Non Uniqueness of Power-Law Flows

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Abstract: We apply the technique of convex integration to obtain non-uniqueness and existence results for power-law fluids, in dimension $d \geq 3$. For the power index q below the compactness threshold, i.e. $q \in (1, \frac{2d}{d+2})$, we show ill-posedness of Leray–Hopf solutions. For a wider class of indices $q \in (1, \frac{3d+2}{d+2})$ we show ill-posedness of distributional (non-Leray–Hopf) solutions, extending the seminal paper of Buckmaster & Vicol [10]. In this wider class we also construct non-unique solutions for every datum in L^2.

1. Introduction

This paper studies non-uniqueness and existence of solutions of the following model of non-Newtonian flows in d dimensions, $d \geq 3$

$$
\begin{aligned}
\partial_t v + \operatorname{div}(v \otimes v) - \operatorname{div} \mathcal{A}(Dv) + \nabla \tilde{\pi} &= 0, \\
\operatorname{div} v &= 0, \\
v_{t=0} &= v_0,
\end{aligned}
\tag{1}
$$

where the velocity field v and and the pressure $\tilde{\pi}$ are the unknowns, $Dv = \frac{1}{2}(\nabla v + \nabla^T v)$, and the non-Newtonian tensor $\mathcal{A}$ is given by the following power law

$$
\mathcal{A}(Q) = (\nu_0 + \nu_1 |Q|)^{q-2} Q,
\tag{2}
$$

for some $v_0, v_1 \geq 0$ and $q \in (1, \infty)$. A natural energy associated with the system (1) is

$$e(t) = \int |v(t)|^2 + 2 \int_0^t \int \mathcal{A}(Dv(s))Dv(s)ds. \tag{3}$$

Let us consider a distributional solution v to (1), (2) with spatial mean zero, on a d-dimensional flat torus. The formula (3) together with $\mathcal{A}(Q)Q \sim |Q|^q$ explains why $v \in L^\infty(L^2) \cap L^q(W^{1,q})$ is called an *energy solution*. If such solution satisfies additionally the energy inequality $e(t) \leq e(0)$ (t-a.e.), then it is called a *Leray–Hopf solution*.

For the problem (1) we show two non-uniqueness and one existence result. In short:

(A) In the regime $1 < q < 2d/(d+2)$: There are non-unique Leray–Hopf solutions.
(B) In the regime $1 < q < (3d+2)/(d+2)$: There are non-unique distributional solutions dissipating the kinetic part of the energy.
(C) In the regime $1 < q < (3d+2)/(d+2)$: For any initial datum $a \in L^2$ there are infinitely many distributional solutions of the Cauchy problem.

Our results are sharp concerning the power-law index q. The regime $1 < q < (3d+2)/(d+2)$ includes the case of the incompressible Navier–Stokes equation in $d \geq 3$. The precise formulations can be found in Sect. 1.3.

1.1. Background of power-law flows. Model (1) with a slightly different choice of $\mathcal{A}(Q)$, namely

$$\mathcal{A}(Q) = (v_0 + v_1|Q|^{q-2})Q, \tag{4}$$

with $q \geq 2$ was introduced to wide mathematical community by Ladyzhenskaya at her 1966 Moscow ICM speech; her formula (30) in [24] corresponds exactly to (1), (4). With $q = 2$, both models (1), (2) and (1), (4) reduce to the (incompressible) Navier–Stokes equations.

The Ladyzhenskaya's choice: (4) with $q \geq 2$ and our (2) with $q \geq 2$ are analytically equivalent. In particular, the non-Newtonian tensor $\mathcal{A}(Q)$ is in both cases nonsingular at $Q = 0$, and distributional solutions are well-defined for velocity fields in the class

$$v \in L^2_{loc}, \quad Dv \in L^q_{loc}. \tag{5}$$

The difference between (4) and (2) plays a role for $q < 2$. Firstly, $v_0 + v_1|Q|^{q-2}$ of (4) is singular at $|Q| = 0$, while our $(v_0 + v_1|Q|)^{q-2}$ for $v_0 > 0$ is not. More importantly, in (4) a *linear* dissipation is present. Thus, distributional solutions to (1)–(4) make sense provided $Dv \in L^2_{loc}$. So the choice (2) isolates the 'pure L^q-dissipation' behaviour, while (4) involves 'L^2-L^q dissipation'.

Ladyzhenskaya's rationale for analysing (1) was twofold: on the one hand, relaxation $q \geq 2$ helps to avoid the traps of the Navier–Stokes case $q = 2$. At the same time, the choice of power-laws for the tensor $\mathcal{A}$ is both consistent with first principles of continuum mechanics and widely used in applications. Let us elaborate on each of these points.

The model (1) with power-law for $\mathcal{A}$ of type (2) or (4) agrees with the constitutive relations for incompressible, viscous fluids. Recall that in deriving the Navier–Stokes equation one restricts the admissible relations between the Cauchy stress tensor $\mathcal{T}$ and D (dictated by the material frame indifference) by the Ansatz of linear dependence between $\mathcal{T}$ and D (i.e. by the Stokes law), cf. [22]. The power law model relaxes this Ansatz, but remains well within the frame indifference principle.

Of course studying an arbitrary model that is merely consistent with the first principles may be applicationally void. This is not the case of (1) however. The power-laws have been proposed independently in 1920's by Norton [33] in metallurgy and by de Waele [19] and Ostwald [34] in polymer chemistry. The related timeline can be found in section 1 of [36]. For details, the interested reader may consult also the monographs [3,28,37,38] and the recent survey [4] with its references. Just in order to fix the hydrodynamical intuition, let us observe that $q < 2$ in (1) models the case when the fluid is more viscous (roughly, 'solid-like') for small shears ('external forces') and less viscous ('liquid-like') for large shears e.g. ice pack, ketchup, emulsion paints, hair gel, whereas $q > 2$ means reverse behavior e.g. cornstarch-water solution, silicone-based solutions.

Let us note that, despite the mathematical interest in $q \geq 2$ in context of gaining regularity compared to Navier–Stokes equations, the 'shear-thinning' case $q \leq 2$ appears to be more meaningful for applications, where models of type (1) with $v_0 > 0$ appear as Bird-Carreau-Yasuda models (or called by a subset of those names). In particular, experimental fits for the threshold value 6/5 and above can be found on p. 174 of [3]. Furthermore, even parameter choices well-into our Leray–Hopf non-uniqueness regime are suggested, cf. p.18 of [39]. (In both [3] and [39] $n = q - 1$, $d = 3$. A discrepancy between appearing there a and our model is insignificant for our results.)

From the applicational perspective, our result may be seen as invalidating certain choices of parameters and data.

1.2. Essential analytical results for power-law fluids. Consider the system (1), (2). For $q > \frac{2d}{d+2}$ the space $W^{1,q}$ of system's energy embeds compactly into L^2_{loc} of the convective term $\mathrm{div}\,(v \otimes v)$. Hence one may expect an existence proof of Leray–Hopf solutions via compactness methods. Indeed, a relevant statement can be found in [20], which is itself the final step in a chain of attempts of many authors, including Frehse and Nečas with collaborators [21,29] to improve the lower bound on q. To be precise, the energy inequality $e(t) \leq e(0)$ is not stated explicitly in [20]; however it can be proven e.g. along the lines of proof of Theorem 3.3 of [4].

Observe that (1) with $v_0 = 0$ is invariant under the scaling

$$v_\lambda := \lambda^\alpha v(\lambda x, \lambda^{\alpha+1}t) \quad \text{with} \quad \alpha = \frac{q-1}{3-q}. \tag{6}$$

Consequently, the energy of v_λ vanishes on small scales iff $q < \frac{3d+2}{d+2}$. This suggests that the case $q \geq \frac{3d+2}{d+2}$ of (1) is a perturbation of the problem (1) without the convective term. Indeed, for $q \geq \frac{3d+2}{d+2}$ uniqueness in the energy class (at least for tame initial data) holds, cf. [28], section 5.4.1; see also [11].

What is known about existence and uniqueness of solutions to (1) can be thus sketched as follows

1.3. Our contribution. The short version of our results presented at the very beginning of the paper, recast graphically to facilitate comparison with Fig. 1, reads

Observe that Fig. 2 complements Fig. 1 sharply with respect to q.

Let us now present the detailed statements of our results. We always consider system (1) on the d-dimensional flat torus $\mathbb{T}^d$, with v having its spatial mean zero, and we define the notion of distributional solution in the usual way.
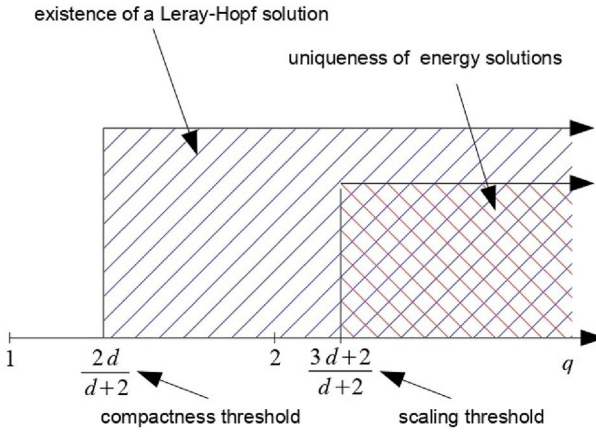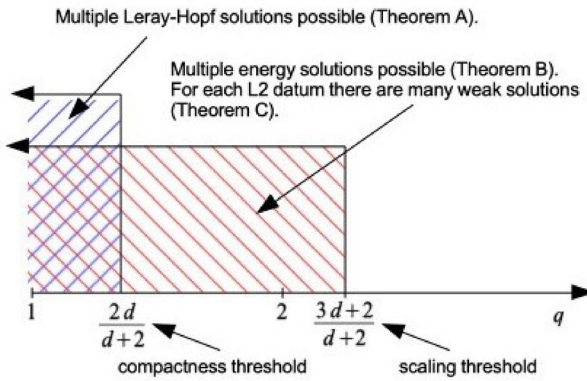
Fig. 1. Known results

Fig. 2. Our results

Definition 1. Let $q \in (1, \infty)$. A vector field $v \in L^2_{loc}(\mathbb{T}^d \times (0, 1))$ with $Dv \in L^{\max\{1, q-1\}}_{loc}(\mathbb{T}^d \times (0, 1))$ is a *distributional solution* to (1), (2) on the space-time domain $\mathbb{T}^d \times (0, 1)$ if

$$\int_0^1 \int_{\mathbb{T}^d} -v \cdot \partial_t \varphi - v \otimes v \nabla \varphi + \mathcal{A}(Dv) \nabla \varphi = 0, \qquad \forall_{t \in [0,1]} \int_{\mathbb{T}^d} v(t) \cdot \nabla \psi = 0$$

for any divergence-free $\varphi \in C^1(\mathbb{T}^d \times [0, 1])$ vanishing at $t = 0$ and $t = 1$, and any $\psi \in C^1(\mathbb{T}^d)$.

The condition $Dv \in L^{\max\{1, q-1\}}_{loc}$ guarantees that $\mathcal{A}(Dv) \in L^1_{loc}$.

1.3.1. Non-uniqueness in the Leray–Hopf class Our first theorem and its corollary show that below the compactness exponent, i.e. for $q < \frac{2d}{d+2}$, multiple Leray–Hopf solutions may emanate from the same L^2 initial data. In fact, we produce solutions $v \in C(L^2) \cap C(W^{1,q})$ with quite arbitrary pre-determined profile e of the (total) energy (3).

Theorem A. *Consider* (1)*,* (2) *on the space-time domain* $\mathbb{T}^d \times (0, 1)$*. Let* $q < \frac{2d}{d+2}$*. Fix an arbitrary* $e \in C^\infty([0, 1]; [1/2, 1])$*. There exists* $v \in C([0, 1]; L^2(\mathbb{T}^d)) \cap C([0, 1]; W^{1,q}(\mathbb{T}^d))$ *such that*

(1) v solves (1) *distributionally;*
(2) the total energy equals e, i.e.

$$\int_{\mathbb{T}^d} |v|^2(t) + 2 \int_0^t \int_{\mathbb{T}^d} \mathcal{A}(Dv)Dv = e(t). \tag{7}$$

Moreover, fix $0 \le T_1 < T \le 1$ *and two energy profiles* e_1, e_2 *as above, such that* $e_1(t) = e_2(t)$ *for* $t \in [0, T]$*. There exists* v_1, v_2 *satisfying (1), (2) and such that* $v_1(t) = v_2(t)$ *for* $t \in [0, T_1]$*. In particular, choosing* $T_1 = 0$*,* $T = 1/2$ *and* e_1, e_2 *to be as above, non-increasing and* $e_1 \not\equiv e_2$*, the corresponding* v_1, v_2 *are two distinct Leray–Hopf solutions with the same initial datum.*

Analysing the proof of Theorem A one realises that choosing an infinite family of non-increasing energy profiles $\{e_\alpha\}_{\alpha \in A}$ with a common C^1 bound, one can produce infinitely many distinct Leray–Hopf solutions with the same initial datum.

1.3.2. Non-uniqueness of distributional solutions If we drop the ambition to control the energy and require only to pre-determine the profile of the kinetic part of the energy $\int_{\mathbb{T}^d} |v|^2(t)$, then we produce non-unique solutions for exponents below the scaling-critical one, i.e. for $q < \frac{3d+2}{d+2}$. Moreover, they enjoy the regularity $v \in C(L^2) \cap C(W^{1,r})$ for any $r < \frac{2d}{d+2}$. This is our second result.

Theorem B. *Consider* (1)*,* (2) *on* $\mathbb{T}^d \times (0, 1)$*. Let* $q < \frac{3d+2}{d+2}$*. Fix any* $e \in C^\infty([0, 1]; [1/2, 1])$ *and* $r \in (\max\{1, q - 1\}, \frac{2d}{d+2})$*. There exists null-mean* $v \in C([0, 1]; L^2(\mathbb{T}^d)) \cap C([0, 1]; W^{1,r}(\mathbb{T}^d))$ *such that*

1) v solves (1) *distributionally;*
2) the kinetic energy equals e, i.e.

$$\int_{\mathbb{T}^d} |v|^2(t) = e(t). \tag{8}$$

Moreover, fix $0 \le T_1 < T \le 1$ and two energy profiles e_1, e_2 as above, such that $e_1(t) = e_2(t)$ for $t \in [0, T]$. There exists v_1, v_2 satisfying 1), 2) and such that $v_1(t) = v_2(t)$ for $t \in [0, T_1]$. In particular, choosing $T_1 = 0$, $T = 1/2$ and e_1, e_2 to be as above, non-increasing and $e_1 \not\equiv e_2$, the corresponding v_1, v_2 are two distinct distributional solutions, which belong to $C(L^2) \cap C(W^{1,r})$, dissipate the kinetic energy, and share the same initial datum.

1.3.3. Existence of multiple solutions for any L^2 data In Theorems A, B the initial data are attained strongly (in particular we can add initial values to the distributional formulas for solutions, extending test functions to non-vanishing ones at $t = 0$), but they are constructed in the convex integration scheme, thus possibly non-generic. This issue is addressed in our third theorem. It shows existence of energy solution emanating from *any* solenoidal vector field in L^2, for power laws below the scaling exponent.

Theorem C. *Consider* (1), (2) *on* $\mathbb{T}^d \times (0, 1)$. *Let* $q < \frac{3d+2}{d+2}$, $r \in (\max\{1, q-1\}, \frac{2d}{d+2})$. *Fix an arbitrary nonzero* $v_0 \in L^2(\mathbb{T}^d)$, *div* $v_0 = 0$. *There exist continuum of* $v \in C((0, 1]; L^2(\mathbb{T}^d)) \cap L^r((0, 1); W^{1,r}(\mathbb{T}^d))$ *such that*

(1) v solves (1) *distributionally;*
(2) $v|_{t=0} = v_0$, in the sense that as $t \to 0^+$, $v(t) \to v_0$ weakly in L^2 and strongly in L^{q_0}, any $q_0 < 2$.

Remark 1. Theorems A, B, C hold for Ladyzhenskaya's choice (4) instead of our choice (2), in the case $q \geq 2$. On the contrary, for $q < 2$, they do not hold for (4), since (4) contains also *linear* dissipation (see Sect. 1.1). However, Ladyzhenskaya's interest was limited to $q \geq 2$ (as a vehicle to mitigate Navier–Stokes difficulties), and in this range the theorems hold both for (2) and for (4), whereas, for $q < 2$, our choice (2) is more common in applications than (4).

1.4. Differences between our non-uniqueness and existence results. The non-uniqueness Theorems A, B, focus on possibly strongest notions of solutions: they allow, respectively, for full- or kinetic energy inequality and strong attainment of a (constructed) initial datum, but they do not produce non-unique solutions for any initial datum. Conversely, Theorem C provides existence of many weak solutions for an arbitrary solenoidal initial datum in L^2. In particular, this is the first existence proof for the case of $q \leq \frac{2d}{d+2}$. The obtained solutions are, however, much weaker than that of Theorems A, B : they do not allow for any kind of energy inequality (in fact, even their kinetic energies are in a sense pathologically large) and the initial datum is attained merely in a weak sense.

For the Euler equations a result similar to our Theorem C (existence of distributional solutions for *any* initial datum, with unnaturally high energy) was first proven in [40]. The difference with our Theorem C is that, in the case of the Euler equations (no dissipation), the problem of controlling higher order derivatives is completely absent.

1.5. The 3d Navier–Stokes case. Theorems B, C cover also the case of non-unique weak solutions of three-dimensional Navier–Stokes equations, first proven in [10]. Our Theorem B shows that $\nabla v \in L^{6/5-}$. This probably holds for solutions constructed in [10] as well, though the best regularity claimed there is curl $v \in L^1$. Theorem C produces infinitely many weak solutions for any divergence-free datum in L^2 (but with unnaturally high energies).

1.6. Methodology and plan. Our approach follows the convex integration methods introduced to inviscid fluid dynamics in [18,25], culminating in [7,23], and extended to the Navier–Stokes case in the important paper [10]. Results on a system involving fractional laplacian can be found in [16,26,35]. Other related interesting results include [2,8,9,12–15].

We stay close to the concentration-oscillation method developed for the transport equation in [31,32], and localised to avoid dimension loss in [30], see also [5].

The basic picture of the construction, as in any convex integration scheme applied to the equations of fluid dynamics, is the following. Given an exact flow (v, π), i.e. a solution to (1), one tries to distinguish the good ('laminar', 'averaged') component of v, i.e. $\langle v \rangle$ and the remainder, thought to be responsible for turbulence (interestingly, the

case $q = 3$ in (2), where scaling (6) fails, is the Smagorinsky model for turbulence). A typical averaging process $\langle \cdot \rangle$ does not commute with nonlinear quantities, thus applying $\langle \cdot \rangle$ to (1) yields for $u = \langle v \rangle$

$$\partial_t u + \mathrm{div}\, (u \otimes u) - \mathrm{div}\, \mathcal{A}(Du) + \nabla \langle \pi \rangle$$
$$= \mathrm{div}\, \big(u \otimes u - \langle v \otimes v \rangle \big) - \mathrm{div}\, \big(\mathcal{A}(Du) - \langle \mathcal{A}(Dv) \rangle \big) =: \mathrm{div}\, R.$$

Above, u is a well-behaved flow and the Reynolds stress R encodes the difference between $u = \langle v \rangle$ and the exact v itself. The rough idea behind producing non-unique solutions to (1) is to reverse-engineer the above picture. We can thus consider the following relaxation of (1)

$$\partial_t u + \mathrm{div}\, (u \otimes u) - \mathrm{div}\, \mathcal{A}(Du) + \nabla \pi = -\mathrm{div}\, R,$$
$$\mathrm{div}\, u = 0. \tag{9}$$

Assume we have identity (9) with certain (u_0, π_0, R_0). It is easy to find at least one smooth solution of (10), since R_0 is at our disposal. If one can produce another u_1, q_1 such that (u_1, q_1, R_1) solves (9) and R_1 is strictly smaller than R_0, there is a hope to iteratively diminish the Reynolds part R_n to 0 with $n \to \infty$. Consequently, in the limit one produces an exact solution v, π. Non uniqueness in the above procedure may be specified in at least two ways:

- either by enforcing v to be equal at some times, say for $t \in [0, 1/3]$, to a given regular solution v_1 and for $t \in [2/3, 1]$ to another regular solution v_2, as for instance in [6] or [31].
- or by specifying a kinetic energy profile, see e.g. [10,18], or the present work.

