mathbb{F}}u_0^{-1}$ satisfies $(*)$ as well.

Proof. Suppose that the torus $uu_0T_{\mathbb{F}}u_0^{-1}u^{-1}$ satisfies condition $(*)$, but the torus $u_0T_{\mathbb{F}}u_0^{-1}$ does not. Since $T_{\mathbb{F}}$ satisfies $(*)$ by assumption, Lemma 4.2.4 implies that the element u_0 is not contained in $G_{\mathcal{D}}(R)$. As the unipotent radicals $U_{\mathbb{F},+}$ and $U_{\mathbb{F},-}$ intersect trivially, we conclude that also the products uu_0 and $u_0^{-1}u^{-1}$ are not contained in $G_{\mathcal{D}}(R)$.

Finally, since every non-trivial element of $N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$ is a product of at least three unipotent elements (cf. [Tit87, Section 3.6]), also $N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$ and the pointwise product $(U_{\mathbb{F},-}).(U_{\mathbb{F},+})$ intersect trivially. In particular, this implies that $u_0^{-1}u^{-1}n \notin G_{\mathcal{D}}(R)$ for all $n \in N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$.

Using Lemma 4.2.3 again, we compute

$$(uu_0)^{-1}(uu_0T_{\mathbb{F}}u_0^{-1}u^{-1} \cap G_{\mathcal{D}}(R))uu_0 = (uu_0)^{-1}(uu_0T_Ru_0^{-1}u^{-1} \cap G_{\mathcal{D}}(R))uu_0 \subseteq T_R,$$

and hence $uu_0T_{\mathbb{F}}u_0^{-1}u^{-1}$ does not satisfy $(*)$, a contradiction. □

In summary, we have shown the following result:

Proposition 4.2.6. *Let R be rank two-rigid, and let $T_{\mathbb{F}}$, $T'_{\mathbb{F}}$ be two maximal $\mathbb{F}$-tori which both satisfy condition $(*)$. Then their respective groups of R-rational points are conjugate in $G_{\mathcal{D}}(R)$.*

Proof. Let $T_{\mathbb{F}}$ and $T'_{\mathbb{F}}$ be two maximal $\mathbb{F}$-tori satisfying $(*)$. By strong transitivity of $G_{\mathcal{D}}(\mathbb{F})$, there exists $g \in G_{\mathcal{D}}(\mathbb{F})$ such that $gT_{\mathbb{F}}g^{-1} = T'_{\mathbb{F}}$.

By Proposition 4.2.2, there exist two unipotent elements u_1, u_2 contained in opposite Borel subgroup such that

$$T'_{\mathbb{F}} = gT_{\mathbb{F}}g^{-1} = u_2u_1T_{\mathbb{F}}u_1^{-1}u_2^{-1}.$$

Moreover, the product u_2u_1 differs from g only by an element of N, i.e. $u_2u_1 = gn$ for some $n \in N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$. Hence the torus $gT_{\mathbb{F}}g^{-1}$ satisfies $(*)$ if and only if the torus $u_2u_1T_{\mathbb{F}}u_1^{-1}u_2^{-1}$ does. By Lemmas 4.2.4 and 4.2.5, the latter is true if and only if the product $u_2u_1 \in G_{\mathcal{D}}(R)$. Hence $gT_{\mathbb{F}}g^{-1}$ satisfies $(*)$ if and only if $u_2u_1 = gn^{-1} \in G_{\mathcal{D}}(R)$.

In particular, the groups T_R and T'_R are conjugate under the product $u_2u_1 \in G_{\mathcal{D}}(R)$. □

4.3 Tori defined over R

Let R be a rank two-rigid ring. Let R be a rank two-rigid ring. We just saw in Proposition 4.2.6 that the groups of R-rational points of two maximal $\mathbb{F}$-tori which satisfy condition

$(*)$ are conjugate in $G_{\mathcal{D}}(R)$. It remains to show that there exist tori which satisfy $(*)$, so that our results apply. We shall prove in this section that every torus defined over R in fact satisfies condition $(*)$. This knowledge will be used in the analysis of isomorphisms, as *definedness over* R is a property on the R-rational points which is preserved by an isomorphism.

Lemma 4.3.1. *Let $B_{\mathbb{F}}$ be a Borel subgroup, let $U_{\mathbb{F}}$ be its unipotent radical, let $T_{\mathbb{F}} \leq B_{\mathbb{F}}$ be a maximal $\mathbb{F}$-torus defined over R, and let $u \in U_{\mathbb{F}}$. Then $uT_Ru^{-1} \subseteq G_{\mathcal{D}}(R)$ if and only if $u \in U_R$.*

Proof. Note that, clearly, $u \in U_R$ implies $uT_Ru^{-1} \subseteq G_{\mathcal{D}}(R)$ so that we can concentrate on the other implication. Write $u = \prod_{i=1}^n x_{\alpha_i}(r_i)$ with α_i simple and n minimal, so that we may assume without loss of generality that $u \in U_R$ if and only if all parameters r_i satisfy $r_i \in R$. We will prove the result by induction on n. Using the additivity of the map x_{α_1}, we compute for $t \in T_R$

$$\begin{aligned} x_{\alpha_1}(r_1)tx_{\alpha_1}(-r_1) &= tt^{-1}x_{\alpha_1}(r_1)tx_{\alpha_1}(-r_1) \\ &\overset{2.5.8(i)}{=} tx_{\alpha_1}(t(\alpha_1)^{-1}r_1)x_{\alpha_1}(-r_1) \\ &= tx_{\alpha_1}((t(\alpha_1)^{-1}-1)r_1). \end{aligned}$$

The element $tx_{\alpha_1}((t(\alpha_1)^{-1}-1)r_1)$ is contained in $G_{\mathcal{D}}(R)$ if and only if $(t(\alpha_1)^{-1}-1)r_1 \in R$. By choosing $u \in R^\times$ with $u^2-1 \in R^\times$, we see that for $t := t_{\alpha_1}(u^{-1})$ the product $(t(\alpha_1)^{-1}-1)r_1 = (u^2-1)r_1 \in R$ if and only if $r_1 \in R$, i.e., if and only if $x_{\alpha_1}(r_1) \in U_R \subseteq G_{\mathcal{D}}(R)$. This establishes the basis of the induction.

Now suppose $n \geq 2$ and let $t' := x_{\alpha_n}(r_n)tx_{\alpha_n}(-r_n)$. By a computation as above there exists $v \in U_{\mathbb{F}}$ such that

$$utu^{-1} = x_{\alpha_1}(r_1)\cdots x_{\alpha_{n-1}}(r_{n-1})t'x_{\alpha_{n-1}}(-r_{n-1})\cdots x_{\alpha_1}(-r_1) = t'v.$$

If $x_{\alpha_n}(r_n) \notin U_R$, then, by our assumption on the parameters r_i, one has $u \notin U_R$. Moreover, by the induction hypothesis, $t' \notin G_{\mathcal{D}}(R)$, whence $utu^{-1} = t'v \notin G_{\mathcal{D}}(R)$. If, on the other hand, $x_{\alpha_n}(r_n) \in U_R$, then $t' \in G_{\mathcal{D}}(R)$, and the claim follows by induction applied to the torus $T'_{\mathbb{F}} := x_{\alpha_n}(r_n)T_{\mathbb{F}}x_{\alpha_n}(-r_n)$. □

Lemma 4.3.2. *Let $B_{\mathbb{F},+}$ and $B_{\mathbb{F},-}$ be opposite Borel subgroups, let $T_{\mathbb{F}} \leq B_{\mathbb{F},+}$ be a maximal $\mathbb{F}$-torus and let $T_R = T_{\mathbb{F}} \cap G_{\mathcal{D}}(R)$ be the group of R-rational points of $T_{\mathbb{F}}$. Let $u_0 \in R_u(B_{\mathbb{F},+})$ be such that the torus $u_0T_Ru_0^{-1} \leq B_{\mathbb{F},+} \cap B_{\mathbb{F},-}$ is not contained in $G_{\mathcal{D}}(R)$. Then for all $u \in R_u(B_{\mathbb{F},-})$ the torus $uu_0T_Ru_0^{-1}u^{-1}$ is not contained in $G_{\mathcal{D}}(R)$.*

Proof. By hypothesis there exists an element $t \in u_0T_Ru_0^{-1}$ which is not contained in $G_{\mathcal{D}}(R)$. Let $u \in R_u(B_{\mathbb{F},-})$. Since the unipotent radical $R_u(B_{\mathbb{F},-})$ is normalised by the torus $u_0T_Ru_0^{-1}$, there exists $v \in R_u(B_{\mathbb{F},-})$ such that the equality

$$\begin{aligned} utu^{-1} &= t(t^{-1}ut)u^{-1} \\ &= tvu^{-1} \end{aligned}$$

holds. Moreover, as $u_0 T_R u_0^{-1}$ and $R_u(B_{\mathbb{F},-})$ intersect trivially, both $t \in u_0 T_R u_0^{-1}$ and $vu^{-1} \in R_u(B_{\mathbb{F},-})$ are uniquely determined. Since $t \notin G_{\mathcal{D}}(R)$ by assumption, it follows that $tvu^{-1} \notin G_{\mathcal{D}}(R)$ and hence $uu_0 T_R u_0^{-1} u^{-1} \nsubseteq G_{\mathcal{D}}(R)$. □

With the help of the above results, we may conclude the following structural result concerning tori.

Proposition 4.3.3. *Let R be a rank two-rigid ring and let $\mathbb{F}$ be its field of fractions. Let $T_{\mathbb{F}}$ be a maximal $\mathbb{F}$-torus of the group $G_{\mathcal{D}}(\mathbb{F})$ which is defined over R.*

Then the torus $T_{\mathbb{F}}$ satisfies condition $(*)$.

Proof. Let $g \in G_{\mathcal{D}}(\mathbb{F})$ and assume that

$$gT_R g^{-1} \subseteq G_{\mathcal{D}}(R). \tag{4.2}$$

We have to show that there exists $n \in N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$ such that $gn \in G_{\mathcal{D}}(R)$.

By Lemma 4.1.4, the torus $T'_{\mathbb{F}} := gT_{\mathbb{F}}g^{-1}$ is the unique maximal $\mathbb{F}$-torus which contains $T'_R := gT_R g^{-1}$. Proposition 4.2.2 now implies the existence of two unipotent elements u_1, u_2 contained in opposite Borel subgroups B_1, B_2 such that $T'_{\mathbb{F}} = u_2 u_1 T_{\mathbb{F}} u_1^{-1} u_2^{-1}$. Moreover, $u_2 u_1 = gn$ for some $n \in N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$. Further, by Lemma 4.1.3 the identity

$$u_2 u_1 T_R u_1^{-1} u_2^{-1} = gnT_R n^{-1} g^{-1} = gT_R g^{-1} = T'_R \tag{4.3}$$

holds. We shall now see that $u_2 u_1 \in G_{\mathcal{D}}(R)$.

By Lemma 4.3.1 we have $u_1 T_R u_1^{-1} \subseteq G_{\mathcal{D}}(R)$ if and only if $u_1 \in R_u(B_1) \cap G_{\mathcal{D}}(R)$. We therefore distinguish two cases.

Case 1: If $u_1 \in G_{\mathcal{D}}(R)$, then $u_1 T_R u_1^{-1} \subseteq G_{\mathcal{D}}(R)$. Another application of Lemma 4.3.1 to the torus $u_1 T_{\mathbb{F}} u_1^{-1}$ implies that $u_2 \in G_{\mathcal{D}}(R)$ if and only if $u_2 u_1 T_R u_1^{-1} u_2^{-1} \overset{(4.3)}{=} T'_R \subseteq G_{\mathcal{D}}(R)$. As the latter is true by assumption, it follows that $u_2 \in G_{\mathcal{D}}(R)$ and hence $gn = u_2 u_1 \in G_{\mathcal{D}}(R)$.

Case 2: If $u_1 \notin G_{\mathcal{D}}(R)$, then by Lemma 4.3.2 the torus $u_2 u_1 T_R u_1^{-1} u_2^{-1} = gT_R g^{-1}$ is not contained in $G_{\mathcal{D}}(R)$, a contradiction to the assumption in (4.2).

Hence the assumption (4.2) implies that there is $n \in N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$ such that $u_2 u_1 = gn \in G_{\mathcal{D}}(R)$. It follows that $T_{\mathbb{F}}$ satisfies condition $(*)$. □

We note the following consequences for later reference.

Corollary 4.3.4. *Let $T_{\mathbb{F}}$ and $T'_{\mathbb{F}}$ be maximal $\mathbb{F}$-tori of $G_{\mathcal{D}}(\mathbb{F})$ defined over R. Then their groups of R-rational points T_R and T'_R are conjugate in $G_{\mathcal{D}}(R)$.*

Proof. Combine Proposition 4.3.3 with Proposition 4.2.6. □

Corollary 4.3.5. *The group $G_{\mathcal{D}}(R)$ is self-normalising in $G_{\mathcal{D}}(\mathbb{F})$ up to centre, i.e. the normaliser of $G_{\mathcal{D}}(R)$ in $G_{\mathcal{D}}(\mathbb{F})$ coincides with $Z(G_{\mathcal{D}}(\mathbb{F})).G_{\mathcal{D}}(R)$.*

Proof. Let $T_{\mathbb{F}}$ be a maximal $\mathbb{F}$-torus of $G_{\mathcal{D}}(\mathbb{F})$ defined over R and let T_R be its subgroup of R-rational points. Given $g \in G_{\mathcal{D}}(\mathbb{F})$, it follows from Proposition 4.3.3 that $gT_Rg^{-1} \subseteq G_{\mathcal{D}}(R)$ if and only if $g \in G_{\mathcal{D}}(R).N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}})$. We conclude that

$$N_{G_{\mathcal{D}}(\mathbb{F})}(G_{\mathcal{D}}(R)) \subseteq \bigcap_{T_{\mathbb{F}} \text{ defined over } R} G_{\mathcal{D}}(R).N_{G_{\mathcal{D}}(\mathbb{F})}(T_{\mathbb{F}}) = G_{\mathcal{D}}(R).Z(G_{\mathcal{D}}(\mathbb{F})).$$

The reverse inclusion is obvious. □

4.4 A sufficient condition for the isomorphism theorem

By Corollary 4.3.4, any two maximal $\mathbb{F}$-tori defined over R have $G_{\mathcal{D}}(R)$-conjugate groups of R-rational points. This will allow to *normalise* an isomorphism, i.e. to assume without loss of generality that under $\varphi\colon G_{\mathcal{D}}(R) \overset{\cong}{\to} G'_{\mathcal{D}'}(R)$ the group of R-rational points of the standard torus $T_{\mathbb{F}}$ is mapped to the group of R-rational points of the standard torus $T'_{\mathbb{F}}$. In this section, we describe the possible isomorphisms, assuming that $\varphi\colon G_{\mathcal{D}}(R) \overset{\cong}{\to} G'_{\mathcal{D}'}(R)$ preserves subgroups of rank one. We continue to assume that the ring R is rank two-rigid.

Proposition 4.4.1. *Let $G_{\mathcal{D}}(R)$ and $G'_{\mathcal{D}'}(R)$ be two-spherical split Kac–Moody groups without residues of type G_2 or direct factors of type A_1 over a rank two-rigid ring R of characteristic not 2. Denote by $A = (a_{ij})_{1\le i,j\le n}$ and $A' = (a'_{ij})_{1\le i,j\le n'}$ their respective generalised Cartan matrices and by I and I' the index sets of A and A', respectively. Let $\varphi\colon G_{\mathcal{D}}(R) \to G'_{\mathcal{D}'}(R)$ be an isomorphism. Assume that $\varphi(T_R) = T'_R$ and that*

$$\begin{aligned} & \{\varphi(g\varphi_i(\mathrm{SL}_2(R))g^{-1}) \mid i \in I, g \in N_{G_{\mathcal{D}}(R)}(T)\} \\ = \; & \{g\varphi'_i(\mathrm{SL}_2(R))g^{-1} \mid i \in I', g \in N_{G'_{\mathcal{D}'}(R)}(T')\}. \end{aligned}$$

Then there exist

(i) an element $n \in N_{G'_{\mathcal{D}'}(R)}(T'_R)$,

(ii) a bijection $\psi\colon I \to I'$ of the index sets of the simple roots satisfying $a_{ij} = a'_{\psi(i)\psi(j)}$,

(iii) for all $i \in I$ an automorphism $\gamma_i \in \mathrm{Aut}(\mathrm{SL}_2(R))$ such that

$$\begin{tikzcd} \mathrm{SL}_2(R) \arrow[r, "\gamma_i"] \arrow[d, "\varphi_i"'] & \mathrm{SL}_2(R) \arrow[d, "\varphi'_{\psi(i)}"] \\ G_{\mathcal{D}}(R) \arrow[r, "c_n\circ\varphi"] & G'_{\mathcal{D}'}(R) \end{tikzcd}$$

commutes, where c_n denotes conjugation with n.

Proof. We may assume without loss of generality that $G_{\mathcal{D}}(R)$ and $G'_{\mathcal{D}'}(R)$ are of rank at least two. Recall that the real roots $\Phi^{re} = W.\Pi$, $(\Phi')^{re} := W'.\Pi'$ are the respective images of the simple roots under the action of the Weyl groups on the root lattices, cf. Section 2.5.

Let $\alpha \in \Phi^{re}$ be a real root, then by definition there exist $w \in W$ and a simple root $\alpha_i \in \Pi$ with $\alpha = w.\alpha_i$. Set $G_\alpha = \tilde{w} G_{\alpha_i} \tilde{w}^{-1}$, where $\tilde{w} \in G_{\mathcal{D}}(R)$ denotes the canonical representative of $w \in W$ in $G_{\mathcal{D}}(R)$, and G_{α_i} is the fundamental rank one group associated to the simple root α_i. This definition of G_α is in fact independent of the choice of i and w and depends only on α. We also have $G_\alpha = G_{-\alpha}$ by construction.

By the assumption made, for all $\alpha \in \Phi^{re}$ there exists a real root $\alpha' \in (\Phi')^{re}$ such that $\varphi(G_\alpha) = G_{\alpha'}$. As above, the root α' is uniquely determined up to sign. It follows that the isomorphism φ induces a map π from pairs of opposite roots of Φ^{re} to those of $(\Phi')^{re}$ via $\{\pm\alpha\} \mapsto \{\pm\alpha'\}$. Since reflections do not change the angle between real roots, conjugating pairs of real roots with elements of W does not change the angle between them. From this, we conclude that π preserves the angles between real roots. Moreover, since φ is bijective, π is bijective as well.

Let T_R and T'_R denote the groups of R-rational points of the standard tori. Again by the assumption, we have that $\varphi(T_R) = T'_R$. Hence by the preceding paragraph π induces a reflection-preserving isomorphism $\pi_0 \colon W \xrightarrow{\cong} W'$ between the Weyl groups of $G_{\mathcal{D}}(R)$ and $G'_{\mathcal{D}'}(R)$. The next step will be to construct a W-equivariant bijection $\Phi^{re} \to (\Phi')^{re}$ between the root systems.

Let $\alpha_i \in \Pi$ be a simple root. Choose a root $\beta' \in (\Phi')^{re}$ with the property that $\pi(\{\pm\alpha_i\}) = \{\pm\beta'\}$ (again, β' is uniquely determined up to sign). Let $w' \in W'$ and $\alpha'_j \in \Pi'$ be such that $w'.\alpha'_j = \beta'$. By construction, there exists an automorphism γ_i which makes the diagram

$$\begin{tikzcd} \mathrm{SL}_2(R) \arrow[r, "\gamma_i"] \arrow[d, "\varphi_i"'] & \mathrm{SL}_2(R) \arrow[d, "c_{\tilde{w}'} \circ \varphi'_j"] \\ G_{\mathcal{D}}(R) \arrow[r, "\varphi"] & G'_{\mathcal{D}'}(R) \end{tikzcd}$$

commutative. By [HO89, Theorem 3.2.35], there is a unique inner automorphism ι_i which is either trivial or conjugation with the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ such that $\iota_i \circ \gamma_i$ is the composition of a diagonal-by-ring automorphism and an automorphism of Reiner type, cf. Example 4.1.9.

The automorphism ι_i determines whether $\pm\alpha_i$ must be mapped to $\pm\beta'$ or to $\mp\beta'$ to obtain an equivariant map. Hence we may define $f \colon \Pi \to (\Phi')^{re}$ by

$$f(\alpha_i) = \begin{cases} \beta', & \text{if } \iota_i \text{ is trivial,} \\ -\beta', & \text{otherwise,} \end{cases}$$

and extend f to a map $\Phi^{re} \to (\Phi')^{re}$ by $\mathbb{Z}$-linearity. By construction f is W-equivariant, which is easily verified. It follows that the set $\{f(\alpha_i) \mid \alpha_i \in \Pi\}$ is a root basis of $(\Phi')^{re}$. However, by [Kac90, Proposition 5.9] any two root bases are conjugate under W' up to sign, and thus there exist $\varepsilon \in \{+,-\}$ and $w' \in W'$ such that

$$\{\varepsilon w'.f(\alpha_i) \mid i \in I\} = \{\alpha'_j \mid j \in I'\}.$$

Consequently, $w' \circ f$ induces the claimed bijection ψ of the index sets I and I'. In particular, f preserves the values of $\alpha_i(\alpha_j^*) = f(\alpha_i)(f(\alpha_j)^*)$, hence the entries of the generalised Cartan matrices satisfy $a_{ij} = a'_{\psi(i)\psi(j)}$.

Finally, choosing a representative $n := \tilde{w}' \in N_{G'_{\mathcal{D}'}(R)}(T'_R)$ of $w' \in W'$, the above construction implies that the equality $(c_n \circ \varphi)(\varphi_i(x)) = \varphi_{\psi(i)}(\gamma_i(x))$ holds for all $x \in \mathrm{SL}_2(R)$ and the claim follows. □

4.5 The isomorphism theorem

This section is devoted to the verifications of the assumptions made in Proposition 4.4.1. We also record a number of consequences of the isomorphism theorem.

We assume that $\mathbb{Q} \subseteq R$. Then clearly $\mathbb{Q} \subseteq R \subseteq \mathbb{F} := \mathrm{quot}(R)$. Recall that a subgroup $H \leq G_{\mathcal{D}}(\mathbb{F})$ is called diagonalisable if H is contained a maximal torus. If H is diagonalisable, then H is called regular if the fixed point set of H on $\Delta(G_{\mathcal{D}}(\mathbb{F}))$ coincides with a single twin apartment.

