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Robust fault isolation observers for non-square systems - a parametric approach

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Abstract: In this article, a linear matrix inequality (LMI)-based design for robust fault isolation observers (FIOs) for linear systems with arbitrary fault detectability indices is presented. A parametric design is used to achieve stable fault isolation in square systems as well as non-square systems. Based on this design, the influence of arbitrary disturbances is attenuated by a proper optimization of the observer gain matrices. The applicability of the proposed design is verified in simulations of a helicopter model.

Keywords: robust fault isolation, observer-based fault isolation, LMIs, parametric design

1. INTRODUCTION

Due to increasing demands on reliability and systems complexity, fault diagnosis has been receiving a lot of attention during the past years. Among other techniques, observer-based fault detection and isolation is used to incorporate knowledge about the system dynamics into the process of fault diagnosis (s. e.g. Chen and Patton (1999), Ding (2008) for an overview).

For practical applications, robustness issues have to be addressed, since in all real-world scenarios systems are subject to uncertain parameters as well as exogenous disturbances. Usually there is a tradeoff between robustness and detection performance, which can be solved by optimization procedures (Mazars et al. (2007), Zhong et al. (2003)). Recently, fault detection in a finite frequency range has also been taken into account (Wang et al. (2007), Li et al. (2011)).

In Liu and Si (1997) duality to non-interacting control is used to design so called *fault isolation observers* (FIOs) that allow to assign a diagonal structure to the transfer matrix relating faults and generated residuals and thus achieve isolation instead of mere detection. Therefore the results of Falb and Wolovich (1967) and subsequent results from the field of non-interacting control can be applied to fault isolation. However, several degrees of freedom are used to achieve isolation and thus less design variables are available to optimize robustness. This can partly be compensated for by employing dynamic observers (Wahrburg and Adamy (2012)). Another possibility is the use of additional measurements resulting in non-square systems. In this case, it is shown by Liu and Si (1997) that there are additional degrees of freedom in the observer design. In Jaimoukha et al. (2006), these are exploited by means of convex, LMI-based optimization to increase disturbance attenuation. Both results however are restricted to systems with fault detectability indices all 1. The approach by Chen and Nagarajaiah (2007)) is based on eigenstructure assignment and also uses LMI-based optimization to account for disturbances.

In this paper, we present a parametric method for designing FIOs, which is similar to the complete modal synthesis introduced by Roppenecker (1986). It is first introduced for square systems, in which the possible number of faults is equal to the number of measurable outputs. The main benefit of the method is the insight it provides into the internal dynamics that might occur. Furthermore, it allows an extension to non-square systems where we emphasize that arbitrary fault detectability indices can be treated. Based on the parametric approach we also present a design procedure similar to the non-interacting controller design by Falb and Wolovich (1967). We further optimize robustness of the FIOs with respect to disturbances by means of an LMI-based design.

Hence, the paper is structured as follows. In the next section, we first summarize some mathematical tools and notation before a proper problem description is given. Section 3 deals with a parametric approach to observer-based fault isolation for both square and non-square systems. In Section 4, we present a time-domain solution to fault-isolation in non-square systems. Based on this design, an LMI-based optimization is proposed that minimizes the influence of disturbances on the residuals while preserving certain performance criteria. The applicability of the proposed methods is shown by simulations in Section 5 before a conclusion is given.

2. PRELIMINARIES

2.1 Notation and mathematical background

The identity matrix of order n is written as $\boldsymbol{I}_n$ while $\boldsymbol{0}$ denotes a matrix of zeros of appropriate dimensions and $\boldsymbol{\phi}_i$ is a unit vector with the i-th element equal to 1 and all other elements 0. Diagonal matrices are written as $\mathbb{R}^{n\times n} \ni \boldsymbol{P} = \operatorname{diag}(p_{11}, \dots, p_{nn})$. The spectrum of a matrix $\boldsymbol{P}$ is given by $\sigma(\boldsymbol{P})$ and the rank is written as $\operatorname{rank}(\boldsymbol{P})$. For a complex number $\lambda \in \mathbb{C}$, $\mathfrak{Re}(\lambda)$ denotes the real part of λ. Positive and negative definiteness of a matrix is denoted by $\boldsymbol{P} \succ \mathbf{0}$ and $\boldsymbol{P} \prec \mathbf{0}$, respectively. For $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ the Hermitian part is written as $\mathrm{He}(\boldsymbol{Q}) = \boldsymbol{Q} + \boldsymbol{Q}^{\mathsf{T}}$. Similar to the kernel of a matrix $\boldsymbol{Q}$ we write the left null space of $\boldsymbol{Q}$ as $\mathrm{lker}(\boldsymbol{Q}) = \left\{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{x}^{\mathsf{T}}\boldsymbol{Q} = \mathbf{0}^{\mathsf{T}}\right\}$. The dimension of a subspace $\mathcal{S}$ is abbreviated by $\dim(\mathcal{S})$. In symmetric matrices, $\star$ denotes symmetric elements. For $\boldsymbol{C} \in \mathbb{R}^{n\times m}$, $\boldsymbol{c}_i^{\mathsf{T}}$ denotes the i-th row of $\boldsymbol{C}$ while the j-th column of $\boldsymbol{C}$ is written as $\boldsymbol{c}_j$. Given a matrix $\boldsymbol{P} \in \mathbb{R}^{n\times m}$, with $n \geq m$ the Moore-Penrose-Inverse of $\boldsymbol{P}$ is denoted by $\boldsymbol{P}^{+} \in \mathbb{R}^{m\times n}$ where $\boldsymbol{P}^{+}\boldsymbol{P} = \boldsymbol{I}_m$ if $\boldsymbol{P}$ is left-invertible. A linear dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$, $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$ is abbreviated by $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$.

To solve the optimization problem we need the well-known Bounded-Real-Lemma (Boyd et al. (1994)) which reads

Lemma 1. Given a stable linear system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u}$, $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$ with transfer matrix $\boldsymbol{G}(s) = \boldsymbol{C}\left(s\boldsymbol{I} - \boldsymbol{A}\right)^{-1}\boldsymbol{B} + \boldsymbol{D}$. Then $||\boldsymbol{G}(s)||_\infty < \gamma$ holds if and only if there exists a real symmetric matrix $\boldsymbol{X} \succ \mathbf{0}$, such that

$$\begin{bmatrix} \mathrm{He}(\boldsymbol{X}\boldsymbol{A}) & \boldsymbol{X}\boldsymbol{B} & \boldsymbol{C}^{\mathsf{T}} \\ \star & -\gamma\boldsymbol{I} & \boldsymbol{D}^{\mathsf{T}} \\ \star & \star & -\gamma\boldsymbol{I} \end{bmatrix} \prec \mathbf{0}.$$

Furthermore, we use the following lemma which can easily be inferred from the rank-nullity theorem (s. Meyer (2001)).