1.7. Organisation of proofs. In Sect. 2, we state the main proposition of the paper, i.e. Proposition 1, which contains the inductive step described above, from (u_0, π_0, R_0) to (u_1, π_1, R_1), with R_1 "much smaller" than R_0. Section 3 gathers preliminary material. In Sect. 4 we introduce a generalisation of Mikado flows that serves as a building block for u_1 given u_0. Next, in Sect. 5, assuming a solution (u_0, π_0, R_0) to (9) is given, we define (u_1, π_1, R_1). Estimates for $(u_1 - u_0)$ and R_1 occupy Sect. 6. Section 7 concludes the proof of the main Proposition 1. Having it in hand, we prove Theorem A in Sect. 8. The proofs of Theorems B–C follow similar lines and therefore are only sketched in Sects. 9–10.

1.8. Notation. We use mostly standard notation, e.g. $\mathbb{T}^d$ denotes the d-dimensional torus $[0, 1]^d$, $\dot{W}^{1,q}$ is a homogenous Sobolev space, $C_0^\infty(\mathbb{T}^d; B)$ are smooth functions with mean zero, domain $\mathbb{T}^d$ and values in set B (the target set will be sometimes omitted). We take $\mathbb{N} = \{1, 2, \dots\}$.

We suppress the variables and the spatial domain of integration, if no confusion arises. We use $|\cdot|$ instead of $\|\cdot\|$ for norms. For L^p-norms on the torus $\mathbb{T}^d$, we will abbreviate $|\cdot|_{L^p(\mathbb{T}^d)}$ to $|\cdot|_{L^p}$ or even to $|\cdot|_p$. In other cases, e.g. when taking the L^p-norm on $\mathbb{R}^d$, we will explicitly write the underlying domain, where the norm is calculated, e.g. $|\cdot|_{L^p(\mathbb{R}^d)}$. The finite-dimensional norm is $|\cdot|$. The projection onto null-mean functions is $P_{\neq 0} f := f - \fint_{\mathbb{T}^d} f$.

We will call $d \times d$ (symmetric) matrices *(symmetric) tensor*. For a tensor T, we denote its traceless part by $\mathring{T} := T - \frac{1}{d}\mathrm{tr}\,(T)\mathrm{Id}$. The space of symmetric tensors will

be denoted by $\mathcal{S}$, its open subset of positive definite tensors by $\mathcal{S}_+$. If R is a symmetric tensor, div R is the usual row-wise divergence.

We use two types of constants M's, which are uniform over iterations, and C's which are not (both possibly with subscripts), for details see Sect. 6.1. All constants may vary between lines.

Further notation is introduced locally when needed.

2. Main Proposition: An Iteration Step

Recall that $\mathcal{S}$ is the space of symmetric tensors.

Definition 2. A solution to *the Non-Newtonian-Reynolds system* is a triple (u, π, R) where

$$u \in C^\infty([0, 1] \times \mathbb{T}^d; \mathbb{R}^d), \quad \pi \in C([0, 1] \times \mathbb{T}^d; \mathbb{R}) \quad R \in C([0, 1] \times \mathbb{T}^d; \mathcal{S})$$

with spatial null-mean u, π, satisfying

$$\partial_t u + \text{div}\,(u \otimes u) - \text{div}\,\mathcal{A}(Du) + \nabla \pi = -\text{div}\,\mathring{R}, \atop \text{div}\, u = 0. \tag{10}$$

in the sense of distributions.

Remark 2. Despite smoothness of u, we can not require that (10) is satisfied in the classical sense or π, R are smooth (in space), because of non-smoothness of $\mathcal{A}(Du)$.

Remark 3. (R vs $\mathring{R}$) Use of the trace-free Reynolds stress simplifies computation, in particular proof of the energy iterate Proposition 14. The difference between R to $\mathring{R}$ is facilitated by the ambiguity of pressure: (u, π, R) solves (10) $\iff$ $(u, \pi - \frac{1}{d} tr R, \mathring{R})$ solves (10).

As observed in the introduction, the crucial point in the convex integration scheme is, given (u_0, q_0, R_0), to produce an appropriate correction (u_1, q_1, R_1) which decreases R_i, improves the energy gap, and retains as much regularity as possible. This single iteration step is given by

Proposition 1. *Let* $\nu_0, \nu_1 \geq 0$ *and* $q < \frac{2d}{d+2}$ *be fixed. Fix an arbitrary* $e \in C^\infty([0, 1]; [\frac{1}{2}, 1])$. *There exist a constant M such that the following holds.*

Let (u_0, π_0, R_0) *be a solution to the Non-Newtonian-Reynolds system* (10), *as in Definition* 2. *Let us choose any* $\delta, \eta, \epsilon \in (0, 1]$. *Assume that*

$$\frac{3}{4}\delta e(t) \leq e(t) - \left(\int_{\mathbb{T}^d} |u_0|^2(t) + 2\int_0^t \int_{\mathbb{T}^d} \mathcal{A}(Du_0)\,Du_0\right) \leq \frac{5}{4}\delta e(t) \tag{11}$$

and

$$|\mathring{R}_0(t)|_{L^1} \leq \frac{\delta}{2^7 d}. \tag{12}$$

Then, there is another solution (u_1, π_1, R_1) *to* (10) *(as in Definition* 2*) such that*

$$|(u_1 - u_0)(t)|_{L^2} \leq M\delta^{\frac{1}{2}} \tag{13a}$$

$$|(u_1 - u_0)(t)|_{W^{1,q}} \leq \eta \tag{13b}$$

$$|R_1(t)|_{L^1} \leq \eta. \tag{13c}$$

Furthermore

$$\frac{3}{8}\delta e(t) \leq e(t) - \left(\int_{\mathbb{T}^d} |u_1|^2(t) + 2 \int_0^t \int_{\mathbb{T}^d} \mathcal{A}(Du_1)Du_1 \right) \leq \frac{5}{8}\delta e(t). \qquad (14)$$

3. Preliminaries

3.1. Control of $\mathcal{A}$. We collect the needed growth estimates for $\mathcal{A}(Q)$ and for $\mathcal{A}(Q)Q$.

Lemma 1 (Growth estimates for $\mathcal{A}$). *Let $\mathcal{A} := (\nu_0 + \nu_1|Q|)^{q-2}Q$, with $\nu_0, \nu_1 \geq 0$. Then*

$$|\mathcal{A}(Q) - \mathcal{A}(P)| \leq \begin{cases} C_{\nu_1}|Q - P|^{q-1} & \text{for } \nu_0 = 0, q \leq 2 \\ C_{\nu_0}|Q - P| & \text{for } \nu_0 > 0, q \leq 2 \\ C_{q,\nu_0,\nu_1}|Q - P| \left(1 + |Q|^{q-2} + |P|^{q-2} \right) & \text{for } q \geq 2 \end{cases}$$

$$(15)$$

$$|\mathcal{A}(Q)Q - \mathcal{A}(P)P| \leq C_{q,\nu_0,\nu_1} \left(1 + |Q|^{q-1} + |P|^{q-1} \right)|Q - P|. \qquad (16)$$

The proof is standard. For convenience of the reader, we added it in Appendix.

Remark 4. Lemma 1 extends to other tensors $\mathcal{A}$, e.g. $(\nu_0 + \nu_1|Q|^2)^{\frac{q-2}{2}} Q$, or to ones given by an appropriate N-function. Consequently, our result extends to such tensors.

3.2. Nash-type decomposition. Let us denote the set of positive-definite $d \times d$ symmetric tensors by $\mathcal{S}_+$. We recall Lemma 2.4 in [17]

Lemma 2. *For any compact set $\mathcal{N} \subset \mathcal{S}_+$ there exists a finite set $K \subset \mathbb{Z}^d$ and smooth functions $\Gamma_k : \mathcal{N} \to [0, 1]$, such that any $R \in \mathcal{N}$ has the following representation:*

$$R = \sum_{k \in K} \Gamma_k^2(R)k \otimes k.$$

3.3. The role of oscillations. The convex integration paradigm is to use fast oscillations of corrector functions (correcting u_i to u_{i+1} in our case, roughly speaking) to inductively diminish error terms (in our case Reynolds stresses R_i). Thus for a function f and $\lambda \in \mathbb{N}$ let us define

$$f_\lambda(x) := f(\lambda x).$$

Observe that f_λ has the same L^p norms as f since we work on $\mathbb{T}^d$, and a factor λ appears for each derivative, i.e.

$$|\nabla^s f_\lambda|_p = \lambda^s |\nabla^s f|_p, \quad s \in \mathbb{N} \cup \{0\}.$$

It holds

Proposition 2 (Mean value). *Let $a \in C^\infty(\mathbb{T}^d; \mathbb{R})$, $v \in C_0^\infty(\mathbb{T}^d; \mathbb{R})$. Then for any $r \in [1, \infty]$*

$$\left| \int_{\mathbb{T}^d} a v_\lambda \right| \leq \lambda^{-1} C_r |\nabla a|_r |v|_{r'} \qquad (17)$$

Proof. The case $r = \infty$ follows the proof of Lemma 2.6 in [31]. For the case $r < \infty$, since v is null-mean, let us solve the Laplace equation div $\nabla h = v$ and define $G := \nabla h$. It holds $(\text{div } G)_\lambda = \lambda^{-1}\text{div}\,(G_\lambda)$ and thus, integrating by parts and using Hölder

$$\left| \int_{\mathbb{T}^d} a v_\lambda \right| = \lambda^{-1} \left| \int_{\mathbb{T}^d} a\text{div } G_\lambda \right| \leq \lambda^{-1} |\nabla a|_r |G_\lambda|_{r'} = \lambda^{-1} |\nabla a|_r |G|_{r'}$$

The Sobolev embedding for the null-mean G yields $|G|_{r'} \leq C|\nabla G|_{L^{\min(r',d+1)}} = C|\nabla^2 h|_{L^{\min(r',d+1)}}$. This is controlled thanks to Calderón-Zygmund theory by $|v|_{L^{\min(r',d+1)}}$. $\qquad\square$

Even when the l.h.s. of (17) is replaced with $\int_{\mathbb{T}^d} |a v_\lambda|$, the decorrelation between frequencies of a and v_λ allows to improve the generic Hölder inequality to (for the proof cf. Lemma 2.1 of [31]):

Proposition 3 (Improved Hölder). *Let f, g be smooth maps on $\mathbb{T}^d$. Let $r \in [1, \infty]$. Then*

$$|fg_\lambda|_r \leq |f|_r |g|_r + C_r \lambda^{-\frac{1}{r}} |f|_{C^1} |g|_r. \tag{18}$$

3.4. Antidivergence operators. We provide now various inverse divergence operators, needed for construction of R_1 in Proposition 1, with appropriate estimates. The purpose of the bilinear inverse divergences below is to extract oscillations of one function, say g_λ, out of the product fg_λ. The last of them, $\mathcal{R}_N^2$, is an operator with symmetric tensor values, such that div div $\mathcal{R}_N^2 f = f$ for every null-mean real function f; it facilitates construction of the R_{lin} term of R_1, cf. (60).

Proposition 4. *Let $p, r, s \in [1, \infty]$ and $\frac{1}{p} = \frac{1}{s} + \frac{1}{r}$.*

(i) (div^{-1}: *symmetric antidivergence*) *There exists* $\text{div}^{-1} : C_0^\infty(\mathbb{T}^d; \mathbb{R}^d) \to C_0^\infty(\mathbb{T}^d; \mathcal{S})$ *such that* div $\text{div}^{-1}u = u$ *and for* $i \geq 0$ *one has*

$$|\nabla^i \text{div}^{-1}u|_p \leq C_{k,p}|\nabla^i u|_p, \tag{19}$$

and for the fast oscillating u_λ

$$|\nabla^i \text{div}^{-1}u_\lambda|_p \leq C_{k,p}\lambda^{i-1}|\nabla^i u|_p. \tag{20}$$

(ii) ($\mathcal{R}_N$: *improved symmetric bilinear antidivergence*) *For any $N \geq 1$ there exists a bilinear operator* $\mathcal{R}_N : C^\infty(\mathbb{T}^d; \mathbb{R}) \times C_0^\infty(\mathbb{T}^d; \mathbb{R}^d) \to C_0^\infty(\mathbb{T}^d; \mathcal{S})$ *such that* div $\mathcal{R}_N(f, u) = fu - \fint fu$ *and*

$$|\mathcal{R}_N(f, u_\lambda)|_p \leq C_{d,p,s,r,N}|u|_s \left(\frac{1}{\lambda}|f|_r + \frac{1}{\lambda^N}|\nabla^N f|_r \right). \tag{21}$$

(iii) ($\tilde{\mathcal{R}}_N$: *improved symmetric bilinear antidivergence on tensors*) *For any $N \geq 1$ there exists a bilinear operator* $\tilde{\mathcal{R}}_N : C^\infty(\mathbb{T}^d; \mathbb{R}^d) \times C_0^\infty(\mathbb{T}^d; \mathbb{R}^{d \times d}) \to C_0^\infty(\mathbb{T}^d; \mathcal{S})$ *such that* div $\tilde{\mathcal{R}}_N(v, T) = Tv - \fint Tv$ *and*

$$|\tilde{\mathcal{R}}_N(v, T_\lambda)|_p \leq C_{d,p,s,r,N}|T|_s \left(\frac{1}{\lambda}|v|_r + \frac{1}{\lambda^N}|\nabla^N v|_r \right). \tag{22}$$

(iv) ($\mathcal{R}_N^2$: *improved symmetric bilinear double antidivergence*) *For any* $N \geq 1$ *there exists a bilinear operator* $\mathcal{R}_N^2 : C^\infty(\mathbb{T}^d; \mathbb{R}) \times C_0^\infty(\mathbb{T}^d; \mathbb{R}) \to C_0^\infty(\mathbb{T}^d; \mathcal{S})$ *such that* $\operatorname{div} \operatorname{div} \mathcal{R}_N^2(f, g) = fg - \fint fg$ *and for any* $j \in \mathbb{N} \cup \{0\}$

$$|\nabla^j \mathcal{R}_N^2(f, g_\lambda)|_p \leq C_{j,d,p,s,r,N} \lambda^j |g|_{W^{j,s}} \left(\frac{1}{\lambda^2}|f|_r + \frac{1}{\lambda^N}|\nabla^N f|_r + \frac{1}{\lambda^{2N+j}}|\nabla^{2N+j} f|_r \right). \tag{23}$$

The proof is standard, cf. [18,30,31] and can be found in Appendix.

Remark 5. ($\mathcal{R}_N^2 \neq \mathcal{R}_N \circ \mathcal{R}_N$) For the operator defined in (iv), we use the notation $\mathcal{R}_N^2$ to denote that this operator acts as a double antidivergence. It does not coincide in general with $\mathcal{R}_N \circ \mathcal{R}_N$.

Remark 6. ($\mathcal{R}_\infty$) The above bilinear antidivergences may be thought of as approximations of 'ideal antidivergence' operators $\mathcal{R}_\infty, \mathcal{R}_\infty^2$ satisfying

$$|\mathcal{R}_\infty(f, u_\lambda)|_p \lesssim \frac{1}{\lambda}|u|_s|f|_r, \qquad |\nabla^j \mathcal{R}_\infty^2(f, g_\lambda)|_p \lesssim \lambda^{j-2}|g|_{W^{j,s}}|f|_r, \tag{24}$$

where the gap between $\mathcal{R}_N$ and $\mathcal{R}_\infty$ closes as $N \to \infty$, similarly for $\mathcal{R}_N^2$ and $\mathcal{R}_\infty^2$.

4. Mikado Flows

In this section we introduce the building blocks of our construction, namely the *concentrated localized traveling Mikado flows*. This section is essentially a rearrangement of known material: *Mikado flows* were first introduced in [17] in the framework of the Euler equations; the *concentrated Mikado flows* (or *fields*) were defined in [31], inspired by the *intermittent Beltrami waves* of [10]; the *traveling* version of concentrated Mikados we present here is close in spirit to the *intermittent jets* introduced for the first time in [6], and resemble the construction in [30].

The original Mikado flows of [17] are fast oscillating pressureless stationary solutions to Euler equations having the form

$$\Psi_\lambda^k(x)k = \Psi^k(\lambda x)k, \tag{25}$$

where $k \in \mathbb{Z}^d$ is a direction. For a finite set of directions K (given by the decomposition Lemma 2) one can choose functions $\Psi^k \in C_0^\infty(\mathbb{T}^d, \mathbb{R})$ so that the following holds for any $k, k' \in K$ and $\lambda \in \mathbb{N}$

$$(i) \qquad \operatorname{div} \Psi_\lambda^k k = 0,$$

$$(ii) \qquad \operatorname{div}(\Psi_\lambda^k k \otimes \Psi_\lambda^k k) = 0,$$

$$(iii) \qquad \fint_{\mathbb{T}^d} (\Psi^k)^2 = 1 \quad \text{thus} \quad \fint_{\mathbb{T}^d} \Psi_\lambda^k k \otimes \Psi_\lambda^k k = k \otimes k,$$

$$(iv) \qquad \Psi_\lambda^k k \quad \text{and} \quad \Psi_\lambda^{k'} k' \quad \text{have disjoint supports for } k \neq k'. \tag{26}$$

Satisfying property (i) is equivalent to choosing Ψ^k so that $\nabla \Psi^k \cdot k \equiv 0$, then also (ii) follows. Having (iii) is a normalisation of $\fint_{\mathbb{T}^d}(\Psi^k)^2$. Disjointness of supports (iv) is ensured in $d \geq 3$ via an appropriate choice of an anchor point ζ_k for the cylinder $B_\rho(0) + \{\zeta_k + sk\}_{s\in\mathbb{R}} + \mathbb{Z}^d$ (which is the periodisation of the cylinder $B_\rho(\zeta_k) + \{sk\}_{s\in\mathbb{R}}$ with radius ρ and axis being the line passing through ζ_k with direction k). Such choice is possible in view of the first part of the following lemma.

Lemma 3. *(Disjoint periodic tubes and blobs)* *There exist $\{\zeta_k\}_{k \in K} \subseteq \mathbb{R}^d$ and $\rho > 0$ such that for all $k, k' \in K$, $k \neq k'$:*

(i) *if $d \geq 3$,*
$$\left(B_\rho(\zeta_k) + sk + \mathbb{Z}^d\right) \cap \left(B_\rho(\zeta_{k'}) + s'k' + \mathbb{Z}^d\right) = \emptyset \qquad (27)$$

for every $s, s' \in \mathbb{R}$;

(ii) *if $d = 2$,*
$$\left(B_\rho(\zeta_k) + sk + \mathbb{Z}^2\right) \cap \left(B_\rho(\zeta_{k'}) + sk' + \mathbb{Z}^2\right) = \emptyset \qquad (28)$$

for every $s \in \mathbb{R}$.

Notice the difference between (27) and (28): in (27) there are two free parameters s, s', whereas in (28) there is only one free parameter s.

