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Fast robust control of linear systems subject to actuator saturation ⋆

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Abstract: This paper deals with the robust stability of soft variable-structure controls. More precisely, the control of linear plants subject to parametric uncertainty and actuator saturation is considered. Earlier works are summarized and new results are presented in this paper. It is shown that for all considered types of soft variable-structure controls, the robustness analysis leads to parameter-dependent Lyapunov inequalities. An overhead crane control is given as an illustrating example.

Keywords: Robustness; soft variable-structure control; actuator saturation.

1. INTRODUCTION

First, we would like to emphasize that soft variable structure controls are completely different from the so-called sliding mode controls. The latter are generally known to utilize sliding modes to meet certain robustness requirements, see e.g. Utkin (1992). These sliding modes occur as a consequence of discontinuous switching between different control laws.

Conversely, soft variable structure controls (SVSC) adjust the control signal continuously, see Adamy and Flemming (2004) for a survey of SVSC. Thus, SVSC avoid stressing the actuator heavily because of high-frequency switching. Although they have been developed from discontinuous VSC and piecewise linear controls, the underlying ideas are different from those of sliding mode controls. The general intention of using SVSC is to achieve nearly time-optimal control performance. So far, there are three types of SVSC known: SVSC with variable saturation (type A); SVSC employing implicit Lyapunov functions (type B); and dynamic SVSC (type C).

This paper aims to show how SVSC can cope with certain robustness problems. We summarize, extend, and improve the results from other authors regarding type B and C, which were published only in German (Niewels and Kiendl, 2003; Franke, 1983). Furthermore, we present a new result for type A, and show that for all the three types parameter-dependent Lyapunov inequalities can be used to solve the problem.

The specific control problem is the robust control of a linear plant subject to actuator saturation and parametric uncertainty. Both actuator saturation and parametric uncertainty are common phenomena in practical control problems. Hence both areas have gained a great deal of attention during the last years, see e.g. the monographs Tarbouriech and Garcia (1997); Kapila and Grigoriadis (2002); Amato (2006).

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2. PRELIMINARIES

2.1 Problem statement

We consider a state space representation of an uncertain linear plant:

$$\dot{\mathbf{x}} = \mathbf{A}\big(\mathbf{q}(t)\big)\mathbf{x} + \mathbf{b}\big(\mathbf{q}(t)\big)u, \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}$ is the control input. The vector function $\mathbf{q} : \mathbb{R} \to \mathcal{Q}$ denotes unknown, possibly time-varying parameters and maps the set of real numbers to the bounded subset $\mathcal{Q} \subset \mathbb{R}^\nu$. The matrix function $\mathbf{A}$ and the vector function $\mathbf{b}$ are of appropriate dimensions and the pair $(\mathbf{A}(\mathbf{q}), \mathbf{b}(\mathbf{q}))$ is completely controllable for all $\mathbf{q} \in \mathcal{Q}$. Furthermore, the control u and the set of initial states $\mathcal{X}_0 \subset \mathbb{R}^n$ are subject to the following constraints:

$$|u| \le u_0, \tag{2}$$

$$\mathcal{X}_0 = \{\mathbf{x}_0 \in \mathbb{R}^n |\, \underline{\mathbf{x}}_0 \le \mathbf{x}_0 \le \overline{\mathbf{x}}_0\}, \tag{3}$$

where $\overline{\mathbf{x}}_0, \underline{\mathbf{x}}_0 \in \mathbb{R}^n$, $u_0 \in \mathbb{R}^+$, and the inequalities are intended component wise.

The control task is to design a controller that leads to a well performing nominal system at constant $\mathbf{q}(t) = \mathbf{q}_0$ while the closed-loop system has to meet two additional conditions:

- (A1) For all possible $\mathbf{q}(t)$ asymptotic stability of the closed-loop system should be assured in a region $\mathcal{G}$ containing the equilibrium state $\mathbf{x} = \mathbf{0}$ and all possible initial states $\mathbf{x}_0 \in \mathcal{X}_0$.
- (A2) The control constraint, $|u| \le u_0$, should be satisfied for all possible $\mathbf{x}(t)$, where $t \ge t_0$ and $\mathbf{x}_0 \in \mathcal{X}_0$.

Below, the argument t of the function $\mathbf{q}(t)$ is omitted to improve readability.

2.2 Soft variable-structure control

The basic concept of SVSC is to vary a parameter-dependent state feedback during the control action so that the available control range is utilized as advantageously as possible. Roughly speaking, the loop gain is increased as the control deviation decreases. This approach leads to nearly time-optimal performance if the control law is suitably chosen.