Lemma 2. Given a matrix $\boldsymbol{P} \in \mathbb{R}^{m\times n}$. Then $\mathrm{rank}(\boldsymbol{P}) + \dim(\mathrm{lker}(\boldsymbol{P})) = m$ holds.

2.2 Problem statement

We consider linear systems described by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{E}\boldsymbol{f} + \boldsymbol{B}_d\boldsymbol{d}, \tag{1a}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}_d\boldsymbol{d}, \tag{1b}$$

with $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^{n_u}$, $\boldsymbol{f} \in \mathbb{R}^{n_f}$, $\boldsymbol{d} \in \mathbb{R}^{n_d}$, $\boldsymbol{y} \in \mathbb{R}^{n_y}$ and matrices of appropriate dimensions. Herein, $\boldsymbol{d}$ describes arbitrary disturbances, which may be exogenous or result from parametric uncertainties in the system matrices. Despite these disturbances, actuator faults $\boldsymbol{f}$ are to be detected and isolated by inspection of residuals $\boldsymbol{r} \in \mathbb{R}^{n_f}$ (s. Fig. 1), which are generated by the FIO

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}\left(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}}\right), \tag{2a}$$
$$\boldsymbol{r} = \boldsymbol{V}\left(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}}\right). \tag{2b}$$

By proper selection of the observer gains $(\boldsymbol{L}, \boldsymbol{V})$, the objective is to render the transfer matrix $\boldsymbol{G}_{\boldsymbol{rf}}(s)$ relating faults and residuals diagonal, i.e.

$$\boldsymbol{G}_{\boldsymbol{rf}}(s) = \mathrm{diag}\left(g_{r_1 f_1}(s), \ldots, g_{r_{n_f} f_{n_f}}(s)\right). \tag{3}$$

In this, $g_{r_i f_i}(s)$ is the transfer channel form fault f_i to residual r_i. Since sensor faults can be recast as pseudo-actuator faults (s. Park et al. (1994)) they are not explicitly considered here. With $\boldsymbol{e}_i$ describing the i-th column of $\boldsymbol{E}$, the *fault detectability indices* δ_i (Liu and Si (1997)) are defined as

$$\delta_i = \min\left\{k\colon \boldsymbol{C}\boldsymbol{A}^{k-1}\boldsymbol{e}_i \neq \mathbf{0},\, k = 1, 2, \ldots\right\} \tag{4}$$

and describe how many times the output has to be differentiated with respect to time for a fault f_i to appear in $\boldsymbol{y}^{(\delta_i)}(t)$. Furthermore define

$$\delta = \sum_{i=1}^{n_f} \delta_i \tag{5}$$

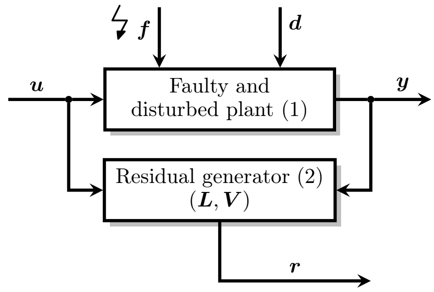


Fig. 1. Observer-based residual generation

and the *fault detectability matrix* $\boldsymbol{D}^* \in \mathbb{R}^{n_y\times n_f}$ with

$$\boldsymbol{D}^* = \left[\boldsymbol{C}\boldsymbol{A}^{\delta_1-1}\boldsymbol{e}_1 \; \cdots \; \boldsymbol{C}\boldsymbol{A}^{\delta_{n_f}-1}\boldsymbol{e}_{n_f}\right]. \tag{6}$$

Note that we do not restrict our results to square systems but consider the more general case of $n_y \geq n_f$ with arbitrary fault detectability indices. We further use

Assumption 1. The fault distribution matrix and output matrix fulfill $\mathrm{rank}(\boldsymbol{E}) = n_f$ and $\mathrm{rank}(\boldsymbol{C}) \geq n_f$.

Assumption 2. The pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable.

Assumption 3. The matrix $\boldsymbol{D}^*$ is left-invertible.

While Assumption 1 and 3 ensure that the system offers enough linearly independent measurements to achieve perfect fault isolation in the nominal case, Assumption 2 allows to arbitrarily place all observer eigenvalues.

3. A PARAMETRIC APPROACH TO OBSERVER-BASED FAULT ISOLATION

In Liu and Si (1997) duality between non-interacting control and fault isolation observer design is exploited to design FIOs. To this end, following Falb and Wolovich (1967) a non-interacting control law is designed for the dual system. The results however require all fault detectability indices to be 1, which severely limits the class of tractable systems. The same limitation holds in the work of Jaimoukha et al. (2006). This is due to internal dynamics that might become unstable in the optimization process. The reason for this is the limited insight the Falb-Wolovich method provides into the internal system dynamics. Here we propose a different approach dual to the complete modal synthesis by Roppenecker and Lohmann (1988), which allows on the one hand to understand why the system might become unstable in the previously mentioned approaches and on the other hand results in a design guaranteeing internal stability for systems with arbitrary fault detectability indices.

3.1 Fault isolation in square systems

Generally, in the absence of disturbances and uncertainties, perfect fault isolation as in (3) can be achieved if $\boldsymbol{D}^*$ is left-invertible (s. Liu and Si (1997)), which is fulfilled in this paper due to Assumption 3. To design an FIO we first determine the dynamics of the observer error $\boldsymbol{\zeta} = \boldsymbol{x} - \hat{\boldsymbol{x}}$, resulting in

$$\dot{\boldsymbol{\zeta}} = \left(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}\right)\boldsymbol{\zeta} + \boldsymbol{E}\boldsymbol{f}, \tag{7a}$$
$$\boldsymbol{r} = \boldsymbol{V}\boldsymbol{C}\boldsymbol{\zeta}. \tag{7b}$$

To further analyze the system dynamics, let $\lambda_{R_i} \in \sigma(\boldsymbol{A}-\boldsymbol{LC})$ be the observer eigenvalues to be assigned and $\boldsymbol{V}_R$ the corresponding matrix of right-eigenvectors. To simplify notation, we make

Assumption 4. No two observer eigenvalues are the same, i.e. $\lambda_{R_i} \neq \lambda_{R_j}\ \forall i,j=1,\ldots,n$.

Assumption 5. No observer eigenvalue is an eigenvalue of $\boldsymbol{A}$, i.e. $\lambda_{R_i} \notin \sigma(\boldsymbol{A})\ \forall i=1,\ldots,n$.

