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Green functions and arithmetic generating series on Hilbert modular surfaces

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Zusammenfassung

In dieser Doktorarbeit definieren und untersuchen wir zwei Typen von Greenfunktionen auf zu reell-quadratischen Zahlkörpern assoziierten Hilbertschen Modulflächen mit logarithmischen Singularitäten entlang von Hirzebruch-Zagier-Divisoren. Dies sind zum einen die *automorphen Greenfunktionen*, ursprünglich eingeführt von Bruinier, und zum anderen die *Kudla-Greenfunktionen*, die auf Kudla zurückgehen. Wir berechnen zugehörige Fourierentwicklungen, untersuchen das Wachstum am Rand, erhalten Integrierbarkeitsaussagen und bestimmen zugehörige Integrale. Speziell für die automorphen Greenfunktionen finden wir eine wertvolle Zerlegung in glatte Funktionen mit vielerlei Anwendungen, aus denen sich erst in Summe die logarithmischen Singularitäten bilden.

Bei der Untersuchung der Kudla-Greenfunktionen stellen wir fest, dass diese nicht in die von Burgos Gil, Kramer und Kühn verallgemeinerte arithmetische Schnitttheorie passen, was an deren zu starkem Wachstum an den Spitzen liegt. Daraufhin stellen wir eine Modifikation vor, die das störende Wachstum mithilfe einer Teilung der Eins am Rand in einer solch eleganten Weise abzieht, dass die resultierenden Funktionen zum einen tatsächlich Greenfunktionen im Sinne von Burgos Gil, Kramer und Kühn sind, und zum anderen die erzeugende Reihe über die abgezogenen Störterme modular ist. Dies benutzen wir, um unser Hauptresultat, nämlich die Modularität der erzeugenden Reihe der arithmetischen Hirzebruch-Zagier-Divisoren versehen mit den modifizierten Kudla-Greenfunktionen, zu beweisen. Dazu führen wir diese Modularität auf die bereits von Bruinier, Burgos Gil und Kühn gezeigte Modularität der erzeugenden Reihe der arithmetischen Hirzebruch-Zagier-Divisoren versehen mit den automorphen Greenfunktionen zurück.

Summary

In this thesis we define and investigate two types of Green functions on Hilbert modular surfaces associated to real quadratic number fields. Both types possess logarithmic singularities along Hirzebruch–Zagier divisors. On the one hand, we consider the *automorphic Green functions*, originally introduced by Bruinier, and on the other hand *Kudla's Green functions*, which go back to Kudla. We calculate associated Fourier expansions, investigate their growth at the boundary, obtain integrability statements and determine associated integrals. Especially for the automorphic Green functions we find a valuable decomposition into smooth functions with many applications.

When examining Kudla's Green functions, we find that they do not fit into the arithmetic intersection theory generalized by Burgos Gil, Kramer and Kühn, which is due to their strong growth at the cusps. We then present a modification that subtracts the undesired growth at the boundary using a partition of unity. This is done in such an elegant way that the resulting functions are not only actual Green functions in the sense of Burgos Gil, Kramer, and Kühn, but the generating series of the subtracted error terms is modular. We use this to prove our main result, the modularity of the generating series of the arithmetic Hirzebruch–Zagier divisors equipped with the modified Green functions. In the proof, we trace its modularity back to the modularity of the generating series of the arithmetic Hirzebruch–Zagier divisors equipped with the automorphic Green functions whose modularity was already shown by Bruinier, Burgos Gil and Kühn.

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For from him and through him and for him are all things.
To him be the glory for ever! Amen.

Romans 11:36

Contents

Chapter 1

Introduction

In 1976, Hirzebruch and Zagier showed that the intersection numbers of certain special divisors, nowadays called *Hirzebruch–Zagier divisors*, on Hilbert modular surfaces can be interpreted as the Fourier coefficients of holomorphic elliptic modular forms of weight 2 (cf. [HZ76]). More precisely, a slight generalization of this result states that the generating series

$$A(\tau) = c_1(\mathcal{M}_{-1/2}(\mathbb{C})) + \sum_{m=1}^{\infty} Z(m)q^m \in \mathbb{Q}[[q]] \otimes_{\mathbb{Q}} \mathrm{CH}^1(\overline{X})_{\mathbb{Q}}$$

is a holomorphic modular form of weight 2, level D and nebentypus χ_D with values in $\mathrm{CH}^1(\overline{X})_{\mathbb{Q}}$. Here, by D we denote the discriminant of the underlying real quadratic number field K, by $c_1(\mathcal{M}_k(\mathbb{C}))$ the first Chern class of the line bundle of modular forms of weight k, by $\overline{X}$ the Hirzebruch compactification of the Hilbert modular surface X associated to K and by $Z(m)$ certain extensions of the Hirzebruch–Zagier divisors $T(m)$ of discriminant m on X to the Hirzebruch compactification $\overline{X}$ (precise definitions are provided in Section 2.5 and Section 2.8). The meaning of $A(\tau)$ being a modular form with values in $\mathrm{CH}^1(\overline{X})_{\mathbb{Q}}$ is that for any linear map $\lambda : \mathrm{CH}^1(\overline{X})_{\mathbb{Q}} \to \mathbb{Q}$ the generating series

$$\lambda(c_1(\mathcal{M}_{-1/2}(\mathbb{C}))) + \sum_{m=1}^{\infty} \lambda(Z(m))q^m$$

is a modular form of weight 2, level D and nebentypus χ_D. In particular, the Hirzebruch–Zagier divisors $Z(m)$ generate in $\mathrm{CH}^1(\overline{X})_{\mathbb{Q}}$ a subspace whose dimension is bounded by the dimension of the space of modular forms of weight 2, level D and nebentypus χ_D.

Kudla and Millson aimed at a generalization of the result from Hirzebruch and Zagier and studied special cycles for the orthogonal group $\mathrm{O}(p,q)$ and the unitary group $\mathrm{U}(p,q)$ in great generality by means of the Weil representation (cf. [KM90]). In the Kudla program one is interested in having arithmetic analogues to the Hirzebruch–Zagier theorem (cf. [Kud02] and [Kud04]). More precisely, instead of proving the modularity of generating series like $A(\tau)$ with coefficients in classical Chow groups, one is interested in proving the modularity of generating series with coefficients in arithmetic Chow groups. The elements of arithmetic Chow groups are arithmetic divisors (or more general arithmetic cycles) up to rational equivalence. An arithmetic divisor in turn is a pair (Z, g), where Z

is a classical divisor (on an integral model of $\overline{X}$) and g is a Green current corresponding to Z (cf. Section 2.9 for details).

Particular cases to study the Kudla program are smooth compactifications of Hilbert modular surfaces. In this thesis we associate to each fractional ideal $\mathfrak{a}$ of a real quadratic number field K a Hilbert modular surface $X(\mathfrak{a})$ (cf. Section 2.5) and its Hirzebruch compactification $\overline{X(\mathfrak{a})}$, a smooth compactification of $X(\mathfrak{a})$ (cf. Section 2.7).

Naturally, one wishes to extend the Hirzebruch–Zagier divisors $T(\mathfrak{a}, m)$ living on $X(\mathfrak{a})$ to divisors $Z(\mathfrak{a}, m)$ living on the compactification $\overline{X(\mathfrak{a})} = X(\mathfrak{a}) \cup E(\mathfrak{a})$. In Section 3.5 we investigate local Borcherds products $\Psi(\mathfrak{a}, m, z)$ to find the right multiplicities of the components of the so-called *exceptional divisor* $E(\mathfrak{a})$. This results in the correct definitions of the divisors $Z(\mathfrak{a}, m)$ which can already be found in Subsection 2.8.5.

In addition to the Hirzebruch–Zagier divisors on $\overline{X(\mathfrak{a})}$, one needs to find appropriate Green functions completing the divisors $Z(\mathfrak{a}, m)$ to arithmetic divisors. In this context it is worth mentioning that a further generalization of the arithmetic intersection theory of Gillet and Soulé (cf. [GS90]), which already generalizes the work of Arakelov to arithmetic varieties, was developed by Burgos Gil, Kramer and Kühn in [BGKK07] and [BGKK05]. This theory allows the Green functions to have so-called *pre-log-log growth* at the exceptional divisor $E(\mathfrak{a})$ (cf. Subsection 2.9.2 for the growth definitions) in addition to the logarithmic singularities along the divisors $Z(\mathfrak{a}, m)$, which is what Gillet and Soulé expect from a Green function.

Within that theory from Burgos Gil, Kramer and Kühn there are different natural choices for appropriate Green functions. In this thesis we discuss two of them. Firstly, there are the almost harmonic automorphic Green functions $\Phi(\mathfrak{a}, m, z)$ introduced by Bruinier in [Bru99]. They are constructed by a regularization of functions $\Phi(\mathfrak{a}, m, s, z)$ with an additional complex parameter s. We discuss the construction and many properties of the automorphic Green functions in Chapter 3. Secondly, in Chapter 4 we discuss Green functions $\Xi(\mathfrak{a}, m, v, z)$ constructed by Kudla (cf. [Kud97]) which are connected to the Kudla–Millson theory.

The arithmetic Hirzebruch–Zagier theorem for automorphic Green functions stating that

$$\sum_{m=0}^{\infty} (Z(\mathfrak{a}, m), G(\mathfrak{a}, m, z))q^m \tag{1.1}$$

is a holomorphic modular form of weight 2, level D and nebentypus χ_D with values in $\widehat{\mathrm{CH}}^1(\overline{X(\mathfrak{a})}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}}$ (cf. Definition 5.5.1) was already proven in [BBGK07] by Bruinier, Burgos Gil and Kühn in the case that D is prime and $\mathfrak{a} = \mathcal{O}_K$ (cf. Section 3.11 where we give more details). Here, the term $G(\mathfrak{a}, m, z)$ denotes a normalized version of $\Phi(\mathfrak{a}, m, z)$ (cf. Section 3.9). However, Bruinier, Burgos Gil and Kühn used in their work an arbitrary smooth compactification of the Hilbert modular surface whose existence is guaranteed by Hironaka in [Hir64]. The extensions of the divisors $T(\mathcal{O}_K, m)$ to the divisors $Z(\mathcal{O}_K, m)$ are abstractly defined by pullback from the singular Baily–Borel compactification $X(\mathcal{O}_K)^*$. To this end, the authors proved that the divisors $T(\mathcal{O}_K, m)$ are $\mathbb{Q}$-Cartier divisors near the cusps without the need of computing any multiplicities for the new components

at the cusps. By determining the divisors $Z(\mathfrak{a}, m)$ for the Hirzebruch compactification $\overline{X(\mathfrak{a})}$ and proving that $G(\mathfrak{a}, m, z)$ are actually Green functions with respect to $Z(\mathfrak{a}, m)$ in Theorem 3.6.5 we make this result more explicit.

However, the main goal of this thesis is to prove an arithmetic Hirzebruch–Zagier theorem for Kudla's Green functions. To that end, we have to overcome a few technical difficulties. Namely, Kudla's Green functions $\Xi(\mathfrak{a}, m, v, z)$ turn out not to be Green functions in the sense of Burgos Gil, Kramer and Kühn which we show in Chapter 4 (cf. Remark 4.3.6) after a detailed analysis of their growth behavior near the exceptional divisor $E(\mathfrak{a})$. During this process we isolate the part of $\Xi(\mathfrak{a}, m, v, z)$ which is growing too strongly and call it $\check{\Xi}(\mathfrak{a}, m, v, z)$. Unfortunately, the function $\check{\Xi}(\mathfrak{a}, m, v, z)$ is only well-defined near the cusp ∞ but not on the whole Hilbert modular surface $X(\mathfrak{a})$. Therefore, we use a partition of unity ρ to subtract $\check{\Xi}(\mathfrak{a}, m, v, z)$ near the cusp ∞ where $\check{\Xi}(\mathfrak{a}, m, v, z)$ is well-defined. This modification is carried out simultaneously for all cusps in Section 4.5 where we come up with the definition of $\widetilde{\Xi}_\rho(\mathfrak{a}, m, v, z)$, which is a Green function in the sense of Burgos Gil, Kramer and Kühn by Theorem 4.5.2. The idea of this modification is due to Berndt and Kühn who investigated the degenerate case $D = 1$ in the article [BK12] (here, instead of a number field one deals with $K = \mathbb{Q} \oplus \mathbb{Q}$ and the analogue of the Hilbert modular group is $\mathrm{SL}_2(\mathbb{Z})^2$). This article also served as inspiration for other ideas concerning Kudla's Green functions. The modification of $\Xi(\mathfrak{a}, m, v, z)$ allows us to consider the generating series

$$\sum_{m \in \mathbb{Z}} (Z(\mathfrak{a}, m), \widetilde{\Xi}_\rho(\mathfrak{a}, m, v, z)) q^m \tag{1.2}$$

with values in the arithmetic Chow group $\widehat{\mathrm{CH}}^1(\overline{X(\mathfrak{a})}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}}$ and to eventually prove the main result of this thesis:

Theorem 1.0.1 (cf. Theorem 5.5.7)**.** *Let D be prime and $\mathfrak{a} = \mathcal{O}_K$. Then the generating series* (1.2) *is a non-holomorphic modular form of weight* 2*, level D and nebentypus χ_D.*

It is a priori not clear how to generalize the notion of being a holomorphic modular form with values in $\widehat{\mathrm{CH}}^1(\overline{X(\mathfrak{a})}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}}$ to the notion of being a *non*-holomorphic modular form with values in $\widehat{\mathrm{CH}}^1(\overline{X(\mathfrak{a})}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}}$. We explain our notion of the latter in Definition 5.5.2. The proof of the theorem transfers the modularity of the series (1.1) to the series (1.2) by investigating the difference of the two series:

Theorem 1.0.2 (cf. Theorem 5.5.6)**.** *Let D be prime. Then*

$$\sum_{m \in \mathbb{Z}} (0, \widetilde{\Xi}_\rho(\mathcal{O}_K, m, v, z) - G(\mathcal{O}_K, m, z)) q^m$$

is a non-holomorphic modular form of weight 2*, level D and nebentypus χ_D.*

Because of that approach which depends on the modularity of (1.1), Theorem 1.0.1 is only proven in the case where D is prime and $\mathfrak{a} = \mathcal{O}_K$ like the arithmetic Hirzebruch–Zagier theorem for the series (1.1). However, apart from this, all intermediate results of

this thesis are proven for all real quadratic number fields (hence, no restriction on D) and all fractional ideals $\mathfrak{a} \in \mathcal{I}_K$.

In order to prove the modularity of the difference series, deep investigations of the automorphic Green functions $\Phi(\mathfrak{a}, m, z)$ and Kudla's Green functions $\Xi(\mathfrak{a}, m, v, z)$ are necessary. Those investigations provide a lot of new insights into the Green functions $\Phi(\mathfrak{a}, m, z)$ and $\Xi(\mathfrak{a}, m, v, z)$ which are of interest on their own. We will name some of the results we develop along the way for both types of Green functions and state where they can be found within this thesis.

We start by listing a few results for the automorphic Green functions: We obtain the Fourier expansion of the unregularized automorphic Green functions $\Phi(\mathfrak{a}, m, s, z)$ (cf. Theorem 3.2.6) and the regularized automorphic Green functions $\Phi(\mathfrak{a}, m, z)$ (cf. Theorem 3.4.1) for arbitrary $\mathfrak{a} \in \mathcal{I}_K$ (whereas in [Bru99] only the case $\mathfrak{a} = \mathcal{O}_K$ is considered). In Section 3.5 we define and investigate local Borcherds products $\Psi(\mathfrak{a}, m, z)$ for all fractional ideals $\mathfrak{a} \in \mathcal{I}_K$. Proposition 3.5.6 determines its vanishing orders along the components of the exceptional divisor $E^\infty(\mathfrak{a})$ explicitly. Another noteworthy result is presented in Section 3.7 where we come up with a decomposition

$$\Phi(\mathfrak{a}, m, s, z) = \sum_{n=0}^{\infty} \Phi_n(\mathfrak{a}, m, s, z) \tag{1.3}$$

of $\Phi(\mathfrak{a}, m, s, z)$ into smooth $\Gamma_\mathfrak{a}$ invariant functions $\Phi_n(\mathfrak{a}, m, s, z)$ without any singularities on $X(\mathfrak{a})$. We partially compute the Fourier coefficients of the functions $\Phi_n(\mathfrak{a}, m, s, z)$ which yields new formulae for the already known coefficients of $\Phi(\mathfrak{a}, m, s, z)$ and reveals new identities (cf. Subsection 3.7.2 and Subsection 3.7.3). Using the decomposition (1.3), we obtain integrability results of $\Phi(\mathfrak{a}, m, z)$ in Section 3.8 and compute the integrals given in the next theorem. For integration we use the volume form ω^2 with ω being the Kähler form on $X(\mathfrak{a})$ (cf. equation (2.14)).

Theorem 1.0.3 (cf. Theorem 3.8.3 and Theorem 3.8.11)**.** *For $m \in \mathbb{N}$ and $\Re(s) > 1$ we have*

$$\int_{X(\mathfrak{a})} \Phi(\mathfrak{a}, m, s, z)\omega^2 = \frac{2\,\mathrm{vol}(T(\mathfrak{a}, m))}{s(s-1)}$$

and

$$\int_{X(\mathfrak{a})} \Phi(\mathfrak{a}, m, z)\omega^2 = -2\,\mathrm{vol}(T(\mathfrak{a}, m)).$$

Further estimates based on the decomposition (1.3) allow us to prove the following convergence theorem in Section 3.10.

Theorem 1.0.4 (cf. Theorem 3.10.1)**.** *Let $q \in \mathbb{C}$ with $|q| < 1$ be fixed. The series*

$$\sum_{m=0}^{\infty} \Phi(\mathfrak{a}, m, z)q^m \quad \textit{and} \quad \sum_{m=0}^{\infty} G(\mathfrak{a}, m, z)q^m$$

converge absolutely for almost all $z \in \mathbb{H}^2$. Furthermore, the series

$$\sum_{m=0}^{\infty} |\Phi(\mathfrak{a}, m, z)q^m| \quad \textit{and} \quad \sum_{m=0}^{\infty} |G(\mathfrak{a}, m, z)q^m|$$

are integrable over $X(\mathfrak{a})$.

Lastly, in Theorem 5.4.1 we show that in case of D being prime and $\mathfrak{a} = \mathcal{O}_K$ the integrals of the $\Phi(\mathfrak{a}, m, z)$ are the coefficients of a modular form.

Let us now continue with a few results about Kudla's Green functions. Here, we compute Fourier expansions (cf. Section 4.2 and Section 4.4) as well. In addition, we compute the following integrals which prove their integrability (in this case nothing more has to be done because Kudla's Green functions do not attain negative values).

Theorem 1.0.5 (cf. Theorem 4.6.2 and Theorem 4.7.2)**.** *For* $m > 0$ *we have*

$$\int_{\Gamma_{\mathfrak{a}}\backslash\mathbb{H}^2} \Xi(\mathfrak{a}, m, v, z)\omega^2 = \frac{\mathrm{vol}(T(\mathfrak{a}, m))}{2\pi v m}.$$

For $m < 0$ *we have if* $N(\varepsilon_0) = -1$ *(for the general case see Remark 4.6.3)*

$$\int_{\Gamma_{\mathfrak{a}}\backslash\mathbb{H}^2} \Xi(\mathfrak{a}, m, v, z)\omega^2 = 2\Gamma(-1, 4\pi v|m|)\,\mathrm{vol}(T(\mathfrak{a}, |m|)).$$

For $m = 0$ *we have*

$$\int_{\Gamma_{\mathfrak{a}}\backslash\mathbb{H}^2} \Xi_*(\mathfrak{a}, m, v, z)\omega^2 = \frac{h_K \log(\varepsilon_0)}{24\pi v\sqrt{D}}.$$

Here, ε_0 is the fundamental unit of the ring of integers of K and h_K is the class number of K. A consequence of the asymptotic growth for large m of the integrals is the next theorem.

Theorem 1.0.6 (cf. Theorem 4.8.2)**.** *Let* $\tau \in \mathbb{H}$ *be fixed. The series*

$$\sum_{m\in\mathbb{Z}} \Xi(\mathfrak{a}, m, v, z)q^m$$

converges absolutely for almost all $z \in \mathbb{H}^2$*. Furthermore, the series*

$$\sum_{m\in\mathbb{Z}} |\Xi(\mathfrak{a}, m, v, z)q^m|$$

is integrable over $X(\mathfrak{a})$.

Lastly, in Section 5.1 we prove the convergence and modularity of the generating series

$$\sum_{m\in\mathbb{Z}} \check{\Xi}(\mathfrak{a}, m, v, z)q^m \tag{1.4}$$

viewed as a function. Recall that the functions $\check{\Xi}(\mathfrak{a}, m, v, z)$ are the error terms we have to subtract near the cusps from Kudla's Green functions in order to make them fit into the theory of Burgos Gil, Kramer and Kühn. The modularity of the generating series (1.4) is a crucial ingredient to the proof of our main theorem.

To finish the introduction we briefly recap the structure of this thesis. In Chapter 2 we discuss all the preliminaries needed to discuss the automorphic Green functions in Chapter 3 and Kudla's Green functions in Chapter 4. In the last chapter, Chapter 5, we then combine the results and prove a variety of modularity related theorems. To keep track of expressions we suggest to use the list of functions and symbols at the end of the thesis.

Chapter 2

Preliminaries

2.1 Quadratic spaces and lattices

In this thesis we encounter rational and real quadratic spaces and lattices. The aim of this section is to discuss the underlying definitions and fundamental properties. Every rational quadratic space V can be embedded into a real quadratic space $V_{\mathbb{R}} := V \otimes_{\mathbb{Z}} \mathbb{R}$. We give definitions to real quadratic spaces only in this section. The same definitions apply to rational quadratic spaces by passing over to $V_{\mathbb{R}}$ first.

Let V be a finite dimensional vector space over $\mathbb{R}$ with a quadratic form $q : V \to \mathbb{R}$. Then

$$(\cdot,\cdot) : V \times V \to \mathbb{R}, \quad (x,y) := q(x+y) - q(x) - q(y)$$

defines a symmetric bilinear form on V. Vice versa, if V is endowed with a symmetric bilinear form $(\cdot,\cdot)$, this induces a quadratic form

$$q : V \to \mathbb{R}, \quad q(x) := (x,x)/2.$$

These mappings are inverse to each other. Therefore, we do not need to distinguish between vector spaces equipped with quadratic forms and vector spaces equipped with symmetric bilinear forms and call the pair (V,q) a *real quadratic space*. If the quadratic form is understood from the context, we may omit q in the notation. The automorphisms of (V,q), i.e., linear maps $\varphi : V \to V$ which are bijective and preserve q, form the group $\mathrm{O}(V)$ called the *orthogonal group of* V.

A real quadratic space V is called *non-degenerate* if the kernel of the map

$$V \to V^*, \quad x \mapsto (y \mapsto (x,y))$$

is trivial (and otherwise *degenerate*). Hence, for non-degenerate real quadratic spaces V we have a canonical isomorphism $V \to V^*$ and we can identify elements of V^* with elements of V.

Non-zero vectors $x \in V$ with $q(x) = 0$ are called *isotropic*, whereas vectors $x \in V$ with $q(x) \neq 0$ are called *anisotropic*. We have the disjoint decomposition

$$V = V^+ \,\dot\cup\, V^- \,\dot\cup\, \mathrm{Iso}(V) \,\dot\cup\, \{0\}\,.$$

Here, we denote by V^+ the set of vectors with positive quadratic form, by V^- the set of vectors with negative quadratic form and by $\mathrm{Iso}(V)$ the set of isotropic vectors. If $V \setminus \{0\} = V^+$, we call V *positive definite*, if $V \setminus \{0\} = V^-$, we call V *negative definite* and if V is positive or negative definite we call V *definite*. A real quadratic space (V, q) is definite if and only if $\mathrm{Iso}(V) = \emptyset$. Definite real quadratic spaces are non-degenerate but the converse is not true.

Each real quadratic space (V, q) possesses an orthogonal basis $b_1, \ldots, b_n$, i.e., $(b_i, b_j) = 0$ for $i \neq j$. Let b^+, b^-, b^0 be the number of basis elements with $q(b_i) > 0$, $q(b_i) < 0$, $q(b_i) = 0$, respectively. Then (b^+, b^-, b^0) does not depend on the choice of the basis and is called the *signature* of (V, q). The space is non-degenerate if and only if $b^0 = 0$. In this case we also call (b^+, b^-) the *signature* of (V, q).

For each non-degenerate real quadratic space we fix as our measure its Haar measure normalized in such a way that a cube spanned by an orthogonal basis $b_1, \ldots, b_n$ with $|(b_i, b_i)| = 1$ (or $|q(b_i)| = 1/2$ equivalently) has volume 1. By this normalization we give the $\mathbb{R}^n$ with the Euclidean scalar product its Lebesgue measure.

A free discrete $\mathbb{Z}$ module $L \subset V$ of rank $\dim(V)$ is called a *lattice* of V. It is called *non-degenerate* if V is non-degenerate. In this case

$$L^\vee := \{w \in V^* : \ w(L) \subset \mathbb{Z}\} = \{x \in V : \ (x, y) \in \mathbb{Z} \text{ for all } y \in \mathbb{Z}\}$$

is called the *dual lattice*. If $b_1, \ldots, b_n$ is a basis of L, the dual basis $b_1^*, \ldots, b_n^*$ with respect to the canonical isomorphism $V \to V^*$ is a basis of $L^\vee$.

For a basis $b_1, \ldots, b_n$ of a non-degenerate lattice L, the determinant

$$\det(L) := \det((b_i, b_j)_{i,j})$$

does not depend on the choice of the basis and is called *determinant of* L. Its sign is given by $(-1)^{b^-}$ and its square root

$$\mathrm{vol}(L) := \sqrt{|\det(L)|} \tag{2.1}$$

is called the *volume of the lattice* L. Both are non-zero and we actually have

$$\mathrm{vol}(L) = \mu(V/L)$$

with μ being the measure of V from above. We have the identity $\det(L^\vee) = \det(L)^{-1}$ and accordingly $\mathrm{vol}(L^\vee) = \mathrm{vol}(L)^{-1}$. If $L' \subset L$ is a sublattice, the quotient L/L' is well-defined and a finite abelian group of order $\mathrm{vol}(L')/\mathrm{vol}(L)$.

Non-degenerate lattices L with $L \subset L^\vee$ (which is equivalent to $(x, y) \in \mathbb{Z}$ for all $x, y \in L$) are called *integral*. From the above it follows that for integral lattices the group $L^\vee/L$ has cardinality $|\det(L)|$. We call non-degenerate lattices L with $q(L) \in \mathbb{Z}$ *even*. They are integral and the quadratic form on V descends to a $\mathbb{Q}/\mathbb{Z}$ valued quadratic form on $L^\vee/L$. This group together with its quadratic form is called the *discriminant group of* L.

An element $x \in L$ is called *primitive* if $ny = x$ with $(n, y) \in \mathbb{Z} \times L$ implies $n \in \{\pm 1\}$. Equivalently, an element $x \in L$ is called *primitive* if it is part of a basis of L. We denote the primitive elements of L by L_0. Later, we will need especially those primitive elements which are isotropic, and denote them by $\mathrm{Iso}(L_0)$.

2.2 Real quadratic number fields

Recall that every real quadratic number field K is of the form $K = \mathbb{Q}(\sqrt{d})$ for a unique squarefree integer $d > 1$. It is a $\mathbb{Q}$ vector space of dimension 2, a possible basis is given by $(1, \sqrt{d})$. We view K as subfield of $\mathbb{R}$ with $\sqrt{d} > 0$. Its discriminant is given by

$$D = \begin{cases} d, & d \equiv 1 \pmod 4, \\ 4d, & d \equiv 2, 3 \pmod 4. \end{cases}$$

The field K possesses one non-trivial automorphism, the so-called *conjugation* which sends $\sqrt{d}$ to $-\sqrt{d}$ and is denoted by $x \mapsto x'$. This allows us to introduce the norm and the trace functions

$$N : K \to \mathbb{Q}, \quad N(x) := xx', \tag{2.2}$$

$$\mathrm{tr} : K \to \mathbb{Q}, \quad \mathrm{tr}(x) := x + x'. \tag{2.3}$$

The trace is a $\mathbb{Q}$ linear map and the norm is a non-degenerate quadratic form turning K into a rational quadratic space of signature $(1, 1)$. Another non-degenerate quadratic form is induced by the $\mathbb{Q}$ bilinear trace form $(x, y) \mapsto \mathrm{tr}(xy)$. The latter is positive definite, i.e., of signature $(2, 0)$. The ring of integers of K is given by

$$\mathcal{O}_K = \mathbb{Z} + \tfrac{D+\sqrt{D}}{2}\mathbb{Z} = \begin{cases} \mathbb{Z} + \sqrt{d}\mathbb{Z}, & D \equiv 0 \pmod 4, \\ \mathbb{Z} + \frac{1+\sqrt{d}}{2}\mathbb{Z}, & D \equiv 1 \pmod 4. \end{cases}$$

By Dirichlet's unit theorem, there exists a unique $\varepsilon_0 > 1$ such that

$$\mathcal{O}_K^\times = \left\{ \pm\varepsilon_0^k : \ k \in \mathbb{Z} \right\}.$$

Analogously, there exists a unique $\varepsilon_1 > 1$ such that

$$\mathcal{O}_K^+ := \mathcal{O}_K^\times \cap K^+ = \left\{ \varepsilon_1^k : \ k \in \mathbb{Z} \right\} \quad \text{with} \quad K^+ := \{x \in K : \ x \gg 0\}.$$

Here $x \gg 0$ being totally positive means $x > 0$ and $x' > 0$. If $N(\varepsilon_0) = 1$, we have $\varepsilon_1 = \varepsilon_0$ and otherwise $\varepsilon_1 = \varepsilon_0^2$. For prime discriminants D we always have $N(\varepsilon_0) = -1$.

2.2.1 Prime ideals in $\mathcal{O}_K$ and the Dirichlet character χ_D

We start with a short disclaimer. With $\mathcal{O}_K$ being a subring of a field the trivial ideal (0) is prime. However, for the whole thesis when we talk about prime ideals we mean prime ideals different from (0).

The ring $\mathcal{O}_K$ is a Dedekind domain. Hence, every non-zero ideal factors into prime ideals. Therefore, it is useful to study the prime ideals (the atoms) of $\mathcal{O}_K$. In our situation where K is a quadratic number field, they are very well understood. There are three kinds of prime ideals whose characterization can be done via the Dirichlet character χ_D which is defined by

$$\chi_D : \mathbb{Z} \to \{1, -1, 0\}, \quad \chi_D(n) := \left(\frac{D}{n}\right), \tag{2.4}$$

where the latter is the Kronecker symbol. The character is periodic with period D, hence it factors through $\mathbb{Z}/D\mathbb{Z}$. Each prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ belongs to a prime number $p \in \mathbb{N}$ via $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$. On the other hand, for each prime number $p \in \mathbb{N}$ there are one or two prime ideals in $\mathcal{O}_K$ that belong to p. Looking at the prime factorisation in $\mathcal{O}_K$ of the principal ideal $(p) = p\mathcal{O}_K$, the following three cases can occur.

(i) We say p *ramifies (completely)*:

$$(p) = \mathfrak{p}^2 \quad \Leftrightarrow \quad \chi_D(p) = 0 \quad \Leftrightarrow \quad p \mid D.$$

(ii) We say p is *inert*:

$$(p) = \mathfrak{p} \quad \Leftrightarrow \quad \chi_D(p) = -1 \quad \Leftrightarrow \quad (p) \text{ is prime in } \mathcal{O}_K.$$

(iii) We say p *splits completely*:

$$(p) = \mathfrak{p}\mathfrak{p}' \text{ with } \mathfrak{p} \neq \mathfrak{p}' \quad \Leftrightarrow \quad \chi_D(p) = 1.$$

The first case happens only for finitely many primes p, the prime divisors of D. The two other cases occur infinitely often by Dirichlet's theorem on arithmetic progressions. They appear equally often because half of the units u in $\mathbb{Z}/D\mathbb{Z}$ satisfy $\chi_D(u) = 1$, while the others satisfy $\chi_D(u) = -1$.

This classifies all prime ideals in $\mathcal{O}_K$ and relates them to prime numbers.

2.2.2 Fractional ideals and the ideal class group

The finitely generated non-zero $\mathcal{O}_K$ modules in K are called *fractional ideals*. They are of the form $x\mathfrak{b}$ with $x \in K^\times$ and $\mathfrak{b} \subset \mathcal{O}_K$ being an (integral) non-trivial ideal of $\mathcal{O}_K$. Each fractional ideal $\mathfrak{a} \subset K$ has a unique decomposition into prime ideals

$$\mathfrak{a} = \prod_{\substack{\mathfrak{p} \subset \mathcal{O}_K \\ \mathfrak{p} \text{ prime}}} \mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})}$$

with $\nu_\mathfrak{p}(\mathfrak{a}) \in \mathbb{Z}$ and $\nu_\mathfrak{p}(\mathfrak{a}) = 0$ for almost all $\mathfrak{p}$. The ideal $\mathfrak{a}$ is integral if and only if $\nu_\mathfrak{p} \geq 0$ for all $\mathfrak{p}$. The fractional ideals form a group denoted by $\mathcal{I}_K$. Hence, $\mathcal{I}_K$ is a free abelian group whose basis consists of all prime ideals $\mathfrak{p} \subset \mathcal{O}_K$. The conjugation on K induces a non-trivial group automorphism on $\mathcal{I}_K$, again called *conjugation*, sending the generators $\mathfrak{p}$ to $\mathfrak{p}'$ (here $\mathfrak{p} \neq \mathfrak{p}'$ if and only if the underlying prime p splits completely which occurs for infinitely many primes as we have seen in the previous subsection). For non-zero integral ideals $\mathfrak{b} \subset \mathcal{O}_K$ the map $N(\mathfrak{b}) := [\mathcal{O}_K : \mathfrak{b}]$ defines a totally multiplicative function which we call the *ideal norm*. There is a unique multiplicative extension to $\mathcal{I}_K$ turning it into the group homomorphism

$$N : \mathcal{I}_K \to \mathbb{Q}^+, \quad N(\mathfrak{a}) := \prod_{\substack{\mathfrak{p} \subset \mathcal{O}_K \\ \mathfrak{p} \text{ prime}}} N(\mathfrak{p})^{\nu_\mathfrak{p}(\mathfrak{a})}.$$

For each principal ideal $\mathfrak{a} = (x)$ with $x \in K^\times$ we have $N(\mathfrak{a}) = |N(x)|$, hence in particular we get $N((x)) = x^2$ for $x \in \mathbb{Q}^\times$. The ideal norm and conjugation on $\mathcal{I}_K$ interact nicely via

$$\mathfrak{a}\mathfrak{a}' = (N(\mathfrak{a})) \quad \text{and therefore} \quad \mathfrak{a}^{-1} = \frac{\mathfrak{a}'}{N(\mathfrak{a})} \tag{2.5}$$

for all $\mathfrak{a} \in \mathcal{I}_K$. An important theorem which more generally holds in all Dedekind domains is the theorem of finite approximation.

Theorem 2.2.1 (Theorem of finite approximation)**.** *Let $\mathfrak{a} \in \mathcal{I}_K$ and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be finitely many prime ideals of $\mathcal{O}_K$. Then there exists an $\alpha \in \mathfrak{a}$ with*

$$\nu_{\mathfrak{p}_i}(\mathfrak{a}) = \nu_{\mathfrak{p}_i}(\alpha) \quad \textit{for all } 1 \leq i \leq n.$$

Corollary 2.2.2. *For $\mathfrak{a} \in \mathcal{I}_K$ and $0 \neq \alpha \in \mathfrak{a}$ there exists $\beta \in \mathfrak{a}$ with $\alpha\mathcal{O}_K + \beta\mathcal{O}_K = \mathfrak{a}$. In particular, every fractional ideal is generated by two elements.*

Proof. There are only finitely many prime ideals $\mathfrak{p}$ with $\nu_{\mathfrak{p}}(\mathfrak{a}) \neq \nu_{\mathfrak{p}}(\alpha)$. We take them in Theorem 2.2.1 to get β. □

2.2.3 The ideal class group and the class number formula

The principal ideals form a subgroup of $\mathcal{I}_K$ which we denote by $\mathcal{P}_K$. The quotient group $\mathrm{Cl}_K := \mathcal{I}_K/\mathcal{P}_K$ is a finite abelian group called the *ideal class group* of K. The conjugation on $\mathcal{I}_K$ fixes $\mathcal{P}_K$ because of $(x)' = (x')$ for $x \in K^\times$, so it descends to a (possibly trivial) automorphism of Cl_K. The cardinality of Cl_K is called the *class number* of K and denoted by h_K. It is strongly related to the Dirichlet L-function

$$L(s, \chi_D) := \sum_{n=1}^{\infty} \chi_D(n) n^{-s} \quad \text{by} \quad h_K = \frac{\sqrt{D}}{2\log(\varepsilon_0)} L(1, \chi_D), \tag{2.6}$$

the so-called *class number formula.* The special value $L(1, \chi_D)$ of the Dirichlet L-function can be interpreted as the residue of the Dedekind zeta function

$$\zeta_K(s) := \sum_{\mathfrak{a} \subset \mathcal{O}_K} N(\mathfrak{a})^{-s}$$

at $s = 1$ since $\zeta_K(s) = \zeta(s) L(s, \chi_D)$.

2.2.4 Fractional ideals as lattices

Each fractional ideal $\mathfrak{a} \in \mathcal{I}_K$ is a free $\mathbb{Z}$ module of rank 2. Equipped with the norm form or the trace form it is a lattice. The two forms are very related so that they can be treated simultaneously. An important ideal is the *different.* It is denoted by $\mathfrak{d}$ and given by $\mathfrak{d} = (\sqrt{D}) = \sqrt{D}\mathcal{O}_K$. Using the different, we can express the duals of $\mathfrak{a}$:

$$\mathfrak{a}^{\vee_N} = (\mathfrak{a}\mathfrak{d})'^{-1} \quad \text{and} \quad \mathfrak{a}^{\vee_{\mathrm{tr}}} = (\mathfrak{a}\mathfrak{d})^{-1}. \tag{2.7}$$

Since $\mathfrak{d}' = \mathfrak{d}$, we see that they coincide if and only if $\mathfrak{a}' = \mathfrak{a}$ which is the case if and only if $\nu_{\mathfrak{p}}(\mathfrak{a}) = \nu_{\mathfrak{p}'}(\mathfrak{a})$ for all prime ideals $\mathfrak{p} \subset \mathcal{O}_K$. The volume of $\mathfrak{a}$ is given by $\mathrm{vol}(\mathfrak{a}) = N(\mathfrak{a})\sqrt{D}$ with respect to both forms. The determinants differ only in the sign, i.e.,

$$\det_N(\mathfrak{a}) = -N(\mathfrak{a})^2 D \quad \text{and} \quad \det_{\mathrm{tr}}(\mathfrak{a}) = N(\mathfrak{a})^2 D.$$

An important additional quadratic form on K defined for each $\mathfrak{a} \in \mathcal{I}_K$ individually turns $\mathfrak{a}$ into an even lattice, namely the normalized norm form:

$$q_{\mathfrak{a}} : K \to \mathbb{Q}, \quad q_{\mathfrak{a}}(x) := \frac{N(x)}{N(\mathfrak{a})}. \tag{2.8}$$

We have $\det_{q_{\mathfrak{a}}}(\mathfrak{a}) = -D$ and $\mathfrak{a}^{\vee q_{\mathfrak{a}}} = \mathfrak{a}\mathfrak{d}^{-1}$, hence the discriminant group contains D elements and is given by $\mathfrak{a}\mathfrak{d}^{-1}/\mathfrak{a}$.

2.3 Hilbert modular groups and their cusps

The symmetry groups which give rise to Hilbert modular surfaces are the so-called *Hilbert modular groups*. In this section we define those groups, prove important properties and analyze their cusps. We start by defining a group action of $\mathrm{GL}_2(K)$ on $\mathbb{P}^1(K)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\alpha : \beta) := (a\alpha + b\beta : c\alpha + d\beta).$$

For subgroups $\Gamma \subset \mathrm{SL}_2(K)$ we call the orbits of the group action restricted to Γ the *cusps of* Γ. Sometimes we call the elements of $\mathbb{P}^1(K)$ *cusps* as well. It will be clear from the context what we mean.

Lemma 2.3.1. *The group* $\mathrm{SL}_2(K)$ *has one cusp, i.e., the group action of* $\mathrm{SL}_2(K)$ *on* $\mathbb{P}^1(K)$ *is transitive.*

Proof. Let $(\alpha : \beta) \in \mathbb{P}^1(K)$ and $\mathfrak{a} := \alpha\mathcal{O}_K + \beta\mathcal{O}_K \in \mathcal{I}_K$. Then we have

$$1 \in \mathcal{O}_K = \mathfrak{a}\mathfrak{a}^{-1} = \alpha\mathfrak{a}^{-1} + \beta\mathfrak{a}^{-1}.$$

Hence, there exist $\alpha^*, \beta^* \in \mathfrak{a}^{-1}$ with $\alpha\beta^* - \beta\alpha^* = 1$. It follows

$$M(1:0) = (\alpha : \beta) \quad \text{with} \quad M := \begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix} \in \mathrm{SL}_2(K).$$

□

Remark 2.3.2. Let $\Gamma \subset \mathrm{SL}_2(K)$ be a subgroup and $M \in \mathrm{GL}_2(K)$. Then the map

$$f : \mathbb{P}^1(K) \to \mathbb{P}^1(K), \quad (\alpha : \beta) \mapsto M^{-1}(\alpha : \beta)$$

induces a bijection between the cusps of Γ and $M^{-1}\Gamma M$.

Definition 2.3.3. For $\mathfrak{a} \in \mathcal{I}_K$ the *Hilbert modular group* associated to $\mathfrak{a}$ is defined by

$$\Gamma_{\mathfrak{a}} := \mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{a}) := \begin{pmatrix} \mathcal{O}_K & \mathfrak{a}^{-1} \\ \mathfrak{a} & \mathcal{O}_K \end{pmatrix} \cap \mathrm{SL}_2(K).$$

We prove in Corollary 2.3.5 that $\Gamma_{\mathfrak{a}}$ is actually a group. In order to do so (and also for many more applications) we shall define for $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_K$

$$M(\mathfrak{a}, \mathfrak{b}) := \begin{pmatrix} \mathfrak{a} & (\mathfrak{a}\mathfrak{b})^{-1} \\ \mathfrak{a}\mathfrak{b} & \mathfrak{a}^{-1} \end{pmatrix} \cap \mathrm{SL}_2(K). \tag{2.9}$$

Then we have $\Gamma_{\mathfrak{a}} = M(\mathcal{O}_K, \mathfrak{a})$. Using the theorem of finite approximation (Theorem 2.2.1), it can be shown that the sets $M(\mathfrak{a}, \mathfrak{b})$ are non-empty. Furthermore, for $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{I}_K$ we have

$$M(\mathfrak{a}_1, \mathfrak{b}_1) = M(\mathfrak{a}_2, \mathfrak{b}_2) \quad \Leftrightarrow \quad \mathfrak{a}_1 = \mathfrak{a}_2 \quad \text{and} \quad \mathfrak{b}_1 = \mathfrak{b}_2.$$

Lemma 2.3.4. *Let $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}, \mathfrak{b} \in \mathcal{I}_K$. Then we have*

(i) $M(\mathfrak{a}, \mathfrak{b})^{-1} = M(\mathfrak{a}^{-1}, \mathfrak{a}^2\mathfrak{b})$ *and*

(ii) $M(\mathfrak{a}_1, \mathfrak{b})M(\mathfrak{a}_2, \mathfrak{a}_1^2\mathfrak{b}) = M(\mathfrak{a}_1\mathfrak{a}_2, \mathfrak{b})$.

Proof. The first equation follows from

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for matrices of determinant 1. To prove the second equation we compute

$$\begin{pmatrix} \mathfrak{a}_1 & (\mathfrak{a}_1\mathfrak{b})^{-1} \\ \mathfrak{a}_1\mathfrak{b} & \mathfrak{a}_1^{-1} \end{pmatrix} \begin{pmatrix} \mathfrak{a}_2 & (\mathfrak{a}_2\mathfrak{a}_1^2\mathfrak{b})^{-1} \\ \mathfrak{a}_2\mathfrak{a}_1^2\mathfrak{b} & \mathfrak{a}_2^{-1} \end{pmatrix} \subset \begin{pmatrix} \mathfrak{a}_1\mathfrak{a}_2 & (\mathfrak{a}_1\mathfrak{a}_2\mathfrak{b})^{-1} \\ \mathfrak{a}_1\mathfrak{a}_2\mathfrak{b} & (\mathfrak{a}_1\mathfrak{a}_2)^{-1} \end{pmatrix}$$

which gives us the inclusion

$$M(\mathfrak{a}_1, \mathfrak{b})M(\mathfrak{a}_2, \mathfrak{a}_1^2\mathfrak{b}) \subset M(\mathfrak{a}_1\mathfrak{a}_2, \mathfrak{b}).$$

The other inclusion follows from an application of what we have already seen:

$$M(\mathfrak{a}_1, \mathfrak{b})^{-1}M(\mathfrak{a}_1\mathfrak{a}_2, \mathfrak{b}) = M(\mathfrak{a}_1^{-1}, \mathfrak{a}_1^2\mathfrak{b})M(\mathfrak{a}_1\mathfrak{a}_2, \mathfrak{b}) \subset M(\mathfrak{a}_2, \mathfrak{a}_1^2\mathfrak{b}).$$

□

Corollary 2.3.5. *The set $\Gamma_{\mathfrak{a}} = \mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{a})$ is a subgroup of $\mathrm{SL}_2(K)$.*

Proof. By Lemma 2.3.4 the set $\Gamma_{\mathfrak{a}} = M(\mathcal{O}_K, \mathfrak{a})$ is closed under inversion and multiplication. □

Corollary 2.3.6. *Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_K$ and $M \in M(\mathfrak{a}, \mathfrak{b})$. Then we have*

$$M^{-1}\,\mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{b})M = \mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{a}^2\mathfrak{b}).$$

Proof. This again follows directly from Lemma 2.3.4. On one hand we have

$$\begin{aligned} M^{-1}\,\mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{b})M &\subset M(\mathfrak{a},\mathfrak{b})^{-1}M(\mathcal{O}_K,\mathfrak{b})M(\mathfrak{a},\mathfrak{b}) \\ &= M(\mathfrak{a}^{-1},\mathfrak{a}^2\mathfrak{b})M(\mathcal{O}_K,\mathfrak{b})M(\mathfrak{a},\mathfrak{b}) \\ &= M(\mathfrak{a}^{-1},\mathfrak{a}^2\mathfrak{b})M(\mathfrak{a},\mathfrak{b}) \\ &= \mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{a}^2\mathfrak{b}). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} M\,\mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{a}^2\mathfrak{b})M^{-1} &\subset M(\mathfrak{a},\mathfrak{b})M(\mathcal{O}_K,\mathfrak{a}^2\mathfrak{b})M(\mathfrak{a},\mathfrak{b})^{-1} \\ &= M(\mathfrak{a},\mathfrak{b})M(\mathfrak{a}^{-1},\mathfrak{a}^2\mathfrak{b}) \\ &= M(\mathcal{O}_K,\mathfrak{b}). \end{aligned}$$

□

Lemma 2.3.7. *Let $\mathfrak{a},\mathfrak{b} \in \mathcal{I}_K$ and $\left(\begin{smallmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{smallmatrix}\right) \in M(\mathfrak{a},\mathfrak{b})$. Then $\alpha\mathcal{O}_K + \beta\mathfrak{b}^{-1} = \mathfrak{a}$. Furthermore, the Hilbert modular group $\mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{b})$ acts transitively on $M(\mathfrak{a},\mathfrak{b})$ by left multiplication.*

Proof. The equation $\alpha\mathcal{O}_K + \beta\mathfrak{b}^{-1} = \mathfrak{a}$ is equivalent to

$$\alpha\mathfrak{a}^{-1} + \beta(\mathfrak{a}\mathfrak{b})^{-1} = \mathcal{O}_K.$$

From $\alpha \in \mathfrak{a}$ and $\beta \in \mathfrak{a}\mathfrak{b}$ we get one inclusion. On the other hand

$$1 = \alpha\beta^* - \beta\alpha^* \in \alpha\mathfrak{a}^{-1} + \beta(\mathfrak{a}\mathfrak{b})^{-1}.$$

Statement (ii) in Lemma 2.3.4 shows

$$\mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{b})M(\mathfrak{a},\mathfrak{b}) = M(\mathfrak{a},\mathfrak{b})$$

which makes the stated operation well-defined. The transitivity follows from

$$M(\mathfrak{a},\mathfrak{b})M(\mathfrak{a},\mathfrak{b})^{-1} = M(\mathfrak{a},\mathfrak{b})M(\mathfrak{a}^{-1},\mathfrak{a}^2\mathfrak{b}) = \mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{b}).$$

□

Lemma 2.3.8. *Let $\mathfrak{b} \in \mathcal{I}_K$ and $(\alpha,\beta) \in K^2$ with $(\alpha : \beta) \in \mathbb{P}^1(K)$. We set $\mathfrak{a} := \alpha\mathcal{O}_K + \beta\mathfrak{b}^{-1}$. Then there exist $\alpha^*, \beta^* \in K$ with*

$$\begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix} \in M(\mathfrak{a},\mathfrak{b}).$$

Proof. We define

$$\mathcal{O} := \alpha\mathfrak{a}^{-1} + \beta(\mathfrak{a}\mathfrak{b})^{-1}$$

and show that $\mathcal{O} = \mathcal{O}_K$ by verifying $\nu_{\mathfrak{p}}(\mathcal{O}) = 0$ for all prime ideals $\mathfrak{p} \subset \mathcal{O}_K$. This implies $1 \in \mathcal{O}$, hence the desired matrix exists. By definition of $\mathcal{O}$ we have

$$\nu_{\mathfrak{p}}(\mathcal{O}) = \min(\nu_{\mathfrak{p}}(\alpha) - \nu_{\mathfrak{p}}(\mathfrak{a}), \nu_{\mathfrak{p}}(\beta) - \nu_{\mathfrak{p}}(\mathfrak{a}) - \nu_{\mathfrak{p}}(\mathfrak{b}))$$

and by definition of $\mathfrak{a}$ we have

$$\nu_{\mathfrak{p}}(\mathfrak{a}) = \min(\nu_{\mathfrak{p}}(\alpha), \nu_{\mathfrak{p}}(\beta) - \nu_{\mathfrak{p}}(\mathfrak{b})).$$

In case $\nu_{\mathfrak{p}}(\mathfrak{a}) = \nu_{\mathfrak{p}}(\alpha)$ we have $\nu_{\mathfrak{p}}(\beta) - \nu_{\mathfrak{p}}(\mathfrak{b}) \geq \nu_{\mathfrak{p}}(\alpha)$ and

$$\nu_{\mathfrak{p}}(\mathcal{O}) = \min(\nu_{\mathfrak{p}}(\alpha) - \nu_{\mathfrak{p}}(\alpha), \nu_{\mathfrak{p}}(\beta) - \nu_{\mathfrak{p}}(\alpha) - \nu_{\mathfrak{p}}(\mathfrak{b})) = 0.$$

In case $\nu_{\mathfrak{p}}(\mathfrak{a}) = \nu_{\mathfrak{p}}(\beta) - \nu_{\mathfrak{p}}(\mathfrak{b})$ we have $\nu_{\mathfrak{p}}(\beta) - \nu_{\mathfrak{p}}(\mathfrak{b}) \leq \nu_{\mathfrak{p}}(\alpha)$ and

$$\nu_{\mathfrak{p}}(\mathcal{O}) = \min(\nu_{\mathfrak{p}}(\alpha) - (\nu_{\mathfrak{p}}(\beta) - \nu_{\mathfrak{p}}(\mathfrak{b})), \nu_{\mathfrak{p}}(\beta) - (\nu_{\mathfrak{p}}(\beta) - \nu_{\mathfrak{p}}(\mathfrak{b})) - \nu_{\mathfrak{p}}(\mathfrak{b})) = 0.$$

□

Proposition 2.3.9. *Let $\mathfrak{b} \in \mathcal{I}_K$. The map*

$$\psi_{\mathfrak{b}} : \mathbb{P}^1(K) \to \mathrm{Cl}_K, \quad (\alpha : \beta) \mapsto [\alpha\mathcal{O}_K + \beta\mathfrak{b}^{-1}]$$

induces a bijection between the cusps of the Hilbert modular group $\mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{b})$ and the ideal class group of K. In particular, $\mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{b})$ has h_K cusps.

Proof. The well-definedness of $\psi_{\mathfrak{b}}$ is clear since a scaling of $(\alpha : \beta)$ translates into a scaling of $\alpha\mathcal{O}_K + \beta\mathfrak{b}^{-1}$. The surjectivity is a direct consequence of Lemma 2.3.7.

Now let $(\alpha : \beta)$ and $(\gamma : \delta)$ be representatives of the same cusp. Hence, after appropriate scaling we find a matrix $A \in \mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{b})$ with

$$A \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}.$$

Let $\mathfrak{a} := \alpha\mathcal{O}_K + \beta\mathfrak{b}^{-1}$. Then by Lemma 2.3.8 there exist $\alpha^*, \beta^* \in K$ with

$$\begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix} \in M(\mathfrak{a}, \mathfrak{b}).$$

We define $\gamma^*, \delta^* \in K$ by the equation

$$\begin{pmatrix} \gamma & \gamma^* \\ \delta & \delta^* \end{pmatrix} := A \begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix}.$$

Because of the operation mentioned in Lemma 2.3.7, we get

$$\begin{pmatrix} \gamma & \gamma^* \\ \delta & \delta^* \end{pmatrix} \in M(\mathfrak{a}, \mathfrak{b}).$$

In particular $\mathfrak{a} = \gamma\mathcal{O}_K + \delta\mathfrak{b}^{-1}$ and hence $\psi_{\mathfrak{b}}(\alpha : \beta) = [\mathfrak{a}] = \psi_{\mathfrak{b}}(\gamma : \delta)$. Now it is left to show that $(\alpha : \beta), (\gamma : \delta) \in \mathbb{P}^1(K)$ with $\psi_{\mathfrak{b}}(\alpha : \beta) = \psi_{\mathfrak{b}}(\gamma : \delta)$ belong to the same cusp. After appropriate scaling we may assume $\mathfrak{a} := \alpha\mathcal{O}_K + \beta\mathfrak{b}^{-1} = \gamma\mathcal{O}_K + \delta\mathfrak{b}^{-1}$. By Lemma 2.3.7 we get matrices

$$\begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix}, \begin{pmatrix} \gamma & \gamma^* \\ \delta & \delta^* \end{pmatrix} \in M(\mathfrak{a}, \mathfrak{b}).$$

Now by the transitivity of the action under $\mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{b})$ they belong to the same orbit, hence $(\alpha : \beta)$ and $(\gamma : \delta)$ belong to the same orbit (also known as cusp) as well. $\square$

2.3.1 The cusp infinity and its stabilizer

In this subsection we have a closer look at the cusp $(1 : 0)$ whose relevance becomes clear by the following remark.

Remark 2.3.10. The cusp $(1 : 0) \in \mathbb{P}^1(K)$ is called the *cusp* ∞. By Proposition 2.3.9 it corresponds in a natural way to the neutral element of Cl_K, the principal ideals. Hence, it is the most natural cusp to look at. In order to understand all cusps of $\mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{b})$ for a fixed $\mathfrak{b} \in \mathcal{I}_K$ Remark 2.3.2, Lemma 2.3.7, Corollary 2.3.6 and Proposition 2.3.9 show that it is equivalent to understand the cusp ∞ of the groups $\mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{a}^2\mathfrak{b})$ for all $\mathfrak{a} \in \mathcal{I}_K$.

The stabilizer of the cusp ∞ is given by

$$\mathrm{SL}_2(K)_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K) : \ c = 0 \right\} = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} : \ x \in K^\times, y \in K \right\}.$$

Hence, we have

$$\Gamma_{\mathfrak{a},\infty} = \left\{ \begin{pmatrix} \varepsilon & \mu \\ 0 & \varepsilon^{-1} \end{pmatrix} : \ \varepsilon \in \mathcal{O}_K^\times, \mu \in \mathfrak{a}^{-1} \right\}.$$

The image $\overline{\Gamma_{\mathfrak{a},\infty}} := \Gamma_{\mathfrak{a},\infty}/\{\pm 1\}$ of $\Gamma_{\mathfrak{a},\infty}$ in $\mathrm{PSL}_2(K)$ can be described as a semidirect product

$$\overline{\Gamma_{\mathfrak{a},\infty}} \cong \mathfrak{a}^{-1} \rtimes (\mathcal{O}_K^\times)^2$$

with respect to the homomorphism

$$(\mathcal{O}_K^\times)^2 \to \mathrm{Aut}(\mathfrak{a}^{-1}), \quad \varepsilon^2 \mapsto (\mu \mapsto \varepsilon^2\mu).$$

The isomorphism is given by

$$\overline{\Gamma_{\mathfrak{a},\infty}} \to \mathfrak{a}^{-1} \rtimes (\mathcal{O}_K^\times)^2, \quad \pm \begin{pmatrix} \varepsilon & \mu \\ 0 & \varepsilon^{-1} \end{pmatrix} \mapsto (\varepsilon\mu, \varepsilon^2)$$

with the inverse

$$\mathfrak{a}^{-1} \rtimes (\mathcal{O}_K^\times)^2 \to \overline{\Gamma_{\mathfrak{a},\infty}}, \quad (\mu, \varepsilon^2) \mapsto \pm \begin{pmatrix} \varepsilon & \varepsilon^{-1}\mu \\ 0 & \varepsilon^{-1} \end{pmatrix}.$$

2.4 Quadratic space and lattices associated to K

Hilbert modular forms can be realized as orthogonal modular forms with respect to a quadratic space of signature $(2,2)$. In this section we introduce this quadratic space and call it V. Furthermore, we define lattices in V which are important for the definition of Hirzebruch–Zagier divisors on the Hilbert modular surface. Next, we investigate relations between those lattices which resemble the relations between the different Hilbert modular groups of the previous section. At the end of the section, we study the isotropic lines of V and their relation to cusps in a separate subsection.

Let us start with the definition of the vector space

$$V := \left\{ \begin{pmatrix} a & \lambda' \\ \lambda & b \end{pmatrix} \in K^{2\times 2} : \ a, b \in \mathbb{Q}, \lambda \in K \right\} = \left\{ A \in K^{2\times 2} : \ A^\top = A' \right\} \tag{2.10}$$

equipped with the quadratic form $\det(A) = ab - N(\lambda)$. The associated bilinear form is given by $(A, B) = \operatorname{tr}(AB^*)$ with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

denoting the adjoint. In terms of entries this unfolds to

$$\left(\begin{pmatrix} a_1 & \lambda_1' \\ \lambda_1 & b_1 \end{pmatrix}, \begin{pmatrix} a_2 & \lambda_2' \\ \lambda_2 & b_2 \end{pmatrix} \right) = a_1 b_2 + a_2 b_1 - (\lambda_1 \lambda_2' + \lambda_1' \lambda_2).$$

This bilinear form turns V into a quadratic space of signature $(2,2)$. The map

$$\mathrm{GL}_2(K) \times V \to V, \quad (M, A) \mapsto M.A := MA(M')^\top$$

defines a linear group action on V which respects the quadratic structure of V in the sense

$$\begin{aligned} (M.A, M.B) &= N(\det(M))(A, B) \\ \text{and} \quad \det(M.A) &= N(\det(M)) \det(A). \end{aligned} \tag{2.11}$$

Hence, restricted to $\mathrm{SL}_2(K)$, the group action is orthogonal. The kernel of the group homomorphism

$$\mathrm{SL}_2(K) \to \mathrm{O}(V), \quad M \mapsto (A \mapsto M.A)$$

is given by ± 1. Thus, we can view $\mathrm{PSL}_2(K) := \mathrm{SL}_2(K)/\{\pm 1\}$ as subgroup of $\mathrm{O}(V)$. For lattices $L \subset V$ and $M \in \mathrm{GL}_2(K)$ equation (2.11) implies

$$(M.L)^\vee = \frac{M.L^\vee}{N(\det(M))}.$$

In particular, we have $(M.L)^\vee = M.L^\vee$ for $M \in \mathrm{SL}_2(K)$.

Definition 2.4.1. Let $\mathfrak{a} \in \mathcal{I}_K$. We define

$$L(\mathfrak{a}) := \left\{ \begin{pmatrix} a & \lambda' \\ \lambda & b \end{pmatrix} \in V : \ a \in \mathbb{Z}, b \in N(\mathfrak{a})\mathbb{Z}, \lambda \in \mathfrak{a} \right\}.$$

It is immediate to see that the quadratic form evaluated on elements of $L(\mathfrak{a})$ takes values in $N(\mathfrak{a})\mathbb{Z}$. Using the knowledge about duals of ideals from Subsection 2.2.4, we obtain

$$L(\mathfrak{a})^\vee = \left\{ \frac{1}{N(\mathfrak{a})} \begin{pmatrix} a & \lambda' \\ \lambda & b \end{pmatrix} \in V : \ a \in \mathbb{Z}, b \in N(\mathfrak{a})\mathbb{Z}, \lambda \in \mathfrak{a}\mathfrak{d}^{-1} \right\}. \tag{2.12}$$

In the dual lattice the quadratic form takes values in $\mathbb{Z}/(N(\mathfrak{a})D)$.

Proposition 2.4.2. *Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_K$ and $M \in M(\mathfrak{a}, \mathfrak{b})$. Then it holds*

$$M.L(\mathfrak{a}^2\mathfrak{b}) = N(\mathfrak{a})L(\mathfrak{b}).$$

In particular, we have

$$M(\mathfrak{a}, \mathfrak{b}).L(\mathfrak{a}^2\mathfrak{b}) = N(\mathfrak{a})L(\mathfrak{b}).$$

The dual statement is

$$M.L(\mathfrak{a}^2\mathfrak{b})^\vee = \frac{L(\mathfrak{b})^\vee}{N(\mathfrak{a})} \quad \textit{and} \quad M(\mathfrak{a}, \mathfrak{b}).L(\mathfrak{a}^2\mathfrak{b})^\vee = \frac{L(\mathfrak{b})^\vee}{N(\mathfrak{a})}.$$

Proof. We first show that it is enough to prove the inclusion

$$M.L(\mathfrak{a}^2\mathfrak{b}) \subset N(\mathfrak{a})L(\mathfrak{b}).$$

We have

$$\begin{aligned} N(\mathfrak{a})L(\mathfrak{b}) \subset M.L(\mathfrak{a}^2\mathfrak{b}) \quad &\Leftrightarrow \quad M^{-1}.L(\mathfrak{b}) \subset N(\mathfrak{a}^{-1})L(\mathfrak{a}^2\mathfrak{b}) \\ &\Leftrightarrow \quad \tilde{M}.L(\tilde{\mathfrak{a}}^2\tilde{\mathfrak{b}}) \subset N(\tilde{\mathfrak{a}})L(\tilde{\mathfrak{b}}) \end{aligned}$$

with $\tilde{\mathfrak{a}} := \mathfrak{a}^{-1}$, $\tilde{\mathfrak{b}} := \mathfrak{a}^2\mathfrak{b}$ and $\tilde{M} := M^{-1}$. Using Lemma 2.3.4 (i), we see $\tilde{M} \in M(\mathfrak{a}^{-1}, \mathfrak{a}^2\mathfrak{b}) = M(\tilde{\mathfrak{a}}, \tilde{\mathfrak{b}})$. Therefore, the last inclusion is of the form of the inclusion from above.

Since the group action on V is linear, it is enough to show the inclusion for a set of vectors in $L(\mathfrak{b})$ which generate $L(\mathfrak{b})$ as $\mathbb{Z}$ module. Let $M = \left(\begin{smallmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{smallmatrix}\right)$. Recall that $M \in M(\mathfrak{a}, \mathfrak{b})$ means

$$\alpha \in \mathfrak{a}, \quad \beta \in \mathfrak{a}\mathfrak{b}, \quad \alpha^* \in (\mathfrak{a}\mathfrak{b})^{-1}, \quad \beta^* \in \mathfrak{a}^{-1}.$$

Then we have

$$\begin{aligned}
M.\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} &= \begin{pmatrix}N(\alpha) & \alpha\beta'\\ \alpha'\beta & N(\beta)\end{pmatrix} \in \begin{pmatrix}N(\mathfrak{a})\mathbb{Z} & N(\mathfrak{a})\mathfrak{b}'\\ N(\mathfrak{a})\mathfrak{b} & N(\mathfrak{a})N(\mathfrak{b})\mathbb{Z}\end{pmatrix}\\
&= N(\mathfrak{a})\begin{pmatrix}\mathbb{Z} & \mathfrak{b}'\\ \mathfrak{b} & N(\mathfrak{b})\mathbb{Z}\end{pmatrix},\\
M.\begin{pmatrix}0 & 0\\ 0 & N(\mathfrak{a}^2\mathfrak{b})\end{pmatrix} &= N(\mathfrak{a}^2\mathfrak{b})\begin{pmatrix}N(\alpha^*) & \alpha^*\beta^{*\prime}\\ \alpha^{*\prime}\beta^* & N(\beta^*)\end{pmatrix}\\
&\in N(\mathfrak{a}^2\mathfrak{b})\begin{pmatrix}N((\mathfrak{a}\mathfrak{b})^{-1})\mathbb{Z} & N(\mathfrak{a}^{-1})\mathfrak{b}^{-1}\\ N(\mathfrak{a}^{-1})\mathfrak{b}'^{-1} & N(\mathfrak{a}^{-1})\mathbb{Z}\end{pmatrix}\\
&= N(\mathfrak{a})\begin{pmatrix}\mathbb{Z} & \mathfrak{b}'\\ \mathfrak{b} & N(\mathfrak{b})\mathbb{Z}\end{pmatrix}.
\end{aligned}$$

Finally, for $\lambda \in \mathfrak{a}^2\mathfrak{b}$ we have

$$M.\begin{pmatrix}0 & \lambda'\\ \lambda & 0\end{pmatrix} = \begin{pmatrix}\mathrm{tr}(\alpha\alpha^{*\prime}\lambda') & \alpha^*\beta'\lambda + \alpha\beta^{*\prime}\lambda'\\ \alpha^{*\prime}\beta\lambda' + \alpha'\beta^*\lambda & \mathrm{tr}(\beta\beta^{*\prime}\lambda')\end{pmatrix}.$$

Let us look at the entries of this matrix using (2.5) (the upper right and lower left are conjugate, therefore one of them is enough):

$$\begin{aligned}
\mathrm{tr}(\alpha\alpha^{*\prime}\lambda') &\in \mathrm{tr}(\mathfrak{a}(\mathfrak{a}\mathfrak{b})'^{-1}\mathfrak{a}'^2\mathfrak{b}') = \mathrm{tr}\left(\frac{\mathfrak{a}^2}{N(\mathfrak{a})}\frac{\mathfrak{b}}{N(\mathfrak{b})}\mathfrak{a}'^2\mathfrak{b}'\right)\\
&= \mathrm{tr}(N(\mathfrak{a})\mathcal{O}_K) \subset N(\mathfrak{a})\mathbb{Z},\\
\mathrm{tr}(\beta\beta^{*\prime}\lambda') &\in \mathrm{tr}(\mathfrak{a}\mathfrak{b}\mathfrak{a}'^{-1}\mathfrak{a}'^2\mathfrak{b}') = \mathrm{tr}\left(\mathfrak{a}\mathfrak{b}\frac{\mathfrak{a}}{N(\mathfrak{a})}\mathfrak{a}'^2\mathfrak{b}'\right)\\
&= \mathrm{tr}\left(N(\mathfrak{a})N(\mathfrak{b})\mathcal{O}_K\right) \subset N(\mathfrak{a}\mathfrak{b})\mathbb{Z},\\
\alpha^{*\prime}\beta\lambda' + \alpha'\beta^*\lambda &\in (\mathfrak{a}\mathfrak{b})'^{-1}\mathfrak{a}\mathfrak{b}(\mathfrak{a}^2\mathfrak{b})' + \mathfrak{a}'\mathfrak{a}^{-1}\mathfrak{a}^2\mathfrak{b}\\
&= \frac{\mathfrak{a}}{N(\mathfrak{a})}\frac{\mathfrak{b}}{N(\mathfrak{b})}\mathfrak{a}\mathfrak{b}\mathfrak{a}'^2\mathfrak{b}' + N(\mathfrak{a})\mathfrak{b} = N(\mathfrak{a})\mathfrak{b}.
\end{aligned}$$

Hence, here as well we get

$$M.\begin{pmatrix}0 & \lambda'\\ \lambda & 0\end{pmatrix} \in N(\mathfrak{a})\begin{pmatrix}\mathbb{Z} & \mathfrak{b}'\\ \mathfrak{b} & N(\mathfrak{b})\mathbb{Z}\end{pmatrix}$$

which finishes the proof of the inclusion. All together we have shown the equality

$$M.L(\mathfrak{a}^2\mathfrak{b}) = N(\mathfrak{a})L(\mathfrak{b})$$

and therefore

$$M(\mathfrak{a},\mathfrak{b}).L(\mathfrak{a}^2\mathfrak{b}) = N(\mathfrak{a})L(\mathfrak{b}).$$

The dual statement follows directly from $(M.L)^\vee = M.L^\vee$ for $M \in \mathrm{SL}_2(K)$ together with the fact that scaling a lattice with a constant factor translates into dividing by that factor when passing over to the dual. □

Corollary 2.4.3. *The lattices $L(\mathfrak{a})$ and $L(\mathfrak{a})^\vee$ are invariant under $\Gamma_\mathfrak{a}$.*

Lemma 2.4.4. *The lower right entry of $A \in V$ is invariant (up to the sign if $N(\varepsilon_0) = -1$) under the operation of $\Gamma_{\mathfrak{a},\infty}$ for $\mathfrak{a} \in \mathcal{I}_K$.*

Proof. Let $A = \left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right)$ and $M = \left(\begin{smallmatrix} \varepsilon & \mu \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right) \in \Gamma_{\mathfrak{a},\infty}$. Then we have

$$\begin{aligned} M.A &= \begin{pmatrix} \varepsilon & \mu \\ 0 & \varepsilon^{-1} \end{pmatrix} \begin{pmatrix} a & \lambda' \\ \lambda & b \end{pmatrix} \begin{pmatrix} \varepsilon' & 0 \\ \mu' & (\varepsilon^{-1})' \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon a + \mu\lambda & \varepsilon\lambda' + b\mu \\ \lambda\varepsilon^{-1} & b\varepsilon^{-1} \end{pmatrix} \begin{pmatrix} \varepsilon' & 0 \\ \mu' & (\varepsilon^{-1})' \end{pmatrix} = \begin{pmatrix} * & * \\ * & N(\varepsilon)b \end{pmatrix}. \end{aligned}$$

□

2.4.1 Isotropic lines

In Section 2.3 we have seen that $\mathrm{SL}_2(K)$ acts transitively on $\mathbb{P}^1(K)$ and that its orbits, when restricted to subgroups $\Gamma \subset \mathrm{SL}_2(K)$, are the cusps of Γ. In this subsection we discuss the connection between this action and the action of $\mathrm{SL}_2(K)$ on isotropic lines of V and its consequences for lattices in V.

Lemma 2.4.5. *The map*

$$\psi_1 : \mathbb{P}^1(K) \to \mathbb{P}^1(\mathrm{Iso}(V)), \quad (\alpha : \beta) \mapsto \left[\begin{pmatrix} N(\alpha) & \alpha\beta' \\ \alpha'\beta & N(\beta) \end{pmatrix}\right]$$

is bijective and compatible with the action of $\mathrm{SL}_2(K)$. Its inverse is given by

$$\psi_2 : \mathbb{P}^1(\mathrm{Iso}(V)) \to \mathbb{P}^1(K), \quad \left[\begin{pmatrix} a & \lambda' \\ \lambda & b \end{pmatrix}\right] \mapsto \begin{cases} (a : \lambda), & (a, \lambda) \neq (0, 0), \\ (\lambda' : b), & (b, \lambda) \neq (0, 0). \end{cases}$$

Proof. It is easy to check that the two maps are well-defined and inverse to each other. To prove the compatibility it is enough to verify

$$\psi_1(M(1 : 0)) = M.\psi_1((1 : 0))$$

for $M \in \mathrm{SL}_2(K)$ because of the transitive action of $\mathrm{SL}_2(K)$ on $\mathbb{P}^1(K)$ by Lemma 2.3.1. Hence, let

$$M = \begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix} \in \mathrm{SL}_2(K).$$

Then we have

$$\psi_1(M(1 : 0)) = \psi_1((\alpha : \beta)) = \left[\begin{pmatrix} N(\alpha) & \alpha\beta' \\ \alpha'\beta & N(\beta) \end{pmatrix}\right]$$

and

$$\begin{aligned} M.\psi_1((1:0)) = M.\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right] &= \left[\begin{pmatrix}\alpha & \alpha^*\\ \beta & \beta^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha' & \beta'\\ \alpha^{*\prime} & \beta^{*\prime}\end{pmatrix}\right] \\ &= \left[\begin{pmatrix}\alpha & \alpha^*\\ \beta & \beta^*\end{pmatrix}\begin{pmatrix}\alpha' & \beta'\\ 0 & 0\end{pmatrix}\right] = \left[\begin{pmatrix}N(\alpha) & \alpha\beta'\\ \alpha'\beta & N(\beta)\end{pmatrix}\right]. \end{aligned}$$

□

Remark 2.4.6. Note that for a lattice $L \subset V$ we have a natural bijection

$$\mathrm{Iso}(L_0)/\{\pm 1\} \to \mathbb{P}^1(\mathrm{Iso}(V)), \quad \pm A \mapsto [A].$$

This is because every line in V (and in particular every isotropic line) contains up to the sign one primitive element in L. Hence, for a subgroup $\Gamma \subset \mathrm{SL}_2(K)$ which acts on $\mathrm{Iso}(L_0)$ the bijection of Lemma 2.4.5 induces a Γ compatible bijection between $\mathrm{Iso}(L_0)/\{\pm 1\}$ and $\mathbb{P}^1(K)$.