A mathematically precise description along the lines of Adamy and Flemming (2004) is as follows: the controller

$$u = \mathscr{F}(\mathbf{x}, p), \tag{4}$$

where $\mathscr{F}$ is a general operator, which depends on the state vector of the plant and a continuous selection parameter, $p \in \mathbb{R}$, which is computed by a selection strategy. This selection strategy is defined by

$$\mathscr{S}(\mathbf{x}, p^{(n)}, \ldots, p) = 0, \tag{5}$$

where $\mathscr{S}$ denotes a continuous real-valued function and $p^{(n)}$ the n-th time derivative of p. Eq. (5) is the general case, which includes the simple case $p = \mathscr{S}(\mathbf{x})$, implicit definitions, and differential equations.

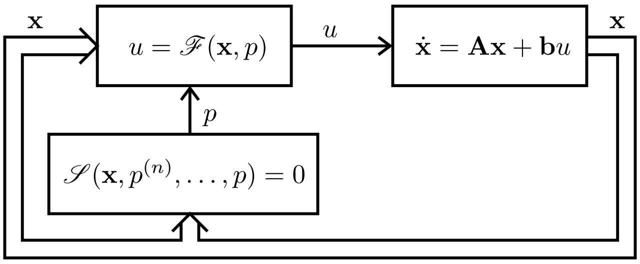

Fig. 1. Block schematic of soft variable structure controls.

3. SOFT VSC WITH VARIABLE SATURATION (TYPE A)

Firstly, we consider Albers' SVSC with variable saturation (Albers, 1983). Since this type of SVSC is described in (Adamy and Flemming, 2004) in detail, we consider only the basics of this control in the following subsection. Afterwards, we focus on robust stability conditions of the closed-loop system.

3.1 Basic definitions

The control law (4) of SVSC with variable saturation is

$$u = -(\mathbf{k}_1 + p \cdot \mathbf{k}_2)^T \mathbf{x}, \tag{6}$$

where the state feedback vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ are in $\mathbb{R}^n$ and the selection parameter is bounded by $p \in [0, 1]$. The selection strategy (5) is determined by

$$p = \frac{u_s(\mathbf{x})}{\mathbf{k}_2^T \mathbf{x}} \operatorname{sat}\left(\frac{\mathbf{k}_2^T \mathbf{x}}{u_s(\mathbf{x})}\right), \tag{7}$$

where $\operatorname{sat}(\cdot)$ denotes the standard saturation function $\operatorname{sat}(y) = \operatorname{sign}(y) \cdot \min\{1, |y|\}$ and

$$\begin{aligned} u_s(\mathbf{x}) &= u_0 - \sqrt{v(\mathbf{x})\, \mathbf{k}_1^T \mathbf{R}^{-1} \mathbf{k}_1}, \\ v(\mathbf{x}) &= \mathbf{x}^T \mathbf{R} \mathbf{x}, \end{aligned} \tag{8}$$

where $\mathbf{R}$ is a positive definite matrix in $\mathbb{R}^{n \times n}$. The function v is a Lyapunov function of the closed loop system.

Moreover, we consider a positively invariant set,

$$\mathcal{G} = \{\mathbf{x} \mid v(\mathbf{x}) < v_G\}, \tag{9}$$

where the positive real parameter v_G is chosen so that

$$\mathcal{X}_0 \subseteq \mathcal{G}. \tag{10}$$

The control constraint (A2) is satisfied if the inequality

$$u_0 \geq \sqrt{v(\mathbf{x})\, \mathbf{k}_1^T \mathbf{R}^{-1} \mathbf{k}_1} \tag{11}$$

holds for all possible states $\mathbf{x}$. This issue is independent of the uncertainty; thus, it is not of our main interest.

3.2 Robust Stability

We focus on the robust stability conditions, which have not been investigated before and are easily obtained by using the notion of quadratic stability (Amato, 2006). From (1) and (6) we derive the system matrix of the overall closed-loop system $\dot{\mathbf{x}} = \widehat{\mathbf{A}}(\mathbf{q}, p)\mathbf{x}$:

$$\widehat{\mathbf{A}}(\mathbf{q}, p) = \mathbf{A}(\mathbf{q}) - \mathbf{b}(\mathbf{q})\,(\mathbf{k}_1 + p\,\mathbf{k}_2)^T. \tag{12}$$

We use v from (8) as a Lyapunov function of the closed-loop system within the ellipsoid $\mathcal{G}$ from (9). The condition $\dot{v}(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$ leads to

$$\mathbf{x}^T \left(\widehat{\mathbf{A}}^T(\mathbf{q}, p)\mathbf{R} + \mathbf{R}\widehat{\mathbf{A}}(\mathbf{q}, p)\right) \mathbf{x} < 0, \tag{13}$$

or the Lyapunov matrix inequality

$$\widehat{\mathbf{A}}^T(\mathbf{q}, p)\mathbf{R} + \mathbf{R}\widehat{\mathbf{A}}(\mathbf{q}, p) < 0, \tag{14}$$

where the notation $\mathbf{M} < 0$ implies the negative definiteness of a symmetric matrix $\mathbf{M}$. This condition must be satisfied for all $\mathbf{q} \in \mathcal{Q}$ and $p \in [0, 1]$.

