Topological Quantum Markov Processes

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Für
Claire

Preface

Introduction

In recent decades humanity gained the technical ability to precisely observe and manipulate quantum systems on the scale of a single atom or a photon. These advances led to new applications, especially in information theory. At the same time they created a new emphasis on the investigation of the dynamics of systems of a few quantum particles and especially their asymptotic behaviour.

The research area of this thesis is *operator-algebraic* or *non-commutative* probability theory and more specifically non-commutative Markov processes (see e.g. [Küm85]), which describe the time evolution of a large class of open quantum systems. One usual method in non-commutative probability is to take established tools from classical probability and see how they can be generalized to the operator-algebraic methods.

In this work, we apply this philosophy to the theory of *topological Markov processes*. Topological Markov processes are a concept from symbolic dynamics and coding theory for example presented in [LM21]. In this context, the word "topological" means that we only describe which trajectories, or sequences of system states, are possible for given dynamics, without tracking the probabilities of a certain trajectory.

We will only consider discrete-time, time-homogeneous *Markov processes*, which are also often called *Markov chains*. In [GKL06, 1.] a definition of "non-commutative topological Markov chain" is presented, which can be seen as an example of the theory we present (see 7.5.5). However, a general theory of non-commutative topological Markov processes does not seem to exist in the literature, for understandable reasons. Since quantum mechanics is a fundamentally stochastic theory, there is no obvious concept of trajectories. However, it is possible to see a classical topological Markov process as an equivalence class of stochastic Markov processes. We lift this idea to the theory of non-commutative Markov processes. While the dynamics of a quantum system still obey the probabilistic laws of quantum mechanics, as they have been discovered about 100 years ago, we describe it just by what we define as its topological properties. This gives us a non-deterministic but non-probabilistic description of the dynamics. We demonstrate that many commonly considered stochastic properties of such dynamics, especially those relevant to the asymptotic behaviour of the system, can be derived completely from this topological description.

The central new concept of this thesis are reach maps, maps on orthogonal projections of an algebra that capture the topological essence of completely positive operators. They turn out to be a useful concrete representation of the previously vague concept of a "topological Markov operator" which we were looking for. Reach maps encode which sequences of events are possible in given dynamics and are exactly the right morphisms to form a category in which we can express non-deterministic topological dynamics.

To define reach maps, we apply methods from non-commutative topology, a theory first introduced by C.A. Akeman with the definition of open projections in [Ake69] (see [AB15] and [Ped18] for modern descriptions). It uses the universal enveloping von Neumann algebra to apply von Neumann algebra methods to C*-algebras, bringing measure theoretic and topological objects closer to each other, like we require for our goal. We adapt this theory to our cause by generalizing some of its foundations to admit other enveloping von Neumann algebras than the universal one.

Besides giving definitions for non-commutative topological dynamics in the form of reach maps and a topological Markov condition, main results in this thesis are a characterization of reach maps via cross-ratios from projective geometry, applying Perron-Frobenius theory to reach maps and the discovery of the surprisingly elegant structure of reach maps of conditional expectations.

This thesis was produced under supervision of and in the research group of Burkhard Kümmerer at the Technical University Darmstadt. In his master's thesis [Bra17] the author built on previous research in the group by A. Gärtner [Gär14] and J. Dörner [Dör10] to discuss definitions like communicating classes, recurrence and transience and the theorem of Perron-Frobenius (see [Sen81]) in the non-commutative probabilistic setting. Those are all examples of concepts which we demonstrate to be topological properties in this work.

Overview

In Chapter 1 we establish the setting of C*-algebras and how they describe classical and quantum systems. We motivate the conceptual idea of reach maps in the classical setting. Then we introduce completely positive operators to give the definition of reach maps, first for non-commutative finite-dimensional systems.

In Chapter 2 we introduce the concepts of enveloped C*-algebras, hereditary subalgebras and open projections. Those definitions are required to give a proper definition of reach maps between infinite-dimensional algebras. Hereditary subalgebras are an established concept in the theory of C*-algebras, the definition of enveloped C*-algebras is a new perspective on well established theories. Our definition of open projections generalizes the usual definition in the literature.

In Chapter 3 we give the general definition of reach maps. We establish their fundamental properties and use them to gain more insight into the "topology" defined by the open projections. We also discuss supports of positive elements and completely positive operators in enveloped C*-algebras. At the end of the chapter we have established the setting of this thesis: The category of reach maps between enveloped C*-algebras.

In the Chapters 4 and 5 we investigate reach maps between finite-dimensional systems in more detail. In Chapter 4 we give different characterizations of reach maps and their corresponding equivalence classes of completely positive operators. First, we give a characterization of reach maps by preservation of cross-ratios, a concept from projective geometry. Then we describe exactly what we mean by “topological properties” of completely positive operators.

In Chapter 5 we harvest the fruits of our work and give the theory of communicating classes, recurrence and transience and a Perron-Frobenius style theorem for reach maps.

In Chapter 6 we show that reach maps of conditional expectations are completely determined by their support and the subalgebra they are projecting onto.

Finally, in Chapter 7 we first discuss how classical shift spaces can be generalized to non-commutative systems. We discuss and define a Markov condition for topological processes. Lastly, we show that every reach map has an associated topological Markov process.

Chapter 1 is introductory. The chapters 2,3,6 and 7 build directly on each other. While the chapters 4 and 5 are central to the theory and give additional context, they are not required for the results in the chapters 6 and 7.

Zusammenfassung

Diese Arbeit generalisiert klassische topologische Markovketten aus der symbolischen Dynamik indem sie sie mit der Theorie nicht-kommutativer Markovprozesse zusammen bringt. Als zentrale neue Definition werden Reach Maps eingeführt. Sie repräsentieren die topologischen Eigenschaften vollständig positiver Operatoren. Für die Definition von Reach Maps in unendlich-dimensionalen C*-Algebren verwenden wir Methoden der nicht-kommutativen Topologie. Wir definieren Reach Maps als Abbildungen auf den sogenannten offen Projektionen bezüglich einer einhüllenden Von-Neumann-Algebra. Dann charakterisieren wir Reach Maps mit Hilfe projektiver Geometrie und beschreiben die Struktur der von ihnen erzeugten Äquivalenzklassen vollständig positiver Operatoren. Wir demonstrieren, dass viele bekannte stochastische Eigenschaften von Markovprozessen, wie kommunizierende Klassen, Rekurrenz und Transienz sowie ein Satz von Perron-Frobenius, topologisch beschrieben werden können. Wir zeigen, dass die Reach Map einer bedingten Erwartung vollständig durch ihren Träger und die Algebra, auf die projiziert wird, festgelegt ist. Die Arbeit endet mit einer Diskussion der Definition topologischer Markovprozesse und der Konstruktion von Markov-Dilatationen.

Danksagung

Ich bin allen Menschen unglaublich dankbar, die es mir, wissentlich oder unwissentlich, ermöglicht haben, diese Arbeit zu schreiben, meine Zeit an der TU Darmstadt bereichert und ihr einen Sinn gegeben haben. Die Zeit an der Universität, am Fachbereich Mathematik und welch Glück es war, diese Gelegenheit zu bekommen, werde ich nie vergessen.

Natürlich haben einige Menschen einen individuellen Dank verdient, beginnend mit der Arbeitsgruppe C^*-Algebren.

Du, lieber Burkhard, hast mich vor über 10 Jahren für die Mathematik begeistert. Seitdem haben wir gemeinsam so viel erlebt und ich von Dir lernen können, weit über die Mathematik hinaus. Ich danke Dir für all die spannenden Gespräche, Deine Unterstützung und die große Freiheit, die Du mir gegeben hast. Du hast mich zu jeder Zeit die Richtung frei wählen lassen. Doch nur Deine Ortskunde hat mir die Möglichkeit gegeben, zu sehen welche Wege es gibt und sie tatsächlich zu befahren.

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Darmstadt, im Oktober 2023

List of Notations

Numbers	$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$	naturals, integers, reals and complex numbers	
	i, j, k, l, m, n, N, d	natural numbers or integers	
	$\lambda, \mu, \alpha, \beta$	real or complex numbers	
Sets	I, J, M, K, L	(index) sets	
	Ω	set, alphabet, topological or probability space	1.1.2
	ω	element of Ω	1.1.2
	$P(\Omega)$	power set of Ω	1.1.4
	χ_B	characteristic function of a set B	1.2.1
Hilbert Spaces	$\mathcal{H}, \mathcal{K}$	Hilbert spaces	1.3.2
	$\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}, \mathcal{K})$	bounded operators between Hilbert spaces	1.3.2
	$\mathcal{K}(\mathcal{H})$	compact operators in $\mathcal{B}(\mathcal{H})$	2.1.2
	ξ, η, ζ	Hilbert space vectors	
	e_i, f_i	basis vectors	
Function Spaces	$C(\Omega), C_0(\Omega)$	continuous functions $\Omega \to \mathbb{C}$ (vanishing at infinity)	2.1.2
	Σ	sigma algebra	
	μ, ν, λ	measures	
	$l^\infty(\Omega)$	bounded functions	2.1.2
	$L^\infty(\Omega, \mu)$	essentially bounded functions	2.1.2
Shift Spaces	S, S_F	shift spaces (with forbidden words F)	7.1.2
	s, σ	shift maps on S and $C(S)$	7.1.1
	v, w	word/block, i.e. finite sequence over an alphabet	7.1.2
	$\omega_{[k,n]}$	finite subsequence of a sequence ω	7.1.2
Algebras	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	C*- or matrix algebras	1.1.1
	x, y, a, b, u	elements of algebras or linear operators	
	$\mathcal{A}^+$	positive cone of $\mathcal{A}$	
	$\mathcal{M}$	von Neumann algebra	
	$\mathcal{A}'$	commutant of $\mathcal{A}$	
	$\mathcal{Z}(\mathcal{M})$	center of a von Neumann algebra	
	$\mathcal{A}^\diamond, \mathcal{B}^\diamond, \mathcal{C}^\diamond$	enveloped C*-algebras	2.1.2
	$\mathcal{M}_\mathcal{A}, \mathcal{M}_\mathcal{B}, \mathcal{M}_\mathcal{C}$	enveloping von Neumann algebras	2.1.2
	$\overline{\mathcal{A}}^{\Vert\cdot\Vert}, \overline{\mathcal{A}}^{\sigma^*}$	closure in the norm/weak topology	2.1.1
	$\mathcal{A}_1, \mathcal{A}_{<1}$	closed/open unit ball of $\mathcal{A}$	
	$\mathcal{A}_p$	reduced algebra	2.2.4
	$C^*(M)$	generated C*-algebra by a subset M of a C*-algebra	

Functionals	$\mathcal{A}^*$	dual of $\mathcal{A}$	
	$\mathcal{M}_*$	predual of $\mathcal{M}$	
	$\mathcal{S}(\mathcal{A}), \mathcal{S}_*(\mathcal{M})$	states on $\mathcal{A}$, normal states on $\mathcal{M}$	1.1.1
	$\varphi, \psi, \tau, \omega$	functionals or states	
	ω_ξ	vector state with $\omega_\xi(x) = \langle x\xi, \xi\rangle$	
	tr	trace functional	
Operators	T, S, P	completely positive operators	1.3.2
	$\mathrm{CP}(\mathcal{A}, \mathcal{B})$	completely positive operators from $\mathcal{A}$ to $\mathcal{B}$	4.2.3
	P, Q	conditional expectations	6.1.1
	i, π, Ψ, α	*-homo-, iso- and automorphisms	
	Ad_a	adjunction with $\mathrm{Ad}_a(x) = a^*xa$	1.3.2
	R, R', R_T	reach maps (of T)	3.1.1
	id	identity map	
Supports	$\mathcal{N}_T$	null algebra of T	3.3.1
	$\mathrm{supp}_{\mathcal{A}^\diamond} T$	support of T in the closed projections of $\mathcal{A}^\diamond$	3.3.1
	$[x]$	support projection of an element x	1.3.4
	$[x]_{\mathcal{A}^\diamond}$	relative support in regard to a subalgebra $\mathcal{A}^\diamond$	6.2.1
Matrices	$M_n, M_{n\times m}$	matrices with complex entries	1.1.2
	a, a_i, b_j	Kraus operators	1.3.2
	$\mathrm{Kr}(T), \mathrm{Kr}(R)$	Kraus operator modules	4.3, 4.4.2
	e_{ij}	matrix units	4.1.4
	$t_{\xi,\eta}$	rank one operator with $t_{\xi,\eta}\zeta = \xi\langle\eta, \zeta\rangle$	4.1.5
Projections	p, q, z, c, r, t	orthogonal projections	
	$[p_1, p_2, p_3, p_4]$	cross-ratio of four rank one projections	4.1.4
	$\mathcal{P}(\mathcal{A})$	orthogonal projections of $\mathcal{A}$	1.2.1
	$\mathcal{P}_1(\mathcal{A})$	minimal projections of a matrix algebra $\mathcal{A}$	4
	$\mathcal{T}(\mathcal{A}^\diamond), \mathcal{T}(\mathcal{A}, \mathcal{M}_\mathcal{A}), \mathcal{T}(\mathcal{A})$	open projections of $\mathcal{A}^\diamond$	2.3.1
	$x \perp y$	x and y are orthogonal positive elements	2.1.3
	$\mathbb{1}, \mathbb{1}_\mathcal{B}$	unit element of a (sub-)algebra	2.1.5
	$\bar{p}, p^\circ$	closure and interior of a projection p	3.2.4
	c	communicating class	5.2.3
	r, t	maximal recurrent and transient projection	5.3.4
	p_u	unital support	5.3.4
Orders	$\vee, \wedge$	supremum and infimum in a lattice	3.1.2
	sup, inf	supremum and infimum in other ordered sets	3.1.2
Spectral	$\sigma(T)$	spectrum of T	
Theory	$\sigma_r(T)$	spectrum of T with absolute value r	
	$r(T)$	spectral radius of T	5.5.1
	$\mathcal{M}_T$	multiplicative algebra of T	5.4.1
	$\mathcal{E}(T), \mathcal{E}(R)$	reversible algebra of T or R	5.4.2

Contents

1 Non-Deterministic Topological Dynamics

In this work, we propose a generalized definition of topological dynamical systems with the primary motivation to analyse the topological properties of Markov processes. We use the word “topological” in contrast to “stochastic” and mean by it that we describe the dynamics of a system by saying which sequences of system states are possible and which are not, without considering how probable a specific sequence of states is. We introduce reach maps as the central tool for the topological description of non-deterministic dynamics.

In this chapter, we will explain how, in our topological approach, reach maps play the role that completely positive operators play in the usual theory of quantum mechanics, which is a stochastic theory. We motivate their definition in three steps:

(1) Reach maps for finite classical systems in Section 1.1.
(2) Reach maps for finite-dimensional quantum systems in Section 1.3.
(3) Reach maps for infinite-dimensional systems in Chapter 3.

We postpone the final definition of reach maps to Chapter 3 because we need more theory to work with infinite-dimensional systems.

This text assumes a certain familiarity with the subjects of C*-algebras, von Neumann algebras and completely positive operators. Those foundations can be found, for example, in [Bla06], [Tak79], [Ped18] or [Stø13]. For convenience, we will still frequently give concrete references.

This introductory chapter contains no new results, but the definition of reach maps is a new concept.

1.1 Topological Dynamics on Classical Systems

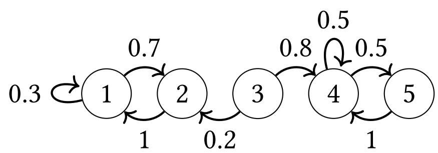
In this section, we talk about the essential idea of reach maps in a classical setting. We start by defining what we mean when we say “classical”.

1.1.1 Classical vs. Quantum Systems

We model all our systems as *C^*-algebras*, usually denoted by $\mathcal{A}$, which we call *matrix algebras* if they are finite-dimensional. We refer to a system as *classical* if its algebra is commutative. In that case, the algebra can be given as a set of functions on a set Ω. We speak of a *quantum* system if we allow the algebra to be non-commutative. Thus, a classical system is a particular case of a quantum system in our framework. The *states* of the system are given by the normed positive functionals $\mathcal{S}(\mathcal{A})$ on $\mathcal{A}$. In the classical case, a state always corresponds to a probability measure on Ω. We can interpret this as the system being in one well-defined point state $\omega \in \Omega$ and the probability measure models our lack of knowledge of which point state the system is in. In a quantum system, we do not generally have point states because of the famous physical concept of superposition. Thus, the probabilistic nature of quantum systems cannot be explained simply by a lack of knowledge about the precise system state.

1.1.2 Topological Dynamics with 0-1-Matrices

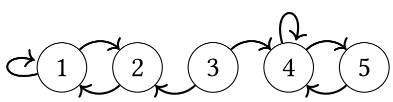
We will now motivate our description of topological dynamics in a finite-dimensional classical system. Let Ω be a finite set with $|\Omega| = n$. We consider a *stochastic matrix* $a \in M_n$, a matrix with non-negative entries such that the sum of every row is 1. It contains the transition probabilities for a Markov process on Ω. We can visualize the transition probabilities as weights of the edges in a directed graph, where the nodes are the elements of Ω. We call a the *adjacency matrix* of the weighted graph. This is an example of non-deterministic dynamics.



The graph of a Markov process on the state space $\Omega = \{1, \ldots, 5\}$ and its adjacency matrix.

$$a = \begin{pmatrix} 0.3 & 0.7 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

We ignore the probabilities to get a topological version of these dynamics. The directed graph now only tells us which transitions are possible. The adjacency matrix marks possible transitions with a 1 and blocked transitions with a 0.



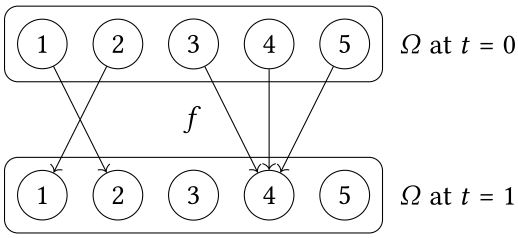
The same non-deterministic dynamics but without probabilities.

$$a = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

This idea of describing topological transitions with 0-1-matrices, easily generalizes to maps between different state spaces. In that case, the matrix is generally not square.

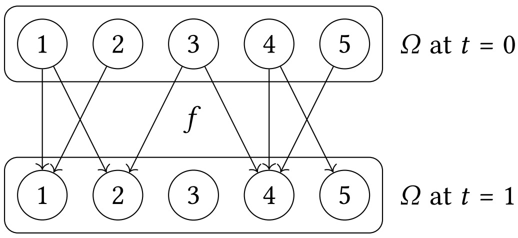
1.1.3 Reaching States

Our guiding question is: How can we generalize the concept of 0-1-matrices as a topological description of non-deterministic dynamics to quantum systems? An entry a_{ij} in the adjacency matrix gives us the information whether a system in point state ω_i can reach the point state ω_j in the next time step. We now discuss different ideas to represent this information to find a representation that does not rely on point states. As a first idea, we could use a function $f\colon \Omega \to \Omega$ mapping every point state to a point state which can be reached in the next time step. This is, in fact, the traditional definition of a classical finite *dynamical system*: A pair (Ω, f) of a finite state space with a function f describing the dynamics. This, however, is too restrictive for our use case because it would imply a deterministic time evolution. It would be equivalent to a 0-1-matrix, which has exactly one 1 per row.



An example of deterministic dynamics for $\Omega = \{1, \ldots, 5\}$.

However, we want to describe also non-deterministic systems. That means that from every point state, we can reach none, one or multiple other point states. We can model this with a set-valued function $f\colon \Omega \to P(\Omega)$.



The function representation of the example of non-deterministic topological dynamics in 1.1.2.

1.1.4 From Points to Subsets

A function from $\Omega \to P(\Omega)$ still relies on point states, though. So, we look for an equivalent representation that does not. We can do this using subsets instead of points in Ω. Instead of mapping a point $\omega \in \Omega$ to the set of points $f(\omega) \subseteq \Omega$ which can be reached from it, we can also equivalently map a set of points $A \subseteq \Omega$ to the set of points $f(A)$ which can be reached from any point in A, giving us a function $g\colon P(\Omega) \to P(\Omega)$ from power set to power set. We can see that f preserves unions of sets: A point $\omega \in \Omega$ can be reached from the set $A \cup B \subseteq \Omega$, i.e. $\omega \in g(A \cup B)$, if and only if either $\omega \in g(A)$, $\omega \in g(B)$ or both. In fact, on finite sets set-valued functions $\Omega \to P(\Omega)$

and union-preserving functions $P(\Omega) \to P(\Omega)$ are in one-to-one correspondence: We can restrict every function $P(\Omega) \to P(\Omega)$ to sets with one element, giving us a function $\Omega \to P(\Omega)$. For the other direction, every union preserving function on $P(\Omega)$ is entirely defined by its action on the atoms of $P(\Omega)$, which are the singleton sets containing each one element of Ω.

1.2 From Sets to Algebras

In this section, we see how we can lift our representation of non-deterministic topological dynamics of a classical system to the matrix algebra representation.

1.2.1 From Subsets to Orthogonal Projections

The algebraic description of a finite state space Ω is given by the algebra of complex-valued (continuous, which is trivial on a discrete space) functions $C(\Omega)$. In contrast to point states, subsets of Ω have a correspondence for quantum systems: For every subset $A \subset \Omega$, the characteristic function $\chi_A \colon \Omega \to \{0, 1\}$ is an *orthogonal projection* in $C(\Omega)$. Conversely, every orthogonal projection in $C(\Omega)$ is a characteristic function of a subset of Ω. We denote the set of orthogonal projections of a C*-algebra $\mathcal{A}$ as $\mathcal{P}(\mathcal{A})$.[1] Now, we can generalize a function $g \colon P(\Omega) \to P(\Omega)$ to a map $R \colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ for any matrix algebra $\mathcal{A}$. For any orthogonal projection p, we call $R(p)$ the *reach* of p. It represents the "part" of $\mathcal{A}$, or when we represent $\mathcal{A}$ on some Hilbert space $\mathcal{H}$, the subspace of $\mathcal{H}$, which can be reached from p.

1.2.2 The Schrödinger Picture

Given a function $g \colon P(\Omega) \to P(\Omega)$ the map $\chi_A \mapsto \chi_{g(A)}$ might be the obvious algebraization of g. That would be the interpretation in the *Schrödinger picture*, where the dynamics are applied to the states of the system while the observables are static. Concretely, given a probability matrix $a \in M_n$ and, as the state of the system, a probability measure μ on Ω which we represent as a row vector in $\mathbb{C}^n$, then μa is the measure in the next step. If we only have g given but do not know a, then we cannot calculate μa. However, when A is the support of μ, i.e. $A = \{\omega \in \Omega : \mu(\{\omega\}) > 0\}$, then g tells us that μa has support $g(A)$. Thus, $\chi_A \mapsto \chi_{g(A)}$ maps supports of states to supports of states.

[1] We often omit the "orthogonal" in "orthogonal projection" because we will not mention any non-orthogonal idempotent elements of C*-algebras in this thesis.

1.2.3 The Heisenberg Picture

We, however, will work in the *Heisenberg picture*, where the dynamics are applied to the *observables* of the system, given by the self-adjoint operators in $\mathcal{A}$, while the states stay fixed. Concretely, for an observable $h\colon \Omega \to \mathbb{R}$, interpreted as a column vector in $\mathbb{C}^n$, ah is an observable, observing the previous time step. This is consistent with the Schrödinger picture because $(\mu a)h = \mu(ah)$, so the resulting observation of a state does not depend on the picture.

First, consider a map $f : \Omega \to \Omega$ describing the deterministic dynamics of a classical dynamical system. Given an observable $h\colon \Omega \to \mathbb{R}$ we algebraize f as $i_f(h) = h \circ f$. The map i_f is a unital *-homomorphism on $C(\Omega)$. We consider the observable χ_A, also called an *event*, because it is an orthogonal projection, that observes whether the state is one of the point states in A. Applying i_f we get $i_f(\chi_A) = \chi_A \circ f = \chi_{\{\omega\in\Omega:f(\omega)\in A\}} = \chi_{f^{-1}[A]}$.

How can we do the same with non-deterministic dynamics $g\colon P(\Omega) \to P(\Omega)$? We cannot simply write "$\chi_A \circ g$". That expression does not make sense. What we can do is map $\chi_A \mapsto \chi_{\{\omega\in\Omega:g(\{\omega\})\cap A\neq\emptyset\}}$. That means an event χ_A can only be observed if, at the previous time step, the system was in a point state ω which has the possibility of reaching A, which is precisely the condition $g(\{\omega\}) \cap A \neq \emptyset$.

To confirm that this is the correct algebraization, we show consistency with the Schrödinger picture. Let μ be a probability measure with support A, and χ_B an observable for the next state of the dynamics. Then χ_B will observe the next state μa exactly if $g(A) \cap B \neq \emptyset$, which needs to be equivalent to the Heisenberg picture with $\mu(\{\omega \in \Omega : g(\{\omega\}) \cap B \neq \emptyset\}) > 0$. We do this via $\mu(\{\omega \in \Omega : g(\{\omega\}) \cap B \neq \emptyset\}) > 0 \Leftrightarrow \{\omega \in \Omega : g(\{\omega\}) \cap B \neq \emptyset\}) \cap A \neq \emptyset \Leftrightarrow \{\omega \in A : g(\{\omega\}) \cap B \neq \emptyset\} \neq \emptyset \Leftrightarrow g(A) \cap B \neq \emptyset$. In the last step, we used the fact that $\bigcup_{\omega\in A} g(\{\omega\}) = g(A)$.

1.3 Topological Dynamics on Quantum Systems

We had motivated $f\colon P(\Omega) \to P(\Omega)$ via 0-1-matrices as containing the topological information of an adjacency matrix with probabilities as entries. To keep this correspondence for quantum systems, a reach map $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ needs to contain the topological information of the non-commutative generalization of such matrices. This leads us to the definition of completely positive operators.

1.3.1 Completely Positive Operators

Definition Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. An operator $T\colon \mathcal{A} \to \mathcal{B}$ is called

(1) *positive* if $0 \le T(x)$ for all $x \in \mathcal{A}^+$,
(2) *k-positive* if $T \otimes \mathrm{id}\colon \mathcal{A} \otimes M_k \to \mathcal{B} \otimes M_k$ is positive for $k \in \mathbb{N}$,
(3) *completely positive* if T is k-positive for all $k \in \mathbb{N}$.

Remark A positive operator is always bounded, i.e., norm continuous (cf. [Stø13, 1.3.3]).

Definition We call a completely positive operator mapping between von Neumann algebras *normal* when it is continuous regarding the weak* topologies (see [Bla06, III.2.2]).

Example 1 Every *-homomorphism i is a positive operator because it maps a positive element xx^* to the positive element $i(x)i(x^*)$. Since $i \otimes \mathrm{id}$ is a *-homomorphism, every *-homomorphism is completely positive.

Example 2 Every positive functional in $\mathcal{A}^{+*}$ is a completely positive operator from $\mathcal{A}$ to $\mathbb{C}$ as every positive map out of or into a commutative C*-algebra is completely positive (cf. [Stø13, 1.2.5]).

Example 3 We consider a matrix $a \in M_{n\times m}$ with positive (but not necessarily strictly positive) entries. On the matrix algebra $\mathbb{C}^m \simeq C(\{1,\dots,m\})$, the linear function a maps positive functions to positive functions and is completely positive because $\mathbb{C}^m$ is commutative. This motivates that completely positive operators can be used as non-commutative generalization of the stochastic version of our transitions.

1.3.2 The Kraus Representation

Because of the central relevance of completely positive operators, we summarize the most fundamental facts about them.

Example Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $a \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ a bounded linear operator between them. Then $\mathrm{Ad}_a\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ defined by $x \mapsto a^*xa$ is a normal completely positive operator.

This example is so general that it can be used to derive a general representation of completely positive operators.

Theorem [Küm86, 1.1.3] *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}), \mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$ be von Neumann algebras represented on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. For a map $T\colon \mathcal{A} \to \mathcal{B}$ the following are equivalent:*

(a) *T is a normal completely positive operator.*

(b) *There exists a family $(a_i)_{i\in I}$ of linear operators $a_i \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that*

$$T(x) = \sum_{i\in I} a_i^* x a_i$$

for all $x \in \mathcal{A}$, where the sum converges in the strong operator topology.

Definition We call the family $(a_i)_{i\in I}$ in (b) a *Kraus representation* of T. We call any operator $a \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ a *Kraus operator* of T if it is an element of a Kraus representation of T.

Remark Kraus operators can be found in any specific representation of $\mathcal{A}$ and $\mathcal{B}$. For every T, there is a representation of $\mathcal{A}$ such that one Kraus operator suffices to implement T. This is called the *Stinespring representation* (see [Bla06, II.6.9.7]) and is a generalization of the Gelfand-Naimark-Segal representation for positive functionals.

1.3.3 The Kadison-Schwarz Inequality

The following inequalities are essential for working with completely positive operators.

Theorem [Pau02, Ex. 3.4] *Let T be 2-positive on a C^*-algebra. Then*

$$T(x^*)T(x) \leq \|T(\mathbb{1})\|\, T(x^*x)$$

and

$$\left\|T(x^*y)\right\|^2 \leq \left\|T(x^*x)\right\| \left\|T(y^*y)\right\|.$$

1.3.4 The Support of Positive Elements

With completely positive operators, we can now define reach maps. For this, we quickly remind ourselves of the following well-known definition.

Definition Let $\mathcal{M}$ be a von Neumann algebra. For any positive element $x \in \mathcal{M}^+$ the *support (projection)* of x, denoted by $[x]$, is the smallest orthogonal projection $p \in \mathcal{P}(\mathcal{M})$ satisfying $pxp = x$.

Remark [Bla06, I.5.2.1] If we represent $\mathcal{M}$ on some Hilbert space $\mathcal{H}$, the support $[x]$ is the orthogonal projection onto the closed range of x in $\mathcal{A}$.

1.3.5 Reach Maps in Finite Dimensions

Definition Let $\mathcal{A}$ and $\mathcal{B}$ be matrix algebras. We call a map $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ a *reach map* if there is a completely positive operator $T\colon \mathcal{A} \to \mathcal{B}$ such that

$$R(p) = [T(p)].$$

Example For a commutative matrix algebra $\mathcal{A} = C(\Omega)$ with a finite Ω, the reach map $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ indeed corresponds to the classical function $g\colon P(\Omega) \to P(\Omega)$ that we discussed in 1.1.3: Let $a \in M_n$ be a stochastic matrix. Then, on the one hand, we have

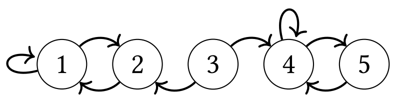
$$R(\chi_{\{j\}})(i) = [a\chi_{\{j\}}](i) = [ae_j](i) = \begin{cases} 1 & \text{if } a_{ij} > 0 \\ 0 & \text{if } a_{ij} = 0 \end{cases}$$

and on the other hand

$$R(\chi_{\{j\}})(i) = \chi_{\{\omega\in\Omega: g(\{\omega\})\cap\{j\}\neq\emptyset\}}(i) = \chi_{\{\omega\in\Omega: j\in g(\{\omega\})\}}(i) = \begin{cases} 1 & \text{if } i \in g(\{j\}) \\ 0 & \text{if } i \notin g(\{j\}) \end{cases}.$$

This gives g exactly the meaning that we intended for it. Reach maps are, therefore, a non-commutative generalization of 0-1-matrices.

Let us consider the specific matrix a from our example in 1.1.2:



$$a = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

We interpret our orthogonal projections as observables since we defined our reach map in the Heisenberg picture. However, the graph displayed in 1.1.2 maps states to states. Thus, to calculate R we follow the arrows backwards and get for example $R(\chi_{\{2\}}) = \chi_{\{1,3\}}$, $R(\chi_{\{1,3\}}) = \chi_{\{1,2,3\}}$ and $R(\chi_{\{1,2,3\}}) = \chi_{\{1,2,3\}}$.

Remark In this introduction, we switched from the Schrödinger to the Heisenberg picture. It is worth pointing out that completely positive operators can also be used in both pictures. Indeed, in physics, using completely positive operators operating on density matrices of states is quite common. In that situation, the completely positive operator in the Schrödinger picture (which is often required to preserve the trace of a matrix) can be seen as an adjoint operator to the operator in the Heisenberg picture (which is equivalently often required to be the identity on $\mathbb{1}$). Both operators contain the same information about the dynamics. In both pictures, a completely positive operator gives rise to a reach map, also hinting at an adjointness relation between reach maps given via $R(p) \perp q \Leftrightarrow p \perp R_*(q)$. While this could be done, we focus solely on the Heisenberg picture for the rest of this work.

In this chapter, we introduced our operator-algebraic setting and completely positive operators. We motivated reach maps via graphs in commutative examples and gave their definition for finite-dimensional systems.

2 Enveloped C*-Algebras and Open Projections

Our journey will lead us through the borderlands between the stochastic and topological realms. In the context of operator algebras, it is well-known that C*-algebras correspond to topological theory, while von Neumann algebras correspond to stochastic theory. Our challenge is that we want to build a topological theory, so applying it to arbitrary C*-algebras should be possible. However, supports are noncontinuous and generally only exist in von Neumann algebras.

In this chapter, we will talk about hereditary C*-subalgebras and open projections, two closely related ways to talk about the supports of positive elements in C*-algebras. Hereditary C*-algebras and open projections are well-known concepts in the literature. However, we will generalize the definition of open projections to work in a von Neumann algebra, which is not the universally enveloping von Neumann algebra of a C*-algebra. For this, we introduce the new definition of an enveloped C*-algebra.

2.1 Enveloped C*-Algebras

This section introduces the category of enveloped C*-algebras, morally C*-algebras with a weak* topology.

2.1.1 Weak Topologies of von Neumann Algebras

We will use the term *von Neumann algebra* for concrete and abstract von Neumann algebras. As a particular case of this rule, we call a weak* closed subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ a von Neumann algebra even if the unit of $\mathcal{M}$ is not the identity on $\mathcal{H}$, as for example in [Ped18, 2.2.6]. In a concrete von Neumann algebra, we can use, additionally to the norm topology, the weak, strong, σ-weak and σ-strong operator topologies (see [Bla06, I.3.1]). Thus, whenever we use, for example, the strong operator topology, we assume that our von Neumann algebra is represented on a Hilbert space. We are, however, trying to stay independent of a specific representation, so we will prefer to use the weak* topology, which is the topology induced by the predual of an abstract von Neumann algebra. It is equivalent to the σ-weak topology whenever we choose a representation (cf. [Tak79, III.3.5]).

2.1.2 Enveloped C*-Algebras

Definition An *enveloped C*-algebra* is a pair $\mathcal{A}^\diamond := (\mathcal{A}, \mathcal{M}_\mathcal{A})$ of a C*-algebra $\mathcal{A}$ and an abstract von Neumann algebra $\mathcal{M}_\mathcal{A}$ where $\mathcal{A}$ is a weak* dense C*-subalgebra of $\mathcal{M}_\mathcal{A}$.

Remark We freely use all the vocabulary implied by this definition. For example $\mathcal{M}_\mathcal{A}$ is enveloping $\mathcal{A}$, and we may call $\mathcal{M}_\mathcal{A}$ an envelope of $\mathcal{A}$. Some literature uses the term *enveloping von Neumann algebra* for the weak* closure in the universal representation. As defined, we use it instead for the weak* closure in any representation of our choice and reserve the term *universal enveloping von Neumann algebra* for the weak* closure $\overline{\mathcal{A}}^{\sigma^*} = \mathcal{A}'' \simeq \mathcal{A}^{**}$ in the universal representation.

Of course, if a given C*-algebra $\mathcal{A}$ is not finite-dimensional, the enveloping von Neumann algebra is not unique up to equivalence and different envelopes can induce different weak* topologies on $\mathcal{A}$.

Definition A representation $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$ is a *normal representation* of $\mathcal{A}^\diamond$ if it extends to a normal *-homomorphism from $\mathcal{M}_\mathcal{A}$ to the weak* closure $\overline{\pi(\mathcal{A})}^{\sigma^*}$ of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$.

Remark Because every von Neumann algebra can be represented on a Hilbert space, every enveloped C*-algebra has a normal faithful representation. In any normal representation, $\mathcal{A}$ is also dense in $\mathcal{M}_\mathcal{A}$ in the weak and strong operator topologies, by the bicommutant theorem (cf., e.g. [Bla06, I.9.1] or [Ped18, 2.2.4]). Every faithful representation $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of a C*-algebra $\mathcal{A}$ on some Hilbert space $\mathcal{H}$ leads us to an enveloping von Neumann algebra $\mathcal{M}_\mathcal{A} := \overline{\pi(\mathcal{A})}^{\sigma^*}$.

Definition Two representations of a C*-algebra $\mathcal{A}$ are *quasi-equivalent* if there is a *-isomorphism between the weak* closures that leaves $\mathcal{A}$ fixed (cf. [Tak79, III.2.10]).

Remark That means choosing an envelope for $\mathcal{A}$ is the same as choosing a faithful representation of $\mathcal{A}$ up to quasi-equivalence.

Example 1 As noted above, the universal enveloping von Neumann algebra is the motivating example for naming this definition. For every C*-algebra $\mathcal{A}$ in its universal representation, the universal enveloping von Neumann algebra $\mathcal{A}'' = \mathcal{A}^{**}$ makes $(\mathcal{A}, \mathcal{A}^{**})$ an enveloped C*-algebra.

Example 2 For every von Neumann algebra $\mathcal{M}$, $(\mathcal{M}, \mathcal{M})$ is an enveloped C*-algebra. We say $\mathcal{M}^\diamond$ is *self-enveloped.* This construction means that most theorems about enveloped C*-algebras are generalizations of theorems for von Neumann algebras and can always be specialized to them.

Example 3 If $\mathcal{A}$ is finite-dimensional, $(\mathcal{A}, \mathcal{A})$ is an enveloped C*-algebra. This is a special case of both examples 1 and 2.

Example 4 The compact operators on a Hilbert space are a C*-algebra, and they are weak* dense in the von Neumann algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$. Thus, $\mathcal{K}(\mathcal{H})^\diamond = (\mathcal{K}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ is an enveloped C*-algebra. Actually, because $\mathcal{K}(\mathcal{H})^* \simeq \mathcal{T}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})^* \simeq \mathcal{B}(\mathcal{H})$ (cf. [Bla06, I.8.6.1]), the envelope $\mathcal{B}(\mathcal{H})$ is isomorphic to the universal envelope of $\mathcal{K}(\mathcal{H})$. So, this is another special case of example 1.

Example 5 Let Ω be a locally compact Hausdorff space. Then the continuous functions vanishing at infinity are weak* dense in the bounded functions $l^\infty(\Omega)$. So $(C_0(\Omega), l^\infty(\Omega))$ is an enveloped C*-algebra.

Proof To see that $C_0(\Omega)$ is indeed dense in $l^\infty(\Omega)$, which is similar to a claim in [GK71, §3], we take any bounded function $f \in l^\infty(\Omega)$ with $\|f\|_\infty = 1$. We define a net Λ by taking all finite subsets $\lambda \subseteq \Omega$. For f_λ we choose a continuous function vanishing at infinity such that for all $\omega \in \lambda$ $f(\omega) = f_\lambda(\omega)$ and $\|f_\lambda\|_\infty \le \|f\|_\infty = 1$. Because Ω is as a locally compact Hausdorff space completely regular, such a function exists. Let $\rho \in l^1(\Omega)$ with $\|\rho\|_1 = 1$ and $\varepsilon > 0$. With $\sum_{\omega\in\Omega} |\rho(\omega)| = 1$ we can find a finite subset $\lambda \subseteq \Omega$ such that $\sum_{\omega\in\lambda} |\rho(\omega)| \ge 1 - \frac{\varepsilon}{2}$. Now for all $\mu \in \Lambda$ with $\lambda < \mu$ we have $|\sum_{\omega\in\Omega}(\rho(f - f_\lambda))(\omega)| \le \sum_{\omega\in\Omega} |\rho(\omega)|\,|(f - f_\lambda)(\omega)| = \sum_{\omega\in\lambda} |\rho(\omega)|\,|(f - f_\lambda)(\omega)| + \sum_{\omega\in\Omega\setminus\lambda} |\rho(\omega)|\,|(f - f_\lambda)(\omega)| \le 0 + \|f - f_\lambda\|_\infty \sum_{\omega\in\Omega\setminus\lambda} |\rho(\omega)| \le 2\frac{\varepsilon}{2} = \varepsilon$. Thus, $f_\lambda \to f$ in the weak* topology of $l^\infty(\Omega)$. □

Remark The definition of an "enveloped C*-algebra" is not an entirely novel concept. We introduce it as a representation theory of C*-algebras up to quasi-equivalence.