Corollary 2.4.7. *Let $\mathfrak{a} \in \mathcal{I}_K$. The primitive isotropic elements up to sign of the lattices $L(\mathfrak{a})$ and $L(\mathfrak{a})^\vee$ decompose into h_K orbits which correspond naturally to the cusps of $\Gamma_\mathfrak{a}$ under the action of $\Gamma_\mathfrak{a}$.*

Proof. This is a direct consequence of Remark 2.4.6 together with Proposition 2.3.9 and Corollary 2.4.3. □

Lemma 2.4.8. *Let $\mathfrak{b} \in \mathcal{I}_K$ and $(\alpha : \beta) \in \mathbb{P}^1(K)$ with fixed representatives $\alpha, \beta \in K$. We set $\mathfrak{a} := \alpha\mathcal{O}_K + \beta\mathfrak{b}^{-1}$ and*

$$M = \begin{pmatrix}\alpha & \alpha^*\\ \beta & \beta^*\end{pmatrix} \in M(\mathfrak{a}, \mathfrak{b}).$$

Then the map

$$F : \Gamma_{\mathfrak{a}^2\mathfrak{b}}/\Gamma_{\mathfrak{a}^2\mathfrak{b},\infty} \to \mathrm{Iso}(L(\mathfrak{b})_0)/\{\pm 1\}, \quad \gamma \mapsto \frac{\pm 1}{N(\mathfrak{a})}(M\gamma).\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

is injective. Its image is the orbit corresponding to $(\alpha : \beta)$ under the operation of $\Gamma_\mathfrak{b}$ on $\mathrm{Iso}(L(\mathfrak{b})_0)/\{\pm 1\}$.

Proof. First of all, Lemma 2.3.8 ensures the existence of the matrix M. We have

$$M(1:0) = (\alpha : \beta)$$

which implies that $F(E)$ (E being the identity matrix) belongs to the line $\psi_1((\alpha : \beta))$. We want to show that $F(E) \in L(\mathfrak{b})_0$. Looking at Definition 2.4.1, we see that

$$E_0 := \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \tag{2.13}$$

is a primitive vector of the lattice $L(\mathfrak{a}^2\mathfrak{b})$ and therefore $E_0/N(\mathfrak{a})$ is a primitive vector of the lattice $L(\mathfrak{a}^2\mathfrak{b})/N(\mathfrak{a})$. With the action of M being an element of $\mathrm{O}(V)$ we know that $M.(E_0/N(\mathfrak{a}))$ is a primitive vector of the lattice

$$M.\left(\frac{L(\mathfrak{a}^2\mathfrak{b})}{N(\mathfrak{a})}\right) = \frac{M.L(\mathfrak{a}^2\mathfrak{b})}{N(\mathfrak{a})} = \frac{N(\mathfrak{a})L(\mathfrak{b})}{N(\mathfrak{a})} = L(\mathfrak{b}).$$

This shows $F(E) \in L(\mathfrak{b})_0$. Now, we want to show that the image of F is actually the full $\Gamma_{\mathfrak{b}}$ orbit. Hence, let $A \in \Gamma_{\mathfrak{b}}$. We want to find a preimage of $A.F(E)$. It is given by $B := M^{-1}AM$ because of

$$\begin{aligned} F(B) &= (MB).(E_0/N(\mathfrak{a})) = (MM^{-1}AM).(E_0/N(\mathfrak{a})) \\ &= (AM).(E_0/N(\mathfrak{a})) = A.(M.(E_0/N(\mathfrak{a}))) = A.F(E). \end{aligned}$$

We actually have $B \in \Gamma_{\mathfrak{a}^2\mathfrak{b}}$ because of Corollary 2.3.6. Injectivity and well-definedness of F are left to the reader. □

Lemma 2.4.9. *For $\mathfrak{b} \in \mathcal{I}_K$ we have*

$$\mathrm{Iso}(L(\mathfrak{b})) = \mathrm{Iso}(N(\mathfrak{b})L(\mathfrak{b})^\vee).$$

Proof. By equation (2.12) we have

$$N(\mathfrak{b})L(\mathfrak{b})^\vee = \left\{ \begin{pmatrix} a & \lambda' \\ \lambda & b \end{pmatrix} \in V : \ a \in \mathbb{Z}, b \in N(\mathfrak{b})\mathbb{Z}, \lambda \in \mathfrak{b}\mathfrak{d}^{-1} \right\}.$$

This looks very similar to the definition of $L(\mathfrak{b})$ (cf. Definition 2.4.1). The difference is that there we have $\lambda \in \mathfrak{b}$ instead of $\lambda \in \mathfrak{b}\mathfrak{d}^{-1}$. This difference matters, hence $N(\mathfrak{b})L(\mathfrak{b})^\vee$ is a proper subset of $L(\mathfrak{b})$. However, the isotropic elements of the two lattices coincide: Take $\lambda \in \mathfrak{b}\mathfrak{d}^{-1}$ belonging to an isotropic element. Then we have $N(\lambda) \in N(\mathfrak{b})\mathbb{Z}$ because the corresponding determinant vanishes. This implies $N(\lambda\mathfrak{b}^{-1}) \in \mathbb{Z}$ which implies $\lambda\mathfrak{b}^{-1} \in \mathcal{O}_K$ because every prime over a prime ideal $\mathfrak{p} \mid \mathfrak{d}$ does not split. We obtain $\lambda \in \mathfrak{b}$. □

Lemma 2.4.10. *Let $\mathfrak{b} \in \mathcal{I}_K$ and $(\alpha : \beta) \in \mathbb{P}^1(K)$ with fixed representatives $\alpha, \beta \in K$. We set $\mathfrak{a} := \alpha\mathcal{O}_K + \beta\mathfrak{b}^{-1}$ and*

$$M = \begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix} \in M(\mathfrak{a}, \mathfrak{b}).$$

Then the map

$$F : \Gamma_{\mathfrak{a}^2\mathfrak{b}}/\Gamma_{\mathfrak{a}^2\mathfrak{b},\infty} \to \mathrm{Iso}((L(\mathfrak{b})^\vee)_0)/\{\pm 1\}, \quad \gamma \mapsto \frac{\pm 1}{N(\mathfrak{a}\mathfrak{b})}(M\gamma).E_0$$

is injective. Its image is the orbit corresponding to $(\alpha : \beta)$ under the operation of $\Gamma_{\mathfrak{b}}$ on $\mathrm{Iso}((L(\mathfrak{b})^\vee)_0)/\{\pm 1\}$.

Proof. This Lemma is simply the analogue of Lemma 2.4.8 for the dual of $L(\mathfrak{b})$. Since by Lemma 2.4.9 the isotropic elements of $L(\mathfrak{b})^\vee$ are the isotropic elements of $L(\mathfrak{b})$ scaled by $N(\mathfrak{b})^{-1}$, there is nothing more to show. □

2.5 Hilbert modular surfaces

The mathematical content of this section is based on the part *Hilbert Modular Forms and Their Applications* in [BvdGHZ08].

2.5.1 Notation

Before we come to the mathematical content, let us fix some notation. For elements $z \in \mathbb{C}^2$ we write z_1 and z_2 for their two components. Each component z_j has real part x_j and imaginary part y_j, i.e., we have $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ with $x_1, x_2, y_1, y_2 \in \mathbb{R}$. We abbreviate $x := (x_1, x_2) \in \mathbb{R}^2$ and $y := (y_1, y_2) \in \mathbb{R}^2$. By $N(z) := z_1 z_2$ we denote the product of the two components, by $\Im(z) := y_1 y_2$ we denote the product of the two imaginary parts and by $\mathrm{tr}(z) := z_1 + z_2$ we denote the sum of the two components. Addition and multiplication of an element $\lambda \in K$ with $z \in \mathbb{C}^2$ ($\mathbb{R}^2$, respectively) is defined as

$$\lambda + z := (\lambda + z_1, \lambda' + z_2) \quad \text{and} \quad \lambda z := (\lambda z_1, \lambda' z_2).$$

Hence, $\mathrm{tr}(\lambda z) = \lambda z_1 + \lambda' z_2$. Note that this behaves well with the definition of the norm and trace for $\lambda \in K$ from Section 2.2. By

$$\mathbb{H} := \{z \in \mathbb{C} : \ \Im(z) > 0\}$$

we address the upper half plane and by $\mathbb{H}^2$ the Cartesian product of $\mathbb{H}$ with itself.

2.5.2 Definition, invariant measure and topology

The transitive action of $\mathrm{GL}_2^+(\mathbb{R})$ on the upper half plane by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az + b}{cz + d}$$

induces a transitive action of $\mathrm{GL}_2^+(\mathbb{R})^2 := \mathrm{GL}_2^+(\mathbb{R}) \times \mathrm{GL}_2^+(\mathbb{R})$ on $\mathbb{H}^2$ by

$$(\gamma_1, \gamma_2) z := (\gamma_1 z_1, \gamma_2 z_2).$$

There is a symmetric $\mathrm{GL}_2^+(\mathbb{R})^2$ invariant Kähler metric on $\mathbb{H}^2$ whose corresponding $(1,1)$-form is given by

$$\omega := \eta_1 + \eta_2 \quad \text{with} \quad \eta_j := \frac{1}{4\pi} \frac{dx_j dy_j}{y_j^2}. \tag{2.14}$$

With the definitions $d := \partial + \overline{\partial}$ and $d^c := (4\pi i)^{-1}(\partial - \overline{\partial})$ it is immediate to see

$$\omega = -dd^c \log(y_1 y_2). \tag{2.15}$$

In this thesis, we use the induced $\mathrm{GL}_2^+(\mathbb{R})^2$ invariant top degree form

$$\omega^2 = \frac{1}{8\pi^2} \frac{dx_1 dy_1}{y_1^2} \frac{dx_2 dy_2}{y_2^2}$$

inducing an $\mathrm{GL}_2^+(\mathbb{R})^2$ invariant measure on $\mathbb{H}^2$ to integrate over Hilbert modular surfaces. By the natural embedding

$$\mathrm{GL}_2^+(K) \to \mathrm{GL}_2^+(\mathbb{R})^2, \quad \gamma \mapsto (\gamma, \gamma')$$

we obtain an action of $\mathrm{GL}_2^+(K)$ on $\mathbb{H}^2$ (where $\mathrm{GL}_2^+(K)$ is the subgroup of matrices in $\mathrm{GL}_2(K)$ with totally positive determinant). With $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(K)$ we get

$$\Im(\gamma z) = \frac{\Im(z) N(\det(\gamma))}{|N(cz+d)|^2}. \tag{2.16}$$

In most cases we are only interested in this action for subgroups of $\mathrm{SL}_2(K)$, namely the Hilbert modular groups $\Gamma_{\mathfrak{a}}$ for $\mathfrak{a} \in \mathcal{I}_K$. In this case the action is properly discontinuous. This means that the set

$$\{\gamma \in \Gamma_{\mathfrak{a}} : \ \gamma W \cap W \neq \emptyset\}$$

is finite for compact $W \subset \mathbb{H}^2$. In particular, for any $a \in \mathbb{H}^2$ the stabilizer $\Gamma_{\mathfrak{a},a} := \{\gamma \in \Gamma_{\mathfrak{a}} : \ \gamma a = a\}$ is a finite subgroup. Let $\overline{\Gamma_{\mathfrak{a},a}}$ be the image of $\Gamma_{\mathfrak{a},a}$ in

$$\mathrm{PSL}_2(K) := \mathrm{SL}_2(K)/\{\pm 1\}.$$

If $|\overline{\Gamma_{\mathfrak{a},a}}| > 1$, then a is called *elliptic fixed point* for $\Gamma_{\mathfrak{a}}$. Its order is defined by $|\overline{\Gamma_{\mathfrak{a},a}}|$. There are only finitely many $\Gamma_{\mathfrak{a}}$ orbits of elliptic fixed points. The quotient $X(\mathfrak{a}) := \Gamma_{\mathfrak{a}} \backslash \mathbb{H}^2$ is a normal complex space and is called *Hilbert modular surface*. Its singularities are given by the elliptic fixed points. Hence, $X(\mathfrak{a})^{\mathrm{ns}}$, i.e., $X(\mathfrak{a})$ without its singular points, is a complex manifold of dimension 2. Because of the $\mathrm{SL}_2(K)$ invariance of ω^2, the top degree form is also well-defined on the quotient $X(\mathfrak{a})$. It has finite volume which is given by a special value of the Dedekind zeta function. Namely, we have

$$\mathrm{vol}(X(\mathfrak{a})) := \int_{X(\mathfrak{a})} \omega^2 = \zeta_K(-1) = L(-1, \chi_D)\zeta(-1) = -\frac{L(-1, \chi_D)}{12}. \tag{2.17}$$

The surface $X(\mathfrak{a})$ is not compact. An important compactification is the so-called *Baily–Borel compactification*. As a set, this is defined to be

$$X(\mathfrak{a})^* := X(\mathfrak{a}) \cup \Gamma_{\mathfrak{a}} \backslash \mathbb{P}^1(K). \tag{2.18}$$

Hence, we add the cusps of $\Gamma_{\mathfrak{a}}$ to $X(\mathfrak{a})$ (whose finite number is given by the class number of K by Proposition 2.3.9). To describe the topology of $X(\mathfrak{a})^*$, we extend the topology of $\mathbb{H}^2$ to a topology of $(\mathbb{H}^2)^* := \mathbb{H}^2 \cup \mathbb{P}^1(K)$ by the following lemma.

Lemma 2.5.1. *On $(\mathbb{H}^2)^*$ there is a unique topology with the following properties.*

(i) The induced topology on $\mathbb{H}^2$ is the usual one.

(ii) The subspace $\mathbb{H}^2$ is open and dense in $(\mathbb{H}^2)^*$.

(iii) The sets $U_C \cup \{\infty\}$ with

$$U_C := \Big\{ z \in \mathbb{H}^2 : \ \Im(z) > C \Big\}$$

for $C \geq 0$ form a base of open neighborhoods of the point ∞.

(iv) If $\kappa \in \mathbb{P}^1(K)$ and $\rho \in \mathrm{SL}_2(K)$ with $\rho\infty = \kappa$, then the sets

$$\rho(U_C \cup \{\infty\}) \quad (C \geq 0)$$

form a base of open neighborhoods of the point κ.

Using this topology on $(\mathbb{H}^2)^*$, the quotient $X(\mathfrak{a})^* = \Gamma_{\mathfrak{a}} \backslash (\mathbb{H}^2)^*$ is a compact Hausdorff space. We want to view $X(\mathfrak{a})^*$ again as normal complex space. For that purpose, we have to describe how the sheaf of functions looks like: Let $V \subset X(\mathfrak{a})^*$ be open and $U \subset (\mathbb{H}^2)^*$ be the preimage of V under the canonical projection. We define $\mathcal{O}_{X(\mathfrak{a})^*}(V)$ to be the ring of continuous functions $f : V \to \mathbb{C}$ such that the pullback $\tilde{f} : U \to \mathbb{C}$ restricted to $U \cap \mathbb{H}^2$ is holomorphic. Then the pair $(X(\mathfrak{a})^*, \mathcal{O}_{X(\mathfrak{a})^*})$ is a normal complex space. In addition to the elliptic points the cusps are singularities.

2.5.3 The Hilbert modular surface at infinity

While it is complicated to give an explicit description of a fundamental domain of $X(\mathfrak{a})^*$ in $\mathbb{H}^2$, it is much easier to describe $X(\mathfrak{a})^*$ near a cusp.

Proposition 2.5.2. *For $C \geq N(\mathfrak{a}^{-1})$ the canonical map*

$$\Gamma_{\mathfrak{a},\infty} \backslash U_C \cup \{\infty\} \to X(\mathfrak{a})^*$$

is an open embedding.

Proof. The proposition follows from the following two statements:

(i) For all $C \geq 0$ the group $\Gamma_{\mathfrak{a},\infty}$ acts on U_C.

(ii) For $C \geq N(\mathfrak{a}^{-1})$ we have that $z \in U_C$ and $\gamma z \in U_C$ with $\gamma \in \Gamma_{\mathfrak{a}}$ implies $\gamma \in \Gamma_{\mathfrak{a},\infty}$.

For $\gamma = \left(\begin{smallmatrix} \varepsilon & \mu \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right) \in \Gamma_{\mathfrak{a},\infty}$ we compute

$$\Im(\gamma z) = \frac{\Im(z)}{|N(0 \cdot z + \varepsilon^{-1})|^2} = \frac{\Im(z)}{|N(\varepsilon^{-1})|^2} = \Im(z).$$

Hence, the operation of $\Gamma_{\mathfrak{a},\infty}$ lets $\Im(z)$ invariant which proves the first statement. Now let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\mathfrak{a}} \setminus \Gamma_{\mathfrak{a},\infty}$ and $z, \gamma z \in U_C$. Without loss of generality, we can assume $C > 0$ since $\min(\Im(z), \Im(\gamma z)) > 0$. We have

$$|N(cz + d)|^2 = \Big((cx_1 + d)^2 + (cy_1)^2\Big)\Big((c'x_2 + d')^2 + (c'y_2)^2\Big) \geq N(c)^2 \Im(z)^2.$$

Hence, since $c \neq 0$

$$C < \Im(\gamma z) = \frac{\Im(z)}{|N(cz+d)|^2} \leq \frac{\Im(z)}{N(c)^2 \Im(z)^2} < \frac{1}{CN(c)^2}.$$

So we have $C|N(c)| < 1$. Remembering that $0 \neq c \in \mathfrak{a}$ we get

$$C|N(c)| < 1 \quad \Rightarrow \quad CN(\mathfrak{a}) < 1 \quad \Leftrightarrow \quad C < N(\mathfrak{a}^{-1})$$

which finishes the proof of the second statement. □

For the quotient $\Gamma_{\mathfrak{a},\infty} \backslash U_C$ it is possible to give an explicit description of a fundamental domain. We want to elaborate this in the following. Since the group action of $\mathrm{SL}_2(K)$ factors through $\mathrm{PSL}_2(K)$, the groups $\Gamma_{\mathfrak{a},\infty}$ and $\overline{\Gamma_{\mathfrak{a},\infty}}$ share the same fundamental domain. As seen in Subsection 2.3.1, $\overline{\Gamma_{\mathfrak{a},\infty}}$ can be viewed as semidirect product $\mathfrak{a}^{-1} \rtimes (\mathcal{O}_K^\times)^2$. The induced operation of $\mathfrak{a}^{-1} \rtimes (\mathcal{O}_K^\times)^2$ on $\mathbb{H}^2$ is given by

$$(\mu, \varepsilon^2).z = \varepsilon^2 z + \mu = (\varepsilon^2 z_1 + \mu, \varepsilon^{-2} z_2 + \mu'). \tag{2.19}$$

This shows that two elements $z, \tilde{z} \in \mathbb{H}^2$ with the same imaginary parts $y = \tilde{y}$ are in the same $\Gamma_{\mathfrak{a},\infty}$ orbit if and only if their real parts are in the same $\mathfrak{a}^{-1}$ orbit, i.e., there exists a $\mu \in \mathfrak{a}^{-1}$ with $x + \mu = \tilde{x}$. On the other hand, in order to decide if the imaginary part of an element $z \in \mathbb{H}^2$ coincides with the imaginary part of an element of the $\Gamma_{\mathfrak{a},\infty}$ orbit of another element $\tilde{z} \in \mathbb{H}^2$, it is only necessary to monitor the $(\mathcal{O}_K^\times)^2$ action of $\tilde{z}$. Since $(\mathcal{O}_K^\times)^2$ is cyclic with generator ε_0^2, we need to check if there exists an $n \in \mathbb{Z}$ with

$$\begin{aligned}
\varepsilon_0^{2n} y = \tilde{y} \quad &\Leftrightarrow \quad (\varepsilon_0^{2n} y_1 = \tilde{y}_1 \quad \wedge \quad \varepsilon_0^{-2n} y_2 = \tilde{y}_2) \\
&\Leftrightarrow \quad \left(\Im(z) = \Im(\tilde{z}) \quad \wedge \quad \varepsilon_0^{2n} = \frac{\tilde{y}_1}{y_1} \right) \\
&\Leftrightarrow \quad \left(\Im(z) = \Im(\tilde{z}) \quad \wedge \quad \varepsilon_0^{2n} = \frac{y_2}{\tilde{y}_2} \right).
\end{aligned}$$

Bringing those two aspects together we have proven the following proposition.

Proposition 2.5.3. *Let $\mathcal{F}$ be a fundamental domain for $\mathbb{R}^2/\mathfrak{a}^{-1}$. Then a fundamental domain of $\Gamma_{\mathfrak{a},\infty} \backslash U_C$ is given by*

$$\left\{ z \in \mathbb{H}^2 : \; y_1 \in [1, \varepsilon_0^2), y_2 > C, x \in \mathcal{F} \right\}.$$

Remark 2.5.4. Since $\mathfrak{a}^{-1} \subset \mathbb{R}^2$ is a lattice in $\mathbb{R}^2$, a fundamental domain of $\mathbb{R}^2/\mathfrak{a}^{-1}$ can be explicitly given by a $\mathbb{Z}$ basis of $\mathfrak{a}^{-1}$. However, in this thesis an explicit construction is not needed. We are rather interested in the volume of the fundamental domain which is given by $\mathrm{vol}(\mathfrak{a}^{-1}) = N(\mathfrak{a}^{-1})\sqrt{D}$ (cf. Subsection 2.2.4).

Lemma 2.5.5. *The volume of $\Gamma_{\mathfrak{a},\infty} \backslash U_C$ with respect to ω^2 is finite if and only if $C > 0$. In that case it holds*

$$\mathrm{vol}(\Gamma_{\mathfrak{a},\infty} \backslash U_C) = \frac{(1 - \varepsilon_0^{-2})\sqrt{D}}{8\pi^2 C N(\mathfrak{a})}.$$

In particular, we get

$$\mathrm{vol}(X(\mathfrak{a})) \geq \frac{(1-\varepsilon_0^{-2})\sqrt{D}}{8\pi^2}.$$

Proof. Together with Proposition 2.5.3 and Remark 2.5.4 we have

$$\begin{aligned}\mathrm{vol}(\Gamma_{\mathfrak{a},\infty}\backslash U_C) = \int_{\Gamma_{\mathfrak{a},\infty}\backslash U_C} \omega^2 &= \frac{1}{8\pi^2}\int_1^{\varepsilon_0^2}\frac{dy_1}{y_1^2}\int_C^\infty \frac{dy_2}{y_2^2}\int_{\mathbb{R}^2/\mathfrak{a}^{-1}} dx_1 dx_2 \\ &= \frac{1}{8\pi^2}(1-\varepsilon_0^{-2})C^{-1}N(\mathfrak{a}^{-1})\sqrt{D} \\ &= \frac{(1-\varepsilon_0^{-2})\sqrt{D}}{8\pi^2 C N(\mathfrak{a})}.\end{aligned}$$

For the additional statement we use $C = N(\mathfrak{a}^{-1})$ and Proposition 2.5.2. □

2.5.4 Siegel domains

As already pointed out in the previous subsection it is quite complicated and technical to make fundamental domains of $\Gamma_{\mathfrak{a}}$ explicit. For our puposes, it is in most cases enough to have an easily described subset $F \subset \mathbb{H}^2$ of finite volume and containing a fundamental domain instead. Such subsets can be constructed with the help of so-called *Siegel domains.*

Definition 2.5.6. For $t > 0$ we define the *Siegel domain*

$$\mathcal{S}_t := \left\{ z \in \mathbb{H}^2 : \ |x_j| < t \quad \text{and} \quad |y_j| > t^{-1} \quad \text{for } j = 1,2 \right\}.$$

We have

$$\mathrm{vol}(\mathcal{S}_t) = \frac{1}{8\pi^2}\left(\int_{t^{-1}}^\infty \frac{dy}{y^2}\right)^2\left(\int_{-t}^t dx\right)^2 = \frac{t^4}{2\pi^2}. \tag{2.20}$$

It is easy to show that there are only finitely many $\gamma \in \Gamma_{\mathfrak{a}}$ with $\gamma\mathcal{S}_t \cap \mathcal{S}_t \neq \emptyset$ (the ideas of the proof appear already in the proof of Proposition 2.5.2). On the other hand, for fixed $C > 0$ and large enough t the Siegel domain $\mathcal{S}_t$ contains a fundamental domain for $\Gamma_{\mathfrak{a},\infty}\backslash U_C$ and hence a neighborhood of the cusp ∞ (cf. again Proposition 2.5.2). In total, this can be put together (with some additional work introducing a distance of points $z \in \mathbb{H}^2$ to cusps $\kappa \in \mathbb{P}^1(K)$) to show the following theorem.

Theorem 2.5.7. *Let $\kappa_1, \ldots, \kappa_{h_K} \in \mathbb{P}^1(K)$ be a set of representatives for the cusps of $\Gamma_{\mathfrak{a}}$ and let $\rho_1, \ldots, \rho_{h_K} \in \mathrm{SL}_2(K)$ be such that $\rho_j\infty = \kappa_j$. Then there exists a $t > 0$ such that*

$$\mathcal{S} := \bigcup_{j=1}^{h_K} \rho_j S_t$$

contains a fundamental domain for $\Gamma_{\mathfrak{a}}$.

2.6 Decomposition and majorant associated to points in $\mathbb{H}^2$

Each $z \in \mathbb{H}^2$ gives rise to an orthogonal decomposition $W_z \oplus \tilde{W}_z$ of $V_{\mathbb{R}} := V \otimes_{\mathbb{Z}} \mathbb{R}$ with respect to the determinant such that the determinant restricted to W_z is negative definite and the determinant restricted to $\tilde{W}_z$ is positive definite. To make that precise we define

$$X_z := \begin{pmatrix} x_1x_2 - y_1y_2 & x_1 \\ x_2 & 1 \end{pmatrix}, \quad Y_z := \begin{pmatrix} x_1y_2 + x_2y_1 & y_1 \\ y_2 & 0 \end{pmatrix}$$

and

$$\tilde{X}_z := \begin{pmatrix} x_1x_2 + y_1y_2 & x_1 \\ x_2 & 1 \end{pmatrix}, \quad \tilde{Y}_z := \begin{pmatrix} x_1y_2 - x_2y_1 & -y_1 \\ y_2 & 0 \end{pmatrix}.$$

It is easy to check that $X_z, Y_z, \tilde{X}_z, \tilde{Y}_z$ form an orthogonal basis of $V_{\mathbb{R}}$. We define W_z to be the plane spanned by X_z and Y_z, and $\tilde{W}_z$ to be the plane spanned by $\tilde{X}_z$ and $\tilde{Y}_z$. It is natural to decompose the quadratic form $\det = q_{W_z} + q_{\tilde{W}_z}$ by

$$q_{W_z}(A) := \det(\pi_{W_z}(A)) \quad \text{and} \quad q_{\tilde{W}_z}(A) := \det(\pi_{\tilde{W}_z}(A))$$

with

$$\pi_{W_z} : V_{\mathbb{R}} \to W_z \quad \text{and} \quad \pi_{\tilde{W}_z} : V_{\mathbb{R}} \to \tilde{W}_z$$

being the corresponding orthogonal projections. This decomposition gives rise to the definition of the *majorant*

$$q_z(A) := -q_{W_z}(A) + q_{\tilde{W}_z}(A).$$

The majorant of the determinant with respect to z is by definition a positive definite quadratic form on $V_{\mathbb{R}}$. The component related to the plane W_z of the decomposition of the determinant is of special interest for later purposes, for example when we come to define Green functions on $X(\mathfrak{a})$. Therefore, we give it a special name $h(A, z) := -q_{W_z}(A)$. Let us summarize the relations. We have

$$\det(A) = q_{\tilde{W}_z}(A) - h(A, z), \quad q_z(A) = q_{\tilde{W}_z}(A) + h(A, z) = \det(A) + 2h(A, z). \tag{2.21}$$

For elements $A = \left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in V$ we want to express $h(A, z)$ in terms of a, b, λ and λ'. For that purpose we express A in terms of the basis $X_z, Y_z, \tilde{X}_z, \tilde{Y}_z$:

$$\begin{aligned} A = &- \frac{b(x_1x_2 - y_1y_2) - \lambda x_1 - \lambda' x_2 + a}{2y_1y_2} X_z \\ &- \frac{b(x_1y_2 + x_2y_1) - \lambda y_1 - \lambda' y_2}{2y_1y_2} Y_z \\ &+ \frac{b(x_1x_2 + y_1y_2) - \lambda x_1 - \lambda' x_2 + a}{2y_1y_2} \tilde{X}_z \\ &+ \frac{b(x_1y_2 - x_2y_1) + \lambda y_1 - \lambda' y_2}{2y_1y_2} \tilde{Y}_z. \end{aligned}$$

Now we have

$$\begin{aligned} h(\eta X_z + \mu Y_z + \tilde{\eta}\tilde{X}_z + \tilde{\mu}\tilde{Y}_z, z) &= -q_{W_z}(\eta X_z + \mu Y_z + \tilde{\eta}\tilde{X}_z + \tilde{\mu}\tilde{Y}_z) \\ &= -\det(\eta X_z + \mu Y_z) = (\eta^2 + \mu^2) y_1 y_2. \end{aligned}$$

Hence,

$$\begin{aligned} h(A, z) &= \left(\frac{b(x_1x_2 - y_1y_2) - \lambda x_1 - \lambda' x_2 + a}{2y_1y_2}\right)^2 y_1y_2 \\ &+ \left(\frac{b(x_1y_2 + x_2y_1) - \lambda y_1 - \lambda' y_2}{2y_1y_2}\right)^2 y_1y_2 \\ &= \frac{|bz_1z_2 - \lambda z_1 - \lambda' z_2 + a|^2}{4y_1y_2}. \end{aligned} \tag{2.22}$$

Analogously, we obtain

$$\begin{aligned} q_{\tilde{W}_z}(A) &= \left(\frac{b(x_1x_2 + y_1y_2) - \lambda x_1 - \lambda' x_2 + a}{2y_1y_2}\right)^2 y_1y_2 \\ &+ \left(\frac{b(x_1y_2 - x_2y_1) + \lambda y_1 - \lambda' y_2}{2y_1y_2}\right)^2 y_1y_2 \\ &= \frac{|b\overline{z}_1z_2 - \lambda\overline{z}_1 - \lambda' z_2 + a|^2}{4y_1y_2}. \end{aligned} \tag{2.23}$$

From the first equation of (2.21) we can now infer

$$|b\overline{z}_1z_2 - \lambda\overline{z}_1 - \lambda' z_2 + a|^2 - |bz_1z_2 - \lambda z_1 - \lambda' z_2 + a|^2 = 4y_1y_2 \det(A).$$

It is an important property of the function $h(A, z)$ to behave well with the operation of $\mathrm{GL}_2^+(K)$ on $\mathbb{H}^2$.

Proposition 2.6.1. *We have*

$$N(\det(M))h(A, z) = h(M.A, Mz)$$

for $M \in \mathrm{GL}_2^+(K)$, $A \in V$ *and* $z \in \mathbb{H}^2$. *In particular*

$$h(A, z) = h(M.A, Mz)$$

for $M \in \mathrm{SL}_2(K)$.

Proof. Let us first fix some notation:

$$A = \begin{pmatrix} a & \lambda' \\ \lambda & b \end{pmatrix}, \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that we have

$$bz_1z_2 - \lambda z_1 - \lambda' z_2 + a = \begin{pmatrix} z_1 & 1 \end{pmatrix} \begin{pmatrix} b & -\lambda \\ -\lambda' & a \end{pmatrix} \begin{pmatrix} z_2 \\ 1 \end{pmatrix}$$

as matrix product. Further, it holds

$$S.A = \begin{pmatrix} b & -\lambda \\ -\lambda' & a \end{pmatrix}.$$

Hence, we have found a different formula for $h(A, z)$:

$$h(A, z) = \frac{\left| \begin{pmatrix} z_1 & 1 \end{pmatrix} (S.A) \begin{pmatrix} z_2 & 1 \end{pmatrix}^\top \right|^2}{4\Im(z)}.$$

It is straightforward to check

$$\begin{pmatrix} Mz_1 \\ 1 \end{pmatrix} = \frac{M \begin{pmatrix} z_1 & 1 \end{pmatrix}^\top}{\gamma z_1 + \delta} \quad \text{and} \quad S^{-1} M^\top S = \det(M) M^{-1}.$$

Prepared with those two identities, we finish the proof with the computation

$$\begin{aligned}
h(A, Mz) &= \frac{\left| \begin{pmatrix} Mz_1 & 1 \end{pmatrix} (S.A) \begin{pmatrix} M'z_2 & 1 \end{pmatrix}^\top \right|^2}{4\Im(Mz)} \\
&= \frac{\left| \frac{\begin{pmatrix} z_1 & 1 \end{pmatrix} M^\top}{\gamma z_1 + \delta} (S.A) \frac{M' \begin{pmatrix} z_2 & 1 \end{pmatrix}^\top}{\gamma' z_2 + \delta'} \right|^2}{4\Im(z) \frac{N(\det(M))}{|N(\gamma z + \delta)|^2}} \\
&= \frac{1}{N(\det(M))} \frac{\left| \begin{pmatrix} z_1 & 1 \end{pmatrix} ((M^\top S).A) \begin{pmatrix} z_2 & 1 \end{pmatrix}^\top \right|^2}{4\Im(z)} \\
&= \frac{1}{N(\det(M))} \frac{\left| \begin{pmatrix} z_1 & 1 \end{pmatrix} ((S \det(M) M^{-1}).A) \begin{pmatrix} z_2 & 1 \end{pmatrix}^\top \right|^2}{4\Im(z)} \\
&= \frac{h((\det(M) M^{-1}).A, z)}{N(\det(M))} = \frac{h(N(\det(M))(M^{-1}.A), z)}{N(\det(M))} \\
&= \frac{N(\det(M))^2 h(M^{-1}.A, z)}{N(\det(M))} = N(\det(M)) h(M^{-1}.A, z).
\end{aligned}$$

□

For anisotropic $A = \left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in V$ the normalized function

$$g(A, z) := \frac{h(A, z)}{\det(A)} = \frac{|bz_1 z_2 - \lambda z_1 - \lambda' z_2 + a|^2}{4 y_1 y_2 \det(A)} \tag{2.24}$$

comes in handy from time to time. Proposition 2.6.1 implies the following corollary.

Corollary 2.6.2. *For anisotropic $A \in V$ and $M \in \mathrm{GL}_2^+(K)$ we have*

$$g(A, z) = g(M.A, Mz).$$

Proof. We simply apply Proposition 2.6.1

$$g(M.A, Mz) = \frac{h(M.A, Mz)}{\det(M.A)} = \frac{N(\det(M))h(A, z)}{N(\det(M))\det(A)} = g(A, z).$$

□

Remark 2.6.3. As a quadratic form, $h(A, z)$ is well-defined for $A \in V/\{\pm 1\}$. The function $g(A, z)$ behaves even better because of the division by the determinant. Namely, it is well-defined for $A \in V^+/\mathbb{R}^\times = \mathbb{P}(V^+)$ and $A \in V^-/\mathbb{R}^\times = \mathbb{P}(V^-)$.

Lemma 2.6.4. *Let $L \subset V_\mathbb{R}$ be a lattice, $m \in \mathbb{R}$, $C > 0$ and $M \subset \mathbb{H}^2$ be compact. Then the set*

$$H_C := \{A \in L : \exists z \in M,\ h(A, z) \leq C,\ \det(A) = m\}$$

is finite. The same holds for

$$G_C := \{A \in L : \exists z \in M,\ g(A, z) \leq C,\ \det(A) = m\}$$

if $m > 0$.

Proof. If $m > 0$, we have by definition of $g(A, z)$

$$G_C = H_{mC}.$$

Therefore, it is enough to prove the finiteness of H_C. Let $A \in H_C$. Then we find a $z \in M$ with $h(A, z) \leq C$. This implies

$$q_z(A) = \det(A) + 2h(A, z) \leq m + 2C.$$

Recall that q_z is a positive definite quadratic form on V. Now let q be any (independent of z) positive definite quadratic form on V. Since all norms on a finite dimensional vector space are equivalent, we find an $\alpha_z > 0$ such that

$$q(x) \leq \alpha_z q_z(x)$$

for all $x \in V$. Because the map $z \mapsto q_z$ is continuous, we can choose the constants $\alpha_z > 0$ in a continuous way, i.e., we may assume that

$$\mathbb{H}^2 \to \mathbb{R}^+, \quad z \mapsto \alpha_z$$

is continuous. In particular, because M is compact, we find an $\alpha > 0$ such that $\alpha_z \leq \alpha$ for all $z \in M$. This implies that we have for all $A \in H_C$

$$q(A) \leq \alpha(m + 2C).$$

We obtain

$$H_C \subset q^{-1}([0, \alpha(m + 2C)])$$

with the latter set being compact. Each compact set contains only finitely many lattice points. Therefore, H_C is finite. □

There is a remarkable relation between the hyperbolic distance and $g(A, z)$ that we want to point out at this place.

Definition 2.6.5 (Hyperbolic distance)**.** The function

$$d : (\mathbb{C} \setminus \mathbb{R})^2 \to \mathbb{R}, \quad d(z_1, z_2) := \frac{|z_1 - z_2|^2}{y_1 y_2}$$

is called the *hyperbolic distance*. For $\gamma \in \mathrm{GL}_2(\mathbb{R})$ we have

$$d(z_1, z_2) = d(\gamma z_1, \gamma z_2).$$

In other sources the hyperbolic distance is often only defined for arguments taken from the upper half plane. In that case one needs to restrict γ to $\mathrm{GL}_2^+(\mathbb{R})$. We, however, need to have it defined for elements of the lower half plane as well.

Lemma 2.6.6. *Let $A \in V$ be anisotropic. Then we have with $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$*

$$g(A, z) = \frac{d(z_1, ASz_2)}{4} = \frac{|z_1 - ASz_2|^2}{4\Im(z_1)\Im(SAz_2)}.$$

Proof. With

$$A = \begin{pmatrix} a & \lambda' \\ \lambda & b \end{pmatrix} \quad \text{we have} \quad AS = \begin{pmatrix} \lambda' & -a \\ b & -\lambda \end{pmatrix}.$$

Hence, we get

$$z_1 - ASz_2 = z_1 - \frac{\lambda' z_2 - a}{bz_2 - \lambda} = \frac{bz_1 z_2 - \lambda z_1 - \lambda' z_2 + a}{bz_2 - \lambda}.$$

Now, the claim follows by the definition (2.24) of $g(A, z)$ with

$$\Im(ASz_2) = \frac{\det(AS) y_2}{|bz_2 - \lambda|^2} = \frac{\det(A) y_2}{|bz_2 - \lambda|^2}. \tag{2.25}$$

□

Another useful relation involving $h(A, z)$ is presented in the next lemma.

Lemma 2.6.7. *With*

$$E_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{we have} \quad h(E_0, z) = \frac{1}{4\Im(z)}.$$

Proof. The equality follows directly from equation (2.22). □

2.7 Hirzebruch compactification

The aim of this section is to describe a different compactification of $X(\mathfrak{a})$ in which the cusps are not singular anymore, i.e., we resolve the cusp singularities of $X(\mathfrak{a})^*$. This resolution was discovered by Hirzebruch in 1971. See [vdG88, Chapter II] for a detailed discussion which is our main reference for this section.

Let $\mathfrak{b} \in \mathcal{I}_K$ and $\kappa = (\alpha : \beta) \in \mathbb{P}^1(K)$. Then there exists a matrix $M \in M(\mathfrak{a}, \mathfrak{b})$ with $\mathfrak{a} := \alpha\mathcal{O}_K + \beta\mathfrak{b}^{-1}$ and $M\infty = \kappa$ by Lemma 2.3.8. Now the map

$$(\mathbb{H}^2)^* \to (\mathbb{H}^2)^*, \quad z \mapsto M^{-1}z$$

induces an isomorphism $X(\mathfrak{b})^* \xrightarrow{\sim} X(\mathfrak{a}^2\mathfrak{b})^*$ mapping the cusp κ of $X(\mathfrak{b})^*$ to the cusp ∞ of $X(\mathfrak{a}^2\mathfrak{b})^*$. Hence, the diagram

$$\begin{tikzcd}
(\mathbb{H}^2)^* \arrow[r, "z\mapsto M^{-1}z"] \arrow[d, "z\mapsto\Gamma_{\mathfrak{b}}z"] & (\mathbb{H}^2)^* \arrow[d, "z\mapsto\Gamma_{\mathfrak{a}^2\mathfrak{b}}z"] \\
X(\mathfrak{b})^* \arrow[r, "\Gamma_{\mathfrak{b}}z\mapsto M^{-1}\Gamma_{\mathfrak{b}}z=\Gamma_{\mathfrak{a}^2\mathfrak{b}}M^{-1}z"] & X(\mathfrak{a}^2\mathfrak{b})^*
\end{tikzcd}$$

commutes and the horizontal maps are isomorphisms. Therefore, it is enough to describe the procedure of desingularizing a cusp for the cusp ∞.

By Proposition 2.5.2 the neighborhoods of ∞ look like the quotients $\Gamma_{\mathfrak{a},\infty}\backslash U_C$. We have already seen that we can replace $\Gamma_{\mathfrak{a},\infty}$ by the semidirect product $\mathfrak{a}^{-1} \rtimes (\mathcal{O}_K^\times)^2$. Let us focus on the factor $\mathfrak{a}^{-1}$ first.

2.7.1 Dividing by the ideal

For shorter notation let $\mathfrak{b} := \mathfrak{a}^{-1}$. Hence, we consider the quotient

$$U_C/\mathfrak{b} \subset \mathbb{H}^2/\mathfrak{b} \subset \mathbb{C}^2/\mathfrak{b}.$$

The latter quotient $\mathbb{C}^2/\mathfrak{b}$ is isomorphic to $(\mathbb{C}^\times)^2 := \mathbb{C}^\times \times \mathbb{C}^\times$, but there is no canonical isomorphism. Each basis (α, β) of the $\mathbb{Z}$ module $\mathfrak{b}$ gives rise to an isomorphism $\mathbb{C}^2/\mathfrak{b} \xrightarrow{\sim} (\mathbb{C}^\times)^2$. Namely, consider the linear map

$$\varphi_{\alpha,\beta} : \mathbb{C}^2 \to \mathbb{C}^2, \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto 2\pi i \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{2.26}$$

The image under $\varphi_{\alpha,\beta}$ of $\mathfrak{b}$ is $2\pi i\mathbb{Z} \times 2\pi i\mathbb{Z}$. More precisely, we have

$$\begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} \mapsto \begin{pmatrix} 2\pi i \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \beta \\ \beta' \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 2\pi i \end{pmatrix}.$$

Hence, $\exp \circ \varphi_{\alpha,\beta} : \mathbb{C}^2 \to (\mathbb{C}^\times)^2$ is a group homomorphism with kernel $\mathfrak{b}$. Therefore, it factors through the quotient $\mathbb{C}^2/\mathfrak{b}$ and induces an isomorphism $\tilde{\varphi}_{\alpha,\beta} : \mathbb{C}^2/\mathfrak{b} \to (\mathbb{C}^\times)^2$. Computing the inverse of $\left(\begin{smallmatrix} \alpha & \beta \\ \alpha' & \beta' \end{smallmatrix}\right)$, it is given by

$$\tilde{\varphi}_{\alpha,\beta} : \mathbb{C}^2/\mathfrak{b} \to (\mathbb{C}^\times)^2, \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} e\left(\frac{\beta' z_1 - \beta z_2}{\alpha\beta' - \alpha'\beta}\right) \\ e\left(\frac{\alpha z_2 - \alpha' z_1}{\alpha\beta' - \alpha'\beta}\right) \end{pmatrix}. \tag{2.27}$$

Let us summarize this in the commutative diagram

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "\varphi_{\alpha,\beta}"] \arrow[d, "\pi"] & \mathbb{C}^2 \arrow[d, "\exp"] \\ \mathbb{C}^2/\mathfrak{b} \arrow[r, "\tilde{\varphi}_{\alpha,\beta}"] & (\mathbb{C}^\times)^2. \end{tikzcd}$$

The inverse $\tilde{\varphi}_{\alpha,\beta}^{-1} : (\mathbb{C}^\times)^2 \to \mathbb{C}^2/\mathfrak{b}$ is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \frac{1}{2\pi i} \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} \begin{pmatrix} \log u \\ \log v \end{pmatrix}.$$

Here choosing a different branch of the logarithm results in a different representative of $\mathbb{C}^2/\mathfrak{b}$. If the branch of the logarithm is chosen independently in each row, every element in the coset can be obtained.

2.7.2 Expressing functions in local coordinates

In this subsection we interrupt the construction of the Hirzebruch compactification in order to apply what we discussed in the previous subsection. We want to express a few $\mathfrak{b}$ invariant exponential functions of interest in local coordinates, i.e., for a given $\mathfrak{b}$ invariant function $f : \mathbb{C}^2 \to \mathbb{C}$ we make the function $\tilde{f} : (\mathbb{C}^\times)^2 \to \mathbb{C}$ induced by

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "f"] \arrow[d, "\pi"] & \mathbb{C} \\ \mathbb{C}^2/\mathfrak{b} \arrow[r, "\tilde{\varphi}_{\alpha,\beta}"] & (\mathbb{C}^\times)^2 \arrow[u, "\tilde{f}"] \end{tikzcd}$$

explicit. Later in this thesis, we will abbreviate this procedure simply by *expressing functions in local coordinates.*

Let $\nu \in \mathfrak{a}\mathfrak{d}^{-1}$, then the function $z \mapsto e(\mathrm{tr}(\nu z))$ is $\mathfrak{b} = \mathfrak{a}^{-1}$ invariant. Namely, for $\mu \in \mathfrak{b}$ we have

$$e(\mathrm{tr}(\nu(z+\mu)) = e(\mathrm{tr}(\nu z))e(\mathrm{tr}(\nu\mu)) = e(\mathrm{tr}(\nu z))$$

because $\mathfrak{a}\mathfrak{d}^{-1}$ is the trace dual of $\mathfrak{b}$ (cf. equation (2.7)). Now let us fix an element $z \in \mathbb{C}^2$. By fixing a respective branch of $\log(u)$ and a possibly different branch of $\log(v)$ we get

$$\begin{pmatrix} 2\pi i z_1 \\ 2\pi i z_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} \begin{pmatrix} \log u \\ \log v \end{pmatrix}. \tag{2.28}$$

Hence, we have

$$\begin{aligned} e(\operatorname{tr}(\nu z)) &= \exp(\nu(\alpha \log(u) + \beta \log(v))) \exp(\nu'(\alpha' \log(u) + \beta' \log(v))) \\ &= \exp((\nu\alpha + \nu'\alpha') \log(u)) \exp((\nu\beta + \nu'\beta') \log(v)) = u^{\operatorname{tr}(\nu\alpha)} v^{\operatorname{tr}(\nu\beta)}. \end{aligned}$$

Since $\alpha, \beta \in \mathfrak{b}$ we see that $\operatorname{tr}(\nu\alpha), \operatorname{tr}(\nu\beta) \in \mathbb{Z}$. Therefore, the final result is independent of the chosen branches of $\log(u)$ and $\log(v)$. Using the same approach, one proves the following lemma.

Lemma 2.7.1. *Let $\nu \in \mathfrak{a}\mathfrak{d}^{-1}$. Then the following functions are $\mathfrak{a}^{-1}$ invariant and can be expressed in local coordinates (u, v) with respect to (α, β):*

$$\begin{aligned} e(\operatorname{tr}(\nu z)) = e(\nu z_1) e(\nu' z_2) &= u^{\operatorname{tr}(\alpha\nu)} v^{\operatorname{tr}(\beta\nu)}, \\ e(\operatorname{tr}(\nu \overline{z})) = e(\nu \overline{z_1}) e(\nu' \overline{z_2}) &= \overline{u}^{-\operatorname{tr}(\alpha\nu)} \overline{v}^{-\operatorname{tr}(\beta\nu)}, \\ e(\nu z_1) e(\nu' \overline{z_2}) &= u^{\alpha\nu} \overline{u}^{-\alpha'\nu'} v^{\beta\nu} \overline{v}^{-\beta'\nu'}, \\ e(\nu \overline{z_1}) e(\nu' z_2) &= u^{\alpha'\nu'} \overline{u}^{-\alpha\nu} v^{\beta'\nu'} \overline{v}^{-\beta\nu}. \end{aligned}$$

The evaluation of the third and fourth line is independent of the chosen branch of the logarithm $\log(u)$ as long as the branch of $\log(\overline{u})$ is chosen accordingly, i.e., $\log(\overline{u}) := \overline{\log(u)}$. The same holds for $\log(v)$ and $\log(\overline{v})$, respectively.

Another important and simple $\mathfrak{b}$ invariant function is $z \mapsto y$. The $\mathfrak{b}$ invariance is immediate because $\mathfrak{b}$ acts only on the real part. In Subsection 2.7.7 we explain how to express y in local coordinates (cf. equation (2.30)).

2.7.3 Gluing

We now continue the construction of the Hirzebruch compactification. Consider a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{C}^2/\mathfrak{b}$ with $\lim_{n\to\infty} \Im(z_n) = \infty$. Looking at the inclusion $(\mathbb{C}^\times)^2 \subset \mathbb{C}^2$, the sequence $\tilde{\varphi}_{\alpha,\beta}(z_n)$ might converge to a point in

$$(\mathbb{C}^\times \times \mathbb{C}^\times)^c = (\{0\} \times \mathbb{C}) \cup (\mathbb{C} \times \{0\}) \subset \mathbb{C}^2.$$

On the other hand, for each point in $(\mathbb{C}^\times \times \mathbb{C}^\times)^c$ we find a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{C}^2/\mathfrak{b}$ with $\lim_{n\to\infty} \Im(z_n) = \infty$ such that $\tilde{\varphi}_{\alpha,\beta}(z_n)$ converges to that point. Therefore, it seems natural to consider $(\mathbb{C}^\times \times \mathbb{C}^\times)^c$ as part of the desingularization of the cusp ∞. However, there are sequences $(z_n)_{n\in\mathbb{N}} \subset \mathbb{C}^2/\mathfrak{b}$ with $\lim_{n\to\infty} \Im(z_n) = \infty$ such that $\tilde{\varphi}_{\alpha,\beta}(z_n)$ does not converge in $\mathbb{C}^2$ for one basis (α, β) but $\tilde{\varphi}_{\tilde{\alpha},\tilde{\beta}}(z_n)$ converges in $\mathbb{C}^2$ for another basis $(\tilde{\alpha}, \tilde{\beta})$. Hence, instead of considering only one basis (α, β) we have to consider a family of

bases simultaneously and glue the resulting spaces $\mathbb{C}^2$ together appropriately. For that purpose the next base change lemma tells us how the induced isomorphsim of $(\mathbb{C}^\times)^2$ looks like when the base is changed.

Lemma 2.7.2 (Base change)**.** *Let (α_1, β_1) and (α_2, β_2) be two bases of $\mathfrak{b}$ and $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ the base change matrix, i.e.,*

$$\begin{pmatrix} \alpha_1 & \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_2 & \beta_2 \end{pmatrix} A.$$

Then the maps $\psi^{\alpha_1,\beta_1}_{\alpha_2,\beta_2}$ and $\tilde{\psi}^{\alpha_1,\beta_1}_{\alpha_2,\beta_2}$ in the commutative diagram

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "\varphi_{\alpha_1,\beta_1}"] \arrow[d, "\mathrm{id}"] & \mathbb{C}^2 \arrow[r, "\exp"] \arrow[d, "\psi^{\alpha_1,\beta_1}_{\alpha_2,\beta_2}"] & (\mathbb{C}^\times)^2 \arrow[d, "\tilde{\psi}^{\alpha_1,\beta_1}_{\alpha_2,\beta_2}"] \\
\mathbb{C}^2 \arrow[r, "\varphi_{\alpha_2,\beta_2}"] & \mathbb{C}^2 \arrow[r, "\exp"] & (\mathbb{C}^\times)^2
\end{tikzcd}$$

are given by

$$\begin{aligned}
\psi^{\alpha_1,\beta_1}_{\alpha_2,\beta_2} : \mathbb{C}^2 \to \mathbb{C}^2, &\quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \\
\tilde{\psi}^{\alpha_1,\beta_1}_{\alpha_2,\beta_2} : (\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^2, &\quad \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u^a v^b \\ u^c v^d \end{pmatrix}.
\end{aligned}$$

Note that in dependence of the coefficient matrix A the map $\tilde{\psi}^{\alpha_1,\beta_1}_{\alpha_2,\beta_2}$ can be holomorphically extended to $u = 0$ or to $v = 0$. We call this extension $\hat{\psi}^{\alpha_1,\beta_1}_{\alpha_2,\beta_2}$. It is defined for $u = 0$ if $a, c \geq 0$ and for $v = 0$ if $b, d \geq 0$.

Now let $(\alpha_i, \beta_i)_{i\in I}$ be a family of bases of $\mathfrak{b}$. We build a complex manifold by starting with $|I|$ copies of $\mathbb{C}^2$, namely

$$Y_I := \dot{\bigcup_{i\in I}} (\mathbb{C}^2)_i.$$

Let us introduce the equivalence relation

$$(u_i, v_i) \sim_I (u_j, v_j) \quad :\Leftrightarrow \quad \hat{\psi}^{\alpha_i,\beta_i}_{\alpha_j,\beta_j}(u_i, v_i) = (u_j, v_j).$$

The quotient $\tilde{Y}_I := Y_I / \sim_I$ is a 2 dimensional complex manifold.

2.7.4 Convex hull

We apply this procedure now to a special family of bases of $\mathfrak{b}$. Namely, consider the embedding

$$\mathfrak{b}^+ \to \mathbb{R}^2, \quad x \mapsto (x, x').$$

By $\mathfrak{b}^+$ we refer to the totally positive elements of $\mathfrak{b}$. Now take the convex hull in $\mathbb{R}^2$ of the image of $\mathfrak{b}^+$ and denote by $(A_k)_{k\in\mathbb{Z}}$ the elements of $\mathfrak{b}^+$ which are mapped to the

boundary of the convex hull. We can arrange them in such a way that for $k < k'$ we have $A_k > A_{k'}$ and hence $A'_k < A'_{k'}$.

It is easy to see that for all $k \in \mathbb{Z}$ the pair (A_k, A_{k+1}) is a basis of $\mathfrak{b}$. Further, for all $k \in \mathbb{Z}$ there exists a $b_k \in \mathbb{N}$ with $A_{k-1} + A_{k+1} = b_k A_k$ and $b_k \geq 2$.

Now we apply the above procedure to the so constructed family of bases. Hence, our index set is $\mathbb{Z}$ and the basis correspoding to $(\mathbb{C}^2)_k$ shall be (A_{k-1}, A_k). We have

$$\begin{pmatrix} A_{k-1} & A_k \end{pmatrix} = \begin{pmatrix} A_k & A_{k+1} \end{pmatrix} \begin{pmatrix} b_k & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence, the base change yields

$$(\mathbb{C}^2)_k \ni (u_k, v_k) \sim_{\mathbb{Z}} (u_k^{b_k} v_k, 1/u_k) = (u_{k+1}, v_{k+1}) \in (\mathbb{C}^2)_{k+1}$$

in $\tilde{Y}_{\mathbb{Z}}$. We define the curve S_k to be

$$S_k := \{v_k = 0\} \cup \{u_{k+1} = 0\} \cong \mathbb{P}^1(\mathbb{C}).$$

Let $p_k := (0,0)_k$ be the origin. Then we have $S_k \cap S_{k-1} = \{p_k\}$. If $|k - l| > 1$, then the curves S_k and S_l do not intersect. Since S_k and S_{k-1} intersect transversally in one point we have $S_k \cdot S_{k-1} = 1$. The self-intersection number S_k^2 can be computed to be $-b_k$.

2.7.5 Dividing by the units

So far we have desingularized the cusp ∞ in the quotient $\mathbb{H}^2/\mathfrak{b}$. However, our goal is to desingularize it in $\Gamma_{\mathfrak{a},\infty} \backslash U_C$ with $\overline{\Gamma_{\mathfrak{a},\infty}} \cong \mathfrak{b} \rtimes (\mathcal{O}_K^\times)^2$. Hence, we still have to factor out $(\mathcal{O}_K^\times)^2 = \langle \varepsilon_0^2 \rangle$. First, let

$$Y_C := \Phi(U_C/\mathfrak{b}) \cup \bigcup_{k \in \mathbb{Z}} S_k$$

with $\Phi : \mathbb{C}^2/\mathfrak{b} \to \tilde{Y}_{\mathbb{Z}}$ be the natural embedding defined by $\tilde{\varphi}_{A_{k-1}, A_k}$

$$\mathbb{C}^2/\mathfrak{b} \to (\mathbb{C}^\times \times \mathbb{C}^\times)_k \hookrightarrow Y_{\mathbb{Z}}$$

for any $k \in \mathbb{Z}$. Now $(\mathcal{O}_K^\times)^2$ acts on $U_C/\mathfrak{b}$. On the other hand, $(\mathcal{O}_K^\times)^2$ acts as well on

$$A := \{A_k : \ k \in \mathbb{Z}\}$$

by multiplication. Since a multiplication with a totally positive unit (for instance ε_0^2) also preserves the order of A, there exists an $r \in \mathbb{N}$ such that

$$A_k = \varepsilon_0^2 A_{k+r} \quad \text{and} \quad b_k = b_{k+r}$$

for all $k \in \mathbb{Z}$. This motivates to define an action of $(\mathcal{O}_K^\times)^2$ on $\tilde{Y}_{\mathbb{Z}}$ by

$$\varepsilon_0^{2n}.(u,v)_k := (u,v)_{k-nr}.$$

On the other hand, from

$$\begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}^{-1} = \begin{pmatrix} \varepsilon_0^2\alpha & \varepsilon_0^2\beta \\ \varepsilon_0^{-2}\alpha' & \varepsilon_0^{-2}\beta' \end{pmatrix}^{-1} \begin{pmatrix} \varepsilon_0^2 & 0 \\ 0 & \varepsilon_0^{-2} \end{pmatrix}$$

it follows $\varphi_{\alpha,\beta}(z) = \varphi_{\varepsilon_0^2\alpha,\varepsilon_0^2\beta}(\varepsilon_0^2 z)$ (cf. definition (2.26)). This shows that the action of $(\mathcal{O}_K^\times)^2$ on $\tilde{Y}_\mathbb{Z}$ resembles the action of $(\mathcal{O}_K^\times)^2$ on $U_C/\mathfrak{b}$ under the embedding Φ. Hence, the quotient $Y_C/(\mathcal{O}_K^\times)^2$ is well-defined and this is what we finally call the *desingularization of* ∞. Therefore, the curves $S_1, \ldots, S_r$ correspond to the original cusp ∞. Taking their sum

$$E^\infty(\mathfrak{a}) := \sum_{k=1}^{r} S_k \tag{2.29}$$

we obtain the so-called *exceptional divisor* at the cusp ∞. The index k in $S_k \subset Y_C/(\mathcal{O}_K^\times)^2$ has to be read as an element of $\mathbb{Z}/r\mathbb{Z}$. The intersection behaviour of S_k and S_{k-1} is the same for $r > 2$ as in $\tilde{Y}_\mathbb{Z}$. For $r = 2$ we have only the two curves S_1 and S_2 corresponding to the cusp ∞. Hence, they touch each other twice, i.e., $S_1 \cdot S_2 = 2$. But still for $r \geq 2$ we have $S_k^2 = -b_k$ as in $\tilde{Y}_\mathbb{Z}$. The curve S_1 is singular if and only if $r = 1$. In this case we have $S_1^2 = 2 - b_1$.

2.7.6 Summary

In the previous subsection we have constructed the desingularization of ∞ for $X(\mathfrak{a})^*$. The cusp is replaced by the exceptional divisor $E^\infty(\mathfrak{a})$ consisting of finitely many $S_k \cong \mathbb{P}^1(\mathbb{C})$ glued together like a pearl necklet. As explained in the introduction of this section, this can be done for every other cusp κ as well by finding an isomorphism between $X(\mathfrak{a})^*$ and a Hilbert modular surface corresponding to a different ideal bringing the cusp $\kappa \in X(\mathfrak{a})^*$ to its cusp ∞. By this procedure we obtain an exceptional divisor $E^\kappa(\mathfrak{a})$ for each cusp $\kappa \in \mathbb{P}^1(K)$. The sum of all exceptional divisors at all cusps

$$E(\mathfrak{a}) := \sum_{\kappa \in \Gamma_\mathfrak{a} \backslash \mathbb{P}^1(K)} E^\kappa(\mathfrak{a})$$

is again called *exceptional divisor*. The resulting object $X(\mathfrak{a})$ together with the exceptional divisor $E(\mathfrak{a})$ is called the *Hirzebruch compactification* of $X(\mathfrak{a})$, denoted by $\overline{X(\mathfrak{a})}$, a compact complex space whose singularities are given by the elliptic fixed points, which are the singularities of $X(\mathfrak{a})$. Being finite quotient singularities, they are mild. Therefore, $X(\mathfrak{a})$ and $\overline{X(\mathfrak{a})}$ are complex orbifolds. The following commutative diagram relates the three complex spaces $X(\mathfrak{a})$, $X(\mathfrak{a})^*$ and $\overline{X(\mathfrak{a})}$.

$$\begin{tikzcd} X(\mathfrak{a}) \arrow[r, hook] \arrow[dr, hook] & \overline{X(\mathfrak{a})} \arrow[d, "\pi"] \\ & X(\mathfrak{a})^* \end{tikzcd}$$

When we talk about the *open Hilbert modular surface* we refer to $X(\mathfrak{a})$ because it is open in both compactifications. Its closure is $X(\mathfrak{a})^*$ or $\overline{X(\mathfrak{a})}$ depending on which space one takes the closure in.

2.7.7 Three coordinate systems for the imaginary part

For a given $z \in \mathbb{H}^2$ the imaginary part $y = (y_1, y_2)$ has two further useful coordinate systems which we want to introduce in this subsection. The first one is quite simple and uses the diffeomorphism

$$\mathbb{R}_+^2 \to \mathbb{R}_+^2, \quad (y_1, y_2) \mapsto (t, r) = \left(\sqrt{y_1 y_2}, \sqrt{y_1/y_2}\right).$$

Its inverse is given by

$$(t, r) \mapsto (tr, t/r) = (y_1, y_2).$$

We denote by t and r without further mentioning those other coordinates of y. We have already seen that some definitions depend only on t (and not on r). For example $\Im(z) = t^2$ and

$$U_C = \left\{ z \in \mathbb{H}^2 : \ t^2 > C \right\}.$$

In this thesis we face definitions which only depend on r as well. That justifies to switch from time to time to the coordinates (t, r). The action of $\overline{\Gamma_{\mathfrak{a},\infty}} \cong \mathfrak{a}^{-1} \rtimes (\mathcal{O}_K^\times)^2$ (cf. equation (2.19)) on $\mathbb{H}^2$ induces the action

$$(\mu, \varepsilon^2).(t, r) = (t, \varepsilon^2 r)$$

on the coordinates t and r.

The other useful coordinate system for $y = (y_1, y_2)$ which we want to explain here is the expression of y using the local coordinates (u, v) of the cusp ∞ with respect to a basis (α, β) described in the antecedent subsections. Interestingly, we only need the absolute values $|u|$ and $|v|$ of u and v to express y. Namely, equation (2.28) directly implies

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= -\frac{1}{2\pi} \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} \begin{pmatrix} \log|u| \\ \log|v| \end{pmatrix} \\ &= -\frac{1}{2\pi} \begin{pmatrix} \alpha \log(|u|) + \beta \log(|v|) \\ \alpha' \log(|u|) + \beta' \log(|v|) \end{pmatrix} = -\frac{1}{2\pi} \begin{pmatrix} \log(|u|^\alpha |v|^\beta) \\ \log(|u|^{\alpha'} |v|^{\beta'}) \end{pmatrix}. \end{aligned} \tag{2.30}$$

On the other hand, it is possible to express $|u|$ and $|v|$ by y:

$$\begin{pmatrix} |u| \\ |v| \end{pmatrix} = \exp\left(-2\pi \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right).$$

Hence, we have (cf. mapping (2.27))

$$|u| = \exp\left(\frac{2\pi(\beta y_2 - \beta' y_1)}{\alpha\beta' - \alpha'\beta} \right) \quad \text{and} \quad |v| = \exp\left(\frac{2\pi(\alpha' y_1 - \alpha y_2)}{\alpha\beta' - \alpha'\beta} \right).$$

Taking the derivatives with respect to u and v of equation (2.30), we obtain

$$\frac{\partial y_1}{\partial u} = -\frac{\alpha}{4\pi u}, \quad \frac{\partial y_1}{\partial v} = -\frac{\beta}{4\pi v}, \quad \frac{\partial y_2}{\partial u} = -\frac{\alpha'}{4\pi u} \quad \text{and} \quad \frac{\partial y_2}{\partial v} = -\frac{\beta'}{4\pi v}. \tag{2.31}$$

Using equation (2.30), we can express t and r in terms of u and v by

$$t = \frac{\sqrt{(\alpha \log(|u|) + \beta \log(|v|))(\alpha' \log(|u|) + \beta' \log(|v|))}}{2\pi}$$

and

$$r = \sqrt{\frac{\alpha \log(|u|) + \beta \log(|v|)}{\alpha' \log(|u|) + \beta' \log(|v|)}}.$$

It follows

$$\lim_{u \to 0} \frac{t}{\log(|u|)} = -\frac{\sqrt{N(\alpha)}}{2\pi} \quad \text{and} \quad \lim_{u \to 0} r = \sqrt{\alpha/\alpha'}. \tag{2.32}$$

In particular, the limit process $u \to 0$ translates in (t, r) coordinates into $t \to \infty$ and $r \to \sqrt{\alpha/\alpha'}$. Taking the derivative of t and r with respect to u, we obtain

$$\frac{\partial t}{\partial u} = \frac{\alpha/r + \alpha' r}{8\pi u} \quad \text{and} \quad \frac{\partial r}{\partial u} = \frac{\alpha - \alpha' r^2}{8\pi u t}. \tag{2.33}$$

2.8 Hirzebruch–Zagier divisors

In this section we introduce a special family of algebraic divisors on $X(\mathfrak{a})$ (on $X(\mathfrak{a})^*$ and $\overline{X(\mathfrak{a})}$ accordingly), the so-called *Hirzebruch–Zagier divisors*. They can be interpreted as Heegner divisors.

2.8.1 Hirzebruch–Zagier divisors on $X(\mathfrak{a})$

There are different approaches how to introduce the Hirzebruch–Zagier divisors. One of them is to introduce them as the zeros of the component $h(A, z)$ of the decomposition of the quadratic form

$$\det(A) = q_{\tilde{W}_z}(A) - h(A, z) \tag{2.34}$$

on V (cf. Section 2.6). For non-zero $A = \left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in V$ we define

$$T_A := \left\{ z \in \mathbb{H}^2 : \ h(A, z) = 0 \right\}. \tag{2.35}$$

There are equivalent expressions for T_A following directly from the definition of $h(A, z)$, for instance

$$\begin{aligned} T_A &= \left\{ z \in \mathbb{H}^2 : \ b z_1 z_2 - \lambda z_1 - \lambda' z_2 + a = 0 \right\} \\ &= \left\{ z \in \mathbb{H}^2 : \ A \in \tilde{W}_z \right\} = \left\{ z \in \mathbb{H}^2 : \ A \in W_z^{\perp} \right\}. \end{aligned}$$

The above decomposition (2.34) of $\det(A)$ shows that T_A is empty for $A \in V^-$. If $A \in \mathrm{Iso}(V)$ with $b \neq 0$, the polynomial

$$bz_1z_2 - \lambda z_1 - \lambda' z_2 + a = b\left(z_1 - \frac{\lambda'}{b}\right)\left(z_2 - \frac{\lambda}{b}\right)$$

decomposes into linear factors. The roots are real, therefore T_A is empty as well. Considered over $\mathbb{C}^2$ however, the curve T_A would be reducible in this case. If $A \in \mathrm{Iso}(V)$ with $b = 0$, the divisor T_A is empty even over $\mathbb{C}^2$. For $A \in V^+$ the curve T_A is non-empty and irreducible. Therefore, we are mainly interested in that case. If $A \in V^+$, there is a further way to express T_A, namely

$$T_A = \{(ASz, z) : \ z \in \mathbb{H}\}. \tag{2.36}$$

This follows directly from Lemma 2.6.6. It is easy to see that we have $T_A = T_B$ for $A, B \in V^+$ if and only if A and B are linearly dependent.

Lemma 2.8.1. *Let* $A = \left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in V^+$. *Then we have*

$$\sup\{\Im(z) : \ z \in T_A\} = \begin{cases} \det(A)/b^2, & b \neq 0, \\ \infty, & b = 0. \end{cases}$$

Proof. We use the description of T_A in equation (2.36). With equation (2.25) we obtain

$$\Im((ASz_2, z_2)) = \frac{y_2^2 \det(AS)}{|bz_2 - \lambda|^2}.$$

For $b = 0$ the statement is clear now. For $b \neq 0$ we continue

$$\Im((ASz_2, z_2)) = \frac{y_2^2 \det(A)}{b^2((x_2 - \lambda/b)^2 + y_2^2)} \leq \frac{\det(A)}{b^2}.$$

Equality holds for $x_2 = \lambda/b$. □

Lemma 2.8.1 shows that T_A runs into the cusp ∞ if and only if $b = 0$.

Lemma 2.8.2. *Let* $0 \neq A \in V$ *and* $M \in \mathrm{GL}_2^+(K)$. *Then we have* $MT_A = T_{M.A}$.

Proof. We have by definition and Proposition 2.6.1

$$\begin{aligned} T_{M.A} &= \{z : \ h(M.A, z) = 0\} = \{Mz : \ h(M.A, Mz) = 0\} \\ &= \{Mz : \ N(\det(M))h(A, z) = 0\} = MT_A. \end{aligned}$$

□

Now we define for $\mathfrak{a} \in \mathcal{I}_K$ and $m \in \mathbb{Z}$

$$T_*(\mathfrak{a}, m) := \sum_{\substack{A \in L(\mathfrak{a})^\vee/\{\pm 1\} \\ \det(A) = m/(N(\mathfrak{a})D)}}' T_A \tag{2.37}$$

and $T(\mathfrak{a}, m) := T_*(\mathfrak{a}, m)$ if $m \neq 0$. The tick at the sum indicates that for $m = 0$ we do not take $A = 0$ since T_0 is undefined. We divide out $\{\pm 1\}$ in order to get every irreducible component with multiplicity 1. The scaling $m/(N(\mathfrak{a})D)$ of the quadratic form is adjusted in such a way that we run through all elements of $L(\mathfrak{a})^\vee$ while running with m through $\mathbb{Z}$ (cf. equation (2.12)).

Note that by this definition $T_*(\mathfrak{a}, m) = 0$ for all $m \leq 0$. For some $m \in \mathbb{N}$ we might have $T_*(\mathfrak{a}, m) = 0$ as well, namely if and only if $L(\mathfrak{a})^\vee$ has no elements of the desired determinant.

We do not define $T(\mathfrak{a}, 0)$ at this place since this definition is a little bit more involved. For the definition of $T(\mathfrak{a}, 0)$ we have to make good for the missing matrix $A = 0$. We do this at the end of the current chapter in Definition 2.9.21.

Proposition 2.8.3. *Let $m \in \mathbb{N}$, $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_K$ and $M \in M(\mathfrak{a}, \mathfrak{b})$. Then we have*

$$MT(\mathfrak{a}^2\mathfrak{b}, m) = T(\mathfrak{b}, m).$$

In particular, $T(\mathfrak{a}, m)$ is invariant under $\Gamma_\mathfrak{a}$.

Proof. For $0 \neq A \in V$ we have by Lemma 2.8.2

$$M.T_A = T_{M.A} = T_{N(\mathfrak{a})M.A}.$$

By Proposition 2.4.2 we have that

$$N(\mathfrak{a})(M.L(\mathfrak{a}^2\mathfrak{b})^\vee) = L(\mathfrak{b})^\vee$$

and for $A \in L(\mathfrak{a}^2\mathfrak{b})^\vee$ with $\det(A) = m/(N(\mathfrak{a}^2\mathfrak{b})D)$ the matrix $M.A$ has the same determinant. Hence, $N(\mathfrak{a})(M.A)$ has determinant $m/(N(\mathfrak{b})D)$ which finishes the proof. □

The $\Gamma_\mathfrak{a}$ invariance of $T(\mathfrak{a}, m)$ justifies to see $T(\mathfrak{a}, m)$ as divisor on $X(\mathfrak{a})$. It is called *Hirzebruch–Zagier divisor of discrimiant m on $X(\mathfrak{a})$*. Regarding the notation, we make no distinction between $T(\mathfrak{a}, m)$ on $\mathbb{H}^2$ and the Hirzebruch–Zagier divisor on $X(\mathfrak{a})$. By

$$T(\mathfrak{a}, m) = \sum_{\substack{A \in \Gamma_\mathfrak{a} \backslash L(\mathfrak{a})^\vee/\{\pm 1\} \\ \det(A) = m/(N(\mathfrak{a})D)}} T_A \tag{2.38}$$

we can define $T(\mathfrak{a}, m)$ directly as finite sum of its irreducible components on the Hilbert modular surface $X(\mathfrak{a})$ if we view T_A as its image under the projection.

Taking the closure of $T(\mathfrak{a}, m)$ in $X(\mathfrak{a})^*$ allows us to define Hirzebruch–Zagier divisors on $X(\mathfrak{a})^*$. We denote them by $T(\mathfrak{a}, m)$ as well. In order to describe the Hirzebruch–Zagier divisors of $\overline{X(\mathfrak{a})}$, however, we need to work a bit more subtle. This is carried out in Subsection 2.8.5.