Finally, we summarize the conditions that are sufficient for robust stability of SVSC with variable saturation in a theorem.

Theorem 1. Let the system (1) subject to the constraints (2) and (3) be controlled by a SVSC according to (6) and (7). Then, the closed-loop system satisfies conditions (A1) and (A2) if the inclusion (10) and the inequalities (11) and (14) hold.

4. SOFT VSC EMPLOYING IMPLICIT LYAPUNOV FUNCTIONS (TYPE B)

We describe SVSC that employ implicit Lyapunov functions. After assembling some basic definitions (see Adamy and Flemming (2004) for details), we provide the robust stability conditions for the closed-loop system.

4.1 Basic definitions

The control law (4) of the implicit SVSC is

$$u = -\mathbf{k}^T(p)\mathbf{x}, \tag{15}$$

where $p \in (0, 1]$ is the selection parameter and $\mathbf{k} : (0, 1] \rightarrow \mathbb{R}^n$ is a vector-valued function. Without loss of generality it is assumed that the nominal system has been transformed into controllable standard form. The controller gain is defined by

$$\mathbf{k}(p) = \mathbf{D}^{-1}(p)\widehat{\mathbf{a}} - \mathbf{a}_0, \tag{16}$$

where $\mathbf{a}_0^T = [a_0, \cdots, a_{n-1}]$ is a vector containing the coefficients of the characteristic polynomial of the uncontrolled nominal system, $\widehat{\mathbf{a}}^T = [\widehat{a}_0, \cdots, \widehat{a}_{n-1}]$ is a vector containing coefficients of the (desired) closed-loop nominal system at $p = 1$, and

$$\mathbf{D}(p) = \operatorname{diag}(p^n, \ldots, p^2, p). \tag{17}$$

Moreover, the selection strategy of (5) is defined by the implicit equation

$$g(p, \mathbf{x}) = 0, \tag{18}$$

where

$$g(p, \mathbf{x}) = e(p)\mathbf{x}^T\mathbf{R}(p)\mathbf{x} - 1, \tag{19}$$

$$e(p) = \frac{1}{u_0^2}\mathbf{k}^T(p)\mathbf{R}^{-1}(p)\mathbf{k}(p),$$

$$\mathbf{R}(p) = \mathbf{D}^{-1}(p)\mathbf{R}_1\mathbf{D}^{-1}(p). \tag{20}$$

$\mathbf{R}_1$ is a positive definite Matrix in $\mathbb{R}^{n\times n}$ and $e(p) > 0$ holds for all $p \in (0, 1]$. Apart from its role as a selection parameter the parameter p also acts as an implicitly defined Lyapunov function, and thus guarantees the stability of the implicit SVSC. Finally,

$$\mathcal{G} = \{\mathbf{x} \mid g(1, \mathbf{x}) < 0\} \tag{21}$$

denotes a Lyapunov region that satisfies

$$\mathcal{X}_0 \subseteq \mathcal{G}. \tag{22}$$

We use the set $\mathcal{G}$ subsequently to bound a region wherein the closed-loop system is asymptotically stable. Inclusion (22) is independent of the uncertainty because the definition of the set $\mathcal{G}$ does not contain the uncertain parameters $\mathbf{q}(t)$. Condition (A2) concerning the actuator saturation is related to the aforementioned choice of the control law (15) and the selection strategy (18), and therefore, this issue is independent of the uncertainty as well.

4.2 Robust stability

The selection parameter $p = v$ in (18) also acts as an implicitly defined Lyapunov function $v(\mathbf{x})$ of the closed-loop system to guarantee stability, that is, the closed-loop system is asymptotically stable if $\dot{v}(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$. Using the implicit Lyapunov function theorem from Adamy (2005), it can be shown that (18) implicitly defines a Lyapunov function $v(\mathbf{x})$ for all $\mathbf{x}$ in $\mathcal{G}$ if

$$\dot{v}(\mathbf{x}) = -\frac{\partial g(v, \mathbf{x}(t))/\partial t}{\partial g(v, \mathbf{x})/\partial v} < 0, \tag{23}$$

which holds if

$$-\infty < \frac{\partial g(v, \mathbf{x})}{\partial v} < 0 \quad \text{and} \quad \frac{\partial g(v, \mathbf{x}(t))}{\partial t} < 0. \tag{24}$$

The first part of (24) leads to two conditions, namely

$$\max_{v\in(0,1]} e'(v) \leq 0 \tag{25}$$

$$\mathbf{N}\mathbf{R}_1 + \mathbf{R}_1\mathbf{N} < 0, \tag{26}$$

where $\mathbf{N} = \mathrm{diag}\ (-n, \ldots, -1)$. The stability conditions (25) and (26) are independent of the robustness issue; therefore, it is not our main interest in this paper. However, the second part of (24),

$$\frac{\partial g(v, \mathbf{x}(t))}{\partial t} = \dot{\mathbf{x}}^T \mathrm{grad}_{\mathbf{x}} g(v, \mathbf{x}) < 0, \tag{27}$$

is affected by the uncertainty. In the following, we extend and improve the results of Niewels and Kiendl (2003).
From (1) and (15) we obtain the system matrix of the closed-loop system $\dot{\mathbf{x}} = \widehat{\mathbf{A}}(\mathbf{q}, v)\mathbf{x}$:

$$\widehat{\mathbf{A}}(\mathbf{q}, v) = \mathbf{A}(\mathbf{q}) - \mathbf{b}(\mathbf{q})\mathbf{k}^T(v). \tag{28}$$

