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Strong periodic solutions to quasilinear parabolic equations: An approach by the Da Prato–Grisvard theorem

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Abstract
This article develops an approach to unique, strong periodic solutions to quasilinear evolution equations by means of the classical Da Prato–Grisvard theorem on maximal L^p-regularity in real interpolation spaces. The method is used to show that quasilinear Keller–Segel systems admit a unique, strong T-periodic solution in a neighborhood of 0 provided the external forces are T-periodic and satisfy certain smallness conditions. A similar assertion applies to a Nernst–Planck–Poisson type system in electrochemistry. The proof for the quasilinear Keller–Segel systems relies also on a new mixed derivative theorem in real interpolation spaces, that is, Besov spaces, which is of independent interest.

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1 | INTRODUCTION

The theory of periodic solutions to ordinary and partial differential equations as well as to evolution equations has a very long history and tradition. In this article, we concentrate on periodic solutions for parabolic evolution equations and more precisely for quasilinear equations. While the situation in the linear and semilinear setting is rather well-understood, this is not the case for quasilinear equations. For approaches within the context of fluid dynamics concerning the *semilinear case*, we refer to fundamental articles by Serrin [41], Kozono and Nakao [28], Galdi [16], Geissert, Hieber and Nguyen [18], and the survey article by Galdi and Kyed [17].

A characterization of strong periodic solutions for *linear* abstract evolution equations within the L^p-setting for $1 < p < \infty$ was obtained by Arendt and Bu in [4]. Their proof is based on operator-valued Fourier multipliers for $L^p(\mathbb{R};X)$, where X denotes a UMD Banach space.

In this article, we are interested in *periodic solutions to quasilinear evolution equations* and ask for conditions such that for a given periodic function f of period $T > 0$, the quasilinear equation

$$\begin{cases} u'(t) + \mathcal{A}(u(t))u(t) = F(t, u(t)) + f(t), & t \in \mathbb{R}, \\ u(t) = u(t+T), & t \in \mathbb{R} \end{cases} \tag{QACP}$$

admits a unique, strong, periodic solution of the same period T within a certain maximal regularity class. A satisfactory answer to this question was given recently by Hieber and Stinner [24] within the context of L^p-spaces for $1 < p < \infty$. The approach in [24] is based on properties of the linear problem

$$u'(t) + \mathcal{A}u(t) = f(t), \quad t \in \mathbb{R}, \qquad u(t) = u(t+T), \quad t \in \mathbb{R} \tag{PACP}$$

with $T = 2\pi$. The operator $\mathcal{A}$ is said to have the property of maximal periodic L^p-regularity, if for each $f \in L^p(0, 2\pi; X)$ the system (PACP) has a unique solution u within the class

$$u \in W^{1,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D(\mathcal{A})).$$

Arendt and Bu [4] characterized this property for the situation of $1 < p < \infty$, X being a UMD-space and $\mathcal{A} : D(\mathcal{A}) \to X$ a closed operator in terms of $i\mathbb{Z} \subset \rho(\mathcal{A})$ and $k(ik - \mathcal{A})^{-1}$ being $\mathcal{R}$-bounded. For general information concerning maximal L^p-regularity, we refer to the monographs and survey articles by Lunardi [32], Amann [2], Denk, Hieber, Prüss [14], Arendt and Bu [5], Kunstmann and Weis [31], and Prüss and Simonett [38]. For the theory of critical spaces within the theory of quasilinear evolution equations, we refer to the article by Prüss, Simonett, and Wilke [39].

In the following, we are in particular interested in the case $p = 1$, which is not covered by the results cited above. Our first aim is to treat quasilinear, periodic problems by means of the classical Da Prato–Grisvard approach [11]. Solutions will be constructed in the maximal regularity space described precisely below in (1.2).

It was shown very recently in [12] that the classical Da Prato–Grisvard theorem for linear problems, being valid in particular for the case $p = 1$, is of crucial importance for global existence results for certain free boundary value problems in the critical space $L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}_+))$. Besides being valid for the case $p = 1$, the Da Prato–Grisvard approach has the further advantage that the underlying linear operator $\mathcal{A}$ only needs to be the generator of a bounded analytic semigroup on an arbitrary Banach space and no conditions on the $\mathcal{R}$-boundedness of $k(ik - \mathcal{A})^{-1}$ are needed.

More precisely, denote by $\mathcal{A}_0 := \mathcal{A}(0)$ the realization of the linear operator associated to (QACP) on a Banach space X with domain $D(\mathcal{A}_0)$, assume that $-\mathcal{A}_0$ is the generator of a bounded analytic semigroup $e^{-t\mathcal{A}_0}$ on X and that $0 \in \rho(\mathcal{A}_0)$. Let $T > 0$, $\theta \in (0,1)$, $1 \leqslant p < \infty$ as well as $f : \mathbb{R} \to D_{\mathcal{A}_0}(\theta, p)$ be T-periodic with $f_{|(0,T)} \in L^p(0, T; D_{\mathcal{A}_0}(\theta, p))$. Here, $D_{\mathcal{A}_0}(\theta, p)$ is defined by

$$D_{\mathcal{A}_0}(\theta, p) := \left\{ x \in X : [x]_{\theta,p} := \left(\int_0^\infty \| t^{1-\theta} \mathcal{A}_0 e^{-t\mathcal{A}_0} x \|_X^p \frac{dt}{t} \right)^{1/p} < \infty \right\}. \tag{1.1}$$

Solutions will be constructed in the maximal regularity space

$$\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0} := \{u \in W^{1,p}(0,T;D_{\mathcal{A}_0}(\theta,p)) : \mathcal{A}_0 u \in L^p(0,T;D_{\mathcal{A}_0}(\theta,p)) \text{ and } u(0) = u(T)\}, \tag{1.2}$$

while the data space $\mathbb{F}_{\mathcal{A}_0}$ is defined as $\mathbb{F}_{\mathcal{A}_0} := L^p(0,T;D_{\mathcal{A}_0}(\theta,p))$. By $\mathbb{B}_\rho := \mathbb{B}^{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}}(0,\rho)$, we denote the centered ball with radius $\rho > 0$ in the maximal regularity space $\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}$.

Roughly speaking, we will show that if $-\mathcal{A}_0$ generates a bounded analytic semigroup on X as well as $0 \in \rho(\mathcal{A}_0)$, if F and f fulfill certain regularity and Lipschitz conditions and if $\mathcal{A}$ also satisfies a Lipschitz condition, then there exists a unique T-periodic solution to (QACP) in $D_{\mathcal{A}_0}(\theta,p)$ provided $F(0)$ and f are sufficiently small. For a precise formulation of our first main result, see Theorem 2.2.

For a different approach to periodic solutions to quasilinear parabolic equations of the form

$$u_t = u^\gamma(\Delta u + u + f),$$

we refer to the work of Giga and Mizoguchi [19, 20], where the existence of unique positive periodic solutions for positive right-hand sides f was proved under certain assumptions on the first Dirichlet eigenvalue of $-\Delta$.

In the following, we apply the Da Prato–Grisvard approach to quasilinear Keller–Segel systems in chemotaxis as well as to a Nernst–Planck–Poisson type system in electrochemistry. This is, however, not a straightforward task.

To explain our strategy, consider first the Keller–Segel model, which is a typical model to describe chemotaxis, that is, the direct movement of cells and organisms in response to chemical gradients, see the original paper by Keller and Segel [27]. More precisely, let $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, be a bounded domain with smooth boundary and consider the quasilinear chemotaxis system given by

$$\begin{cases} n_t = \nabla\cdot((n+1)^m\nabla n) - \nabla\cdot(n\nabla c) + f_n, & x\in\Omega,\ t\in\mathbb{R}, \\ c_t = \Delta c - c + n + f_c, & x\in\Omega,\ t\in\mathbb{R}, \\ \frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0, & x\in\partial\Omega,\ t\in\mathbb{R} \end{cases} \tag{PQKS}$$

for time periodic functions f_n, f_c.

We now choose $X_0 := L^q_0(\Omega) \times W^{1,q}(\Omega)$ as the underlying ground space, where the first component $L^q_0(\Omega) := \{g \in L^q(\Omega) : \int_\Omega g\,\mathrm{d}x = 0\}$ denotes the space of functions in $L^q(\Omega)$ with mean value zero, and let $A_0 := A(0)$ be the realization of the linear operator associated to (PQKS) in X_0. Define $D_{A_0}(\theta,p)$ as well as the solution space $\mathbb{E}^{\mathrm{per}}_{A_0}$ accordingly. An application of our approach to the Keller–Segel equations above is now based on embedding properties of the first component of the maximal regularity space $\mathbb{E}^{\mathrm{per},1}_{A_0}$ of the form

$$\mathbb{E}^{\mathrm{per},1}_{A_0} \hookrightarrow L^\infty(0,T;B^{2\theta+1}_{qp}(\Omega))$$

for certain values of θ. Here $B^{2\theta+1}_{qp}(\Omega)$ denotes the Besov space of order $2\theta+1$. These spaces appear naturally at this point because gradients of functions are estimated in $B^{2\theta}_{qp}(\Omega)$, and the latter spaces coincide with the real interpolation spaces in the present setting.

We will obtain embeddings of this desired form by developing a general mixed derivative theorem for sectorial operators A in real interpolation spaces in Section 3. The latter reads as

$$W^{1,p}(J;D_A(\theta,q))\cap L^p(J;E_1(A))\hookrightarrow H^{\alpha,p}(J;(D_A(\theta,q),E_1(A))_{1-\alpha}),$$

where $\alpha\in(0,1)$, and $(\cdot,\cdot)_\alpha$ represents the complex interpolation functor. Moreover, $E_1(A)$ denotes the domain of the realization of a sectorial operator A on the space $D_A(\theta,q)$, that is,

$$E_1(A):=\{u\in D(A):Au\in D_A(\theta,q)\}.$$

For a precise statement of the general mixed derivative theorem in real interpolation spaces, see Theorem 3.1. This result is of independent interest and can be considered as the counterpart of the classical mixed derivative theorem for L^p-spaces, now in the context of Besov spaces. In the case of $\Omega=\mathbb{R}^d$, it reads as

$$W^{1,p}(J;B^{2\theta}_{qp}(\mathbb{R}^d))\cap L^p(J;B^{2\theta+2}_{qp}(\mathbb{R}^d))\hookrightarrow H^{\alpha,p}(J;B^{2\theta+2-2\alpha}_{qp}(\mathbb{R}^d)),$$

where J denotes a time interval, $\alpha\in(0,1)$, $1<p,q<\infty$ and $0<\theta<1$.

Combining our approach to quasilinear periodic equations with the mixed derivative theorem in the situation of boundary conditions enables us to show the following result on strong, periodic solutions for (PQKS): If f_n and f_c are T-periodic and small enough, then there exists a unique, periodic solution $w=(n,c)^{\mathrm{T}}:\mathbb{R}\to D_{A_0}(\theta,p)$ of (PQKS) with the same period T and which fulfills $w_{|(0,T)}\in\overline{\mathbb{B}}_R$ for some $R>0$.