2.8.2 The volume

The aim of this subsection is to express the volume of the Hirzebruch–Zagier divisors $T(\mathfrak{a}, m)$ by the volume of certain discrete quotients of $\mathbb{H}$. By equation (2.38) we see that the volume of $T(\mathfrak{a}, m)$ is given by the sum of the volumes of its finitely many components

$$\operatorname{vol}(T(\mathfrak{a}, m)) = \sum_{\substack{A \in \Gamma_{\mathfrak{a}} \backslash L(\mathfrak{a})^\vee / \{\pm 1\} \\ \det(A) = m/(N(\mathfrak{a})D)}} \operatorname{vol}(T_A). \tag{2.39}$$

Fix a representative $A \in L(\mathfrak{a})^\vee$ and consider the map

$$\varphi_A : \mathbb{H} \to \mathbb{H}^2, \quad z \mapsto (ASz, z).$$

The image of φ_A is T_A (understood as subset of $\mathbb{H}^2$) by equation (2.36). Here, we are interested in a fundamental domain $\mathcal{F}_A \subset \mathbb{H}$ of the equivalence relation

$$z \sim \tilde{z} \quad :\Leftrightarrow \quad \pi(\varphi_A(z)) = \pi(\varphi_A(\tilde{z})) \quad \Leftrightarrow \quad \exists M \in \Gamma_{\mathfrak{a}} : \ M\varphi_A(z) = \varphi_A(\tilde{z})$$

on $\mathbb{H}$ where $\pi : \mathbb{H}^2 \to X(\mathfrak{a})$ is the canonical projection. Then

$$\operatorname{vol}(T_A) = \operatorname{vol}(\mathcal{F}_A) = \int_{\mathcal{F}_A} \eta \quad \text{with} \quad \eta := \frac{1}{4\pi} \frac{dxdy}{y^2}.$$

The last equivalence can be unfolded using the next lemma.

Lemma 2.8.4. *Let $M \in \mathrm{GL}_2^+(K)$ and $A \in V^+$. Then we have*

$$M\varphi_A(z) = \varphi_{M.A}(M'z).$$

Proof. We have to show

$$\begin{aligned} (MASz, M'z) = ((M.A)SM'z, M'z) \quad &\Leftrightarrow \quad MASz = MA(M')^\top SM'z \\ &\Leftrightarrow \quad Sz = (M')^\top SM'z \\ &\Leftrightarrow \quad z = S(M')^\top SM'z. \end{aligned}$$

Computing the matrix product shows that for arbitrary 2×2 matrices M over any commutative ring we have

$$SM^\top SM = -\det(M)E_2$$

which proves the statement. □

Hence, we have

$$z \sim \tilde{z} \quad \Leftrightarrow \quad \exists M \in \Gamma_{\mathfrak{a}} : \ \varphi_A(\tilde{z}) = \varphi_{M.A}(M'z).$$

Regarding the second component, this implies $\tilde{z} = M'z$. Now looking at the first component, we obtain $AS\tilde{z} = (M.A)S\tilde{z}$. Neglecting those $\tilde{z}$ which are elliptic fixed points with respect to $\Gamma_{\mathfrak{a}}S$, this implies $M.A = A$ or $M.A = -A$. We can neglect the fixed

points because there are only countably many due to $\Gamma_{\mathfrak{a}}$ being countable. Hence, they are of measure zero. In total it turns out that

$$\Gamma'_{\mathfrak{a},\pm A} := \{M' : \ M \in \Gamma_{\mathfrak{a}} \quad \text{and} \quad M.A \in \{\pm A\}\}$$

is the subgroup of $\Gamma_{\mathfrak{a}}$ sending $z \in \mathbb{H}$ to equivalent $\tilde{z} \in \mathbb{H}$. Therefore, we have

$$\mathrm{vol}(T_A) = \mathrm{vol}(\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}) = \int_{\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}} \eta \tag{2.40}$$

and

$$\mathrm{vol}(T(\mathfrak{a}, m)) = \sum_{\substack{A\in\Gamma_{\mathfrak{a}}\backslash L(\mathfrak{a})^\vee/\{\pm 1\} \\ \det(A)=m/(N(\mathfrak{a})D)}} \mathrm{vol}(\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}).$$

To actually compute those volumes, one could explicitly compute fundamental domains $\mathcal{F}_A$ for $\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}$ in sepcial cases. For our purposes, however, it is enough to obtain the proven identity.

Remark 2.8.5. For $m \in -\mathbb{N}$ we have $T(\mathfrak{a}, m) = 0$ by definition and thus $\mathrm{vol}(T(\mathfrak{a}, m)) = 0$. However,

$$\sum_{\substack{A\in\Gamma_{\mathfrak{a}}\backslash L(\mathfrak{a})^\vee/\{\pm 1\} \\ \det(A)=m/(N(\mathfrak{a})D)}} \mathrm{vol}(\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H})$$

is still meaningful and in general it is different from 0. In case $N(\varepsilon_0) = -1$ there is an easy bijection between

$$\{A \in L(\mathfrak{a})^\vee : \ \det(A) = m/(N(\mathfrak{a})D)\}$$

and

$$\{A \in L(\mathfrak{a})^\vee : \ \det(A) = -m/(N(\mathfrak{a})D)\}$$

given by

$$A \mapsto E_0.A \quad \text{with} \quad E_0 := \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.41}$$

We have

$$\Gamma_{\mathfrak{a},\pm E_0.A} = E_0\Gamma_{\mathfrak{a},\pm A}E_0^{-1}.$$

Because of $E_0 \in \Gamma_{\mathfrak{a}}$, two elements $A, B \in L(\mathfrak{a})^\vee$ are in the same $\Gamma_{\mathfrak{a}}$ orbit if and only if $E_0.A$ and $E_0.B$ are. We conclude

$$\sum_{\substack{A\in\Gamma_{\mathfrak{a}}\backslash L(\mathfrak{a})^\vee/\{\pm 1\} \\ \det(A)=m/(N(\mathfrak{a})D)}} \mathrm{vol}(\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}) = \mathrm{vol}(T(\mathfrak{a}, |m|)).$$

2.8.3 Hirzebruch–Zagier divisors at infinity

In this subsection we have a closer look at the Hirzebruch–Zagier divisors near the cusp ∞. This leads in Subsection 2.8.5 to the definition of $Z(\mathfrak{a}, m)$, the Hirzebruch–Zagier divisor on $\overline{X(\mathfrak{a})}$. Since in $\overline{X(\mathfrak{a})}$ the cusp ∞ is replaced by the exceptional divisor $E^\infty(\mathfrak{a})$, we have to clarify if the components of $E^\infty(\mathfrak{a})$ contribute to the Hirzebruch–Zagier divisor $Z(\mathfrak{a}, m)$ and if yes, with which multiplicities. Recall from the end of Subsection 2.8.1 that we have defined the Hirzebruch–Zagier divisor of discriminant m in the Baily–Borel compactification $X(\mathfrak{a})^*$ simply to be the closure of $T(\mathfrak{a}, m)$. Because of this simple definition we do not distinguish in the notation, i.e., $T(\mathfrak{a}, m)$ denotes both the Hirzebruch–Zagier divisor of discriminant m on $X(\mathfrak{a})$ and on $X(\mathfrak{a})^*$. On $\overline{X(\mathfrak{a})}$, however, we give the Hirzebruch–Zagier divisor of discriminant m the new name $Z(\mathfrak{a}, m)$ because it contains new components.

We define

$$\Lambda(\mathfrak{a}, m) := \left\{ \lambda \in \mathfrak{a}\mathfrak{d}^{-1} : \ N(\lambda) = -\frac{mN(\mathfrak{a})}{D} \right\}, \tag{2.42}$$

$$\Lambda^+(\mathfrak{a}, m) := \{\lambda \in \Lambda(\mathfrak{a}, m) : \ \lambda > 0\}, \tag{2.43}$$

$$\Lambda^-(\mathfrak{a}, m) := \{\lambda \in \Lambda(\mathfrak{a}, m) : \ \lambda < 0\} = -\Lambda^+(\mathfrak{a}, m).$$

This definition makes sense for $m \in \mathbb{Z}$ even though in this subsection we are only interested in the case $m \in \mathbb{N}$ because $T_*(\mathfrak{a}, m) = 0$ otherwise. In Chapter 4, however, when we talk about Kudla's Green functions, we use these definitions for $m \in \mathbb{Z}$.

In order to undestand $T(\mathfrak{a}, m)$ near the cusp ∞, only the components running into the cusp ∞ are of concern. We have already concluded from Lemma 2.8.1 that T_A with $A = \left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in V^+$ runs into the cusp ∞ if and only if $b = 0$. That leads to the definition

$$T^\infty(\mathfrak{a}, m) := \sum_{\substack{A = \left(\begin{smallmatrix} a & \lambda' \\ \lambda & 0 \end{smallmatrix}\right) \in L(\mathfrak{a})^\vee / \{\pm 1\} \\ \det(A) = m/(N(\mathfrak{a})D)}} T_A = \sum_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \sum_{a \in \mathbb{Z}} \left\{ z \in \mathbb{H}^2 : \ \operatorname{tr}(\lambda z) = a \right\}. \tag{2.44}$$

We conclude that $T(\mathfrak{a}, m)$ considered on $X(\mathfrak{a})^*$ contains the cusp ∞ if and only if $\Lambda^+(\mathfrak{a}, m) \neq \emptyset$ (for other cusps we have to consider the sets $\Lambda^+(\mathfrak{a}\mathfrak{b}^2, m)$ with $\mathfrak{b} \in \mathcal{I}_K$ chosen accordingly). Only in that case $Z(\mathfrak{a}, m)$ contains components of $E^\infty(\mathfrak{a})$. Therefore, we assume $\Lambda^+(\mathfrak{a}, m) \neq \emptyset$ and $m \in \mathbb{N}$ for the rest of the section.

Neglecting the real part in the definition of $T^\infty(\mathfrak{a}, m)$, we make the further definition

$$S(\mathfrak{a}, m) := \bigcup_{\lambda \in \Lambda^+(\mathfrak{a}, m)} S_\lambda \quad \text{with} \quad S_\lambda := \left\{ z \in \mathbb{H}^2 : \ \operatorname{tr}(\lambda y) = 0 \right\}. \tag{2.45}$$

Note that the connected components of $S(\mathfrak{a}, m)$ are given by $(S_\lambda)_{\lambda \in \Lambda^+(\mathfrak{a}, m)}$. An equivalent definition of S_λ is given by

$$S_\lambda := \left\{ z \in \mathbb{H}^2 : \ \operatorname{tr}(\lambda z) \in \mathbb{R} \right\}.$$

Hence, S_λ summarizes all components of $T^\infty(\mathfrak{a}, m)$ belonging to λ and interpolates between them by allowing $\operatorname{tr}(\lambda z)$ to be real instead of integral. The components of

$T^\infty(\mathfrak{a}, m)$ are 2 dimensional over $\mathbb{R}$, whereas the components of $S(\mathfrak{a}, m)$ are 3 dimensional over $\mathbb{R}$.

The group $(\mathcal{O}_K^\times)^2$ acts on $\Lambda^+(\mathfrak{a}, m)$ by multiplication (or division). This action can be recovered in some sense by the action of $\Gamma_\mathfrak{a}$ on $T(\mathfrak{a}, m)$ as we are going to explain now: The action of $\Gamma_\mathfrak{a}$ on $T(\mathfrak{a}, m)$ induces an action of $\Gamma_{\mathfrak{a},\infty}$ on $T^\infty(\mathfrak{a}, m)$. As we have seen, this action factors through $\overline{\Gamma_{\mathfrak{a},\infty}}$ and can be decomposed using $\overline{\Gamma_{\mathfrak{a},\infty}} \cong \mathfrak{a}^{-1} \rtimes (\mathcal{O}_K^\times)^2$ into an action of $\mathfrak{a}^{-1}$ and an action of $(\mathcal{O}_K^\times)^2$. Looking at the definition of the action of $\mathfrak{a}^{-1} \rtimes (\mathcal{O}_K^\times)^2$ (cf. equation (2.19)), it is clear that $\overline{\Gamma_{\mathfrak{a},\infty}}$ acts on $S(\mathfrak{a}, m)$ as well. Since for fixed $\gamma \in \overline{\Gamma_{\mathfrak{a},\infty}}$ the map $z \mapsto \gamma z$ is continuous, an action of $\overline{\Gamma_{\mathfrak{a},\infty}}$ on the connected components of $S(\mathfrak{a}, m)$ is induced. Each component S_λ is stabilized by $\mathfrak{a}^{-1}$. Hence, the action of $\overline{\Gamma_{\mathfrak{a},\infty}}$ on the connected components of $S(\mathfrak{a}, m)$ factors through $(\mathcal{O}_K^\times)^2$. The action of $(\mathcal{O}_K^\times)^2$ on $\Lambda^+(\mathfrak{a}, m)$ is now recovered by

$$\begin{aligned}\varepsilon^2 S_\lambda &= \left\{\varepsilon^2 z : \ \mathrm{tr}(\lambda y) = 0\right\} = \left\{z : \ \mathrm{tr}(\lambda(\varepsilon^{-2} y)) = 0\right\} \\ &= \left\{z : \ \mathrm{tr}(\varepsilon^{-2}\lambda y) = 0\right\} = S_{\varepsilon^{-2}\lambda}\end{aligned}$$

for all $\varepsilon^2 \in (\mathcal{O}_K^\times)^2$ and $\lambda \in \Lambda^+(\mathfrak{a}, m)$. The number of $(\mathcal{O}_K^\times)^2$ orbits of $\Lambda^+(\mathfrak{a}, m)$ is finite. For later use we need to understand the growth behavior of the number of orbits for large m. The next lemma provides an estimate.

Lemma 2.8.6. *Let $\alpha > 0$ and $\mathfrak{a} \in \mathcal{I}_K$ be fixed. Then we have*

$$\left|\Lambda(\mathfrak{a}, m)/(\mathcal{O}_K^\times)^2\right| = O(m^\alpha).$$

Proof. Let $n_\mathfrak{a} \in \mathbb{N}$ with $n_\mathfrak{a}\mathfrak{a} \subset \mathcal{O}_K$. Then we can define a map

$$\Lambda(\mathfrak{a}, m)/(\mathcal{O}_K^\times)^2 \to \left\{\mathfrak{b} \subset \mathcal{O}_K : \ N(\mathfrak{b}) = m n_\mathfrak{a}^2 N(\mathfrak{a})\right\}$$

by

$$\lambda(\mathcal{O}_K^\times)^2 \mapsto (n_\mathfrak{a}\lambda\sqrt{D}).$$

This map is four-to-one because of $[\mathcal{O}_K^\times : (\mathcal{O}_K^\times)^2] = 4$. The number of integral ideals of a given norm $k \in \mathbb{N}$ is bounded by $d(k)$. Therefore, we have

$$\left|\Lambda(\mathfrak{a}, m)/(\mathcal{O}_K^\times)^2\right| \leq 4d(m n_\mathfrak{a}^2 N(\mathfrak{a})).$$

The claim follows now with $d(k) = O(k^\alpha)$. □

Using the coordinates t and r for y introduced in Subsection 2.7.7, there is another natural description of the components S_λ. With the convention

$$R_\mathfrak{a}^m := \sqrt{\frac{mN(\mathfrak{a})}{D}} \tag{2.46}$$

we have

$$S_\lambda = \left\{z \in \mathbb{H}^2 : \ r = \frac{R_\mathfrak{a}^m}{\lambda}\right\} \quad \text{or equivalently} \quad S_\lambda = \left\{z \in \mathbb{H}^2 : \ r = \frac{-\lambda'}{R_\mathfrak{a}^m}\right\}$$

for all $\lambda \in \Lambda^+(\mathfrak{a}, m)$.

2.8.4 Weyl chambers and vectors

The set $\Lambda^+(\mathfrak{a}, m)$ considered as subset of $\mathbb{R}^+$ is totally ordered and discrete. Hence, it is isomorphic to $\mathbb{Z}$ (as total order). For fixed $w \in (\mathbb{R}^+)^2$ the map

$$\Lambda^+(\mathfrak{a}, m) \to \mathbb{R}, \quad \lambda \mapsto \operatorname{tr}(\lambda w)$$

is strictly increasing. Its image is discrete in $\mathbb{R}$ and neither bounded from above nor from below.

We call $\lambda \in \Lambda^+(\mathfrak{a}, m)$ *reduced* with respect to w if λ is minimal with $\operatorname{tr}(\lambda w) \geq 0$ in its $(\mathcal{O}_K^\times)^2$ orbit. The set of all reduced $\lambda \in \Lambda^+(\mathfrak{a}, m)$ with respect to w is denoted by $R(\mathfrak{a}, m, w)$. The set $R(\mathfrak{a}, m, w)$ is finite and it is a set of representatives of $\Lambda^+(\mathfrak{a}, m)/(\mathcal{O}_K^\times)^2$. An easy way to check if a $\lambda \in \Lambda^+(\mathfrak{a}, m)$ is reduced with respect to w is

$$\lambda \in R(\mathfrak{a}, m, w) \quad \Leftrightarrow \quad \operatorname{tr}(\lambda w) \geq 0 \quad \text{and} \quad \operatorname{tr}(\varepsilon_0^{-2} \lambda w) < 0.$$

We define

$$\rho(\mathfrak{a}, m, w) := \sum_{\lambda \in R(\mathfrak{a}, m, w)} \frac{\lambda}{\varepsilon_0^2 - 1} \tag{2.47}$$

and call it *Weyl vector* with respect to w. Note that this also defines $R(\mathfrak{a}, m, \nu)$ and $\rho(\mathfrak{a}, m, \nu)$ for $\nu \in K$ with $\nu \gg 0$. With $\varepsilon^2 w := (\varepsilon^2 w_1, \varepsilon^{-2} w_2)$ for $\varepsilon^2 \in (\mathcal{O}_K^\times)^2$ one sees $R(\mathfrak{a}, m, \varepsilon^2 w) = \varepsilon^{-2} R(\mathfrak{a}, m, w)$ and accordingly $\rho(\mathfrak{a}, m, \varepsilon^2 w) = \varepsilon^{-2} \rho(\mathfrak{a}, m, w)$.

Lemma 2.8.7. *The Weyl vector $\rho(\mathfrak{a}, m, w)$ is totally positive.*

Proof. We prove that each summand

$$\frac{\lambda}{\varepsilon_0^2 - 1}$$

is totally positive. In order to do so we do not need that λ is reduced. It is enough to know $\lambda \in \Lambda^+(\mathfrak{a}, m)$. That makes the numerator positive and because of $\varepsilon_0 > 1$ the denominator is positive as well. The conjugate is given by

$$\left(\frac{\lambda}{\varepsilon_0^2 - 1}\right)' = \frac{\lambda'}{\varepsilon_0^{-2} - 1} = \frac{-\lambda'}{1 - \varepsilon_0^{-2}}.$$

Now again numerator and denominator are positive. □

Taking the predecessor $\tilde{\lambda} \in \Lambda^+(\mathfrak{a}, m)$ to a $\lambda \in \Lambda^+(\mathfrak{a}, m)$, the set

$$W_\lambda := \left\{ z \in \mathbb{H}^2 : \ \frac{R_{\mathfrak{a}}^m}{\lambda} < r < \frac{R_{\mathfrak{a}}^m}{\tilde{\lambda}} \right\} \tag{2.48}$$

is a connected component of $\mathbb{H}^2 \setminus S(\mathfrak{a}, m)$. Each connected component of $\mathbb{H}^2 \setminus S(\mathfrak{a}, m)$ is of such shape. Those components are called *Weyl chambers* of index m. We denote the set of all those Weyl chambers by

$$W(\mathfrak{a}, m) := \left\{ W_\lambda : \ \lambda \in \Lambda^+(\mathfrak{a}, m) \right\}. \tag{2.49}$$

From the discussion of the previous subsection it is immediate that $(\mathcal{O}_K^\times)^2$ acts on $W(\mathfrak{a}, m)$. Again this action can be viewed in two different ways: Either we look at the action of $(\mathcal{O}_K^\times)^2$ on the defining $\lambda \in \Lambda^+(\mathfrak{a}, m)$ or we regard the Weyl chambers as subset of $\mathbb{H}^2$ and obtain an induced action of $(\mathcal{O}_K^\times)^2$ by considering the action of $\overline{\Gamma_{\mathfrak{a},\infty}} \cong \mathfrak{a}^{-1} \rtimes (\mathcal{O}_K^\times)^2$ on $\mathbb{H}^2$. We have

$$\varepsilon^2 W_\lambda = W_{\varepsilon^{-2}\lambda}$$

for all $\varepsilon^2 \in (\mathcal{O}_K^\times)^2$ and $\lambda \in \Lambda^+(\mathfrak{a}, m)$.

For $\lambda \in \Lambda(\mathfrak{a}, m)$ and $W \in W(\mathfrak{a}, m)$ we write

$$(\lambda, W) > 0 \quad \text{if} \quad \operatorname{tr}(\lambda y) > 0 \quad \forall z \in W.$$

If $\lambda \in \Lambda^+(\mathfrak{a}, m)$, this is equivalent to

$$r > \frac{R_{\mathfrak{a}}^m}{\lambda} \quad \forall z \in W.$$

Moreover, $\lambda \in \Lambda^+(\mathfrak{a}, m)$ is called *reduced* with respect to W if

$$(\lambda, W) > 0 \quad \text{and} \quad (\varepsilon_0^{-2}\lambda, W) < 0.$$

Note that being reduced with respect to W is equivalent to being reduced with respect to y with $z \in W$. One way using the coordinate r to write down this condition is

$$W \subset \left\{ z \in \mathbb{H}^2 : \ \frac{R_{\mathfrak{a}}^m}{\lambda} < r < \varepsilon_0^2 \frac{R_{\mathfrak{a}}^m}{\lambda} \right\}.$$

By our definition each $\lambda \in \Lambda^+(\mathfrak{a}, m)$ is reduced with respect to the Weyl chamber W_λ. On the other hand, for each $W \in W(\mathfrak{a}, m)$ and $\lambda \in \Lambda^+(\mathfrak{a}, m)$ we find exactly one reduced $\tilde{\lambda}$ in the $(\mathcal{O}_K^\times)^2$ orbit

$$\left\{ \varepsilon^2 \lambda : \ \varepsilon^2 \in (\mathcal{O}_K^\times)^2 \right\}$$

of λ. We denote by $R(\mathfrak{a}, m, W)$ the set of all $\lambda \in \Lambda^+(\mathfrak{a}, m)$ which are reduced with respect to W. Note that we have

$$R(\mathfrak{a}, m, W) = R(\mathfrak{a}, m, y)$$

with $z \in W$. For W_λ with $\lambda \in \Lambda^+(\mathfrak{a}, m)$ we can make the set precise by

$$R(\mathfrak{a}, m, W_\lambda) = \left\{ \tilde{\lambda} \in \Lambda^+(\mathfrak{a}, m) : \ \lambda \leq \tilde{\lambda} < \varepsilon_0^2 \lambda \right\}.$$

It follows directly $R(\mathfrak{a}, m, \varepsilon^2 W) = \varepsilon^{-2} R(\mathfrak{a}, m, W)$ for all $\varepsilon^2 \in (\mathcal{O}_K^\times)^2$. Analogously to above we define the so-called *Weyl vector*

$$\rho(\mathfrak{a}, m, W) := \sum_{\lambda \in R(\mathfrak{a}, m, W)} \frac{\lambda}{\varepsilon_0^2 - 1}. \tag{2.50}$$

Hence, $\rho(\mathfrak{a}, m, W) = \rho(\mathfrak{a}, m, y)$ with $z \in W$. We have $\rho(\mathfrak{a}, m, \varepsilon^2 W) = \varepsilon^{-2} \rho(\mathfrak{a}, m, W)$ for all $\varepsilon^2 \in (\mathcal{O}_K^\times)^2$.

2.8.5 Hirzebruch–Zagier divisors on $\overline{X(\mathfrak{a})}$

In the previous subsection we developed the correct notions to be now able to define the Hirzebruch–Zagier divisors on $\overline{X(\mathfrak{a})}$. To distinguish them from the Hirzebruch–Zagier divisors $T(\mathfrak{a}, m)$ on $X(\mathfrak{a})$, we use a different notation and call them $Z(\mathfrak{a}, m)$. We define

$$Z(\mathfrak{a}, m) := T(\mathfrak{a}, m) + \sum_{\kappa \in \Gamma_{\mathfrak{a}} \backslash \mathbb{P}^1(K)} Z^{\kappa}(\mathfrak{a}, m)$$

where κ runs through all cusps of $\Gamma_{\mathfrak{a}}$ and $Z^{\infty}(\mathfrak{a}, m)$ is defined by

$$Z^{\infty}(\mathfrak{a}, m) := \sum_{k=1}^{r} \operatorname{tr}(\rho(\mathfrak{a}, m, A_k) A_k) S_k. \tag{2.51}$$

Recall that $r \in \mathbb{N}$ is defined to fulfill

$$A_k = \varepsilon_0^2 A_{k+r} \quad \text{and} \quad b_k = b_{k+r}$$

and the index k in S_k has to be understood as element of $\mathbb{Z}/r\mathbb{Z}$ (cf. Subsection 2.7.5). For other cusps than ∞, the divisor $Z^{\kappa}(\mathfrak{a}, m)$ is given by the image of $Z^{\infty}(\mathfrak{a}\mathfrak{b}^2, m)$ under the isomorphism $\overline{X(\mathfrak{a}\mathfrak{b}^2)} \xrightarrow{\sim} \overline{X(\mathfrak{a})}$ (here $\mathfrak{b} \in \mathcal{I}_K$ has to be chosen appropriately, cf. the introduction of Section 2.7).

2.9 Green functions

There are several definitions of Green functions serving different purposes. In this section we introduce our definition of a Green function, namely the pre-log-log Green function, developed by Burgos Gil, Kramer and Kühn, which is the central object of this thesis. For that purpose we explain the classical Green currents, their differential equation, growth conditions and logarithmic singularities.

2.9.1 Classical Green currents

The classical Green currents and their intersection theory were developed by Gillet and Soulé in 1990 (cf. [GS90]). In this subsection we briefly introduce the notion of a classical Green current which is discussed in more detail in [Sou92, Chapter II: Green currents].

For a compact complex manifold X of dimension d (more generally we allow complex orbifolds to comprise Hilbert modular surfaces with their elliptic fixed points) we denote by $A^{p,q}(X)$ the space of smooth $\mathbb{C}$ valued differential forms of type (p, q). We consider the inclusion

$$A^{p,q}(X) \to D^{p,q}(X), \quad \omega \mapsto [\omega].$$

Here, $D^{p,q}(X)$ is the topological dual to the space of differential forms $A^{d-p,d-q}(X)$ (cf. [Sou92] for the topology). The functional $[\omega]$ is defined by

$$[\omega](\alpha) := \int_X \omega \wedge \alpha \quad \text{for any } \alpha \in A^{d-p,d-q}(X).$$

The elements of $D^{p,q}(X)$ are called *currents*. By applying differential operators ∂, $\overline{\partial}$, $d := \partial + \overline{\partial}$ and $d^c := (4\pi i)^{-1}(\partial - \overline{\partial})$ to the argument we obtain those operators for the space $D^{p,q}(X)$ as well. To make the respective diagrams

$$\begin{tikzcd} A^{p,q}(X) \arrow[r, hook] \arrow[d, "\partial"] & D^{p,q}(X) \arrow[d, "\partial"] \\ A^{p+1,q}(X) \arrow[r, hook] & D^{p+1,q}(X) \end{tikzcd}$$

commutative, one has to multiply the operators by $(-1)^{p+q+1}$. For example in the case of ∂ we have

$$(\partial g)(\alpha) := g((-1)^{p+q+1}\partial\alpha) \quad \text{for } g \in D^{p,q}(X) \text{ and } \alpha \in A^{d-p-1,d-q}(X).$$

Now, for every divisor T on X we define the *Dirac current* $\delta_T \in D^{1,1}(X)$ by

$$\delta_T(\alpha) := \int_T \alpha \quad \text{for all } \alpha \in A^{d-1,d-1}(X). \tag{2.52}$$

Definition 2.9.1 (Green current)**.** A *Green current* for a divisor T is a current $g \in D^{0,0}(X)$ such that

$$dd^c g + \delta_T = [\omega]$$

for some form $\omega \in A^{1,1}(X)$. The pair (T, g) is called *arithmetic divisor*.

2.9.2 Differential forms with growth conditions

For our purpose the definition of a classical Green current is too strict. In our case of Hilbert modular surfaces we want to allow mild singularities of ω at the cusps. A precise theory of such generalized Green currents was developed by Burgos Gil, Kramer and Kühn in [BGKK07] and [BGKK05]. They allow ω to be a *pre-log-log growth form*. We make that notion precise in this subsection and recommend [BBGK07, Section 1.1] for details.

Definition 2.9.2 (Log-log growth, log-log growth form and pre-log-log growth forms)**.** A function f defined in a neighborhood of D on $X \setminus D$ has *log-log growth along* D if we have

$$|f(z_1, \ldots, z_d)| \leq C \prod_{i=1}^{k} \log(\log(1/|z_i|))^M$$

in local coordinates in which D is given by $z_1 \cdots z_k = 0$ for a constant $C > 0$ and some positive integer M. Differential forms generated by those functions together with the differentials

$$\begin{aligned} \frac{dz_i}{z_i \log(1/|z_i|)}, \ \frac{d\overline{z}_i}{\overline{z}_i \log(1/|z_i|)}, &\quad \text{for } i = 1, \ldots, k, \\ dz_i, \ d\overline{z}_i, &\quad \text{for } i = k+1, \ldots, d, \end{aligned}$$

are called *log-log growth forms*. A log-log growth form ω such that $\partial\omega$, $\overline{\partial}\omega$ and $\partial\overline{\partial}\omega$ are log-log growth forms as well is called a *pre-log-log growth form*.

Remark 2.9.3. A function f is a pre-log-log growth form if and only if

$$f, \quad w_1 \log(|w_1|)\frac{\partial f}{\partial w_1}, \quad w_1 w_2 \log(|w_1|)\log(|w_2|)\frac{\partial^2 f}{\partial w_1 \partial w_2}$$

have log-log growth for $w_1, w_2 \in \{z_1, \dots, z_k, \overline{z}_1, \dots, \overline{z}_k\}$ and $w_1 \neq w_2$. In most cases where we have to prove a function f to be a pre-log-log growth form we will see that the given terms even go to 0 for small z_i $(1 \leq i \leq k)$.

Lemma 2.9.4. *The function*

$$f(z_1, \dots, z_d) = \prod_{i=1}^{k} \log(\log(1/|z_i|))^M$$

from Definition 2.9.2 is a pre-log-log growth form.

Proof. Clearly, by definition f is of log-log growth. Hence, it is left to verify that ∂f, $\overline{\partial} f$ and $\partial\overline{\partial} f$ are log-log growth forms as well. One computes

$$\frac{\partial}{\partial z}\log(\log(1/|z|))^M = \frac{M\log(\log(1/|z|))^{M-1}}{2z\log(|z|)}$$

and the conjugate for $\partial/\partial\overline{z}$. Hence, ∂f and $\overline{\partial} f$ grow accordingly. We further compute

$$\begin{aligned}&\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\log(\log(1/|z|))^M\\ =&\frac{M\log(\log(1/|z|))^{M-2}}{4z\overline{z}\log(|z|)^2}\left(M-1-\log(\log(1/|z|))\right)\end{aligned}$$

which shows the desired growth of $\partial\overline{\partial} f$. □

Definition 2.9.5 (Log growth, log growth form and pre-log growth forms)**.** Analogously to Definition 2.9.2, a function f defined in a neighborhood of D on $X \setminus D$ has *log growth along D* if we have

$$|f(z_1, \dots, z_d)| \leq C \prod_{i=1}^{k} \log(1/|z_i|)^M$$

in local coordinates in which D is given by $z_1 \cdots z_k = 0$ for a constant $C > 0$ and some positive integer M. Differential forms generated by those functions together with the differentials

$$\begin{aligned}&\frac{dz_i}{z_i}, \ \frac{d\overline{z}_i}{\overline{z}_i}, \quad \text{for } i = 1, \dots, k,\\ &dz_i, \ d\overline{z}_i, \quad \text{for } i = k+1, \dots, d,\end{aligned}$$

are called *log growth forms*. A log growth form ω such that $\partial\omega$, $\overline{\partial}\omega$ and $\partial\overline{\partial}\omega$ are log growth forms as well is called a *pre-log growth form*.

Definition 2.9.6 (Mixed growth form and mixed form)**.** Let D_1 and D_2 be normal crossing divisors. A differential form with log growth along D_1 and log-log growth along D_2 is called *mixed growth form along* (D_1, D_2). A mixed growth form ω such that $\partial\omega$, $\overline{\partial}\omega$ and $\partial\overline{\partial}\omega$ are also mixed growth forms is called a *mixed form.*

2.9.3 Pre-log-log Green functions

In this subsection we first want to give the definition of pre-log-log Green functions and then apply them to our situation with Hilbert modular surfaces and Hirzebruch–Zagier divisors.

Definition 2.9.7 (Pre-log-log Green functions)**.** A *pre-log-log Green function* for divisors (D_1, D_2) is a mixed 0-form f along (D_1, D_2) such that $g := [f]$ satisfies

$$dd^c g + \delta_{D_1} = [\omega]$$

for some pre-log-log growth $(1,1)$-form ω along D_2. In this context the arithmetic divisor (D_1, f) defines a class in the first arithmetic Chow group $\widehat{\mathrm{CH}}^1(X, \mathcal{D}_{\mathrm{pre}})$.

Remark 2.9.8. Note that the multiplicities of the components of D_2 do not play any role in Definition 2.9.7 since D_2 is only needed for the growth behavior. The multiplicities of the divisor D_1, however, are important because they influence the Dirac current δ_{D_1}.

Now we want to apply the definition to our situation. We have $X = \overline{X(\mathfrak{a})}$ and hence dimension $d = 2$. The divisor D_1 is the Hirzebruch–Zagier divisor $Z(\mathfrak{a}, m)$ with $m \in \mathbb{Z}$ fixed where D_2 is the exceptional divisor $E(\mathfrak{a})$. The $(1,1)$-form ω is given by $dd^c f$. Therefore, the demanded current equation looks like

$$dd^c[f] + \delta_{D_1} = [dd^c f].$$

Let $\eta \in A^{1,1}(X)$ be a test form. Then we have

$$(dd^c[f])(\eta) = (d^c[f])(d\eta) = [f](-d^c d\eta) = [f](dd^c\eta) = \int_X f \wedge dd^c\eta.$$

Hence, we obtain for the demanded current equation expressed using integrals

$$\int_X f \wedge dd^c\eta + \int_{D_1} \eta = \int_X dd^c f \wedge \eta. \tag{2.53}$$

We call this equation *Green's differential equation* or *Green equation* for short. The equation is also known as the dd^c *equation.*

Remark 2.9.9. In the remainder of this thesis we will consider different Green functions f for the Hirzebruch–Zagier divisors $Z(\mathfrak{a}, m)$ and want to prove that they are actually pre-log-log Green functions for the divisors $(Z(\mathfrak{a}, m), E(\mathfrak{a}))$ according to Definition 2.9.7 (later we often only mention $Z(\mathfrak{a}, m)$ since the exceptional divisor $E(\mathfrak{a})$ is independent of m and we never regard any other divisor for the pre-log-log growth). To do so it is

enough to show that f can be decomposed into $f = f_1 + f_2$ such that f_1 has logarithmic singularities along $-Z(\mathfrak{a}, m)$ (the exact meaning of that is made precise in the next subsection) and f_2 is a pre-log-log growth form along $E(\mathfrak{a})$. Namely, the upcoming Lemma 2.9.15 and Corollary 2.9.18 imply that f then satisfies the Green equation. Further, Remark 2.9.11 ensures that f has the correct growth behavior.

2.9.4 Logarithmic singularities and the Green equation

In this subsection we give a precise meaning to the expression *having logarithmic singularities along divisors* and investigate the Green equations of functions with logarithmic singularities and the Green equation of functions which are pre-log-log growth forms.

Definition 2.9.10 (Logarithmic singularity)**.** (i) Let $Y \subset X$ be an irreducible analytic subvariety of X of codimension 1. We say a function $f : X \setminus Y \to \mathbb{C}$ has a *logarithmic singularity* along Y if for each $p \in X$ there exists an open neighborhood $U_p \subset X$ of p such that

$$f(z) = g(z) + \log(|h(z)|^2)$$

with $g \in C^2(U_p)$ and $h : U_p \to \mathbb{C}$ holomorphic with divisor $U_p \cap Y$ (hence, h has a simple zero at $U_p \cap Y$).

(ii) Let $D \subset X$ be a divisor of X. We can write D as a finite sum

$$D = \sum_{i \in I} \lambda_i Y_i$$

with Y_i being irreducible analytic subvarieties of X of codimension 1. We say $f : X \setminus D \to \mathbb{C}$ has a *logarithmic singularity* along D if we can decompose f into

$$f(z) = \sum_{i \in I} \lambda_i f_i(z)$$

with f_i having a logarithmic singularity along Y_i for all $i \in I$.

Remark 2.9.11. It is important to notice that a function $f : X \setminus D \to \mathbb{C}$ with a logarithmic singularity along D is a pre-log growth form with respect to the divisor D.

Lemma 2.9.12. *Let $h : X \to \mathbb{C}$ be a non-zero meromorphic function. Then*

$$f(z) := \log(|h(z)|^2)$$

has a logarithmic singularity along $\operatorname{div}(h)$*. Furthermore, $dd^c f = 0$.*

Proof. The first part follows directly from Definition 2.9.10. The second part is a consequence of

$$f(z) = \log(h(z)) + \overline{\log(h(z))}$$

which holds locally outside of $\operatorname{div}(h)$ by choosing appropriate branches of the natural logaritm. Hence, f is locally the sum of a holomorphic and an antiholomorphic function. Therefore, it is in the kernel of dd^c. □

Lemma 2.9.13. *Let $f \in C^2(\mathbb{C}^2)$ and $\eta \in A_c^{1,1}(\mathbb{C}^2)$. Then we have*

$$\int_{\mathbb{C}^2} f \wedge dd^c\eta = \int_{\mathbb{C}^2} dd^c f \wedge \eta.$$

Proof. Let σ be a compactly supported 3-form on $\mathbb{C}^2$. Then we have by Stokes' theorem

$$\int_{\mathbb{C}^2} d\sigma = 0.$$

We apply this twice in the following computation:

$$\begin{aligned}
\int_{\mathbb{C}^2} f \wedge dd^c\eta &= \int_{\mathbb{C}^2} d(f \wedge d^c\eta) - \int_{\mathbb{C}^2} df \wedge d^c\eta \\
&= \int_{\mathbb{C}^2} d^c f \wedge d\eta \\
&= -\int_{\mathbb{C}^2} d(d^c f \wedge \eta) + \int_{\mathbb{C}^2} dd^c f \wedge \eta \\
&= \int_{\mathbb{C}^2} dd^c f \wedge \eta.
\end{aligned}$$

□

Lemma 2.9.14. *Let $\eta \in A_c^{1,1}(\mathbb{C}^2)$. Then we have*

$$\int_{(u,v)\in\mathbb{C}^2} \log(|v|^2) \wedge dd^c\eta = \int_{u\in\mathbb{C}, v=0} \eta.$$

Proof. The left integral looks quite similar to the integral in the previous lemma. However, our function $f(u,v) := \log(|v|)$ has a logarithmic singularity along $v = 0$. Hence, we cannot apply the previous lemma. Instead we use

$$\int_{\mathbb{C}^2} \log(|v|^2) \wedge dd^c\eta = \lim_{\varepsilon\to 0} \int_{|v|\geq\varepsilon} \log(|v|^2) \wedge dd^c\eta.$$

Here we can apply Stokes' theorem again but in contrast to the previous lemma, our domain has a boundary, namely $|v| = \varepsilon$. Therefore, we have by Stokes' theorem

$$\int_{|v|\geq\varepsilon} d\sigma = -\int_{|v|=\varepsilon} \sigma.$$

We compute

$$\begin{aligned}
\int_{|v|\geq\varepsilon} \log(|v|^2) \wedge dd^c\eta &= \int_{|v|\geq\varepsilon} d(\log(|v|^2) \wedge d^c\eta) - \int_{|v|\geq\varepsilon} d\log(|v|^2) \wedge d^c\eta \\
&= -\int_{|v|=\varepsilon} \log(|v|^2) \wedge d^c\eta + \int_{|v|\geq\varepsilon} d^c\log(|v|^2) \wedge d\eta.
\end{aligned}$$

The first summand goes to zero for small ε because we have

$$\begin{aligned}\left|-\int_{|v|=\varepsilon}\log(|v|^2)\wedge d^c\eta\right| &\leq 2|\log(\varepsilon)|\left|\int_{|v|=\varepsilon}d^c\eta\right|\\ &\leq 4\pi\varepsilon|\log(\varepsilon)|C\end{aligned}$$

with $C>0$ appropriate. Since $dd^c\log(|v|^2)=0$ by Lemma 2.9.12, for the second summand we get

$$\begin{aligned}\int_{|v|\geq\varepsilon}d^c\log(|v|^2)\wedge d\eta &= -\int_{|v|\geq\varepsilon}d(d^c\log(|v|^2)\wedge\eta)+\int_{|v|\geq\varepsilon}dd^c\log(|v|^2)\wedge\eta\\ &= \int_{|v|=\varepsilon}d^c\log(|v|^2)\wedge\eta.\end{aligned}$$

We now write $v=\varepsilon e^{i\theta}$ and express dv and $d\overline{v}$ by $d\theta$ and $d\varepsilon$:

$$dv=\varepsilon ie^{i\theta}d\theta+e^{i\theta}d\varepsilon \quad\text{and}\quad d\overline{v}=-\varepsilon ie^{-i\theta}d\theta+e^{-i\theta}d\varepsilon.$$

Recall $d^c:=(4\pi i)^{-1}(\partial-\overline{\partial})$ to see

$$\begin{aligned}d^c\log(|v|^2) &= \frac{1}{4\pi i}\left(\frac{dv}{v}-\frac{d\overline{v}}{\overline{v}}\right)\\ &= \frac{1}{4\pi i}\left(\frac{\varepsilon ie^{i\theta}d\theta+e^{i\theta}d\varepsilon}{\varepsilon e^{i\theta}}-\frac{-\varepsilon ie^{-i\theta}d\theta+e^{-i\theta}d\varepsilon}{\varepsilon e^{-i\theta}}\right)=\frac{d\theta}{2\pi}.\end{aligned}$$

Hence, we finally get

$$\lim_{\varepsilon\to 0}\int_{|v|=\varepsilon}d^c\log(|v|^2)\wedge\eta=\lim_{\varepsilon\to 0}\int_{u\in\mathbb{C},\theta\in[0,2\pi),v=\varepsilon e^{i\theta}}\frac{d\theta}{2\pi}\wedge\eta=\int_{u\in\mathbb{C},v=0}\eta.$$

□

Lemma 2.9.15. *Let X be two dimensional over $\mathbb{C}$ and $f:X\setminus D\to\mathbb{C}$ have a logarithmic singularity along a divisor $-D\subset X$ and $\eta\in A_c^{1,1}(X)$. Then we have*

$$\int_X f\wedge dd^c\eta+\int_D\eta=\int_X dd^cf\wedge\eta$$

or equivalently

$$\int_X f\wedge dd^c\eta=\int_X dd^cf\wedge\eta+\int_{-D}\eta.$$

Proof. Let us consider first the case with D being an irreducible analytic subvariety Y. Let $p\in X$. In case $p\notin Y$ we find an open neighborhood U_p of p with $U_p\subset X\setminus Y$. In case $p\in Y$ we find an open neighborhood U_p of p and a chart $\varphi:U_p\to\mathbb{C}^2$ with

$$\varphi(p)=(0,0)\quad\text{and}\quad\varphi(Y\cap U_p)=\Big\{(u,v)\in\mathbb{C}^2:\ v=0\Big\}\cap\varphi(U_p).$$

In addition, this neighborhood shall have the local decomposition of f according to Definition 2.9.10. Let $\tilde{\eta} \in A_c^{1,1}(U_p)$. Then in the first case where $p \notin Y$ we have

$$\int_X f \wedge dd^c\tilde{\eta} + \int_D \tilde{\eta} = \int_X dd^c f \wedge \tilde{\eta}$$

by Lemma 2.9.13 because $\int_D \tilde{\eta}$ vanishes. In the second case where $p \in Y$ we can express $f(z)$ as

$$f(z) = g(z) - \log(|h(z)|^2)$$

for all $z \in U_p$ with $g \in C^2(U_p, \mathbb{R})$ and $h : U_p \to \mathbb{C}$ holomorphic with a simple zero at $Y \cap U_p$. The function h can be expressed in local coordinates with respect to the chart φ as $h = v \cdot \tilde{h}$ with a holomorphic, nowhere vanishing function $\tilde{h} : U_p \to \mathbb{C}^\times$. Hence, we locally have

$$f = g - \log(|v \cdot \tilde{h}|^2) = g - \log(|\tilde{h}|^2) - \log(|v|^2).$$

Here $\tilde{g} := g - \log(|\tilde{h}|^2)$ is C^2. By Lemma 2.9.12 we get

$$dd^c f = dd^c g = dd^c \tilde{g}.$$

Therefore, we get

$$\begin{aligned}
\int_X f \wedge dd^c\tilde{\eta} &= \int_X (\tilde{g} - \log(|v|^2)) \wedge dd^c\tilde{\eta} \\
&= \int_X \tilde{g} \wedge dd^c\tilde{\eta} - \int_X \log(|v|^2) \wedge dd^c\tilde{\eta} \\
&= \int_X dd^c\tilde{g} \wedge \tilde{\eta} - \int_D \tilde{\eta} \\
&= \int_X dd^c f \wedge \tilde{\eta} - \int_D \tilde{\eta}
\end{aligned}$$

by Lemma 2.9.13 and Lemma 2.9.14 or equivalently

$$\int_X f \wedge dd^c\tilde{\eta} + \int_D \tilde{\eta} = \int_X dd^c f \wedge \tilde{\eta}.$$

Now using that η has compact support and the theory of smooth partitions of unity, there exists a finite subset $P \subset X$ and $\eta_p \in A_c^{1,1}(U_p)$ for all $p \in P$ such that

$$\sum_{p \in P} \eta_p = \eta.$$

Hence, by linearity of the integral we get

$$\int_X f \wedge dd^c\eta + \int_D \eta = \int_X dd^c f \wedge \eta.$$

This proves the simplified version where D is irreducible. Let us now consider the more general case where D is a linear combination of irreducible analytic subvarieties. We write D as finite sum

$$D = \sum_{i \in I} \lambda_i Y_i$$

with Y_i being irreducible analytic subvarieties of X of codimension 1 and

$$f(z) = \sum_{i \in I} \lambda_i f_i(z),$$

respectively, such that f_i has logarithmic singularities along $-Y_i$. Using this decomposition of f the result follows again by using the linearity of the integral. □

Lemma 2.9.16. *Let $U \subset \mathbb{C}^2$ be open and let $f : U \setminus \{v = 0\} \to \mathbb{C}$ be two times continuously differentiable and locally integrable on U. Further, let*

$$\lim_{v \to 0} v \frac{\partial f}{\partial v} = \lim_{v \to 0} v \frac{\partial f}{\partial u} = \lim_{v \to 0} \overline{v} \frac{\partial f}{\partial \overline{v}} = \lim_{v \to 0} \overline{v} \frac{\partial f}{\partial \overline{u}} = 0.$$

Then we have for all $\eta \in A_c^{1,1}(U)$

$$\int_U f \wedge dd^c \eta = \int_U dd^c f \wedge \eta.$$

Proof. We reuse the idea of Lemma 2.9.14:

$$\int_U f \wedge dd^c \eta = \lim_{\varepsilon \to 0} \int_{|v| \geq \varepsilon} f \wedge dd^c \eta.$$

The first integral (and hence the limit of the second integral) exists because of the local integrability of f and the compact support of η. We compute

$$\begin{aligned} \int_{|v| \geq \varepsilon} f \wedge dd^c \eta &= \int_{|v| \geq \varepsilon} d(f \wedge d^c \eta) - \int_{|v| \geq \varepsilon} df \wedge d^c \eta \\ &= - \int_{|v| = \varepsilon} f \wedge d^c \eta + \int_{|v| \geq \varepsilon} d^c f \wedge d\eta. \end{aligned}$$

As in the proof of Lemma 2.9.14, the first summand goes to 0 for small ε. Hence, we are left with

$$\begin{aligned} \int_{|v| \geq \varepsilon} d^c f \wedge d\eta &= - \int_{|v| \geq \varepsilon} d(d^c f \wedge \eta) + \int_{|v| \geq \varepsilon} dd^c f \wedge \eta \\ &= \int_{|v| = \varepsilon} d^c f \wedge \eta + \int_{|v| \geq \varepsilon} dd^c f \wedge \eta. \end{aligned}$$

Clearly,

$$\lim_{\varepsilon \to 0} \int_{|v| \geq \varepsilon} dd^c f \wedge \eta = \int_U dd^c f \wedge \eta.$$

Therefore, we have to show that

$$\lim_{\varepsilon \to 0} \int_{|v| = \varepsilon} d^c f \wedge \eta = 0.$$

We will compute now $d^c f \wedge \eta$ in detail. First, we have

$$4\pi i (d^c f) = \partial f - \overline{\partial} f = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv - \frac{\partial f}{\partial \overline{u}} d\overline{u} - \frac{\partial f}{\partial \overline{v}} d\overline{v}.$$

Now we decompose η into

$$\eta = \eta_1 du \wedge d\overline{u} + \eta_2 du \wedge d\overline{v} + \eta_3 dv \wedge d\overline{u} + \eta_4 dv \wedge d\overline{v}$$

and get

$$\begin{aligned}
&4\pi i(d^c f \wedge \eta) \\
=&\frac{\partial f}{\partial u} du \wedge (\eta_3 dv \wedge d\overline{u} + \eta_4 dv \wedge d\overline{v}) + \frac{\partial f}{\partial v} dv \wedge (\eta_1 du \wedge d\overline{u} + \eta_2 du \wedge d\overline{v}) \\
-&\frac{\partial f}{\partial \overline{u}} d\overline{u} \wedge (\eta_2 du \wedge d\overline{v} + \eta_4 dv \wedge d\overline{v}) - \frac{\partial f}{\partial \overline{v}} d\overline{v} \wedge (\eta_1 du \wedge d\overline{u} + \eta_3 dv \wedge d\overline{u}) \\
=&\left(\frac{\partial f}{\partial u}\eta_3 - \frac{\partial f}{\partial v}\eta_1\right) du \wedge dv \wedge d\overline{u} + \left(\frac{\partial f}{\partial u}\eta_4 - \frac{\partial f}{\partial v}\eta_2\right) du \wedge dv \wedge d\overline{v} \\
+&\left(\frac{\partial f}{\partial \overline{u}}\eta_2 - \frac{\partial f}{\partial \overline{v}}\eta_1\right) du \wedge d\overline{u} \wedge d\overline{v} + \left(\frac{\partial f}{\partial \overline{u}}\eta_4 - \frac{\partial f}{\partial \overline{v}}\eta_3\right) dv \wedge d\overline{u} \wedge d\overline{v}.
\end{aligned}$$

To compute the integral, we write $v = \varepsilon e^{i\theta}$ like in the proof of Lemma 2.9.14 and express dv and $d\overline{v}$ by $d\theta$ and $d\varepsilon$:

$$dv = \varepsilon i e^{i\theta} d\theta + e^{i\theta} d\varepsilon \quad \text{and} \quad d\overline{v} = -\varepsilon i e^{-i\theta} d\theta + e^{-i\theta} d\varepsilon.$$

Because we integrate over a domain in which ε is fixed, the $d\varepsilon$ part of dv and $d\overline{v}$ is irrelevant. Hence, in the integrand we may replace dv by $ivd\theta$ and $d\overline{v}$ by $-i\overline{v}d\theta$. Therefore, we lose the two terms with $dv \wedge d\overline{v}$ and our integrand looks like

$$\left(v\frac{\partial f}{\partial v}\eta_1 - v\frac{\partial f}{\partial u}\eta_3 + \overline{v}\frac{\partial f}{\partial \overline{v}}\eta_1 - \overline{v}\frac{\partial f}{\partial \overline{u}}\eta_2\right) i d\theta \wedge du \wedge d\overline{u}.$$

Because of our assumption and because of the boundedness of η, this expression goes to zero for $\varepsilon \to 0$. Hence, we have shown

$$\lim_{\varepsilon \to 0} \int_{|v|=\varepsilon} d^c f \wedge \eta = 0.$$

□

Corollary 2.9.17. *Let $U \subset \mathbb{C}^2$ be open and let $f : U \setminus \{uv = 0\} \to \mathbb{C}$ be two times continuously differentiable and locally integrable on U. Further, let*

$$\lim_{u \to 0} u\frac{\partial f}{\partial u} = \lim_{u \to 0} u\frac{\partial f}{\partial v} = \lim_{u \to 0} \overline{u}\frac{\partial f}{\partial \overline{u}} = \lim_{u \to 0} \overline{u}\frac{\partial f}{\partial \overline{v}} = 0$$

and

$$\lim_{v \to 0} v\frac{\partial f}{\partial u} = \lim_{v \to 0} v\frac{\partial f}{\partial v} = \lim_{v \to 0} \overline{v}\frac{\partial f}{\partial \overline{u}} = \lim_{v \to 0} \overline{v}\frac{\partial f}{\partial \overline{v}} = 0.$$

Then we have for all $\eta \in A_c^{1,1}(U)$

$$\int_U f \wedge dd^c \eta = \int_U dd^c f \wedge \eta.$$

Corollary 2.9.18. *Let $U \subset \mathbb{C}^2$ be open and let $f : U \setminus \{uv = 0\} \to \mathbb{C}$ be a pre-log-log growth form along $uv = 0$. Then we have for all $\eta \in A_c^{1,1}(U)$*

$$\int_U f \wedge dd^c\eta = \int_U dd^c f \wedge \eta.$$

Proof. We simply have to verify the conditions of Corollary 2.9.17. Because f itself is of log-log growth, it is locally integrable on U. Now Remark 2.9.3 together with the facts

$$\lim_{x\to\infty} \frac{\log(\log(x))^M}{\log(1/x)} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{\log(\log(x))^M}{x} = 0$$

implies that the limit conditions of Corollary 2.9.17 are satisfied. □

2.9.5 Hilbert modular forms induce Green functions

Definition 2.9.19 (Hilbert modular form)**.** A meromorphic function $f : \mathbb{H}^2 \to \mathbb{C}$ is called a *meromorphic Hilbert modular form* of weight $k \in \mathbb{Z}$ for $\Gamma_{\mathfrak{a}}$ if it satisfies

$$f(\gamma z) = N(cz + d)^k f(z)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\mathfrak{a}}$. If f is holomorphic on $\mathbb{H}^2$, then it is called a *holomorphic Hilbert modular form.*

If $k \neq 0$ and $f \neq 0$ the function f is not invariant under $\Gamma_{\mathfrak{a}}$ and therefore it does not define a function $\Gamma_{\mathfrak{a}} \to \mathbb{C}$. But since $N(cz + d)^k$ is nowhere vanishing on $\mathbb{H}^2$, the function f defines a divisor on $X(\mathfrak{a})$.

Remark 2.9.20. A shorter notation for the transformation law uses the *Petersson slash operator* $|_k$ which is defined by

$$(f \mid_k \gamma)(z) := N(cz + d)^{-k} f(\gamma z)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})^2$ and functions $f : \mathbb{H}^2 \to \mathbb{C}$. The Petersson slash operator defines a right group action on the set of all functions $f : \mathbb{H}^2 \to \mathbb{C}$, i.e., we have

$$(f \mid_k \gamma_1) \mid_k \gamma_2 = (f \mid_k \gamma_1\gamma_2)$$

for all $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{R})^2$. By using the Petersson slash operator, the transformation law of Definition 2.9.19 breaks down to $f \mid_k \gamma = f$ for all $\gamma \in \Gamma_{\mathfrak{a}}$.

Let f be a Hilbert modular form. The transformation law implies, independent of the weight, an invariance of f under the translation by elements of $\mathfrak{b} := \mathfrak{a}^{-1}$. Hence, if f is holomorphic, it has a normally convergent Fourier expansion

$$f(z) = \sum_{\nu \in \mathfrak{b}^\vee} a_\nu e(\mathrm{tr}(\nu z)).$$

The Fourier coefficients are given by

$$a_\nu = \frac{1}{\mathrm{vol}(\mathfrak{b})} \int_{\mathbb{R}^2/\mathfrak{b}} f(z) e(-\operatorname{tr}(\nu z)) dx_1 dx_2.$$

By the Götzky–Koecher principle holomorphic Hilbert modular forms are automatically holomorphic at the cusps. For the cusp ∞, this means $a_\nu \neq 0$ implies $\nu = 0$ or $\nu \gg 0$.

Because of the $\mathfrak{b}$ invariance, we can express f in local coordinates (u, v) with respect to a totally positive basis (α, β) of $\mathfrak{b}$. From Lemma 2.7.1 we get

$$f(z) = \sum_{\nu \in \mathfrak{b}^\vee} a_\nu u^{\operatorname{tr}(\alpha\nu)} v^{\operatorname{tr}(\beta\nu)}.$$

We see that f extends to a holomorphic function on the S_k. Therefore, f defines not only a divisor on $X(\mathfrak{a})$ but a divisor on $\overline{X(\mathfrak{a})}$ as well. Meromorphic Hilbert modular forms f define a divisor on $\overline{X(\mathfrak{a})}$ as well but naturally they might have poles at cusps.

Definition 2.9.21. For this thesis we fix a weight $k \in \mathbb{N}$ and for each $\mathfrak{a} \in \mathcal{I}_K$ we fix a non-zero (meromorphic) Hilbert modular form $F(\mathfrak{a}, z)$ for $\Gamma_\mathfrak{a}$ of weight k such that they are compatible with each other under conjugation, i.e., we have

$$F(\mathfrak{a}^2\mathfrak{b}, z) = (F(\mathfrak{b}, \cdot) \mid_k M)(z) \tag{2.54}$$

for $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_K$ and $M \in M(\mathfrak{a}, b)$ (cf. Corollary 2.3.6). We define

$$T(\mathfrak{a}, 0) := -\frac{\operatorname{div}(F(\mathfrak{a}, \cdot))}{2k} \quad \text{and} \quad Z(\mathfrak{a}, 0) := -\frac{\operatorname{div}(F(\mathfrak{a}, \cdot))}{2k}$$

where in the definition of $T(\mathfrak{a}, 0)$ we consider $F(\mathfrak{a}, \cdot)$ on $X(\mathfrak{a})$ and in the definition of $Z(\mathfrak{a}, 0)$ we consider $F(\mathfrak{a}, \cdot)$ on $\overline{X(\mathfrak{a})}$.

Note once again that even though $F(\mathfrak{a}, z)$ does neither define a function on $X(\mathfrak{a})$ nor on $\overline{X(\mathfrak{a})}$, the divisor of Definition 2.9.21 is well-defined.

The divisors $T(\mathfrak{a}, 0)$ and $Z(\mathfrak{a}, 0)$ depend on the choice of the Hilbert modular form $F(\mathfrak{a}, z)$. However, different choices lead to divisors which are rational equivalent since the quotient defines a meromorphic Hilbert modular form of weight 0, i.e., a meromorphic function on $\overline{X(\mathfrak{a})}$.

Definition 2.9.22 (Petersson norm)**.** For a meromorphic Hilbert modular form f we define

$$||f(z)|| := (16\pi^2 y_1 y_2)^{k/2} |f(z)|,$$

the so-called *Petersson norm* of f.

Remark 2.9.23. The factor $(y_1 y_2)^{k/2}$ makes $||f(z)||$ invariant under $\Gamma_\mathfrak{a}$. Therefore, $||f(z)||$ is well-defined on $X(\mathfrak{a})$.

Proposition 2.9.24. *The function*

$$G(\mathfrak{a}, 0, z) := \log\left(||F(\mathfrak{a}, z)||^{1/k}\right)$$

has logarithmic singularities along the divisor $-Z(\mathfrak{a}, 0)$ *and is a pre-log-log Green function on* $\overline{X(\mathfrak{a})}$ *with respect to the divisor* $Z(\mathfrak{a}, 0)$*. Further, we have*

$$dd^c G(\mathfrak{a}, 0, z) = -\frac{\omega}{2}$$

and

$$G(\mathfrak{b}, 0, Mz) = G(\mathfrak{a}^2\mathfrak{b}, 0, z)$$

for $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_K$ *and* $M \in M(\mathfrak{a}, b)$*. Here* ω *denotes the Kähler form (cf. equation* (2.14)*).*

Proof. By Remark 2.9.23 $G(\mathfrak{a}, 0, z)$ is well-defined on $X(\mathfrak{a})$. We have

$$\begin{aligned} G(\mathfrak{a}, 0, z) &= \log\left(4\pi(y_1y_2)^{1/2}|F(\mathfrak{a}, z)|^{1/k}\right) \\ &= \log(4\pi) + \frac{\log(y_1y_2)}{2} + \frac{\log(|F(\mathfrak{a}, z)|^2)}{2k}. \end{aligned} \tag{2.55}$$

This proves

$$dd^c G(\mathfrak{a}, 0, z) = -\frac{\omega}{2}$$

together with equation (2.15) and Lemma 2.9.12. It also proves that $G(\mathfrak{a}, 0, z)$ has a logarithmic singularity along the divisor

$$\frac{\operatorname{div}(F(\mathfrak{a}, \cdot))}{2k} = -Z(\mathfrak{a}, 0).$$

Since there is no other growth apart from the logarithmic singularity, we conclude that $G(\mathfrak{a}, 0, z)$ is a pre-log-log Green function with respect to the divisor $Z(\mathfrak{a}, 0)$.

We come to the proof of the transformation law. By equation (2.54) we have

$$F(\mathfrak{b}, Mz) = N(cz + d)^k F(\mathfrak{a}^2\mathfrak{b}, z).$$

Hence, we obtain with equation (2.55) and equation (2.16)

$$\begin{aligned} G(\mathfrak{b}, 0, Mz) &= \log(4\pi) + \frac{\log(\Im(Mz))}{2} + \frac{\log(|F(\mathfrak{b}, Mz)|^2)}{2k} \\ &= \log(4\pi) + \frac{\log(\Im(z)/|N(cz+d)|^2)}{2} + \frac{\log(\left|N(cz+d)^k F(\mathfrak{a}^2\mathfrak{b}, z)\right|^2)}{2k} \\ &= \log(4\pi) + \frac{\log(\Im(z))}{2} + \frac{\log(|F(\mathfrak{a}^2\mathfrak{b}, z)|^2)}{2k} = G(\mathfrak{a}^2\mathfrak{b}, 0, z). \end{aligned}$$

□

Definition 2.9.25. We define

$$L(\mathfrak{a},0) := \frac{1}{2} - \int_{X(\mathfrak{a})} \frac{G(\mathfrak{a},0,z)}{\mathrm{vol}(X(\mathfrak{a}))}\omega^2 \quad \text{and} \quad \Phi(\mathfrak{a},0,z) := G(\mathfrak{a},0,z) + L(\mathfrak{a},0).$$

At this point the definitions of $\Phi(\mathfrak{a},0,z)$, $L(\mathfrak{a},0)$ and $G(\mathfrak{a},0,z)$ appear unmotivated. In the next chapter we will define $\Phi(\mathfrak{a},m,z)$, $L(\mathfrak{a},m)$ and $G(\mathfrak{a},m,z)$ for $m \in \mathbb{N}$ and see their relation in equation (3.9) which is true for $m = 0$ as well by Definition 2.9.25. While the purpose of the definition of $G(\mathfrak{a},0,z)$ becomes clear in Section 3.11 where we talk about the arithmetic Hirzebruch–Zagier theorem for the automorphic Green functions, the choice of the constant $L(\mathfrak{a},0)$ will not be justified before Theorem 5.4.1.

Chapter 3

Automorphic Green functions

The definition of automorphic Green functions on Hilbert modular surfaces associated to real quadratic number fields goes back to Bruinier in the late 90s who defined and investigated them in [Bru99] and later publications. In this chapter we generalize his definition by defining them for Hilbert modular surfaces $X(\mathfrak{a})$ for arbitary ideals $\mathfrak{a}$, compute the Fourier expansion, investigate its growth behavior near the cusps, show that it is a pre-log-log Green function, present a valuable decomposition into smooth functions, work a lot with this decomposition, compute associated integrals, and eventually present the arithmetic Hirzebruch–Zagier theorem which is made more explicit around the cusps by our work.

3.1 Motivation, definition, convergence and invariance of the unregularized Green function

In the upcoming sections we elaborate on [Bru99, Section 3]. We generalize the definition of the automorphic Green functions to arbitrary ideals and formulate the respective results. In particular, we develop the Fourier expansions of the generalized automorphic Green functions.

The naive idea for the definition of the Green function $\Phi(\mathfrak{a}, m, z)$ associated to the Hirzebruch–Zagier divisor $T(\mathfrak{a}, m)$ on $X(\mathfrak{a})$ for $m \in \mathbb{N}$ is to set

$$\Phi(\mathfrak{a}, m, z) := \sum_{\substack{A=\left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in L(\mathfrak{a})^\vee \\ \det(A)=m/(N(\mathfrak{a})D)}} \log \left| \frac{bz_1\overline{z_2} - \lambda z_1 - \lambda'\overline{z_2} + a}{bz_1 z_2 - \lambda z_1 - \lambda' z_2 + a} \right|. \tag{3.1}$$

Green functions associated to a divisor shall have logarithmic singularities along the negative of its divisor and shall be smooth elsewhere. The denominator of (3.1) is the polynomial which is used for the definition of the Hirzebruch–Zagier divisor $T(\mathfrak{a}, m)$ (cf. Subsection 2.8.1). Hence, we expect logarithmic singularities along $-T(\mathfrak{a}, m)$. The numerator in (3.1) is then used to make the sum formally invariant under $\Gamma_{\mathfrak{a}}$. Namely,

we have

$$\log\left|\frac{bz_1\overline{z_2}-\lambda z_1-\lambda'\overline{z_2}+a}{bz_1z_2-\lambda z_1-\lambda' z_2+a}\right| = \frac{1}{2}\log\left(\frac{\det(A)+h(A,z)}{h(A,z)}\right) = \frac{1}{2}\log\left(\frac{g(A,z)+1}{g(A,z)}\right)$$

for $A = \left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right)$ using the equations (2.21), (2.22) and (2.23). Now, the $\mathrm{GL}_2^+(K)$ invariance of $g(A,z)$ (cf. Corollary 2.6.2) together with the $\Gamma_{\mathfrak{a}}$ invariance of $L(\mathfrak{a})^\vee$ implies the formal invariance of $\Phi(\mathfrak{a},m,z)$ under $\Gamma_{\mathfrak{a}}$. However, this is a formal invariance only, since the above series defining $\Phi(\mathfrak{a},m,z)$ does not converge if $T(\mathfrak{a},m)\neq 0$. In case $T(\mathfrak{a},m)=0$ we have $\Phi(\mathfrak{a},m,z)=0$ since the defining sum is empty. Therefore, from now onwards we assume $m\in\mathbb{N}$ to be such that $T(\mathfrak{a},m)\neq 0$ when we talk about divergence and simple poles.

The divergence of (3.1) causes the actual definition of $\Phi(\mathfrak{a},m,z)$ to be more complicated. It involves a regularization process. For that purpose we introduce a new complex variable s.

Definition 3.1.1. For $\mathfrak{a}\in\mathcal{I}_K$, $m\in\mathbb{N}$, $s\in\mathbb{C}$ with $\Re(s)>1$ and $z\in\mathbb{H}^2\setminus T(\mathfrak{a},m)$ we define

$$\Phi(\mathfrak{a},m,s,z) := \sum_{\substack{A\in L(\mathfrak{a})^\vee \\ \det(A)=m/(N(\mathfrak{a})D)}} Q_{s-1}\left(1+2g(A,z)\right).$$

Here, $Q_{s-1}(z)$ is the Legendre function of the second kind (cf. [OLBC10, 14.12.6]), defined by

$$Q_{s-1}(x) := \int_0^\infty \left(x+\sqrt{x^2-1}\cosh t\right)^{-s} dt$$

for $x>1$ and $\Re(s)>0$.

By the argument from above the $\Gamma_{\mathfrak{a}}$ invariance of $\Phi(\mathfrak{a},m,s,z)$ is clear as long as the series converges absolutely. To prove this absolute convergence and a more general transformation law than the $\Gamma_{\mathfrak{a}}$ invariance we introduce two lemmata first.

Lemma 3.1.2. *Let $m\in\mathbb{N}$, $\alpha>1$ and $f:\mathbb{R}^+\to\mathbb{C}$ be a continuous function which satisfies $f(x)=O(x^{-\alpha})$ for large x. Then the series*

$$\sum_{\substack{A\in L(\mathfrak{a})^\vee \\ \det(A)=m/(N(\mathfrak{a})D)}} f\left(g(A,z)\right)$$

converges normally for $z\in\mathbb{H}^2\setminus T(\mathfrak{a},m)$. If f extends continuously to $x=0$, the statement holds for all $z\in\mathbb{H}^2$.

Proof. First of all, by definition of $T(\mathfrak{a},m)$ we have

$$z\in T(\mathfrak{a},m) \quad\Leftrightarrow\quad \exists A\in L(\mathfrak{a})^\vee \text{ with } \det(A)=\frac{m}{N(\mathfrak{a})D} \text{ and } g(A,z)=0.$$

This explains why we have to exclude $z \in T(\mathfrak{a}, m)$ if $f(0)$ is not properly defined. Now, let $B \subset \mathbb{H}^2$ be a compact subset. Then by Lemma 2.6.4 the set

$$\left\{ A \in L(\mathfrak{a})^\vee : \det(A) = \frac{m}{N(\mathfrak{a})D} \text{ with } g(A,z) \leq C \text{ for a } z \in B \right\}$$

is finite for any $C > 0$. Hence, we are done by showing normal convergence of

$$\sum_{\substack{A \in L(\mathfrak{a})^\vee \\ \det(A) = m/(N(\mathfrak{a})D) \\ g(A,z) > C, \forall z \in B}} f\left(g(A,z)\right) \tag{3.2}$$

for $C > 0$ chosen large. We define $\tilde{f}(x) := f(x-1)$. Then of course $\tilde{f}(x) = O(x^{-\alpha})$ as well. Hence, for $A = \left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in L(\mathfrak{a})^\vee$ with $\det(A) = m/(N(\mathfrak{a})D)$ we obtain by the first equation in (2.21) and equation (2.23)

$$\begin{aligned} f\left(g(A,z)\right) = \tilde{f}\left(1 + g(A,z)\right) = \tilde{f}\left(\frac{\det(A) + h(A,z)}{\det(A)}\right) &= \tilde{f}\left(q_{\tilde{W}_z}(A)\frac{N(\mathfrak{a})D}{m}\right) \\ &= \tilde{f}\left(\frac{|b\overline{z}_1 z_2 - \lambda\overline{z}_1 - \lambda' z_2 + a|^2}{4y_1y_2}\frac{N(\mathfrak{a})D}{m}\right). \end{aligned}$$

Within the compact set B the factor

$$\frac{N(\mathfrak{a})D}{4y_1y_2m}$$

is bounded from below. Therefore, the normal convergence of (3.2) follows from the normal convergence of

$$\sum_{\substack{A\left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in L(\mathfrak{a})^\vee \\ \det(A) = m/(N(\mathfrak{a})D)}} \frac{1}{|b\overline{z}_1 z_2 - \lambda\overline{z}_1 - \lambda' z_2 + a|^{2\alpha}}$$

for $\alpha > 1$ which is well known (cf. [Zag75]). □

Lemma 3.1.3. *Let $m > 0$ and $f : \mathbb{R}_0^+ \to \mathbb{C} \cup \{\infty\}$ be a function. We define for $\mathfrak{a} \in \mathcal{I}_K$ and $z \in \mathbb{H}^2$*

$$F(\mathfrak{a}, z) := \sum_{\substack{A \in L(\mathfrak{a})^\vee \\ \det(A) = m/(N(\mathfrak{a})D)}} f\left(g(A,z)\right).$$

Now let $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_K$ and $M \in M(\mathfrak{a}, \mathfrak{b})$. Then for any $z \in \mathbb{H}^2$ the series $F(\mathfrak{b}, Mz)$ converges absolutely if and only if the series $F(\mathfrak{a}^2\mathfrak{b}, z)$ converges absolutely. In that case we have

$$F(\mathfrak{b}, Mz) = F(\mathfrak{a}^2\mathfrak{b}, z).$$

Proof. We show that the two series sum up the same values. We have

$$\begin{aligned}
F(\mathfrak{a}^2\mathfrak{b}, z) &= \sum_{\substack{A\in L(\mathfrak{a}^2\mathfrak{b})^\vee \\ \det(A)=m/(N(\mathfrak{a}^2\mathfrak{b})D)}} f\left(g(A,z)\right) \\
&\overset{(i)}{=} \sum_{\substack{A\in L(\mathfrak{a}^2\mathfrak{b})^\vee \\ \det(A)=m/(N(\mathfrak{a})^2N(\mathfrak{b})D)}} f\left(g(M.A,Mz)\right) \\
&\overset{(ii)}{=} \sum_{\substack{A\in L(\mathfrak{b})^\vee/N(\mathfrak{a}) \\ \det(A)=m/(N(\mathfrak{a})^2N(\mathfrak{b})D)}} f\left(g(A,Mz)\right) \\
&= \sum_{\substack{A\in L(\mathfrak{b})^\vee \\ \det(A/N(\mathfrak{a}))=m/(N(\mathfrak{a})^2N(\mathfrak{b})D)}} f\left(g(A/N(\mathfrak{a}),Mz)\right) \\
&\overset{(iii)}{=} \sum_{\substack{A\in L(\mathfrak{b})^\vee \\ \det(A)=m/(N(\mathfrak{b})D)}} f\left(g(A,Mz)\right) \\
&= F(\mathfrak{b}, Mz).
\end{aligned}$$

In step (i) we used Corollary 2.6.2. In step (ii) we used

$$M.L(\mathfrak{a}^2\mathfrak{b})^\vee = \frac{L(\mathfrak{b})^\vee}{N(\mathfrak{a})}$$

from Proposition 2.4.2. In step (iii) we used Remark 2.6.3. □

Proposition 3.1.4. *The series defining $\Phi(\mathfrak{a}, m, s, z)$ converges normally for $\Re(s) > 1$ and $z \in \mathbb{H}^2 \setminus T(\mathfrak{a}, m)$ to a function which is $\Gamma_\mathfrak{a}$ invariant and holomorphic in s. Further, we have for $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_K$ and $M \in M(\mathfrak{a}, \mathfrak{b})$*

$$\Phi(\mathfrak{b}, m, s, Mz) = \Phi(\mathfrak{a}^2\mathfrak{b}, m, s, z).$$

Proof. By [OLBC10, 14.8.15] we have

$$Q_{s-1}(x) \sim \frac{\Gamma(s)}{\Gamma\left(s+\frac{1}{2}\right)} \frac{\sqrt{\pi}}{(2x)^s}$$

for large x which implies $Q_{s-1}(x) = O(x^{-\Re(s)})$. Therefore, we can apply Lemma 3.1.2 to get the convergence statement for all $z \in \mathbb{H}^2 \setminus T(\mathfrak{a}, m)$. Because for each compact subset

$$B \subset \{s \in \mathbb{C} : \Re(s) > 1\}$$

there exists an $\alpha > 1$ with $\Re(s) > \alpha$ for all $s \in B$, we obtain normal convergence in s as well. This implies the holomorphicity in s. The transformation law follows from Lemma 3.1.3. The $\Gamma_\mathfrak{a}$ invariance is a special instance of this more general transformation law using

$$M \in M(\mathcal{O}_K, \mathfrak{a}) = \Gamma_\mathfrak{a}.$$

□

By [OLBC10, 14.5.9] we have

$$Q_0(x) = \frac{1}{2}\log\left(\frac{x+1}{x-1}\right).$$

Applied to the argument $1+2g(A,z)$ of Q_{s-1} in the definition of $\Phi(\mathfrak{a},m,s,z)$, we get

$$Q_0\left(1+2g(A,z)\right) = \frac{1}{2}\log\left(\frac{(1+2g(A,z))+1}{(1+2g(A,z))-1}\right) = \frac{1}{2}\log\left(\frac{g(A,z)+1}{g(A,z)}\right).$$

Therefore, $\Phi(\mathfrak{a},m,1,z)$ formally gives us the naive non-converging definition of $\Phi(\mathfrak{a},m,z)$ in equation (3.1) back.

