TECHNISCHE
UNIVERSITÄT
DARMSTADT

Derived F-zips

Vom Fachbereich Mathematik
der Technischen Universität Darmstadt
zur Erlangung des Grades eines
Doktor der Naturwissenschaften
(Dr.rer.nat.)
genehmigte Dissertation

Tag der Einreichung: 20. April 2022
Tag der mündlichen Prüfung: 11. Mai 2022

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Darmstadt, D17
2022

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Date of submission: 20.04.2022
Date of thesis defense 11.05.2022

Darmstadt – D17

Jahr der Veröffentlichung der Dissertation auf TUprints: 2022
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Zusammenfassung

Diese Dissertation beschäftigt sich mit der Definition von derivierten F-zips und der Analyse dereen Modulraums. Durch Benutzung deriviert algebraischer Methoden assoziieren wir zu jedem eigentlich glatten Morphismus von Schemata einen derivierten F-zip via der zugehörigen de Rham Hyperkohomologie. Wir analysieren den Zusammenhang zwischen derivierten F-zips und klassischen F-zips im Fall, dass die Hodge-de Rham Spektralsequenz ausartet. Ein Beispiel von geometrischen Objekten, deren Hodge-de Rham Specktralsequenz nicht ausartet, sind supersinguläre Enriques-Flächen in Characteristik 2. Wir nutzen unsere Theorie der derivierten F-zips um Aussagen über die Geometrie des Modulraums der Enriques-Flächen zu beweisen, die mit der klassischen Theorie der F-zips nicht möglich ist.

Um den Sachverhalt besser zu verstehen, fassen wir wichtige Aspekte der derivierte algebraischen Geometrie zusammen und wiederholen den Beweis, dass der derivierte Stack der perfekten Komplexe lokal geometrisch ist im Fall der ∞-Kategorien nach den Resultaten von Antieu-Gepner und Toën-Vaquié.

Abstract

We define derived versions of F-zips and associate a derived F-zip to any proper, smooth morphism of schemes in positive characteristic. We analyze the stack of derived F-zips and certain substacks. We make a connection to the classical theory and look at problems that arise when trying to generalize the theory to derived G-zips and derived F-zips associated to lci morphisms. As an application, we look at Enriques-surfaces and analyze the geometry of the moduli stack of Enriques-surfaces via the associated derived F-zips. As there are Enriques-surfaces in characteristic 2 with non-degenerate Hodge-de Rham spectral sequence, this gives a new approach, which could previously not be obtained by the classical theory of F-zips.

For this we also recall important aspects of derived algebraic geometry and the proof that the derived stack of perfect complexes is locally geometric, using the results of Antieau-Gepner and Toën-Vaquié.

Contents

1 Introduction

The notion of F-zips was introduced by Moonen and Wedhorn in [MW04]. Before starting with the positive characteristic case discussed in [MW04], let us first look at the characteristic 0 case, after [Wed08].

Let $X \to \mathrm{Spec}(\mathbb{C})$ be a proper smooth morphism of schemes. By general GAGA principals, we can associate a compact complex manifold X^{an} to X in a "universal" way (we do not make this explicit here). Important for us, is that the algebraic de Rham cohomology $H^n_{\mathrm{dR}}(X/\mathbb{C})$ of X is isomorphic to the complex de Rham cohomology $H^n_{\mathrm{dR}}(X^{\mathrm{an}})$ of X^{an}. Complex de Rham cohomology computes the singular cohomology of X^{an} with complex coefficients, i.e. $H^n_{\mathrm{dR}}(X^{\mathrm{an}}) \cong H^n_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C}) \cong H^n_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$. Thus, complex de Rham cohomology comes equipped with an integral structure. The $\mathbb{C}$-vector space given by H^1_{dR} with its integral structure characterizes for example abelian varieties over $\mathbb{C}$ (called the global Torelli property of abelian varieties).