Inserting $\dot{\mathbf{x}} = \widehat{\mathbf{A}}(\mathbf{q}, v)\mathbf{x}$ and (19) into (27), we get

$$e(v)\mathbf{x}^T[\widehat{\mathbf{A}}^T(\mathbf{q}, v)\mathbf{R}(v) + \mathbf{R}(v)\widehat{\mathbf{A}}(\mathbf{q}, v)]\mathbf{x} < 0, \tag{29}$$

which is equivalent to

$$\widehat{\mathbf{A}}^T(\mathbf{q}, v)\mathbf{R}(v) + \mathbf{R}(v)\widehat{\mathbf{A}}(\mathbf{q}, v) < 0 \tag{30}$$

for all $(\mathbf{q}, v)$. This is the crucial condition concerning the robustness.
We now simplify Condition (30). For this purpose the plant (1) will be split into a constant nominal and a parameter dependent part

$$\dot{\mathbf{x}} = [\mathbf{A}_0 + \Delta\mathbf{A}(\mathbf{q})]\mathbf{x} + [\mathbf{b}_0 + \Delta\mathbf{b}(\mathbf{q})]u, \tag{31}$$

where $\mathbf{A}_0 = \mathbf{A}(\mathbf{q}_0)$ and $\mathbf{b}_0 = \mathbf{b}(\mathbf{q}_0)$. Using (15), (16), and (31), the system matrix (28) may be written in the form [1]

$$\begin{aligned}\widehat{\mathbf{A}}(\mathbf{q}, v) =& v^{-1}\mathbf{D}(v)\widehat{\mathbf{A}}_1\mathbf{D}^{-1}(v) + \Delta\mathbf{A}(\mathbf{q}) \\ &- \Delta\mathbf{b}(\mathbf{q})[\widehat{\mathbf{a}}^T\mathbf{D}^{-1}(v) - \mathbf{a}_0^T],\end{aligned} \tag{32}$$

where $\widehat{\mathbf{A}}_1 = \mathbf{A}_0 - \mathbf{b}_0\mathbf{k}^T(v = 1)$. Then, applying (20) and (32) to (30) we obtain an equivalent version:

$$\widetilde{\mathbf{A}}^T(\mathbf{q}, v)\mathbf{R}_1 + \mathbf{R}_1\widetilde{\mathbf{A}}(\mathbf{q}, v) < 0, \tag{33}$$

where

$$\begin{aligned}\widetilde{\mathbf{A}}(\mathbf{q}, v) =& \widehat{\mathbf{A}}_1 - v\mathbf{D}^{-1}(v)\left[\Delta\mathbf{b}(\mathbf{q})\,\widehat{\mathbf{a}}^T\right. \\ &\left. - \left(\Delta\mathbf{A}(\mathbf{q}) + \Delta\mathbf{b}(\mathbf{q})\,\mathbf{a}_0^T\right)\mathbf{D}(v)\right].\end{aligned} \tag{34}$$

Note that if matrix $\widetilde{\mathbf{A}}$ does not depend on $\mathbf{q}$, that is $\Delta\mathbf{A} \equiv \mathbf{0}$ and $\Delta\mathbf{b} \equiv \mathbf{0}$, we have $\widetilde{\mathbf{A}} \equiv \widehat{\mathbf{A}}_1$. Thus inequality (33) reduces to the simple parameter independent expression $\widehat{\mathbf{A}}_1^T\mathbf{R}_1 + \mathbf{R}_1\widehat{\mathbf{A}}_1 < \mathbf{0}$, which was intended in the original works as one of the stability conditions.

Thus, we conclude this subsection with a theorem that summarizes the four conditions that are sufficient for robust stability of implicit SVSC.

Theorem 2. Let the system (1) subject to the constraints (2) and (3) be controlled by a SVSC according to (15) and (18). Then the closed-loop system satisfies conditions (A1) and (A2) if the inclusion (22) and the inequalities (25), (26), and (33) hold.

5. DYNAMIC SVSC (TYPE C)

In the following, we consider dynamic SVSC from Franke (1983). Again, we only briefly deal with the basic definitions (see Adamy and Flemming (2004) for details) before we return to robustness.