2.1.3 Orthogonal Positive Elements

We introduced enveloped C*-algebras because we were missing supports in C*-algebras. In the context of an enveloped C*-algebra $\mathcal{A}^\diamond$ for any $x \in \mathcal{A}^+$, we have $[x] \in \mathcal{M}_\mathcal{A}$. We might wonder how arbitrary the choice of the envelope is. Maybe supports in an enveloped C*-algebra are just an entirely additional structure not giving us information about the C*-algebra. The following lemma should give us hope that this is not the case. It might seem elementary, but we emphasize it here because it will be used often in this thesis.

Lemma *Let $\mathcal{A}^\diamond$ be an enveloped C*-algebra. For two positive elements $x, y \in \mathcal{A}^+$ the following are equivalent:*

(a) $[x][y] = 0$
(b) $yxy = 0$
(c) $xy = 0$

Proof (a) $\Rightarrow$ (b) With $[x][y] = 0$ we calculate $yxy = yx[x][y]y = yx0y = 0$.

(b) $\Rightarrow$ (c) If $0 = \|yxy\| = \left\|\sqrt{x}y\right\|^2$ we get $xy = \sqrt{x}\sqrt{x}y = \sqrt{x}0 = 0$.

(c) $\Rightarrow$ (a) If $xy = 0$ then [Bla06, I.5.2.1] says that $[x]y = 0$. Applying that argument a second time, we arrive at $[x][y] = 0$. □

Definition If those conditions are true, we call x and y *orthogonal* and write $x \perp y$.

Remark This definition coincides with the well-known definition for orthogonal projections and is also common in the literature (cf., e.g. [Bla06, II.3.1.13]). The most valuable property of this equivalence is that (b) and (c) are just equations in the C*-algebra $\mathcal{A}$. Thus, the property for two elements to have orthogonal supports is independent of the envelope. We can apply this theorem to all positive elements of a von Neumann algebra by self-enveloping it like in example 2.1.2 2.

2.1.4 Enveloped Subalgebras

Next, we look at subalgebras of enveloped C*-algebras and their units.

Remark Let $\mathcal{A}^\diamond$ be an enveloped C*-algebra and $\mathcal{B}$ a C*-subalgebra of $\mathcal{A}$, then $\mathcal{B}$ and its weak* closure $\mathcal{M}_\mathcal{B} := \overline{\mathcal{B}}^{\sigma^*}$ in $\mathcal{M}_\mathcal{A}$ form an enveloped C*-algebra $\mathcal{B}^\diamond := (\mathcal{B}, \mathcal{M}_\mathcal{B})$, where $\mathcal{M}_\mathcal{B}$ is a von Neumann subalgebra of $\mathcal{M}_\mathcal{A}$.

Definition We call $\mathcal{B}^\diamond$ a *subalgebra* of $\mathcal{A}^\diamond$.

2.1.5 Units of Subalgebras

A subalgebra $\mathcal{B}^\diamond \subseteq \mathcal{A}^\diamond$ will generally not have a unit $\mathbb{1}_\mathcal{B} \in \mathcal{B}$. Nevertheless, $\mathcal{M}_\mathcal{B}$ as a von Neumann algebra has a unit. Let us see how this unit relates to the C*-algebra $\mathcal{B}$.

Definition An approximate unit of a C*-algebra $\mathcal{A}$ is a net $(u_i)_{i\in I}$ in $\mathcal{A}_1^+$ so that $i < j$ implies $u_i \leq u_j$ and $\lim_{i\in I} u_i x = x$ for all $x \in \mathcal{A}$.

Remark Like in [Ped18, 1.4.1], we include the condition that an approximate unit has to consist of monotonically increasing positive elements in our definition.

For any subalgebra $\mathcal{B}^\diamond$ of an enveloped C*-algebra, we write $\mathbb{1}_\mathcal{B}$ for the unit of $\mathcal{M}_\mathcal{B}$. It should not be confused to mean that the C*-algebra $\mathcal{B}$ has a unit, but whenever $\mathcal{B}$ has a unit, it will be $\mathbb{1}_\mathcal{B}$.

Lemma *Let $\mathcal{B}^\diamond$ be a subalgebra of an enveloped C*-algebra $\mathcal{A}^\diamond$.*

(1) *All approximate units of $\mathcal{B}$ converge strongly to $\mathbb{1}_\mathcal{B} \in \mathcal{M}_\mathcal{B}$.*
(2) $\mathbb{1}_\mathcal{B} = \sup \mathcal{B}_1^+$.

Proof (1) Let $(u_i)_{i\in I}$ be an approximate unit of $\mathcal{B}$. Then by monotone convergence (cf. [Bla06, I.3.2.5]) the net $(u_i)_{i\in I}$ has a strong limit $u \in \mathcal{M}_\mathcal{B}$. By one-sided continuity of multiplication, we have $ux = xu = x$ for all $x \in \mathcal{B}$. Because $\mathcal{B}$ is strongly dense in $\mathcal{M}_\mathcal{B}$, we can use one-sided continuity of multiplication again to achieve $ux = xu = x$ for all $x \in \mathcal{M}_\mathcal{B}$. For $x = \mathbb{1}_\mathcal{B}$ we get $u = \mathbb{1}_\mathcal{B} u = \mathbb{1}_\mathcal{B}$.

(2) By a standard proof for the existence of approximate units, as, for example, given in [Ped18, 1.4.2], the open unit ball $\mathcal{B}^+_{<1}$ is an approximate unit of $\mathcal{B}$. The set $\mathcal{B}^+_{<1}$ has the supremum $\mathbb{1}_\mathcal{B}$. All elements of the closed unit ball $\mathcal{B}^+_1$ are also dominated by $\mathbb{1}_\mathcal{B}$, thus, $\mathcal{B}^+_1$ which contains $\mathcal{B}^+_{<1}$ has the same supremum. □

2.1.6 Morphisms Between Enveloped C*-Algebras

As morphisms in our category, we want completely positive operators compatible with the structure of the enveloped C*-algebra. For this, we choose completely positive operators between C*-algebras, that have a normal extension onto the weak* closures. We proceed to make this more rigorous:

Lemma *Let $\mathcal{A}^\diamond$ and $\mathcal{B}^\diamond$ be enveloped C*-algebras and $T\colon \mathcal{M}_\mathcal{A} \to \mathcal{M}_\mathcal{B}$ a weak* continuous map. If $T|_\mathcal{A}$ is a completely positive operator from $\mathcal{A}$ to $\mathcal{B}$, then*

(1) *T is a normal completely positive operator uniquely determined by $T|_\mathcal{A}$,*
(2) $\|T\| = \|T|_\mathcal{A}\|$,
(3) *the following are equivalent:*
  (a) *T is a *-homomorphism*
  (b) *$T|_\mathcal{A}$ is a *-homomorphism*

Proof (1) $T|_\mathcal{A}$ uniquely determines T because $\mathcal{A}$ is weak* dense in $\mathcal{M}_\mathcal{A}$. By weak* continuity of addition and scalar multiplication, T is linear. By the theorem of Kaplansky (cf. [Ped18, 2.3.3]) we know that any $x \in (\mathcal{M}_\mathcal{A} \otimes M_k)^+$ can be weak* approximated by a bounded net in $(\mathcal{A} \otimes M_k)^+_{\|x\|}$. Since T is completely positive, we have $(T \otimes \mathrm{id})((\mathcal{A} \otimes M_k)^+) \subseteq (\mathcal{A} \otimes M_k)^+$. Because M_k is finite-dimensional, $T \otimes \mathrm{id}$ is weak* continuous, and we get $(T \otimes \mathrm{id})((\mathcal{M}_\mathcal{A} \otimes M_k)^+) \subseteq (\mathcal{M}_\mathcal{A} \otimes M_k)^+$ for all $k \in \mathbb{N}$ and T is completely positive.

(2) By the definition of the operator norm, it is clear that $\|T|_\mathcal{A}\| \le \|T\|$ and we rescale T so that $\|T\| = \|T(\mathbb{1})\| = 1$. We take an approximate unit $(u_i)_{i\in I}$ of $\mathcal{A}$ and choose a normal representation of $\mathcal{B}^\diamond$ on some Hilbert space $\mathcal{H}$. T is σ-strongly continuous (see [Bla06, III.2.2.2]), and because it is bounded, it is strongly continuous on $\mathcal{A}_1$ (see [Bla06, I.3.1.4]). $T(u_i)$ converges strongly to $T(\mathbb{1})$. So for all $\xi \in \mathcal{H}$ and $\varepsilon > 0$ there is an $i \in I$ such that $\|T(\mathbb{1})\xi - T(u_i)\xi\| < \frac{\varepsilon}{2}$. Because $\|T(\mathbb{1})\| = 1$, there is for every $\varepsilon > 0$ a $\xi \in \mathcal{H}_1$ such that $\|T(\mathbb{1})\xi\| > 1 - \frac{\varepsilon}{2}$. For that $\xi \in \mathcal{H}_1$ and an appropriate $i \in I$ we

get $\|T(u_i)\xi\| = \|T(u_i - \mathbb{1})\xi + T(\mathbb{1})\xi\| \geq \big|\|T(u_i)\xi - T(\mathbb{1})\xi\| - \|T(\mathbb{1})\xi\|\big| \geq \left|(1 - \frac{\varepsilon}{2}) - \frac{\varepsilon}{2}\right| = |1 - \varepsilon|$. Thus $\|T(u_i)\| \longrightarrow 1$ and we get $1 \leq \|T|_{\mathcal{A}}\| = \|T\|$.

(3) If T is a *-homomorphism, $T|_{\mathcal{A}}$ is too. If $T|_{\mathcal{A}}$ is a *-homomorphism, we consider $x, y \in \mathcal{M}_{\mathcal{A}}$ with nets $(x_j)_{j\in\mathcal{J}}, (y_k)_{k\in K} \subseteq \mathcal{A}$ with $x_j \longrightarrow x$ and $y_k \longrightarrow y$ in the weak* topology. Then we can calculate $T(xy) = T((\lim_{j\in\mathcal{J}} x_j)y) = T(\lim_{j\in\mathcal{J}} x_j y) = \lim_{j\in\mathcal{J}} T(x_j y) = \lim_{j\in\mathcal{J}} T(x_j(\lim_{k\in K} y_k)) = \lim_{j\in\mathcal{J}}(\lim_{k\in K} T(x_j y_k)) = \lim_{j\in\mathcal{J}} \lim_{k\in K} T(x_j)T(y_k) = \lim_{j\in\mathcal{J}} T(x_k)T(y) = T(x)T(y)$ by one-sided weak* continuity of multiplication and weak* continuity of T. □

Definition Let $\mathcal{A}^\diamond$ and $\mathcal{B}^\diamond$ be enveloped C*-algebras.

(1) We call $T\colon \mathcal{A}^\diamond \longrightarrow \mathcal{B}^\diamond$ a *normal completely positive operator*, if T is a normal completely positive operator from $\mathcal{M}_{\mathcal{A}}$ to $\mathcal{M}_{\mathcal{B}}$ with $T(\mathcal{A}) \subseteq \mathcal{B}$.
(2) We define the *category of enveloped C*-algebras*, with enveloped C*-algebras as objects and normal completely positive operators between them as morphisms.
(3) We call such a morphism a **-homomorphism* if it fulfils the conditions of the statement (3) in the above lemma.
(4) We call it a **-isomorphism* if $T\colon \mathcal{M}_{\mathcal{A}} \longrightarrow \mathcal{M}_{\mathcal{B}}$ and $T|_{\mathcal{A}}\colon \mathcal{A} \longrightarrow \mathcal{B}$ are *-isomorphisms.
(5) A **-automorphism* is (as usual) a *-isomorphism from $\mathcal{A}^\diamond$ to $\mathcal{A}^\diamond$.

2.1.7 Isomorphic Envelopes

We call two enveloped C*-algebras $\mathcal{A}^\diamond$ and $\mathcal{B}^\diamond$ isomorphic or equivalent and write $\mathcal{A}^\diamond \simeq \mathcal{B}^\diamond$ if there is a *-isomorphism between them. Two different representations of a C*-algebra $\mathcal{A}$ are quasi-equivalent if and only if their induced enveloped C*-algebras are isomorphic. Then, in the spirit of a subrepresentation, we can define a subenvelope.

Proposition *If von Neumann algebras $\mathcal{N}$ and $\mathcal{M}$ are envelopes of a C*-algebra $\mathcal{A}$ then the following are equivalent:*

(a) *There is a central orthogonal projection $z \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ such that $(z\mathcal{A}, z\mathcal{M}) \simeq (\mathcal{A}, \mathcal{N})$.*
(b) *There is a *-homomorphism $\pi\colon (\mathcal{A}, \mathcal{M}) \longrightarrow (\mathcal{A}, \mathcal{N})$ that is an extension of the identity on $\mathcal{A}$ and surjective from $\mathcal{M}$ onto $\mathcal{N}$.*

Proof (a) $\Rightarrow$ (b) is clear, because up to isomorphism $x \mapsto zx$ is exactly the desired surjective *-homomorphism.

(b) $\Rightarrow$ (a) Let $z_{\mathcal{N}}, z_{\mathcal{M}} \in \mathcal{P}(\mathcal{Z}(\mathcal{A}^{**}))$ such that $(\mathcal{A}, \mathcal{N}) = (z_{\mathcal{N}}\mathcal{A}, z_{\mathcal{N}}\mathcal{A}^{**})$ and $(\mathcal{A}, \mathcal{M}) = (z_{\mathcal{M}}\mathcal{A}, z_{\mathcal{M}}\mathcal{A}^{**})$. Then by [Bla06, III.5.1.3] we have $z_{\mathcal{N}} \leq z_{\mathcal{M}}$. □

Definition In this case, we call $\mathcal{N}$ a *subenvelope* of $\mathcal{M}$.

Remark In the previous proof we used that every enveloped C*-algebra $\mathcal{A}^\diamond$ is isomorphic to $(z\mathcal{A}, z\mathcal{A}^{**})$ for some central projection $z \in \mathcal{P}(\mathcal{Z}(\mathcal{A}^{**}))$ in the universal enveloping von Neumann algebra $\mathcal{A}^{**}$. In other words, every envelope of $\mathcal{A}$ is a subenvelope of the universal envelope. There is a one-to-one correspondence between envelopes of $\mathcal{A}$ and such $z \in \mathcal{P}(\mathcal{Z}(\mathcal{A}^{**}))$ for which $x \mapsto zx$ is a faithful representation (cf. [Ped18, 3.8.2]).

2.1.8 Interplay of Isomorphisms Between an Algebra and its Envelope

We have defined an isomorphism $i\colon \mathcal{M}_\mathcal{A} \to \mathcal{M}_\mathcal{B}$ by requiring that $i|_\mathcal{A}\colon \mathcal{A} \to \mathcal{B}$ and $i\colon \mathcal{M}_\mathcal{A} \to \mathcal{M}_\mathcal{B}$ are *-isomorphisms. It turns out that both conditions do not imply each other:

Example 1 Let $\mathcal{A}^\diamond$ be any enveloped C*-algebra with $\mathcal{A} \neq \mathcal{M}_\mathcal{A}$. Then the identity map $\mathrm{id}\colon \mathcal{A}^\diamond \to (\mathcal{M}_\mathcal{A}, \mathcal{M}_\mathcal{A})$ is a morphism in the category of enveloped C*-algebras. The same map on the envelope $\mathrm{id}\colon \mathcal{M}_\mathcal{A} \to \mathcal{M}_\mathcal{A}$ is a *-isomorphism, but the restriction to the C*-algebras $\mathrm{id}\colon \mathcal{A} \to \mathcal{M}_\mathcal{A}$ is not. Hence, a *-isomorphism on the weak* closure does not need to be a *-isomorphism on the C*-algebra.

Example 2 Whenever a C*-algebra $\mathcal{A}$ has an envelope $\mathcal{M}_\mathcal{A}$ so that $(\mathcal{A}, \mathcal{M}_\mathcal{A})$ is not equivalent to $(\mathcal{A}, \mathcal{A}^{**})$ we have a surjective map $\pi\colon \mathcal{A}^{**} \to \mathcal{M}_\mathcal{A}$ that is a *-isomorphism on $\mathcal{A}$ but not from $\mathcal{A}^{**} \to \mathcal{M}_\mathcal{A}$. A concrete example is $(C([0,1]), l^\infty([0,1]))$ (Example 5 in 2.1.2). To see that $l^\infty([0,1])$ is not equivalent to the universal envelope, we consider that every functional of $C([0,1])$ is an element of the predual of $C([0,1])^{**}$. The predual of $l^\infty([0,1])$ is $l^1([0,1])$, which, for example, cannot contain a density function for the state induced by the Lebesgue measure.

2.1.9 Functorial Relations

There are two fundamental examples of how to find morphisms between enveloped C*-algebras.

Example 1 Between universally enveloped C*-algebras $(\mathcal{A}, \mathcal{A}^{**})$ and $(\mathcal{B}, \mathcal{B}^{**})$ all completely positive operators $T : \mathcal{A} \to \mathcal{B}$ have a normal extension $T^{**}\colon \mathcal{A}^{**} \to \mathcal{B}^{**}$. It is given by $T^{**}(x)(\varphi) = x(\varphi \circ T)$ for $x \in \mathcal{A}^{**}$ and all $\varphi \in \mathcal{A}^*$. T^{**} is normal because if $x_i(\varphi) \to x(\varphi)$ for all $\varphi \in \mathcal{A}^*$, then $x_i(\varphi \circ T) \to x(\varphi \circ T)$.

Example 2 Between self-enveloped von Neumann algebras, all normal completely positive operators on the von Neumann algebras are morphisms in the category of enveloped C*-algebras.

Remark With these two examples, we have seen the categorical background for our considerations given by the following commuting diagram of functors. We have the category of matrix algebras (MA) contained in the category of C*-algebras (C*) and the category of von Neumann algebras (vNA), and our new category of enveloped C*-algebras (env-C*). Our morphisms are completely positive operators (CPO), which must be normal (σ^*-CPO) if a von Neumann algebra is involved.

$$\begin{tikzcd}
& (\mathrm{MA}, \mathrm{CPO}) \arrow[dl, hook, "\mathrm{id}"'] \arrow[dr, hook, "\mathrm{id}"] & \\
(\mathrm{C}^*, \mathrm{CPO}) \arrow[dr, "\text{universal envelope}"'] & & (\mathrm{vNA}, \sigma^*\text{-}\mathrm{CPO}) \arrow[dl, "\text{self-envelope}"] \\
& (\text{env-}\mathrm{C}^*, \sigma^*\text{-}\mathrm{CPO}) &
\end{tikzcd}$$

2.2 Hereditary Subalgebras

During our discussion in 1.2.1, we established the correspondence between orthogonal projections and subsets of a state space. This correspondence can lose its footing when we go to infinite-dimensional spaces. Consider, for example, the algebra $C([0,1])$ of continuous functions on the interval [0,1]. It contains no non-trivial orthogonal projections and thus cannot have support projections for most positive functions. This is a problem for us because a reach map $R\colon \mathcal{P}(C([0,1])) \to \mathcal{P}(C([0,1]))$ can only be very trivial.

An established method to keep up this analogy are hereditary subalgebras, which are a decent stand-in for orthogonal projections in a lot of contexts. They are, though, just one perspective onto a collection of closely related concepts as there are in a C*-algebra $\mathcal{A}$ correspondences between

(a) hereditary subalgebras,
(b) open projections (in the universal envelope),
(c) closed left (or right) ideals,
(d) faces in the cone of positive functionals $(\mathcal{A}^*)^+$,
(e) faces in the set of states $\mathcal{S}(\mathcal{A})$.

We will, however, only explore some of those correspondences and restrict ourselves mainly to the relation of (a) and (b) with a little bit of (c). More details can be found in [Ped18, 1.5, 3.10, 3.11].

2.2.1 Hereditary Subalgebras

Definition

A C*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is called *hereditary* if for every $b \in \mathcal{B}^+$ and $a \in \mathcal{A}^+$ $a \leq b$ implies $a \in \mathcal{B}$.

Proposition [Ped18, 1.5.2]

Let $\mathcal{A}$ be a C-algebra.*

(1) *For any hereditary C*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$, the set $\{ x \in \mathcal{A} : x^*x \in \mathcal{B} \}$ is a closed left ideal.*
(2) *For any closed left ideal $I \subseteq \mathcal{A}$, $I \cap I^*$ is a hereditary C*-subalgebra.*
(3) *These two maps are one-to-one and inverses of each other.*

Remark [Ped18, 2.5.4]

In any von Neumann algebra $\mathcal{M}$, every orthogonal projection $p \in \mathcal{P}(\mathcal{M})$ is the unit of the hereditary von Neumann subalgebra $p\mathcal{M}p$. All hereditary von Neumann subalgebras are of this form, but there can be more hereditary C*-subalgebras of $\mathcal{M}$, which are not weak* closed. So, in von Neumann algebras, where there are enough orthogonal projections, we see a correspondence between projections and hereditary subalgebras.

Example 1

For every C*-algebra $\mathcal{A}$, trivially $\mathcal{A}$ and $\{ 0 \}$ are hereditary.

Example 2

In a commutative C*-algebra $C_0(\Omega)$ (with Ω locally compact and Hausdorff), every left ideal is a two-sided ideal. Thus, the hereditary C*-subalgebras are exactly the closed ideals. A closed ideal in $C_0(\Omega)$ is of the form $\{ f \in C_0(\Omega) : f(x) = 0 \text{ for all } x \in \Omega \setminus X \}$ for some open set $X \subseteq \Omega$. Hence, the hereditary C*-subalgebras correspond to the open sets of Ω.

2.2.2 Infima from Suprema

In the theory of von Neumann algebras, it is well-known that the orthogonal projections form a *complete lattice*. This means that they have a partial (i.e. generally non-total) order for which arbitrary sets have a supremum, i.e. a smallest upper bound, denoted by $\vee$ and an infimum, i.e. a largest lower bound, denoted by $\wedge$, which are elements of the lattice. As we will see, the hereditary C*-subalgebras and the open projections are also lattices. To show this, we will need the following general result from the theory of lattices.

Theorem [Bly05, Thm. 2.11]

If an ordered set L has suprema for infinite sets and a minimum, then it also has infima for infinite sets, and they are given by

$$\bigwedge_{i \in I} x_i = \bigvee \{ x \in L : x \leq x_i \text{ for all } i \in I \} \quad \text{for all} \quad (x_i)_{i \in I} \subseteq L.$$

Thus, L is a complete lattice. The dual result (switching infima and suprema, with a maximum) also holds.

REMARK This theorem formalizes the commonly used fact that the largest lower bound is the same as the supremum of all lower bounds. It can be used to describe a lot of situations where we define generated objects. For example, the closed subspaces of a Hilbert space form a lattice by inclusion. The span of a set of elements A in $\mathcal{H}$ is the intersection of all subspaces containing A. Which is the same as the supremum of the set of one-dimensional subspaces $\{\mathbb{C}\xi : \xi \in A\}$.

EXAMPLE This result can also lead to slightly unintuitive lattices. Consider, for example, a topological space $(\Omega, \mathcal{T})$. As a subset of the power set $P(\Omega)$, the topology $\mathcal{T}$ is ordered by inclusion. By definition, $\mathcal{T}$ is closed under unions and contains the minimum $\emptyset$. Thus, Theorem 2.2.2 tells us that $\mathcal{T}$ also has infima for arbitrary sets. However, topologies generally are not closed under the intersection of infinite sets. Instead, the lattice definition of the infimum for a family of open sets A is the largest open set contained in every open set in A, which is the interior of their intersections, but not necessarily their intersection itself.

2.2.3 The Lattice of Hereditary Subalgebras

PROPOSITION *If $\mathcal{A}$ is a C*-algebra, then the ordered set of hereditary C*-subalgebras of $\mathcal{A}$ form a complete lattice with the infimum given by intersection.*

PROOF The C*-subalgebras of $\mathcal{A}$ can be ordered by inclusion. The intersection of an arbitrary number of hereditary algebras is again a hereditary C*-subalgebra by straightforward calculation. Using $\mathcal{A}$ as the maximum, we apply Theorem 2.2.2. □

REMARK The supremum we receive by applying Theorem 2.2.2 is the smallest hereditary C*-subalgebra containing all the algebras we are calculating the supremum over. It is, however, not generally the smallest C*-subalgebra containing all of them because that subalgebra does not have to be hereditary.

2.2.4 Hereditary C*-subalgebras by Reduction

We have announced hereditary C*-subalgebras as a kind of replacement for orthogonal projections. To make this notion precise, we look at the relation between hereditary C*-subalgebras and projections in the envelope.

DEFINITION Let $\mathcal{A}^\diamond$ be an enveloped C*-algebra. We write $\mathcal{A}_p \coloneqq \mathcal{A} \cap p\mathcal{M}_{\mathcal{A}}p$ for any $p \in \mathcal{P}(\mathcal{M}_{\mathcal{A}})$.

REMARK In the case of a self-enveloped von Neumann algebra $(\mathcal{M}, \mathcal{M})$, the definition of $\mathcal{M}_p$ as $\mathcal{M} \cap p\mathcal{M}p = p\mathcal{M}p$ coincides with the usual definition of a reduced von Neumann algebra (cf. [Tak79, II. 3.11]). For the next lemma we remember from Definition 2.1.5 that $\mathbb{1}_{\mathcal{A}_p}$ is the unit of the envelope of the enveloped subalgebra $\mathcal{A}_p^\diamond$ and therefore an element of $\mathcal{M}_{\mathcal{A}_p}$ but not necessarily of $\mathcal{A}_p$.

Lemma *If $p \in \mathcal{P}(\mathcal{M}_{\mathcal{A}})$ is an orthogonal projection, then*

(1) $\mathcal{A}_p = \mathcal{A} \cap p\mathcal{A}p$,
(2) $\mathcal{A}_p$ *is a hereditary C*-subalgebra of* $\mathcal{A}$,
(3) $\mathbb{1}_{\mathcal{A}_p} \leq p$.

Proof (1) We show the equality by proving both inclusions. The inclusion $\mathcal{A} \cap p\mathcal{A}p \subseteq \mathcal{A}_p = \mathcal{A} \cap p\mathcal{M}_{\mathcal{A}}p$ follows from $p\mathcal{A}p \subseteq p\mathcal{M}_{\mathcal{A}}p$. On the other hand, for every $x \in \mathcal{A}_p$, we have $x = pxp$ and $x \in \mathcal{A}$, which means that $x \in \mathcal{A} \cap p\mathcal{A}p$.

(2) The reduced algebra $\mathcal{A}_p$ is the intersection of two C*-algebras and thus a C*-algebra. To see that it is hereditary, let $x \in \mathcal{A}$ and $y \in \mathcal{A}_p$ with $x \leq y$. We know that $x \leq y \perp p^{\perp}$, and so $x = pxp$ and $x \in \mathcal{A}_p$.

(3) From Lemma 2.1.5 we know that $\mathbb{1}_{\mathcal{A}_p} = \bigvee(\mathcal{A}_p)^+_1$. Since $(\mathcal{A}_p)^+_1 \subseteq \mathcal{M}_{\mathcal{A}}$ we can conclude $\mathbb{1}_{\mathcal{A}_p} \in p\mathcal{M}_{\mathcal{A}}p$. □

Remark Statement (3) leaves the possibility that the unit of $\mathcal{A}_p$ can be smaller than p. So, some projections can be characterized by their reduced hereditary C*-subalgebra, while others cannot. This leads us to the definition of open projections.

2.3 Open Projections

We will now introduce open projections because we need them as domains of reach maps.

2.3.1 Open Projections

Proposition *Let $\mathcal{A}^{\diamond}$ be an enveloped C*-algebra. For an orthogonal projection $p \in \mathcal{P}(\mathcal{M}_{\mathcal{A}})$ the following are equivalent:*

(a) $p = \mathbb{1}_{\mathcal{A}_p}$.
(b) $p = \mathbb{1}_{\mathcal{B}}$ *for a subalgebra* $\mathcal{B}^{\diamond} \subseteq \mathcal{A}^{\diamond}$.
(c) *There is a monotonically increasing net of positive elements* $(x_i)_{i \in I} \subseteq \mathcal{A}^+$ *with* $\sup_{i \in I} x_i = p$.

Proof (a) $\Rightarrow$ (b) This step is trivial because the reduced algebra $\mathcal{A}_p$ is a C*-subalgebra of $\mathcal{A}$ (cf. Lemma 2.2.4).

(b) $\Rightarrow$ (c) All approximate units of $\mathcal{B}$ are monotonically increasing nets converging to $\mathbb{1}_{\mathcal{B}}$ according to Lemma 2.1.5.

(c) $\Rightarrow$ (a) We know from Lemma 2.2.4 that $\mathbb{1}_{\mathcal{A}_p} \leq p$ and need to show equality. It follows from the inclusion $(x_i)_{i \in I} \subseteq (\mathcal{A}_p)^+_1$. □

Definition We call p an *open projection* of $\mathcal{A}^\diamond$ if the above conditions hold. With $\mathcal{T}(\mathcal{A}^\diamond) = \mathcal{T}(\mathcal{A}, \mathcal{M}_\mathcal{A})$ we denote the set of all open projections of $\mathcal{A}^\diamond$. We often write $\mathcal{T}(\mathcal{A}) := \mathcal{T}(\mathcal{A}^\diamond)$ if it is clear which envelope we are talking about. The orthogonal complement of an open projection is a *closed* projection. If a projection is closed and can be majorized by an element in $\mathcal{A}^+$, we call it *compact*.

Remark This definition of open projections is more general than usual in the literature. For example, in the original definition in [Ake69, II.1] or [Ped18, 3.11.10], only the universal envelope (or equivalently, the "atomic" envelope, see Lemma 3.2.3) is used for the definition of open projections. In that case, a one-to-one correspondence exists between open projections and hereditary C*-subalgebras (cf. Proposition 3.2.1). In other envelopes, different hereditary C*-subalgebras $\mathcal{B}_1, \mathcal{B}_2$ can lead to the same open projection with $\mathbb{1}_{\mathcal{B}_1} = \mathbb{1}_{\mathcal{B}_2}$. [AB15, 2.3] starts with our more general definition but quickly returns to the more special case in [AB15, 2.5].

The notation $\mathcal{T}(\mathcal{A})$ is chosen in analogy to the topology of a topological space. We will explore in the next chapter how well this analogy works.

Example 1 In a self-enveloped von Neumann algebra, every orthogonal projection is open.

Example 2 In every enveloped C*-algebra $\mathcal{A}^\diamond$, 0 and $\mathbb{1}_\mathcal{A}$ are open.

Remark We will see more examples of open projections (see Sections 3.2.5, 3.2.6), how they form a lattice (see Section 3.2.4) and further topological properties (see Section 3.4) in the next chapter.

2.3.2 Every support is an open projection

Our aim was a possibility to talk about the supports of elements in a C*-algebra. Open projections are precisely the right tool to do this:

Proposition *Let $\mathcal{A}^\diamond$ be an enveloped C*-algebra with $h \in \mathcal{A}_1^+$ and $\mathcal{B} := \overline{h\mathcal{A}h}^{\|\cdot\|}$.*

(i) *$\mathcal{B}$ is the smallest hereditary C*-subalgebra of $\mathcal{A}$ which contains h.*
(ii) *$(h^{\frac{1}{n}})_{n\in\mathbb{N}}$ is an approximate unit of $\mathcal{B}$.*
(iii) *$[h] = \mathbb{1}_\mathcal{B}$.*
(iv) *If $\sigma(h) \subseteq \{0\} \cup (\varepsilon, \infty)$, for some $\varepsilon > 0$, $\mathcal{B} = h\mathcal{A}h$ and $[h] \in \mathcal{B}$.*

Further for every separable C-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ the following are equivalent:*

(a) *$\mathcal{B}$ is hereditary.*
(b) *There is $h \in \mathcal{A}_1^+$ with $\mathcal{B} = \overline{h\mathcal{A}h}^{\|\cdot\|}$.*

All of the statements in this proposition are proven in the literature: The first four statements can be found in (i) [Mur90, 3.2.4], (ii) [Bla06, II.4.2.1], (iii) [Ped18, 3.10.5] and (iv) [Bla06, II.3.2.9 and II.3.2.11]. The final equivalence is given in [Mur90, 3.2.5 Theorem].

Remark For a certain subalgebra $\mathcal{B}$ there are many h with $\overline{h\mathcal{A}h}^{\|\cdot\|} = \mathcal{B}$. On the other hand, only if $\mathcal{A}$ is separable we know that every open projection is the support of an element in $\mathcal{A}^+$.

Since we will use reach maps to prove properties about open projections, we will postpone further investigation of open projections to the next chapter.

In this chapter, we introduced enveloped C*-algebras and their morphisms, a representation theory up to quasi-equivalence. Then we introduced substitutes for orthogonal projections in C*-algebras, primarily hereditary C*-subalgebras and open projections in the envelope.

3 Reach Maps

This chapter continues our investigation of reach maps and open projections. We will start with the general definition of reach maps and show that they preserve suprema. Then we discuss the lattice of open projections and its analogy to topology. Ultimately, we will be able to define the category of reach maps.

The most important results of this chapter are that reach maps preserve suprema (Theorem 3.1.3) and that our generalized definition of open projections forms a lattice as well (Proposition 3.2.4). Also noteworthy is the slightly intricate analysis of supports of completely positive operators in enveloped C*-algebras (Proposition 3.3.1) and the proof of a Hausdorff property for our generalized open projections (Proposition 3.4.3).

For this chapter, let $\mathcal{A}^\diamond$ and $\mathcal{B}^\diamond$ be enveloped C*-algebras and $T\colon \mathcal{A}^\diamond \to \mathcal{B}^\diamond$ a normal completely positive operator between them.

3.1 Reach Maps, Suprema and Infima

We have seen reach maps on matrix algebras in Definition 1.3.5. Now, we give the general definition on enveloped C*-algebras.

3.1.1 Reach Maps

Definition We define the *reach map* R_T of T as

$$R_T\colon \mathcal{T}(\mathcal{A}) \ni p \mapsto [T(p)] \in \mathcal{P}(\mathcal{M}_\mathcal{B})$$

Remark We will prove in Proposition 3.5.1 that, in fact, $R_T(p) \in \mathcal{T}(\mathcal{B})$ for all $p \in \mathcal{T}(\mathcal{A})$, thus a reach map maps open projections to open projections. Until we have completed that proof, we will only write $R_T(p) \in \mathcal{M}_\mathcal{B}$.

We explore reach maps of different completely positive operators throughout this whole work. We have looked at commutative finite-dimensional systems in Chapter 1 and will examine the non-commutative finite-dimensional case in Chapter 4. See Chapter 6 for reach maps of conditional expectations.

Example 1 One special case of a completely positive operator is a normal positive functional $\varphi\colon \mathcal{M} \to \mathbb{C}$. Then the reach map $R_\varphi\colon \mathcal{T}(\mathcal{M},\mathcal{M}) \to \mathcal{P}(\mathbb{C})$ is a function into the set $\{0,1\}$. We have $R_\varphi(p) = 1$ if and only if $\varphi(p) \neq 0$ and correspondingly $R_\varphi(p) = 0$ exactly if $\varphi(p) = 0$.

Example 2 If $i\colon \mathcal{A}^\diamond \to \mathcal{B}^\diamond$ is a *-homomorphism, then $R_i(p) = [i(p)] = i(p)$ for all $p \in \mathcal{T}(\mathcal{A})$. Therefore, we often write i for R_i. Also, if $p \perp q$ for $p, q \in \mathcal{T}(\mathcal{A})$ then $R_i(p) \perp R_i(q)$. We will discuss in 4.2.4 that reach maps preserving orthogonality are always induced by a *-homomorphism.

3.1.2 A Special Reach Map: The Support

One perspective on the support function $x \mapsto [x]$ is that it is the reach map of the identity operator $\mathrm{id}\colon \mathcal{A}^\diamond \to \mathcal{A}^\diamond$. Many calculations with reach maps come down to operating with supports. Thus, we collect a few basic properties of the support function. A note on notation: We use "$\bigvee$" for suprema in the lattice of orthogonal projections and "sup" for suprema of general positive elements when we know that they exist, most often for bounded monotonically increasing nets.

Lemma *If $x, y \in {\mathcal{M}_\mathcal{A}}^+$, then*

(1) $x \leq y \Rightarrow [x] \leq [y]$.
(2) *If $(x_i)_{i\in I} \subseteq {\mathcal{M}_\mathcal{A}}^+$ is a bounded, increasing net, then* $[\sup_{i\in I} x_i] = \sup_{i\in I}[x_i]$.
(3) $[\sum_{i\in\mathbb{N}} x_i] = \bigvee_{i\in\mathbb{N}}[x_i]$ *for all strongly convergent sums of $(x_i)_{i\in\mathbb{N}} \subseteq {\mathcal{M}_\mathcal{A}}^+$.*
(4) $[T(x)] = [T([x])]$ *for every normal completely positive operator $T\colon \mathcal{A}^\diamond \to \mathcal{B}^\diamond$.*
(5) *If $xy = yx$, then* $[x][y] = [y][x] = [yx] = [xy]$.

Proof (1) If $p := [y]^\perp \in \mathcal{M}_\mathcal{A}$, then $p \perp y$ and $pxp \leq pyp = 0$. So $p \perp [x]$ and thus $[x] \leq [y]$.

Before we prove statement (2), we show the auxiliary statement $T(x) = 0 \iff T([x]) = 0$, which is a special case of (4).
For every normal functional $\varphi \in \mathcal{S}_*(\mathcal{M}_\mathcal{A})$, $\psi := \varphi \circ T$ is normal. With $p := \operatorname{supp}\psi$ we get $\psi(x) = 0 \Leftrightarrow pxp = 0 \Leftrightarrow [x] \perp p \Leftrightarrow \psi([x]) = 0$. Since $\mathcal{S}_*(\mathcal{M}_\mathcal{A})$ is separating for $\mathcal{M}_\mathcal{A}$ we have shown the auxiliary statement $T(x) = 0 \Leftrightarrow T([x]) = 0$.

(2) For every $p \in \mathcal{P}(\mathcal{M}_\mathcal{A})$ we have

$$p \perp [\sup_{i\in I} x_i] \Leftrightarrow 0 = p(\sup_{i\in I} x_i)p = \sup_{i\in I} px_ip \Leftrightarrow \forall i \in I\ x_i \perp p$$
$$\Leftrightarrow 0 = \sup_{i\in I} p[x_i]p = p(\sup_{i\in I}[x_i])p \Leftrightarrow p \perp \sup_{i\in I}[x_i].$$

Which gives us $\sup x = \sup_{i\in I}[x_i]$.

(3) First, we show the equality for finite sums. Since $x \leq x+y$ and $y \leq x+y$ we have $[x] \leq [x+y]$ and $[y] \leq [x+y]$ and so $[x] \vee [y] \leq [x+y]$. Let $p := ([x] \vee [y])^\perp \in \mathcal{P}(\mathcal{M}_\mathcal{A})$ then $0 = pxp + pyp = p(x+y)p$ so $p \perp [x+y]$ and $[x+y] \leq [x] \vee [y]$.

