Arch. Math. 117 (2021), 79–85

0003-889X/21/010079-7
published online March 25, 2021
https://doi.org/10.1007/s00013-021-01594-0

Archiv der Mathematik

The extension problem for fractional Sobolev spaces with a partial vanishing trace condition

Sebastian Bechtel

Abstract. We construct whole-space extensions of functions in a fractional Sobolev space of order $s \in (0,1)$ and integrability $p \in (0,\infty)$ on an open set O which vanish in a suitable sense on a portion D of the boundary ∂O of O. The set O is supposed to satisfy the so-called *interior thickness condition in* $\partial O \backslash D$, which is much weaker than the global interior thickness condition. The proof works by means of a reduction to the case $D = \emptyset$ using a geometric construction.

Mathematics Subject Classification. Primary 46E35; Secondary 26D15.

Keywords. (Fractional) Sobolev spaces, Kondratiev spaces, Measure density condition, Extension operators, Hardy's inequality.

1. Introduction and main results. Let $O \subseteq \mathbb{R}^d$ be open. For $s \in (0,1)$ and $p \in (0,\infty)$, the fractional Sobolev space $\mathrm{W}^{s,p}(O)$ consists of those $f \in \mathrm{L}^p(O)$ for which the seminorm

$$[f]_{\mathrm{W}^{s,p}(O)} := \Bigg(\iint\limits_{\substack{x,y\in O \\ |x-y|<1}} \frac{|f(x)-f(y)|^p}{|x-y|^{sp+d}} \,\mathrm{d}y\,\mathrm{d}x \Bigg)^{\frac{1}{p}}$$

is finite. Under the interior thickness condition

$$\forall x \in O, r \in (0,1]: \quad |\mathrm{B}(x,r) \cap O| \gtrsim |\mathrm{B}(x,r)|, \tag{ITC}$$

whole-space extensions for $\mathrm{W}^{s,p}(O)$ were constructed by Zhou [11]. Though the mapping is in general not linear, extensions depend boundedly on the data. The case $p \geq 1$ was already treated earlier by Jonsson and Wallin [8], and their extension operator is moreover linear. In fact, Zhou has shown that the interior thickness condition is equivalent for $\mathrm{W}^{s,p}(O)$ to admit whole-space extensions. If we impose a vanishing trace condition on ∂O in a suitable sense, zero extension is possible, so in this case no geometric quality of O is needed. It is now natural to ask what happens if a vanishing trace condition is only imposed on a portion $D \subseteq \partial O$.

To be more precise, we consider the space $\mathrm{W}^{s,p}_D(O)$ given by $\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp})$, where d_D is the distance function to D. The fractional Hardy term in there models the vanishing trace condition on D, compare with [3–6,9]. Spaces of this kind were also recently investigated in [2] and have a history of successful application in the theory of elliptic regularity, see for example [7].
The present paper seeks minimal geometric requirements under which functions in $\mathrm{W}^{s,p}_D(O)$ can be boundedly extended to whole-space functions. We will see in Lemma 2.2 that in (ITC) we could equivalently consider balls centered in ∂O instead of O. Put $N := \partial O \setminus D$. In Definition 2.1, we introduce the *interior thickness condition in* N, which requires that $|\mathrm{B}(x,r) \cap O| \gtrsim |\mathrm{B}(x,r)|$ only holds for $x \in N$ and $r \in (0,1]$. For $D = \emptyset$, this is just the usual interior thickness condition in virtue of the aforementioned Lemma 2.2. It is the main result of this article to show that the interior thickness condition in N is sufficient for the $\mathrm{W}^{s,p}_D(O)$-extension problem.

A major obstacle is that the interior thickness condition in N does not provide thickness in *any* neighborhood around N, which makes localization techniques not applicable. An example for this is a self-touching cusp, see Example 2.3. Our construction is as follows. The extension procedure decomposes into a zero extension from O to some suitable superset $\boldsymbol{O}$ of O, which is an enlargement of O near D, followed by an application of Zhou's construction on $\boldsymbol{O}$. Hence, suitability of $\boldsymbol{O}$ is measured by two properties: First, the zero extension can be bounded in $\mathrm{W}^{s,p}(\boldsymbol{O})$ with the aid of the fractional Hardy term. Second, $\boldsymbol{O}$ satisfies (ITC), so that Zhou's result is applicable. A similar construction of $\boldsymbol{O}$ was performed by the author together with M. Egert and R. Haller-Dintelmann in [1]. The main result then reads as follows.