As shown by Roppenecker (1986) for the dual problem of non-interacting controller design, both assumptions do not impose a loss of generality and can be relaxed by adequate extensions of the method. Those are omitted here due to space restrictions. With this in mind, (7) is formulated as

$$\begin{aligned}\boldsymbol{G_{rf}}(s) &= \boldsymbol{VC}\left(s\boldsymbol{I}_n-(\boldsymbol{A}-\boldsymbol{LC})\right)^{-1}\boldsymbol{E}\\ &= \boldsymbol{VCV}_R\boldsymbol{V}_R^{-1}\left(s\boldsymbol{I}_n-(\boldsymbol{A}-\boldsymbol{LC})\right)^{-1}\boldsymbol{V}_R\boldsymbol{V}_R^{-1}\boldsymbol{E}\\ &= \boldsymbol{VCV}_R\mathrm{diag}\left(\frac{1}{s-\lambda_{R_1}},\ldots,\frac{1}{s-\lambda_{R_n}}\right)\boldsymbol{V}_R^{-1}\boldsymbol{E}\\ &= \sum_{k=1}^{n}\frac{\boldsymbol{VCv}_{R_k}\cdot\boldsymbol{w}_{R_k}^{\mathsf{T}}\boldsymbol{E}}{s-\lambda_{R_k}} \end{aligned} \tag{8}$$

Note that the diagonalization is possible due to Assumption 4. In (8), $\boldsymbol{v}_{R_k}$ are the right-eigenvectors while $\boldsymbol{w}_{R_k}^{\mathsf{T}}$ describe the left-eigenvectors corresponding to λ_{R_k}. To achieve the diagonal structure prescribed in (3), the first δ observer eigenvalues are denoted by $\lambda_{R_{ij}}$ in the following. The first index $i=1,\ldots,n_f$ characterizes the transfer channel the eigenvalue is assigned to, while the second index $j=1,\ldots,\delta_i$ numbers the eigenvalues in channel i. Because of (5), there remain $n-\delta$ eigenvalues λ_{R_k} that are not to appear in $\boldsymbol{G_{rf}}(s)$.

It follows from (8) that $\lambda_{R_{ij}}$ appears solely in the i-th column of $\boldsymbol{G_{rf}}(s)$ if

$$\boldsymbol{w}_{R_{ij}}^{\mathsf{T}}\boldsymbol{E} = \boldsymbol{\phi}_i^{\mathsf{T}} = [0\ \ldots\ 0\ \underbrace{1}_{i\text{-th column}}\ 0\ \ldots\ 0] \tag{9}$$

is fulfilled. From the definition of left-eigenvectors we have

$$\boldsymbol{w}_{R_{ij}}^{\mathsf{T}}\left(\lambda_{R_{ij}}\boldsymbol{I}_n-(\boldsymbol{A}-\boldsymbol{LC})\right)=\boldsymbol{0}^{\mathsf{T}}. \tag{10}$$

Introducing so called *parameter vectors* $\boldsymbol{p}_{ij}^{\mathsf{T}} = -\boldsymbol{w}_{R_{ij}}^{\mathsf{T}}\boldsymbol{L}$, this can be written as

$$\boldsymbol{w}_{R_{ij}}^{\mathsf{T}}\left(\lambda_{R_{ij}}\boldsymbol{I}_n-\boldsymbol{A}\right)=\boldsymbol{p}_{ij}^{\mathsf{T}}\boldsymbol{C}. \tag{11}$$

Based on Assumption 5, we can further write

$$\boldsymbol{w}_{R_{ij}}^{\mathsf{T}}=\boldsymbol{p}_{ij}^{\mathsf{T}}\boldsymbol{C}\left(\lambda_{R_{ij}}\boldsymbol{I}_n-\boldsymbol{A}\right)^{-1}. \tag{12}$$

Inserting (12) into (9) leads to

$$\boldsymbol{p}_{ij}^{\mathsf{T}}\underbrace{\boldsymbol{C}\left(\lambda_{R_{ij}}\boldsymbol{I}_n-\boldsymbol{A}\right)^{-1}\boldsymbol{E}}_{\boldsymbol{\Pi}(\lambda_{R_{ij}})}=\boldsymbol{\phi}_i^{\mathsf{T}}. \tag{13}$$

If $\lambda_{R_{ij}}$ is selected such that it is not a transmission zero of $(\boldsymbol{A},\boldsymbol{E},\boldsymbol{C})$, then $\boldsymbol{\Pi}\left(\lambda_{R_{ij}}\right)$ is invertible and the parameter vector is found as

$$\boldsymbol{p}_{ij}^{\mathsf{T}}=\boldsymbol{\phi}_i^{\mathsf{T}}\boldsymbol{\Pi}\left(\lambda_{R_{ij}}\right)^{-1}. \tag{14}$$

The first δ parameter vectors are thus given by $\boldsymbol{p}_1^{\mathsf{T}} = \boldsymbol{p}_{11}^{\mathsf{T}},\ldots,\boldsymbol{p}_{\delta_1}^{\mathsf{T}} = \boldsymbol{p}_{1\delta_1}^{\mathsf{T}},\ldots,\boldsymbol{p}_{\delta}^{\mathsf{T}} = \boldsymbol{p}_{n_f\delta_{n_f}}^{\mathsf{T}}$ and therefore in case of $\delta=n$, all parameter vectors are characterized. The observer matrix $\boldsymbol{L}_p$ resulting from the parametric approach is given by

$$\boldsymbol{L}_p=-\boldsymbol{V}_R\boldsymbol{P}, \tag{15a}$$

$$\boldsymbol{P}=\begin{bmatrix}\boldsymbol{p}_1^{\mathsf{T}}\\ \vdots\\ \boldsymbol{p}_n^{\mathsf{T}}\end{bmatrix},\ \boldsymbol{V}_R=\begin{bmatrix}\boldsymbol{p}_1^{\mathsf{T}}\boldsymbol{C}\left(\lambda_1\boldsymbol{I}_n-\boldsymbol{A}\right)^{-1}\\ \vdots\\ \boldsymbol{p}_n^{\mathsf{T}}\boldsymbol{C}\left(\lambda_n\boldsymbol{I}_n-\boldsymbol{A}\right)^{-1}\end{bmatrix}^{-1}. \tag{15b}$$

The existence of the inverse can be shown by similar arguments as in Roppenecker and Lohmann (1988).