Now, with the equality shown for finite sums, the claim for infinite sums follows directly from (2): $[\sup_{n\in\mathbb{N}} \sum_{i=1}^{n} x_i] = \sup_{n\in\mathbb{N}} [\sum_{i=1}^{n} x_i] = \sup_{n\in\mathbb{N}} \bigvee_{i=1}^{n} [x_i] = \bigvee_{i\in\mathbb{N}} [x_i]$.

(4) First, we consider any Kraus operator $a \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ of T where $\mathcal{M}_{\mathcal{A}} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{M}_{\mathcal{B}} \subseteq \mathcal{B}(\mathcal{K})$. Now for all $x' \in \mathcal{B}(\mathcal{H})^+, y' \in \mathcal{B}(\mathcal{K})^+$ we have $a^*x'a \perp y' \Rightarrow a^*x'ay' = 0 \Rightarrow ay'a^*x'ay'a^* = 0 \Rightarrow x' \perp ay'a^*$ and by symmetry of the argument we get $a^*x'a \perp y' \Leftrightarrow x' \perp ay'a^*$. Applying that to any $p \in \mathcal{P}(\mathcal{M}_{\mathcal{B}})$ we get $a^*xa \perp p \Leftrightarrow x \perp apa^* \Leftrightarrow [x] \perp apa^* \Leftrightarrow a^*[x]a \perp p$ and so $[a^*xa] = [a^*[x]a]$. Now for a Kraus representation $(a_i)_{i\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{K}, \mathcal{H})$ of T we can compute $[T(x)] = [\sum_{i=1}^{\infty} a_i^* x a_i] = \bigvee_{i\in\mathbb{N}} [a_i^* x a_i] = \bigvee_{i\in\mathbb{N}} [a_i^*[x]a_i] = [\sum_{i=1}^{\infty} a_i^*[x]a_i] = [T([x])]$.

(5) First, x commutes with $[y]$ because it commutes with everything in $\{y\}''$. With the same argument, $[y]$ also commutes with $[x]$. As a product of two commuting orthogonal projections, $[x][y]$ is also an orthogonal projection. By (4), we know that $[a^*xa] = [a^*[x]a]$ and thus we get $[yx] = [y^{\frac{1}{2}}xy^{\frac{1}{2}}] = [y^{\frac{1}{2}}[x]y^{\frac{1}{2}}] = [y[x]] = [[x]y[x]] = [[x][y][x]] = [[x][y]] = [x][y]$. □

Remark While the support map is continuous from below, it is very non-continuous from above. Consider $\frac{1}{n}x$ for any $x \in \mathcal{A}^+$. That sequence converges in norm against zero while the support converges against $[x]$.

Statement (4), an essential ingredient for working with reach maps, can already be found in [Gär14, 2.4.1].

3.1.3 Reach Maps Preserve Suprema

The following theorem will later give us, for example, $R_T(p \vee q) = R_T(p) \vee R_T(q)$ for $p, q \in \mathcal{T}(\mathcal{A})$. Since we have not yet shown that $p \vee q$ is an open projection, we formulate this theorem for all $\mathcal{P}(\mathcal{M}_{\mathcal{A}})$.

Theorem *If $(p_i)_{i\in I} \subseteq \mathcal{P}(\mathcal{M}_{\mathcal{A}})$ is a family of orthogonal projections, then $[T(\bigvee_{i\in I} p_i)] = \bigvee_{i\in I} [T(p_i)]$.*

Proof We define the projection $p := \bigvee_{i\in I} p_i$, the set $M := \{\mathcal{J} \subseteq I : |\mathcal{J}| < \infty\}$ and a partial order on M by set inclusion. Define $q_{\mathcal{J}} := \bigvee_{i\in\mathcal{J}} p_i$ for all $\mathcal{J} \in M$. Then $(q_{\mathcal{J}})_{\mathcal{J}\in M}$ is an increasing, bounded net with $\sup_{\mathcal{J}\in M} q_{\mathcal{J}} = \bigvee_{i\in I} p_i = p$. Also, since every $\mathcal{J} \in M$ is finite, the sum $\sum_{i\in\mathcal{J}} p_i$ exists, and we have $[T(q_{\mathcal{J}})] = [T(\bigvee_{i\in\mathcal{J}} p_i)] = [T([\sum_{i\in\mathcal{J}} p_i])] = [T(\sum_{i\in\mathcal{J}} p_i)] = [\sum_{i\in\mathcal{J}} T(p_i)] = \bigvee_{i\in\mathcal{J}} [T(p_i)]$. We can conclude $[T(p)] = [T(\sup_{\mathcal{J}\in M} q_{\mathcal{J}})] = [\sup_{\mathcal{J}\in M} T(q_{\mathcal{J}})] = \sup_{\mathcal{J}\in M} [T(q_{\mathcal{J}})] = \sup_{\mathcal{J}\in M} \bigvee_{i\in\mathcal{J}} [T(p_i)] = \bigvee_{i\in I} [T(p_i)]$. □

Remark Because of $R_T(0) = [T(0)] = 0$ reach maps map 0 to 0, which is also a consequence of the above theorem. With $0 = \bigvee \emptyset$ we get $R_T(0) = R_T(\bigvee \emptyset) = \bigvee \emptyset = 0$.

3.1.4 Monotone Maps

Corollary (1) *If* $p \leq q \in \mathcal{P}(\mathcal{M}_{\mathcal{A}})$, *then* $[T(p)] \leq [T(q)]$.
(2) *If* $(p_i)_{i \in I} \subseteq \mathcal{P}(\mathcal{M}_{\mathcal{A}})$, *then* $[T(\bigwedge p_i)] \leq \bigwedge [T(p_i)]$.

Proof (1) All maps R which preserve suprema are monotone: Let $q \leq p$, then $R(q) \leq R(q) \vee R(p) = R(q \vee p) = R(p)$.

(2) All monotone maps R fulfil this property: $R(\bigwedge p_i) \leq R(p_i)$ for all $i \in I$, therefore $R(\bigwedge p_i) \leq \bigwedge R(p_i)$. □

3.2 The Lattice of Open Projections

Now, we will see that the set of open projections is closed under suprema. That fact is known in the literature for the universal envelope, and we will use the reach map of representations to transfer the result to other envelopes.

3.2.1 Open Projections in the Universal Representation

Proposition [Ped18, 3.11.10] *Let* $\mathcal{A}^\diamond := (\mathcal{A}, \mathcal{A}^{**})$ *be a universally enveloped C*-algebra, then* $\mathcal{B} \mapsto \mathbb{1}_{\mathcal{B}}$ *is a bijection from the hereditary C*-subalgebras of* $\mathcal{A}$ *to the open projections* $\mathcal{T}(\mathcal{A})$.

3.2.2 The Universal Atomic Envelope

As discussed in Section 2.1.7, every envelope of a C*-algebra $\mathcal{A}$ is a subenvelope of the universal envelope. We want to single out one other envelope of $\mathcal{A}$ which is particularly natural for working with open projections. We say a projection is *minimal* if it is non-zero and has no other non-zero projection below it. A von Neumann algebra $\mathcal{M}$ is called *atomic* if below every non-zero projection in $\mathcal{P}(\mathcal{M})$ there is a minimal projection.

Definition Let $\mathcal{A}^\diamond$ be an enveloped C*-algebra. If $z_a \in \mathcal{P}(\mathcal{Z}(\mathcal{A}^{**}))$ is the smallest orthogonal projection in $\mathcal{P}(\mathcal{Z}(\mathcal{A}^{**}))$ such that $\varphi(z_a) = 1$ for all pure states $\varphi \in \mathcal{S}(\mathcal{A})$, then we call $\mathcal{M}_{\mathcal{A}}$ the *universal atomic* envelope of $\mathcal{A}$ if $(\mathcal{A}, \mathcal{M}_{\mathcal{A}}) \simeq (z_a\mathcal{A}, z_a\mathcal{A}^{**})$.

Remark Every atomic envelope is a subenvelope of the universal atomic envelope (see [Tak79, III.6.36]). The universal atomic envelope is isomorphic to the envelope induced by the universal atomic representation, which is defined as the direct sum of all irreducible representations of $\mathcal{A}$ (see [Tak79, III.6.35]). It is also equivalent to the envelope induced by the reduced atomic representation, which is defined as the direct sum of one representative for every class of equivalent irreducible representations of $\mathcal{A}$ (see [Ped18, 4.3.7]).

3.2.3 Open Projections in Other Representations

Lemma *If $\pi\colon (\mathcal{A}, \mathcal{A}^{**}) \to (\mathcal{A}, \mathcal{M}_{\mathcal{A}})$ is a surjective *-homomorphism with $\pi|_{\mathcal{A}} = \mathrm{id}$, then:*

(1) *The reach map R_π is surjective onto the open projections $\mathcal{T}(\mathcal{A}, \mathcal{M}_{\mathcal{A}})$.*

(2) *If the universal atomic envelope is a subenvelope of $\mathcal{M}_{\mathcal{A}}$, then R_π is bijective.*

Proof (1) First, we need to show that R_π only maps to open projections. For that, we consider any open projection $p \in \mathcal{T}(\mathcal{A}, \mathcal{M}_{\mathcal{A}})$, which is the supremum of an approximating unit $(u_i)_{i\in I}$ of a hereditary subalgebra of $\mathcal{A}$. Then $(\pi(u_i))_{i\in I} = (u_i)_{i\in I}$ is the same approximating unit, which has a supremum in $\mathcal{A}^{**}$ and therefore $R_\pi(p) = \pi(\sup_{i\in I} u_i) = \sup_{i\in I} \pi(u_i) \in \mathcal{T}(\mathcal{A}, \mathcal{M}_{\mathcal{A}})$ is an open projection.

Second, we show surjectivity by considering an open projection $p \in \mathcal{T}(\mathcal{A}, \mathcal{M}_{\mathcal{A}})$. It is the supremum of an approximating unit $(u_i)_{i\in I}$ of a hereditary subalgebra in $\mathcal{A}$ as a subalgebra of $\mathcal{M}_{\mathcal{A}}$. Of course $(u_i)_i$ also has a supremum $\sup_{i\in I} u_i$ in $\mathcal{T}(\mathcal{A}, \mathcal{A}^{**})$ and because π is normal we have $\pi(\sup_{i\in I} u_i) = p$.

(2) See [AB15, 2.5]. □

3.2.4 The Lattice of Open Projections

Proposition (1) *The supremum of open projections is an open projection.*

(2) *The open projections imbued with the order of orthogonal projections are a complete lattice.*

(3) *The map $\mathcal{B} \mapsto \mathbb{1}_{\mathcal{B}}$ from the hereditary C*-subalgebras to the open projections is surjective, monotone and preserves suprema.*

Proof (1) is shown in [GK71, Theorem 3.3] for open projections of the universal atomic envelope, and therefore also for the universal envelope. For all other envelopes we can conclude it from Lemma 3.2.3: Any family of open projections in $(p_i)_{i\in I} \subseteq \mathcal{T}(\mathcal{A}, \mathcal{M}_{\mathcal{A}})$ is an image of a family of open projections in $\mathcal{T}(\mathcal{A}, \mathcal{A}^{**})$ under a *-homomorphism $\pi\colon (\mathcal{A}, \mathcal{A}^{**}) \to \mathcal{A}^{\diamond}$ like in Lemma 3.2.3. Since $\bigvee_{i\in I} p_i$ is an open projection in $\mathcal{T}(\mathcal{A}, \mathcal{A}^{**})$ its image $\pi(\bigvee_{i\in I} p_i) = \bigvee_{i\in I} \pi(p_i)$ is an open projection in $\mathcal{T}(\mathcal{A}, \mathcal{M}_{\mathcal{A}})$.

(2) follows from (1) via Theorem 2.2.2.

(3) The map is surjective because it can be written as the concatenation of the bijective map $\mathcal{B} \to \mathbb{1}_{\mathcal{B}}$ onto $\mathcal{T}(\mathcal{A}, \mathcal{A}^{**})$ (Proposition 3.2.1) and the surjective map $R_\pi\colon \mathcal{T}(\mathcal{A}, \mathcal{A}^{**}) \to \mathcal{T}(\mathcal{A}, \mathcal{M}_{\mathcal{A}})$ (Lemma 3.2.3). That the map is monotone is clear from the definition. When we consider a family of hereditary C*-subalgebras $(\mathcal{B}_i)_{i\in I}$ of $\mathcal{A}$, then $\bigvee_{i\in I} \mathcal{B}_i =: \mathcal{B}$ belongs to a unique open projection $\mathbb{1}_{\mathcal{B}} \in \mathcal{T}(\mathcal{A}, \mathcal{A}^{**})$. When we consider the open projection $p := \bigvee_{i\in I} \mathbb{1}_{\mathcal{B}_i}$ then because of $\mathbb{1}_{\mathcal{B}_i} \leq \mathbb{1}_{\mathcal{B}}$, we have $p \leq \mathbb{1}_{\mathcal{B}}$. Also, p belongs to a hereditary C*-subalgebra $\mathcal{A}_p$ which includes all $\mathcal{B}_i$ and thus $\mathcal{B} \leq \mathcal{A}_p$ and $\mathbb{1}_{\mathcal{B}} \leq p$. We conclude that $\mathbb{1}_{\mathcal{B}} = p$ and consequently the map

$\mathcal{B} \mapsto \mathbb{1}_{\mathcal{B}} \in \mathcal{T}(\mathcal{A}, \mathcal{A}^{**})$ preserves suprema. For other envelopes, we need to add that the reach map $R_\pi \colon \mathcal{T}(\mathcal{A}, \mathcal{A}^{**}) \to \mathcal{T}(\mathcal{A}, \mathcal{M}_{\mathcal{A}})$ preserves suprema. □

Definition For $p \in \mathcal{P}(\mathcal{M}_{\mathcal{A}})$ we call $p^\circ := \bigvee \{ q \in \mathcal{T}(\mathcal{A}^\diamond) \mid q \le p \} \in \mathcal{T}(\mathcal{A}^\diamond)$ the *interior* of p. Similarly, $\bar{p} = ((p^\perp)^\circ)^\perp$ is the *closure* of p.

Remark We have shown in this proposition that, like all unions of open sets are open sets, the suprema of open projections yield open projections. On the other hand, we know that intersections of open sets do, in general, not result in an open set. The same is valid for open projections. The infimum in the set of orthogonal projections (i.e. the intersection of Hilbert subspaces) of open projections is not generally an open projection. Still, the open sets and the open projections form a complete lattice. We only must be careful to take the infimum in the right lattice. For a family of open sets, their infimum is the interior of their intersection. The infimum of a family of open projections is the largest open projection below their intersection. In the language of lattice theory, the open projections are a complete sub $\vee$-semilattice but not a complete sub-lattice of the orthogonal projections.

The parallel between open sets and open projections has its limits. There are examples of the infimum of two non-commuting open projections not being open (see [Ake69, II.6]). However, at least in the universal envelope, the intersection of commuting open projections is open (see [Ake69, II.7]).

We have seen in the preceding theorems that the hereditary C*-subalgebras give a structure to a C*-algebra by which we can analyse it. When we choose a “small” envelope, we lose some of this structure because multiple hereditary C*-subalgebras induce the same open projection. The “bigger” we make our envelope, the larger the set of open projections becomes. In terms of topology, our topology becomes finer. However, once the envelope contains the universal atomic envelope, we have achieved the finest topology. Making the envelope even bigger only makes the von Neumann algebra bigger without gaining more distinct open projections. Thus, we can see the universal atomic envelope as a sweet spot. Sometimes, we want a coarser topology. For example, in a self-enveloped von Neumann algebra $(\mathcal{M}, \mathcal{M})$, the structure given by all projections $\mathcal{P}(\mathcal{M})$ is sufficient to inspect reach maps on $\mathcal{M}$. If instead we used the universal envelope $(\mathcal{M}, \mathcal{M}^{**})$, a rather unusual construct, we would get many more open projections, one for every hereditary C*-subalgebra. In a von Neumann algebra, the hereditary von Neumann subalgebras (a subset of the hereditary C*-subalgebras) give a more helpful structure.

3.2.5 Open Projections in Commutative Algebras

Example 1 In a commutative C*-algebra $C_0(\Omega)$ with Ω locally compact and Hausdorff, we had discussed in Example 2 in 2.2.1 that the hereditary C*-subalgebras correspond precisely to the open sets in Ω. Consequently, in the universal envelope of $C_0(\Omega)$, the open projections are exactly the characteristic functions of open sets (cf. [Ped18, 3.11.10]), motivating the name open projections.

Example 2 If we choose another envelope, the characteristic functions of different open sets could lead to the same open projection. We can observe that the point evaluation $f \mapsto f(\omega)$ for every $\omega \in \Omega$ is a pure state. In an envelope where each of these pure states is normal, for example $(C_0(\Omega), l^\infty(\Omega))$, we can separate characteristic functions of different open sets. This is an example of how the universal atomic envelope is enough to make the map from the hereditary C*-subalgebras to the open projections bijective.

Example 3 In contrast, let us look at $C([0,1]) \subseteq L^\infty([0,1],\lambda)$ where we use the Lebesgue measure as a probability measure on the unit interval. Different open sets, e.g. $[0,\frac{1}{2})$ and $(0,\frac{1}{2})$ or $[0,1]$ and $[0,\frac{1}{2}) \cup (\frac{1}{2},1]$, lead us to the same open projection. Two open sets give us the same open projection if and only if they only differ by a set of measure zero.

Example 4 We consider the Cantor space $\Omega = \Omega_0^{\mathbb{Z}}$ for a finite Ω_0. We envelope it with $L^\infty(\Omega,\mu)$ for any product measure μ. Again, we know that the characteristic functions of open sets are open projections. We can say more in this case: A cylinder set is defined by an $n \in \mathbb{N}$ and $w \in \Omega_0^{2n}$ to be $\mathcal{Z}_w := \{\omega \in \Omega \mid \omega_{[-n,n]} = w\}$, where $\omega_{[-n,n]}$ is the subsequence $(\omega_{-n}, \omega_{-n+1}, \dots, \omega_{n-1}, \omega_n)$. Every cylinder set is clopen in Ω. Because Ω is a topological product space, finite unions of cylinder sets give us all clopen sets in Ω (cf. [LM21, Ex. 6.1.7.]). So, the orthogonal projections in $\mathcal{P}(C(\Omega))$ are the characteristic functions of a finite union of cylinder sets. Further, every open set is given as a union of cylinder sets. A topological space like this, with a topological base of clopen sets, is also called *zero dimensional* (see [Eng89, 6.2]). This property means that we cannot only approximate every open projection in $\mathcal{P}(L^\infty(\Omega,\mu))$ with an increasing net in $C(\Omega)^+$, but that we can choose the increasing net to consist of orthogonal projections in $\mathcal{P}(C(\Omega))$. This means the open projections are the closure under suprema of the orthogonal projections in the algebra. Again, if two open sets only differ by a null set, they will belong to the same projection. Which pairs of open sets only differ by a null set will heavily depend on the chosen product measure.

3.2.6 Examples for Open Projections in Non-Commutative Algebras

Example 1 For the compact operators on a Hilbert space $\mathcal{K}(\mathcal{H})$ we know that $\mathcal{K}(\mathcal{H})^{**} = \mathcal{B}(\mathcal{H})$. Every orthogonal projection in $\mathcal{P}(\mathcal{B}(\mathcal{H}))$ can be written as a supremum of minimal projections and thus as an increasing sequence of finite-dimensional orthogonal projections. So, we have $\mathcal{T}(\mathcal{K}(\mathcal{H}), \mathcal{B}(\mathcal{H})) = \mathcal{P}(\mathcal{B}(\mathcal{H}))$. Since every possible projection is open, this non-commutative topology could be called discrete.

Example 2 We consider the open projections of the CAR-algebra $\mathcal{A} := M_2^{\otimes \mathbb{Z}}$. This is the non-commutative version of the Cantor space. $\mathcal{A}$ has *real rank zero*, which means that all hereditary C*-subalgebras have an approximate unit consisting of orthogonal projections. This property is a non-commutative analogue to a zero-dimensional topological space (see [Bla06, V.3.2.7)] and is also what we have observed for the Cantor space. Again, it means for separable algebras that every open projection can be approximated by a monotone sequence of orthogonal projections in $\mathcal{A}$, or consequently, every open projection can be written as a supremum of projections in the algebra. The Cantor space $\{0,1\}^{\mathbb{Z}}$ can be embedded on the diagonal of $\mathcal{A}$. Thus, the previous example gives all open projections on the diagonal. For every unitary $u \in \mathcal{A}$ and every open projection $p \in \mathcal{T}(C(\{0,1\}^{\mathbb{Z}}))$, we have that $u^* p u$ is also an open projection. By Ad_u being normal, it maps the sequence approximating p to $u^* p u$.

3.3 Supports of Completely Positive Operators

Besides positive elements and functionals, completely positive operators also have a *support*. It is a valuable tool we introduce now because we will need it to analyse open projections and reach maps further. Supports, especially of functions, are a widely used concept in von Neumann algebras. In the context of enveloped C*-algebras, we notice, however, that there are two different notions of support for a normal completely positive operator.

3.3.1 The Support of a Completely Positive Operator

Proposition *Let $\mathcal{A}^\diamond$ and $\mathcal{B}^\diamond$ be enveloped C*-algebras and $T\colon \mathcal{A}^\diamond \to \mathcal{B}^\diamond$ a normal completely positive operator.*

(1) *The set $\mathcal{N}_T := \{ x \in \mathcal{A} : T(xx^*) = T(x^*x) = 0 \}$ is a hereditary C*-subalgebra.*

Also for the orthogonal projections $q := \mathbb{1}_{\mathcal{N}_T}$ and $p := \operatorname{supp} T := q^\perp$ in $\mathcal{P}(\mathcal{M}_\mathcal{A})$ the following statements hold:

(2) *q is open and p is closed.*
(3) *$T(q) = 0$ and $T(p) = T(\mathbb{1})$.*
(4) *$\mathcal{N}_T = \mathcal{A}_q$.*
(5) *$q = \bigvee \{ q' \in \mathcal{T}(\mathcal{A}) : T(q') = 0 \}$.*
(6) *$T(pxp) = T(xp) = T(px) = T(x)$ for all $x \in \mathcal{A}$.*
(7) *$T(x) = 0 \Leftrightarrow x \perp p$ for all $x \in \mathcal{A}^+$.*
(8) $\operatorname{supp}_{(\mathcal{M}_\mathcal{A}, \mathcal{M}_\mathcal{A})} T \leq \operatorname{supp}_{\mathcal{A}^\diamond} T = \overline{\operatorname{supp}_{(\mathcal{M}_\mathcal{A}, \mathcal{M}_\mathcal{A})} T}$.

In statement (8), we use the definition of $\operatorname{supp} T$ for two different enveloped C*-algebras, once for $\mathcal{A}^\diamond$ and once for $\mathcal{M}_\mathcal{A}$ self-enveloped. As in Definition 3.2.4, the closure is the smallest dominating closed projection regarding the topology $\mathcal{T}(\mathcal{A}, \mathcal{M}_\mathcal{A})$.

Proof (1) We show that $I := \{x \in \mathcal{A} : T(x^*x) = 0\}$ is a closed left ideal. For $x, y \in I$ and $\lambda \in \mathbb{C}$ we have $T((\lambda x^*)(\lambda x)) = |\lambda|^2 T(x^*x) = 0$. Also, $\|T((x+y)^*(x+y))\| \le \|T(x^*x)\| + \|T(y^*y)\| + \|T(x^*y)\| + \|T(y^*x)\|$. Proposition 1.3.3 says that $\|T(x^*y)\|^2 \le \|T(x^*x)\| \, \|T(y^*y)\| = 0$ and we conclude with $\|T(x+y)(x+y)^*\| = 0$ that I is a subspace. If $x \in I$, $y \in \mathcal{A}$, then $T((xy)^*(xy)) = T(x^*y^*yx)) \le T(x^* \|y^*y\| \mathbb{1} x) = 0$ and I is an ideal. To show that I is closed, we choose a uniform convergent net $(x_j)_{j\in\mathcal{J}} \to x$ in I. Then $(x_j^* x_j)_{j\in\mathcal{J}}$ converges against x^*x, and because T is continuous, $T(x^*x) = 0$. Since I is a closed left ideal, we can conclude with Proposition 2.2.1 that $\mathcal{N}_T = I^* \cap I$ is a hereditary C*-subalgebra.

(2) q is open by definition and thus p is closed.

(3) q can be written as a weak*-limit of an approximating unit of $\mathcal{N}_T$ and because T is normal $T(q) = 0$. $T(p) = T(\mathbb{1})$ follows directly.

(4) For all elements $x \in \mathcal{N}_T^+ \subseteq \mathcal{A}$ we have $pxp = x$ and so $x \in \mathcal{A}_q$. On the other hand, if $x \in \mathcal{A}_q^+$ then $x \le \|x\| \, q$ and so $T(x) \le \|x\| \, T(q) = 0$, which implies $x \in \mathcal{N}_T^+$. Since both C*-subalgebras have the same positive cone, they are the same.

(5) q is open with $T(q) = 0$, so q is one of the open projections in the supremum we are calculating. On the other hand, any open $q' \in \mathcal{T}(\mathcal{A})$ with $T(q') = 0$ can be approximated from below by elements in $\mathcal{N}_T$, thus $q' \le q$.

(6) With Proposition 1.3.3 we get $T(xq) = T(qx) = 0$. Hence, $T(x) = T((p+q)x) = T(px)$ and similar for the other terms.

(7) Let $T(x) = 0$, then $T([x]) = 0$ which means $[x] \le q$ and $x \perp p$. On the other hand if $x \perp p$ then $T(x) = T(xp) = T(0) = 0$.

(8) We set $q_{\mathcal{M}_\mathcal{A}} := (\operatorname{supp}_{\mathcal{M}_\mathcal{A}} T)^\perp$. Since $T(q) = 0$ and $q \in \mathcal{M}_\mathcal{A}$ we know from the definition of $q_{\mathcal{M}_\mathcal{A}}$ that $q \le q_{\mathcal{M}_\mathcal{A}}$. On the other hand, for every $q' \in \mathcal{T}(\mathcal{A})$ with $q' \le q_{\mathcal{M}_\mathcal{A}}$ we have $T(q') \le T(q_{\mathcal{M}_\mathcal{A}}) = 0$, thus $q = q^\circ_{\mathcal{M}_\mathcal{A}}$. □

Definition We call $\mathcal{N}_T$ the *null algebra* of T, the open projection q the *null space* of T, and the closed projection $p = \operatorname{supp} T$ the *support of* T. If $\operatorname{supp} T = \mathbb{1}$, we call T *faithful*. Further we call $\operatorname{supp}_{(\mathcal{M}_\mathcal{A}, \mathcal{M}_\mathcal{A})} T$ the *support on the envelope of* T and if $\operatorname{supp}_{(\mathcal{M}_\mathcal{A}, \mathcal{M}_\mathcal{A})} T = \mathbb{1}$, we call T *faithful on the envelope*.

Remark This definition is designed to coincide with the usual definition of the support as, for example, given in [Stø13, 1.4.1] for self-enveloped von Neumann algebras.

Example 1 For an example where the support and the support on the envelope are very different, we look again at the enveloped C*-algebra $(C([0,1]), l^\infty([0,1]))$. We can consider the functional φ defined by $\varphi := \sum_{i=1}^{\infty} \frac{1}{n^2}\omega_{q_i}$, where $(q_i)_{i\in\mathbb{N}}$ is an enumeration of the rational numbers in $[0,1]$ and ω_{q_i} the point evaluation $f \mapsto f(q_i)$ at q_i. If a positive continuous function on $[0,1]$ is zero on all rational points, it is zero. Thus, the support of φ is $\mathbb{1}$. On the other hand, all functions in $l^\infty([0,1])$, that are only non-zero on irrational numbers, are sent to zero by φ. The support on the envelope of φ is $\chi_{\mathbb{Q}\cap[0,1]}$ which is significantly smaller than $\mathbb{1}$.

Example 2 In our graph picture in Chapter 1, the support of an operator is the characteristic function of all points reached by an arrow.

3.4 An Excursion into Topology

This section explores a few more properties of non-commutative topology. Most of the results here are not required for the following chapters.

3.4.1 Open Projections in C*-Algebras with Added Unit

Lemma *If $\mathcal{A}^\diamond$ is an enveloped C*-algebra, then $\mathcal{T}(\mathcal{A}, \mathcal{M}_\mathcal{A}) = \mathcal{T}(\mathcal{A} + \mathbb{C}\mathbb{1}, \mathcal{M}_\mathcal{A})$.*

Proof *First inclusion, $T(\mathcal{A}, \mathcal{M}_\mathcal{A}) \subseteq \mathcal{T}(\mathcal{A} + \mathbb{C}\mathbb{1}, \mathcal{M}_\mathcal{A})$:* Let $\sup_{i\in I} x_i = p$ with $(x_i)_{i\in I} \in \mathcal{A}^+$, then $x_i \in \mathcal{A} + \mathbb{C}\mathbb{1}$ and thus $p \in \mathcal{T}(\mathcal{A} + \mathbb{C}\mathbb{1})$.
Second inclusion, $\mathcal{T}(\mathcal{A} + \mathbb{C}\mathbb{1}, \mathcal{M}_\mathcal{A}) \subseteq \mathcal{T}(\mathcal{A}, \mathcal{M}_\mathcal{A})$: First let $x \in (\mathcal{A} + \mathbb{C}\mathbb{1})_{+,1}$, we show $[x] \in \mathcal{T}(\mathcal{A}^\diamond)$. We pick an approximate unit $(u_i)_{i\in I}$ of $\mathcal{A}$. Because of $x^{\frac{1}{2}} \in \mathcal{A} + \mathbb{C}\mathbb{1}$ we have $x^{\frac{1}{2}} u_i x^{\frac{1}{2}} \in \mathcal{A}$. By monotone convergence and one-sided strong-continuity of multiplication, we have $\sup_{i\in I} x^{\frac{1}{2}} u_i x^{\frac{1}{2}} = x$. Because the support map preserves suprema we get $\sup_{i\in I}[x^{\frac{1}{2}} u_i x^{\frac{1}{2}}] = [x^{\frac{1}{2}} \mathbb{1} x^{\frac{1}{2}}] = [x] \in \mathcal{T}(\mathcal{A}^\diamond)$. For any $p \in \mathcal{T}(\mathcal{A} + \mathbb{C}\mathbb{1}, \mathcal{M}_\mathcal{A})$ with $\sup_{i\in I} x_i = p$ we get $\sup_{i\in I}[x_i] = p \in \mathcal{T}(\mathcal{A}^\diamond)$, because $[x_i] \in \mathcal{T}(\mathcal{A}^\diamond)$. □

This proof was inspired by a trick in the proof of [GK71, Lemma 3.2].

This lemma shows a complication with the approach chosen in this work to allow arbitrary envelopes for the definition of $\mathcal{T}(\mathcal{A})$. A commutative C*-algebra $C_0(\Omega)$ is unital if Ω is compact and non-unital if Ω is only locally compact. This fits the introduced definition of compactness for projections. If $\mathbb{1}$ can be majorized by an element in $\mathcal{A}$, $\mathbb{1}$ is compact and $\mathcal{A}$ unital. In the universal envelope, whether $\mathcal{A}$ is unital can be deduced from the topology $\mathcal{T}(\mathcal{A}, \mathcal{A}^{**})$ (see [AB15, 3.]). However, when we fix the envelope and adjoin a $\mathbb{1}$ like in this lemma, the topology stays the same and $\mathcal{A}^{**}$ is generally not the universal envelope of $\mathcal{A} + \mathbb{C}\mathbb{1}$. So whether $\mathbb{1}$ is compact and $\mathcal{A}$ is unital can only be determined from $\mathcal{T}(\mathcal{A})$ if it is the topology in the universal envelope.

3.4.2 Open Spectral Projections

The following result gives us an additional method to find open projections.

Proposition *Let $U \subseteq \mathbb{R}$ be an open subset of the real numbers and $x \in \mathcal{A}_{sa}$ a self-adjoint element of an enveloped C^*-algebra $\mathcal{A}^\diamond$. Then the spectral projection $\chi_U(x)$ is an open projection.*

Proof We use a shortened version of the proof of [GK71, Lemma 3.2] but confirm that we do not require the envelope to be atomic, as is required there. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function with support U. Then we have $\chi_U(x) = [f(x)]$. From the spectral calculus (cf., e.g. [Mur90, 2.1.13]) we know that $f(x) \in C^*(x, \mathbb{1}) \subseteq C^*(\mathcal{A}, \mathbb{1})$, thus $\chi_U(x) = [f(x)] \in \mathcal{T}(\mathcal{A} + \mathbb{C}\mathbb{1}) = \mathcal{T}(\mathcal{A})$. □

3.4.3 A Hausdorff Property

One of the most important properties a topological space can have is the Hausdorff property. Usually, it is used to separate points. We give a Hausdorff property by separating supports of pure states, the closest thing to a point we have available.

Lemma *If $\varphi \in \mathcal{S}(\mathcal{A})$ is a pure state on $\mathcal{A}$ with normal extension onto $\mathcal{M}_\mathcal{A}$, then* $\operatorname{supp}\varphi$ *is minimal in $\mathcal{P}(\mathcal{M}_\mathcal{A})$.*

Proof Since $\mathcal{M}_\mathcal{A}$ is a subenvelope of $\mathcal{A}^{**}$, there is a *-homomorphism $\pi\colon (\mathcal{A}, \mathcal{A}^{**}) \to (\mathcal{A}, \mathcal{M}_\mathcal{A})$ which extends id on $\mathcal{A}$. First, we show that $\operatorname{supp}\varphi$ is minimal in $\mathcal{P}(\mathcal{A}^{**})$. Because φ is pure, there is, as shown in [Ped18, 3.13.6], an open projection p in $\mathcal{T}(\mathcal{A}, \mathcal{A}^{**})$ with $\varphi(p) = 0$ and $p^\perp$ minimal in $\mathcal{P}(\mathcal{A}^{**})$. We can see that $p^\perp$ is already the support of φ: From $\varphi(p) = 0$ we conclude $p \leq (\operatorname{supp}\varphi)^\perp$ and $\operatorname{supp}\varphi \leq p^\perp$. Now $\operatorname{supp}\varphi = p^\perp$ holds because $p^\perp$ is minimal. To look at $\mathcal{M}_\mathcal{A}$ again, we observe that the support is defined as $\mathbb{1} - \mathbb{1}_{\mathcal{N}_\varphi}$ and thus $\pi(\operatorname{supp}\varphi) = \operatorname{supp}\varphi$. Since π is surjective, $\operatorname{supp}\varphi$ is also minimal in $\mathcal{M}_\mathcal{A}$. □

Proposition *Let $\mathcal{A}^\diamond$ be an enveloped C^*-algebra. If $\varphi, \psi \in \mathcal{S}(\mathcal{A})$ are pure states with normal extension onto $\mathcal{M}_\mathcal{A}$ and* $\operatorname{supp}\varphi \perp \operatorname{supp}\psi$, *then there are open projections $p \perp q \in \mathcal{T}(\mathcal{A})$ with* $\operatorname{supp}\varphi \leq p$ *and* $\operatorname{supp}\psi \leq q$.

Proof *First: Let $\mathcal{A}$ be unital and universally enveloped.*
Since $\mathcal{A}$ is unital, $\operatorname{supp}\varphi$ and $\operatorname{supp}\psi$ are compact projections. Then by the non-commutative Urysohn lemma (cf. [Ped18, 3.11.12]) there is an $x \in \mathcal{A}^+$ with $\operatorname{supp}\varphi \leq x \leq (\operatorname{supp}\psi)^\perp$. So we can pick two disjoint open neighbourhoods U_0 and U_1 of 0 and 1 to get $\chi_{U_0}(x) \geq \operatorname{supp}\psi$ and $\chi_{U_1}(x) \geq \operatorname{supp}\varphi$.

Second: Let $\mathcal{A}$ be unital, with any envelope.
Since $\mathcal{M}_\mathcal{A}$ is a subenvelope of $\mathcal{A}^{**}$ there is a *-homomorphism $\pi\colon (\mathcal{A}, \mathcal{A}^{**}) \to (\mathcal{A}, \mathcal{M}_\mathcal{A})$ which extends id on $\mathcal{A}$. The states φ and ψ have an extension to normal

states on $\mathcal{A}^{**}$ with $\varphi \circ \pi = \varphi$ and $\psi \circ \pi = \psi$. To apply the result from the previous step, we need to show that $\operatorname{supp}\varphi \perp \operatorname{supp}\psi \in \mathcal{P}(\mathcal{A}^{**})$. Since $\operatorname{supp}\varphi$ is closed, we know that there is a monotonically increasing net $(x_i)_{i\in I} \subseteq \mathcal{A}^+$ with $\sup_{i\in I} x_i = (\operatorname{supp}\varphi)^\perp$. Because we have given that $\operatorname{supp}\varphi \perp \operatorname{supp}\psi$ in $\mathcal{M}_\mathcal{A}$, we know that $\psi(\operatorname{supp}\varphi) = 0$ and $\psi(\sup_{i\in I} x_i) = 1$. Because $\mathcal{A}$ is weak* dense in $\mathcal{M}_\mathcal{A}$ and in $\mathcal{A}^{**}$, $\psi(\sup_{i\in I} x_i) = 1$ also holds for $\psi \in \mathcal{S}_*(\mathcal{A}^{**})$. From $1 = \psi(\operatorname{supp}\varphi^\perp) = \psi(\operatorname{supp}\psi \operatorname{supp}\varphi^\perp \operatorname{supp}\psi)$ we conclude $\operatorname{supp}\psi \operatorname{supp}\varphi^\perp \operatorname{supp}\psi = \operatorname{supp}\psi$ and that $\operatorname{supp}\varphi \perp \operatorname{supp}\psi$ in $\mathcal{A}^{**}$. Now, we can find the desired open projections p and q in $\mathcal{T}(\mathcal{A}, \mathcal{A}^{**})$, map them with π into $\mathcal{T}(\mathcal{A}^\diamond)$ and π preserves orthogonality.

Third: Let $\mathcal{A}$ be non-unital. Now for any non-unital $\mathcal{A}$, φ and ψ are also pure states on $\mathcal{A} + \mathbb{C}\mathbb{1}$ and because $\mathcal{T}(\mathcal{A} + \mathbb{C}\mathbb{1}) = \mathcal{T}(\mathcal{A})$ we can apply the theorem for the unital case. □

For the case of the universal (atomic) envelope, this theorem can be found in [GK71, Theorem 3.6].

Corollary *Let $\mathcal{A}^\diamond$ be an atomically enveloped C^*-algebra. Then all $p \in \mathcal{P}(\mathcal{M}_\mathcal{A})$ can be written as $p = \bigwedge\{ q \in \mathcal{T}(\mathcal{A}) : p \leq q\}$.*

Proof Since $\mathcal{M}_\mathcal{A}$ is atomic we have enough pure normal states so that for $p \in \mathcal{P}(\mathcal{M}_\mathcal{A})$ we have $p = \bigvee\{\operatorname{supp}\omega : \omega \in \mathcal{S}_*(\mathcal{M}_\mathcal{A}), \omega \text{ pure}, \operatorname{supp}\omega \leq p\}$ and equivalently for $p^\perp$. From the Hausdorff property for any pure normal φ with $\operatorname{supp}\varphi \leq p^\perp$ we get that $p \leq \bigvee\{ q \in \mathcal{T}(\mathcal{A}) : \varphi(q) = 0\}$. We get the result if we intersect those projections for all pure normal φ with $\operatorname{supp}\varphi \leq p^\perp$. □

3.4.4 The Open Projections are Weak* Total

Theorem *Let $\mathcal{A}^\diamond$ be an enveloped C^*-algebra and $(\mathcal{M}_\mathcal{A})_*$ separable, then $\mathcal{T}(\mathcal{A})'' = \mathcal{M}_\mathcal{A}$.*

Proof First, we observe that the result is valid for all atomic envelopes. Since $\mathcal{T}(\mathcal{A})''$ is, as a von Neumann algebra, closed under infima of orthogonal projections, we have $\mathcal{P}(\mathcal{M}_\mathcal{A}) \subseteq \mathcal{T}(\mathcal{A})$ and $\mathcal{P}(\mathcal{M}_\mathcal{A})$ is weak* total in $\mathcal{M}_\mathcal{A}$.