5.1 Basic definitions

The control law (4) of dynamic SVSC is

$$u = -(\mathbf{k}_1 + p\cdot\mathbf{k}_2)^T\mathbf{x}, \tag{35}$$

where the state feedbacks $\mathbf{k}_1$ and $\mathbf{k}_2$ are vectors in $\mathbb{R}^n$ and p is the selection parameter in $\mathbb{R}$.
The selection strategy (5) is determined by a differential equation,

$$\dot{p} = \gamma^{-1}\left(\mathbf{x}^T\mathbf{R}\mathbf{b}_0\mathbf{k}_2^T\mathbf{x} - p\cdot r(p, \mathbf{x})\right), \tag{36}$$

where $\mathbf{b}_0 = \mathbf{b}(\mathbf{q}_0)$, $\mathbf{R}$ is a positive definite matrix in $\mathbb{R}^{n\times n}$, γ is a positive real number, and $r(p, \mathbf{x})$ is a positive real function.

[1] Since $\mathbf{A}_0$ and $\mathbf{b}_0$ are in controllable standard form we have $\mathbf{A}_0 + \mathbf{b}_0\mathbf{k}^T(v) = \mathbf{A}_0 + \mathbf{b}_0\mathbf{a}_0^T - \mathbf{b}_0\widehat{\mathbf{a}}^T\mathbf{D}^{-1}(v) = v^{-1}\mathbf{D}(v)[\mathbf{A}_0 + \mathbf{b}_0\mathbf{a}_0^T - \mathbf{b}_0\widehat{\mathbf{a}}^T]\mathbf{D}^{-1}(v) = v^{-1}\mathbf{D}(v)\widehat{\mathbf{A}}_1\mathbf{D}^{-1}(v)$.