Theorem 1.1. *Let $O \subseteq \mathbb{R}^d$ and let $D \subseteq \partial O$, $p \in (0,\infty)$, and $s \in (0,1)$. If O satisfies the interior thickness condition in $\partial O \setminus D$, then there exists a bounded extension mapping*

$$\mathrm{E} : \mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp}) \to \mathrm{W}^{s,p}_D(\mathbb{R}^d).$$

If $p \geq 1$, then E *can be chosen to be linear.*

We will also comment on the sharpness of our result in Section 4.

Finally, a remark on the case $p = \infty$ is in order. In this situation, the fractional Sobolev space is substituted by the Hölder space of order $s \in (0,1)$. Then the Whitney extension theorem [10, Thm. 3, p. 174] provides a linear extension operator without any geometric requirements. In particular, the fractional Hardy term is not needed, though it is easily seen that $\|f\, \mathrm{d}_D^{-s}\|_\infty$ can only be finite if f vanishes identically on D, and the same is of course true for the extension.

(Non-)Standard notation. We write $\mathrm{B}(x,r)$ for the open ball around x with radius r. The closure of a set A is denoted by $\overline{A}$ and the Lebesgue measure of A is denoted by $|A|$. If we integrate with respect to the Lebesgue measure, we write $\mathrm{d}x$, $\mathrm{d}y$, and so on. For diameter and distance induced by the Euclidean metric, we write $\operatorname{diam}(\cdot)$ and $\mathrm{d}(\cdot,\cdot)$. Also, the shorthand notation $\mathrm{d}_E(x) := \mathrm{d}(\{x\}, E)$ is used. We employ the notation $\lesssim$ and $\gtrsim$ for estimates up to an implicit constant that does not depend on the quantified objects. If two quantities satisfy both $\lesssim$ and $\gtrsim$, we write $\approx$.

2. Geometry.

Definition 2.1. Let $E \subseteq \mathbb{R}^d$. Say that E satisfies the *interior thickness condition* if

$$\forall x \in E, r \in (0,1] : \quad |\mathrm{B}(x,r) \cap E| \gtrsim |\mathrm{B}(x,r)|.$$

Moreover, if $F \subseteq \partial E$, then E satisfies the *interior thickness condition in* F if

$$\forall x \in F, r \in (0,1] : \quad |\mathrm{B}(x,r) \cap E| \gtrsim |\mathrm{B}(x,r)|.$$

In the special case $E = O$, the condition (ITC) from the introduction just means that O satisfies the interior thickness condition, and the following lemma shows the equivalence between (ITC) and the interior thickness condition in ∂O for O already mentioned in the introduction. Though its proof is simple, we include it for good measure.

Lemma 2.2. *Let $E \subseteq \mathbb{R}^d$. Then E satisfies the interior thickness condition if and only if E satisfies the interior thickness condition in ∂E.*

Proof. Assume (ITC) and let $x \in \partial E$, $r \in (0,1]$. Then pick some $y \in \mathrm{B}(x, r/2) \cap E$ and calculate

$$|\mathrm{B}(x,r) \cap E| \geq |\mathrm{B}(y,r/2) \cap E| \gtrsim |\mathrm{B}(y,r/2)| \approx |\mathrm{B}(x,r)|.$$

Conversely, let $x \in E$, $r \in (0,1]$ and assume that E is interior thick in ∂E. If $\mathrm{B}(x,r/2) \subseteq E$, then the claim follows immediately. Otherwise, pick again some $y \in \mathrm{B}(x,r/2) \cap \partial E$ and argue as above. □

The following simple example shows that a set can satisfy the interior thickness condition in some closed subset of the boundary but fails to have it in any neighborhood of it.

Example 2.3. Consider $O = \{(x,y) \in \mathbb{R}^2 \colon |y| < x^2, x < 0\} \cup \{(x,y) \in \mathbb{R}^2 \colon x > 0\}$. This means that O consists of the right half-plane touched by a cusp from the left. Put D to be the boundary of the cusp and N is the y-axis except the origin. Then the (ITC) estimate holds in N since each ball centered in N hits the half-plane with half of its area, but any proper neighborhood around N would contain a region around the tip of the cusp, in which thickness does not hold (consider a sequence that approximates the tip of the cusp and test with balls that do not reach N).