For results on local and global existence as well as blow-up criteria for solutions to the classical Keller–Segel system, we refer to the articles [6, 25, 26, 30] and the references therein. Existence results for global, strong solutions have been studied, for example, in [29, 34, 35]. For global existence and blow-up results for quasilinear Keller–Segel systems with nonlinear diffusion, we refer to [8] and [6, section 3.6].

As in [25, section 2.5], we consider a version of the quasilinear Keller–Segel system, where the classical cross diffusion term depends linearly on the cell density. For $m<0$ the nonlinear diffusion term appears in a version (see [9, section 4]) of the so-called volume filling models that have been derived in [37].

Finally, we turn our attention to periodic solutions to Nernst–Planck–Poisson type equations in electrochemistry. For a bounded domain $\Omega\subset\mathbb{R}^d$, $d\in\mathbb{N}$, with smooth boundary consider the system of equations

$$\begin{cases} u_t=\mu_u\Delta u+\nabla\cdot(u\nabla w)+g_u, & x\in\Omega,\ t\in\mathbb{R},\\ v_t=\mu_v\Delta v-\nabla\cdot(v\nabla w)+g_v, & x\in\Omega,\ t\in\mathbb{R},\\ w_t=\Delta w+u-v+g_w, & x\in\Omega,\ t\in\mathbb{R},\\ \frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=\frac{\partial w}{\partial\nu}=0, & x\in\partial\Omega,\ t\in\mathbb{R}, \end{cases}\tag{PNPP}$$

where g_u, g_v and g_w are given time periodic functions. Here u and v represent concentrations of oppositely charged ions, and w denotes the induced electrical potential. For more information on the Nernst–Planck–Poisson system we refer, for example, to [36] or to [40], where the second reference emphasizes the electrodiffusion of ions in fluids. In the sequel we presume that $\mu_u,\mu_v>0$ are constant, and we set $\mu_u=\mu_v=1$ for simplicity. We note that Prüss, Simonett, and Wilke [39] obtained solutions for the initial value problem for a Nernst–Planck–Poisson type system in critical spaces. In [10] Constantin and Ignatova proved global existence and stability results for large data for a Nernst–Planck–Navier–Stokes system in bounded domains in two dimensions, describing ionic electrodiffusion in fluids. Bothe, Fischer, and Saal showed in [7] the local existence of solutions to electrokinetic flows and proved global well-posedness in two dimensions.

In our third main result, we show that if g_u, g_v, g_w are T-periodic and small enough, then there exists a unique, strong solution $z = (u, v, w)^{\mathrm{T}} : \mathbb{R} \to D_B(\theta, p)$ of (PNPP) with the same period T that satisfies $z_{|(0,T)} \in \overline{\mathbb{B}}_R$ for some $R > 0$. Here $D_B(\theta, p)$ is defined analogously to $D_{A_0}(\theta, p)$, see (1.1), with B being the realization of the linear operator associated to (PNPP) in the ground space $X_0 := L^q_0(\Omega) \times L^q_0(\Omega) \times W^{2,q}_N(\Omega) \cap L^q_0(\Omega)$.

The structure of this article is as follows: In Section 2, we briefly recall the Da Prato–Grisvard theory in the linear setting before presenting its extension to the quasilinear periodic setting. We then develop a mixed derivative theorem to evolution equations acting in real interpolation spaces in Section 3. Section 4 is dedicated to quasilinear chemotaxis systems; the main result for this model as well as an analogous result for a physically relevant variant of this model are stated there. In Section 5, we present our main result for the Nernst–Planck–Poisson type system. Finally, the proofs of the results concerning the Keller–Segel system as well as the Nernst–Planck–Poisson type system are given in Section 6.

2 | A QUASILINEAR EXTENSION OF THE DA PRATO–GRISVARD APPROACH

We start this section by recalling the periodic version of the linear Da Prato–Grisvard theorem from [23] and will then study the associated quasilinear problem.

To this end, let $-\mathcal{A}$ be the generator of a bounded analytic semigroup $e^{-t\mathcal{A}}$ on a Banach space X with domain $D(\mathcal{A})$ and assume that $0 \in \rho(\mathcal{A})$. Let $0 < T < \infty, \theta \in (0,1)$ and $1 \leqslant p < \infty$. Consider the space $D_{\mathcal{A}}(\theta, p)$ defined as in (1.1). Then for $f : \mathbb{R} \to D_{\mathcal{A}}(\theta, p)$ T-periodic, we consider the inhomogeneous Cauchy problem

$$\begin{cases} u'(t) + \mathcal{A}u(t) = f(t), & t \in \mathbb{R}, \\ u(t) = u(t+T), & t \in \mathbb{R}. \end{cases} \tag{PACP}$$

Formally, a candidate for a solution to (PACP) is given by

$$u(t) := \int_{-\infty}^{t} e^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}s. \tag{2.1}$$

Furthermore, we define the solution space $\mathbb{E}^{\mathrm{per}}_{\mathcal{A}}$ for (PACP) by

$$\mathbb{E}^{\mathrm{per}}_{\mathcal{A}} := \{u \in W^{1,p}(0,T; D_{\mathcal{A}}(\theta,p)) : \mathcal{A}u \in L^p(0,T; D_{\mathcal{A}}(\theta,p)) \text{ and } u(0) = u(T)\}, \tag{2.2}$$

and we equip it with the norm

$$\|u\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}}} := \|u\|_{W^{1,p}(0,T;D_{\mathcal{A}}(\theta,p))} + \|\mathcal{A}u\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))}.$$

The underlying data space is defined as

$$\mathbb{F}_{\mathcal{A}} := L^p(0,T;D_{\mathcal{A}}(\theta,p)).$$

The following proposition presents a summary of the most essential results of [23, section 2].

Proposition 2.1. *Let $\theta \in (0,1)$, $1 \leqslant p < \infty$, $0 < T < \infty$ and $f : \mathbb{R} \to D_{\mathcal{A}}(\theta,p)$ such that $f_{|(0,T)} \in L^p(0,T;D_{\mathcal{A}}(\theta,p))$. Then u defined by (2.1) is the unique, strong solution to (PACP), that is, u is the unique T-periodic function in $C(\mathbb{R};X)$ that is differentiable in t for almost every $t \in \mathbb{R}$, satisfies $u(t) \in D(\mathcal{A})$ as well as $\mathcal{A}u \in L^p(0,T;X)$, and u solves*

$$u'(t) + \mathcal{A}u(t) = f(t).$$

In addition, there exists a constant $C > 0$ such that

$$\|u\|_{\mathbb{E}^{\text{per}}_{\mathcal{A}}} \leqslant C\|f\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))}. \tag{2.3}$$

We now turn to the quasilinear situation. As above, let $-\mathcal{A}_0 := -\mathcal{A}(0)$ be the generator of a bounded analytic semigroup $e^{-t\mathcal{A}_0}$ on a Banach space X with domain $D(\mathcal{A}_0)$ as well as $0 \in \rho(\mathcal{A}_0)$, and let $T > 0$, $\theta \in (0,1)$, $1 \leqslant p < \infty$ and $f : \mathbb{R} \to D_{\mathcal{A}_0}(\theta,p)$ T-periodic such that $f_{|(0,T)} \in L^p(0,T;D_{\mathcal{A}_0}(\theta,p))$. Recall that the space $D_{\mathcal{A}_0}(\theta,p)$ is defined by

$$D_{\mathcal{A}_0}(\theta,p) = \left\{ x \in X : [x]_{\theta,p} := \left(\int_0^\infty \|t^{1-\theta}\mathcal{A}_0 e^{-t\mathcal{A}_0}x\|_X^p \frac{\mathrm{d}t}{t} \right)^{1/p} < \infty \right\}.$$

Equipped with the norm $\|x\|_{\theta,p} := \|x\| + [x]_{\theta,p}$, the space $D_{\mathcal{A}_0}(\theta,p)$ is a Banach space and coincides with the real interpolation space $(X, D(\mathcal{A}_0))_{\theta,p}$. Note that $[\cdot]_{\theta,p}$ is equivalent to the real interpolation space norm because $0 \in \rho(\mathcal{A}_0)$, see [22, Corollary 6.5.5].

Consider the quasilinear periodic Cauchy problem

$$\begin{cases} u'(t) + \mathcal{A}(u(t))u(t) = F(t,u(t)) + f(t), & t \in \mathbb{R}, \\ u(t) = u(t+T), & t \in \mathbb{R}. \end{cases} \tag{QACP}$$

Solutions will be constructed in the adjusted version of the space in (2.2), namely, in

$$\mathbb{E}^{\text{per}}_{\mathcal{A}_0} := \{u \in W^{1,p}(0,T;D_{\mathcal{A}_0}(\theta,p)) : \mathcal{A}_0 u \in L^p(0,T;D_{\mathcal{A}_0}(\theta,p)) \text{ and } u(0) = u(T)\},$$

while the data space is

$$\mathbb{F}_{\mathcal{A}_0} = L^p(0,T;D_{\mathcal{A}_0}(\theta,p)).$$

By $\mathbb{B}_\rho := \mathbb{B}^{\mathbb{E}^{\text{per}}_{\mathcal{A}_0}}(0,\rho)$ we denote the centered ball with radius $\rho > 0$ in the space of maximal regularity $\mathbb{E}^{\text{per}}_{\mathcal{A}_0}$, and M represents the infimum of all constants $C > 0$ fulfilling (2.3). To simplify notation, we define the spaces E_1 and E_γ by

$$E_1 := \{u \in D(\mathcal{A}_0) : \mathcal{A}_0 u \in D_{\mathcal{A}_0}(\theta,p)\} \text{ and } E_\gamma := (D_{\mathcal{A}_0}(\theta,p), E_1)_{1-\frac{1}{p},p},$$

that is, we denote by E_1 the domain of the realization of the operator $\mathcal{A}_0$ on the space $D_{\mathcal{A}_0}(\theta, p)$ and by E_γ the trace space in the resulting setting.

For the sake of clarity we include the assumptions concerning $\mathcal{A}_0$ once again, and we also state our assumptions on the right-hand side F as well as on the quasilinear operator $\mathcal{A}(\cdot)$.

Assumption Q. Assume that $\mathcal{A} : E_\gamma \to \mathcal{L}(E_1, D_{\mathcal{A}_0}(\theta, p))$ is a family of closed linear operators such that

$$-\mathcal{A}_0 \text{ generates a bounded analytic semigroup on } X \text{ and } 0 \in \rho(\mathcal{A}_0), \tag{G0}$$

and assume that there exist $R > 0$ and $L > 0$ such that

$$F(\cdot, v(\cdot)) \in \mathbb{F}_{\mathcal{A}_0} \text{ for all } v \in \overline{\mathbb{B}}_R, \tag{F1}$$

$$\|F(\cdot, v(\cdot)) - F(\cdot, \overline{v}(\cdot))\|_{\mathbb{F}_{\mathcal{A}_0}} \leqslant \frac{1}{4M} \|v - \overline{v}\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}}, \tag{F2}$$

where M denotes the infimum of all constants $C > 0$ fulfilling (2.3), and

$$\|\mathcal{A}(v(\cdot))w(\cdot) - \mathcal{A}(\overline{v}(\cdot))w(\cdot)\|_{\mathbb{F}_{\mathcal{A}_0}} \leqslant L\|v - \overline{v}\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} \|w\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} \text{ for all } v, \overline{v}, w \in \overline{\mathbb{B}}_R. \tag{A1}$$

Our first main result reads then as follows.