If $\delta<n$, the eigenvalues λ_{R_k} with $k=\delta+1,\ldots,n$ must not appear in the closed-loop transfer function $\boldsymbol{G_{rf}}(s)$ which is guaranteed by $\boldsymbol{w}_{R_k}^{\mathsf{T}}\boldsymbol{E}=\boldsymbol{0}^{\mathsf{T}}$. Since furthermore $\boldsymbol{w}_{R_k}^{\mathsf{T}}\left(\lambda_{R_k}\boldsymbol{I}_n-\boldsymbol{A}\right)-\boldsymbol{p}_k^{\mathsf{T}}\boldsymbol{C}=\boldsymbol{0}^{\mathsf{T}}$ holds,

$$\begin{bmatrix}\boldsymbol{w}_{R_k}^{\mathsf{T}} & \boldsymbol{p}_k^{\mathsf{T}}\end{bmatrix}\underbrace{\begin{bmatrix}\lambda_{R_k}\boldsymbol{I}_n-\boldsymbol{A} & \boldsymbol{E}\\ -\boldsymbol{C} & \boldsymbol{0}\end{bmatrix}}_{\boldsymbol{\Sigma}(\lambda_{R_k})}=\boldsymbol{0}^{\mathsf{T}} \tag{16}$$

has to be fulfilled. In square systems we have $n_y=n_f$ and thus the Rosenbrock matrix $\boldsymbol{\Sigma}\left(\lambda_{R_k}\right)\in\mathbb{R}^{(n+n_f)\times(n+n_f)}$. By Assumption 1, $\mathrm{rank}\left(\boldsymbol{\Sigma}\left(\lambda_{R_k}\right)\right)=n+n_f$ for almost all λ_{R_k} and it follows from Lemma 2 that (16) only has a non-trivial solution if λ_{R_k} is an invariant zero of $(\boldsymbol{A},\boldsymbol{E},\boldsymbol{C})$. If $(\boldsymbol{A},\boldsymbol{E},\boldsymbol{C})$ is non-minimum phase this causes unstable eigenvalues λ_{R_k}, which is the reason why only minimum phase square systems can be stably fault isolated by means of a static observer in general.

Once $\boldsymbol{L}_p$ is calculated, we have $\boldsymbol{\Lambda}=\boldsymbol{C}(\boldsymbol{A}-\boldsymbol{L}_p\boldsymbol{C})^{-1}\boldsymbol{E}$ and the filter matrix $\boldsymbol{V}$ is selected such that $\boldsymbol{G_{rf}}(s)$ achieves static gains of $\tilde{z}_i$, i.e. $\boldsymbol{G_{rf}}(s=0)=\mathrm{diag}\left(\tilde{z}_1,\ldots,\tilde{z}_{n_f}\right)$. Thus

$$\boldsymbol{V}_p=-\mathrm{diag}\left(\tilde{z}_1,\ldots,\tilde{z}_{n_f}\right)\boldsymbol{\Lambda}^{-1}. \tag{17}$$

3.2 Fault isolation in non-square systems

Intuitively, fault isolation should benefit from additional measurements. However, in the design of FIOs in case of $n_y>n_f$ there arise problems regarding stability and the approaches in Liu and Si (1997) and Jaimoukha et al. (2006) are restricted to systems with $\delta_i=1\,\forall i=1,\ldots,n_f$. Again, complete modal synthesis offers deeper insight into the system dynamics and allows to design stable FIOs for arbitrary δ_i if $\boldsymbol{D}^*$ is left-invertible.

Considering (13), it is obvious that $\boldsymbol{\Pi}\left(\lambda_{R_{ij}}\right)\in\mathbb{R}^{n_y\times n_f}$ for non-square systems. As for square systems, $\lambda_{R_{ij}}$ can be placed arbitrarily but not at transmission zeros of $(\boldsymbol{A},\boldsymbol{E},\boldsymbol{C})$. Then $\boldsymbol{\Pi}\left(\lambda_{R_{ij}}\right)$ is left-invertible and (13) is fulfilled by all parameter vectors given by

$$\boldsymbol{p}_{ij}^{\mathsf{T}}=\boldsymbol{\phi}_i^{\mathsf{T}}\left(\boldsymbol{\Pi}_{ij}^{+}+\boldsymbol{\Gamma}_{ij}\left(\boldsymbol{I}_{n_y}-\boldsymbol{\Pi}_{ij}\cdot\boldsymbol{\Pi}_{ij}^{+}\right)\right), \tag{18}$$

where $\boldsymbol{\Gamma}_{ij}\in\mathbb{R}^{n_f\times n_y}$ has arbitrary real elements. Note that $\boldsymbol{\Pi}_{ij}=\boldsymbol{\Pi}\left(\lambda_{R_{ij}}\right)$ due to notational purposes. Multiplying (18) with $\boldsymbol{\Pi}_{ij}$ from the right results in

$$\begin{aligned}\boldsymbol{p}_{ij}^{\mathsf{T}}\boldsymbol{\Pi}_{ij}&=\boldsymbol{\phi}_i^{\mathsf{T}}\boldsymbol{\Pi}_{ij}^{+}\boldsymbol{\Pi}_{ij}+\boldsymbol{\phi}_i^{\mathsf{T}}\boldsymbol{\Gamma}_{ij}\left(\boldsymbol{\Pi}_{ij}-\boldsymbol{\Pi}_{ij}\cdot\boldsymbol{\Pi}_{ij}^{+}\boldsymbol{\Pi}_{ij}\right)\\ &=\boldsymbol{\phi}_i^{\mathsf{T}}+\boldsymbol{\phi}_i^{\mathsf{T}}\boldsymbol{\Gamma}_{ij}\left(\boldsymbol{\Pi}_{ij}-\boldsymbol{\Pi}_{ij}\right)=\boldsymbol{\phi}_i^{\mathsf{T}},\end{aligned} \tag{19}$$

which is equal to (13) and proves the claim. For $k=\delta+1,\ldots,n$ we have to solve (16) again, but with a non-square $\boldsymbol{\Sigma}\left(\lambda_{R_k}\right)\in\mathbb{R}^{(n+n_y)\times(n+n_f)}$. Because of $\mathrm{rank}\left(\boldsymbol{\Sigma}\left(\lambda_{R_k}\right)\right)=n+n_f$ for almost all λ_{R_k}, $\dim\left(\mathrm{lker}\left(\boldsymbol{\Sigma}\right)\left(\lambda_{R_k}\right)\right)=n_y-n_f=r>0$ follows from Lemma 2 for almost all λ_{R_k}. Hence λ_{R_k} can be placed regardless of possibly non-minimum phase invariant zeros opposed to the square case. Therefore, the remaining parameter- and left-eigenvectors are chosen such that

$$\left[\boldsymbol{w}_{R_k}^\mathsf{T}\ \boldsymbol{p}_k^\mathsf{T}\right] \in \operatorname{lker}\left(\boldsymbol{\Sigma}\left(\lambda_{R_k}\right)\right), k = \delta+1,\ldots,n. \quad (20)$$