3. The extension operator. In this section, we prove Theorem 1.1. We follow the strategy outlined in the introduction. First, we construct an auxiliary set $\boldsymbol{O}$ and show that it is interior thick. Second, we show that the zero extension to $\boldsymbol{O}$ is bounded using a simple geometric argument. Finally, we patch everything together to conclude. Throughout, O and D are as in Theorem 1.1 and we put $N := \partial O \setminus D$ for convenience.

3.1. Embedding into an interior thick set. We construct an open set $\boldsymbol{O} \subseteq \mathbb{R}^d$ with $O \subseteq \boldsymbol{O}$, $\partial O \subseteq \partial \boldsymbol{O}$ and that satisfies (ITC). According to the assumption on N and Lemma 2.2, it suffices to check that $\boldsymbol{O}$ is interior thick in D and the "added" boundary. Of course we could take $\boldsymbol{O}$ as $\mathbb{R}^d \setminus \partial O$ in this step but this would make zero extension in Section 3.2 impossible. Therefore, our construction will be in such a way that moreover $|x-y| \gtrsim \mathrm{d}_D(x)$ whenever $x \in O$ and $y \in \boldsymbol{O} \setminus O$, see Lemma 3.1, which will do the trick in step two.
Let $\{Q_j\}_j$ be a Whitney decomposition for the complement of $\overline{N}$ in $\mathbb{R}^d$, which means that the Q_j are disjoint dyadic open cubes such that

$$\text{(i)} \quad \bigcup_j \overline{Q_j} = \mathbb{R}^d \setminus \overline{N}, \qquad \text{(ii)} \quad \operatorname{diam}(Q_j) \leq \mathrm{d}(Q_j, N) \leq 4 \operatorname{diam}(Q_j).$$

Using the Whitney decomposition, we define

$$\Sigma := \{Q_j : \overline{Q_j} \cap \overline{O} \neq \emptyset\} \qquad \text{and} \qquad \boldsymbol{O} := O \cup \Big(\bigcup_{Q \in \Sigma} Q \setminus D \Big).$$

Note that for $Q \in \Sigma$, one has $Q \setminus D = Q \setminus \partial O$. Then all claimed properties of $\boldsymbol{O}$ except (ITC) follow immediately by definition. So, let $x \in \partial \boldsymbol{O}$ and $r \in (0,1]$. If $x \in \overline{N}$, then we are done by assumption (argue as in the proof of Lemma 2.2 to even get the interior thickness condition in $\overline{N}$ instead of merely in N). Otherwise, either $x \in D$ or $x \in \partial Q$ for some $Q \in \Sigma$ (to see this, use that the Whitney decomposition is locally finite). But if $x \in D$, then $x \in \overline{Q}$ for some $Q \in \Sigma$ by property (i) of the Whitney decomposition and the definition of Σ. Hence, in either case $x \in \overline{Q}$ for some $Q \in \Sigma$. Now we make a case distinction on the radius size compared to the size of Q. If $r \geq 4\,\mathrm{d}(Q,N)$, pick $y \in \overline{Q}$ and $z \in \overline{N}$ with $\mathrm{d}(Q,N) = |y-z|$. Then, with (ii), we get

$$|x-z| \leq |x-y| + |y-z| \leq \operatorname{diam}(Q) + \mathrm{d}(Q,N) \leq 2\,\mathrm{d}(Q,N) \leq r/2,$$

hence $\mathrm{B}(x,r)$ contains a ball of radius $r/2$ centered in $\overline{N}$ and we are done. Otherwise, if $r < 4\,\mathrm{d}(Q,N)$, then, by (ii), we get $r < 16 \operatorname{diam}(Q)$ and the claim follows from (ITC) for Q.

3.2. Zero extension. Let $\boldsymbol{O}$ denote the set constructed in the previous step. We define the zero extension operator E_0 from O to $\boldsymbol{O} \cup D$ and claim that it is $\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp}) \to \mathrm{W}^{s,p}(\boldsymbol{O})$ bounded. We start with a preparatory lemma.

Lemma 3.1. *One has* $|x-y| \geq \frac{1}{2}\,\mathrm{d}_D(x)$ *whenever* $x \in O$ *and* $y \in \boldsymbol{O} \setminus O$.