Proposition 4.5.1. *Let $\mathcal{D}$ and $\mathcal{D}'$ be Kac–Moody root data and let $\varphi\colon G_{\mathcal{D}}(R) \to G'_{\mathcal{D}'}(R)$ be an isomorphism. Then there exists a subgroup $X \subseteq \mathbb{Q}^\times$ of finite index such that for $T_X := \langle\, \varphi_i(\mathrm{diag}(x, x^{-1})) \mid i \in I, x \in X \,\rangle$ the group $\varphi(T_X)$ is diagonalisable and regular.*

Moreover, there exists an inner automorphism ν of $G'_{\mathcal{D}'}(R)$ such that $(\nu \circ \varphi)(T_R) = T'_R$, the standard torus of $G'_{\mathcal{D}'}(R)$.

Proof. It follows from [Cap09, Corollary 5.12] that there exists a finite index subgroup $X \subseteq \mathbb{Q}^\times$ such that T_X and $\varphi(T_X)$ are both diagonalisable subgroups of $G_{\mathcal{D}}(\mathbb{F})$ and $G'_{\mathcal{D}'}(\mathbb{F})$, respectively. Since by [Cap09, Lemma 4.9 (iii)] T_X is regular in $G_{\mathcal{D}}(\mathbb{F})$, the only maximal torus of $G_{\mathcal{D}}(\mathbb{F})$ containing T_X is the standard torus $T_{\mathbb{F}}$.

We shall next prove that $\varphi(T_X)$ is regular. Since this property is invariant under automorphisms, by conjugating $\varphi(T_X)$ with an element $g \in G'_{\mathcal{D}'}(\mathbb{F})$ we may assume without loss of generality that $\varphi(T_X)$ is contained in $T'_{\mathbb{F}}$, the standard torus of $G'_{\mathcal{D}'}(T_{\mathbb{F}})$. Since each morphism $\mathrm{SL}_2(\mathbb{Q}) \to G'_{\mathcal{D}'}(\mathbb{F})$ has bounded image ([Cap09, Theorem 5.11]), by [Cap09, Lemma 5.9] (see also [Hai10, Proposition 5.7]) each morphism $\varphi \circ \varphi_i\colon \mathrm{SL}_2(\mathbb{Q}) \to G'_{\mathcal{D}'}(\mathbb{F})$ is the restriction of a morphism defined over $\mathbb{F}$, it follows that $\varphi(T_X)$ is algebraically isomorphic to $X^n \subseteq (\mathbb{Q}^\times)^n$. This shows that the Zariski closure of $\varphi(T_X)$ inside $T'_{\mathbb{F}}$ is a torus of rank at least n. Hence we may conclude that $\mathrm{rk}(\mathcal{D}') \geq n$. Suppose now that

$\varphi(T_X)$ is not regular. Then by [Cap09, Proposition 4.6] there exists a non-trivial image Y of $\mathrm{SL}_2(\mathbb{Q})$ in the group $C_{G'_{\mathcal{D}'}(\mathbb{F})}(\varphi(T_X))$. In particular, the rank one torus of Y does not intersect $gG'_{\mathcal{D}'}(R)g^{-1}$. Since this torus is also contained in $T'_{\mathbb{F}}$, we conclude that $\mathrm{rk}(\mathcal{D}') > n$. Applying the above arguments to the isomorphism $\varphi^{-1}\colon G'_{\mathcal{D}'}(R) \to G_{\mathcal{D}}(R)$ and the standard torus T'_R, we obtain that $\mathrm{rk}(\mathcal{D}) \geq \mathrm{rk}(\mathcal{D}') > n = \mathrm{rk}(\mathcal{D})$, a contradiction. Hence $\varphi(T_X)$ is regular.

Since $\varphi(T_X)$ is regular in $G'_{\mathcal{D}'}(\mathbb{F})$, it is contained in a unique maximal $\mathbb{F}$-torus $T^0_{\mathbb{F}}$ of $G'_{\mathcal{D}'}(\mathbb{F})$ ([Cap09, Section 4.2.4]). Taking the respective Zariski closures again, it follows that $\varphi(T_R) \subseteq T^0_{\mathbb{F}} \cap G'_{\mathcal{D}'}(R) = T^0_R$. Arguing similar with respect to the torus T^0_R and the isomorphism φ^{-1}, we see that $\varphi^{-1}(T^0_R) \subseteq T_R$, and consequently, $\varphi(T_R) = T^0_R$. Moreover, we conclude that $T^0_{\mathbb{F}}$ is defined over R. Hence by Corollary 4.3.4 there exists $g \in G'_{\mathcal{D}'}(R)$ such that for $\varphi' := c_g \circ \varphi$ the group $\varphi'(T_R)$ coincides with the group of R-rational points of the standard torus of $G'_{\mathcal{D}'}(\mathbb{F})$. □

Lemma 4.5.2. *Let $\alpha \in \Pi$, and let $u \in G_{\alpha,R}$ be a unipotent element. Then there exists a morphism $\chi : \mathrm{SL}_2(\mathbb{Q}) \to G_{\mathcal{D}}(\mathbb{F})$ such that $\chi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = u$ and $\mathrm{Im}(\chi)$ is normalised by T_R.*

Proof. This follows from [Hai10, Lemma 6.13]. □

Lemma 4.5.3. *Let $\alpha \in \Pi$, and let $u \in G_{\alpha,R}$ be a unipotent element. Then $\varphi(u)$ is contained in L^J_R, a Levi factor with J of finite type. Moreover, $\varphi(u)$ is unipotent.*

Proof. Let χ be as in Lemma 4.5.2. Then by [Cap09, Proposition 5.7 and Theorem 5.11] $\varphi(u)$ is a unipotent element of $L^J_{\mathbb{F}}$. Since also $\varphi(u) \in G'_{\mathcal{D}'}(R)$, it follows that $\varphi(u) \in L^J_{\mathbb{F}} \cap G'_{\mathcal{D}'}(R) = L^J_R$. □

Lemma 4.5.4. *Let $u \in U_\alpha$, where α is a simple root. Then $\varphi(u) \in U'_{\alpha',R}$ for some root $\alpha' \in \Phi'$.*

Proof. In view of Lemma 4.5.3, we may apply [Hai10, Theorem 6.18] and conclude that $\varphi(u) \in U'_{\alpha',\mathbb{F}} \cap G'_{\mathcal{D}'}(R) = U'_{\alpha',R}$. The claim follows. □

Theorem 4.5.5. *Let R be a rank two-rigid ring containing $\mathbb{Q}$. Let $G_{\mathcal{D}}(R)$ and $G'_{\mathcal{D}'}(R)$ be two-spherical split Kac–Moody groups without G_2-residue or direct factor of type A_1 over R and let $\varphi\colon G_{\mathcal{D}}(R) \to G'_{\mathcal{D}'}(R)$ be an isomorphism. Denote by $A = (a_{ij})_{1\leq i,j\leq n}$ and $A' = (a'_{ij})_{1\leq i,j\leq n'}$ their respective generalised Cartan matrices. Then there exist*

(i) a bijection $\pi\colon I \to I'$ of the index sets such that $a_{ij} = a'_{\pi(i)\pi(j)}$,

(ii) an inner automorphism ν of $G'_{\mathcal{D}'}(R)$,

(iii) for all distinct $i, j \in I$ in the same connected component of the Dynkin diagram an automorphism γ_{ij} of the rank two group $X_{ij} = X(R)$ with $X \in \{A_1 \times A_1, A_2, B_2\}$

such that the diagram

$$\begin{tikzcd} X_{ij} \arrow[r, "\gamma_{ij}"] \arrow[d, "\mathrm{id}"'] & X_{\pi(i)\pi(j)} \arrow[d, "\mathrm{id}"] \\ G_{\mathcal{D}}(R) \arrow[r, "\nu\circ\varphi"] & G'_{\mathcal{D}'}(R) \end{tikzcd}$$

commutes. Moreover, γ_{ij} is the composition of a diagonal-by-ring-by-sign automorphism.

Proof. By Lemma 4.5.4 the isomorphism φ maps root groups to root groups. Hence φ induces a bijection on the set of subgroups of rank one or two. It follows that φ' satisfies the assumptions of Proposition 4.4.1. Moreover, φ' induces an automorphism of the rank two groups $X_{ij} \in \{A_1(R) \times A_1(R), A_2(R), B_2(R)\}$. As by [HO89, Theorem 3.2.31] for groups of type A_2 and [O'M68, Theorem C] for groups of type B_2 these automorphisms are diagonal-by-ring-by-sign, the result follows. □

Corollary 4.5.6 (Mostow rigidity)**.** *Assume that $\mathbb{Q} \subseteq R$ and let $(R, \mathbb{F})$ be a rank two-rigid pair and assume that $\mathcal{D}$ and $\mathcal{D}'$ have the same rank. Let $G_{\mathcal{D}}(\mathbb{F})$, $G'_{\mathcal{D}'}(\mathbb{F})$ be two-spherical split Kac–Moody groups without G_2-residue or direct factor of type A_1 over $\mathbb{F}$ and let $G_{\mathcal{D}}(R)$ and $G'_{\mathcal{D}'}(R)$ be their respective subgroups defined over R.*

Then for any isomorphism $\varphi\colon G_{\mathcal{D}}(R) \to G'_{\mathcal{D}'}(R)$ there exists a unique isomorphism $\psi\colon G_{\mathcal{D}}(\mathbb{F}) \to G'_{\mathcal{D}'}(\mathbb{F})$ with the property that

$$\psi|_{G_{\mathcal{D}}(R)}^{G'_{\mathcal{D}'}(R)} = \varphi.$$

In particular, $G_{\mathcal{D}}(R)$ is Mostow-rigid in $G_{\mathcal{D}}(\mathbb{F})$.

Proof. By Corollary 4.3.5, the normaliser of $G_{\mathcal{D}}(R)$ in $G_{\mathcal{D}}(\mathbb{F})$ coincides with the group $Z(G_{\mathcal{D}}(\mathbb{F})).G_{\mathcal{D}}(R)$. Consequently, any inner automorphism of $G_{\mathcal{D}}(R)$ lifts uniquely to an inner automorphism of $G_{\mathcal{D}}(\mathbb{F})$.

Since the outer automorphism groups of $G_{\mathcal{D}}(R)$ and $G_{\mathcal{D}}(\mathbb{F})$ coincide by Theorem 4.5.5 and [Cap09, Theorem 4.1], any isomorphism $G_{\mathcal{D}}(R) \overset{\cong}{\to} G'_{\mathcal{D}'}(R)$ uniquely lifts to an isomorphism $G_{\mathcal{D}}(\mathbb{F}) \overset{\cong}{\to} G'_{\mathcal{D}'}(\mathbb{F})$ of the ambient Kac–Moody groups defined over $\mathbb{F}$. □

A similar result to Corollary 4.5.6 has been obtained by R. Gramlich and the author in [GM09, Main Result] for the unitary form with respect to the twisted Chevalley involution as subgroup of an infinite split Kac–Moody group over a finite field of square order.

Corollary 4.5.7. *Let R be a rank two-rigid ring containing $\mathbb{Q}$ and let $\mathcal{D}$ and $\mathcal{D}'$ denote the respective two-spherical Kac–Moody root data without residues of type G_2 or direct factor of type A_1 of two split Kac–Moody groups $G_{\mathcal{D}}(R)$ and $G'_{\mathcal{D}'}(R)$ of the same rank. Let $\varphi\colon G_{\mathcal{D}}(R) \to G'_{\mathcal{D}'}(R)$ be an isomorphism.*

Then φ induces an isomorphism of root data $\mathcal{D} \cong \mathcal{D}'$.

Proof. Let $\mathbb{F}$ be the field of fractions of R. By Corollary 4.5.6, φ uniquely lifts to an isomorphism $G_{\mathcal{D}}(\mathbb{F}) \to G'_{\mathcal{D}'}(\mathbb{F})$ which preserves subgroups of rank one and two. Hence by [CM06, Theorem 5.1] the root data $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic. □

4.6 Open problems

(i) Our proof does not cover the ring of integers, for example. Inspired by the result in the spherical case, it seems natural to ask whether this question also has a positive answer. Certainly, the strategy presented here cannot cover this case, as there is no one-to-one correspondence between maximal tori in $G_{\mathcal{D}}(\mathbb{Z})$ and twin apartments of $\Delta(G_{\mathcal{D}}(\mathbb{R}))$.

Question 4.6.1. *Does every automorphism of $G_{\mathcal{D}}(\mathbb{Z})$ uniquely lift to an automorphism of $G_{\mathcal{D}}(\mathbb{R})$?*

(ii) The above proof of Mostow rigidity might rather be called an observation instead of proof. Typically, to show Mostow rigidity (or other types of rigidity), one uses the fact that given a locally compact group G and a lattice $\Gamma \leq G$, i.e. a discrete subgroup of finite covolume, Mostow rigidity (of Γ in G) holds. Of course, the statement *finite covolume* only makes sense if there is a left-invariant Haar measure on G, which in turn is true if and only if G is locally compact. But a non-spherical split Kac–Moody group defined over a locally compact or k_ω-ring R equipped with the Kac–Peterson topology is not locally compact.

Question 4.6.2. *Let R be an integral domain and let $\mathbb{F}$ be its field of fractions. Does there exist a measure μ on $G_{\mathcal{D}}(\mathbb{F})$ such that $G_{\mathcal{D}}(R)$ is a discrete subgroup with finite covolume in $G_{\mathcal{D}}(\mathbb{F})$?*

However, there is no such measure known to the author. Hence (compared to the above) a completely different strategy could be to construct a measure on $G_{\mathcal{D}}(\mathbb{F})$ and apply the theory developed by Margulis.

(iii) The general method of proof presented in this chapter uses the fact that $G_{\mathcal{D}}(R)$ embeds into $G_{\mathcal{D}}(\mathbb{F})$, where $\mathbb{F}$ is the field of fractions of R. The following sketches (roughly) an idea how the method could be adapted to cover rings with zero divisors.

Let R_1 be a ring, $S \subseteq R_1$ the set of all non-zero divisors and let $R := S^{-1}R_1$ be the localisation at S. Assuming that the ring R has trivial Jacobson radical, every $0 \neq r \in R_1$ maps to a non-trivial element of the ring

$$\prod_{\mathfrak{m} \in \mathfrak{M}} R/\mathfrak{m},$$

where $\mathfrak{M}$ denotes the set of all maximal ideals of R. The latter is a direct product of fields, while R_1 embeds diagonally into it. The problem then is to establish that maximal tori are conjugate and then adapt the strategy of Section 4.5.

CHAPTER 5

Orbit structures in topological buildings

The first part of this chapter is devoted to establishing a link between split Kac–Moody groups over k_ω-fields, equipped with the Kac–Peterson topology, and topological twin buildings as described in Section 2.4, see also Appendix A and [Har06]. In this setup, we prove [Har06, Conjecture 4.3.14].

The remainder of this chapter then provides applications of this result by analysing the topological orbit structure of some special subgroups of $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$. This generalises known results in case that $\mathcal{D}$ is spherical.

We shall assume throughout this chapter that $\mathbb{F}$ is a k_ω-field.

5.1 Buildings of split Kac–Moody groups over k_ω-fields

The main result of this section is Theorem 5.1.6, stating that the canonical twin building of a split Kac–Moody group $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$, equipped with the quotient topology, is a topological twin building in the sense of Definition 2.4.1.

This result allows to apply the theory developed in [Har06]. A summary of some of the results in loc. cit. is given in Appendix A, as it seems that the work is not publicly accessible.

First, we recall from Proposition 3.1.8 the defining sequence for the topology on $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$. We considered a pair $(i, \overline{\alpha}) \in \{0,1\} \times (\Phi^{re})^k$ and the product map

$$\begin{aligned} p_{(i,\overline{\alpha})} \colon T^i \times G_{\alpha_1} \times \ldots \times G_{\alpha_k} &\to G_{\mathcal{D}}(\mathbb{F}), \\ (t, g_1, \ldots, g_k) &\mapsto t g_1 \cdots g_k, \end{aligned}$$

which induces a quotient topology $\tau_{(i,\overline{\alpha})}$ on its image $G_{(i,\overline{\alpha})}$.

By Corollary 3.1.9, the direct limit of the spaces $(G_{(i,\overline{\alpha})}, \tau_{(i,\overline{\alpha})})$ coincides with the topological group $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$. This also implies that the quotient topology on $G_{(i,\overline{\alpha})} = \mathrm{Im}(p_{(i,\overline{\alpha})})$ induced by $p_{(i,\overline{\alpha})}$ coincides with the subspace topology from $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$.

Proposition 5.1.1. *Let B_+, B_- be the standard Borel subgroups of $G_{\mathcal{D}}(\mathbb{F})$. Let $k \in \mathbb{N}$ and $(i, \overline{\alpha}) \in \{0,1\} \times (\Phi^{re})^k$. Then the set $G_{(i,\overline{\alpha})}$ is contained in*

$$G_k^{\varepsilon} := \bigcup_{l(w) \leq k} B_{\varepsilon} w B_{\varepsilon}$$

and hence also in $G_k := G_k^+ \cap G_k^-$.

Proof. We prove the result by induction on k. For $k = 0$, as $T^i \subseteq B_{\varepsilon}$, there is nothing to show.

Assume that $|\overline{\alpha}| = k$ and that for all $(j, \overline{\beta})$ with $|\overline{\beta}| < |\overline{\alpha}|$ the space $G_{(j,\overline{\beta})}$ is contained in $G^{\varepsilon}_{|\overline{\beta}|}$. Let $\overline{\alpha_0}$ be the subsequence $(\alpha_1, \ldots, \alpha_{k-1})$ of $\overline{\alpha}$. Then by the induction hypothesis $G_{(i,\overline{\alpha_0})} \subseteq G^{\varepsilon}_{|\overline{\alpha_0}|}$. Moreover, by the Bruhat decomposition $G_{\alpha_k} = B^{\varepsilon}_{\alpha_k} \cup B^{\varepsilon}_{\alpha_k} s_{\alpha_k} B^{\varepsilon}_{\alpha_k}$ (where $B^{\varepsilon}_{\alpha_k} := B_{\varepsilon} \cap G_{\alpha_k}$) it follows that

$$\begin{aligned}
G_{(i,\overline{\alpha})} = G_{(i,\overline{\alpha_0})}.G_{\alpha_k} \quad &\subseteq \quad \left(\bigcup_{l(w) \leq k-1} B_{\varepsilon} w B_{\varepsilon} \right) . (B_{\varepsilon}^{\alpha_k} \cup B_{\varepsilon}^{\alpha_k} s_{\alpha_k} B_{\varepsilon}^{\alpha_k}) \\
&\subseteq \quad \left(\bigcup_{l(w) \leq k-1} B_{\varepsilon} w B_{\varepsilon} \right) . (B_{\varepsilon} \cup B_{\varepsilon} s_{\alpha_k} B_{\varepsilon}) \\
&\overset{(TBN1)}{\subseteq} \quad \bigcup_{l(w) \leq k-1} B_{\varepsilon} w B_{\varepsilon} \cup B_{\varepsilon} w s_{\alpha_k} B_{\varepsilon} \\
&\subseteq \quad \bigcup_{l(w) \leq k} B_{\varepsilon} w B_{\varepsilon}.
\end{aligned}$$

Hence we conclude that for all $\overline{\alpha}$ with $|\overline{\alpha}| = k$ the inclusion $G_{(i,\overline{\alpha})} \subseteq G_k^{\varepsilon}$ holds. Since ε was arbitrary, the claim follows. □

From this fact, we obtain an equivalent description of the Kac–Peterson topology on $G_{\mathcal{D}}(\mathbb{F})$.

Corollary 5.1.2. *Let $\mathbb{F}$ be a k_{ω}-field and let $\mathcal{D}$ be simply connected or adjoint. Equip the spaces G_k^+, G_k^- and $G_k^+ \cap G_k^-$ with the subspace topology induced from $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$.*

Then the Kac–Peterson topology τ_{KP} on the group $G_{\mathcal{D}}(\mathbb{F})$ is the direct limit topology for the filtrations $\{G_k^+\}_{k \in \mathbb{N}}$, $\{G_k^-\}_{k \in \mathbb{N}}$ and $\{G_k^+ \cap G_k^-\}_{k \in \mathbb{N}}$.

Proof. By Proposition 5.1.1, for $k := |\overline{\alpha}|$ the space $G_{(i,\overline{\alpha})}$ is a subspace of G_k^+, G_k^- and $G_k^+ \cap G_k^-$. Moreover, the inclusions $G_{(i,\overline{\alpha})} \hookrightarrow G_k^{\varepsilon}$ are continuous. Hence by Proposition A.3.1 all the direct limit topologies agree and by Corollary 3.1.9 they coincide with the Kac–Peterson topology. □

From this alternative characterisation, we may conclude that Borel subgroups are closed. In the spherical case of Lie groups this is known (cf. [BS87, Theorem 3.12]), and if there exists a matrix representation, this is easy to prove.

Remark 5.1.3. Note that none of $\{G_k^+\}_{k\in\mathbb{N}}$, $\{G_k^-\}_{k\in\mathbb{N}}$ and $\{G_k^+ \cap G_k^-\}_{k\in\mathbb{N}}$ is a k_ω-sequence in general.

Lemma 5.1.4. *The torus T and all root groups U_α are closed in $(G_\mathcal{D}(\mathbb{F}), \tau_{KP})$.*

Proof. Since $\varphi_\alpha(\mathrm{SL}_2(\mathbb{F})) = G_\alpha$ is closed in $(G_\mathcal{D}(\mathbb{F}), \tau_{KP})$ and $U_\alpha \leq G_\alpha$ is closed, it follows that U_α is closed in $(G_\mathcal{D}(\mathbb{F}), \tau_{KP})$. Hence $N_{G_\mathcal{D}(\mathbb{F})}(U_\alpha)$ is closed (cf. also the remark after [HM06, Definition 5.53]), and consequently, $T = \bigcap_{\alpha\in(\Phi)^{re}} N_{G_\mathcal{D}(\mathbb{F})}(U_\alpha)$ is closed. □

Proposition 5.1.5. *Let $(G_\mathcal{D}(\mathbb{F}), \tau_{KP})$ be a simply connected or adjoint split Kac–Moody group equipped with the Kac–Peterson topology. Then the Borel subgroups B_ε are closed.*

Proof. By Corollary 5.1.2, it suffices to show that the Borel subgroup B_ε is relatively closed in each of the spaces $G_k^{-\varepsilon}$.