We also have a descending filtration $C^\bullet$ on this complex vector space $H^n_{\mathrm{dR}}(X^{\mathrm{an}})$ which is induced by the Hodge spectral sequence. It is known that this spectral sequence degenerates. Therefore, successive quotients are computed by the Hodge-cohomologies $C^i/C^{i+1} = H^{n-i}(X^{\mathrm{an}}, \Omega^i_{X^{\mathrm{an}}})$. The real structure on the singular cohomology, together with the complex conjugation on $\mathbb{C}$, induce an $\mathbb{R}$-linear endomorphism on $H^n_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C})$. The image of the C^i under this map are again complex vector spaces and induce an ascending filtration on $H^n_{\mathrm{dR}}(X^{\mathrm{an}})$ by $D_i := \overline{C^{n-i}}$. One can show that $D_{i-1} \oplus C^i = H^n_{\mathrm{dR}}(X^{\mathrm{an}})$ for all $i \in \mathbb{Z}$. These data, together with the integral structure obtained via the comparison with the singular cohomology, endow $H^n_{\mathrm{dR}}(X/\mathbb{C})$ with an *integral Hodge-structure*. The study of integral Hodge-structures and its moduli can be found in [BP96] and lead to the notion of *Griffiths' period domains*. The study of these data, enables us to analyze the moduli of geometric objects via linear-algebra data.

These results can be extended to arbitrary smooth proper families in characteristic zero. In characteristic $p > 0$ however, we do not have a complex conjugation. But still, we have an analogous structure on the de Rham cohomology.

Let us fix an $\mathbb{F}_p$-algebra A and a smooth proper A-scheme X. Contrary to the characteristic zero case, we have a second spectral sequence on the de Rham cohomology, the *conjugate spectral sequence*. This spectral sequence, endows the de Rham cohomology $H^n_{\mathrm{dR}}(X/A)$ with a second filtration $D_\bullet$. We also have an analogue of the Poincaré lemma, the Cartier isomorphism. The Cartier isomorphism links the graded pieces of the Hodge filtration $C^\bullet$ with $D_\bullet$. If we assume that the Hodge cohomologies are finite projective and the Hodge-de Rham spectral sequence is degenerate, then the successive quotients are *isomorphic up to Frobenius twist*, i.e. we have $C^i/C^{i+1} \otimes_{A,\mathrm{Frob}} A \cong D_i/D_{i-1}$, where $\mathrm{Frob}\colon A \to A$ denotes the Frobenius endomorphism $a \mapsto a^p$. Putting all of these data together, we have a finite projective A-module $H^n_{\mathrm{dR}}(X/A)$ equipped with two filtrations $C^\bullet$ and $D_\bullet$ and isomorphisms $\varphi_i\colon C^i/C^{i+1} \otimes_{A,\mathrm{Frob}} A \cong D_i/D_{i-1}$. In good cases Generalizing this by replacing the de Rham cohomology by an arbitrary finite projective module, we get the following definition.

An *F-zip over a scheme S* of characteristic $p > 0$ is a tuple $(M, C^\bullet, D_\bullet, \varphi_\bullet)$, where M is a finite locally free $\mathcal{O}_S$-module, $C^\bullet$ is a descending filtration on M, $D_\bullet$ is an ascending filtration on M and $\varphi_\bullet\colon (\operatorname{gr}^\bullet_C M)^{(1)} \xrightarrow{\sim} \operatorname{gr}^\bullet_D M$ are isomorphisms, where the index (1) denotes the Frobenius twist[1]. The associated stack of F-zips is then a rather combinatorical object. Associating an F-zip to a geometric object, such as abelian schemes, one can analyze its moduli by analyzing the moduli of F-zips.

With this method Moonen-Wedhorn extended results of Ekedahl and Oort on the stratification of families of abelian schemes in positive characteristic. They defined new stratifications on families of proper smooth morphisms satisfying certain conditions, which generalized the known results for abelian varieties in positive characteristic (see [MW04]). By attaching extra structure to F-zips, the theory of F-zips can also be generalized to the theory of so called *G-zips*, defined in [PWZ15]. Goldring and Koskivirta used the theory of G-zips to construct group-theoretical Hasse invariants on Ekedahl-Oort stratum closures of a general Hodge-type Shimura variety (see [GK19]). They apply this in different cases to the Langlands program and prove for example a conjecture of Oort.