It can be shown that the first two termwise derivatives of the series defining $\Phi(\mathfrak{a},m,s,z)$ converge normally as well. Hence, $\Phi(\mathfrak{a},m,s,z)$ is two times continuously differentiable and using the differential equation of $Q_{s-1}(x)$, one can deduce that $\Phi(\mathfrak{a},m,s,z)$ is an eigenfunction with respect to the hyperbolic Laplace operators

$$\Delta_1 := y_1^2\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_1^2}\right) \quad \text{and} \quad \Delta_2 := y_2^2\left(\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial y_2^2}\right).$$

Namely, we have for $j\in\{1,2\}$

$$\Delta_j\Phi(\mathfrak{a},m,s,z) = s(s-1)\Phi(\mathfrak{a},m,s,z). \tag{3.3}$$

3.2 Fourier expansion of the unregularized Green function

We write $\Phi(\mathfrak{a},m,s,z)$ in the form

$$\Phi(\mathfrak{a},m,s,z) = \sum_{b\in\mathbb{Z}}\Phi^b(\mathfrak{a},m,s,z)$$

with

$$\Phi^b(\mathfrak{a},m,s,z) := \sum_{\substack{A=\left(\begin{smallmatrix} a & \lambda' \\ \lambda & b\end{smallmatrix}\right)\in L(\mathfrak{a})^\vee \\ \det(A)=m/(N(\mathfrak{a})D)}} Q_{s-1}\left(1+2g(A,z)\right).$$

Note that the lower right entry of the elements of $L(\mathfrak{a})^\vee$ is actually integral by equation (2.12). Hence, it makes sense to sum over $b\in\mathbb{Z}$. By Remark 2.6.3 we have $\Phi^b(\mathfrak{a},m,s,z) = \Phi^{-b}(\mathfrak{a},m,s,z)$. Therefore, we can assume $b\in\mathbb{N}_0$. Individually, the functions $\Phi^b(\mathfrak{a},m,s,z)$ converge normally for $\Re(s)>1/2$. In particular, for $s=1$ they are well-defined. They are in general not invariant under the full group $\Gamma_{\mathfrak{a}}$ anymore, however Lemma 2.4.4 and Remark 2.6.3 imply that they are still invariant under $\Gamma_{\mathfrak{a},\infty}$. Hence, they are also defined in a neighborhood of the cusp ∞ of $X(\mathfrak{a})$. The invariance under $\Gamma_{\mathfrak{a},\infty}$ implies in particular an $\mathfrak{a}^{-1}$ periodicity. Therefore, they might be expressible as Fourier series.

Lemma 2.8.1 implies that for $b\in\mathbb{N}$ the function $\Phi^b(\mathfrak{a},m,s,z)$ has no singularity for arguments $z\in\mathbb{H}^2$ with $\Im(z)>m/(N(\mathfrak{a})Db^2)$. Together with the normal convergence this

implies that $\Phi^b(\mathfrak{a}, m, s, z)$ is continuous in this region and possesses an almost everywhere converging Fouier series. Later we will see that $\Phi^b(\mathfrak{a}, m, s, z)$ is actually smooth, hence the Fourier series converges everywhere on $\Im(z) > m/(N(\mathfrak{a})Db^2)$.

We treat the cases $b = 0$ and $b \in \mathbb{N}$ separately and start with the latter. Let us fix $b \in \mathbb{N}$ and abbreviate $B := m/(N(\mathfrak{a})Db^2)$. Then we get

$$\begin{aligned}\Phi^b(\mathfrak{a}, m, s, z) &= \sum_{\substack{a\in\mathbb{Z}/N(\mathfrak{a}),\, \lambda\in\mathfrak{a}\mathfrak{d}^{-1}/N(\mathfrak{a})\\ ab-N(\lambda)=m/(N(\mathfrak{a})D)}} Q_{s-1}\left(1+\frac{|bz_1z_2-\lambda z_1-\lambda' z_2+a|^2}{2y_1y_2m/(N(\mathfrak{a})D)}\right)\\ &= \sum_{\substack{a\in\mathbb{Z}/N(\mathfrak{a}),\, \lambda\in\mathfrak{a}\mathfrak{d}^{-1}/N(\mathfrak{a})\\ ab-N(\lambda)=m/(N(\mathfrak{a})D)}} Q_{s-1}\left(1+\frac{|(z_1-\lambda'/b)(z_2-\lambda/b)+B|^2}{2y_1y_2B}\right).\end{aligned}$$

The possible λ occurring in the sum index have a nice periodicity revealed by the next lemma.

Lemma 3.2.1. *For fixed $b \in \mathbb{N}$ we have*

$$\begin{aligned}&\left\{\lambda \in \mathfrak{a}\mathfrak{d}^{-1}/N(\mathfrak{a}) :\ \exists a \in \mathbb{Z}/N(\mathfrak{a}), ab - N(\lambda) = m/(N(\mathfrak{a})D)\right\}\\ &= \left\{\frac{\lambda+b\mu}{N(\mathfrak{a})} :\ \mu \in \mathfrak{a}, \lambda \in \mathfrak{a}\mathfrak{d}^{-1}, \frac{N(\sqrt{D}\lambda)}{N(\mathfrak{a})} \equiv m \pmod{bD}\right\}.\end{aligned}$$

Proof. Let λ be in the first set with $a \in \mathbb{Z}/N(\mathfrak{a})$ given. Then $\lambda N(\mathfrak{a}) \in \mathfrak{a}\mathfrak{d}^{-1}$. Taking $\mu = 0$, it is only left to show the congruence relation for proving the first inclusion:

$$\begin{aligned}\frac{N(\sqrt{D}\lambda N(\mathfrak{a}))}{N(\mathfrak{a})} &= -DN(\mathfrak{a})N(\lambda) = DN(\mathfrak{a})(m/(N(\mathfrak{a})D) - ab)\\ &= m - DN(\mathfrak{a})ab \equiv m \pmod{bD}.\end{aligned}$$

For the other inclusion we have to show that $\frac{\lambda+b\mu}{N(\mathfrak{a})}$ given from the second set satisfies

$$N\left(\frac{\lambda+b\mu}{N(\mathfrak{a})}\right) \in -\frac{m}{N(\mathfrak{a})D} + \frac{b}{N(\mathfrak{a})}\mathbb{Z}.$$

Let $c \in \mathbb{Z}$ with

$$\frac{N(\sqrt{D}\lambda)}{N(\mathfrak{a})} = m + cbD \quad\Leftrightarrow\quad N(\lambda) = -\frac{N(\mathfrak{a})m}{D} - N(\mathfrak{a})cb.$$

Now we compute

$$\begin{aligned}N\left(\frac{\lambda+b\mu}{N(\mathfrak{a})}\right) &= \frac{1}{N(\mathfrak{a})^2}\left(N(\lambda) + b\,\mathrm{tr}(\lambda'\mu) + b^2N(\mu)\right)\\ &= \frac{1}{N(\mathfrak{a})^2}\left(-\frac{N(\mathfrak{a})m}{D} - N(\mathfrak{a})cb + b\,\mathrm{tr}(\lambda'\mu) + b^2N(\mu)\right)\\ &= -\frac{m}{N(\mathfrak{a})D} + \frac{b}{N(\mathfrak{a})}\left(-c + \mathrm{tr}\left(\frac{\lambda'}{N(\mathfrak{a})}\mu\right) + b\frac{N(\mu)}{N(\mathfrak{a})}\right).\end{aligned}$$

We are left to show that the latter bracket is integral. With $\mu \in \mathfrak{a}$ we get $N(\mu) \in N(\mathfrak{a})\mathbb{Z}$. Further, by equation (2.7) we have

$$\frac{\lambda'}{N(\mathfrak{a})} \in \frac{\mathfrak{a}'\mathfrak{d}^{-1}}{N(\mathfrak{a})} = (\mathfrak{a}\mathfrak{d})^{-1} = \mathfrak{a}^{\vee_{\mathrm{tr}}}$$

which completes the proof. □

Let $R^b(\mathfrak{a}, m)$ be a set of representatives of

$$\left\{\lambda \in \mathfrak{a}\mathfrak{d}^{-1}/b\mathfrak{a} : \ \frac{N(\sqrt{D}\lambda)}{N(\mathfrak{a})} \equiv m \pmod{bD}\right\}.$$

Then we have

$$\begin{aligned}
\Phi^b(\mathfrak{a}, m, s, z) &= \sum_{\substack{a \in \mathbb{Z}/N(\mathfrak{a}),\, \lambda \in \mathfrak{a}\mathfrak{d}^{-1}/N(\mathfrak{a}) \\ ab - N(\lambda) = m/(N(\mathfrak{a})D)}} Q_{s-1}\left(1 + \frac{|(z_1 - \lambda'/b)(z_2 - \lambda/b) + B|^2}{2y_1y_2B}\right) \\
&= \sum_{\lambda \in R^b(\mathfrak{a},m)} \sum_{\mu \in \mathfrak{a}} Q_{s-1}\left(1 + \frac{\left|\left(z_1 - \frac{\lambda' + b\mu'}{N(\mathfrak{a})b}\right)\left(z_2 - \frac{\lambda + b\mu}{N(\mathfrak{a})b}\right) + B\right|^2}{2y_1y_2B}\right) \\
&= \sum_{\lambda \in R^b(\mathfrak{a},m)} \sum_{\mu \in \mathfrak{a}^{-1}} Q_{s-1}\left(1 + \frac{\left|\left(z_1 + \mu + \frac{\lambda'}{N(\mathfrak{a})b}\right)\left(z_2 + \mu' + \frac{\lambda}{N(\mathfrak{a})b}\right) + B\right|^2}{2y_1y_2B}\right).
\end{aligned}$$

Hence, the problem is deduced to computing the Fourier expansion of the $\mathfrak{a}^{-1}$ periodic function $H_s^B(\mathfrak{a}^{-1}, z)$ with

$$H_s^B(\mathfrak{b}, z) := \sum_{\mu \in \mathfrak{b}} Q_{s-1}\left(1 + \frac{|(z_1 + \mu)(z_2 + \mu') + B|^2}{2y_1y_2B}\right). \tag{3.4}$$

Namely, let

$$H_s^B(\mathfrak{b}, z) = \sum_{\nu \in (\mathfrak{b}\mathfrak{d})^{-1}} b_s^B(\mathfrak{b}, \nu, y) e(\mathrm{tr}(\nu x))$$

be the Fourier expansion of $H_s^B(\mathfrak{b}, z)$. Then we have

$$\begin{aligned}
\Phi^b(\mathfrak{a}, m, s, z) &= \sum_{\lambda \in R^b(\mathfrak{a},m)} \sum_{\nu \in \mathfrak{a}\mathfrak{d}^{-1}} b_s^B(\mathfrak{a}^{-1}, \nu, y) e\left(\mathrm{tr}\left(\nu\left(x + \tfrac{\lambda'}{N(\mathfrak{a})b}\right)\right)\right) \\
&= \sum_{\nu \in \mathfrak{a}\mathfrak{d}^{-1}} \left(\sum_{\lambda \in R^b(\mathfrak{a},m)} e\left(\mathrm{tr}\left(\tfrac{\nu\lambda'}{N(\mathfrak{a})b}\right)\right)\right) b_s^B(\mathfrak{a}^{-1}, \nu, y) e(\mathrm{tr}(\nu x)) \\
&= \sum_{\nu \in \mathfrak{a}\mathfrak{d}^{-1}} G^b(\mathfrak{a}, m, \nu) b_s^B(\mathfrak{a}^{-1}, \nu, y) e(\mathrm{tr}(\nu x))
\end{aligned}$$

with the finite exponential sum

$$G^b(\mathfrak{a}, m, \nu) := \sum_{\substack{\lambda \in \mathfrak{a}\mathfrak{d}^{-1}/b\mathfrak{a} \\ \frac{N(\lambda)}{N(\mathfrak{a})} \equiv -\frac{m}{D} \ (b\mathbb{Z})}} e\left(\operatorname{tr}\left(\frac{\nu\lambda'}{N(\mathfrak{a})b}\right)\right). \tag{3.5}$$

Remark 3.2.2. For fixed $\mathfrak{a} \in \mathcal{I}_K$ the exponential sum $G^b(\mathfrak{a}, m, \nu)$ is defined for $\nu \in \mathfrak{a}\mathfrak{d}^{-1}$. The relation

$$G^b(\mathfrak{a}, m, \nu) = G^b(\mu\mathfrak{a}, m, \mu\nu)$$

for all $\mu \in K^\times$ is directly apparent from the definition. This implies that the sum is essentially defined for ideal classes $[\mathfrak{a}] \in \mathrm{Cl}_K$.

Definition 3.2.3. For shorter notation we define for $\nu \in K^\times$

$$\mathcal{I}_\kappa^\nu(z) := \begin{cases} I_\kappa(z), & N(\nu) > 0, \\ J_\kappa(z), & N(\nu) < 0. \end{cases}$$

Here, $I_\kappa(z)$ and $J_\kappa(z)$ denote the respective Bessel functions, i.e., $I_\kappa(z)$ is the modified Bessel function of the first kind (cf. [OLBC10, 10.25.2]) and $J_\kappa(z)$ is the Bessel function of the first kind (cf. [OLBC10, 10.2.2]). By $K_\kappa(z)$ we denote the modified Bessel function of the second kind (cf. [OLBC10, 10.25.3]).

Lemma 3.2.4. *Let $\mathfrak{b} \in \mathcal{I}_K$ and $B > 0$. The function $H_s^B(\mathfrak{b}, z)$ defined by equation* (3.4) *converges normally for $\Re(s) > 1/2$ and for those $z \in \mathbb{H}^2$ at which no term in the series has a singularity, i.e., the arguments of all Q_{s-1} are greater than* 1*. For $y_1 y_2 > B$ this is the case and the series has the Fourier expansion*

$$H_s^B(\mathfrak{b}, z) = \sum_{\nu \in (\mathfrak{b}\mathfrak{d})^{-1}} b_s^B(\mathfrak{b}, \nu, y) e(\operatorname{tr}(\nu x))$$

with

$$\begin{aligned} b_s^B(\mathfrak{b}, 0, y) =& \frac{\pi\Gamma(s-1/2)^2}{2\sqrt{D}N(\mathfrak{b})\Gamma(2s)}(4B)^s(y_1y_2)^{1-s}, \\ b_s^B(\mathfrak{b}, \nu, y) =& \frac{4\pi}{N(\mathfrak{b})}\sqrt{\frac{By_1y_2}{D}}\mathcal{I}_{2s-1}^\nu\left(4\pi\sqrt{B|N(\nu)|}\right)K_{s-1/2}(2\pi|\nu|y_1) \\ & \times K_{s-1/2}(2\pi|\nu'|y_2), \quad \text{if } \nu \neq 0. \end{aligned}$$

Proof. An easy computation shows that for fixed $\mu \in K$ and fixed $y_1, y_2 > 0$ the equation

$$(z_1 + \mu)(z_2 + \mu') + B = 0$$

has a solution in terms of $x_1, x_2 \in \mathbb{R}$ if and only if $y_1 y_2 \leq B$. This shows that in case $y_1 y_2 > B$ all terms of the series are well-defined. Using [Bru99, Lemma 2] it is only left to note that by Poisson summation we have

$$b_s^B(\mathfrak{b}, \nu, y) = \frac{1}{\mathrm{vol}(\mathfrak{b})}\int_{\mathbb{R}^2} Q_{s-1}\left(1 + \frac{|z_1z_2 + B|^2}{2y_1y_2B}\right)e(-\operatorname{tr}(\nu x))dx_1dx_2$$

with $\mathrm{vol}(\mathfrak{b}) = N(\mathfrak{b})\sqrt{D}$. Hence, the integral is solved up to the factor $N(\mathfrak{b})$ in the cited source. $\square$

We are left with the analysis of $\Phi^0(\mathfrak{a}, m, s, z)$. We have

$$\begin{aligned}
\Phi^0(\mathfrak{a}, m, s, z) &= \sum_{\substack{A=\left(\begin{smallmatrix} a & \lambda' \\ \lambda & 0 \end{smallmatrix}\right)\in L(\mathfrak{a})^\vee \\ \det(A)=m/(N(\mathfrak{a})D)}} Q_{s-1}\left(1 + 2g(A, z)\right) \\
&= \sum_{\substack{a\in\mathbb{Z}/N(\mathfrak{a}),\ \lambda\in\mathfrak{a}\mathfrak{d}^{-1}/N(\mathfrak{a}) \\ -N(\lambda)=m/(N(\mathfrak{a})D)}} Q_{s-1}\left(1 + \frac{|-\lambda z_1 - \lambda' z_2 + a|^2}{2y_1y_2m/(N(\mathfrak{a})D)}\right) \\
&= 2\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\sum_{a\in\mathbb{Z}} Q_{s-1}\left(1 + \frac{|\lambda z_1 + \lambda' z_2 + a|^2}{2y_1y_2mN(\mathfrak{a})/D}\right).
\end{aligned}$$

Let us define for $r_1, r_2 \in \mathbb{R}$

$$\alpha(r_1, r_2) := \max(|r_1|, |r_2|) \quad \text{and} \quad \beta(r_1, r_2) := \min(|r_1|, |r_2|). \tag{3.6}$$

Lemma 3.2.5. *The series*

$$\Phi^0(\mathfrak{a}, m, s, z) = 2\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\sum_{a\in\mathbb{Z}} Q_{s-1}\left(1 + \frac{|\lambda z_1 + \lambda' z_2 + a|^2}{2y_1y_2mN(\mathfrak{a})/D}\right)$$

converges normally for $z \in \mathbb{H}^2 \setminus T^\infty(\mathfrak{a}, m)$ and $\Re(s) > 1/2$. Moreover, on $\mathbb{H}^2 \setminus S(\mathfrak{a}, m)$ one has the Fourier expansion

$$\begin{aligned}
\Phi^0(\mathfrak{a}, m, s, z) = \frac{4\pi}{2s-1}\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)} &\alpha(\lambda y_1, \lambda' y_2)^{1-s}\beta(\lambda y_1, \lambda' y_2)^s \\
+ 4\pi\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\sum_{n=1}^{\infty} &\sqrt{|\lambda\lambda' y_1 y_2|}\, I_{s-1/2}(2\pi n\beta(\lambda y_1, \lambda' y_2)) \\
\times\ K_{s-1/2}(2\pi n\alpha(\lambda y_1, \lambda' y_2))&\left(e(n\,\mathrm{tr}(\lambda x)) + e(-n\,\mathrm{tr}(\lambda x))\right).
\end{aligned}$$

Proof. Clearly $z \in T^\infty(\mathfrak{a}, m)$ is equivalent to having one term in the series undefined (cf. representation (2.44) of $T^\infty(\mathfrak{a}, m)$). If all terms in the series are well-defined, the convergence statement can be easily verified. Let $\lambda \in \Lambda^+(\mathfrak{a}, m)$ be fixed. Then we have

$$\begin{aligned}
&\sum_{a\in\mathbb{Z}} Q_{s-1}\left(1 + \frac{|\lambda z_1 + \lambda' z_2 + a|^2}{2y_1y_2mN(\mathfrak{a})/D}\right) \\
=&\sum_{a\in\mathbb{Z}} Q_{s-1}\left(1 + \frac{(\lambda x_1 + \lambda' x_2 + a)^2 + (\lambda y_1 + \lambda' y_2)^2}{2y_1y_2|\lambda\lambda'|}\right) \\
=&\sum_{a\in\mathbb{Z}} Q_{s-1}\left(\frac{(\lambda x_1 + \lambda' x_2 + a)^2 + \lambda^2 y_1^2 + \lambda'^2 y_2^2}{2y_1y_2|\lambda\lambda'|}\right) \\
=&h_{\alpha(\lambda y_1, \lambda' y_2), \beta(\lambda y_1, \lambda' y_2)}(s, \mathrm{tr}(\lambda x))
\end{aligned}$$

with

$$h_{\alpha,\beta}(s,x) := \sum_{a\in\mathbb{Z}} Q_{s-1}\left(\frac{(x+a)^2+\alpha^2+\beta^2}{2\alpha\beta}\right).$$

By [Bru99, proof of Lemma 1] the $\mathbb{Z}$ periodic function $h_{\alpha,\beta}(s,x)$ has for $\alpha > \beta > 0$ and $\Re(s) > 1/2$ the Fourier expansion

$$h_{\alpha,\beta}(s,x) = \sum_{n\in\mathbb{Z}} a_{\alpha,\beta}(s,n)e(nx)$$

with

$$a_{\alpha,\beta}(s,n) = \begin{cases} \frac{2\pi}{2s-1}\alpha^{1-s}\beta^s, & n=0,\\ 2\pi\sqrt{\alpha\beta}K_{s-1/2}(2\pi|n|\alpha)I_{s-1/2}(2\pi|n|\beta), & n\neq 0.\end{cases}$$

This finishes the proof. Note that $\alpha(\lambda y_1, \lambda' y_2) > \beta(\lambda y_1, \lambda' y_2)$ is equivalent to $z \in \mathbb{H}^2$ not being in the component S_λ of $S(\mathfrak{a}, m)$. □

We summarize the results of this section in the following theorem.

Theorem 3.2.6. *The Fourier expansion of $\Phi(\mathfrak{a}, m, s, z)$ is given by*

$$\sum_{\nu\in\mathfrak{a}\mathfrak{d}^{-1}} u_\nu(\mathfrak{a}, m, s, y)e(\mathrm{tr}(\nu x))$$

with

$$\begin{aligned} u_0(\mathfrak{a}, m, s, y) =& \frac{4\pi}{2s-1}\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)} \alpha(\lambda y_1, \lambda' y_2)^{1-s}\beta(\lambda y_1, \lambda' y_2)^s \\ &+ \frac{\pi\Gamma(s-1/2)^2}{\sqrt{D}\Gamma(2s)}(4m/D)^s\,(N(\mathfrak{a})y_1y_2)^{1-s}\sum_{b=1}^{\infty} G^b(\mathfrak{a}, m, 0)b^{-2s} \end{aligned}$$

and

$$\begin{aligned} u_\nu(\mathfrak{a}, m, s, y) =& \tilde{u}_\nu(\mathfrak{a}, m, s, y) + \delta_{-\frac{N(\nu)D}{N(\mathfrak{a})m}=\square} 4\pi\sqrt{mN(\mathfrak{a})y_1y_2/D} \\ &\times I_{s-1/2}(2\pi\beta(\nu y_1, \nu' y_2))K_{s-1/2}(2\pi\alpha(\nu y_1, \nu' y_2)) \end{aligned}$$

with

$$\begin{aligned} \tilde{u}_\nu(\mathfrak{a}, m, s, y) :=& \frac{8\pi}{D}\sqrt{mN(\mathfrak{a})y_1y_2}K_{s-1/2}(2\pi|\nu|y_1)K_{s-1/2}(2\pi|\nu'|y_2) \\ &\times \sum_{b=1}^{\infty}\frac{G^b(\mathfrak{a}, m, \nu)}{b}\mathcal{I}^{\nu}_{2s-1}\left(\frac{4\pi}{b}\sqrt{\frac{m|N(\nu)|}{N(\mathfrak{a})D}}\right) \end{aligned}$$

for $\nu \neq 0$. Note that $u_\nu(\mathfrak{a}, m, s, y) = \tilde{u}_\nu(\mathfrak{a}, m, s, y)$ if and only if $-\frac{N(\nu)D}{N(\mathfrak{a})m}$ is not a square number which is only possible for $N(\nu) < 0$. The Fourier series converges to $\Phi(\mathfrak{a}, m, s, z)$ for $\Re(s) > 1$, $z \in \mathbb{H}^2 \setminus S(\mathfrak{a}, m)$ and $y_1y_2 > m/(N(\mathfrak{a})D)$.

3.3 Regularization

In this section we regularize $\Phi(\mathfrak{a}, m, s, z)$ at $s = 1$ and compute the Fourier expansion of the regularization.

In the previous section we have decomposed $\Phi(\mathfrak{a}, m, s, z)$ into

$$\Phi(\mathfrak{a}, m, s, z) = \Phi^0(\mathfrak{a}, m, s, z) + 2\sum_{b=1}^{\infty} \Phi^b(\mathfrak{a}, m, s, z)$$

and computed the Fourier expansion for each $\Phi^b(\mathfrak{a}, m, s, z)$. We saw that they are convergent and well-defined for $\Re(s) > 1/2$, hence in particular for $s = 1$. Therefore, the convergence issue of $\Phi(\mathfrak{a}, m, s, z)$ at $s = 1$ is not related to a particular $\Phi^b(\mathfrak{a}, m, s, z)$ but arises solely from their infinite sum. The Fourier expansions of the $\Phi^b(\mathfrak{a}, m, s, z)$ contain finite exponential sums $G^b(\mathfrak{a}, m, \nu)$ (cf. equation (3.5) for their definition). In order to understand which parts of the Fourier expansion are mild and which cause the divergence at $s = 1$ we need to understand the growth behaviour of $G^b(\mathfrak{a}, m, \nu)$ for growing $b \in \mathbb{N}$. It turns out that we have to distinguish between $\nu \in \mathfrak{a}\mathfrak{d}^{-1} \setminus \{0\}$ and $\nu = 0$. By generalizing the results of Zagier in [Zag75, §4 Proposition] to arbitrary $\mathfrak{a} \in \mathcal{I}_K$, we can infer the following two lemmata as presented in [Bru99, p. 65–66] which answer the two cases respectively.

Lemma 3.3.1. *For each $\mathfrak{a} \in \mathcal{I}_K$ there exists a constant $C > 0$ such that*

$$|G^b(\mathfrak{a}, m, \nu)| \leq Cd(b)\sqrt{b|N(\nu)|}$$

for all $m \in \mathbb{Z}$, $\nu \in \mathfrak{a}\mathfrak{d}^{-1} \setminus \{0\}$ and $b \in \mathbb{N}$.

Lemma 3.3.2. *The series*

$$\sum_{b=1}^{\infty} G^b(\mathfrak{a}, m, 0) b^{-2s}$$

converges for $\Re(s) > 1$ and has a meromorphic continuation to $\Re(s) > 3/4$ with a simple pole at $s = 1$.

Proposition 3.3.3. *Let $b_0 \in \mathbb{N}$. Then the series*

$$\sum_{\substack{\nu \in \mathfrak{a}\mathfrak{d}^{-1} \\ \nu \neq 0}} \tilde{u}_\nu^{b_0}(\mathfrak{a}, m, s, y) e(\mathrm{tr}(\nu x))$$

with

$$\begin{aligned}\tilde{u}_\nu^{b_0}(\mathfrak{a}, m, s, y) :=& \frac{8\pi}{D}\sqrt{mN(\mathfrak{a})y_1y_2}K_{s-1/2}(2\pi|\nu|y_1)K_{s-1/2}(2\pi|\nu'|y_2) \\ &\times \sum_{b=b_0}^{\infty} \frac{G^b(\mathfrak{a}, m, \nu)}{b} \mathcal{I}_{2s-1}^{\nu}\left(\frac{4\pi}{b}\sqrt{\frac{m|N(\nu)|}{N(\mathfrak{a})D}}\right)\end{aligned}$$

converges for $\Im(z) > m/(DN(\mathfrak{a})b_0^2)$ and $\Re(s) > 3/4$ normally to a smooth function in z and a holomorphic function in s.

Proof. We first show that $\tilde{u}_\nu^{b_0}(\mathfrak{a}, m, s, y)$ is well-defined for $\Re(s) > 3/4$ by showing that

$$\sum_{b=b_0}^{\infty} \frac{G^b(\mathfrak{a}, m, \nu)}{b} \mathcal{I}_{2s-1}^{\nu}\left(\frac{4\pi}{b}\sqrt{\frac{m|N(\nu)|}{N(\mathfrak{a})D}}\right)$$

converges for fixed $\nu \neq 0$. For large b the argument of the Bessel function approaches 0. By [OLBC10, 10.30.1 and 10.7.3] we have

$$\mathcal{I}_\kappa^\nu \sim \frac{(z/2)^\kappa}{\Gamma(\kappa+1)}$$

for $\kappa \notin -\mathbb{N}$ and $z \to 0$. Hence, the Bessel factor behaves like

$$\frac{1}{\Gamma(2s)}\left(\frac{2\pi}{b}\sqrt{\frac{m|N(\nu)|}{N(\mathfrak{a})D}}\right)^{2s-1}$$

for large b. With Lemma 3.3.1 there exists a $C > 0$ with

$$\left|\frac{G^b(\mathfrak{a}, m, \nu)}{b}\right| \leq Cd(b)\sqrt{|N(\nu)|/b}.$$

Therefore, in total it is enough to investigate the convergence of

$$\begin{aligned}&\sum_{b=b_0}^{\infty} Cd(b)\sqrt{|N(\nu)|/b}\frac{1}{\Gamma(2s)}\left(\frac{2\pi}{b}\sqrt{\frac{m|N(\nu)|}{N(\mathfrak{a})D}}\right)^{2s-1}\\ =&\frac{C(2\pi)^{2s-1}}{\Gamma(2s)}\left(\frac{m}{N(\mathfrak{a})D}\right)^{s-1/2}|N(\nu)|^s\sum_{b=1}^{\infty} d(b)b^{1/2-2s}.\end{aligned}$$

Clearly, the series $\sum_{b=b_0}^{\infty} d(b)b^{1/2-2s}$ converges if and only if $\Re(s) > 3/4$ because $d(b)$ grows slower than b^ε for all $\varepsilon > 0$.

The above argument showed that $\tilde{u}_\nu^{b_0}(\mathfrak{a}, m, s, y)$ converges for fixed ν. Now we have to show that $\tilde{u}_\nu^{b_0}(\mathfrak{a}, m, s, y)$ decays faster than any polynomial for growing (ν, ν'). This implies by Fourier theory the convergence and smoothness of the above series in x. The smoothness in y follows since our next argument also holds for all derivatives in y. The normal convergence in s implies the holomorphicity.

By [OLBC10, 10.25.3] we have

$$K_\kappa(z) \sim \sqrt{\frac{\pi}{2z}}\exp(-z)$$

for $z \to \infty$. Hence, for large (ν, ν') we have

$$K_{s-1/2}(2\pi|\nu|y_1)K_{s-1/2}(2\pi|\nu'|y_2) \sim \frac{\exp(-2\pi\operatorname{tr}(|\nu|y))}{4\sqrt{|N(\nu)y_1y_2|}}$$

which gives us an exponential decay in (ν, ν'). However, the estimate we did above for

$$\sum_{b=b_0}^{\infty} \frac{G^b(\mathfrak{a}, m, \nu)}{b} I_{2s-1}\left(\frac{4\pi}{b}\sqrt{\frac{m|N(\nu)|}{N(\mathfrak{a})D}}\right)$$

cannot be applied in this situation since we fixed ν above and we were only interested in convergence (so we could ignore the magnitude of finitely many terms with a relatively small b). However, if the argument of the I_κ Bessel function is large, it has exponential growth

$$I_\kappa(z) \sim \frac{\exp(z)}{\sqrt{2\pi z}}$$

for $z \to \infty$ (cf. [OLBC10, 10.30.4]). The magnitude of the whole sum over b is hence determined by the first term. We neglect the factors decorating the exponentials since the latter themselves dictate the growth or decay behavior. Therefore, we consider

$$\exp(-2\pi \operatorname{tr}(|\nu|y)) \exp\left(\frac{4\pi}{b_0}\sqrt{\frac{m|N(\nu)|}{N(\mathfrak{a})D}}\right).$$

Now let

$$c := \sqrt{\frac{m}{DN(\mathfrak{a})b_0^2 y_1 y_2}}.$$

Then we have

$$\begin{aligned}
&\exp\left(-2\pi \operatorname{tr}(|\nu|y) + \frac{4\pi}{b_0}\sqrt{\frac{m|N(\nu)|}{N(\mathfrak{a})D}}\right) \\
&= \exp\left(-2\pi\left(|\nu|y_1 + |\nu'|y_2 - 2c\sqrt{|N(\nu)|y_1 y_2}\right)\right) \\
&= \exp(-2\pi(1-c)\operatorname{tr}(|\nu|y)) \exp\left(-2\pi c\left(\sqrt{|\nu|y_1} - \sqrt{|\nu'|y_2}\right)^2\right) \\
&\leq \exp(-2\pi(1-c)\operatorname{tr}(|\nu|y)).
\end{aligned}$$

In case $c < 1$, which is equivalent to the condition $\Im(z) > m/(DN(\mathfrak{a})b_0^2)$ in the statement of the proposition, this gives us the exponential decay in (ν, ν') we aimed for.

The respective treatment of the J_κ Bessel function in case $N(\nu) < 0$ is no problem since the J_κ Bessel function is bounded. □

Theorem 3.3.4. *The function $\Phi(\mathfrak{a}, m, s, z)$ has a meromorphic continuation in s to $\{s \in \mathbb{C} : \Re(s) > 3/4\}$ for all $z \in \mathbb{H}^2 \setminus T(\mathfrak{a}, m)$. Up to a simple pole at $s = 1$ it is holomorphic in this domain.*

Proof. First of all, let $b \in \mathbb{N}_0$. Recall that $\Phi^b(\mathfrak{a}, m, s, z)$ is holomorphic in s for $\Re(s) > 1/2$ for $z \in \mathbb{H}^2$ outside the singularities along T_A with $A \in L(\mathfrak{a})^\vee$ and $\det(A) = m/(N(\mathfrak{a})D)$ and $\pm b$ being the lower right entry of A. Therefore, we do not have to care about

finitely many $\Phi^b(\mathfrak{a}, m, s, z)$. Let $z \in \mathbb{H}^2 \setminus T(\mathfrak{a}, m)$ be fixed. Then we find $b_0 \in \mathbb{N}$ with $\Im(z) > m/(DN(\mathfrak{a}b_0^2))$. Proposition 3.3.3 implies that the underlying Fourier series of

$$\sum_{b=b_0}^{\infty} \Phi^b(\mathfrak{a}, m, s, z)$$

converges normally for $\Re(s) > 3/4$ to a holomorphic function in s when the constant Fourier coefficients are neglected. Hence, we are left with the analysis of the series of the constant Fourier coefficients

$$\frac{\pi\Gamma(s-1/2)^2}{\sqrt{D}\Gamma(2s)} (4m/D)^s (N(\mathfrak{a})y_1y_2)^{1-s} \sum_{b=b_0}^{\infty} G^b(\mathfrak{a}, m, 0)b^{-2s}.$$

The factors in front of the G^b series are nicely holomorphic in s and the G^b series itself is treated in Lemma 3.3.2 which finishes the proof. □

Theorem 3.3.4 allows us now to define the regularized automorphic Green function $\Phi(\mathfrak{a}, m, z)$.

Definition 3.3.5. We define

$$\Phi(\mathfrak{a}, m, z) := \mathcal{C}_{s=1} \left[\Phi(\mathfrak{a}, m, s, z)\right]$$

to be the constant term in the Laurent expansion of $\Phi(\mathfrak{a}, m, s, z)$ at $s = 1$.

By construction $\Phi(\mathfrak{a}, m, z)$ is $\Gamma_{\mathfrak{a}}$ invariant and we expect logarithmic singularities along $-T(\mathfrak{a}, m)$. A formal proof of the latter is given in Proposition 3.6.1.

3.4 Fourier expansion of the regularized Green function

In this section we state the Fourier expansion of the regularized Green function $\Phi(\mathfrak{a}, m, z)$. This follows straightforward from Theorem 3.2.6. Subsequently, we do some refactoring of the parts essential for the growth behavior at the cusp ∞.

Theorem 3.4.1. *The residue of $\Phi(\mathfrak{a}, m, s, z)$ at $s = 1$ is independent of z. Hence, we have*

$$\Phi(\mathfrak{a}, m, z) = \lim_{s\to 1} \left(\Phi(\mathfrak{a}, m, s, z) - \frac{q(\mathfrak{a}, m)}{s-1}\right)$$

with

$$q(\mathfrak{a}, m) := \operatorname{res}_{s=1} \left(\Phi(\mathfrak{a}, m, s, z)\right).$$

Furthermore, there exists a constant $L(\mathfrak{a}, m)$ such that the Fourier expansion of $\Phi(\mathfrak{a}, m, z)$ is given for $z \in \mathbb{H}^2 \setminus S(\mathfrak{a}, m)$ with $\Im(z) > m/(DN(\mathfrak{a}))$ by

$$\begin{aligned}
&\Phi(\mathfrak{a}, m, z) = L(\mathfrak{a}, m) - q(\mathfrak{a}, m)\log(16\pi^2 y_1 y_2) \\
&+ 4\pi \sum_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \beta(\lambda y_1, \lambda' y_2) \\
&+ \sum_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \sum_{n=1}^{\infty} \frac{e^{-2\pi n|\operatorname{tr}(\lambda y)|} - e^{-2\pi n(\lambda y_1 - \lambda' y_2)}}{n} \left(e(n\operatorname{tr}(\lambda x)) + e(-n\operatorname{tr}(\lambda x))\right) \\
&+ \sum_{\substack{\nu \in \mathfrak{a}\mathfrak{d}^{-1} \\ \nu \neq 0}} \frac{2\pi}{D}\sqrt{\frac{mN(\mathfrak{a})}{|N(\nu)|}} \exp(-2\pi \operatorname{tr}(|\nu| y)) \\
&\times \sum_{b=1}^{\infty} \frac{G^b(\mathfrak{a}, m, \nu)}{b} \mathcal{I}_1^{\nu}\left(\frac{4\pi}{b}\sqrt{\frac{m|N(\nu)|}{N(\mathfrak{a})D}}\right) e(\operatorname{tr}(\nu x)).
\end{aligned}$$

Remark 3.4.2. The residue $q(\mathfrak{a}, m)$ of $\Phi(\mathfrak{a}, m, s, z)$ at $s = 1$ can be made explicit in terms of a generalized divisor sum. This is carried out in the special case of a prime discriminant D with $\mathfrak{a} = \mathcal{O}_K$ in [BBGK07, Section 2.3]. We discuss the results of that special case in Section 3.9. It follows $q(\mathfrak{a}, m) = O(m^2)$. Respectively, the constant $L(\mathfrak{a}, m)$ can be expressed using a generalized divisor sum and its derivative. It follows $L(\mathfrak{a}, m) = O(m^2 \log(m))$. Later, in Theorem 3.8.11 we show that $q(\mathfrak{a}, m)$ is proportional to the volume of $T(\mathfrak{a}, m)$ which then gives explicit formulae for $\operatorname{vol}(T(\mathfrak{a}, m))$. This again implies that $q(\mathfrak{a}, m)$ grows polynomially in m because the volumes can be interpreted as coefficents of vector valued Eisenstein series of weight 2 by [Kud03, Theorem I].

Proof of Theorem 3.4.1. The theorem follows from Theorem 3.2.6 by plugging in $s = 1$ for those terms which converge and determining the constant term of the Laurent expansion for the diverging part. We do this now step by step and start with the diverging series

$$\frac{\pi\Gamma(s - 1/2)^2}{\sqrt{D}\Gamma(2s)} (4m/D)^s (N(\mathfrak{a})y_1 y_2)^{1-s} \sum_{b=1}^{\infty} G^b(\mathfrak{a}, m, 0) b^{-2s}. \tag{3.7}$$

This expression can be written by $f(s)(y_1 y_2)^{1-s}$ with a meromorphic function $f(s)$ independent of z having a simple pole at $s = 1$ (cf. Lemma 3.3.2). Therefore, we have

$$q(\mathfrak{a}, m) = \operatorname{res}_{s=1}(\Phi(\mathfrak{a}, m, s, z)) = \operatorname{res}_{s=1}\left(f(s)(y_1 y_2)^{1-s}\right) = \operatorname{res}_{s=1}(f(s)).$$

Now, the constant term of $f(s)(y_1 y_2)^{1-s}$ computes to

$$\begin{aligned}
&\mathcal{C}_{s=1}[f(s)]\left[(y_1 y_2)^{1-s}\right]_{s=1} + \operatorname{res}_{s=1}[f(s)]\left[-\log(y_1 y_2)(y_1 y_2)^{1-s}\right]_{s=1} \\
&= \mathcal{C}_{s=1}[f(s)] - q(\mathfrak{a}, m)\log(y_1 y_2) \\
&= \underbrace{\mathcal{C}_{s=1}[f(s)] + q(\mathfrak{a}, m)\log(16\pi^2)}_{=:L(\mathfrak{a}, m)} - q(\mathfrak{a}, m)\log(16\pi^2 y_1 y_2).
\end{aligned}$$

This explains the part $L(\mathfrak{a},m)-q(\mathfrak{a},m)\log(16\pi^2y_1y_2)$. The sum over $\beta(\lambda y_1,\lambda' y_2)$ follows directly from the left over part of $u_0(\mathfrak{a},m,s,y)$.

The next term has its origin from the non-constant Fourier coefficients of $\Phi^0(\mathfrak{a},m,1,z)$. For general s they are computed in Lemma 3.2.5. By [OLBC10, 10.39.1 and 10.39.2] we have

$$I_{1/2}(z)=\sqrt{\frac{2}{\pi z}}\sinh(z)=\frac{e^z-e^{-z}}{\sqrt{2\pi z}}\quad\text{and}\quad K_{1/2}(z)=\sqrt{\frac{\pi}{2z}}\exp(-z).$$

Let $\lambda\in\Lambda^+(\mathfrak{a},m)$ and $n\in\mathbb{N}$ be fixed. We abbreviate $\alpha(\lambda y_1,\lambda' y_2)$ with α and $\beta(\lambda y_1,\lambda' y_2)$ with β and compute for $s=1$

$$\begin{aligned}&I_{s-1/2}(2\pi n\beta(\lambda y_1,\lambda' y_2))K_{s-1/2}(2\pi n\alpha(\lambda y_1,\lambda' y_2))\\ =&\frac{\exp(2\pi n\beta)-\exp(-2\pi n\beta)}{\sqrt{4\pi^2 n\beta}}\cdot\frac{\exp(-2\pi n\alpha)}{\sqrt{4n\alpha}}\\ =&\frac{\exp(-2\pi n(\alpha-\beta))-\exp(-2\pi n(\alpha+\beta)}{4\pi n\sqrt{\alpha\beta}}.\end{aligned}$$

Note that

$$\alpha\beta=|\lambda\lambda' y_1y_2|,\quad \alpha+\beta=\lambda y_1-\lambda' y_2,\quad \alpha-\beta=|\lambda y_1+\lambda' y_2|=|\operatorname{tr}(\lambda y)|.$$

Therefore, we have

$$\begin{aligned}&4\pi\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\sum_{n=1}^{\infty}\sqrt{|\lambda\lambda' y_1y_2|}I_{s-1/2}(2\pi n\beta(\lambda y_1,\lambda' y_2))\\ \times\ &K_{s-1/2}(2\pi n\alpha(\lambda y_1,\lambda' y_2))\left(e(n\operatorname{tr}(\lambda x))+e(-n\operatorname{tr}(\lambda x))\right)\\ =&\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\sum_{n=1}^{\infty}\frac{e^{-2\pi n|\operatorname{tr}(\lambda y)|}-e^{-2\pi n(\lambda y_1-\lambda' y_2)}}{n}\left(e(n\operatorname{tr}(\lambda x))+e(-n\operatorname{tr}(\lambda x))\right).\end{aligned}$$

What is left over are the contributions of the non-constant Fourier coefficients for $\Phi^b(\mathfrak{a},m,1,z)$ with $b\in\mathbb{N}$. We use the above identity of $K_{1/2}(z)$ one more time and see

$$K_{1/2}(2\pi|\nu|y_1)K_{1/2}(2\pi|\nu'|y_2)=\frac{\exp(-2\pi\operatorname{tr}(|\nu|y))}{4\sqrt{|N(\nu)|y_1y_2}}.$$

Therefore, we have

$$\begin{aligned}\tilde{u}_\nu(\mathfrak{a},m,1,y)=&\frac{2\pi}{D}\sqrt{\frac{mN(\mathfrak{a})}{|N(\nu)|}}\exp(-2\pi\operatorname{tr}(|\nu|y))\\ &\times\sum_{b=1}^{\infty}\frac{G^b(\mathfrak{a},m,\nu)}{b}\mathcal{I}_1^\nu\left(\frac{4\pi}{b}\sqrt{\frac{m|N(\nu)|}{N(\mathfrak{a})D}}\right)\end{aligned}$$

which finishes the proof. □

Lemma 3.4.3. *We have for $z \in \mathbb{H}^2 \setminus S(\mathfrak{a}, m)$*

$$\sum_{\lambda \in \Lambda^+(\mathfrak{a},m)} \sum_{n=1}^{\infty} \frac{e^{-2\pi n|\operatorname{tr}(\lambda y)|} - e^{-2\pi n(\lambda y_1 - \lambda' y_2)}}{n} \left(e(n \operatorname{tr}(\lambda x)) + e(-n \operatorname{tr}(\lambda x))\right)$$
$$= -4\pi \sum_{\lambda \in \Lambda^+(\mathfrak{a},m)} \beta(\lambda y_1, \lambda' y_2) + 2 \log \prod_{\lambda \in \Lambda^+(\mathfrak{a},m)} \left| \frac{1 - e(|\lambda| z_1)\overline{e(|\lambda'| z_2)}}{e(|\lambda| z_1) - e(|\lambda'| z_2)} \right|.$$

Proof. We compute the inner sum on the left hand side for fixed $\lambda \in \Lambda^+(\mathfrak{a}, m)$. Because the fraction is real and $\overline{e(n \operatorname{tr}(\lambda x))} = e(-n \operatorname{tr}(\lambda x))$, the value of the whole sum is twice the real part of half of the sum (considering only one factor of the two conjugate factors). Recall the power series of the logarithm

$$\sum_{n=1}^{\infty} \frac{q^n}{n} = -\log(1 - q)$$

for $|q| < 1$. Let us assume $\operatorname{tr}(\lambda y) > 0$ first (we can exclude $\operatorname{tr}(\lambda y) = 0$ because that is equivalent to $z \in S_\lambda \subset S(\mathfrak{a}, m)$). Then we have

$$\sum_{n=1}^{\infty} \frac{e^{-2\pi n|\operatorname{tr}(\lambda y)|}}{n} e(n \operatorname{tr}(\lambda x)) = \sum_{n=1}^{\infty} \frac{e(n \operatorname{tr}(\lambda z))}{n} = -\log(1 - e(\operatorname{tr}(\lambda z))).$$

Note that $\log(|e(\tilde{z})|) = -2\pi \tilde{y}$ for $\tilde{z} \in \mathbb{C}$. Twice the real part is hence

$$\begin{aligned} -2\Re(\log(1 - e(\operatorname{tr}(\lambda z)))) &= -2 \log(|1 - e(\lambda z_1) e(\lambda' z_2)|) \\ &= -2 \log(|e(\lambda' z_2)| \cdot |e(-\lambda' z_2) - e(\lambda z_1)|) \\ &= 4\pi \lambda' y_2 - 2 \log(|e(|\lambda| z_1) - e(|\lambda'| z_2)|). \end{aligned}$$

Now let us consider $\operatorname{tr}(\lambda y) < 0$. Then we have

$$\sum_{n=1}^{\infty} \frac{e^{-2\pi n|\operatorname{tr}(\lambda y)|}}{n} e(-n \operatorname{tr}(\lambda x)) = \sum_{n=1}^{\infty} \frac{e(-n \operatorname{tr}(\lambda z))}{n} = -\log(1 - e(-\operatorname{tr}(\lambda z))).$$

Twice the real part is now

$$\begin{aligned} -2 \log(|1 - e(-\operatorname{tr}(\lambda z))|) &= -2 \log(|e(-\lambda z_1)| \cdot |e(\lambda z_1) - e(-\lambda' z_2)|) \\ &= -4\pi \lambda y_1 - 2 \log(|e(|\lambda| z_1) - e(|\lambda'| z_2)|). \end{aligned}$$

In total we have seen

$$\sum_{n=1}^{\infty} \frac{e^{-2\pi n|\operatorname{tr}(\lambda y)|}}{n} \left(e(n \operatorname{tr}(\lambda x)) + e(-n \operatorname{tr}(\lambda x))\right)$$
$$= -4\pi \beta(\lambda y_1, \lambda' y_2) - 2 \log(|e(|\lambda| z_1) - e(|\lambda'| z_2)|).$$

Let us now consider the missing part

$$\sum_{n=1}^{\infty} \frac{-e^{-2\pi n(\lambda y_1 - \lambda' y_2)}}{n} \left(e(n\operatorname{tr}(\lambda x)) + e(-n\operatorname{tr}(\lambda x))\right).$$

We have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{-e^{-2\pi n(\lambda y_1 - \lambda' y_2)}}{n} e(n\operatorname{tr}(\lambda x)) &= \sum_{n=1}^{\infty} \frac{-e(n\lambda z_1)e(n\lambda'\overline{z_2})}{n} \\ &= \log(1 - e(\lambda z_1)e(\lambda'\overline{z_2})) \\ &= \log(1 - e(\lambda z_1)\overline{e(-\lambda' z_2)}).\end{aligned}$$

Twice the real part is now

$$2\log(|1 - e(|\lambda| z_1)\overline{e(|\lambda'| z_2)}|).$$

The statement of the lemma follows now from the functional equation of the logarithm. □

Lemma 3.4.3 gives rise to the following simplification of Theorem 3.4.1.

Theorem 3.4.4. *The Green function $\Phi(\mathfrak{a}, m, z)$ is given for $z \in \mathbb{H}^2 \setminus S(\mathfrak{a}, m)$ with $\Im(z) > m/(DN(\mathfrak{a}))$ by*

$$\begin{aligned}\Phi(\mathfrak{a}, m, z) = {} & L(\mathfrak{a}, m) - q(\mathfrak{a}, m)\log(16\pi^2 y_1 y_2) \\ & + 2\log \prod_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \left| \frac{1 - e(|\lambda| z_1)\overline{e(|\lambda'| z_2)}}{e(|\lambda| z_1) - e(|\lambda'| z_2)} \right| \\ & + \sum_{\substack{\nu \in \mathfrak{a}\mathfrak{d}^{-1} \\ \nu \gg 0}} \frac{2\pi}{D} \sqrt{\frac{mN(\mathfrak{a})}{|N(\nu)|}} \sum_{b=1}^{\infty} \frac{G^b(\mathfrak{a}, m, \nu)}{b} I_1\left(\frac{4\pi}{b}\sqrt{\frac{m|N(\nu)|}{N(\mathfrak{a})D}}\right) \\ & \times \left(e(\operatorname{tr}(\nu z)) + \overline{e(\operatorname{tr}(\nu z))}\right) \\ & + \sum_{\substack{\nu \in \mathfrak{a}\mathfrak{d}^{-1} \\ \nu > 0,\, \nu' < 0}} \frac{2\pi}{D} \sqrt{\frac{mN(\mathfrak{a})}{|N(\nu)|}} \sum_{b=1}^{\infty} \frac{G^b(\mathfrak{a}, m, \nu)}{b} J_1\left(\frac{4\pi}{b}\sqrt{\frac{m|N(\nu)|}{N(\mathfrak{a})D}}\right) \\ & \times \left(e(\nu z_1)\overline{e(-\nu' z_2)} + \overline{e(\nu z_1)}e(-\nu' z_2)\right).\end{aligned}$$

Proof. Starting from Theorem 3.4.1, the main work was done in Lemma 3.4.3. For the different notation of the exponentials in the last lines verify for $\nu \in K^\times$

$$e(\operatorname{tr}(\nu x))e(i\operatorname{tr}(|\nu| y)) = \begin{cases} e(\operatorname{tr}(\nu z)), & \nu \gg 0, \\ \overline{e(-\operatorname{tr}(\nu z))}, & \nu \ll 0, \\ e(\nu z_1)\overline{e(-\nu' z_2)}, & \nu > 0,\ \nu' < 0, \\ \overline{e(-\nu z_1)}e(\nu' z_2), & \nu < 0,\ \nu' > 0. \end{cases}$$

Finally, by definition (3.5) of the exponential sum $G^b(\mathfrak{a}, m, \nu)$ we have

$$G^b(\mathfrak{a}, m, \nu) = G^b(\mathfrak{a}, m, -\nu)$$

since the index set of the sum is invariant under multiplication with -1. □

Proposition 3.4.5. *The regularized Green function $\Phi(\mathfrak{a}, m, z)$ is real analytic and satisfies for $j \in \{1, 2\}$*

$$\Delta_j \Phi(\mathfrak{a}, m, z) = q(\mathfrak{a}, m).$$

Proof. We want to give two independent arguments for the Laplace equation. The first one considers equation (3.3)

$$\Delta_j \Phi(\mathfrak{a}, m, s, z) = s(s-1)\Phi(\mathfrak{a}, m, s, z)$$

which holds a priori for $\Re(s) > 1$. However, the right hand side is defined for $\Re(s) > 3/4$ as well by analytic continuation and is even holomorphic there. Therefore,

$$\Delta_j \Phi(\mathfrak{a}, m, z) = [s(s-1)\Phi(\mathfrak{a}, m, s, z)]_{s=1} = \operatorname{res}_{s=1}(\Phi(\mathfrak{a}, m, s, z)) = q(\mathfrak{a}, m).$$

The other argument is based on the representation of $\Phi(\mathfrak{a}, m, z)$ in Theorem 3.4.4. We see that all terms except for $-q(\mathfrak{a}, m)\log(16\pi^2 y_1 y_2)$ are the real part of a holomorphic function (in z_1 or z_2, respectively). This proves that $\Phi(\mathfrak{a}, m, z)$ is real analytic and the Laplace equation follows with

$$\Delta_j \log(16\pi^2 y_1 y_2) = \Delta_j \log(y_j) = -1.$$

□

3.5 Local Borcherds product

In this section, we define for each ideal $\mathfrak{a} \in \mathcal{I}_K$ the local Borcherds product $\Psi(\mathfrak{a}, m, z)$ at infinity in $\overline{X(\mathfrak{a})}$, obtain interesting representations and express it in local coordinates to determine its vanishing orders along the components of the exceptional divisor $E^\infty(\mathfrak{a})$. The motivation is that the logarithmic singularities of $\Phi(\mathfrak{a}, m, z)$ at and near infinity match up to a factor the logarithm of $|\Psi(\mathfrak{a}, m, z)|$. The latter is analyzed in Corollary 3.5.7.

Definition 3.5.1. Let

$$\sigma : \Lambda^+(\mathfrak{a}, m) \to \{\pm 1\}$$

be a sign function with

$$\lim_{\lambda \to 0} \sigma(\lambda) = +1 \quad \text{and} \quad \lim_{\lambda \to \infty} \sigma(\lambda) = -1.$$

We define for $z \in \mathbb{H}^2$

$$\Psi_\sigma(\mathfrak{a}, m, z) := \prod_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \sigma(\lambda)\psi_\lambda(z) \quad \text{with} \quad \psi_\lambda(z) := e(|\lambda| z_1) - e(|\lambda'| z_2).$$

Remark 3.5.2. The function σ in the definition of $\Psi_\sigma(\mathfrak{a}, m, z)$ is there for technical reasons only to make the product convergent. Namely, for fixed $z \in \mathbb{H}^2$ we have

$$\lim_{\lambda \to 0} \psi_\lambda(z) = +1 \quad \text{and} \quad \lim_{\lambda \to \infty} \psi_\lambda(z) = -1.$$

By the equivalence relation

$$\sigma_1 \sim \sigma_2 \quad :\Leftrightarrow \quad \prod_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \sigma_1(\lambda)\sigma_2(\lambda) = 1$$

we partition the set of all admissible sign functions σ into two classes. Note that the product defining the equivalence relation is well-defined since almost all factors are equal to 1. We have

$$\Psi_{\sigma_1}(\mathfrak{a}, m, z) = \Psi_{\sigma_2}(\mathfrak{a}, m, z) \quad \Leftrightarrow \quad \sigma_1 \sim \sigma_2$$

and

$$\Psi_{\sigma_1}(\mathfrak{a}, m, z) = -\Psi_{\sigma_2}(\mathfrak{a}, m, z) \quad \Leftrightarrow \quad \sigma_1 \not\sim \sigma_2.$$

There is no canonical choice for the sign function σ, that is why we have to include it in the definition of $\Psi_\sigma(\mathfrak{a}, m, z)$. Later we are mostly interested in $|\Psi(\mathfrak{a}, m, z)|$ where the original sign of the product does not matter anymore. Whenever the sign is unimportant we simply write $\Psi(\mathfrak{a}, m, z)$.

Proposition 3.5.3. *The product $\Psi_\sigma(\mathfrak{a}, m, z)$ is a holomorphic function on $\mathbb{H}^2$ with simple roots at $T^\infty(\mathfrak{a}, m)$. Let $n \in 2\mathbb{N}$ with*

$$\frac{n}{1 - \varepsilon_0^2} \in \mathcal{O}_K.$$

Then $\Psi(\mathfrak{a}, m, z)^n$ is invariant under $\Gamma_{\mathfrak{a},\infty}$.

Proof. Clearly, each $\psi_\lambda(z)$ for $\lambda \in \Lambda^+(\mathfrak{a}, m)$ is holomorphic. Consider

$$\begin{aligned} \psi_\lambda(z) = 0 \quad &\Leftrightarrow \quad e(\lambda z_1) = e(-\lambda' z_2) \\ &\Leftrightarrow \quad e(\mathrm{tr}(\lambda z)) = 1 \\ &\Leftrightarrow \quad \mathrm{tr}(\lambda z) \in \mathbb{Z} \end{aligned}$$

to see that $\psi_\lambda(z)$ vanishes if and only if z lies in the components of $T^\infty(\mathfrak{a}, m)$ belonging to λ (cf. representation (2.44) of $T^\infty(\mathfrak{a}, m)$). Further, from $e(z)$ having a non-vanishing derivative it follows that all zeros of $\psi_\lambda(z)$ are simple. Hence, the normal convergence of the product proves that $\Psi(\mathfrak{a}, m, z)$ is a holomorphic function on $\mathbb{H}^2$ with simple roots at $T^\infty(\mathfrak{a}, m)$.

To prove the $\Gamma_{\mathfrak{a},\infty}$ invariance we make use of the decomposition

$$\overline{\Gamma_{\mathfrak{a},\infty}} \cong \mathfrak{a}^{-1} \rtimes (\mathcal{O}_K^\times)^2$$

and show the invariance for both factors individually. For $\varepsilon^2 \in (\mathcal{O}_K^\times)^2$ it is immediate by the definition of $\psi_\lambda(z)$ that we have

$$\psi_\lambda(\varepsilon^2 z) = \psi_{\varepsilon^2 \lambda}(z).$$

Because n is even we do not have to bother about the sign. Hence, the factors are only permuted by the action of $(\mathcal{O}_K^\times)^2$. However, for $\mu \in \mathfrak{a}^{-1}$ we have

$$\begin{aligned}\psi_\lambda(z+\mu) &= e(\lambda(z_1+\mu)) - e(-\lambda'(z_2+\mu')) \\ &= e(\lambda z_1)e(\lambda\mu) - e(-\lambda' z_2)e(-\lambda'\mu') \\ &= e(\lambda\mu)\left(e(\lambda z_1) - e(-\lambda' z_2)e(-\lambda\mu)e(-\lambda'\mu')\right) \\ &= e(\lambda\mu)\psi_\lambda(z).\end{aligned}$$

Here we used $\operatorname{tr}(\lambda\mu) \in \mathbb{Z}$ which is true because $\mathfrak{a}\mathfrak{d}^{-1}$ is the trace dual of $\mathfrak{a}^{-1}$. Analogously, we can factor $e(-\lambda'\mu')$ out to obtain

$$\psi_\lambda(z+\mu) = e(-\lambda'\mu')\psi_\lambda(z).$$

In particular, we have $e(\lambda\mu) = e(-\lambda'\mu')$ which can also be seen directly using $\operatorname{tr}(\lambda\mu) \in \mathbb{Z}$. The set $\Lambda^+(\mathfrak{a}, m)$ decomposes into finitely many $(\mathcal{O}_K^\times)^2$ orbits. For each orbit we have

$$\prod_{k\in\mathbb{Z}} \psi_{\varepsilon_0^{2k}\lambda}(z+\mu)^n = \prod_{k\in\mathbb{Z}} \psi_{\varepsilon_0^{2k}\lambda}(z)^n \cdot \prod_{k\in\mathbb{Z}} e(\lambda\varepsilon_0^{2k}\mu)^n.$$

To compute the later product we use

$$\prod_{k\in\mathbb{Z}} e(\lambda\varepsilon_0^{2k}\mu) = \prod_{k=1}^{\infty} e(\lambda\varepsilon_0^{-2k}\mu) \cdot \prod_{k=0}^{\infty} e(-\lambda'\varepsilon_0^{-2k}\mu').$$

Using the functional equation, this boils down to computing the sum

$$\begin{aligned}\sum_{k=1}^{\infty} \lambda\varepsilon_0^{-2k}\mu - \sum_{k=0}^{\infty} \lambda'\varepsilon_0^{-2k}\mu' &= \lambda\mu\frac{\varepsilon_0^{-2}}{1-\varepsilon_0^{-2}} - \lambda'\mu'\frac{1}{1-\varepsilon_0^{-2}} \\ &= \lambda\mu\frac{1}{\varepsilon_0^2-1} - \lambda'\mu'\left(\frac{1}{1-\varepsilon_0^2}\right)' = \operatorname{tr}\left(\frac{\lambda\mu}{\varepsilon_0^2-1}\right).\end{aligned}$$

Hence, we have proven

$$\prod_{k\in\mathbb{Z}} e(\lambda\varepsilon_0^{2k}\mu)^n = e\left(\operatorname{tr}\left(\frac{n\lambda\mu}{\varepsilon_0^2-1}\right)\right).$$

By the choice of n we have

$$\frac{n\lambda}{\varepsilon_0^2-1} \in \mathfrak{a}\mathfrak{d}^{-1}$$

which proves

$$\operatorname{tr}\left(\frac{n\lambda\mu}{\varepsilon_0^2-1}\right) \in \mathbb{Z}.$$

Hence, the infinite product

$$\prod_{k\in\mathbb{Z}} \psi_{\varepsilon_0^{2k}\lambda}(z)^n$$

is invariant under translation by $\mathfrak{a}^{-1}$ and therefore invariant under $\Gamma_{\mathfrak{a},\infty}$. The same holds for $\Psi(\mathfrak{a}, m, z)^n$ which is a finite product of such factors. $\square$

An easy way to come up with an admissible sign function σ is to partition the set $\Lambda^+(\mathfrak{a}, m)$ into a lower and an upper part with respect to a fixed $w \in (\mathbb{R}^+)^2$ using the trace by

$$\sigma_w : \Lambda^+(\mathfrak{a}, m) \to \{\pm 1\}, \quad \sigma_w(\lambda) := \begin{cases} +1, & \operatorname{tr}(\lambda w) < 0, \\ -1, & \operatorname{tr}(\lambda w) \geq 0. \end{cases}$$

The next proposition states a useful representation of Ψ_{σ_w}.

Proposition 3.5.4. *Let $w \in (\mathbb{R}^+)^2$ and let*

$$\Lambda_w := \left\{\lambda \in \Lambda^+(\mathfrak{a}, m) : \operatorname{tr}(\lambda w) \geq 0\right\} \cup \left\{\lambda \in \Lambda^-(\mathfrak{a}, m) : \operatorname{tr}(\lambda w) > 0\right\}.$$

Then we have

$$\Psi_{\sigma_w}(\mathfrak{a}, m, z) = e\left(\operatorname{tr}\left(\rho(\mathfrak{a}, m, w) z\right)\right) \prod_{\lambda \in \Lambda_w} \left(1 - e(\operatorname{tr}(\lambda z))\right).$$

Proof. Note that Λ_w is a set of representatives of $\Lambda(\mathfrak{a}, m)/\{\pm 1\}$. Let $\lambda \in \Lambda^+(\mathfrak{a}, m) \cap \Lambda_w$. Then we have $\sigma_w(\lambda) = -1$ and

$$\sigma_w(\lambda)\psi_\lambda(z) = e(-\lambda' z_2) - e(\lambda z_1) = e(-\lambda' z_2)(1 - e(\operatorname{tr}(\lambda z))).$$

On the other hand, for $\lambda \in \Lambda^-(\mathfrak{a}, m) \cap \Lambda_w$ we have $-\lambda \in \Lambda^+(\mathfrak{a}, m)$, $\sigma_w(-\lambda) = 1$ and

$$\sigma_w(-\lambda)\psi_{-\lambda}(z) = e(-\lambda z_1) - e(\lambda' z_2) = e(-\lambda z_1)(1 - e(\operatorname{tr}(\lambda z))).$$

Hence, this proves

$$\Psi_{\sigma_w}(\mathfrak{a}, m, z) = \prod_{\lambda \in \Lambda_w} \left(1 - e(\operatorname{tr}(\lambda z))\right) \times \begin{cases} e(-\lambda' z_2), & \lambda > 0, \\ e(-\lambda z_1), & \lambda < 0. \end{cases} \tag{3.8}$$

Using $R(\mathfrak{a}, m, w)$, the set of reduced $\lambda \in \Lambda^+(\mathfrak{a}, m)$ with respect to w, we can express all elements of Λ_w by

$$\bigcup_{\lambda \in R(\mathfrak{a}, m, w)} \left\{\lambda \varepsilon_0^{2k} : k \in 2\mathbb{N}_0\right\} \cup \left\{-\lambda \varepsilon_0^{-2k} : k \in 2\mathbb{N}\right\}.$$

Therefore, the product of the second factors of (3.8) is given by

$$\prod_{\lambda \in R(\mathfrak{a}, m, w)} \left(\prod_{k=0}^{\infty} e(-(\lambda \varepsilon_0^{2k})' z_2) \times \prod_{k=1}^{\infty} e(-(-\lambda \varepsilon_0^{-2k}) z_1)\right). \tag{3.9}$$

We compute both inner products for a fixed $\lambda \in R(\mathfrak{a}, m, w)$ using the functional equation

$$\begin{aligned} \sum_{k=0}^{\infty} -\lambda' \varepsilon_0^{-2k} z_2 + \sum_{k=1}^{\infty} \lambda \varepsilon_0^{-2k} z_1 &= -\lambda' z_2 \frac{1}{1 - \varepsilon_0^{-2}} + \lambda z_1 \frac{\varepsilon_0^{-2}}{1 - \varepsilon_0^{-2}} \\ &= \frac{\lambda}{\varepsilon_0^2 - 1} z_1 - \left(\frac{\lambda}{1 - \varepsilon_0^2}\right)' z_2 \\ &= \operatorname{tr}\left(\frac{\lambda}{\varepsilon_0^2 - 1} z\right). \end{aligned}$$

Using the definition of the Weyl vector (cf. equation (2.47)), we obtain

$$\sum_{\lambda \in R(\mathfrak{a},m,w)} \operatorname{tr}\left(\frac{\lambda}{\varepsilon_0^2 - 1} z\right) = \operatorname{tr}\left(\rho(\mathfrak{a}, m, w) z\right)$$

and hence

$$e\left(\operatorname{tr}\left(\rho(\mathfrak{a}, m, w) z\right)\right)$$

as the result of the product (3.9) in accordance to the statement of the proposition which finishes the proof. □

The classic approach introducing the local Borcherds product makes use of Weyl chambers (cf. [BvdGHZ08, p. 153, eq. (3.13)]). The next corollary shows that the resulting product is the same.

Corollary 3.5.5. *Let $W \in W(\mathfrak{a}, m)$ be a Weyl chamber of index m. Let us fix one $z_0 \in W$ to define $\sigma(\lambda) := -\operatorname{sgn}(\operatorname{tr}(\lambda y_0))$. Then we have*

$$\Psi_\sigma(\mathfrak{a}, m, z) = e\left(\operatorname{tr}\left(\rho(\mathfrak{a}, m, W) z\right)\right) \prod_{\substack{\lambda \in \Lambda(\mathfrak{a},m) \\ (\lambda, W) > 0}} \left(1 - e(\operatorname{tr}(\lambda z))\right).$$

Proof. Using $w := y_0$, we have $\sigma = \sigma_w$, $\rho(\mathfrak{a}, m, W) = \rho(\mathfrak{a}, m, w)$ and

$$\{\lambda \in \Lambda(\mathfrak{a}, m) : \ (\lambda, W) > 0\} = \Lambda_w$$

with Λ_w defined as in Proposition 3.5.4. Hence, the result is nothing but a direct application of Proposition 3.5.4. □

Proposition 3.5.6. *Let (α, β) be a totally positive basis of $\mathfrak{a}^{-1}$ and $n \in \mathbb{N}$ with*

$$\frac{n}{1 - \varepsilon_0^2} \in \mathcal{O}_K.$$

Then $\Psi(\mathfrak{a}, m, z)^n$ is invariant under $\mathfrak{a}^{-1}$ and possesses a holomorphic extension to $u = 0$ and $v = 0$ in local coordinates (u, v) with respect to (α, β). At $u = 0$ ($v = 0$, respectively) the product vanishes. Its order of vanishing along u (v, respectively) is given by $n \operatorname{tr}(\rho(\mathfrak{a}, m, \alpha)\alpha)$ ($n \operatorname{tr}(\rho(\mathfrak{a}, m, \beta)\beta)$, respectively).

Proof. Since α and β (and hence u and v) are interchangeable, we prove the result for v only. By Proposition 3.5.4 the Borcherds product is expressible as

$$e\left(\operatorname{tr}\left(\rho(\mathfrak{a}, m, \beta) z\right)\right) \prod_{\lambda \in \Lambda_\beta} \left(1 - e(\operatorname{tr}(\lambda z))\right).$$

By Lemma 2.7.1 each factor of the product is $\mathfrak{a}^{-1}$ invariant and we have

$$\prod_{\lambda \in \Lambda_\beta} \left(1 - e(\operatorname{tr}(\lambda z))\right) = \prod_{\lambda \in \Lambda_\beta} \left(1 - u^{\operatorname{tr}(\lambda\alpha)} v^{\operatorname{tr}(\lambda\beta)}\right)$$

in local coordinates. We list some facts we know about the exponents of u and v:

(i) We have $\mathrm{tr}(\lambda\alpha) \in \mathbb{Z}$ and $\mathrm{tr}(\lambda\beta) \in \mathbb{N}_0$ for all $\lambda \in \Lambda_\beta$.

(ii) For each $m \in \mathbb{Z}$ there are at most two $\lambda \in \Lambda_\beta$ with $\mathrm{tr}(\lambda\alpha) = m$ ($\mathrm{tr}(\lambda\beta) = m$ respecively).

(iii) There are only finitely many $\lambda \in \Lambda_\beta$ with $\mathrm{tr}(\lambda\alpha) < 0$.

Those facts imply that the product converges normally to a holomorphic function in u and v in the domain

$$\Big\{(u,v) \in \mathbb{C}^2 : \ 0 < |u| < 1, |v| < 1\Big\}$$

and that it does not vanish at $v = 0$. Hence, we are left with inspecting the factor in front of the product $e(\mathrm{tr}(\rho z))$ (for simplicity we abbreviate $\rho := \rho(\mathfrak{a}, m, \beta)$ for the rest of the proof). This factor might not be $\mathfrak{a}^{-1}$ invariant but the n-th power is because we have $e(\mathrm{tr}(\rho z))^n = e(\mathrm{tr}(n\rho z))$. Now by assumption on n and the definition of the Weyl vector ρ (cf. (2.47)) we have $n\rho \in \mathfrak{a}\mathfrak{d}^{-1}$. Hence, Lemma 2.7.1 again implies the $\mathfrak{a}^{-1}$ invariance of $e(\mathrm{tr}(n\rho z))$ and

$$e(\mathrm{tr}(n\rho z)) = u^{\mathrm{tr}(n\rho\alpha)} v^{\mathrm{tr}(n\rho\beta)}.$$

By Lemma 2.8.7 the Weyl vector ρ is totally positive. That makes $\mathrm{tr}(\rho\beta)$ positive which finishes the proof. □

Corollary 3.5.7. *The function*

$$\log |\Psi(\mathfrak{a}, m, z)|^2$$

is well-defined in a neighborhood of the exceptional divisor $E^\infty(\mathfrak{a}) \subset \overline{X(\mathfrak{a})}$ and has logarithmic singularities along the divisor $T^\infty(\mathfrak{a}, m) + Z^\infty(\mathfrak{a}, m)$.