We need a bit of machinery for the rest of the proof because we need to consider three different envelopes of $\mathcal{A}$. We consider the universal envelope $(\mathcal{A}, \mathcal{A}^{**})$, the universal atomic envelope $(\mathcal{A}, \mathcal{M}_a)$ and the envelope $\mathcal{A}^\diamond = (\mathcal{A}, \mathcal{M}_\mathcal{A})$ given in the statement of the theorem. Since $(\mathcal{M}_\mathcal{A})_*$ is separable we can represent $\mathcal{A}^\diamond$ so that $\mathcal{M}_\mathcal{A}$ is σ-finite. Since $\mathcal{M}_a$ and $\mathcal{M}_\mathcal{A}$ are subenvelopes of $\mathcal{A}^{**}$ we have two normal surjective representations $\pi_a\colon (\mathcal{A}, \mathcal{A}^{**}) \to (\mathcal{A}, \mathcal{M}_a)$ and $\pi\colon (\mathcal{A}, \mathcal{A}^{**}) \to (\mathcal{A}, \mathcal{M}_\mathcal{A})$.

We consider the set $\mathcal{U}(\mathcal{A}) \subseteq \mathcal{A}^{**}$ of the *universally measurable* elements in the universal envelope of $\mathcal{A}$. For a definition of $\mathcal{U}(\mathcal{A})$ see [Ped18, 4.3.11]. We just need to know three things about $\mathcal{U}(\mathcal{A})$:

(1) $\mathcal{T}(\mathcal{A}, \mathcal{A}^{**}) \subseteq \mathcal{U}(\mathcal{A})$ (cf. [Ped18, 4.3.13]).
(2) The atomic representation π_a is faithful on $\mathcal{U}(\mathcal{A})$ (cf. [Ped18, 4.3.15]).
(3) The σ-finite representation π maps $\pi(\mathcal{U}(\mathcal{A})) = \mathcal{M}_{\mathcal{A}\mathrm{sa}}$ (cf. [Tak79, III.6.39])

Since $\mathcal{T}(\mathcal{A}, \mathcal{M}_a)$ is weak* total in $\mathcal{M}_a$ we can approximate $\pi_a(x) \in \mathcal{M}_a$ with elements from $\mathcal{T}(\mathcal{A}, \mathcal{M}_a)$ for every $x \in \mathcal{U}(\mathcal{A})$. Because π_a is surjective, maps open projections to open projections and is faithful on $\mathcal{U}(\mathcal{A})$, the set of open projections $\mathcal{T}(\mathcal{A}, \mathcal{A}^{**})$ is weak* total in $\mathcal{U}(\mathcal{A})$. And again, because π maps open projections to open projections, this means that $\mathcal{T}(\mathcal{A}, \mathcal{M}_{\mathcal{A}})$ is weak* total in $\mathcal{M}_{\mathcal{A}}$. □

3.5 The Category of Reach Maps

With this, we conclude our analysis of open projections and focus on reach maps. We now have all the structure to do this.

3.5.1 The Reach is Open

Proposition *If $p \in \mathcal{T}(\mathcal{A})$ then $R_T(p) \in \mathcal{T}(\mathcal{B})$.*

Proof With $\sup_{i\in I} x_i = p$ we show $R_T(p) = [T(\sup_{i\in I} x_i)] = [\sup_{i\in I} T(x_i)] = \sup_{i\in I}[T(x_i)] = \bigvee_{i\in I}[T(x_i)]$. Because $\mathcal{T}(\mathcal{B})$ is closed under suprema (cf. Proposition 3.2.4), this is an open projection. □

3.5.2 The Category of Reach Maps

In sum, we have established that reach maps are suprema (or *join*) preserving maps between lattices of open projections. Reach maps do in general not preserve infima. For example, for the graph

$$1 \leftarrow 2 \rightarrow 3$$

we have $0 = R(\chi_{\{2\}}) = R(\chi_{\{1,2\}} \wedge \chi_{\{2,3\}}) < R(\chi_{\{1,2\}}) \wedge R(\chi_{\{2,3\}}) = \chi_{\{2\}}$. Remember that our reach map is defined in the Heisenberg picture, while we interpret the graph in the Schrödinger picture. This means that for example the observable $\chi_{\{2,3\}}$ will only be reached by a state in point 2, so $R(\chi_{\{2,3\}}) = \chi_{\{2\}}$.

Definition We define the *category of reach maps* with enveloped C*-algebras as objects and reach maps as morphisms.

Remark In the theory of complete lattices, there are three common categories. The category of complete lattices with completely join and meet preserving maps is often too restrictive. Thus, there are also the category of complete lattices with maps which preserve only joins and the category of complete lattices which preserve only meets.

By replacing the enveloped C*-algebra $\mathcal{A}^\diamond$ with its lattice of open projections $\mathcal{T}(\mathcal{A})$, we can see the category of reach maps as a sub-category of the category of complete lattices with completely join preserving maps.

3.5.3 Domains of Reach Maps

We have defined $R_T(p) = [T(p)]$ for all open projections $p \in \mathcal{T}(\mathcal{A})$. We could have chosen a larger domain for our reach maps, for example, $R_T(x) = [T(x)]$ for all $x \in \mathcal{M}_\mathcal{A}{}^+$. Let us discuss the implications of choosing different domains.

Proposition *For two normal completely positive operators $T, S\colon \mathcal{A}^\diamond \to \mathcal{B}^\diamond$ the following are equivalent:*

(a) $R_T = R_S$
(b) $[T(x)] = [S(x)]$ *for all* $x \in \mathcal{A}^+$.

Proof (a) $\Rightarrow$ (b) For $x \in \mathcal{A}^+$ we calculate $[T(x)] = [T([x])] = R_T([x]) = R_S([x]) = [S(x)]$.

(b) $\Rightarrow$ (a) For any open projection $p = \sup_{i\in I} x_i$ with a monotonically increasing net $(x_i)_{i\in I} \subseteq \mathcal{A}^+$ we have $R_T(p) = \sup_{i\in I}[T(x_i)] = \sup_{i\in I}[S(x_i)] = R_S(p)$. □

Thus, a reach map R_T has a unique extension to a map $R_T\colon \mathcal{T}(\mathcal{A}) \cup \mathcal{A}^+ \to \mathcal{T}(\mathcal{A})$ and is uniquely determined by the behaviour of that extension on $\mathcal{A}^+$. We don't know whether $R_T = R_S$ implies $[T(x)] = [S(x)]$ for all $x \in \mathcal{M}_\mathcal{A}{}^+$. That means two different reach maps from $(\mathcal{M}_\mathcal{A}, \mathcal{M}_\mathcal{A})$ to $(\mathcal{M}_\mathcal{B}, \mathcal{M}_\mathcal{B})$ might belong to the same reach map from $(\mathcal{A}, \mathcal{M}_\mathcal{A})$ to $(\mathcal{B}, \mathcal{M}_\mathcal{B})$. This differs from the category of completely positive normal operators, where an operator $T\colon \mathcal{A}^\diamond \to \mathcal{B}^\diamond$ is uniquely determined on all of $\mathcal{M}_\mathcal{A}$ by its behaviour on $\mathcal{A}$.

3.5.4 The Functor

Proposition *We can define a functor R from the category of* enveloped C*-algebras with normal completely positive operators *into* enveloped C*-algebras with reach maps *with the object part being the identity and the morphism part being the mapping to the reach map of an operator.*

Proof It is clear that $R_{\mathrm{id}} = \mathrm{id}$. We use Lemma 3.1.2 to show $(R_T \circ R_S)(p) = [T([S(p)])] = [T(S(p)] = R_{T\circ S}(p)$ for all $p \in \mathcal{T}(\mathcal{A})$. □

Overall, we have now established the functorial relations between the categories we are interested in:

$$
\begin{tikzcd}
& (\mathrm{MA}, \mathrm{CPO}) \arrow[dl, hook, "\mathrm{id}"] \arrow[dr, hook, "\mathrm{id}"] & \\
(\mathrm{C}^*, \mathrm{CPO}) \arrow[dr, "\text{universal envelope}"'] & & (\mathrm{vNA}, \sigma^*\text{-}\mathrm{CPO}) \arrow[dl, "\text{self-envelope}"] \\
& (\text{env-}\mathrm{C}^*, \sigma^*\text{-}\mathrm{CPO}) \arrow[d, "R"] & \\
& (\text{env-}\mathrm{C}^*, \text{Reach Maps}) &
\end{tikzcd}
$$

In this chapter, we deepened our understanding of reach maps, giving their definition on the open projections and showing that the open projections form a lattice. This led us to the definition of the category of reach maps. In between, we defined the two notions of support of a completely positive operator between enveloped C*-algebras and discussed the "topology" created by the open projections, proving for example a Hausdorff property.

4 Characterization of Reach Maps

This chapter investigates different characterizations of the information encoded by a reach map. For this, we look mainly at three different perspectives. First, we look at reach maps of rank one as projectivities. Next, we look at reach maps as equivalence classes of completely positive operators, and lastly, we investigate the Kraus operators associated with a reach map. We investigate this for finite-dimensional systems.

The deepest result of this chapter is the characterization of reach maps of rank one in 4.1.6. After that, our investigation becomes increasingly more concrete, ending in a practical criterion to pinpoint reach maps in 4.4.7.

For this chapter, $\mathcal{A} \subseteq M_n, \mathcal{B} \subseteq M_m$ are matrix algebras. Since $\mathcal{A}$ is finite-dimensional, $(\mathcal{A}, \mathcal{A})$ is an enveloped C*-algebra, and $\mathcal{P}(\mathcal{A}) = \mathcal{T}(\mathcal{A})$. Because in finite dimensions the technical details of open projections are irrelevant, we will say that R is a reach map from $\mathcal{P}(\mathcal{A})$ to $\mathcal{P}(\mathcal{B})$ in this chapter. With $\mathcal{P}_1(\mathcal{A})$, we denote the minimal projections in $\mathcal{A}$. Every matrix algebra is isomorphic to a direct sum of M_{n_i}. If n is so small that $\mathcal{A} = \bigoplus_{i=1}^k M_{n_i} \subseteq M_n$ with $n = \sum_{i=1}^k n_i$, then all minimal projections have rank one in M_n. In that case, we call $\mathcal{A} \subseteq M_n$ *minimally represented*. All minimal representations of $\mathcal{A}$ are unitarily equivalent.

4.1 A Characterization via Cross-Ratios

Until now, our only way to determine whether a suprema preserving map between lattices of projections is a reach map was to find a completely positive operator which induces the reach map. That is unsatisfying on multiple levels. For one, it can be hard to do so. In particular, with further criteria, it might be easier to show when such an operator does *not* exist. Also, it would be nice if the category of reach maps could become more independent of its obvious "parent" category, the completely positive operators.

4.1.1 Preserving Suprema is Not Everything

The obvious first question when one wants to define reach maps without finding a concrete, completely positive operator is whether all suprema-preserving maps are reach maps. We construct an example for a suprema-preserving map $\mathcal{P}(M_2) \to \mathcal{P}(M_2)$ which is not a reach map.

Example Consider a map $f : \mathcal{P}(M_2) \to \mathcal{P}(M_2)$ with $f(0) = 0, f(\mathbb{1}) = \mathbb{1}$. The lattice $\mathcal{P}(M_2)$ has a very simple structure. The projection 0 is the smallest element, $\mathbb{1}$ is the largest, and all other projections are incomparable with each other, unless they are equal. On the one-dimensional projections, f can be any arbitrary injective map into $\mathcal{P}(M_2)\setminus\{0\}$ and would still preserve suprema. From continuity considerations, it seems implausible that every injective map on the one-dimensional projections would make f a reach map. Indeed, there are more constraints on f if it is supposed to be a reach map: We consider the 6 rank-1 projections

$$p_1 \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad p_2 \coloneqq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad p_3 \coloneqq \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$
$$p_4 \coloneqq \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad p_5 \coloneqq \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad p_6 \coloneqq \frac{1}{5}\begin{pmatrix} 1 & -2i \\ 2i & 4 \end{pmatrix},$$

and assume $f(p_i) = p_i$ for $1 \leq i \leq 6$. Many injective maps on $\mathcal{P}_1(M_2)$ fulfil this constraint. If f is a reach map, we have a completely positive operator T with $f = R_T$. For T it holds by our assumption that $T(p_i) = \lambda_i p_i$ with $\lambda_i \in \mathbb{R}_{>0}$ for $i \in \{1, \dots, 6\}$. The p_i span M_2, thus T is completely defined by the λ_i. We want to show that T is determined up to scalar multiples. We observe $5p_4 = 4p_3 - p_1 + 2p_2$, similarly $5p_6 = 4p_5 - p_1 + 2p_2$. If we apply T to those equations we receive $5\lambda_4 p_4 = 4\lambda_3 p_3 - \lambda_1 p_1 + 2\lambda_2 p_2$ and $5\lambda_6 p_6 = 4\lambda_5 p_5 - \lambda_1 p_1 + 2\lambda_2 p_2$. The solution of this system of eight linear equations gives us that all λ_i must be equal, and thus $T = \lambda_1 \mathrm{id}$ and $f = R_{\mathrm{id}}$.

4.1.2 Joining Reach Maps

We will decompose reach maps into components that are simpler to characterize. For that, we need to know how we can join reach maps together.

Definition If $R, R' : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ are reach maps, then we define $R \vee R' : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ by $R \vee R'(p) \coloneqq R(p) \vee R'(p)$ for all $p \in \mathcal{P}(\mathcal{A})$. Further, we define an order on the set of reach maps between $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{B})$ by setting $R \leq R'$ if $R(p) \leq R'(p)$ for all $p \in \mathcal{P}(\mathcal{A})$.

Proposition *The map $R \vee R'$ is a reach map, which is the supremum of R and R' in the partially ordered set of reach maps.*

Proof There are completely positive operators T and T' with $R = R_T$ and $R' = R_{T'}$. Consequently, $R \vee R'(p) = R(p) \vee R'(p) = [T(p)] \vee [T'(p)] = [(T + T')(p)] = R_{T+T'}(p)$ is a reach map. The fact that $R \vee R'$ is the supremum in the pointwise order is generally true for maps between partially ordered sets: First, by the pointwise definition, we have $R \leq R \vee R'$ and $R' \leq R \vee R'$. On the other hand for any $\tilde{R}$ with $R \leq \tilde{R}$ and $R' \leq \tilde{R}$ we have $(R \vee R')(p) \leq \tilde{R}(p)$ for all $p \in \mathcal{P}(\mathcal{A})$. □

4.1.3 The Rank of a Reach Map

Now, we can decompose our reach map.

Definition A reach map $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ has *rank one* if in the minimal representation $\mathcal{A} \subseteq M_n$ of $\mathcal{A}$ with $\mathcal{B} \subseteq M_m$ there is a Kraus operator $a \in M_{n\times m}$ with $R = R_{\mathrm{Ad}_a}$.

Remark We must use the minimal representation here. As mentioned in 1.3.2 if we allowed arbitrary representations, every reach map would have rank one.

Proposition *For a map $\mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ the following are equivalent:*

(a) *R is a reach map.*
(b) *R is the supremum of a finite family of rank one reach maps.*

Definition The *rank of R* is the smallest cardinality of a finite family fulfilling (b).

Proof (a) ⇒ (b) Let $R = R_T$ with $T = \sum_{i=1}^{d} \mathrm{Ad}_{a_i}$ then $R = \bigvee_{i=1}^{d} R_{\mathrm{Ad}_{a_i}}$.

(b) ⇒ (a) We have just shown in Proposition 4.1.2 that the supremum of reach maps is a reach map. □

Remark Every rank one reach map maps minimal projections to minimal projections or zero. However, not every reach map fulfilling this condition has rank one. An example would be $T\colon M_n \to M_n$ with $n \geq 2$ and $T(x) = e_{11}\varphi(x)$ with a faithful state φ. R_T maps minimal projections to minimal projections but is not of rank one.

While a succinct direct criterion to tell whether a map $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ is a reach map would be desirable, we have reduced the problem of characterizing reach maps to the characterization of rank one reach maps.

4.1.4 The Cross-Ratio

Every reach map preserves suprema and is thus uniquely determined by its behaviour on the minimal projections. We will use this fact to describe reach maps of rank one.

We consider a map $R\colon \mathcal{P}_1(M_2) \to \mathcal{P}_1(M_2)$. This is simply a map on the Bloch sphere and is especially well known in quantum mechanics if $R = R_{\mathrm{Ad}_u}$ for a unitary $u \in M_2$. However, we assume that $R = R_{\mathrm{Ad}_a}$ is a rank one reach map for a not necessarily unitary but invertible $a \in M_2$. The question is how much information is necessary to uniquely determine a or equivalently R. For linear maps, it is sufficient to analyse their behaviour on a basis. However, $\mathcal{P}_1(M_2)$ is not a linear space, and R is not a linear map. Let $(e_{ij})_{1\leq i,j\leq 2}$ be *matrix units* for an orthonormal basis $e_1, e_2 \in \mathbb{C}^2$. If we know the value of $R(e_{11})$ then the value of $R(e_{22})$ is not completely determined. If we additionally know the value $R(e_{22})$ then that does still not determine the value of $R(\frac{1}{2}(e_{11} + e_{12} + e_{21} + e_{22}))$ (the projection onto $\frac{1}{\sqrt{2}}(e_1 + e_2)$). For example, we can have $a_1 e_1 = 2e_1$ and $a_1 e_2 = e_1$ or $a_2 e_1 = e_1$ and $a_2 e_2 = 2e_2$. Both maps $R_{\mathrm{Ad}_{a_1}}$ and $R_{\mathrm{Ad}_{a_2}}$ will behave the same on e_{11} and e_{22} but differently on $\frac{1}{2}(e_{11} + e_{12} + e_{21} + e_{22})$.

We aim to describe how the reach map R is uniquely determined if we evaluate it at three distinct points in $\mathcal{P}_1(M_2)$. As a quotient of the vector space $\mathbb{C}^2 \setminus \{0\}$, the set of rank one projections $\mathcal{P}_1(M_2)$ is a complex *projective space*. Since the global phase and scale of a vector representing a state are unobservable in quantum mechanics, it is well-known that it is sensible to consider pure quantum states as points in a projective space (see [Var07]). Three distinct points in $\mathcal{P}_1(M_2)$ form a so-called *projective frame*, the projective equivalent of a basis. While sensible morphisms between vector spaces need to be linear, i.e. preserve linear combinations, morphisms between projective spaces need to preserve so-called cross-ratios, which we will now define. To get an intuitive grasp of what a cross-ratio is, it might help to already look at statement (3) of Corollary 4.1.5 while reading on.

As a projective space $\mathcal{P}_1(M_2)$ has dimension 1 (the projective dimension is the dimension of the underlying vector space, in this case $\mathbb{C}^2$, minus 1), consequently, it is called a *projective line*. An - or rather *the* - canonical example for a projective line is the projective space $\tilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ (see [Ber94, 4.2.5]). We can imagine it by continuously wrapping $\mathbb{C}$ around the Bloch sphere and tying it together with a point called ∞ on the opposite side. A map f between projective spaces is called a *projectivity* (or homography or isomorphism between projective spaces) if it is induced by an invertible linear map a via $f(\mathbb{C}\xi) = \mathbb{C}a\xi$ for all vectors ξ from the underlying vector space (cf. [Ber94, 4.5.2]).

Let p_1,p_2,p_3, p_4 be points on a complex projective line L, with p_1,p_2 and p_3 being distinct.

Proposition [Ber94, 4.6.9] *There is a unique projectivity $f : L \to \tilde{\mathbb{C}}$ with $f(p_1) = \infty$, $f(p_2) = 0$ and $f(p_3) = 1$.*

Definition [Ber94, 6.1.1] We define the *cross-ratio* of the four points p_1 to p_4, as $[p_1, p_2, p_3, p_4] := f(p_4) \in \tilde{\mathbb{C}}$. We also write $[p_i] := [p_1, p_2, p_3, p_4]$.

Remark The cross-ratio can be understood as the position of p_4 on the projective line in coordinates fixed by the three other points. In a linear space, a line, i.e. a one-dimensional subspace, would be determined by one vector ξ. Then any point on that line could be addressed as $\lambda\xi$ with $\lambda \in \mathbb{C}$. So, this simple coordinate would be the linear analogue of a cross-ratio. In an affine space, a line is spanned by two vectors ξ_1 and ξ_2. Again, any point on the line can be addressed as $\lambda\xi_1 + (1 - \lambda)\xi_2$ with $\lambda \in \mathbb{C}$. This would be the affine analogue to a cross-ratio.

4.1.5 Cross-Ratios for Rank One Projections

There is a concrete formula to calculate the cross-ratio for rank one projections in a matrix algebra. We formulate it here not because it is convenient to calculate, but to demonstrate that cross-ratios are a concrete property of the lattice of projections.

Proposition *Let $p_1, p_2, p_3, p_4 \in \mathcal{P}_1(\mathcal{A})$ four minimal projections such that p_1, p_2 and p_3 are pairwise distinct and $p_3, p_4 \leq p_1 \vee p_2$. We define $p_1' := (p_1 \vee p_2) - p_1$ and $\lambda_i \in \tilde{\mathbb{C}}$ so that $p_1' p_i p_1' = \lambda_i p_1'$.*

Then p_1, p_2, p_3, p_4 lie on a projective line and the cross-ratio $\lambda := [p_1, p_2, p_3, p_4]$ fulfils

$$p_1 \left(\frac{p_4}{\lambda_4} - \frac{p_2}{\lambda_2} \right) p_1' = \lambda \, p_1 \left(\frac{p_3}{\lambda_3} - \frac{p_2}{\lambda_2} \right) p_1'.$$

For $\xi, \eta \in \mathbb{C}^n$ we use the notation $t_{\xi,\eta}$ for the rank one matrix $t_{\xi,\eta}\zeta := \langle \eta, \zeta \rangle \xi$.

Proof Since it contains at least three distinct orthogonal projection of rank one but $p_1 \vee p_2$ has at most rank 2 the reduced algebra $\mathcal{A}_{p_1 \vee p_2}$ is isomorphic to M_2. Consequently, the rank one projections $\mathcal{P}_1(\mathcal{A}_{p_1 \vee p_2})$ form a projective line.

We can find an orthonormal basis $e_1, e_2 \in (p_1 \vee p_2)\mathcal{H}$ in which

$$\begin{aligned} p_1\mathcal{H} &= \mathbb{C}e_1 & p_2\mathcal{H} &= \mathbb{C}(\alpha_2 e_1 + \beta_2 e_2) \\ p_3\mathcal{H} &= \mathbb{C}(\alpha_3 e_1 + \beta_3 e_2) & p_4\mathcal{H} &= \mathbb{C}(\alpha_4 e_1 + \beta_4 e_2) \end{aligned}$$

with $\alpha_i, \beta_i \in \mathbb{C}$ and $|\alpha_i|^2 + |\beta_i|^2 = 1$ for $i \in \{2, 3, 4\}$. By our definition we have $\lambda_i = |\beta_i|^2$ and $p_1 p_i p_1' = \alpha_i \overline{\beta}_i t_{e_1,e_2}$. If we insert this into our formula for λ, we get

$$\lambda = \frac{\frac{\alpha_4}{\beta_4} - \frac{\alpha_2}{\beta_2}}{\frac{\alpha_3}{\beta_3} - \frac{\alpha_2}{\beta_2}}.$$

The projections p_i for $i \in \{1, \ldots, 4\}$ are points on the projective line $\mathcal{P}_1(\mathcal{A}_{p_1 \vee p_2})$. By definition, they all can be given in coordinates of e_1 and e_2, up to a scalar factor. So, according to [Ber94, 6.2.3], their cross-ratio is given by

$$\frac{-\beta_3 \cdot \begin{vmatrix} \alpha_4 & \alpha_2 \\ \beta_4 & \beta_2 \end{vmatrix}}{\begin{vmatrix} \alpha_3 & \alpha_2 \\ \beta_3 & \beta_2 \end{vmatrix} \cdot -\beta_4} = \frac{\beta_3(\alpha_4\beta_2 - \beta_4\alpha_2)}{(\alpha_3\beta_2 - \beta_3\alpha_2)\beta_4} = \lambda.$$

□

Corollary

(1) *The* cross-ratio *is ∞ iff $p_4 = p_1$, 0 iff $p_4 = p_2$ and 1 iff $p_4 = p_3$.*

(2) *If $p_1 \perp p_2$ then the cross-ratio simplifies to*

$$p_1 p_4 p_2 = \frac{\lambda \lambda_4}{\lambda_3} p_1 p_3 p_2$$

.

(3) *If p_i is the projection onto $\mathbb{C}\xi_i$ with $\xi_1 = e_1$, $\xi_2 = e_2$, $\xi_3 = \frac{1}{\sqrt{2}}(e_1 + e_2)$ and $\xi_4 = \alpha e_1 + \beta e_2$ then $[p_1, p_2, p_3, p_4] = \frac{\alpha}{\beta}$.*

PROOF (1) Is obvious from the definition of the cross-ratio.

(2) With $p_2 = p_1'$ we get $\beta_2 = 1$ and so $\lambda_2 = 1$. With $p_2 p_1 = 0$, the given formula follows.

(3) With $\alpha_2 = 0$ and $\alpha_3 = \beta_3$ we can see $[p_i] = \frac{\alpha_4}{\beta_4}$ in the formula for λ in the above proof. □

4.1.6 Characterization of Rank One Reach Maps

DEFINITION Let $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ be a suprema preserving function. We define the *support* of R as $\operatorname{supp} R := \mathbb{1} - \bigvee\{\, p \in \mathcal{P}(\mathcal{A}) : R(p) = 0 \,\}$.

REMARK This definition is crafted so that if $R = R_T$ for a completely positive operator T, we have $\operatorname{supp} R = \operatorname{supp} T$.

THEOREM *Let $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ be a map on the lattice of orthogonal projections in $\mathcal{A}$. Then the following are equivalent:*

(a) *R is a reach map of rank one.*

(b) *R fulfils the following conditions:*

  (1) *R preserves suprema.*

  (2) *$R|_D$ is an injective, cross-ratio preserving map into $\mathcal{P}_1(\mathcal{B})$ with $D := \{\, p \in \mathcal{P}_1(\mathcal{A}) : p \leq \operatorname{supp} R \,\}$.*

  (3) *$R(p) = R([(\operatorname{supp} R)p(\operatorname{supp} R)])$ for all $p \in \mathcal{P}_1(\mathcal{A})$.*

The more interesting conditions in (b) are (1) and (2). Condition (3) is just necessary so that R extends sensibly outside its support.

To prove this result, we first show the core argument in the following lemma:

LEMMA *For a map $R\colon \mathcal{P}_1(M_n) \to \mathcal{P}_1(M_n)$ the following are equivalent:*

(a) *R is a projectivity.*

(b) *There is a matrix $a \in \mathrm{GL}(n)$ with $R = R_{\mathrm{Ad}_a}|_{\mathcal{P}_1(M_n)}$.*

(c) *R is a bijection which preserves suprema and cross-ratios.*

REMARK By [Fis01, 3.2.1], the linear map $a \in \mathrm{GL}(n)$ in statement (b) is unique up to scalar multiples.

PROOF (b) $\Rightarrow$ (a) Via $\mathrm{Ad}_a(t_{\xi,\xi}) = t_{a^*\xi,a^*\xi}$ the reach map of Ad_a is the projectivity on the projective space $\mathcal{P}_1(M_n)$ which is induced by the invertible linear map a^*.

(a) $\Rightarrow$ (b) If R is a projectivity, then by definition, there is an invertible linear map $a \in \mathrm{GL}(n)$ which induces R. We can implement it as $R = R_{\mathrm{Ad}_{a^*}}$.

(a) $\Rightarrow$ (c) Projectivities are bijective and preserve cross-ratios (cf., e.g. [Fis01, 3.3.1]).

(c) $\Rightarrow$ (a) *We show the implication via case distinction over the dimension $n \in \mathbb{N}$.*

Let $n = 1$. Then $\mathcal{P}_1(M_1)$ has exactly one point, and the result is clear.
Let $n = 2$. Then $\mathcal{P}_1(M_2)$ is a projective line. Since R preserves cross-ratios, it is a projectivity by [Fis01, 3.3.1].
Let $n \geq 3$. Then we can apply the main theorem of projective geometry (cf. [Fis01, 3.3.9]). It says that since R fulfils $R(p \vee q) \leq R(p) \vee R(q)$ it is a so-called *semi-projectivity*, where a semi-projectivity is a map on a projective space which is induced by a bijective semi-linear map, i.e. a map $a\colon \mathbb{C}^m \to \mathbb{C}^m$ with $a(\xi + \eta) = a\xi + a\eta$ and $a(\lambda\xi) = \alpha(\lambda)a\xi$ for some field-automorphism $\alpha\colon \mathbb{C} \to \mathbb{C}$. Since R preserves cross-ratios, we know that R is a projectivity when we restrict it to any projective line, as shown in the $n = 2$ step. So, since the underlying semi-linear map a is a linear map on every 2-dimensional subspace, it is a linear map on all of $\mathbb{C}^n$. Thus, R is a projectivity. ◻

Now we prove the actual theorem:

Proof For the whole proof, we represent $\mathcal{A} \subseteq M_n$ and $\mathcal{B} \subseteq M_m$ minimally.

(a) $\Rightarrow$ (b) We have a Kraus operator $a \in M_{n \times m}$ with $R = R_{\mathrm{Ad}_a}$. We check the conditions in (b).

(1) As a reach map, R is join preserving.

(2) For every $p \in D$ a^*pa is not zero. Since we have represented $\mathcal{A}$ minimally, p has rank one. Hence, a^*pa can only have rank one and $[a^*pa]$ must be in $\mathcal{P}_1(\mathcal{B})$. $R|_D$ is injective and preserves cross-ratios because it is a restriction of the projectivity $R_{\mathrm{Ad}_a}\colon \mathcal{P}_1((\mathrm{supp}R)M_n(\mathrm{supp}R)) \to \mathcal{P}_1(R(\mathbb{1})M_mR(\mathbb{1}))$.

(3) This condition is true for all reach maps.

(b) $\Rightarrow$ (a) Our task is to find a Kraus operator a which induces R. We do this by splitting $\mathrm{supp}R$ along the minimally central projections of $\mathcal{A}$ into a family of orthogonal projections $(p_i)_{1 \leq i \leq d}$ with $\mathrm{supp}R = \sum_{i=1}^d p_i$. Now for each $i \in \{1, \ldots, d\}$, the reduced algebra $p_i\mathcal{A}p_i$ is isomorphic to M_{k_i} for some $k_i \in \mathbb{N}$. We want to show that R is a projectivity on each of these factors. Then we glue the resulting Kraus operators together into one.

Auxiliary statement: For any $p \leq \mathrm{supp}R$ *such that* $p\mathcal{A}p \simeq M_k$ *for a* $k \in \mathbb{N}$ *we have* $R(p)\mathcal{B}R(p) \simeq M_k$ *and* R *maps* $\mathcal{P}_1(p\mathcal{A}p)$ *bijective onto* $\mathcal{P}_1(R(p)\mathcal{B}R(p))$. First, we observe that p can be written as a sum of minimal projections, which get mapped to minimal projections. Consequently, we know that $\dim R(p) \leq k$.
We show the rest of the statement via induction on k.
For $k = 1$ the statement is obvious because p and $R(p)$ have rank one and $\mathcal{P}_1(\mathbb{C})$ consists only of one point.
For $k = 2$ we see by injectivity, that three distinct points below p get mapped to three distinct points below $R(p)$. So, $R(p)$ has at least and at most rank 2. $R(p)\mathcal{B}R(p)$ can

also not be commutative because then R could not be injective, thus $R(p)\mathcal{B}R(p) \simeq M_2$. Because every point in $\mathcal{P}_1(M_2) \subseteq \mathcal{B}$ can be given a coordinate as a cross-ratio regarding the image of the three distinct points we chose earlier, R is surjective onto $\mathcal{P}_1(M_2)$. *Now let p have rank $k+1$* and let $q \leq p$ be a projection of rank k. Then $\mathcal{P}_1(q\mathcal{A}q)$ gets mapped surjectively into a subspace of the form $\mathcal{P}_1(M_k)$. Two further points in $\mathcal{P}_1(p\mathcal{A}p) \setminus \mathcal{P}_1(q\mathcal{A}q)$ will, because of injectivity, be mapped to two different points outside of $R(q)\mathcal{B}R(q)$. So $R(p)$ needs to have rank $k+1$ and $R(p)\mathcal{B}R(p)$ needs to be isomorphic to M_{k+1} because that is the smallest subalgebra of $R(p)M_mR(p)$ which contains $R(q)\mathcal{B}R(q)$ and two different projections of rank 1. Again, every point in $\mathcal{P}_1(R(p)\mathcal{B})R(p))$ can be given a coordinate as a cross-ratio and thus is in $R(\mathcal{P}_1(p\mathcal{A}p))$. This proves the auxiliary statement.

In conclusion, we have shown that R can be written as a combination of different projectivities on D. Because $R|_D$ is injective, the projectivities have disjoint images. For every projectivity on a p_i we find a Kraus operator a_i with $a_i^*\colon p_i\mathbb{C}^n \to \mathbb{C}^m$. We can add them to define $a := \sum_{i=1}^d a_i$. For every $p \in \mathcal{P}_1(\mathcal{A})$ we get $R(p) = R_{\mathrm{Ad}_a}(p)$. □

Remark A welcome property of the characterization of rank one reach maps is that the criteria are representation-independent.

In commutative matrix algebras, no projective lines exist, because they have no subalgebra isomorphic to M_2. So, every map trivially preserves cross-ratios. The conditions (2) and (3) become trivial. Thus, in commutative matrix algebras, the reach maps are precisely the suprema preserving maps.

This characterization of reach maps does not rely on completely positive operators, which is the beauty of this theorem. However, explicitly calculating the cross-ratios is not very practical. In fact, we only use the preceding result once again in this thesis. It makes the proof of Proposition 4.4.3 more elegant, although an elementary (but tedious) proof which does not rely on projective geometry can be given for that proposition. Besides that exception, we put the perspective from projective geometry aside and continue to investigate reach maps from the point of view of completely positive operators.

4.2 Topological Equivalence

Reach maps describe the topological properties of a completely positive operator. Different completely positive operators can have the same topological properties. We will consider those operators equivalent which lead to the same reach map. We explore this equivalence in this section.

4.2.1 Topological Comparison

In Proposition 4.1.2, we have established a partial order (transitive, reflexive, antisymmetric) on reach maps. We can pull back this order through the functor $T \mapsto R_T$ to get a pre-order (transitive, reflexive, but not antisymmetric) on completely positive operators.

Proposition *For all completely positive operators $S, T\colon \mathcal{A} \to \mathcal{B}$ the following are equivalent:*

(a) $R_S \leq R_T$.

(b) *For all completely positive operators $P : \mathcal{B} \to \mathcal{B}$, $Q\colon \mathcal{A} \to \mathcal{A}$, we have $PTQ = 0 \Rightarrow PSQ = 0$.*

(c) *For all orthogonal projections $p \in \mathcal{P}(\mathcal{A})$, $q \in \mathcal{P}(\mathcal{B})$, we have $T(p) \perp q \Rightarrow S(p) \perp q$.*

(d) *For all minimal orthogonal projections $p \in \mathcal{P}_1(\mathcal{A})$, $q \in \mathcal{P}_1(\mathcal{B})$, we have $T(p) \perp q \Rightarrow S(p) \perp q$.*

Proof (a) $\Rightarrow$ (b) For all $p \in \mathcal{P}(\mathcal{A})$, $PTQ = 0$ means $0 = P(R_T(Q(p))) \geq P(R_S(Q(p)))$ and $0 = [P(R_S(Q(p)))] = [P(S(Q(p)))]$.

(b) $\Rightarrow$ (c) Let $p \in \mathcal{P}(\mathcal{A})$, $q \in \mathcal{P}(\mathcal{B})$ with $S(q)p = 0$, then we have $\mathrm{Ad}_p \circ S \circ \mathrm{Ad}_q = 0$. With (b) we can conclude $\mathrm{Ad}_p \circ T \circ \mathrm{Ad}_q = 0$ and therefore $T(q)p = 0$.

(c) $\Rightarrow$ (d) Is clear because $\mathcal{P}_1(\mathcal{A}) \subseteq \mathcal{P}(\mathcal{A})$.

(d) $\Rightarrow$ (a) Pick any minimal orthogonal projections $p \in \mathcal{P}_1(\mathcal{A})$, $q \in \mathcal{P}_1(\mathcal{B})$ with $R_T(p) \perp q$. Then from $T(p) \perp q$ we get $S(p) \perp q$. This holds for any minimal q orthogonal to $R_T(p)$. Together, those cover $R_T(p)^\perp$ which gives us $R_S(p) \leq R_T(p)$. Any non-minimal $p \in \mathcal{P}(\mathcal{A})$ can be written as a union of minimal projections $p_1, \ldots, p_k$. Since reach maps preserve suprema, we get $R_S(p_i) \leq R_T(p)$ and $R_S(p) \leq R_T(p)$. □

4.2.2 Topological Equivalence

We can now pick any of the criteria above to define an equivalence relation on the completely positive operators.

Corollary *For two completely positive operators $S, T\colon \mathcal{A} \to \mathcal{B}$ the following are equivalent:*

(a) $R_S \leq R_T$ *and* $R_T \leq R_S$.

(b) $R_S = R_T$.

(c) $S(p) \perp q \Leftrightarrow T(p) \perp q$ *for all* $p \in \mathcal{P}(\mathcal{A})$, $q \in \mathcal{P}(\mathcal{B})$.

Proof (a) $\Rightarrow$ (b) Because the $\leq$ relation on reach maps is a partial order it is antisymmetric. This gives us exactly the wanted implication.

(b) $\Rightarrow$ (c) $S(p) \perp q \Leftrightarrow R_S(p) \perp q \Leftrightarrow R_T(p) \perp q \Leftrightarrow T(p) \perp q$.

(c) $\Rightarrow$ (a) Follows from Proposition 4.2.1. □

Definition Given those conditions, we say that S and T are *topologically equivalent*.

Since $T \mapsto R_T$ maps a pre-order to a partial order, it maps, by our definition, an equivalence class of completely positive operators to a single reach map.

4.2.3 Reach Maps as Faces in the Cone of Completely Positive Operators

We might wonder whether the topological equivalence classes of completely positive operators have a geometric structure. Indeed, we can find a closely related face in the cone of completely positive operators for every equivalence class. We write $\mathrm{CP}(\mathcal{A}, \mathcal{B})$ for the cone of completely positive operators.

Definition A *face* in $\mathrm{CP}(\mathcal{A}, B)$ is a subset $F \subseteq \mathrm{CP}(\mathcal{A}, B)$ such that for all $T = S + Q$ with $T, S, Q \in \mathrm{CP}(\mathcal{A}, \mathcal{B})$ the following are equivalent:

(a) $T \in F$.
(b) $S \in F$ and $Q \in F$.