By the Bruhat decomposition of $G_\mathcal{D}(\mathbb{F})$, we see that $\bigcup_{l(w)\leq k} B_{-\varepsilon} w B_{-\varepsilon}$ intersects U_ε exactly in those root groups U_α of U_ε with $|\mathrm{ht}(\alpha)| \leq k$. Hence we obtain that

$$B_\varepsilon \cap G_k^{-\varepsilon} = B_\varepsilon \cap \left(\bigcup_{l(w)\leq k} B_{-\varepsilon} w B_{-\varepsilon} \right) \cong T \times \prod_{\alpha\in\Phi^\varepsilon, |\mathrm{ht}(\alpha)|\leq k} U_\alpha.$$

Now by Lemma 5.1.4 the groups T and U_α are closed, hence the above product is closed. Thus B_ε is closed in the space $\lim_\to G_k^{-\varepsilon}$, which by Corollary 5.1.2 is $(G_\mathcal{D}(\mathbb{F}), \tau_{KP})$. □

Here is the main result of this section.

Theorem 5.1.6. *Let $\mathbb{F}$ be a k_ω-field and let $(G_\mathcal{D}(\mathbb{F}), \tau_{KP})$ be a simply connected or adjoint split Kac–Moody group over $\mathbb{F}$. Then the canonical twin building*

$$\Delta = \Delta(G_\mathcal{D}(\mathbb{F})) = ((G_\mathcal{D}(\mathbb{F})/B_+, \delta_+), (G_\mathcal{D}(\mathbb{F})/B_-, \delta_-), \delta^*)$$

associated to $(G_\mathcal{D}(\mathbb{F}), \tau_{KP})$, equipped with the quotient topology, is a topological twin building.

Proof. In order to apply Theorem A.2.7, we need to check its hypotheses

(i) that B_ε is closed,

(ii) the set B_-B_+ is open,

(iii) that $(G_\mathcal{D}(\mathbb{F}), \tau_{KP}) = \lim_\to G_k$,

(iv) that the multiplication map $m\colon U_+ \times T \times U_- \to B_+B_-$ is open,

(v) and that $(G_\mathcal{D}(\mathbb{F}), \tau_{KP})$ is Hausdorff.

We have shown (i) in Proposition 5.1.5, (iii) in Corollary 5.1.2 and (v) in Proposition 3.1.8 or Theorem 3.1.11. The remaining two technical assertions will be shown in Propositions 5.1.7 and 5.1.8 below. □

Proposition 5.1.7. *The multiplication maps $m\colon U_+ \times T \times U_- \to B_+B_-$ and $m'\colon U_- \times T \times U_+ \to B_-B_+$ are homeomorphisms.*

Proof. By [KP85, Corollary 4.2 (b)], the map $m\colon U_+ \times T \times U_- \to B_+B_-$ is bijective. Hence it suffices to show that m^{-1} is continuous and open. Defining the canonical projections

$$\begin{aligned} &\mathrm{pr}^+_{U_+}\colon U_+ \times T \to U_+, \quad \mathrm{pr}^+_T\colon U_+ \times T \to T,\\ &\mathrm{pr}^-_{U_-}\colon U_- \times T \to U_-, \quad \mathrm{pr}^-_T\colon U_- \times T \to T, \end{aligned}$$

we may write m^{-1} as

$$\begin{aligned} B_+B_- &\to U_+ \times T \times U_-,\\ b_+b_- &\mapsto (\mathrm{pr}^+_{U_+}(b_+), \mu(\mathrm{pr}^+_T(b_+), \mathrm{pr}^-_T(b_-)), \mathrm{pr}^-_{U_-}(b_-)), \end{aligned}$$

where μ denotes the multiplication map $T \times T \to T$ of the topological group T.

Hence m^{-1}, being a product of compositions of continuous and open maps, is continuous and open. It follows that m is a homeomorphism, the map m' is treated analogously. □

Note that Proposition 5.1.7 gives a positive answer to [Har06, Conjecture 4.3.14] in the setting of split Kac–Moody groups over k_ω-fields.

Proposition 5.1.8. *Let $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ be a simply connected or adjoint split Kac–Moody group. Then the set B_-B_+ is open in $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$.*

Proof. In view of Proposition A.2.4 and using our above results we may apply Proposition A.2.5, which implies that in $\Delta_+ = G_{\mathcal{D}}(\mathbb{F})/B_+$, the B_--orbit $B_-.c_+$ of the fundamental chamber $c_+ = B_+$ is open. Since $q_+\colon G_{\mathcal{D}}(\mathbb{F}) \to G_{\mathcal{D}}(\mathbb{F})/B_+$ is continuous, it follows that $q_+^{-1}(B_-.c_+) = B_-B_+$ is open in $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP}))$. □

We close this section with an observation answering the question about the existence of compact panels.

Lemma 5.1.9. *Let $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ be a split Kac–Moody group, assume that $\mathbb{F}$ is a non-discrete locally compact k_ω-field and let $c \in \Delta_\varepsilon(G_{\mathcal{D}}(\mathbb{F})) = G_{\mathcal{D}}(\mathbb{F})/B_\varepsilon$ be a chamber. Let $s \in S$. Then $P_s(c)$ is homeomorphic to the projective line $\mathbb{P}^1(\mathbb{F}) \cong \overline{\mathbb{F}} = \mathbb{F} \cup \{\infty\}$.*

In particular, $P_s(c)$ is compact.

Proof. Let G_{α_s} be the fundamental rank one subgroup and let P_s be the standard parabolic subgroup of type s. Then $P_s(c) \cong P_s/B_\varepsilon \cong G_{\alpha_s}/B_{\alpha_s} \cong \mathbb{P}^1(\mathbb{F})$. The result then follows from [SBG+95, Proposition 14.5 and Corollary 14.7]. □

The last result also shows that the panels of split Kac–Moody groups are pairwise homeomorphic. Since every split Kac–Moody is locally split (cf. [Cap09, Section 1.2.2]), the Moufang sets and hence the panels must be isomorphic. Moreover, there exists a compact panel of $\Delta(G_{\mathcal{D}}(\mathbb{F}))$ if $\mathbb{F}$ is non-discrete locally compact, cf. [Har06, Definition 3.1.1]. In this case, every panel is compact.

5.2 Orbits of Borel subgroups

The algebraic structure of the double coset spaces $B_\varepsilon \backslash G_{\mathcal{D}}(\mathbb{F})/B_\varepsilon$ and $B_{-\varepsilon} \backslash G_{\mathcal{D}}(\mathbb{F})/B_\varepsilon$ is given by the Bruhat and Birkhoff decompositions of $G_{\mathcal{D}}(\mathbb{F})$, cf. Chapter 2. Namely, the elements of the double coset spaces correspond bijectively to elements of the Weyl group $W = N_{G_{\mathcal{D}}(\mathbb{F})}(T)/C_{G_{\mathcal{D}}(\mathbb{F})}(T) = N_{G_{\mathcal{D}}(\mathbb{F})}(T)/T$.

However, in the spherical case, there is also a known topological description: Let $\leq$ be the Bruhat order on W. For a Lie group G and $w \in W$ we then have the closure relation

$$\overline{BwB} = \bigsqcup_{w' \leq w} Bw'B,$$

with respect to the Lie group topology, see for example [Ste68, Theorem 23] or [KP83b, Lemma 3.4].

We shall generalise and extend this result to the case of split Kac–Moody groups over a non-discrete k_ω-field. For the sake of simplicity, we assume without loss of generality that $\varepsilon = +$ in the sequel, as the case $\varepsilon = -$ follows by symmetry.

First of all, we note that the group $B_- = \mathrm{Stab}_{G_{\mathcal{D}}(\mathbb{F})}(c_-)$ acts transitively on the set of chambers at codistance w from the chamber c_- by Lemma 2.7.2.

Lemma 5.2.1. *Let B_-wB_+ be an orbit of B_- on Δ_+. Then B_- also acts on the closure of B_-wB_+.*

In particular, $\overline{B_-wB_+}$ is a union of B_--orbits.

Proof. Let x be in the closure of $X := B_-wB_+$ and $b \in B_-$. Then there exists a net $(x_n)_{n\in J}$ with $x_n \in X$ and $\lim x_n = x$. Since

$$\begin{aligned} B_- \times G_{\mathcal{D}}(\mathbb{F})/B_+ &\to G_{\mathcal{D}}(\mathbb{F})/B_+, \\ (b, c) &\mapsto b.c \end{aligned}$$

is jointly continuous and each element $b.x_n$ lies in X, it follows that

$$b.x = b. \lim x_n = \lim b.x_n \in \overline{X}.$$

Hence B_- acts on $\overline{X}$ and the claim follows. □

Remark 5.2.2. In the following we will speak about openness of sets of the form YB_ε. This set can be seen both as a subset of $G_{\mathcal{D}}(\mathbb{F})$ and as a subset of $G_{\mathcal{D}}(\mathbb{F})/B_\varepsilon$. These two viewpoints are in fact equivalent due to q_ε being continuous and open, as shown in the commutative diagram below, where ι denotes inclusion of subspaces.

$$\begin{tikzcd}
YB_\varepsilon \arrow[dr, "\iota"] \arrow[dd, "\iota"'] & \\
& G_{\mathcal{D}}(\mathbb{F})/B_\varepsilon \\
G_{\mathcal{D}}(\mathbb{F}) \arrow[ur, "q_\varepsilon"'] &
\end{tikzcd}$$

Hence $YB_\varepsilon \subseteq G_{\mathcal{D}}(\mathbb{F})$ is open if and only if $YB_\varepsilon \subseteq \Delta_\varepsilon$ is open. We therefore may and will take sets as subsets in the building or the group as appropriate.

From Proposition 5.1.8, we already know that the B_--orbit B_-B_+ of the fundamental chamber B_+ is open. We shall now describe a criterion for openness of (unions of) orbits.

Proposition 5.2.3. *Let $c_- \in \Delta_-$, $d \in \Delta_+$ be chambers with $\delta^*(c_-, d) = w$, let B_- be the Borel subgroup associated to c_- and let B_+ be a Borel subgroup opposite B_-. For $v \in W$ consider an orbit B_-vB_+ such that $w \not\geq v$ in the Bruhat order of W.*

Then there exists an open neighbourhood of d in Δ_+ disjoint from B_-vB_+.

Proof. Let Σ be a twin apartment containing d and c_-. Choose a representative $\tilde{w}^{-1} \in \mathrm{Stab}_{G_{\mathcal{D}}(\mathbb{F})}(\Sigma)$ of w^{-1}, which maps d to the (unique) chamber of Σ opposite c_-. Then by construction the codistance of $\tilde{w}^{-1}.d$ and c_- is equal to 1_W. Similarly, we put $X := \tilde{w}^{-1}B_-vB_+$.

Let $x \in X$. Then Lemma 2.7.4 implies that

$$\delta^*(c_-, x) = \delta^*(c_-, \tilde{w}^{-1}.d) \in \{w_1 v \mid w_1 \text{ is a subexpression of } w^{-1}\}.$$

Since $w \not\geq v$, we therefore conclude that $1_W \notin \delta^*(c_-, X)$. In particular, X does not intersect the space B_-B_+ and $A := \delta^*(c_-, X)$ is a finite set of cardinality at most $2^{l(w)}$.

For every $a \in A$, an application of Lemma A.2.1 (ii) yields an open neighbourhood U_a of $\tilde{w}^{-1}.d$ which does not intersect $\tilde{w}^{-1}B_-aB_+$. Hence the set

$$U := \bigcap_{a \in A} U_a,$$

is an open neighbourhood of $\tilde{w}^{-1}.d$ which does not intersect X. Since $\tilde{w} \in G_{\mathcal{D}}(\mathbb{F})$ is a homeomorphism, it follows that $\tilde{w}.U$ is an open neighbourhood of $\tilde{w}\tilde{w}^{-1}.d = d$ which intersects $\tilde{w}X = B_-vB_+$ trivially. $\square$

Lemma 5.2.4. *Let $\mathbb{F}$ be a non-discrete k_ω-field, let $w \in W$ and $s \in S$.*

(i) If $l(ws) > l(w)$, then the inclusions

(a) $\overline{B_-wB_+} \supseteq B_-wsB_+$ *and*

(b) $\overline{B_+wB_-} \supseteq B_+wsB_-$ *hold.*

(ii) If $l(sw) > l(w)$*, then the inclusions*

(a) $\overline{B_-wB_+} \supseteq B_-swB_+$ *and*

(b) $\overline{B_+wB_-} \supseteq B_+swB_-$ *hold.*

Proof. (i) We prove (a), as (b) follows by symmetry.

Let c_- be the fundamental chamber in Δ_-. Choose a chamber $c \in \Delta_+$ with the property that $\delta^*(c_-, c) = w$, i.e. a representative of the B_--orbit B_-wB_+. Let $s \in S$ be such that $l(sw) > l(w)$ and consider the s-panel $P_s(c)$ around c.

Define $d := \mathrm{proj}_{P_s(c)}(c_-)$, then by Definition 2.3.7 d is the unique chamber of $P_s(c)$ that has codistance ws from c_-. Since $G_{\mathcal{D}}(\mathbb{F})$ is split, the stabiliser of $P_s(c)$ in B_- is isomorphic to the root group associated to the chamber $d \in P_s(c)$. In particular, the group $\mathrm{Stab}_{B_-}(P_s(c))$ acts transitively on $P_s(c) \setminus \{d\}$.

Now the space $P_s(c) \setminus \{d\}$ is homeomorphic to $\mathbb{A}^1(\mathbb{F}) \cong \mathbb{P}^1(\mathbb{F}) \setminus \{x\}$ for some $x \in \mathbb{P}^1(\mathbb{F})$. Since $\mathbb{F}$ is non-discrete, it follows that d is contained in $\overline{P_s(c) \setminus \{d\}}$, hence also in $\overline{B_-wB_+}$. As $s \in S$ was arbitrary, the closure of B_-wB_+ intersects every orbit B_-swB_+ with $l(sw) > l(w)$. Since by Lemma 5.2.1 this closure is a union of orbits, it follows that

$$\overline{B_-wB_+} \supseteq B_-wsB_+$$

for all $s \in S$ with $l(ws) > l(w)$.

(ii) As inversion is a homeomorphism of $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$, we conclude that $\overline{B_\varepsilon wB_{-\varepsilon}} \supseteq B_\varepsilon swB_{-\varepsilon}$ if and only if $\overline{B_{-\varepsilon}w^{-1}B_\varepsilon} \supseteq B_{-\varepsilon}w^{-1}sB_\varepsilon$. Since moreover $l(w^{-1}s) = l(w^{-1}s^{-1}) = l(sw) > l(w) = l(w^{-1})$, the result follows from part (i) above.

□

These results allow us to give the announced topological description of orbits of Borel subgroups.

Theorem 5.2.5. *Let* $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ *be a simply connected or adjoint split Kac–Moody group over a non-discrete* k_ω*-field* $\mathbb{F}$*, let* W *be its Weyl group, let* $\leq$ *be the Bruhat order of* W *and consider the orbits of the action of* B_- *on* Δ_+*. Then the following hold:*

(i) For $w \in W$ *the closure relation*

$$\overline{B_-wB_+} = \bigsqcup_{w' \geq w} B_-w'B_+$$

holds.

(ii) Let $w \in W$. The unique smallest open B_--invariant set containing the B_--orbit B_-wB_+ is

$$\bigsqcup_{w' \le w} B_-w'B_+.$$

In particular, it consists of finitely many B_--orbits.

(iii) The set $\bigsqcup_{l(w)\le n} B_-wB_+$ is open in $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$.

In particular, $\left\{\bigsqcup_{l(w)\le n} B_-wB_+\right\}_{n\in\mathbb{N}}$ is a filtration of $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ consisting of open sets.

Proof. (i) If for all $s \in S$ the conditions $l(sw) < l(w)$ and $l(ws) < l(w)$ hold, then w is the longest element in W, while B_-wB_+ reduces to a single chamber, and there is nothing to prove (note that it also follows that W is spherical in this case). An induction using Lemma 5.2.4 shows that

$$\overline{B_-wB_+} \supseteq \bigsqcup_{w' \ge w} B_-w'B_+.$$

To show the converse inclusion, let x be a point in the complement of $\bigsqcup_{w'\ge w} B_-w'B_+$, which coincides with the set $X := \bigsqcup_{w' \not\ge w} B_-w'B_+$. We show that if x is contained in $\bigcup_{l(v)\le n} B_+vB_+$, then x is an interior point of $X \cap \bigcup_{l(v)\le n} B_+vB_+$.

Let $n \in \mathbb{N}$ be large enough such that $x \in \bigcup_{l(v)\le n} B_+vB_+$. The intersection of $\bigsqcup_{w'\ge w} B_-w'B_+$ with $\bigcup_{l(v)\le n} B_+vB_+$ meets finitely many B_--orbits, hence $A := \{a \in W \mid a \ge w, B_-aB_+ \cap X \ne \emptyset\}$ is finite. For every $a \in A$, we may apply Proposition 5.2.3 to obtain an open neighbourhood U_a of x in the space $\bigcup_{l(v)\le n} B_+vB_+$ not intersecting B_-aB_+.

Then $U := \bigcap_{a\in A} U_a \cap \bigcup_{l(v)\le n} B_+vB_+$ is open in $\bigcup_{l(v)\le n} B_+vB_+$, contains x and intersects $\bigsqcup_{w'\ge w} B_-w'B_+$ trivially. Hence $X \cap \bigcup_{l(v)\le n} B_+vB_+$ is relatively open in $\bigcup_{l(v)\le n} B_+vB_+$ for each $n \in \mathbb{N}$ and since $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP}) = \lim_{\rightarrow} \bigcup_{l(v)\le n} B_+vB_+$ by Corollary 5.1.2, we conclude that X is open in $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$.

(ii) Define $X := \{sw' \mid w' \le w, sw' \not\le w\}$. Then X is finite and by (i) we have

$$\bigsqcup_{w'\le w} B_-w'B_+ = (G_{\mathcal{D}}(\mathbb{F}), \tau_{KP}) \setminus \bigcup_{x\in X} \overline{B_-xB_+}.$$

Since $\bigcup_{x\in X} \overline{B_-xB_+}$ is a finite union of closed sets, it follows that its complement, $\bigsqcup_{w'\le w} B_-w'B_+$, is open.

Finally, if U is an open B_--invariant set containing B_-wB_+, then by (i) for each $w' \le w$ we necessarily have $B_-w'B_+ \subseteq U$.

(iii) Note that $\bigsqcup_{l(w)\le n} B_-wB_+ = \bigcup_{l(w)\le n} \left(\bigsqcup_{w'\le w} B_-w'B_+\right)$ and apply (ii). □

Remark 5.2.6. Let $(W, \leq)$ be the Weyl group equipped with the Bruhat order of W. Equip W with the Alexandrov discrete topology with respect to the reversed partial order of $\leq$. That is, a set $X \subseteq W$ is open if and only if it is a lower set with respect to the partial order $\leq$. It then follows from Theorem 5.2.5 that the map $\Delta_+ \to W, c \mapsto \delta^*(c, c_-)$ is a quotient map.

The name Alexandrov *discrete* topology originates from the fact that with respect to this topology, an arbitrary intersection of open sets is open again.

Note that this topology on W does not coincide with the topology on W induced from $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ by taking the quotient N/T. The latter topology is discrete.

We also record the following immediate consequences of the last result.

Corollary 5.2.7. *There exists a unique open B_--orbit in Δ_+, namely the big cell B_-B_+, which is dense.* □

Corollary 5.2.8. *There exists a closed B_--orbit if and only if W is spherical, in which case the (unique) closed orbit is (the chamber) $B_-w_0B_+$, where w_0 is the longest word in W.*

Proof. The Bruhat order of W has a maximal element if and only if W is spherical. By Theorem 5.2.5 (i) an orbit B_-wB_+ is closed if and only if w is a maximal element. □

The result of Theorem 5.2.5 may also be generalised to orbits of parabolic subgroups, we start with an algebraic description.

Proposition 5.2.9. *Let $J \subseteq S$ be a subset, let P_J be the corresponding negative standard parabolic subgroup of $G_{\mathcal{D}}(\mathbb{F})$, whose Weyl group is denoted by $W_J = \langle\, J \,\rangle \leq W$. Choose a chamber $\tilde{w}B_+ = c \in B_-wB_+$. Then*

$$q_+^{-1}(P_J.c) = B_-W_JwB_+.$$

Proof. By the axioms for twin BN-pairs, we have that $B_-sB_-wB_+ = B_-swB_+$ if $l(sw) < l(w)$ and $B_-sB_-wB_+ = B_-wB_+ \cup B_-swB_+$ if $l(sw) > l(w)$. Since P_J admits a Bruhat decomposition $P_J = \bigcup_{v \in W_J} B_-vB_-$, by an induction on the length of v we obtain the identities

$$\begin{aligned} q_+^{-1}(P_J.c) &= \bigcup_{v \in W_J} B_-vB_-wB_+ \\ &= \bigsqcup_{v \in W_J} B_-vwB_+ \\ &= B_-W_JwB_+, \end{aligned}$$

which is what we wanted to show. □

This description enables us to apply our results regarding B_--orbits from Theorem 5.2.5.

Corollary 5.2.10. *Let X be the set of elements in $W_J w$ of minimal length. Then*

$$\overline{P_J.c} = \overline{\bigsqcup_{x \in X} B_- x B_+} = \bigcup_{x \in X} \overline{B_- x B_+}.$$

Similarly, if Y is the set of elements in $W_J w$ of maximal length, then the smallest open P_J-invariant (or equivalently, B_--invariant) set containing $P_J.c$ is the union of all B_--orbits $B_- v B_+$ for which there is $y \in Y$ with $v \leq y$.

Proof. Combine Theorem 5.2.5 and Proposition 5.2.9. □

5.3 Orbits of the unitary form

Let θ be a flip of $(G_\mathcal{D}(\mathbb{F}), \tau_{KP})$ and consider the unitary form $G_\theta \leq G_\mathcal{D}(\mathbb{F})$ with respect to θ. We shall prove in this section that there is a natural G_θ-action on the filtration with respect to θ-codistance and use the results from Section 5.2 to present a topological description of the G_θ-orbits on the twin building of $G_\mathcal{D}(\mathbb{F})$.