Theorem 2.2. *Let $T > 0$, $0 < \theta < 1$ as well as $1 \leqslant p < \infty$. Assume Assumption Q is valid and that $f : \mathbb{R} \to D_{\mathcal{A}_0}(\theta, p)$ is T-periodic. Then there exist constants $r \leqslant R$ and $c = c(T, \theta, p, r) > 0$ such that if $\|F(\cdot, 0)\|_{\mathbb{F}_{\mathcal{A}_0}} \leqslant c$ and $\|f\|_{\mathbb{F}_{\mathcal{A}_0}} \leqslant c$, then there exists a unique solution $u : \mathbb{R} \to D_{\mathcal{A}_0}(\theta, p)$ to (QACP) with the same period T and which fulfills $u_{|(0,T)} \in \overline{\mathbb{B}}_r$.*

Proof. We denote by $Q : \overline{\mathbb{B}}_R \to \mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}, v \mapsto u$ the solution operator of the linear equation

$$u'(t) + \mathcal{A}(0)u(t) = \mathcal{A}(0)v(t) - \mathcal{A}(v(t))v(t) + F(t, v(t)) + f(t) \text{ in } (0, T), \quad u(0) = u(T).$$

The assumptions (F1) and (A1) ensure that the above equation is well-defined, and Proposition 2.1 in conjunction with (G0) implies that there is a unique solution $u \in \mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}$.

Let now $v \in \overline{\mathbb{B}}_r$ for some $r \in (0, R]$. Making use of (2.3), (A1), and (F2), we get

$$\begin{aligned}
\|Q(v)\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} &\leqslant M\Big(\|(\mathcal{A}(0) - \mathcal{A}(v(\cdot)))v(\cdot)\|_{\mathbb{F}_{\mathcal{A}_0}} + \|F(\cdot, v(\cdot))\|_{\mathbb{F}_{\mathcal{A}_0}} + \|f\|_{\mathbb{F}_{\mathcal{A}_0}}\Big) \\
&\leqslant ML\|v\|^2_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} + M\|F(\cdot, v(\cdot)) - F(\cdot, 0)\|_{\mathbb{F}_{\mathcal{A}_0}} + M\left(\|F(\cdot, 0)\|_{\mathbb{F}_{\mathcal{A}_0}} + \|f\|_{\mathbb{F}_{\mathcal{A}_0}}\right) \\
&\leqslant ML\|v\|^2_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} + \frac{1}{4}\|v\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} + M\left(\|F(\cdot, 0)\|_{\mathbb{F}_{\mathcal{A}_0}} + \|f\|_{\mathbb{F}_{\mathcal{A}_0}}\right) \\
&\leqslant MLr^2 + \frac{r}{4} + M\left(\|F(\cdot, 0)\|_{\mathbb{F}_{\mathcal{A}_0}} + \|f\|_{\mathbb{F}_{\mathcal{A}_0}}\right) \\
&\leqslant \frac{r}{4} + \frac{r}{4} + \frac{r}{2} \leqslant r
\end{aligned}$$

by setting $r := \min\{\frac{1}{4ML}, \frac{R}{2}\}$ and assuming additionally that $\|F(\cdot,0)\|_{\mathbb{F}_{\mathcal{A}_0}}, \|f\|_{\mathbb{F}_{\mathcal{A}_0}} \leqslant c$ for $c := \frac{r}{4M}$. Hence, we have proved that $Q(\overline{\mathbb{B}}_r) \subset \overline{\mathbb{B}}_r$.

For the above r let $v_1, v_2 \in \overline{\mathbb{B}}_r$. The choice of $r \leqslant \frac{R}{2}$ guarantees that

$$\|v_1 - v_2\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} \leqslant \|v_1\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} + \|v_2\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} \leqslant 2r \leqslant R,$$

so we may use (A1) in the case of $w = v_1 - v_2$. We then obtain, additionally employing (2.3) and (F2),

$$\begin{aligned}
\|Q(v_1) - Q(v_2)\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} \leqslant M\Big(& \|(\mathcal{A}(0) - \mathcal{A}(v_1(\cdot)))(v_1(\cdot) - v_2(\cdot))\|_{\mathbb{F}_{\mathcal{A}_0}} + \|(\mathcal{A}(v_2(\cdot)) - \mathcal{A}(v_1(\cdot)))v_2(\cdot)\|_{\mathbb{F}_{\mathcal{A}_0}} \\
& + \|F(\cdot, v_1(\cdot)) - F(\cdot, v_2(\cdot))\|_{\mathbb{F}_{\mathcal{A}_0}}\Big) \\
\leqslant\ & ML\|v_1\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}}\|v_1 - v_2\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} + ML\|v_1 - v_2\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}}\|v_2\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} + \frac{1}{4}\|v_1 - v_2\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} \\
\leqslant\ & 2MLr\|v_1 - v_2\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} + \frac{1}{4}\|v_1 - v_2\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}} \\
\leqslant\ & \frac{3}{4}\|v_1 - v_2\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}_0}}
\end{aligned}$$

by the choice of r. The solution operator Q is thus a contraction on $\overline{\mathbb{B}}_r$, and the contraction mapping theorem yields the existence of a unique fixed point in $\overline{\mathbb{B}}_r$. Denoting the latter one by u, we can extend u periodically to the whole real line again, as $Q(u) = u$ and $u(0) = u(T)$, finishing the proof. $\square$

For convenience, we briefly discuss the semilinear case that can be viewed as a particular instance of the quasilinear one. The setting basically remains the same apart from the step back to $\mathcal{A}$ with the same properties as in the linear case. We consider $f : \mathbb{R} \to D_{\mathcal{A}}(\theta, p)$ T-periodic with $f_{|(0,T)} \in L^p(0, T; D_{\mathcal{A}}(\theta, p))$. The semilinear abstract Cauchy problem is then given by

$$\begin{cases} u'(t) + \mathcal{A}u(t) = F(t, u(t)) + f(t), & t \in \mathbb{R}, \\ u(t) = u(t+T), & t \in \mathbb{R}. \end{cases} \tag{SACP}$$

As in the linear case, the solution u will be considered in the space $\mathbb{E}^{\mathrm{per}}_{\mathcal{A}}$ from (2.2), and the data space is again $\mathbb{F}_{\mathcal{A}} = L^p(0, T; D_{\mathcal{A}}(\theta, p))$.

In the sequel, we will again use the notation $\mathbb{B}_\rho := \mathbb{B}^{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}}}(0, \rho)$ to denote the ball in $\mathbb{E}^{\mathrm{per}}_{\mathcal{A}}$ with center 0 and radius $\rho > 0$. By $M > 0$ we still denote the infimum of all constants $C > 0$ that satisfy (2.3). Compared to Assumption Q, we need to adapt the spaces involved slightly, and the Lipschitz constant of the right-hand side can be increased by the factor 2. The adjusted assumptions then are as follows:

$$-\mathcal{A} \text{ generates a bounded analytic semigroup on } X \text{ and } 0 \in \rho(\mathcal{A}), \tag{G}$$

and there is $R > 0$ such that

$$F(\cdot, v(\cdot)) \in \mathbb{F}_{\mathcal{A}} \text{ for all } v \in \overline{\mathbb{B}}_R, \text{ and} \tag{S1}$$

$$\|F(\cdot, v_1(\cdot)) - F(\cdot, v_2(\cdot))\|_{\mathbb{F}_{\mathcal{A}}} \leqslant \frac{1}{2M}\|v_1 - v_2\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}}} \text{ for all } v_1, v_2 \in \overline{\mathbb{B}}_R. \tag{S2}$$

The existence result for the semilinear setting can be proved as above, where (G0), (F1), and (F2) are being replaced by (G), (S1), and (S2), respectively.

Corollary 2.3. *Let $T > 0$, $0 < \theta < 1$ and $1 \leqslant p < \infty$. Assume (G), (S1) as well as (S2), and let $f : \mathbb{R} \to D_A(\theta, p)$ be T-periodic. Then there exist constants $r \leqslant R$ and $c = c(T, \theta, p, R) > 0$ such that if $\|F(\cdot, 0)\|_{\mathbb{F}_A} \leqslant c$ and $\|f\|_{\mathbb{F}_A} \leqslant c$, there is a unique solution $u : \mathbb{R} \to D_A(\theta, p)$ of (SACP) with the same period T and $u_{|(0,T)} \in \overline{\mathbb{B}}_R$.*

We remark that in contrast to Theorem 2.2, it is possible to choose $r = R$ in the preceding corollary.

3 | THE MIXED DERIVATIVE THEOREM FOR EVOLUTION EQUATIONS IN REAL INTERPOLATION SPACES

This section presents the mixed derivative theorem for evolution equations in real interpolation spaces. Apart from its general interest, mixed derivative results for the Laplacian in real interpolation spaces will be needed in our approach to the Keller–Segel system in Section 4.

In the sequel, we denote by $\mathcal{S}(X)$ the class of sectorial operators in X, and given $A \in \mathcal{S}(X)$, we define the spectral angle ϕ_A of A by

$$\phi_A := \inf\{\phi : \rho(-A) \supset \Sigma_{\pi-\phi}, \sup_{\lambda \in \Sigma_{\pi-\phi}} \|\lambda(\lambda + A)^{-1}\| < \infty\}.$$

For details concerning the concepts of operators with bounded $\mathcal{H}^\infty$-calculus, the class of operators with bounded imaginary powers $\mathcal{BIP}$ as well as $\mathcal{R}$-sectorial operators we refer, for example, to [14, sections 2, 3, and 4] as well as to [38, chapters 3 and 4].

First, let $J = (0, T)$ with $0 < T < \infty$, or $J = \mathbb{R}_+$, Y be a Banach space with the UMD-property and $p \in [1, \infty]$. Then the negative derivative operator B_p in $L^p(J; Y)$ is defined by

$$(B_p u)(t) = -u'(t), \quad t \in J, \quad u \in D(B_p) = W^{1,p}(J; Y). \tag{3.1}$$

For $J = (0, T)$ or $J = \mathbb{R}_+$, a Banach space X and a differential operator A on X, we still denote the realization of A on the space $D_A(\theta, q)$, which is defined as in (1.1), and the canonical extension of A to $L^p(J; X)$ by A for convenience. Moreover, similarly as in Section 2, we introduce the notation $E_1(A)$ to denote the domain of the realization of the operator A on the space $D_A(\theta, q)$, that is,

$$E_1(A) := \{u \in D(A) : Au \in D_A(\theta, q)\}.$$

The main result of this section reads as follows.