Proof. Let $n \in \mathbb{N}$ be like in Proposition 3.5.3. Then $\Psi(\mathfrak{a}, m, z)^n$ is invariant under $\Gamma_{\mathfrak{a},\infty}$. With Proposition 2.5.2 this shows that $\Psi(\mathfrak{a}, m, z)^n$ is well-defined on a punctured neighborhood of ∞ in $X(\mathfrak{a})^*$ and holomorphic there. With Proposition 3.5.6 we obtain that $\Psi(\mathfrak{a}, m, z)^n$ is well-defined on $E^\infty(\mathfrak{a})$ as well, hence on a neighborhood of $E^\infty(\mathfrak{a})$ in $\overline{X(\mathfrak{a})}$, and that this extension is holomorphic. With $\Psi(\mathfrak{a}, m, z)^n$ being well-defined, of course also

$$\log |\Psi(\mathfrak{a}, m, z)|^2 = \frac{1}{n} \log |\Psi(\mathfrak{a}, m, z)^n|^2$$

is well-defined. Now, we come to prove the stated logarithmic singularities. For this we have to show that the divisor of the holomorphic function $\Psi(\mathfrak{a}, m, z)^n$ agrees with $n(T^\infty(\mathfrak{a}, m) + Z^\infty(\mathfrak{a}, m))$. By Proposition 3.5.3 the function $\Psi(\mathfrak{a}, m, z)^n$ vanishes of order n at $T^\infty(\mathfrak{a}, m)$ in $X(\mathfrak{a})$. By Proposition 3.5.6 the divisor $nZ^\infty(\mathfrak{a}, m)$ provides the correct multiplicities for the vanishing of $\Psi(\mathfrak{a}, m, z)^n$ along $E^\infty(\mathfrak{a})$. To see that, recall definition (2.51) of $Z^\infty(\mathfrak{a}, m)$ with $(\alpha, \beta) := (A_{k-1}, A_k)$ to realize that the multiplicities of the components S_k of $nZ^\infty(\mathfrak{a}, m)$ are precisely defined to match the multiplicites of the zeros of $\Psi(\mathfrak{a}, m, z)^n$ along S_k. □

Until now we assumed $m > 0$ which is quite natural for the investigation of $\Phi(\mathfrak{a}, m, z)$ since $\Phi(\mathfrak{a}, m, z)$ is the regularization of $\Phi(\mathfrak{a}, m, s, z)$ for $m \in \mathbb{N}$. However, when we come to Chapter 4 we will need a definition of $\Psi(\mathfrak{a}, m, z)$ for $m \in -\mathbb{N}$ as well. Therefore, let us discuss this case briefly.

Definition 3.5.8. For $m \in -\mathbb{N}$ we define

$$\Psi(\mathfrak{a}, m, z) := \prod_{\lambda \in \Lambda^+(\mathfrak{a}, m)} (1 - e(\mathrm{tr}(\lambda z)))\,.$$

Each single factor of that product is invariant under translation by $\mathfrak{a}^{-1}$. The operation of $(\mathcal{O}_K^\times)^2$ permutes the factors. Because $\lambda > 0$ and $\lambda' > 0$, each single factor equals 1 at the cusp ∞. With respect to a totally positive basis (α, β) of $\mathfrak{a}^{-1}$ it has the representation

$$\Psi(\mathfrak{a}, m, z) = \prod_{\lambda \in \Lambda^+(\mathfrak{a}, m)} (1 - u^{\mathrm{tr}(\lambda\alpha)} v^{\mathrm{tr}(\lambda\beta)}).$$

Here $\mathrm{tr}(\lambda\alpha)$ and $\mathrm{tr}(\lambda\beta)$ are natural numbers. This proves the following proposition.

Proposition 3.5.9. *For $m \in -\mathbb{N}$ the local Borcherds product $\Psi(\mathfrak{a}, m, z)$ defines a holomorphic nowhere vanishing function on $\mathbb{H}^2$ which is invariant under $\Gamma_{\mathfrak{a},\infty}$. It possesses a holomorphic extension to the Hirzebruch desingularization of the cusp ∞ in the quotient $\Gamma_{\mathfrak{a},\infty} \backslash \mathbb{H}^2$ by being set to 1 at the cusp.*

Remark 3.5.10. Note that Corollary 3.5.7 holds for $m \in -\mathbb{N}$ as well. In this case $\Psi(\mathfrak{a}, m, z)$ is holomorphic and different from 0 in a neighborhood of $E^\infty(\mathfrak{a}) \subset \overline{X(\mathfrak{a})}$. Therefore

$$\log |\Psi(\mathfrak{a}, m, z)|^2$$

has no singularity at all in a neighborhood of $E^\infty(\mathfrak{a}) \subset \overline{X(\mathfrak{a})}$. This suits well since $T^\infty(\mathfrak{a}, m) = 0$ and $Z^\infty(\mathfrak{a}, m) = 0$ by definition.

3.6 Growth analysis

In this section we prove that the automorphic Green functions $\Phi(\mathfrak{a}, m, z)$ are actual Green functions, i.e., $\Phi(\mathfrak{a}, m, z)$ is a pre-log-log Green function (cf. Definition 2.9.7) on $\overline{X(\mathfrak{a})}$ with respect to the divisor $Z(\mathfrak{a}, m)$.

Since $\Phi(\mathfrak{a}, m, z)$ has logarithmic singularities along $-T(\mathfrak{a}, m)$ (this is formally proven in the next proposition), it follows by the groundwork of Subsection 2.9.4 that $[\Phi(\mathfrak{a}, m, z)]$ is a classical Green current on $X(\mathfrak{a})$. The part at the desingularization of the cusps is more sophisticated since here we do not have purely logarithmic singularities but additional pre-log-log growth. The goal of this section is to separate the logarithmic singularities from the pre-log-log growth following Remark 2.9.9 to eventually obtain one of the main theorems of this chapter, Theorem 3.6.5, which states that $\Phi(\mathfrak{a}, m, z)$ is a pre-log-log Green function on $\overline{X(\mathfrak{a})}$ with respect to the divisor $Z(\mathfrak{a}, m)$.

Proposition 3.6.1. *The regularized automorphic Green function $\Phi(\mathfrak{a}, m, z)$ living on $\mathbb{H}^2$ has logarithmic singularities along $-T(\mathfrak{a}, m)$.*

Proof. Let $z_0 \in \mathbb{H}^2$. If $z_0 \notin T(\mathfrak{a}, m)$, we know that there exists a small neighborhood $U \subset \mathbb{H}^2$ of z_0 with $U \cap T(\mathfrak{a}, m) = \emptyset$. Hence, by Proposition 3.4.5 $\Phi(\mathfrak{a}, m, z)$ is smooth on U and nothing remains to show.

However, if $z_0 \in T(\mathfrak{a}, m)$, by Lemma 2.6.4 we find a small neighborhood $U \subset \mathbb{H}^2$ of z_0 such that only finitely many

$$A \in L := \left\{A \in L(\mathfrak{a})^\vee : \det(A) = m/(N(\mathfrak{a})D)\right\}$$

satisfy $h(A, z) = 0$ for any $z \in U$. Let us collect those matrices in the finite set $F \subset L$. Then we have

$$\Phi(\mathfrak{a}, m, z) = \mathcal{C}_{s=1}\left[\sum_{A \in L \setminus F} Q_{s-1}\left(1 + 2g(A, z)\right)\right] + \sum_{A \in F} Q_0\left(1 + 2g(A, z)\right).$$

The first part is smooth in U and for the second part we have (cf. Section 3.1)

$$\begin{aligned}\sum_{A \in F} Q_0\left(1 + 2g(A, z)\right) &= \sum_{\left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in F} \log\left|\frac{bz_1\overline{z_2} - \lambda z_1 - \lambda'\overline{z_2} + a}{bz_1 z_2 - \lambda z_1 - \lambda' z_2 + a}\right| \\ &= \sum_{\left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in F/\{\pm 1\}} \log\left|\frac{bz_1\overline{z_2} - \lambda z_1 - \lambda'\overline{z_2} + a}{bz_1 z_2 - \lambda z_1 - \lambda' z_2 + a}\right|^2. \end{aligned} \tag{3.10}$$

For $A = \left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in F$ we have

$$\frac{\left|bz_1\overline{z_2} - \lambda z_1 - \lambda'\overline{z_2} + a\right|^2}{4y_1 y_2} = q_{\tilde{W}_z}(A) = \det(A) + h(A, z) \geq \det(A) = \frac{m}{N(\mathfrak{a})D} > 0$$

by (2.23) and (2.21). This implies that the numerator of (3.10) never vanishes for $z \in \mathbb{H}^2$ and is therefore smooth in z. However, the denominator gives us logarithmic singularities. The irreducible components of $T(\mathfrak{a}, m)$ intersecting U are given by

$$\sum_{A \in F/\{\pm 1\}} T_A = \sum_{\left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in F/\{\pm 1\}} \left\{bz_1 z_2 - \lambda z_1 - \lambda' z_2 + a = 0\right\}.$$

Thus, the function $\Phi(\mathfrak{a}, m, z)$ satisfies the conditions of Definition 2.9.10 of logarithmic singularities along $-T(\mathfrak{a}, m)$. □

Lemma 3.6.2. *Let $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{R}$ with*

$$\alpha + a_1 + a_2 > 0 \quad \textit{and} \quad \beta + b_1 + b_2 > 0.$$

Then the function

$$f : (\mathbb{C}^\times)^2 \to \mathbb{C}, \quad f(u, v) = u^{a_1}\overline{u}^{a_2}|u|^\alpha \cdot v^{b_1}\overline{v}^{b_2}|v|^\beta$$

is a pre-log-log growth form along $uv = 0$.

Proof. We work with Remark 2.9.3. The function f has a continuous extension to $uv = 0$ with $f(u, v) = 0$. This is due to the positivity condition on the exponents. The continuity of f implies that f itself is of log-log growth. Now, one computes

$$\begin{aligned}
\frac{\partial f}{\partial u} &= \left(a_1 + \frac{\alpha}{2}\right) u^{a_1-1}\overline{u}^{a_2}|u|^{\alpha} \cdot v^{b_1}\overline{v}^{b_2}|v|^{\beta}, \\
\frac{\partial f}{\partial v} &= \left(b_1 + \frac{\beta}{2}\right) u^{a_1}\overline{u}^{a_2}|u|^{\alpha} \cdot v^{b_1-1}\overline{v}^{b_2}|v|^{\beta}, \\
\frac{\partial f}{\partial \overline{u}} &= \left(a_2 + \frac{\alpha}{2}\right) u^{a_1}\overline{u}^{a_2-1}|u|^{\alpha} \cdot v^{b_1}\overline{v}^{b_2}|v|^{\beta}, \\
\frac{\partial f}{\partial \overline{v}} &= \left(b_2 + \frac{\beta}{2}\right) u^{a_1}\overline{u}^{a_2}|u|^{\alpha} \cdot v^{b_1}\overline{v}^{b_2-1}|v|^{\beta}.
\end{aligned}$$

The derivatives multiplied with the respective prefactors from Remark 2.9.3 vanish at $uv = 0$ (again due to the positivity condition on the exponents and the fact that the logarithm grows slower than powers with positive exponents). The same holds for the second order derivatives. □

Lemma 3.6.3. *Let $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{R}$ with*

$$\alpha + a_1 + a_2 > 0 \quad \text{and} \quad \beta + b_1 + b_2 > 0.$$

Then the function

$$f : (\mathbb{C}^{\times})^2 \to \mathbb{C}, \quad f(u, v) = \log\left|1 - u^{a_1}\overline{u}^{a_2}|u|^{\alpha} \cdot v^{b_1}\overline{v}^{b_2}|v|^{\beta}\right|^2$$

is a pre-log-log growth form along $uv = 0$.

Proof. The proof is analogue to the proof of Lemma 3.6.2. Again, f has a continuous extension to $uv = 0$ with $f(u, v) = 0$. Using

$$\frac{\partial}{\partial z}\log(|g(z)|^2) = \frac{\frac{\partial g}{\partial z}}{g} + \frac{\frac{\partial \overline{g}}{\partial z}}{\overline{g}},$$

we see

$$\begin{aligned}
\frac{\partial f}{\partial u} &= \frac{2a_1 + \alpha}{2}\frac{-u^{a_1-1}\overline{u}^{a_2}|u|^{\alpha}v^{b_1}\overline{v}^{b_2}|v|^{\beta}}{1 - u^{a_1}\overline{u}^{a_2}|u|^{\alpha}v^{b_1}\overline{v}^{b_2}|v|^{\beta}} + \frac{2a_2 + \alpha}{2}\frac{-u^{a_2-1}\overline{u}^{a_1}|u|^{\alpha}v^{b_2}\overline{v}^{b_1}|v|^{\beta}}{1 - u^{a_2}\overline{u}^{a_1}|u|^{\alpha}v^{b_2}\overline{v}^{b_1}|v|^{\beta}}, \\
\frac{\partial f}{\partial v} &= \frac{2b_1 + \beta}{2}\frac{-u^{a_1}\overline{u}^{a_2}|u|^{\alpha}v^{b_1-1}\overline{v}^{b_2}|v|^{\beta}}{1 - u^{a_1}\overline{u}^{a_2}|u|^{\alpha}v^{b_1}\overline{v}^{b_2}|v|^{\beta}} + \frac{2b_2 + \beta}{2}\frac{-u^{a_2}\overline{u}^{a_1}|u|^{\alpha}v^{b_2-1}\overline{v}^{b_1}|v|^{\beta}}{1 - u^{a_2}\overline{u}^{a_1}|u|^{\alpha}v^{b_2}\overline{v}^{b_1}|v|^{\beta}}.
\end{aligned}$$

Respectively for $\partial f/\partial\overline{u}$ and $\partial f/\partial\overline{v}$. For $u \to 0$ or $v \to 0$ the denominator goes to 1 and the numerator to 0 after being multiplied with the respective factor (cf. Remark 2.9.3). The same holds for the second order derivatives. □

Lemma 3.6.4. *Let (α, β) be a totally positive basis of $\mathfrak{a}^{-1}$. The $\mathfrak{a}$ invariant function*

$$f : \mathbb{H}^2 \to \mathbb{C}, \quad f(z) := \log(y_1 y_2)$$

expressed in local coordinates (u, v) with respect to (α, β) is a pre-log-log growth form along $uv = 0$.

Proof. By equation (2.30) we have

$$y_1 = -\frac{\log(|u|^\alpha |v|^\beta)}{2\pi} \quad \text{and} \quad y_2 = -\frac{\log(|u|^{\alpha'} |v|^{\beta'})}{2\pi}.$$

It follows

$$\begin{aligned}\log(y_1 y_2) = &-\log(4\pi^2) + \log\left(\alpha \log(1/|u|) + \beta \log(1/|v|)\right) \\ &+ \log\left(\alpha' \log(1/|u|) + \beta' \log(1/|v|)\right).\end{aligned}$$

Analogously to Lemma 2.9.4 this is a pre-log-log growth form. □

Theorem 3.6.5. *The function $\Phi(\mathfrak{a}, m, z)$ is a pre-log-log Green function on $\overline{X(\mathfrak{a})}$ with respect to the divisor $Z(\mathfrak{a}, m)$.*

Proof. We proceed as suggested in Remark 2.9.9. First of all, Proposition 3.6.1 ensures that we do not have to care about $X(\mathfrak{a})$ anymore. Therefore, the focus of this proof lies on the cusps and by the often repeated argument it is enough to consider the cusp ∞. We write

$$\Phi(\mathfrak{a}, m, z) = f_1(z) + f_2(z) + f_3(z) + f_4(z) + f_5(z) + f_6(z)$$

near the cusp ∞ as sum of six parts according to Theorem 3.4.4. Following Remark 2.9.9, we show that the functions f_j with $j \in \{1, 2, 3, 4, 5\}$ are pre-log-log growth forms along $E^\infty(\mathfrak{a})$ and that f_6 has logarithmic singularities along the divisor $-(T^\infty(\mathfrak{a}, m) + Z^\infty(\mathfrak{a}, m))$. Note that the divisor $T^\infty(\mathfrak{a}, m) + Z^\infty(\mathfrak{a}, m)$ is the part of $Z(\mathfrak{a}, m)$ in small neighborhoods of $E^\infty(\mathfrak{a})$. For proving the pre-log-log growth we express f_j in local coordinates (u, v) with respect to a totally positive basis (α, β) of $\mathfrak{a}^{-1}$. Let us make our decomposition of $\Phi(\mathfrak{a}, m, z)$ precise:

$$\begin{aligned}
f_1(z) &:= L(\mathfrak{a}, m), \\
f_2(z) &:= -q(\mathfrak{a}, m) \log(16\pi^2 y_1 y_2), \\
f_3(z) &:= \sum_{\substack{\nu \in \mathfrak{a}\mathfrak{d}^{-1} \\ \nu \gg 0}} \frac{2\pi}{D} \sqrt{\frac{mN(\mathfrak{a})}{|N(\nu)|}} \sum_{b=1}^{\infty} \frac{G^b(\mathfrak{a}, m, \nu)}{b} I_1\left(\frac{4\pi}{b}\sqrt{\frac{m|N(\nu)|}{N(\mathfrak{a})D}}\right) \\
&\quad \times \left(e(\operatorname{tr}(\nu z)) + \overline{e(\operatorname{tr}(\nu z))}\right), \\
f_4(z) &:= \sum_{\substack{\nu \in \mathfrak{a}\mathfrak{d}^{-1} \\ \nu > 0,\, \nu' < 0}} \frac{2\pi}{D} \sqrt{\frac{mN(\mathfrak{a})}{|N(\nu)|}} \sum_{b=1}^{\infty} \frac{G^b(\mathfrak{a}, m, \nu)}{b} J_1\left(\frac{4\pi}{b}\sqrt{\frac{m|N(\nu)|}{N(\mathfrak{a})D}}\right) \\
&\quad \times \left(e(\nu z_1)\overline{e(-\nu' z_2)} + \overline{e(\nu z_1)} e(-\nu' z_2)\right), \\
f_5(z) &:= \log \prod_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \left|1 - e(|\lambda| z_1)\overline{e(|\lambda'| z_2)}\right|^2 = \sum_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \log\left|1 - e(\lambda z_1)\overline{e(-\lambda' z_2)}\right|^2, \\
f_6(z) &:= -\log \prod_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \left|e(|\lambda| z_1) - e(|\lambda'| z_2)\right|^2 = -\log|\Psi(\mathfrak{a}, m, z)|^2.
\end{aligned}$$

The function f_1 is constant, hence it is a pre-log-log growth form. The function f_2 was considered (up to constants) in Lemma 3.6.4. The function f_3 is real analytic even at $uv = 0$ because of

$$e(\mathrm{tr}(\nu z)) = u^{\mathrm{tr}(\alpha\nu)} v^{\mathrm{tr}(\beta\nu)} \quad \text{and} \quad \overline{e(\mathrm{tr}(\nu z))} = \overline{u}^{\mathrm{tr}(\alpha\nu)} \overline{v}^{\mathrm{tr}(\beta\nu)}$$

by Lemma 2.7.1. Note that $\mathrm{tr}(\alpha\nu), \mathrm{tr}(\beta\nu) \in \mathbb{N}$. Hence, it is a pre-log-log growth form. Unfortunately, the function f_4 is not even differentiable at $uv = 0$ but at least continuous. We have by Lemma 2.7.1

$$\begin{aligned} e(\nu z_1)\overline{e(-\nu' z_2)} &= u^{\alpha\nu} \overline{u}^{-\alpha'\nu'} v^{\beta\nu} \overline{v}^{-\beta'\nu'} \\ &= u^{\mathrm{tr}(\alpha\nu)} |u|^{-2\alpha'\nu'} v^{\mathrm{tr}(\beta\nu)} |v|^{-2\beta'\nu'} \end{aligned}$$

and

$$\overline{e(\nu z_1)} e(-\nu' z_2) = \overline{u}^{\mathrm{tr}(\alpha\nu)} |u|^{-2\alpha'\nu'} \overline{v}^{\mathrm{tr}(\beta\nu)} |v|^{-2\beta'\nu'}.$$

The advantage of having integer powers on u, $\overline{u}$, v and $\overline{v}$ is that it is well-defined without specifying a branch of the logarithm. Since $\nu > 0$ and $\nu' < 0$, we have

$$\mathrm{tr}(\alpha\nu) - 2\alpha'\nu' = \alpha\nu - \alpha'\nu' > 0 \quad \text{and} \quad \mathrm{tr}(\beta\nu) - 2\beta'\nu' = \beta\nu - \beta\nu' > 0.$$

Hence, the claim for f_4 follows by Lemma 3.6.2. Considering f_5, we see that we can write each summand in local coordinates using the same identity and get

$$\log \left| 1 - e(\lambda z_1)\overline{e(-\lambda' z_2)} \right|^2 = \log \left| 1 - u^{\mathrm{tr}(\alpha\lambda)} |u|^{-2\alpha'\lambda'} v^{\mathrm{tr}(\beta\lambda)} |v|^{-2\beta'\lambda'} \right|^2.$$

Because of $\lambda > 0$ and $\lambda' < 0$ we can apply Lemma 3.6.3 to achieve the claim for f_5. Now, we are left with

$$f_6(z) = -\log |\Psi(\mathfrak{a}, m, z)|^2$$

for which we have proven the claim already in Corollary 3.5.7. □

3.7 A valuable representation using the hypergeometric function

In this section we follow the idea (for example present in [BEY21]) to express $\Phi(\mathfrak{a}, m, s, z)$ using the hypergeometric function ${}_2F_1(a, b; c; z)$. This yields a valuable decomposition

$$\Phi(\mathfrak{a}, m, s, z) = \sum_{n=0}^{\infty} \Phi_n(\mathfrak{a}, m, s, z)$$

into smooth, $\Gamma_{\mathfrak{a}}$ invariant functions $\Phi_n(\mathfrak{a}, m, s, z)$. Using this decomposition, a lot of already known results about $\Phi(\mathfrak{a}, m, s, z)$ and $\Phi(\mathfrak{a}, m, z)$ can be reproven. Some of those proofs reveal new perspectives on the old results. For example, computing the Fourier expansions of the functions $\Phi_n(\mathfrak{a}, m, s, z)$ yields new formulae for the Fourier coefficients

of $\Phi(\mathfrak{a}, m, s, z)$. However, the motivation for the author to look at this decomposition was to prove the integrability of $\Phi(\mathfrak{a}, m, z)$ and understand the growth behavior of

$$\int_{X(\mathfrak{a})} |\Phi(\mathfrak{a}, m, z)|\omega^2$$

for large m which is essential for the main result of this thesis. Those two results can be found in Theorem 3.8.10 and Theorem 3.8.13 in Section 3.8. The main work towards these theorems is done in the current section.

Let us first explain the differences between $\Phi(\mathfrak{a}, m, s, z)$ and $\Phi_n(\mathfrak{a}, m, s, z)$. The function $\Phi(\mathfrak{a}, m, s, z)$ has logarithmic singularities along $-T(\mathfrak{a}, m)$. That is why our Fourier expansion does not converge on all of $\mathbb{H}^2$ but only for those $z \in \mathbb{H}^2$ with $\Im(z) > m/(DN(\mathfrak{a}))$ because that is the region where only the $T(\mathfrak{a}, m)$ components of $T^\infty(\mathfrak{a}, m)$ live in. To exclude $T^\infty(\mathfrak{a}, m)$, we consider only $z \in \mathbb{H}^2$ with $z \notin S(\mathfrak{a}, m)$. In contrast, the functions $\Phi_n(\mathfrak{a}, m, s, z)$ are smooth on all of $\mathbb{H}^2$. Therefore, they have an everywhere converging Fourier series. Unfortunately, it is more complicated to compute the Fourier coefficients explicitly. The logarithmic singularities along $-T^\infty(\mathfrak{a}, m)$ occur not in a single $\Phi_n(\mathfrak{a}, m, s, z)$ but in the sum over all of them. Adding the Fourier coefficients of $\Phi_n(\mathfrak{a}, m, s, z)$ gives us back the Fourier coefficients of $\Phi(\mathfrak{a}, m, s, z)$.

3.7.1 The decomposition

The main ingredient in Definition 3.1.1 of $\Phi(\mathfrak{a}, m, s, z)$ is $Q_{s-1}(x)$, the Legendre function of the second kind. This however has the nice representation

$$Q_{s-1}(x) = \frac{\Gamma(s)^2}{2\Gamma(2s)} \left(\frac{2}{1+x}\right)^s {}_2F_1\left(s, s; 2s; \frac{2}{1+x}\right) \tag{3.11}$$

using the hypergeometric function ${}_2F_1(a, b; c; z)$ which follows from [OLBC10, 14.3.7 and 15.8.13] together with the Legendre duplication formula. The hypergeometric function again is defined by its power series

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!} \tag{3.12}$$

which implies

$$Q_{s-1}(x) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\Gamma(s+n)^2}{\Gamma(2s+n)} \frac{1}{n!} \left(\frac{2}{1+x}\right)^{n+s}.$$

Plugged into Definition 3.1.1 we get

$$\Phi(\mathfrak{a}, m, s, z) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\Gamma(s+n)^2}{\Gamma(2s+n)} \frac{1}{n!} \sum_{\substack{A \in L(\mathfrak{a})^\vee \\ \det(A) = m/(N(\mathfrak{a})D)}} (1 + g(A, z))^{-(n+s)}.$$

Defining

$$\Psi(\mathfrak{a}, m, s, z) := \sum_{\substack{A \in L(\mathfrak{a})^\vee \\ \det(A) = m/(N(\mathfrak{a})D)}} (1 + g(A, z))^{-s}, \tag{3.13}$$

we get

$$\Phi(\mathfrak{a}, m, s, z) = \sum_{n=0}^{\infty} \underbrace{\frac{\Gamma(s+n)^2}{\Gamma(2s+n)} \frac{\Psi(\mathfrak{a}, m, s+n, z)}{2n!}}_{=:\Phi_n(\mathfrak{a},m,s,z)}.$$

The convergence of $\Psi(\mathfrak{a}, m, s, z)$ for $\Re(s) > 1$ follows directly from the convergence of $\Phi(\mathfrak{a}, m, s, z)$ but can also be seen as a further application of Lemma 3.1.2. Here, $\Psi(\mathfrak{a}, m, s, z)$ is even well-defined for $z \in T(\mathfrak{a}, m)$ and smooth in z since $(1+x)^{-s}$ has no singularity at $x = 0$. Furthermore, $\Psi(\mathfrak{a}, m, s, z)$ is holomorphic in s.

It follows that the functions $\Phi_n(\mathfrak{a}, m, s, z)$ are holomorphic in s, $\Gamma_{\mathfrak{a}}$ invariant and smooth in z on $\mathbb{H}^2$ for $\Re(s) > 1 - n$. Inductively, one can show that for all $N \in \mathbb{N}_0$

$$\sum_{n=N}^{\infty} \Phi_n(\mathfrak{a}, m, s, z)$$

converges for $\Re(s) > 1 - N$ to a $\Gamma_{\mathfrak{a}}$ invariant and smooth function on $\mathbb{H}^2 \setminus T(\mathfrak{a}, m)$ which is holomorphic in s (in particular $N = 1$ implies convergence for $\Re(s) > 0$). By Theorem 3.3.4 we know that $\Phi(\mathfrak{a}, m, s, z)$ has a meromorphic extension to $\Re(s) > 3/4$ for $z \in \mathbb{H}^2 \setminus T(\mathfrak{a}, m)$ with simple pole at $s = 1$ of residue $q(\mathfrak{a}, m)$. It follows that $\Phi_0(\mathfrak{a}, m, s, z)$ has a meromorphic extension to $\Re(s) > 3/4$ with simple pole at $s = 1$ of residue $q(\mathfrak{a}, m)$. We define

$$\Phi_0(\mathfrak{a}, m, z) := \mathcal{C}_{s=1}\left[\Phi_0(\mathfrak{a}, m, s, z)\right]$$

and get

$$\Phi(\mathfrak{a}, m, z) = \Phi_0(\mathfrak{a}, m, z) + \sum_{n=1}^{\infty} \Phi_n(\mathfrak{a}, m, 1, z).$$

3.7.2 Fourier expansion of $\Psi(\mathfrak{a}, m, s, z)$

We proceed analogously to Section 3.2 and write

$$\Psi(\mathfrak{a}, m, s, z) = \Psi^0(\mathfrak{a}, m, s, z) + 2\sum_{b=1}^{\infty} \Psi^b(\mathfrak{a}, m, s, z)$$

with

$$\Psi^b(\mathfrak{a}, m, s, z) := \sum_{\substack{A=\left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in L(\mathfrak{a})^\vee \\ \det(A) = m/(N(\mathfrak{a})D)}} (1 + g(A, z))^{-s}.$$

The functions $\Psi^b(\mathfrak{a}, m, s, z)$ are invariant under $\Gamma_{\mathfrak{a},\infty}$ as $\Phi^b(\mathfrak{a}, m, s, z)$ in Section 3.2. Hence, they are $\mathfrak{a}^{-1}$ periodic and possess a Fourier expansion. Again, we treat the cases $b = 0$ and $b \in \mathbb{N}$ separately and start with $b \in \mathbb{N}$. We have with $B := m/(N(\mathfrak{a})Db^2)$ and

$R^b(\mathfrak{a}, m)$ defined as in Section 3.2

$$\begin{aligned}\Psi^b(\mathfrak{a}, m, s, z) &= \sum_{\substack{a\in\mathbb{Z}/N(\mathfrak{a}),\, \lambda\in\mathfrak{a}\mathfrak{d}^{-1}/N(\mathfrak{a})\\ ab-N(\lambda)=m/(N(\mathfrak{a})D)}} \left(1+\frac{|bz_1z_2-\lambda z_1-\lambda' z_2+a|^2}{4y_1y_2m/(N(\mathfrak{a})D)}\right)^{-s}\\ &= \sum_{\substack{a\in\mathbb{Z}/N(\mathfrak{a}),\, \lambda\in\mathfrak{a}\mathfrak{d}^{-1}/N(\mathfrak{a})\\ ab-N(\lambda)=m/(N(\mathfrak{a})D)}} \left(1+\frac{|(z_1-\lambda'/b)(z_2-\lambda/b)+B|^2}{4y_1y_2B}\right)^{-s}\\ &= \sum_{\lambda\in R^b(\mathfrak{a},m)}\sum_{\mu\in\mathfrak{a}^{-1}} \left(1+\frac{\left|\left(z_1+\mu+\frac{\lambda'}{N(\mathfrak{a})b}\right)\left(z_2+\mu'+\frac{\lambda}{N(\mathfrak{a})b}\right)+B\right|^2}{4y_1y_2B}\right)^{-s}.\end{aligned}$$

Hence, the problem is reduced to computing the Fourier expansion of the $\mathfrak{a}^{-1}$ periodic function $\tilde{H}_s^B(\mathfrak{a}^{-1}, z)$ with

$$\tilde{H}_s^B(\mathfrak{b}, z) := \sum_{\mu\in\mathfrak{b}} \left(1+\frac{|(z_1+\mu)(z_2+\mu')+B|^2}{4y_1y_2B}\right)^{-s}.$$

Namely, let

$$\tilde{H}_s^B(\mathfrak{b}, z) = \sum_{\nu\in(\mathfrak{b}\mathfrak{d})^{-1}} \tilde{b}_s^B(\mathfrak{b}, \nu, y)e(\operatorname{tr}(\nu x))$$

be the Fourier expansion of $\tilde{H}_s^B(\mathfrak{b}, z)$. Then we have

$$\begin{aligned}\Psi^b(\mathfrak{a}, m, s, z) &= \sum_{\lambda\in R^b(\mathfrak{a},m)}\sum_{\nu\in\mathfrak{a}\mathfrak{d}^{-1}} \tilde{b}_s^B(\mathfrak{a}^{-1}, \nu, y)e\left(\operatorname{tr}\left(\nu\left(x+\tfrac{\lambda'}{N(\mathfrak{a})b}\right)\right)\right)\\ &= \sum_{\nu\in\mathfrak{a}\mathfrak{d}^{-1}}\left(\sum_{\lambda\in R^b(\mathfrak{a},m)} e\left(\operatorname{tr}\left(\tfrac{\nu\lambda'}{N(\mathfrak{a})b}\right)\right)\right)\tilde{b}_s^B(\mathfrak{a}^{-1}, \nu, y)e(\operatorname{tr}(\nu x))\\ &= \sum_{\nu\in\mathfrak{a}\mathfrak{d}^{-1}} G^b(\mathfrak{a}, m, \nu)\tilde{b}_s^B(\mathfrak{a}^{-1}, \nu, y)e(\operatorname{tr}(\nu x))\end{aligned}$$

with $G^b(\mathfrak{a}, m, \nu)$ defined as in equation (3.5). By Poisson summation the Fourier coefficients are then given by

$$\tilde{b}_s^B(\mathfrak{b}, \nu, y) = \frac{1}{\operatorname{vol}(\mathfrak{b})}\int_{\mathbb{R}^2}\left(1+\frac{|z_1z_2+B|^2}{4y_1y_2B}\right)^{-s} e(-\operatorname{tr}(\nu x))dx_1dx_2. \tag{3.14}$$

For $\nu \neq 0$, the double integral is too complicated to be solved explicitly. Only one of the integrals can be solved explicitly, for the second one the author did not come up with an explicit solution. However, for our purpose it is enough to estimate $|\tilde{b}_1^B(\mathfrak{b}, \nu, y)|$. Nevertheless, for $\nu = 0$ an estimate of $\tilde{b}_1^B(\mathfrak{b}, 0, y)$ is not enough because the series

$$\sum_{b=1}^{\infty} G^b(\mathfrak{a}, m, 0)\tilde{b}_s^{m/(N(\mathfrak{a})Db^2)}(\mathfrak{a}^{-1}, 0, y) \tag{3.15}$$

diverges at $s = 1$. Rather, we have to determine $\tilde{b}_s^B(\mathfrak{b}, 0, y)$ explicitly to compute the meromorphic continuation at $s = 1$ of (3.15) and extract (or estimate) the constant term.

Lemma 3.7.1. *Let $B > 0$, $\mathfrak{b} \in \mathcal{I}_K$ and $\nu \in (\mathfrak{b}\mathfrak{d})^{-1}$. Then we have (cf. equation* (3.6) *for the definition of $\alpha(\cdot, \cdot)$)*

$$\left|\tilde{b}_1^B(\mathfrak{b}, \nu, y)\right| \leq \frac{4B\pi^2}{\mathrm{vol}(\mathfrak{b})} \exp(-2\pi\alpha(\nu y_1, \nu' y_2)).$$

Proof. We have to estimate the integral given by equation (3.14) at $s = 1$:

$$\begin{aligned}\tilde{b}_1^B(\mathfrak{b}, \nu, y) =& \frac{1}{\mathrm{vol}(\mathfrak{b})} \int_{\mathbb{R}^2} \left(1 + \frac{|z_1 z_2 + B|^2}{4y_1 y_2 B}\right)^{-1} e(-\operatorname{tr}(\nu x)) dx_1 dx_2 \\ =& \frac{4y_1 y_2 B}{\mathrm{vol}(\mathfrak{b})} \int_{\mathbb{R}^2} \left(4y_1 y_2 B + |z_1 z_2 + B|^2\right)^{-1} e(-\operatorname{tr}(\nu x)) dx_1 dx_2.\end{aligned}$$

Now, using the identity

$$4y_1 y_2 B + |z_1 z_2 + B|^2 = |z_2|^2 \left(\left(x_1 + \frac{Bx_2}{|z_2|^2}\right)^2 + \left(y_1 + \frac{By_2}{|z_2|^2}\right)^2\right)$$

the double integral is given by

$$\begin{aligned}&\int_{\mathbb{R}} |z_2|^{-2} \int_{\mathbb{R}} \left(\left(x_1 + \frac{Bx_2}{|z_2|^2}\right)^2 + \left(y_1 + \frac{By_2}{|z_2|^2}\right)^2\right)^{-1} e(-\operatorname{tr}(\nu x)) dx_1 dx_2 \\ =& \int_{\mathbb{R}} |z_2|^{-2} \left(\int_{\mathbb{R}} \left(x_1^2 + a(y_1, z_2)^2\right)^{-1} e(-\nu x_1) dx_1\right) e\left(\frac{\nu B x_2}{|z_2|^2} - \nu x_2\right) dx_2\end{aligned}$$

with $a(y_1, z_2) =: y_1 + \frac{By_2}{|z_2|^2}$. Using [EMOT54, p. 8, eq. (11)] (which holds for $\nu = 0$ as well, even though this case is omitted in the reference), we get for the inner integral

$$\begin{aligned}&\int_{\mathbb{R}} \left(x_1^2 + a(y_1, z_2)^2\right)^{-1} e(-\nu x_1) dx_1 \\ =& 2\int_0^\infty \left(x_1^2 + a(y_1, z_2)^2\right)^{-1} \cos(2\pi|\nu| x_1) dx_1 \\ =& \pi \frac{\exp(-2\pi|\nu| a(y_1, z_2))}{a(y_1, z_2)}.\end{aligned}$$

Coming back to our double integral, we estimate

$$\begin{aligned}&\left|\int_{\mathbb{R}} |z_2|^{-2} \left(\pi \frac{\exp(-2\pi|\nu| a(y_1, z_2))}{a(y_1, z_2)}\right) e\left(\frac{\nu B x_2}{|z_2|^2} - \nu x_2\right) dx_2\right| \\ \leq& \pi \int_{\mathbb{R}} \frac{\exp(-2\pi|\nu| a(y_1, z_2))}{a(y_1, z_2)|z_2|^2} dx_2 \\ \leq& \frac{\pi \exp(-2\pi|\nu| y_1)}{y_1} \int_{\mathbb{R}} \frac{1}{x_2^2 + y_2^2} dx_2 = \frac{\pi^2 \exp(-2\pi|\nu| y_1)}{y_1 y_2}.\end{aligned}$$

Hence, in total we have shown

$$\left|\tilde{b}_1^B(\mathfrak{b},\nu,y)\right| \leq \frac{4B\pi^2}{\mathrm{vol}(\mathfrak{b})}\exp(-2\pi|\nu|y_1).$$

For symmetry reasons we have

$$\left|\tilde{b}_1^B(\mathfrak{b},\nu,y)\right| \leq \frac{4B\pi^2}{\mathrm{vol}(\mathfrak{b})}\exp(-2\pi|\nu'|y_2)$$

as well which proves the claim. □

In order to compute $\tilde{b}_s^B(\mathfrak{b},0,y)$ explicitly, we use the following two lemmata.

Lemma 3.7.2. *Let $a>0$ and $s\in\mathbb{C}$ with $\Re(s)>1/2$. Then we have*

$$\int_{\mathbb{R}}(x^2+a^2)^{-s}dx = a^{1-2s}\,\mathrm{B}(\tfrac{1}{2},s-\tfrac{1}{2}).$$

Here, by $\mathrm{B}(x,y)$ we denote the beta function

$$\mathrm{B}(x,y) := \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}. \tag{3.16}$$

Proof. We obtain by the substitution $x\mapsto ax$

$$\int_{\mathbb{R}}(x^2+a^2)^{-s}dx = \int_{\mathbb{R}}((ax)^2+a^2)^{-s}adx = a^{1-2s}\int_{\mathbb{R}}(x^2+1)^{-s}dx.$$

Now we use of the integral representation (cf. [OLBC10, 5.12.3])

$$\mathrm{B}(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt \tag{3.17}$$

which holds for $\Re(x),\Re(y)>0$:

$$\mathrm{B}(\tfrac{1}{2},s-\tfrac{1}{2}) = \int_0^\infty \frac{t^{1/2-1}}{(1+t)^s}dt = \int_0^\infty \frac{(t^2)^{-1/2}}{(1+t^2)^s}2tdt = 2\int_0^\infty (1+t^2)^{-s}dt.$$

□

Lemma 3.7.3. *Let $\Re(s)>1/2$ and $b>0$. Then we have*

$$\int_{\mathbb{R}}\frac{(x^2+b^2)^{1-2s}}{(x^2+1)^{1-s}}dx = \mathrm{B}\left(\tfrac{1}{2},s-\tfrac{1}{2}\right){}_2F_1\left(2s-1,s-\tfrac{1}{2};s;1-b^2\right).$$

Proof. This integral was solved with Maple and Mathematica independently. However, the result Mathematica gave to the integral was of a much more complex shape but could be reduced to the Maple result by a cumbersome computation using functional equations of the hypergeometric, the gamma and the beta function (all stated in [OLBC10]). □

Lemma 3.7.4. *For $B > 0$, $\mathfrak{b} \in \mathcal{I}_K$ and $\Re(s) > 1/2$ the constant Fourier coefficient of $\tilde{H}_s^B(\mathfrak{b}, z)$ is given by*

$$\tilde{b}_s^B(\mathfrak{b}, 0, y) = \frac{(4B)^s (y_1 y_2)^{1-s} \, \mathrm{B}(\frac{1}{2}, s - \frac{1}{2})^2}{\mathrm{vol}(\mathfrak{b})} {}_2F_1(2s-1, s-\tfrac{1}{2}; s; -B/(y_1 y_2)).$$

Remark 3.7.5. Note that we get with $\mathrm{B}(\frac{1}{2}, \frac{1}{2}) = \pi$ and equation (3.19)

$$\tilde{b}_1^B(\mathfrak{b}, 0, y) = \frac{4B\pi^2}{\mathrm{vol}(\mathfrak{b})} {}_2F_1\left(1, \tfrac{1}{2}; 1; -B/(y_1 y_2)\right) = \frac{4B\pi^2}{\mathrm{vol}(\mathfrak{b})} \frac{1}{\sqrt{1 + B/(y_1 y_2)}}$$

in accordance with the upper bound by Lemma 3.7.1. If we take the limit

$$\lim_{\Im(z) \to \infty} \tilde{b}_1^B(\mathfrak{b}, 0, y) = \frac{4B\pi^2}{\mathrm{vol}(\mathfrak{b})},$$

we actually reach the upper bound.

Proof of Lemma 3.7.4. We can copy the proof of Lemma 3.7.1 until the point of the substitution in the inner integral of the double integral. By that we get

$$\tilde{b}_s^B(\mathfrak{b}, 0, y) = \frac{(4y_1 y_2 B)^s}{\mathrm{vol}(\mathfrak{b})} \int_{\mathbb{R}} |z_2|^{-2s} \int_{\mathbb{R}} \left(x_1^2 + a(y_1, z_2)^2\right)^{-s} dx_1 dx_2.$$

Now using Lemma 3.7.2, the inner integral computes to

$$a(y_1, z_2)^{1-2s} \, \mathrm{B}(\tfrac{1}{2}, s - \tfrac{1}{2}).$$

The integrand of the outer integral is then, up to the beta function factor, given by

$$\begin{aligned}
|z_2|^{-2s} a(y_1, z_2)^{1-2s} &= \frac{1}{|z_2|^{2s}} \left(\frac{y_1 |z_2|^2 + B y_2}{|z_2|^2} \right)^{1-2s} \\
&= \frac{(y_1 |z_2|^2 + B y_2)^{1-2s}}{|z_2|^{2s} |z_2|^{2-4s}} \\
&= \frac{(y_1 (x_2^2 + y_2^2) + B y_2)^{1-2s}}{(x_2^2 + y_2^2)^{1-s}} \\
&= y_1^{1-2s} \frac{(x_2^2 + y_2^2 + B y_2 / y_1)^{1-2s}}{(x_2^2 + y_2^2)^{1-s}}.
\end{aligned}$$

It follows

$$\begin{aligned}
&\int_{\mathbb{R}} |z_2|^{-2s} a(y_1, z_2)^{1-2s} dx_2 \\
&= y_1^{1-2s} \int_{\mathbb{R}} \frac{(x_2^2 + y_2^2 + By_2/y_1)^{1-2s}}{(x_2^2 + y_2^2)^{1-s}} dx_2 \\
&= y_1^{1-2s} \int_{\mathbb{R}} \frac{((x_2y_2)^2 + y_2^2 + By_2/y_1)^{1-2s}}{((x_2y_2)^2 + y_2^2)^{1-s}} y_2 dx_2 \\
&= y_1^{1-2s} y_2^{2(1-2s)-2(1-s)+1} \int_{\mathbb{R}} \frac{(x_2^2 + 1 + B/(y_1y_2))^{1-2s}}{(x_2^2+1)^{1-s}} dx_2 \\
&= (y_1y_2)^{1-2s} \int_{\mathbb{R}} \frac{(x^2 + b(y)^2)^{1-2s}}{(x^2+1)^{1-s}} dx
\end{aligned}$$

with $b(y)^2 = 1 + B/(y_1y_2)$. The last integral is given using Lemma 3.7.3 by

$$\mathrm{B}(\tfrac{1}{2}, s - \tfrac{1}{2})_2F_1(2s-1, s-\tfrac{1}{2}; s; -B/(y_1y_2)).$$

Collecting the omitted prefactors, we get the stated result. □

Now we come to the case $b = 0$. Hence, we determine the Fourier expansion of $\Psi^0(\mathfrak{a}, m, s, z)$. We have

$$\begin{aligned}
\Psi^0(\mathfrak{a}, m, s, z) &= \sum_{\substack{a \in \mathbb{Z}/N(\mathfrak{a}),\, \lambda \in \mathfrak{a}\mathfrak{d}^{-1}/N(\mathfrak{a}) \\ -N(\lambda) = m/(N(\mathfrak{a})D)}} \left(1 + \frac{|-\lambda z_1 - \lambda' z_2 + a|^2}{4y_1y_2m/(N(\mathfrak{a})D)}\right)^{-s} \\
&= 2 \sum_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \sum_{a \in \mathbb{Z}} \left(1 + \frac{|\lambda z_1 + \lambda' z_2 + a|^2}{4y_1y_2mN(\mathfrak{a})/D}\right)^{-s}.
\end{aligned}$$

Lemma 3.7.6. *The series*

$$\Psi^0(\mathfrak{a}, m, s, z) = 2 \sum_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \sum_{a \in \mathbb{Z}} \left(1 + \frac{|\lambda z_1 + \lambda' z_2 + a|^2}{4y_1y_2mN(\mathfrak{a})/D}\right)^{-s}$$

converges normally for $z \in \mathbb{H}^2$ and $\Re(s) > 1/2$ and has the Fourier expansion

$$\begin{aligned}
\Psi^0(\mathfrak{a}, m, s, z) = 2 &\left(\frac{4y_1y_2mN(\mathfrak{a})}{D}\right)^s \mathrm{B}(\tfrac{1}{2}, s - \tfrac{1}{2}) \sum_{\lambda \in \Lambda^+(\mathfrak{a}, m)} (\lambda y_1 - \lambda' y_2)^{1-2s} \\
&+ \frac{4\pi^s}{\Gamma(s)} \left(\frac{4y_1y_2mN(\mathfrak{a})}{D}\right)^s \sum_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \sum_{n=1}^{\infty} \left(\frac{n}{\lambda y_1 - \lambda' y_2}\right)^{s-1/2} \\
&\times K_{s-1/2}(2\pi n(\lambda y_1 - \lambda' y_2)) \left(e(n \operatorname{tr}(\lambda x)) + e(-n \operatorname{tr}(\lambda x))\right).
\end{aligned}$$

Proof. As in Lemma 3.2.5, we investigate the series over a for each $\lambda \in \Lambda^+(\mathfrak{a}, m)$ individually:

$$\begin{aligned}
&\sum_{a\in\mathbb{Z}} \left(1 + \frac{|\lambda z_1 + \lambda' z_2 + a|^2}{4y_1y_2mN(\mathfrak{a})/D}\right)^{-s} \\
&= \sum_{a\in\mathbb{Z}} \left(1 + \frac{(\mathrm{tr}(\lambda x) + a)^2 + \mathrm{tr}(\lambda y)^2}{-4y_1y_2\lambda\lambda'}\right)^{-s} \\
&= \left(\frac{4y_1y_2mN(\mathfrak{a})}{D}\right)^s \sum_{a\in\mathbb{Z}} \Big((\mathrm{tr}(\lambda x) + a)^2 + (\lambda y_1 - \lambda' y_2)^2\Big)^{-s}.
\end{aligned}$$

Hence, we are interested in the Fourier expansion of the $\mathbb{Z}$ periodic function

$$h_\gamma(s, x) := \sum_{a\in\mathbb{Z}} \Big((x + a)^2 + \gamma^2\Big)^{-s}$$

with $\gamma > 0$ (note that actually always $\gamma := |\lambda y_1 - \lambda' y_2| > 0$ since $\lambda y_1 \neq \lambda' y_2$ due to $N(\lambda) < 0$). It is straightforward to see that $h_\gamma(s, x)$ converges if and only if $\Re(s) > 1/2$. We have

$$h_\gamma(s, x) = \sum_{n\in\mathbb{Z}} a_\gamma(s, n)e(nx)$$

with

$$a_\gamma(s, n) = \int_{\mathbb{R}} (x^2 + \gamma^2)^{-s} e(-nx)dx.$$

Lemma 3.7.2 yields

$$a_\gamma(s, 0) = \mathrm{B}(\tfrac{1}{2}, s - \tfrac{1}{2})\gamma^{1-2s}.$$

For $n \neq 0$ we use [EMOT54, p. 11, eq. (7)] (valid for $\Re(s) > 0$)

$$\begin{aligned}
a_\gamma(s, n) &= 2\int_0^\infty (x^2 + \gamma^2)^{-s} \cos(2\pi|n|x)dx \\
&= 2\left(\frac{\pi|n|}{\gamma}\right)^{s-1/2} \sqrt{\pi}\Gamma(s)^{-1} K_{s-1/2}(2\pi\alpha|n|) \\
&= \frac{2\pi^s}{\Gamma(s)} \left(\frac{|n|}{\gamma}\right)^{s-1/2} K_{s-1/2}(2\pi\gamma|n|).
\end{aligned}$$

We obtain

$$\begin{aligned}
&\sum_{a\in\mathbb{Z}} \left(1 + \frac{|\lambda z_1 + \lambda' z_2 + a|^2}{4y_1y_2mN(\mathfrak{a})/D}\right)^{-s} = \left(\frac{4y_1y_2mN(\mathfrak{a})}{D}\right)^s h_{|\lambda y_1 - \lambda' y_2|}(s, \mathrm{tr}(\lambda x)) \\
&= \left(\frac{4y_1y_2mN(\mathfrak{a})}{D}\right)^s \mathrm{B}(\tfrac{1}{2}, s - \tfrac{1}{2})|\lambda y_1 - \lambda' y_2|^{1-2s} \\
&+ \frac{2\pi^s}{\Gamma(s)} \left(\frac{4y_1y_2mN(\mathfrak{a})}{D}\right)^s \sum_{n\in\mathbb{Z}}{}' \left|\frac{n}{\lambda y_1 - \lambda' y_2}\right|^{s-1/2} K_{s-1/2}(2\pi|n||\lambda y_1 - \lambda' y_2|)e(\mathrm{tr}(\lambda nx))
\end{aligned}$$

which proves the lemma. Here, the tick at the sum indicates that we do not sum over $n = 0$. □

Corollary 3.7.7. *The function $\Psi^0(\mathfrak{a}, m, 1, z)$ has the Fourier expansion*

$$\frac{8\pi y_1 y_2 m N(\mathfrak{a})}{D} \sum_{\lambda \in \Lambda^+(\mathfrak{a},m)} \sum_{n \in \mathbb{Z}} \frac{\exp(-2\pi|n|(\lambda y_1 - \lambda' y_2))}{\lambda y_1 - \lambda' y_2} e(n \operatorname{tr}(\lambda x)).$$

Proof. Plugging in $s = 1$ into Lemma 3.7.6 with the identity

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} \exp(-x)$$

yields the Fourier expansion. □

3.7.3 New identities for the Fourier coefficients of $\Phi(\mathfrak{a}, m, s, z)$

In the previous subsection we determined most of the Fourier coefficients of $\Psi^b(\mathfrak{a}, m, s, z)$ explicitly. Using the decomposition

$$\Phi^b(\mathfrak{a}, m, s, z) = \sum_{n=0}^{\infty} \frac{\Gamma(s+n)^2}{\Gamma(2s+n)} \frac{\Psi^b(\mathfrak{a}, m, s+n, z)}{2n!}, \tag{3.18}$$

we get new formulae for the Fourier coefficients of $\Phi^b(\mathfrak{a}, m, s, z)$ (and therefore also for the Fourier coefficients of $\Phi(\mathfrak{a}, m, s, z)$). Recall that we determined the Fourier coefficients of $\Phi^b(\mathfrak{a}, m, s, z)$ already in Section 3.2. Hence, we now have two different formulae for most of the Fourier coefficients of $\Phi^b(\mathfrak{a}, m, s, z)$ which give rise to new identities. The author proved most of the new identities for verification and curiosity reasons for the Fourier coefficents of $\Psi^b(\mathfrak{a}, m, s, z)$ already directly. We invite the inclined reader to do the same and omit those proofs in this thesis.

We start by comparing the Fourier coefficients of $\Phi^0(\mathfrak{a}, m, s, z)$. By Lemma 3.2.5 the constant Fourier coefficient is given by

$$\frac{4\pi}{2s-1} \sum_{\lambda \in \Lambda^+(\mathfrak{a},m)} \alpha(\lambda y_1, \lambda' y_2)^{1-s} \beta(\lambda y_1, \lambda' y_2)^s.$$

Lemma 3.7.6 together with the decomposition (3.18) yields the representation

$$\sum_{n=0}^{\infty} \frac{\Gamma(s+n)^2}{\Gamma(2s+n)n!} \left(\frac{4y_1 y_2 m N(\mathfrak{a})}{D}\right)^{s+n} \mathrm{B}\left(\tfrac{1}{2}, s+n-\tfrac{1}{2}\right) \times \sum_{\lambda \in \Lambda^+(\mathfrak{a},m)} (\lambda y_1 - \lambda' y_2)^{1-2(s+n)}$$

for the same coefficient. In both cases we have a sum over all $\lambda \in \Lambda^+(\mathfrak{a}, m)$ and the equality actually holds termwise for all $s \in \mathbb{C}$ with $\Re(s) > 1/2$:

$$\begin{aligned} &\frac{4\pi}{2s-1} \alpha(\lambda y_1, \lambda' y_2)^{1-s} \beta(\lambda y_1, \lambda' y_2)^s \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(s+n)^2}{\Gamma(2s+n)n!} \left(\frac{4y_1 y_2 m N(\mathfrak{a})}{D}\right)^{s+n} \mathrm{B}\left(\tfrac{1}{2}, s+n-\tfrac{1}{2}\right) (\lambda y_1 - \lambda' y_2)^{1-2(s+n)}. \end{aligned}$$

Lemma 3.2.5 also tells us that the Fourier coefficient of $\Phi^0(\mathfrak{a}, m, s, z)$ of index $k\lambda$ with $\lambda \in \Lambda^+(\mathfrak{a}, m)$ and $k \in \mathbb{N}$ is given by

$$4\pi\sqrt{|\lambda\lambda' y_1 y_2|} I_{s-1/2}(2\pi k\beta(\lambda y_1, \lambda' y_2)) K_{s-1/2}(2\pi k\alpha(\lambda y_1, \lambda' y_2)).$$

By Lemma 3.7.6 together with the decomposition (3.18) we obtain

$$2\sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{\Gamma(2s+n)n!} \left(\frac{4\pi y_1 y_2 m N(\mathfrak{a})}{D}\right)^{s+n} \left(\frac{k}{\lambda y_1 - \lambda' y_2}\right)^{s+n-1/2} \times K_{s+n-1/2}(2\pi k(\lambda y_1 - \lambda' y_2)).$$

The two identities we obtained with that comparison involve number theoretic magnitudes like $\lambda \in \Lambda^+(\mathfrak{a}, m)$ and the norm of the ideal $\mathfrak{a}$. We can get rid of them and obtain two purely analytic identities:

Proposition 3.7.8. *Let $\alpha \geq \beta > 0$ and $\Re(s) > 1/2$. Then we have*

$$\sum_{n=0}^{\infty} \frac{\Gamma(s+n)^2}{\Gamma(2s+n)} \frac{(4\alpha\beta)^{s+n}}{n!} \mathrm{B}(\tfrac{1}{2}, s+n-\tfrac{1}{2})(\alpha+\beta)^{1-2(s+n)} = \frac{4\pi\alpha^{1-s}\beta^s}{2s-1}$$

and

$$\sum_{n=0}^{\infty} \frac{\Gamma(s+n)(4\pi\alpha\beta)^{s+n}}{\Gamma(2s+n)n!(\alpha+\beta)^{s+n-1/2}} K_{s+n-1/2}(2\pi(\alpha+\beta)) = 2\pi\sqrt{\alpha\beta} I_{s-1/2}(2\pi\beta) K_{s-1/2}(2\pi\alpha).$$

A last identity we can obtain comes from the constant Fourier coefficient of $\Phi^b(\mathfrak{a}, m, s, z)$ for $b \in \mathbb{N}$. Recall that we did not compute but only estimated the non-constant coefficients of $\Psi^b(\mathfrak{a}, m, s, z)$. That is why we do not get an identity involving those. By Lemma 3.2.4 together with the preceding work the constant Fourier coefficient of $\Phi^b(\mathfrak{a}, m, s, z)$ is given by

$$\frac{\pi N(\mathfrak{a})\Gamma(s-\frac{1}{2})^2}{2\sqrt{D}\Gamma(2s)} \left(\frac{4m}{N(\mathfrak{a})Db^2}\right)^s (y_1 y_2)^{1-s} G^b(\mathfrak{a}, m, 0).$$

Using Lemma 3.7.4 and decomposition (3.18), we obtain

$$\sum_{n=0}^{\infty} \frac{\Gamma(s+n)^2}{2\Gamma(2s+n)n!} \left(\frac{4m}{N(\mathfrak{a})Db^2}\right)^{s+n} \frac{(y_1 y_2)^{1-(s+n)} N(\mathfrak{a})\, \mathrm{B}(\frac{1}{2}, s+n-\frac{1}{2})^2}{\sqrt{D}} \times {}_2F_1\left(2(s+n)-1, s+n-\tfrac{1}{2}; s+n; -\tfrac{m}{N(\mathfrak{a})Db^2 y_1 y_2}\right) G^b(\mathfrak{a}, m, 0)$$

instead. By canceling identical factors and resolving the beta function we obtain the following identity.

Proposition 3.7.9. *For all $x > 0$ and $s \in \mathbb{C}$ with $\Re(s) > 1/2$ we have*

$$\frac{\Gamma(s-\frac{1}{2})^2}{\Gamma(2s)} = \sum_{n=0}^{\infty} \frac{\Gamma(s+n-\frac{1}{2})^2}{\Gamma(2s+n)n!} (4x)^n \, {}_2F_1\left(2(s+n)-1, s+n-\tfrac{1}{2}; s+n; -x\right).$$

3.7.4 Fourier coefficients of $\Phi_0(\mathfrak{a}, m, z)$

In this subsection we mostly put together the work of the previous subsection to obtain the Fourier coefficients (or good enough estimates) of $\Phi_0(\mathfrak{a}, m, z)$, the regularization of $\Phi_0(\mathfrak{a}, m, s, z)$ at $s = 1$.

Lemma 3.7.10. *For $\Re(s) > 0$ the series*

$$\sum_{b=1}^{\infty} G^b(\mathfrak{a}, m, 0) b^{-2s} \left(1 - {}_2F_1\left(2s-1, s-\tfrac{1}{2}; s; -\frac{m}{N(\mathfrak{a})Db^2 y_1 y_2}\right)\right)$$

converges normally and is hence holomorphic in s. At $s = 1$ the value of the series is bounded by

$$\frac{m\pi^2}{6N(\mathfrak{a})y_1 y_2}.$$

Proof. Let y be fixed. Then the function

$$f(s, x) := 1 - {}_2F_1\left(2s-1, s-\tfrac{1}{2}; s; -\frac{mx}{N(\mathfrak{a})Dy_1 y_2}\right)$$

is analytic in x at $x = 0$. Therefore, we have the power series representation

$$f(s, x) = \sum_{n=1}^{\infty} a_n(s) x^n$$

with coefficients $a_n(s)$ continuous in s. Hence, for fixed $s \in \mathbb{C}$ there exist constants $C_s > 0$ and $x_s > 0$ such that for all $|x| \leq x_s$ we have

$$|f(s, x)| \leq C_s |x|.$$

Let $b_0 \in \mathbb{N}$ be large enough such that for all $|x| \leq 1/b_0^2$ the above estimate holds. Then we have

$$\begin{aligned}
&\left|\sum_{b=b_0}^{\infty} G^b(\mathfrak{a}, m, 0) b^{-2s} \left(1 - {}_2F_1\left(2s-1, s-\tfrac{1}{2}; s; -\frac{m}{N(\mathfrak{a})Db^2 y_1 y_2}\right)\right)\right| \\
&= \left|\sum_{b=b_0}^{\infty} G^b(\mathfrak{a}, m, 0) b^{-2s} f(s, 1/b^2)\right| \leq C_s \sum_{b=b_0}^{\infty} G^b(\mathfrak{a}, m, 0) b^{-2(\Re(s)+1)}.
\end{aligned}$$

If $\Re(s) > 0$, this series converges by Lemma 3.3.2. The constants C_s and x_s can be chosen continuously in s which implies normal convergence for the series in consideration.

For $s = 1$ we use the identity

$$ {}_2F_1\left(1, \tfrac{1}{2}; 1; -x\right) = \frac{1}{\sqrt{x+1}}. \tag{3.19}$$

Note that for $x \geq 0$ we have

$$0 \leq 1 - \sqrt{\frac{1}{1+x}} \leq x.$$

Therefore, we get

$$\begin{aligned}
0 &\leq \sum_{b=1}^{\infty} G^b(\mathfrak{a}, m, 0) b^{-2} \left(1 - {}_2F_1\left(1, \tfrac{1}{2}; 1; -\frac{m}{N(\mathfrak{a}) D b^2 y_1 y_2}\right)\right) \\
&\leq \sum_{b=1}^{\infty} G^b(\mathfrak{a}, m, 0) b^{-2} \frac{m}{N(\mathfrak{a}) D b^2 y_1 y_2} \\
&= \frac{m}{N(\mathfrak{a}) D y_1 y_2} \sum_{b=1}^{\infty} G^b(\mathfrak{a}, m, 0) b^{-4} \\
&\leq \frac{m}{N(\mathfrak{a}) D y_1 y_2} \sum_{b=1}^{\infty} D b^{-2} = \frac{m \pi^2}{6 N(\mathfrak{a}) y_1 y_2}.
\end{aligned}$$

□

Theorem 3.7.11. *The function $\Phi_0(\mathfrak{a}, m, z)$ has a Fourier expansion of the form*

$$\Phi_0(\mathfrak{a}, m, z) = \frac{\Psi^0(\mathfrak{a}, m, 1, z)}{2} + \sum_{\nu \in \mathfrak{a}\mathfrak{d}^{-1}} u_\nu(\mathfrak{a}, m, y) e(\operatorname{tr}(\nu x)).$$

The Fourier expansion of $\Psi^0(\mathfrak{a}, m, 1, z)$ is given by Corollary 3.7.7. The Fourier coefficients $u_\nu(\mathfrak{a}, m, y)$ satisfy

$$|u_0(\mathfrak{a}, m, y)| \leq |L(\mathfrak{a}, m) - q(\mathfrak{a}, m) \log(16\pi^2 y_1 y_2)| + \frac{2m^2\pi^4}{3D^{3/2} N(\mathfrak{a}) y_1 y_2}$$

and

$$|u_\nu(\mathfrak{a}, m, y)| \leq C m \sqrt{|N(\nu)|} \exp(-2\pi\alpha(\nu y_1, \nu' y_2))$$

for $\nu \neq 0$. Here, $C > 0$ is a constant only dependent on the ideal $\mathfrak{a}$ (and in particular independent of m).

Proof. Recall that we have

$$\begin{aligned}
\Phi_0(\mathfrak{a}, m, z) &= \mathcal{C}_{s=1}\left[\Phi_0(\mathfrak{a}, m, s, z)\right] \\
&= \mathcal{C}_{s=1}\left[\frac{\Gamma(s)^2}{2\Gamma(2s)}\left(\Psi^0(\mathfrak{a}, m, s, z) + 2\sum_{b=1}^{\infty} \Psi^b(\mathfrak{a}, m, s, z)\right)\right].
\end{aligned}$$

The individual $\Psi^b(\mathfrak{a}, m, s, z)$ are well-defined and holomorphic for $\Re(s) > 1/2$. Thus, we obtain

$$\Phi_0(\mathfrak{a}, m, z) = \frac{\Psi^0(\mathfrak{a}, m, 1, z)}{2} + \mathcal{C}_{s=1}\left[\frac{\Gamma(s)^2}{\Gamma(2s)} \sum_{b=1}^{\infty} \Psi^b(\mathfrak{a}, m, s, z)\right].$$

This explains the $\Psi^0(\mathfrak{a}, m, 1, z)$ part in the Fourier expansion discussed in Corollary 3.7.7. Accordingly,

$$\sum_{\nu \in \mathfrak{a}\mathfrak{d}^{-1}} u_\nu(\mathfrak{a}, m, y) e(\mathrm{tr}(\nu x))$$

is the Fourier expansion of

$$\begin{aligned}
&\mathcal{C}_{s=1}\left[\frac{\Gamma(s)^2}{\Gamma(2s)} \sum_{b=1}^{\infty} \Psi^b(\mathfrak{a}, m, s, z)\right] \\
=&\mathcal{C}_{s=1}\left[\frac{\Gamma(s)^2}{\Gamma(2s)} \sum_{\nu \in \mathfrak{a}\mathfrak{d}^{-1}} \left(\sum_{b=1}^{\infty} G^b(\mathfrak{a}, m, \nu) \tilde{b}_s^{m/(N(\mathfrak{a})Db^2)}(\mathfrak{a}^{-1}, \nu, y)\right) e(\mathrm{tr}(\nu x))\right].
\end{aligned}$$

As in the proof of Theorem 3.4.1, the coefficients $u_\nu(\mathfrak{a}, m, y)$ for $\nu \neq 0$ possess a holomorphic extension to $\Re(s) > 3/4$ and we can simply plug in $s = 1$ and obtain

$$u_\nu(\mathfrak{a}, m, y) = \sum_{b=1}^{\infty} G^b(\mathfrak{a}, m, \nu) \tilde{b}_1^{m/(N(\mathfrak{a})Db^2)}(\mathfrak{a}^{-1}, \nu, y).$$

To get the estimates of $u_\nu(\mathfrak{a}, m, y)$ for $\nu \neq 0$ of the statement, we use Lemma 3.7.1 and obtain

$$\left|\tilde{b}_1^{m/(N(\mathfrak{a})Db^2)}(\mathfrak{a}^{-1}, \nu, y)\right| \leq \frac{4\pi^2 m}{b^2 D^{3/2}} \exp(-2\pi\alpha(\nu y_1, \nu' y_2)).$$

Using Lemma 3.3.1 to estimate $|G^b(\mathfrak{a}, m, \nu)|$, we get

$$\begin{aligned}
|u_\nu(\mathfrak{a}, m, y)| &\leq \sum_{b=1}^{\infty} \left|G^b(\mathfrak{a}, m, \nu) \tilde{b}_1^{m/(N(\mathfrak{a})Db^2)}(\mathfrak{a}^{-1}, \nu, y)\right| \\
&\leq \frac{4\pi^2 m C}{D^{3/2}} \sqrt{|N(\nu)|} \exp(-2\pi\alpha(\nu y_1, \nu' y_2)) \sum_{b=1}^{\infty} d(b) b^{-3/2} \\
&= \frac{4\pi^2 m C}{D^{3/2}} \zeta(3/2)^2 \sqrt{|N(\nu)|} \exp(-2\pi\alpha(\nu y_1, \nu' y_2))
\end{aligned}$$

with a constant $C > 0$ only depending on $\mathfrak{a}$ by Lemma 3.3.1.

Now we are left with proving the stated estimate of

$$u_0(\mathfrak{a}, m, y) = \mathcal{C}_{s=1}\left[\frac{\Gamma(s)^2}{\Gamma(2s)} \sum_{b=1}^{\infty} G^b(\mathfrak{a}, m, 0) \tilde{b}_s^{m/(N(\mathfrak{a})Db^2)}(\mathfrak{a}^{-1}, 0, y)\right].$$

Using Lemma 3.7.4, we can rewrite this series to

$$\begin{aligned}
&\frac{\Gamma(s)^2}{\Gamma(2s)} \sum_{b=1}^{\infty} G^b(\mathfrak{a}, m, 0) \frac{(4m/(N(\mathfrak{a})Db^2))^s (y_1 y_2)^{1-s} \,\mathrm{B}(\frac{1}{2}, s - \frac{1}{2})^2}{\mathrm{vol}(\mathfrak{a}^{-1})} \\
&\qquad \times\ {}_2F_1\left(2s - 1, s - \tfrac{1}{2}; s; -(m/(N(\mathfrak{a})Db^2))/(y_1 y_2)\right) \\
&= \frac{\pi \Gamma(s - \frac{1}{2})^2}{\sqrt{D}\Gamma(2s)} (4m/D)^s (N(\mathfrak{a}) y_1 y_2)^{1-s} \sum_{b=1}^{\infty} G^b(\mathfrak{a}, m, 0) b^{-2s} \\
&\qquad \times\ {}_2F_1\left(2s - 1, s - \tfrac{1}{2}; s; -\frac{m}{N(\mathfrak{a})Db^2 y_1 y_2}\right).
\end{aligned}$$

This looks surprisingly similar to the meromorphic part (3.7) of $\Phi(\mathfrak{a}, m, s, z)$. The only difference is the additional hypergeometric factor at the end. In the proof of Theorem 3.4.1 we have seen that (3.7) has a meromorphic continuation to $\Re(s) > 3/4$ with simple pole at $s = 1$ of residue $q(\mathfrak{a}, m)$. We make use of this computation and consider the difference of the two series. The pole should cancel out and we expect a holomorphic function for $\Re(s) > 3/4$. The truth is even better, Lemma 3.7.10 shows that the difference has a holomorphic continuation to $\Re(s) > 0$. Now $u_0(\mathfrak{a}, m, y)$ is given by that holomorphic difference evaluated at $s = 1$ plus the constant term in the Laurent expansion of (3.7) which we computed in Theorem 3.4.1 to be $L(\mathfrak{a}, m) - q(\mathfrak{a}, m)\log(16\pi^2 y_1 y_2)$. Hence, we get with the estimate of Lemma 3.7.10

$$|u_0(\mathfrak{a}, m, y)| \leq |L(\mathfrak{a}, m) - q(\mathfrak{a}, m)\log(16\pi^2 y_1 y_2)| + \frac{2m^2\pi^4}{3D^{3/2}N(\mathfrak{a})y_1 y_2}.$$

□

3.8 Integrability and integrals

In this section we compute the integral of $\Phi(\mathfrak{a}, m, z)$. In order to do so, we compute the integrals of $\Psi(\mathfrak{a}, m, s, z)$ and $\Phi(\mathfrak{a}, m, s, z)$ as well. Our method of computing $\Phi(\mathfrak{a}, m, z)$ requires the integrability of $\Phi(\mathfrak{a}, m, z)$ first which is much more demanding to show than computing the actual integral afterwards. In the process we prove that the growth of

$$\int_{X(\mathfrak{a})} |\Phi(\mathfrak{a}, m, z)|\omega^2$$

is polynomial in m which is an important ingredient for the main theorem of this thesis.

We start with computing the integral of $\Psi(\mathfrak{a}, m, s, z)$ for which we need the following lemma.

Lemma 3.8.1. *Let $\Re(s) > 1$. Then we have*

$$\int_{\mathbb{H}} \left(1 + \frac{|z-i|^2}{4y}\right)^{-s} \frac{dxdy}{y^2} = \frac{4\pi}{s-1}.$$

Proof. We have by Lemma 3.7.2

$$\begin{aligned}
\int_{\mathbb{H}} \left(1 + \frac{|z-i|^2}{4y}\right)^{-s} \frac{dxdy}{y^2} &= 4^s \int_{\mathbb{H}} \left(4y + (x^2 + (y-1)^2)\right)^{-s} \frac{dxdy}{y^{2-s}} \\
&= 4^s \int_0^\infty \int_{-\infty}^\infty \left(x^2 + (y+1)^2\right)^{-s} dx\ y^{s-2} dy \\
&= 4^s \int_0^\infty B(\tfrac{1}{2}, s - \tfrac{1}{2})(y+1)^{1-2s} y^{s-2} dy \\
&= 4^s B(\tfrac{1}{2}, s - \tfrac{1}{2}) \int_0^\infty \frac{y^{s-2}}{(y+1)^{2s-1}} dy \\
&= 4^s B(\tfrac{1}{2}, s - \tfrac{1}{2}) B(s-1, s).
\end{aligned}$$

The last identity is due to equation (3.17) where we need $\Re(s) > 1$. Now we make use of the Legendre duplication formula and the functional equation of the gamma function:

$$\begin{aligned}
4^s B(\tfrac{1}{2}, s-\tfrac{1}{2}) B(s-1, s) &= 4^s \frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})\Gamma(s-1)\Gamma(s)}{\Gamma(s)\Gamma(2s-1)} \\
&= 4^s \sqrt{\pi} \frac{\Gamma(s-\frac{1}{2})\Gamma(s-1)}{\Gamma(2s-1)} \\
&= 2^{2s} \sqrt{\pi} \frac{\sqrt{\pi} 2^{1-2(s-1)} \Gamma(2s-2)}{\Gamma(2s-1)} \\
&= \frac{8\pi}{2s-2} = \frac{4\pi}{s-1}.
\end{aligned}$$

□

In the next theorem we compute the integral of $\Psi(\mathfrak{a}, m, s, z)$ over $X(\mathfrak{a})$. We can do that by unfolding the integral without determining a fundamental domain for $X(\mathfrak{a})$ explicitly. This approach is explained in the upcoming proof in detail. In later proofs we will shorten the argument.

Theorem 3.8.2. *For $\Re(s) > 1$ we have*

$$\int_{X(\mathfrak{a})} \Psi(\mathfrak{a}, m, s, z) \omega^2 = \frac{4}{s-1} \operatorname{vol}(T(\mathfrak{a}, m)).$$

Proof. In this proof we will freely interchange integration and summation. Looking at the definition

$$\Psi(\mathfrak{a}, m, s, z) := \sum_{\substack{A \in L(\mathfrak{a})^\vee \\ \det(A) = m/(N(\mathfrak{a})D)}} (1 + g(A, z))^{-s},$$

we see that for $s \in \mathbb{R}$ this is justified by Tonelli's theorem because all summands are positive. For $s \in \mathbb{C}$ we see

$$|\Psi(\mathfrak{a}, m, s, z)| \leq \Psi(\mathfrak{a}, m, \Re(s), z)$$

using the triangle inequality, hence Fubini's theorem can be applied by Lebesgue's dominated convergence theorem.