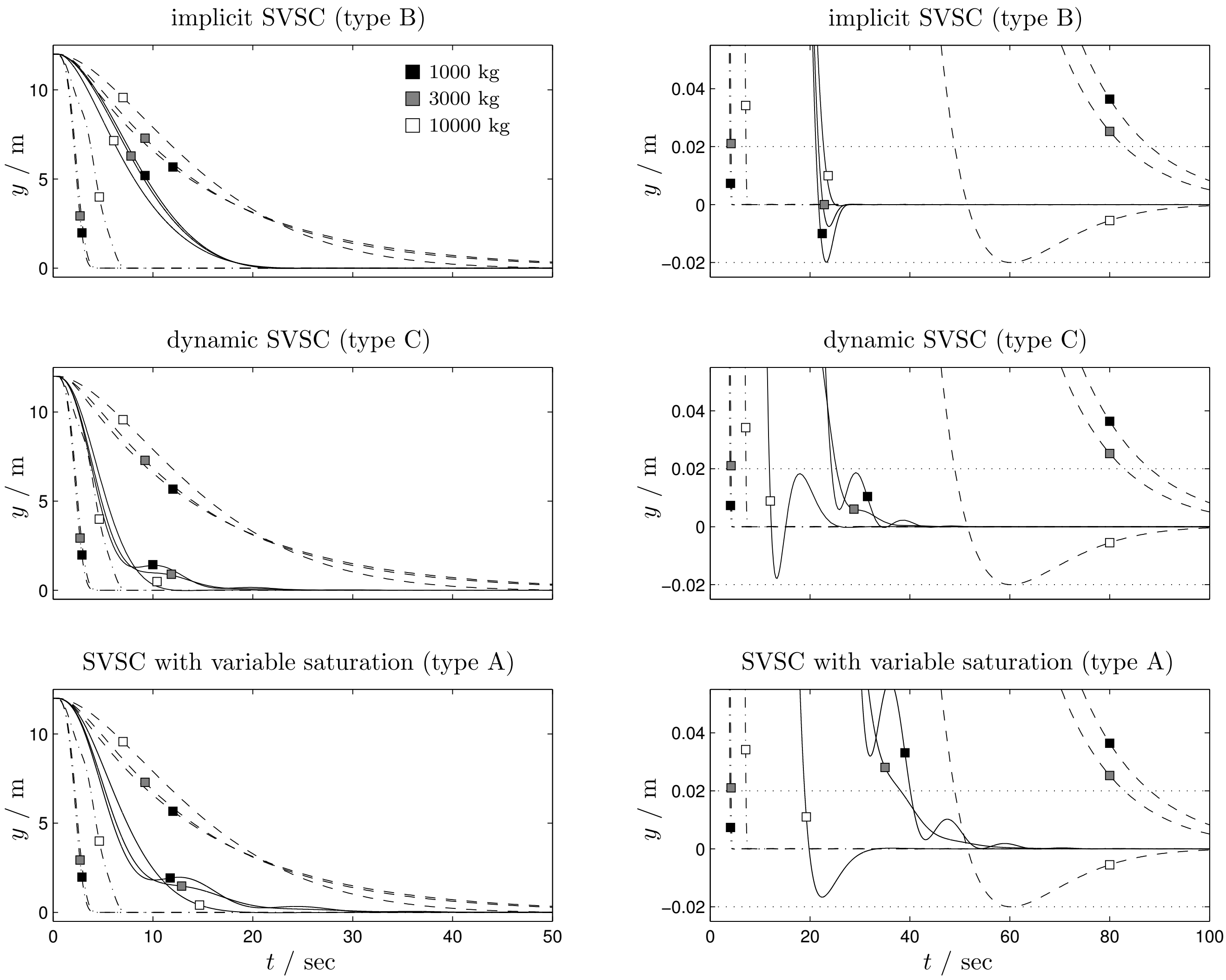

Fig. 2. Load position y of the overhead crane example for minimal, nominal, and maximal load. The subfigures on the left-hand side display the whole range. Close-ups on the tolerance band $|y| \leq 2$ cm are shown on the right-hand side. Each subfigure contains the linear control (dashed) and the time optimal feedforward control (dash-dotted) for comparison. The solid lines indicate the SVSC.

Eq. (36) is similar to an anti-windup system that constrains the selection parameter p. This anti-windup system guarantees that Condition (A2) is fulfilled, if

$$|\mathbf{k}_1^T \mathbf{x}_0| \leq u_0 \quad \text{for all} \quad \mathbf{x}_0 \in \mathcal{X}_0 \tag{37}$$

holds.

5.2 Robust stability

Starting with the considerations from Franke (1983), we simplify them in order to get an easier examination method.

From the definitions above, we obtain the following description of the overall closed-loop system:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \left(\widehat{\mathbf{A}}(\mathbf{q}) - p \cdot \mathbf{b}(\mathbf{q})\mathbf{k}_2^T\right)\mathbf{x} \\ \gamma^{-1}\left(\mathbf{x}^T \mathbf{R}\mathbf{b}_0 \mathbf{k}_2^T \mathbf{x} - p\, r(p, \mathbf{x})\right) \end{bmatrix}, \tag{38}$$

where $\widehat{\mathbf{A}}(\mathbf{q}) = \mathbf{A}(\mathbf{q}) - \mathbf{b}(\mathbf{q})\mathbf{k}_1^T$. The proposed robust stability proof of system (38) is subject to two restrictions on the parameter dependency of the input vector $\mathbf{b}(\mathbf{q})$:

(B1) The parameter dependency of the input vector of system (1) is summarized in a single positive scalar function $\kappa(\mathbf{q})$, that is

$$\mathbf{b}(\mathbf{q}) = \kappa(\mathbf{q})\mathbf{b}_0, \tag{39}$$

where $\kappa(\mathbf{q}) > 0$.

(B2) The input vector of system (1) is not time-varying, that is $\dot{\kappa}(\mathbf{q}) = 0$.

The first restriction is not very hard because in the vast majority of practical cases the input gain of a plant does not change its sign. The second one is more demanding and excludes time-varying input vectors. Nevertheless, the class of the remaining systems is large.

Now, a quadratic Lyapunov function defined by

$$v(p, \mathbf{x}) = \mathbf{x}^T \mathbf{R}\mathbf{x} + \gamma\kappa(\mathbf{q})p^2 \tag{40}$$

is utilized to guarantee robust stability of the closed-loop system (38). The system's equilibrium point $(p, \mathbf{x}) = (0, \mathbf{0})$ is asymptotically stable if $\dot{v}(p, \mathbf{x}) < 0$ for all $(p, \mathbf{x}) \neq (0, \mathbf{0})$. Computing $\dot{v}(p, \mathbf{x})$ yields

$$\dot{v}(p,\mathbf{x}) = \dot{\mathbf{x}}^T\mathbf{R}\mathbf{x} + \mathbf{x}^T\mathbf{R}\dot{\mathbf{x}} + \gamma\,\kappa(\mathbf{q})2p\,\dot{p}$$
$$= \mathbf{x}^T\left[\widehat{\mathbf{A}}^T(\mathbf{q})\mathbf{R} + \mathbf{R}\widehat{\mathbf{A}}(\mathbf{q})\right]\mathbf{x}$$
$$- 2p^2\kappa(\mathbf{q})\,r(p,\mathbf{x}) < 0. \quad (41)$$

Since $r(p,\mathbf{x}) > 0$ and $\kappa(\mathbf{q}) > 0$, inequality (41) is satisfied if

$$\widehat{\mathbf{A}}^T(\mathbf{q})\mathbf{R} + \mathbf{R}\widehat{\mathbf{A}}(\mathbf{q}) < 0. \quad (42)$$

Again, we summarize the conditions that are sufficient for the robust stability of dynamic SVSC in a theorem.

Theorem 3. Let the system (1) subject to the constraints (2) and (3) and confined by the additional restrictions (B1) and (B2) be controlled by a SVSC according to (35) and (36). Then the closed-loop system satisfies conditions (A1) and (A2) if the inequalities (37) and (42) hold.