Theorem 3.1. *Let X be a Banach space with the UMD-property, and let $A \in \mathcal{S}(X)$ with spectral angle $\phi_A < \frac{\pi}{2}$ be invertible. Let further $J = (0, T)$ or $J = \mathbb{R}_+$, $1 < p < \infty$ and $1 < q < \infty$. Moreover, let B_p in $L^p(J; D_A(\theta, q))$ be defined as in (3.1), and suppose that the canonical extension of A to $L^p(J; D_A(\theta, q))$, still denoted by A, and B_p commute. Then, for every $\alpha \in (0, 1)$ it holds that*

$$W^{1,p}(J; D_A(\theta, q)) \cap L^p(J; E_1(A)) \hookrightarrow H^{\alpha,p}(J; (D_A(\theta, q), E_1(A))_{1-\alpha}).$$

The first consequence of the preceding theorem concerns the Laplacian on $\mathbb{R}^d$, $d \in \mathbb{N}$. More precisely, for $1 < q < \infty$, let the negative Laplacian $-\Delta$ in $L^q(\mathbb{R}^d)$ be defined by $Au = -\Delta u$, with $D(A) = W^{2,q}(\mathbb{R}^d)$. It is well-known that $A \in \mathcal{S}(L^q(\mathbb{R}^d))$ with $\phi_A = 0$, see, for example, [14]. The same remains valid when considering a shift of the Laplacian, that is, we define A_I by

$$A_I u = (I - \Delta)u, \quad D(A_I) = D(A) = W^{2,q}(\mathbb{R}^d).$$

This implies $D_{A_I}(\theta, q) = B^{2\theta}_{qp}(\mathbb{R}^d)$ with equivalent norms in view of [38, Proposition 3.4.4]. The identification of the real interpolation space as a Besov space is classical and can be found, for example, in [42, Section 2.4.2]. Moreover, A_I is invertible. The fact that $E_1(A_I) = B^{2\theta+2}_{qp}(\mathbb{R}^d)$ and an application of Theorem 3.1 then yield the following corollary.

Corollary 3.2. *Let $T > 0$, $J = (0,T)$ or $J = \mathbb{R}_+$, $0 < \theta < 1$, $1 < p < \infty$ and $1 < q < \infty$. Then for every $\alpha \in (0,1)$ we have the embedding*

$$W^{1,p}(J; B^{2\theta}_{qp}(\mathbb{R}^d)) \cap L^p(J; B^{2\theta+2}_{qp}(\mathbb{R}^d)) \hookrightarrow H^{\alpha,p}(J; B^{2\theta+2-2\alpha}_{qp}(\mathbb{R}^d)).$$

The second application concerns the Dirichlet and Neumann Laplacian on bounded domains with smooth boundaries. In fact, let $G \subset \mathbb{R}^d$ be a bounded domain with boundary of class C^2. For $1 < q < \infty$ and $L^q(G)$ we consider the negative Dirichlet Laplacian $-\Delta_D$ and the negative Neumann Laplacian defined by

$$\Delta_D u = \Delta u, D(\Delta_D) = W^{2,q}(G) \cap W^{1,q}_0(G) \text{ and } \Delta_N u = \Delta u, D(\Delta_N) = \{u \in W^{2,q}(G) : \partial_\nu u = 0 \text{ on } \partial G\},$$

respectively. The results in [14] imply that $-\Delta_D, -\Delta_N \in \mathcal{S}(L^q(G))$ with $\phi_{-\Delta_D} = \phi_{-\Delta_N} = 0$, and it is well-known that $D(\Delta_D)$ and $D(\Delta_N)$ are UMD-spaces. As the Neumann Laplacian is not invertible on $L^q(G)$, we use again a shift and set

$$A_{N,I} u = (I - \Delta)u, \text{ with } D(A_{N,I}) = D(\Delta_N).$$

Then $A_{N,I} \in \mathcal{S}(L^q(G))$ as well as $\phi_{A_{N,I}} = 0$.

In the sequel, we use the prescripts 0 and N to denote Dirichlet and Neumann boundary conditions in the spaces involved, respectively. For $0 < \theta < 1$ and $1 < q < \infty$, [38, Proposition 3.4.4] in conjunction with [1, chapter 5] leads to $D_{-\Delta_D}(\theta, q) = B^{2\theta}_{qp}(G)$ if $2\theta < \frac{1}{q}$ and $D_{A_{N,I}}(\theta, q) = B^{2\theta}_{qp}(G)$ provided $2\theta < 1 + \frac{1}{q}$. In addition, we observe that

$$E_1(-\Delta_D) = \{u \in B^{2\theta+2}_{qp}(G) : u = 0 \text{ on } \partial G\} =: {}_0B^{2\theta+2}_{qp}(G), \text{ and}$$

$$E_1(A_{N,I}) = \{u \in B^{2\theta+2}_{qp}(G) : \partial_\nu u = 0 \text{ on } \partial G\} =: {}_N B^{2\theta+2}_{qp}(G).$$

Employing Theorem 3.1 again, we infer the subsequent result by virtue of [21, Theorem 2.3].

Corollary 3.3. *Let $T > 0$, $J = (0,T)$ or $J = \mathbb{R}_+$, $0 < \theta < 1$, $1 < p < \infty$ and $1 < q < \infty$.*

(a) If $2\theta < \frac{1}{q}$, then for each $\alpha \in (0,1)$, $\alpha \neq \theta + 1 - \frac{1}{2q}$,

$$W^{1,p}(J; B^{2\theta}_{qp}(G)) \cap L^p(J; {}_0B^{2\theta+2}_{qp}(G)) \hookrightarrow \begin{cases} H^{\alpha,p}(J; B^{2\theta+2-2\alpha}_{qp}(G)), & \text{if } 2\theta + 2 - 2\alpha < \frac{1}{q}, \\ H^{\alpha,p}(J; {}_0B^{2\theta+2-2\alpha}_{qp}(G)), & \text{if } 2\theta + 2 - 2\alpha > \frac{1}{q}. \end{cases}$$

(b) If $2\theta < 1 + \frac{1}{q}$*, then*

$$W^{1,p}(J; B_{qp}^{2\theta}(G)) \cap L^p(J; {}_N B_{qp}^{2\theta+2}(G)) \hookrightarrow \begin{cases} H^{\alpha,p}(J; B_{qp}^{2\theta+2-2\alpha}(G)), & \text{if } 2\theta + 2 - 2\alpha < 1 + \frac{1}{q}, \\ H^{\alpha,p}(J; {}_N B_{qp}^{2\theta+2-2\alpha}(G)), & \text{if } 2\theta + 2 - 2\alpha > 1 + \frac{1}{q} \end{cases}$$

for all $\alpha \in (0,1)$*,* $\alpha \neq \theta + \frac{1}{2} - \frac{1}{2q}$*.*

The remainder of this section is dedicated to proving Theorem 3.1. We first collect several useful properties in the next lemma.

Lemma 3.4. *(a) Let* $p \in (1,\infty)$*,* $J = (0,T)$ *or* $J = \mathbb{R}_+$*,* Y *be a Banach space with the UMD-property, and let* B_p *be defined as in (3.1). Then* $B_p \in \mathcal{H}^\infty(L^p(J;Y))$ *with* $\mathcal{H}^\infty$*-angle* $\phi_{B_p}^\infty = \frac{\pi}{2}$*.*

(b) If X *is an arbitrary Banach space,* $A \in \mathcal{S}(X)$ *is invertible,* $\theta \in (0,1)$ *and* $1 \leqslant q < \infty$*, then it holds that* $A \in \mathcal{H}^\infty(D_A(\theta,q))$ *with* $\mathcal{H}^\infty$*-angle equal to* ϕ_A*.*

(c) For $A \in \mathcal{S}(X)$ *define* X_α *by* $X_\alpha = X_{A^\alpha} = (D(A^\alpha), \|\cdot\|_\alpha)$*, with* $\|x\|_\alpha = \|x\| + \|A^\alpha x\|$ *for* $0 < \alpha < 1$*. If* $A \in \mathcal{BIP}(X)$*, then* $X_\alpha \cong (X, X_A)_\alpha$*,* $\alpha \in (0,1)$*, where* $(X, X_A)_\alpha$ *denotes the complex interpolation space between* X *and* $X_A \hookrightarrow X$ *of order* α*.*

(d) Let X *and* Y *be Banach spaces with the UMD-property as well as* $\theta \in (0,1)$ *and* $1 < q < \infty$*. Then* $(X,Y)_{\theta,q}$ *also has the UMD-property.*

Proof of Lemma 3.4. Concerning (a) we refer to [33, Theorem 2.7] for $J = \mathbb{R}_+$. The case $J = (0,T)$ readily follows by the subsequent observations. Denoting by $E_{(0,T)}$ and $R_{(0,T)}$ the extension and restriction operator, from $L^p(0,T;Y) \to L^p(\mathbb{R}_+;Y)$ and $L^p(\mathbb{R}_+;Y) \to L^p(0,T;Y)$, respectively, we see that for $\lambda \in \mathbb{C}$ with $|\arg\lambda| > \frac{\pi}{2}$, the resolvent $R(\lambda, B_{p,(0,T)})$ of $B_{p,(0,T)}$ can be represented by

$$R(\lambda, B_{p,(0,T)}) = R_{(0,T)} R(\lambda, B_{p,\mathbb{R}_+}) E_{(0,T)}.$$

In conjunction with

$$f(B_{p,(0,T)})v = R_{(0,T)} f(B_{p,\mathbb{R}_+}) E_{(0,T)} v, \text{ for } v \in L^p(0,T;Y) \text{ and } f \in \mathcal{H}_0^\infty(\Sigma_\theta),$$

this yields the claim for $J = (0,T)$.

Property (b) is a result due to Dore [15]. For (c), we refer, for example, to [14, Theorem 2.5], while (d) can be found in [2, Theorem III.4.5.2 (vii)]. □

Prior to proving Theorem 3.1, we recall the following relations, see, for instance, [14, section 4.4], and the references therein:

$$A \in \mathcal{H}^\infty(X) \subset \mathcal{BIP}(X) \subset \mathcal{RS}(X) \subset \mathcal{S}(X) \text{ and } \phi_A^\infty \geqslant \theta_A \geqslant \phi_A^{\mathcal{R}} \geqslant \phi_A. \tag{3.2}$$

Proof of Theorem 3.1. We already know that $D_A(\theta,q) = (X, D(A))_{\theta,q}$ with equivalent norms. Thanks to $0 \in \rho(A)$, the norm of the real interpolation space is equivalent to the homogeneous norm $[\cdot]_{\theta,q}$ by [22, Corollary 6.5.5]. We remark that $D(A)$ also has the UMD-property, as $A^{-1} : X \to D(A)$ is an isomorphism, transferring the UMD-property from X to $D(A)$. Thus, Lemma 3.4(d) implies that $D_A(\theta,q))$ has the UMD-property, so Lemma 3.4(a) yields that the negative derivative operator satisfies $B_p \in \mathcal{H}^\infty(L^p(J; D_A(\theta,q)))$ with $\phi_{B_p}^\infty = \frac{\pi}{2}$.