Example 5.3.1. Consider the case where $\mathbb{F} = \mathbb{C}$ and θ is the Chevalley involution twisted with complex conjugation: Then the fact that the group $\mathrm{SU}_2(\mathbb{C})$ acts transitively on the complex projective line implies that the Kac–Moody group $G_\mathcal{D}(\mathbb{C})$ admits a **generalised Iwasawa decomposition** $G_\mathcal{D}(\mathbb{C}) = G_\theta B_+ = G_\theta B_-$, cf. [DMGH09, Corollary 3]. Hence the action of G_θ on Δ_+ (and similarly, also on Δ_-) is transitive, so the orbit structure is trivial.

Of course, the situation is not as well-behaved as in Example 5.3.1 in general.

We now introduce the filtration of Δ_+ to be studied. Let

$$\Delta_w := \{c \in \Delta_+ \mid \delta^\theta(c) = \delta^*(c, \theta(c)) = w \in W\}$$

be the set of chambers of Δ_+ of θ-codistance w. Then the filtration $\{\Delta_w \mid w \in W\}$ may be partially ordered as to $\Delta_w \leq \Delta_{w'}$ if and only if $w \leq w'$ with respect to the Bruhat order of W.

The group G_θ preserves this filtration. Indeed, as $G_\theta \leq G_\mathcal{D}(\mathbb{F})$ acts via isometries, for given $c \in \Delta_w$ and $g \in G_\theta$ it follows that

$$\delta^\theta(g.c) = \delta^*(g.c, \theta(g.c)) = \delta^*(g.c, g.\theta(c)) = \delta^*(c, \theta(c)) = \delta^\theta(c) = w.$$

We call Δ_w a **filtration segment** if $\Delta_w \neq \emptyset$, .

In Example 5.3.1, the group G_θ acts transitively on the building, in particular on each set Δ_w. However, in general G_θ does not act transitively on each set Δ_w, so these may consist of multiple G_θ-orbits.

Lemma 5.3.2. *Assume that Δ_w is non-empty. Then $w = w^{-1}$, i.e. w is an involution. In particular, $\delta^\theta(c) \in \mathrm{Inv}(W) := \{w \in W \mid w = w^{-1}\}$ for all chambers $c \in \Delta_+ \cup \Delta_-$.*

Proof. Choose a chamber $c \in \Delta_w$. Then by the axioms for twin buildings and the fact that θ is an isometry, we obtain that

$$w = \delta^*(c, \theta(c)) = \delta^*(\theta(c), c)^{-1} = \theta(\delta^*(c, \theta(c))^{-1}) = \delta^*(c, \theta(c))^{-1} = w^{-1}. \qquad \square$$

Define $\mathrm{Cod}(\theta) := \{w \in W \mid \Delta_w \neq \emptyset\} \subseteq \mathrm{Inv}(W)$, i.e. $\mathrm{Cod}(\theta)$ consists of those involutions $w \in W$, for which there exists a chamber of θ-codistance w.

Recall from Section 2.9 the definition of the Lang map $\tau_\theta \colon G_{\mathcal{D}}(\mathbb{F}) \to G_{\mathcal{D}}(\mathbb{F}), x \mapsto x^{-1}\theta(x)$.

Proposition 5.3.3. *Let θ be a flip of $G_{\mathcal{D}}(\mathbb{F})$ and assume that $\theta(B_+) = B_-$. Let $w \in W$ and let $c = xB_+ \in \Delta_+$ be a chamber. Then $\delta^\theta(c) = w$ if and only if $x \in \tau_\theta^{-1}(B_+wB_-)$.*

Proof. By construction of the twin building associated to $G_{\mathcal{D}}(\mathbb{F})$ the following chain of equivalences follows:

$$\begin{aligned} \delta^\theta(c) = w &\iff \delta^*(c, \theta(c)) = w \\ &\iff x^{-1}\theta(x) \in B_+wB_- \\ &\iff \tau_\theta(x) \in B_+wB_- \\ &\iff x \in \tau_\theta^{-1}(B_+wB_-). \end{aligned} \qquad \square$$

Note that Proposition 5.3.3 also implies that for given $w \in \mathrm{Cod}(\theta)$ the equality $\Delta_w = q_+(\tau_\theta^{-1}(B_+wB_-)$ holds.

We now present the topological description of G_θ-orbits. Again, we shall apply the topological description of Borel subgroups which we obtained in Theorem 5.2.5.

Theorem 5.3.4. *Assume that $\mathbb{F}$ is a non-discrete k_ω-field and let $G_{\mathcal{D}}(\mathbb{F})$ be a simply connected or adjoint split Kac–Moody group over $\mathbb{F}$. Let θ be a flip of $G_{\mathcal{D}}(\mathbb{F})$, assume that $\theta(B_+) = B_-$ and consider the filtration $\{\Delta_w \mid w \in \mathrm{Cod}(\theta)\}$ of Δ_+. Let $X \subseteq \mathrm{Cod}(\theta)$. Then the following hold:*

(i) The union

$$\Delta_X := \bigcup_{w \in X} \Delta_w$$

is open if and only if X is a lower set in $\mathrm{Cod}(\theta)$ with respect to the order induced by the Bruhat order on W, i.e. if and only if X is open with respect to reversed Alexandrov discrete topology on $\mathrm{Cod}(\theta)$ as in Remark 5.2.6.

(ii) The set Δ_X is closed if and only if X is an upper set in $\mathrm{Cod}(\theta)$ with respect to the order induced by the Bruhat order on W, i.e. if and only if X is closed with respect to reversed Alexandrov discrete topology on $\mathrm{Cod}(\theta)$.

(iii) Let $w \in \mathrm{Cod}(\theta)$. Then the closure relation

$$\overline{\Delta_w} = \Delta_{\geq w} := \bigcup_{w' \geq w} \Delta_{w'}$$

holds.

(iv) Let $w \in \mathrm{Cod}(\theta)$. Then the smallest open G_θ-invariant set containing Δ_w coincides with

$$\Delta_{\leq w} = \bigcup_{w' \leq w} \Delta_{w'}.$$

Proof. (i) & (ii) Let $q_+ \colon G_{\mathcal{D}}(\mathbb{F}) \to G_{\mathcal{D}}(\mathbb{F})/B_+ \cong \Delta_+$ be the canonical continuous and open quotient map and consider the Lang map $\tau_\theta \colon G_{\mathcal{D}}(\mathbb{F}) \to G_{\mathcal{D}}(\mathbb{F}), x \mapsto x^{-1}\theta(x)$. Since θ is continuous and open, so is τ_θ.

By Proposition 5.3.3, $\Delta_X = q_+(\tau_\theta^{-1}(B_- X B_+))$. It follows that Δ_X is open if and only if $B_- X B_+$ is open. By Theorem 5.2.5, the latter set is open if and only if X is a lower set with respect to the Bruhat order, i.e. if and only if

$$\Delta_X = \Delta_{\leq X} := \bigsqcup_{(\exists w \in X): w' \leq w} \Delta_{w'} = q_+\left(\tau_\theta^{-1}\left(\bigsqcup_{(\exists w \in X): w' \leq w} B_+ w' B_-\right)\right).$$

Similarly, it follows that Δ_X is closed if and only if X is an upper set.

(iii) & (iv) Combine (i) and (ii) with Theorem 5.2.5, respectively. □

Remark 5.3.5. As in Remark 5.2.6, Theorem 5.3.4 implies that the map $\Delta_+ \to \mathrm{Cod}(\theta), c \mapsto \delta^\theta(c)$ is a quotient map, where $\mathrm{Cod}(\theta)$ is equipped with the Alexandrov discrete topology with respect to the reversed Bruhat order on $\mathrm{Cod}(\theta)$.

Remark 5.3.6. Assume that θ is a (σ-twisted) Chevalley involution. The fact that the closure of Δ_w contains all sets $\Delta_{w'}$ with $w' \geq w$ can also be seen using a local analysis and induction as conducted in Lemma 5.2.4 or the proof of Theorem 5.2.5 by investigating the action of suitable stabilisers in G_θ on the (θ-parallel) panels and the associated two-dimensional vector spaces over $\mathbb{F}$ which are equipped with a σ-sesquilinear form.

Corollary 5.3.7. *Let θ be a flip of $G_{\mathcal{D}}(\mathbb{F})$ and consider the filtration $\{\Delta_w \mid w \in \mathrm{Cod}(\theta)\}$ of Δ_+. Then the following hold:*

(i) The filtration $\{\Delta_w \mid w \in \mathrm{Cod}(\theta)\}$ contains a unique open filtration segment Δ_1, which is dense.

(ii) There exists a closed filtration segment if and only if there exists a maximal element in $\mathrm{Cod}(\theta)$. In this case, the closed filtration segments are Δ_w, where w is maximal in $\mathrm{Cod}(\theta)$.

In particular, if $\mathcal{D}$ is spherical, then there exists a closed filtration segment.

Proof. (i) By Theorem 5.3.4 (i), the filtration segment Δ_1 is open and dense, provided it is non-empty. Moreover, by (iv) any other non-empty union of filtration segments which is open contains Δ_1. Note that the fact that $\Delta_1 \neq \emptyset$ is ensured by the assumption on θ to map a Borel subgroup to an opposite one, i.e. a Phan chamber to exist.

(ii) Combine Theorem 5.3.4 with Theorem 5.2.5. □

Note however that contrary to the case of Borel subgroups acting, if there exists a closed filtration segment, then it it not necessarily unique.

Example 5.3.8. Let $G_\mathcal{D}(\mathbb{C})$ be a non-spherical split Kac–Moody group over $\mathbb{C}$ and let θ be the Chevalley involution on $G_\mathcal{D}(\mathbb{C})$. Then, choosing a chamber c with θ-codistance w, a local analysis of the panels around c shows that if $l(sw) > l(w)$ and sw is an involution, then there is a chamber $d \in P_s(c)$ with θ-codistance sw. Hence $\mathrm{Cod}(\theta) = \mathrm{Inv}(W)$ in this case. In particular, there is no closed filtration segment since W has no involution which is maximal in the Bruhat order of $\mathrm{Cod}(\theta)$.

Assume that $\mathcal{D}$ is spherical and let $w_0 \in W$ be the longest element. Then it is not necessarily true that $\Delta_{w_0} \neq \emptyset$. This can be seen in Example 5.3.1, where all chambers are Phan chambers, i.e. have θ-codistance 1, independent of $\mathcal{D}$.

We also note that there is a geometric description of the τ_θ-inverse images of orbits of parabolic subgroups modulo G_θ and B_+.

Proposition 5.3.9 (cf. [GHM, Proposition 6.7])**.** *Let θ be a flip of $G_\mathcal{D}(\mathbb{F})$, let Σ be the fundamental twin apartment containing c_+ and assume that every chamber of Δ_+ is contained in a θ-stable twin apartment. Let $J \subseteq S$ be a subset, let P_J be the corresponding negative standard parabolic subgroup and let c be a chamber of Δ_+ at codistance w from c_-. Then*

$$\tau_\theta^{-1}(q_+^{-1}(P_J.c_+))/B_+ \cong \{g\mathrm{Fix}_{G_\mathcal{D}(\mathbb{F})}(\Sigma) \mid \tau_\theta(g) \in \mathrm{Stab}_{G_\mathcal{D}(\mathbb{F})}(\Sigma) \cap \mathrm{Stab}_{G_\mathcal{D}(\mathbb{F})}(R_J(w.c_+))\}.$$

via $gB_+ \mapsto g\mathrm{Fix}_{G_\mathcal{D}(\mathbb{F})}(\Sigma)$. Consequently,

$$G_\theta\backslash\tau_\theta^{-1}(q^{-1}(P_J.c_+))/B_+ \cong \{G_\theta g\mathrm{Fix}_G(\Sigma) \mid \tau_\theta(g) \in \mathrm{Stab}_G(\Sigma) \cap \mathrm{Stab}_G(R_J(w.c_+))\}.$$

via the map $G_\theta gB_+ \mapsto G_\theta g\mathrm{Fix}_G(\Sigma)$.

Proof. By Proposition 5.2.9, the equality $q_+^{-1}(P_J.c_+) = B_-W_JwB_+$ holds. Hence the set $\tau_\theta^{-1}(q_+^{-1}(P_J.c_+))$ consists exactly of the (q_+-preimages of) chambers having θ-codistance in the coset $W_Jw \subseteq W$.

Let $h \in \tau_\theta^{-1}(q_+^{-1}(P_J.c_+))$. By assumption, we may choose Σ' to be a θ-stable twin apartment containing the translate $h.c_+$. Since $G_\mathcal{D}(\mathbb{F})$ acts strongly transitively on Δ_+, there exists $g \in G_\mathcal{D}(\mathbb{F})$ such that $g.\Sigma = \Sigma'$ and $g.c_+ = h.c_+$. The latter equality implies that $gB_+ = hB_+$, hence $\delta^\theta(gB_+) = \delta^\theta(hB_+)$ and consequently $g \in \tau_\theta^{-1}(q^{-1}(P_J.c_+))$. Moreover, we have that

$$g.\Sigma = \Sigma' = \theta(g.\Sigma) = \theta(g).\Sigma,$$

hence $\tau_\theta(g) \in \mathrm{Stab}_{G_{\mathcal{D}}(\mathbb{F})}(\Sigma)$. Since the θ-codistance of gB_+ is contained in $W_J w$, we also have that $g^{-1}\theta(g) \in \mathrm{Stab}_{G_{\mathcal{D}}(\mathbb{F})}(R_J(w.c_+))$. Finally, since g is unique up to right translation with elements from $T = \mathrm{Fix}_{G_{\mathcal{D}}(\mathbb{F})}(\Sigma)$, we get that $g\mathrm{Fix}_{G_{\mathcal{D}}(\mathbb{F})}(\Sigma) \mapsto gB_+$ defines an isomorphism of double cosets, as claimed. □

The question whether every chamber is contained in a θ-stable twin apartment has been discussed in [GHM, Section 6].

We would like to emphasise that for the results concerning G_θ-orbits presented here, the Lang map τ_θ plays an important role, as it links the two filtrations with respect to the codistance to a fixed chamber and with respect to the θ-codistance.

5.4 Concluding remarks & open problems

(i) In general it is not true that G_θ acts transitively on each filtration segment. It follows that each filtration segment Δ_w possibly consists of more than one G_θ-orbit. In this sense, Theorem 5.3.4 should not be called a result about the structure of G_θ-orbits, but rather on the θ-codistance. In order to be able to talk about orbits, one has to deduce the structure of G_θ-orbits on each filtration segment. This problem is connected to the parametrisation of norm classes of the defining field. A similar question was solved in [DMGH09] using the strategy described above.

(ii) The famous Solomon–Tits Theorem (see [AB08, 4.127]) states that the geometric realisation of a (discrete) spherical building is homotopy equivalent to a wedge of n-spheres, where n is the rank of the building. Moreover, H. Abels and P. Abramenko ([AA93], [Abr96]) have shown that the geometry consisting of chambers opposite one fixed chamber is n-spherical as well, provided that the defining field is large enough. Taking the geometric realisation of a topological twin building defined over $\mathbb{R}$ or $\mathbb{C}$, it is known that if Δ is spherical, then $|\Delta|$ is homeomorphic to a sphere of high dimension; if Δ is non-spherical, then $|\Delta|$ is contractible (see [Har06, Section 3.3], [Kra02, Corollary 7.11]).

Over $\mathbb{R}$ or $\mathbb{C}$ (the only connected local fields), the following is known to the author.

Proposition 5.4.1. *Let Δ be a spherical topological (twin) building of arbitrary type defined over $\mathbb{R}$ or $\mathbb{C}$, let $c \in \Delta$ be a chamber and let $J \subseteq S$ be a subset. Denote by c_J^{op} the subcomplex of Δ consisting of J-simplices opposite c.*

Then $|c_J^{op}|$, the realisation of c_J^{op}, is contractible. □

For $J = S$, this implies that the geometric realisation of the opposites geometry, is contractible. The non-spherical case or other base fields, however, cannot be covered yet. This results from the fact that there is no CW-decomposition of the topological twin building (unless defined over $\mathbb{R}$ or $\mathbb{C}$) known, while in the non-spherical real or complex case the homotopy type of the CW-decomposition could not be determined yet.

In [KK86], [Nis02], [Kum85, Chapter 1] or [Kum02, Chapters VII and XI], the (co-)homology of complex Kac–Moody groups has been studied. Similary, for example in [Kit08] a similar analysis was conducted with respect to (co-)homology of complex unitary forms.

To the knowledge of the author, there are hardly any results in this direction for Kac–Moody groups defined over local fields. The concept of cyclic cohomology which has been used in the literature to analyse Lie groups defined over non-archimedean local fields (which are totally disconnected) might be used in this context. However, the main difficulty certainly is to transfer the concept to the non-spherical case. See for example [BCH94], [BHP93], [BB92], [HN96], [LQ84], [Nis93], [Sch98] or [Sch96] for details on cyclic (co-)homology.

Question 5.4.2. *Let $\mathbb{F}$ be a (totally disconnected) local field and let $G_{\mathcal{D}}(\mathbb{F})$ be a Kac–Moody group over $\mathbb{F}$. Is it possible to compute the cyclic (co-)homology of $G_{\mathcal{D}}(\mathbb{F})$?*

(iii) Continuing (ii) above, the question about the homotopy type of the collection of all Phan chambers Δ_θ (with the subspace topology) arises. In the discrete case, there are results available, see [GW, Theorem 4.1], [Gra09, Section 5.3] or [DGM09, Theorem 6.6]. For the non-discrete case discussed here, there are no results about this question in either the spherical or non-spherical case known to the author. A direct computation of the homotopy type seems awkward, as locally (i.e. in a panel) a flip-flop system cannot be described as easy as an opposites geometry. In the latter case, all chambers except one (namely the projection of the fixed chamber) will be in the opposites geometry again, whereas in the first case, the geometry may be described as the set of non-isotropic one-dimensional subspaces of the natural module of the group SU_2.

Another approach might be to analyse the Lang map τ_θ (which is continuous and open) and its fibres, then apply a certain amount of algebraic topology, reducing the question to the geometry opposite a fixed chamber. In view of [Spa66, Chapter 6, Section 9, Theorem 18], this approach heavily depends on the fibres, which are unitary groups. However, the unitary groups do not admit a cell decomposition consisting of cells of minimal dimension *large enough*, see for example [Yok57]. Another difficulty in this analysis is that τ_θ maps a chamber to an orbit of a Borel subgroup. Hence τ_θ does not induce a map on the level of chambers.

Question 5.4.3. *Let $\mathbb{F}$ be a (totally disconnected) local field and let θ be a flip of $G_{\mathcal{D}}(\mathbb{F})$. Is it possible to compute the cyclic (co-)homology of the group $G_\theta = \mathrm{Stab}_{G_{\mathcal{D}}(\mathbb{F})}(\theta)$?*

(iv) Our proofs on the topological orbit structure of a topological twin building (Theorems 5.2.5 and 5.3.4) rely heavily on the fact that there exists a group acting via homeomorphisms on the building. However, as the orbits B_-wB_+ can also be described in a purely algebraic way as the set of all chambers which have codistance w from the chamber c_+ associated to B_+, the question arises whether the group action is necessary. While in Theorems 5.2.5 and 5.3.4 one inclusion of the respective closure relations may be deduced from a local analysis by looking at panels, the other inclusion relies on the existence of enough homeomorphisms of the twin building (see the proof of Theorem 5.2.5).

(v) Our results in this chapter use the fact that the Kac–Moody groups $G_{\mathcal{D}}(\mathbb{F})$ we are working with is split. This raises the question of what can be said about twin buildings of quasi-split groups. Certainly, the map m from Proposition 5.1.7 is not as easily described as in the present case. However, as already remarked in [Har06, Conjecture 4.3.14], by looking at examples the result appears to be true in general.

Once one knows that the twin building is a topological twin building, the (topological) orbit structure may be analysed. We have used the fact that $G_{\mathcal{D}}(\mathbb{F})$ is split in the proof of Theorem 5.2.5 conducting the local analysis by looking at panels. More precisely, we used that the stabiliser of a panel in a Borel subgroup is isomorphic to the root group corresponding to a specific chamber (the projection). This argument would need a replacement.

See also Section 3.4 (ii), where we raised a similar question regarding the topology on quasi-split groups.

CHAPTER 6

Central extensions of Kac–Moody groups over rings

The aim of this chapter is to determine the universal central extension of a centred split Kac–Moody group $G_{\mathcal{D}}(R)$ defined over a ring R, with some restrictions on the ring R. As application, under very strong assumptions on R, we show that the universal central extension and the universal cover of $G_{\mathcal{D}}(R)$ coincide, as it is the case in the theory of finite-dimensional connected semisimple Lie groups.

6.1 Assumptions and comments

We fix a centred two-spherical Kac–Moody root datum $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ without direct factors of type A_1 for the remainder of this chapter. Moreover, we introduce the following conditions on R (in addition to being commutative and unital).

Definition 6.1.1. We say that a commutative unital ring R has **nice units (for $\mathcal{D}$)** if R

(i) contains for every $i \in \{2, 4-\alpha(\beta^*)\beta(\alpha^*) \mid \alpha \in \Pi, \{\alpha, \beta\} \subseteq \Phi^{re}$ prenilpotent and $\alpha(\beta^*) \neq 0\} \setminus \{0\}$ a unit d such that $1 - d^i$ is a unit,

(ii) contains a unit v such that $1 - v + v^2$ is a unit and v, $1 - v$ are squares of units,

(iii) has the property that either $1 + 1 = 0$ or $1 + 1 \in R^\times$. □

The above hypotheses on R will be required in the proof of Theorem 6.2.2 to conclude that for every central extension of the Steinberg group, the defining relation (B) can be lifted to a relation in the extension. Note that (iii) is an empty condition if R is a field.

Remark 6.1.2. If $\mathcal{D}$ is spherical, then condition (i) of Definition 6.1.1 collapses to the requirement that for $i \in \{1, 2, 3\}$ there is a unit $v \in R^\times$ such that $1 - v^i$ is again a unit because $\alpha(\beta^*)\beta(\alpha^*) \leq 4$ is valid for all roots.