Proof. We consider $y \in \boldsymbol{O} \setminus O$ and pick some $Q \in \Sigma$ that contains y. We distinguish whether or not x and y are far away from each other in relation to $\operatorname{diam}(Q)$.
Case 1: $|x-y| < \operatorname{diam}(Q)$. Fix a point $z \in \partial O$ on the line segment connecting x with y. Assume for the sake of contradiction that $z \in N$. Then, using (ii), we calculate

$$\mathrm{d}(Q,N) \leq |y-z| \leq |x-y| < \operatorname{diam}(Q) \leq \mathrm{d}(Q,N),$$

hence we must have $z \in D$. Thus, $|x-y| \geq |x-z| \geq \mathrm{d}_D(x)$.

Case 2: $|x-y| \geq \operatorname{diam}(Q)$. By definition of Σ and since $y \notin O$, we can pick $z \in \overline{Q} \cap D$. Then

$$|x-z| \leq |x-y| + |y-z| \leq |x-y| + \operatorname{diam}(Q) \leq 2|x-y|,$$

hence $|x-y| \geq \frac{1}{2} \mathrm{d}_D(x)$. □

This enables us to estimate E_0. Clearly, we only have to estimate the $\mathrm{W}^{s,p}(O)$–seminorm since extension by zero is always isometric on L^p. Let $f \in \mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp})$. Then

$$\begin{aligned} \iint\limits_{\substack{x,y \in \boldsymbol{O} \\ |x-y|<1}} \frac{|\mathrm{E}_0 f(x) - \mathrm{E}_0 f(y)|^p}{|x-y|^{sp+d}} \,\mathrm{d}y\,\mathrm{d}x \leq & \iint\limits_{\substack{x,y \in O \\ |x-y|<1}} \frac{|f(x)-f(y)|^p}{|x-y|^{sp+d}} \,\mathrm{d}x\,\mathrm{d}y \\ & + 2 \iint\limits_{\substack{x \in O, y \in (\boldsymbol{O} \setminus O) \\ |x-y|<1}} \frac{|f(x)|^p}{|x-y|^{sp+d}} \,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{1}$$

The first term is bounded by $\|f\|^p_{W^{s,p}(O)}$, so it only remains to bound the second term. Using Lemma 3.1 and calculating in polar coordinates, we find

$$\int\limits_{\substack{y \in (\boldsymbol{O} \setminus O) \\ |x-y|<1}} |x-y|^{-sp-d} \,\mathrm{d}y \lesssim \mathrm{d}_D(x)^{-sp}.$$

Plugging this back into (1) yields that we can bound the second term therein by the Hardy term $\|f\|^p_{\mathrm{L}^p(O, \mathrm{d}_D^{-sp})}$.

3.3. Proof of Theorem 1.1. We combine the results from the previous sections with the extension procedure of Zhou to conclude.

Proof of Theorem 1.1. Put $\mathrm{E} = \boldsymbol{E} \circ \mathrm{E}_0$, where $\boldsymbol{E}$ is the (non-linear) extension operator of Zhou and E_0 is the zero extension operator from the previous step. Clearly, E_0 is linear, and we have seen in Section 3.2 that it is $\mathrm{W}^{s,p}_D(O) \to \mathrm{W}^{s,p}(\boldsymbol{O})$ bounded. Since $\boldsymbol{O}$ satisfies (ITC) by Section 3.1, $\boldsymbol{E}$ is well-defined on $\mathrm{W}^{s,p}(\boldsymbol{O})$ and bounded into $\mathrm{W}^{s,p}(\mathbb{R}^d)$ by Zhou's result.

The claim for $p < 1$ then follows already by composition. In the case $p \geq 1$, note that $\boldsymbol{E}$ can be constructed to be linear, see also [8]. □

4. On the sharpness of our result. In this final section, we take a look on how close to a characterization our condition is. We will see in Example 4.1 that the interior thickness condition in N is not necessary for the extension problem, but that our construction might fail without it. Afterwards, we will introduce a *degenerate interior thickness condition in* N, which is necessary for the extension problem, but is not sufficient for our construction.

Example 4.1. Consider the upper half-plane in $\mathbb{R}^2$. A Whitney decomposition can be constructed from layers of dyadic cubes. Let O be a "cusp" that is build from those Whitney cubes which intersect the area below the graph of the exponential function, and let N be its lower boundary given by the real line in $\mathbb{R}^2$. It is eminent that O is not interior thick in N. Moreover, our construction of $\boldsymbol{O}$ just adds another layer of cubes, so $\boldsymbol{O}$ is of the same geometric quality. Hence, our construction does not work in this situation. But zero extension to the upper half-plane is still possible, so with $\boldsymbol{O}$ chosen as the upper half-plane, we can construct an extension procedure for $\mathrm{W}^{s,p}_D(O)$ in this configuration. This shows that the interior thickness condition in N is not necessary for $\mathrm{W}^{s,p}_D(O)$ to admit whole-space extensions, but is "necessary" for our construction to work.