We start by some rewriting of $\Psi(\mathfrak{a}, m, s, z)$. Subsequently, we explain every step.

$$\begin{aligned}
\Psi(\mathfrak{a}, m, s, z) &\overset{(i)}{=} 2 \sum_{\substack{A \in L(\mathfrak{a})^\vee/\{\pm 1\} \\ \det(A) = m/(N(\mathfrak{a})D)}} (1 + g(A, z))^{-s} \\
&\overset{(ii)}{=} 2 \sum_{\substack{A \in \Gamma_\mathfrak{a} \backslash L(\mathfrak{a})^\vee/\{\pm 1\} \\ \det(A) = m/(N(\mathfrak{a})D)}} \sum_{M \in \Gamma_\mathfrak{a}/\Gamma_{\mathfrak{a},\pm A}} (1 + g(M.A, z))^{-s} \\
&\overset{(iii)}{=} 2 \sum_{\substack{A \in \Gamma_\mathfrak{a} \backslash L(\mathfrak{a})^\vee/\{\pm 1\} \\ \det(A) = m/(N(\mathfrak{a})D)}} \sum_{M \in \Gamma_\mathfrak{a}/\Gamma_{\mathfrak{a},\pm A}} \left(1 + g(A, M^{-1}z)\right)^{-s} \\
&\overset{(iv)}{=} 2 \sum_{\substack{A \in \Gamma_\mathfrak{a} \backslash L(\mathfrak{a})^\vee/\{\pm 1\} \\ \det(A) = m/(N(\mathfrak{a})D)}} \sum_{M \in \Gamma_{\mathfrak{a},\pm A} \backslash \Gamma_\mathfrak{a}} (1 + g(A, Mz))^{-s}.
\end{aligned}$$

In step (i) we use the sign invariance of $g(A, z)$ (cf. Remark 2.6.3). In step (ii) we group up the summands. We factor out the action of $\Gamma_\mathfrak{a}$ on $L(\mathfrak{a})^\vee/\{\pm 1\}$. The resulting quotient is finite and each element corresponds to one component of $T(\mathfrak{a}, m)$ viewed as divisor of $X(\mathfrak{a})$. Now for each element in the quotient we have to sum over the whole $\Gamma_\mathfrak{a}$ orbit to obtain all original summands back. This is what the inner sum does. We have to factor out the stabilizer

$$\Gamma_{\mathfrak{a},\pm A} := \{M : \ M \in \Gamma_\mathfrak{a} \quad \text{and} \quad M.A \in \{\pm A\}\}$$

in order to obtain every element in the orbit once. In step (iii) we use the invariance of $g(A, z)$ introduced in Corollary 2.6.2. In step (iv) we invert $\Gamma_\mathfrak{a}/\Gamma_{\mathfrak{a},\pm A}$ which turns the left cosets into right cosets. We make good for the inversion by inverting M^{-1} as well.

Because the inner sum is invariant under $\Gamma_\mathfrak{a}$ for each fixed $A \in L(\mathfrak{a})^\vee$ with $\det(A) = m/(N(\mathfrak{a})D)$, we can compute the integral of that inner sum over $X(\mathfrak{a}) = \Gamma_\mathfrak{a} \backslash \mathbb{H}^2$ first on its own. Again, we explain the equations step by step after the computation.

$$\begin{aligned}
&\int_{\Gamma_\mathfrak{a} \backslash \mathbb{H}^2} \sum_{M \in \Gamma_{\mathfrak{a},\pm A} \backslash \Gamma_\mathfrak{a}} (1 + g(A, Mz))^{-s} \, \omega^2 \\
&\overset{(v)}{=} \int_{\Gamma_{\mathfrak{a},\pm A} \backslash \mathbb{H}^2} (1 + g(A, z))^{-s} \, \omega^2 \\
&\overset{(vi)}{=} 2 \int_{z_2 \in \Gamma'_{\mathfrak{a},\pm A} \backslash \mathbb{H}} \int_{z_1 \in \mathbb{H}} (1 + g(A, z))^{-s} \, \eta_1 \eta_2 \\
&\overset{(vii)}{=} 2 \int_{z_2 \in \Gamma'_{\mathfrak{a},\pm A} \backslash \mathbb{H}} \int_{z_1 \in \mathbb{H}} \left(1 + \frac{d(z_1, ASz_2)}{4}\right)^{-s} \eta_1 \eta_2 \\
&\overset{(viii)}{=} 2 \int_{z_2 \in \Gamma'_{\mathfrak{a},\pm A} \backslash \mathbb{H}} \int_{z_1 \in \mathbb{H}} \left(1 + \frac{d(z_1, i)}{4}\right)^{-s} \eta_1 \eta_2 \\
&\overset{(ix)}{=} \frac{2}{s-1} \int_{z_2 \in \Gamma'_{\mathfrak{a},\pm A} \backslash \mathbb{H}} \eta_2 \overset{(x)}{=} \frac{2}{s-1} \operatorname{vol}(T_A).
\end{aligned}$$

In step (v) the actual unfolding takes place. Instead of integrating a sum of $\Gamma_{\mathfrak{a},\pm A}\backslash\Gamma_{\mathfrak{a}}$ shifted functions over $\Gamma_{\mathfrak{a}}\backslash\mathbb{H}^2$ it is possible to integrate over $\Gamma_{\mathfrak{a},\pm A}\backslash\mathbb{H}^2$ in the first place and skip the sum and shifting. In step (vi) we use the fact that up to a set of measure zero a fundamental domain of $\Gamma_{\mathfrak{a},\pm A}\backslash\mathbb{H}^2$ is given by $\mathbb{H}\times\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}$. Further, we use that $\omega^2 = 2\eta_1\eta_2$ (cf. equation (2.14)). Step (vii) is an application of Lemma 2.6.6. Step (viii) is a consequence of the $\mathrm{GL}_2^+(\mathbb{R})$ invariance of η_1 and the hyperbolic distance. Since we integrate over all of $\mathbb{H}$ in the first argument, the reference point in the second argument is arbitrary. Step (ix) is an application of Lemma 3.8.1 (note the scaling of η with $(4\pi)^{-1}$ in equation (2.14)). Step (x) is based on equation (2.40).

In total we have

$$\int_{X(\mathfrak{a})}\Psi(\mathfrak{a},m,s,z)\omega^2 = 2\sum_{\substack{A\in\Gamma_{\mathfrak{a}}\backslash L(\mathfrak{a})^\vee/\{\pm1\}\\ \det(A)=m/(N(\mathfrak{a})D)}}\frac{2}{s-1}\operatorname{vol}(T_A) = \frac{4}{s-1}\operatorname{vol}(T(\mathfrak{a},m))$$

by equation (2.39). □

This allows us to compute the integral of $\Phi(\mathfrak{a},m,s,z)$.

Theorem 3.8.3. *For $\Re(s)>1$ we have*

$$\int_{X(\mathfrak{a})}\Phi(\mathfrak{a},m,s,z)\omega^2 = \frac{2\operatorname{vol}(T(\mathfrak{a},m))}{s(s-1)}.$$

Proof. To compute the integral we use the decomposition

$$\Phi(\mathfrak{a},m,s,z) = \sum_{n=0}^{\infty}\frac{\Gamma(s+n)^2}{\Gamma(2s+n)}\frac{\Psi(\mathfrak{a},m,s+n,z)}{2n!}$$

and Theorem 3.8.2. The reason why we are allowed to interchange summation and integration is the same as in the proof of Theorem 3.8.2. We get

$$\begin{aligned}\int_{X(\mathfrak{a})}\Phi(\mathfrak{a},m,s,z) &= \sum_{n=0}^{\infty}\frac{\Gamma(s+n)^2}{\Gamma(2s+n)}\frac{\frac{4}{s+n-1}\operatorname{vol}(T(\mathfrak{a},m))}{2n!}\\ &= 2\operatorname{vol}(T(\mathfrak{a},m))\sum_{n=0}^{\infty}\frac{\Gamma(s+n)^2}{\Gamma(2s+n)}\frac{1}{n!}\frac{1}{s+n-1}.\end{aligned}$$

Now using the functional equation

$$\Gamma(s+n) = (s+n-1)\Gamma(s+n-1)$$

of the gamma function we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\Gamma(s+n)^2}{\Gamma(2s+n)}\frac{1}{n!}\frac{1}{s+n-1} &= \sum_{n=0}^{\infty}\frac{\Gamma(s+n)\Gamma(s+n-1)}{\Gamma(2s+n)}\frac{1}{n!}\\ &= \frac{\Gamma(s)\Gamma(s-1)}{\Gamma(2s)}F(s,s-1,2s,1)\\ &= \frac{\Gamma(s)\Gamma(s-1)}{\Gamma(2s)}\frac{\Gamma(2s)\Gamma(2s-s-(s-1))}{\Gamma(2s-s)\Gamma(2s-(s-1))}\\ &= \frac{\Gamma(s-1)}{\Gamma(s+1)} = \frac{1}{s(s-1)}\end{aligned}$$

where we used the power series expansion of the hypergeometric function (3.12) and [OLBC10, 15.4.2] for

$$F(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

□

Remark 3.8.4. Using the identity

$$\int_{\mathbb{H}} Q_{s-1}\left(1+\frac{d(z,i)}{2}\right)\eta = \frac{1}{2s(s-1)}$$

for $\Re(s) > 1$, Theorem 3.8.3 can be proven without the decomposition of $\Phi(\mathfrak{a},m,s,z)$ into the $\Psi(\mathfrak{a},m,s,z)$ by using the representation given in Definition 3.1.1 of $\Phi(\mathfrak{a},m,s,z)$ and following the proof of Theorem 3.8.2. However, one way to prove the integral identity of Q_{s-1} would be to use the representation (3.11) together with the power series of the hypergeometric function. Therefore, in the end it is the same argument.

Now we come to show the integrability of $\Phi(\mathfrak{a},m,z)$. We need to prove a few lemmata first.

Lemma 3.8.5. *The function $\Psi^0(\mathfrak{a},m,1,z)$ is integrable over the Siegel domain $\mathcal{S}_C$ for all $C>0$. For all $\alpha>0$ we have*

$$\int_{\mathcal{S}_C}\left|\Psi^0(\mathfrak{a},m,1,z)\right|\omega^2 = O(m^{1+\alpha}).$$

for large m.

Proof. Recall the definition of $R_{\mathfrak{a}}^m$ in equation (2.46). By Corollary 3.7.7 we have

$$\begin{aligned}
&\int_{\mathcal{S}_C} \left|\Psi^0(\mathfrak{a}, m, 1, z)\right| \omega^2 \\
&\leq \frac{(2C)^2}{8\pi^2} \int_{C^{-1}}^{\infty} \int_{C^{-1}}^{\infty} \frac{8\pi y_1 y_2 m N(\mathfrak{a})}{D} \sum_{\lambda \in \Lambda^+(\mathfrak{a},m)} \sum_{n \in \mathbb{Z}} \frac{e^{-2\pi|n|(\lambda y_1 - \lambda' y_2)}}{\lambda y_1 - \lambda' y_2} \frac{dy_1 dy_2}{y_1^2 y_2^2} \\
&= \frac{4C^2 (R_{\mathfrak{a}}^m)^2}{\pi} \int_{C^{-1}}^{\infty} \int_{C^{-1}}^{\infty} \sum_{\lambda \in \Lambda^+(\mathfrak{a},m)} \sum_{n \in \mathbb{Z}} \frac{e^{-2\pi|n|(\lambda y_1 - \lambda' y_2)}}{(\lambda y_1 - \lambda' y_2) y_1 y_2} dy_1 dy_2 \\
&\leq \frac{8C^2 (R_{\mathfrak{a}}^m)^2}{\pi} \int_{C^{-1}}^{\infty} \int_{C^{-1}}^{\infty} \sum_{\lambda \in \Lambda^+(\mathfrak{a},m)} \sum_{n=0}^{\infty} \frac{e^{-2\pi|n|(\lambda y_1 - \lambda' y_2)}}{(\lambda y_1 - \lambda' y_2) y_1 y_2} dy_1 dy_2 \\
&= \frac{8C^2 (R_{\mathfrak{a}}^m)^2}{\pi} \int_{C^{-1}}^{\infty} \int_{C^{-1}}^{\infty} \sum_{\lambda \in \Lambda^+(\mathfrak{a},m)} \frac{\left(1 - e^{-2\pi(\lambda y_1 - \lambda' y_2)}\right)^{-1}}{(\lambda y_1 - \lambda' y_2) y_1 y_2} dy_1 dy_2 \\
&\leq \frac{8C^2 (R_{\mathfrak{a}}^m)^2}{\pi(1 - e^{-4\pi R_{\mathfrak{a}}^m / C})} \int_{C^{-1}}^{\infty} \int_{C^{-1}}^{\infty} \sum_{\lambda \in \Lambda^+(\mathfrak{a},m)} \frac{1}{(\lambda y_1 - \lambda' y_2) y_1 y_2} dy_1 dy_2 \\
&= \frac{8C^3 (R_{\mathfrak{a}}^m)^2}{\pi(1 - e^{-4\pi R_{\mathfrak{a}}^m / C})} \sum_{\lambda \in \Lambda(\mathfrak{a},m)} \frac{1}{|\lambda'|} \log\left(\frac{|\lambda - \lambda'|}{|\lambda|}\right).
\end{aligned}$$

In the last estimate we used

$$\lambda y_1 - \lambda' y_2 \geq 2\sqrt{|\lambda\lambda'| y_1 y_2} \geq 2R_{\mathfrak{a}}^m / C.$$

In the last equation we used the integral equation

$$\int_{C^{-1}}^{\infty} \int_{C^{-1}}^{\infty} \frac{1}{(ax + by)xy} dx dy = C\left(\frac{1}{b} \log\left(\frac{a+b}{a}\right) + \frac{1}{a} \log\left(\frac{a+b}{b}\right)\right)$$

which holds for $a, b > 0$. Because of $(R_{\mathfrak{a}}^m)^2 = mN(\mathfrak{a})/D$, the prefactor

$$\frac{8C^3 (R_{\mathfrak{a}}^m)^2}{\pi(1 - e^{-4\pi R_{\mathfrak{a}}^m / C})}$$

is in $O(m)$ for large m. Now for any $\lambda_0 \in \Lambda(\mathfrak{a}, m)$ and $\varepsilon \in \mathcal{O}_K^+$ one can estimate

$$\sum_{\lambda \in \left\{\lambda_0 \varepsilon^k \colon k \in \mathbb{Z}\right\}} \frac{1}{|\lambda'|} \log\left(\frac{|\lambda - \lambda'|}{|\lambda|}\right) \leq \frac{1}{\varepsilon - 1}\left(\frac{\varepsilon}{R_{\mathfrak{a}}^m} + \log(2) + 2\log(\varepsilon)\frac{\varepsilon}{\varepsilon - 1}\right)$$

using a suitable splitting of the series, properties of the logarithm and the geometric series. In particular, for $\varepsilon := \varepsilon_1$ the sum is bounded in m. Totally, we get for a constant $\tilde{C} > 0$ independent of m

$$\sum_{\lambda \in \Lambda(\mathfrak{a},m)} \frac{1}{|\lambda'|} \log\left(\frac{|\lambda - \lambda'|}{|\lambda|}\right) \leq \tilde{C}\, |\Lambda(\mathfrak{a}, m)/\langle \varepsilon_1 \rangle| = \tilde{C} \left|\Lambda(\mathfrak{a}, m)/(\mathcal{O}_K^\times)^2\right|.$$

However, the size of $\Lambda(\mathfrak{a}, m)/(\mathcal{O}_K^\times)^2$ depends on m. By Lemma 2.8.6 we have $|\Lambda(\mathfrak{a}, m)/\langle\varepsilon\rangle| = O(m^\alpha)$ which finishes the proof. □

Lemma 3.8.6. *The function* $\log(y_1y_2)$ *is integrable over the Siegel domain* $\mathcal{S}_C$ *for all* $C > 0$. *We have*

$$\int_{\mathcal{S}_C} |\log(y_1y_2)|\,\omega^2 = O(C^4\log(C))$$

for large C.

Proof. We have

$$\int_{\mathcal{S}_C} |\log(y_1y_2)|\,\omega^2 = \frac{(2C)^2}{8\pi^2}\int_{C^{-1}}^\infty\int_{C^{-1}}^\infty |\log(y_1y_2)|\,\frac{dy_1dy_2}{y_1^2y_2^2}.$$

Now for $C \geq 1$ we get

$$\begin{aligned}\int_{C^{-1}}^\infty\int_{C^{-1}}^\infty |\log(y_1y_2)|\frac{dy_1dy_2}{y_1^2y_2^2} &\leq \int_{C^{-1}}^\infty\int_{C^{-1}}^\infty |\log(y_1)| + |\log(y_2)|\frac{dy_1dy_2}{y_1^2y_2^2}\\ &= 2\int_{C^{-1}}^\infty\int_{C^{-1}}^\infty |\log(y_1)|\frac{dy_1dy_2}{y_1^2y_2^2}\\ &= 2C\int_{C^{-1}}^\infty |\log(y_1)|\frac{dy_1}{y_1^2}\\ &= 2C^2\log(C) - 2C^2 + 4C\end{aligned}$$

and for $0 < C \leq 1$

$$\int_{C^{-1}}^\infty\int_{C^{-1}}^\infty |\log(y_1y_2)|\frac{dy_1dy_2}{y_1^2y_2^2} \leq 2\int_1^\infty\int_1^\infty |\log(y_1)|\frac{dy_1dy_2}{y_1^2y_2^2} = 2.$$

□

The next lemma is a generalization of equation (2.20), the volume formula of $\mathcal{S}_C$.

Lemma 3.8.7. *The function* $\Im(z)^{-\alpha}$ *is integrable for* $\alpha > -1$ *over the Siegel domain* $\mathcal{S}_C$ *for all* $C > 0$. *We have*

$$\int_{\mathcal{S}_C} \Im(z)^{-\alpha}\omega^2 = \frac{C^{2\alpha+4}}{2\pi^2(\alpha+1)^2}.$$

Hence, the integral is in $O(C^{2\alpha+4})$ *for large* C.

Proof. We compute

$$\begin{aligned}\int_{\mathcal{S}_C} \Im(z)^{-\alpha}\omega^2 &= \frac{(2C)^2}{8\pi^2}\int_{C^{-1}}^\infty\int_{C^{-1}}^\infty \frac{1}{y_1^\alpha y_2^\alpha}\frac{dy_1dy_2}{y_1^2y_2^2}\\ &= \frac{C^2}{2\pi^2}\left(\left[\frac{y^{-\alpha-1}}{-\alpha-1}\right]_{C^{-1}}^\infty\right)^2\\ &= \frac{C^2C^{2(\alpha+1)}}{2\pi^2(\alpha+1)^2} = \frac{C^{2\alpha+4}}{2\pi^2(\alpha+1)^2}.\end{aligned}$$

□

Lemma 3.8.8. *The function*

$$\sum_{\nu\in\mathfrak{a}\mathfrak{d}^{-1}} \sqrt{|N(\nu)|}\exp(-2\pi\alpha(\nu y_1,\nu' y_2))$$

is integrable over the Siegel domain $\mathcal{S}_C$ for all $C>0$. We have

$$\int_{\mathcal{S}_C}\sum_{\nu\in\mathfrak{a}\mathfrak{d}^{-1}} \sqrt{|N(\nu)|}\exp(-2\pi\alpha(\nu y_1,\nu' y_2))\omega^2 = O(C^8)$$

for large C.

Proof. We have

$$\begin{aligned}
&\int_{\mathcal{S}_C}\sum_{\nu\in\mathfrak{a}\mathfrak{d}^{-1}} \sqrt{|N(\nu)|}\exp(-2\pi\alpha(\nu y_1,\nu' y_2))\omega^2\\
=&\frac{(2C)^2}{8\pi^2}\int_{C^{-1}}^{\infty}\int_{C^{-1}}^{\infty}\sum_{\nu\in\mathfrak{a}\mathfrak{d}^{-1}} \sqrt{|N(\nu)|}\exp(-2\pi\alpha(\nu y_1,\nu' y_2))\frac{dy_1dy_2}{y_1^2y_2^2}\\
\leq&\frac{C^4}{2\pi^2}\sum_{\nu\in\mathfrak{a}\mathfrak{d}^{-1}} \sqrt{|N(\nu)|}\exp(-2\pi\alpha(\nu,\nu')/C).
\end{aligned}$$

The remaining series is convergent for all $C>0$ and in $O(C^4)$ (without the prefactor $C^4/(2\pi^2)$ which itself is in $O(C^4)$). □

Proposition 3.8.9. *The function $\Phi_0(\mathfrak{a},m,z)$ is integrable over the Siegel domain $\mathcal{S}_C$ for all $C>0$. We have*

$$\int_{\mathcal{S}_C}|\Phi_0(\mathfrak{a},m,z)|\,\omega^2 = O(m^2\log(m))$$

for large m.

Proof. The proof of this proposition relies on decomposition of $\Phi_0(\mathfrak{a},m,z)$ and estimates of the components by Theorem 3.7.11 and on the preceding lemmata. With Lemma 3.8.5 we get that the integral of

$$\left|\frac{\Psi^0(\mathfrak{a},m,1,z)}{2}\right|$$

is in $O(m^2)$. The integrability of

$$|u_0(\mathfrak{a},m,y)| \leq |q(\mathfrak{a},m)\log(16\pi^2y_1y_2)| + |L(\mathfrak{a},m)| + \frac{2m^2\pi^4}{3D^{3/2}N(\mathfrak{a})y_1y_2}$$

is ensured by Lemma 3.8.6 and Lemma 3.8.7. With the growth behavior of $|L(\mathfrak{a},m)|$ and $|q(\mathfrak{a},m)|$ mentioned in Remark 3.4.2 the integral turns out to be in $O(m^2\log(m))$. We are left with the integral of

$$\left|\sum_{\nu\in\mathfrak{a}\mathfrak{d}^{-1}}{}' u_\nu(\mathfrak{a},m,y)e(\mathrm{tr}(\nu x))\right| \leq \sum_{\nu\in\mathfrak{a}\mathfrak{d}^{-1}}{}' |u_\nu(\mathfrak{a},m,y)|$$

which is in $O(m)$ by Theorem 3.7.11 and Lemma 3.8.8. The three bounds taken together end up in the stated bound. □

Theorem 3.8.10. *The function $\Phi(\mathfrak{a}, m, z)$ is integrable.*

Proof. We use the decomposition

$$\Phi(\mathfrak{a}, m, z) = \Phi_0(\mathfrak{a}, m, z) + \sum_{n=1}^{\infty} \Phi_n(\mathfrak{a}, m, 1, z)$$

and prove that both parts are integrable. The series

$$\sum_{n=1}^{\infty} \Phi_n(\mathfrak{a}, m, s, z)$$

is integrable for $\Re(s) > 0$ and the integral is holomorphic in s by Theorem 3.8.2 and arguments used in the proof of Theorem 3.8.3. To see the integrability of $\Phi_0(\mathfrak{a}, m, z)$, we use Theorem 2.5.7 which implies that it is enough to show that $\Phi_0(\mathfrak{a}, m, z)$ is integrable over every Siegel domain and their translates to other cusps. However, by Lemma 3.1.3 integrating $\Phi_0(\mathfrak{a}, m, z)$ over translated Siegel domains is equivalent to integrating $\Phi_0(\mathfrak{a}\mathfrak{b}^2, m, z)$, for $\mathfrak{b} \in \mathcal{I}_K$ chosen appropriately, over actual Siegel domains. Therefore, it is enough to consider original Siegel domains. We have done this in Proposition 3.8.9 which finishes the proof. □

Theorem 3.8.11. *We have*

$$\int_{X(\mathfrak{a})} \Phi(\mathfrak{a}, m, z)\omega^2 = -2\,\mathrm{vol}(T(\mathfrak{a}, m)) = -q(\mathfrak{a}, m)\zeta_K(-1).$$

In particular

$$q(\mathfrak{a}, m) = \frac{2\,\mathrm{vol}(T(\mathfrak{a}, m))}{\zeta_K(-1)}.$$

Proof. By Theorem 3.8.10 we know that $\Phi(\mathfrak{a}, m, z)$ is integrable. This allows us to apply Lebesgue's dominated convergence theorem

$$\begin{aligned}\int_{X(\mathfrak{a})} \Phi(\mathfrak{a}, m, z)\omega^2 &= \lim_{s\to 1}\left(\int_{X(\mathfrak{a})} \Phi(\mathfrak{a}, m, s, z)\omega^2 - \int_{X(\mathfrak{a})} \frac{q(\mathfrak{a}, m)}{s-1}\omega^2\right)\\ &= \lim_{s\to 1}\left(\frac{2\,\mathrm{vol}(T(\mathfrak{a}, m))}{s(s-1)} - \frac{q(\mathfrak{a}, m)}{s-1}\zeta_K(-1)\right).\end{aligned}$$

In the second step we used Theorem 3.8.3 and equation (2.17). Since the integral is finite by Theorem 3.8.10 the only possibility is

$$2\,\mathrm{vol}(T(\mathfrak{a}, m)) = q(\mathfrak{a}, m)\zeta_K(-1)$$

which proves the stated identity about $q(\mathfrak{a}, m)$. We get with L'Hôpital's rule

$$\begin{aligned}\int_{X(\mathfrak{a})} \Phi(\mathfrak{a}, m, z)\omega^2 &= \lim_{s\to 1} \frac{2\operatorname{vol}(T(\mathfrak{a}, m)}{s-1}\left(\frac{1}{s}-1\right)\\ &= 2\operatorname{vol}(T(\mathfrak{a}, m) \lim_{s\to 1} \frac{\frac{1}{s}-1}{s-1} = -2\operatorname{vol}(T(\mathfrak{a}, m).\end{aligned}$$

□

Corollary 3.8.12. *We have*

$$\int_{X(\mathfrak{a})} \Phi_0(\mathfrak{a}, m, z)\omega^2 = -4\operatorname{vol}(T(\mathfrak{a}, m))$$

and

$$\int_{X(\mathfrak{a})} \sum_{n=1}^{\infty} \Phi_n(\mathfrak{a}, m, 1, z)\omega^2 = 2\operatorname{vol}(T(\mathfrak{a}, m)).$$

Proof. Clearly,

$$\int_{X(\mathfrak{a})} \Phi_0(\mathfrak{a}, m, z)\omega^2 = \int_{X(\mathfrak{a})} \left(\Phi(\mathfrak{a}, m, z) - \sum_{n=1}^{\infty} \Phi_n(\mathfrak{a}, m, 1, z)\right)\omega^2.$$

As explained in the proof of Theorem 3.8.10, the series

$$\sum_{n=1}^{\infty} \Phi_n(\mathfrak{a}, m, s, z)$$

is actually integrable for $\Re(s) > 0$ and the integral is holomorphic in s. For $\Re(s) > 1$ we can express the integral by the difference

$$\begin{aligned}&\int_{X(\mathfrak{a})} \Phi(\mathfrak{a}, m, s, z)\omega^2 - \int_{X(\mathfrak{a})} \Phi_0(\mathfrak{a}, m, s, z)\omega^2\\ &= \frac{2\operatorname{vol}(T(\mathfrak{a}, m))}{s(s-1)} - \frac{\Gamma(s)^2}{2\Gamma(2s)}\frac{4}{s-1}\operatorname{vol}(T(\mathfrak{a}, m))\\ &= \frac{2\operatorname{vol}(T(\mathfrak{a}, m)}{s-1}\left(\frac{1}{s} - \frac{\Gamma(s)^2}{\Gamma(2s)}\right).\end{aligned}$$

Using L'Hôpital's rule we get

$$\lim_{s\to 1} \frac{\frac{1}{s} - \frac{\Gamma(s)^2}{\Gamma(2s)}}{s-1} = 1$$

which implies the second identity. Therefore,

$$\begin{aligned}\int_{X(\mathfrak{a})} \Phi_0(\mathfrak{a}, m, z)\omega^2 &= \int_{X(\mathfrak{a})} \left(\Phi(\mathfrak{a}, m, z) - \sum_{n=1}^{\infty} \Phi_n(\mathfrak{a}, m, 1, z)\right)\omega^2\\ &= -2\operatorname{vol}(T(\mathfrak{a}, m)) - 2\operatorname{vol}(T(\mathfrak{a}, m))\\ &= -4\operatorname{vol}(T(\mathfrak{a}, m)).\end{aligned}$$

□

Theorem 3.8.13. *We have*

$$\int_{X(\mathfrak{a})} |\Phi(\mathfrak{a}, m, z)|\, \omega^2 = O(m^2 \log(m))$$

for large m.

Proof. As in the proof of Theorem 3.8.10, we use the decomposition

$$\Phi(\mathfrak{a}, m, z) = \Phi_0(\mathfrak{a}, m, z) + \sum_{n=1}^{\infty} \Phi_n(\mathfrak{a}, m, 1, z).$$

Note that

$$\sum_{n=1}^{\infty} \Phi_n(\mathfrak{a}, m, 1, z) \geq 0$$

by its definition. Hence, the integral coincides with the integral of the absolute value. By Corollary 3.8.12 we have

$$\int_{X(\mathfrak{a})} \sum_{n=1}^{\infty} \Phi_n(\mathfrak{a}, m, 1, z)\omega^2 = 2\,\mathrm{vol}(T(\mathfrak{a}, m)) = q(\mathfrak{a}, m)\zeta_K(-1) = O(m^2).$$

We still have to argue that the integral of $|\Phi_0(\mathfrak{a}, m, z)|$ over $X(\mathfrak{a})$ is in $O(m^2 \log(m))$. By the arguments presented in the proof of Theorem 3.8.10 it is enough to see that for $\mathfrak{b} \in \mathcal{I}_K$ the integral of $|\Phi_0(\mathfrak{b}, m, z)|$ over Siegel domains is in $O(m^2 \log(m))$ which is the result of Proposition 3.8.9. □

3.9 Normalized automorphic Green functions

To make the generating series of the arithmetic Hirzebruch–Zagier divisors equipped with the automorphic Green functions modular, we need to normalize the automorphic Green functions $\Phi(\mathfrak{a}, m, z)$ appropriately:

$$G(\mathfrak{a}, m, z) := \Phi(\mathfrak{a}, m, z) - L(\mathfrak{a}, m). \tag{3.20}$$

Note that we do that for $m \in \mathbb{N}$. For $m = 0$ the Green function $G(\mathfrak{a}, m, z)$ is already defined by Proposition 2.9.24. By Definition 2.9.25 equation (3.20) is still satisfied in the case $m = 0$. For $m \in -\mathbb{N}$ we define $G(\mathfrak{a}, m, z) := 0$ in accordance with $\Phi(\mathfrak{a}, m, z) = 0$. Looking at the Fourier expansion of $\Phi(\mathfrak{a}, m, z)$ (cf. Theorem 3.4.1), the subtraction of $L(\mathfrak{a}, m)$ in (3.20) is very natural, since it removes the constant independent of z.

Of course, Proposition 3.6.1 and Theorem 3.6.5 holds for the normalized version of $\Phi(\mathfrak{a}, m, z)$ as well, i.e., the function $G(\mathfrak{a}, m, z)$ has logarithmic singularities along $-T(\mathfrak{a}, m)$ and $G(\mathfrak{a}, m, z)$ is a pre-log-log Green function on $\overline{X(\mathfrak{a})}$ with respect to the divisor $Z(\mathfrak{a}, m)$.

Note that in [BBGK07] the function $G_m(z)$ is $1/2$ of our $G(\mathcal{O}_K, m, z)$. We do not apply this scaling with $1/2$, because the way we define it fits better our definition of having logarithmic singularities.

With Theorem 3.8.11 we obtain

$$\begin{aligned}\int_{X(\mathfrak{a})} G(\mathfrak{a}, m, z)\omega^2 &= \int_{X(\mathfrak{a})} \Phi(\mathfrak{a}, m, z)\omega^2 - \operatorname{vol}(X(\mathfrak{a}))L(\mathfrak{a}, m) \\ &= -2\operatorname{vol}(T(\mathfrak{a}, m)) - \zeta_K(-1)L(\mathfrak{a}, m) \\ &= -2\operatorname{vol}(T(\mathfrak{a}, m))\left(1 + \frac{L(\mathfrak{a}, m)}{q(\mathfrak{a}, m)}\right).\end{aligned}$$

In the case where the discriminant D is prime in [BBGK07, Section 2.3] the constants $L(\mathcal{O}_K, m)$ and $q(\mathcal{O}_K, m)$ are made explicit using the generalized divisor sum

$$\sigma_m(s) := m^{(1-s)/2} \sum_{d|m} d^s(\chi_D(d) + \chi_D(m/d)). \tag{3.21}$$

Namely, we have with

$$\varphi_m(s) := -\frac{\Gamma(s - 1/2)\sigma_m(2s - 1)}{\Gamma(3/2 - s)L(1 - 2s, \chi_D)}$$

the identities

$$L(\mathcal{O}_K, m) = \varphi_m'(1) - \varphi_m(1)(2\Gamma'(1) - \log(16D)) \quad \text{and} \quad q(\mathcal{O}_K, m) = \varphi_m(1).$$

It follows

$$L(\mathcal{O}_K, m) = \varphi_m(1)\left(2\frac{L'(-1, \chi_D)}{L(-1, \chi_D)} - 2\frac{\sigma_m'(-1)}{\sigma_m(-1)} + \log(D)\right)$$

and

$$q(\mathcal{O}_K, m) = -\frac{\sigma_m(-1)}{L(-1, \chi_D)}. \tag{3.22}$$

Note that $\sigma_m(s) = \sigma_m(-s)$, in particular $\sigma_m(1) = \sigma_m(-1)$. Therefore, we obtain

$$\begin{aligned}\int_{X(\mathfrak{a})} G(\mathcal{O}_K, m, z)\omega^2 &= -2\operatorname{vol}(T(\mathcal{O}_K, m))\left(1 + \frac{L(\mathcal{O}_K, m)}{q(\mathcal{O}_K, m)}\right) \\ = -2\operatorname{vol}(T(\mathcal{O}_K, m))&\left(1 + 2\frac{L'(-1, \chi_D)}{L(-1, \chi_D)} - 2\frac{\sigma_m'(-1)}{\sigma_m(-1)} + \log(D)\right).\end{aligned} \tag{3.23}$$

Another useful relation coming from Theorem 3.8.11 and equation (3.22) is

$$\begin{aligned}\operatorname{vol}(T(\mathcal{O}_K, m)) &= \frac{q(\mathcal{O}_K, m)\zeta_K(-1)}{2} = -\frac{\sigma_m(-1)}{L(-1, \chi_D)}\frac{\zeta_K(-1)}{2} \\ &= -\frac{\sigma_m(-1)\zeta(-1)}{2} = \frac{\sigma_m(-1)}{24}.\end{aligned} \tag{3.24}$$

3.10 Generating series

For the main result of the thesis we have to deal with the generating series

$$\sum_{m=0}^{\infty} G(\mathfrak{a}, m, z) q^m. \tag{3.25}$$

Here, $q \in \mathbb{C}$ with $|q| < 1$. Later, we will introduce another variable $\tau \in \mathbb{H}$ and we define $q := e(\tau)$. Even though the single $G(\mathfrak{a}, m, z)$ have logarithmic singularities along the divisors $-T(\mathfrak{a}, m)$ and the union

$$\bigcup_{m \in \mathbb{N}} T(\mathfrak{a}, m) \tag{3.26}$$

lies dense in $\mathbb{H}^2$, the next theorem shows that we still have many values $z \in \mathbb{H}^2$ for which the series (3.25) is absolutely convergent.

Theorem 3.10.1. *Let $q \in \mathbb{C}$ with $|q| < 1$ be fixed. The series*

$$\sum_{m=0}^{\infty} \Phi(\mathfrak{a}, m, z) q^m \quad \textit{and} \quad \sum_{m=0}^{\infty} G(\mathfrak{a}, m, z) q^m$$

converge absolutely for almost all $z \in \mathbb{H}^2$. Furthermore, the series

$$\sum_{m=0}^{\infty} |\Phi(\mathfrak{a}, m, z) q^m| \quad \textit{and} \quad \sum_{m=0}^{\infty} |G(\mathfrak{a}, m, z) q^m|$$

are integrable over $X(\mathfrak{a})$.

Proof. Using the ratio test, the series

$$\sum_{m=0}^{\infty} p(m) q^m$$

converges absolutely with $p(m)$ being a polynomial in m. Since $\Phi(\mathfrak{a}, m, z)$ and $G(\mathfrak{a}, m, z)$ differ only by a constant $L(\mathfrak{a}, m)$ (cf. equation (3.20)) growing polynomially in m, the generating series of $\Phi(\mathfrak{a}, m, z)$ converges for fixed $z \in \mathbb{H}^2$ if and only if the generating series of $G(\mathfrak{a}, m, z)$ converges. Analogously, the series of $|\Phi(\mathfrak{a}, m, z) q^m|$ is integrable if and only if the series of $|G(\mathfrak{a}, m, z) q^m|$ is integrable. By Tonelli's theorem we have

$$\int_{X(\mathfrak{a})} \sum_{m=0}^{\infty} |\Phi(\mathfrak{a}, m, z) q^m| \, \omega^2 = \sum_{m=0}^{\infty} \left(\int_{X(\mathfrak{a})} |\Phi(\mathfrak{a}, m, z)| \, \omega^2 \right) |q|^m. \tag{3.27}$$

Hence, the integrability of the series of $|\Phi(\mathfrak{a}, m, z) q^m|$ is equivalent to the finiteness of the sum on the right hand side in (3.27). This sum, however, is finite by Theorem 3.8.13

together with the remark from above about the ratio test. Because the left hand side of (3.27) is finite, the set of all $z \in \mathbb{H}^2$ satisfying

$$\sum_{m=0}^{\infty} |\Phi(\mathfrak{a}, m, z) q^m| = \infty$$

has measure zero. This proves the absolute convergence of

$$\sum_{m=0}^{\infty} \Phi(\mathfrak{a}, m, z) q^m$$

for almost all $z \in \mathbb{H}^2$. □

Even though Theorem 3.10.1 shows that the series (3.25) converges almost everywhere, we cannot expect the series to be continuous in a single point $z \in \mathbb{H}^2$ because the union (3.26) is a dense subset of $\mathbb{H}^2$.

3.11 An arithmetic Hirzebruch–Zagier theorem

At this point we want to cite the arithmetic Hirzebruch–Zagier theorem for the automorphic Green functions $G(\mathfrak{a}, m, z)$. It was proven in [BBGK07] by Bruinier, Burgos Gil and Kühn in the special case of K having a prime discriminant and $\mathfrak{a} = \mathcal{O}_K$, but is expected to be true for all real quadratic number fields and all $\mathfrak{a} \in \mathcal{I}_K$. We suggest reading the article [Bru04] for a good overview. There, the result can be found in Theorem 8.4.

Our Hilbert modular surface can be embedded into projective space. Thus, by Chow's lemma it can be understood as a projective algebraic variety. By the work of Rapoport, Deligne and Pappas (cf. [Rap78] and [DP94]) the Hilbert modular surface possesses a canonical integral model over $\operatorname{Spec}(\mathbb{Z})$. The Hirzebruch–Zagier divisors possess natural integral models as well. By taking the base change to the complex numbers the Hilbert modular surface together with its Hirzebruch–Zagier divisors can be recovered again. The arithmetic Hirzebruch–Zagier theorem holds in that wider context for integral models. Viewed over the complex numbers it states that the arithmetic generating series

$$\sum_{m=0}^{\infty} (Z(\mathcal{O}_K, m), G(\mathcal{O}_K, m, z)) q^m$$

is a holomorphic modular form of weight 2, level D and nebentypus χ_D with values in $\widehat{\mathrm{CH}}^1(\overline{X(\mathcal{O}_K)}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}}$ (cf. Definition 5.5.1). Note that in the original statement one reads $\hat{c}_1(\mathcal{M}_{-1/2})$ instead of $(Z(\mathcal{O}_K, 0), G(\mathcal{O}_K, 0, z))$. However, by our definition of $G(\mathcal{O}_K, 0, z)$ and $Z(\mathcal{O}_K, 0)$ (cf. Proposition 2.9.24) they define the same element in $\widehat{\mathrm{CH}}^1(\overline{X(\mathcal{O}_K)}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{Q}}$.

In this thesis $\overline{X(\mathcal{O}_K)}$ is the Hirzebruch compactification (cf. Section 2.7) and the divisors $Z(\mathcal{O}_K, m)$ are explicitly defined (cf. Subsection 2.8.5). This explicit construction of the divisors corresponding to the Green functions $G(\mathcal{O}_K, m, z)$ is new work.

Bruinier, Burgos Gil and Kühn used in their work an arbitrary desingularization of the Baily–Borel compactification $X(\mathcal{O}_K)^*$ whose existence is ensured by Hironaka in [Hir64]. The extensions of the divisors $T(\mathcal{O}_K, m)$ to the divisors $Z(\mathcal{O}_K, m)$ are abstractly defined by pullback from the Baily–Borel compactification $X(\mathcal{O}_K)^*$. To this end the authors proved that the divisors $T(\mathcal{O}_K, m)$ are $\mathbb{Q}$-Cartier divisors near the cusps but they did not need to compute any multiplicities of the components of the exceptional divisor $E(\mathcal{O}_K)$ at the cusps.

The main goal of this thesis is to prove an analogue of the arithmetic Hirzebruch–Zagier theorem for Kudla's Green functions. Those functions are introduced and discussed in the next chapter.

Chapter 4

Kudla's Green functions

In this chapter we consider a different type of Green functions for the Hirzebruch–Zagier divisors constructed by Kudla (cf. [Kud97, p. 603 a.s.]). We call those functions *Kudla's Green functions* and denote them by $\Xi(\mathfrak{a}, m, v, z)$. In addition to a fractional ideal $\mathfrak{a} \in \mathcal{I}_K$ and an integer $m \in \mathbb{Z}$, they depend on a parameter $v > 0$. Later, when we come to generating series involving Kudla's Green functions, this parameter will be interpreted as the imaginary part of a complex variable $\tau \in \mathbb{H}$ but for most of this chapter, it is enough to regard it as positive real parameter.

During the course of this chapter we investigate the growth of $\Xi(\mathfrak{a}, m, v, z)$ at the cusps by looking at its Fourier expansion and come to the conclusion that $\Xi(\mathfrak{a}, m, v, z)$ is not a pre-log-log Green function on $\overline{X(\mathfrak{a})}$. However, in Section 4.5 we find a slight modification involving a partition of unity ρ called $\widetilde{\Xi}_\rho(\mathfrak{a}, m, v, z)$ which is pre-log-log Green function on $\overline{X(\mathfrak{a})}$. Near the end of this chapter we compute integrals over Kudla's Green functions and start discussing its generating series.

A great source of inspiration for coming up with the right ideas was the article [BK12] from Berndt and Kühn. In that source, the authors consider the respective object in the degenerate case $D = 1$, $K = \mathbb{Q} \oplus \mathbb{Q}$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})^2$.

4.1 Definition, convergence and invariance

Recall the definition

$$E_1(x) := \int_x^\infty \exp(-t)\frac{dt}{t} = \int_1^\infty \exp(-xt)\frac{dt}{t} \tag{4.1}$$

for $x > 0$ of the E_1 function which is related to the exponential integral by $E_1(x) = -\operatorname{Ei}(-x)$. It has a logarithmic singularity at $x = 0$. More precisely, the function

$$E_1(x) + \log(x), \quad x > 0$$

has a holomorphic extension to the whole complex plane. Its power series is given by

$$-\gamma - \sum_{k=1}^\infty \frac{(-x)^k}{kk!} = -\gamma + \operatorname{Ein}(x) \tag{4.2}$$

where γ is the Euler-Mascheroni constant. The logarithmic singularity of $E_1(x)$ at $x = 0$ provides the logarithmic singularity along the Hirzebruch–Zagier divisor of Kudla's Green functions.

Definition 4.1.1. For $\mathfrak{a} \in \mathcal{I}_K$, $m \in \mathbb{Z}$, $v > 0$ and $z \in \mathbb{H}^2 \setminus T(\mathfrak{a}, m)$ we define

$$\Xi(\mathfrak{a}, m, v, z) := \Xi_*(\mathfrak{a}, m, v, z) + \delta_{m,0}\Xi_0(\mathfrak{a}, v, z)$$

with

$$\Xi_*(\mathfrak{a}, m, v, z) := \frac{1}{2} \sideset{}{'}\sum_{\substack{A \in L(\mathfrak{a})^\vee \\ \det(A) = m/(N(\mathfrak{a})D)}} E_1\left(4\pi v D N(\mathfrak{a}) h(A, z)\right)$$

and

$$\Xi_0(\mathfrak{a}, v, z) := G(\mathfrak{a}, 0, z) - \frac{\log(4\pi v D/N(\mathfrak{a})) + \gamma}{2}.$$

Note the tick at the sum defining $\Xi_*(\mathfrak{a}, m, v, z)$ which indicates that for $m = 0$ we do not include $A = 0$ (recall Proposition 2.9.24 for the definition of $G(\mathfrak{a}, 0, z)$). We need to exclude $A = 0$ because

$$E_1\left(4\pi v D N(\mathfrak{a}) h(0, z)\right) = E_1(0)$$

is not defined. To compensate for that missing term, we add $\Xi_0(\mathfrak{a}, v, z)$ in case $m = 0$. Later, we will understand in which sense this is a good compensation.

Proposition 4.1.2. *The series defining Kudla's Green function $\Xi(\mathfrak{a}, m, v, z)$ converges for all $z \in \mathbb{H}^2 \setminus T(\mathfrak{a}, m)$ normally to a $\Gamma_\mathfrak{a}$ invariant function with logarithmic singularities along $-T(\mathfrak{a}, m)$. For ideals $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_K$ and $M \in M(\mathfrak{a}, \mathfrak{b})$ Kudla's Green functions are related by*

$$\Xi_*(\mathfrak{b}, m, v, Mz) = \Xi_*(\mathfrak{a}^2\mathfrak{b}, m, v, z) \quad \textit{and} \quad \Xi_0(\mathfrak{b}, v, Mz) = \Xi_0(\mathfrak{a}^2\mathfrak{b}, v, z) - \log(N(\mathfrak{a})).$$

Proof. To prove the first statement, it is enough by Proposition 2.9.24 to prove the analogue statement for $\Xi_*(\mathfrak{a}, m, v, z)$ with respect to the divisor $-T_*(\mathfrak{a}, m)$. By the estimate

$$E_1(x) = \int_x^\infty \exp(-t)\frac{dt}{t} \leq \frac{1}{x}\int_x^\infty \exp(-t)dt = \frac{\exp(-x)}{x} \tag{4.3}$$

we see that $E_1(x)$ has exponential decay for large x (that implies $E_1(x) = O(x^{-\alpha})$ for all $\alpha \in \mathbb{R}$). Hence, the convergence follows in case $m > 0$ directly from Lemma 3.1.2. It is legit to apply Lemma 3.1.2 here since $g(A, z)$ and $h(A, z)$ differ in this situation only by the multiplicative constant $\det(A) = m/(N(\mathfrak{a})D)$. However, Lemma 3.1.2 does not cover the case $m \leq 0$. We now present a new proof which is valid for all $m \in \mathbb{Z}$. We have to show the normal convergence of

$$\sideset{}{'}\sum_{\substack{A \in L(\mathfrak{a})^\vee \\ \det(A) = m/(N(\mathfrak{a})D)}} E_1\left(4\pi v D N(\mathfrak{a}) h(A, z)\right). \tag{4.4}$$

Let $B \subset \mathbb{H}^2$ be compact. Then by Lemma 2.6.4 only a finite number of $A \in L(\mathfrak{a})^\vee$ with $\det(A) = m/(N(\mathfrak{a})D)$ satisfy

$$4\pi v D N(\mathfrak{a}) h(A, z) < 1$$

for some $z \in B$. Using

$$E_1(x) \leq \exp(-x), \quad \text{for } x \geq 1,$$

which is a direct consequence of estimate (4.3) we get for the other $A \in L(\mathfrak{a})^\vee$ with $\det(A) = m/(N(\mathfrak{a})D)$

$$\begin{aligned} E_1\left(4\pi v D N(\mathfrak{a}) h(A, z)\right) \exp(-2\pi v m) &\leq \exp(-4\pi v D N(\mathfrak{a}) h(A, z)) \exp(-2\pi v m) \\ &\leq \exp(-2\pi v D N(\mathfrak{a})(2h(A, z) + \det(A))) \\ &= \exp(-2\pi v D N(\mathfrak{a}) q_z(A)). \end{aligned}$$

Recall the last equation of (2.21) for the last step. This shows that $\exp(-2\pi v m)$ times the series (4.4) can be majorized up to finitely many terms by the theta series

$$\sum_{A \in L(\mathfrak{a})^\vee} \exp(-2\pi v D N(\mathfrak{a}) q_z(A)).$$

Because $q_z(A)$ is a positive definite quadratic form this implies the normal convergence.

The $\Gamma_{\mathfrak{a}}$ invariance is a consequence of the transformation law

$$\Xi_*(\mathfrak{b}, m, v, Mz) = \Xi_*(\mathfrak{a}^2\mathfrak{b}, m, v, z),$$

the second statement of the proposition. This is proven for $\Xi_*(\mathfrak{a}, m, v, z)$ analogously to the proof of Lemma 3.1.3: If $g(A, z)$ is replaced by $h(A, z)$ the proof works for $m \leq 0$ as well. However, it is important to have the prefactor $N(\mathfrak{a})$ in front of $h(A, z)$ to make the rescaling of the lattice work.

For the transformation law of Ξ_0 we use Proposition 2.9.24 again and see

$$\begin{aligned} \Xi_0(\mathfrak{a}^2\mathfrak{b}, v, z) &= G(\mathfrak{a}^2\mathfrak{b}, 0, z) - \frac{\log(4\pi v D/N(\mathfrak{a}^2\mathfrak{b})) + \gamma}{2} \\ &= G(\mathfrak{b}, 0, Mz) - \frac{\log(4\pi v D/N(\mathfrak{b})) + \gamma}{2} + \log(N(\mathfrak{a})) \\ &= \Xi_0(\mathfrak{b}, v, z) + \log(N(\mathfrak{a})). \end{aligned}$$

Now, we come to the logarithmic singularities of $\Xi_*(\mathfrak{a}, m, v, z)$. Here we only have to consider $m > 0$ since $T_*(\mathfrak{a}, m) = 0$ otherwise. Let $A \in L(\mathfrak{a})^\vee \cap V^+$ be fixed. Then locally the matrices A and $-A$ contribute to the divisor component $T_A = T_{-A}$. With $A = \left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right)$

we have

$$\begin{aligned}
&\frac{1}{2}E_1\left(4\pi vDN(\mathfrak{a})h(A,z)\right)+\frac{1}{2}E_1\left(4\pi vDN(\mathfrak{a})h(-A,z)\right)\\
=&E_1\left(4\pi vDN(\mathfrak{a})\frac{|bz_1z_2-\lambda z_1-\lambda' z_2+a|^2}{4y_1y_2}\right)\\
=&-\gamma+\mathrm{Ein}\left(\pi vDN(\mathfrak{a})\frac{|bz_1z_2-\lambda z_1-\lambda' z_2+a|^2}{y_1y_2}\right)\\
&-\log\left(\pi vDN(\mathfrak{a})\frac{|bz_1z_2-\lambda z_1-\lambda' z_2+a|^2}{y_1y_2}\right)\\
=&-\gamma+\mathrm{Ein}\left(\pi vDN(\mathfrak{a})\frac{|bz_1z_2-\lambda z_1-\lambda' z_2+a|^2}{y_1y_2}\right)\\
&-\log\left(\frac{\pi vDN(\mathfrak{a})}{y_1y_2}\right)-\log\left(|bz_1z_2-\lambda z_1-\lambda' z_2+a|^2\right).
\end{aligned}$$

This shows that the terms in the definition of $\Xi_*(\mathfrak{a},m,v,z)$ (cf. Definition 4.1.1) indexed by A and $-A$ have a logarithmic singularity along $-T_A$. Because of the good convergence behavior, in total they sum up to logarithmic singularities along $-T_*(\mathfrak{a},m)$. □

The logarithmic singularities along $-T(\mathfrak{a},m)$ promote the question whether the analogue of Theorem 3.6.5 is true for $\Xi(\mathfrak{a},m,v,z)$ as well, i.e., whether $\Xi(\mathfrak{a},m,v,z)$ is a pre-log-log Green function on $\overline{X(\mathfrak{a})}$ with respect to the divisor $Z(\mathfrak{a},m)$. In this chapter we show that, unfortunately, this is not the case. However, in Section 4.5 we describe a modification $\widetilde{\Xi}_\rho(\mathfrak{a},m,v,z)$ of $\Xi(\mathfrak{a},m,v,z)$ which is a pre-log-log Green function. Since $\Xi(\mathfrak{a},m,v,z)$ has logarithmic singularities along $-T(\mathfrak{a},m)$, the behavior of $\Xi(\mathfrak{a},m,v,z)$ which makes it fail Theorem 3.6.5 occurs near the cusps.

Analogously as for $\Phi(\mathfrak{a},m,s,z)$, we write $\Xi_*(\mathfrak{a},m,v,z)$ in the form

$$\Xi_*(\mathfrak{a},m,v,z)=\sum_{b\in\mathbb{Z}}\Xi_*^b(\mathfrak{a},m,v,z)$$

with

$$\Xi_*^b(\mathfrak{a},m,v,z):=\frac{1}{2}\sideset{}{'}\sum_{\substack{A=\left(\begin{smallmatrix}a&\lambda'\\\lambda&b\end{smallmatrix}\right)\in L(\mathfrak{a})^\vee\\ \det(A)=m/(N(\mathfrak{a})D)}}E_1\left(4\pi vDN(\mathfrak{a})h(A,z)\right).$$

For $b\neq 0$ or $m\neq 0$ we define

$$\Xi^b(\mathfrak{a},m,v,z):=\Xi_*^b(\mathfrak{a},m,v,z).$$

Further, we set

$$\Xi^0(\mathfrak{a},0,v,z):=\Xi_*^0(\mathfrak{a},0,v,z)+\Xi_0(\mathfrak{a},v,z).$$

Hence, we get for all $m\in\mathbb{Z}$

$$\Xi(\mathfrak{a},m,v,z)=\sum_{b\in\mathbb{Z}}\Xi^b(\mathfrak{a},m,v,z).$$

As for $\Phi(\mathfrak{a}, m, s, z)$ the functions $\Xi_*^b(\mathfrak{a}, m, v, z)$ are invariant under translation by $\mathfrak{a}^{-1}$ and admit a Fourier expansion for those y at which no singularities occur.

Analogously to the situation with the functions $\Phi^b(\mathfrak{a}, m, s, z)$, the imaginary part $\Im(z) = y_1 y_2$ of the singularities of the functions $\Xi^b(\mathfrak{a}, m, v, z)$ is bounded by $m/(N(\mathfrak{a})Db^2)$ for $b \neq 0$ and $m > 0$. Even more, using the exponential decay of $E_1(x)$ and its derivatives it can be shown that the series of the $\Xi^b(\mathfrak{a}, m, v, z)$ with $b \neq 0$ is tame at the cusp ∞:

Proposition 4.1.3. *The series*

$$\sum_{b\in\mathbb{Z}}{}' \Xi^b(\mathfrak{a}, m, v, z) = 2\sum_{b=1}^{\infty} \Xi^b(\mathfrak{a}, m, v, z) \tag{4.5}$$

is well-defined in a neighborhood of the cusp ∞ and defines a pre-log-log growth form along the exceptional divisor $E^\infty(\mathfrak{a})$.

Proof. The convergence of the series (4.5) follows from Proposition 4.1.2. Each individual $\Xi^b(\mathfrak{a}, m, v, z)$ is invariant under the stabilizer of the cusp ∞ by Lemma 2.4.4 and Remark 2.6.3. Therefore, the series (4.5) is invariant under the stabilizer of the cusp ∞ as well and is well-defined by Proposition 2.5.2 on $X(\mathfrak{a})$ in a neighborhood of the cusp ∞. Furthermore, because we omit $\Xi^0(\mathfrak{a}, m, v, z)$, we do not have any singularities for all $z \in \mathbb{H}^2$ with $\Im(z) > m/(N(\mathfrak{a})D)$. Thus, we are left with controlling the growth of (4.5) and its derivatives near the cusp ∞. We want to apply Remark 2.9.3 and suggest the reader come back to the proof here after having read Section 4.3. There, we present the notation including the variable switch to t and r together with the application of Remark 2.9.3 in more detail and want to shorten the proof here a bit.

The main ingredient to the proof is the exponential decay (4.3) of $E_1(x)$ which we have by

$$E_1'(x) = -\frac{\exp(-x)}{x} \quad \text{and} \quad E_1''(x) = \frac{\exp(-x)(x+1)}{x^2} \tag{4.6}$$

for its derivatives as well. Therefore, the handling of the derivatives is similar to the handling of the original series and it is enough to consider the latter only. After a variable switch to t and r it is sufficient to prove

$$\lim_{t\to\infty} \sum_{b=1}^{\infty} \Xi^b(\mathfrak{a}, m, v, z) = 0$$

for fixed r. We have

$$2\sum_{b=1}^{\infty} \Xi^b(\mathfrak{a}, m, v, z) = \sum_{\substack{A=\left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right)\in L(\mathfrak{a})^\vee \\ \det(A)=m/(N(\mathfrak{a})D) \\ b>0}} E_1\left(4\pi v D N(\mathfrak{a}) h(A, z)\right). \tag{4.7}$$

Because of

$$\lim_{t\to\infty} \frac{h(A, z)}{t^2} = \frac{b^2}{4}$$

for $A = \left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in L(\mathfrak{a})^\vee$ with $b \neq 0$ (cf. equation (2.22)), we infer that each single term of the series on the right hand side of (4.7) goes to 0 for $t \to \infty$. That being said, it is fine to exclude finitely many terms of the series. It is easy to see that only a finite number of $A \in L(\mathfrak{a})^\vee$ with $\det(A) = m/(N(\mathfrak{a})D)$ and $b \neq 0$ satisfy

$$4\pi v D N(\mathfrak{a}) h(A, z) < 1$$

for large t. Therefore, we exclude those from now on and try to prove the limit of the remaining series (4.7). We proceed like in the proof of Proposition 4.1.2 and end up with the series

$$\sum_{\substack{A=\left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in L(\mathfrak{a})^\vee \\ b>0}} \exp(-2\pi v D N(\mathfrak{a}) q_z(A)). \tag{4.8}$$

Using the fact that $q_z(A)$ is a positive definite quadratic form, one now proves that (4.8) goes to 0 for $t \to \infty$. We demonstrate this in the special case $x_1 = x_2 = 0$:

$$\begin{aligned} q_z(A) = 2h(A, z) + \det(A) &= \frac{|bz_1z_2 - \lambda z_1 - \lambda' z_2 + a|^2}{2y_1y_2} + ab - \lambda\lambda' \\ &= \frac{(-by_1y_2 + a)^2 + (-\lambda y_1 - \lambda' y_2)^2}{2t^2} + \frac{2ab - 2\lambda\lambda'}{2} \\ &= \frac{(bt - a/t)^2 + (\lambda r - \lambda'/r)^2 + 2ab - 2\lambda\lambda'}{2} \\ &= \frac{b^2t^2 + a^2/t^2 + \lambda^2 r^2 + (\lambda')^2/r^2}{2}. \end{aligned}$$

With the definition (2.12) of $L(\mathfrak{a})^\vee$ we have

$$\begin{aligned} &\sum_{\substack{A=\left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right) \in L(\mathfrak{a})^\vee \\ b>0}} \exp(-2\pi v D N(\mathfrak{a}) q_z(A)) \\ &= \left(\sum_{b=1}^{\infty} \exp(-\pi v D N(\mathfrak{a}) b^2 t^2)\right) \left(\sum_{a \in \mathbb{Z}} \exp\left(-\frac{\pi v D a^2}{t^2 N(\mathfrak{a})}\right)\right) \\ &\times \left(\sum_{\lambda \in \mathfrak{a}\mathfrak{d}^{-1}}^{\infty} \exp\left(-\frac{\pi v D(\lambda^2 r^2 + (\lambda')^2/r^2)}{N(\mathfrak{a})}\right)\right). \end{aligned} \tag{4.9}$$

The map

$$\lambda \mapsto \lambda^2 r^2 + (\lambda')^2/r^2$$

is a positive definite quadratic form on K. Therefore, the series in line (4.9) converges. The other line can be expressed as

$$\frac{\theta(ivDN(\mathfrak{a})t^2) - 1}{2} \theta\left(\frac{ivD}{t^2 N(\mathfrak{a})}\right),$$

using the theta function

$$\theta : \mathbb{H} \to \mathbb{C}, \quad \theta(z) := \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z}. \tag{4.10}$$

By its transformation law

$$\theta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{i}}\theta(z) \tag{4.11}$$

we obtain

$$\theta\left(\frac{ivD}{t^2 N(\mathfrak{a})}\right) = t\sqrt{\frac{N(\mathfrak{a})}{vD}}\theta\left(\frac{it^2 N(\mathfrak{a})}{vD}\right).$$

Using the estimate

$$|\theta(it) - 1| \leq 2\exp(-\pi t), \tag{4.12}$$

for large $t > 0$ we get

$$\begin{aligned}
&\lim_{t\to\infty} \frac{\theta(ivDN(\mathfrak{a})t^2) - 1}{2}\theta\left(\frac{ivD}{t^2 N(\mathfrak{a})}\right) \\
&= \left(\lim_{t\to\infty} t\sqrt{\frac{N(\mathfrak{a})}{vD}}\frac{\theta(ivDN(\mathfrak{a})t^2) - 1}{2}\right)\left(\lim_{t\to\infty}\theta\left(\frac{it^2 N(\mathfrak{a})}{vD}\right)\right) \\
&= 0 \cdot 1 = 0.
\end{aligned}$$

□

By Proposition 4.1.3 we are left with the investigation of $\Xi^0_*(\mathfrak{a}, m, v, z)$. This part has singularities if and only if $m > 0$ and $\Lambda^+(\mathfrak{a}, m) \neq 0$. Those singularities run into the cusp ∞ and one would wish $\Xi^0(\mathfrak{a}, m, v, z)$ to have logarithmic singularities at the cusp ∞ along the divisor $-Z^\infty(\mathfrak{a}, m)$ with possibly some additional term of pre-log-log growth.

4.2 Fourier expansion of $\Xi^0(\mathfrak{a}, m, v, z)$ for $m \neq 0$

For the course of this section we assume $m \in \mathbb{Z}\backslash\{0\}$. We compute the Fourier expansion of $\Xi^0(\mathfrak{a}, m, v, z)$ (cf. Section 4.4 for the case $m = 0$). Let us start by rewriting $\Xi^0(\mathfrak{a}, m, v, z)$ in the following way:

$$\begin{aligned}
\Xi^0(\mathfrak{a}, m, v, z) &= \frac{1}{2} \sum_{\substack{A=\left(\begin{smallmatrix} a & \lambda' \\ \lambda & 0 \end{smallmatrix}\right) \in L(\mathfrak{a})^\vee \\ \det(A)=m/(N(\mathfrak{a})D)}} E_1\left(4\pi v D N(\mathfrak{a}) h(A, z)\right) \\
&= \frac{1}{2} \sum_{\substack{a \in \mathbb{Z}/N(\mathfrak{a}),\, \lambda \in \mathfrak{a}\mathfrak{d}^{-1}/N(\mathfrak{a}) \\ -N(\lambda)=m/(N(\mathfrak{a})D)}} E_1\left(\frac{4\pi v D N(\mathfrak{a})|-\lambda z_1 - \lambda' z_2 + a|^2}{4 y_1 y_2}\right) \\
&= \sum_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \sum_{a \in \mathbb{Z}} E_1\left(\frac{\pi v D |\lambda z_1 + \lambda' z_2 + a|^2}{N(\mathfrak{a}) y_1 y_2}\right) \\
&= \sum_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \sum_{a \in \mathbb{Z}} E_1\left(\frac{\pi v D}{N(\mathfrak{a}) y_1 y_2}\left((\mathrm{tr}(\lambda x) + a)^2 + \mathrm{tr}(\lambda y)^2\right)\right).
\end{aligned}$$

Note that for negative m we have $\mathrm{tr}(\lambda y) > 0$ for all y with $z \in \mathbb{H}^2$. The rewriting of $\Xi^0(\mathfrak{a}, m, v, z)$ shows that we need to compute the Fourier expansion of the $\mathbb{Z}$ periodic function

$$h_{\alpha,\beta}(x) := \sum_{a \in \mathbb{Z}} E_1(\alpha^2((x+a)^2 + \beta^2)).$$

To express the constant Fourier coefficient of $h_{\alpha,\beta}(x)$, we make use of the function

$$\beta(x) := \frac{1}{16\pi} \int_1^\infty u^{-3/2} e^{-xu} du = \frac{1}{8\pi}\left(\exp(-x) - \sqrt{\pi x}\,\mathrm{erfc}(\sqrt{x})\right) \tag{4.13}$$

which was introduced in [HZ76, p. 91] for $x \geq 0$. It is neither to be confused with the function $\beta(r_1, r_2)$ defined in the last chapter in equation (3.6) nor with the parameter β in $h_{\alpha,\beta}(x)$.

Lemma 4.2.1. *Let $\alpha, \beta > 0$. Then we have*

$$h_{\alpha,\beta}(x) = \sum_{n \in \mathbb{Z}} a_{\alpha,\beta}(n) e(nx)$$

with

$$a_{\alpha,\beta}(0) = \frac{16\pi^{3/2}}{\alpha} \beta(\alpha^2 \beta^2)$$

and

$$a_{\alpha,\beta}(n) = \frac{e^{-2\pi|n|\beta}}{|n|} - \frac{e^{2\pi|n|\beta}\,\mathrm{erfc}\left(\pi|n|/\alpha + \alpha\beta\right) + e^{-2\pi|n|\beta}\left(\mathrm{erfc}\left(\pi|n|/\alpha - \alpha\beta\right)\right)}{2|n|}$$

for $n \neq 0$.

Proof. By Poisson summation we have

$$\begin{aligned}
a_{\alpha,\beta}(n) &= \int_{-\infty}^{\infty} E_1(\alpha^2(x^2+\beta^2))e(-nx)dx \\
&= \int_{-\infty}^{\infty}\int_1^{\infty} \exp(-\alpha^2(x^2+\beta^2)t)\frac{dt}{t}e(-nx)dx \\
&= \int_1^{\infty}\int_{-\infty}^{\infty} \exp(-\alpha^2x^2t)e(-nx)dx\exp(-\alpha^2\beta^2t)\frac{dt}{t} \\
&= 2\int_1^{\infty}\int_0^{\infty} \exp(-\alpha^2x^2t)\cos(2\pi nx)dx\exp(-\alpha^2\beta^2t)\frac{dt}{t}.
\end{aligned}$$

For solving the inner integral we use the cosine transform [EMOT54, p. 15, eq. (11)]

$$\int_0^{\infty} \exp(-ax^2)\cos(xy)dx = \frac{\sqrt{\pi}}{2\sqrt{a}}\exp\left(-\frac{y^2}{4a}\right)$$

which holds for $\Re(a) > 0$. In the reference it allows only $y > 0$, but of course this is true for all $y \in \mathbb{R}$. Therefore, we get

$$\begin{aligned}
a_{\alpha,\beta}(n) &= 2\int_1^{\infty} \frac{\sqrt{\pi}}{2\alpha\sqrt{t}}\exp\left(-\frac{4\pi^2n^2}{4\alpha^2t}\right)\exp(-\alpha^2\beta^2t)\frac{dt}{t} \\
&= \frac{\sqrt{\pi}}{\alpha}\int_1^{\infty} \exp\left(-\frac{\pi^2n^2}{\alpha^2t} - \alpha^2\beta^2t\right)\frac{dt}{t^{3/2}} \\
&= \frac{\sqrt{\pi}}{\alpha}\int_0^{1} \exp\left(-\frac{\pi^2n^2}{\alpha^2}t - \alpha^2\beta^2/t\right)\frac{dt}{t^{1/2}}.
\end{aligned}$$

To evaluate the integral for $n = 0$, we do not need the last substitution and have by equation (4.13)

$$a_{\alpha,\beta}(0) = \frac{\sqrt{\pi}}{\alpha}\int_1^{\infty} \exp\left(-\alpha^2\beta^2t\right)\frac{dt}{t^{3/2}} = \frac{16\pi^{3/2}}{\alpha}\beta(\alpha^2\beta^2).$$

For the rest of the proof let us assume $n \neq 0$. Recall the definition of the error function

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)dt.$$

Now let us compute the derivative of

$$f_{a,b}(t) := \frac{\sqrt{\pi}}{2a}\left(e^{2ab}\operatorname{erf}\left(a\sqrt{t}+b/\sqrt{t}\right) + e^{-2ab}\operatorname{erf}\left(a\sqrt{t}-b/\sqrt{t}\right)\right)$$

for $a, b > 0$:

$$\begin{aligned}
f'_{a,b}(t) &= \frac{\sqrt{\pi}}{2a}e^{2ab}\frac{2}{\sqrt{\pi}}e^{-(a\sqrt{t}+b/\sqrt{t})^2}\left(\frac{a}{2\sqrt{t}} - \frac{b}{2t^{3/2}}\right) \\
&\quad + \frac{\sqrt{\pi}}{2a}e^{-2ab}\frac{2}{\sqrt{\pi}}e^{-(a\sqrt{t}-b/\sqrt{t})^2}\left(\frac{a}{2\sqrt{t}} + \frac{b}{2t^{3/2}}\right) \\
&= \frac{\exp\left(-a^2t - b^2/t\right)}{\sqrt{t}}.
\end{aligned}$$

Note that using

$$\lim_{x\to-\infty} \operatorname{erf}(x) = -1 \quad \text{and} \quad \lim_{x\to\infty} \operatorname{erf}(x) = 1,$$

the function $f_{a,b}(t)$ can be continuously extended to $t = 0$ by

$$f_{a,b}(0) = \frac{\sqrt{\pi}}{2a}\left(e^{2ab} - e^{-2ab}\right).$$

We simplify using $\operatorname{erfc}(x) := 1 - \operatorname{erf}(x)$

$$\begin{aligned} &f_{a,b}(1) - f_{a,b}(0) \\ &\quad = \frac{\sqrt{\pi}}{2a}\left(e^{2ab}\operatorname{erf}(a+b) + e^{-2ab}\operatorname{erf}(a-b)\right) - \frac{\sqrt{\pi}}{2a}\left(e^{2ab} - e^{-2ab}\right) \\ &\quad = -\frac{\sqrt{\pi}}{2a}\left(e^{2ab}\operatorname{erfc}(a+b) + e^{-2ab}\left(\operatorname{erfc}(a-b) - 2\right)\right). \end{aligned}$$

This helps us to solve the integral

$$\begin{aligned} a_{\alpha,\beta}(n) &= \frac{\sqrt{\pi}}{\alpha}\int_0^1 f'_{\pi|n|/\alpha,\alpha\beta}(t)dt \\ &= \frac{\sqrt{\pi}}{\alpha}\left(f_{\pi|n|/\alpha,\alpha\beta}(1) - f_{\pi|n|/\alpha,\alpha\beta}(0)\right) \\ &= -\frac{e^{2\pi|n|\beta}\operatorname{erfc}(\pi|n|/\alpha + \alpha\beta) + e^{-2\pi|n|\beta}\left(\operatorname{erfc}(\pi|n|/\alpha - \alpha\beta) - 2\right)}{2|n|}. \end{aligned}$$

□

Theorem 4.2.2. *The function $\Xi^0(\mathfrak{a}, m, v, z)$ is given for values $z \in \mathbb{H}^2 \setminus S(\mathfrak{a}, m)$ by the Fourier series*

$$\begin{aligned} &16\pi\sqrt{\frac{N(\mathfrak{a})y_1y_2}{vD}} \sum_{\lambda\in\Lambda^+(\mathfrak{a},m)} \beta\left(\frac{\pi vD\operatorname{tr}(\lambda y)^2}{N(\mathfrak{a})y_1y_2}\right) \\ &+ \sum_{\lambda\in\Lambda^+(\mathfrak{a},m)} \sum_{n\in\mathbb{Z}}{}' \frac{e^{-2\pi|n||\operatorname{tr}(\lambda y)|}}{|n|} e(n\operatorname{tr}(\lambda x)) \\ &- \sum_{\lambda\in\Lambda^+(\mathfrak{a},m)} \sum_{n\in\mathbb{Z}}{}' \frac{e^{2\pi|n||\operatorname{tr}(\lambda y)|}\operatorname{erfc}\left(\sqrt{\frac{\pi N(\mathfrak{a})y_1y_2n^2}{vD}} + \sqrt{\frac{\pi vD\operatorname{tr}(\lambda y)^2}{N(\mathfrak{a})y_1y_2}}\right)}{2|n|} e(n\operatorname{tr}(\lambda x)) \\ &- \sum_{\lambda\in\Lambda^+(\mathfrak{a},m)} \sum_{n\in\mathbb{Z}}{}' \frac{e^{-2\pi|n||\operatorname{tr}(\lambda y)|}\operatorname{erfc}\left(\sqrt{\frac{\pi N(\mathfrak{a})y_1y_2n^2}{vD}} - \sqrt{\frac{\pi vD\operatorname{tr}(\lambda y)^2}{N(\mathfrak{a})y_1y_2}}\right)}{2|n|} e(n\operatorname{tr}(\lambda x)). \end{aligned}$$

Proof. Through the rewriting of $\Xi^0(\mathfrak{a}, m, v, z)$ in the beginning of the current section we have seen

$$\Xi^0(\mathfrak{a}, m, v, z) = \sum_{\lambda\in\Lambda^+(\mathfrak{a},m)} h_{\sqrt{\frac{\pi vD}{N(\mathfrak{a})y_1y_2}},|\operatorname{tr}(\lambda y)|}(\operatorname{tr}(\lambda x)).$$

The function $h_{\alpha,\beta}(x)$ is smooth for $\alpha, \beta > 0$ and its Fourier expansion is given in Lemma 4.2.1. The statement of the theorem is derived by using that Fourier expansion. For $\beta = 0$ the function $h_{\alpha,\beta}(x)$ has singularities. They correspond to the $z \in \mathbb{H}^2$ with $\operatorname{tr}(\lambda y) = 0$, hence $z \in S(\mathfrak{a}, m)$. For negative m we have $S(\mathfrak{a}, m) = \emptyset$ and hence no singularities. □

4.3 Growth analysis of $\Xi^0(\mathfrak{a}, m, v, z)$ for $m \neq 0$

In this section we use the Fourier expansion of $\Xi^0(\mathfrak{a}, m, v, z)$ stated in Theorem 4.2.2 to investigate the growth behavior of the function near the cusp ∞, or more precisely, near the exceptional divisor $E^\infty(\mathfrak{a})$. To do so, we fix a totally positive basis (α, β) of $\mathfrak{a}^{-1}$ and let (u, v) be the local coordinates with respect to this basis. Note that the variable v serves as real parameter in $\Xi^0(\mathfrak{a}, m, v, z)$ as well. To reduce confusion we mostly talk about u (v can be treated analogously). In the situations we still use v it is clear from the context which v is meant. In addition to the local variables (u, v), we use the variables (t, r) to express parts of $\Xi^0(\mathfrak{a}, m, v, z)$ (cf. Subsection 2.7.7) and use the relations between (u, v) and (t, r) developed in the end of the referenced subsection.