Example 6.1.3. Examples of rings which satisfy the above conditions are rings which contain an infinite field as a subring, e.g. the rings $\mathbb{F}[t], \mathbb{F}[t, t^{-1}]$ with $\mathbb{F}$ infinite. Hence also the adèle ring $\mathbb{A}_\mathbb{F}$ of a global field $\mathbb{F}$ is an example. The integers $\mathbb{Z}$, however, do not satisfy any of (i), (ii) or (iii) of Definition 6.1.1.

Indeed, assume that $\mathbb{F} \subseteq R$ with $|\mathbb{F}| = \infty$. Then clearly, (i) is satisfied, as there are infinitely many units and for any prenilpotent pair $\{\alpha, \beta\}$ of real roots, the assertion reduces to finding an element which is not a root of the polynomial determined by the pair $\{\alpha, \beta\}$.

If $\mathrm{char}(\mathbb{F}) \neq 2$, being able to choose a unit x such that $x^2 - 1 \neq 0$ and $x^2 + 1 \neq 0$ suffices. Then $a := (\frac{1+x^2}{2})^2 = \frac{1}{4} + \frac{x^2}{2} + \frac{x^4}{4}$ and $b := \frac{1}{4} - \frac{x^2}{2} + \frac{x^4}{4} = (\frac{1-x^2}{2})^2$ are squares of units whose difference is x^2. Hence $v := ax^{-1}$ and $1 - v$ are squares of units.

For $\mathrm{char}(\mathbb{F}) = 2$, choose $w \in \mathbb{F}$ such that $w \notin \{0, 1\}$ and set $v := w^2$. Then v is a square of a unit and $1 - w = 1 - v^2 = (1 - v)^2$ is a square of a unit as well.

Moreover, the first condition in (ii) translates to $1 - v + v^2 \neq 0$. As this polynomial has at most two roots in $\mathbb{F}$, in summary every field with more than six units satisfies (ii) above.

Remark 6.1.4. P.-E. Caprace has shown in [Cap07] that the Steinberg group is the universal central extension of a two-spherical split Kac–Moody group defined over a field, provided that this is true for the (fundamental) subgroups of rank two. The latter result was obtained in [DMT09]. However, the approach used to determine the universal central extension of rank two groups in [DMT09] makes use of the fact that the group acts strongly transitively on a Moufang n-gon. Hence this strategy does not adapt immediately to the present situation.

In the work [MR90] the authors deal with the situation of split Kac–Moody groups defined over fields. They determine the universal central extension and give a description of the Schur multiplier in terms of generators and relations. However, their strategy uses the Bruhat decomposition of $G_\mathcal{D}(R)$, which does not exist unless R is a field.

Our strategy of lifting generators and relations is rather based on [Ste68] and [Ste62]. The assumptions we made in Definition 6.1.1 also show up in [MR90] and [DMT09].

6.2 The Steinberg group as central extension

We first show that the kernel of the map $\rho\colon \mathrm{St}_A(R) \to G_\mathcal{D}(R)$ is central in $\mathrm{St}_A(R)$.

Proposition 6.2.1. *Let R be commutative and unital and assume that $\mathcal{D}$ is centred, simply connected, two-spherical and without direct factors of type A_1. Then the Steinberg group $\mathrm{St}_A(R)$ is a central extension of $G_\mathcal{D}(R)$. Moreover, the kernel of the natural map*

$\rho\colon \mathrm{St}_A(R) \to G_{\mathcal{D}}(R)$ is generated by the elements $\{u,v\}_\alpha := h_\alpha(u)h_\alpha(v)h_\alpha(uv)^{-1}$ for $\alpha \in \Pi$, where $u, v \in R^\times$, $h_\alpha(u) := s_\alpha(u)s_\alpha(-1)$ and $s_\alpha(u) := x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u)$.

Proof. We first show surjectivity of the map $\rho\colon \mathrm{St}_A(R) \hookrightarrow \mathrm{St}_A(R) \rtimes \mathrm{Hom}(\Lambda, R^\times) \to G_{\mathcal{D}}(R)$. Because $\{h_i\}$ is a basis of $\Lambda^\vee$, every generator of $\mathrm{Hom}(\Lambda, R^\times)$ may be written as $u^{h_{\alpha_i}} = (\lambda \mapsto u^{h_{\alpha_i}(\lambda)})$. Let $u^{h_{\alpha_i}} \in \mathrm{Hom}(\Lambda, R^\times)$. Since the map $\mathrm{Hom}(\Lambda, R^\times) \to G_{\mathcal{D}}(R)$ is injective by (KMG2) of Definition 2.5.2, we identify $u^{h_{\alpha_i}}$ with its image in $G_{\mathcal{D}}(R)$. Now by relation (iii) of Theorem 2.5.8 we know that

$$u^{h_{\alpha_i}} = s_{\alpha_i}(u)s_{\alpha_i}(-1) = h_{\alpha_i}(u).$$

Since $s_{\alpha_i}(u)s_{\alpha_i}(-1) \in \mathrm{Im}(\rho)$, this implies that $u^{h_{\alpha_i}} \in \mathrm{Im}(\rho)$. By $\mathbb{Z}$-linearity, it follows that $\mathrm{Hom}(\Lambda, R^\times) = \langle\, u^{h_{\alpha_i}} \,\rangle \in \mathrm{Im}(\rho)$. Hence ρ is surjective.

We now show that the kernel of the quotient map $\psi\colon \mathrm{St}_A(R) \rtimes \mathrm{Hom}(\Lambda, R^\times) \to G_{\mathcal{D}}(R)$ centralises $\mathrm{St}_A(R)$, from which the result will follow. Consider the following element in $G_{\mathcal{D}}(R)$:

$$s_{\alpha_i}(u)x_\alpha(r)s_{\alpha_i}(u)^{-1}x_{s_{\alpha_i}(\alpha)}(\varepsilon u^{-h_{\alpha_i}(\alpha)}r)^{-1}, \tag{6.1}$$

where ε is a sign as in relation (B') of Proposition 2.5.7, cf. Remark 2.5.6.

We claim that it is equal to $1 \in G_{\mathcal{D}}(R)$ and hence belongs to the kernel of $\mathrm{St}_A(R) \to G_{\mathcal{D}}(R)$. In fact, using the action of $\mathrm{Hom}(\Lambda, R^\times)$ on $\mathrm{St}_A(R)$ as well as (i), (iii) and (iv) as in Theorem 2.5.8, we obtain

$$\begin{aligned} s_{\alpha_i}(u)x_\alpha(r)s_{\alpha_i}(u)^{-1} &\overset{2.5.8(iii)}{=} u^{h_{\alpha_i}}s_{\alpha_i}x_\alpha(r)s_{\alpha_i}^{-1}(u^{h_{\alpha_i}})^{-1} \\ &\overset{2.5.8(iv)}{=} u^{h_{\alpha_i}}x_{s_{\alpha_i}(\alpha)}(\varepsilon r)(u^{h_{\alpha_i}})^{-1} \\ &\overset{2.5.8(i)}{=} x_{s_{\alpha_i}(\alpha)}(\varepsilon u^{h_{\alpha_i}(s_{\alpha_i}(\alpha))}r) \\ &= x_{s_{\alpha_i}(\alpha)}(\varepsilon u^{s_{\alpha_i}(h_{\alpha_i})(\alpha))}r) \\ &= x_{s_{\alpha_i}(\alpha)}(\varepsilon u^{-h_{\alpha_i}(\alpha)}r). \end{aligned}$$

Now by [Tit87, Remark 3.7 (a_4)], under the assumption of two-sphericity on W, the element in (6.1) already equals 1 in $\mathrm{St}_A(R)$ and, in particular, relation (B') holds in $\mathrm{St}_A(R)$ if A is two-spherical. Moreover, this implies that the subgroup of $\mathrm{St}_A(R)$ generated by the elements as in Theorem 2.5.8 (iv) is trivial.

We now show that the elements as in 2.5.8 (ii) and (iii) centralise each root group $U_\alpha \subseteq \mathrm{St}_A(R)$. For this, we conjugate every generator $x_\alpha(r)$ of $\mathrm{St}_A(R)$ by those elements.

$$\begin{aligned} (ii): \quad & s_{\alpha_i}ts_{\alpha_i}^{-1}s_{\alpha_i}(t)^{-1}x_\alpha(r)s_{\alpha_i}(t)s_{\alpha_i}t^{-1}s_{\alpha_i}^{-1} \\ \overset{2.5.8(i)}{=} \quad & s_{\alpha_i}ts_{\alpha_i}^{-1}x_\alpha(s_{\alpha_i}(t)^{-1}(\alpha)r)s_{\alpha_i}t^{-1}s_{\alpha_i}^{-1} \\ \overset{2.5.8(iv)}{=} \quad & s_{\alpha_i}tx_{s_{\alpha_i}(\alpha)}(\varepsilon s_{\alpha_i}(t)^{-1}(\alpha)r)t^{-1}s_{\alpha_i}^{-1} \\ \overset{2.5.8(i)}{=} \quad & s_{\alpha_i}x_{s_{\alpha_i}(\alpha)}(\varepsilon t(s_{\alpha_i}(\alpha))s_{\alpha_i}(t)^{-1}(\alpha)r)s_{\alpha_i}^{-1} \\ \overset{2.5.8(iv)}{=} \quad & x_\alpha(r), \end{aligned}$$

because of the contragredient action of the Weyl group on the inner product $\Lambda \times \Lambda^\vee \to R$, which implies $\langle\, t, s_{\alpha_i}(\alpha)\,\rangle = \langle\, s_{\alpha_i}(t), \alpha\,\rangle$.

$$\begin{aligned}
(iii): \quad & s_{\alpha_i} s_{\alpha_i}(u)^{-1} u^{h_{\alpha_i}} x_\alpha(r) \left(u^{h_{\alpha_i}}\right)^{-1} s_{\alpha_i}(u) s_{\alpha_i}^{-1} \\
\overset{2.5.8(i)}{=} \quad & s_{\alpha_i} s_{\alpha_i}(u)^{-1} x_\alpha(u^{h_{\alpha_i}(\alpha)} r) s_{\alpha_i}(u) s_{\alpha_i}^{-1} \\
\overset{(6.1)}{=} \quad & s_{\alpha_i} x_{s_{\alpha_i}(\alpha)}(\varepsilon u^{-h_{\alpha_i}(\alpha)} u^{h_{\alpha_i}(\alpha)} r) s_{\alpha_i}^{-1} \\
\overset{2.5.8(iv)}{=} \quad & x_\alpha(\varepsilon^2 r) \\
= \quad & x_\alpha(r).
\end{aligned}$$

Hence $\ker(\mathrm{St}_A(R) \rtimes \mathrm{Hom}(\Lambda, R^\times) \to G_{\mathcal{D}}(R))$ centralises $\mathrm{St}_A(R)$, from which it follows that the kernel of the map $\rho\colon \mathrm{St}_A(R) \to G_{\mathcal{D}}(R)$ is central in $\mathrm{St}_A(R)$.

Since $\mathcal{D}$ is simply connected, the group $G_{\mathcal{D}}(R)$ is isomorphic to the one described in Proposition 2.5.7. Hence $\ker(\rho)$ is generated by the elements $h_\alpha(u)h_\alpha(v)h_\alpha(uv)^{-1}$. Because $h_\alpha(u)h_\alpha(v)h_\alpha(uv)^{-1} \in Z(\mathrm{St}_A(R))$, the group span of the elements $h_\alpha(u)h_\alpha(v)h_\alpha(uv)^{-1}$ coincides with their normal span. Hence the second claim follows from relation (C) of Proposition 2.5.7. □

It remains to show that the central extension $\rho\colon \mathrm{St}_A(R) \to G_{\mathcal{D}}(R)$ is universal.

Theorem 6.2.2. *Assume that $\mathcal{D}$ is centred and the Weyl group of $\mathcal{D}$ is two-spherical and has no direct factors of type A_1. Let R be a ring having nice units.*

Then the universal central extension of the split Kac–Moody group $G_{\mathcal{D}}(R)$ is the Steinberg group $\mathrm{St}_A(R)$.

Proof. We first show that the groups $\mathrm{St}_2(R)$ and hence $\mathrm{St}_A(R)$ are perfect. Indeed, for $a \in R^\times$ such that $c := a^2 - 1 \in R^\times$ (which exists by Definition 6.1.1 (i)), we have that $[h_\alpha(a), x_\alpha(c^{-1}r)] = x_\alpha((a^2-1)c^{-1}r) = x_\alpha(r)$. Hence $\mathrm{St}_2(R) = \langle\, x_{\pm\alpha}(r)\,\rangle \le [\mathrm{St}_2(R), \mathrm{St}_2(R)]$.

Step 1: Reduction to the case where $\mathcal{D}$ is simply connected.

Proposition 6.2.1 implies that the map $\mathrm{St}_A(R) \to G_{\mathcal{D}^{sc}}(R)$ is a central extension if $\mathcal{D}^{sc}$ denotes the simply connected root datum associated to the Cartan matrix A. However, by Proposition 2.5.10 the kernel of the map $G_{\mathcal{D}^{sc}}(R) \to G_{\mathcal{D}}(R)$ is central. It follows that $\mathrm{St}_A(R) \to G_{\mathcal{D}}(R)$ is a central extension for the root datum $\mathcal{D}$.

Hence by Theorem 2.11.4 it suffices to show that the central extension $\mathrm{St}_A(R) \to G_{\mathcal{D}^{sc}}(R)$ is universal.

Step 2: Definition of $\hat{\psi}$.

Since $\mathrm{St}_A(R)$ is perfect, by Theorem 2.11.4 we have to show that every central extension of $\mathrm{St}_A(R)$ admits a section.

Let $1 \to C \to E \xrightarrow{\psi} \mathrm{St}_A(R) \to 1$ be a central extension of the Steinberg group.

Choose an element $a \in R^\times$ with the property that $c := a^2 - 1 \in R^\times$ (Definition 6.1.1 (i)). Given $\alpha \in \Phi^{re}$ and $r \in R$, we define $x'_\alpha(r) \in E$ by

$$x'_\alpha(r) := [y', x'],$$

where $x' \in \psi^{-1}(x_\alpha(c^{-1}r))$ and $y' \in \psi^{-1}(h_\alpha(a))$ are arbitrary. Define $\hat{\psi}\colon \mathrm{St}_A(R) \to E$ by $\hat{\psi}(x_\alpha(r)) := x'_\alpha(r)$.

Step 3: A commutator depends only on classes mod $Z(E)$.

Let $x, y \in E$ and $c, d \in Z(E)$. Then

$$[cx, dy] = cxdyx^{-1}c^{-1}y^{-1}d^{-1} = xyx^{-1}y^{-1} = [x, y].$$

Hence $x'_\alpha(r)$ is in fact independent of the choice of x' and y', and we may conclude that $\hat{\psi}\colon \mathrm{St}_A(R) \to E, x_\alpha(r) \mapsto x'_\alpha(r)$ is well-defined.

Step 4: If the map $\hat{\psi}$ extends to a homomorphism, then it is a section.

To prove this, we need to show that $\psi(x'_\alpha(r)) = x_\alpha(r)$.

Indeed, we have that

$$\begin{aligned}
\psi(x'_\alpha(r)) &= \psi([y', x']) \\
&= [\psi(y'), \psi(x')] \\
&= [h_\alpha(a), x_\alpha(c^{-1}r)] \\
&= x_\alpha((a^2 - 1)c^{-1}r) \\
&= x_\alpha(r).
\end{aligned}$$

Therefore it suffices to show that with respect to the elements $x'_\alpha(r)$ the defining relations (A) and (B) of the Steinberg group $\mathrm{St}_A(R)$ can be lifted to relations in E in order to conclude that $\hat{\psi}\colon \mathrm{St}_A(R) \to E, x_\alpha(r) \mapsto x'_\alpha(r)$ is a well-defined homomorphism.

Step 5: Lifting of the action of $h_\beta(v)$ on $x_\alpha(r)$.

Using the above construction of the elements $x'_\alpha(r)$, we define

$$\begin{aligned}
s'_\alpha(u) &:= x'_\alpha(u)x'_{-\alpha}(-u^{-1})x'_\alpha(u) \in \psi^{-1}(s_\alpha(u)), \\
h'_\alpha(u) &:= s'_\alpha(u)s'_\alpha(1)^{-1} \in \psi^{-1}(h_\alpha(u)),
\end{aligned}$$

in analogy to the elements $s_\alpha(u), h_\alpha(u) \in \mathrm{St}_A(R)$.

In $\mathrm{St}_A(R)$, the relation $h_\beta(u)x_\alpha(r)h_\beta(u)^{-1} = x_\alpha(u^{\alpha(\beta^*)}r)$ holds. We claim that in E the analogous relation $h'_\beta(u)x'_\alpha(r)h'_\beta(u)^{-1} = x'_\alpha(u^{\alpha(\beta^*)}r)$ is true.

Conjugating the equation $[h'_\alpha(a), x'_\alpha(c^{-1}r)] = x'_\alpha(r)$, with $h'_\beta(u) \in \psi^{-1}(h_\beta(u))$ we obtain

$$\begin{aligned}
h'_\beta(u)x'_\alpha(r)h'_\beta(u)^{-1} &= h'_\beta(u)[h'_\alpha(a), x'_\alpha(c^{-1}r)]h'_\beta(u)^{-1} \\
&\overset{\text{Step 3}}{=} [h'_\alpha(a), h'_\beta(u)x'_\alpha(c^{-1}r)h'_\beta(u)^{-1}] \\
&\overset{\text{Step 3}}{=} [h'_\alpha(a), x'_\alpha(u^{\alpha(\beta^*)}c^{-1}r)] \\
&= x'_\alpha(u^{\alpha(\beta^*)}r),
\end{aligned}$$

from which we conclude that $h'_\beta(u)x'_\alpha(r)h'_\beta(u)^{-1} = x'_\alpha(u^{\alpha(\beta^*)}r)$ for all $r \in R$, $u \in R^\times$ and $\alpha, \beta \in \Phi^{re}$.

Step 6: Lifting of the action of $s_\beta(v)$ on $x_\alpha(r)$.

To show that conjugation of $x_\alpha(r)$ by $s_\beta(u)$ in E yields the same value as in $\mathrm{St}_A(R)$, we first compute the conjugate of $h_\alpha(u)$ by $s_\beta(v)$. We obtain:

$$\begin{aligned}
s_\beta(v)h_\alpha(a)s_\beta(v)^{-1} &= s_\beta(v)s_\alpha(a)s_\alpha^{-1}s_\beta(v)^{-1} \\
&= s_\beta(v)x_\alpha(a)x_{-\alpha}(-a^{-1})x_\alpha(a)\,(x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1))^{-1}\,s_\beta(v)^{-1} \\
&\overset{\text{2.5.7(B')}}{=} x_{s_\beta(\alpha)}(\varepsilon a v^{-\alpha(\beta^*)})x_{-s_\beta(\alpha)}(-\varepsilon a^{-1}v^{\alpha(\beta^*)})x_{s_\beta(\alpha)}(\varepsilon a v^{-\alpha(\beta^*)}) \\
&\quad \left(x_{s_\beta(\alpha)}(\varepsilon v^{-\alpha(\beta^*)})x_{-s_\beta(\alpha)}(-\varepsilon v^{\alpha(\beta^*)})x_{s_\beta(\alpha)}(\varepsilon v^{-\alpha(\beta^*)})\right)^{-1} \\
&= s_{s_\beta(\alpha)}(\varepsilon a v^{-\alpha(\beta^*)})s_{s_\beta(\alpha)}(\varepsilon v^{-\alpha(\beta^*)})^{-1} \\
&= h_{s_\beta(\alpha)}(\varepsilon a v^{-\alpha(\beta^*)})h_{s_\beta(\alpha)}(\varepsilon v^{-\alpha(\beta^*)})^{-1}.
\end{aligned} \tag{6.2}$$

Hence, conjugating the equation $[h'_\alpha(a), x'_\alpha(c^{-1}r)] = x'_\alpha(r)$, with $s'_\beta(v) \in \psi^{-1}(s_\beta(v))$ we obtain

$$\begin{aligned}
& s'_\beta(v)x'_\alpha(r)s'_\beta(v)^{-1} \\
={}& s'_\beta(v)[h'_\alpha(a), x'_\alpha(c^{-1}r)]s'_\beta(v)^{-1} \\
={}& [s'_\beta(v)h'_\alpha(a)s'_\beta(v)^{-1}, s'_\beta(v)x'_\alpha(c^{-1}r)s'_\beta(v)^{-1}] \\
\overset{\text{Step 3,(6.2)}}{=}{}& [h'_{s_\beta(\alpha)}(\varepsilon a v^{-\alpha(\beta^*)})h'_{s_\beta(\alpha)}(-\varepsilon v^{-\alpha(\beta^*)})^{-1}, x'_{s_\beta(\alpha)}(\varepsilon v^{-\alpha(\beta^*)}c^{-1}r)] \\
\overset{\text{Step 5}}{=}{}& h'_{s_\beta(\alpha)}(\varepsilon a v^{-\alpha(\beta^*)}))x'_{s_\beta(\alpha)}(\varepsilon v^{2\alpha(\beta^*)}v^{-\alpha(\beta^*)}c^{-1}r) \\
& h'_{s_\beta(\alpha)}(\varepsilon a v^{-\alpha(\beta^*)})x'_{s_\beta(\alpha)}(\varepsilon v^{-\alpha(\beta^*)}c^{-1}r) \\
\overset{\text{Step 5}}{=}{}& x'_{s_\beta(\alpha)}(\varepsilon \underbrace{v^{2\alpha(\beta^*)}v^{-2\alpha(\beta^*)}}_{=1}a^2v^{-\alpha(\beta^*)}c^{-1}r)x'_{s_\beta(\alpha)}(\varepsilon v^{-\alpha(\beta^*)}c^{-1}r) \\
\overset{\text{Step 5}}{=}{}& [h'_{s_\beta(\alpha)}(a), x'_{s_\beta(\alpha)}(\varepsilon v^{-\alpha(\beta^*)}c^{-1}r)] \\
={}& x'_{s_\beta(\alpha)}(\varepsilon v^{-\alpha(\beta^*)}r).
\end{aligned}$$

and hence $s'_\beta(v)x'_\alpha(r)s'_\beta(v)^{-1} = x'_{s_\beta(\alpha)}(\varepsilon v^{-\alpha(\beta^*)}r)$, as claimed.