Part (ii) of the statement is not strictly needed for our proof (because we are always assuming $d \geq 3$). We think however that Part (ii) could be of interest in view of future applications to two dimensional problems, and thus we decided to include it in the statement. Proof of the Lemma can be found in Appendix.

The convex integration approach uses the properties (i)-(iv) to diminish a given Reynolds stress R_0 of a given solution (u_0, π_0, R_0) to the Non-Newtonian-Reynolds system (10) by correcting u_0 roughly as follows. Thanks to (iii), we can decompose R_0 via the Nash Lemma 2 into

$$\sum_k \Gamma_k^2(R_0)k \otimes k = \sum_k \Gamma_k^2(R_0) \fint_{\mathbb{T}^d} \Psi_\lambda^k k \otimes \Psi_\lambda^k k. \qquad (29)$$

Let us add to u_0 the corrector $\tilde{u} = \Gamma_k(R_0)\Psi_\lambda^k k$. Recall the notation $P_{\neq 0} f := f - \fint_{\mathbb{T}^d} f$. Thanks to (iv) and (ii)

$$\text{div}\left(\tilde{u} \otimes \tilde{u} - R_0\right) = \sum_k P_{\neq 0}\left(\Psi_\lambda^k k \otimes \Psi_\lambda^k k\right) \nabla \Gamma_k^2(R_0).$$

Since the term $P_{\neq 0}\left(\Psi_\lambda^k k \otimes \Psi_\lambda^k k\right)$ is λ-periodic and null mean, applying $\mathcal{R}_\infty$ of (24) to the r.h.s. above yields R_1 of order λ^{-1}, such that $\text{div } R_1 = \text{div}\left(\tilde{u} \otimes \tilde{u} - R_0\right)$. So picking λ large, i.e. letting Ψ_λ^k oscillate fast, allows to deal with the error R_0. The property (i) allows to control $\text{div } \tilde{u}$.

4.1. Concentrated Mikado flows. Since

$$|\nabla^i \Psi_\lambda^k k|_{L^p(\mathbb{T}^d)} = \lambda^i |\nabla^i \Psi^k k|_{L^p(\mathbb{T}^d)} = C\lambda^i \to \infty \quad \text{as} \quad \lambda \to \infty,$$

fast oscillations, in general, blow up derivatives of the corrector $\tilde{u}$. Thus controlling Sobolev norms of velocity fields appearing over convex integration steps seems problematic. This issue may be circumvented by a *concentration* mechanism, introduced in [31] and critically inspired by [10].

Let us briefly explain it. For $n \leq d$, take a compactly supported smooth function $f : \mathbb{R}^n \to \mathbb{R}$, rescale it to $f_\mu(x) = \mu^a f(\mu x)$, $\mu \geq 1$, and periodize without renaming to $f_\mu : \mathbb{T}^d \to \mathbb{R}$. This is *concentrating* and results in

$$|\nabla^i f_\mu|_{L^p(\mathbb{T}^d)} = \mu^{a+i-\frac{n}{p}} |\nabla^i f|_{L^p(\mathbb{R}^n)} = C\mu^{a+i-\frac{n}{p}} \quad \text{and} \quad |\nabla^i f_{\mu,\lambda}|_{L^p(\mathbb{T}^n)} = C\lambda^i \mu^{a+i-\frac{n}{p}}.$$

This procedure yields the 'concentrated Mikado' $\Psi_\mu^k k$ satisfying

$$|\nabla^i \Psi_\mu^k k|_{L^p(\mathbb{T}^d)} = C\lambda^i \mu^{a+i-\frac{n}{p}}.$$

Having now an interplay between λ and μ one can expect to control certain Sobolev norms by choosing a, p appropriately. However, to preserve the properties (i) and (ii) of (26), i.e. $\nabla f_\mu \cdot k \equiv 0$ (or in other words: $\Psi_\mu^k k$ being the Euler flow), the underlying function f_μ cannot depend on the direction k. It means that the underlying real function is not compactly supported in $\mathbb{R}^d$, but at best in $\mathbb{R}^{d-1}$. Thus at best $n = d - 1$, but then

$$|\nabla^i \Psi_\mu^k k|_{L^p(\mathbb{T}^d)} = C\lambda^i \mu^{a+i-\frac{d-1}{p}}.$$

The quantity $\fint_{\mathbb{T}^d} \Psi_\mu^k k \otimes \Psi_\mu^k k$ shall be of order $k \otimes k$, cf. (29). Therefore $\fint |\Psi_\mu^k|^2$ should be λ- and μ-independent, cf. (iii) of (26). This leads to the choice $a = \frac{d-1}{2}$ above and consequently to

$$(v) \qquad |\nabla^i \Psi_\mu^k k|_{L^p(\mathbb{T}^d)} = C\lambda^i \mu^{\frac{d-1}{2}+i-\frac{d-1}{p}}. \qquad (30)$$

Scaling (30) would force us to prove our results with d substituted by $d - 1$, so that e.g. q could vary only in the interval $1 < q < \frac{2(d-1)}{(d-1)+2}$ (which in particular requires $d \geq 4$, as in [27]).

Summing up, the concentrated Mikado $\Psi_\mu^k k$ satisfies properties (26) (i)–(iv) of the original Mikado, but has unsatisfactory scaling (v).

4.2. Concentrated localized Mikado flows.

A natural idea to deal with the 'loss of dimension' in (30) is to localise the Ansatz (25). Let us thus take a smooth radial cutoff function ϕ and define $\Phi : \mathbb{R}^d \to \mathbb{R}$ via $\Phi(x) = \phi(|x|)$. We want to retain gains stemming from concentrating, and since now $\Phi : \mathbb{R}^d \to \mathbb{R}$, while Ψ of (25) allowed merely for $d - 1$ concentrations, it is better to concentrate in Φ, thus producing Φ_μ. We periodize this function without renaming it and allow to oscillate at an independent frequency λ_1. Hence our new Ansatz reads

$$\Psi_{\lambda_2}^k \Phi_{\mu,\lambda_1} k. \qquad (31)$$

Let us now state and prove a result gathering needed properties of the cutoff Φ_{μ,λ_1}.

Lemma 4. *Let $K \subset \mathbb{Z}^d$ be a fixed finite set of directions. There exists $\rho > 0$ such that for every $\lambda_1 \in \mathbb{N}$, $\mu \in \mathbb{N}$, $\mu \geq \rho^{-1}$ there is $\Phi_{\mu,\lambda_1}^k \in C^\infty(\mathbb{T}^d; \mathbb{R})$ with the following properties*

$$\fint_{\mathbb{T}^d} (\Phi_{\mu,\lambda_1}^k)^2 dx = 1, \qquad |\Phi_{\mu,\lambda_1}^k|_{W^{i,r}(\mathbb{T}^d)} \leq M_{i,r,k}\lambda_1^i \mu^{i+\frac{d}{2}-\frac{d}{r}}, \qquad (32)$$

$$\operatorname{supp} \Phi_{\mu,\lambda_1}^k(\cdot - sk) \cap \operatorname{supp} \Phi_{\mu,\lambda_1}^{k'}(\cdot - sk') = \varnothing \quad \textit{for all} \quad k, k' \in K, k \neq k', s \in \mathbb{R}. \qquad (33)$$

Proof. Take $\Phi \in C_c^\infty(\mathbb{R}^d)$, with $\operatorname{supp} \Phi \subseteq B_1(0) \subseteq \mathbb{R}^d$ such that $\int_{\mathbb{R}^d} \Phi^2 = 1$. Let us concentrate Φ to $\Phi_\mu : \mathbb{T}^d \to \mathbb{R}$, hence $\fint_{\mathbb{T}^d} (\Phi_\mu)^2 = \mu^{a-\frac{d}{2}}$. Choosing $a = \frac{d}{2}$ yields the desired $\fint_{\mathbb{T}^d} \Phi_\mu^2 dx = 1$ and $|\Phi_\mu|_{W^{i,r}(\mathbb{T}^d)} \leq C_{i,r}\mu^{i+\frac{d}{2}-\frac{d}{r}}$. Concentration gives also $\operatorname{supp} \Phi_\mu \subseteq B_{1/\mu}(0) + \mathbb{Z}^d$.

Let $\rho > 0$ and $\{\zeta_k\}_{k \in K}$ be given by Lemma 3. We now set for every $k \in K$ and $\lambda_1 \in \mathbb{N}$,

$$\Phi^k_{\mu,\lambda_1}(x) := \Phi_\mu(\lambda_1(x - \zeta_k)).$$

Property (32) now follows from the scaling properties of Φ_μ, whereas (33) from supp $\Phi_\mu \subseteq B_{1/\mu}(0) + \mathbb{Z}^d$, (27), and assumed $\mu \geq \rho^{-1}$.   □

4.3. Concentrated localized traveling Mikado flow. Unsurprisingly, introducing d-dimensional cutoff Φ destroys the properties (i)–(iii) of standard Mikados. The most severe loss, due to its critical scaling, is not having (ii) anymore. A crucial idea how to handle this issue, introduced in [10], is to let the cutoff function Φ travel in time along l^k with speed ω. This leads to a corrector term Y^k (see below), whose time derivative compensates lack of (ii). At the same time Y^k is of order $\frac{1}{\omega}$, so it can be controlled by choosing ω large.

The concentrated localized traveling Mikado flow is our final Ansatz. It will be denoted by W^k, but it is important to bear in mind that it is determined by the parameters

$$\mu, \lambda_1, \lambda_2, \omega \in \mathbb{N}.$$

The next proposition concerns our final Mikado flows W^k and Mikado correctors Y^k.

Proposition 5. *Let $K \subset \mathbb{Z}^d$ be a fixed finite set of directions. Let $\Psi^k_{\lambda_2}$ be the function used to produce the standard Mikado (25) with its properties (i)–(iv). Let Φ_{μ,λ_1} be the localisation provided by Lemma 4.*

Define the functions $W^k : \mathbb{T}^d \times [0,1] \to \mathbb{R}^d$, $Y^k : \mathbb{T}^d \times [0,1] \to \mathbb{R}^d$ by

$$W^k(x,t) := \left(\Psi^k_{\lambda_2} \Phi^k_{\mu,\lambda_1} k\right)(x - \omega t k), \qquad Y^k(x,t) := \left(\frac{1}{\omega}(\Psi^k_{\lambda_2})^2 (\Phi^k_{\mu,\lambda_1})^2 k\right)(x - \omega t k). \tag{34}$$

There exists $\rho > 0$ such that for every $\mu, \lambda_1, \lambda_2, \omega \in \mathbb{N}$ satisfying

$$\mu \geq \frac{1}{\rho}, \quad \frac{\lambda_2}{\lambda_1} \in \mathbb{N} \quad and \quad \frac{\lambda_1 \mu}{\lambda_2} < \frac{1}{2} \tag{35}$$

the functions W^k, Y^k are spatially λ_1-periodic are have the following properties:

$$(v') \quad |W^k(t)|_{W^{i,r}(\mathbb{T}^d)} \leq M_{i,r} \lambda_2^i \mu^{\frac{d}{2} - \frac{d}{r}}, \qquad |Y^k(t)|_{W^{i,r}(\mathbb{T}^d)} \leq M_{i,r} \frac{\lambda_2^i \mu^{d - \frac{d}{r}}}{\omega}; \tag{36}$$

$$(iii') \quad \left|\fint W^k(t) \otimes W^k(t) - k \otimes k\right| \leq M_1 \frac{\lambda_1 \mu}{\lambda_2}; \tag{37}$$

$$(iv') \quad for\ k, k' \in K,\ k \neq k'\ \text{supp}\ W^k \cap \text{supp}\ W^{k'} = \emptyset;$$

$$(ii') \quad \partial_t Y^k + \text{div}\,(W^k \otimes W^k) = 0. \tag{38}$$

Proof. The spatial λ_1-periodicity of W^k, Y^k follows from the assumption $\lambda_2/\lambda_1 \in \mathbb{N}$. Since W^k, Y^k are obtained from stationary functions by means of a Galilean shift, for

(36) and (37) it suffices to estimate the respective stationary functions.

$$|\Psi_{\lambda_2}^k \Phi_{\mu,\lambda_1}^k k|_{W^{i,r}(\mathbb{T}^d)} \leq M_k \sum_{j=0}^{i} |\Psi_{\lambda_2}|_{W^{i-j,\infty}(\mathbb{T}^d)} |\Phi_{\mu,\lambda_1}|_{W^{j,r}(\mathbb{T}^d)}$$

$$\leq M_{i,r,k} \sum_{j=0}^{i} \lambda_1^j \mu^{j+\frac{d}{2}-\frac{d}{r}} \lambda_2^{i-j} \tag{39}$$

with the second inequality due to (25) and (32). Since by assumption $\lambda_1 \mu < \lambda_2$, we obtain (36) estimate for W^k. A similar computation yields the estimate for Y^k. For (37) we compute

$$\left| \fint (\Psi_{\lambda_2}^k)^2 (\Phi_{\mu,\lambda_1}^k)^2 (k \otimes k) - (k \otimes k) \right| = \left| (k \otimes k) \fint (\Phi_{\mu,\lambda_1}^k)^2 \left((\Psi_{\lambda_2}^k)^2 - 1 \right) \right| =: I$$

with the equality valid because the normalisation of (32) holds. Since the normalisation (iii) of the standard Mikado Ψ^k implies that $(\Psi_{\lambda_2}^k)^2 - 1$ is null-mean, and it oscillates at the frequency λ_2, by Proposition 2 we have

$$I \leq \frac{M_k}{\lambda_2} \left| \nabla \left((\Phi_{\mu,\lambda_1}^k)^2 \right) \right|_1 |(\Psi^k)^2 - 1|_\infty \leq M_k \frac{\lambda_1 \mu}{\lambda_2}$$

via (32). We reached (37).

Disjointness of supports of follows from (33): Assume that for some $(x, t) \in \mathbb{T}^d \times [0, 1]$ and $k, k' \in K$, $k \neq k'$, $W^k(x, t) W^{k'}(x, t) \neq 0$. Hence in view of the definition (34)

$$x - \omega t k \in \text{supp } \Phi_{\mu,\lambda_1}^k, \quad x - \omega t k' \in \text{supp } \Phi_{\mu,\lambda_1}^{k'},$$

thus contradicting (33).

The Mikado functions have the form

$$W^k(x, t) = F(x - \omega t k)k, \quad Y^k(x, t) = \frac{1}{\omega} G(x - \omega t k)k,$$

with $F^2 = G$. Therefore $\text{div}(W^k \otimes W^k) = (\nabla G(x - \omega t k) \cdot k)k$, whereas $\partial_t Y^k = -(\nabla G(x - \omega t k) \cdot k)k$. Hence (38). □

Remark 7. Let us compare our W^k with the concentrated Mikado.

(a) W^k is not divergence free, i.e. (i) does not hold. Furthermore W^k is now time dependent.
(b) W^k does not satisfy (ii). There appears Mikado corrector Y^k to compensates this deficiency, see (38). This means however that the new term Y^k must be appropriately estimated.
(c) Property (iii) holds approximately, see (37).
(d) Supports are pairwise disjoint (now in space-time).
(e) The scaling with a dimension loss (30) is now improved to (36), which is our main gain.

Proposition 5 yields the following estimates in relation to (b), (c), (e):

$$\left| Y^k(t) \right|_{L^2(\mathbb{T}^d)} \sim \frac{\mu^{d/2}}{\omega}, \qquad\qquad \left| \nabla Y^k(t) \right|_{L^q(\mathbb{T}^d)} \sim \lambda_2 \frac{\mu^{d-\frac{d}{q}}}{\omega},$$

$$\left| \fint W^k(t) \otimes W^k(t) - k \otimes k \right| \sim \frac{\lambda_1 \mu}{\lambda_2}, \qquad |\nabla W^k(t)|_{L^q(\mathbb{T}^d)} \sim \lambda_2 \mu^{\frac{d}{2}-\frac{d}{q}}.$$

In order to deal with (a), let us recall the heuristics of an ideal antidivergence operator $\mathcal{R}_\infty$ of Remark 6. A short computation involving (34), (36), and (32) yields

$$\left| \mathcal{R}_\infty(\text{div } W^k) \right|_{L^2(\mathbb{T}^d)} \sim \frac{\lambda_1 \mu}{\lambda_2}, \qquad \left| \mathcal{R}_\infty \partial_t W^k \right|_{L^1(\mathbb{T}^d)} \sim \frac{\lambda_1 \mu}{\lambda_2} \cdot \frac{\omega}{\mu^{d/2}}$$

Therefore, seeking smallness of

$$\frac{\mu^{d/2}}{\omega}, \quad \frac{\lambda_1 \mu}{\lambda_2}, \quad \lambda_2 \mu^{\frac{d}{2}-\frac{d}{q}}, \quad \frac{\lambda_1 \mu}{\lambda_2} \cdot \frac{\omega}{\mu^{d/2}} \tag{40}$$

will motivate the choice of the relations between the parameters μ, λ_1, λ_2, ω in Sect. 7.

Notice that we did not add the term $\lambda_2 \frac{\mu^{d-\frac{d}{q}}}{\omega}$ to the list (40), as it is the product of the first and the third term in (40), and thus its smallness is implied by smallness of these terms. □

5. Definition of (u_1, π_1, R_1)

Let (u_0, π_0, R_0) be a solution to the Non-Newtonian-Reynolds system (10), and $\delta, \eta \in (0, 1]$ as in Proposition 1. We define

$$u_1 := u_0 + u_p + u_c, \quad \pi_1 := \pi_0 + \pi_p,$$

where

- u_p is a perturbation based on the Mikado flows of Sect. 4, aimed at decreasing R_0,
- u_c is a corrector restoring solenoidality of u_1 and compensating for our Mikado flows not solving Euler equations.

Since (u_0, π_0, R_0) solves (10), it holds in the sense of distributions

$$\begin{aligned}
&\partial_t u_1 + \text{div } (u_1 \otimes u_1) - \text{div } \mathcal{A}(Du_1) + \nabla \pi_1 \\
&= \partial_t(u_p + u_c) + \text{div } (u_0 \otimes u_p + u_p \otimes u_0) \\
&\quad + \text{div } (u_0 \otimes u_c + u_c \otimes u_0 + u_p \otimes u_c + u_c \otimes u_p + u_c \otimes u_c) \\
&\quad + \text{div } (u_p \otimes u_p - \mathring{R}_0) \\
&\quad - \text{div } \left(\mathcal{A}\big(D(u_0 + u_p + u_c)\big) - \mathcal{A}(Du_0) \right) + \nabla \pi_p.
\end{aligned} \tag{41}$$

5.1. Decomposition of R_0 and energy control. In general, R_0 is only continuous (recall Definition 2 and Remark 2). Since it is convenient to work with smooth objects, we regularize R_0 (extended for times outside [0, 1] by $R_0(x, 0)$ and $R_0(x, 1)$, respectively) with the standard mollifier ϕ_ϵ in space and time. Thus

$$\mathring{R}_0^\epsilon := \mathring{R}_0 * \phi_\epsilon. \tag{42}$$

Now R_0^ϵ is smooth and (12) implies

$$|\mathring{R}_0^\epsilon(t)|_1 \leq \frac{\delta}{2^7 d}, \quad \text{for every } t \in [0, 1]. \tag{43}$$

Next, we decompose $\mathring{R}_0^\epsilon$ into basic directions. In order to stay within $\mathcal{S}_+$ of Lemma 2, we shift and normalise $\mathring{R}_0^\epsilon$ via

$$\text{Id} + \frac{\mathring{R}_0^\epsilon(x, t)}{\varrho(x, t)}, \quad \text{with}$$

$$\varrho(x, t) := 2\sqrt{\epsilon^2 + |\mathring{R}_0^\epsilon(x, t)|^2} + \gamma_0(t), \tag{44}$$

$$\gamma_0(t) := \frac{e(t)\left(1 - \frac{\delta}{2}\right) - \left(\int_{\mathbb{T}^d} |u_0|^2(t) + 2 \int_0^t \int_{\mathbb{T}^d} \mathcal{A}(Du_0) Du_0\right)}{d}. \tag{45}$$

The role of $\sqrt{\epsilon^2 + ..}$ is to avoid the degeneracy $|\mathring{R}_0(x, t)| = 0$, whereas the role of γ_0 is to pump energy into the system, thus facilitating the step (11) $\to$ (14). Observe that $\gamma_0 > 0$ because of (11). The choice (44) yields in particular $\varrho \geq 2|\mathring{R}_0^\epsilon(x, t)|$ and hence

$$\frac{1}{2}\text{Id} \leq \text{Id} + \frac{\mathring{R}_0^\epsilon}{\varrho} \leq \frac{3}{2}\text{Id}. \tag{46}$$

Remark 8. The set $\mathcal{N} \subset \mathcal{S}_+$ of Lemma 2 is fixed by (46) uniformly over the convex integration iterations.