Proposition *For any reach map* $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ *the set* $\{\, T \in \mathrm{CP}(\mathcal{A}, \mathcal{B}) : R_T \leq R\}$ *is a face in the cone* $\mathrm{CP}(\mathcal{A}, \mathcal{B})$ *of completely positive operators.*

Proof Let $T = S + Q$ with $S, Q \in \mathrm{CP}(\mathcal{A}, \mathcal{B})$ we need to show that the following are equivalent:

(a) $R_T \leq R$.
(b) $R_S \leq R$ and $R_Q \leq R$.

(a) $\Rightarrow$ (b) For all $p \in \mathcal{P}(\mathcal{A})$, $q \in \mathcal{P}(\mathcal{B})$ we see that $T(p) \perp q$ implies $S(p) \leq T(p) \perp q$ and $Q(p) \perp q$.

(b) $\Rightarrow$ (a) For all $p \in \mathcal{P}(\mathcal{A})$, $q \in \mathcal{P}(\mathcal{B})$ we can conclude from $S(p) \perp q$ and $Q(p) \perp q$ that $T(p) = S(p) + Q(p) \perp q$. □

Remark The face generated by R_T contains the equivalence class of T. Not all elements of the face generated by R_T are equivalent to T because they might belong to a strictly smaller reach map, which would generate a strict sub-face. The face generated by T is always contained in the face generated by R_T. However, this inclusion can be strict. As we will see in example 4.4.5 (once we have more tools to talk about faces), there can be a completely positive operator S which does not belong to the face generated by T but $R_S \leq R_T$.

4.2.4 Homomorphisms from Reach Maps

When we talk about equivalence classes of completely positive operators, it is interesting to know when the operator is up to scalar multiples uniquely determined by the reach map. One such class are the *-homomorphisms. In example 2 in 3.1.1, we have already seen that reach maps of *-homomorphisms preserve orthogonality. In finite dimensions, this is an equivalent characterization.

Proposition *If $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ is a reach map, then the following are equivalent:*

(a) *$p \perp q$ implies $R(p) \perp R(q)$ for all $p, q \in \mathcal{P}(\mathcal{A})$.*
(b) *$R = R_i$ for a *-homomorphism $i\colon \mathcal{A} \to \mathcal{B}$.*

Proof We pick a $T : \mathcal{A} \to \mathcal{B}$ with $R = R_T$. Then for $x, y \in \mathcal{A}^+$ with $[x] \perp [y]$ we have $[T(x)] = R_T(x) \perp R_T(y) = [T(y)]$, which means $T(y) \perp T(x)$. By [WZ09, Theorem 3.3] this means that there is a *-homomorphism $\pi\colon \mathcal{A} \to \mathcal{B} \cap \{T(\mathbb{1})\}'$ with $T(x) = T(\mathbb{1})\pi(x)$ for all $x \in \mathcal{A}$. We define $i\colon x \mapsto [T(\mathbb{1})]\pi(x)$, which is also a *-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ because π takes values in $\mathcal{B} \cap \{T(\mathbb{1})\}'$. With statement (5) from Lemma 3.1.2 get $R_i(p) = [[T(\mathbb{1})]\pi(p)] = [T(\mathbb{1})\pi(p)] = [T(p)] = R(p)$ for all $p \in \mathcal{P}(\mathcal{A})$. □

4.2.5 Uniqueness of Homomorphisms

Proposition *For any two *-homomorphisms $i, j : \mathcal{A} \to \mathcal{B}$ the following are equivalent:*

(a) *$i = j$.*
(b) *$R_i = R_j$.*

Proof (a) $\Rightarrow$ (b) Clear.

(b) $\Rightarrow$ (a) Since $\mathcal{P}(\mathcal{A})$ spans $\mathcal{A}$ and $i|_{\mathcal{P}(\mathcal{A})} = R_i = R_j = j|_{\mathcal{P}(\mathcal{A})}$ we have $i = j$. □

Remark As in the category of normal completely positive operators (which do not necessarily preserve stationary states) *-homomorphisms are not the monomorphisms in the category of reach maps.

Not every reach map with an inverse reach map belongs to a *-isomorphism or needs to be orthogonality preserving. Consider Ad_a for any full rank but non-normal $a \in M_2$.

4.3 Blocked Transitions

In Corollary 4.2.2, we have seen that we can pinpoint a reach map by knowing all pairs of projections $p \in \mathcal{A}$ and $q \in \mathcal{B}$ with $R(p) \perp q$. We will call this a blocked transition. This is merely a change of perspective but will be very relevant when investigating dynamics.

For a completely positive operator $T\colon M_n \supseteq \mathcal{A} \to \mathcal{B} \subseteq M_m$ we define the set $\mathrm{Kr}(T) := \mathbb{C}\{\, a_i \in M_{n\times m} \mid a_i \text{ is a Kraus operator of } T\,\}$.

4.3.1 Transitions

Before we look at blocked transition, we can have a more general look for context. If p is a minimal projection we have $pT(q)p = \lambda p$ for some $\lambda \in \mathbb{R}_{\geq 0}$. Thus, we could consider labelling the transition from q to p with the value λ. In fact, labelling all transitions this way already completely determines T since there is a basis of $\mathcal{A}$ consisting of minimal orthogonal projections.

Proposition *If $T, S\colon \mathcal{A} \to \mathcal{B}$ are two completely positive operators, then the following are equivalent:*

(a) $T = S$

(b) $qT(p)q = qS(p)q$ *for all $p \in \mathcal{P}_1(\mathcal{A}), q \in \mathcal{P}_1(\mathcal{B})$.*

(c) $q_jT(p_i)q_j = q_iS(p_j)q_i$ *for all p_i, q_j in bases of $\mathcal{A}$ and $\mathcal{B}$ consisting of minimal orthogonal projections.*

Remark If we pick some bases of minimal projections for $\mathcal{A}$ and $\mathcal{B}$, then T is uniquely determined by its positive value $q_jT(p_i)q_j = \lambda_{ij}q_j$ on the pairs (p_i, q_j) like in condition (c). However, not all families of λ_{ij} result in a completely positive operator.

4.3.2 Blocked Transitions

Coming back to blocked transitions, like in the classical case, we ignore the exact numeric amount of the λ_{ij} for a topological investigation. We only care whether a transition can happen at all, that means whether it is blocked or not. We now give many relations which all describe blocked transitions. Some of them are very similar, but we spell out all of them because it is helpful to have all of them handy in day-to-day calculations.

Proposition *Let $\mathcal{A} \subseteq M_n, \mathcal{B} \subseteq M_m$ be matrix algebras and $T\colon \mathcal{A} \to \mathcal{B}$ a completely positive contraction. For two orthogonal projections $p \in \mathcal{P}(\mathcal{A})$, $q \in \mathcal{P}(\mathcal{B})$ the following are equivalent:*

(a) $R_T(p) \perp q$.

(b) $T(p) \leq q^\perp$.

(c.i) $qT(p)q = 0$.	(d.i) $paq = 0$ *for all $a \in \mathrm{Kr}(T)$.*
(c.ii) $T(p)q = 0$.	(d.ii) $pa = paq^\perp$ *for all $a \in \mathrm{Kr}(T)$.*
(c.iii) $qT(p) = 0$.	(d.iii) $aq = p^\perp aq$ *for all $a \in \mathrm{Kr}(T)$.*
(e.i) $qT(x)q = qT(p^\perp xp^\perp)q$.	(f.i) $T(pxp) = q^\perp T(pxp)q^\perp$.
(e.ii) $qT(x) = qT(p^\perp x)$.	(f.ii) $T(px) = q^\perp T(px)$.
(e.iii) $T(x)q = T(xp^\perp)q$.	(f.iii) $T(xp) = T(xp)q^\perp$.

And if $T(\mathbb{1}) = \mathbb{1}$:

(g) $T(p^\perp) \geq q$.

Definition In this case, we call the tuple (p, q) a *blocked transition* for R_T or T.

Remark Condition (d.i) means that all Kraus operators have a “corner” in which they are zero, as hinted at by this matrix representation:

$$a_i = \begin{array}{c c} & \begin{array}{cc} q & q^\perp \end{array} \\ \begin{array}{c} p \\ p^\perp \end{array} & \left(\begin{array}{cc} 0 & * \\ * & * \end{array} \right) \end{array} .$$

Proof (a) ⇔ (b) ⇔ (c) All those criteria are slightly different formulations of $T(p) \perp q$ (see 2.1.3).

(c.i) ⇒ (d.i) If $0 = qT(p)q = \sum_{i=1}^d qa_i^* pa_i q = 0$ we can follow $qa^* paq = 0$ for all $a \in \mathrm{Kr}(T)$, which means $paq = 0$.

(d.i) ⇔ (d.ii) ⇔ (d.iii) The three statements are equivalent because $pa = paq + paq^\perp$ and $aq = paq + p^\perp aq$.

(d.iii) ⇒ (e.*) For example $qT(x) = q\sum_{i=1}^d a_i^* x a_i = q\sum_{i=1}^d a_i^* p^\perp x a_i = qT(p^\perp x)$.

(d.ii) ⇒ (f.*) For example $pT(x) = \sum_{i=1}^d a_i^* pxa_i = q^\perp \sum_{i=1}^d a_i^* pxa_i = q^\perp T(px)$.

(e.*) ⇒ (c.*) For example $qT(p)q = qT(p^\perp pp^\perp q = qT(0)q = 0$.

(f.*) ⇒ (c.*) For example $qT(p)q = qq^\perp T(p)q^\perp q = 0$.

(b) ⇒ (g) $T(p^\perp) = T(\mathbb{1}) - T(p) = \mathbb{1} - T(p) \geq \mathbb{1} - q^\perp = q$.

(g) ⇒ (b) This step actually only requires that T is a contraction: $T(p) = T(\mathbb{1}) - T(p^\perp) \leq \mathbb{1} - T(p^\perp) \leq \mathbb{1} - q = q^\perp$. □

Remark This proposition is a generalization of [Bra17, 2.1.1]. There, as we will see in Chapter 5, T is defined as an endomorphism and p and q are orthogonal to each other. It was the suggestion by Burkhard Kümmerer to drop the orthogonality in that proposition to analyse topological Markov chains which started the endeavour culminating in this thesis.

Saying “the transition from p to q is blocked” is again spoken in the Heisenberg picture. It might sometimes be more intuitive to say that a state transition from q to p is blocked.

If (p, q) is blocked, then for all $p' \leq p$ and $q' \leq q$, the transition (p', q') is also blocked. It is, thus, in some sense sufficient to only know the “largest” blocked transitions of an operator T. This intuition led to the definition of reach maps which capture precisely this information because $(p, R(p)^\perp)$ is blocked. The reach map R maps p to the smallest projection q for which $(p, q^\perp)$ is a blocked transition.

Example 1 If p or q are zero (p, q) is always blocked.

Example 2 For the zero reach map $R\colon \mathcal{A} \to \mathcal{B}$ with $R(p) = 0$, all transitions are blocked.

Example 3 If R has a non-zero null space $p \in \mathcal{P}(\mathcal{A})$, then for all $q \in \mathcal{P}(\mathcal{B})$, the transition (p, q) is blocked.

Example 4 Let $R = R_{\mathrm{Ad}_a}$ for some $a \in M_{n\times m}$. Then for every (unit) vector $\xi \in \mathbb{C}^n$, $a^* t_{\xi,\xi} a$ is one-dimensional. If we write q for the projection onto $(a^*\xi)^\perp$, then $(t_{\xi,\xi}, q)$ is blocked.

4.4 Kraus Operator Modules of Reach Maps

This section looks at the most concrete representation of reach maps. We have already established a connection between reach maps and faces in the cone of completely positive operators. As a next step, we connect faces in the cone of completely positive operators to submodules of the linear spaces in which the Kraus operators live. Those are different representations of reach maps which can be easier to analyse.

4.4.1 Kraus Operators and Faces of Completely Positive Operators

Let $T\colon M_n \supseteq \mathcal{A} \to \mathcal{B} \subseteq M_m$ be a completely positive operator.

Definition We call a subspace $K \subseteq M_{n\times m}$ an *$\mathcal{A}'$-$\mathcal{B}'$-submodule* if $a'xb' \in K$ for all $x \in K$, $a \in \mathcal{A}'$ and $b \in \mathcal{B}'$.

Remark The face of the cone $\mathrm{CP}(\mathcal{A}, \mathcal{B})$ generated by T is the smallest face which contains T. It is spanned by operators majorized by T. Every face C of the cone $\mathrm{CP}(\mathcal{A}, \mathcal{B})$ has a (non-unique) representative operator generating it. To see this, we start with any operator T. If $S \in C$ is not in the face generated by T, we can switch to the face generated by $S + T$. This procedure terminates because $\mathrm{CP}(\mathcal{A}, \mathcal{B})$ is finite-dimensional.

Proposition (1) *The set* $\mathrm{Kr}(T) \subseteq M_{n\times m}$ *is an* $\mathcal{A}'$-$\mathcal{B}'$*-submodule of* $M_{n\times m}$.

(2) *The set* $\{ S \in \mathrm{CP}(\mathcal{A}, \mathcal{B}) : a \in \mathrm{Kr}(T)$ *for all Kraus operators* a *of* $S\}$ *is the face generated by* T.

This relation establishes a one-to-one correspondence between $\mathcal{A}'$-$\mathcal{B}'$*-submodules of* $M_{n\times m}$ *and faces of* $\mathrm{CP}(\mathcal{A}, \mathcal{B})$.

Proof Let $(a_i)_{1\le i\le d}$ be a Kraus representation of T. This proof relies mainly on the following theorem: By [Küm86, 1.1.12] for any operator the following are equivalent:

(a) There is $\lambda > 0$ such that $T - \lambda S$ is completely positive, i.e. S is an element of the face generated by T.

(b) All Kraus operators S are of the form $\sum_{i=1}^d a_i' a_i$ with $a_i' \in \mathcal{A}'$.

(1) By the above statement, $\mathrm{Kr}(T)$ is closed under addition. Let $(a_i)_{i\in I}$ be a Kraus representation of a T and some unitaries $u'_a \in \mathcal{A}'$, $u'_b \in \mathcal{B}'$ given. Then $(u'_b a_i u'_a)_{i\in I}$ is also a Kraus representation of T via $\sum_{i=1} u'^*_b a^*_i u'^*_a x u'_a a_i u'_b = \sum_{i=1} a^*_i x u'^*_a u'_a a_i u'^*_b u'_b = T(x)$ for all $x \in \mathcal{A}$. Since $\mathrm{Kr}(T)$ is closed under addition and $\mathcal{A}'$ and $\mathcal{B}'$ are spanned by their unitary elements, $\mathrm{Kr}(T)$ is an $\mathcal{A}'$-$\mathcal{B}'$-submodule.

(2) Again, by the given equivalence, all Kraus operators of every element of the face generated by T are elements of $\mathrm{Kr}(T)$ and all operators with Kraus operators out of $\mathrm{Kr}(T)$ are part of the face generated by T.

One-to-one correspondence: If K is an $\mathcal{A}'$-$\mathcal{B}'$-submodule of $M_{n\times m}$ then since it also is a linear subspace, we can pick a basis of K and use it as a Kraus representation to define an operator T. Then $\mathrm{Kr}(T)$ is an $\mathcal{A}'$-$\mathcal{B}'$-submodule which contains a basis of K and thus K. Nevertheless, since every Kraus operator for an operator in the side generated by T is an $\mathcal{A}'$ sum over the previous basis, K is all of $\mathrm{Kr}(T)$. Thus, every $\mathcal{A}'$-$\mathcal{B}'$-submodule corresponds to a face in $\mathrm{CP}(\mathcal{A},\mathcal{B})$ and their correspondence embeds the $\mathcal{A}'$-$\mathcal{B}'$-submodules into the faces of $\mathrm{CP}(\mathcal{A},\mathcal{B})$. Since any face is generated by a completely positive operator, our mapping is one-to-one. □

4.4.2 The Kraus Operator Module of a Reach Map

We already know that reach maps generate faces in the cone of completely positive operators. Therefore, we can now define a corresponding $\mathcal{A}'$-$\mathcal{B}'$-submodule of Kraus operators belonging to a reach map.

Definition Let $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ be a reach map. We define the *Kraus operator module* of R as

$$\mathrm{Kr}(R) := \bigcup \{\mathrm{Kr}(T) : R = R_T,\, T \in \mathrm{CP}(\mathcal{A},\mathcal{B})\}.$$

Corollary $\mathrm{Kr}(R)$ *is the* $\mathcal{A}'$-$\mathcal{B}'$*-submodule of* $M_{n\times m}$ *belonging to the face* $\{T \in \mathrm{CP}(\mathcal{A},\mathcal{B}) : R_T \le R\}$. *It is also given by* $\mathrm{Kr}(R) = \{a \in M_{n\times m} : \forall p \in \mathcal{P}(\mathcal{A}), q \in \mathcal{P}(\mathcal{B}), R(p) \perp q \Rightarrow paq = 0\}$.

Remark As discussed, the face generated by R has a representative operator T such that $\mathrm{Kr}(R) = \mathrm{Kr}(T)$. This operator, T, has precisely those blocked transitions which define R. There are always operators in the face generated by R with more blocked transitions and smaller generated faces.

We now have at least four equivalent ways to describe the topological properties of completely positive operators: Reach maps, blocked transitions, specific faces in the cone of completely positive operators and certain Kraus operator modules. The relations between those different perspectives is monotonous in the following way: A larger reach map is equivalent to a larger face, a larger Kraus operator module and fewer blocked transitions.

4.4.3 Kraus Operator Modules of Rank One Reach Maps

Since we characterized reach maps by looking at reach maps of rank one, it makes sense to analyse what their Kraus operator module looks like.

Proposition *Let $\mathcal{A} \subseteq M_n, \mathcal{B} \subseteq M_m$ minimally represented and $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ a rank one reach map then there is a matrix $a \in M_{n\times m}$ such that $\mathrm{Kr}(R) = \mathcal{A}' a \mathcal{B}'$.*

Proof Let T be a completely positive operators with $R_T = R$ and $T = \sum_{i=1}^{d} \mathrm{Ad}_{a_i}$. We split the support of T along the central projections of $\mathcal{A}$ into $\operatorname{supp} T = \sum_{j=1}^{k} p_j$. By the Lemma in 4.1.6, a_j is up to scalar multiples uniquely determined on $p_j\mathbb{C}^n$. That means there are $b_j\colon p_j\mathbb{C}^n \to \mathbb{C}^m$ such that for all a_i there are λ_{ij} with $a_i^* = \sum_{j=1}^{k} \lambda_{ij} b_j z_j = \sum_{j=1}^{k} b_j z_j \sum_{j=1}^{k} \lambda_{ij} z_j$. Since $\sum_{i=1}^{k} \lambda_{ij} z_j \in \mathcal{A}'$ with $a := \sum_{j=1}^{k} b_j^*$ we get the result. □

Corollary *If $\mathcal{A} = M_n$ and $\mathcal{B} = M_m$, then $\mathrm{Kr}(R) = \mathbb{C}a$.*

Proof If $\mathcal{A} = M_n$ and $\mathcal{B} = M_m$, then $\mathcal{A}' = \mathbb{C}\mathbb{1}$ and $\mathcal{B}' = \mathbb{C}\mathbb{1}$. □

4.4.4 The Full Reach Map

Now that we have collected all perspectives on reach maps, we can look at a particular example:

Example For a reach map $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ the following are equivalent:

(a) $R(p) = \mathbb{1}$ for all $p \in \mathcal{P}(\mathcal{A}) \setminus \{0\}$.
(b) $R = R_T$ with a faithful state φ on $\mathcal{A}$ and $T\colon \mathcal{A} \to \mathcal{B}$ with $T(x) = \mathbb{1}\varphi(x)$ for all $x \in \mathcal{A}$.
(c) No non-zero transitions $(p, q) \in \mathcal{P}(\mathcal{A}) \setminus \{0\} \times \mathcal{P}(\mathcal{B}) \setminus \{0\}$ from $\mathcal{A}$ to $\mathcal{B}$ are blocked for R.

We call this reach map the full reach map. If $\mathcal{A} = M_n$ and $\mathcal{B} = M_m$, then the full reach map is defined by $\mathrm{Kr}(R) = M_{n\times n}$. In the lattice of reach maps, the constant 0 reach map is the minimum, and the full reach map is the maximum.

4.4.5 A Submodule That Does Not Belong to a Reach Map

As promised, we give an example of a face in the cone of completely positive operators not induced by a reach map.

Example Let $T\colon M_2 \to M_2$ be given by the Kraus operators $e_{11}, e_{22}, e_{12} + e_{21}$. Then $\mathrm{Kr}(T) = \mathrm{span}(e_{11}, e_{22}, e_{12} + e_{21}) \neq M_2 = \mathrm{Kr}(R_T)$.

Proof We look for blocked transitions $R_T(p) \perp q$. They need to fulfil $S(p) \perp q$ for all S in the face generated by T. Since the identity map can be given as $\mathrm{id} = \mathrm{Ad}_{\mathbb{1}}$ and $\mathbb{1} = e_{11} + e_{22}$ is in the generated face of T a blocked transition $R_T(p) \perp q$ has to fulfil $p = \mathrm{id}(p) \le R_T(p) \perp q$. This means that in M_2, any blocked transition must be of the form $(p, p^\perp)$ with $p \in \mathcal{P}_1(M_2)$. For $\mathrm{Ad}_{e_{11}}$ and $\mathrm{Ad}_{e_{22}}$ the only blocked transitions which fulfil this are (e_{11}, e_{22}) and (e_{22}, e_{11}). Those transitions are however not blocked for $\mathrm{Ad}_{e_{12}+e_{21}}$. Thus, R_T has no non-zero blocked transitions and is the full reach map. □

Remark So, in this case, the face generated by T has all blocked transitions of R_T, but the face generated by R_T does not have more blocked transitions. Thus, the face generated by T is not induced by a reach map.

4.4.6 Submodules That Cannot Belong to the Full Reach Map

Since it can apparently easily happen that a face in the cone of completely positive operators has no blocked transitions, we give one sufficient criterion for when there must be blocked transitions.

Proposition *Let $a_1, \dots, a_n$ be n matrices in M_n. Then the smallest reach map $R\colon \mathcal{P}(M_n) \to \mathcal{P}(M_n)$ with $\mathrm{span}(a_1, \dots, a_n) \subseteq \mathrm{Kr}(R)$ is not the full reach map.*

Proof The function $f\colon \mathbb{C} \to \mathbb{C}\colon \lambda \mapsto \det(\lambda a_1 + a_2)$ is a complex polynomial and therefore has a root. So there is a λ such that $\lambda a_1 + a_2$ does not have full rank, and with that, a $\xi \in \mathbb{C}^n$ with $\lambda a_1 \xi + a_2 \xi = 0$ so $a_1\xi$ and $a_2\xi$ are linearly dependent. This means that $\dim(\mathrm{span}\{\, a_i\xi : 1 \le i \le n\}) \le n$. We can construct a blocked transition for $\mathrm{span}(a_1, \dots, a_n)$ from this. □

4.4.7 Identifying Faces Belonging to Reach Maps

Given a subspace of M_n, we know it belongs to a face in the cone of completely positive operators $\mathrm{CP}(M_n, M_m)$. However, the face might not be generated by a reach map.

Theorem *For an $\mathcal{A}'$-$\mathcal{B}'$-submodule $K \subseteq M_{n\times m}$ the following are equivalent:*

(a) *$K = \mathrm{Kr}(R)$ for some reach map $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$.*

(b) *There is a set $B \subseteq \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{B})$ such that $K = \{\, a \in M_{n\times m} : paq = 0$ for all $(p, q) \in B\}$.*

Proof (a) ⟹ (b) Let $K = \mathrm{Kr}(R)$ then by Proposition 4.2.2 R is completely determined by its blocked transitions.

(b) ⟹ (a) Let T be a representative of the face generated by K, i.e. $\mathrm{Kr}(T) = K$. We know that $K = \mathrm{Kr}(T) \le \mathrm{Kr}(R_T)$. Every blocked transition $(p, q) \in B$ implies a blocked transition $T(p) \perp q$ of T and blocked transitions of T are blocked transitions of R_T. Consequently, every $a \in \mathrm{Kr}(R_T)$ also fulfils $paq = 0$ for all $(p, q) \in B$, and we get $K = \mathrm{Kr}(R_T)$. □

Remark While this observation might not look particularly exciting, it is beneficial to pinpoint concrete reach maps in practice. It means that $R \mapsto \mathrm{Kr}(R)$ is a bijective map from reach maps to submodules fulfilling condition (b). We can define a reach map R from a set B with only very few blocked transitions. However, in that case, R often has a lot more blocked transitions than we required.

For reach maps on M_n, we can describe this in another way. For this, we use an inner product, known as the *Hilbert-Schmidt inner product*, on M_n via the trace $\mathrm{tr}\colon M_m \to \mathbb{C}$ as $\langle x, y\rangle := \mathrm{tr}(y^* x)$ for all $x, y \in M_n$.

Corollary *For a subspace $K \subseteq M_{n\times n}$ the following are equivalent:*

(a) $K = \mathrm{Kr}(R)$ *for some reach map* $R\colon \mathcal{P}(M_n) \to \mathcal{P}(M_n)$.

(b) K *is the Hilbert-Schmidt orthogonal complement of a set of rank one matrices in* M_n.

Proof (b) $\Rightarrow$ (a) If $t_{\eta,\xi}$ is a rank one matrix, then for all $a \in M_{n\times m}$ we have $t_{\eta,\eta} a t_{\xi,\xi} = 0$ if and only if $\mathrm{tr}(t^*_{\eta,\xi} a) = 0$. Thus, the orthogonal complement defines K by a set of blocked transitions.

(a) $\Rightarrow$ (b) By Proposition 4.2.1, R is already determined from its blocked transitions of rank one. □

In this chapter, we have characterized reach maps in finite-dimensional systems. First, we gave a definition of reach maps of rank one via preservation of cross-ratios. Then we associated reach maps with certain faces in the cone of completely positive operators. We connected those faces to submodules in the space of possible Kraus operators between algebras and characterized when such submodules belong to a reach map.

5 Perron-Frobenius Theory for Reach Maps

In this chapter, we finally see how the topological properties of a completely positive operator already determine quite a lot of its behaviour including even its peripheral spectrum. For a fixed reach map, we decompose the matrix algebra into communicating classes (Theorem 5.2.4), define recurrence and transience (Definition 5.3.3) and present a Perron-Frobenius style theorem for reach maps (Theorem 5.5.2). This generalizes the classical theory for positive matrices as presented in e.g. [Sen81]. This chapter can be considered an improved version of [Bra17], while the new part here is that everything presented can be done on the level of reach maps.

In this chapter, again, $\mathcal{A}$ will be a matrix algebra, i.e. finite-dimensional, and R a reach map. But now instead of going from $\mathcal{A}$ to $\mathcal{B}$ our reach map $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ is an endomorphism in the category of reach maps.

5.1 Irreducibility

The motivational idea for the decomposition theory in this chapter is "from where to where in the algebra" states can move under the dynamics. This motivation already shows the connection to blocked transitions from the last chapter which will come into play here. A special case is when states can "go everywhere". For that, we introduce the concept of irreducibility.

5.1.1 Unital Representatives

Much of the theory in this chapter traditionally only applies to, or is at least significantly easier for, unital completely positive operators (also called Markov operators). Their use is often well justified because they describe a probability-preserving dynamic. Every reach map has representatives that are not unital. In fact, it is even worse. We cannot present a non-trivial criterion to tell whether a given reach map has a unital representative. There are, however, examples where this is not the case:

Example — If $a \in M_n$, then by Corollary 4.4.3 we have $\mathrm{Kr}(R_{\mathrm{Ad}_a}) = \mathbb{C}a$. Thus, R_{Ad_a} has a unital representative if and only if a is a multiple of a unitary matrix.

At least in the commutative case, we can always find a unital representative.

Lemma *Let $\mathcal{A}$ be commutative, then any reach map $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ with $R(\mathbb{1}) = \mathbb{1}$ has a unital representative.*

Proof A commutative reach map corresponds to a matrix with 0-1-entries. Because of $R(\mathbb{1}) = \mathbb{1}$, the matrix has at least one non-zero entry in every row. We assign a Laplace distribution to the non-zero entries to create a stochastic matrix belonging to the correct equivalence class. □

5.1.2 Subinvariant projections

The core tool for discussing irreducibility, communicating classes, recurrence and transience are subinvariant projections.

Definition Let $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ be a reach map. Then we call an orthogonal projection $p \in \mathcal{P}(\mathcal{A})$ *subinvariant* for R if $R(p) \leq p$.

Remark If $T\colon \mathcal{A} \to \mathcal{B}$ is a completely positive operator with $R = R_T$, we will also say that p is subinvariant for T. From our perspective, a subinvariant projection is a special case of a blocked transition. Since $R(p) \perp p^\perp$ the transition $(p, p^\perp)$ is blocked. Thus, all the calculation rules given in Proposition 4.3.2 can be applied to subinvariant projections and become simpler for that case. For example, we have $T(xp) = T(xp)p$ for all $x \in \mathcal{A}$ and subinvariance means $pap^\perp = 0$ for all $a \in \mathrm{Kr}(T)$.

Subinvariant projections are often called superharmonic, for example, in [Gär14] and [Bra17]. Then their orthogonal complement is called subharmonic.

For example, the following result can also be found in [Gär14, 2.4.3]. We get it nearly for free from our general theory of reach maps.

Proposition *The subinvariant projections are closed under suprema and infima.*

Proof We consider a family $(p_i)_{i\in I} \subseteq \mathcal{P}(\mathcal{A})$ with $R(p_i) \leq p_i$ for all $i \in I$. Reach maps preserve suprema and are majorized by infima, so we have $R(\bigvee_i p_i) = \bigvee_i R(p_i) \leq \bigvee_i p_i$ and $R(\bigwedge_i p_i) \leq \bigwedge_i R(p_i) \leq \bigwedge_i p_i$. □

Remark Thus, the subinvariant projections are a sublattice of the orthogonal projections.

Example 1 Trivially, $\mathbb{1}$ and 0 are subinvariant for every reach map.

Example 2 For $a \in M_n$ and $\mathrm{Ad}_a\colon M_n \to M_n$, we can find an orthonormal basis in which a has upper triangular form. This means there are at least n different projections with $pap^\perp = 0$.

5.1.3 Irreducibility

Definition We call a reach map $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ irreducible if 0 and $\mathbb{1}$ are the only subinvariant projections of R. We call an operator irreducible if its reach map is irreducible.

Example 1 In the case $\mathcal{A} = \mathbb{C}$, every reach map, including the zero reach map, is irreducible. In all other cases, the zero reach map has a lot of subinvariant projections and hence is not irreducible.

Example 2 The full reach map is irreducible.

Example 3 The commutative reach map $R\colon \mathcal{P}(\mathbb{C}^2) \to \mathcal{P}(\mathbb{C}^2)$ given by the 0-1-matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is irreducible. $\mathbb{C}^2$ has only two non-trivial projections, e_1 and e_2. Since $R(e_1) = e_2$ and $R(e_2) = e_1$, both of them are not subinvariant.

Example 4 Let $R\colon \mathcal{P}(M_2) \to \mathcal{P}(M_2)$ with $\mathrm{Kr}(R) = \mathrm{span}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}^{\perp}$. All non-trivial orthogonal projections have rank one. We have $R(e_{11}) = e_{22}$, $R(e_{22}) = e_{11}$ and for all other one-dimensional projections $p \in \mathcal{P}(M_2)$ we get $R(p) = \mathbb{1}$. Hence, R is irreducible.

We can immediately make one observation about irreducible reach maps:

Lemma *Let $R \neq 0$ be an irreducible reach map, then $R(\mathbb{1}) = \mathbb{1}$.*

Proof Let $R(\mathbb{1}) = p$, then $R(p) \leq R(\mathbb{1}) = p$ thus p is subinvariant. Since $R \neq 0$, p must be $\mathbb{1}$. □

5.2 Decomposition into Communicating Classes

In the classical theory of Markov chains, two points in the state space communicate with each other if there is a non-zero probability of going from one to the other in both directions. This gives us an equivalence relation which can be used to decompose the state space. We generalize this theory to reach maps.

5.2.1 Reduced Reach Maps

To decompose the algebra $\mathcal{A}$ into parts along projections in $\mathcal{P}(\mathcal{A})$, we formalize the concept of restricting a reach map $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ and a completely positive operator $T\colon \mathcal{A} \to \mathcal{A}$ to a hereditary subalgebra.

Definition For any orthogonal projection p we define the *reduced* reach map $R_p\colon \mathcal{P}(\mathcal{A}_p) \to \mathcal{P}(\mathcal{A}_p)$ as $R_p(q) := [pR(q)p]$ for all $q \in \mathcal{P}(\mathcal{A}_p)$ and the *reduced* completely positive operator $T_p\colon \mathcal{A}_p \to \mathcal{A}_p$ as $T_p(x) := pT(x)p$ for all $x \in \mathcal{A}_p$.

Remark Sometimes it is convenient to extend R_p to all orthogonal projections $q \in \mathcal{P}(\mathcal{A})$ as $R_p(q) = [pR([pqp])p]$ and for $x \in \mathcal{A}$ as $T_p(x) = pT(pxp)p$.

If $R = R_T$ then we have $R_p = R_{T_p}$.

5.2.2 Projections Closed Under Communication

To define equivalence classes of communicating points, we need two properties of a projection. First, no point (or more generally, projection) inside the class should communicate with something outside the class, and this is the concept we introduce here:

Definition Let $q \leq p$ be two subinvariant projections, then we call the projection $p - q$ *closed under communication.*

Example If p is subinvariant, p and $p^\perp$ are closed under communication, since $p = p - 0$ and $p^\perp = \mathbb{1} - p$.

Since we want to restrict the dynamics to reduced algebras with projections which are closed under communication, we need to confirm that this plays nicely with the concept of subinvariance.

Lemma *Let $q \leq p$ be two subinvariant projections, for $q' \leq c := p - q$ the following are equivalent:*

(a) *q' is subinvariant for R_c.*
(b) *$q' + q$ is subinvariant for R.*

Proof (a) $\Rightarrow$ (b) We have $R_c(q') \leq q'$. With $[cR(q')c] \perp q'^\perp$ we get $R(q') \perp c - q'$, so $R(q') \leq \mathbb{1} - (c - q')$, but we also know that $R(q') \leq R(p) \leq p$ so $R(q') \leq \mathbb{1} - (c - q') \wedge p = p - (c - q') = p - (p - q - q') = q + q'$. Consequently, $R(q' + q) = R(q') \vee R(q) \leq (q + q') \vee q = q + q'$.

(b) $\Rightarrow$ (a) Let $R(q' + q) \leq q' + q$ apply on both sides R_{Ad_c} then $R_c(q') \leq [cR(q' + q)c] \leq q'$. □

Corollary *If the projection c is a projection which is closed under communication, then for a dominated projection $c' \leq c$ the following are equivalent:*

(a) *c' is closed under communication for R_c.*
(b) *c' is closed under communication for R.*

5.2.3 Communicating Classes

Additionally to no point in the class communicating with the outside, we need everything inside a communicating class to communicate with each other.

Definition We call an orthogonal projection c a communicating class if it is minimally closed under communication.

Proposition *If a projection c is closed under communication, then the following are equivalent:*

(a) *c is a communicating class.*
(b) *R_c is irreducible.*

Proof We pick subinvariant projections p and q such that $c = p - q$.

(a) $\Rightarrow$ (b) Let q' be subinvariant for R_c. Then $q' + q$ is subinvariant. Thus, $p - (q' + q)$ is closed under communication below c, so we have either $q' = c$ or $q' = 0$.

(b) $\Rightarrow$ (a) Let $c' \leq c$ be closed under communication. Then $c' = p' - q'$ with $q \leq q' \leq p' \leq p$. Since R_c is irreducible q' and p' have to be either q or p. So c' is either 0 or c. □

5.2.4 Decompositions of One

Definition We call a family of orthogonal projections $p_1, \ldots, p_k$ a *decomposition of* $\mathbb{1}$ if $p_1 \vee \cdots \vee p_k = \mathbb{1}$ and $p_i \wedge p_j = 0$ for all $i \neq j$. We call the decomposition orthogonal if $p_i \perp p_j$ for all $i \neq j$.

Theorem *Let $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ be a reach map. There is an orthogonal decomposition of $\mathbb{1}$ consisting of communicating classes $c_1, \ldots, c_k$ such that $\sum_{i=1}^{l} c_i$ is subinvariant for all $1 \leq l \leq k$.*

Proof Assume $c_1, \ldots, c_k$ is an orthogonal decomposition of $\mathbb{1}$, fulfiling that $\sum_{i=1}^{l} c_i$ is subinvariant for all $1 \leq l \leq k$, but without the condition that the c_i are irreducible. Then we can pick an arbitrary reducible c_i and a R_{c_i}-subinvariant projection p and split c_i into $c_i - p$ and p. Since $\sum_{j=1}^{i-1} c_j + p$ is subinvariant, the new decomposition fulfils the same conditions but is finer. Since $\mathcal{A}$ is finite-dimensional, we can iterate this until all c_i are irreducible. We can start with the decomposition $k = 1$ and $c_1 = \mathbb{1}$. □

Remark The version of this theorem for completely positive contractions was shown in [Bra17, 3.3.2]. This theorem can also be understood as bringing the (not necessarily self-adjoint) algebra generated by $\mathrm{Kr}(R)$ in a blockwise upper triangular form. In this form, the statement is very similar to [GLR06, Theorem 11.2.2]. That book talks about linear algebra from the point of view of invariant subspaces, which are directly connected to subinvariant projections.

In the same way in which matrices do not have unique upper triangular forms, the decomposition into communicating classes is not unique.

Example For $\mathrm{id}\colon \mathcal{A} \to \mathcal{A}$, every projection is subinvariant. Consequently, every orthogonal decomposition of $\mathbb{1}$ into minimal projections satisfies the theorem.

5.3 Recurrence and Transience

Recurrence and transience are important properties of the asymptotic dynamics of a system. Classically recurrent points are states into which the dynamics always return, while transient points will be left at some point and then never returned to. We will introduce stochastic and topological recurrence, where stochastic recurrence is a property of projections in regard to a completely positive operator, and topological recurrence is a property in regard to a reach map.

5.3.1 Supports of Eigenstates

Stochastic recurrence can most elegantly be described by the existence of invariant states. To prepare this, we take a quick look at the eigenstates of an operator.

Lemma *If p is the support of an eigenstate of a completely positive operator $T\colon \mathcal{A} \to \mathcal{A}$, then $p^\perp$ is subinvariant for T.*

Proof Let φ be an eigenstate of T with eigenvalue λ. Then $\varphi(T(p^\perp)) = \lambda\varphi(p^\perp) = 0$ which yields $T(p^\perp) \perp p$ and $T(p^\perp) \leq p^\perp$. □

5.3.2 The Eigenvalue Theorem for Irreducible Operators

Lemma *Any irreducible completely positive operator $T \neq 0$ has spectral radius $r(T) > 0$, and there is a faithful state $\varphi \in \mathcal{S}(\mathcal{A})$ with $\varphi \circ T = r(T)\varphi$.*

Proof That the spectral radius is strictly positive is, for example, shown in [Gro81, 2.3]. By the eigenvalue theorem (cf. [Gro81, 2.1]) an eigenstate for the eigenvalue $r(T)$ exists. It is faithful because T is irreducible together with the previous lemma. □

5.3.3 Topological Recurrence

Definition We call a communicating class c *topologically recurrent* for R if $c^\perp$ is subinvariant for R. Otherwise, we call it *topologically transient*.

We call R *recurrent* if all communicating classes are topologically recurrent.