Step 7: Definition of the elements $f_{\alpha\beta}(r,s)$.

Let $\{\alpha,\beta\} \subseteq (\Phi)^{re}$ be a prenilpotent pair of roots. Define

$$f_{\alpha\beta}(r,s) := [x'_\alpha(r), x'_\beta(s)] \prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij} r^i s^j)^{-1}. \tag{6.3}$$

Then by relation (B) it follows that $\psi(f_{\alpha\beta}(r,s)) = 1$ and hence $f_{\alpha\beta}(r,s) \in C$ is central in E.

Note that the product in (6.3) involves only finitely many roots.

Step 8: For all $v \in R^\times$ and $\gamma \in \Phi^{re}$ the identity $f_{\alpha\beta}(r,s) = f_{\alpha\beta}(rv^{\alpha(\gamma^*)}, sv^{\beta(\gamma^*)})$ holds.

Let $\gamma \in \Phi^{re}$, $v \in R^\times$. Then taking the conjugate of (6.3) with $h'_\gamma(v)$, we obtain

$$\begin{aligned} f_{\alpha\beta}(r,s) &= h'_\gamma(v) f_{\alpha\beta}(r,s) h'_\gamma(v)^{-1} \\ &= h'_\gamma(v) x'_\alpha(r) x'_\beta(s) x'_\alpha(r)^{-1} x'_\beta(s)^{-1} \\ &\quad \prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij} r^i s^j)^{-1} h'_\gamma(v)^{-1} \\ &\overset{\text{Step 5}}{=} x'_\alpha(v^{\alpha(\gamma^*)} r) x'_\beta(v^{\beta(\gamma^*)} s) x'_\alpha(v^{\alpha(\gamma^*)} r)^{-1} x'_\beta(v^{\beta(\gamma^*)} s)^{-1} \\ &\quad \prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij} v^{(i\alpha+j\beta)(\gamma^*)} r^i s^j)^{-1} \\ &= x'_\alpha(v^{\alpha(\gamma^*)} r) x'_\beta(v^{\beta(\gamma^*)} s) x'_\alpha(v^{\alpha(\gamma^*)} r)^{-1} x'_\beta(v^{\beta(\gamma^*)} s)^{-1} \\ &\quad \prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij} (v^{\alpha(\gamma^*)} r)^i (v^{\beta(\gamma^*)} s)^j)^{-1} \\ &= f_{\alpha\beta}(rv^{\alpha(\gamma^*)}, sv^{\beta(\gamma^*)}). \end{aligned}$$

Step 9: For all real roots γ, we have $f_{\alpha\beta}(r,s) = f_{s_\gamma(\alpha)s_\gamma(\beta)}(r,s)$.

This follows from the relations

$$\begin{aligned} f_{\alpha\beta}(r,s) &= s'_\gamma f_{\alpha\beta}(r,s) {s'_\gamma}^{-1} \\ &= s'_\gamma [x'_\alpha(r), x'_\beta(s)] \prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij} r^i s^j) {s'_\gamma}^{-1} \\ &\overset{\text{Step 6}}{=} [x'_{s_\gamma(\alpha)}(r), x'_{s_\gamma(\beta)}(s)] \prod_{i,j>0} x'_{is_\gamma(\alpha)+js_\gamma(\beta)}(c_{\alpha\beta ij} r^i s^j) \\ &= f_{s_\gamma(\alpha)s_\gamma(\beta)}(r,s), \end{aligned}$$

as $c_{\alpha\beta ij} = c_{s_\gamma(\alpha)s_\gamma(\beta)ij}$.

Let $\{\alpha,\beta\} \subseteq \Phi^{re}$ be a prenilpotent pair and consider the following conditions.

(D, k) $f_{\alpha\beta}(r_1+r_2, s) = f_{\alpha\beta}(r_1,s) f_{\alpha\beta}(r_2,s)$ if $|i\alpha + j\beta \cap \Phi^{re}| \leq k$, (E, k) $f_{\alpha\beta}(r, s_1+s_2) = f_{\alpha\beta}(r, s_1)f_{\alpha\beta}(r, s_2)$ if $|i\alpha + j\beta \cap \Phi^{re}| \leq k$,

(F, k) $f_{\alpha\beta}(r, s) = 1$ for all r, s and $|i\alpha + j\beta \cap \Phi^{re}| \leq k$.

Step 10: Relations (D, 0) and (E, 0) hold.

For (D, 0) we consider the case that α, β are orthogonal roots. We have the following identities:

$$\begin{aligned}
f_{\alpha\beta}(r_1+r_2, s) &= [x'_\alpha(r_1+r_2), x'_\beta(s)] \\
&\overset{\text{Step 3}}{=} [x'_\alpha(r_1)x'_\alpha(r_2), x'_\beta(s)] \\
&= x'_\alpha(r_1)x'_\alpha(r_2)x'_\beta(s)x'_\alpha(r_2)^{-1}x'_\alpha(r_1)^{-1}x'_\beta(s)^{-1} \\
&= x'_\alpha(r_1)[x'_\alpha(r_2), x'_\beta(s)]x'_\beta(s)x'_\alpha(r_1)^{-1}x'_\beta(s)^{-1} \\
&= x'_\alpha(r_1)f_{\alpha\beta}(r_2, s)x'_\beta(s)x'_\alpha(r_1)^{-1}x'_\beta(s)^{-1} \\
&= f_{\alpha\beta}(r_2, s)[x'_\alpha(r_1), x'_\beta(s)] \\
&= f_{\alpha\beta}(r_1, s)f_{\alpha\beta}(r_2, s).
\end{aligned}$$

Relation (E, 0) is shown similarly.

Step 11: Relation (F, $k-1$) implies both (D, k) and (E, k).

The argument is the same as in Step 10. Note however that here the formulas involve products of root group elements of the form $\prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij}(r_1+r_2)^i s^j)$. Then, however, the assumption (F, $k-1$) applies when interchanging x_α and $\prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij}(r_1+r_2)^i s^j)$, as there are fewer roots of the form $k\alpha + l(i\alpha + j\beta)$ with $i, j, k, l > 0$.

We show that (F, $k-1$) implies (D, k). The other implication stated above is obtained similarly. Let $\{\alpha, \beta\}$ be a prenilpotent pair with $|(i\alpha + j\beta)| \cap \Phi^{re} = k$. The main key is that the computation below is true in $\mathrm{St}_A(R)$, and by (F, $k-1$)

may be spelled out the very same way in E.

$$\begin{aligned}
& f_{\alpha\beta}(r_1+r_2,s) \\
= \quad & [x'_\alpha(r_1+r_2),x'_\beta(s)]\prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij}(r_1+r_2)^i s^j)^{-1} \\
\overset{\text{Step 3}}{=} \quad & [x'_\alpha(r_1)x'_\alpha(r_2),x'_\beta(s)]\prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij}(r_1+r_2)^i s^j)^{-1} \\
= \quad & x'_\alpha(r_1)x'_\alpha(r_2)x'_\beta(s)x'_\alpha(r_2)^{-1}x'_\alpha(r_1)^{-1}x'_\beta(s)^{-1} \\
& \prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij}(r_1+r_2)^i s^j)^{-1} \\
= \quad & x'_\alpha(r_1)[x'_\alpha(r_2),x'_\beta(s)]x'_\beta(s)x'_\alpha(r_1)^{-1}x'_\beta(s)^{-1} \\
& \prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij}(r_1+r_2)^i s^j)^{-1} \\
= \quad & x'_\alpha(r_1)f_{\alpha\beta}(r_2,s)\prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij}r_2^i s^j)x'_\beta(s)x'_\alpha(r_1)^{-1}x'_\beta(s)^{-1} \\
& \prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij}(r_1+r_2)^i s^j)^{-1}
\end{aligned}$$

$$\begin{aligned}
\overset{(F,k-1)}{=} \quad & f_{\alpha\beta}(r_2,s)\prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij}r_2^i s^j)\prod_{i,j,k,l>0} x'_{k\alpha+l(i\alpha+j\beta)}(c_{\alpha\beta kl}c^l_{\alpha\beta ij}r_1^k r_2^{il}s^{lj}) \\
& [x'_\alpha(r_1),x'_\beta(s)]\prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij}(r_1+r_2)^i s^j)^{-1} \\
= \quad & f_{\alpha\beta}(r_2,s)f_{\alpha\beta}(r_1,s),
\end{aligned}$$

as, again by (F, $k-1$), the products vanish.

Step 12: Relations (D, k) and (E, k) imply (F, k).

By Step 9, we may assume without loss of generality that α is simple.

If $\alpha(\beta^*)=0$, we apply Step 8 with $\gamma=\alpha$ and $u=v$ to obtain $f_{\alpha\beta}(r,s)=f_{\alpha\beta}(rv^2,s)$. Hence $f_{\alpha\beta}(r(1-v^2),s)=1$ and since v can be chosen so that $1-v^2\in R^\times$ (Definition 6.1.1), it follows that $f_{\alpha\beta}(r,s)=1$ for all $r,s\in R$.

If $\alpha(\beta^*)\neq 0$, then Step 8 applied with $\gamma=\alpha$ and $u=v^2$ yields $f_{\alpha\beta}(r,s)=f_{\alpha\beta}(rv^4,sv^{2\beta(\alpha^*)})$. Another application of Step 8 with $\gamma=\beta$, $u=v^{-\beta(\alpha^*)}$ to $f_{\alpha\beta}(rv^4,sv^{2\beta(\alpha^*)})$ yields $f_{\alpha\beta}(rv^4,sv^{2\beta(\alpha^*)})=f_{\alpha\beta}(rv^d,s)$, with $d:=4-\alpha(\beta^*)\beta(\alpha^*)$. Summarising, we conclude that

$$f_{\alpha\beta}(r,s)=f_{\alpha\beta}(rv^d,s).$$

Now if $d\neq 0$, choosing an element $v\in R^\times$ with $1-v^d\in R^\times$ (which exists by Definition 6.1.1) yields $f_{\alpha\beta}(r,s)=1$ for all $r,s\in R$.

On the other hand, if $d = 0$, then there are three possibilities to obtain $\alpha(\beta^*)\beta(\alpha^*) = 4$, namely $(\alpha(\beta^*), \beta(\alpha^*)) \in \{(2,2), (1,4), (4,1)\}$.

If $(\alpha(\beta^*), \beta(\alpha^*)) = (1,4)$, we apply Step 8 with $\gamma = \beta$, $u = v$ to obtain $f_{\alpha\beta}(r,s) = f_{\alpha\beta}(rv, sv^2)$. By assumption, either $2 \in R^\times$ or $2 = 0$.

For if $2 \in R^\times$, this implies by choosing $v := -1$ that $f_{\alpha\beta}(r,s) = f_{\alpha\beta}(-r,s)$, and hence $f_{\alpha\beta}(2r,s) = 1$ holds. From this we conclude $f_{\alpha\beta}(r,s) = 1$.

On the other hand, if $2 = 0$, we may choose $v \in R^\times$ such that $1 - v \in R^\times$ and $1 - v + v^2 \in R^\times$. This yields

$$\begin{aligned} & f_{\alpha\beta}(r(v - v^2), s) = f_{\alpha\beta}(r, s(v - v^2)^{-2}) = f_{\alpha\beta}(r, sv^{-2}(1-v)^{-2}) \\ \overset{(*)}{=} \; & f_{\alpha\beta}(r, sv^{-2}) f_{\alpha\beta}(r, s(1-v)^{-2}) = f_{\alpha\beta}(rv, s) f_{\alpha\beta}(r(1-v), s) = f_{\alpha\beta}(r,s), \end{aligned}$$

where we have used the decomposition into partial fractions $\frac{s}{v^2(1-v)^2} = \frac{s}{v^2} + \frac{s}{(1-v)^2}$ in $(*)$.

Hence we obtain $f_{\alpha\beta}(r,s) = 1$. The case $(\alpha(\beta^*), \beta(\alpha^*)) = (4,1)$ is treated analogously, using Step 8 with $\gamma = \alpha$, $u = v$ and additivity of $f_{\alpha\beta}(r,s)$ in the second component.

Assume $(\alpha(\beta^*), \beta(\alpha^*)) = (2,2)$. Then applying Step 8 with $\gamma = \alpha, u = v$ yields $f_{\alpha\beta}(r,s) = f_{\alpha\beta}(rv^2, sv^2)$. Choosing v and $1 - v$ to be squares of units, this implies that $f_{\alpha\beta}(r,s) = 1$ as in the case $2 = 0$ above.

Hence we have shown that relation (D, k) holds.

Step 13: Lifting of relation (A).

Let $x := x'_\alpha(rc^{-1})x'_\alpha(sc^{-1})x'_\alpha((r+s)c^{-1})^{-1} \in C$. Then $h'_\alpha(a)xh'_\alpha(a)^{-1} = x$ and hence

$$\begin{aligned} x &= x'_\alpha(rc^{-1})x'_\alpha(sc^{-1})x'_\alpha((r+s)c^{-1})^{-1} \\ &= h'_\alpha(a)x'_\alpha(rc^{-1})h'_\alpha(a)^{-1}h'_\alpha(a)x'_\alpha(sc^{-1})h'_\alpha(a)^{-1} \\ &\quad\; h'_\alpha(a)x'_\alpha((r+s)c^{-1})^{-1}h'_\alpha(a)^{-1} \\ &= [h'_\alpha(a), x'_\alpha(rc^{-1})]x'_\alpha(rc^{-1})[h'_\alpha(a), x'_\alpha(sc^{-1})]x'_\alpha(sc^{-1}) \\ &\quad\; [h'_\alpha(a), x'_\alpha((r+s)c^{-1})^{-1}]x'_\alpha((r+s)c^{-1})^{-1} \\ &= x'_\alpha(r)x'_\alpha(rc^{-1})x'_\alpha(s)x'_\alpha(sc^{-1})x'_\alpha(r+s)^{-1}x'_\alpha((r+s)c^{-1})^{-1} \\ &\overset{(\mathrm{F},0)}{=} xx'_\alpha(r)x'_\alpha(s)x'_\alpha(r+s)^{-1}, \end{aligned}$$

from which $x'_\alpha(r)x'_\alpha(s) = x'_\alpha(r+s)$ follows.

Step 14: Lifting of relation (B).

By Step 12 we have that $[x'_\alpha(r), x'_\beta(s)] = \prod_{i,j>0} x'_{i\alpha+j\beta}(c_{\alpha\beta ij}r^is^j)$ for all $\{\alpha,\beta\} \subseteq \Phi$ prenilpotent and $r, s \in R$. We may therefore conclude that relation (B) holds in E with respect to the elements $x'_\alpha(r)$.

Step 15: The map $\hat{\psi}$ is a section.

It follows from Steps 13 and 14 that $\hat{\psi}\colon \mathrm{St}_A(R) \to E$ mapping $x_\alpha(s)$ to $x'_\alpha(s)$ is a homomorphism and hence a section (Step 4). In particular, $1 \to C \to E \to \mathrm{St}_A(R) \to 1$ splits and by Theorem 2.11.4 $\psi \circ \hat{\psi}$ is the identity on $\mathrm{St}_A(R)$.

Hence the central extension $\mathrm{St}_A(R) \to G_{\mathcal{D}}(R)$ is universal. This completes the proof. □

6.3 The Schur multiplier

A natural question to ask is about the structure of the kernel of the natural map $\rho\colon \mathrm{St}_A(R) \to G_{\mathcal{D}^{sc}}(R)$. If ρ is a universal central extension, then $\ker(\rho)$ is called the **Schur multiplier of** $G_{\mathcal{D}^{sc}}(R)$.

Definition 6.3.1. Let L be the abelian group generated by symbols $c_\alpha(u,v)$ for all $\alpha \in \Pi$ and all $u, v \in R^\times$, subject to the following defining relations, where $\alpha \neq \beta \in \Pi, t, u, v \in R^\times$ and $c_{\alpha\beta}(u,v) := c_\beta(u^{\beta(\alpha^*)}, v)$:

(M1) $c_\alpha(t,u)c_\alpha(tu,v) = c_\alpha(t,uv)c_\alpha(u,v)$,

(M2) $c_\alpha(1,1) = 1$,

(M3) $c_\alpha(u,v) = c_\alpha(u^{-1}, v^{-1})$,

(M4) if $1-u \in R^\times$, then $c_\alpha(u,v) = c_\alpha(u,(1-u)v)$,

(M5) $c_\alpha(u, v^{\alpha(\beta^*)}) = c_\beta(v, u^{\beta(\alpha^*)})^{-1}$,

(M6) $c_{\alpha\beta}(tu,v) = c_{\alpha\beta}(t,v)c_{\alpha\beta}(u,v)$,

(M7) $c_{\alpha\beta}(t,uv) = c_{\alpha\beta}(t,u)c_{\alpha\beta}(t,v)$. □

Remark 6.3.2. A number of consequences of the relations in Definition 6.3.1 are the following:

(C1) $c_\alpha(u,v^2)c_\alpha(u,w) = c_\alpha(u,v^2w)$,

(C2) $c_\alpha(u,v^2)c_\alpha(w,v^2) = c_\alpha(uw,v^2)$,

(C3) $c_\alpha(u,v^{-1}) = c_\alpha(v,u) = c_\alpha(u^{-1},v)$,

(C4) $c_\alpha(u,v)c_\alpha(u,-v^{-1}) = c_\alpha(u,-1)$,

(C5) $c_\alpha(u,v^2) = c_\alpha(u,v)c_\alpha(v,u)^{-1} = c_\alpha(u^2,v)$, (C6) $c_\alpha(u,v) = c_\alpha(u(1-v),v)$,

(C7) $c_\alpha(u,1) = c_\alpha(1,u) = 1$.

See for example [vdK77, Section 3] for details.

We obtain the following result.

Theorem 6.3.3. *Let R be commutative and unital. Assume that the Weyl group of $G_{\mathcal{D}}(R)$ is two-spherical, has no direct factors of type A_1 and that $\mathcal{D}$ is centred and simply connected. Then the kernel of the natural map $\rho\colon \mathrm{St}_A(R) \to G_{\mathcal{D}^{sc}}(R)$ is a quotient of L.*

Proof. The kernel of $\mathrm{St}_2^\alpha(R) \to \mathrm{SL}_2^\alpha(R)$ is generated by the elements $\{u,v\}_\alpha$ with $u,v \in R^\times$ by Proposition 6.2.1 (see also [HO89, 1.5.2 and the Remark after 1.5.5]). More precisely, each $\{u,v\}_\alpha$ corresponds to the element $h_\alpha(u)h_\alpha(v)h_\alpha(uv)^{-1}$ in the rank one group $\mathrm{St}_2^\alpha(R)$ of $\mathrm{St}_A(R)$. Hence $L_\alpha := \langle\, \{u,v\}_\alpha \,\rangle$ embeds into $\mathrm{St}_2^\alpha(R) \leq \mathrm{St}_A(R)$. We shall identify the groups L_α as subgroups of $\mathrm{St}_A(R)$ in this sense.

Moreover, for all $\alpha \in \Pi$ the group L_α in $\mathrm{St}_A(R)$ lies in the kernel of the map $\rho\colon \mathrm{St}_A(R) \to G_{\mathcal{D}}(R)$ by Proposition 6.2.1. Hence $L' := \langle\, L_\alpha \,\rangle \leq \ker(\rho)$. On the other hand, since $\ker(\rho)$ is generated by the central elements $\{u,v\}_\alpha \in L'$ (Proposition 6.2.1 again), it follows that $\ker(\rho) = \langle\, \{u,v\}_\alpha \,\rangle \leq L'$. Hence $\ker(\rho) = L' = \langle\, \{u,v\}_\alpha \mid \alpha \in \Pi \,\rangle$, with $\alpha \in \Pi$ and $u,v \in R^\times$. We shall prove that L' is a quotient of L via the map $c_\alpha(u,v) \mapsto \{u,v\}_\alpha$.

Let $\alpha \in \Pi$. We first show that L_α satisfies relations (M1) – (M4). For (M1), it follows that

$$\begin{aligned} \{t,uv\}_\alpha\{u,v\}_\alpha &= h_\alpha(t)h_\alpha(uv)h_\alpha(tuv)^{-1}\{u,v\}_\alpha \\ &= h_\alpha(t)\{u,v\}_\alpha h_\alpha(uv)h_\alpha(tuv)^{-1} \\ &= h_\alpha(t)h_\alpha(u)h_\alpha(v)h_\alpha(tuv)^{-1} \\ &= h_\alpha(t)h_\alpha(u)h_\alpha(tu)^{-1}h_\alpha(tu)h_\alpha(v)h_\alpha(tuv)^{-1} \\ &= \{t,u\}_\alpha\{tu,v\}_\alpha. \end{aligned}$$

Relation (M2) is easy, as $\{1,1\}_\alpha = h_\alpha(1) = s_\alpha(1)s_\alpha(1)^{-1} = 1$.

To show identity (M3), we note that $s_\alpha h_\alpha(u)s_\alpha^{-1} = h_\alpha(u^{-1})$, as

$$\begin{aligned} s_\alpha h_\alpha(u)s_\alpha^{-1} &= s_\alpha s_\alpha(u)s_\alpha^{-1}s_\alpha^{-1} \\ &= s_\alpha(u^{-1})s_\alpha^{-1} \\ &= h_\alpha(u^{-1}). \end{aligned}$$

It follows that

$$\begin{aligned} \{u,v\}_\alpha &= s_\alpha h_\alpha(u)h_\alpha(v)h_\alpha(uv)^{-1}s_\alpha^{-1} \\ &= h_\alpha(u^{-1})h_\alpha(v^{-1})h_\alpha((uv)^{-1})^{-1} \\ &= \{u^{-1},v^{-1}\}_\alpha, \end{aligned}$$

which is (M3).