Define

$$a_k(x, t) := \varrho^{\frac{1}{2}}(x, t)\Gamma_k\left(\text{Id} + \frac{\mathring{R}_0^\epsilon(x, t)}{\varrho(x, t)}\right). \tag{47}$$

Thanks to Lemma 2 it holds

$$\varrho\text{Id} + \mathring{R}_0^\epsilon = \sum_{k \in K} \varrho\Gamma_k^2\left(\text{Id} + \frac{\mathring{R}_0^\epsilon}{\varrho}\right) k \otimes k = \sum_{k \in K} a_k^2 \, k \otimes k. \tag{48}$$

5.2. Choice of u_p. Now we choose the principal corrector, motivated by the corrector $\tilde{u}$ that appeared in the initial part of Sect. 4. Let W^k be the Mikado flow of Proposition 5 with a_k defined by (47). Let

$$u_p(x, t) := \sum_{k \in K} a_k(x, t) W^k(x, t). \tag{49}$$

The disjoint supports of $W^k(t)$, $W^{k'}(t)$, $k \neq k'$ imply

$$u_p \otimes u_p = \sum_{k \in K} a_k^2 W^k \otimes W^k. \tag{50}$$

Recall the notation $P_{\neq 0} f := f - \fint_{\mathbb{T}^d} f$. Use (48) and (50) to write

$$
\begin{aligned}
u_p \otimes u_p - \mathring{R}_0^\epsilon &= \varrho \mathrm{Id} + \sum_{k \in K} a_k^2 \left(W^k \otimes W^k - k \otimes k \right) \\
&= \varrho \mathrm{Id} + \sum_{k \in K} a_k^2 P_{\neq 0}(W^k \otimes W^k) + a_k^2 \left(\fint W^k \otimes W^k - k \otimes k \right)
\end{aligned}
\tag{51}
$$

We therefore have

$$
\begin{aligned}
\mathrm{div}\,(u_p \otimes u_p - \mathring{R}_0^\epsilon) = &\nabla \varrho + \sum_{k \in K} P_{\neq 0}(W^k \otimes W^k) \nabla a_k^2 \\
&+ \sum_{k \in K} \left(\fint W^k \otimes W^k - k \otimes k \right) \nabla a_k^2 + \sum_{k \in K} a_k^2 \mathrm{div}\,(W^k \otimes W^k).
\end{aligned}
\tag{52}
$$

Remark 9. Observe that for the original Mikados, or their concentrated version, the second line of (52) vanishes. These additional terms will be taken care of by (37) and (38).

In order to avoid troublesome solenoidality correctors of the last Σ in (52), let us (Helmholtz) project it onto divergence free vectors by $P_H = Id - \nabla \Delta^{-1} \mathrm{div}$ and balance the identity by incorporating $\nabla \Delta^{-1} \mathrm{div}$ into the pressure, with the new pressure

$$\tilde{\varrho} := \varrho + \Delta^{-1} \mathrm{div} \sum_{k \in K} a_k^2 \, \mathrm{div} \left(W^k \otimes W^k \right).$$

Applying $P_{\neq 0}$ to both sides of the resulting identity, we arrive at

$$
\begin{aligned}
\mathrm{div}\,(u_p \otimes u_p - \mathring{R}_0^\epsilon) = &\nabla \tilde{\varrho} + P_{\neq 0} \sum_{k \in K} P_{\neq 0}(W^k \otimes W^k) \nabla a_k^2 \\
&+ P_{\neq 0} \sum_{k \in K} \left(\fint W^k \otimes W^k - k \otimes k \right) \nabla a_k^2 + P_{\neq 0} P_H \sum_{k \in K} a_k^2 \mathrm{div}\,(W^k \otimes W^k).
\end{aligned}
\tag{53}
$$

5.3. Choice of u_c. The corrector term u_c has the following roles: (i) to cancel the highest-order bad term 'div $(W^k \otimes W^k)$' of (53) via (38), (ii) to render the entire perturbation $u_p + u_c$ solenoidal and (iii) null-mean.

For (i), observe that (38) implies

$$P_{\neq 0} P_H \sum_{k \in K} a_k^2 \partial_t Y^k + P_{\neq 0} P_H \sum_{k \in K} a_k^2 \text{div}\,(W^k \otimes W^k) = 0. \tag{54}$$

Thus taking

$$u_c^I := P_{\neq 0} P_H \sum_{k \in K} a_k^2 Y^k \tag{55}$$

will allow to cancel, with a part of the time derivative of u_c^I, the bad term 'div $(W^k \otimes W^k)$' of (53).

For (ii), observe that thanks to P_H, u_c^I is already solenoidal. Therefore it suffices to compensate lack of solenoidality of u_p. We will now define u_c^{II} accordingly. By the definition (49) of u_p, the definition (34) of W^k, and since $\text{div}\,\Psi_{\lambda_2}^k k = 0$ (cf. the property (i) of (26)) we have

$$\text{div}\,u_p(x, t) = \sum_{k \in K} \text{div}\,\left(a_k(x, t) \left(\Psi_{\lambda_2}^k \Phi_{\mu, \lambda_1}^k k \right) (x - \omega t k) \right)$$

$$= \sum_{k \in K} \Psi_{\lambda_2}^k(x)\, k \cdot \nabla\big(a_k \Phi_{\mu, \lambda_1}^k (x - \omega t k) \big).$$

Therefore we define

$$u_c^{II}(t, \cdot) := -\text{div} \sum_{k \in K} \mathcal{R}_N^2 \Big(k \cdot \nabla \big(a_k(t) \Phi_{\mu, \lambda_1}^k (\cdot - \omega t k) \big),\ \Psi_{\lambda_2}^k \Big), \tag{56}$$

where $\mathcal{R}_N^2$ is the double antidivergence given by Proposition 4, and N will be fixed later (see the discussion at the beginning of Sect. 6).

Since div div $\mathcal{R}_N^2 = \text{Id}\,P_{\neq 0}$, div $u_c^{II} + \text{div}\,u_p = 0$.

As u_c^I, u_c^{II} are null-mean, to take into account the condition (iii), it suffices to define

$$u_c := (u_c^I + u_c^{II}) - \fint u_p. \tag{57}$$

5.4. Reynolds stresses. Let us distribute $\partial_t (u_c^I + u_c^{II})$ and $\mathring{R}_0^\epsilon$ in (41) as follows

$$\partial_t u_1 + \text{div}\,(u_1 \otimes u_1) - \text{div}\,\mathcal{A}(Du_1) + \nabla \pi_1$$

$$= \partial_t \left(P_{\neq 0} u_p + u_c^{II} \right) + \text{div}\,(u_0 \otimes u_p + u_p \otimes u_0)$$

$$+ \text{div}\,(u_0 \otimes u_c + u_c \otimes u_0 + u_p \otimes u_c + u_c \otimes u_p + u_c \otimes u_c)$$

$$+ \partial_t u_c^I + \text{div}\,(u_p \otimes u_p - \mathring{R}_0^\epsilon) \tag{58}$$

$$+ \text{div}\,(\mathring{R}_0^\epsilon - \mathring{R}_0)$$

$$- \text{div}\,\big(\mathcal{A}(D(u_0 + u_p + u_c)) - \mathcal{A}(Du_0) \big) - \nabla \pi_p$$

We rewrite the r.h.s. of (58) further, recasting it into a divergence form.

(i). First line of r.h.s. of (58)) The definition (49) of u_p, the definition (34) of W^k, and $0 = k \cdot \nabla \Psi_{\lambda_2}^k(x)$ via property (i) of (26), give together

$$\partial_t u_p(t) = \sum_{k \in K} \Psi_{\lambda_2}^k k \, \partial_t \left(a_k(\cdot, t) \Phi_{\mu, \lambda_1}^k (\cdot - \omega k t) \right). \tag{59}$$

Using the above formula and the definition (56) of u_c^{II}, we define the antidivergence of the first line of r.h.s. of (58)

$$
\begin{aligned}
R_{lin} := &\sum_{k \in K} \mathcal{R}_N \left(\partial_t \left(a_k(\cdot, t) \Phi_{\mu, \lambda_1}^k (\cdot - \omega k t) \right), \Psi_{\lambda_2}^k k \right) \\
&- \mathcal{R}_N^2 \left(k \cdot \partial_t \left(a_k(\cdot, t) \Phi_{\mu, \lambda_1}^k (\cdot - \omega k t) \right), \Psi_{\lambda_2}^k \right) \\
&+ (u_0 \otimes u_p + u_p \otimes u_0).
\end{aligned} \tag{60}
$$

(ii). Second line of r.h.s. of (58))

$$R_{corr} := u_0 \otimes u_c + u_c \otimes u_0 + u_p \otimes u_c + u_c \otimes u_p + u_c \otimes u_c. \tag{61}$$

(iii). Third line of r.h.s. of (58)) Here we use an important idea of [10]. Via the definition (55) of u_c^I and the property (54) we have

$$\partial_t u_c^I = -P_{\neq 0} P_H \sum_{k \in K} a_k^2 \operatorname{div} \left(W^k \otimes W^k \right) + P_{\neq 0} P_H \sum_{k \in K} (\partial_t a_k^2) Y^k.$$

Adding the above identity to (53) that expresses div $(u_p \otimes u_p - \mathring{R}_0^\epsilon)$, the 'div $(W^k \otimes W^k)$' term cancel out and one has

$$
\begin{aligned}
&\partial_t u_c^I + \operatorname{div} (u_p \otimes u_p - \mathring{R}_0^\epsilon) - \nabla \tilde{\varrho} \\
&= \sum_{k \in K} \left(W^k \otimes W^k - k \otimes k \right) \nabla a_k^2 + P_{\neq 0} P_H \sum_{k \in K} (\partial_t a_k^2) Y^k.
\end{aligned}
$$

Let us thus define, leaving out $\nabla \tilde{\varrho}$ since it will be accounted for by the pressure perturbation q_p,

$$R_{quadr} := \sum_{k \in K} \tilde{\mathcal{R}}_1 \left(\nabla a_k^2, P_{\neq 0} \left(W^k \otimes W^k \right) \right) + a_k^2 \left(f \, W^k \otimes W^k - k \otimes k \right) \tag{62}$$

$$+ \operatorname{div}^{-1} P_{\neq 0} P_H \left((\partial_t a_k^2) Y^k \right), \tag{63}$$

so that div $R_{quadr} = \partial_t u_c^I + \operatorname{div} (u_p \otimes u_p - \mathring{R}_0^\epsilon) - \nabla \tilde{\varrho}$.

(iv. Fourth line on r.h.s. of (58)) Let

$$R_{moll} := \mathring{R}_0^\epsilon - \mathring{R}_0. \tag{64}$$

(v. Last line of r.h.s. of (58)) Let

$$R_{\mathcal{A}} := -\left(\mathcal{A}(D(u_0 + u_p + u_c)) - \mathcal{A}(Du_0) \right). \tag{65}$$

5.5. Pressure. In order to balance for $\nabla\tilde{\varrho}$ and ensure null-trace of $\mathring{R}_1$, we choose π_p such that

$$\nabla\pi_p \;=\; \nabla\tilde{\varrho} + \mathrm{div}\,\frac{1}{d}tr(R_{lin} + R_{corr} + R_{quadr} + R_{\mathcal{A}})\mathrm{Id}, \qquad \text{i.e.}$$

$$\pi_p(x,t) + c(t) := \tilde{\varrho} + \frac{1}{d}tr(R_{lin} + R_{corr} + R_{quadr} + R_{\mathcal{A}}),$$

having freedom in choosing $c(t)$, we set it so that $\pi_p(x,t)$ is null-mean.

5.6. Conclusion. Comparing our choices (60), (61), (62), (64) and (65) for r.h.s. of (58) with its l.h.s. we have reached

$$\partial_t u_1 + \mathrm{div}\,(u_1 \otimes u_1) - \mathrm{div}\,\mathcal{A}(Du_1) + \nabla\pi_1 = -\mathrm{div}\,\mathring{R}_1 \tag{66}$$

with

$$R_1 := -(R_{lin} + R_{corr} + R_{quadr} + R_{moll} + R_{\mathcal{A}}). \tag{67}$$

Notice that tensor R_1 is symmetric, since its components are symmetric (cf. respective definitions and Proposition 4).

6. Estimates

We continue the proof of the main Proposition 1. In the previous section, given (u_0, π_0, R_0) solving the non-Newtonian-Reynolds system, we defined $u_1 = u_0 + u_p + u_c$, π_1, and the new Reynolds stress R_1, required by Proposition 1. The perturbation u_p, the corrector u_c and the error R_1 depend on the six parameters

$$\epsilon > 0, \quad \mu, \lambda_1, \lambda_2, \omega \in \mathbb{N}, \quad N \in \mathbb{N}, \tag{68}$$

which satisfy the condition (35). The mollification parameter ϵ helps to avoid degeneracies or singularities of $\mathcal{A}$. Let us immediately fix it so that

$$\epsilon \leq \frac{\delta}{2^7 d}, \qquad |\mathring{R}_0^\epsilon(t) - \mathring{R}_0(t)|_1 \leq \frac{\eta}{2} \quad \text{for every } t \in [0,1], \tag{69}$$

where $\eta > 0$ is the parameter appearing in the statement of the main Proposition 1.

  In this section we estimate u_p, u_c, R_1 and the energy gap of the new solution u_1 in terms of the remaining five parameters $\mu, \lambda_1, \lambda_2, \omega, N$. They will be appropriately chosen in Sect. 7 so that (13a)–(14) hold, thus concluding the proof of the main Proposition 1.

Remark 10. (Silent assumptions) For results of this section, we assume without writing it explicitly at each occasion: (u_1, π_1, R_1) is the triple constructed in the previous section, the set of directions K is fixed by Remark 8, the relations (35) hold, the assumptions of Proposition 1 hold, the choice (69) holds.

6.1. Constants. We distinguish two types of constants: the uniform ones (*M*'s) and the usual ones (*C*'s). None depend on μ, λ_1, λ_2, ω.

More precisely, we denote by *M* any constant depending only on the following parameters

ν_0, ν_1 the parameters entering in the definition of the non-Newtonian tensor field $\mathcal{A}$

q the exponent entering in the definition of the non-Newtonian tensor field $\mathcal{A}$

e the energy profile fixed in the assumptions of Proposition 1

Φ, Ψ the profiles used in the definition of the Mikado functions in Sect. 4

K the fixed set of directions, cf. Remark 8.

$$(70)$$

Consequently, any universal constant *M* remains uniform over the convex integration iteration. We will not explicitly write the dependence of *M*'s on the objects in (70).

On the other side, we will denote by *C* (possibly with subscripts) any constant depending not only on the universal quantities (70), but also on

$$(u_0, \pi_0, R_0), \delta, \eta \quad \text{given in the assumptions of Proposition 1,} \qquad (71)$$

$$i, r \quad \text{if we are estimating the } W^{i,r} \text{ norm}$$
$$N \quad \text{the (not yet fixed) parameter in (67),} \qquad (72)$$

Constants *C* will be controlled within each iteration step (i.e. in the proof of Proposition 1) by appropriate choices μ, λ_1, λ_2, ω.

6.2. Preliminary estimates: control of a_k.

Proposition 6. *Coefficients a_k defined via* (47) *satisfy*

$$|a_k(t)|_2 \leq 2\delta^{\frac{1}{2}}, \qquad (73)$$

$$|a_k|_{C^i_{x,t}} \leq C_i \quad \text{for } i \geq 0 \qquad (74)$$

Proof. The definition (45) of γ_0, assumption (11), and the assumed bounds on the energy profile $e \in [\frac{1}{2}, 1]$ yield $\gamma_0(t) \in [\frac{\delta}{8d}, \delta]$. This and the choice (69) of ϵ give, via the definition (44) of ϱ,

$$\varrho(x,t) \leq 2\epsilon + 2|\mathring{R}_0^\epsilon(x,t)| + \gamma_0(t) \leq 3(|\mathring{R}_0^\epsilon(x,t)| + \delta).$$

Therefore, since $a_k^2 = \varrho \Gamma_k^2$ by its definition (47), whereas $\Gamma_k \leq 1$ by Lemma 2,

$$|a_k(t)|_2^2 = \int_{\mathbb{T}^d} \varrho \Gamma_k^2 \leq 3(\delta + |\mathring{R}_0^\epsilon(t)|_1),$$

The last inequality together with (43) yields (73).

Using smoothness of Γ_k supported in the compact set (46) and that $\varrho \geq \gamma_0 \geq \frac{\delta}{8d}$, one has (74). $\qquad \square$

6.3. Estimates for velocity increments. We will use now the improved Hölder inequality
(18) and the preliminary estimates to control u_p, u_c.

Proposition 7 (Estimates for the principal increment u_p). *For every $r \in [1, \infty]$*

$$|u_p(t)|_r \le C_r \mu^{\frac{d}{2} - \frac{d}{r}}, \tag{75}$$

$$|u_p(t)|_{W^{1,r}} \le C_r \lambda_2 \mu^{\frac{d}{2} - \frac{d}{r}}. \tag{76}$$

Moreover, there is a universal constant $M > 0$ such that

$$|u_p(t)|_2 \le M \delta^{1/2} + C \lambda_1^{-\frac{1}{2}}. \tag{77}$$

Proof. The definition (49) of u_p yields

$$|u_p(t)|_r \le \sum_{k \in K} |a_k(t) W^k(t)|_r. \tag{78}$$

Using (74) to control $|a_k|_\infty$ and (36) to control $|W^k(t)|_r$, we obtain (75). An analogous
computation gives (76).

Notice that for $r = 2$ the power of μ in (75) is 0: for this reason, to reach (77), one
needs more care. Recall from Proposition 5 that W^k is λ_1-periodic. Therefore we may
apply improved Hölder inequality (Proposition 3) to the r.h.s. of (78) and use (36) for
$|W^k(t)|_2$ to obtain

$$|u_p(t)|_2 \le M \sum_{k \in K} |a_k(t)|_2 + \lambda_1^{-\frac{1}{2}} |a_k(t)|_{C_x^1}$$

Let us now use (73), (74) to control a_k, reaching (77). □

Now we deal with the corrector u_c. In order to shorten the related formulas, let us
introduce

$$L(r) := \frac{\mu^{d - \frac{d}{r}}}{\omega} + \frac{\lambda_1 \mu}{\lambda_2} \mu^{\frac{d}{2} - \frac{d}{r}} \left[1 + \lambda_2^2 \left(\frac{\lambda_1 \mu}{\lambda_2} \right)^N \right] \tag{79}$$

Observe that $r \mapsto L(r)$ is a non-decreasing map.