Moreover, Lemma 3.4(b) implies that $A \in \mathcal{H}^\infty(D_A(\theta,q))$ with $\mathcal{H}^\infty$-angle less than $\frac{\pi}{2}$, so employing the natural extension, we infer that $A \in \mathcal{H}^\infty(L^p(J;D_A(\theta,q)))$ with $\mathcal{H}^\infty$-angle less than $\frac{\pi}{2}$. We then deduce from (3.2) that

$$A \in \mathcal{BIP}(L^p(J;D_A(\theta,q))) \text{ and } A \in \mathcal{RS}(L^p(J;D_A(\theta,q))), \text{ with } \mathcal{R}\text{-angle less than } \tfrac{\pi}{2}. \tag{3.3}$$

Summing up, we have verified that $B_p \in \mathcal{H}^\infty(L^p(J;D_A(\theta,q)))$, $A \in \mathcal{RS}(L^p(J;D_A(\theta,q)))$ as well as $\phi_{B_p}^\infty + \phi_A^{\mathcal{R}} < \pi$, and B_p and A commute by assumption. Therefore, denoting by A the operator A on $L^p(J;D_A(\theta,q))$, we conclude by the mixed derivative theorem in the version of [38, Corollary 4.5.10] that

$$W^{1,p}(J;D_A(\theta,q)) \cap L^p(J;E_1(A)) = D(B_p) \cap D(A) \hookrightarrow D(B_p^\alpha A^{1-\alpha}).$$

We argue that the last embedding in the statement of the theorem is valid by an application of Lemma 3.4(c). The application is legit by recalling (3.3) and in view of $B_p \in \mathcal{BIP}(L^p(J;D_A(\theta,q)))$ by (3.2). □

4 | PERIODIC SOLUTIONS TO THE QUASILINEAR CHEMOTAXIS SYSTEM

In this section, we show how Theorem 2.2 can be applied to quasilinear chemotaxis systems, that is, we show the existence of strong time periodic solutions to (PQKS). In addition, we discuss a slight transform of the model to take into account the physical background.

Throughout this section, let $d \in \mathbb{N}$, $d \geqslant 2$, and denote by $\Omega \subset \mathbb{R}^d$ a bounded domain with smooth boundary. Let $f = (f_n, f_c)^{\mathrm{T}}$ be a given T-periodic function. Assume first that $(n,c)^{\mathrm{T}}$ is a T-periodic solution to (PQKS). Integrating the first equation in (PQKS) and using the divergence theorem as well as the Neumann boundary condition, we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n(t,x)\,\mathrm{d}x = \int_\Omega f_n(t,x)\,\mathrm{d}x = |\Omega|\overline{f_n}(t)$$

is valid for all $t > 0$, where $\overline{f_n}$ denotes the spatial average of f_n. For $M(t) := \frac{1}{|\Omega|}\int_\Omega n(t,x)\,\mathrm{d}x$ we thus get $M(t) = M(0) + \int_0^t \overline{f_n}(s)\,\mathrm{d}s$, so T-periodicity of n yields that $M(0) = M(T)$ and hence $\int_0^T \overline{f_n}(t)\,\mathrm{d}t = 0$. The spatial average $\overline{f_n}$ of f_n thus has to have mean 0 in our further analysis. In fact, with $L_0^q(\Omega)$ denoting the space of functions in $L^q(\Omega)$ with spatial average 0, that is, $L_0^q(\Omega) := \{g \in L^q(\Omega) : \int_\Omega g\,\mathrm{d}x = 0\}$, we will show in Lemma 6.1 that $D_{A_0}^1(\theta,p) = B_{qp}^{2\theta}(\Omega) \cap L_0^q(\Omega)$, so the assumption that $f = (f_n, f_c)^{\mathrm{T}} : \mathbb{R} \to D_{A_0}(\theta,p)$ is T-periodic in Theorem 4.2 especially yields that $\int_0^T \overline{f_n}(t)\,\mathrm{d}t = 0$ in view of $f_n(t,\cdot) \in L_0^q(\Omega)$.

For $p,q \in (1,\infty)$ and $k \in \mathbb{N}$ we consider $L_0^q(\Omega)$ as above and the Sobolev space with Neumann boundary condition $W_N^{k,q}(\Omega) := \{f \in W^{k,q}(\Omega) : \frac{\partial f}{\partial \nu} = 0 \text{ on } \partial\Omega\}$. The ground space is given by

$$X_0 := L_0^q(\Omega) \times W^{1,q}(\Omega). \tag{4.1}$$

We denote by Δ_N the Neumann Laplacian on $L_0^q(\Omega)$ with domain $D(\Delta_N) = W_N^{2,q}(\Omega) \cap L_0^q(\Omega)$, and $(\Delta_N - 1)^1$ represents the translated Neumann Laplacian on $W^{1,q}(\Omega)$ with domain given by $D((\Delta_N - 1)^1) = W_N^{3,q}(\Omega)$. As we consider the quasilinear setting, for n sufficiently smooth we also introduce the linearized operator $\nabla \cdot ((n+1)^m \nabla)$ on $L^q_0(\Omega)$, and it has the domain $D(\nabla \cdot ((n+1)^m \nabla)) = W^{2,q}_N(\Omega) \cap L^q_0(\Omega)$.

For $w = (n,c)^{\mathrm{T}}$, we now define the operator $A(w)$ and the right-hand side $F(w)$ by

$$A(w) := -\begin{pmatrix} \nabla \cdot ((n+1)^m \nabla) & 0 \\ 1 & (\Delta_N - 1)^1 \end{pmatrix}, \quad F(w) := \begin{pmatrix} -\nabla \cdot (n \nabla c) \\ 0 \end{pmatrix}. \tag{4.2}$$

For $w = (n,c)^{\mathrm{T}}$ and $z = (z_1, z_2)^{\mathrm{T}}$, we thus have

$$A(0) = -\begin{pmatrix} \Delta_N & 0 \\ 1 & (\Delta_N - 1)^1 \end{pmatrix} \text{ and } A(w)z = -\begin{pmatrix} \nabla \cdot ((n+1)^m \nabla z_1) \\ z_1 + (\Delta_N - 1)^1 z_2 \end{pmatrix}. \tag{4.3}$$

For a T-periodic function $f = (f_n, f_c)^{\mathrm{T}}$, we may thus rewrite (PQKS) as

$$\begin{cases} w'(t) + A(w(t))w(t) = F(w(t)) + f(t), & t \in \mathbb{R}, \\ w(t) = w(t+T), & t \in \mathbb{R}. \end{cases}$$

We next verify that $A_0 := A(0)$ defined as in (4.3) is within the scope of Section 2.

Lemma 4.1. *Let X_0 be as in (4.1). Then the operator $-A_0$ defined as in (4.3) generates a bounded analytic semigroup e^{-tA_0} and $0 \in \rho(A_0)$, that is, it satisfies aspect (G0) of Assumption Q.*

Proof. First, we observe that $0 \in \rho(\Delta_N)$, which is due to the underlying space $L^q_0(\Omega)$, as well as $0 \in \rho((\Delta_N - 1)^1)$ in view of $0 \in \rho(\Delta_N - 1)$ and Banach scale arguments, see, for example, [2, Theorem V.1.5.1]. The triangular structure of A_0 yields that $0 \in \rho(A_0)$.

From results in [14], it follows that the Neumann Laplacian Δ_N generates a bounded analytic semigroup of angle $\frac{\pi}{2}$ on $L^q(\Omega)$, that is, see, for example, [3, Theorem 3.7.11], $\Sigma_\pi \subset \rho(\Delta_N)$ and

$$\sup_{\lambda \in \Sigma_{\pi-\varepsilon}} \|\lambda R(\lambda, \Delta_N)\|_{\mathcal{L}(L^q(\Omega))} < \infty \tag{4.4}$$

for all $\varepsilon > 0$. We define the projection $P : L^q(\Omega) \to L^q_0(\Omega)$ onto the closed subspace $L^q_0(\Omega)$ by

$$Pu := u - \frac{1}{|\Omega|} \int_\Omega u \, \mathrm{d}x.$$

As for all $u \in D(\Delta_N)$, we have $Pu \in D(\Delta_N)$ and

$$P(\Delta_N u) = \Delta u - \frac{1}{|\Omega|} \int_\Omega \Delta u \, \mathrm{d}x = \Delta u - \frac{1}{|\Omega|} \int_{\partial\Omega} \partial_\nu u \, \mathrm{d}s = \Delta u = \Delta_N(Pu)$$

by virtue of the divergence theorem and the Neumann boundary condition, we deduce the identity $R(\lambda, \Delta_N)P = PR(\lambda, \Delta_N)$ for all $\lambda \in \rho(\Delta_N)$, see, for example, [3, Proposition B.7]. Given any $f \in L^q_0(\Omega) \subset L^q(\Omega)$, there exists a unique $u \in D(\Delta_N)$ such that $(\lambda - \Delta_N)u = f$, so

$$u = R(\lambda, \Delta_N)f = R(\lambda, \Delta_N)Pf = PR(\lambda, \Delta_N)f \in D(\Delta_N) \cap L^q_0(\Omega).$$

We thus get for $\lambda \in \Sigma_{\pi-\varepsilon}$, $\varepsilon > 0$, that

$$\|\lambda R(\lambda, \Delta_N)f\|_{L^q_0(\Omega)} = \|\lambda R(\lambda, \Delta_N)f\|_{L^q(\Omega)},$$

so by (4.4) and [3, Theorem 3.7.11], Δ_N generates a bounded analytic semigroup on $L^q_0(\Omega)$.

On the other hand, we deduce from results in [13] that the translated Neumann Laplacian is in the class of operators with bounded imaginary powers on $L^q(\Omega)$, that is, $-\Delta_N + 1 \in \mathcal{BIP}(L^q(\Omega))$. For $X_\alpha = (D((-\Delta_N + 1)^\alpha), \|\cdot\|_\alpha)$ as in Lemma 3.4(c), $\alpha \in \mathbb{R}$, additionally making use of $0 \in \rho(-\Delta_N + 1)$, we then infer by [2, Proposition V.1.5.5] that $(-\Delta_N + 1)^\alpha \in \mathcal{BIP}(X_\alpha)$. This is in particular valid for $\alpha = \frac{1}{2}$, and for convenience we denote the corresponding operator by $(-\Delta_N + 1)^1$. By Lemma 3.4(c), it follows that $D((-\Delta_N + 1)^1) \cong (L^q(\Omega), W^{2,q}_N(\Omega))_{1/2}$. Together with [1, Theorem 5.2 and (5.2)], this yields that

$$D((-\Delta_N + 1)^1) = (L^q(\Omega), W^{2,q}_N(\Omega))_{1/2} = H^{1,q}(\Omega) = W^{1,q}(\Omega).$$

Concatenating the previous steps, we argue that $(-\Delta_N + 1)^1 \in \mathcal{BIP}(W^{1,q}(\Omega))$, so it follows that $-(-\Delta_N + 1)^1 = (\Delta_N - 1)^1$ generates a bounded analytic semigroup on $W^{1,q}(\Omega)$.