We now show that $\{u,(1-u)v\}_\alpha = \{u,v\}_\alpha$. Recall that

$$s_\alpha(u) = x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u) = x_{-\alpha}(-u^{-1})x_\alpha(u)x_{-\alpha}(-u^{-1}) = s_{-\alpha}(-u^{-1}). \tag{6.4}$$

It follows that

$$\begin{aligned}
& h_\alpha(uv-u^2v)h_\alpha(v-uv)^{-1}\\
= \;& s_\alpha(uv-u^2v)s_\alpha(v-uv)^{-1}\\
= \;& s_\alpha(uv(1-u))s_\alpha(v(1-u))^{-1}\\
= \;& \underbrace{x_\alpha(uv)x_\alpha(-uv)}_{=1}s_\alpha(uv(1-u))s_\alpha(-v(1-u))\\
\overset{(2.1)}{=} \;& x_\alpha(uv)s_\alpha(uv(1-u))x_{-\alpha}((uv)^{-1}(1-u)^{-2})s_\alpha(-v(1-u))\\
\overset{(6.4)}{=} \;& x_\alpha(uv)x_{-\alpha}(-(uv)^{-1}(1-u)^{-1})x_\alpha(uv(1-u))\\
& x_{-\alpha}(-(uv)^{-1}(1-u)^{-1})x_{-\alpha}((uv)^{-1}(1-u)^{-2})s_\alpha(-v(1-u))\\
= \;& x_\alpha(uv)x_{-\alpha}(-(uv)^{-1}(1-u)^{-1})x_\alpha(uv(1-u))\\
& x_{-\alpha}(v^{-1}(1-u)^{-2})s_\alpha(-v(1-u))\\
\overset{(2.1)}{=} \;& x_\alpha(uv)x_{-\alpha}(-(uv)^{-1}(1-u)^{-1})s_\alpha(-v(1-u))x_{-\alpha}(-uv^{-1}(1-u)^{-1})x_\alpha(-v)\\
\overset{(6.4)}{=} \;& x_\alpha(uv)x_{-\alpha}(-(uv)^{-1}(1-u)^{-1})x_{-\alpha}(v^{-1}(1-u)^{-1})x_\alpha(-v(1-u))\\
& x_{-\alpha}(v^{-1}(1-u)^{-1})x_{-\alpha}(-uv^{-1}(1-u)^{-1})x_\alpha(-v)\\
= \;& x_\alpha(uv)x_{-\alpha}(-(uv)^{-1})x_\alpha(uv-v)\\
& x_{-\alpha}(v^{-1}(1-u)^{-1})x_{-\alpha}(-uv^{-1}(1-u)^{-1})x_\alpha(-v)\\
= \;& x_\alpha(uv)x_{-\alpha}(-(uv)^{-1})x_\alpha(uv)x_\alpha(-v)x_{-\alpha}(v^{-1})x_\alpha(-v)\\
\overset{(6.4)}{=} \;& s_\alpha(uv)s_\alpha(-v)\\
= \;& s_\alpha(uv)s_\alpha(v)^{-1}\\
= \;& h_\alpha(uv)h_\alpha(v)^{-1}.
\end{aligned}$$

Hence $h_\alpha(uv-u^2v)h_\alpha(u-uv)^{-1} = h_\alpha(uv)h_\alpha(v)^{-1} = \{u,v\}_\alpha^{-1}h_\alpha(u)$, implying relation (M4).

It therefore remains to show that the elements $\{u,v\}_\alpha \in \mathrm{St}_A(R)$ satisfy relations (M5) – (M7).

Now since $s_\alpha(u)x_\beta(r)s_\alpha(-u) = x_{s_\alpha(\beta)}(\varepsilon r u^{-\beta(\alpha^*)})$, applying the definition of $h_\alpha(u)$ we conclude that $h_\alpha(u)h_\beta(v)h_\alpha(u)^{-1} = h_\beta(u^{\beta(\alpha^*)}v)h_\beta(u^{\beta(\alpha^*)})^{-1}$. In particular, we obtain

$$\{u,v\}_{\alpha\beta} := [h_\alpha(u),h_\beta(v)] = h_\alpha(u)h_\alpha(v^{\alpha(\beta^*)})h_\alpha(uv^{\alpha(\beta^*)})^{-1} = \{u,v^{\alpha(\beta^*)}\}_\alpha.$$

On the other hand, we have that

$$\{u,v\}_{\alpha\beta} = [h_\alpha(u),h_\beta(v)] = h_\beta(u^{\beta(\alpha^*)}v)h_\beta(u^{\beta(\alpha^*)})^{-1}h_\beta(v)^{-1} = \{v,u^{\beta(\alpha^*)}\}_\beta^{-1},$$

from which (M5) follows.

For (M6), we obtain

$$\begin{aligned}\{t,u\}_\alpha^{-1}h_\alpha(t)h_\alpha(u)h_\beta(v) &= h_\alpha(tu)h_\beta(v)\\ &= \{tu,v\}_{\alpha\beta}h_\beta(v)h_\alpha(tu)\\ &= \{tu,v\}_{\alpha\beta}h_\beta(v)\{t,u\}_\alpha^{-1}h_\alpha(t)h_\alpha(u),\end{aligned}$$

hence it follows that $h_\alpha(t)h_\alpha(u)h_\beta(v) = \{tu,v\}_{\alpha\beta}h_\beta(v)h_\alpha(t)h_\alpha(u)$ (recall that the elements $\{u,v\}_\alpha$ are central). From this, we obtain

$$h_\beta(v)h_\alpha(t)h_\alpha(u)\{t,v\}_{\alpha\beta}\{u,v\}_{\alpha\beta} = \{tu,v\}_{\alpha\beta}h_\beta(v)h_\alpha(t)h_\alpha(u),$$

which immediately implies that (M6) holds. Relation (M7) is treated similarly.

This shows that the map $L \to L'$, $c_\alpha(u,v) \mapsto \{u,v\}_\alpha$ is a morphism. As it maps a generating set to a generating set, it is also surjective. Hence $L' = \ker(\rho)$ is a quotient of L. □

Before moving on, we briefly comment on the references [Mat69, Theorem 5.10], [vdK77, Theorems 3.4 and 3.7], [vdK80, Section 3] and [vdKMS75, Remark 2.8 (4)] we shall use. In [Mat69, Theorem 5.10], it it shown that $\ker(\mathrm{St}_A(R) \to G_{\mathcal{D}}(R))$ is isomorphic to L if $\mathcal{D}$ is spherical and R is a field. More generally, the results [vdK77, Theorems 3.4 and 3.7] and [vdK80, Section 3] generalise this fact to the case of rings which are a direct limit of a directed system of direct or restricted product of fields or local rings whose residue field contain at least six elements.

It is also shown that, again if $\mathcal{D}$ is spherical, $\ker(\mathrm{St}_A(R) \to G_{\mathcal{D}}(R))$ is generated by the set $\{c_{\alpha_0}(u,v) \mid u,v \in R^\times, \alpha_0 \in \Pi \text{ a long root}\}$.

Corollary 6.3.4. *With the assumptions of Theorem 6.3.3, assume additionally that R is a direct limit of a directed system of direct or restricted product of fields or local rings whose residue field contain at least six elements.*

Then the kernel of ρ is isomorphic to L.

Note that the term direct limit is meant in the category of rings, for example a local ring need not dominate its predecessors. To name a few examples, among the rings which satisfy the stronger assumptions are the class of connected locally compact rings without compact factor or the adèle ring $A_{\mathbb{F}}$ of a global field $\mathbb{F}$.

Proof of Corollary 6.3.4. Let α, β be adjacent simple roots and let $\mathrm{St}^{\alpha\beta}(R)$, $G_{\alpha\beta}$ be the associated rank two groups of $\mathrm{St}_A(R)$ and $G_{\mathcal{D}}(R)$, respectively. By our assumption on R, we may apply [Mat69, Theorem 5.10], [vdK77, Theorems 3.4 and 3.7], [vdK80, Section 3] and [vdKMS75, Remark 2.8 (4)], yielding that the kernel of the natural map $\mathrm{St}^{\alpha\beta}(R) \to G_{\alpha\beta}(R)$ is isomorphic to the subgroup of L generated by $c_\alpha(u,v)$ and $c_\beta(u,v)$. Moreover, by Theorem 6.3.3 the kernel of $\rho\colon \mathrm{St}_A(R) \to G_{\mathcal{D}}(R)$ is a quotient of L. It therefore remains to show that there are no relations in $\ker(\rho)$ involving three or more simple roots.

Suppose that $\{u,v\}_\alpha = \{u_1,v_1\}_{\alpha_1}\{u_2,v_2\}_{\alpha_2}$ is a non-trivial relation. If both α_1 and α_2 are not adjacent to α, then we conclude that $G_\alpha \cap G_{\alpha_1\alpha_2} = \{1\}$, hence $\langle G_\alpha, G_{\alpha_1\alpha_2}\rangle \cong G_\alpha \times G_{\alpha_1\alpha_2}$. Since central extensions commute with direct products, we obtain

$$\langle \operatorname{St}^\alpha(R), \operatorname{St}^{\alpha_1\alpha_2}(R) \rangle \cong \operatorname{St}^\alpha(R) \times \operatorname{St}^{\alpha_1\alpha_2}(R)$$

and hence $\{u,v\}_\alpha \in \operatorname{St}^\alpha \cap \operatorname{St}^{\alpha_1\alpha_2} = \{1\}$, a contradiction.

Now suppose that one of α_1, α_2 is adjacent to α. Without loss of generality, assume this root is α_1. Then the element $\{u,v\}_\alpha\{u_1,v_1\}_{\alpha_1}^{-1}$ is contained in $\operatorname{St}^{\alpha\alpha_1}(R)$. Now applying [Mat69, Theorem 5.10], [vdK77, Theorems 3.4 and 3.7], [vdK80, Section 3] and [vdKMS75, Remark 2.8 (4)] again, this implies that $\{u,v\}_\alpha\{u_1,v_1\}_{\alpha_1}^{-1}$ can be written as $\{u_0,v_0\}_{\alpha_0}$, where α_0 is a long simple root in the (spherical) root subsystem generated by α, α_1. Hence the above relation $\{u,v\}_\alpha = \{u_1,v_1\}_{\alpha_1}\{u_2,v_2\}_{\alpha_2}$ reduces to a relation involving only two simple roots. Now an induction on the number of roots occurring shows that there are no non-trivial relations involving three or more roots. In other words, the map $L \to \ker(\rho)$ is injective.

It follows that the span of the groups L_α inside $\operatorname{St}_A(R)$ satisfies the same relations as L does, and hence that the map $L \to \ker(\rho) = L'$ given by $c_\alpha(u,v) \mapsto \{u,v\}_\alpha = h_\alpha(u)h_\alpha(v)h_\alpha(uv)^{-1}$ extends to an isomorphism, which is what we wanted to show. □

Corollary 6.3.5. *Assume that the Weyl group of $G_\mathcal{D}(R)$ is two-spherical, has no direct factors of type A_1, that $\mathcal{D}$ is centred and simply connected. Let R be a ring having nice units. Then the Schur multiplier is a quotient of L.*

If moreover R is a direct product, restricted product or direct limit of fields or local rings with residue field containing at least six elements, then the Schur multiplier of $G_\mathcal{D}(R)$ is isomorphic to L. Hence the sequence

$$1 \to L \to \operatorname{St}_A(R) \to G_\mathcal{D}(R) \to 1$$

is exact.

Proof. By the assumptions on the ring R, Theorem 6.2.2 implies that $\rho\colon \operatorname{St}_A(R) \to G_{\mathcal{D}^{sc}}(R)$ is a universal central extension. Hence the Schur multiplier of $G_{\mathcal{D}^{sc}}(R)$ coincides with $\ker(\rho)$. Now Theorem 6.3.3 and Corollary 6.3.4 apply and yield the claim. □

Remark 6.3.6. It is in fact possible to reduce the generating set of the presentation for L in the setup of Corollary 6.3.4. Given the Dynkin diagram, the algorithms works as follows.

(i) Take the simple roots $\{\alpha_1,\ldots,\alpha_n\}$ which are locally of maximal length, i.e. those which are not adjacent to a simple root which is strictly longer and consider the subdiagram spanned by their respective nodes.

(ii) If there is a simply-laced path joining α_i and α_j, delete one of them.

(iii) If two simple roots α_i and α_j can be joined by a monotone path with at least one multiple edge, delete the root which is shorter. Let Π' be the remaining roots.

Then L is generated by the $c_\alpha(u, v)$ where $\alpha \in \Pi'$. This follows from the description of the Schur multiplier given in [Mat69, Theorem 5.10], [vdK77, Theorems 3.4 and 3.7], [vdK80, Section 3] and [vdKMS75, Remark 2.8 (4)], which we used in the proof of Corollary 6.3.4. Here two-sphericity is important for applying the cited results to the rank two-case. See Figure 6.1 for an example.

(a)

(b)

(c)

(d)

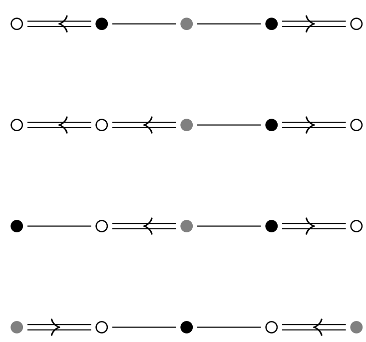

Figure 6.1.: Examples for (the algorithm presented in) Remark 6.3.6. White nodes indicate deleted simple roots after the first step of the algorithm. Gray nodes indicate a possible minimal set of simple roots necessary for the presentation of L.

We close this section with a comment on results available in the literature.

Remark 6.3.7. (i) In classical algebraic K-theory, the name *Steinberg group* together with the symbol $\mathrm{St}(R)$ is commonly used for the direct limit of the groups $\mathrm{St}_n(R)$ with canonical inclusion morphisms. In the notation used here, this group coincides with the direct limit of $\mathrm{St}_A(R)$, where A is the classical Cartan matrix of type A_{n-1}.

(ii) A description of the Schur multiplier in terms of generators and relations for the classical group $\mathrm{St}(R)$ can be found for example in [HO89, Sections 1.4 and 1.5].

(iii) Similar to $\mathrm{St}(R)$, there is also a notion of *orthogonal* or *unitary Steinberg group* over a ring R with involution, given by generators and relations. The stable orthogonal or unitary groups $O(R)$ and $U(R)$ are obtained as direct limit of the corresponding groups $O_{2n}(R)$ and $U_{2n}(R)$. Their central extensions and the relations to the orthogonal or unitary Steinberg group, respectively, are studied in [KM70a] and [KM70b].

(iv) Finally, for the class of rings considered the spherical analogues to Theorem 6.3.3 and Corollary 6.3.5 are [Mat69, Theorem 5.10] and [vdK77, Theorems 3.4 and 3.7].

(v) Further work on this subject can be found (among others) in [Keu81], [vdKMS75], [Dun76], [HO89], [Kol84], [DS75], [Ste73], [Ste71] and [vdKS77].

6.4 Open problems

(i) In Definition 6.1.1, we imposed conditions on the ring R depending on the root system of $\mathcal{D}$. Precisely, for any prenilpotent pair $\{\alpha, \beta\} \subseteq \Phi$ of real roots with α being simple, we assumed R to have a unit v such that $1 - v^{\alpha(\beta^*)\beta(\alpha^*)}$ is a unit again. This assumption was used in Step 12 of the proof of Theorem 6.2.2 in order to conclude that every central extension of the Steinberg group splits. This raises the question whether the assumption is necessary.

(ii) Our description of the kernel of the map $\rho\colon \mathrm{St}_A(R) \to G_{\mathcal{D}}(R)$ uses the strategy of reduction to the same question for the fundamental groups of rank one or two. We then apply results available in the literature. The proof (for local rings with infinite residue field) presented in [vdK77] uses a specific topology on the ring R for which R is irreducible. Van der Kallen then uses arguments for the big cell as *variety* in R^n for some n. These arguments should be familiar to the reader with knowledge in the theory of algebraic groups. For a direct generalisation of the arguments, one would have to understand this topology on a non-spherical Steinberg or Kac–Moody group over such rings.

(iii) The strategy used to compute the groups L_α for a ring more general as in (ii) (i.e. a local ring with residue field containing at least six elements, see for example [vdKMS75, Remark 2.8 (4)]) uses the concept of n-fold stability of a ring.

There is no result for a general commutative unital ring known to the author. As the arguments we developed above are local-to-global, stronger results for the local case (irreducible rank two group schemes over R) may be used to show Corollaries 6.3.4 and 6.3.5 with less assumptions on the ring R.

APPENDIX A

Topological twin buildings

In this chapter, we collect a number of facts on topological twin buildings as shown in T. Hartnick's Master's thesis [Har06]. It seems that his work is not publicly accessible, so we include complete proofs, occasionally adapted to the setting of this thesis.

A.1 The projection map

Recall from Definition 2.3.7 that the projection of a chamber $c \in \Delta_\varepsilon$ onto a spherical residue $R \subseteq \Delta_{-\varepsilon}$ is the unique chamber of R which maximises the codistance from c. However, this is a purely algebraic description which is not easily applicable to topological questions. On the other hand, a basic feature of topological twin buildings is that the projection maps onto panels are continuous on certain sets of chambers, cf. Definition 2.4.1. We therefore aim to bring these two concepts together. Throughout this section, $G_{\mathcal{D}}(\mathbb{F})$ denotes a split Kac–Moody group over a field $\mathbb{F}$.

In order to prove continuity of projections onto s-panels (using Proposition A.2.5), we need a precise description of the projection map on the group-theoretic level. The following three results deal with that problem.

Given $w \in W$ and a representative $w \in N$, we define $O^w := w^{-1}U_+w$ and $O^w_\varepsilon := O^w \cap U_\varepsilon$. Then it holds that $O^w = O^w_+ O^w_-$.

Lemma A.1.1. *Let $w \in W$. Then $w^{-1}B_+wB_- \subseteq B_+B_-$.*

Proof. We proceed by an induction on $l := l(w)$. The case $l = 0$ is trivial.

Let $w = sw_0$ with $l(w) = l(w_0) + 1$. Let $x \in B_+wB_-$, then $sx \in B_+sB_+wB_- = B_+swB_- = B_+w_0B_-$, which implies that

$$x^{-1}s \in B_-w_0^{-1}sB_+sB_+ = B_-w_0^{-1}B_+.$$

Applying the induction hypothesis to w_0, we see that

$$w^{-1}x = w_0^{-1}sx \in w_0^{-1}B_+w_0B_- \subseteq B_+B_-,$$

and the claim follows. $\square$

Let $G_{\mathcal{D}}(\mathbb{F})$ be a split Kac–Moody group, then it holds that $B_\varepsilon = T \ltimes U_\varepsilon$ and the multiplication map $m\colon U_+ \times T \times U_- \to B_+B_-$ is bijective, cf. [KP85, Corollary 4.2 (b)].

It follows that the map $\psi\colon B_- \hookrightarrow B_+B_- \to U_+\backslash B_+B_-$ is a bijection and hence

$$\begin{aligned} \pi\colon U_+\backslash B_+B_- &\to B_-, \\ x &\mapsto \psi^{-1}(U_+x) \end{aligned}$$

is a well-defined and bijective map.

For given $w \in W$ we obtain a map which will be used to describe projections explicitly:

$$\begin{aligned} \rho_w\colon B_+wB_- &\to B_-, \\ x &\mapsto \pi(w^{-1}x). \end{aligned}$$

This map is well-defined for each $w \in W$ because of Lemma A.1.1.

Remark A.1.2. We note that if m is open, also ψ is open and therefore π and ρ_w are continuous.

Proposition A.1.3. *Let $x \in B_+wB_-$. Then $x \in B_+w\rho_w(x)$.*

Proof. By the Birkhoff decomposition of $G_{\mathcal{D}}(\mathbb{F})$ (see Section 2.2), we may write $x = u_+wtu_-$. Then we have that $w^{-1}u_+w \in O^w = O^w_+O^w_-$. Hence there exist $u^1_\varepsilon \in O^w_\varepsilon$ such that $w^{-1}u_+w = u^1_+u^1_-$, and it follows that

$$\begin{aligned} x &= ww^{-1}u_+wtu_- \\ &= wu^1_+u^1_-tu_-. \end{aligned}$$

Thus $w^{-1}x = u^1_+u^1_-tu_-$ which implies that $\rho_w(x) = u^1_-tu_-$ and therefore $x = wu^1_+\rho_w(x)$.

But $u^1_+ \in O^w$, hence by definition there exists $u_2 \in U_+$ such that $u^1_+ = w^{-1}u_2w$. Hence it follows that

$$x = wu^1_+\rho_w(x) = u_2w\rho_w(x) \in B_+w\rho_w(x),$$

which is what we wanted to show. $\square$

Theorem A.1.4. *Let $\Delta = \Delta(G)$ be the twin building of type (W, S) associated to some group G with root group datum $(G, \{U_\alpha\}_{\alpha\in\Phi})$. Let $c_+ = gB_+ \in \Delta_+$ and $c_- = hB_- \in \Delta_-$ be two chambers with $\delta^*(c_+, c_-) = w \in W$. Let $s \in S$ such that $l(ws) > l(w)$. Then*

$$\operatorname{proj}_{P_s(c_-)}(c_+) = h\rho_w(g^{-1}h)^{-1}sB_-.$$

Proof. Define $x := g^{-1}h$ and $k := h\rho_w(x)^{-1}s$. With this notation, the claim is that kB_- is the chamber having codistance ws from c_+ and distance s from c_-.

Since $\delta^*(c_+, c_-) = w$, we have that $x \in B_+wB_-$. Applying Proposition A.1.3 we conclude that

$$x = b_+w\rho_w(x), \quad g^{-1} = xh^{-1} = b_+w\rho_w(x)h^{-1}$$

for some $b_+ \in B_+$. Hence we obtain the following inclusion:

$$g^{-1}k = (b_+w\rho_w(x)h^{-1})(h\rho_w(x)^{-1}s) = b_+ws \in B_+wsB_-,$$

which shows that $\delta^*(c_+, kB_-) = \delta^*(gB_+, kB_-) = ws$.