Proposition 8 (Estimates for the corrector u_c). *For every $r \in (1, \infty)$ it holds*

$$|u_c(t)|_{\dot{W}^{i,r}} \le C_{i,d,r,N} \lambda_2^i L(r). \tag{80}$$

Proof. Recall that $u_c = (u_c^I + u_c^{II}) - \fint_{\mathbb{T}^d} u_p$ by its definition (57), with u_c^I defined by
(55) and u_c^{II} defined by (56).

The Calderón-Zygmund estimate, (74), and (36) give

$$|u_c^I(t)|_{\dot{W}^{i,r}} \le C_r \frac{\lambda_2^i \mu^{d - \frac{d}{r}}}{\omega} \tag{81}$$

For the estimate of u_c^{II} in $\dot{W}^{i,r}$ we use the inequality (23) for $j = i + 1, s = \infty$, which
yields

$$|u_c^{II}|_{\dot{W}^{i,r}} \le C_{j,d,r,N} \lambda_2^{i+1}$$
$$\times \left(\frac{1}{\lambda_2^2} |\nabla(a_k \Phi_{\mu,\lambda_1}^k)|_r + \frac{1}{\lambda_2^N} |\nabla^{N+1}(a_k \Phi_{\mu,\lambda_1}^k)|_r + \frac{1}{\lambda_2^{2N+i+1}} |\nabla^{2N+i+2}(a_k \Phi_{\mu,\lambda_1}^k)|_r \right).$$

Now we estimate a_k by (74) and Ψ^k_{μ,λ_1} by (32). Using the assumption (35), we arrive at

$$|u^{II}_c|_{\dot{W}^{i,r}} \leq C_{i,d,r,N} \lambda_2^i \frac{\lambda_1 \mu}{\lambda_2} \mu^{\frac{d}{2}-\frac{d}{r}} \left[1 + \lambda_2^2 \left(\frac{\lambda_1 \mu}{\lambda_2} \right)^N \right]. \tag{82}$$

Recall $u_c = (u^I_c + u^{II}_c) - \fint_{\mathbb{T}^d} u_p$ by its definition (57). Therefore, putting together (81), (82), and

$$\left| \fint u_p \right| \leq \frac{\lambda_1 \mu}{\lambda_2} \mu^{-\frac{d}{2}} \leq \frac{\lambda_1 \mu}{\lambda_2} \mu^{\frac{d}{2}-\frac{d}{r}}$$

valid via (17) with $r = 1$ and $v = \Psi^k$, yields (80). $\square$

6.4. Estimates on the Reynolds stress. Recall (67)

$$R_1 := -(R_{lin} + R_{corr} + R_{quadr} + R_{moll} + R_{\mathcal{A}}).$$

In this section we estimate each term of R_1. For our further purposes L^1 estimates suffice, but due to using Calderón-Zygmund theory, some estimates are phrased as L^r ones.

Proposition 9 (Estimates on the principal Reynolds R_{quadr}). *For every $r \in (1, \infty)$, it holds*

$$|R_{quadr}(t)|_r \leq C_r \left((\omega^{-1} + \lambda_1^{-1}) \mu^{d-\frac{d}{r}} + \lambda_2^{-1} \lambda_1 \mu \right) \tag{83}$$

Proof. Recall the definition (62) of R_{quadr}. Let us estimate its three terms in order of their appearance.

(i) The first term of R_{quadr} is the sum over k of $\tilde{\mathcal{R}}_1(\nabla a_k^2, P_{\neq 0}(W^k \otimes W^k))$. The term $P_{\neq 0}(W^k \otimes W^k))$ is null mean and it oscillates at the frequency λ_1, since W^k does. Therefore (21) with (74) and (36) give

$$|\tilde{\mathcal{R}}_1(\nabla a_k^2, P_{\neq 0}(W^k \otimes W^k))|_r \leq C_r \lambda_1^{-1} |P_{\neq 0}(W^k \otimes W^k))|_r |\nabla^2 (a_k^2)|_\infty \leq C_r \lambda_1^{-1} \mu^{d-\frac{d}{r}}. \tag{84}$$

(ii) The second term of R_{quadr} is the sum over k of $a_k^2 (\fint W^k \otimes W^k - k \otimes k)$. We use estimate (74) to control a_k terms and (37) to control the Mikado terms

$$\left| a_k^2 \left(\fint W^k \otimes W^k - k \otimes k \right) \right|_r \leq |a_k^2|_r \left| \fint W^k \otimes W^k - k \otimes k \right| \leq C_r \frac{\lambda_1 \mu}{\lambda_2}. \tag{85}$$

(iii) The last term is the sum over k of $\text{div}^{-1} P_H \left(P_{\neq 0}(\partial_t a_k^2) Y^k \right)$. We deal with div^{-1} via (19) and with P_H via Calderón-Zygmund, control $\partial_t a_k$ using (74), and use (36) to estimate Y^k. Hence

$$\left| \text{div}^{-1} P_H \left(P_{\neq 0}(\partial_t a_k^2) Y^k \right) \right|_r (t) \leq C_r \left| P_{\neq 0}(\partial_t a_k^2) Y^k \right|_r (t) \leq C_r |Y^k(t)|_r \frac{\mu^{d-\frac{d}{r}}}{\omega}. \tag{86}$$

Together, (84), (85), (86) yield (83). $\square$

Proposition 10 (Estimate on R_{lin}). *It holds*

$$|R_{lin}|_1(t) \leq C_N \left[\mu^{-\frac{d}{2}} + \frac{\lambda_1 \mu^{1-\frac{d}{2}} \omega}{\lambda_2} \left(1 + \lambda_2^2 \left(\frac{\lambda_1 \mu}{\lambda_2} \right)^N \right) \right]. \tag{87}$$

Proof. Recall the definition (60) of R_{lin}. It involves three terms, which we estimate in order of their appearance.

(i) The first term of R_{lin} is the sum over k of

$$\mathcal{R}_N\big((\partial_t a_k)\Phi^k_{\mu,\lambda_1} + \omega a_k k \cdot \nabla \Phi^k_{\mu,\lambda_1}, \Psi^k_{\lambda_2}k\big).$$

Using (21) with $|u|_s = |\Psi^k|_\infty$, the assumption (35), and disposing of a_k as usual, one has

$$|\mathcal{R}_N\left(\partial_t\left(a_k(\cdot,t)\Phi^k_{\mu,\lambda_1}(\cdot-\omega kt)\right), \Psi^k_{\lambda_2}k\right)|_1(t) \le C_{d,N}\frac{\omega\lambda_1\mu^{1-d/2}}{\lambda_2}\Big[1+\lambda_2\Big(\frac{\lambda_1\mu}{\lambda_2}\Big)^N\Big]. \tag{88}$$

(ii) The second term of R_{lin} is the sum over k of

$$\mathcal{R}^2_N(k\cdot\nabla\partial_t(a_k\Phi^k_{\mu,\lambda_1}(\cdot-\omega kt), \Psi^k_{\lambda_2}).$$

We observe that

$$|\nabla^{i+1}\partial_t\left(a_k\Phi^k_{\mu,\lambda_1}(\cdot-\omega kt)\right)|_1 \le |\partial_t a_k\Phi^k_{\mu,\lambda_1}|_{W^{i+1,1}} + \omega|k||a_k\nabla\Phi^k_{\mu,\lambda_1}|_{W^{i+1,1}}$$
$$\le C\omega\lambda_1^{i+2}\mu^{i+2-\frac{d}{2}}. \tag{89}$$

Using the computation (89) in (23) with $j = 0$ and $|u|_s = |\Psi^k|_\infty$, we get

$$\big|\mathcal{R}^2_N\left(k\cdot\nabla\partial_t\left(a_k\Phi^k_{\mu,\lambda_1}(\cdot-\omega kt)\right), \Psi^k_{\lambda_2}\right)\big|_1 \le C_{d,N}\,\omega\mu^{-\frac{d}{2}}\left(\frac{\lambda_1\mu}{\lambda_2}\right)^2\left[1+\lambda_2^2\Big(\frac{\lambda_1\mu}{\lambda_2}\Big)^N\right]. \tag{90}$$

(iii) The third term of R_{lin} equals $u_0 \otimes u_p + u_p \otimes u_0$, so we write using (75)

$$|u_0|_\infty |u_p|_1 \le C\mu^{-\frac{d}{2}}. \tag{91}$$

Putting together (88), (90), (91), and observing that the right-hand sides of both (88) and (90) are estimated by $C_{d,N}\,\omega\mu^{-\frac{d}{2}}\frac{\lambda_1\mu}{\lambda_2}[1+\lambda_2^2(\frac{\lambda_1\mu}{\lambda_2})^N]$ thanks to (35), one has (87). □

Proposition 11 (Estimates on R_{corr}). *Let* $L(2)$ *be given by* (79). *It holds*

$$|R_{corr}(t)|_1 \le C_N\big(L(2) + L^2(2)\big). \tag{92}$$

Proof. By the definition (61) of R_{corr} we have

$$|R_{corr}|_1(t) \le C(|u_0|_2|u_c|_2 + |u_p|_2|u_c|_2 + |u_c|_2^2)(t),$$

since $|u_p|_2 \le C$ via (75) and $|u_c|_2 \le C_N L(2)$ via (80), we have (92).   □

Proposition 12 (Estimates on the dissipative Reynolds $R_\mathcal{A}$). *For* $q \in (1,\infty)$ *being the growth parameter of* $\mathcal{A}$ *it holds*

$$|R_\mathcal{A}(t)|_r \le \begin{cases} C\big((\lambda_2\mu^{\frac{d}{2}-\frac{d}{r}})^{q-1} + (\lambda_2 L(r))^{q-1}\big) & \text{for } v_0 = 0, q \le 2, \quad \text{any } r > 1, \\ C\big(\lambda_2\mu^{\frac{d}{2}-\frac{d}{r}} + \lambda_2 L(r)\big) & \text{for } v_0 > 0, q \le 2, \quad \text{any } r > 1, \\ C\big(\lambda_2\mu^{\frac{d}{2}-\frac{d}{r(q-1)}} + \big(\lambda_2\mu^{\frac{d}{2}-\frac{d}{r(q-1)}}\big)^{q-1} \\ \quad + \lambda_2 L(r(q-1)) + (\lambda_2 L(r(q-1)))^{q-1}\big) & \text{for } q \ge 2, \quad \text{any } r(q-1) > 1. \end{cases} \tag{93}$$

Proof. By definition (65), we have

$$|R_{\mathcal{A}}| = |\mathcal{A}(Du_0 + Du_p + Du_c) - \mathcal{A}(Du_0)|.$$

Therefore the inequality (15) gives the pointwise estimate

$$|R_{\mathcal{A}}| \leq \begin{cases} C_{\nu_1}|Du_p + Du_c|^{q-1} & \text{for } \nu_0 = 0, q \leq 2 \\ C_{\nu_0}|Du_p + Du_c| & \text{for } \nu_0 > 0, q \leq 2 \\ C_{q,\nu_0,\nu_1}|Du_p + Du_c|\left(1 + |Du_0|^{q-2} + |Du_1|^{q-2}\right) & \text{for } q \geq 2 \end{cases}$$

Using Jensen inequality and $\frac{1}{q-1} \geq 1$ in the first case, and Hölder inequality with $q-1$, $\frac{q-1}{q-2}$ in the last case, one has

$$|R_{\mathcal{A}}|_r \leq \begin{cases} C|Du_p + Du_c|_r^{q-1} & \text{for } \nu_0 = 0, q \leq 2 \\ C|Du_p + Du_c|_r & \text{for } \nu_0 > 0, q \leq 2 \\ C|Du_p + Du_c|_{r(q-1)}\left(1 + |Du_0|_{r(q-1)}^{q-2} + |Du_1|_{r(q-1)}^{q-2}\right) & \text{for } q \geq 2 \end{cases}$$

For any $s \in (1, \infty)$ the estimate (80) controls $|Du_c|_s$ via $\lambda_2 L(s)$, whereas $\lambda_2 \mu^{d/2-d/s}$ controls $|Du_p|_s$ thanks to (76). This closes the case $q < 2$ of (93). Recalling that $u_1 = u_0 + u_p + u_c$ and that C may contain norms of u_0, we obtain the case $q \geq 2$. $\quad\square$

Remark 11. For the current purpose of proving Proposition 1 and thus Theorem A, the case $q \leq 2$ of (93) suffices. We included already the case $q \geq 2$, because it is needed to prove Theorem B.

Immediately from the definition of R_{moll} in (64) and the choice of ϵ in (69) we have

Proposition 13 (Estimate on R_{moll}). *It holds*

$$|R_{moll}(t)|_1 \leq \eta/2 \tag{94}$$

where η is the parameter appearing in the assumptions of Proposition 1.

6.5. Estimates on the energy increment. We intend to approach the desired energy profile $e(t)$, i.e. perform the step (11) $\rightarrow$ (14). Let us thence define δE as follows

$$\delta E(t) := \left| e(t)\left(1 - \frac{\delta}{2}\right) - \left(\int |u_1|^2(t) + 2\int_0^t \int \mathcal{A}(Du_1)Du_1\right) \right|. \tag{95}$$

Recall quantities L of (79). We will show

Proposition 14 (energy iterate). *For $q \in (1, \infty)$ being the growth parameter of $\mathcal{A}$ it holds*

$$\delta E(t) \leq \frac{\delta}{16}e(t) + C_N\left(\lambda_1^{-1} + \mu^{-\frac{d}{2}} + L(2) + L(2)^2 + \lambda_2\mu^{\frac{d}{2}-\frac{d}{q}} + \left(\lambda_2\mu^{\frac{d}{2}-\frac{d}{q}}\right)^q\right.$$
$$\left. + \lambda_2 L(q) + \left(\lambda_2 L(q)\right)^q\right). \tag{96}$$

Proof. Recall (51). Taking its trace and recalling that $\mathring{R}_0^\epsilon$ is traceless we have

$$|u_p|^2 = d\varrho + \sum_{k \in K} a_k^2 P_{\neq 0}|W^k|^2 + a_k^2 \left(\fint |W^k|^2 - |k|^2 \right). \tag{97}$$

By the definition (44) it holds $d\varrho = 2d\sqrt{\epsilon^2 + |\mathring{R}_0^\epsilon|^2} + d\gamma_0$, therefore

$$|u_p|^2 - d\gamma_0 = 2d\sqrt{\epsilon^2 + |\mathring{R}_0^\epsilon|^2} + \sum_{k \in K} a_k^2 P_{\neq 0}|W^k|^2 + a_k^2 \left(\fint |W^k|^2 - |k|^2 \right)$$

Integrating and using $\sqrt{\epsilon^2 + |x|^2} \leq \epsilon + |x|$, we have

$$\left| \int |u_p|^2 - d\gamma_0 \right| \leq 2d\epsilon + 2d|\mathring{R}_0^\epsilon(t)|_1 + \sum_{k \in K} \left| \int a_k^2 P_{\neq 0}|W^k|^2 \right| + |a_k|_2^2 \left| \fint |W^k|^2 - |k|^2 \right| \tag{98}$$

We estimate the first two terms of the r.h.s. of (98) using (69) and (43) as follows

$$2d\epsilon + 2d|\mathring{R}_0^\epsilon(t)|_1 \leq \frac{\delta}{2^6} + \frac{\delta}{2^6} \leq \frac{\delta}{2^4} e(t),$$

where in the second inequality we used the assumption $e(t) \geq \frac{1}{2}$. This in (98) yields

$$\left| \int |u_p|^2 - d\gamma_0 \right| \leq \frac{\delta}{2^4} e(t) + \sum_{k \in K} \left| \int a_k^2 P_{\neq 0}|W^k|^2 \right| + |a_k|_2^2 \left| \fint |W^k|^2 - |k|^2 \right| \tag{99}$$

The first integral of r.h.s. of (99) involves a λ_1-oscillating function $P_{\neq 0}|W^k|^2$, recall Proposition 5. Therefore, using (17), then (74) to control a_k and (36) for W^k, we have

$$\left| \int a_k^2 P_{\neq 0}|W^k|^2 \right| \leq C\lambda_1^{-1}|a_k^2|_\infty|(P_{\neq 0}|W^k|)|_2^2 \leq C\lambda_1^{-1}. \tag{100}$$

For the integral following the second sum in (99), we use (37) and (74) to get

$$|a_k|_2^2 \left| \fint |W^k|^2 - |k|^2 \right| \leq C\frac{\lambda_1 \mu}{\lambda_2}. \tag{101}$$

We plug (100) and (101) to (99) and obtain

$$\left| \int |u_p|^2 - d\gamma_0 \right| \leq \frac{\delta}{2^4} e(t) + C\left(\frac{1}{\lambda_1} + \frac{\lambda_1 \mu}{\lambda_2} \right). \tag{102}$$

Use $u_1 = u_0 + u_p + u_c$ in the definition (95) of E to write for the time instant t

$$\delta E(t) \leq \left| \int |u_p|^2 - d\gamma_0 \right| + \left| \int |u_c|^2 + 2(u_0 u_c + u_0 u_p + u_p u_c) \right|$$

$$+ 2\left| \int_0^t \int \mathcal{A}(Du_1)Du_1 - \mathcal{A}(Du_0)Du_0 \right|,$$

where a cancellation occurs, thanks to the definition (45) of γ_0. Inequality (102) allows to control the first term of the r.h.s. above. For the last term we use (16), next Hölder inequality with q, $\frac{q}{q-1}$, and finally $u_1 = u_0 + u_p + u_c$ to get

$$\int |\mathcal{A}(Du_1)Du_1 - \mathcal{A}(Du_0)Du_0| \leq C \left(|Du_p + Du_c|_q + |Du_p + Du_c|_q^q \right).$$

Thus, integrating in time over $[0, t] \subseteq [0, 1]$,

$$\int_0^t \int |\mathcal{A}(Du_1)Du_1 - \mathcal{A}(Du_0)Du_0|$$
$$\leq C \sup_{\tau \in [0,1]} \left(|Du_p(\tau) + Du_c(\tau)|_q + |Du_p(\tau) + Du_c(\tau)|_q^q \right).$$

Consequently

$$\delta E(t) \leq \frac{\delta}{2^4} e(t) + C \left(\frac{1}{\lambda_1} + \frac{\lambda_1 \mu}{\lambda_2} \right)$$
$$+ C \left(|u_c|_2^2 + |u_0|_2 |u_c|_2 + |u_0|_\infty |u_p|_1 + |u_c|_2 |u_p|_2 \right)(t)$$
$$+ C \sup_{\tau \in [0,1]} \left(|Du_p(\tau) + Du_c(\tau)|_q + |Du_p(\tau) + Du_c(\tau)|_q^q \right).$$

The terms in the second line above are estimated, using (80) for u_c and (75) for u_p, by $C_N(L(2)^2 + L(2) + \mu^{-d/2})$. Observe that $L(2)$ of this term can absorb $\lambda_2^{-1}\lambda_1\mu$ of the first line. The terms in the last line are estimated by $\lambda_2 \mu^{\frac{d}{2}-\frac{d}{q}} + \left(\lambda_2 \mu^{\frac{d}{2}-\frac{d}{q}}\right)^q + \lambda_2 L(q) + (\lambda_2 L(q))^q$, using (76) for Du_p, (80) for Du_c. We thus arrived at (96). $\qquad \square$

7. Proof of the Main Proposition 1

Having at hand the estimates of the previous section, we are ready to show that (u_1, q_1, R_1) constructed in Sect. 5 satisfy the inequalities (13a)–(14).