Exploiting the triangular structure and the invertibility of the operators involved, we conclude that $-A_0$ generates a bounded analytic semigroup on X_0. □

We recall that

$$D(A_0) = D(\Delta_N) \times D((\Delta_N - 1)^1) = W^{2,q}_N(\Omega) \cap L^q_0(\Omega) \times W^{3,q}_N(\Omega).$$

Following Section 2, for X_0 as in (4.1) and A_0 as in (4.3) we define the space $D_{A_0}(\theta, p)$ by

$$D_{A_0}(\theta, p) := \left\{ x \in X_0 : [x]_{\theta,p} := \left(\int_0^\infty \|t^{1-\theta} A_0 e^{-tA_0} x\|^p_{X_0} \frac{\mathrm{d}t}{t} \right)^{1/p} < \infty \right\}.$$

Accordingly, we define the solution space by

$$\mathbb{E}^{\mathrm{per}}_{A_0} := \{w \in W^{1,p}(0,T; D_{A_0}(\theta, p)) : A_0 w \in L^p(0,T; D_{A_0}(\theta, p)) \text{ and } w(0) = w(T)\} \tag{4.5}$$

with norm

$$\|w\|_{\mathbb{E}^{\mathrm{per}}_{A_0}} := \|w\|_{W^{1,p}(0,T;D_{A_0}(\theta,p))} + \|A_0 w\|_{L^p(0,T;D_{A_0}(\theta,p))},$$

and the underlying data space is $\mathbb{F}_{A_0} := L^p(0,T; D_{A_0}(\theta, p))$. In the sequel for $r > 0$, we denote by $\mathbb{B}_r$ the open ball with center 0 and radius r in $\mathbb{E}^{\mathrm{per}}_{A_0}$.

We require some additional assumptions on p and q. For some fixed $r \in (1, \infty)$, suppose that

$$p > 2r \text{ and } q > \frac{r}{r-1}(d-1). \tag{4.6}$$

Our result on strong periodic solutions to the quasilinear chemotaxis system reads as follows.

Theorem 4.2. *Let $T > 0$, $0 < \theta < 1$, $1 < p < \infty$ and $1 < q < \infty$ subject to (4.6) and such that $\theta < 1/2 + 1/(2q)$ and $2\theta > d/q + 1/r$. Then there are $R > 0$ and $c = c(T, \theta, p, R) > 0$ such that if $f = (f_n, f_c)^{\mathrm{T}} : \mathbb{R} \to D_{A_0}(\theta, p)$ is T-periodic with $\|f\|_{\mathbb{F}_{A_0}} \leqslant c$, there exists a unique solution $w : \mathbb{R} \to D_{A_0}(\theta, p)$ of (PQKS) with the same period T and $w_{|(0,T)} \in \overline{\mathbb{B}}_R$.*

Remark 4.3. An analogous result to Theorem 4.2 can be shown for the classical *semilinear* Keller–Segel system. We remark that not only the additional assumption (4.6) can then be removed, but it is also possible to consider $p = 1$ in this case. In total, we require that $\theta \in (0,1)$, $p \in [1,\infty)$ and $q \in (1,\infty)$ satisfy $\theta < 1/2 + 1/(2q)$ and $2\theta > d/q$ or, if $p = 1$, $2\theta \geqslant d/q$.

As n and c denote a density and a concentration, respectively, it is natural to demand that $n, c \geqslant 0$. Let $f = (f_n, f_c)^{\mathrm{T}}$ be a given T-periodic function and consider a T-periodic solution (N, C) to (PQKS), where N and C are nonnegative, implying that $N(\cdot,0)$ and $C(\cdot,0)$ are in particular nonnegative at time $t = 0$. The comparison principle yields that f_n and f_c must be nonnegative to guarantee the nonnegativity of N and C. If we assume that f_n and f_c are nonnegative, we integrate the first equation in (PQKS) and make use of the Neumann boundary conditions of N and C together with the divergence theorem to deduce that for any $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega N(t,x)\,\mathrm{d}x = \int_\Omega f_n(t,x)\,\mathrm{d}x = |\Omega|\overline{f_n}(t),$$

where $\overline{f_n}$ represents the spatial average of f_n as above. Setting $V(t) := \frac{1}{|\Omega|}\int_\Omega N(t,x)\,\mathrm{d}x$, we observe that $V(t) = V(0) + \int_0^t \overline{f_n}(t)\,\mathrm{d}t$. From the T-periodicity of N it follows that $V(T) = V(0)$, resulting in $\int_0^T \overline{f_n}(t)\,\mathrm{d}t = 0$, which in turns implies that $f_n \equiv 0$ by nonnegativity of f_n. As a result, it holds that $V(t) = V(0) =: V$ is a constant for each $t \geqslant 0$.

Having clarified the situation for nonnegative density and concentration, we set $n(t,x) := N(t,x) - V$ as well as $c(t,x) := C(t,x) - V$, so it follows that $\int_\Omega n(t,x)\,\mathrm{d}x = 0$, and $(n,c)^{\mathrm{T}}$ is a solution to

$$\begin{cases} n_t = \nabla\cdot((n+V+1)^m\nabla n) - \nabla\cdot((n+V)\nabla c), & x\in\Omega,\ t\in\mathbb{R},\\ c_t = \Delta c - c + n + f_c(t), & x\in\Omega,\ t\in\mathbb{R},\\ \frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = 0, & x\in\partial\Omega,\ t\in\mathbb{R}. \end{cases} \tag{PQKS-V}$$

The following result can be shown analogously as Theorem 4.2.

Proposition 4.4. *Let $T > 0$, $0 < \theta < 1$, $1 < p < \infty$ and $1 < q < \infty$ subject to (4.6) and such that $\theta < 1/2 + 1/(2q)$ and $2\theta > d/q$. Then there are $R > 0$, $c = c(T,\theta,p,R) > 0$ and V_0 such that if $f = (0, f_c)^{\mathrm{T}} : \mathbb{R} \to D_{A_0}(\theta,p)$ is T-periodic with $\|f\|_{\mathbb{F}_{A_0}} \leqslant c$ and $V < V_0$, there exists a unique solution $w : \mathbb{R} \to D_{A_0}(\theta,p)$ of (PQKS-V) with the same period T and $w_{|(0,T)} \in \overline{\mathbb{B}}_R$.*

5 | PERIODIC SOLUTIONS TO A NERNST–PLANCK–POISSON TYPE SYSTEM

This section is devoted to the study of strong, time periodic solutions to (PNPP) by means of Corollary 2.3.

Consider $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, a bounded domain with smooth boundary, and let $g = (g_u, g_v, g_w)^{\mathrm{T}}$ denote a given T-periodic function. As in Section 4, we see that $\int_0^T \overline{g_u}(t)\,\mathrm{d}t = \int_0^T \overline{g_v}(t)\,\mathrm{d}t = 0$, $\overline{g_u}$ and $\overline{g_v}$ being the respective spatial averages, is a necessary condition for the existence of strong solutions, and that it is satisfied for T-periodic $g = (g_u, g_v, g_w)^{\mathrm{T}} : \mathbb{R} \to D_B(\theta,p)$. The ground space in this context is

$$X_0 := L^q_0(\Omega) \times L^q_0(\Omega) \times W^{2,q}_N(\Omega) \cap L^q_0(\Omega). \tag{5.1}$$

Again, Δ_N represents the Neumann Laplacian on $L^q_0(\Omega)$ with domain $D(\Delta_N) = W^{2,q}_N(\Omega) \cap L^q_0(\Omega)$. Moreover, we denote by Δ_N^2 the Neumann Laplacian on $D(\Delta_N)$, so its domain is given by

$$\{w \in W^{4,q}_N(\Omega) : \partial_\nu \Delta w = 0 \text{ on } \partial\Omega\} \cap L^q_0(\Omega).$$

For $z = (u, v, w)^{\mathrm{T}}$ we define the operator B and the right-hand side term $G(z)$ for the Nernst–Planck–Poisson type system by

$$B := -\begin{pmatrix} \Delta_N & 0 & 0 \\ 0 & \Delta_N & 0 \\ 1 & -1 & \Delta_N^2 \end{pmatrix}, \quad G(z) := \begin{pmatrix} \nabla \cdot (u\nabla w) \\ -\nabla \cdot (v\nabla w) \\ 0 \end{pmatrix}. \tag{5.2}$$

Thus, given a T-periodic function $g = (g_u, g_v, g_w)^{\mathrm{T}}$, we rewrite (PNPP) as

$$\begin{cases} z'(t) + Bz(t) = G(z(t)) + g(t), & t \in \mathbb{R}, \\ z(t) = z(t+T), & t \in \mathbb{R}. \end{cases}$$

We have already argued in Section 4 that Δ_N is invertible and generates a bounded analytic semigroup in the given setting. It then follows by Banach scale arguments as presented in [2, chapter 5] that $0 \in \rho(\Delta_N^2)$ is also valid and that Δ_N^2 generates a bounded analytic semigroup as well. Making use of the triangular structure of B as in (5.2), we conclude that $0 \in \rho(B)$ holds true and that $-B$ generates a bounded analytic semigroup e^{-tB} on X_0. In summary, we obtain the following.

Lemma 5.1. *Let X_0 be as in (5.1). Then the operator B defined as in (5.2) fulfills assumption (G) as in Section 2.*

The associated trace space is then defined by

$$D_B(\theta, p) := \left\{ x \in X_0 : [x]_{\theta,p} := \left(\int_0^\infty \|t^{1-\theta} B e^{-tB} x\|_{X_0}^p \, \frac{\mathrm{d}t}{t} \right)^{1/p} < \infty \right\}.$$

The data space is $\mathbb{F}_B := L^p(0, T; D_B(\theta, p))$, while the corresponding solution space is given by

$$\mathbb{E}_B^{\mathrm{per}} := \{z \in W^{1,p}(0, T; D_B(\theta, p)) : Bz \in L^p(0, T; D_B(\theta, p)) \text{ and } z(0) = z(T)\}.$$

It is endowed with the norm

$$\|z\|_{\mathbb{E}_B^{\mathrm{per}}} := \|z\|_{W^{1,p}(0,T;D_B(\theta,p))} + \|Bz\|_{L^p(0,T;D_B(\theta,p))}.$$

In the subsequent main result of this section, $\mathbb{B}_r$ denotes the open ball with center 0 and radius r in $\mathbb{E}_B^{\mathrm{per}}$.

Theorem 5.2. *Let $T > 0$, $0 < \theta < 1$, $1 \leqslant p < \infty$ and $1 < q < \infty$ such that $\theta < 1/2 + 1/(2q)$. In addition, let $2\theta > d/q$ or, if $p = 1$, let $2\theta \geqslant d/q$. Then there are $R > 0$ and $c = c(T, \theta, p, R) > 0$ such that if $g = (g_u, g_v, g_w)^{\mathrm{T}} : \mathbb{R} \to D_B(\theta, p)$ is T-periodic with $\|g\|_{\mathbb{F}_B} \leqslant c$, there exists a unique solution $z : \mathbb{R} \to D_B(\theta, p)$ of (PNPP) with the same period T and $z_{|(0,T)} \in \overline{\mathbb{B}}_R$.*

Note that it is possible to find $\theta \in (0,1)$ and $q \in (1,\infty)$ as required in Theorem 5.2 provided $q > \frac{d}{2}$ and $q > d-1$.

In contrast to Theorem 4.2, we can deal with the case $p = 1$ in Theorem 5.2. The reason is that the proof of the latter theorem does not rely on an application of the mixed derivative theorem in real interpolation spaces, Theorem 3.1, which is in turn needed for the estimates in the proof of Theorem 4.2. We refer to Lemma 6.3 for further details.