Similarly, we have that

$$h^{-1}k = \rho_w(x)^{-1}s \in B_-sB_-,$$

which implies $\delta_-(c_-, kB_-) = \delta_-(hB_-, kB_-) = s$. This completes the proof. □

Corollary A.1.5. *Let Δ be the twin building of type (W, S) associated to some group G with root group datum $(G, \{U_\alpha\}_{\alpha\in\Phi})$. Let $c_+ = gB_+ \in \Delta_+$ and $c_- = hB_- \in \Delta_-$ be two opposite chambers. Finally, let $s \in S$. Then*

$$\mathrm{proj}_{P_s(c_-)}(c_+) = h\pi(g^{-1}h)^{-1}sB_-.$$

Proof. Note that $\rho_1 = \pi$. Hence the claim follows from Theorem A.1.4. □

A.2 The twin building of a split Kac–Moody group

Lemma A.2.1. *Let (Δ, τ) be a twin building with a topology which satisfies (TTB1). Let $1 \neq w = s_1 \cdots s_k \in W$ and assume that this expression is reduced. Let $c_\varepsilon \in \Delta_\varepsilon$ be opposite chambers and assume that for each $d \in \Delta_\varepsilon$ and $s \in S$, the map $\mathrm{proj}_d\colon E_1^*(d) \to \Delta_\varepsilon, c \mapsto \mathrm{proj}_{P_s(d)}(c)$ is continuous. Then the following hold:*

(i) There exists a chamber $d \in \Delta_-$ with the properties $\delta^*(c_+, d) = 1$ and $\delta^*(E_w^*(c_-), d) = \{s_k\}$.

(ii) There exists an open neighbourhood of c_+ which does not intersect $E_w^*(c_-)$.

Proof. (i) By the definition of projections there is a unique chamber a_0 in the s_1-panel around c_- having codistance s_1 from c_+. Since Δ is thick, we may choose a chamber $a_1 \in P_{s_1}(c_-) \setminus \{c_-, a_0\}$. Then a_1 is opposite to c_+ and by axiom (Tw2), we have that $\delta^*(x, a_1) = s_2 \cdots s_k$ for all $x \in E_w^*(c_-)$.

Hence by an induction on k we obtain a gallery $(a_1, \dots, a_{k-1})$ with $\delta^*(c_+, a_i) = 1$ and for all $x \in E_w^*(c_-)$ it holds that $\delta^*(x, a_i) = s_{i+1} \cdots s_k$. Thus $d := a_{k-1}$ is the chamber we are looking for.

(ii) Let d be the chamber with $\delta^*(c_+, d) = 1$ and $\delta^*(E^*_w(c_-), d) = \{s_k\}$ provided by (i) and consider the s_k-panel P around d.

By thickness we may choose $d' \in P \setminus \{d\}$ such that $d' \neq \mathrm{proj}_P(c_+)$ and c_+ and d' are opposite. In particular, the union $\{c_+\} \cup E^*_w(c_-)$ consists only of chambers opposite d'. Hence by assumption the map $\mathrm{proj}_{d'} \colon E^*_w(c_-) \cup \{c_+\} \to P, c \mapsto \mathrm{proj}_P(c)$ is continuous. Now $\mathrm{proj}_P(E^*_w(c_-)) = \{d\}$ and $\mathrm{proj}_P(c_+) \neq d$, hence by the Hausdorff axiom we find two disjoint open neighbourhoods U, V with $c_+ \in U$ and $\mathrm{proj}_P(c_+) \in V$. Hence the proj-preimage of V is an open neighbourhood (in the set of chambers opposite d) of c_+ which does not intersect $E^*_w(c_-) \subseteq \mathrm{proj}_P^{-1}(U)$. Hence, by definition of the subspace topology, there exists an open neighbourhood of c_+ in the topological twin building not intersecting $E^*_w(c_-)$, as claimed. □

Proposition A.2.2. *Let (Δ, τ) be a twin building with a topology which satisfies the axioms (TTB1), (TTB3). Let $c_- \in \Delta_-$ and assume that the map $\mathrm{proj}_{c_-} \colon E^*_1(c_-) \to \Delta_-$ is continuous. Then the set of chambers opposite c_- is open.*

Proof. By axiom (TTB3) it suffices to show that there exists a chamber c_0 such that the set of chambers opposite c_- is relatively open in each set $E_{\leq w}(c_0)$, $w \in W$. We will show this by proving that each chmaber c_0 opposite c_- admits an open neighbourhood whose intersection with $E_{\leq w}(c_-)$ is contained in the set of chambers opposite c_-.

For fixed $w \in W$, the set $\delta^*(c_-, E_{\leq w}(c_0))$ is finite, i.e. $\delta^*(c_-, E_{\leq w}(c_0)) = \{w_1, \dots, w_n\}$ for suitable $w_i \in W$. Applying Lemma A.2.1 (ii) to each of the $w_i \neq 1$, we find an open neighbourhood U_i of c_0 which does not intersect $E^*_{w_i}(c_-)$. Hence $\bigcap_{i=1}^n U_i$ is an open neighbourhood of c_0 contained in the set of chambers opposite c_-. Consequently, the set of chambers opposite c_- is relatively open in each set $E_{\leq w}(c_0)$ and the result follows. □

We next prove some sufficient conditions on the group level in order to obtain a topological twin building by taking the quotient $G_{\mathcal{D}}(\mathbb{F})/B_\varepsilon$ with the natural distance and codistance functions (as described in Section 2.2). These results are applied in Section 5.1. In the sequel, $\Delta = \Delta(G_{\mathcal{D}}(\mathbb{F})) := ((G_{\mathcal{D}}(\mathbb{F})/B_+, \delta_+), (G_{\mathcal{D}}(\mathbb{F})/B_-, \delta_-), \delta^*)$ is the twin building associated to $G_{\mathcal{D}}(\mathbb{F})$ equipped with the quotient topology.

Lemma A.2.3. *The building Δ is Hausdorff if and only if both B_+ and B_- are closed in $G_{\mathcal{D}}(\mathbb{F})$.*

Proof. The spaces $\Delta_+ = G_{\mathcal{D}}(\mathbb{F})/B_+$ and $\Delta_- = G_{\mathcal{D}}(\mathbb{F})/B_-$ equipped with the quotient topology induced from the Hausdorff group $G_{\mathcal{D}}(\mathbb{F})$ are Hausdorff if and only if the point stabilisers and their conjugates, i.e. if and only if B_+ and B_- are closed subgroups of $G_{\mathcal{D}}(\mathbb{F})$. □

Proposition A.2.4. *Let $\mathbb{F}$ be a k_ω-field, let $G_{\mathcal{D}}(\mathbb{F})$ be a split Kac–Moody group over $\mathbb{F}$, equipped with the Kac–Peterson topology and let B_+, B_- be the standard Borel subgroups. Assume that B_+ and B_- are closed, that $G_{\mathcal{D}}(\mathbb{F}) = \lim_\to G_n$ and that the bijective product map $m \colon U_+ \times T \times U_- \to B_+B_-$ is open.*

Then for each $s \in S$, the map $\mathrm{proj}_{c_+} \colon E^*_1(c_+) \to P_s(c_+)$ is continuous. Moreover, B_+B_- is open in $G_{\mathcal{D}}(\mathbb{F})$.

Proof. Let c_ε the fundamental chambers associated to B_ε, respectively.

Since the map m is open by assumption, it follows from Theorem A.1.4 that for the chamber $c_+ = B_+$, the map $\mathrm{proj} : E_1^*(c_+) \to \Delta_-$, $d \mapsto \mathrm{proj}_{P_s(c_+)}(d)$ is continuous. Hence by Proposition A.2.2 the set B_+B_- of chambers opposite $c_+ = B_+$ is open. Since $q_- : G_\mathcal{D}(\mathbb{F}) \to \Delta_-$ is continuous, it follows that $q_-^{-1}(B_+B_-) = B_+B_-$ is open in $G_\mathcal{D}(\mathbb{F})$. □

Proposition A.2.5. *If B_+B_- is open and m is open, then Δ satisfies (TTB2).*

Proof. We first show that the set of opposite chambers is open. Let $\mu\colon G_\mathcal{D}(\mathbb{F})\times G_\mathcal{D}(\mathbb{F}), (x,y) \mapsto x^{-1}y$. Since B_+B_- is open by assumption, it follows that $\mu^{-1}(B_+B_-) = \{(gB_+, hB_-) \mid (g,h) \in \mu^{-1}B_+B_-\}$ is open with respect to the product topology. By the construction of twin buildings (Remark 2.3.6), this is exactly the set of opposite chambers.

Let $s \in S$. If $l(ws) > l(w)$, then $p_s(gB_+, hB_-) = \mathrm{proj}_{P_s(hB_-)}(gB_+) = hB_-$ is continuous, as it coincides with the projection onto the second coordinate. Otherwise if $l(ws) < l(w)$, by Theorem A.1.4 we have that $p_s(gB_+, hB_-) = \mathrm{proj}_{P_s(hB_-)}(gB_+) = h\rho_w(g^{-1}h)^{-1}sB_-$. Since openness of m implies continuity of ρ_w, the claim follows. □

Let $n \in \mathbb{N}$. For the next proposition, we fix the notation

$$G_n := \left(\bigcup_{l(w)\leq n} B_+wB_+\right) \cap \left(\bigcup_{l(w)\leq n} B_-wB_-\right).$$

By the Bruhat decomposition, we have that $G_\mathcal{D}(\mathbb{F}) = \bigcup_{n\in\mathbb{N}} G_n$. Moreover, for $w \in W$ we set

$$G_w := \left(\bigcup_{v\leq w} B_+vB_+\right) \cap \left(\bigcup_{v\leq w} B_-vB_-\right).$$

Proposition A.2.6. *Assume that $G_\mathcal{D}(\mathbb{F}) = \lim_\to G_n$. Then its associated building satisfies (TTB3).*

Proof. Note that for each n there exists w such that $G_n \subseteq G_w$. Similarly, for each w there exists n such that $G_w \subseteq G_n$.

Hence these two sequences are both filtrations for $G_\mathcal{D}(\mathbb{F})$. Since $G_\mathcal{D}(\mathbb{F}) = \lim_\to G_n$ by assumption, this implies that

$$G_\mathcal{D}(\mathbb{F}) = \lim_{\to} G_w = \lim_{\to} \bigcup_{v\leq w} B_+vB_+ = \lim_{\to} \bigcup_{v\leq w} B_-vB_-.$$

From this we may conclude that

$$\Delta_\varepsilon = G_\mathcal{D}(\mathbb{F})/B_\varepsilon = \lim_{\to} \bigcup_{v\leq w} B_\varepsilon vB_\varepsilon = \lim_{\to} E_{\leq w}(c_\varepsilon),$$

where c_ε are the fundamental chambers in $\Delta_\varepsilon = G/B_\varepsilon$. □

Taking all the results of this appendix together, we have proved the following.

Theorem A.2.7 ([Har06, Theorem 4.3.13])**.** *Let $(G_{\mathcal{D}}(\mathbb{F}), \tau_{KP})$ be a split Kac–Moody group equipped with the Kac–Peterson topology. Let Δ be its canonical twin building and equip $\Delta_\varepsilon = G/B_\varepsilon$ with the quotient topology τ_Δ. Assume further that B_ε is closed, the big cell B_+B_- is open, that $G = \lim_\rightarrow G_n$, and that the multiplication map $m\colon U_+ \times T \times U_- \to B_+B_-$ is open.*

Then (Δ, τ_Δ) is a topological twin building.

Proof. Since B_ε is closed, τ_Δ is Hausdorff and satisfies (TTB1), cf. Lemma A.2.3. Moreover, the assumptions of Propositions A.2.5 and A.2.6 are fulfilled, showing that τ_Δ also satisfies (TTB2) and (TTB3). □

Finally, [Har06, Conjecture 4.3.14] conjectures that the map m is always open if G is of Kac–Moody type. We give a proof of this conjecture for split Kac–Moody groups in Proposition 5.1.7.

A.3 Some basic topology

We collect some results from basic topology for reference.

Proposition A.3.1. *Let X be a topological space and assume that $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are filtrations of X such that $A_n \subseteq B_n$ for all $n \in \mathbb{N}$ and, moreover, that $X = \lim_\rightarrow A_n$.*

Then also $X = \lim_\rightarrow B_n$.

Proof. Let τ_A and τ_B denote the respective direct limit topologies on X. For all $n \in \mathbb{N}$ the inclusion $B_n \to (X, \tau_A)$ is continuous, hence by the universal property of direct limits we have that

$$\tau_A \subseteq \tau_B.$$

For the converse inclusion let $U \in \tau_B$ and $n \in \mathbb{N}$. Then $U \cap B_n$ is open in B_n and it follows that $U \cap A_n \cap B_n$ is open in A_n. Hence

$$\lim_{\rightarrow} U \cap B_n = U$$

is open in $\lim_\rightarrow A_n = X$. □

Lemma A.3.2. *Let X be a topological space and $\{U_i\}$ be a family of connected subspaces of X such that $U_i \cap U_j$ is non-empty for all i, j.*

Then $\bigcup_i U_i$ is connected.

Proof. Let $A \subseteq C := \bigcup_i U_i$ be a non-empty clopen set. Then A, being non-empty, intersects one of the U_i. Hence $A \cap U_i$ is a non-empty clopen set in U_i. Now connectedness of U_i implies that $A \cap U_i = U_i$. In particular, A intersects every U_j. Again by connectedness, we see that $A \cap U_j = U_j$ for all j. It follows that $A = C$, hence C is connected. $\square$

Lemma A.3.3. *Let $\{X_n\}_{n\in\mathbb{N}}, \{Y_n\}_{n\in\mathbb{N}}, Z$ be topological spaces and let $q_n\colon X_n \to Y_n$ be an open quotient map. Let $f_n\colon X_n \to Z$ be continuous, let $Y := \lim_{\to} Y_n$ and $g_n\colon Y \to Z$ be a family of continuous maps such that for all n the diagram*

$$\begin{tikzcd} X_n \arrow[d, "q_n"'] \arrow[dr, "f_n"] & \\ Y_n \arrow[r, "g_n"'] & Z \end{tikzcd}$$

is commutative.

Then $g := \lim_{\to} g_n\colon Y \to Z$ is continuous.

Proof. Let $U \subseteq Z$ be open. Since f_n is continuous and q_n is open, we obtain that $U_n := (g_n)^{-1}(U) = q_n((f_n)^{-1}(U))$ is open in Y_n. It follows that $g^{-1}(U) = \lim_{\to}(g_n)^{-1}(U) = \lim_{\to} U_n$ is open in $\lim_{\to} Y_n = Y$. Hence g is continuous. $\square$

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Index

Glossary

$(*)$	a condition on the conjugates of the R-rational points of an $\mathbb{F}$-torus, 44
A	generalised Cartan matrix, 15
$\mathbb{A}_{\mathbb{F}}$	adèle ring of a global field $\mathbb{F}$, 74
Ad	adjoint representation, 16, 22
$\mathbb{A}^1(\mathbb{F})$	affine line over $\mathbb{F}$, 63
$\mathcal{A}$	amalgam (of groups), 27
$\mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_R)$	automorphism group of $\mathcal{U}_R$ preserving the grading, 21
(B, N)	BN-pair, 10
(B_+, B_-, N)	twin BN-pair, 10
B	Borel subgroup, 11
$c_\alpha(u, v)$	Steinberg cocycles, 83
c_i	simple roots in the root lattice, 16
$c_{\alpha\beta ij}$	structure constants, 17
$\mathrm{Cod}(\theta)$	θ-codistances of chambers occurring with respect to θ, 67
c_g	conjugation with g, 32, 50
$c^{\mathbf{op}}$	the subset of chambers which are opposite c, also known as opposites geometry with respect to the chamber c, 70
(D, k)	additivity condition, 79
$\mathcal{D}$	Kac–Moody root datum, 16
(Δ, δ)	building, 12
$\Delta_w^{\pm}$	chambers at codistance w, 13
δ^*	codistance function of a twin building, 13
δ	distance function of a building, 12
δ_{ij}	Kronecker's delta, i.e. $\delta_{ij} = 1$ if and only if $i = j$, 15

Δ_θ	flip-flop system, 27
Δ_1	flip-flop system, 27
$((\Delta_+,\delta_+),(\Delta_-,\delta_-),\delta^*)$	twin building, 13
Δ_w	set of chambers at θ-codistance w, 66
Δ_X	set of chambers at θ-codistance $X \subseteq \mathrm{Inv}(W)$, 67
$E^*_w(c_\varepsilon)$	chambers at codistance w from c_ε, 13
$E^*_{\leq w}(c_\varepsilon)$	chambers at codistance $\leq w$ from c_ε, 13
$E_w(c_\varepsilon)$	chambers at distance w from c_ε, 13
$E_{\leq w}(c_\varepsilon)$	chambers at distance $\leq w$ from c_ε, 13
(E,k)	additivity condition, 79
ε	as convention, we take $\varepsilon \in \{+,-\}$, 11
$\mathrm{eval_w}$	evaluation map at w, 33
$f_{\alpha\beta}(r,s)$	certain elements in a central extension of the Steinberg group, 78
(F,k)	a condition on the central elements $f_{\alpha\beta}(r,s)$ to be trivial, 79
G'	derived group of G, 28
$G_{(i,\overline{\alpha})}$	image of $T^i \times G_{\alpha_1} \times \ldots \times G_{\alpha_k}$ under the product map, 31
G_k	a defining sequence for $(G_\mathcal{D}(\mathbb{F}),\tau_{KP})$ given by $G^\varepsilon_k \cap G^{-\varepsilon}_k$, 58
G^ε_k	a defining sequence for $(G_\mathcal{D}(\mathbb{F}),\tau_{KP})$, 58
$\mathfrak{g}$	Kac–Moody algebra, 15
$G_\mathcal{D}(R)$	split Kac–Moody group of type $\mathcal{D}$ over R, 16
$G_{\mathcal{D}^{ad}}(R)$	adjoint Kac–Moody group of type A over R, 22
$G_{\mathcal{D}^{sc}}(R)$	simply connected Kac–Moody group of type A over R, 18
G_α	rank one-subgroup, 9
$(G,\{U_\alpha\}_{\alpha\in\Phi},T)$	RGD system, 9
G_θ	unitary form (with respect to a flip θ), 26
$\mathcal{G}$	Tits functor, 16
(G,B,N,S)	Tits system, 10
(G,B_+,B_-,N,S)	twin Tits system, 11
Γ	Dynkin diagram, 27
$\mathfrak{g}_\alpha$	rank one subalgebra, 15
$\mathfrak{h}$	Cartan subalgebra, 15
$h_\alpha(u)$	an element in $\mathrm{St}_A(R)$ contained in the preimage of the standard torus, 18, 75

h_i	simple coroots in the coroot lattice, 16
$\mathrm{Hom}(\Lambda, R^\times)$	one possible identification of the torus, 19
$(i, \overline{\alpha})$	a pair in $\{0,1\} \times (\Phi^{re})^k$, 31
k_ω**-space**	name for a topological space which is a direct limit of compact spaces, 24
l	word length in a Coxeter group, 12
Λ	root lattice, 16
$\mu(u)$	an element of $U_{-\alpha} u U_{-\alpha}$ acting as reflection, 9
m'	multiplication map $U_- \times T \times U_+ \to B_- B_+$, 59
m	multiplication map $U_+ \times T \times U_- \to B_+ B_-$, 59
$\mathcal{O}_{pw}$	topology of pointwise convergence, 31
ω	Chevalley involution, 26
$p_{(i,\overline{\alpha})}$	quotient map with image $G_{(i,\overline{\alpha})}$, 31
$P_s(c)$	s-panel around the chamber c, 12
P_ε	standard parabolic subgroup, 11
φ_α	embedding of the rank one-subgroup associated to α, 26
$(\Phi)^{re}$	real roots of the root system Φ, 15
Φ	root system, 15
Π	basis of simple roots, 15
$\mathrm{proj}_R(c)$	projection of c onto R, 12, 13
p_s	restricted projection map, 14
$\mathbb{P}^1(\mathbb{F})$	the projective line over the field $\mathbb{F}$, 60
q_ε	quotient map $G_{\mathcal{D}}(\mathbb{F}) \to G_{\mathcal{D}}(\mathbb{F})/B_\varepsilon$, 61
$(R, \mathbb{K})$	rank two-rigid pair, 42
$\rho_{\Sigma,c}$	retraction onto Σ centred at c, 23
ρ	the natural map $\mathrm{St}_A(R) \to G_{\mathcal{D}}(R)$, 74
$s_\alpha(u)$	reflection associated to α with parameter u, 18
σ	ring involution, 26
$(\Sigma_+.\Sigma_-)$	twin apartment, 13
$\mathfrak{sl}_2(\mathbb{C})$	Lie algebra of traceless 2×2-matrices, 15
$\mathrm{St}_A(R)$	Steinberg group of type A over R, 17

$\tau_{(i,\overline{\alpha})}$	quotient topology on $G_{(i,\overline{\alpha})}$, 31
τ_θ	Lang map (with respect to a flip θ), 26
θ	flip of a Kac–Moody group or a twin building, 26
τ_{KP}	Kac–Peterson topology, 25
$\delta^\theta(c)$	θ-codistance of the chamber c, 27
T	standard torus, 11
$(\{U_\alpha\}_{\alpha\in\Phi}, T)$	root group datum, 9
U_α	root group, 9
$\{u, v\}_\alpha$	central elements of $\mathrm{St}_A(R)$ lying in the kernel of $\mathrm{St}_A(R) \to G_{\mathcal{D}}(R)$, 75
$R_u(B)$	unipotent radical of the Borel subgroup B, 45
$\mathcal{U} = \mathcal{U}(\mathfrak{g})$	universal enveloping algebra of the Kac–Moody algebra, 21
$(U(\mathcal{A}), \nu)$	universal enveloping group of an amalgam, 27
$\mathcal{U}_R$	tensor-product of $\mathcal{U}_\mathbb{Z}$ with the ring R, 21
$\mathcal{U}_\mathbb{Z}$	a $\mathbb{Z}$-form of the universal enveloping algebra of a Kac–Moody algebra, 21
w_0	the longest element in a spherical Coxeter group, 12
(W, S)	Coxeter system, 10
$\tilde{w}$	a representative of $w \in W$ in $N_{G_{\mathcal{D}}(\mathbb{F})}(T) \subseteq G_{\mathcal{D}}(\mathbb{F})$, 32
W	Weyl group, 9

Wissenschaftlicher Werdegang

15. April 1983	Geburt in Frankfurt/Main
1994 – 2002	Gymnasium in Frankfurt/Main
2002	Abitur
2003 – 2007	Studium der Mathematik (MCS-Diplom) an der TU Darmstadt
2007	Diplom in Mathematik (MCS) an der TU Darmstadt
2008 – 2011	Promotionsstudium an der TU Darmstadt
2008 – 2011	Wissenschaftlicher Mitarbeiter in Forschung und Lehre am Fachbereich Mathematik der TU Darmstadt
2011	Promotion