Proposition (1) *For a communicating class c the following are equivalent:*

(a) *c is topologically recurrent.*

(b) *For all representatives T of R, T has an eigenstate φ for a positive eigenvalue with $c = \operatorname{supp}\varphi$.*

(2) *If R is recurrent, then for all projections p the following are equivalent:*

(a) *p is subinvariant.*

(b) *$p^\perp$ is subinvariant.*

(c) *p is closed under communication.*

Proof (1) (a) $\Rightarrow$ (b) Let T be a representative of R. We consider the restriction T_c, which is irreducible and thus has a faithful eigenstate φ. We can extend φ as $\varphi \circ \mathrm{Ad}_c$ onto all of $\mathcal{A}$ and $\varphi \circ \mathrm{Ad}_c$ has support c. Since $c^\perp$ is subinvariant by Proposition 4.3.2 we have $\mathrm{Ad}_c \circ T = T_c$. We can conclude $\varphi \circ T = \varphi \circ \mathrm{Ad}_c \circ T = \varphi \circ T_c = r(T_c)\varphi$.

(1) (b) $\Rightarrow$ (a) Let φ be an eigenstate for a strictly positive eigenvalue of T with support c. Then $\varphi(T(c^\perp)) = \lambda\varphi(c^\perp) = 0$, so $T(c^\perp) \perp \operatorname{supp}\varphi$, or $T(c^\perp) \leq c^\perp$. This was the only property we needed to check to show that c is topological recurrent.

(2) (a) $\Rightarrow$ (b) The projection p has an orthogonal decomposition into communicating classes. Their complements are all subinvariant, so $p^\perp$ as the intersection of them is subinvariant.

(2) (b) $\Rightarrow$ (c) As mentioned in 5.2.2, any subinvariant projection and its complement are closed under communication.

(2) (c) $\Rightarrow$ (a) Since it is closed under communication, p has a decomposition into communicating classes. Thus, with the argument from (a) $\Rightarrow$ (b) $p^\perp$ is subinvariant. This means that $p^\perp$ is also closed under communication. With the same argument again, we conclude that p is subinvariant. □

Example 1 Let $R\colon \mathcal{A} \to \mathcal{B}$ be defined by the single blocked transition $(p, p^\perp)$ for any non-trivial orthogonal projection $p \in \mathcal{P}(\mathcal{A})$. p is subinvariant, thus p and $p^\perp$ are closed under communication. R_p and $R_{p^\perp}$ are irreducible. Hence, p and $p^\perp$ are communicating classes. Since p is subinvariant and $p^\perp$ is not, p is topologically transient and $p^\perp$ is topologically recurrent.

Example 2 Let $R\colon \mathcal{A} \to \mathcal{B}$ be defined by the blocked transitions $(p, p^\perp)$ and $(p^\perp, p)$ for any non-trivial orthogonal projection $p \in \mathcal{P}(\mathcal{A})$. This is the same case as before, but p and $p^\perp$ are subinvariant. Hence, both communicating classes are topologically recurrent, and R is recurrent.

5.3.4 Stochastic Recurrence

While the definition of topological recurrence is new here, there is already an established definition of recurrence in regard to completely positive operators. Inconveniently, topological recurrence in regard to a reach map R only implies stochastic recurrence in regard to an operator T with $R = R_T$ if T is unital. We now establish this relation in detail:

Definition For a completely positive contraction T, we call a projection *stochastically recurrent* if it is below the support of a stationary state of T. We call the projection *stochastically transient* if it is orthogonal to the supports of all stationary states of T.

We call an operator T *recurrent* if $\mathbb{1}$ is stochastically recurrent for T.

Further, we define the *unital support* of T as $p_u := \bigvee \{ p \in \mathcal{P}(\mathcal{A}) : T(p) \geq p \}$.

Remark Relevant properties of the unital support are $T(p_u) \geq p_u$, its complement $p_u^\perp$ is subinvariant, and T is unital if and only if $p_u = \mathbb{1}$.

Lemma *If a positive element $a \in \mathcal{A}^+$ and projection $p \in \mathcal{P}(\mathcal{A})$ fulfil $\|a\| \leq 1$ and $pap = p$, then it holds that $p \leq a$.*

Proof With $p \leq paap = pap^\perp ap + papap = pap^\perp ap + p$ we get $pap^\perp = 0$ and thus $a = pap + p^\perp ap^\perp \geq pap = p$. □

Proposition *Let $T \colon \mathcal{A} \to \mathcal{A}$ be a completely positive contraction.*

(1) *There is a maximal stochastically recurrent projection $r \in \mathcal{P}(\mathcal{A})$ and a stationary state with support r.*

(2) *For a communicating class $c \in \mathcal{P}(\mathcal{A})$ the following are equivalent:*
   (a) *$c \leq r$, i.e. c is stochastically recurrent.*
   (b) *$T(c) \geq c$.*
   (c) *c is topologically recurrent and $c \leq p_u$.*

(3) *With $t := r^\perp$ for a projection $p \in \mathcal{P}(\mathcal{A})$ the following are equivalent:*
   (a) *$p \leq t$, i.e. p is stochastically transient.*
   (b) *$T^n(p) \to 0$.*

(4) *Every communicating classes $c \in \mathcal{P}(\mathcal{A})$ has exactly one of these three properties:*
   (A) *topologically and stochastically recurrent*
   (B) *topologically and stochastically transient*
   (C) *topologically recurrent but stochastically transient.*

Proof (1) First, we recognize that every stochastically recurrent projection is dominated by the support of a stationary state. So, to show the statement, it is sufficient to show that the supremum of supports of stationary states again is the support of a stationary state. If we consider two stationary states φ and ψ, $\omega := \frac{1}{2}(\varphi + \psi)$

is also stationary. For $0 < x \leq \operatorname{supp}\varphi$ we have $\omega(x) > 0$, so $\operatorname{supp}\varphi \leq \operatorname{supp}\omega$ and the same argument holds for $\operatorname{supp}\psi$. On the other hand $\omega(\operatorname{supp}\varphi \vee \operatorname{supp}\psi) = \frac{1}{2}(\varphi(\operatorname{supp}\varphi \vee \operatorname{supp}\psi) + \psi(\operatorname{supp}\varphi \vee \operatorname{supp}\psi)) = \frac{1}{2}(1+1) = 1$. That yields $\operatorname{supp}\varphi \vee \operatorname{supp}\psi \geq \operatorname{supp}\omega$. Since any chain of projections in $\mathcal{P}(\mathcal{A})$ is finite, we can reduce arbitrary suprema to finite suprema. Thus, we receive r as the supremum over all supports of stationary states, and it is the support of a stationary state.

(2) (a) $\Rightarrow$ (b) Let $p \leq r$ with $T_r(p^\perp) \leq p^\perp$. Then $\varphi(p^\perp - T_r(p^\perp)) = \varphi(p^\perp - p^\perp) = 0$ and thus $T_r(p^\perp) = p^\perp$ and $T_r(p) = p$. We can complete c to a decomposition of r where, by the previous argument, all summands are fixed points of T_r and get $T_r(c) = c$. This way, we see $cT(c)c = crT(c)rc = cT_r(c)c = c$ and thus $T(c) \geq c$.
(2) (b) $\Rightarrow$ (c) Since T is a contraction $T(c) \geq c$ implies $R(c^\perp) \leq c^\perp$.
(2) (c) $\Rightarrow$ (a) There is an eigenstate φ with support c and eigenvalue $\lambda \leq 1$. We show $\lambda = \lambda\varphi(\mathbb{1}) = \varphi(T(\mathbb{1})) \geq \varphi(T(p_u)) \geq \varphi(p_u) = 1$. Thus, φ is stationary and c is stochastically recurrent.

Auxiliary statement. Let p be subinvariant. Then either $T^n(p) \longrightarrow 0$ or there is a stochastically recurrent communicating class below p.
Because $T(p) \leq p$, the sequence $T^n(p)$ converges monotonically against some fixed point $h \in \mathcal{A}_p^+$. If $h = 0$, we are finished. So, we assume $h > 0$. Then T_p has spectral radius 1 and thus a stationary state. From the stationary state, we get a stochastically recurrent communicating class c for T_p. With $T(c) = T(pcp) = T_p(c) \geq c$, c is also stochastically recurrent for T. This shows the auxiliary statement.

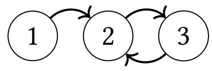
(3) (b) $\Rightarrow$ (a) Let $T^n(p) \longrightarrow 0$, then $\varphi(p) = \varphi(T^n(p)) \longrightarrow 0$ and so $p \perp r$ and p is transient.
(3) (a) $\Rightarrow$ (b) t is subinvariant. Since, by definition, there are no stochastically recurrent communicating classes below it, we have $T^n(t) \longrightarrow 0$.

(4) By definition, a communicating class is either topologically recurrent or transient. From (3), we know that stochastic recurrence implies topological recurrence. Thus, we only need to show that every communicating class is either stochastically recurrent or stochastically transient. It is clear that the cases are mutually exclusive, but we need to show that always one of them is true. Let c be not stochastically recurrent and let p be minimal such that $c + p$ is subinvariant. We choose any orthogonal ordered decomposition of p. Since p was minimal, none of the summands in the decomposition are topologically recurrent. We show by induction that all summands of the decomposition of p and c are stochastically transient. We start with the fact that 0 is transient. As induction step, consider a communicating class c and a transient, subinvariant p such that $c + p$ is also subinvariant. With our lemma, either $\varphi(c + p) = \varphi(T^n(c + p)) \longrightarrow 0$, which means c is transient, or there is a stochastically recurrent class below $c + p$. Since it must be orthogonal to the stochastically transient p, it must be c. $\square$

Corollary *If T is unital, then stochastic and topological recurrence are the same.*

Proof This is a direct consequence of (3) with $p_u = \mathbb{1}$. □

Example Let us look at a simple commutative example to discuss the different recurrence properties:



In this graph, or rather for its corresponding reach map, point 1 is topologically transient while the communicating class $\{2,3\}$ is topologically recurrent. We can see that for any probabilities assigned to the edges, the probability that a state could move out of the set $\{2,3\}$ is zero. However, from this topological analysis we cannot tell whether a specific stochastic transition will preserve probabilities in $\{2,3\}$. Only if we know that we choose a unital operator, then every edge gets assigned the probability 1 and then $\{2,3\}$ is certainly stochastically recurrent. The stationary measure would be the Laplace distribution on $\{2,3\}$ extended to point 1 with probability 0.

Corollary *All projections that are closed under communication commute with the maximal stochastically recurrent projection r.*

Proof Take any projection p closed under communication and decompose it into communicating classes c_i. Every communicating class is either below r or t. Thus, $pr = \sum_{c_i \text{ recurrent}} c_i = rp$. □

5.3.5 Dealing with Leakage

In the previous example, we have discussed that non-unital operators are not probability preserving. In a classical probability graph, we can interpret a situation where the probabilities of outgoing edges do not sum up to 1 as the possibility that the state gets lost into the environment. We could say the state leaks out of the system. For any non-unital completely positive operator $T\colon \mathcal{A} \to \mathcal{A}$, we can extend our algebra to find a canonical unital extension of T. In the introduced analogy, we can interpret this as explicitly adding the environment into which a state could have been lost into our algebra.

Proposition *Let $T(\mathbb{1}) < \mathbb{1}$, then there is $\tilde{T}\colon \mathbb{C} \oplus \mathcal{A} \to \mathbb{C} \oplus \mathcal{A}$ unital such that $\tilde{T}_{\mathbb{1}_{\mathcal{A}}} = T$ and for all $p \in \mathcal{A}$ the following are equivalent:*

(a) *p is subinvariant for T.*
(b) *$0 \oplus p$ is subinvariant for $\tilde{T}$.*

Proof We define $\tilde{T}$ as $\tilde{T}(\alpha \oplus x) = \alpha \oplus (T(x) + \alpha(\mathbb{1} - T(\mathbb{1})))$.

(a) $\Rightarrow$ (b) $\tilde{T}(0 \oplus p) = 0 \oplus T(p) \leq 0 \oplus p$.

(b) $\Rightarrow$ (a) We get $0 \oplus T(p) = \tilde{T}(0 \oplus p) \leq 0 \oplus p$ and thus $T(p) \leq p$. □

5.4 The Reversible Algebra

Recurrence and transience are properties to investigate the asymptotic behaviour of dynamics. Another central tool for that is the reversible algebra. This is the only part of the algebra which is asymptotically relevant for the dynamics, and on it, the dynamics operate as a *-automorphism; hence the name, *reversible*.

5.4.1 The Multiplicative Algebra

To capture the multiplicative aspect of an operator, we look at the so-called multiplicative algebra. We get it by converting the Kadison inequality to an equality.

Definition For a completely positive contraction $T\colon \mathcal{A} \to \mathcal{A}$ we define

$$\mathcal{M}_T := \left\{ x \in \mathcal{A} : T(xx^*) = T(x)T(x^*), T(x^*x) = T(x^*)T(x) \right\}$$

as the *multiplicative algebra* of T.

Remark By [Stø13, 2.1.5 and 2.1.6] $\mathcal{M}_T$ is indeed a *-subalgebra of $\mathcal{A}$ and for all $x \in \mathcal{M}_T$ and $y \in \mathcal{A}$ we have $T(xy) = T(x)T(y)$ and $T(yx) = T(y)T(x)$.

The core trick of the rest of this chapter is that the multiplicative algebra and the behaviour of the operator on it are topological properties. This might not be very surprising because we had already observed this for the case $\mathcal{A} = \mathcal{M}_T$, which means that T is a *-homomorphism.

Proposition *Let S and T be unital completely positive operators with $R_S \leq R_T$, then $S(x) = T(x)$ for all $x \in \mathcal{M}_T$.*

Proof As a matrix algebra, $\mathcal{M}_T$ has a basis consisting of orthogonal projections, so it suffices to show this for orthogonal projections. Let $p \in \mathcal{P}(\mathcal{M}_T)$, then we have $R_T(p) = T(p)$. Since T is unital, $\mathbb{1}$ is also in $\mathcal{M}_T$ and thus $p^\perp \in \mathcal{M}_T$. Again, since T is unital, we have $T(p^\perp) = \mathbb{1} - T(p) = T(p)^\perp$. So, we also have $R_T(p^\perp) = T(p)^\perp$. Thus, we have $S(p) \leq R_S(p) \leq R_T(p) = T(p)$ and $S(p^\perp) \leq T(p)^\perp$. But since $S(p + p^\perp) = \mathbb{1}$, this means $S(p) = T(p)$ (and $S(p^\perp) = T(p)^\perp = T(p^\perp)$). □

5.4.2 The Reversible Algebra

Definition We define the *reversible algebra* of T as $\mathcal{E}(T) := \operatorname{span}\{\, x : T(x) = \lambda x, |\lambda| = 1\,\}$.

Proposition *Let $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ be a recurrent reach map with unital representatives T and S, then*

(1) *$\mathcal{E}(T)$ is a *-subalgebra of $\mathcal{A}$*
(2) *T is a *-automorphism on $\mathcal{E}(T)$.*
(3) *$\mathcal{E}(T) = \mathcal{E}(S)$*
(4) *$T(x) = S(x)$ for all $x \in \mathcal{E}(T)$.*
(5) *$\sigma_1(T) = \sigma_1(S)$.*

Proof For the proof of (1) and (2), see [Bra17, 5.1.3].

Since $\mathcal{E}(T) \subseteq \mathcal{M}_T = \mathcal{M}_S \supseteq \mathcal{E}(S)$, we have $T(x) = S(x)$ for $x \in \mathcal{E}(T)$ and $x \in \mathcal{E}(S)$. Now (3), (4) and (5) follow directly. $\square$

Definition If R has a unital representative T, we say that $\mathcal{E}(R) := \mathcal{E}(T)$ is the reversible algebra of R.

5.5 Perron-Frobenius Theory for Reach Maps

For the asymptotic behaviour of completely positive contractions, the peripheral spectrum is of special importance. The theorem of Perron-Frobenius is the main theorem about the peripheral spectrum of linear maps. We present a version of this theorem for reach maps.

5.5.1 Equivalence to Unital Operators from Irreducibility

Lemma [Gro82, 2.2] *Let T be a non-zero irreducible completely positive operator with spectral radius $r(T)$. Then there is an invertible completely positive operator U such that $r(T)^{-1}(U \circ T \circ U^{-1})$ is a unital operator.*

5.5.2 The Perron-Frobenius Theorem for Reach Maps

Definition A *cycle* for a reach map R is a sequence of projections $p_1, \dots, p_k$, such that $R(p_i) \geq p_{(i \bmod k)+1}$ for all $1 \leq i \leq k$. An *exact cycle* fulfils $R(p_i) = p_{(i \bmod k)+1}$.

Remark For the commutative case, this definition of a cycle is compatible with the definition of a cycle in a directed graph.

Theorem *Let $R > 0$ be an irreducible reach map.*

(1) *There is a maximal k such that there is an exact cycle $p_1, \dots, p_k$ which is a decomposition of $\mathbb{1}$. For $\mathcal{A} \subseteq M_n$ we have $k \leq n$.*

Let T be any representative of R with spectral radius $r = r(T)$.

(2) $\sigma_r(T) = r\left\{ e^{\frac{i2\pi l}{k}} : l \in \{1, \dots, k\} \right\}$.

(3) *All eigenvalues in $\sigma_r(T)$ are simple and $\frac{1}{r}\sigma_r(T) \cdot \sigma(T) = \sigma(T)$.*

(4) *r is the unique eigenvalue of T with a positive eigenvector. This eigenvector is strictly positive.*

If T is unital, then

(4) *The decomposition $p_1, \dots, p_k$ is orthogonal.*

(5) $\mathcal{E}(R) = \operatorname{span}\{p_1, \dots, p_k\}$ *and the peripheral eigenspaces are spanned by unitaries.*

Proof *First we assume that T is a unital representative of R.* Then statements (2), (3) and (4) are directly given by [Gro81, 3.1] when we consider that $\sigma_r(T)$ must be finite. We set $k = |\sigma_r(T)|$.

By [Gro81, 3.3], a decomposition consisting of k projections like in (1) exists. Their span is a commutative *-subalgebra of dimension k and lies in $\mathcal{E}(T) = \mathcal{E}(R)$, which also has dimension k. The unitary eigenvectors are given by $\sum_{i=0}^{k} \lambda^{-i} p_i$ for $\lambda \in \sigma_1(T)$, proving (5).

On the other hand, assume a decomposition like in (1). Since $R(p_i) = p_{i+1}$, $T(p_i) \leq p_{i+1}$ and T is recurrent, we have $T(p_i) = p_{i+1}$. Since all the p_i get each mapped to orthogonal projections, they are part of $\mathcal{M}_T$. Consider the *-algebra $\mathcal{B}$ generated by the p_i. T maps $\mathcal{B}$ into $\mathcal{B}$ and T^k is the identity on $\mathcal{B}$. That means T is a *-automorphism on $\mathcal{B}$ with inverse T^{k-1}. Because T is a *-automorphism on $\mathcal{B}$, we have $\mathcal{B} \subseteq \mathcal{E}(T)$. Since $\mathcal{E}(T)$ is commutative, the p_i must be orthogonal, and since $\mathcal{E}(T)$ has dimension k, this limits the number of p_i. Thus, a decomposition as in (1) exists. The k is maximal and is always orthogonal, proving (4).

Now we assume T to not be unital. By Lemma 5.5.1, there is an invertible completely positive operator U such that $S = \frac{1}{r}(U \circ T \circ U^{-1})$ is a unital operator. We have $\sigma(T) = r\sigma(T)$ and U preserves positivity, proving again (2), (3) and (4).

Let p_i be an exact cycle orthogonal decomposition of S, then $T(U^{-1}(p_i)) = rU^{-1}(p_{i+1})$, so $R([U^{-1}(p_i)]) = [U^{-1}(p_{i+1})]$. So the $[U^{-1}(p_i)]$ form an exact cycle for R. Since U^{-1} is invertible $[U^{-1}(\mathbb{1})] = \mathbb{1}$, and thus $\mathbb{1} = [U^{-1}(\mathbb{1})] = [U^{-1}(\sum_{i=0}^{k} p_i)] = [\sum_{i=0}^{k} U^{-1}(p_i)] = \bigvee_{i=0}^{k}[U^{-1}(p_i)]$. At the same time we have $[U([U^{-1}(p_i)] \wedge [U^{-1}(p_j)])] \leq [U([U^{-1}(p_i)])] \wedge [U([U^{-1}(p_j)])] = p_i \wedge p_j = 0$. Since U has to be faithful, we get that the $[U^{-1}(p_i)]$ are a decomposition of $\mathbb{1}$. Thus, the decomposition of (1) exists even in the absence of a unital representative of R. On the other hand, if a decomposition like that of any size exists, it will translate to an orthogonal decomposition for T and can thus maximally have size k. □

Definition We call k the period of R.

Remark As is apparent from the proof, in the unital case the Perron-Frobenius theorem for positive operators can easily be transferred, since we know that all unital representatives coincide on the reversible algebra. The strength of this theorem lies in the non-unital case. Especially the concept of a non-orthogonal exact cycle for a non-unital irreducible operator is only sensibly expressible via the reach map and thus probably a new contribution to the Perron-Frobenius theory of completely positive operators.

We see that the peripheral spectrum of any representative of R can be determined from R. This is, however, not possible for the rest of the spectrum.

For irreducible reach maps, this theorem gives as a necessary condition for the existence of a unital representative: If we find an exact cycle, which is a decomposition of $\mathbb{1}$ with the length of the period, it must be an orthogonal decomposition.

Example Let e_1, e_2 and f_1, f_2 be two bases of $\mathbb{C}^2$ with matrix units e_{ij} and f_{ij} such that $e_{11} \neq f_{11}$. Let $R\colon \mathcal{P}(M_2) \to \mathcal{P}(M_2)$ be a reach map defined by the blocked transitions (e_{11}, f_{22}) and (f_{11}, e_{22}). Then $\operatorname{Kr}(R) = \left\{ t_{e_1,f_2}, t_{f_1,e_2} \right\}^{\perp}$ with the Hilbert-Schmidt inner product on M_2. A basis of $\operatorname{Kr}(R)$ can be given by e_{21} and f_{21}. We can calculate $R(e_{11}) = R_{\mathrm{Ad}_{e_{21}}}(e_{11}) \vee R_{\mathrm{Ad}_{f_{21}}}(e_{11}) = 0 \vee f_{11} = f_{11}$ and $R(f_{11}) = e_{11}$. For all other rank one projections p, we have $R(p) = \mathbb{1}$ since they are not orthogonal to e_{22} or f_{22}. Hence, R is irreducible and since it has the exact cycle e_{11}, f_{11} it has period 2. If we are in the special case $e_1 \in \mathbb{C}f_2$ and $f_1 \in \mathbb{C}e_2$, then the basis of $\operatorname{Kr}(R)$ simplifies to e_{12}, e_{21} and R has a unital representative. Consistent with our theorem we have $e_{11} \perp e_{22} = f_{11}$ which means that the exact cycle e_{11}, e_{22} is an orthogonal decomposition of $\mathbb{1}$.

5.5.3 Aperiodicity

A special case of irreducible operators are the ones with $|\sigma_1(T)| = 1$. For them, the asymptotic behaviour is completely independent of the initial state of the system. We can also describe this class of operators on the level of reach maps.

Proposition *For a non-zero irreducible reach map R the following are equivalent:*

(a) *The period of R is* 1.
(b) *For all $p \in \mathcal{P}(\mathcal{A})$ there is a $k \in \mathbb{N}$ such that $R^k(p) = \mathbb{1}$.*
(c) *R^k is irreducible for all $k \in \mathbb{N}$.*

Definition In this case, we call the reach map R *aperiodic*.

Proof (a) $\Rightarrow$ (b) We have $R^k(p) = [S^k(U^{-1}(p))]$ where S has a unique stationary state and $S^k(x) \longrightarrow \mathbb{1}\varphi(x)$. So $R^k(p) \longrightarrow \mathbb{1}$, where, by dimensional arguments, the limit must be reached after a finite number of steps.

(b) $\Rightarrow$ (c) Assume $R^k(p) \leq p$, then $R^{kn}(p) \leq p$ for all $n \in \mathbb{N}$ thus $R^l(p)$ can never become $\mathbb{1}$.

(c) $\Rightarrow$ (a) We prove this implication by contraposition. Let R have period $k > 1$, then there is a $p \neq 0, \mathbb{1}$ with $R^k(p) = p$. So R^k is not irreducible. □

5.5.4 Structure of the Reversible Algebra

There is more to say about the reversible algebra of a reach map. We only give the central theorem here. Since the perspective of reach maps does not add anything new to this investigation, we refer to [Bra17] for further discussion.

Theorem *Let R be a recurrent reach map with a unital representative. Then there are orthogonal projections $p_1, \ldots, p_k$ that are an (up to order) unique orthogonal decomposition of $\mathbb{1}$ so that for all $i \in \{1, \ldots, k\}$*

(1) *$p_i \in \mathcal{Z}(\mathcal{E}(R))$,*
(2) *There is an $n_i \in \mathbb{N}$, a matrix algebra $\mathcal{B}_i$, a *-automorphism $\alpha_i \colon M_{n_i} \longrightarrow M_{n_i}$ and an irreducible reach map $R_i' \colon \mathcal{P}(\mathcal{B}_i) \longrightarrow \mathcal{P}(\mathcal{B}_i)$ such that $\mathcal{A}_{p_i} \simeq M_{n_i} \otimes \mathcal{B}_i$ and $R_{p_i} \simeq \alpha_i \otimes R_i'$.*

Proof This structure of the reversible algebra of a unital recurrent operator is the main result of [Bra17, 5.2.8]. The idea of the proof is to show that every peripheral eigenvector can be decomposed into partial isometries. The uniqueness of the decomposition is given in [Bra17, 5.3.2]. A very similar result has independently been shown in [GFY18, Theorem 2]. Because the structure of $\mathcal{E}(R)$ is independent of the concrete representative T, the result holds for reach maps as well. □

Remark The central projections p_i in this theorem are not the minimal central projections which determine the structure of $\mathcal{E}(R)$ as a direct sum of M_{n_i}. However, to every restricted reach map R_i', there belongs an exact cycle, and the projections in this cycle are, in fact, minimal central projections of $\mathcal{E}(R)$.

5.5.5 Asymptotics

Also, in [Bra17, Chapter 6], it is shown that the asymptotic time behaviour of any completely positive operator can be given by $T^k - T^k \circ P \longrightarrow 0$, where P projects onto $\mathcal{E}(T)$ on which T acts as an automorphism. While the automorphism behaviour of T on $\mathcal{E}(T)$ is uniquely determined by R, the exact form of P, but not its reach map, depends on the representative. If additionally T is not recurrent, the form of

$\mathcal{E}(T)$ (which is not simply a subalgebra in this case) also depends on the concrete representative.

In this chapter, we presented a Perron-Frobenius theory for reach maps. First, we decomposed the algebra along communicating classes. Then we sorted those classes into recurrent and transient classes. Finally, we described the dynamics of a reach map restricted to one of those classes.

6 Conditional Expectations

Conditional expectations are a special kind of completely positive operators. They are essential for describing how a system can be a subsystem of a larger system. We will need conditional expectations to project onto a single time step of a Markov process on an algebra representing the system's state for all times. Since we want to describe processes with infinitely many time steps, we need to consider infinite-dimensional systems. Thus, for the rest of this work, we return to the category of enveloped C*-algebras.

In this chapter, we show that the reach map of a conditional expectation is determined by its support (on the envelope) and the subalgebra onto which the expectation is projecting. This is described in generality in Theorem 6.2.2, which is the central result of this chapter. The main takeaway, that reach maps of faithful conditional expectations are completely determined by the subalgebra, can be found in Corollary 6.2.3.

6.1 Conditional Expectations and Expected Subalgebras

We start with a quick summary of conditional expectations and their properties in enveloped C*-algebras.

6.1.1 Definition

First, we look at the established definition of conditional expectations.

Proposition [Bla06, II.6.10.1] *Let $\mathcal{A} \subseteq \hat{\mathcal{A}}$ be C*-algebras, consider a linear map $P : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ with $\mathcal{P}(\hat{\mathcal{A}}) = \mathcal{A}$ and $\|P\| = 1$. Then the following are equivalent:*

(a) *P is completely positive with*

$$P(axb) = aP(x)b \quad \text{for all } x \in \hat{\mathcal{A}}, a, b \in \mathcal{A}.$$

(b) *$P^2 = P$.*

Definition We call an operator like that a *conditional expectation*. It is also called a *projection of norm one* in the literature.

Example The most important example for a conditional expectation for us will be $P_\varphi\colon \mathcal{A}\otimes\mathcal{B}\to\mathcal{A}\otimes\mathcal{B}$ for two unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$ defined with some state $\varphi\in\mathcal{S}(\mathcal{B})$ via $P(x\otimes y)=\varphi(y)x\otimes\mathbb{1}$ for all $x\in\mathcal{A}$ and $y\in\mathcal{B}$.

6.1.2 Enveloped Conditional Expectations

For enveloped C*-algebras, we have to specify what a conditional expectation is.

Proposition *Let $\hat{\mathcal{A}}^\diamond$ be an enveloped C*-algebra and $P:\hat{\mathcal{A}}^\diamond\to\mathcal{A}^\diamond\subseteq\hat{\mathcal{A}}^\diamond$ a normal completely positive operator. If $P|_{\hat{\mathcal{A}}}\colon\hat{\mathcal{A}}\to\mathcal{A}\subseteq\hat{\mathcal{A}}$ is a conditional expectation onto $\mathcal{A}$ then $P:\mathcal{M}_{\hat{\mathcal{A}}}\to\mathcal{M}_{\mathcal{A}}\subseteq\mathcal{M}_{\hat{\mathcal{A}}}$ is a conditional expectation onto $\mathcal{M}_{\mathcal{A}}$.*

Proof By Lemma 2.1.6, P is completely positive and of norm 1. Since $\mathcal{A}$ is weak* dense in $\mathcal{M}_{\mathcal{A}}$ and P normal, we get $P(\mathcal{M}_{\mathcal{A}})=\mathcal{M}_{\mathcal{A}}$. Also, by continuity we have $P|_{\mathcal{M}_{\mathcal{A}}}=\mathrm{id}|_{\mathcal{M}_{\mathcal{A}}}$. □

Definition In this case, P is called a *conditional expectation* onto $\mathcal{A}^\diamond$.

6.1.3 Properties of Conditional Expectations

We will need a few properties about how conditional expectations work in enveloped C*-algebras. We need to be careful again because of the difference between the support on $\mathcal{A}$ and $\mathcal{M}_{\mathcal{A}}$.

The first lemma shows that commutation relations are stable under taking the closure or the interior of a projection.

Lemma *Let $x\in\mathcal{A}$, $p\in\mathcal{P}(\mathcal{M}_{\mathcal{A}})$ and $xp=px$. Then $x\bar{p}=\bar{p}x$ and $xp^\circ=p^\circ x$.*

Proof Because of the commutation relation, we can see that $x^*p^\perp x\perp p$. Since $(p^\perp)^\circ=\bar{p}^\perp\le p^\perp$ we also know $x^*\bar{p}^\perp x\perp p$ and $[x^*\bar{p}^\perp x]\perp p$. Ad_x is an endomorphism on the enveloped C*-algebra $(\mathcal{A},\mathcal{M}_{\mathcal{A}})$. As the orthogonal complement of a closed projection $\bar{p}^\perp$ is an open projection, so $[x^*\bar{p}^\perp x]=R_{\mathrm{Ad}_x}(\bar{p}^\perp)$ is also an open projection. This allows us to switch to the closure on the other side of the orthogonality and get $[x^*\bar{p}^\perp x]\perp\bar{p}$. So $\bar{p}x^*\bar{p}^\perp x\bar{p}=0$, $\bar{p}^\perp x\bar{p}=0$, and $x\bar{p}=\bar{p}x\bar{p}$. With the same argument for x^* we get $\bar{x}p=\bar{p}x\bar{p}=\bar{p}x$. When x commutes with p it also commutes with $p^\perp$ and thus also with $\overline{(p^\perp)}=(p^\circ)^\perp$ and p°. □

Lemma *Let $P:\hat{\mathcal{A}}^\diamond\to\mathcal{A}^\diamond\subseteq\hat{\mathcal{A}}^\diamond$ be a normal conditional expectation in an enveloped C*-algebra. With $r:=\mathrm{supp}P$ we get*

1. $r\in\mathcal{A}'$
2. $P(r)=\mathbb{1}_{\mathcal{A}}\ge r$
3. $r\mathcal{A}\ni x\mapsto P(x)\in\mathcal{A}$ *is a *-isomorphism.*
4. $r\mathcal{A}\ni x\mapsto r^\perp P(x)\in r^\perp\mathcal{A}$ *is a *-homomorphism.*

Proof (1) We consider $p \coloneqq \operatorname{supp}_{(\mathcal{M}_{\mathcal{A}},\mathcal{M}_{\mathcal{A}})} P$. Let $x \in \mathcal{A}$. Then $P(xx^*) = P(P(x)x^*) = P(P(xp)x^*) = P(xpx^*)$ and consequently $xp^{\perp}x^* \perp p$. Because $px = xp$ we also have for $r = \bar{p}$, $rx = xr$.

(2) We have $P(\mathbb{1}) \in \mathcal{M}_{\mathcal{A}}$. For any $x \in \mathcal{M}_{\mathcal{A}}$ we get $xP(\mathbb{1}) = P(x\mathbb{1}) = P(\mathbb{1}x) = P(\mathbb{1})x$. Since also $P(\mathbb{1})^2 = P(\mathbb{1})P(\mathbb{1}) = P(\mathbb{1}P(\mathbb{1})) = P(P(\mathbb{1})) = P(\mathbb{1})$ we can conclude $\mathbb{1}_{\mathcal{A}} = P(\mathbb{1}) = P(r)$, where the last step uses Proposition 3.3.1 (4). Since r is closed, to show $\mathbb{1}_{\mathcal{A}} \geq r$ it is enough to show that $x \perp \mathbb{1}_{\mathcal{A}}$ entails $x \perp r$ for all $x \in \hat{\mathcal{A}}$. If $x\mathbb{1}_{\mathcal{A}} = 0$, we have $P(x) = \mathbb{1}_{\mathcal{A}}P(x) = P(\mathbb{1}_{\mathcal{A}}x) = P(0) = 0$. That implies $x \perp r$.

(3) $\mathcal{A} \ni x \mapsto rx \in r\mathcal{A}$ is a *-homomorphism, because r commutes with $\mathcal{A}$. Because of $x = P(x) = P(rx)$ and $rx = rP(x) = rP(rx)$ for $x \in \mathcal{A}$ it is the inverse map of $r\mathcal{A} \ni x \mapsto P(x) \in \mathcal{A}$.

(4) Let $x, y \in \mathcal{A}$ then $P(rxry)r^{\perp} = P(rxy)r^{\perp} = P(xP(y))r^{\perp} = P(rx)r^{\perp}P(ry)r^{\perp} = P(rx)r^{\perp}P(ry)r^{\perp}$. □

Remark This lemma is inspired by the investigation of idempotent Markov operators in [Gär14, Chapter 4] and picks the properties we need from the bigger picture painted there.

6.1.4 Expected Subalgebras

We define expected subalgebras to make the notion of a subsystem concrete.

Lemma *Let $P\colon \hat{\mathcal{A}}^{\diamond} \to \mathcal{A}^{\diamond} \subseteq \hat{\mathcal{A}}^{\diamond}$ be a normal conditional expectation, then the following are equivalent:*

(a) $\operatorname{supp} P = \mathbb{1}_{\mathcal{A}}$.
(b) $\mathcal{N}_P \perp \mathcal{A}$.

Proof (a) $\Rightarrow$ (b) Let $x \in \mathcal{N}_P^+$ and $y \in \mathcal{A}_1^+$, thus $P(x) = 0$ and $x \perp \mathbb{1}_{\mathcal{A}} \geq y$.

(b) $\Rightarrow$ (a) Since $\operatorname{supp} P \coloneqq \mathbb{1}_{\mathcal{N}_P}^{\perp}$ and $\mathbb{1}_{\mathcal{A}} \perp \mathbb{1}_{\mathcal{N}_P}$ we have $\operatorname{supp} P \geq \mathbb{1}_{\mathcal{A}}$. However, since we always have $\operatorname{supp} P \leq \mathbb{1}_{\mathcal{A}}$, we get (a). □

Definition We call a C*-subalgebra $\mathcal{A} \subseteq \hat{\mathcal{A}}$ *expected* if there is a conditional expectation P onto $\mathcal{A}$ with $\mathcal{N}_P \perp \mathcal{A}$. We call a subalgebra $\mathcal{A}^{\diamond} \subseteq \hat{\mathcal{A}}^{\diamond}$ of an enveloped C*-subalgebra *expected* if $\mathcal{A}$ is expected and the conditional expectation P has a normal extension.

Remark The weakest sensible definition for an expected subalgebra would be to just require the existence of any conditional expectation. That however would be too weak, since very non-faithful conditional expectations preserve too little structure to be useful. On the other hand if we would require that P be faithful, then every expected subalgebra would be required to have the same $\mathbb{1}$ as the surrounding algebra. That would be too restrictive for our use case. The condition here is the correct weakening of faithfulness for subalgebras which do not share the unit of $\hat{\mathcal{A}}$.

Morally, the faithfulness condition does not exclude a lot of subalgebras. If we have $\mathrm{supp} P < \mathbb{1}_{\mathcal{A}}$ we can switch to P_r projecting onto $\mathcal{A}_r$ and $\mathcal{A}_r$ is isomorphic to the original $\mathcal{A}$. There is, however, one technical problem with this. In general, r is not an element of $\mathcal{A}$, so $\mathcal{A}_r$ might not be a subalgebra of $\mathcal{A}$. In that case one would need to see if it is feasible to adjust the algebra accordingly.

6.2 Reach Maps of Conditional Expectations

Now, we can look at the reach map of a conditional expectation. We will see that they are often completely determined by the algebra we are projecting on.

6.2.1 The Relative Support

Definition Let $x \in \hat{\mathcal{A}}^+$ be a positive element of an enveloped C*-algebra $\hat{\mathcal{A}}^\diamond$ and $\mathcal{A}^\diamond \subseteq \hat{\mathcal{A}}^\diamond$ a subalgebra. Then we define the *(relative) support of x in $\mathcal{A}^\diamond$* as $[x]_{\mathcal{A}^\diamond} := \bigwedge \{ q \in \mathcal{P}(\mathcal{M}_{\mathcal{A}}) \mid x \leq \|x\|\, q \}$.

Proposition *Let $x \in \hat{\mathcal{A}}^+$, then*

(1) $[x] \leq [x]_{\mathcal{A}^\diamond}$.
(2) $[x] = [x]_{\hat{\mathcal{A}}^\diamond}$.

Proof (1) Follows from the definition.

(2) $[x] \geq [x]_{\hat{\mathcal{A}}^\diamond}$ is true, because $x \leq \|x\|\,[x]$. On the other hand $[x]_{\hat{\mathcal{A}}^\diamond} x [x]_{\hat{\mathcal{A}}^\diamond} = x$ and thus $[x] \leq [x]_{\hat{\mathcal{A}}^\diamond}$. □

Remark By definition, there is no reason at this point to assume that $[x]_{\mathcal{A}}$ is an open projection.

6.2.2 Reach Maps of Conditional Expectations

We come to the main theorem about reach maps of conditional expectations.