6 | PROOF OF THEOREMS 4.2 AND 5.2

In the context of the chemotaxis system and the Nernst–Planck–Poisson type system, we consider a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, with smooth boundary, and we suppose that in addition $d \geqslant 2$ for the chemotaxis system. For convenience we denote the domain by Ω in both cases. In the sequel, we also use the notation

$$D_{A_0}(\theta,p) = D^1_{A_0}(\theta,p) \times D^2_{A_0}(\theta,p) \text{ as well as } D_B(\theta,p) = D^1_B(\theta,p) \times D^2_B(\theta,p) \times D^3_B(\theta,p).$$

By virtue of sectoriality of the operators involved, the trace spaces coincide with the real interpolation spaces, see, for example, [38, Proposition 3.4.4]. For the following results, we refer, for example, to [1, 2], see also [39, section 5].

Lemma 6.1. *Let $\theta \in (0,1)$, $p \in (1,\infty)$ and $q \in (1,\infty)$. For $D_B(\theta,p)$ consider also $p = 1$.*

(a) If $\theta < \frac{1}{2} + \frac{1}{2q}$, then

$$D^1_{A_0}(\theta,p) = B^{2\theta}_{qp}(\Omega) \cap L^q_0(\Omega),\quad D^1_B(\theta,p) = D^2_B(\theta,p) = B^{2\theta}_{qp}(\Omega) \cap L^q_0(\Omega) \text{ and } D^3_B(\theta,p) = {}_N B^{2\theta+2}_{qp}(\Omega)$$

with equivalent norms, and the prescript N denotes a Neumann boundary condition, see Section 3.

(b) If $2\theta > \frac{d}{q}$, then

$$D^2_{A_0}(\theta,p) = \{c \in B^{2\theta+1}_{qp}(\Omega) : \partial_\nu c = 0 \text{ on } \partial\Omega\} \text{ with equivalent norms.}$$

The following lemma is concerned with estimates for nonlinear terms. For related results on less regular domains, we refer to [23, Lemma 5.2].

Lemma 6.2. *Let G be a bounded domain with smooth boundary, $\theta \in (0,1)$, $p \in [1,\infty)$ as well as $q \in (1,\infty)$. If $\theta > \frac{d}{2q}$ or, if $\theta \geqslant \frac{d}{2q}$ in the case $p = 1$, then $B^{2\theta}_{qp}(G)$ is a Banach algebra.*

Recalling the definitions of $\mathbb{E}^{\mathrm{per}}_{A_0}$ in (4.5), of A and $A(0)$ from (4.2) and (4.3), respectively, we deduce from Lemma 6.1 that

$$\mathbb{E}^{\mathrm{per},1}_{A_0} \hookrightarrow L^p(0,T;B^{2\theta+2}_{qp}(\Omega)) \text{ and } \mathbb{E}^{\mathrm{per},2}_{A_0} \hookrightarrow L^p(0,T;B^{2\theta+2}_{qp}(\Omega)). \tag{6.1}$$

By a similar argument, and additionally using the embedding $W^{1,p}(0,T;X) \hookrightarrow L^\infty(0,T;X)$, which is true for arbitrary Banach spaces X, we conclude that

$$\begin{aligned}\mathbb{E}^{\mathrm{per},1}_{A_0} &\hookrightarrow W^{1,p}(0,T;B^{2\theta}_{qp}(\Omega)) \hookrightarrow L^\infty(0,T;B^{2\theta}_{qp}(\Omega)) \text{ and} \\ \mathbb{E}^{\mathrm{per},2}_{A_0} &\hookrightarrow W^{1,p}(0,T;B^{2\theta+1}_{qp}(\Omega)) \hookrightarrow L^\infty(0,T;B^{2\theta+1}_{qp}(\Omega)).\end{aligned} \tag{6.2}$$

Analogously, using that $D_B^3(\theta,p)$ is in particular a subspace of $B_{qp}^{2\theta+2}(\Omega)$, we infer that

$$\mathbb{E}_B^{\mathrm{per},1}, \mathbb{E}_B^{\mathrm{per},2} \hookrightarrow L^p(0,T;B_{qp}^{2\theta+1}(\Omega)) \text{ and } \mathbb{E}_B^{\mathrm{per},3} \hookrightarrow L^p(0,T;B_{qp}^{2\theta+2}(\Omega)) \tag{6.3}$$

and

$$\begin{aligned} \mathbb{E}_B^{\mathrm{per},1}, \mathbb{E}_B^{\mathrm{per},2} &\hookrightarrow W^{1,p}(0,T;B_{qp}^{2\theta}(\Omega)) \hookrightarrow L^\infty(0,T;B_{qp}^{2\theta}(\Omega)) \text{ as well as} \\ \mathbb{E}_B^{\mathrm{per},3} &\hookrightarrow W^{1,p}(0,T;B_{qp}^{2\theta+1}(\Omega)) \hookrightarrow L^\infty(0,T;B_{qp}^{2\theta+1}(\Omega)). \end{aligned} \tag{6.4}$$

As $B_{qp}^{2\theta}(\Omega) \hookrightarrow L^\infty(\Omega)$ if and only if $2\theta > \frac{d}{q}$, we combine Sobolev's embedding with (6.2) for the case $2\theta > \frac{d}{q}$ to obtain

$$\mathbb{E}_{A_0}^{\mathrm{per},1} \hookrightarrow L^\infty(0,T;L^\infty(\Omega)) \text{ and } \mathbb{E}_{A_0}^{\mathrm{per},2} \hookrightarrow L^\infty(0,T;L^\infty(\Omega)). \tag{6.5}$$

The next embeddings are more delicate, and their derivation relies on the mixed derivative Theorem 3.1.

Lemma 6.3. *Let* $0 < \theta < 1$, $2 < p < \infty$, $1 < q < \infty$ *such that* $\theta < 1/2 + 1/(2q)$ *and* $2\theta > d/q$.
(a) Then

$$\mathbb{E}_{A_0}^{per,1} \hookrightarrow L^\infty(0,T;B_{qp}^{2\theta+1}(\Omega)). \tag{6.6}$$

(b) If for some fixed $r \in (1,\infty)$ *the condition (4.6) holds and if* $2\theta > d/q + 1/r$, *then*

$$\mathbb{E}_{A_0}^{per,1} \hookrightarrow L^\infty(0,T;W^{2,\infty}(\Omega)). \tag{6.7}$$

Proof. It is well-known that $L_0^q(\Omega)$ is a UMD Banach space. Furthermore, we have already seen in the proof of Lemma 4.1 that Δ_N generates a bounded analytic semigroup on $L_0^q(\Omega)$ and that $0 \in \rho(\Delta_N)$; hence $-\Delta_N \in \mathcal{S}(L_0^q(\Omega))$ with $\phi_{-\Delta_N} < \frac{\pi}{2}$, and it is invertible. In addition, Δ_N and the negative derivative operator B_p commute on $L^p(0,T;D_{A_0}^1(\theta,p))$.

Recall that $(B_{qp}^{2\theta}(\Omega), B_{qp}^{2\theta+2}(\Omega))_{1-\alpha} = B_{qp}^{2\theta+2-2\alpha}(\Omega)$ is valid for all $\alpha \in (0,1)$. Applying Theorem 3.1 and making use of Lemma 6.1, we infer that

$$\mathbb{E}_{A_0}^{\mathrm{per},1} \hookrightarrow H^{\alpha,p}(0,T;B_{qp}^{2\theta+2-2\alpha}(\Omega)), \tag{6.8}$$

and we aim for an embedding of the latter space into $L^\infty(0,T;B_{qp}^{2\theta+1}(\Omega))$. By Sobolev's embedding this is possible provided $\alpha > \frac{1}{p}$ and $\alpha \leqslant \frac{1}{2}$. Hence, we find such $\alpha \in (0,1)$ if $p > 2$, proving the claim (a).

Concerning (b), we first proceed as in the proof of (a) to deduce that (6.8) is valid. The space $H^{\alpha,p}(0,T;B_{qp}^{2\theta+2-2\alpha}(\Omega))$ then is supposed to embed into $L^\infty(0,T;W^{2,\infty}(\Omega))$. This is valid provided $\alpha > \frac{1}{p}$ and $2\theta - 2\alpha > \frac{d}{q}$. It hence remains to verify that we can find such $\alpha \in (0,1)$ under the assumptions of the lemma.

By $p > 2r$, $r \in (1,\infty)$, there is $\alpha \in (0,1)$ with $\frac{1}{p} < \alpha < \frac{1}{2r}$. Moreover, we derive from the condition on q that $\frac{d}{q} + \frac{1}{r} < 1 + \frac{1}{q}$, so it is possible to find $\theta \in (0,1)$ such that $\frac{d}{q} + \frac{1}{r} < 2\theta < 1 + \frac{1}{q}$. On the other hand, we conclude that for the above choice of $\alpha \in (0,1)$ it holds that

$$2\theta - 2\alpha > \frac{d}{q} + \frac{1}{r} - 2\alpha > \frac{d}{q},$$

so (b) is indeed true. $\square$

Proof of Theorem 4.2. We observe that we have already checked (G0) in Lemma 4.1, and we continue with the verification of (F1). For $R > 0$ let $w = (n, c)^{\mathrm{T}} \in \bar{\mathbb{B}}_R$. Observe first the identity $\int_\Omega \nabla \cdot (n \nabla c)\,\mathrm{d}x = 0$ by an application of the divergence theorem together with the Neumann boundary conditions, so

$$\nabla \cdot (n\nabla c) \in L^q_0(\Omega) \tag{6.9}$$

holds true for $w \in \overline{\mathbb{B}}_R$.

For the rest of this proof, C represents a generic constant. By virtue of Lemma 6.1, (6.9), the Leibniz rule, Lemma 6.2, (6.2), and (6.1), we obtain

$$\begin{aligned}
\|F(w(\cdot))\|_{\mathbb{F}_{A_0}} &\leqslant \|-\nabla\cdot(n\nabla c)\|_{L^p(0,T;B^{2\theta}_{qp}(\Omega))} \\
&\leqslant \|\nabla n\|_{L^p(0,T;B^{2\theta}_{qp}(\Omega))}\|\nabla c\|_{L^\infty(0,T;B^{2\theta}_{qp}(\Omega))} + \|n\|_{L^\infty(0,T;B^{2\theta}_{qp}(\Omega))}\|\Delta c\|_{L^p(0,T;B^{2\theta}_{qp}(\Omega))} \\
&\leqslant C\Big(\|n\|_{L^p(0,T;B^{2\theta+1}_{qp}(\Omega))}\|c\|_{W^{1,p}(0,T;B^{2\theta+1}_{qp}(\Omega))} + \|n\|_{W^{1,p}(0,T;B^{2\theta}_{qp}(\Omega))}\|c\|_{L^p(0,T;B^{2\theta+2}_{qp}(\Omega))}\Big) \\
&\leqslant 2C\|n\|_{\mathbb{E}^{\mathrm{per},1}_{A_0}}\|c\|_{\mathbb{E}^{\mathrm{per},2}_{A_0}} < \infty,
\end{aligned}$$

showing the validity of (F1) for every $R > 0$.