Let $\{\psi_{R_{k,1}}^\mathsf{T},\ldots,\psi_{R_{k,r}}^\mathsf{T}\}$ be an r-dimensional basis spanning $\operatorname{lker}(\boldsymbol{\Sigma})(\lambda_{R_k})$. Then with $\boldsymbol{\Gamma}_k \in \mathbb{R}^{1\times r}$ all possible parameter- and left-eigenvectors are given by

$$\left[\boldsymbol{w}_{R_k}^\mathsf{T}\ \boldsymbol{p}_k^\mathsf{T}\right] = \underbrace{\left[\gamma_{k,1}\ \cdots\ \gamma_{k,r}\right]}_{\boldsymbol{\Gamma}_k}\begin{bmatrix}\psi_{R_{k,1}}^\mathsf{T}\\ \vdots\\ \psi_{R_{k,r}}^\mathsf{T}\end{bmatrix}, k = \delta+1,\ldots,n. \quad (21)$$

Once all parameter- and left-eigenvectors are calculated, the observer matrix $\boldsymbol{L}_p$ is again given by (15) and similar to (17) we calculate

$$\boldsymbol{V}_p = -\operatorname{diag}\left(\tilde{z}_1,\ldots,\tilde{z}_{n_f}\right)\boldsymbol{\Lambda}^+ + \boldsymbol{\Gamma}_V\left(\boldsymbol{I}_{n_y} - \boldsymbol{\Lambda}\boldsymbol{\Lambda}^+\right) \quad (22)$$

with $\boldsymbol{\Lambda} = \boldsymbol{C}(\boldsymbol{A}-\boldsymbol{L}_p\boldsymbol{C})^{-1}\boldsymbol{E}$ and an arbitrary matrix $\boldsymbol{\Gamma}_V \in \mathbb{R}^{n_f\times n_y}$.

While Theorem 3.1 in Liu and Si (1997) requires $\delta_i = 1\,\forall i = 1,\ldots,n_f$, we emphasize that our results allow stable fault isolation in non-square systems with arbitrary fault detectability indices. The resulting internal eigenvalues can be arbitrarily placed by the proposed parametric design. Apart from that, $\boldsymbol{\Gamma}_{ij}$, $\boldsymbol{\Gamma}_k$, and $\boldsymbol{\Gamma}_V$ provide additional degrees of freedom in the design. While the parametric approach allows the design of a stable FIO, a Falb-Wolovich-based design is better suited to utilize the degrees of freedom to optimize robustness, which is shown in the following Section.

4. TIME DOMAIN SOLUTION TO OBSERVER-BASED FAULT ISOLATION

4.1 General design procedure

To exploit the additional degrees of freedom resulting from $n_y > n_f$, we optimize the solution obtained by the parametric approach described in Section 3.2 increasing robustness with respect to exogenous disturbances. Therefore, we first present a Theorem that extends the results by Liu and Si (1997) and Jaimoukha et al. (2006).

Theorem 1. Given a system (1) with arbitrary fault detectability indices $\delta_i \geq 1$ and left-invertible $\boldsymbol{D}^*$, all observers parameterized by

$$\boldsymbol{L} = \boldsymbol{L}_0 + \boldsymbol{R}_M\boldsymbol{Z},$$
$$\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{R}_N\boldsymbol{Z}$$

with $\boldsymbol{L}_0 = \boldsymbol{M}\boldsymbol{D}^{*+}$, $\boldsymbol{V}_0 = \boldsymbol{N}\boldsymbol{D}^{*+}$, $\boldsymbol{Z} = \boldsymbol{I}_{n_y} - \boldsymbol{D}^*\boldsymbol{D}^{*+}$, arbitrary matrices $\boldsymbol{R}_M \in \mathbb{R}^{n\times n_y}$ and $\boldsymbol{R}_N \in \mathbb{R}^{n_f\times n_y}$, and

$$\boldsymbol{M} = \begin{bmatrix}\left(\boldsymbol{A}^{\delta_1}\boldsymbol{e}_{a_1} + \sum_{k=0}^{\delta_1-1} q_{1k}\boldsymbol{A}^k\boldsymbol{e}_{a_1}\right)^\mathsf{T}\\ \vdots\\ \left(\boldsymbol{A}^{\delta_{n_f}}\boldsymbol{e}_{a_{n_f}} + \sum_{k=0}^{\delta_{n_f}-1} q_{n_f k}\boldsymbol{A}^k\boldsymbol{e}_{a_{n_f}}\right)^\mathsf{T}\end{bmatrix}^\mathsf{T},$$

$$\boldsymbol{N} = \operatorname{diag}\left(z_{10},\ldots,z_{n_f 0}\right)$$

result in a diagonal transfer matrix $\boldsymbol{G}_{\boldsymbol{rf}}(s) = \operatorname{diag}\left(g_{r_1f_1}(s),\ldots,g_{r_{n_f}f_{n_f}}\right)$ with

$$g_{r_if_i}(s) = \frac{z_{i0}}{s^{\delta_i} + q_{i\delta_i-1}s^{\delta_i-1} + \ldots + q_{i1}s + q_{i0}}.$$

Proof 1. To prove Theorem 1 consider (7) again. The fact that $(\boldsymbol{L},\boldsymbol{V})$ achieves fault isolation is equivalent to $\left(\boldsymbol{L}^\mathsf{T},\boldsymbol{V}^\mathsf{T}\right)$ being a non-interacting controller for the dual system of (7), which is given by

$$\dot{\boldsymbol{\zeta}} = \left(\boldsymbol{A}^\mathsf{T} - \boldsymbol{C}^\mathsf{T}\boldsymbol{L}^\mathsf{T}\right)\overline{\boldsymbol{\zeta}} + \boldsymbol{C}^\mathsf{T}\boldsymbol{V}^\mathsf{T}\boldsymbol{f}, \quad (23a)$$
$$\boldsymbol{r} = \boldsymbol{E}^\mathsf{T}\overline{\boldsymbol{\zeta}}. \quad (23b)$$

To ease the notation, we further write $\boldsymbol{A}^\mathsf{T} = \overline{\boldsymbol{A}}$, $\boldsymbol{C}^\mathsf{T} = \overline{\boldsymbol{B}}$, $\boldsymbol{E}^\mathsf{T} = \overline{\boldsymbol{C}}$, $\boldsymbol{L}^\mathsf{T} = \overline{\boldsymbol{K}}$ and $\boldsymbol{V}^\mathsf{T} = \overline{\boldsymbol{F}}$, resulting in

$$\dot{\boldsymbol{\zeta}} = \left(\overline{\boldsymbol{A}} - \overline{\boldsymbol{B}}\,\overline{\boldsymbol{K}}\right)\overline{\boldsymbol{\zeta}} + \overline{\boldsymbol{B}}\,\overline{\boldsymbol{F}}\boldsymbol{f}, \quad (24a)$$
$$\boldsymbol{r} = \overline{\boldsymbol{C}}\,\overline{\boldsymbol{\zeta}}. \quad (24b)$$