Theorem *Let $\hat{\mathcal{A}}^\diamond$ be an enveloped C^*-algebra and $P\colon \hat{\mathcal{A}}^\diamond \to \mathcal{A}^\diamond \subseteq \hat{\mathcal{A}}^\diamond$ be a normal conditional expectation.*

(1) *$R_P = R_P^2$ and R_P is surjective onto $\mathcal{T}(\mathcal{A}^\diamond)$.*
(2) *If $r := \mathrm{supp}_{(\mathcal{M}_{\hat{\mathcal{A}}}, \mathcal{M}_{\hat{\mathcal{A}}})} P$, then $R_P(p) = [rpr]_{\mathcal{A}^\diamond}$ for all open projections $p \in \mathcal{T}(\hat{\mathcal{A}}^\diamond)$.*
(3) *If $\mathcal{A} = \mathcal{M}_{\mathcal{A}}$ and $r := \mathrm{supp}_{\hat{\mathcal{A}}^\diamond} P$, then $R_P(p) = [rpr]_{\mathcal{A}^\diamond}$ for all open projections $p \in \mathcal{T}(\hat{\mathcal{A}}^\diamond)$.*

Proof (1) For any $p \in \hat{\mathcal{A}}$ we have $R_P(R_P(p)) = [P([P(p)])] = [P(P(p))] = [P(p)] = R_P(p)$ and $p = P(p) = [P(p)] = R_P(p)$ for any $p \in \mathcal{T}(\mathcal{A})$.

In the rest of the proof, we show (2) and (3) in lockstep. Let $q := R_P(p) \vee p - R_P(p)$. First we show $P(q) = 0$.
Because $R_P(p) \leq R_P(p) \vee p$ the defined q is an orthogonal projection. We note $q \leq R_P(p) \vee p$. We can apply R_P, which is join-preserving, monotone and idempotent, to both sides of this inequality to get $R_P(q) \leq R_P(p) \vee R_P(p) = R_P(p)$. We continue by observing that $q \perp R_P(p)$, which, with the just proven $R_P(p) \geq R_P(q)$, gives us $q \perp R_P(q)$ and so $qP(q) = 0$. Now we can conclude $0 = P(0) = P(qP(q)) = P(q)P(q) = P(q)^2$ and $P(q) = 0$.

We show $q \perp r$ for the case of statement (2). With $r := \operatorname{supp}_{(\mathcal{M}_{\hat{\mathcal{A}}},\mathcal{M}_{\hat{\mathcal{A}}})} P$ we can directly conclude $r \perp q$ with Proposition 3.3.1 (4).

We show $q \perp r$ for the case of statement (3). With $r := \operatorname{supp}_{\hat{\mathcal{A}}^\diamond} P$ we need to show that q is open. We consider an increasing net in $(x_i)_{i\in I} \subseteq \hat{\mathcal{A}}^+$ with $\sup_{i\in I} x_i = R_P(p) \vee p$. Since $R_P(p) \in \mathcal{A} \subseteq \hat{\mathcal{A}}$ by the extra condition for statement (3), we can see that $R_P(p)^\perp x_i R_P(p)^\perp \in \hat{\mathcal{A}}$. Thus, $\sup_{i\in I} R_P(p)^\perp x_i R_P(p)^\perp = R_P(p)^\perp (R_P(p) \vee p) R_P(p)^\perp = (R_P(p) \vee p)(\mathbb{1} - R_P(p)) = R_P(p) \vee p - (R_P(p) \vee p)R_P(p)) = R_P(p) \vee p - R_P(p) = q$ is open and thus $q \perp r$ again by Proposition 3.3.1 (4).

Next we show $rpr \leq R_P(p)$. For both definitions of r, we have $r \in \mathcal{A}'$; thus $R_P(p)r = rR_P(p)$ is an orthogonal projection. From $q \perp r$ we get $r(R_P(p) \vee p - R_P(p))r = 0$ and so $rR_P(p) = r(R_P(p) \vee p)r$ is an orthogonal projection. So $rR_P(p)r = r(R_P(p) \vee p)r = [r(R_P(p) \vee p)r] = r(R_P(p))r \vee [rpr]$ and thus $rpr \leq [rpr] \leq R_P(p)r \leq R_P(p)$.

Finally, we show $R_P(p) = [rpr]_{\mathcal{A}^\diamond}$. We already know that $rpr \leq R_P(p)$ and thus $[rpr]_{\mathcal{A}^\diamond} \leq R_P(p)$. On the other hand we consider $q' \in \mathcal{P}(\mathcal{M}_{\mathcal{A}})$ with $rpr \leq q' \leq R_P(p)$. We define a new $q := R_P(p) - q'$ and want to show $q = 0$. Since we know $rpr \perp q$ and $q \in \mathcal{A}''$, we get $0 = P(qrpr) = qP(p)$, which means $q \perp R_P(p)$. With $q \leq R_P(p)$ we conclude $q \perp q$ and thus $q = 0$. This means $R_P(p) = [rpr]_{\mathcal{A}^\diamond}$. □

Remark This means the reach map of a conditional expectation is uniquely determined by its support projection and the subalgebra onto which we are projecting.

One consequence of this theorem is that we can project open projections onto a subalgebra without knowing a specific conditional expectation. We can always calculate the relative support for any subalgebra. However, suppose we do not know whether there is a conditional expectation. In that case, we also do not know if the relative support is a reach map and, for example, whether it maps into the open projections.

6.2.3 Faithful Conditional Expectations

The theorem becomes particularly elegant for faithful conditional expectations.

Corollary *If P is faithful on the envelope, or P is faithful and $\mathcal{A} = \mathcal{M}_{\mathcal{A}}$ then*

$$p \leq R_P(p) = [p]_{\mathcal{A}^\diamond}$$

for all $p \in \mathcal{T}(\hat{\mathcal{A}}^\diamond)$.

This corollary is, for example, always true for expected finite-dimensional subalgebras with $\mathbb{1}_{\mathcal{A}} = \mathbb{1}_{\hat{\mathcal{A}}}$.

6.2.4 Conditional Expectations onto Finite Subalgebras

When working with topological Markov chains we want that the properties which we are discussing are independent of the specific chosen envelope. For this we need the following slightly technical proposition which moves the perspective onto the hereditary subalgebras, to abstract from a specific envelope. It says that if $\mathcal{A}$ is finite-dimensional, we can show for arbitrary supports of the conditional expectation that the behaviour of the reach map is independent of the envelope.

Proposition *Let $\mathcal{A}$ be a finite-dimensional C^*-subalgebra of a C^*-algebra $\hat{\mathcal{A}}$ and $\mathcal{M}_P$ and $\mathcal{M}_Q$ different envelopes of $\hat{\mathcal{A}}$. If we have two normal conditional expectations $P\colon (\hat{\mathcal{A}}, \mathcal{M}_P) \to \mathcal{A} \subseteq (\hat{\mathcal{A}}, \mathcal{M}_P)$ and $Q\colon (\hat{\mathcal{A}}, \mathcal{M}_Q) \to \mathcal{A} \subseteq (\hat{\mathcal{A}}, \mathcal{M}_Q)$ with the same null algebra, then for every hereditary C^*-subalgebra of $\mathcal{B} \subseteq \hat{\mathcal{A}}$ we have*

$$R_P(\mathbb{1}_{\mathcal{B}}) = R_Q(\mathbb{1}_{\mathcal{B}}).$$

Proof Let π_P and π_Q be the normal surjective representations mapping $\mathcal{A}^{**}$ to $\mathcal{M}_P$ and $\mathcal{M}_Q$. P and Q have unique normal continuations onto $\mathcal{A}^{**}$ and because of normality we have $\pi_P \circ P = P \circ \pi_P$ (and the same for Q). Now for $\mathbb{1}_{\mathcal{B}} \in \mathcal{A}^{**}$ we have $R_P(\mathbb{1}_{\mathcal{B}}) = [\mathrm{supp} P \mathbb{1}_{\mathcal{B}} \mathrm{supp} P]_{\mathcal{A}} = [\mathrm{supp} Q \mathbb{1}_{\mathcal{B}} \mathrm{supp} Q]_{\mathcal{A}} = R_Q(\mathbb{1}_{\mathcal{B}}) \in \mathcal{A}^{**}$.

Then $\mathcal{M}_P \ni R_P(\mathbb{1}_{\mathcal{B}}) = [P(\pi_P(\mathbb{1}_{\mathcal{B}}))] = [\pi_P(P(\mathbb{1}_{\mathcal{B}}))] = [\pi_P([P(\mathbb{1}_{\mathcal{B}})])] = \pi_P([P(\mathbb{1}_{\mathcal{B}})]) = \pi_P(R_P(\mathbb{1}_{\mathcal{B}})) \in \mathcal{A} \subseteq \hat{\mathcal{A}}^{**}$. Since we can do the same equality chain for R_Q, we get $\mathcal{M}_P \ni R_P(\mathbb{1}_{\mathcal{B}}) = R_Q(\mathbb{1}_{\mathcal{B}}) \in \mathcal{M}_Q$. □

6.3 Examples of Conditional Expectations

Before we move on to discuss topological Markov chains, we look at a few examples of Corollary 6.2.3 in action.

6.3.1 Classical Finite Product Systems

Example We start with a finite set Ω and consider the algebra $\mathcal{A} = C(\Omega)$ embedded via $i(f)(\omega_1, \omega_2) = f(\omega_1)$ into $\hat{\mathcal{A}} = C(\Omega \times \Omega)$. We identify $\mathcal{A}$ with $i(\mathcal{A})$. A conditional expectation $P\colon \hat{\mathcal{A}} \to \mathcal{A} \subseteq \hat{\mathcal{A}}$ can be used to lift a measure μ on $\mathcal{A}$ to a measure $\hat{\mu}$ on $\hat{\mathcal{A}}$ via $\varphi_{\hat{\mu}} = \varphi_\mu \circ P$. We assume that P is faithful. Then it lifts faithful measures to faithful measures. We can describe the reach map of P on the set level. Let $M \subseteq \Omega \times \Omega$, then $R_P(\chi_M) = \chi_N$ with $N = \{\omega_1 \in \Omega : \exists \omega_2 \in \Omega . (\omega_1, \omega_2) \in M\}$. If we consider the coordinate projection $\pi_1 : \Omega \times \Omega \to \Omega$, then this means simply $N = \pi_1(M)$. Thus, R_P is just lifting of the coordinate projection to orthogonal projections. Our theorem applies because N is the smallest subset of Ω such that $M \subseteq N \times \Omega$.

This example is a special case of the next two examples, in which we will manually demonstrate that our result works.

6.3.2 Commutative Conditional Expectations

Example We consider a probability space (Ω, Σ, μ) and a finite sub σ-algebra $\Sigma_0 \subseteq \Sigma$, without null-sets. The sub σ-algebra Σ_0 gives us a subalgebra $L^\infty(\Omega, \Sigma_0) \subseteq L^\infty(\Omega, \Sigma)$.

Let us consider the conditional expectation $P\colon L^\infty(\Omega, \Sigma) \to L^\infty(\Omega, \Sigma_0) \subseteq L^\infty(\Omega, \Sigma)$ which preserves μ. Let Σ_0 be generated by k atoms $S_i \in \Sigma_0$ for $1 \leq i \leq k$. Then $L^\infty(\Omega, \Sigma_0) \simeq \mathbb{C}^k$. And for $f \in L^\infty(\Omega, \Sigma)$ we have

$$P(f)_i = \int_{S_i} f \,\mathrm{d}\mu \quad \text{for all } i \in \{1, \dots, k\}.$$

If we consider the special case of an orthogonal projection $p = \chi_S \in L^\infty(\Omega)$ with $S \in \Sigma$, we get

$$P(p)_i = \int_{S_i} \chi_S \,\mathrm{d}\mu = \int_{S \cap S_i} 1 \,\mathrm{d}\mu = \mu(S \cap S_i) \quad \text{for all } i \in \{1, \dots, k\}.$$

Now, let us look at the support of the image. For all $i \in \{1, \dots, k\}$ we get

$$R_P(p)_i = \begin{cases} 0, \text{ if } \mu(S \cap S_i) = 0 \\ 1, \text{ if } \mu(S \cap S_i) > 0. \end{cases}$$

This is consistent with our theorem. If we regard $L^\infty(\Omega, \Sigma)$ as self-enveloped we get

$$\begin{aligned} R_P(\chi_S) = [\chi_S]_{L^\infty(\Omega, \Sigma_0)} &= \bigwedge \left\{ q : q \in \mathcal{P}(L^\infty(\Omega, \Sigma_0)), q \geq \chi_S \right\} \\ &= \bigwedge \{ \chi_K : K \in \Sigma_0, \chi_K \geq \chi_S \} \\ &= \chi_{\bigcup \{ S_i : \mu(S \cap S_i) > 0,\ 1 \leq i \leq k \}}. \end{aligned}$$

Which is the same result.

6.3.3 Tensor Conditional Expectations

Example Consider the algebra $M_n \otimes M_k = M_k(M_n)$ and any faithful state $\varphi \in \mathcal{S}(M_k)$. We want to look at the behaviour of the conditional expectation $P := \mathrm{id} \otimes \varphi$. Without loss of generality we pick φ to have a diagonal density $(\lambda_i)_{1\leq i\leq k}$ in M_k. Then for any projection $q = (a_{ij})_{1\leq i,j\leq k} \in \mathcal{P}(M_k(M_n))$ we have

$$P((a_{ij})_{1\leq i,j\leq k}) = \sum_i \lambda_i a_{ii} \in M_n^+.$$

We can calculate the reach map to be $R_P(q) = [\sum_i \lambda_i a_{ii}] = \bigvee_i [a_{ii}] \in \mathcal{P}(M_n)$. We are still free to pick a basis for M_n. Let us pick one, such that $p := R_P(q) \in \mathcal{P}(M_n)$ is diagonal and sorted, so that it looks like

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

Then, because $p \geq [a_{ii}]$, the a_{ii} are only different from zero in the upper left corner with ones on the diagonal. For the matrix $(a_{ij})_{ij}$ to be positive, the same has to hold also for the off-diagonal entries $a_{ij} \in M_n$. Thus, $q \leq p \otimes \mathbb{1}$.

We could wonder if there could be a smaller $p' \in \mathcal{P}(M_n)$ such that $q \leq p' \otimes \mathbb{1} \leq p \otimes \mathbb{1}$. However, from that, we could conclude $[a_{ii}] \leq p'$ for all i and thus, by definition of p we have $p \leq p'$ and $p = p'$.

6.3.4 A Peek at Tensor Dilations

Example The independence of a specific conditional expectation applied to a tensor product becomes very relevant when we look at dilations of Markov operators, like we do in the next chapter: We consider two matrix algebras $\mathcal{A}$ and $\mathcal{C}$, a *-automorphism $\alpha\colon \mathcal{A} \otimes \mathcal{C} \to \mathcal{A} \otimes \mathcal{C}$ and a faithful state $\psi \in \mathcal{S}(\mathcal{C})$. Then we can define a completely positive unital operator $T_\psi\colon \mathcal{A} \to \mathcal{A}$ as $T_\psi(x) = (\mathrm{id} \otimes \psi)(\alpha(x \otimes \mathbb{1}))$. We can see the map $\mathrm{id} \otimes \psi$ as a conditional expectation $P_\psi\colon \mathcal{A} \otimes \mathcal{C} \to \mathcal{A} \otimes \mathbb{1} \subseteq \mathcal{A} \otimes \mathcal{C}$, which is faithful, since ψ is. Thus, the reach map of T_ψ given by $R_{T_\psi}(p) = R_{P_\psi}(R_\alpha(p \otimes \mathbb{1}))$ for all $p \in \mathcal{P}(\mathcal{A})$ is independent of ψ, as long as $\psi \in \mathcal{S}(\mathcal{C})$ is faithful. As we have seen, this means that a lot of properties of T_ψ, like the structure of communicating classes and the peripheral spectrum, are completely determined by $\mathcal{A}, \mathcal{C}$ and α.

In this chapter, we showed that the reach map of a conditional expectation is determined completely by its support on the envelope and the subalgebra we are projecting onto. We showed the usefulness of this result in multiple examples.

7 Topological Markov Processes

In this chapter, we finally join the concepts of topological Markov processes and quantum Markov processes. We do this by unifying both theories in the category of reach maps. In classical symbolic dynamics and coding theory, topological Markov chains are a special case of shift spaces. We mirror this by first picking a suitable generalization of a non-commutative shift space. Then we discuss which properties such a shift space needs to have, to make it a Markov process. In the end we will discuss how every topological Markov process is a dilation of a reach map and that every reach map has a dilation.

The central result in this chapter is the comparison of Markov conditions in Theorem 7.4.2.

In analogy to classical shift spaces all processes we discuss in this chapter are time-discrete, homogeneous and two-sided.

7.1 Classical Shift Spaces

First we introduce classical shift spaces. They are the topological equivalent of a stationary stochastic process. Shift Spaces are shift-invariant (hence the name) subspaces of the so-called full shift. Thus, we start by defining full shifts.

7.1.1 Full Shifts

Definition Let Ω be a finite set. We call it an *alphabet*. Then the *full shift* over the alphabet Ω is the set of paths

$$\Omega^{\mathbb{Z}} = \{(\omega_i)_{i\in\mathbb{Z}} : \omega_i \in \Omega \quad \text{for all } i \in \mathbb{Z}\}.$$

The *(left) shift* is the map $s\colon \Omega^{\mathbb{Z}} \to \Omega^{\mathbb{Z}}$ with $s((\omega_i)_{i\in\mathbb{Z}}) = (\omega_{i+1})_{i\in\mathbb{Z}}$.

Remark Using the product topology of the discrete topology on Ω, we get a metrizable topology on $\Omega^{\mathbb{Z}}$, which makes it a compact space.

Example The simplest interesting full shift is the space of all 0-1-sequences.

7.1.2 Shift Spaces

Definition Let $F \subseteq \Omega^* := \bigcup_{i\in\mathbb{N}} \Omega^i$. We define the *shift space* or a *sub shift* with the *forbidden words* or *forbidden blocks* F as

$$S_F := \left\{ \omega \in \Omega^{\mathbb{Z}} : \nexists\, w \in F, i \in \mathbb{Z} \text{ such that } w = \omega_{[i,i+|f|-1]} \right\}.$$

Theorem [LM21, 6.1.21] *For a subset $S \subseteq \Omega^{\mathbb{Z}}$ of a full shift $\Omega^{\mathbb{Z}}$ the following are equivalent:*

(a) *$S = S_F$ for some set $F \subseteq \Omega^*$.*
(b) *S is compact and $s(S) = S$.*

Remark For any shift space S over the alphabet Ω there can be different sets of forbidden words $F_1, F_2 \subseteq \Omega^*$ with $F_1 \neq F_2$ and $S = S_{F_1} = S_{F_2}$.

Example 1 Every full shift is a subshift with $F = \emptyset$.

Example 2 We pick the full shift $\{0,1\}^{\mathbb{Z}}$ and one forbidden block $F = \{000\}$. Now S_F contains all sequences of any form $\ldots 0100101110101011 \ldots$ which never contain more than two zeros in a row. We can see here that the set of forbidden words is not unique. If we define the set $\tilde{F} = \{000, 0001\}$ we get $S_F = S_{\tilde{F}}$. Clearly, in this case the forbidden word 0001 is unnecessary.

Example 3 We consider a 0-1-matrix $a \in M_n$ on the alphabet $\Omega := \{0, \ldots, n\}$. We define the set of forbidden words as $F := \left\{ ij \in \Omega \times \Omega : a_{ij} = 0 \right\}$. The adjacency matrix a describes which transitions are possible in the shift space. We will come back to this example, when we define topological Markov chains.

7.2 Non-Commutative Shift Spaces

As a next step, we generalize shift spaces to non-commutative systems.

7.2.1 Homogeneous Processes

Definition A *process* is a tuple $(\hat{\mathcal{A}}, \mathcal{A}, (i_n)_{n\in\mathbb{Z}})$ where $\mathcal{A}$ and $\hat{\mathcal{A}}$ are C*-algebras and $i_n \colon \mathcal{A} \to \hat{\mathcal{A}}$ is a family of injective *-homomorphisms, such that all $i_n(\mathcal{A}) \subseteq \hat{\mathcal{A}}$ are expected. A *dynamical system* is a tuple $(\hat{\mathcal{A}}, \alpha)$ of a C*-algebra $\hat{\mathcal{A}}$ and a *-automorphism $\alpha \colon \hat{\mathcal{A}} \to \hat{\mathcal{A}}$. A process is called *homogeneous* if there is a dynamical system $(\hat{\mathcal{A}}, \alpha)$ such that $i_n = \alpha^n \circ i_0$ for all $n \in \mathbb{Z}$. We call these processes and dynamical systems *stochastic* if all involved algebras are von Neumann algebras and all morphisms are normal, otherwise, we call them *topological.*

Remark For a homogeneous process, we can identify $\mathcal{A}$ with its image under i_0 in $\hat{\mathcal{A}}$. Then the process is completely determined by the dynamical system $(\hat{\mathcal{A}}, \alpha)$ and the expected subalgebra $\mathcal{A}$. Hence, we write $(\hat{\mathcal{A}}, \alpha, \mathcal{A})$ for a homogeneous process. This slightly hides the fact, that there are potentially multiple different conditional expectations onto $\mathcal{A}$, but since we will mainly care about reach maps, this choice does not matter to us.

Technically speaking, all processes are topological and some of them stochastic, but often it is not very useful to consider a stochastic process as a topological process, akin to how the universal enveloping von Neumann algebra of a von Neumann algebra is a strange concept.

Definition We write $\mathcal{A}_I := C^*(\bigcup_{n\in I} i_n(\mathcal{A}_0))$ for all index sets $I \subseteq \mathbb{Z}$. A topological process is called *minimal* if $\mathcal{A}_{\mathbb{Z}} = \hat{\mathcal{A}}$ and a stochastic process is called *minimal* if $\overline{\mathcal{A}_{\mathbb{Z}}}^{\sigma^*} = \mathcal{A}$.

We say $\mathcal{A}$ is the process's algebra of *values*. If $\mathcal{A}$ is finite-dimensional, we speak of a process with *finite-dimensional values*.

Two homogeneous processes $(\hat{\mathcal{A}}_1, \alpha_1, \mathcal{A}_1)$ $(\hat{\mathcal{A}}_2, \alpha_2, \mathcal{A}_2)$ are *equivalent* if there is a *-isomorphism $\Phi\colon \hat{\mathcal{A}}_1 \to \hat{\mathcal{A}}_2$ with

$$\Phi \circ \alpha_1 = \alpha_2 \circ \Phi \quad \text{and} \quad \Phi(\mathcal{A}_1) = \mathcal{A}_2.$$

Remark From now on, all our processes will be homogeneous with finite-dimensional values.

Example We consider a stochastic process with values in a finite set Ω on a canonical path space $(\Omega^{\mathbb{Z}}, \Sigma, \mu)$. The process is then given by the random variables $X_n\colon (\Omega^{\mathbb{Z}}, \mu) \to \Omega$, which are just the coordinate projections onto time $n \in \mathbb{Z}$. With the shift $s\colon \Omega^{\mathbb{Z}} \to \Omega^{\mathbb{Z}}$, we get $X_n = X_0 \circ s^n$ for all $n \in \mathbb{Z}$. We define $\hat{\mathcal{A}} := L^\infty(\Omega^{\mathbb{Z}}, \mu)$ and the algebra of values $\mathcal{A} := L^\infty(\Omega)$. We embed $\mathcal{A}$ into $\hat{\mathcal{A}}$ via the *-homomorphism $i_0\colon \mathcal{A} \to \hat{\mathcal{A}}$ with $i_0(f) = f \circ X_0$ and identify $\mathcal{A}$ with $i_0(\mathcal{A})$. Then there is a classical conditional expectation onto $\mathcal{A}$, which leaves μ invariant, making $\mathcal{A}$ an expected subalgebra. With the algebraic shift σ, $(\hat{\mathcal{A}}, \sigma, \mathcal{A})$ is a stochastic homogeneous process by our definition. We have $i_n(f) = \sigma^n \circ i_0(f) = f \circ X_0 \circ s^n$ for all $n \in \mathbb{Z}$.

7.2.2 Enveloped Processes

In principle, it would be nice for topological Markov processes, to be simply describable in the category of C*-algebras. But we have seen that we will need to embed the topological process into an envelope so that we can talk about supports of elements and use reach maps. For this reason we take a look at how a topological process behaves together with an envelope.

We remind ourselves of Definition 2.1.6: A normal *-isomorphism between enveloped C*-algebras is a normal *-isomorphism between the enveloping von Neumann algebras, such that the restriction on the C*-algebras is also a *-isomorphism.

Proposition *Let $\hat{\mathcal{A}}^\diamond := (\hat{\mathcal{A}}, \mathcal{M}_{\hat{\mathcal{A}}})$ be an enveloped C*-algebra, $\mathcal{A} \subseteq \hat{\mathcal{A}}$ an expected finite-dimensional subalgebra and $\alpha\colon \hat{\mathcal{A}}^\diamond \to \hat{\mathcal{A}}^\diamond$ a normal *-automorphism. Then*

(1) *$(\hat{\mathcal{A}}, \alpha, \mathcal{A})$ is a topological process.*
(2) *$(\mathcal{M}_{\hat{\mathcal{A}}}, \alpha, \mathcal{A})$ a stochastic process..*
(3) *If $(\hat{\mathcal{A}}, \alpha, \mathcal{A})$ is a minimal topological process, $(\mathcal{M}_{\hat{\mathcal{A}}}, \alpha, \mathcal{A})$ is a minimal stochastic process.*
(4) *For a normal *-isomorphism $\Psi\colon \hat{\mathcal{A}}^\diamond \to \hat{\mathcal{B}}^\diamond$ with $(\hat{\mathcal{B}}, \beta, \mathcal{B})$ constructed as $(\hat{\mathcal{A}}, \alpha, \mathcal{A})$, the following are equivalent:*
   (a) *Ψ is a topological process equivalence.*
   (b) *Ψ is a stochastic process equivalence.*

Proof (1) and (2) are clear by definition.

(3) If $\mathcal{A}_{\mathbb{Z}} = \hat{\mathcal{A}}$ then clearly $\overline{\mathcal{A}_{\mathbb{Z}}}^{\sigma^*} = \mathcal{M}_{\hat{\mathcal{A}}}$.

(4) Follows from restriction and continuity. □

Definition In this situation, we call $(\hat{\mathcal{A}}^\diamond, \alpha, \mathcal{A})$ an *enveloped process*, in the case of (3) a *minimal enveloped process* and in (4) we call the enveloped processes *equivalent*.

Remark For every homogeneous topological process $(\hat{\mathcal{A}}, \alpha, \mathcal{A})$, we can find an envelope so that it becomes an enveloped process. The other way around, let $(\hat{\mathcal{A}}, \alpha, \mathcal{A})$ be a stochastic process. Then $\mathcal{A}_{\mathbb{Z}}$ is a weak* dense subalgebra of $\overline{\mathcal{A}_{\mathbb{Z}}}^{\sigma^*}$. Thus, $(\mathcal{A}_{\mathbb{Z}}, \overline{\mathcal{A}_{\mathbb{Z}}}^{\sigma^*})$ is an enveloped C*-algebra. This way, we can find a topological process inside every stochastic process.

7.2.3 Algebraization of Shift Spaces

We now want to give a definition for non-commutative shift spaces. The number one condition for any non-commutatitve generalization is that we can recover the classical definition in the commutative case. First we show how a classical shift space is a topological process.

Example Let S be a shift space over an alphabet Ω, then we can define C*-algebras $\mathcal{A} := C(\Omega)$ and $\hat{\mathcal{A}} := C(S)$. Via $i\colon \mathcal{A} \to \hat{\mathcal{A}}, f \mapsto f(\omega_0)$ we can embed $\mathcal{A}$ into $\hat{\mathcal{A}}$, thus we will consider $\mathcal{A}$ a subalgebra of $\hat{\mathcal{A}}$. The shift $s\colon S \to S$ lifts to a *-automorphism $\sigma\colon \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ which makes $(\hat{\mathcal{A}}, \sigma, \mathcal{A})$ a topological process. Thus, every shift space gives rise to a canonical topological process.

7.2.4 Commutative Topological Processes are Shift Spaces

Now we show that under the right conditions all commutative topological processes belong to a shift space.

Theorem *Let $(\hat{\mathcal{A}}, \alpha, \mathcal{A})$ be a commutative, minimal, topological process with $\mathcal{A} \simeq C(\Omega_n)$ for a finite set Ω_n and $\hat{\mathcal{A}}$ unital with the same $\mathbb{1}$ as $\mathcal{A}$. If we define a set of forbidden words*

$$F := \left\{ w \in \Omega_n^* \;\middle|\; \prod_{1 \leq i \leq |w|} \alpha^i(\chi_{w_i}) = 0 \right\}$$

and the resulting shift space $S := S_F$ then with the canonical shift σ on $C(S)$ we have

$$(\hat{\mathcal{A}}, \alpha, \mathcal{A}) \simeq (C(S), \sigma, C(\Omega_n))$$

Here $\chi_{w_i} \in C(\Omega_n) \simeq \mathcal{A} \subseteq \hat{\mathcal{A}}$ denotes the characteristic function of the singleton set $\{ w_i \}$, where w_i is the i-th element of the word w.

Definition We call S the *associated shift space* of the topological process.

Proof We represent $\hat{\mathcal{A}}$ as $C(\Omega)$ on a compact Hausdorff space Ω. Then α induces a homeomorphism on Ω and $\mathcal{A}$ induces a partition of clopen sets $(P_j)_{j \in \Omega_n}$ of Ω. For $(\omega_i)_{i \in \mathbb{Z}} \in S$ we note that there is exactly one $\omega \in \Omega$ with $\alpha^i(\omega) \in P_{\omega_i}$ for all $i \in \mathbb{Z}$. It exists because $(\omega_i)_i$ would otherwise contain a forbidden word. It is unique because $\mathcal{A}_{\mathbb{Z}} = \hat{\mathcal{A}}$. By [LM21, 6.5.8] the map $S \ni (\omega_i)_i \mapsto \omega \in \Omega$ is continuous, surjective and conjugates between (S, s) and (Ω, α). However, in contrast to the conditions in [LM21], our partition is disjoint, making the map injective. Because S is compact and Ω Hausdorff by [Mun00, Theorem 26.6] our map is as a bijective continuous function a homeomorphism. We can lift that homeomorphism to a *-isomorphism between the C*-algebras, giving us the desired topological process equivalence. □

Now we can give the general non-commutative definition of a shift space.

Definition We call a minimal, homogeneous topological process $(\hat{\mathcal{A}}, \alpha, \mathcal{A})$ with finite-dimensional values and $\mathbb{1}_{\mathcal{A}} = \mathbb{1}_{\hat{\mathcal{A}}}$ a *shift space*.

Remark For our definition of topological Markov processes, we will not need the conditions of minimality nor that $\mathbb{1}_{\mathcal{A}} = \mathbb{1}_{\hat{\mathcal{A}}}$. That is why we use the more general definition of topological processes instead of only talking about shift spaces. Shift spaces remain our most important example, though.

7.2.5 Stochastic Processes on Shift Spaces

Example Let $(\mathcal{M}, \alpha, \mathcal{A})$ be a commutative stochastic process with a faithful normal (not necessarily invariant) state φ. We know $(C(S), \sigma, C(\Omega_n)) \simeq (\mathcal{A}_{\mathbb{Z}}, \alpha, \mathcal{A})$ and can consider φ on $C(S)$. In the cyclic representation with regard to φ we get $C(S)'' = L^\infty(S, \mu_\varphi)$ (compare [Mur90, 4.4.1 Example]). For any finite set $I \subseteq \mathbb{Z}$ we see that $(\mathcal{A}_I, \varphi)$ is equivalent to $(C(S_I), \varphi)$. This means that as a classical stochastic process, this is an equivalent representation. Thus, we have proven the classical Daniell-Kolmogorov extension theorem for constructing the canonical path space for this special case.

7.3 Subshifts of Finite Type

There are many different kinds of shift spaces. One large and interesting class is given by the so-called subshifts of finite type, of which again topological Markov chains are a special case. We now introduce subshifts of finite type and how they can be algebraized.

7.3.1 Forbidden Words

We have already talked about forbidden words, but here we give a more algebraic definition.

Definition Let $(\hat{\mathcal{A}}, \sigma, \mathcal{A})$ be a topological process. We call tuples of orthogonal projections $(p_0, \dots, p_n) \in \mathcal{P}(\mathcal{A})^* := \bigcup_{n\in\mathbb{N}} \mathcal{P}(\mathcal{A})^n$ *words*. We call a word $(p_0, \dots, p_n)$ *forbidden* if

$$\prod_{0\le i\le n} \sigma^i(p_i) = 0.$$

7.3.2 Full Shifts

We have seen that a classical shift space $S \subseteq \Omega^{\mathbb{Z}}$ is called full if $S = \Omega^{\mathbb{Z}}$. Let us see if we can describe this in our algebraic language.

Proposition *For a commutative shift space $(\hat{\mathcal{A}}, \sigma, \mathcal{A})$ the following are equivalent:*

(a) *The associated subshift S is a full shift.*
(b) *No word in $\mathcal{P}(\mathcal{A})^*$ is forbidden.*

Proof We have $A = C(\Omega_n)$ for some alphabet Ω_n. Because we did not require the projections in a word in $\mathcal{P}(\mathcal{A})^*$ to be minimal, one word in $\mathcal{P}(\mathcal{A})^*$ corresponds to a set of words in Ω_n^*. As remarked in example 1 in 7.1.2, a shift space without forbidden words is a full shift and a full shift has no forbidden words. □

7.3.3 Subshifts of Finite Type

Classically a subshift of finite type is defined like this:

Definition A classical shift space $S \subseteq \Omega^{\mathbb{Z}}$ is called an *M-step subshift of finite type* for $M \in \mathbb{N}$ if $S = S_F$ for a finite set F of forbidden words of length at most $M + 1$.

A 1-step classical subshift of finite type is called a *topological Markov chain.*

Example 1 A full shift is a 0-step subshift of finite type and thus also a topological Markov chain.

Example 2 Our example with $F := \{ 000 \}$ is a 2-step subshift of finite type.

Remark We can see that the step number of a subshift of finite type implies how much "memory" is required to determine if a certain sequence belongs to the subshift. To determine whether a sequence belongs to the full shift, we do not need to do anything. To see if a sequence belongs to S_F from example 2 we need to go through the seqence and always remember if the last two points we have visited were 0.

Example 3 Again, we consider a 0-1-matrix $a \in M_n$ on the alphabet $\Omega := \{ 0, \dots, n \}$ and define the set of forbidden words by the blocked transitions of a as $F := \left\{ ij \in \Omega \times \Omega : a_{ij} = 0 \right\}$. All forbidden words are of length 2. Thus, S_F is a 1-step subshift of finite type, or, as expected, a topological Markov chain.

Example 4 We discuss which topological Markov chains exist on the alphabet $\{ 0, 1 \}$. There are 16 different possible 0-1-matrices $a \in M_2$. With 4 zeros in a, our shift space is empty. With 4 ones, we get the full shift. With 3 zeros, the shift space is either empty or contains one constant sequence if 00 or 11 is allowed. With 2 zeros, either we have exactly one outgoing edge for every node, giving us a deterministic chain, or only one node has outgoing edges meaning that only that node can appear in the sequence, giving us again a constant sequence. Thus, only the case with only one blocked transition in a gives us a non-trivial Markov chain. Here there are effectively two cases. The first case is that one of the two node switches 01 and 10 is forbidden. This means that we get stuck in one node. If, for example, 01 is blocked, all sequences (besides the constant sequences) are of the form $\dots 11111100000 \dots$ and the only question is when the switch from 0 to 1 happens. By the way, we can see here, that 0 is recurrent and 1 transient. The last case is where 00 or 11 is blocked. If 00 is blocked we can have sequences of the form $\dots 110101111010111 \dots$, where everything is allowed but two zeros in a row. Of course if we switch 0 and 1 the topological Markov chain with 11 blocked is the same. This means that there is essentially only one "interesting" topological Markov chain with a two-letter alphabet. This process is recurrent.

Now, we attempt to generalize the definition of subshifts of finite type to the non-commutative case.

Definition We call a shift space $(\hat{\mathcal{A}}, \sigma, \mathcal{A})$ an *M-step subshift of finite type* for $M \in \mathbb{N}$ if every forbidden word consisting of minimal projections contains a forbidden word of length at most $M + 1$.

And, of course, we show that this is a proper generalization.

Proposition *For a commutative shift space $(\hat{\mathcal{A}}, \sigma, \mathcal{A})$ the following are equivalent:*

(a) *The associated subshift S is an M-step subshift of finite type.*
(b) *$(\hat{\mathcal{A}}, \sigma, \mathcal{A})$ is an M-step subshift of finite type.*

Proof Since we require in the algebraic definition of an M-step subshift of finite type that the projections in a forbidden word are minimal, the classical and the algebraic definitions of forbidden words coincide.

(a) $\Rightarrow$ (b) Let w be a forbidden word of the associated subshift S with $k := |w| > M + 1$. We consider the two subwords of $w_{[1,k-1]}$ and $w_{[2,k]}$ of w where we drop the first and the last letter respectively. They have a common subword $w_{[2,k-1]}$ of length greater or equal to $|M|$. Thus, by [LM21, Theorem 2.1.8] one of those two subwords has to be forbidden. We can iterate this to find a subword of length at most $M + 1$.

(b) $\Rightarrow$ (a) If we have a forbidden word w, then any forbidden subword of w forbids the same or more sequences. Thus, given any, possibly infinite, set of forbidden words for an algebraic subshift of finite type, we can shorten those words to length $M + 1$ without allowing more sequences. We arrive at a set of forbidden words of length at most $M + 1$. This set must be finite because the alphabet is. $\square$

We could use the non-commutative definition of a 1-step subshift of finite type to define topological Markov processes. However, as so often, many different equivalent characterizations in the commutative setting become inequivalent in the more general setting. Thus, in the next section, we will inspect different possible characterizations and pick one.

7.4 Markov Processes

Now we get to the definition of topological Markov processes. For inspiration and analogy we first give the stochastic definition of a non-commutative Markov process. Then we discuss the different possible Markov criteria.

7.4.1 Stochastic Markov Processes

The stochastic Markov condition unsurprisingly applies to stochastic processes, so we need to introduce their definition.

Definition We call $(\hat{\mathcal{M}}, \alpha, \mathcal{M}, \varphi)$ a *stationary stochastic process*, if $\mathcal{M} \subseteq \hat{\mathcal{M}}$ are von Neumann algebras, α is a normal *-automorphism on $\hat{\mathcal{M}}$, φ is a faithful normal state on $\hat{\mathcal{M}}$ which is invariant for α and there is a normal conditional expectation preserving φ onto $\mathcal{M}$.

Proposition [Küm85, 2.1.3 & 2.2.3] *Let $(\hat{\mathcal{M}}, \alpha, \mathcal{M}, \varphi)$ be a stationary stochastic process, then for every $I \subseteq \mathbb{Z}$ the conditional expectation P_I onto $\mathcal{A}_I$ exists and the following are equivalent:*

(a) *For any $n, m \in \mathbb{N}$ $P_0(x) = P_{[-m,0]}(x)$ for every $x \in \mathcal{M}_{[0,n]}$.*
(b) *$P_0(x) = P_{0]}(x)$ for every $x \in \mathcal{M}_{[0}$.*
(c) *For any $n \in \mathbb{Z}$ $P_n(x) = P_{n]}(x)$ for every $x \in \mathcal{M}_{[n}$.*

Definition If any of these conditions is true, we call the stochastic process a *Markov* process.

7.4.2 Markov Conditions

For the definition of a non-commutative topological Markov condition, we use the following criteria:

- It should actually be a generalization of the commutative definition (shown in the following theorem).
- If we use an envelope for the topological process the condition should be independent of the process (shown in the following theorem).
- The Markov condition should ensure that the process is a *dilation* of a reach map (defined and shown in 7.5.1).
- The topological process belonging to a stochastic Markov processes should be a topological Markov process (shown in 7.4.4).
- Ideally the definition is symmetric in time (shown in 7.4.5).