We now take $w, \overline{w} \in \overline{\mathbb{B}}_R$ into account. Similar arguments as for the verification of (F1) and the choice $R \leqslant \frac{1}{16MC}$, where M is defined as in Assumption Q, imply that

$$\begin{aligned}
\|F(w(\cdot)) - F(\overline{w}(\cdot))\|_{\mathbb{F}_{A_0}} &\leqslant \|\nabla n\cdot(\nabla c - \nabla\overline{c})\|_{L^p(0,T;B^{2\theta}_{qp}(\Omega))} + \|(\nabla n - \nabla\overline{n})\cdot\nabla\overline{c}\|_{L^p(0,T;B^{2\theta}_{qp}(\Omega))} \\
&\quad + \|n(\Delta c - \Delta\overline{c})\|_{L^p(0,T;B^{2\theta}_{qp}(\Omega))} + \|(n-\overline{n})\Delta\overline{c}\|_{L^p(0,T;B^{2\theta}_{qp}(\Omega))} \\
&\leqslant 2C\Big(2\|n\|_{\mathbb{E}^{\mathrm{per},1}_{A_0}}\|c-\overline{c}\|_{\mathbb{E}^{\mathrm{per},2}_{A_0}} + 2\|n-\overline{n}\|_{\mathbb{E}^{\mathrm{per},1}_{A_0}}\|\overline{c}\|_{\mathbb{E}^{\mathrm{per},2}_{A_0}}\Big) \\
&\leqslant 4CR\|w-\overline{w}\|_{\mathbb{E}^{\mathrm{per}}_{A_0}} \\
&\leqslant \frac{1}{4M}\|w-\overline{w}\|_{\mathbb{E}^{\mathrm{per}}_{A_0}},
\end{aligned}$$

so (F2) holds true as well.

We have seen in Lemma 6.1 that $n \in D^1_{A_0}(\theta,p)$ has mean value zero and that $D^2_{A_0}(\theta,p)$ includes a homogeneous Neumann boundary condition under the present assumptions. By E_1 we denote the domain of A on $D_{A_0}(\theta,p)$ and by E_γ we denote the trace space in the resulting setting. We have to verify that for $w \in E_\gamma$ and $z \in E_1$ the first component of $A(w)z$ has mean value zero and the second component satisfies the Neumann boundary condition. It follows from $z = (z_1, z_2)^{\mathrm{T}} \in E_1$ that $z_1 \in B^{2\theta+2}_{qp}(\Omega)$ with $\partial_\nu z_1 = 0$ on $\partial\Omega$ as well as $z_2 \in B^{2\theta+3}_{qp}(\Omega)$ with $\partial_\nu z_2 = \partial_\nu \Delta z_2 = 0$ on $\partial\Omega$.

Recalling (4.3), we then deduce by virtue of the divergence theorem that

$$\int_\Omega \nabla\cdot((n+1)^m\nabla z_1)\,\mathrm{d}x = \int_{\partial\Omega}(n+1)^m\nabla z_1\cdot\nu\,dS = 0,$$

while the second component of $A(w)z$ is given by $-(z_1+(\Delta_N-1)^1 z_2)$, and the boundary condition is thus fulfilled. Therefore, $A : E_\gamma \to \mathcal{L}(E_1, D_{A_0}(\theta,p))$ is a well-defined family of closed linear operators.

We now check that there is $R>0$ such that (A1) is satisfied. First, by (6.5) there is $\tilde{c}>0$ such that for $w=(n,c)^{\mathrm{T}}\in\bar{\mathbb{B}}_R$

$$\|(n,c)^{\mathrm{T}}\|_{(L^\infty(\Omega\times(0,T)))^2} \leqslant \tilde{c}\|(n,c)^{\mathrm{T}}\|_{\mathbb{E}^{\mathrm{per}}_{A_0}} \leqslant \tilde{c}R.$$

Setting $R_0 := \frac{1}{2\tilde{c}}$ and considering $R\in(0,R_0)$, we derive that

$$\frac{1}{2} \leqslant n+1 \leqslant \frac{3}{2}. \tag{6.10}$$

By interpolation, we conclude that $W^{2,q}(\Omega)\hookrightarrow B^{2\theta}_{qp}(\Omega)$, and we also observe the embedding $W^{2,\infty}(\Omega)\hookrightarrow W^{2,q}(\Omega)$ for $q\in(1,\infty)$. A calculation reveals that $m(n+1)^{m-1}\in L^\infty(0,T;W^{2,\infty}(\Omega))$ is valid provided $n+1>0$ and $n\in L^\infty(0,T;W^{2,\infty}(\Omega))$. For $R\in(0,R_0)$ and $(n,c)^{\mathrm{T}}\in\bar{\mathbb{B}}_R$ we then have by (6.10) and (6.7) that

$$\|m(n+1)^{m-1}\|_{L^\infty(0,T;B^{2\theta}_{qp}(\Omega))} < \infty. \tag{6.11}$$

Similar arguments as above show that the first component of the difference $A(w)z - A(\overline{w})z$ is contained in $L^q_0(\Omega)$ for $w,\overline{w},z\in\bar{\mathbb{B}}_R, z=(z_1,z_2)^{\mathrm{T}}$. Making use of the latter observation, Lemma 6.1, Lemma 6.2, (6.11), and the mean value theorem, we get for $w,\overline{w},z\in\overline{B}_R$

$$\begin{aligned}
&\|A(w(\cdot))z(\cdot)-A(\overline{w}(\cdot))z(\cdot)\|_{\mathbb{F}_{A_0}} \leqslant \|\nabla\cdot((n+1)^m\nabla z_1)-\nabla\cdot((\overline{n}+1)^m\nabla z_1)\|_{L^p(0,T;B^{2\theta}_{qp}(\Omega))}\\
&\quad\leqslant \Big(\|m(n+1)^{m-1}(\nabla n-\nabla\overline{n})\nabla z_1\|_{L^p(0,T;B^{2\theta}_{qp}(\Omega))}\\
&\qquad + \|(m(n+1)^{m-1}-m(\overline{n}+1)^{m-1})\nabla\overline{n}\cdot\nabla z_1\|_{L^p(0,T;B^{2\theta}_{qp}(\Omega))}\\
&\qquad + \|((n+1)^m-(\overline{n}+1)^m)\Delta z_1\|_{L^p(0,T;B^{2\theta}_{qp}(\Omega))}\Big)\\
&\quad\leqslant C\Big(\|\nabla n-\nabla\overline{n}\|_{L^p(0,T;B^{2\theta}_{qp}(\Omega))}\|\nabla z_1\|_{L^\infty(0,T;B^{2\theta}_{qp}(\Omega))}\\
&\qquad + \|n-\overline{n}\|_{L^\infty(0,T;B^{2\theta}_{qp}(\Omega))}\|\nabla\overline{n}\|_{L^p(0,T;B^{2\theta}_{qp}(\Omega))}\|\nabla z_1\|_{L^\infty(0,T;B^{2\theta}_{qp}(\Omega))}\\
&\qquad + \|n-\overline{n}\|_{L^\infty(0,T;B^{2\theta}_{qp}(\Omega))}\|\Delta z_1\|_{L^p(0,T;B^{2\theta}_{qp}(\Omega))}\Big)\\
&\quad\leqslant C\Big(\|n-\overline{n}\|_{L^p(0,T;B^{2\theta+1}_{qp}(\Omega))}\|z_1\|_{L^\infty(0,T;B^{2\theta+1}_{qp}(\Omega))}\\
&\qquad + \|n-\overline{n}\|_{L^\infty(0,T;B^{2\theta}_{qp}(\Omega))}\|\overline{n}\|_{L^p(0,T;B^{2\theta+1}_{qp}(\Omega))}\|z_1\|_{L^\infty(0,T;B^{2\theta+1}_{qp}(\Omega))}\\
&\qquad + \|n-\overline{n}\|_{L^\infty(0,T;B^{2\theta}_{qp}(\Omega))}\|z_1\|_{L^p(0,T;B^{2\theta+2}_{qp}(\Omega))}\Big).
\end{aligned}$$

An application of (6.1), (6.2), and (6.6) together with the shape of $\mathbb{E}_{A_0}^{\mathrm{per},1}$ then yields that

$$\begin{aligned}\|A(w(\cdot))z(\cdot)-A(\overline{w}(\cdot))z(\cdot)\|_{\mathbb{F}_{A_0}} &\leqslant C\Big(\|n-\overline{n}\|_{\mathbb{E}_{A_0}^{\mathrm{per},1}}\|z_1\|_{\mathbb{E}_{A_0}^{\mathrm{per},1}}+\|n-\overline{n}\|_{\mathbb{E}_{A_0}^{\mathrm{per},1}}\|\overline{n}\|_{\mathbb{E}_{A_0}^{\mathrm{per},1}}\|z_1\|_{\mathbb{E}_{A_0}^{\mathrm{per},1}}\\ &\qquad+\|n-\overline{n}\|_{\mathbb{E}_{A_0}^{\mathrm{per},1}}\|z_1\|_{\mathbb{E}_{A_0}^{\mathrm{per},1}}\Big)\\ &\leqslant C(R+2)\|w-\overline{w}\|_{\mathbb{E}_{A_0}^{\mathrm{per}}}\|z\|_{\mathbb{E}_{A_0}^{\mathrm{per}}}.\end{aligned}$$

In summary, we have shown that (A1) is satisfied for $L=L(R)=C(R+2)$. The assertion of Theorem 4.2 then follows by an application of Theorem 2.2 and upon noting that $F(0)=0$. □

Proof of Theorem 5.2. We verify Assumptions (S1) and (S2) to apply Corollary 2.3, as we have already checked (G) in Lemma 5.1. The same argument as in the proof of Theorem 4.2 yields that for $z=(u,v,w)^{\mathrm{T}}\in\bar{\mathbb{B}}_R$, $R>0$, it holds that $\nabla\cdot(u\nabla w),\nabla\cdot(v\nabla w)\in L_0^q(\Omega)$. Thus, arguing as in the verification of (F1), this time using (6.3) and (6.4), we get for $z\in\overline{\mathbb{B}}_R$

$$\|G(z(\cdot))\|_{\mathbb{F}_B}=\|\nabla\cdot(u\nabla w)\|_{\mathbb{F}_B^1}+\|-\nabla\cdot(v\nabla w)\|_{\mathbb{F}_B^2}\leqslant 2c\left(\|u\|_{\mathbb{E}_B^{\mathrm{per},1}}+\|v\|_{\mathbb{E}_B^{\mathrm{per},2}}\right)\|w\|_{\mathbb{E}_B^{\mathrm{per},3}}<\infty,$$

where $c>0$, as for the remainder of the proof, denotes a generic constant.

Considering $z_1,z_2\in\overline{\mathbb{B}}_R$, rewriting terms suitably as in the verification of (F2) in the proof of Theorem 4.2 and choosing $R\leqslant\frac{1}{16Mc}$, we obtain

$$\|G(z_1(\cdot))-G(z_2(\cdot))\|_{\mathbb{F}_B}\leqslant 8cR\|z_1-z_2\|_{\mathbb{E}_B^{\mathrm{per}}}\leqslant\frac{1}{2M}\|z_1-z_2\|_{\mathbb{E}_B^{\mathrm{per}}},$$

implying that (S2) is also satisfied.

We finally deduce the assertion of Theorem 5.2 by observing that $G(0)=0$ and by then applying Corollary 2.3. □

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