Considering the i-th residual r_i, we can deduce

$$r_i = \overline{\boldsymbol{c}}_i^\mathsf{T}\overline{\boldsymbol{\zeta}},\ \ \dot{r}_i = \overline{\boldsymbol{c}}_i^\mathsf{T}\overline{\boldsymbol{A}}\,\overline{\boldsymbol{\zeta}}, \quad (25a)$$
$$\vdots$$
$$r_i^{(\delta_i-1)} = \overline{\boldsymbol{c}}_i^\mathsf{T}\overline{\boldsymbol{A}}^{\delta_i-1}\overline{\boldsymbol{\zeta}}, \quad (25b)$$
$$r_i^{(\delta_i)} = \overline{\boldsymbol{c}}_i^\mathsf{T}\overline{\boldsymbol{A}}^{\delta_i}\overline{\boldsymbol{\zeta}} - \overline{\boldsymbol{c}}_i^\mathsf{T}\overline{\boldsymbol{A}}^{\delta_i-1}\overline{\boldsymbol{B}}\,\overline{\boldsymbol{K}}\,\overline{\boldsymbol{\zeta}} + \overline{\boldsymbol{c}}_i^\mathsf{T}\overline{\boldsymbol{A}}^{\delta_i-1}\overline{\boldsymbol{B}}\,\overline{\boldsymbol{F}}\boldsymbol{f}, \quad (25c)$$

since $\overline{\boldsymbol{c}}_i^\mathsf{T}\overline{\boldsymbol{A}}^k\overline{\boldsymbol{B}} = \boldsymbol{0}^\mathsf{T}\,\forall k = 0,\ldots,\delta_i-2$ due to (4). The observer is to assign the dynamics

$$r_i^{(\delta_i)} + q_{i\delta_i-1}r_i^{(\delta_i-1)} + \ldots + q_{i0}r_i = z_{i0}f_i \quad (26)$$

to each transfer channel, which is an equivalent description of $g_{r_if_i}(s)$ in the time domain. Substituting (25) into (26) gives

$$\left(\overline{\boldsymbol{c}}_i^\mathsf{T}\overline{\boldsymbol{A}}^{\delta_i} + \sum_{k=0}^{\delta_i-1} q_{ik}\overline{\boldsymbol{c}}_i^\mathsf{T}\overline{\boldsymbol{A}}^k\right)\overline{\boldsymbol{\zeta}} - \overline{\boldsymbol{c}}_i^\mathsf{T}\overline{\boldsymbol{A}}^{\delta_i-1}\overline{\boldsymbol{B}}\,\overline{\boldsymbol{K}}\,\overline{\boldsymbol{\zeta}} + \ldots$$
$$+ \overline{\boldsymbol{c}}_i^\mathsf{T}\overline{\boldsymbol{A}}^{\delta_i-1}\overline{\boldsymbol{B}}\,\overline{\boldsymbol{F}}\boldsymbol{f} = z_{i0}f_i. \quad (27)$$

Summarizing these equations for all transfer channels results in the condition

$$\overline{\boldsymbol{M}}\,\overline{\boldsymbol{\zeta}} - \overline{\boldsymbol{D}}^*\overline{\boldsymbol{K}}\,\overline{\boldsymbol{\zeta}} + \overline{\boldsymbol{D}}^*\overline{\boldsymbol{F}}\boldsymbol{f} = \overline{\boldsymbol{N}}\boldsymbol{f}, \quad (28)$$

which has to be fulfilled to achieve fault isolation. Note that therein, $\overline{\boldsymbol{M}} = \boldsymbol{M}^\mathsf{T}$, $\overline{\boldsymbol{D}}^* = \boldsymbol{D}^{*\mathsf{T}}$, and $\overline{\boldsymbol{N}} = \boldsymbol{N}^\mathsf{T} = \boldsymbol{N}$ are used. The observer gains proposed in Theorem 1 lead to

$$\overline{\boldsymbol{K}} = \boldsymbol{L}^\mathsf{T} = \overline{\boldsymbol{D}}^{*+}\overline{\boldsymbol{M}} + \left(\boldsymbol{I}_{n_y} - \overline{\boldsymbol{D}}^{*+}\overline{\boldsymbol{D}}^*\right)\overline{\boldsymbol{R}}_M, \quad (29a)$$
$$\overline{\boldsymbol{F}} = \boldsymbol{V}^\mathsf{T} = \overline{\boldsymbol{D}}^{*+}\overline{\boldsymbol{N}} + \left(\boldsymbol{I}_{n_y} - \overline{\boldsymbol{D}}^{*+}\overline{\boldsymbol{D}}^*\right)\overline{\boldsymbol{R}}_N, \quad (29b)$$

with $\overline{\boldsymbol{R}}_M = \boldsymbol{R}_M^\mathsf{T}$ and $\overline{\boldsymbol{R}}_N = \boldsymbol{R}_N^\mathsf{T}$. Substituting (29) into (28) we have

$$\overline{\boldsymbol{M}}\,\overline{\boldsymbol{\zeta}} - \overline{\boldsymbol{D}}^*\left(\overline{\boldsymbol{D}}^{*+}\overline{\boldsymbol{M}} + \left(\boldsymbol{I}_{n_y} - \overline{\boldsymbol{D}}^{*+}\overline{\boldsymbol{D}}^*\right)\overline{\boldsymbol{R}}_M\right)\overline{\boldsymbol{\zeta}} + \ldots$$
$$+ \overline{\boldsymbol{D}}^*\left(\overline{\boldsymbol{D}}^{*+}\overline{\boldsymbol{N}} + \left(\boldsymbol{I}_{n_y} - \overline{\boldsymbol{D}}^{*+}\overline{\boldsymbol{D}}^*\right)\overline{\boldsymbol{R}}_N\right)\boldsymbol{f} = \overline{\boldsymbol{N}}\boldsymbol{f}. \quad (30)$$

Using $\overline{\boldsymbol{D}}^*\overline{\boldsymbol{D}}^{*+} = \boldsymbol{I}_{n_f}$ we further obtain

$$\overline{\boldsymbol{M}}\,\overline{\boldsymbol{\zeta}} - \left(\overline{\boldsymbol{M}} + \left(\overline{\boldsymbol{D}}^* - \overline{\boldsymbol{D}}^*\right)\overline{\boldsymbol{R}}_M\right)\overline{\boldsymbol{\zeta}} + \ldots$$
$$\left(\overline{\boldsymbol{N}} + \left(\overline{\boldsymbol{D}}^* - \overline{\boldsymbol{D}}^*\right)\overline{\boldsymbol{R}}_N - \overline{\boldsymbol{N}}\right)\boldsymbol{f} = \boldsymbol{0}, \quad (31)$$

which is fulfilled and thus completes the proof. □

Remark 1. To achieve the same static gain of $g_{r_if_i}(s)$ as in the parametric approach, $z_{i0} = \tilde{z}_i q_{i0}$ has to be chosen.