For the next theorem we keep in mind, that if $\mathcal{A}$ is expected in $\hat{\mathcal{A}}$, then there is an envelope (namely at least the universal envelope) such that $\mathcal{A}^\diamond$ is expected in $\hat{\mathcal{A}}^\diamond$ We write P_I for a conditional expectation onto the subalgebra $\mathcal{A}_I$ and R_I for R_{P_I}.

Theorem *Let $(\hat{\mathcal{A}}, \sigma, \mathcal{A})$ be a topological process with finite-dimensional values. Let $\mathcal{M}_{\hat{\mathcal{A}}}$ be an envelope of $\hat{\mathcal{A}}$ such that $\mathcal{A}^\diamond$ is expected in $(\hat{\mathcal{A}}, \mathcal{M}_{\hat{\mathcal{A}}})$. If $(\hat{\mathcal{A}}, \sigma, \mathcal{A})$ is a commutative shift space the following are equivalent:*

(a) *The process is 1-step.*
(b) *$p \perp q$ implies $R_0(p) \perp R_0(q)$ for all $n \in \mathbb{N}$, $p \in \mathcal{T}(\mathcal{A}^\diamond_{[-n,0]})$ and $q \in \mathcal{T}(\mathcal{A}^\diamond_{[0,n]})$.*
(c) *$R_{[-n,0]}(q) = R_0(q)$ for all $n \in \mathbb{N}$, $q \in \mathcal{T}(\mathcal{A}^\diamond_{[0,n]})$.*
(d) *$p \perp q$ implies $R_0(p) \perp R_0(q)$ for all $p \in \mathcal{T}(\mathcal{A}^\diamond_{0]})$, $q \in \mathcal{T}(\mathcal{A}^\diamond_{[0})$.*
(e) *$p \perp q$ implies $R_n(p) \perp R_n(q)$ for all $n \in \mathbb{N}$, $p \in \mathcal{T}(\mathcal{A}^\diamond_{n]})$, $q \in \mathcal{T}(\mathcal{A}^\diamond_{[n})$.*

In the non-commutative case, the following implications hold:

$$\begin{tikzcd}
(a) & (b) \arrow[l, Rightarrow] \arrow[r, Rightarrow, bend right, "(***)"'] \arrow[d, Rightarrow, shift right, "(**)"'] & (d) \arrow[l, Rightarrow, bend right] \arrow[r, Leftrightarrow] & (e). \\
& (c)^{(*)} \arrow[u, Rightarrow, shift right] & &
\end{tikzcd}$$

(*) We only consider condition (c) if $\mathcal{A}^\diamond_{[-n,0]}$ is expected for all $n \in \mathbb{N}$.
(**) If for finite intervals $I \subseteq \mathbb{Z}$ all $\mathcal{A}_I$ are finite-dimensional and have the same $\mathbb{1}$.
(***) If all open projections in $\mathcal{T}(\mathcal{A}^\diamond_{0]})$ and $\mathcal{T}(\mathcal{A}^\diamond_{[0})$ are increasing limits of elements in $\bigcup_{n\in\mathbb{N}} \mathcal{A}^+_{[-n,0]}$ and $\bigcup_{n\in\mathbb{N}} \mathcal{A}^+_{[0,n]}$.

Also, for all five conditions, assuming (*) and (**) for condition (c), the following are equivalent:

(i) *The condition is true for all envelopes $\mathcal{M}_{\hat{\mathcal{A}}}$.*
(ii) *The condition is true for some envelope $\mathcal{M}_{\hat{\mathcal{A}}}$.*

Proof *We will first show the implications in the general setting, then the equivalence in the commutative case and, lastly, the independence of the envelope $\mathcal{M}_{\hat{\mathcal{A}}}$.*

(e) $\Rightarrow$ (d) Is clear with $n = 0$.

(d) $\Rightarrow$ (e) Follows from the fact, that σ is an automorphism and σ^{-n} maps e.g. $\mathcal{A}_{n]}$ to $\mathcal{A}_{0]}$ and $\mathcal{A}_n$ to $\mathcal{A}_0$.

(d) $\Rightarrow$ (b) Is clear because (b) is quantified over fewer elements than (d).

(c) $\Rightarrow$ (b) We show $pq = 0 \Rightarrow 0 = P_{[-n,0]}(pq) = pP_{[-n,0]}(q) = pP_0(q) \Rightarrow 0 = P_0(pP_0(q)) = P_0(p)P_0(q) \Rightarrow R_0(p) \perp R_0(q)$.

(b) $\Rightarrow$ (a) Consider a forbidden word $w \in \mathcal{P}(\mathcal{A})^*$ of length at least 3. We shift the word w so that $w_{0]}$ and $w_{[0}$ have both at least length 2 and embed them into $\hat{\mathcal{A}}$. Then $0 = w = w_{0]} w_{[0}$ and so $w^*_{0]} w_{0]} w_{[0} w^*_{[0} = 0$. We have $w^*_{0]} w_{0]} \leq w_0$ and $w_{[0} w^*_{[0} \leq w_0$. Since $w_0 \leq \mathbb{1}_{\mathcal{A}_0}$ is minimal in $\mathcal{A}_0$ this means $R_0([w^*_{0]} w_{0]}]) = [w^*_{0]} w_{0]}]_{\mathcal{A}^\diamond_0} \in \{0, w_0\}$ and $R_0([w_{[0} w^*_{[0}]) = [w_{[0} w^*_{[0}]_{\mathcal{A}^\diamond_0} \in \{0, w_0\}$. Since we haven given from (b) that $R_0([w^*_{0]} w_{0]}])R_0([w_{[0} w^*_{[0}]) = 0$ and R_0 is faithful on $\mathbb{1}_{\mathcal{A}}$ this means either $w^*_{0]} w_{0]} = 0$

or $w_{[0} w_{[0}^* = 0$. So w contains a forbidden subword. By induction on the length of w, it contains a forbidden subword of length 2.

(b) $\Rightarrow$ (c) *with (**)* For $q \in \mathcal{T}(\mathcal{A}_{[0,n]}^\diamond)$ we put $p := \mathbb{1}_{\mathcal{A}} - R_{[-n,0]}(q)$. We generally know that $R_{[-n,0]}(q) = [q]_{\mathcal{A}_{[-n,0]}^\diamond} \leq [q]_{\mathcal{A}_0^\diamond} = R_0(q)$. On the other hand $p \perp R_{[-n,0]}(q) \geq q$ and thus by (b) $p \leq R_0(p) \perp R_0(q) \geq R_{[-n,0]}(q)$. But since $p + R_{[-n,0]}(q) = \mathbb{1}_{\mathcal{A}}$ this means that $R_0(p) + R_0(q) = \mathbb{1}_{\mathcal{A}}$, $p = R_0(p)$ and $R_0(q) = R_{[-n,0]}(q)$.

(b) $\Rightarrow$ (d) *with (***)* Let $\sup_{i \in I} x_i = p$ and $\sup_{j \in J} y_j = q$ for $(x_i)_{i \in I} \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{A}_{[-n,0]}^+$ and $(y_j)_{j \in J} \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{A}_{[0,n]}^+$. That implies $x_i \perp y_j$ for all $i \in I$ and $j \in J$. Because $R_0(x_i) \perp R_0(y_j)$ for all $i \in I$ and $j \in J$ we can conclude $R_0(p) \perp R_0(q)$.

Commutative case.
If $\hat{\mathcal{A}}$ is a commutative shift space then the conditions of (*), (**) and (***) are fulfilled and thus the only missing implication is (a) $\Rightarrow$ (b). If $\hat{\mathcal{A}}$ is commutative $\mathcal{A}_{[-n,0]}$ and $\mathcal{A}_{[0,n]}$ are spanned by words. We choose minimal $w_1, \ldots, w_n \in \mathcal{P}_1(\mathcal{A}_{[-n,0]})$ and $v_i, \ldots, v_n \in \mathcal{P}_1(A_{[0,n]})$ with $p = \sum w_i$ and $q = \sum v_j$ with $w_i \neq 0 \neq v_j$ for all i, j. Since $w_i v_j$ is a forbidden word, but $w_i \neq 0 \neq v_j$ we have $w_{i0} \neq v_{j0}$ and thus $R_0(w_i) \leq w_{i0} \perp v_{j0} \geq R_0(v_j)$. So $R_0(p) R_0(q) = R_0(\bigvee w_i) R_0(\bigvee v_j) = \bigvee R_0(w_i) \bigvee R_0(v_j) = \bigvee R_0(w_i) R_0(v_j) = 0$.

Envelope independence.
(i) $\Rightarrow$ (ii) is clear since there always is at least one envelope which we quantify over. So we need to show (ii) $\Rightarrow$ (i). Condition (a) is clearly independent of $\mathcal{M}_{\hat{\mathcal{A}}}$. For (b),(d) and (e) we consider hereditary C*-subalgebras $\mathcal{B}_p$ and $\mathcal{B}_q$ each of the corresponding $\mathcal{A}_I$ in question. Since $R_0(\mathbb{1}_{\mathcal{B}_p}) \perp R_0(\mathbb{1}_{\mathcal{B}_q})$ in some envelope by Proposition 6.2.4 it holds for all permissible conditional expectations in all envelopes. If we assume (*) and (**), (c) is equivalent to (b) and thus itself independent of the envelope. $\square$

Definition If a topological process fulfils condition (d) (and equivalently (e)), we call it a *topological Markov process*.

Remark We present option (a) to show the parallel to the classical definition and option (c) to show the close analogy to the stochastic Markov condition.

The other sensible choice for the Markov condition instead of (d)/(e) would have been option (b). It can be shown that both conditions fulfil all the presented criteria. Their closest analogies in the stochastic case are equivalent, which we did not achieve to show for the topological case here. We use the stronger condition (d) because of aesthetical preference.

7.4.3 Properties of the Markov Condition

In the case that $\mathbb{1}_{\mathcal{A}} = \mathbb{1}_{\hat{\mathcal{A}}}$ the Markov condition automatically strengthens to $R_0(p) \perp R_0(q) \Leftrightarrow p \perp q$ for all $p \in \mathcal{T}(\mathcal{A}^{\diamond}_{0]})$, $q \in \mathcal{T}(\mathcal{A}^{\diamond}_{[0})$. More generally we have the following relations:

Proposition *Let R_0 be the reach map of a conditional expectation onto an expected subalgebra $\mathcal{A}^{\diamond} \subseteq \hat{\mathcal{A}}^{\diamond}$ and $p, q \in \mathcal{T}(\hat{\mathcal{A}}^{\diamond})$.*

(1) *The following are equivalent:*
  (a) $R_0(p) \perp R_0(q)$.
  (b) $R_0(p) \perp q$.
  (c) $p \perp R_0(q)$.

(2) *If $p \leq \mathbb{1}_{\mathcal{A}}$ or $q \leq \mathbb{1}_{\mathcal{A}}$ the conditions in (1) imply $p \perp q$.*

Proof (1) (a) $\Rightarrow$ (b) $R_0(p) \perp R_0(q) \Rightarrow 0 = R_0(p)R_0(q)R_0(p) \geq R_0(p)[\mathbb{1}_{\mathcal{A}} q \mathbb{1}_{\mathcal{A}}]R_0(p) \Rightarrow 0 = R_0(p)\mathbb{1}_{\mathcal{A}} q \mathbb{1}_{\mathcal{A}} R_0(p) = R_0(p)qR_0(p) \Rightarrow R_0(p) \perp q$.
(1) (b) $\Rightarrow$ (a) $R_0(p) \perp q \Rightarrow R_0(p)q = 0 \Rightarrow P(R_0(p)q) = 0 \Rightarrow R_0(p)P(q) = 0 \Rightarrow R_0(p)R_0(q) = 0 \Rightarrow R_0(p) \perp R_0(q)$.
The equivalence (a) $\Leftrightarrow$ (c) follows if we swap p and q.

(2) Let $p \perp R_0(q)$ then $pR_0(q)p = 0$ with $[\mathbb{1}_{\mathcal{A}} q \mathbb{1}_{\mathcal{A}}] \leq R_0(q)$ we have $p[\mathbb{1}_{\mathcal{A}} q \mathbb{1}_{\mathcal{A}}]p = 0$ and $0 = p\mathbb{1}_{\mathcal{A}} q \mathbb{1}_{\mathcal{A}} p = pqp$ so $p \perp q$. □

7.4.4 Stochastic Markov Processes are Topological Markov Processes

Now we go on to prove the different criteria that we mentioned before.

Proposition *Let $\hat{\mathcal{A}}^{\diamond}$ be an enveloped C^*-algebra, $\mathcal{A} \subseteq \hat{\mathcal{A}}$ finite-dimensional and $(\mathcal{M}_{\hat{\mathcal{A}}}, \sigma, \mathcal{A}, P, \varphi)$ a stationary stochastic Markov process. Then $(\hat{\mathcal{A}}, \sigma, \mathcal{A})$ is a faithful topological Markov process.*

Proof If we have a stochastic Markov process given, then by applying the support map to both sides of the Markov condition, we get $R_{0]}(p) = R_0(p)$ for all $p \in \mathcal{T}(\mathcal{A}^{\diamond}_{[0})$. This implies (d) in the same way that (c) implied (b) in the last proof. □

7.4.5 Time Symmetry

Proposition *Let $(\hat{\mathcal{A}}, \sigma, \mathcal{A})$ be a topological Markov process, then $(\hat{\mathcal{A}}, \sigma^{-1}, \mathcal{A})$ is also a topological Markov process.*

Proof Clear by the definition of the Markov condition. □

7.5 Markov Dilations

As our last point of order we tie back Markov processes to our analysis of reach maps. First, we show how every Markov process gives us a reach map. Then we construct Markov processes for arbitrary reach maps.

7.5.1 Markov Dilations

Proposition *Let $(\hat{\mathcal{A}}, \sigma, \mathcal{A})$ be a topological Markov process with any envelope. For the reach map $R\colon \mathcal{T}(\mathcal{A}^\diamond) \to \mathcal{T}(\mathcal{A}^\diamond)$ defined by $R := R_0 \circ \sigma$ the following diagram commutes for all $n \in \mathbb{N}$.*

$$\begin{tikzcd} \mathcal{T}(\mathcal{A}^\diamond) \arrow[r, "R^n"] \arrow[d, "\mathrm{id}"] & \mathcal{T}(\mathcal{A}^\diamond) \\ \mathcal{T}(\hat{\mathcal{A}}^\diamond) \arrow[r, "\sigma^n"] & \mathcal{T}(\hat{\mathcal{A}}^\diamond) \arrow[u, "R_0"] \end{tikzcd}$$

Proof Let $p \in \mathcal{T}(\mathcal{A}^\diamond)$. We show the statement by induction over n. For $n = 0$, it is fulfilled by definition. Let $q \in \mathcal{P}(\mathcal{M}_{\mathcal{A}})$ with $q \perp R_0(\sigma^n(p))$. Then $q \perp \sigma^n(p)$ and thus $\sigma^{-1}(q) \perp \sigma^{n-1}(p)$. Now $\sigma^{-1}(q) \perp R_0(\sigma^{n-1}(p)) = R^{n-1}(p)$. So $q \perp \sigma(R^{n-1}(p))$ and $q \perp R_0(\sigma(R^{n-1}(p))) = R^n(p)$. Thus, $R^n(p) \leq R_0(\sigma^n(p))$. □

Definition We call $(\hat{\mathcal{A}}, \sigma, \mathcal{A})$ a *Markov dilation* of R.

7.5.2 1-Dilations for Rank One Reach Maps

To construct Markov dilations, we will first construct a 1-dilation for one Kraus operator.

Definition Let $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ be a reach map on a matrix algebra $\mathcal{A}$. Then we call $(\hat{\mathcal{A}}, \alpha, \mathcal{A})$ a 1-dilation of R if $\mathcal{A}$ is expected in a C*-algebra $\hat{\mathcal{A}}$ via a conditional expectation $P\colon \hat{\mathcal{A}} \to \mathcal{A}$, $\alpha\colon \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ a *-automorphism and $R = R_{(P\circ\alpha)|_{\mathcal{A}}}$.

Proposition *Let $\mathcal{A} \subseteq M_n$ and $a \in M_n$ with $\mathrm{Ad}_a(\mathcal{A}) \subseteq \mathcal{A}$. Then there is a finite-dimensional 1-dilation for the reach map $R_{\mathrm{Ad}_a}\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$.*

Proof We embed $\mathcal{A}$ as $\mathcal{A} \otimes e_{11}$ into $M_n \otimes M_2$. Then we define $u := \begin{pmatrix} a & \sqrt{\mathbb{1}_n - aa^*} \\ \sqrt{\mathbb{1}_n - a^*a} & -a^* \end{pmatrix}$ which fulfils

$$u^* u = \begin{pmatrix} a^* & \sqrt{\mathbb{1} - a^* a} \\ \sqrt{\mathbb{1} - aa^*} & -a \end{pmatrix} \begin{pmatrix} a & \sqrt{\mathbb{1} - aa^*} \\ \sqrt{\mathbb{1} - a^* a} & -a^* \end{pmatrix}$$
$$= \begin{pmatrix} a^* a + \sqrt{\mathbb{1} - a^* a}^2 & a^* \sqrt{\mathbb{1} - aa^*} - \sqrt{\mathbb{1} - a^* a}\, a a^* \\ \sqrt{\mathbb{1} - aa^*}\, a - a\sqrt{\mathbb{1} - a^* a} & \sqrt{\mathbb{1} - aa^*}^2 + aa^* \end{pmatrix} = \mathbb{1}_{n \otimes 2}.$$

We have used that $af(a^* a) = f(aa^*)a$ for all polynomials f to see that the off-diagonal entries are zero. The other identity $uu^* = \mathbb{1}$ follows immediately if we swap a^* with a in our definition of u. Thus, u is a unitary element.

Now

$$[(\mathrm{id} \otimes \omega_{11})(u^*(p \otimes e_{11})u) \otimes e_{11}] = [(\mathrm{id} \otimes \omega_{11})(u^*(p \otimes e_{11})u)] \otimes e_{11}$$
$$= \left[(\mathrm{id} \otimes \omega_{11}) \left(\begin{pmatrix} a^* & \sqrt{\mathbb{1} - a^* a} \\ \sqrt{\mathbb{1} - aa^*} & -a \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & \sqrt{\mathbb{1} - aa^*} \\ \sqrt{\mathbb{1} - a^* a} & -a^* \end{pmatrix} \right) \right] \otimes e_{11}$$
$$= \left[(\mathrm{id} \otimes \omega_{11}) \left(\begin{pmatrix} a^* pa & \cdots \\ \cdots & \cdots \end{pmatrix} \right) \right] \otimes e_{11}$$
$$= [a^* pa] \otimes e_{11}.$$

Thus, our dilation of R_{Ad_a} is given by $(M_n \otimes M_2, \mathrm{Ad}_u, \mathcal{A} \otimes e_{11})$. □

7.5.3 Structure of 1-Dilations

To continue our construction of dilations, we show that the construction we used in the last proof is actually the general structure of 1-dilations.

Lemma *Let $\mathcal{A}$ be a matrix algebra and $R : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ a reach map with a finite-dimensional 1-dilation. Then there are minimal natural numbers $n, d \in \mathbb{N}$, a unitary matrix $u \in \mathcal{U}(M_n \otimes M_d)$ and an orthogonal projection $p \in M_d$ so that the 1-dilation can be extended to a 1-dilation $(M_n \otimes M_d, \mathrm{Ad}_u, \mathcal{A} \otimes p)$ of R.*

Proof Let the given 1-dilation be $(\hat{\mathcal{A}}, \alpha, \mathcal{A})$. First, we take the smallest n such that we can represent $\mathcal{A} \subseteq M_n$. Second, we choose the smallest $d \in \mathbb{N}$ such that we can embed $\hat{\mathcal{A}} \subseteq M_{nd}$ (where possibly $\mathbb{1}_{\hat{\mathcal{A}}} \leq \mathbb{1}_{nd}$) and we extend α to a *-automorphism Ad_u of M_{nd}. Now we can extend the embedding $i\colon \mathcal{A} \to \hat{\mathcal{A}} \subseteq M_{nd}$ to an embedding $i\colon M_n \to M_{nd}$. For a set of matrix units e_{jk} of M_n, all projections $i(e_{jj})$ have the same dimension, say l, in M_{nd}. So, $i(\mathbb{1}_n)$ has dimension $n \cdot l$. We split $\mathbb{1}_{nd} - i(\mathbb{1}_n)$ into n orthogonal projections of dimension $l - d$ and extend them to matrix units f_{jk}. Now we can define a third set of matrix units $g_{jk} := i(e_{jk}) + f_{jk}$, with $\sum_{j=1}^n g_{jj} = \mathbb{1}_{nd}$. We can now factor $M_{nd} = M_n \otimes M_d$ as a tensor product along those matrix units. Then there is a $p \in M_d$ such that $i(1_n) = \sum_{j=1}^d e_{jj} = \mathbb{1}_n \otimes p$. By construction, we have $i(R_{\mathcal{A}}(\alpha(q))) = R_{\mathcal{A} \otimes p}(u^*(q \otimes p)u)$ for all $q \in \mathcal{P}(\mathcal{A})$. □

7.5.4 1-Dilations for Suprema

We have already constructed 1-dilations for reach maps induced by one Kraus operator. Technically, by the Stinespring representation, this covers all reach maps, but we can also construct a 1-dilation in any representation we are given by combining the reach maps of the single Kraus operators:

Proposition *Let R_1 and R_2 be reach maps on Matrix algebra $\mathcal{A}$ with finite-dimensional 1-dilations. Then there is a finite-dimensional 1-dilation for $R := R_1 \vee R_2$.*

Proof Let $(M_n \otimes M_{d_i}, \mathrm{Ad}_{u_i}, \mathcal{A} \otimes p_i)$ be the 1-dilation of R_i, then $(M_n \otimes (M_{d_1} \oplus M_{d_2}), \mathrm{Ad}_{u_1 \oplus u_2}, \mathcal{A} \otimes (p_1 \oplus p_2))$ is a 1-dilation of R:

$$\begin{aligned}
&[(\mathrm{id} \otimes \mathrm{tr}((p_1 \oplus p_2)\cdot))((u_1^* \oplus u_2^*)(p \otimes (p_1 \oplus p_2))(u_1 \oplus u_2))] \otimes (p_1 \oplus p_2)\\
&= [(\mathrm{id} \otimes \mathrm{tr}((p_1 \oplus p_2)\cdot))(u_1^*(p \otimes p_1)u_1) \oplus (u_2^*(p \otimes p_2)u_2)] \otimes (p_1 \oplus p_2)\\
&= [(\mathrm{id} \otimes \mathrm{tr}(p_1\cdot))(u_1^*(p \otimes p_1)u_1) + (\mathrm{id} \otimes \mathrm{tr}(p_2\cdot))(u_2^*(p \otimes p_2)u_2)] \otimes (p_1 \oplus p_2)\\
&= ([(\mathrm{id} \otimes \mathrm{tr}(p_1\cdot))(u_1^*(p \otimes p_1)u_1)] \vee [(\mathrm{id} \otimes \mathrm{tr}(p_2\cdot))(u_2^*(p \otimes p_2)u_2)]) \otimes (p_1 \oplus p_2)\\
&= (R_1(p) \vee R_2(p)) \otimes (p_1 \oplus p_2)
\end{aligned}$$ □

7.5.5 Dilations for Infinitely Many Steps

In [GKL06, 1.1] a definition of a “non-commutative topological Markov chain” is presented. Because they construct a one-sided process, they use a slightly more general 1-dilation given by a *-homomorphism $\mathcal{J}\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$. Then they use the usual *coupling to a shift* construction to build a dilation for infinitely many steps. We do exactly the same in the next theorem, but for a two-sided process, demonstrating that the definition of a “non-commutative topological Markov chain” from [GKL06] is subsumed by our definition.

In this construction we will use the von Neumann tensor product $\bar{\otimes}$ (see [Sak98, 1.22.10]) and the infinite tensor project of von Neumann algebras $\bigotimes$ along a product state (see [Sak98, 4.4.1]).

Lemma *Let $\mathcal{A} \subseteq M_n$, $\mathcal{C}_{0]}$, $\mathcal{C}_{[0}$ and $\hat{\mathcal{A}} := \mathcal{C}_{0]} \bar{\otimes} M_n \bar{\otimes} \mathcal{C}_{[0}$ be von Neumann algebras, $\psi_{0]}$ and $\psi_{[0}$ normal states on $\mathcal{C}_{0]}$ and $\mathcal{C}_{[0}$ and α a *-automorphism on $\hat{\mathcal{A}}$. With $p_{0]} := \mathrm{supp}\psi_{0]}$, $p_{[0} := \mathrm{supp}\psi_{[0}$ and $\mathcal{A}_0 := p_{0]} \bar{\otimes} \mathcal{A} \bar{\otimes} p_{0]} \subseteq \hat{\mathcal{A}}$ we get a topological process $(\hat{\mathcal{A}}, \alpha, \mathcal{A}_0)$. If $\mathcal{A}_{0]} \subseteq \mathcal{C}_{0]} \bar{\otimes} \mathcal{A} \bar{\otimes} p_{[0}$ and $\mathcal{A}_{[0} \subseteq p_{0]} \bar{\otimes} \mathcal{A} \bar{\otimes} \mathcal{C}_{[0}$ the process is Markov.*

Proof We can check the Markov condition in any envelope, so it is an obvious choice to check it for $\mathcal{T}(\hat{\mathcal{A}}, \hat{\mathcal{A}}) = \mathcal{P}(\hat{\mathcal{A}})$. We pick any conditional expectation $P\colon M_n \to \mathcal{A} \subseteq \mathcal{M}_n$. Further we write $p_{0]}\psi_{0]}$ for the map $\mathcal{C}_{0]} \ni x \mapsto p_{0]}\psi_{0]}(x)$ and $p_{[0}\psi_{[0}$ analogous. Then we have in $(\hat{\mathcal{A}}, \hat{\mathcal{A}})$ the normal conditional expectations $P_{0]} = \mathrm{id} \otimes P \otimes p_{[0}\psi_{[0}$, and $P_0 = p_{0]}\psi_{0]} \otimes \mathrm{id} \otimes p_{[0}\psi_{[0}$. Let $p \in \mathcal{P}(\mathcal{A}_{0]})$ and $q \in \mathcal{P}(\mathcal{A}_{[0})$ with $q \perp p$. By assumption there is a $q' \in \mathcal{A} \otimes \mathcal{C}_{[0}$ with $q = p_{0]} \otimes q'$ and thus $P_{0]}(q) = P_{0]}(p_{0]} \otimes q) = \mathrm{id}(p_{0]}) \otimes (P \otimes p_{[0}\psi_{[0})(q') = p_{0]}\psi_{0]}(p_{0]}) \otimes (P \otimes p_{[0}\psi_{[0})(q') = P_0(q)$. Now we have $0 = pq = P_{0]}(pq) = pP_{0]}(q) = pP_0(q)$ and thus the topological process is Markov. □

Theorem *Let $\mathcal{A}$ be a matrix algebra and $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ a reach map with a 1-dilation. Then there is a Markov dilation for R.*

Proof Let $(M_n \otimes M_d, \mathrm{Ad}_u, \mathcal{A} \otimes p)$ be the 1-dilation of R. We define the state $\varphi\colon x \mapsto \frac{1}{\mathrm{tr}(p)}\mathrm{tr}(px)$ on M_d. We define $(\mathcal{C}_{0]}, \psi_{0]}) := \bigotimes_{\mathbb{Z}_{<0}}(M_d, \varphi)$ and $(\mathcal{C}_{[0}, \psi_{[0}) := \bigotimes_{\mathbb{Z}_{\geq 0}}(M_d, \varphi)$, so that $(\mathcal{C}, \psi) := (\mathcal{C}_{0]}\bar{\otimes}\mathcal{C}_{[0}, \psi_{0]} \otimes \psi_{[0}) = \bigotimes_{\mathbb{Z}}(M_d, \varphi)$. Now, we define the topological process like in the lemma. For this we define $\alpha = \mathrm{Ad}_{\hat{u}} \circ (\mathrm{id}_{\mathcal{A}} \otimes S)$ where S is the tensor (right) shift on $\mathcal{C}$ and $\hat{u}$ is $u \otimes \bigotimes_{\mathbb{Z}_{\neq 0}} \mathbb{1}_d$. We need to show that $\mathcal{A}_{0]} \subseteq \mathcal{C}_{0]}\bar{\otimes}\mathcal{A}\bar{\otimes}p_{[0}$ and $\mathcal{A}_{[0} \subseteq p_{0]}\bar{\otimes}\mathcal{A}\bar{\otimes}\mathcal{C}_{[0}$ so that the process is Markov as well as a dilation of R (instead of for a different reach map). For both we observe that $p_{[0} = p \otimes p_{[0}$ and $p_{0]} = p_{0]} \otimes p$. By induction $A_{[-n,0]} \subseteq \mathcal{C}_{0]}\bar{\otimes}\mathcal{A}\bar{\otimes}p_{[0}$ and $A_{[0,n]} \subseteq \mathcal{C}_{0]}\bar{\otimes}\mathcal{A}\bar{\otimes}p_{[0}$ for all $n \in \mathbb{N}$. Also, we have $R_0(\alpha(p_{0]} \otimes q \otimes p_{[0})) = R_0(\mathrm{Ad}_{\hat{u}}(S(p_{0]} \otimes q \otimes p_{[0}))) = R_0(\mathrm{Ad}_{\hat{u}}(p_{0]} \otimes q \otimes p_{[0})) = R_0(p_{0]} \otimes \mathrm{Ad}_u(q \otimes p) \otimes p_{[0}) = p_{0]} \otimes R_{A\otimes p}(\mathrm{Ad}_u(q \otimes p)) \otimes p_{[0} = p_{0]} \otimes R(q) \otimes p_{[0}$. □

7.5.6 Dilations for Everyone

This brings our journey to an end.

Corollary *Let $\mathcal{A}$ be a matrix algebra and $R\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ a reach map, then there is a Markov dilation for R.*

Proof Every reach map on a matrix algebra can be given as the supremum of a finite number of reach maps of rank one. Thus, there are 1-dilations and so dilations for any reach map. □

Remark The category we designed and our definition of dilation has been deliberately crafted in such a general way that we could arrive at this result. It would not be as easy if we required $\mathbb{1}_{\mathcal{A}} = \mathbb{1}_{\hat{\mathcal{A}}}$. Also, the underlying completely positive operators which we construct will often have norm strictly smaller than one. Thus, these dilations would not be very useful as dilations of those completely positive operators.

With this we conclude the final proper chapter, in which we discussed generalizations of shift spaces, subshifts of finite type and topological Markov processes to non-commutative systems. We compared different possible Markov conditions and which criteria we used to give the final definition. Then we defined and constructed Markov dilations for reach maps.

8 Conclusion and Open Questions

In the seven preceding chapters, we have laid the foundation for a theory of non-commutative topological Markov processes. However, a lot of questions had to remain unanswered, of which we present a selection here, after a quick summary.

8.1 Summary

We established the theory necessary to apply the category of reach maps between enveloped C*-algebras to our task. To compensate for the lack of orthogonal projections in C*-algebras we enriched them with surrounding von Neumann algebras to define the lattice of open projections. Reach maps are suprema preserving maps between such lattices, defined from a completely positive operator together with the support map. Simultaneously, we discussed different properties of the "topology" given by the open projections and proved a Hausdorff property. It also became apparent that the support of a completely positive operator in an enveloped C*-algebra is a closed projection and not the same as the well-known support of the same operator in regard to the enveloping von Neumann algebra.

We gave different characterizations of reach maps between finite-dimensional algebras. Reach maps of rank one are suprema preserving maps which preserve cross-ratios on their support. Also, reach maps relate to those faces in the cone of completely positive operators which are defined by their blocked transitions. We gave a criterion to determine which Kraus operator submodules of the $M_{n\times m}$ matrices belong to a reach map and discussed various examples of concrete reach maps and their submodules.

Then we applied our methods to investigate the communication structure of endomorphisms in the category of reach maps. We defined communicating classes and discussed the difference between topological and stochastic recurrence and transience, which coincide for unital completely positive operators. A Perron-Frobenius style theorem for reach maps was presented, including a property of cycles of orthogonal projections even for non-unital operators.

We saw that conditional expectations of reach maps are completely determined by the subalgebra they are projecting on and their support.

Finally, we discussed and defined non-commutative topological Markov processes in the category of reach maps. We showed how a Markov process relates to a reach map and how every reach map belongs to a Markov process.

8.2 Open Questions

We first collect a few questions concerning the topics of specific chapters, then we open our discussion to further ideas. As a collection of loose threads, this part does not try to present a closed narrative.

8.2.1 Chapter 1

As mentioned in the beginning, we only described our theory in the Heisenberg picture. The concept of an adjoint reach map could certainly be fleshed out and a discussion of reach maps in the Schrödinger picture might be helpful, especially when trying to apply our results to literature in theoretical physics.

The name of the concept reach map should not be considered set in stone. Frankly, the name was partially chosen because the letter R was not yet overused in our context. Many different names have been considered, for example "adjancency functions", since reach maps are a generalization of adjacency matrices. The term "next neighbour function" from graph theory was also a candidate.

8.2.2 Chapters 2 and 3

With the definition of enveloped C*-algebras we intentionally chose a very general setting for the definition of reach maps. Applications of the theory might reveal whether this is too much generality. On the one hand it might be helpful to specialize the theory to von Neumann algebras. On the other hand it often might suffice to use the universal envelope. In that case, reach maps are essentially maps on the lattice of hereditary C*-algebras. While a map for which the points are algebras is probably not the most intuitive concept, it might be interesting to explore this perspective on reach maps.

Our definition of reach maps used the open projections as domain. For a given completely positive operator we can extend the domain to all orthogonal projections of the envelope. We could not present an example for two different completely positive operators which induce the same reach map on the open projections but different reach maps on all orthogonal projections.

8.2.3 Chapters 4 and 5

An obvious route of investigation would be to figure out which aspects of chapters 4 and 5 can be generalized to infinite-dimensional algebras. One of the places where this could be possible is the result that an orthogonality preserving reach map is the reach map of a *-isomorphism. At least for von Neumann algebras this result seems likely to be true with the tools from [WZ09]. For some parts of the theory this will certainly become more intricate. For example in the finite-dimensional case we did not need to differentiate between positive recurrence and null recurrence.

But even in the finite-dimensional case a lot of open questions remain. Are there better sufficient criteria to detect whether a suprema preserving map is a reach map? Are there better criteria to tell when a Kraus operator module belongs to a reach map? For example, for which dimensions of a linear subspace of $K \subseteq M_n$ can we say for certain that there is a reach map $R\colon \mathcal{P}(M_n) \to \mathcal{P}(M_n)$ with $K = \mathrm{Kr}(T)$. For dimensions $\dim K = 1$ and $\dim K = n^2$ this is the case, we have seen examples for other dimensions where it is not.

Another open question is to find sufficient criteria to say that a reach map with $R(\mathbb{1}) = \mathbb{1}$ has a unital representative. As discussed after Theorem 5.5.2, for irreducible reach maps the orthogonality of a decomposition which forms an exact cycle is a necessary criterion, but we are not aware of interesting sufficient conditions for non-commutative algebras.

In Proposition 5.3.4, we showed that there is a maximal stochastically recurrent projection and a maximal stochastically transient projection which sum up to $\mathbb{1}$. To complete the picture in that theorem, it would be nice to show that the same can be said about topological recurrence and transience.

8.2.4 Chapter 6

In our theorem about reach maps of conditional expectations we needed to use the support on the envelope in condition (2). It remains to be seen whether the theorem can be strengthened to use the proper support of P in $\hat{\mathcal{A}}^{\diamond}$.

8.2.5 Chapter 7

For the whole chapter, we restricted ourselves to processes with finite-dimensional values, which was consistent with the fact that chapters 4 and 5 used finite-dimensional algebras. This is, as remarked for those chapters, a natural restriction to lift. However, with the methods we developed, a generalization of for example Theorem 7.4.2 about Markov conditions and their envelope independence seems infeasible since we would lose the envelope independence given by Proposition 6.2.4.

We omitted deepening our theory of non-commutative shift spaces beyond Markov processes. For example, it would probably be possible to give at least definitions of *sofic* shifts (see [LM21, Chapter 3]).

For commutative irreducible topological Markov processes, it is possible to find a measure on the process which converts it into a stationary stochastic Markov process (see [LM21, 13.3]). For any non-commutative Markov property, it would be an accolade if we could show that there exists an envelope and a state on that envelope which converts the process into a stochastic Markov process. Moreover, in the classical case, the measure can be chosen so that it maximizes the entropy of the stochastic Markov chain, giving a possible value for an entropy of the topological Markov chain. It would be interesting to investigate possible entropy definitions for reach maps or non-commutative topological Markov chains and how they relate to existing stochastic definitions of entropy.

Some of the conditions in Theorem 7.4.2 are quite technical, however the author did not succeed in strengthening the theorem. It would be nice to find interesting classes of processes for which these conditions become equivalent or show that for example condition (***) is unnecessary.

As discussed in Proposition 7.5.5, in [GKL06] a non-commutative topological Markov process is constructed. Then the process is endowed with a state to define the concept of *asymptotical completeness*. If a stochastic Markov dilation which is constructed from two matrix algebras $\mathcal{A}$ and $\mathcal{C}$ and a *-homomorphism $\mathcal{J}\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ is asymptotically complete, the operator $T_\psi \coloneqq \mathrm{id} \otimes \psi \circ \mathcal{J}$ for a faithful state $\psi \in \mathcal{S}(\mathcal{C})$ is aperiodic irreducible. From our investigation we know that the reach map of T_ψ is independent of ψ as long as the state is faithful. That means that the question whether T_ψ is aperiodic irreducible is independent of ψ. In fact, in [GKL06, 4.4] it is shown that $\mathcal{J}$ is asymptotically complete together with a specific faithful state ψ if and only if it is asymptotically complete with any faithful state ψ. It is therefore clear that asymptotic completeness is just a property of $\mathcal{J}$ and $\mathcal{A} \otimes \mathcal{C}$. Since $\mathcal{J}$ is a *-homomorphism, all information about it is contained in its reach map $R_{\mathcal{J}}$. Further, the reach map of the conditional expectation $\mathrm{id} \otimes \psi\colon \mathcal{A} \otimes \mathcal{C} \to \mathcal{A} \otimes \mathbb{1} \subseteq \mathcal{A} \otimes \mathcal{C}$ is uniquely determined by $\mathcal{A} \otimes \mathcal{C}$. Thus, it seems very plausible that asymptotic completeness is a purely topological property. Consequently, an investigation into giving a topological definition of asymptotic completeness seems very fruitful.

8.2.6 Further Questions and Speculations

Having a theory of topological Markov processes opens a lot of further questions. Basically, we can consider every concept of classical shift spaces as in [LM21] and ask ourselves whether we can generalize it to the non-commutative case. This applies, for example, to the different kinds of codes discussed there. In [Ste08], Lisa Steiner presented a quantum coding theory. She used a different approach to generalize quantum shift spaces, defining them via infinite tensor products, in close analogy to classical shift spaces. She observed then that so-called higher block coders for non-commutative algebras are in conflict with the no-cloning theorem of quantum mechanics. We do not know whether our approach would encounter the same problem.

A more far-fetched idea is related to the theory of quantum error correction. Coding theory is exactly the setting in which classical error correction in information systems can be described. It would certainly be interesting to see if our perspective on topological processes can be helpful in understanding quantum error correction.

Also in [LM21] there is a wide discussion of different equivalence relations on shift spaces. One of those relations is the so-called *flow equivalence*. Cuntz and Krieger famously constructed C*-algebras for shift spaces which are invariant under flow equivalence and, with a bit of additional structure, can be made complete invariants of the equivalence. A particularly far-fetched but intriguing question would be if it is possible to define something like a Cuntz-Krieger algebra for a given reach map and if it could provide any invariants for some equivalences.

With this, we conclude our collection of open questions and this thesis.

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