We introduce the aforementioned modified version of the interior thickness condition in $N \subseteq \partial O$ that degenerates near $\partial O \setminus N$.

Definition 4.2. Say that O satisfies the *degenerate interior thickness condition in* N if $O \subseteq \mathbb{R}^d$ is open, $N \subseteq \partial O$, and they fulfill

$$\forall x \in N, r \leq \min(1, \mathrm{d}_{\partial O \setminus N}(x))\colon |\mathrm{B}(x,r) \cap O| \gtrsim |\mathrm{B}(x,r)|.$$

In fact, this condition is necessary for the $\mathrm{W}^{s,p}(O) \cap \mathrm{L}^p(O, \mathrm{d}_D^{-sp})$-extension problem. The technique to show this is due to Zhou [11]. By the restriction in radii, the test functions used in Zhou's proof belong to $\mathrm{W}^{s,p}_D(O)$, and then his proof applies *verbatim*, hence we omit the details.

Proposition 4.3. *Let $O \subseteq \mathbb{R}^d$ be open, $D \subseteq \partial O$, $p \in (0,\infty)$, $s \in (0,1)$, and put $N := \partial O \setminus D$. If $\mathrm{W}^{s,p}_D(O)$ admits whole-space extensions, then O satisfies the degenerate interior thickness condition in N.*

Remark 4.4. In Example 4.1, we have seen a configuration which admits whole-space extensions for $\mathrm{W}^{s,p}_D(O)$-functions, so in this situation, O satisfies the degenerate interior thickness condition in N by Proposition 4.3 (of course, this can also be seen directly). On the other hand, we have seen in that example that in this configuration our construction does not work. Hence, the degenerate interior thickness condition in N is too weak for our proof of Theorem 1.1.

Acknowledgements. The author thanks his PhD advisor Robert Haller-Dintelmann for his support, the "Studienstiftung des deutschen Volkes" for financial and academic support, Joachim Rehberg for suggesting the topic, and Juha Lehrbäck for valuable discussions.

Funding Open Access funding enabled and organized by Projekt DEAL.

References

[1] Bechtel, S., Egert, M., Haller-Dintelmann, R.: The Kato square root problem on locally uniform domains. Adv. Math. **375**, 107410 (2020)

[2] Dahlke, S., Hansen, M., Schneider, C., Sickel, W.: Properties of Kondratiev spaces. Preprint (2019)

[3] Dyda, B., Vähäkangas, A.: A framework for fractional Hardy inequalities. Ann. Acad. Sci. Fenn. Math. **39**(2), 675–689 (2014)

[4] Egert, M., Haller-Dintelmann, R., Rehberg, J.: Hardy's inequality for functions vanishing on a part of the boundary. Potential Anal. **43**, 49–78 (2015)

[5] Egert, M., Tolksdorf, P.: Characterizations of Sobolev functions that vanish on a part of the boundary. Discrete Contin. Dyn. Syst. Ser. S **10**(4), 729–743 (2017)

[6] Hajłasz, P.: Pointwise Hardy inequalities. Proc. Amer. Math. Soc. **127**(2), 417–423 (1999)

[7] Hansen, M.: Nonlinear approximation rates and Besov regularity for elliptic PDEs on polyhedral domains. Found. Comput. Math. **15**(2), 561–589 (2015)

[8] Jonsson, A., Wallin, H.: Function spaces on subsets of $\mathbb{R}^n$. Math. Rep. **2**(1), xiv+221 (1984)

[9] Lehrbäck, J.: Pointwise Hardy inequalities and uniformly fat sets. Proc. Amer. Math. Soc. **136**(6), 2193–2200 (2008)

[10] Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)

[11] Zhou, Y.: Fractional Sobolev extension and imbedding. Trans. Amer. Math. Soc. **367**(2), 959–979 (2015)

Sebastian Bechtel
Fachbereich Mathematik
Technische Universität Darmstadt
Schlossgartenstr. 7
64289 Darmstadt
Germany
e-mail: bechtel@mathematik.tu-darmstadt.de

Received: 7 January 2021

Revised: 1 February 2021

Accepted: 16 February 2021.