Note that according to Theorem 1, only δ eigenvalues are assigned to the transfer channels. Hence, there remain $n-\delta$ uncontrollable eigenvalues, which are not visible in $\boldsymbol{G_{rf}}(s)$ and may result in an unstable FIO depending on the choice of $\boldsymbol{R}_M$. In summary, Theorem 1 guarantees fault isolation but not stability of the observer. However, it is valid to use the solution $(\boldsymbol{L}_p, \boldsymbol{V}_p)$ generated by the parametric approach instead of $(\boldsymbol{L}_0, \boldsymbol{V}_0)$ in Theorem 1, since it is shown in Section 3.2 that it achieves fault isolation and thus $\boldsymbol{L}_p\boldsymbol{D}^* = \boldsymbol{M}$ is fulfilled. Since furthermore $\boldsymbol{L}_0 = \boldsymbol{M}\boldsymbol{D}^{*+}$, we conclude $(\boldsymbol{L}_0 - \boldsymbol{L}_p)\,\boldsymbol{D}^* = \boldsymbol{0}$. Thus $(\boldsymbol{L}_0 - \boldsymbol{L}_p) \in \operatorname{lker}(\boldsymbol{D}^*)$. Since $\boldsymbol{Z}$ is a projector onto $\operatorname{lker}(\boldsymbol{D}^*)$, it is always possible to find an $\tilde{\boldsymbol{R}}_M$, such that $\boldsymbol{L}_0 = \boldsymbol{L}_p + \tilde{\boldsymbol{R}}_M\boldsymbol{Z}$. Therefore we have

Corollary 1. For systems treated in Theorem 1, it is always possible to find an $\boldsymbol{R}_M$, such that fault isolation is achieved and $\mathfrak{Re}\,(\lambda_{R_i}) < 0\,\forall\lambda_{R_i} \in \sigma\,(\boldsymbol{A} - \boldsymbol{LC})$.

4.2 Optimizing robustness with respect to disturbances

Since the existence of a stable FIO is guaranteed by Corollary 1 this section focuses on optimizing robustness with respect to disturbances. To this end, $||\boldsymbol{G_{rd}}(s)||_\infty$ is to be minimized using $\boldsymbol{R}_M$ and $\boldsymbol{R}_N$ as design variables as also proposed in Jaimoukha et al. (2006) and Chen and Nagarajaiah (2007). The system relating disturbances and residuals is

$$\dot{\boldsymbol{\xi}} = (\boldsymbol{A} - \boldsymbol{LC})\,\boldsymbol{\xi} + (\boldsymbol{B}_d - \boldsymbol{LD}_d)\,\boldsymbol{d}, \tag{32a}$$

$$\boldsymbol{r} = \boldsymbol{VC\xi} + \boldsymbol{VD}_d\boldsymbol{d}. \tag{32b}$$

Using Lemma 1, the resulting optimization problem with $\|G_{\boldsymbol{rd}}(s)\|_\infty < \gamma$ can be written as

$$\text{minimize } \gamma, \text{ subject to} \tag{33a}$$

$$\mathbb{R}^{n\times n} \ni \boldsymbol{X} = \boldsymbol{X}^\mathsf{T} \succ \boldsymbol{0}, \tag{33b}$$

$$\begin{bmatrix} \mathrm{He}\,(\boldsymbol{X}(\boldsymbol{A} - \boldsymbol{LC})) & \boldsymbol{X}(\boldsymbol{B}_d - \boldsymbol{LD}_d) & (\boldsymbol{VC})^\mathsf{T} \\ \star & -\gamma\boldsymbol{I}_{n_d} & (\boldsymbol{VD}_d)^\mathsf{T} \\ \star & \star & -\gamma\boldsymbol{I}_{n_f} \end{bmatrix} \prec \boldsymbol{0}. \tag{33c}$$

Due to multiplicative terms $\boldsymbol{XR}_M$, (33) is non-convex. Introducing $\boldsymbol{Y} = \boldsymbol{XR}_M \in \mathbb{R}^{n\times n_y}$, (33c) can be written as (34) resulting in a convex optimization problem, which can efficiently be treated by LMI solvers. The FIO is then parameterized by $\boldsymbol{L} = \boldsymbol{L}_0 + \boldsymbol{X}^{-1}\boldsymbol{YZ}$ and $\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{R}_N\boldsymbol{Z}$. However, the uncontrollable eigenvalues λ_{R_k} of $(\boldsymbol{A} - \boldsymbol{LC}, \boldsymbol{E})$ are not constrained. The optimization (33a), (33b), (34) might therefore result in very slow decays of initial observer errors $\boldsymbol{e}(0) \neq \boldsymbol{0}$ if there are eigenvalues with $\mathfrak{Re}(\lambda_{R_k}) \approx 0$. Furthermore, eigenvalues λ_{R_k} with too large absolute values might cause numerical problems in the optimization as well as in the implementation. To this end we propose the following design.

(1) Design an initial solution $(\boldsymbol{L}_p, \boldsymbol{V}_p)$ guaranteeing a stable FIO using the parametric approach presented in Section 3.2.
(2) Set $\boldsymbol{L}_0 = \boldsymbol{L}_p$ and $\boldsymbol{V}_0 = \boldsymbol{V}_p$ and solve the convex optimization problem (33a), (33b), (34) with additional constraints

$$\mathrm{He}\,(\boldsymbol{X}(\boldsymbol{A} - \boldsymbol{L}_0\boldsymbol{C}) - \boldsymbol{YZC}) - 2\alpha\boldsymbol{X} \prec \boldsymbol{0}, \tag{35a}$$

$$\begin{bmatrix} -\beta\boldsymbol{X} & \boldsymbol{X}(\boldsymbol{A} - \boldsymbol{L}_0\boldsymbol{C}) - \boldsymbol{YZC} \\ \star & -\beta\boldsymbol{X} \end{bmatrix} \prec \boldsymbol{0}. \tag{35b}$$

While (35a) ensures $\mathfrak{Re}(\lambda_{R_k}) < \alpha$, (35b) constrains the eigenvalues to lie in a circular region with radius β around the origin. Care has to be taken that the eigenvalues $\lambda_{R_{ij}}$ assigned to the transfer channels also fulfill (35a) and (35b). Of course, other regions in the complex plane could also be specified (s. Chilali et al. (1999) for details).