6. A UNIFYING FRAMEWORK

The preceding sections revealed that robust stability with respect to parametric uncertainty of all types of SVSC can be proved by parameter-dependent Lyapunov inequalities of the form:

$$\widehat{\mathbf{A}}^T(\tilde{\mathbf{q}})\mathbf{P} + \mathbf{P}\widehat{\mathbf{A}}(\tilde{\mathbf{q}}) < 0, \quad (43)$$

where $\mathbf{P}$ is a constant positive definite matrix and $\tilde{\mathbf{q}}$ is a vector of real parameters. This inequality is similar to (14), (33), and (42). The matrix $\mathbf{P}$ corresponds to matrices $\mathbf{R}$ and $\mathbf{R}_1$. The vector $\tilde{\mathbf{q}}$ contains the uncertain parameters $\mathbf{q}$ and the selection parameter p or v in case of type A or B, respectively, for example $\tilde{\mathbf{q}}^T = [\mathbf{q}^T, v]$.

Generally, ensuring the negative definiteness of the matrix on the left-hand side of (43) is difficult because we cannot check it for each possible $\tilde{\mathbf{q}}$. However, this task can be simplified by reasonable assumptions on the parametric uncertainty. If, for instance, vector $\tilde{\mathbf{q}}$ ranges within a convex polytope $\mathcal{Q}_P$ and the entries of matrix $\widehat{\mathbf{A}}$ depend multi-affinely on $\tilde{\mathbf{q}}$, then (43) holds for all $\tilde{\mathbf{q}} \in \mathcal{Q}_P$ if and only if

$$\widehat{\mathbf{A}}^T(\tilde{\mathbf{q}}^{(i)})\mathbf{P} + \mathbf{P}\widehat{\mathbf{A}}(\tilde{\mathbf{q}}^{(i)}) < 0, \quad (44)$$

holds for all $\tilde{\mathbf{q}}^{(i)}$, $i = 1, \ldots, k$, where $\tilde{\mathbf{q}}^{(i)}$ are the vectors of the k vertices of $\mathcal{Q}_P$ (Horisberger and Belanger, 1976; Garofalo et al., 1993). Thus, checking the definiteness of a finite number of matrices in (44) is sufficient. The literature oh the stability of uncertain linear systems provides a great variety of similar numerically tractable methods to test (43), see e.g. (Barmish and Kang, 1993; Amato, 2006) and the linear matrix inequalities (LMI) standard reference from Boyd et al. (1994).

On this basis, we may say that for practical cases a posteriori analysis of stability robustness against parametric uncertainty of all types of SVSC can be reduced to a numerically tractable problem: It is sufficient to check the negative definiteness of a finite number of matrices.

Remark: In LPV control the Lyapunov equalities are often extended by introducing a parameter-dependent Matrix $\mathbf{P}(\tilde{\mathbf{q}})$ to obtain less conservative results. Since the corresponding matrices $\mathbf{R}$ and $\mathbf{R}_1$ are used during the online calculation of the control input and the parameters are unknown this extension is not possible here.

7. ILLUSTRATING EXAMPLE: AN OVERHEAD CRANE CONTROL

The plant to be controlled is the well-known model of an overhead crane taken from Ackermann (2002). In this case, the crab mass is 1 tonne, the rope length is 10 meters, and the load mass q can vary over the range of 1...10 tonnes with a nominal value $q_0 = 3$ tonnes. The resulting system is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 9.81q & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.981(q+1) & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 10^{-3} \\ 0 \\ -10^{-4} \end{bmatrix}u,$$

where x_1 and x_2 are the position and the velocity of the crab whereas x_3 and x_4 are the angle and the angular velocity of the rope, respectively. The constraints regarding (2) and (3) are specified by

$$|u| \le 10^4 \text{ N} \quad \text{and}$$
$$\mathcal{X}_0 = \{\mathbf{x}_0 \in \mathbb{R}^4 |\, |x_{1,0}| \le 12 \text{ m},\ |x_{2,0}| \le 1 \text{ m/s},$$
$$|x_{3,0}| \le 2^\circ \tfrac{\pi}{180^\circ}, |x_{4,0}| \le 0.5^\circ \tfrac{\pi}{180^\circ \cdot \text{s}}\}.$$

The control task is to drive the position of the load,

$$y = [1\ \ 0\ \ 10\ \ 0]\,\mathbf{x}, \quad (45)$$

from the initial state $\mathbf{x}_0 = [12\,\text{m}, 0, 0, 0]^T$ to the tolerance band $|y| \le 2\,\text{cm}$ in minimum settling time T_S. The position y must not undershoot -2 cm in order to avoid accidents. Additionally, the controls have to meet the conditions (A1) and (A2) as stated in Section 2.