The estimates of the previous section have at their right-hand sides two type of terms: ones where the parameters $\lambda_2, \lambda_1, \mu, \omega$ are intertwined, and the remaining ones. These remaining ones can be made small simply by choosing the relevant parameters large. The terms with $\lambda_2, \lambda_1, \mu, \omega$ interrelated need more care, so let us focus on them. They contain two little technical nuisances: (i) appearance of N and (ii) estimates for some parts of R not holding in L^1. Let us ignore these nuisances for a moment, which is easily acceptable after recalling (i) $\mathcal{R}_\infty$ of Remark 6 (which heuristically cancels the terms involving N) and that (ii) estimates for R hold in L^r for any $r > 1$, whereas an ϵ of room is assured by the assumed sharp inequality $q < \frac{2d}{d+2}$. So for a moment let us consider estimates of Sect. 6 allowing for $|R|_1$ and disregarding the terms with N. After inspection, we see that smallness of their right-hand sides where $\lambda_2, \lambda_1, \mu, \omega$ are intertwined, needed for Proposition 1 is precisely the smallness of (40), Remark 7. Therefore we will proceed as follows.

Firstly, guided by (40), we will choose relation between magnitudes of $\lambda_2, \lambda_1, \mu, \omega$. To this end we postulate

$$\lambda_1 := \lambda, \quad \mu := \lambda^a, \quad \omega := \lambda^b, \quad \lambda_2 := \lambda^c, \tag{103}$$

and choosing relation between magnitudes means picking a, b, c so that (40) are strictly decreasing in λ.

Secondly, we will need to make sure that when $r > 1$ and N appear, the relations between magnitudes do not change. This will be achieved by choosing N large and r small in relation to a, b, c.

Finally, we will send $\lambda \to \infty$ to reach (13a)–(14).

7.1. Picking magnitudes a, b, c. The requirement that powers in (40) rewritten in terms of (103) are negative reads

$$
\begin{aligned}
\frac{\lambda_1 \mu}{\lambda_2} &= \lambda^{1+a-c} & i.e. && 1+a-c &< 0, \\[2mm]
\frac{\lambda_1 \mu}{\lambda_2} \cdot \frac{\omega}{\mu^{d/2}} &= \lambda^{1+(1-\frac{d}{2})a-c+b} & i.e. && 1+\left(1-\frac{d}{2}\right)a - c + b &< 0, \\[2mm]
\frac{\mu^{d/2}}{\omega} &= \lambda^{ad/2-b} & i.e. && \frac{d}{2}a - b &< 0, \\[2mm]
\lambda_2 \mu^{\frac{d}{2}-\frac{d}{q}} &= \lambda^{c+(\frac{d}{2}-\frac{d}{q})a} & i.e. && c - \left(\frac{d}{q} - \frac{d}{2}\right)a &< 0.
\end{aligned}
\tag{104}
$$

These conditions on a, b, c can be simultaneously achieved as follows.

(1) The conditions not involving b amount to the requirement

$$
1 + a < c < \left(\frac{d}{q} - \frac{d}{2}\right)a.
\tag{105}
$$

From the assumption $q < \frac{2d}{d+2}$ of Proposition 1 it follows that $d/q - d/2 > 1$. Therefore satisfying (105) is possible with a large. More precisely, let us pick

$$
a > \frac{3}{d\left(\frac{1}{q} - \frac{d+2}{2d}\right)}.
\tag{106}
$$

Then between $1 + a$ and $(d/q - d/2)a$ there are at least two natural numbers. We then fix $c \in \mathbb{N}$ as the largest natural number satisfying (105). Notice that there is still at least one natural number between $1 + a$ and c.

(2) Let us fix $b \in \mathbb{N}$ so that

$$
\frac{d}{2}a < b < \left(\frac{d}{2} - 1\right)a + c - 1.
\tag{107}
$$

This is possible, because, as observed in point (1), there is at least one natural number between $1+a$ and c and thus also between $ad/2$ and $(d/2-1)a+c-1$. The condition (107) automatically verifies the two conditions concerning b.

Let us denote by $-\zeta < 0$ the largest power of those appearing in (104). We have just showed

$$
\frac{\mu^{d/2}}{\omega} \leq \lambda^{-\zeta}, \quad \lambda_2 \frac{\mu^{d-\frac{d}{q}}}{\omega} \leq \lambda^{-\zeta}, \quad \frac{\lambda_1 \mu}{\lambda_2} \leq \lambda^{-\zeta}, \quad \lambda_2 \mu^{\frac{d}{2}-\frac{d}{q}} \leq \lambda^{-\zeta}, \quad \frac{\lambda_1 \mu}{\lambda_2} \cdot \frac{\omega}{\mu^{d/2}} \leq \lambda^{-\zeta}.
\tag{108}
$$

7.2. Fixing N and $r_0 > 1$. Let us fix $N \in \mathbb{N}$ so that

$$c - \zeta N \leq 0. \tag{109}$$

This choice yields

$$1 + \lambda_2^2 \left(\frac{\lambda_1 \mu}{\lambda_2}\right)^N \leq 1 + \lambda^{2c-2\zeta N} \leq 2. \tag{110}$$

Using the definition (79) of L with (108) and (110), one has

$$L(q) \leq C\lambda^{-\zeta}, \quad L(2) \leq C\lambda^{-\zeta}, \quad \lambda_2 L(q) \leq C\lambda^{-\zeta}. \tag{111}$$

Importantly, fixing the gauge N freezes all C_N's in estimates to C.

Let us fix also an exponent $r_0 \in (1, \infty)$ (close to 1) such that

$$\left(d - \frac{d}{r_0}\right) a \leq \frac{1}{2}. \tag{112}$$

This is possible because the l.h.s. above vanishes as $r_0 \to 1$.

7.3. Obtaining (13a)–(14). Recall that δ, η are given small numbers. Since $u_1 - u_0 = u_p + u_c$, we have by (77) and (80)

$$|(u_1 - u_0)(t)|_{L^2} \leq M\delta^{\frac{1}{2}} + C\left(\lambda_1^{-1/2} + L(2)\right) \leq M\delta^{\frac{1}{2}} + C\left(\lambda^{-1/2} + \lambda^{-\zeta}\right), \tag{113}$$

recalling for the latter inequality that $\lambda_1 = \lambda$ via (103), and (111). Choose $\lambda \in \mathbb{N}$ large in relation to C we thus have

$$|(u_1 - u_0)(t)|_{L^2} \leq 2M\delta^{\frac{1}{2}} = M_0 \delta^{\frac{1}{2}},$$

defining $M_0 := \frac{M}{2}$, hence (13a). Notice that M_0 depends only on the universal constant M. Thus M_0 itself is universal, i.e. it may depend on the quantities (70), but not on the quantities (71).

Similarly to obtaining (113), using (76), (79) and (80) we have

$$|u_1 - u_0|_{W^{1,q}} \leq C\left(\lambda_2 \mu^{d/2-d/q} + L(q) + \lambda_2 L(q)\right) \leq C\lambda^{-\zeta}, \tag{114}$$

where for the term $\lambda_2 \mu^{d/2-d/q}$ we used (108). Estimate (13b) follows by choosing λ big enough.

Recall that $R_1 = -(R_{lin} + R_{corr} + R_{quadr} + R_{moll} + R_{\mathcal{A}})$ by its definition (67). By (87) we have, with C_N now fixed to C by the choice (109) of N

$$|R_{lin}|_1(t) \leq C\left[\mu^{-\frac{d}{2}} + \frac{\lambda_1 \mu^{1-\frac{d}{2}}\omega}{\lambda_2}\left(1 + \lambda_2^2 \left(\frac{\lambda_1 \mu}{\lambda_2}\right)^N\right)\right] \leq C(\lambda^{-ad/2} + \lambda^{-\zeta}2), \tag{115}$$

where for the second inequality we invoked $\mu = \lambda^a$ by (103), (108), and (110).

Similarly, using (92) and (111)

$$|R_{corr}(t)|_1 \leq C\lambda^{-\zeta}. \tag{116}$$

For the L^1-estimate of R_{quadr} we need to switch to the L^{r_0} estimate, where r_0 was fixed in (112). We have, using (83)

$$|R_{quadr}(t)|_1 \leq |R_{quadr}(t)|_{r_0} \leq C(\omega^{-1} + \lambda_1^{-1})\mu^{d-\frac{d}{r_0}} + C\lambda^{-1}\lambda_1\mu$$
$$\leq C(\lambda^{(d-d/r_0)a-b} + \lambda^{(d-d/r_0)a-1}) + C\lambda^{-\zeta}.$$

Thanks to the choice of r_0 in (112), we hence have

$$|R_{quadr}(t)|_1 \leq C\lambda^{-\frac{1}{2}} + C\lambda^{-\zeta}. \tag{117}$$

Similarly, for the estimate of $R_{\mathcal{A}}$ we use (93) with $q \in (1, 2)$, obtaining

$$|R_{\mathcal{A}}(t)|_1 \leq |R_{\mathcal{A}}(t)|_q \leq \begin{cases} C\left(\left(\lambda_2\mu^{\frac{d}{2}-\frac{d}{q}}\right)^{q-1} + \lambda_2 L(q)\right)^{q-1} & \text{for } \nu_0 = 0, q < 2, \\ C\left(\lambda_2\mu^{\frac{d}{2}-\frac{d}{q}} + \lambda_2 L(q)\right) & \text{for } \nu_0 > 0, q < 2. \end{cases}$$

Therefore by (108), (111) and $q - 1 \in (0, 1)$

$$|R_{\mathcal{A}}(t)|_1 \leq C\lambda^{-\zeta(q-1)}. \tag{118}$$

Together, the terms $R_{lin}, R_{corr}, R_{quadr}, R_{\mathcal{A}}$ are bounded in view of, respectively, (115), (116), (117), and (118) by $C\lambda^{-\zeta'}$ with certain $\zeta' > 0$:

$$|R_{lin}(t)|_1 + |R_{corr}(t)|_1 + |R_{quadr}(t)|_1 + |R_{\mathcal{A}}(t)|_1 \leq C\lambda^{-\zeta'}$$

Therefore, using for the remaining R_{moll} the estimate (94), we have

$$|R_1(t)|_1 \leq \frac{\eta}{2} + C\lambda^{-\zeta'}. \tag{119}$$

thus showing (13c) by taking λ large.

Let us show the last remaining inequality (14). By (96), with C_N fixed to C by the choice (109) of N, we have in view of (108) and (111)

$$\delta E(t) \leq \frac{\delta}{16}e(t) + C\left(\lambda^{-1} + \lambda^{-\frac{ad}{2}} + \lambda^{-\zeta} + \lambda^{-2\zeta} + \lambda^{-\zeta} + \lambda^{-q\zeta}\right) \leq \frac{\delta}{16}e(t) + \frac{\delta}{32} \leq \frac{\delta}{8}e(t) \tag{120}$$

The proof of Proposition 1 is concluded.

8. Proof of Theorem A

We will iterate Proposition 1. Let us start at the trivial solution $(u_0, \pi_0, R_0) \equiv 0$ with $\delta_0 = 1$. At the nth step we take $\delta_n := 2^{-n}$ and $\eta_n := \frac{\delta_{n+1}}{2^8 d}$, hence $|R_{n+1}(t)|_{L^1} \leq \eta_{n+1} = \frac{1}{2}\frac{\delta_{n+1}}{2^7 d}$. This and $|\mathring{A}| \leq |A| + \frac{1}{d}|tr A||\text{Id}| \leq |A| + \frac{1}{d}\sqrt{d}|A|\sqrt{d}$ give $|\mathring{R}_{n+1}(t)|_{L^1} \leq \frac{\delta_{n+1}}{2^7 d}$, which is the assumption (12) of the step $n + 1$. Similarly, for any $t \in [0, 1]$ at the n-th step we get, by (14)

$$\frac{3}{8}\delta_n e(t) \leq e(t) - \left(\int |u_{n+1}|^2(t) + 2\int_0^t \int \mathcal{A}(Du_{n+1})Du_{n+1}\right) \leq \frac{5}{8}\delta_n e(t)$$

which is the assumption (11) of the step $n + 1$, since

$$\frac{3}{8}\delta_n e(t) = \frac{3}{4}\delta_{n+1} e(t), \qquad \frac{5}{8}\delta_n e(t) = \frac{5}{4}\delta_{n+1} e(t).$$

Consequently we obtain iteratively, as $\eta_n \leq 2^{-n}$,

$$\sup_{t\in[0,1]} |(u_{n+1} - u_n)(t)|_{L_2} \leq M_0 2^{-n/2},$$

$$\sup_{t\in[0,1]} |(u_{n+1} - u_n)(t)|_{W^{1,q}} \leq 2^{-(n+1)}, \qquad (121)$$

$$\sup_{t\in[0,1]} |R_{n+1}(t)|_{L_1} \leq 2^{-(n+1)}$$

Inequalities (121) mean that $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $C(L^2) \cap C(W^{1,q})$. Denote its limit by $v \in C(L^2) \cap C(W^{1,q})$. Send $n \to \infty$ in the distributional formulation of (10). In particular, in order to pass to the limit in the dissipative term, take a test function φ and use (15) for $q < \frac{2d}{d+2} < 2$ and the Hölder inequality to obtain

$$\left| \int_0^t \int [\mathcal{A}(Du_n) - \mathcal{A}(Dv)] \nabla\varphi \, dx dt \right|$$

$$\leq C \begin{cases} |\nabla\varphi|_{L_t^1 L_x^q} \displaystyle\sup_{t\in[0,1]} |Du_n - Dv|_{L^q}^{q-1}(t) & \text{for } v_0 = 0, \\[2ex] |\nabla\varphi|_{L_t^1 L_x^{q'}} \displaystyle\sup_{t\in[0,1]} |Du_n - Dv|_{L^q}(t) & \text{for } v_0 > 0. \end{cases}$$

The right-hand sides tend to 0 as $n \to \infty$ thanks to (121). Consequently we see that v satisfies the distributional formulation of (1).

For the $2 \int_0^t \int \mathcal{A}(Du_n) Du_n$ term of energy we use (16) to write

$$\int_0^t \int |\mathcal{A}(Du_n) Du_n - \mathcal{A}(Dv) Dv| \leq C \int_0^t \int \left(1 + |Du_n|^{q-1} + |Dv|^{q-1}\right)|Du_n - Dv|.$$

which via Hölder inequality and (121) allows to pass with $n \to \infty$. This and $\lim_{n\to\infty} |u_n - v|_{C(L^2)} = 0$ provided by (121) yields (7).

Let us now focus on proving the last part of Theorem A, i.e. the non-uniqeness statement. Let us take the two energy profiles e_1, e_2 and the respective triples (u_n^1, π_n^1, R_n^1) and (u_n^2, π_n^2, R_n^2) of our convex integration scheme (in what follows, superscripts denote the cases of e_1, e_2, respectively). At each iteration step $n \to n + 1$ one picks value of λ_n^i (= λ of Sect. 7.3) that works for (u_n^i, π_n^i, R_n^i). Observe that choosing $\bar{\lambda}_n = \max(\lambda_n^1, \lambda_n^2)$ works simultaneously for both triples. Thus, without renaming the triples, let us make the choice $\bar{\lambda}_n$ for both (u_n^i, π_n^i, R_n^i), $i = 1, 2$. It results in using identical Mikado flows W^k for both iterations.

Now we want to inductively argue that, thanks to the assumed $e_1(t) = e_2(t)$ for $t \in [0, T]$, it holds $u_n^1(t) = u_n^2(t)$ for every n and $t \in [0, T - \frac{1}{2^7 d}]$. Let us assume thence that $u_n^1(t) = u_n^2(t)$ and $\mathring{R}_n^1(t) = \mathring{R}_n^2(t)$ for times $t \in [0, T - \sum_{i=0}^{n} \frac{2^{-i}}{2^7 d}]$ (This holds for $n = 0$, since we begin with the zero triple). Formula (47), with (44) and (45) shows that $a_{k,n+1}^i(t)$ (i.e. every $a_k^i(t)$, $k \in K$ at the step $n \to n + 1$) depends on $e(t)$, $\mathring{R}_n^{i,\epsilon}(t)$ and $u_{n|[0,t]}^i$, with $\epsilon \leq \frac{2^{-(n+1)}}{2^7 d}$ being the mollification parameter, cf. (69) with the

choice $\delta_{n+1} := 2^{-(n+1)}$, and the u_n^i-dependence being nonlocal due to the dissipative term in (45). So by our inductive assumption we see that $a_{k,n+1}^1(t) = a_{k,n+1}^2(t)$ for times $t \in [0, T - \sum_{i=0}^{n+1} \frac{2^{-i}}{2^7 d}]$. Consequently, via the definition (49), the principal perturbations $u_p^i(t)$, $i = 1, 2$ at the step $n \to n+1$ are identical for times $t \in [0, T - \sum_{i=0}^{n+1} \frac{2^{-i}}{2^7 d}]$. Therefore $u_{n+1}^1(t) = u_{n+1}^2(t)$ and $\mathring{R}_{n+1}^1(t) = \mathring{R}_{n+1}^2(t)$ for $t \in [0, T - \sum_{i=0}^{n+1} \frac{2^{-i}}{2^7 d}]$, since the correctors and the new errors are defined pointwisely in time.

Under the assumption that e_1, e_2 are identical on $[0, T]$, we produced iteratively $u_n^1(t), u_n^2(t)$ that agree for $t \in [0, T - \frac{1}{2^7 d}]$ thus also their limits satisfy $v^1(t) \equiv v^2(t)$ for $t \in [0, T - \frac{1}{2^7 d}]$.

Replacing $T - \frac{1}{2^7 d}$ with any fixed number T_1 strictly smaller than T requires only mollifying at the scales below $T - T_1$ instead of $\frac{1}{2^7 d}$.

9. Sketch of the Proof of Theorem B

Let us indicate changes needed in proofs of Proposition 1 and Theorem A to reach Theorem . Now, we extend the allowed range of growths of $\mathcal{A}$ to $q \in (1, \frac{3d+2}{d+2})$ at the cost of abandoning the control over the dissipative term $2 \int_0^t \int \mathcal{A}(Dv) Dv$ of the energy. Recall that $r \in (\max\{1, q-1\}, \frac{2d}{d+2})$ is an additional exponent, fixed in the assumptions of Theorem B. The main observation is that

$$q - 1 \leq r < \frac{2d}{d+2}$$

is subcritical in the sense of choices made in Sect. 7.1.

Let us first consider modifications in proof Proposition 1. Replacing in Sects. 7.1, 7.2 q of Proposition 1 with r implies the following analogue of (111):

$$L(r) \leq C\lambda^{-\zeta}, \quad L(2) \leq C\lambda^{-\zeta}, \quad \lambda_2 L(r) \leq C\lambda^{-\zeta}.$$

for some positive $\zeta > 0$. Consequently, (114) holds now also with r in place of q. Next, since q now may exceed 2, to control $|R_{\mathcal{A}}(t)|_1$ we use the entire (93) to write

$$|R_{\mathcal{A}}(t)|_1 \leq \begin{cases} |R_{\mathcal{A}}(t)|_r \leq C\left(\left(\lambda_2 \mu^{\frac{d}{2}-\frac{d}{r}}\right)^{q-1} + (\lambda_2 L(r))^{q-1}\right) & \text{for } v_0 = 0, \ q \leq 2, \\[2mm] |R_{\mathcal{A}}(t)|_r \leq C\left(\lambda_2 \mu^{\frac{d}{2}-\frac{d}{r}} + \lambda_2 L(r)\right) & \text{for } v_0 > 0, \ q \leq 2, \\[2mm] |R_{\mathcal{A}}(t)|_1 \leq C\left(\lambda_2 \mu^{\frac{d}{2}-\frac{d}{q-1}} + \left(\lambda_2 \mu^{\frac{d}{2}-\frac{d}{q-1}}\right)^{q-1}\right. \\[2mm] \qquad \left. + \lambda_2 L(q-1) + (\lambda_2 L(q-1))^{q-1}\right) & \text{for } q > 2. \end{cases}$$

In any of these cases, right-hand sides are controlled by powers of $\lambda_2 \mu^{\frac{d}{2}-\frac{d}{r}}$ and $\lambda_2 L(r)$, therefore we can reach (119). Finally, since we abandon the control over the dissipative term in the energy inequality, (45) and δE of (95) (let us rename it to $\delta \tilde{E}$) loose their dissipative terms. The consequence of the latter is that (96) simplifies to

$$\delta \tilde{E}(t) \leq \frac{\delta}{16} e(t) + C_N\left(\lambda_1^{-1} + \mu^{-\frac{d}{2}} + L(2) + L(2)^2\right). \tag{122}$$

Inequality (122) allows to prove (120) for $\delta \tilde{E}$ as in Sect. 7.