5. EXAMPLE

To show the applicability of the results obtained above we design an FIO for a CE-150 model-helicopter (Horayek (2003)). The linearized model with $n = 6$ and $n_u = 2$ describes the coupled pitch- and yaw-dynamics as well as actuator dynamics with

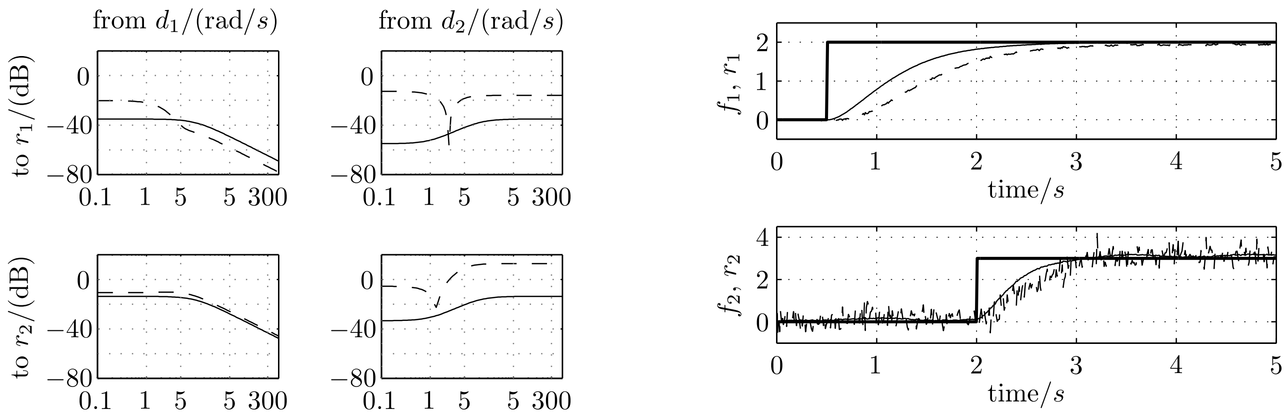
$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_{12}}{J_n} & -\frac{B_{n,1}}{J_n} & \frac{k_{11}}{J_nT_1} & 0 & -\frac{k_{v2}}{J_n} & 0 \\ 0 & 0 & -\frac{1}{T_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{k_{v1}}{J_gT_1} & 0 & -\frac{B_{g,1}}{J_g} & \frac{k_{21}}{J_gT_2} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{T_2} \end{bmatrix}. \tag{36}$$

For a detailed description of the parameters, we refer to Horayek (2003). The system is assumed to be disturbed by $\boldsymbol{d} = [d_1\, d_2]^\mathsf{T} \in \mathbb{R}^2$, where d_1 is a low-frequency sinusoidal signal with constant offset affecting the system dynamics and d_2 is white Gaussian noise with zero mean and power 0.07 affecting the sensors. First it is assumed that only the pitch- and yaw-angle were measurable, i.e. $n_y = 2$ with $y_1 = x_1$ and $y_2 = x_4$. Both actuators can be subject to faults, i.e. $n_f = 2$ with

$$\boldsymbol{E} = \boldsymbol{B} = \begin{bmatrix} 0 & 0 & K_e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_e \end{bmatrix}^\mathsf{T}. \tag{37}$$

The fault detectability indices are $\delta_1 = \delta_2 = 3$ and a standard FIO is designed assigning eigenvalues $\lambda_1 = -2.5$, $\lambda_2 = -2.8$, and $\lambda_3 = -3$ for channel 1 and $\lambda_4 = -4$, $\lambda_5 = -5$, and $\lambda_6 = -6$ for channel 2. If in addition to the angles the corresponding angular velocities can be measured, we have $n_y = 4$ with $y_3 = x_2$, $y_4 = x_5$ and $\delta_1 = \delta_2 = 2$. To this end, a robust FIO for the resulting non-square system is designed following the method proposed in Section 4. With the same initial eigenvalues as in the square case, the observer gains are optimized, where λ_2 and λ_5 are not assigned to any transfer channel and the eigenvalues are constrained by $\alpha = -1$ and $\beta = 10$. The resulting eigenvalues not assigned to any transfer channel are $\lambda_5^* = -7.426$ and $\lambda_6^* = -9.973$. The optimization was solved using YALMIP (Löfberg (2004)) with solver SDPT3 (Toh et al. (1999)), resulting in $\gamma_{\text{square}} = 4.341$ and $\gamma_{\text{non-square}} = 0.208$. In Fig. 2(a) the bode plots of

$$\begin{bmatrix} \mathrm{He}\,(\boldsymbol{X}(\boldsymbol{A} - \boldsymbol{L}_0\boldsymbol{C}) - \boldsymbol{YZC}) & \boldsymbol{X}(\boldsymbol{B}_d - \boldsymbol{L}_0\boldsymbol{D}_d) - \boldsymbol{YZD}_d & (\boldsymbol{V}_0\boldsymbol{C} + \boldsymbol{R}_N\boldsymbol{ZC})^\mathsf{T} \\ \star & -\gamma\boldsymbol{I}_{n_d} & (\boldsymbol{V}_0\boldsymbol{D}_d + \boldsymbol{R}_N\boldsymbol{ZD}_d)^\mathsf{T} \\ \star & \star & -\gamma\boldsymbol{I}_{n_f} \end{bmatrix} \prec \boldsymbol{0}. \tag{34}$$



(a) Bode plots of $\boldsymbol{G_{rd}}(s)$ in square (−−) and optimized non-square case (−)

(b) Faults (−) and residuals in square (−−) and optimized non-square case (−)

Fig. 2. Simulation results

$\boldsymbol{G_{rd}}(s)$ of the standard FIO and the optimized non-square FIO are compared. The optimized observer with $n_y > n_f$ achieves better disturbance attenuation regarding both the low-frequency disturbance d_1 as well as d_2. Fig. 2(b) depicts the evaluation over time of the two possible faults and the corresponding residuals of both FIOs. Obviously, the robust FIO in the non-square case generates much smoother and less disturbed residuals. In addition to that, its detection speed is higher due to the decreased relative degree of $g_{r_i f_i}(s)$ with similar pole locations.

6. CONCLUSION

In this contribution we presented a technique to design fault isolation observers for square and non-square linear systems. Existing results are extended since we provide a design that guarantees stability of the observer eigenvalues for arbitrary fault detectability indices while minimizing the influence of disturbances on the residuals at the same time. Considering uncertain parameters not only as disturbances but explicitly in the design is subject to ongoing research.

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