The controller parameters for all three types of SVSC were obtained by an evolutionary algorithm (Schwefel, 1995), where the optimization problem consists of i) constraints resulting from the corresponding theorems and ii) a quadratic cost functional, which is calculated by

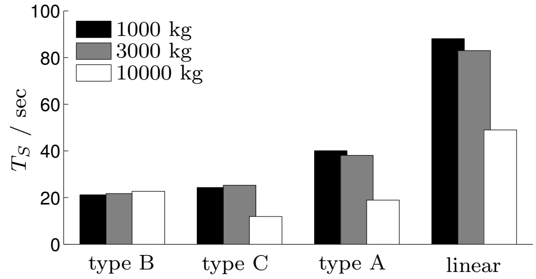

Fig. 3. Comparison of the settling time, T_S, of the different controllers for minimal, nominal, and maximal load. The settling time is defined by the time that the control loops require to drive the load into the tolerance band $|y| \le 2\,\text{cm}$ from the initial state $\mathbf{x}_0 = [12\,\text{m}, 0, 0, 0]^T$.(A): variable saturation, (B): implicit, (C): dynamic.

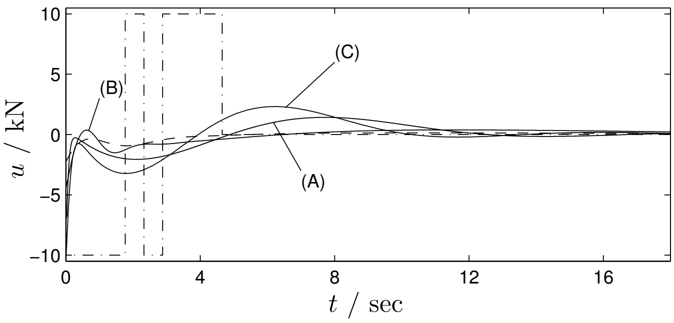

Fig. 4. Control action of the different controllers for nominal load. (A): variable saturation, (B): implicit, (C): dynamic, (dashed): linear, (dash-dotted): time optimal feedforward control.

numerical simulations. For this purpose the problem is reformulated as an unconstrained one be means of a barrier function. The obtained parameters can be found in the appendix.

For comparison, we use a robust linear control $u = -\mathbf{k}^T\mathbf{x}$, whose parameters are also given in the appendix. By using LMI-techniques, it can be proven that this control also meets the conditions (A1) and (A2), Boyd et al. (1994). Fig. 2 shows the output for the three different types of SVSC. The subfigures on the left-hand side depict the whole range while the ones on the right focus on the tolerance band $|y| \leq 2\,\text{cm}$. Additionally, in the same figure the results using a robust linear control $u = -\mathbf{k}^T\mathbf{x}$ and the time-optimal feedforward control are plotted. As it can be seen from Fig. 2 the performance of all three types of SVSC is significantly higher than that of the linear control. To quantify this result, Fig. 3 considers the achieved settling times, T_S, obtained from the simulations. The SVSC's control is "soft" – in contrast to the time optimal control input. This is exemplarily shown in Fig. 4 for the nominal case. For an assessment of the computational burden for calculating the control input we refer the reader to the survey paper (Adamy and Flemming, 2004).

Appendix A. PARAMETERS OF SIMULATION EXAMPLE

Soft VSC with variable saturation:

$$\mathbf{R} = \begin{bmatrix} 0.1364 & 0.8095 & -3.97 & 7.461 \\ 0.8095 & 7.453 & -37.87 & 69.01 \\ -3.97 & -37.87 & 2108 & -180 \\ 7.461 & 69.01 & -180 & 873.7 \end{bmatrix} \cdot 10^{-3},$$

$$\mathbf{k}_1^T = [239.9\ 2019\ -29790\ 2389],$$

$$\mathbf{k}_2^T = [634.3\ 4197\ -16880\ -107300].$$

Soft VSC employing implicit Lyapunov functions:

$$\mathbf{R}_1 = \begin{bmatrix} 1.59 & 18.49 & 20.25 & 0.6329 \\ 18.49 & 255.1 & 248.8 & 16.95 \\ 20.25 & 248.8 & 831.6 & 131.5 \\ 0.6329 & 16.95 & 131.5 & 124.8 \end{bmatrix} \cdot 10^{-3},$$

$$\widehat{\mathbf{a}}^T = [0.1593\ 1.997\ 9.543\ 2.526].$$

Dynamic SVSC:

$$\mathbf{R} = \begin{bmatrix} 6.996 & 11.51 & -90.01 & 82.07 \\ 11.51 & 52.34 & -238 & 450.1 \\ -90.01 & -238 & 10500 & -433.8 \\ 82.07 & 450.1 & -433.8 & 4690 \end{bmatrix}$$

$$\mathbf{k}_1^T = [466.3\ 3509\ -2739\ -31030],$$

$$\mathbf{k}_2^T = [-121.1\ -510.1\ 5260\ 2544] \cdot 10^3,$$

$$\mu = 10^9, \quad \mu_0 = 10^{-3}, \quad \gamma = 2 \cdot 10^6, \quad P = 1.$$

Linear controller:

$$\mathbf{k}^T = [187.35\ 2869.3\ -26048\ 2455.3].$$

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