The above modifications allow to prove Theorem B along Sect. 8 with the difference that now $\{u_n\}_{n=0}^{\infty}$ forms a Cauchy sequence in $C(L^2) \cap C(W^{1,r})$ and, in the case $q > 2$, we pass to the limit in the dissipative term via (15) for $q \geq 2$ and the Hölder inequality that give

$$\left| \int_0^t \int [\mathcal{A}(Du_n) - \mathcal{A}(Dv)] \nabla \varphi \right| \leq C |Du_n - Dv|_{L^{q-1}} \left(1 + |Du_n|_{L^{q-1}}^{q-2} + |Dv|_{L^{q-1}}^{q-2} \right).$$

10. Sketch of the Proof of Theorem C

Let us first introduce the following modification of Definition 2

Definition 3. Fix $a \in L^2(\mathbb{T}^d)$. A solution to *the Non-Newtonian-Reynolds Cauchy problem* is a triple (u, π, R) where

$$u \in L^\infty(L^2) \cap L^q(W^{1,q}), \quad \pi \in \mathcal{D}, \quad R \in L^1$$

with spatial null-mean u, solving the Cauchy problem

$$\begin{aligned} \partial_t u + \operatorname{div}(u \otimes u) - \operatorname{div}\mathcal{A}(Du) + \nabla \pi &= -\operatorname{div}\mathring{R}, \\ \operatorname{div} u &= 0, \\ u(0) &= a, \end{aligned} \tag{123}$$

in the sense of distributions, where the data are attained in the weak L^2-sense.

The drop of regularity between objects of Definition 3 and objects of Definition 2 stems from a different starting point for our iterations. To prove Theorems A, B, we started the iteration at the smooth triple $(u_0, \pi_0, R_0) = (0, 0, 0)$ and added smooth perturbations in each iteration. To prove Theorem C we will start iterations with $(v_a, \tilde{\pi}_a, -v_a \otimes v_a)$, where $v_a, \tilde{\pi}_a$ solves the Cauchy problem of a non-Newtonian-Stokes system:

$$\begin{aligned} \partial_t v_a - \operatorname{div}\mathcal{A}(Dv_a) + \nabla \tilde{\pi}_a &= 0 \\ \operatorname{div} v_a &= 0 \\ v_a(0) &= a \end{aligned} \tag{124}$$

The smoothness of such $v_a, \tilde{\pi}_s$ is in general false, so even though at each step we again add smooth perturbations, the regularity at each step cannot be better than that of $v_a, \tilde{\pi}_a$ solving (124).

The starting point of our iterations is given by

Proposition 15 (Leray–Hopf solutions for non-Newtonian Stokes). *Fix $a \in L^2(\mathbb{T}^d)$. There is*

$$v_a \in L^\infty(L^2) \cap L^q(W^{1,q}), \quad \tilde{\pi}_a \in \mathcal{D}$$

solving the Cauchy problem (124) *in the sense of distributions, so that* $\lim_{t \to 0} |v_a(t) - a|_2 = 0$ *and*

$$\int_{\mathbb{T}^d} |v_a|^2(t) + 2\int_0^t \int_{\mathbb{T}^d} \mathcal{A}(Dv_a)Dv_a \leq \int_{\mathbb{T}^d} |a|^2.$$

The proof uses monotonicity of $\mathcal{A}$ and for strong attainment of the initial datum, the energy inequality.

The main ingredient of proof of Theorem C is a version of Proposition 1 tailored to deal with the Cauchy problem.

Roughly speaking, given a solution to the Non-Newtonian-Reynolds system (123), which assume the given initial datum, we construct another solution to (123) *with the same initial datum*, and with a well-controlled Reynolds stress. The price we pay to keep the initial datum intact is growth of energy. More precisely, energy of the ultimately produced solution to the Cauchy problem with datum a for (1) is *much above* an energy of a non-Newtonian Stokes emanating from the same a. Hence we cannot reach energy inequality, even for merely the kinetic energy in the range $q < \frac{2d}{d+2}$. This is why we do not distinguish the subcompact range $q < \frac{2d}{d+2}$ in Theorem C.

We are ready to state

Proposition 16. *Let $v_0, v_1 \geq 0$ and $q < \frac{3d+2}{d+2}$, $r \in (\max\{1, q-1\}, \frac{2d}{d+2})$ be fixed. Fix an arbitrary nonzero initial datum $a \in L^2(\mathbb{T}^d)$, $\operatorname{div} a = 0$. There exist a constant M such the following holds.*

Let (u_0, π_0, R_0) be a solution to the Non-Newtonian-Reynolds Cauchy problem with datum a. Let us choose any $\delta, \eta, \sigma \in (0, 1]$ and $\gamma > 0$. Assume that

$$|\mathring{R}_0(t)|_{L^1} \leq \frac{\delta}{2^7 d} \quad \text{for all } t \in [2\sigma, 1]. \tag{125}$$

Then, there is a solution (u_1, π_1, R_1) to Non-Newtonian-Reynolds Cauchy problem with same datum a such that

$$|(u_1 - u_0)(t)|_{L^2} \leq \begin{cases} M\delta^{\frac{1}{2}} + M\gamma^{\frac{1}{2}} & t \in [4\sigma, 1], \\ M(\delta + \sup_{\tau \in (t-\sigma/4, t+\sigma/4)} |\mathring{R}_0(\tau)|_1 + \gamma)^{\frac{1}{2}} & t \in [\sigma/2, 4\sigma], \\ 0 & t \in [0, \sigma/2], \end{cases} \tag{126a}$$

$$|(u_1 - u_0)(t)|_{W^{1,r}} \leq \begin{cases} \eta \quad \text{for all } t \in [0, 1], \\ 0 \quad t \in [0, \frac{\sigma}{2}]. \end{cases} \tag{126b}$$

$$|R_1(t)|_{L^1} \leq \begin{cases} \eta, & t \in [\sigma, 1], \\ |R_0(t)|_{L^1} + \eta, & t \in [\sigma/2, \sigma], \\ |R_0(t)|_{L^1}, & t \in [0, \sigma/2]. \end{cases} \tag{126c}$$

and

$$\left| |u_1|_2^2 - |u_0|_2^2 - d\gamma \right|(t) \leq \frac{\delta}{2^4} \quad t \in [4\sigma, 1], \tag{127}$$

Proof. (Sketch of the proof of Proposition 16) Let us indicate the changes we need to make in the proof of Proposition 1.

The constant γ will be used instead of the energy pump $\gamma_0(t)$ of (45). This changes (44) and gives

$$\varrho(x, t) \leq 2\epsilon + 2|\mathring{R}_0^\epsilon(x, t)| + \gamma$$

and thus alters (73) to

$$|a_k(t)|_2 \leq 2(\delta + |\mathring{R}_0(t)|_1 + \gamma)^{\frac{1}{2}}. \tag{128}$$

Define u_p, u_c by (49), (57) respectively (with the new ϱ). Let us introduce a smooth cutoff

$$\chi(t) \begin{cases} = 0 & t \leq \sigma/2, \\ \in [0, 1] & t \in (\sigma/2, \sigma) \\ = 1, & t \geq \sigma. \end{cases}$$

and define the perturbations $\tilde{u}_p$, $\tilde{u}_c$ as follows

$$\tilde{u}_p(t) := \chi(t)u_p(t), \qquad \tilde{u}_c(t) := \chi^2(t)u_c(t).$$

Due to (128), the new version of (77) reads

$$|u_p(t)|_2 \leq M(\delta + |\mathring{R}_0(t)|_1 + \gamma)^{1/2} + \lambda_1^{-\frac{1}{2}} C. \tag{129}$$

Since (125) holds only on $[2\sigma, 1]$, and recalling the fact that $\mathring{R}_0^\epsilon$ is the mollification in space *and* time of $\mathring{R}_0$, we can follow the lines of Proposition 1 only on $[4\sigma, 1]$. This gives the first line of (126a). Concerning the case $[\sigma/2, 4\sigma]$ of (126a) let us compute, using (128)

$$|\tilde{u}_p(t)|_{L^2}^2 \leq |u_p(t)|_{L^2}^2 \leq M(\delta + |\mathring{R}_0(t)|_1 + \gamma)$$

Our assumption now does not control $\mathring{R}_0^\epsilon(t)$ for $t \leq 2\sigma$, but we can always write, choosing $\epsilon \ll \sigma$

$$|\tilde{u}_p(t)|_{L^2}^2 \leq M\left(\delta + \sup_{\tau \in (t-\sigma/4, t+\sigma/4)} |\mathring{R}_0(\tau)|_1 + \gamma\right),$$

which, together with the smallness of $|\tilde{u}_c(t)|_{L^2}$, cf. (80), gives the case $[\sigma/2, 4\sigma]$ of (126a). On $[0, \sigma/2]$, it holds $u_1 = u_0$, as there $\chi \equiv 0$, thence the respective part of (126a).

The estimate (126b) holds on the whole time interval, because the Mikado flows are small in $W^{1,r}$ for $r < \frac{2d}{d+2}$ by construction.

The estimate on the new Reynolds stress (126c) on $[\sigma, 1]$ is analogue to the corresponding estimate (13c) of Proposition 1, as on $[\sigma, 1]$ the time cutoff $\chi \equiv 1$. The estimate on $[0, \sigma/2]$ is trivially satisfied, as on this time interval the cutoff $\chi \equiv 0$ (here $u_0 = u_1$, so $R_0 = R_1$). On the intermediate time interval $[\sigma/2, \sigma]$, R_0 is decomposed as $R_0 = R_0(1 - \chi^2) + R_0\chi^2$. The term $R_0\chi^2$ is canceled by $\tilde{u}_p \otimes \tilde{u}_p = \chi^2(u_p \otimes u_p)$, as in the proof of Proposition 1, thus giving the η in the second line of (126c), whereas the term $R_0(1 - \chi^2)$ is responsible for $|R_0(t)|_{L^1}$ in the second line of (126c). There is, however, in the new Reynolds stress R_1 an additional term coming from the time derivative of the cutoff:

$$\text{div } R_{cutoff}(t) := \chi'(t)u_p(t) + (\chi^2)'(t)u_c(t). \tag{130}$$

For the first term in (130) we just use (75) with $r = 1$:

$$|\text{div}^{-1}(\chi'(t)u_p(t))|_1 \leq |\chi'(t)u_p(t)|_1 \leq \frac{C}{\sigma}\mu^{-d/2} \leq \frac{C}{\sigma}\lambda^{-ad/2},$$

where we used the choice $\mu = \lambda^a$, as in Sect. 7. For the second term in (130) we have, invoking (19)

$$|\text{div}^{-1}\big((\chi^2)'(t)u_c(t)\big)|_1 \leq |\text{div}^{-1}\big((\chi^2)'(t)u_c(t)\big)|_2 \leq |(\chi^2)'(t)u_c(t)|_2 \leq \frac{C}{\sigma}L(2)$$

Since by (111) $L(2) \leq C\lambda^{-\zeta}$, both terms of (130) are estimated by negative powers of λ. Thus they can be made as small as we wish by picking λ big enough.

Let us now justify (127) along the proof of Proposition 14. In (98) γ_0 changes to γ. Now we do not control $|\mathring{R}_0^\epsilon(t)|_1$ on the entire time interval $[0, 1]$, only on $[4\sigma, 1]$ via (125). On this interval $\chi = 1$ and hence for any $t \in [4\sigma, 1]$ one has the following counterpart of (102)

$$\left| |\tilde{u}_p|_2^2 - d\gamma \right| \leq \frac{\delta}{2^5} + C\Big(\frac{1}{\lambda_1} + \frac{\lambda_1\mu}{\lambda_2}\Big). \tag{131}$$

We use now $u_1 = u_0 + \tilde{u}_p + \tilde{u}_c$ to write for any $t \in [4\sigma, 1]$

$$\left| |u_1|_2^2 - |u_0|_2^2 - d\gamma \right| \leq \frac{\delta}{2^5} + C\Big(\frac{1}{\lambda_1} + \frac{\lambda_1\mu}{\lambda_2}\Big) + \left| |\tilde{u}_c|_2^2 + 2\int(u_0\tilde{u}_c + u_0\tilde{u}_p + \tilde{u}_p\tilde{u}_c)\right|. \tag{132}$$

The r.h.s. above can be made arbitrarily small in view of (131) and arguments analogous to that leading to Proposition 1, which yields (127).   □

Iterating Proposition 16, we can now complete the proof of Theorem C. Let us choose $\sigma_n := 2^{-n}$ along the iteration. We will choose $\delta_n = 2^{-n}$ and $\eta_n = 2^{-(n+9)}d^{-1}$ as in proofs of Theorems A, B. There are two main differences between the current iterations and the iterations leading to Theorems A, B. Firstly, we initiate the iterations with the triple $(v_a, \tilde{\pi}_a, -v_a \otimes v_a)$, where $v_a, \tilde{\pi}_a$ are given by Proposition 15. Secondly, we will choose the now additional free parameter γ just to distinguish between different solutions. Namely, let us choose $\gamma_n = d^{-1}\delta_n$ *except for* γ_3, which we require it to be a large constant, say K.

The condition (125) for the initial triple is void (empty interval where it shall hold) and over iterations it is satisfied thanks to the first case of (126c) and our choices for $\eta_n, \delta_n, \sigma_n$. The third iteration produces u_3 out of u_2 such that

$$\left| |u_3|_2^2 - |u_2|_2^2 - dK\right|(t) \leq 2^{-7} \qquad t \in [1/2, 1].$$

At this step the energies of the iteratively produced solutions branch: choosing two K's that considerably differ, we will see that the kinetic energies on $t \in [1/2, 1]$ of the finally produced solutions differ considerably.

From step $n = 4$ onwards $\gamma_n = \delta_n$, thus the first line of (126a) is analogous to (13a). Iterating Proposition 16 we thus obtain convergence of the sequence $\{u_n - v_a\}_n$ to some

$$v_\infty \in C((0, 1]; L^2(\mathbb{T}^d)) \cap C([0, 1]; W^{1,r}(\mathbb{T}^d)).$$

Note the open side of an interval above. Taking into account the regularity class of v_a and $r < q$, we thus have

$$u_n \to v_a + v_\infty := v \qquad \text{strongly in} \quad L^\infty((0, 1); L^2(\mathbb{T}^d)) \cap L^r((0, 1); W^{1,r}(\mathbb{T}^d)),$$

which allows to pass to the limit in the distributional formulation of (124), since by choice $r > \max(1, q - 1)$.

Concerning the attainment of the initial datum a, for any $q_0 < 2$ the estimate (126b) yields $|v_\infty|_{q_0}(t) \to 0$ as $t \to 0$. Therefore $v = v_\infty + v_a$ satisfies $|v - a|_{q_0}(t) \to 0$ as $t \to 0$. From this and the fact that the L^2 norm of $v(t)$ is uniformly bounded in time on $[0, 1]$, it follows that $v(t) \rightharpoonup a$ weakly in L^2 as $t \to 0^+$.

Let us finally argue for multiplicity of solutions. At the step $n = 3$ let us choose two different K, K'. Let us distinguish the resulting u_n's and their limits v by, respectively, u_n and u'_n, and v and v'. On $t \in [1/2, 1]$ (127) yields for $n \geq 4$

$$\left| |u_n|_2^2 - |u_{n-1}|_2^2 \right|(t) \leq 2^{-(4+n)} + 2^{-n} \qquad \left| |u'_n|_2^2 - |u'_{n-1}|_2^2 \right|(t) \leq 2^{-(4+n)} + 2^{-n},$$

whereas

$$\left| |u_3|_2^2 - |u_2|_2^2 - dK \right|(t) \leq 2^{-7} \qquad \left| |u'_3|_2^2 - |u'_2|_2^2 - dK' \right|(t) \leq 2^{-7}$$

So, since $u_2 = u'_2$

$$\left| |u_n|_2^2 - |u_2|_2^2 - dK \right|(t) \leq 1/2 \qquad \left| |u'_n|_2^2 - |u_2|_2^2 - dK' \right|(t) \leq 1/2.$$

The same inequalities hold for the strong limits v, v'. Therefore, for $d|K - K'| > 1$ they must differ.

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11. Appendix

11.1. Proof of Lemma 1. Let us first consider a scalar function $f : \mathbb{R} \to \mathbb{R}$

$$f(t) := (v_0 + v_1|t|)^{q-2} t.$$

It is Lipschitz for $v_1 = 0$. In the range $q \in (1, 2]$ f is $(q - 1)$-Hölder continuous for $v_0 = 0, v_1 > 0$; and locally Lipschitz for $v_0 > 0, v_1 > 0$. By the last statement we mean that

$$|f(t) - f(s)| \leq C_q |t - s|(v_0 + v_1(|t| + |s|))^{q-2}. \tag{133}$$

It is proven by writing

$$|f(t) - f(s)| \leq |q - 1||t - s| \int_0^1 (v_0 + v_1|s + x(t - s)|)^{q-2} dx$$

$$\leq |q - 1||t - s|(v_0 + v_1(|t| + |s|))^{q-1-\delta} \int_0^1 (v_0 + v_1|s + x(t - s)|)^{\delta-1} dx, \tag{134}$$

with the second inequality given by splitting $q - 2 = (q - 1 - \delta) + (\delta - 1)$ so that $q - 1 - \delta > 0$ and thus the function $t \to t^{q-1-\delta}$ is increasing. For the integral in the second line of (134) we use that for $\gamma \in (-1, 0]$ and $|t| + |s| > 0$ it holds

$$\int_0^1 (v_0 + v_1|s + x(t - s)|)^{\delta-1} dx \leq C_\gamma (v_0 + v_1(|t| + |s|))^\gamma. \tag{135}$$

Computation for (135) is contained in proof of Lemma 2.1 in [1]. Using (135) in (134) we obtain (133).

In the range $q \geq 2$ for f, the function $t \to t^{q-2}$ is increasing and integrable at 0, so (133) holds for any $v_0 \geq 0$, $v_1 \geq 0$. Altogether

$$|f(t) - f(s)| \leq \begin{cases} C_{v_1} |t - s|^{q-1} & \text{for } v_0 = 0, q < 2 \\ C_{v_0} |t - s| & \text{for } v_0 > 0, q < 2 \\ C_{q,v_0,v_1} |t - s| \left(1 + |t|^{q-2} + |s|^{q-2}\right) & \text{for } q \geq 2 \end{cases} \tag{136}$$

Replacing scalars with tensors in (136) will give (15). Details follow.