Lemma 4.3.1. *Let $w > 0$. Then the series*

$$\sum_{\lambda \in \Lambda^+(\mathfrak{a},m)} |\operatorname{tr}(\alpha\lambda)| \sum_{n=1}^{\infty} \log(|u|) \exp\left(\pm 2\pi n |\operatorname{tr}(\lambda y)|\right) \operatorname{erfc}\left(\sqrt{\frac{\pi y_1 y_2 n^2}{w}} \pm \sqrt{\frac{\pi w \operatorname{tr}(\lambda y)^2}{y_1 y_2}}\right)$$

goes to 0 for $u \to 0$. The same holds for the series without the factor $|\operatorname{tr}(\alpha\lambda)|$.

Proof. We translate the series into coordinates (t, r). Recall from (2.32) that the limit process $u \to 0$ translates in (t, r) coordinates into $t \to \infty$ and $r \to \sqrt{\alpha/\alpha'}$. Therefore, for the rest of the proof we will regard r as constant. Furthermore, by the first limit in (2.32) t grows like $\log(|u|)$, hence it is equivalent to replace the factor $\log(|u|)$ by t. Then the inner series is given by

$$t \sum_{n=1}^{\infty} \exp\left(\pm 2\pi n t |\lambda r + \lambda'/r|\right) \operatorname{erfc}\left(tn\sqrt{\pi/w} \pm |\lambda r + \lambda'/r|\sqrt{\pi w}\right).$$

If the argument of erfc is positive, which is always true in case $\pm = +$, we can use the estimate

$$\operatorname{erfc}(x) \leq \exp(-x^2), \quad x \geq 0,$$

and obtain

$$\begin{aligned}
&\exp\left(\pm 2\pi n t |\lambda r + \lambda'/r|\right) \operatorname{erfc}\left(tn\sqrt{\pi/w} \pm |\lambda r + \lambda'/r|\sqrt{\pi w}\right) \\
\leq &\exp\left(\pm 2\pi n t |\lambda r + \lambda'/r|\right) \exp\left(-\left(tn\sqrt{\pi/w} \pm |\lambda r + \lambda'/r|\sqrt{\pi w}\right)^2\right) \\
= &\exp\left(-t^2 n^2 \pi/w\right) \exp\left(-\pi w (\lambda r + \lambda'/r)^2\right).
\end{aligned}$$

Therefore, we are left with

$$\left(\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)} |\mathrm{tr}(\alpha\lambda)| \exp\left(-\pi w(\lambda r+\lambda'/r)^2\right)\right)\left(t\sum_{n=1}^{\infty}\exp\left(-t^2n^2\pi/w\right)\right).$$

The left series is independent of t and can be shown to converge (with and without the factor $|\mathrm{tr}(\alpha\lambda)|$). The right series is clearly convergent and goes to zero for $t\to\infty$.

Hence, we are left with the case $\pm=-$. If we fix some $\lambda\in\Lambda^+(\mathfrak{a},m)$, we still can apply the above argument to the inner series since we can pick t big enough to make

$$tn\sqrt{\pi/w}-|\lambda r+\lambda'/r|\sqrt{\pi w},$$

the argument of erfc, positive. Unfortunately, this cannot be done uniformly in λ. Therefore, we still have to deal with almost all $\lambda\in\Lambda^+(\mathfrak{a},m)$. Note that for almost all $\lambda\in\Lambda^+(\mathfrak{a},m)$ we have

$$|\lambda r+\lambda'/r|>C$$

for any fixed constant C. Therefore, for almost all $\lambda\in\Lambda^+(\mathfrak{a},m)$ we have

$$\exp\left(-2\pi t|\lambda r+\lambda'/r|\right)\le\frac{1}{2}.$$

We obtain by the simple estimate $\mathrm{erfc}(x)\le 2$

$$\begin{aligned}
&t\sum_{n=1}^{\infty}\exp\left(-2\pi nt|\lambda r+\lambda'/r|\right)\mathrm{erfc}\left(tn\sqrt{\pi/w}-|\lambda r+\lambda'/r|\sqrt{\pi w}\right)\\
&\le 2t\sum_{n=1}^{\infty}\exp\left(-2\pi nt|\lambda r+\lambda'/r|\right)\\
&=\frac{2t\exp\left(-2\pi t|\lambda r+\lambda'/r|\right)}{1-\exp\left(-2\pi t|\lambda r+\lambda'/r|\right)}\le\ 4t\exp\left(-2\pi t|\lambda r+\lambda'/r|\right).
\end{aligned}$$

Hence, the assertion of the lemma follows by the claim

$$\lim_{t\to\infty}\sum_{\substack{\lambda\in\Lambda^+(\mathfrak{a},m)\\ |\lambda r+\lambda'/r|>C}} t\,|\mathrm{tr}(\alpha\lambda)|\exp\left(-2\pi t|\lambda r+\lambda'/r|\right)=0$$

for the constant C chosen appropriately. This claim is indeed true since for each of the finitely many $\langle\varepsilon_1\rangle$ orbits of $\Lambda^+(\mathfrak{a},m)$ we have that $|\lambda r+\lambda'/r|$ grows like $\varepsilon_1^{|k|}$ with $k\in\mathbb{Z}$. □

Proposition 4.3.2. *The series*

$$\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\sum_{n\in\mathbb{Z}}{}'\frac{e^{2\pi|n||\mathrm{tr}(\lambda y)|}\,\mathrm{erfc}\left(\sqrt{\frac{\pi N(\mathfrak{a})y_1y_2n^2}{vD}}+\sqrt{\frac{\pi vD\,\mathrm{tr}(\lambda y)^2}{N(\mathfrak{a})y_1y_2}}\right)}{2|n|}e(n\,\mathrm{tr}(\lambda x)) \tag{4.14}$$

$$+\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\sum_{n\in\mathbb{Z}}{}'\frac{e^{-2\pi|n||\mathrm{tr}(\lambda y)|}\,\mathrm{erfc}\left(\sqrt{\frac{\pi N(\mathfrak{a})y_1y_2n^2}{vD}}-\sqrt{\frac{\pi vD\,\mathrm{tr}(\lambda y)^2}{N(\mathfrak{a})y_1y_2}}\right)}{2|n|}e(n\,\mathrm{tr}(\lambda x)) \tag{4.15}$$

converges on $\mathbb{H}^2$ to a smooth function with pre-log-log growth along the exceptional divisor $E^\infty(\mathfrak{a})$.

Proof. The proof of this proposition is a bit technical and lengthy even though we outsourced already an important estimate into Lemma 4.3.1. In order to keep the argument clear and not to bore the reader with almost repetitive computations, we skip some similar looking computations but try to present the expedient ideas.

For example, for the proof of smoothness we leave it to the reader to verify by induction on the order of derivative that

$$\begin{aligned} &e^{2\pi|n||\operatorname{tr}(\lambda y)|}\operatorname{erfc}\left(\sqrt{\frac{\pi N(\mathfrak{a})y_1y_2n^2}{vD}}+\sqrt{\frac{\pi vD\operatorname{tr}(\lambda y)^2}{N(\mathfrak{a})y_1y_2}}\right)\\ +\ &e^{-2\pi|n||\operatorname{tr}(\lambda y)|}\operatorname{erfc}\left(\sqrt{\frac{\pi N(\mathfrak{a})y_1y_2n^2}{vD}}-\sqrt{\frac{\pi vD\operatorname{tr}(\lambda y)^2}{N(\mathfrak{a})y_1y_2}}\right)\end{aligned}$$

is smooth. Note that each line on its own is not even differentiable at $z \in S_\lambda$. The smoothness is attained by considering the sum.

For the proof of the pre-log-log growth we follow Remark 2.9.3 with f being the series of our proposition. To apply this remark we have to regard f in terms of the local coordinates (u, v). To be not confused with the real parameter variable v coming from Kudla's Green functions we write $w := vD/N(\mathfrak{a})$ and prefer using w instead of v for that real parameter.

In the application of Remark 2.9.3 we abbreviate a bit by considering only f and $\partial f/\partial u$. The other derivatives of first order work analogously and even for the second order derivatives no new ideas are needed.

The appropriate growth behavior of f follows directly from Lemma 4.3.1 because of

$$\begin{aligned} &\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\sum_{n\in\mathbb{Z}}{}'\left|\frac{e^{\pm2\pi|n||\operatorname{tr}(\lambda y)|}\operatorname{erfc}\left(\sqrt{\frac{\pi N(\mathfrak{a})y_1y_2n^2}{vD}}\pm\sqrt{\frac{\pi vD\operatorname{tr}(\lambda y)^2}{N(\mathfrak{a})y_1y_2}}\right)}{2|n|}e(n\operatorname{tr}(\lambda x))\right|\\ \le&\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\sum_{n=1}^{\infty}\exp\left(\pm2\pi n|\operatorname{tr}(\lambda y)|\right)\operatorname{erfc}\left(\sqrt{\frac{\pi y_1y_2n^2}{w}}\pm\sqrt{\frac{\pi w\operatorname{tr}(\lambda y)^2}{y_1y_2}}\right).\end{aligned}$$

The missing factor $\log(|u|)$ (compared with Lemma 4.3.1) only accelerates the convergence to 0 for $u \to 0$. Therefore, we are left with $\partial f/\partial u$. To lighten the proof, we consider $n \in \mathbb{N}$ and $\pm = -$ only (the case we present is the more difficult one). Hence, we have to compute the derivative of

$$\begin{aligned} &\frac{e^{-2\pi n|\operatorname{tr}(\lambda y)|}\operatorname{erfc}\left(\sqrt{\frac{\pi N(\mathfrak{a})y_1y_2n^2}{vD}}-\sqrt{\frac{\pi vD\operatorname{tr}(\lambda y)^2}{N(\mathfrak{a})y_1y_2}}\right)}{2n}e(n\operatorname{tr}(\lambda x))\\ =&\frac{1}{2n}\operatorname{erfc}\left(tn\sqrt{\pi/w}-|\lambda r+\lambda'/r|\sqrt{\pi w}\right)e(n\operatorname{tr}(\lambda x)+in|\operatorname{tr}(\lambda y)|)\end{aligned}$$

with respect to u. In case $\operatorname{tr}(\lambda y) \geq 0$ we have by Lemma 2.7.1

$$e(n \operatorname{tr}(\lambda x) + in|\operatorname{tr}(\lambda y)|) = e(n \operatorname{tr}(\lambda z)) = u^{n \operatorname{tr}(\alpha\lambda)} v^{n \operatorname{tr}(\beta\lambda)}.$$

Otherwise, we get by the same lemma

$$e(n \operatorname{tr}(\lambda x) + in|\operatorname{tr}(\lambda y)|) = e(n \operatorname{tr}(\lambda \overline{z})) = \overline{u}^{-n \operatorname{tr}(\alpha\lambda)} \overline{v}^{-n \operatorname{tr}(\beta\lambda)}.$$

In the first case, using the product rule we have

$$\begin{aligned}
&\frac{\partial}{\partial u} \frac{1}{2n} \operatorname{erfc}\left(tn\sqrt{\pi/w} - |\lambda r + \lambda'/r|\sqrt{\pi w}\right) u^{n \operatorname{tr}(\alpha\lambda)} v^{n \operatorname{tr}(\beta\lambda)} \\
&= \frac{1}{2n}\left(\frac{\partial}{\partial u} \operatorname{erfc}\left(tn\sqrt{\pi/w} - |\lambda r + \lambda'/r|\sqrt{\pi w}\right)\right) u^{n \operatorname{tr}(\alpha\lambda)} v^{n \operatorname{tr}(\beta\lambda)} \\
&+ \frac{\operatorname{tr}(\alpha\lambda)}{2} \operatorname{erfc}\left(tn\sqrt{\pi/w} - |\lambda r + \lambda'/r|\sqrt{\pi w}\right) u^{n \operatorname{tr}(\alpha\lambda)-1} v^{n \operatorname{tr}(\beta\lambda)}.
\end{aligned}$$

Following Remark 2.9.3, we have to show that $u \log(|u|) \partial f / \partial u$ has at most log-log growth. We do that for the two lines separately. For the second line we have to consider

$$\log(|u|) \sum_{\substack{\lambda \in \Lambda^+(\mathfrak{a}, m) \\ \operatorname{tr}(\lambda y) \geq 0}} \sum_{n=1}^{\infty} \frac{\operatorname{tr}(\alpha\lambda)}{2} \operatorname{erfc}\left(tn\sqrt{\pi/w} - |\lambda r + \lambda'/r|\sqrt{\pi w}\right) u^{n \operatorname{tr}(\alpha\lambda)} v^{n \operatorname{tr}(\beta\lambda)}$$

which goes to 0 for $u \to 0$ by Lemma 4.3.1. Because of

$$\frac{\partial}{\partial u} \overline{u}^{-n \operatorname{tr}(\alpha\lambda)} \overline{v}^{-n \operatorname{tr}(\beta\lambda)} = 0$$

in the second case $\operatorname{tr}(\lambda y) < 0$ the second line vanishes. Therefore, we can simultaneously consider the two cases by considering the line

$$\frac{1}{2n}\left(\frac{\partial}{\partial u} \operatorname{erfc}\left(tn\sqrt{\pi/w} - |\lambda r + \lambda'/r|\sqrt{\pi w}\right)\right) e^{-2\pi n|\operatorname{tr}(\lambda y)|} e(n \operatorname{tr}(\lambda x)).$$

Using

$$\operatorname{erfc}'(x) = -\frac{2}{\sqrt{\pi}} \exp(-x^2)$$

we obtain

$$\begin{aligned}
&\frac{\partial}{\partial u} \operatorname{erfc}\left(tn\sqrt{\pi/w} - |\lambda r + \lambda'/r|\sqrt{\pi w}\right) \\
&= -\frac{2}{\sqrt{\pi}} \exp\left(-\left(tn\sqrt{\pi/w} - |\lambda r + \lambda'/r|\sqrt{\pi w}\right)^2\right)\left(n\sqrt{\pi/w}\frac{\partial t}{\partial u} \pm \left(\lambda - \lambda'/r^2\right)\sqrt{\pi w}\frac{\partial r}{\partial u}\right) \\
&= -\frac{2}{\sqrt{w}} \exp\left(-t^2 n^2 \pi/w + 2\pi n t|\lambda r + \lambda'/r| - \pi w|\lambda r + \lambda'/r|^2\right) \\
&\quad \times \left(n\frac{\partial t}{\partial u} \pm \left(\lambda - \lambda'/r^2\right) w\frac{\partial r}{\partial u}\right).
\end{aligned}$$

Again, we have to multiply everything by $u\log(|u|)$. Using (2.32) and (2.33), we obtain

$$\lim_{u\to 0} u\frac{\partial t}{\partial u} = \frac{\sqrt{N(\alpha)}}{4\pi} \quad \text{and} \quad \lim_{u\to 0} u\log(|u|)\frac{\partial r}{\partial u} = 0.$$

Therefore, we are left with showing the log-log growth of the series of

$$-\frac{\log(|u|)}{2n}\frac{2}{\sqrt{w}}\exp\left(-t^2n^2\pi/w + 2\pi nt|\lambda r + \lambda'/r| - \pi w|\lambda r + \lambda'/r|^2\right)$$
$$\times n\frac{\sqrt{N(\alpha)}}{4\pi}e^{-2\pi n|\operatorname{tr}(\lambda y)|}e(n\operatorname{tr}(\lambda x)).$$

Taking absolute values of this expression, we get

$$|\log(|u|)|\,\frac{\sqrt{N(\alpha)}}{4\pi\sqrt{w}}\exp\left(-t^2n^2\pi/w\right)\exp\left(-\pi w|\lambda r + \lambda'/r|^2\right).$$

Therefore, the log-log growth of the series follows by the convergence of

$$\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\exp\left(-\pi w|\lambda r + \lambda'/r|^2\right)$$

and the limit

$$\lim_{t\to\infty} t\sum_{n=1}^{\infty}\exp\left(-t^2n^2\pi/w\right) = 0.$$

Recall that because of (2.32) we can replace $|\log(|u|)|$ by t. □

Lemma 4.3.3. *We have*

$$\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\sum_{n\in\mathbb{Z}}{}' \frac{e^{-2\pi|n||\operatorname{tr}(\lambda y)|}}{|n|}e(n\operatorname{tr}(\lambda x))$$
$$= -2\log|\Psi(\mathfrak{a},m,z)| - 4\pi\delta_{m>0}\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\beta(\lambda y_1,\lambda' y_2).$$

Proof. The actual work for $m>0$ was already carried out in the previous chapter in Lemma 3.4.3 where we investigated the automorphic Green functions. In that case we have

$$\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\sum_{n\in\mathbb{Z}}{}' \frac{e^{-2\pi|n||\operatorname{tr}(\lambda y)|}}{|n|}e(n\operatorname{tr}(\lambda x))$$
$$= \sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\sum_{n=1}^{\infty}\frac{e^{-2\pi n|\operatorname{tr}(\lambda y)|}}{n}\left(e(n\operatorname{tr}(\lambda x)) + e(-n\operatorname{tr}(\lambda x))\right)$$
$$= -4\pi\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\beta(\lambda y_1,\lambda' y_2) - 2\log\prod_{\lambda\in\Lambda^+(\mathfrak{a},m)}\left|e(|\lambda|z_1) - e(|\lambda'|z_2)\right|$$
$$= -4\pi\sum_{\lambda\in\Lambda^+(\mathfrak{a},m)}\beta(\lambda y_1,\lambda' y_2) - 2\log|\Psi(\mathfrak{a},m,z)|.$$

For the second equation look into the proof of Lemma 3.4.3. For $m < 0$, however, we can only adopt the first equation but still the ideas of Lemma 3.4.3 apply. Because of $\operatorname{tr}(\lambda y) > 0$ we have for $m < 0$

$$\sum_{n=1}^{\infty} \frac{e^{-2\pi n|\operatorname{tr}(\lambda y)|}}{n} e(n \operatorname{tr}(\lambda x)) = -\log(1 - e(\operatorname{tr}(\lambda z))$$

and

$$\sum_{n=1}^{\infty} \frac{e^{-2\pi n|\operatorname{tr}(\lambda y)|}}{n} e(-n \operatorname{tr}(\lambda x)) = \overline{-\log(1 - e(\operatorname{tr}(\lambda z))}.$$

In total we obtain twice the real part

$$-2 \log |1 - e(\operatorname{tr}(\lambda z))| .$$

Definition 3.5.8 of $\Psi(\mathfrak{a}, m, z)$ for $m < 0$ yields the stated result. □

The lemma uncovers the logarithm of the absolute value of the local Borcherds product $\Psi(\mathfrak{a}, m, z)$ in $\Xi^0(\mathfrak{a}, m, v, z)$ providing the logarithmic singularities along $-Z^\infty(\mathfrak{a}, m)$ in a neighborhood of $E^\infty(\mathfrak{a})$ (cf. Corollary 3.5.7). That is what we were looking for. Now we would like to show that the remainder of $\Xi^0(\mathfrak{a}, m, v, z)$ is of pre-log-log growth at $E^\infty(\mathfrak{a})$ to make $\Xi(\mathfrak{a}, m, v, z)$ a pre-log-log Green function with respect to the divisor $Z(\mathfrak{a}, m)$. Unfortunately, this is not the case as we see in the next pages of this section. Let us treat the remainder of the first two lines of the Fourier expansion given in Theorem 4.2.2 now and give it the name

$$\begin{aligned} \check{\Xi}(\mathfrak{a}, m, v, z) := 16\pi \sqrt{\frac{N(\mathfrak{a}) y_1 y_2}{vD}} \sum_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \beta\left(\frac{\pi v D \operatorname{tr}(\lambda y)^2}{N(\mathfrak{a}) y_1 y_2}\right) \\ - 4\pi \delta_{m>0} \sum_{\lambda \in \Lambda^+(\mathfrak{a}, m)} \beta(\lambda y_1, \lambda' y_2). \end{aligned} \tag{4.16}$$

Note that $\check{\Xi}(\mathfrak{a}, m, v, z)$ depends only on $y = (y_1, y_2)$ but not on the real part $x = (x_1, x_2)$ of $z = (z_1, z_2)$.

Proposition 4.3.4. *The function $\check{\Xi}(\mathfrak{a}, m, v, z)$ can be written in (t, r) coordinates (cf. Subsection 2.7.7) by*

$$\check{\Xi}(\mathfrak{a}, m, v, z) = t \sum_{\lambda \in \Lambda^+(\mathfrak{a}, m)} F_\lambda(r)$$

with analytic functions $F_\lambda : \mathbb{R}^+ \to \mathbb{R}$. In particular, $\check{\Xi}(\mathfrak{a}, m, v, z)$ is smooth in z.

Proof. We have

$$16\pi \sqrt{\frac{N(\mathfrak{a}) y_1 y_2}{vD}} \beta\left(\frac{\pi v D \operatorname{tr}(\lambda y)^2}{N(\mathfrak{a}) y_1 y_2}\right) = 16\pi t \sqrt{\frac{N(\mathfrak{a})}{vD}} \beta\left(\frac{\pi v D (\lambda r + \lambda'/r)^2}{N(\mathfrak{a})}\right).$$

This proves the statement for $m < 0$ since $\beta(x)$ is analytic for arguments $x > 0$ and $\lambda r + \lambda'/r > 0$ in case $\lambda \in \Lambda^+(\mathfrak{a}, m)$ with $m < 0$. The main idea for $m > 0$ is to write $\check{\Xi}(\mathfrak{a}, m, v, z)$ in terms of the functions

$$g_\lambda(r) := |\lambda r + \lambda'/r|$$

for $\lambda \in \Lambda^+(\mathfrak{a}, m)$. The function $g_\lambda(r)$ itself is not differentiable but $g_\lambda(r)^2$ is even analytic. Note that we have

$$|\mathrm{tr}(\lambda y)| = t g_\lambda(r) \quad \text{and} \quad 2\beta(\lambda y_1, \lambda' y_2) = t(\lambda r - \lambda'/r - g_\lambda(r)).$$

With $C := \sqrt{vD/N(\mathfrak{a})}$ and equation (4.13) we get

$$\begin{aligned} &16\pi\sqrt{\frac{N(\mathfrak{a})y_1y_2}{vD}} \sum_{\lambda\in\Lambda^+(\mathfrak{a},m)} \beta\left(\frac{\pi v D\,\mathrm{tr}(\lambda y)^2}{N(\mathfrak{a})y_1y_2}\right) \\ &= 2t \sum_{\lambda\in\Lambda^+(\mathfrak{a},m)} C^{-1}\left(\exp(-\pi C^2 g_\lambda(r)^2) - \pi C g_\lambda(r)\,\mathrm{erfc}(\sqrt{\pi} C g_\lambda(r))\right) \end{aligned}$$

and

$$-4\pi \sum_{\lambda\in\Lambda^+(\mathfrak{a},m)} \beta(\lambda y_1, \lambda' y_2) = -2t \sum_{\lambda\in\Lambda^+(\mathfrak{a},m)} \pi(\lambda r - \lambda' r - g_\lambda(r)).$$

Hence, we are left with proving that

$$g_\lambda(r)\,\mathrm{erfc}\left(\sqrt{\pi} C g_\lambda(r)\right) - g_\lambda(r) = -g_\lambda(r)\,\mathrm{erf}\left(\sqrt{\pi} C g_\lambda(r)\right)$$

is analytic in r. However, this follows from $\mathrm{erf}(z)$ being analytic and odd. Namely, we have

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}.$$

Therefore,

$$g_\lambda(r)\,\mathrm{erf}\left(\sqrt{\pi} C g_\lambda(r)\right) = \sum_{n=1}^{\infty} a_n(C) g_\lambda(r)^{2n}$$

consists only of even powers of $g_\lambda(r)$. Hence, it is the composition of two analytic functions and therefore analytic. □

Proposition 4.3.5. *Let $m \neq 0$. The difference*

$$\Xi^0(\mathfrak{a}, m, v, z) - \check{\Xi}(\mathfrak{a}, m, v, z)$$

is well-defined in a neighborhood of $E^\infty(\mathfrak{a}) \subset \overline{X(\mathfrak{a})}$ and decomposes into a sum of a part having logarithmic singularities along the divisor

$$-(T^\infty(\mathfrak{a}, m) + Z^\infty(\mathfrak{a}, m))$$

and a part being a pre-log-log growth form along the divisor $E^\infty(\mathfrak{a})$.

Proof. The well-definedness is clear with Proposition 2.5.2 because the function is invariant under the stabilizer $\Gamma_{\mathfrak{a},\infty}$. By Theorem 4.2.2, Lemma 4.3.3 and the definition of $\check{\Xi}(\mathfrak{a}, m, v, z)$ we have

$$\begin{aligned}
&\Xi^0(\mathfrak{a}, m, v, z) - \check{\Xi}(\mathfrak{a}, m, v, z) = -2\log|\Psi(\mathfrak{a}, m, z)| \\
&- \sum_{\lambda\in\Lambda^+(\mathfrak{a},m)} \sum_{n\in\mathbb{Z}}{}' \frac{e^{2\pi|n||\operatorname{tr}(\lambda y)|}\operatorname{erfc}\left(\sqrt{\frac{\pi N(\mathfrak{a})y_1y_2n^2}{vD}} + \sqrt{\frac{\pi vD\operatorname{tr}(\lambda y)^2}{N(\mathfrak{a})y_1y_2}}\right)}{2|n|} e(n\operatorname{tr}(\lambda x)) \\
&- \sum_{\lambda\in\Lambda^+(\mathfrak{a},m)} \sum_{n\in\mathbb{Z}}{}' \frac{e^{-2\pi|n||\operatorname{tr}(\lambda y)|}\operatorname{erfc}\left(\sqrt{\frac{\pi N(\mathfrak{a})y_1y_2n^2}{vD}} - \sqrt{\frac{\pi vD\operatorname{tr}(\lambda y)^2}{N(\mathfrak{a})y_1y_2}}\right)}{2|n|} e(n\operatorname{tr}(\lambda x)).
\end{aligned}$$

By Corollary 3.5.7 the function

$$-2\log|\Psi(\mathfrak{a}, m, z)| = -\log|\Psi(\mathfrak{a}, m, z)|^2$$

has logarithmic singularities along $-(T^\infty(\mathfrak{a}, m) + Z^\infty(\mathfrak{a}, m))$. The remainder is smooth on $X(\mathfrak{a})$ and a pre-log-log growth form along $E^\infty(\mathfrak{a})$ by Proposition 4.3.2. □

Remark 4.3.6. At this place we want to point out that the subtraction of $\check{\Xi}(\mathfrak{a}, m, v, z)$ in Proposition 4.3.5 is essential in case $\check{\Xi}(\mathfrak{a}, m, v, z)$ is different from the zero function (which is equivalent to $\Lambda(\mathfrak{a}, m) \neq \emptyset$): Recall

$$\lim_{u\to 0} \frac{t}{\log(|u|)} = -\frac{\sqrt{N(\alpha)}}{2\pi}$$

from (2.32). Hence, by Proposition 4.3.4 the function $\check{\Xi}(\mathfrak{a}, m, v, z)$ has log growth but not log-log growth at the exceptional divisor $E^\infty(\mathfrak{a})$. Therefore, $\Xi^0(\mathfrak{a}, m, v, z)$ is not of pre-log-log growth at $E^\infty(\mathfrak{a})$. Together with Theorem 4.5.1 which we prove in Section 4.5 this implies that $\Xi(\mathfrak{a}, m, v, z)$ is not a pre-log-log Green function on $\overline{X(\mathfrak{a})}$ with respect to the divisor $Z(\mathfrak{a}, m)$.

4.4 Fourier expansion and growth analysis of $\Xi^0(\mathfrak{a}, 0, v, z)$

We compute

$$\begin{aligned}
\Xi^0_*(\mathfrak{a}, 0, v, z) &= \frac{1}{2} \sum_{\substack{A=\left(\begin{smallmatrix} a & \lambda' \\ \lambda & 0 \end{smallmatrix}\right)\in L(\mathfrak{a})^\vee \\ \det(A)=0}}{}' E_1\left(4\pi vDN(\mathfrak{a})h(A, z)\right) \\
&= \frac{1}{2} \sum_{a\in\mathbb{Z}/N(\mathfrak{a})}{}' E_1\left(\frac{4\pi vDN(\mathfrak{a})a^2}{4y_1y_2}\right) = \sum_{a=1}^{\infty} E_1\left(\frac{\pi vDa^2}{N(\mathfrak{a})y_1y_2}\right).
\end{aligned}$$

Hence, we see that $\Xi^0_*(\mathfrak{a}, 0, v, z)$ is independent of the real part $x = (x_1, x_2)$ of $z \in \mathbb{H}^2$. In particular, its Fourier expansion consists only of the constant term $\Xi^0_*(\mathfrak{a}, 0, v, z)$ itself.

To understand the growth of $\Xi^0_*(\mathfrak{a}, 0, v, z)$ near the cusp ∞ we need two lemmata:

Lemma 4.4.1. *For small $x > 0$ we have the asymptotics*

$$\sum_{n=1}^{\infty} E_1(n^2 x) \sim_a \sqrt{\frac{\pi}{x}} + \frac{\log(x)}{2} + \frac{\gamma}{2} - \log(2\pi).$$

With $\sim_a$ we mean that the difference of the two sides goes to 0 for $x \to 0$.

Proof. It holds

$$\sum_{n=1}^{\infty} E_1(n^2 x) = \sum_{n=1}^{\infty} \int_1^{\infty} e^{-tn^2 x} \frac{dt}{t}.$$

For $s \in \mathbb{C}$ with $\Re(s) > 1$ we have

$$\sum_{n=1}^{\infty} \int_1^{\infty} e^{-tn^2 x} t^s \frac{dt}{t} = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-tn^2 x} t^s \frac{dt}{t} - \sum_{n=1}^{\infty} \int_0^{1} e^{-tn^2 x} t^s \frac{dt}{t}.$$

The left side of the equation is holomorphic at $s = 0$ and it is just the expression we are interested in. Both integrals of the right hand side, however, do not converge at $s = 0$ but for $\Re(s) > 1$. Therefore, we compute an analytic continuation to arrive at $s = 0$. It holds

$$\begin{aligned}
&\sum_{n=1}^{\infty} \int_0^{\infty} e^{-tn^2 x} t^s \frac{dt}{t} = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-t} \left(\frac{t}{n^2 x}\right)^s \frac{dt}{t} \\
&= x^{-s} \sum_{n=1}^{\infty} n^{-2s} \int_0^{\infty} e^{-t} t^{s-1} dt = x^{-s} \zeta(2s) \Gamma(s) \\
&= \Big(1 - \log(x) s + O(s^2)\Big) \left(-\frac{1}{2} - \log(2\pi) s + O(s^2)\right) \left(\frac{1}{s} - \gamma + O(s)\right) \\
&= -\frac{1}{2s} + \left(1 \cdot \left(-\frac{1}{2}\right)(-\gamma) + 1 \cdot (-\log(2\pi)) \cdot 1 + (-\log(x)) \left(-\frac{1}{2}\right) \cdot 1\right) + O(s) \\
&= -\frac{1}{2s} + \left(\frac{\gamma}{2} - \log(2\pi) + \frac{\log(x)}{2}\right) + O(s).
\end{aligned}$$

For the other integral we use the theta function (4.10) and its transformation law (4.11).

We obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\int_0^1 e^{-tn^2x}t^s\frac{dt}{t} &= \int_0^1 \frac{\theta\left(\frac{txi}{\pi}\right)-1}{2}t^s\frac{dt}{t}\\
&= \frac{1}{2}\int_0^1 \theta\left(\frac{txi}{\pi}\right)t^{s-1}dt - \frac{1}{2}\int_0^1 t^{s-1}dt\\
&= \frac{1}{2}\int_1^{\infty} \theta\left(\frac{xi}{\pi t}\right)t^{1-s}t^{-2}dt - \frac{1}{2}\left[\frac{t^s}{s}\right]_0^1\\
&= \frac{1}{2}\int_1^{\infty} \sqrt{\frac{\pi it}{xi}}\theta\left(\frac{\pi it}{x}\right)t^{-s-1}dt - \frac{1}{2s}\\
&= -\frac{1}{2s} + \frac{1}{2}\sqrt{\frac{\pi}{x}}\int_1^{\infty} \theta\left(\frac{\pi it}{x}\right)t^{-s-1/2}dt\\
&= -\frac{1}{2s} + \frac{1}{2}\sqrt{\frac{\pi}{x}}\int_1^{\infty} t^{-s-1/2}dt + \varepsilon(s,x)\\
&= -\frac{1}{2s} + \frac{1}{2}\sqrt{\frac{\pi}{x}}\left[\frac{t^{-s+1/2}}{-s+1/2}\right]_1^{\infty} + \varepsilon(s,x)\\
&= -\frac{1}{2s} + \frac{1}{2s-1}\sqrt{\frac{\pi}{x}} + \varepsilon(s,x)
\end{aligned}$$

with

$$\varepsilon(s,x) := \sqrt{\frac{\pi}{x}}\int_1^{\infty}\sum_{n=1}^{\infty}\exp\left(-\frac{\pi^2n^2t}{x}\right)t^{-s-1/2}dt.$$

We get

$$\begin{aligned}
&\int_1^{\infty}\sum_{n=1}^{\infty} e^{-tn^2x}t^s\frac{dt}{t}\\
&= -\frac{1}{2s} + \left(\frac{\gamma}{2} - \log(2\pi) + \frac{\log(x)}{2}\right) + O(s) - \left(-\frac{1}{2s} + \frac{1}{2s-1}\sqrt{\frac{\pi}{x}} + \varepsilon(s,x)\right)\\
&= \frac{\gamma}{2} - \log(2\pi) + \frac{\log(x)}{2} + \frac{1}{1-2s}\sqrt{\frac{\pi}{x}} - \varepsilon(s,x) + O(s).
\end{aligned}$$

At the limit $s \to 0$ it remains

$$\sqrt{\frac{\pi}{x}} + \frac{\log(x)}{2} + \frac{\gamma}{2} - \log(2\pi) - \varepsilon(0,x).$$

Hence, to finish the proof it is left to show

$$\lim_{x\to 0}\varepsilon(0,x) = 0.$$

We have

$$\begin{aligned}
\varepsilon(0,x) &= \sqrt{\frac{\pi}{x}} \int_1^\infty \sum_{n=1}^\infty \exp\left(-\frac{\pi^2 n^2 t}{x}\right) t^{-1/2} dt \\
&\leq \sqrt{\frac{\pi}{x}} \sum_{n=1}^\infty \int_1^\infty \exp\left(-\frac{\pi^2 n^2 t}{x}\right) dt \\
&= \sqrt{\frac{\pi}{x}} \sum_{n=1}^\infty \left[\frac{\exp\left(-\frac{\pi^2 n^2 t}{x}\right)}{-\frac{\pi^2 n^2}{x}}\right]_1^\infty \\
&= \sqrt{\frac{\pi}{x}} \frac{x}{\pi^2} \sum_{n=1}^\infty \frac{\exp\left(-\frac{\pi^2 n^2}{x}\right)}{n^2} \\
&\leq \frac{\sqrt{x\pi}}{\pi^2} \exp\left(-\frac{\pi^2}{x}\right) \sum_{n=1}^\infty \frac{1}{n^2} \\
&= \frac{\sqrt{x\pi}}{\pi^2} \exp\left(-\frac{\pi^2}{x}\right) \frac{\pi^2}{6} \\
&= \frac{\sqrt{x\pi}}{6} \exp\left(-\frac{\pi^2}{x}\right).
\end{aligned}$$

This vanishes for $x \to 0$. □

Lemma 4.4.2. *For large $t > 0$ we have the asymptotics*

$$2 \sum_{n=1}^\infty \exp\left(-\frac{\pi n^2}{t^2}\right) \sim_a t - 1.$$

Proof. Similar to the previous proof we use the theta function (4.10) and its transformation law (4.11) to obtain

$$\sum_{n\in\mathbb{Z}} \exp\left(-\frac{\pi n^2}{t^2}\right) = \theta(i/t^2) = t\theta(it^2) \sim_a t. \tag{4.17}$$

The asymptotics $t\theta(it^2) \sim_a t$ follows from estimate (4.12). Now, the statement of the lemma follows from (4.17) by subtracting 1. □

In analogy to equation (4.16), we define, written in (t, r) coordinates (cf. Subsection 2.7.7),

$$\check{\Xi}(\mathfrak{a}, 0, v, z) := t\sqrt{\frac{N(\mathfrak{a})}{vD}}.$$

Remark 4.4.3. This definition of $\check{\Xi}(\mathfrak{a}, 0, v, z)$ is no exception for $m = 0$. We just have to rewrite equation (4.16) with $\Lambda(\mathfrak{a}, m)$ instead of $\Lambda^+(\mathfrak{a}, m)$ and obtain because of

$\beta(0) = 1/(8\pi)$ and $\Lambda(\mathfrak{a}, 0) = \{0\}$

$$\begin{aligned}\check{\Xi}(\mathfrak{a}, m, v, z) = 8\pi t \sqrt{\frac{N(\mathfrak{a})}{vD}} \sum_{\lambda \in \Lambda(\mathfrak{a}, m)} \beta\left(\frac{\pi v D(\lambda r + \lambda'/r)^2}{N(\mathfrak{a})}\right) \\ - 2\pi t\ \delta_{m>0} \sum_{\lambda \in \Lambda(\mathfrak{a}, m)} \beta(\lambda r, \lambda'/r)\end{aligned} \tag{4.18}$$

for all $m \in \mathbb{Z}$. With this reformulation Proposition 4.3.4 holds for $m = 0$ as well. Here, $F_0(r) = \check{\Xi}(\mathfrak{a}, m, v, z)/t$ is even a constant.

Proposition 4.4.4. *The difference*

$$\Xi_*^0(\mathfrak{a}, 0, v, z) - \check{\Xi}(\mathfrak{a}, 0, v, z)$$

is a pre-log-log growth form along the divisor $E^\infty(\mathfrak{a})$.

Proof. As presented a few times already we stick to Remark 2.9.3 and discuss the two cases f and $\partial f/\partial u$ only with f representing the difference function from the statement. For the case f we apply Lemma 4.4.1 and obtain

$$\begin{aligned}\Xi_*^0(\mathfrak{a}, 0, v, z) - \check{\Xi}(\mathfrak{a}, 0, v, z) &= \sum_{a=1}^{\infty} E_1\left(\frac{\pi v D a^2}{N(\mathfrak{a}) t^2}\right) - t\sqrt{\frac{N(\mathfrak{a})}{vD}} \\ &\sim_a \sqrt{\frac{\pi}{\frac{\pi v D}{N(\mathfrak{a}) t^2}}} + \frac{\log\left(\frac{\pi v D}{N(\mathfrak{a}) t^2}\right)}{2} + \frac{\gamma}{2} - \log(2\pi) - t\sqrt{\frac{N(\mathfrak{a})}{vD}} \\ &= \frac{\gamma}{2} + \frac{\log\left(\frac{vD}{4\pi N(\mathfrak{a})}\right)}{2} - \log(t).\end{aligned} \tag{4.19}$$

Since $\log(t)$ is of log-log growth by Lemma 3.6.4, this proves the case f. Let us now consider the case $\partial f/\partial u$. We have by equation (4.6)

$$\begin{aligned}\frac{\partial}{\partial u}\left(\Xi_*^0(\mathfrak{a}, 0, v, z) - \check{\Xi}(\mathfrak{a}, 0, v, z)\right) &= \left(-\sum_{a=1}^{\infty} \frac{\exp\left(-\frac{\pi v D a^2}{N(\mathfrak{a}) t^2}\right)}{\frac{\pi v D a^2}{N(\mathfrak{a}) t^2}} \frac{-2\pi v D a^2}{N(\mathfrak{a}) t^3} - \sqrt{\frac{N(\mathfrak{a})}{vD}}\right) \frac{\partial t}{\partial u} \\ &= \left(2\sum_{a=1}^{\infty} \exp\left(-\frac{\pi v D a^2}{N(\mathfrak{a}) t^2}\right) - t\sqrt{\frac{N(\mathfrak{a})}{vD}}\right) \frac{1}{t}\frac{\partial t}{\partial u}.\end{aligned}$$

Now, by Lemma 4.4.2 we obtain for the inner bracket the limit

$$\lim_{t\to\infty}\left(2\sum_{a=1}^{\infty} \exp\left(-\frac{\pi v D a^2}{N(\mathfrak{a}) t^2}\right) - t\sqrt{\frac{N(\mathfrak{a})}{vD}}\right) = \lim_{t\to\infty}\left(t\sqrt{\frac{N(\mathfrak{a})}{vD}} - 1 - t\sqrt{\frac{N(\mathfrak{a})}{vD}}\right) = -1.$$

Further, using the equations (2.32) and (2.33), we get

$$\lim_{u\to 0} u\log(|u|)\frac{1}{t}\frac{\partial t}{\partial u} = \lim_{u\to 0} u\frac{\log(|u|)}{t}\frac{\sqrt{N(\alpha)}}{4\pi u} = -\frac{2\pi}{\sqrt{N(\alpha)}}\frac{\sqrt{N(\alpha)}}{4\pi} = -\frac{1}{2}.$$

Combined we have shown

$$\lim_{u\to 0} u\log(|u|)\frac{\partial}{\partial u}\left(\Xi^0_*(\mathfrak{a},0,v,z)-\check{\Xi}(\mathfrak{a},0,v,z)\right)=\frac{1}{2}$$

which proves the case $\partial f/\partial u$ of Remark 2.9.3. □

Remark 4.4.5. It is also interesting to look at the asymptotics of

$$\Xi^0(\mathfrak{a},0,v,z)-\check{\Xi}(\mathfrak{a},0,v,z)$$

near the cusp ∞. Using (4.19) and (2.55), we obtain

$$\begin{aligned}
&\Xi^0(\mathfrak{a},0,v,z)-\check{\Xi}(\mathfrak{a},0,v,z)\\
\sim_a\ &\Xi_0(\mathfrak{a},v,z)+\frac{\gamma}{2}+\frac{\log\left(\frac{vD}{4\pi N(\mathfrak{a})}\right)}{2}-\log(t)\\
=\ &G(\mathfrak{a},0,z)-\frac{\log(4\pi vD/N(\mathfrak{a}))+\gamma}{2}+\frac{\gamma}{2}+\frac{\log\left(\frac{vD}{4\pi N(\mathfrak{a})}\right)}{2}-\log(t)\\
=\ &G(\mathfrak{a},0,z)-\log(4\pi)-\log(t)\\
=\ &\frac{\log(|F(\mathfrak{a},z)|^2)}{2k}.
\end{aligned}$$

Hence, we see that even the log-log growth $\log(t)$ cancels out. Only the logarithmic singularities of $G(\mathfrak{a},0,z)$ along the divisor $-Z(\mathfrak{a},0)$ remain.

4.5 Modification at the cusps using partitions of unity

We summarize the insights of Section 4.1, Section 4.3 and Section 4.4 in the following theorem.

Theorem 4.5.1. *For all $m\in\mathbb{Z}$ the difference*

$$\Xi(\mathfrak{a},m,v,z)-\check{\Xi}(\mathfrak{a},m,v,z) \tag{4.20}$$

is well-defined in a neighborhood of $E^\infty(\mathfrak{a})\subset\overline{X(\mathfrak{a})}$ and decomposes into a sum of a part having logarithmic singularities along the divisor $-Z(\mathfrak{a},m)$ and a part being a pre-log-log growth form along the divisor $E^\infty(\mathfrak{a})$.

Proof. By Proposition 4.1.3 it is equivalent to consider

$$\Xi^0(\mathfrak{a},m,v,z)-\check{\Xi}(\mathfrak{a},m,v,z)$$

instead. In case $m\neq 0$ this was done in Proposition 4.3.5 for the divisor

$$-(T^\infty(\mathfrak{a},m)+Z^\infty(\mathfrak{a},m))$$

instead of the divisor $-Z(\mathfrak{a}, m)$. However, in a small enough neighborhood of $E^\infty(\mathfrak{a}) \subset \overline{X(\mathfrak{a})}$ the two divisors coincide. In case $m = 0$ we have

$$\Xi^0(\mathfrak{a}, 0, v, z) - \check{\Xi}(\mathfrak{a}, 0, v, z) = (\Xi^0_*(\mathfrak{a}, 0, v, z) - \check{\Xi}(\mathfrak{a}, 0, v, z)) + \Xi_0(\mathfrak{a}, v, z).$$

For $\Xi^0_*(\mathfrak{a}, 0, v, z) - \check{\Xi}(\mathfrak{a}, 0, v, z)$ the work was done in Proposition 4.4.4 and the second component

$$\Xi_0(\mathfrak{a}, v, z) = G(\mathfrak{a}, 0, z) - \frac{\log(4\pi v D/N(\mathfrak{a})) + \gamma}{2}.$$

has logarithmic singularities along $-Z(\mathfrak{a}, 0)$ according to Proposition 2.9.24. □

However, we cannot simply define a Green function on $\overline{X(\mathfrak{a})}$ by equation (4.20) for two reasons: First, the subtraction of $\check{\Xi}(\mathfrak{a}, m, v, z)$ solves the problem only for the cusp ∞ and if $h_K > 1$ there are other cusps. Second, the function $\check{\Xi}(\mathfrak{a}, m, v, z)$ (and therefore the difference as well) is not defined on all of $X(\mathfrak{a})$. It is only defined in a neighborhood of the cusp ∞. That is because $\check{\Xi}(\mathfrak{a}, m, v, z)$ as a function on $\mathbb{H}^2$ is only invariant under the stabilizer of ∞ but not under the whole Hilbert modular group.

We start with solving the second problem first. Since the difference (4.20) is well-defined in small enough neighborhoods of the cusp ∞, we find two such small neighborhoods $U^0 \subset U^1$ of the cusp ∞ with a smooth function

$$\rho : X(\mathfrak{a}) \to [0, 1]$$

by the theory of partitions of unity which satisfies

$$\rho|_{U_0} \equiv 1 \quad \text{and} \quad \operatorname{supp}(\rho) \subset U_1.$$

Because $\check{\Xi}(\mathfrak{a}, m, v, z)$ is a well-defined function on U_1 the relation $\operatorname{supp}(\rho) \subset U_1$ implies that $\rho(z)\check{\Xi}(\mathfrak{a}, 0, v, z)$ is well-defined on $X(\mathfrak{a})$. Therefore,

$$\tilde{\Xi}_\rho(\mathfrak{a}, m, v, z) := \Xi(\mathfrak{a}, m, v, z) - \rho(z)\check{\Xi}(\mathfrak{a}, m, v, z)$$

is well-defined on $X(\mathfrak{a})$ and by construction Theorem 4.5.1 implies that $\tilde{\Xi}_\rho(\mathfrak{a}, m, v, z)$ behaves as desired at the cusp ∞.

Now we come to solving the first problem. If $h_K \neq 1$ the process has to be done for the other cusps as well to finally obtain a pre-log-log Green function. Let us now explain the process for simultaneously correcting $\Xi(\mathfrak{b}, m, v, z)$ at all cusps. We switch here in the notation from $\Xi(\mathfrak{a}, m, v, z)$ to $\Xi(\mathfrak{b}, m, v, z)$ to be more consistent with the referenced results. Let $\sigma_1, \ldots, \sigma_{h_K}$ be representatives of the cusps of $\Gamma_\mathfrak{b}$. We find $\mathfrak{a}_1, \ldots, \mathfrak{a}_{h_K} \in \mathcal{I}_K$ such that the isomorphism

$$X(\mathfrak{b}) \to X(\mathfrak{a}_j^2\mathfrak{b}), \quad z \mapsto M_j^{-1} z$$

with $M_j \in M(\mathfrak{a}_j, \mathfrak{b})$ maps the cusp $\sigma_j \in X(\mathfrak{b})^*$ to the cusp $\infty \in X(\mathfrak{a}_j^2\mathfrak{b})^*$. Note that we have

$$\Xi_*(\mathfrak{b}, m, M_j z) = \Xi_*(\mathfrak{a}_j^2\mathfrak{b}, m, z)$$

by Proposition 4.1.2. Now let $U_j^0 \subset U_j^1$ be small enough neighborhoods of the cusp σ_j in $X(\mathfrak{b})$ and let

$$\rho_j : X(\mathfrak{b}) \to [0,1]$$

be smooth functions with

$$\rho_j|_{U_j^0} \equiv 1 \quad \text{and} \quad \operatorname{supp}(\rho_j) \subset U_j^1.$$

Clearly, all U_j^1 are disjoint and we can define

$$\rho : X(\mathfrak{b}) \to [0,1] \quad \text{by} \quad \rho(z) := \sum_{j=1}^{h_k} \rho_j(z). \tag{4.21}$$

Finally, we define

$$\widetilde{\Xi}_\rho(\mathfrak{b}, m, v, z) := \Xi(\mathfrak{b}, m, v, z) - \sum_{j=1}^{h_k} \rho_j(z) \check{\Xi}(\mathfrak{a}_j^2 \mathfrak{b}, m, v, M_j^{-1} z). \tag{4.22}$$

By construction we obtain the next theorem.

Theorem 4.5.2. *The function $\widetilde{\Xi}_\rho(\mathfrak{a}, m, v, z)$ defined by equation (4.22) is a pre-log-log Green function on $\overline{X(\mathfrak{a})}$ with respect to the divisor $Z(\mathfrak{a}, m)$.*

4.6 Integrals for $m \neq 0$

In this section we compute the integral

$$\int_{\Gamma_{\mathfrak{a}} \backslash \mathbb{H}^2} \Xi(\mathfrak{a}, m, v, z) \omega^2$$

for $m \neq 0$. We need to recall the definition

$$\Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t} \, dt \tag{4.23}$$

of the *upper incomplete gamma function* for the first lemma.

Lemma 4.6.1. *Let $v > 0$. Then we have*

$$\int_{z \in \mathbb{H}} E_1\left(\pi v \frac{|z-i|^2}{y}\right) \frac{dxdy}{y^2} = \frac{1}{v}$$

and

$$\int_{z \in \mathbb{H}} E_1\left(\pi v \frac{|z+i|^2}{y}\right) \frac{dxdy}{y^2} = 4\pi \Gamma(-1, 4\pi v).$$

Proof. We have

$$\begin{aligned}
&\int_{z\in\mathbb{H}} E_1\left(\pi v\frac{|z\pm i|^2}{y}\right)\frac{dxdy}{y^2}\\
=&\int_{z\in\mathbb{H}}\int_1^\infty \exp\left(-\pi vt\frac{x^2+(y\pm 1)^2}{y}\right)\frac{dt}{t}\frac{dxdy}{y^2}\\
=&\int_1^\infty\int_0^\infty\int_{-\infty}^\infty \exp\left(-\pi vt\frac{x^2}{y}\right)dx\exp\left(-\pi vt\frac{(y\pm 1)^2}{y}\right)\frac{dy}{y^2}\frac{dt}{t}\\
=&\int_1^\infty\int_0^\infty \sqrt{\frac{\pi}{\pi vt/y}}\exp\left(-\pi vt\frac{(y\pm 1)^2}{y}\right)\frac{dy}{y^2}\frac{dt}{t}\\
=&\frac{1}{\sqrt{v}}\int_1^\infty\int_0^\infty \exp\left(-\pi vt\frac{(y\pm 1)^2}{y}\right)\frac{dy}{y^{3/2}}\frac{dt}{t^{3/2}}\\
=&\frac{1}{\sqrt{v}}\int_1^\infty \exp(\mp 2\pi vt)\int_0^\infty \exp\left(-\pi vt(y+y^{-1})\right)\frac{dy}{y^{3/2}}\frac{dt}{t^{3/2}}\\
\overset{(i)}{=}&\frac{1}{\sqrt{v}}\int_1^\infty \exp(\mp 2\pi vt)\cdot 2K_{-1/2}(2\pi vt)\frac{dt}{t^{3/2}}\\
\overset{(ii)}{=}&\frac{1}{\sqrt{v}}\int_1^\infty \exp(\mp 2\pi vt)\cdot (vt)^{-1/2}\exp(-2\pi vt)\frac{dt}{t^{3/2}}\\
=&\frac{1}{v}\int_1^\infty t^{-2}\exp(\mp 2\pi vt)\exp(-2\pi vt)dt.
\end{aligned}$$

In step (i) we used

$$\int_0^\infty \exp\left(-a(y+y^{-1})\right)y^{s-1}dy = 2K_s(2a)$$

by [EMOT54, p. 313, eq. (17)] which holds for $a>0$. In step (ii) we used

$$K_{-1/2}(z)=\sqrt{\frac{\pi}{2z}}e^{-z}$$

by [OLBC10, 10.39.2]. Now we have to distinguish the two cases. In the case $\pm=-$ we have

$$\frac{1}{v}\int_1^\infty t^{-2}\exp(+2\pi vt)\exp(-2\pi vt)dt=\frac{1}{v}\int_1^\infty t^{-2}dt=\frac{1}{v}.$$

In the case $\pm=+$ we get

$$\begin{aligned}
\frac{1}{v}\int_1^\infty t^{-2}\exp(-2\pi vt)\exp(-2\pi vt)dt &= \frac{1}{v}\int_1^\infty t^{-2}\exp(-4\pi vt)dt\\
&= \frac{1}{v}\frac{1}{4\pi v}\int_{4\pi v}^\infty (t/4\pi v)^{-2}\exp(-t)dt\\
&= 4\pi\int_{4\pi v}^\infty t^{-2}\exp(-t)dt\\
&= 4\pi\Gamma(-1,4\pi v).
\end{aligned}$$

□

Theorem 4.6.2. *For $m > 0$ we have*

$$\int_{\Gamma_{\mathfrak{a}}\backslash\mathbb{H}^2} \Xi(\mathfrak{a}, m, v, z)\omega^2 = \frac{\operatorname{vol}(T(\mathfrak{a}, m))}{2\pi v m}.$$

For $m < 0$ we have if $N(\varepsilon_0) = -1$

$$\int_{\Gamma_{\mathfrak{a}}\backslash\mathbb{H}^2} \Xi(\mathfrak{a}, m, v, z)\omega^2 = 2\Gamma(-1, 4\pi v|m|)\operatorname{vol}(T(\mathfrak{a}, |m|)).$$

Remark 4.6.3. For $m < 0$ we get in case $N(\varepsilon_0) = 1$ in the upcoming proof still a solution for the integral, namely

$$\int_{\Gamma_{\mathfrak{a}}\backslash\mathbb{H}^2} \Xi(\mathfrak{a}, m, v, z)\omega^2 = \sum_{\substack{A\in\Gamma_{\mathfrak{a}}\backslash L(\mathfrak{a})^\vee/\{\pm 1\}\\ \det(A)=m/(N(\mathfrak{a})D)}} 2\Gamma(-1, 4\pi v|m|)\operatorname{vol}(\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}),$$

but we cannot associate it to the volume of a Hirzebruch–Zagier divisor. However, it is possible to define Hirzebruch–Zagier divisors as divisors on $\mathbb{C}^2$ instead of $\mathbb{H}^2$. In that case the integral is independently of $N(\varepsilon_0)$ given by

$$\int_{\Gamma_{\mathfrak{a}}\backslash\mathbb{H}^2} \Xi(\mathfrak{a}, m, v, z)\omega^2 = 2\Gamma(-1, 4\pi v|m|)\operatorname{vol}(T(\mathfrak{a}, m))$$

and one has in case $N(\varepsilon_0) = -1$

$$\operatorname{vol}(T(\mathfrak{a}, m)) = \operatorname{vol}(T(\mathfrak{a}, -m)).$$

This is proven by using the bijection presented in Remark 2.8.5.

Proof of Theorem 4.6.2. We proceed analogously as in Theorem 3.8.2 and skip therefore the explanations which coincide with the explanations given there. We start by rewriting Kudla's Green function

$$\begin{aligned}
\Xi(\mathfrak{a}, m, v, z) &= \sum_{\substack{A\in L(\mathfrak{a})^\vee/\{\pm 1\}\\ \det(A)=m/(N(\mathfrak{a})D)}} E_1\left(4\pi v D N(\mathfrak{a}) h(A, z)\right)\\
&= \sum_{\substack{A\in\Gamma_{\mathfrak{a}}\backslash L(\mathfrak{a})^\vee/\{\pm 1\}\\ \det(A)=m/(N(\mathfrak{a})D)}} \sum_{M\in\Gamma_{\mathfrak{a}}/\Gamma_{\mathfrak{a},\pm A}} E_1\left(4\pi v D N(\mathfrak{a}) h(M.A, z)\right)\\
&= \sum_{\substack{A\in\Gamma_{\mathfrak{a}}\backslash L(\mathfrak{a})^\vee/\{\pm 1\}\\ \det(A)=m/(N(\mathfrak{a})D)}} \sum_{M\in\Gamma_{\mathfrak{a}}/\Gamma_{\mathfrak{a},\pm A}} E_1\left(4\pi v D N(\mathfrak{a}) h(A, M^{-1}z)\right)\\
&= \sum_{\substack{A\in\Gamma_{\mathfrak{a}}\backslash L(\mathfrak{a})^\vee/\{\pm 1\}\\ \det(A)=m/(N(\mathfrak{a})D)}} \sum_{M\in\Gamma_{\mathfrak{a},\pm A}\backslash\Gamma_{\mathfrak{a}}} E_1\left(4\pi v D N(\mathfrak{a}) h(A, Mz)\right).
\end{aligned}$$

Now we have by Tonelli's theorem (note that the integrand is positive)

$$\begin{aligned}
&\int_{\Gamma_{\mathfrak{a}}\backslash\mathbb{H}^2} \sum_{M\in\Gamma_{\mathfrak{a},\pm A}\backslash\Gamma_{\mathfrak{a}}} E_1\left(4\pi v DN(\mathfrak{a})h(A,Mz)\right)\omega^2 \\
&= \int_{\Gamma_{\mathfrak{a},\pm A}\backslash\mathbb{H}^2} E_1\left(4\pi v DN(\mathfrak{a})h(A,z)\right)\omega^2 \\
&= 2\int_{z_2\in\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}}\int_{z_1\in\mathbb{H}} E_1\left(4\pi v DN(\mathfrak{a})h(A,z)\right)\eta_1\eta_2 \\
&= 2\int_{z_2\in\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}}\int_{z_1\in\mathbb{H}} E_1\left(4\pi v DN(\mathfrak{a})\det(A)g(A,z)\right)\eta_1\eta_2 \\
&= 2\int_{z_2\in\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}}\int_{z_1\in\mathbb{H}} E_1\left(\pi v m d(z_1,ASz_2)\right)\eta_1\eta_2 \\
&= 2\int_{z_2\in\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}}\int_{z_1\in\mathbb{H}} E_1\left(\pi v m d(z_1,\pm i)\right)\eta_1\eta_2 \\
&= 2\int_{z_2\in\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}}\int_{z_1\in\mathbb{H}} E_1\left(\pi v|m|\frac{|z_1\mp i|^2}{y_1}\right)\eta_1\eta_2.
\end{aligned}$$

Here the sign $\pm$ is given by $\operatorname{sgn}(m)$. In both cases this is the moment to apply Lemma 4.6.1. We start with $m>0$ and have

$$\begin{aligned}
&2\int_{z_2\in\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}}\int_{z_1\in\mathbb{H}} E_1\left(\pi v m\frac{|z_1-i|^2}{y_1}\right)\eta_1\eta_2 \\
&= 2\int_{z_2\in\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}}\frac{1}{4\pi v m}\eta_2 = \frac{\operatorname{vol}(\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H})}{2\pi v m}.
\end{aligned}$$

Hence, in total we get

$$\int_{\Gamma_{\mathfrak{a}}\backslash\mathbb{H}^2}\Xi(\mathfrak{a},m,v,z)\omega^2 = \sum_{\substack{A\in\Gamma_{\mathfrak{a}}\backslash L(\mathfrak{a})^\vee/\{\pm1\}\\ \det(A)=m/(N(\mathfrak{a})D)}}\frac{\operatorname{vol}(\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H})}{2\pi v m} = \frac{\operatorname{vol}(T(\mathfrak{a},m))}{2\pi v m}.$$

In case $m<0$ we have

$$\begin{aligned}
&2\int_{z_2\in\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}}\int_{z_1\in\mathbb{H}} E_1\left(\pi v|m|\frac{|z_1+i|^2}{y_1}\right)\eta_1\eta_2 \\
&= 2\int_{z_2\in\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}}\Gamma(-1,4\pi v|m|)\eta_2 = 2\Gamma(-1,4\pi v|m|)\operatorname{vol}(\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}).
\end{aligned}$$

Hence, in total we get by Remark 2.8.5 (and this is the only time we need $N(\varepsilon_0)=-1$)

$$\begin{aligned}
\int_{\Gamma_{\mathfrak{a}}\backslash\mathbb{H}^2}\Xi(\mathfrak{a},m,v,z)\omega^2 &= \sum_{\substack{A\in\Gamma_{\mathfrak{a}}\backslash L(\mathfrak{a})^\vee/\{\pm1\}\\ \det(A)=m/(N(\mathfrak{a})D)}} 2\Gamma(-1,4\pi v|m|)\operatorname{vol}(\Gamma'_{\mathfrak{a},\pm A}\backslash\mathbb{H}) \\
&= 2\Gamma(-1,4\pi v|m|)\operatorname{vol}(T(\mathfrak{a},|m|)).
\end{aligned}$$

□

4.7 Integral for $m = 0$

In this section we compute the integral

$$\int_{\Gamma_{\mathfrak{a}}\backslash\mathbb{H}^2} \Xi_*(\mathfrak{a}, 0, v, z)\omega^2.$$

For that purpose we prove one auxiliary lemma first.

Lemma 4.7.1. *Let $v > 0$. Then we have*

$$\int_0^\infty E_1\left(\frac{v}{y}\right)\frac{dy}{y^2} = \frac{1}{v}.$$

Proof. We compute

$$\begin{aligned}
\int_0^\infty E_1\left(\frac{v}{y}\right)\frac{dy}{y^2} &= \int_0^\infty \int_1^\infty \exp\left(-\frac{vt}{y}\right)\frac{dt}{t}\frac{dy}{y^2} \\
&= \int_1^\infty \int_0^\infty \exp\left(-\frac{vt}{y}\right)\frac{dy}{y^2}\frac{dt}{t} \\
&= \int_1^\infty \int_\infty^0 \exp\left(-\frac{vt}{y^{-1}}\right)\left(-y^{-2}\right)\frac{dy}{y^{-2}}\frac{dt}{t} \\
&= \int_1^\infty \int_0^\infty \exp\left(-vty\right)dy\frac{dt}{t} \\
&= \int_1^\infty \frac{1}{vt}\frac{dt}{t} \\
&= \frac{1}{v}\int_1^\infty \frac{dt}{t^2} = \frac{1}{v}.
\end{aligned}$$

□

Theorem 4.7.2. *We have*

$$\int_{\Gamma_{\mathfrak{a}}\backslash\mathbb{H}^2} \Xi_*(\mathfrak{a}, 0, v, z)\omega^2 = \frac{h_K \log(\varepsilon_0)}{24\pi v\sqrt{D}}.$$

In particular, the value of the integral is independent of $\mathfrak{a} \in \mathcal{I}_K$.

Proof. For the proof we rename the ideal $\mathfrak{a}$ from the statement to $\mathfrak{b}$. This is because we need the letter $\mathfrak{a}$ for other ideals for the sake of consistency with referenced results.

It holds

$$\begin{aligned}
\Xi_*(\mathfrak{b},0,v,z) &= \frac{1}{2}\sideset{}{'}\sum_{\substack{A\in L(\mathfrak{b})^\vee\\ \det(A)=0}} E_1\left(4\pi v D N(\mathfrak{b})h(A,z)\right)\\
&= \frac{1}{2}\sum_{A\in \mathrm{Iso}(L(\mathfrak{b})^\vee)} E_1\left(4\pi v D N(\mathfrak{b})h(A,z)\right)\\
&= \frac{1}{2}\sum_{A\in \mathrm{Iso}((L(\mathfrak{b})^\vee)_0)}\sum_{n=1}^{\infty} E_1\left(4\pi v D N(\mathfrak{b})h(nA,z)\right)\\
&= \sum_{n=1}^{\infty}\sum_{A\in \mathrm{Iso}((L(\mathfrak{b})^\vee)_0)/\{\pm 1\}} E_1\left(4\pi v n^2 D N(\mathfrak{b})h(A,z)\right).
\end{aligned}$$

Because $v>0$ is arbitrary, it is enough to solve the integral for the case $n=1$. We have

$$\begin{aligned}
&\sum_{A\in \mathrm{Iso}((L(\mathfrak{b})^\vee)_0)/\{\pm 1\}} E_1\left(4\pi v D N(\mathfrak{b})h(A,z)\right)\\
&= \sum_{A\in \Gamma_{\mathfrak{b}}\backslash \mathrm{Iso}((L(\mathfrak{b})^\vee)_0)/\{\pm 1\}}\ \sum_{\gamma\in\Gamma_{\mathfrak{b}}/\Gamma_{\mathfrak{b},\pm A}} E_1\left(4\pi v D N(\mathfrak{b})h(\gamma.A,z)\right)\\
&\overset{(i)}{=} \sum_{\substack{[\mathfrak{a}]\in \mathrm{Cl}_K\\ M_{\mathfrak{a}}\in M(\mathfrak{a},\mathfrak{b})}}\ \sum_{\gamma\in\Gamma_{\mathfrak{a}^2\mathfrak{b}}/\Gamma_{\mathfrak{a}^2\mathfrak{b},\infty}} E_1\left(4\pi v D N(\mathfrak{b})h\left(\frac{\pm 1}{N(\mathfrak{a}\mathfrak{b})}(M_{\mathfrak{a}}\gamma).E_0,z\right)\right)\\
&= \sum_{\substack{[\mathfrak{a}]\in \mathrm{Cl}_K\\ M_{\mathfrak{a}}\in M(\mathfrak{a},\mathfrak{b})}}\ \sum_{\gamma\in\Gamma_{\mathfrak{a}^2\mathfrak{b}}/\Gamma_{\mathfrak{a}^2\mathfrak{b},\infty}} E_1\left(\frac{4\pi v D}{N(\mathfrak{a}^2\mathfrak{b})}h\left((M_{\mathfrak{a}}\gamma).E_0,z\right)\right)\\
&\overset{(ii)}{=} \sum_{\substack{[\mathfrak{a}]\in \mathrm{Cl}_K\\ M_{\mathfrak{a}}\in M(\mathfrak{a},\mathfrak{b})}}\ \sum_{\gamma\in\Gamma_{\mathfrak{a}^2\mathfrak{b}}/\Gamma_{\mathfrak{a}^2\mathfrak{b},\infty}} E_1\left(\frac{4\pi v D}{N(\mathfrak{a}^2\mathfrak{b})}h\left(\gamma.E_0,M_{\mathfrak{a}}^{-1}z\right)\right)\\
&\overset{(iii)}{=} \sum_{\substack{[\mathfrak{a}]\in \mathrm{Cl}_K\\ M_{\mathfrak{a}}\in M(\mathfrak{a},\mathfrak{b})}}\ \sum_{\gamma\in\Gamma_{\mathfrak{a}^2\mathfrak{b},\infty}\backslash\Gamma_{\mathfrak{a}^2\mathfrak{b}}} E_1\left(\frac{4\pi v D}{N(\mathfrak{a}^2\mathfrak{b})}h\left(E_0,\gamma M_{\mathfrak{a}}^{-1}z\right)\right).
\end{aligned}$$

In step (i) we use Lemma 2.4.10. The first sum runs over the finitely many ideal classes. For each class $[\mathfrak{a}]\in \mathrm{Cl}_K$ we pick *one* $M_{\mathfrak{a}}\in M(\mathfrak{a},\mathfrak{b})$ and do not run over all elements of $M(\mathfrak{a},\mathfrak{b})$. In step (ii) and (iii) we use Proposition 2.6.1.

Now let us fix a class $[\mathfrak{a}] \in \mathrm{Cl}_K$ and compute the integral of the inner sum.

$$\begin{aligned}
&\int_{\Gamma_{\mathfrak{b}}\backslash\mathbb{H}^2} \sum_{\gamma\in\Gamma_{\mathfrak{a}^2\mathfrak{b},\infty}\backslash\Gamma_{\mathfrak{a}^2\mathfrak{b}}} E_1\left(\frac{4\pi vD}{N(\mathfrak{a}^2\mathfrak{b})} h\left(E_0, \gamma M_{\mathfrak{a}}^{-1} z\right)\right)\omega^2 \\
\overset{(iv)}{=} &\int_{M_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{b}}M_{\mathfrak{a}}\backslash\mathbb{H}^2} \sum_{\gamma\in\Gamma_{\mathfrak{a}^2\mathfrak{b},\infty}\backslash\Gamma_{\mathfrak{a}^2\mathfrak{b}}} E_1\left(\frac{4\pi vD}{N(\mathfrak{a}^2\mathfrak{b})} h\left(E_0, \gamma z\right)\right)\omega^2 \\
\overset{(v)}{=} &\int_{\Gamma_{\mathfrak{a}^2\mathfrak{b}}\backslash\mathbb{H}^2} \sum_{\gamma\in\Gamma_{\mathfrak{a}^2\mathfrak{b},\infty}\backslash\Gamma_{\mathfrak{a}^2\mathfrak{b}}} E_1\left(\frac{4\pi vD}{N(\mathfrak{a}^2\mathfrak{b})} h\left(E_0, \gamma z\right)\right)\omega^2 \\
\overset{(vi)}{=} &\int_{\Gamma_{\mathfrak{a}^2\mathfrak{b},\infty}\backslash\mathbb{H}^2} E_1\left(\frac{4\pi vD}{N(\mathfrak{a}^2\mathfrak{b})} h\left(E_0, z\right)\right)\omega^2 \\
\overset{(vii)}{=} &\int_{\Gamma_{\mathfrak{a}^2\mathfrak{b},\infty}\backslash\mathbb{H}^2} E_1\left(\frac{\pi vD}{N(\mathfrak{a}^2\mathfrak{b})\Im(z)}\right)\omega^2 \\
= &\frac{\mathrm{vol}((\mathfrak{a}^2\mathfrak{b})^{-1})}{8\pi^2}\int_1^{\varepsilon_0^2}\int_0^{\infty} E_1\left(\frac{\pi vD}{N(\mathfrak{a}^2\mathfrak{b})\Im(z)}\right)\frac{dy_1 dy_2}{y_1^2 y_2^2} \\
\overset{(viii)}{=} &\frac{\mathrm{vol}((\mathfrak{a}^2\mathfrak{b})^{-1})}{8\pi^2}\int_1^{\varepsilon_0^2}\frac{N(\mathfrak{a}^2\mathfrak{b})}{\pi vDy_2}dy_2 = \frac{\log(\varepsilon_0)}{4\pi^3 v\sqrt{D}}.
\end{aligned}$$

In step (iv) we translated the integration region with $M_{\mathfrak{a}}$ in order to get rid of it in the argument of h. In step (v) we used Corollary 2.3.6. In step (vi) we unfolded. In step (vii) we used Lemma 2.6.7. In step (viii) we used Lemma 4.7.1.

Now the theorem follows by noting that so far we only computed the integral for one ideal class $[\mathfrak{a}] \in \mathrm{Cl}_K$ instead of h_K many and by accounting for the $n > 1$ with

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Hence, the final result is

$$h_K \cdot \frac{\pi^2}{6}\cdot\frac{\log(\varepsilon_0)}{4\pi^3 v\sqrt{D}} = \frac{h_K\log(\varepsilon_0)}{24\pi v\sqrt{D}}.$$

□

4.8 Generating series

Analogously to Section 3.10 we want to consider the generating series

$$\sum_{m\in\mathbb{Z}} \Xi(\mathfrak{a}, m, v, z)q^m. \tag{4.24}$$

Different from the situation there the Green function $\Xi(\mathfrak{a}, m, v, z)$ and the variable q are not independent. Namely, for $\tau = u + iv \in \mathbb{H}$ with $u, v \in \mathbb{R}$ we define

$$q := e(\tau) = \exp(2\pi i\tau).$$

Hence, the variable v in $\Xi(\mathfrak{a}, m, v, z)$ is in relation with q. We have

$$|q| = |\exp(2\pi i\tau)| = \exp(-2\pi v) < 1.$$

Before stating and proving our convergence theorem we need to provide a little lemma.

Lemma 4.8.1. *Let $n \in \mathbb{N}$, $c > 0$ and $q \in \mathbb{C}$ with $\exp(-c) < |q|$. Then*

$$\sum_{m=1}^{\infty} \Gamma(-1, cm) m^n q^{-m}$$

converges absolutely.

Proof. For $x > 0$ we have

$$\Gamma(-1, x) = \int_x^{\infty} t^{-2} \exp(-t) dt \leq x^{-2} \int_x^{\infty} \exp(-t) dt = x^{-2} \exp(-x).$$

It follows

$$\begin{aligned} \sum_{m=1}^{\infty} \Gamma(-1, cm) m^n |q|^{-m} &\leq \sum_{m=1}^{\infty} (cm)^{-2} \exp(-cm) m^n |q|^{-m} \\ &= c^{-2} \sum_{m=1}^{\infty} m^{n-2} \left(\frac{\exp(-c)}{|q|} \right)^m < \infty. \end{aligned}$$

□

Theorem 4.8.2. *Let $\tau \in \mathbb{H}$ be fixed. The series*

$$\sum_{m \in \mathbb{Z}} \Xi(\mathfrak{a}, m, v, z) q^m$$

converges absolutely for almost all $z \in \mathbb{H}^2$. Furthermore, the series

$$\sum_{m \in \mathbb{Z}} |\Xi(\mathfrak{a}, m, v, z) q^m|$$

is integrable over $X(\mathfrak{a})$.