Let us consider first the (global Hölder) case $v_0 = 0$, $q < 2$, i.e. the first line of (136). We show the respective first line of (15) by considering two cases: (i) Q, P lie along a line passing through the origin and (ii) Q, P lie on a sphere centered at the origin. The case (i) is $Q = tQ_0$, $P = sQ_0$ for some $s, t \in \mathbb{R}$ and $|Q_0| = 1$, thus (15) here follows immediately from (136). The case (ii) is $|Q| = |P| = l$ for some $l > 0$. Here

$$\mathcal{A}(Q) = Q(v_1 l)^{q-2}, \quad \mathcal{A}(P) = P(v_1 l)^{q-2},$$

so

$$\frac{|\mathcal{A}(Q) - \mathcal{A}(P)|}{|Q - P|^{q-1}} = \frac{|Q - P|}{(v_1 l)^{2-q}|Q - P|^{q-1}} = \left(\frac{|Q - P|}{v_1 l}\right)^{2-q} \leq \left(\frac{2l}{v_1 l}\right)^{2-q}$$

and the latter is a l-independent constant. Both cases (i), (ii) yield for every nonzero Q, P

$$|\mathcal{A}(Q) - \mathcal{A}(P)| \leq \underbrace{\left|\mathcal{A}(Q) - \mathcal{A}\left(\frac{|Q|}{|P|}P\right)\right|}_{\text{arguments lie: on the sphere } \partial B_{|Q|}(0)} + \underbrace{\left|\mathcal{A}\left(\frac{|Q|}{|P|}P\right) - \mathcal{A}(P)\right|}_{\text{on the same line through origin}}$$

$$\leq C_{v_1}\left(\left|Q - \frac{|Q|}{|P|}P\right|^{q-1} + \left|\frac{|Q|}{|P|}P - P\right|^{q-1}\right) \leq C_{v_1}|P - Q|^{q-1}.$$

We have just proven the first $v_0 = 0$, $q < 2$ case of (15).

The second (global Lipschitz) case $v_0 > 0$, $q < 2$, i.e. the tensorial version of the second line of (136), can be proven analogously. But in fact the computation leading to (133) works well when applied immediately to tensor mappings both in the case $v_0 > 0$, $q < 2$ and $q \geq 2$, giving the remainder of (15). Estimate (16) follows from an argument that gave the case $q \geq 2$ in (15), applied to $\tilde{f}(t) = (v_0 + v_1|t|)^{q-2}t^2$.

11.2. Proof of Proposition 4 on antidivergence operators.

(i). $\mathrm{div}^{-1} : C_0^\infty(\mathbb{T}^d; \mathbb{R}^d) \to C_0^\infty(\mathbb{T}^d; \mathcal{S}))$ Let $\Delta^{-1}v$ denote the null-mean solution u to $\Delta u = v$ on $\mathbb{T}^d$. Recall that $Df := \frac{\nabla f + \nabla^T f}{2}$ (symmetric gradient), and recall $P_H f := f - \nabla \Delta^{-1} \mathrm{div}\, f$ (Helmholtz projector). Take

$$\mathrm{div}^{-1}v := D\Delta^{-1}v + DP_H\Delta^{-1}v,$$

which is symmetric, because D is symmetric. (Compare div^{-1} with the inverse divergence $\frac{3}{2}D\Delta^{-1}v + \frac{1}{2}DP_H\Delta^{-1}v - \frac{1}{2}\mathrm{div}\,\Delta^{-1}v\mathrm{Id}$ of Definition 4.2 [18], which is automatically traceless. We use the simpler choice for div^{-1}, since trace zero is provided by a pressure shift.) Since $2\mathrm{div}\,D = \Delta + \nabla\mathrm{div}$, we have for $\Delta u = v$

$$\mathrm{div}\,(Du + DP_Hu) = \frac{1}{2}\left(\Delta u + \nabla\mathrm{div}\,u + \Delta(u - \nabla\Delta^{-1}\mathrm{div}\,u) + \nabla\mathrm{div}\,(u - \nabla\Delta^{-1}\mathrm{div}\,u)\right)$$
$$= \Delta u = v.$$

Estimates (19), (20) follow from arguments analogous to that of [31], proof of Lemma 2.2, so we only sketch them. The estimate (19) for $p > 1$ follows from Calderón-Zygmund theory, suboptimally, because div^{-1} is -1-homogenous. This suboptimality yields the borderline cases. In particular $p = \infty$ holds, since $\nabla\mathrm{div}^{-1}$ is 0-homogenous thus, via Sobolev embedding and Calderón-Zygmund for $\nabla\mathrm{div}^{-1}$ one has $|\mathrm{div}^{-1}f|_\infty \leq C|\nabla\mathrm{div}^{-1}f|_{d+1} \leq C|f|_{d+1} \leq C|f|_\infty$. The other borderline case $p = 1$ follows from the fact that the operator dual to div^{-1} is -1-homogenous and from duality argument. The claim (20) uses (19), -1-homogeneity of div^{-1} that yields

$$\mathrm{div}^{-1}(u_\lambda) = \lambda^{-1}(\mathrm{div}^{-1}u)_\lambda, \tag{137}$$

and that we are on a torus, so the L^p norms remain unchanged under oscillations.

(ii). $\mathcal{R}_N : C^\infty(\mathbb{T}^d; \mathbb{R}) \times C_0^\infty(\mathbb{T}^d; \mathbb{R}^d) \to C_0^\infty(\mathbb{T}^d; \mathcal{S}))$ (Compare [18], Proposition 5.2 and Corollary 5.3.) We construct the two-argument improved symmetric antidivergence iteratively upon div^{-1}. Let us commence with

$$\mathcal{R}_0(f, u) := \mathrm{div}^{-1}\left(fu - \fint fu\right). \tag{138}$$

Our aim is an antidivergence that extracts oscillations of u_λ out of the product fu_λ. Therefore (137) suggests to apply div^{-1} to u_λ, and correct the remainder. Let us thence compute

$$\mathrm{div}\left(f\mathrm{div}^{-1}u\right) = fu + \sum_{k=1}^{d}\partial_k f(\mathrm{div}^{-1}u)e_k \tag{139}$$

and define $\mathcal{R}_1(f, u) := f\mathrm{div}^{-1}u - \sum_{k=1}^{d}\mathcal{R}_0\left(\partial_k f, (\mathrm{div}^{-1}u)e_k\right)$. The choice (138) of $\mathcal{R}_0$ and (139) yield

$$\mathrm{div}\,\mathcal{R}_1(f, u) = fu - \fint fu. \tag{140}$$

Define inductively $\mathcal{R}_N(f, u) := f\mathrm{div}^{-1}u - \sum_{k=1}^{d}\mathcal{R}_{N-1}\left(\partial_k f, (\mathrm{div}^{-1}u)e_k\right)$, per analogiam with the construction $\mathcal{R}_0 \to \mathcal{R}_1$. Using induction over N, one proves now that $\mathcal{R}_N$ is bilinear, symmetric, $\mathrm{div}\,\mathcal{R}_N(f, u) = fu - \fint_{\mathbb{T}^d}fu$, satisfies Leibniz rule:

$$\partial_k\mathcal{R}_N(f, u) = \mathcal{R}_N(f, \partial_k u) + \mathcal{R}_N(\partial_k f, u)$$

and estimates (21), similarly to proof of Lemma 3.5 of [30]. For instance, to show $\operatorname{div} \mathcal{R}_N(f, u) = fu - \fint_{\mathbb{T}^d} fu$ we have (140) as the initial step. Assuming $\operatorname{div} \mathcal{R}_{N-1}(f, u) = fu - \fint_{\mathbb{T}^d} fu$, one has

$$\operatorname{div} \mathcal{R}_N(f, u) = \operatorname{div}(f \operatorname{div}^{-1} u) - \sum_{k=1}^{d} \left(\partial_k f (\operatorname{div}^{-1} u) e_k - \fint \partial_k f \left((\operatorname{div}^{-1} u) e_k \right) \right)$$

$$= fu - \fint fu$$

with the second equality due to (139).

The estimate (21) holds for $N = 1$ since (19) and Jensen imply $|\mathcal{R}_0(f, u)|_p \le C|f|_r |u|_s$ and thus

$$|\mathcal{R}_1(f, u_\lambda)|_p \le C|f|_r |\operatorname{div}^{-1} u_\lambda|_s + \sum_{k=1}^{d} |\mathcal{R}_0\left(\partial_k f, (\operatorname{div}^{-1} u_\lambda) e_k \right)|_p$$

$$\le C|f|_r |\operatorname{div}^{-1} u_\lambda|_s + C|\nabla f|_r |(\operatorname{div}^{-1} u_\lambda)|_s \le C|u|_s \left(\frac{1}{\lambda} |f|_r + \frac{1}{\lambda} |\nabla f|_r \right),$$

with the last inequality due to (20). For the inductive step $N - 1 \to N$, let us compute, using (137)

$$|\mathcal{R}_N(f, u_\lambda)|_p \le C|f|_r |\operatorname{div}^{-1} u_\lambda|_s + \frac{1}{\lambda} \sum_{k=1}^{d} |\mathcal{R}_{N-1}\left(\partial_k f, ((\operatorname{div}^{-1} u)(\lambda \cdot)) e_k \right)|_p$$

$$\le C|u|_s \frac{1}{\lambda} |f|_r + C_{d,p,s,r,N-1} \sum_{k=1}^{d} |\operatorname{div}^{-1} u|_s \frac{1}{\lambda} \left(\frac{1}{\lambda} |\partial_k f|_r + \frac{1}{\lambda^{N-1}} |\nabla^{N-1} \partial_k f|_r \right),$$

with the second inequality valid via the inductive assumption, i.e. (21) for $N - 1$. The estimate (19) and interpolation yield the inductive thesis.

(iii). $\tilde{\mathcal{R}}_N : C^\infty(\mathbb{T}^d; \mathbb{R}^d) \times C_0^\infty(\mathbb{T}^d; \mathbb{R}^{d \times d}) \to C_0^\infty(\mathbb{T}^d; \mathcal{S}))$ Take

$$\tilde{\mathcal{R}}_N(v, T) := \sum_{k=1}^{d} \mathcal{R}_N(v_k, T e_k).$$

Then $\operatorname{div} \tilde{\mathcal{R}}_N(v, T) = \sum_{k=1}^{d} v_k T e_k - \fint_{\mathbb{T}^d} v_k T e_k = Tv - \fint_{\mathbb{T}^d} Tv$ and since it is a linear combination of $\mathcal{R}_N$'s, it retains all its properties.

(iv). $\mathcal{R}_N^2 : C^\infty(\mathbb{T}^d; \mathbb{R}) \times C_0^\infty(\mathbb{T}^d; \mathbb{R}) \to C_0^\infty(\mathbb{T}^d; \mathcal{S}))$ We redo the reasoning of (i)–(iii), starting from the following 'standard double antidivergence' $\operatorname{div}^{-2} : C_0^\infty(\mathbb{T}^d; \mathbb{R}) \to C_0^\infty(\mathbb{T}^d; \mathcal{S})$

$$\operatorname{div}^{-2} v := \nabla^2 \Delta^{-2} v,$$

where ∇^2 is the (symmetric) tensor of second derivatives, and thus $\operatorname{div} \operatorname{div} \operatorname{div}^{-2} f = f$. Analogously as for div^{-1} we have $|\operatorname{div}^{-2} v|_{W^{1,p}} \le |v|_p C_p$. Since it is -2-homogenous, it holds $\operatorname{div}^{-2}(v_\lambda) = \lambda^{-2}(\operatorname{div}^{-2} v)_\lambda$, thus

$$|\nabla^i \operatorname{div}^{-2}(v_\lambda)|_{L^p} \le \lambda^{i-2} |\nabla^i v|_{L^p} C_{k,p}. \tag{141}$$

Upon div^{-2} we build now $\mathcal{R}_N^2$, starting with

$$\mathcal{R}_0^2(a, b) := \mathrm{div}^{-2}\left(ab - \fint ab\right) \tag{142}$$

and the computation

$$\mathrm{div}\,\mathrm{div}\,\left(a\,\mathrm{div}^{-2}b\right) = ab + \sum_{k,l=1}^d 2\partial_l(\mathrm{div}^{-2}b)^{kl}\partial_k a + (\mathrm{div}^{-2}b)^{kl}\partial_{kl}^2 a, \tag{143}$$

where we used symmetry of div^{-2}. Hence let us define by recursion

$$\mathcal{R}_N^2(a, b) := a\,\mathrm{div}^{-2}b - \sum_{k,l=1}^d \left[2\mathcal{R}_{N-1}^2\left(\partial_k a, \partial_l(\mathrm{div}^{-2}b)^{kl}\right) + \mathcal{R}_{N-1}^2\left(\partial_{kl}^2 a, (\mathrm{div}^{-2}b)^{kl}\right)\right].$$

Let us inductively prove that

$$\mathrm{div}\,\mathrm{div}\,\mathcal{R}_N^2(a, b) = ab - \fint ab. \tag{144}$$

The initial statement for $N = 0$ is (142). Assuming (144) for $N - 1$, we use (143) to compute

$$\mathrm{div}\,\mathrm{div}\,\mathcal{R}_N^2(a, b) = ab + \fint \sum_{k,l=1}^d \left(2\partial_k a\,\partial_l(\mathrm{div}^{-2}b)^{kl} + \partial_{kl}^2 a(\mathrm{div}^{-2}b)^{kl}\right).$$

We see again via (143) that the mean value above equals $-\fint ab$.

Via induction over N, one proves that $\mathcal{R}_N^2$ is bilinear, symmetric, and satisfies Leibniz rule. Concerning the estimate (23), let us first prove it for $j = 0$. The proof is by induction on N. For $N = 1$, the estimate is true, since $|\mathcal{R}_0^2(a, b)|_p \leq C|a|_r|b|_s$ and thus

$$|\mathcal{R}_1^2(a, b_\lambda)|_p \leq C|a|_r|\mathrm{div}^{-2}b_\lambda|_s + C|\nabla a|_r|(\nabla\mathrm{div}^{-2}b_\lambda)|_s + C|\nabla^2 a|_r|\mathrm{div}^{-2}b_\lambda|_s$$

$$\leq C|b|_s\left(\frac{1}{\lambda^2}|a|_r + \frac{1}{\lambda}|\nabla a|_r + \frac{1}{\lambda^2}|\nabla^2 a|_r\right)$$

with the last inequality due to (141). For the inductive step $N - 1 \to N$, let us compute using -2-homogeneity of div^{-2}, -1-homogeneity of its derivatives, bilinearity of $\mathcal{R}^2$

$$|\mathcal{R}_N^2(a, b_\lambda)|_p \leq C|b|_s\frac{1}{\lambda^2}|a|_r$$

$$+ \sum_{k,l=1}^d 2\frac{1}{\lambda}|\mathcal{R}_{N-1}^2\left(\partial_k a, \partial_l(\mathrm{div}^{-2}b)_\lambda^{kl}\right)|_p + \frac{1}{\lambda^2}|\mathcal{R}_{N-1}^2\left(\partial_{kl}^2 a, (\mathrm{div}^{-2}b)_\lambda^{kl}\right)|_p,$$

using to the r.h.s. above the $j = 0$ inductive assumption (23) for $N - 1$ and interpolation yields (23) for $j = 0$. Finally, estimate (23) for any $j \in \mathbb{N}$ is proven by induction on j. We already know that (23) holds for $j = 0$. Assuming it is valid for j, one has by Leibniz rule and linearity

$$\nabla^{j+1}\mathcal{R}_N^2(a, b_\lambda) = \nabla^j\mathcal{R}_N^2(\nabla a, b_\lambda) + \lambda\nabla^j\mathcal{R}_N^2(a, (\nabla b)_\lambda),$$

which via the inductive assumption yields (23) for $j + 1$.

11.3. Proof of Lemma 3. The proof of part (i) is by induction on $|K|$. For $|K| = 1$ the proof is trivial. Let us now assume $|K| \geq 2$. Let us write $K = K' \cup \{k\}$, where $|K'| = |K| - 1$. By inductive assumption, $\{\zeta_{k'}\}_{k' \in K'}$ and $\rho' > 0$ are already defined, so that the periodization of the cylinders with radius ρ' and axis $\{\zeta_{k'} + sk'\}_{s \in \mathbb{R}}$, for $k' \in K'$, are pairwise disjoints. It is then enough to find $\zeta_k \in \mathbb{R}^d$ and $\rho \in (0, \rho')$ such that (27) holds for all $k' \in K'$. Notice that (27) is equivalent to

$$B_\rho(\zeta_k) \cap \Big(B_\rho(0) + \big\{\zeta_{k'} + sk + sk'\big\}_{s,s' \in \mathbb{R}} + \mathbb{Z}^d \Big) = \emptyset.$$

Since $d \geq 3$, the countable union of planes $\{\zeta_{k'} + sk + s'k'\}_{s,s' \in \mathbb{R}} + \mathbb{Z}^d$ has zero measure and, since $k, k' \in \mathbb{Z}^d$, it is closed in $\mathbb{R}^d$. Therefore, for any $\rho \in (0, \rho')$, also

$$\bigcup_{k' \in K'} \overline{B_\rho(0)} + \big\{\zeta_{k'} + sk + s'k'\big\}_{s,s' \in \mathbb{R}} + \mathbb{Z}^d$$

is closed in $\mathbb{R}^d$ and, if ρ small enough, it is strictly contained in $\mathbb{R}^d$. We can thus find ζ_k and $\rho \in (0, \rho')$ such that

$$B_\rho(\zeta_k) \subseteq \mathbb{R}^d \setminus \Big(\bigcup_{k' \in K'} \overline{B_\rho(0)} + \big\{\zeta_{k'} + sk + s'k' : s, s' \in \mathbb{R}\big\} + \mathbb{Z}^d \Big),$$

with the superset being open, thus concluding the proof of part (i).

For part (ii), consider the set of directions in $\mathbb{Z}^3$ given by

$$\tilde{K} := \{(k, 1) : k \in K\} \subseteq \mathbb{Z}^3.$$

By part (i), we can find $\{\tilde{\zeta}_k\}_{k \in K} \subseteq \mathbb{R}^3$ and $\rho > 0$ such that (in $\mathbb{R}^3$),

$$\Big(B_\rho(\tilde{\zeta}_k) + s \underbrace{(k, 1)}_{\in \tilde{K}} + \mathbb{Z}^3 \Big) \cap \Big(B_\rho(\tilde{\zeta}_{k'}) + s' \underbrace{(k', 1)}_{\in \tilde{K}} + \mathbb{Z}^3 \Big) = \emptyset \qquad (145)$$

for all $k, k' \in K$ and $s \in \mathbb{R}$ (the balls above are in $\mathbb{R}^3$). Observe that, for every $(k, 1) \in \tilde{K}$, the axis of the cylinder $B_\rho(\tilde{\zeta}_k) + s(k, 1) + \mathbb{Z}^3 \subseteq \mathbb{R}^3$ is never lying on the horizontal plane x_1, x_2. Therefore we can assume w.l.o.g. that each $\tilde{\zeta}_k$ has the form

$$\tilde{\zeta}_k = (\zeta_k, 0).$$

Let us now show (28), with such choice of $\{\zeta_k\}_{k \in K}$. Assume by contradiction that there are two distinct directions $k, k' \in K$ and $s \in \mathbb{R}$ such that

$$\Big(B_\rho(\zeta_k) + sk + \mathbb{Z}^2 \Big) \cap \Big(B_\rho(\zeta_{k'}) + sk' + \mathbb{Z}^2 \Big) \neq \emptyset.$$

Then there are $x \in B_\rho(\zeta_k) \subseteq \mathbb{R}^2$, $y \in B_\rho(\zeta_{k'}) \subseteq \mathbb{R}^2$, $l, l' \in \mathbb{Z}^2$, such that

$$x + sk + l = y + sk' + l' \in \mathbb{R}^2.$$

This implies that

$$\underbrace{(x, 0)}_{\in B_\rho(\tilde{\zeta}_k)} + s(k, 1) + \underbrace{(l, 0)}_{\in \mathbb{Z}^3} = \underbrace{(y, 0)}_{\in B_\rho(\tilde{\zeta}_{k'})} + s(k', 1) + \underbrace{(l', 0)}_{\in \mathbb{Z}^3} \in \mathbb{R}^3,$$

contradicting (145).

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