Proof. The integrability of the series

$$\sum_{m \in \mathbb{Z}} |\Xi(\mathfrak{a}, m, v, z) q^m|$$

implies the statement about the convergence. By Tonelli's theorem we obtain

$$\int_{X(\mathfrak{a})} \sum_{m \in \mathbb{Z}} |\Xi(\mathfrak{a}, m, v, z) q^m| \, \omega^2 = \sum_{m \in \mathbb{Z}} \left(\int_{X(\mathfrak{a})} |\Xi(\mathfrak{a}, m, v, z)| \, \omega^2 \right) |q|^m.$$

Therefore, it is enough to prove the finiteness of the right hand side. We split the sum up into positive and negative m. By Definition 4.1.1 we have $|\Xi(\mathfrak{a}, m, v, z)| = \Xi(\mathfrak{a}, m, v, z)$ for $m \neq 0$, hence by Theorem 4.6.2 we have to consider

$$\sum_{m=1}^{\infty} \frac{\mathrm{vol}(T(\mathfrak{a}, m))}{2\pi v m} |q|^m \quad \text{and} \quad \sum_{m=1}^{\infty} 2\Gamma(-1, 4\pi v m)\, \mathrm{vol}(T(\mathfrak{a}, m))|q|^{-m}. \tag{4.25}$$

Technically, in the second series we should take

$$\sum_{\substack{A \in \Gamma_{\mathfrak{a}} \backslash L(\mathfrak{a})^{\vee} / \{\pm 1\} \\ \det(A) = -m/(N(\mathfrak{a})D)}} \mathrm{vol}(\Gamma'_{\mathfrak{a}, \pm A} \backslash \mathbb{H})$$

instead of $\mathrm{vol}(T(\mathfrak{a}, m))$ in case $N(\varepsilon_0) = 1$, but here we only use that the volumes grow polynomially in m which is true in both cases. This polynomial growth directly implies the finiteness of the first series of (4.25). For the second series the convergence follows from Lemma 4.8.1 with $c = 4\pi v$ and

$$\exp(-4\pi v) < \exp(-2\pi v) = |q|.$$

□

Theorem 4.8.3. *The series*

$$\sum_{m \in \mathbb{Z}} \widetilde{\Xi}_{\rho}(\mathfrak{a}, m, v, z) q^m$$

converges absolutely for almost all $z \in \mathbb{H}^2$ and all $\tau \in \mathbb{H}$. Furthermore, the series

$$\sum_{m \in \mathbb{Z}} \left| \widetilde{\Xi}_{\rho}(\mathfrak{a}, m, v, z) q^m \right|$$

is integrable over $X(\mathfrak{a})$.

Proof. Formally, we have by definition (4.22)

$$\sum_{m \in \mathbb{Z}} \widetilde{\Xi}_{\rho}(\mathfrak{b}, m, v, z) q^m = \sum_{m \in \mathbb{Z}} \Xi(\mathfrak{b}, m, v, z) q^m - \sum_{m \in \mathbb{Z}} \sum_{j=1}^{h_k} \rho_j(z) \check{\Xi}(\mathfrak{a}_j^2 \mathfrak{b}, m, v, M_j^{-1} z) q^m.$$

Therefore, by Theorem 4.8.2 it is only left to show that

$$\rho_j(z) \sum_{m \in \mathbb{Z}} \check{\Xi}(\mathfrak{a}_j^2 \mathfrak{b}, m, v, M_j^{-1} z) q^m$$

converges absolutely and that

$$\rho_j(z) \sum_{m \in \mathbb{Z}} \left| \check{\Xi}(\mathfrak{a}_j^2 \mathfrak{b}, m, v, M_j^{-1} z) q^m \right|$$

is integrable over $X(\mathfrak{b})$. In Section 5.1 we investigate the generating series

$$\sum_{m \in \mathbb{Z}} \check{\Xi}(\mathfrak{a}, m, v, z) q^m.$$

Theorem 5.1.3 provides the convergence and integrability results we need. □

Chapter 5

Modularity

In this last chapter of the thesis we discuss many modularity results for generating series. In particular, we prove our main theorem, the modularity of

$$\sum_{m\in\mathbb{Z}} (Z(\mathfrak{a},m), \widetilde{\Xi}_\rho(\mathfrak{a},m,v,z))q^m$$

for prime discriminant D and $\mathfrak{a} = \mathcal{O}_K$. The transformation law we have in mind when we talk about scalar valued modularity is the one of the following definition. However, we consider non-holomorphic modular forms as well.

Definition 5.0.1. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a *modular form of weight k, level D and nebentypus χ_D* if f is holomorphic at the cusps and satisfies

$$f(\gamma\tau) = \chi_D(a)(c\tau + d)^k f(z) \tag{5.1}$$

for all $\tau \in \mathbb{H}$ and all

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \ c \equiv 0 \pmod{D} \right\}.$$

The space of all modular forms of weight k, level D and nebentypus χ_D is denoted by $M_k(D, \chi_D)$.

5.1 Modularity of the correction term

In this section we prove that

$$\sum_{m\in\mathbb{Z}} \check{\Xi}(\mathfrak{a},m,v,z)q^m \tag{5.2}$$

converges and is modular in $\tau \in \mathbb{H}$. Recall that $q = e(\tau)$ and $v = \Im(\tau)$. In accordance with [HZ76, p. 98], we define for $\lambda, \lambda' \in \mathbb{R}$

$$\begin{aligned} U_\tau(\lambda,\lambda') &:= 2v^{-1/2}\beta(\pi v(\lambda-\lambda')^2)e(\lambda\lambda'\tau), \\ V_\tau(\lambda,\lambda') &:= \tfrac{1}{2}\delta_{\lambda\lambda'>0}\beta(\lambda,\lambda')e(\lambda\lambda'\tau), \\ W_\tau(\lambda,\lambda') &:= U_\tau(\lambda,\lambda') - V_\tau(\lambda,\lambda'). \end{aligned}$$

The main ingredient of our modularity proof of the series (5.2) is the transformation law (5) in [HZ76, p. 98, Proposition 1]. Let us start with some preliminary work.

Lemma 5.1.1. *Let $a > 0$. Then we have*

$$W_{a^2\tau}(\lambda/a, \lambda'/a) = W_\tau(\lambda, \lambda')/a.$$

Proof. We show the relations separately for U and V:

$$\begin{aligned} U_{a^2\tau}(\lambda/a, \lambda'/a) &= 2(a^2 v)^{-1/2}\beta(\pi a^2 v(\lambda/a - \lambda'/a)^2)e(\lambda/a \cdot \lambda'/a \cdot a^2\tau) \\ &= 2\frac{v^{-1/2}}{a}\beta(\pi v(\lambda - \lambda')^2)e(\lambda\lambda' \cdot \tau) = U_\tau(\lambda, \lambda')/a \end{aligned}$$

and

$$\beta(\lambda/a, \lambda'/a)e(\lambda/a \cdot \lambda'/a \cdot a^2\tau) = \frac{1}{a}\beta(\lambda, \lambda')e(\lambda\lambda'\tau).$$

□

Lemma 5.1.2. *We have*

$$\sum_{m\in\mathbb{Z}} \check{\Xi}(\mathfrak{a}, m, v, z)q^m = \frac{4\pi t}{\sqrt{D}} \sum_{\lambda\in\mathfrak{a}} W_{\tau/N(\mathfrak{a})}(\lambda r, \lambda'/r).$$

Proof. We do not talk about convergence in this proof, this is covered in Theorem 5.1.3. Note that

$$\mathfrak{a} = \bigcup_{m\in\mathbb{Z}} \{\lambda \in \mathfrak{a} : \ N(\lambda) = mN(\mathfrak{a})\}.$$

Therefore, it is enough to show for all $m \in \mathbb{Z}$

$$\check{\Xi}(\mathfrak{a}, m, v, z)q^m = \frac{4\pi t}{\sqrt{D}} \sum_{\substack{\lambda\in\mathfrak{a} \\ N(\lambda)=mN(\mathfrak{a})}} W_{\tau/N(\mathfrak{a})}(\lambda r, \lambda'/r).$$

We split the right hand side up into the U and V part and consider them separately. Note that for $\lambda \in \Lambda(\mathfrak{a}, m)$ we have

$$q^m = e(m\tau) = e(-\lambda\lambda' D\tau/N(\mathfrak{a}))$$

by equation (2.42). Using

$$\sqrt{D}\Lambda(\mathfrak{a}, m) = \{\lambda \in \mathfrak{a} : \ N(\lambda) = mN(\mathfrak{a})\},$$

we obtain

$$\begin{aligned}
&\frac{4\pi t}{\sqrt{D}} \sum_{\substack{\lambda\in\mathfrak{a}\\ N(\lambda)=mN(\mathfrak{a})}} U_{\tau/N(\mathfrak{a})}(\lambda r, \lambda'/r)\\
=&\frac{4\pi t}{\sqrt{D}} \sum_{\lambda\in\Lambda(\mathfrak{a},m)} U_{\tau/N(\mathfrak{a})}(\lambda\sqrt{D}r, -\lambda'\sqrt{D}/r)\\
=&\frac{4\pi t}{\sqrt{D}} \sum_{\lambda\in\Lambda(\mathfrak{a},m)} 2(v/N(\mathfrak{a}))^{-1/2}\beta(\pi(v/N(\mathfrak{a}))(\lambda\sqrt{D}r+\lambda'\sqrt{D}/r)^2)e(-\lambda\lambda' D\tau/N(\mathfrak{a}))\\
=&8\pi t\sqrt{\frac{N(\mathfrak{a})}{vD}} \sum_{\lambda\in\Lambda(\mathfrak{a},m)} \beta\left(\frac{\pi v D(\lambda r+\lambda'/r)^2}{N(\mathfrak{a})}\right) q^m.
\end{aligned}$$

On the other hand, we have for $m > 0$

$$\begin{aligned}
&\frac{4\pi t}{\sqrt{D}} \sum_{\substack{\lambda\in\mathfrak{a}\\ N(\lambda)=mN(\mathfrak{a})}} V_{\tau/N(\mathfrak{a})}(\lambda r, \lambda'/r)\\
=&\frac{4\pi t}{\sqrt{D}} \sum_{\lambda\in\Lambda(\mathfrak{a},m)} V_{\tau/N(\mathfrak{a})}(\lambda\sqrt{D}r, -\lambda'\sqrt{D}/r)\\
=&\frac{2\pi t}{\sqrt{D}} \sum_{\lambda\in\Lambda(\mathfrak{a},m)} \beta(\lambda\sqrt{D}r, -\lambda'\sqrt{D}/r)e(-\lambda\lambda' D\tau/N(\mathfrak{a}))\\
=&2\pi t \sum_{\lambda\in\Lambda(\mathfrak{a},m)} \beta(\lambda r, \lambda'/r)q^m.
\end{aligned}$$

Taking the difference, we get the representation of $\check{\Xi}(\mathfrak{a}, m, v, z)$ given in equation (4.18) which finishes the proof. □

Theorem 5.1.3. *Let $\tau \in \mathbb{H}$ be fixed. The series*

$$\sum_{m\in\mathbb{Z}} \check{\Xi}(\mathfrak{a}, m, v, z)q^m$$

is absolutely convergent for all $z \in \mathbb{H}^2$ and

$$\sum_{m\in\mathbb{Z}} \left|\check{\Xi}(\mathfrak{a}, m, v, z)q^m\right|$$

is integrable over the Siegel domain $\mathcal{S}_C$ for arbitrary $C > 0$.

Proof. Since

$$\mathbb{H}^2 = \bigcup_{C>0} \mathcal{S}_C,$$

we are allowed to fix $C > 0$ for the whole proof and only consider $z \in \mathcal{S}_C$. We make use of the representation from Lemma 5.1.2 which implies

$$\sum_{m\in\mathbb{Z}} \left|\check{\Xi}(\mathfrak{a}, m, v, z)q^m\right| \leq \frac{4\pi t}{\sqrt{D}} \left(\sum_{\lambda\in\mathfrak{a}} \left|U_{\tau/N(\mathfrak{a})}(\lambda r, \lambda'/r)\right| + \sum_{\lambda\in\mathfrak{a}} \left|V_{\tau/N(\mathfrak{a})}(\lambda r, \lambda'/r)\right| \right).$$

Now let $R > \max(r^2, r^{-2})$ for all possible values of r in the Siegel domain $\mathcal{S}_C$. Since τ is fixed, we can neglect the scaling by $1/N(\mathfrak{a})$ and consider (cf. definition (4.13))

$$\begin{aligned}
\left|U_\tau(\lambda r, \lambda'/r)\right| &= 2v^{-1/2}\beta(\pi v(\lambda r - \lambda'/r)^2)\left|e(\lambda\lambda'\tau)\right| \\
&\leq 2v^{-1/2}\frac{\exp(-\pi v(\lambda r - \lambda'/r)^2)}{8\pi}\exp(-2\pi N(\lambda)v) \\
&= \frac{\exp(-\pi v(r^2\lambda^2 + r^{-2}(\lambda')^2))}{4v^{1/2}\pi} \\
&\leq \frac{\exp(-\pi v R\,\mathrm{tr}(\lambda^2))}{4v^{1/2}\pi}.
\end{aligned}$$

Using the fact that the trace form is a positive definite quadratic form on K, we obtain that

$$\sum_{\lambda\in\mathfrak{a}} \left|U_{\tau/N(\mathfrak{a})}(\lambda r, \lambda'/r)\right|$$

is bounded on the Siegel domain $\mathcal{S}_C$. We continue with

$$\left|V_\tau(\lambda r, \lambda'/r)\right| = \frac{1}{2}\beta(\lambda r, \lambda'/r)\left|e(\lambda\lambda'\tau)\right| = \frac{1}{2}\beta(\lambda r, \lambda'/r)\exp(-2\pi N(\lambda)v)$$

in case $N(\lambda) > 0$. Now, for each $\lambda_0 \in \mathfrak{a}$ with $N(\lambda_0) > 0$ there are infinitely $\lambda \in \mathfrak{a}$ with $N(\lambda) = N(\lambda_0)$ because $\mathcal{O}_K^+ = \langle\varepsilon_1\rangle$ acts freely on the set of all such λ. We have, using the geometric series,

$$\sum_{k\in\mathbb{Z}} \beta((\lambda\varepsilon_1^k)r, (\lambda\varepsilon_1^k)'/r) \leq \frac{2\sqrt{N(\lambda)}}{1-\varepsilon_1^2}.$$

By Lemma 2.8.6 we obtain that the series

$$\sum_{\lambda\in\mathfrak{a}} \left|V_{\tau/N(\mathfrak{a})}(\lambda r, \lambda'/r)\right|$$

can be be estimated up to a factor by the converging series

$$\sum_{m=1}^{\infty} m\exp(-cm)$$

for an appropriately chosen $c > 0$. Now, with

$$\sum_{\lambda\in\mathfrak{a}} \left|U_{\tau/N(\mathfrak{a})}(\lambda r, \lambda'/r)\right| \quad \text{and} \quad \sum_{\lambda\in\mathfrak{a}} \left|V_{\tau/N(\mathfrak{a})}(\lambda r, \lambda'/r)\right|$$

being bounded on $\mathcal{S}_C$, Lemma 3.8.7 provides us with $\alpha = -1/2$ the integrability of

$$\frac{4\pi t}{\sqrt{D}}\left(\sum_{\lambda\in\mathfrak{a}}\left|U_{\tau/N(\mathfrak{a})}(\lambda r,\lambda'/r)\right| + \sum_{\lambda\in\mathfrak{a}}\left|V_{\tau/N(\mathfrak{a})}(\lambda r,\lambda'/r)\right|\right).$$

□

Since $\mathfrak{a}$ defines an even lattice with respect to the quadratic form $q_{\mathfrak{a}}$ (cf. Subsection 2.2.4), we can talk about vector valued modular forms with respect to the lattice $\mathfrak{a}$. These are functions

$$f : \mathbb{H} \to \mathbb{C}[\mathfrak{a}^{\vee_{q_{\mathfrak{a}}}}/\mathfrak{a}] = \bigoplus_{\lambda\in\mathfrak{a}^{\vee_{q_{\mathfrak{a}}}}/\mathfrak{a}} \mathbb{C}\mathfrak{e}_\lambda$$

satisfying the respective transformation law dictated by the Weil representation. See [Sch15, Section 2] for further details.

Theorem 5.1.4. *We define for fixed $r > 0$ and all $\nu \in \mathfrak{a}^{\vee_{q_{\mathfrak{a}}}}/\mathfrak{a}$*

$$W^{\mathfrak{a}}_\nu(\tau) := \sum_{\lambda\in\mathfrak{a}} W_{\tau/N(\mathfrak{a})}((\lambda+\nu)r,(\lambda'+\nu')/r).$$

Then

$$W^{\mathfrak{a}}(\tau) := \sum_{\nu\in\mathfrak{a}^{\vee_{q_{\mathfrak{a}}}}/\mathfrak{a}} W^{\mathfrak{a}}_\nu(\tau)\mathfrak{e}_\nu = \sum_{\lambda\in\mathfrak{a}^{\vee_{q_{\mathfrak{a}}}}} W_{\tau/N(\mathfrak{a})}(\lambda r,\lambda'/r)\mathfrak{e}_\lambda$$

is a non-holomorphic vector valued modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight 2 with respect to the Weil representation.

Proof. Recall from equation (2.8) and the subsequent discussion that $q_{\mathfrak{a}}(x) = N(x)/N(\mathfrak{a})$ and that we have $\mathfrak{a}^{\vee_{q_{\mathfrak{a}}}} = \mathfrak{a}\mathfrak{d}^{-1}$. Since $\mathrm{SL}_2(\mathbb{Z})$ is generated by T and S, it suffices to prove the following two transformation laws:

$$\begin{aligned} W^{\mathfrak{a}}_\gamma(\tau+1) &= e(q_{\mathfrak{a}}(\gamma))W^{\mathfrak{a}}_\gamma(\tau),\\ W^{\mathfrak{a}}_\gamma(-1/\tau) &= D^{-1/2}\tau^2 \sum_{\delta\in\mathfrak{a}^{\vee_{q_{\mathfrak{a}}}}/\mathfrak{a}} e(\mathrm{tr}(\delta\gamma')/N(\mathfrak{a}))W^{\mathfrak{a}}_\delta(\tau). \end{aligned}$$

We have

$$\begin{aligned} W^{\mathfrak{a}}_\nu(\tau+1) &= \sum_{\lambda\in\mathfrak{a}} W_{(\tau+1)/N(\mathfrak{a})}((\lambda+\nu)r,(\lambda'+\nu')/r)\\ &= \sum_{\lambda\in\mathfrak{a}} W_{\tau/N(\mathfrak{a})}((\lambda+\nu)r,(\lambda'+\nu')/r)e(N(\lambda+\nu)/N(\mathfrak{a}))\\ &= \sum_{\lambda\in\mathfrak{a}} W_{\tau/N(\mathfrak{a})}((\lambda+\nu)r,(\lambda'+\nu')/r)e(q_{\mathfrak{a}}(\lambda+\nu))\\ &= e(q_{\mathfrak{a}}(\nu))W^{\mathfrak{a}}_\nu(\tau) \end{aligned}$$

which proves the first transformation law. Now we define for $x \in \mathbb{R}^2$

$$X_\tau(x) := W_\tau(x_1 r, x_2/r).$$

Therefore, we have

$$W_\nu^{\mathfrak{a}}(\tau) = \sum_{\lambda \in \mathfrak{a}} X_{\tau/N(\mathfrak{a})}(\lambda + \nu).$$

Using Poisson summation with respect to the usual trace form, we obtain

$$W_\nu^{\mathfrak{a}}(\tau) = \frac{1}{\mathrm{vol}(\mathfrak{a})} \sum_{\lambda \in \mathfrak{a}^{\vee \mathrm{tr}}} \hat{X}_{\tau/N(\mathfrak{a})}(\lambda) e(\mathrm{tr}(\lambda\nu))$$

with

$$\begin{aligned}
\hat{X}_{\tau/N(\mathfrak{a})}(\lambda) &= \int_{\mathbb{R}^2} X_{\tau/N(\mathfrak{a})}(x) e(-\mathrm{tr}(\lambda x)) dx \\
&= \int_{\mathbb{R}^2} W_{\tau/N(\mathfrak{a})}(x_1 r, x_2/r) e(-(\lambda/r)(x_1 r) - (\lambda' r)(x_2/r)) dx \\
&\overset{(i)}{=} \int_{\mathbb{R}^2} W_{\tau/N(\mathfrak{a})}(x_1, x_2) e(-(\lambda/r)x_1 - (\lambda' r)x_2) dx \\
&\overset{(ii)}{=} \hat{W}_{\tau/N(\mathfrak{a})}(\lambda/r, \lambda' r) \\
&\overset{(iii)}{=} (\tau/N(\mathfrak{a}))^{-2} W_{-N(\mathfrak{a})/\tau}(\lambda/r, \lambda' r) \\
&\overset{(iv)}{=} N(\mathfrak{a})\tau^{-2} W_{-1/(\tau N(\mathfrak{a}))}(N(\mathfrak{a})\lambda/r, N(\mathfrak{a})\lambda' r).
\end{aligned}$$

Here, in step (i) we use integration by substitution with $(x_1, x_2) \mapsto (x_1/r, x_2 r)$. In step (ii) we identify the Fourier transform. In step (iii) we use [HZ76, Proposition 1, p. 98] and in step (iv) we apply Lemma 5.1.1. Now, we obtain for $W_\nu^{\mathfrak{a}}(\tau)$

$$\begin{aligned}
&\frac{1}{\sqrt{D} N(\mathfrak{a})} \sum_{\lambda \in \mathfrak{a}'\mathfrak{d}^{-1}/N(\mathfrak{a})} N(\mathfrak{a})\tau^{-2} W_{-1/(\tau N(\mathfrak{a}))}(N(\mathfrak{a})\lambda/r, N(\mathfrak{a})\lambda' r) e(\mathrm{tr}(\lambda\nu)) \\
=& D^{-1/2}\tau^{-2} \sum_{\lambda \in \mathfrak{a}'\mathfrak{d}^{-1}} W_{-1/(\tau N(\mathfrak{a}))}(\lambda/r, \lambda' r) e(\mathrm{tr}(\lambda\nu/N(\mathfrak{a}))) \\
\overset{(v)}{=}& D^{-1/2}\tau^{-2} \sum_{\lambda \in \mathfrak{a}\mathfrak{d}^{-1}} W_{-1/(\tau N(\mathfrak{a}))}(\lambda r, \lambda'/r) e(\mathrm{tr}(\lambda'\nu/N(\mathfrak{a}))) \\
=& D^{-1/2}\tau^{-2} \sum_{\delta \in \mathfrak{a}^{\vee q_{\mathfrak{a}}}/\mathfrak{a}} W_\delta^{\mathfrak{a}}(-1/\tau) e(\mathrm{tr}(\delta\nu')/N(\mathfrak{a})).
\end{aligned}$$

In step (v) we used the symmetry of W_τ in its arguments. The substitution $\tau \mapsto -1/\tau$ proves the second transformation law. □

In [Sch09, p. 13], Scheithauer attaches to each discriminant form $\tilde{D}$ a quadratic character $\chi_{\tilde{D}}$ by

$$\chi_{\tilde{D}} : \mathbb{Z} \to \{1, -1, 0\}, \quad \chi_{\tilde{D}}(a) := \left(\frac{a}{|\tilde{D}|}\right) e\left((a-1)\,\mathrm{oddity}(\tilde{D})/8\right).$$

Even though for fixed real quadratic field K with discriminant D the discriminant forms $\mathfrak{a}^{\vee q_{\mathfrak{a}}}/\mathfrak{a}$ have the same cardinality D, they are not isomorphic as discriminant forms when

running over the fractional ideals $\mathfrak{a} \in \mathcal{I}_K$. However, it turns out that they all induce the same character $\chi_{\mathfrak{a}^\vee q_\mathfrak{a}/\mathfrak{a}}$ and it holds $\chi_D = \chi_{\mathfrak{a}^\vee q_\mathfrak{a}/\mathfrak{a}}$ for all $\mathfrak{a} \in \mathcal{I}_K$. This can be shown using the generalized law of quadratic reciprocity to the Kronecker symbol and by computing appropriate oddities and p-excesses. Here, χ_D is defined by equation (2.4) as throughout the whole thesis.

Theorem 5.1.5. *For every fixed $z \in \mathbb{H}^2$ the function*

$$\mathbb{H} \to \mathbb{C}, \quad \tau \mapsto \sum_{m \in \mathbb{Z}} \check{\Xi}(\mathfrak{a}, m, v, z) q^m$$

is a non-holomorphic modular form of weight 2, level D and nebentypus χ_D.

Proof. By Lemma 5.1.2 we can prove instead that

$$\sum_{\lambda \in \mathfrak{a}} W_{\tau/N(\mathfrak{a})}(\lambda r, \lambda'/r)$$

which is the zero component of $W^\mathfrak{a}(\tau)$ is a non-holomorphic modular form of weight 2, level D and nebentypus χ_D (note that the prefactor $4\pi t/\sqrt{D}$ is independent of τ). However, this follows from [Sch09, Proposition 4.5] applied to Theorem 5.1.4 together with the above insight $\chi_D = \chi_{\mathfrak{a}^\vee q_\mathfrak{a}/\mathfrak{a}}$. □

5.2 Modularity of two generating series of differential forms

Definition 5.2.1. We define

$$\omega(\mathfrak{a}, m, z) := dd^c G(\mathfrak{a}, m, z) \quad \text{and} \quad \varphi(\mathfrak{a}, m, v, z) := dd^c \Xi(\mathfrak{a}, m, v, z).$$

Since $G(\mathfrak{a}, m, z)$ and $\Xi(\mathfrak{a}, m, v, z)$ are smooth apart from the logarithmic singularities along the divisors $-T(\mathfrak{a}, m)$ on $X(\mathfrak{a})$, we obtain with Lemma 2.9.12 that $\omega(\mathfrak{a}, m, \cdot)$ and $\varphi(\mathfrak{a}, m, v, \cdot)$ are $(1,1)$-forms on all of $X(\mathfrak{a})$, i.e., they are elements of $A^{1,1}(X(\mathfrak{a}))$.

By the work of Zagier (cf. [Zag75]) we obtain the following theorem.

Theorem 5.2.2. *The generating series*

$$\Omega(\mathfrak{a}, \tau, z) := \sum_{m=0}^{\infty} \omega(\mathfrak{a}, m, z) q^m$$

converges to a smooth $(1,1)$-form, i.e., $\Omega(\mathfrak{a}, \tau, \cdot) \in A^{1,1}(X(\mathfrak{a}))$. The form $\Omega(\mathfrak{a}, \cdot, z)$ is a holomorphic modular form of weight 2, level D and nebentypus χ_D with values in $A^{1,1}(X(\mathfrak{a}))$.

A similar result holds for the generating series of the $\varphi(\mathfrak{a}, m, v, z)$ by the Kudla–Millson theory (cf. [KM90]). Namely, the respective series can be identified as the Kudla–Millson theta series. To explain the modified zero term $\Xi(\mathfrak{a}, 0, v, z)$ (cf. Definition 4.1.1), we need the next lemma.

Lemma 5.2.3. *Let $v > 0$. Then we have*

$$\lim_{A\to 0} dd^c E_1\left(vh(A,z)\right) = -\omega.$$

Here, ω is the Kähler form (cf. equation (2.15)*).*

Proof. We use the representation of $E_1(x)$ from equation (4.2) and get

$$E_1(x) = -\gamma + \mathrm{Ein}(x) - \log(x).$$

Since $h(0,z) = 0$ is independent of z, it follows with $A = \left(\begin{smallmatrix} a & \lambda' \\ \lambda & b \end{smallmatrix}\right)$

$$\begin{aligned}
\lim_{A\to 0} dd^c E_1\left(vh(A,z)\right) &= -\lim_{A\to 0} dd^c \log(vh(A,z)) \\
&= -\lim_{A\to 0} dd^c \log\left(\frac{|bz_1z_2 - \lambda z_1 - \lambda' z_2 + a|^2}{4y_1y_2}\right) \\
&= dd^c \log(y_1y_2) = -\omega.
\end{aligned}$$

From the second to the third line we used Lemma 2.9.12. □

Theorem 5.2.4. *The generating series*

$$\varphi(\mathfrak{a},\tau,z) := \sum_{m\in\mathbb{Z}} \varphi(\mathfrak{a},m,v,z)q^m$$

converges to a smooth $(1,1)$*-form, i.e.,* $\varphi(\mathfrak{a},\tau,\cdot) \in A^{1,1}(X(\mathfrak{a}))$*. The form* $\varphi(\mathfrak{a},\cdot,z)$ *is a non-holomorphic modular form of weight* 2*, level* D *and nebentypus* χ_D *with values in* $A^{1,1}(X(\mathfrak{a}))$*.*

Proof. By Definition 4.1.1 the series $\varphi(\mathfrak{a},\tau,z)$ is given by

$$\sum_{m\in\mathbb{Z}} dd^c\Xi(\mathfrak{a},m,v,z)q^m = dd^c\Xi_0(\mathfrak{a},v,z) + \sum_{m\in\mathbb{Z}} dd^c\Xi_*(\mathfrak{a},m,v,z)q^m.$$

By Proposition 2.9.24 we have

$$dd^c\Xi_0(\mathfrak{b},v,z) = -\frac{\omega}{2}.$$

Using Definition 4.1.1, we obtain

$$\sum_{m\in\mathbb{Z}} dd^c\Xi_*(\mathfrak{a},m,v,z)q^m = \frac{1}{2}\sum_{A\in L(\mathfrak{a})^\vee}{}' dd^c E_1\left(4\pi v DN(\mathfrak{a})h(A,z)\right) q^{\det(A)DN(\mathfrak{a})}.$$

Now, Lemma 5.2.3 suggests that we can interpret $dd^c\Xi_0(\mathfrak{b},v,z)$ as the omitted term for $A = 0$. Therefore, with this interpretation of the right hand side we have

$$\varphi(\mathfrak{a},\tau,z) = \frac{1}{2}\sum_{A\in L(\mathfrak{a})^\vee} dd^c E_1\left(4\pi v DN(\mathfrak{a})h(A,z)\right) q^{\det(A)DN(\mathfrak{a})}.$$

With this interpretation the right hand side is the Kudla–Millson theta series which is modular by construction using the general Kudla–Millson theory. As source for the general Kudla–Millson theory see [KM90]. In this particular case see [Kud03, p. 329–330]. □

5.3 The Eisenstein series $E_2^+(\tau, s)$ and its derivative

In this section we restrict ourselves to real quadratic number fields K of prime discriminant D. This implies that ε_0 has a negative norm and that $\Gamma_0(D)$ has two cusps, namely 0 and ∞ (not to be confused with $\mathrm{SL}_2(\mathcal{O}_K)$ whose cusp number is h_k). Therefore, there are two non-holomorphic Eisenstein series $E_2^0(\tau, s)$ and $E_2^\infty(\tau, s)$ of weight 2, level D and nebentypus χ_D corresponding to the cusps, respectively. A certain linear combination gives rise to the Eisenstein series $E_2^+(\tau, s)$ satisfying the so-called *plus space condition*, i.e., the m-th Fourier coefficient vanishes whenever $\chi_D(m) = -1$. Details can be found in [BY06, Section 2]. We quote now Theorem 2.2 of [BY06] for weight $k = 2$ which makes the Fourier expansion of $E_2^+(\tau, s)$ explicit. Recall from equation (3.21) the definition of the generalized divisor sum $\sigma_m(s)$. We lighten the notation of the usual W-Whittaker function $W_{\nu,\mu}(z)$ (cf. [OLBC10, 13.14.3]) by defining

$$\mathcal{W}_s(v) := |v|^{-1} W_{\mathrm{sgn}(v),-1/2-s}(|v|) \tag{5.3}$$

for $s \in \mathbb{C}$ and $v \in \mathbb{R}^\times$.

Theorem 5.3.1. *The Eisenstein series $E_2^+(\tau, s)$ has the Fourier expansion*

$$E_2^+(\tau, s) = c(0, s, v) + \sum_{m\in\mathbb{Z}}{}' C(m, s)\mathcal{W}_s(4\pi m v)e(mu),$$

where

$$\begin{aligned}
c(0, s, v) &= (vD)^s - 2^{-2s}\pi D^{-3/2-s} v^{-1-s} \frac{\Gamma(2s+1)}{\Gamma(s+2)\Gamma(s)} \frac{L(2s+1, \chi_D)}{L(2s+2, \chi_D)}, \\
C(m, s) &= 2\left(\frac{D}{4\pi}\right)^s \frac{\cos(\pi s)\Gamma(2s+2)\sigma_{|m|}(-1-2s)}{\Gamma(s)L(-1-2s, \chi_D)}, \quad \text{if } m < 0, \\
C(m, s) &= 2\left(\frac{D}{4\pi}\right)^s \frac{\cos(\pi s)\Gamma(2s+2)\sigma_m(-1-2s)}{\Gamma(s+2)L(-1-2s, \chi_D)}, \quad \text{if } m > 0.
\end{aligned}$$

Corollary 5.3.2. *We have*

$$E_2^+(\tau, 0) = 1 + \frac{2}{L(-1, \chi_D)} \sum_{m=1}^{\infty} \sigma_m(-1)q^m.$$

Proof. The statement follows with $\mathcal{W}_0(v) = e^{-v/2}$ directly from Theorem 5.3.1 by plugging in $s = 0$. The terms of negative index m vanish because $1/\Gamma(s)$ has a simple zero at $s = 0$. □

Proposition 5.3.3. *The derivative of $E_2^+(\tau, s)$ with respect to s at $s = 0$ is given by*

$$\frac{d}{ds} E_2^+(\tau, s) \mid_{s=0} = \sum_{m\in\mathbb{Z}} \tilde{c}(m, v)q^m,$$

where

$$\begin{aligned}
\tilde{c}(0,v) &= \log(vD) + \frac{h_K \log(\varepsilon_0)}{\pi v\sqrt{D}L(-1,\chi_D)},\\
\tilde{c}(m,v) &= \frac{2\sigma_{|m|}(-1)}{L(-1,\chi_D)}\Gamma(-1,4\pi|m|v), \quad \textit{if } m<0,\\
\tilde{c}(m,v) &= \frac{2\sigma_m(-1)}{L(-1,\chi_D)}\\
&\quad\times\left(\log(D/4\pi)+1-\gamma-2\frac{\sigma_m'(-1)}{\sigma_m(-1)}+2\frac{L'(-1,\chi_D)}{L(-1,\chi_D)}+\frac{1}{4\pi mv}\right), \quad \textit{if } m>0.
\end{aligned}$$

Proof. We use that for holomorphic $f(s)$ we have

$$\lim_{s\to 0}\frac{d}{ds}\frac{f(s)}{\Gamma(s)} = f(0). \tag{5.4}$$

Applied to the constant term of $E_2^+(\tau,s)$ this implies

$$\frac{d}{ds}c(0,s,v)\mid_{s=0} = \log(vD) - \pi D^{-3/2}v^{-1}\frac{L(1,\chi_D)}{L(2,\chi_D)}.$$

Using the functional equation of the L-function $L(s,\chi_D)$, we obtain

$$L(2,\chi_D) = -\frac{2\pi^2}{D^{3/2}}L(-1,\chi_D).$$

By the class number formular (2.6) we have

$$h_K = \frac{\sqrt{D}}{2\log(\varepsilon_0)}L(1,\chi_D).$$

We obtain

$$-\pi D^{-3/2}v^{-1}\frac{L(1,\chi_D)}{L(2,\chi_D)} = \frac{h_K\log(\varepsilon_0)}{\pi v\sqrt{D}L(-1,\chi_D)}.$$

For $m<0$ we obtain using (5.4) a further time

$$\frac{d}{ds}C(m,s)\mathcal{W}_s(4\pi mv)e(mu)\mid_{s=0} = (\Gamma(s)C(m,s))\mid_{s=0}\mathcal{W}_0(4\pi mv)e(mu). \tag{5.5}$$

We have

$$(\Gamma(s)C(m,s))\mid_{s=0} = 2\left(\frac{D}{4\pi}\right)^0\frac{\cos(\pi\cdot 0)\Gamma(2)\sigma_{|m|}(-1)}{L(-1,\chi_D)} = \frac{2\sigma_{|m|}(-1)}{L(-1,\chi_D)}.$$

Together with $\mathcal{W}_0(v) = e^{-v/2}\Gamma(-1,|v|)$ for $v<0$ (recall equation (4.23) for the definition of the upper incomplete gamma function) we obtain

$$\frac{2\sigma_{|m|}(-1)}{L(-1,\chi_D)}\Gamma(-1,4\pi|m|v)q^m$$

as result of equation (5.5).

For $m > 0$ we compute the logarithmic derivative of

$$C(m,s) = 2\left(\frac{D}{4\pi}\right)^s \frac{\cos(\pi s)\Gamma(2s+2)\sigma_m(-1-2s)}{\Gamma(s+2)L(-1-2s,\chi_D)}$$

and evaluate it directly at $s = 0$:

$$\log(D/4\pi) + 2\Gamma'(2) - 2\frac{\sigma_m'(-1)}{\sigma_m(-1)} - \Gamma'(2) + 2\frac{L'(-1,\chi_D)}{L(-1,\chi_D)}.$$

This implies with $\Gamma'(2) = 1 - \gamma$

$$C'(m,0) = \frac{2\sigma_m(-1)}{L(-1,\chi_D)}\left(\log(D/4\pi) + 1 - \gamma - 2\frac{\sigma_m'(-1)}{\sigma_m(-1)} + 2\frac{L'(-1,\chi_D)}{L(-1,\chi_D)}\right).$$

Using

$$\frac{d}{ds}\mathcal{W}_s(v)\mid_{s=0} = \frac{e^{-v/2}}{v}$$

for $v > 0$, we obtain

$$\begin{aligned}&\frac{d}{ds}C(m,s)\mathcal{W}_s(4\pi m v)e(mu)\mid_{s=0}\\ =&\frac{2\sigma_m(-1)}{L(-1,\chi_D)}\left(\log(D/4\pi) + 1 - \gamma - 2\frac{\sigma_m'(-1)}{\sigma_m(-1)} + 2\frac{L'(-1,\chi_D)}{L(-1,\chi_D)} + \frac{1}{4\pi m v}\right)q^m.\end{aligned}$$

□

5.4 Modularity of the integrals

Again, in this section we restrict ourselves to real quadratic number fields K of prime discriminant D.

Theorem 5.4.1. *We have*

$$\begin{aligned}-\frac{L(-1,\chi_D)}{24}E_2^+(\tau,0) &= \sum_{m=0}^{\infty}\left(\int_{X(\mathcal{O}_K)}\Phi(\mathcal{O}_K,m,z)\omega^2\right)q^m\\ &= \int_{X(\mathcal{O}_K)}\left(\sum_{m=0}^{\infty}\Phi(\mathcal{O}_K,m,z)q^m\right)\omega^2.\end{aligned}$$

Hence, the integral of the generating series of the $\Phi(\mathcal{O}_K,m,z)$ is a holomorphic modular form of weight 2, level D and nebentypus χ_D.

Proof. The second equality holds because of Fubini–Tonelli combined with Theorem 3.10.1. By Corollary 5.3.2 we have

$$-\frac{L(-1,\chi_D)}{24}E_2^+(\tau,0) = -\frac{L(-1,\chi_D)}{24} - \frac{1}{12}\sum_{m=1}^{\infty}\sigma_m(-1)q^m.$$

Theorem 3.8.11 with equation (3.24) implies

$$\int_{X(\mathcal{O}_K)} \Phi(\mathcal{O}_K, m, z)\omega^2 = -\frac{\sigma_m(-1)}{12}$$

for $m \in \mathbb{N}$. For $m = 0$ we have by Definition 2.9.25 and equation (2.17)

$$\begin{aligned}
&\int_{X(\mathcal{O}_K)} \Phi(\mathcal{O}_K, 0, z)\omega^2 \\
=&\int_{X(\mathcal{O}_K)} \left(L(\mathcal{O}_K, 0) + G(\mathcal{O}_K, 0, z)\right)\omega^2 \\
=&\operatorname{vol}(X(\mathcal{O}_K))L(\mathcal{O}_K, 0) + \int_{X(\mathcal{O}_K)} G(\mathcal{O}_K, 0, z)\omega^2 \\
=&\frac{\operatorname{vol}(X(\mathcal{O}_K))}{2} - \operatorname{vol}(X(\mathcal{O}_K)) \int_{X(\mathcal{O}_K)} \frac{G(\mathfrak{a}, 0, z)}{\operatorname{vol}(X(\mathcal{O}_K))}\omega^2 \\
&+ \int_{X(\mathcal{O}_K)} G(\mathcal{O}_K, 0, z)\omega^2 = -\frac{L(-1, \chi_D)}{24}
\end{aligned}$$

which finishes the proof. □

Theorem 5.4.2. *We have*

$$\begin{aligned}
&\frac{L(-1, \chi_D)}{24}\left((\log(4\pi) + \gamma)E_2^+(\tau, 0) + \frac{d}{ds}E_2^+(\tau, s)\mid_{s=0}\right) \\
=&\sum_{m\in\mathbb{Z}} \int_{X(\mathcal{O}_K)} \left(\Xi(\mathcal{O}_K, m, v, z) - G(\mathcal{O}_K, m, z)\right)\omega^2\ q^m \\
=&\int_{X(\mathcal{O}_K)} \left(\sum_{m\in\mathbb{Z}} \Xi(\mathcal{O}_K, m, v, z)q^m - \sum_{m=0}^{\infty} G(\mathcal{O}_K, m, z)q^m\right)\omega^2.
\end{aligned}$$

Hence, the integral of the differences of the two generating series is a non-holomorphic modular form of weight 2*, level* D *and nebentypus* χ_D*.*

Proof. The second equality holds because of Fubini–Tonelli combined with Theorem 3.10.1 and Theorem 4.8.2. By Corollary 5.3.2 and Proposition 5.3.3 we have

$$\frac{L(-1, \chi_D)}{24}\left((\log(4\pi) + \gamma)E_2^+(\tau, 0) + \frac{d}{ds}E_2^+(\tau, s)\mid_{s=0}\right) = \sum_{m\in\mathbb{Z}} \hat{c}(m, v)q^m$$

with

$$\begin{aligned}
\hat{c}(0, v) &= \frac{L(-1, \chi_D)}{24}\left(\log(4\pi v D) + \gamma\right) + \frac{h_K \log(\varepsilon_0)}{24\pi v\sqrt{D}}, \\
\hat{c}(m, v) &= \frac{\sigma_{|m|}(-1)}{12}\Gamma(-1, 4\pi|m|v), \quad \text{if } m < 0, \\
\hat{c}(m, v) &= \frac{\sigma_{|m|}(-1)}{12}\left(\log(D) + 1 - 2\frac{\sigma_m'(-1)}{\sigma_m(-1)} + 2\frac{L'(-1, \chi_D)}{L(-1, \chi_D)} + \frac{1}{4\pi m v}\right), \quad \text{if } m > 0.
\end{aligned}$$

Using equation (3.24), we get

$$\frac{\sigma_{|m|}(-1)}{12} = 2\,\mathrm{vol}(T(\mathcal{O}_K, |m|)).$$

Now, equation (3.23) with Theorem 4.6.2 shows the equality for $m > 0$. Theorem 4.6.2 alone shows the equality for $m < 0$ (note that D being prime implies $N(\varepsilon_0) = -1$). For $m = 0$ we have to address the two terms in $\hat{c}(0, v)$. By Definition 4.1.1 we have

$$\Xi(\mathcal{O}_K, 0, v, z) - G(\mathcal{O}_K, 0, z) = \Xi_*(\mathcal{O}_K, 0, v, z) - \frac{\log(4\pi v D) + \gamma}{2}.$$

Theorem 4.7.2 states that the integral of the first term coincides with

$$\frac{h_K \log(\varepsilon_0)}{24\pi v\sqrt{D}}.$$

Equation (2.17) implies that the integral of the second term coincides with

$$L(-1, \chi_D)\frac{\log(4\pi v D) + \gamma}{24}.$$

□

5.5 An arithmetic Hirzebruch–Zagier theorem for Kudla's Green functions

This section is devoted to our main theorem which states that the generating series

$$\sum_{m\in\mathbb{Z}} (Z(\mathcal{O}_K, m), \widetilde{\Xi}_\rho(\mathcal{O}_K, m, v, z))q^m \tag{5.6}$$

is modular for D prime. However, we first have to define what that means. The definition in the sense of the arithmetic Hirzebruch–Zagier theorem from Section 3.11 is not applicable to our situation here. To explain why let us first recall that definition.

Definition 5.5.1. A generating series

$$\sum_{m\in\mathbb{Z}} \hat{Z}(m)q^m$$

with $\hat{Z}(m) \in \widehat{\mathrm{CH}}^1(\overline{X(\mathfrak{a})}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}}$ is called a *holomorphic modular form of weight k, level D and nebentypus χ_D with values in* $\widehat{\mathrm{CH}}^1(\overline{X(\mathfrak{a})}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}}$ if for every linear map

$$\lambda : \widehat{\mathrm{CH}}^1(\overline{X(\mathcal{O}_K)}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}} \to \mathbb{C}$$

we have that

$$\sum_{m=0}^{\infty} \lambda(\hat{Z}_m)q^m \tag{5.7}$$

is a holomorphic modular form of weight k, level D and nebentypus χ_D (cf. Definition 5.0.1).

The series (5.6) is surely not a holomorphic modular form because it has coefficients of negative index. Furthermore, the coefficients depend on v, the imaginary part of τ. That is also a typical behavior of classicial non-holomorphic modular forms.

As a consequence we need to generalize Definition 5.5.1 to non-holomorphic modular forms. Motivated by [ES18, Definition 4.11] we give the following definition.

Definition 5.5.2. The generating series

$$\sum_{m\in\mathbb{Z}} \hat{Z}(m,v)q^m$$

with $\hat{Z}(m,v) \in \widehat{\mathrm{CH}}^1(\overline{X(\mathfrak{a})}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}}$ is called a *non-holomorphic modular form of weight k, level D and nebentypus χ_D with values in* $\widehat{\mathrm{CH}}^1(\overline{X(\mathfrak{a})}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}}$ if the following four conditions are satisfied.

(i) There exists a decomposition

$$\hat{Z}(m,v) = \hat{Z}_1(m) + \hat{Z}_2(m,v)$$

in $\widehat{\mathrm{CH}}^1(\overline{X(\mathfrak{a})}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}}$.

(ii) The generating series

$$\sum_{m\in\mathbb{Z}} \hat{Z}_1(m)q^m$$

is a holomorphic modular form of weight k, level D and nebentypus χ_D with values in $\widehat{\mathrm{CH}}^1(\overline{X(\mathfrak{a})}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}}$ (cf. Definition 5.5.1).

(iii) The elements $\hat{Z}_2(m,v)$ can be represented by arithmetic divisors $(0, g(m,v,z))$.

(iv) For fixed $\tau \in \mathbb{H}$ the series

$$\sum_{m\in\mathbb{Z}} g(m,v,z)q^m$$

converges almost everywhere to an integrable function. Therefore, it induces a current. In addition, regarded as current, it is modular of weight k, level D and nebentypus χ_D, i.e., for all smooth $\eta \in A^4(\overline{X(\mathfrak{a})})$

$$\int_{X(\mathfrak{a})} \left(\sum_{m\in\mathbb{Z}} g(m,v,z)q^m \right) \wedge \eta$$

satisfies the transformation law (5.1) in τ.

Remark 5.5.3. Condition (iv) of Definition 5.5.2 is slightly stronger than the respective condition in [ES18, Definition 4.11] in the sense that there they only demand the convergence in the weak sense, namely

$$\lim_{M\to\infty} \sum_{|m|\leq M} \int_{X(\mathfrak{a})} g(m,v,z)q^m \wedge \eta.$$

Proposition 5.5.4. *Let $\mathfrak{a} \in \mathcal{I}_K$ and $\tau \in \mathbb{H}$. There exists a smooth function*

$$A_\tau : X(\mathfrak{a}) \to \mathbb{C}$$

such that for almost all $z \in X(\mathfrak{a})$ we have

$$A_\tau(z) = \sum_{m \in \mathbb{Z}} \left(\Xi(\mathfrak{a}, m, v, z) - G(\mathfrak{a}, m, z)\right) q^m.$$

Furthermore, $dd^c A_\tau(z)$ is a non-holomorphic modular form of weight 2, level D and nebentypus χ_D with values in $A^{1,1}(X(\mathfrak{a}))$.

Proof. We define

$$B_\tau(z) := \sum_{m \in \mathbb{Z}} \left(\Xi(\mathfrak{a}, m, v, z) - G(\mathfrak{a}, m, z)\right) q^m$$

wherever the series converges. In Theorem 4.8.2 and Theorem 3.10.1 we have seen that the series

$$\sum_{m \in \mathbb{Z}} \Xi(\mathfrak{a}, m, v, z) q^m \quad \text{and} \quad \sum_{m \in \mathbb{Z}} G(\mathfrak{a}, m, z) q^m$$

are absolutely convergent for almost all $z \in X(\mathfrak{a})$. Hence, $B_\tau(z)$ is well-defined for almost all $z \in X(\mathfrak{a})$. Further, we have seen in Theorem 4.8.2 and Theorem 3.10.1 as well that the series

$$\sum_{m \in \mathbb{Z}} |\Xi(\mathfrak{a}, m, v, z) q^m| \quad \text{and} \quad \sum_{m \in \mathbb{Z}} |G(\mathfrak{a}, m, z) q^m|$$

are integrable over $X(\mathfrak{a})$. Therefore, $B_\tau(z)$ is integrable as well and $[B_\tau]$ is a well-defined current. Let $\eta \in A_c^2(X(\mathfrak{a}))$. Then we have with the notation introduced in Section 5.2

$$\begin{aligned}
(dd^c[B_\tau])(\eta) &= \int_{X(\mathfrak{a})} B_\tau \wedge dd^c \eta \\
&= \int_{X(\mathfrak{a})} \left(\sum_{m \in \mathbb{Z}} \left(\Xi(\mathfrak{a}, m, v, z) - G(\mathfrak{a}, m, z)\right) q^m \right) \wedge dd^c \eta \\
&= \sum_{m \in \mathbb{Z}} q^m \int_{X(\mathfrak{a})} \left(\Xi(\mathfrak{a}, m, v, z) - G(\mathfrak{a}, m, z)\right) \wedge dd^c \eta \\
&= \sum_{m \in \mathbb{Z}} q^m \int_{X(\mathfrak{a})} dd^c \left(\Xi(\mathfrak{a}, m, v, z) - G(\mathfrak{a}, m, z)\right) \wedge \eta \\
&= \sum_{m \in \mathbb{Z}} q^m \int_{X(\mathfrak{a})} \left(\varphi(\mathfrak{a}, m, v, z) - \omega(\mathfrak{a}, m, z)\right) \wedge \eta \\
&= \int_{X(\mathfrak{a})} \left(\varphi(\mathfrak{a}, \tau, z) - \Omega(\mathfrak{a}, \tau, z)\right) \wedge \eta \\
&= [\varphi(\mathfrak{a}, \tau, \cdot) - \Omega(\mathfrak{a}, \tau, \cdot)](\eta).
\end{aligned}$$

By Fubini–Tonelli we are allowed to interchange summation and integration. Bringing the dd^c to the other side is valid in this case since

$$\Xi(\mathfrak{a}, m, v, z) - G(\mathfrak{a}, m, z)$$

is smooth on $X(\mathfrak{a})$ (cf. Lemma 2.9.13), the logarithmic singularities of $\Xi(\mathfrak{a}, m, v, z)$ and $G(\mathfrak{a}, m, z)$ cancel out because they belong to the same divisor $-T(\mathfrak{a}, m)$.

This shows that the current $dd^c[B_\tau]$ is represented by $[\varphi(\mathfrak{a}, \tau, \cdot) - \Omega(\mathfrak{a}, \tau, \cdot)]$ with $\varphi(\mathfrak{a}, \tau, \cdot)$ and $\Omega(\mathfrak{a}, \tau, \cdot)$ being smooth differential forms by Theorem 5.2.2 and Theorem 5.2.4.

Since $X(\mathfrak{a})$ is a Kähler manifold, the Laplace operator and the dd^c operator on functions are strongly related (cf. [Bal06, Exercise 5.51 1)]). This allows us to apply the regularization theorem from [GH94, p. 378] for the dd^c operator which is formulated in the cited source for the Laplace operator only. Hence, there exists a smooth function $A_\tau : X(\mathfrak{a}) \to \mathbb{C}$ such that $dd^c A_\tau(z) = \varphi(\mathfrak{a}, \tau, z) - \Omega(\mathfrak{a}, \tau, z)$ and $[A_\tau] = [B_\tau]$. The first equality proves the modularity of $dd^c A_\tau(z)$ (cf. Theorem 5.2.2 and Theorem 5.2.4). The latter equality implies that the functions A_τ and B_τ agree almost everywhere which finishes the proof of the proposition. □

Lemma 5.5.5. *Let $f : X(\mathfrak{a}) \to \mathbb{C}$ be a smooth function with $dd^c f = 0$. Then f is constant.*

Proof. The equation $dd^c f = 0$ implies that f has a pluriharmonic real part and a pluriharmonic imaginary part (cf. [Bru02, p. 82, Definition 3.12]). This implies by Lemma 3.13 of the cited source that there exist holomorphic functions

$$h_1, h_2 : \mathbb{H}^2 \to \mathbb{C}$$

with

$$\Re(f) = \Re(h_1) \quad \text{and} \quad \Im(f) = \Re(h_2).$$

Note that in order to apply Lemma 3.13, we had to pull back f to $\mathbb{H}^2$. We get that the real parts of h_1 and h_2 are $\Gamma_\mathfrak{a}$ invariant and want to prove that this implies that h_1 and h_2 are $\Gamma_\mathfrak{a}$ invariant themselves. Hence, let $h : \mathbb{H}^2 \to \mathbb{C}$ be a holomorphic function with $\Gamma_\mathfrak{a}$ invariant real part. We define for each $\gamma \in \Gamma_\mathfrak{a}$ the map

$$g_\gamma : \mathbb{H}^2 \to \mathbb{C}, \quad g_\gamma(z) = h(z) - h(\gamma z).$$

Since g_γ is holomorphic with vanishing real part, we infer that g_γ is constant. Therefore, we obtain a group homomorphism

$$\varphi : \Gamma_\mathfrak{a} \to i\mathbb{R}, \quad \gamma \mapsto g_\gamma.$$

Since the group $i\mathbb{R}$ is abelian, the group homomorphism φ has to factor through the abelianization of $\Gamma_\mathfrak{a}$. Because the abelianization of $\Gamma_\mathfrak{a}$ is a torsion group and $i\mathbb{R}$ is torsion free, we obtain $\ker(\varphi) = \Gamma_\mathfrak{a}$. This proves the $\Gamma_\mathfrak{a}$ invariance of h.

Now that we know that h_1 and h_2 are $\Gamma_\mathfrak{a}$ invariant they define Hilbert modular forms of weight 0 on $X(\mathfrak{a})$. Hence, they are constant. This proves that f is constant as well. □

Theorem 5.5.6. *Let D be prime. Then*

$$\sum_{m \in \mathbb{Z}} (0, \widetilde{\Xi}_\rho(\mathcal{O}_K, m, v, z) - G(\mathcal{O}_K, m, z)) q^m$$

is a non-holomorphic modular form of weight 2, level D and nebentypus χ_D.

Proof. First of all, note that $\widetilde{\Xi}_\rho(\mathcal{O}_K, m, v, z) - G(\mathcal{O}_K, m, z)$ is smooth on $X(\mathcal{O}_K)$ because it is the difference of two pre-log-log Green functions for the same divisor $Z(\mathcal{O}_K, m)$. Hence,

$$(0, \widetilde{\Xi}_\rho(\mathcal{O}_K, m, v, z) - G(\mathcal{O}_K, m, z))$$

actually defines an element in $\widehat{\mathrm{CH}}^1(\overline{X(\mathcal{O}_K)}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}}$.

Now, coming to the actual proof, according to Definition 5.5.2, we have to decompose the arithmetic divisors. Since the first components of our arithmetic divisors are already 0, we can take the zero series for the holomorphic part and (i), (ii) and (iii) of Definition 5.5.2 are trivially satisfied. Therefore, we are left with (iv) and have to deal with the series

$$\sum_{m\in\mathbb{Z}} \left(\widetilde{\Xi}_\rho(\mathcal{O}_K, m, v, z) - G(\mathcal{O}_K, m, z)\right) q^m \tag{5.8}$$

$$= \sum_{m\in\mathbb{Z}} \left(\Xi(\mathcal{O}_K, m, v, z) - G(\mathcal{O}_K, m, z)\right) q^m \tag{5.9}$$

$$- \sum_{m\in\mathbb{Z}} \sum_{j=1}^{h_k} \rho_j(z) \check{\Xi}(\mathfrak{a}_j^2, m, v, M_j^{-1} z) q^m. \tag{5.10}$$

The decomposition of $\widetilde{\Xi}_\rho(\mathcal{O}_K, m, v, z)$ is due to definition (4.22) of $\widetilde{\Xi}_\rho(\mathcal{O}_K, m, v, z)$. The almost everywhere convergence to an integrable function of (5.8) is given by Theorem 4.8.3 and Theorem 3.10.1. The splitting of (5.8) into (5.9) and (5.10) is allowed because of Theorem 4.8.2 and Theorem 3.10.1. For each j the series

$$\rho_j(z) \sum_{m\in\mathbb{Z}} \check{\Xi}(\mathfrak{a}_j^2, m, v, M_j^{-1} z) q^m$$

is an in z continuous non-holomorphic modular form of weight 2, level D and nebentypus χ_D in τ due to Theorem 5.1.5. Note that z is fixed for that argument. Therefore, the scaling with $\rho_j(z)$ and the evaluation at $M_j^{-1} z$ (instead of z) is irrelevant and we can actually apply Theorem 5.1.5. By this argument we get the modularity of (5.10) as function. Since it is integrable, it defines a modular current. Therefore, we are left with proving the modularity of

$$\sum_{m\in\mathbb{Z}} \left(\Xi(\mathcal{O}_K, m, v, z) - G(\mathcal{O}_K, m, z)\right) q^m$$

as current. Since this series coincides with the function $A_\tau(z)$ from Proposition 5.5.4 for almost all $z \in X(\mathfrak{a})$, the current of this series is given by $[A_\tau]$ and it is enough to prove the modularity of $A_\tau(z)$ as function in τ.

For this purpose we fix $\tau \in \mathbb{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(D)$ and define

$$f(z) := A_{\gamma\tau}(z) - \chi_D(a)(c\tau + d)^2 A_\tau(z).$$

By the modularity of $dd^c A_\tau(z)$ from Proposition 5.5.4 we obtain

$$\begin{aligned} dd^c f(z) &= dd^c \left(A_{\gamma\tau}(z) - \chi_D(a)(c\tau + d)^2 A_\tau(z)\right) \\ &= dd^c A_{\gamma\tau}(z) - \chi_D(a)(c\tau + d)^2 dd^c A_\tau(z) \\ &= \chi_D(a)(c\tau + d)^2 dd^c A_\tau(z) - \chi_D(a)(c\tau + d)^2 dd^c A_\tau(z) = 0. \end{aligned}$$

Lemma 5.5.5 implies that f is constant. Say $f = c$. Then we have

$$\begin{aligned} c \cdot \mathrm{vol}(X(\mathcal{O}_K)) &= \int_{X(\mathcal{O}_K)} f(z)\omega^2 \\ &= \int_{X(\mathcal{O}_K)} A_{\gamma\tau}(z)\omega^2 - \chi_D(a)(c\tau + d)^2 \int_{X(\mathcal{O}_K)} A_\tau(z)\omega^2 \\ &= 0 \end{aligned}$$

because of the modularity of

$$\int_{X(\mathcal{O}_K)} A_\tau(z)\omega^2$$

by Theorem 5.4.2. Therefore, $c = 0$ which implies $f = 0$ which again implies the modularity of $A_\tau(z)$.

□

Theorem 5.5.7. *Let D be prime. Then*

$$\sum_{m\in\mathbb{Z}} (Z(\mathcal{O}_K, m), \widetilde{\Xi}_\rho(\mathcal{O}_K, m, v, z))q^m$$

is a non-holomorphic modular form of weight 2, level D and nebentypus χ_D.

Proof. Again, according to Definition 5.5.2, we have to decompose the arithmetic divisors. We do that by

$$\begin{aligned} &(Z(\mathcal{O}_K, m), \widetilde{\Xi}_\rho(\mathcal{O}_K, m, v, z)) \\ =&(Z(\mathcal{O}_K, m), G(\mathcal{O}_K, m, z)) + (0, \widetilde{\Xi}_\rho(\mathcal{O}_K, m, v, z) - G(\mathcal{O}_K, m, z)). \end{aligned}$$

This already proves (i) and (iii) in Definition 5.5.2. The generating series

$$\sum_{m\in\mathbb{Z}} (Z(\mathcal{O}_K, m), G(\mathcal{O}_K, m, z))q^m = \sum_{m=0}^{\infty} (Z(\mathcal{O}_K, m), G(\mathcal{O}_K, m, z))q^m$$

is a holomorphic modular form of weight 2, level D and nebentypus χ_D with values in $\widehat{\mathrm{CH}}^1(\overline{X(\mathcal{O}_K)}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}}$ by the arithmetic Hirzebruch–Zagier theorem (cf. Section 3.11). Therefore, (ii) is satisfied as well. The left over condition (iv) is treated in Theorem 5.5.6.

□

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List of functions and symbols

- $\sim_a$: The a stands for *additive*. The meaning of this symbol is that the difference of the two sides goes to 0 for the variable in question approaching the indicated value.
- $A^{p,q}(X)$: Space of smooth (p,q)-forms on X.
- $A_c^{p,q}(X)$: Space of smooth compactly supported (p,q)-forms on X.
- $\alpha(r_1, r_2)$: Certain maximum function (cf. equation (3.6)).
- $\mathrm{B}(x,y)$: Beta function (cf. equation (3.16)).
- $\beta(x)$: Integral function introduced in [HZ76] (cf. equation (4.13)).
- $\beta(r_1, r_2)$: Certain minimum function (cf. equation (3.6)).
- χ_D: Dirichlet character (cf. Definition 2.4).
- $\widehat{\mathrm{CH}}^1(\overline{X(\mathfrak{a})}, \mathcal{D}_{\mathrm{pre}})_{\mathbb{C}}$: First arithmetic Chow group. Elements are represented by pairs (T, g) with g being a pre-log-log Green function with respect to the divisors T (cf. Definition 2.9.7).
- Cl_K: Ideal class group of K (cf. Subsection 2.2.3).
- D: Discriminant of the real quadratic number field K (cf. Section 2.2).
- $\mathfrak{d}$: Ideal called the *different* (cf. Subsection 2.2.4).
- $d(n)$: Number of positive divisors of n.
- $\delta_T(\alpha)$: Dirac current (cf. equation (2.52)).
- $\mathrm{div}(f)$: Divisor associated to a meromorphic function f. The function f is holomorphic if and only if $\mathrm{div}(f) \geq 0$.
- E_0: Context based one of two different matrices (cf. equation (2.13) and equation (2.41)).
- E_1: Function related to the exponential integral Ei (cf. equation (4.1)).
- $E(\mathfrak{a})$: Exceptional divisor of $\overline{X(\mathfrak{a})}$ (cf. Subsection 2.7.6).

- $E^{\infty}(\mathfrak{a})$: Exceptional divisor of $\overline{X(\mathfrak{a})}$ at the cusp ∞ (cf. equation (2.29)).
- $e(z)$: Abbreviation for $e(2\pi i z)$.
- Ein: Entire function related to the exponential integral (cf. equation (4.2)).
- ε_0: Fundamental unit greater 1 of $\mathcal{O}_K^{\times}$ (cf. Section 2.2).
- ε_1: Fundamental unit greater 1 of $\mathcal{O}_K^{+}$ (cf. Section 2.2).
- η_j: $(1,1)$-form occuring in the definition of the Kähler form (cf. equation (2.14)).
- ${}_2F_1(a,b;c;z)$: Hypergeometric function (cf. equation (3.12)).
- $G(\mathfrak{a},m,z)$: Normalized automorphic Green function (cf. equation (3.20)).
- $G^b(\mathfrak{a},m,\nu)$: Finite exponential sum (cf. equation (3.5)).
- $g(A,z)$: Normalized version of $h(A,z)$ (cf. equation (2.24)).
- $\Gamma_{\mathfrak{a}}$: Hilbert modular group associated to $\mathfrak{a}$ (cf. Definition 2.3.3).
- $\Gamma(s)$: Gamma function.
- $\Gamma(s,x)$: Upper incomplete gamma function (cf. equation (4.23)).
- $\mathbb{H}$: Upper complex half plane (cf. Subsection 2.5.1).
- $h(A,z)$: Projection of the determinant in $V_{\mathbb{R}}$ with respect to $z \in \mathbb{H}^2$ (cf. Section 2.6).
- h_K: Class number of number field K (cf. Subsection 2.2.3).
- $\Im(z)$: For $z \in \mathbb{H}^2$ we have $\Im(z) := \Im(z_1)\Im(z_2)$ (cf. Subsection 2.5.1).
- $\mathcal{I}_K$: Group of fractional ideals of K (cf. Subsection 2.2.2).
- $\mathcal{I}_{\kappa}^{\nu}(z)$: One of the Bessel functions $I_{\kappa}(z)$ and $J_{\kappa}(z)$ (cf. Definition (3.2.3)).
- $I_{\kappa}(z)$: Modified Bessel function of the first kind (cf. [OLBC10, 10.25.2]).
- $\mathrm{Iso}(V)$: Isotropic elements of V (in this thesis $0 \in V$ is not isotropic).
- $J_{\kappa}(z)$: Bessel function of the first kind (cf. [OLBC10, 10.2.2]).
- K: Real quadratic number field of discriminant D with fixed embedding into $\mathbb{R}$ (cf. Section 2.2).
- K^{+}: Elements of K which are totally positive (cf. Section 2.2).
- $K_{\kappa}(z)$: Modified Bessel function of the second kind (cf. [OLBC10, 10.25.3]).
- $L(\mathfrak{a})$: Lattice in quadratic space V associated to the fractional ideal $\mathfrak{a}$ (cf. Definition 2.4.1).

- $L(\mathfrak{a}, m)$: Constant occurring in the Fourier expansion of $\Phi(\mathfrak{a}, m, z)$ (cf. Theorem 3.4.1).
- $L(s, \chi_D)$: Dirichlet L-function associated to the Dirichlet character χ_D (cf. equation (2.6)).
- $\Lambda(\mathfrak{a}, m)$: Elements in $\mathfrak{a}\mathfrak{d}^{-1}$ of fixed norm $-mN(\mathfrak{a})/D$ (cf. equation (2.42)).
- $\Lambda^+(\mathfrak{a}, m)$: Positive elements in $\Lambda(\mathfrak{a}, m)$ (cf. equation (2.43)).
- $M(\mathfrak{a}, \mathfrak{b})$: Certain subset of matrices in $\mathrm{SL}_2(K)$ (cf. equation (2.9)).
- N: Either the norm in K (cf. equation (2.2)) or the product of the two entries for arguments in $\mathbb{C}^2$ (cf. Subsection 2.5.1).
- $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.
- $\mathcal{O}_K$: Ring of integers of the number field K.
- $\mathcal{O}_K^+ = \mathcal{O}_K^\times \cap K^+$ (cf. Section 2.2).
- ω: Kähler form (cf. equation (2.14)).
- $\omega(\mathfrak{a}, m, z)$: Differential form related to the automorphic Green function $G(\mathfrak{a}, m, z)$ (cf. Definition 5.2.1).
- $\Omega(\mathfrak{a}, \tau, z)$: Generating series of the $\omega(\mathfrak{a}, m, z)$ (cf. Theorem 5.2.2).
- $\mathbb{P}^1(K)$: Projective line over K.
- $\Phi(\mathfrak{a}, m, z)$: Regularized automorphic Green function (Definition 3.3.5).
- $\Phi(\mathfrak{a}, m, s, z)$: Unregularized automorphic Green function (Definition 3.1.1).
- $\Phi_n(\mathfrak{a}, m, s, z)$: Part of the decomposition of $\Phi(\mathfrak{a}, m, s, z)$ (cf. Section 3.7).
- $\varphi(\mathfrak{a}, \tau, z)$: Generating series of the $\varphi(\mathfrak{a}, m, v, z)$ (cf. Theorem 5.2.4).
- $\varphi(\mathfrak{a}, m, v, z)$: Differential form related to Kudla's Green function $\Xi(\mathfrak{a}, m, v, z)$ (cf. Definition 5.2.1).
- $\mathcal{P}_K$: Group of principal ideals of K (cf. Subsection 2.2.3).
- $\Psi(\mathfrak{a}, m, z)$: Local Borcherds product up to sign (cf. Section 3.5).
- $\Psi_\sigma(\mathfrak{a}, m, z)$: Local Borcherds product with sign function σ (cf. Definition 3.5.1).
- $\Psi(\mathfrak{a}, m, s, z)$: Essential building block of $\Phi_n(\mathfrak{a}, m, s, z)$ (cf. equation (3.13)).
- q: $e(\tau)$ for $\tau \in \mathbb{H}$.

- $q_{\mathfrak{a}}$: Certain quadratic form on V (cf. equation (2.8)).
- $q(\mathfrak{a}, m)$: Residue of $\Phi(\mathfrak{a}, m, s, z)$ at $s = 1$ (cf. Theorem 3.4.1).
- $Q_s(x)$: The Legendre function of the second kind. The italic non-boldface version in [OLBC10, 14.2(ii)].
- $R_{\mathfrak{a}}^m$: Positive constant defined in equation (2.46).
- $R(\mathfrak{a}, m, w)$: Set of all reduced $\lambda \in \Lambda^+(\mathfrak{a}, m)$ with respect to a $w \in (\mathbb{R}^+)^2$ (cf. Subsection 2.8.4).
- $R(\mathfrak{a}, m, W)$: Set of all reduced $\lambda \in \Lambda^+(\mathfrak{a}, m)$ with respect to a Weyl chamber $W \in W(\mathfrak{a}, m)$ (cf. Subsection 2.8.4).
- ρ: Certain partition of unity living on a Hilbert modular surface (cf. equation (4.21)).
- $\rho(\mathfrak{a}, m, w)$: Weyl vector with respect to a $w \in (\mathbb{R}^+)^2$ (cf. equation (2.47)).
- $\rho(\mathfrak{a}, m, W)$: Weyl vector with respect to a Weyl chamber $W \in W(\mathfrak{a}, m)$ (cf. equation (2.50)).
- $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.
- $S(\mathfrak{a}, m)$: Complement of Weyl chambers (cf. equation (2.45)).
- S_λ: Component of $S(\mathfrak{a}, m)$ (cf. equation (2.45)).
- $\mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{a})$: Hilbert modular group associated to $\mathfrak{a}$ (cf. Definition 2.3.3).
- $\sum'$: The tick at the sum sign indicates that we omit to sum over 0 (whenever we use the tick at the sum sign the index set we sum over is a group).
- T_A: Divisor component (cf. equation (2.35)).
- $T(\mathfrak{a}, m)$: Hirzebruch–Zagier divisor (cf. equation (2.37) for $m \neq 0$ and Definition 2.9.21 for $m = 0$).
- $T_*(\mathfrak{a}, m)$: Hirzebruch–Zagier divisor without special treatment of $m = 0$ (cf. equation (2.37)).
- tr: Either the trace in K (cf. equation (2.3)) or the sum of the two entries for arguments in $\mathbb{C}^2$ (cf. Subsection 2.5.1).
- V: In most cases the real quadratic space defined in (2.10).
- V^+: The elements in V with positive quadratic form (cf. Section 2.1).

- v: Has two meanings within this thesis: One is the imaginary part of $\tau \in \mathbb{H}$ (for example when we consider $\Xi(\mathfrak{a}, m, v, z)$) and the other one is the second coordinate of the local complex coordinates (u, v) of the cusp ∞.
- $\mathrm{vol}(L)$: Volume of a non-degenerate lattice L (cf. equation (2.1)).
- $W(\mathfrak{a}, m)$: Weyl chambers of index m (cf. equation (2.49)).
- W_λ: Weyl chamber (cf. equation (2.48)).
- $\mathcal{W}_s(v)$: A lightened notation for the usual W-Whittaker function $W_{\nu,\mu}(z)$ (cf. equation (5.3)).
- $X(\mathfrak{a})$: Open (uncompactified) Hilbert modular surface (cf. Subsection 2.5.2).
- $X(\mathfrak{a})^*$: Baily–Borel compactification of $X(\mathfrak{a})$ (cf. equation (2.18)).
- $\overline{X(\mathfrak{a})}$: Hirzebruch compactification of $X(\mathfrak{a})$ (cf. Subsection 2.7.6).
- $\Xi(\mathfrak{a}, m, v, z)$: Kudla's Green function (cf. Definition 4.1.1).
- $\Xi_*(\mathfrak{a}, m, v, z)$: Principal part of Kudla's Green function (cf. Definition 4.1.1).
- $\Xi_0(\mathfrak{a}, v, z)$: Non-principal part of Kudla's Green function $\Xi(\mathfrak{a}, 0, v, z)$ (cf. Definition 4.1.1).
- $\check{\Xi}(\mathfrak{a}, m, v, z)$: Error term of $\Xi(\mathfrak{a}, m, v, z)$ (cf. equation (4.18)).
- $\widetilde{\Xi}_\rho(\mathfrak{a}, m, v, z)$: Modified version of $\Xi(\mathfrak{a}, m, v, z)$ with the correct growth behavior at the cusps (cf. Section 4.5).
- $Z(\mathfrak{a}, m)$: Hirzebruch–Zagier divisor on $\overline{X(\mathfrak{a})}$ (cf. Subsection 2.8.5).
- $Z^\infty(\mathfrak{a}, m)$: Components of $Z(\mathfrak{a}, m)$ at $E^\infty(\mathfrak{a})$.

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