One major tool to associate an F-zip to a geometric object is by its de Rham cohomology. Namely, the Hodge and conjugate spectral sequence induce two filtrations on the k-th de Rham cohomology. If the Hodge-de Rham spectral sequence degenerates and the Hodge cohomologies are finite projective (and thus also the conjugate spectral sequence[2]) the graded pieces are isomorphic up to Frobenius twist via the Cartier isomorphism. The hypothesis on the Hodge-de Rham spectral sequence restricts us to a certain class of geometric objects, which for example include abelian schemes, K3-surfaces, smooth proper curves and smooth complete intersections in the projective space. But since for example the Hodge-de Rham spectral sequence does not degenerate for supersingular Enriques surfaces in characteristic 2 (see [Lan95, Thm. 2]), we cannot use the theory of F-zips to analyze their moduli stack.

One possible solution to this problem is going to the derived world. The idea is straightforward. If we replace the k-th de Rham cohomology by its hypercohomology, we get a perfect complex with two filtrations and the Cartier isomorphism still applies to the graded pieces[3]. But taking the derived category naively leads us to problems, since we can not glue in the ordinary derived category. To apply geometric methods, we want descent on the derived category. This problem is solved by introducing the language of ∞-categories. So in particular, the idea of this thesis is to use homotopy theoretical methods to analyze derived versions of F-zips.

A homotopy theoretical version of algebraic geometry was developed in [TV08]. In the reference Toën-Vezzosi work in the model categorical setting. They use simplicial commutative rings as a replacement for commutative rings and presheaves of spaces as a replacement for presheaves of sets (or groupoids). They define model structures on those (actually in a more general setting, see [TV05]) and use Grothendieck topologies (in their setting) to define derived versions of stacks, schemes and affine schemes as fibrant objects in the corresponding model category. They analyze certain properties such as geometricity, smoothness and the cotangent complex. In this context a derived

[1]For an $\mathcal{O}_S$-module $\mathcal{F}$, we set $\mathcal{F}^{(1)} := \mathcal{F} \otimes_{\mathcal{O}_S, \mathrm{Frob}_S} \mathcal{O}_S$.

[2]See [Kat72, (2.3.2)].

[3]See Section 6.1 for the notion of filtrations in the derived category and the graded pieces.

stack is n-geometric if it has a $(n-1)$-geometric atlas by a coproduct of derived affine schemes. This notion allows us to define notions like smoothness, flatness and étale by using the atlas and defining it on the level of animated rings. The notion of higher geometricity comes into play if one wants to work with stacks that take values in higher groupoids. In [TV07], Toën and Vaquié gave an important example of a geometric stack, namely the derived stack of perfect complexes. In [AG14], Antieau and Gepner recalled this fact in detail in the setting of spectral algebraic geometry. This shows that one can glue perfect complexes and can cover it by affine derived schemes (in a suitable sense) and therefore the notion of derived F-zips indicated above should also behave in a similar fashion.

The translation to the world of ∞-categories is rather straightforward using Lurie's works [Lur09],[Lur17] and [Lur18]. Since Toën-Vezzosi defined their version of derived algebraic geometry using fibrant objects in model categories, we get analogous notions if we look at the ∞-categories associated to the corresponding model categories. Nevertheless, we will recall a lot of the definition ins [TV08] and [TV07] without using much of the model structure and prove the results purely in the world of ∞-categories. This shows that the definitions and results obtained this way do not rely on the chosen model structure but on the underlying ∞-category.

Derived algebraic geometry

The first part of this paper focuses on reformulating the results of [TV08], [TV07] and [Lur18] in the language of animated rings. The ∞-category of *animated rings* $\mathrm{AR}_{\mathbb{Z}}$ is given by freely adjoining sifted colimits to polynomial algebras. Looking at over categories for any animated ring A, we can define the ∞-category of *animated A-algebras* $\mathrm{AR}_A := (\mathrm{AR}_{\mathbb{Z}})_{A/}$. The benefit of this definition is that a lot of questions about functors from $\mathrm{AR}_{\mathbb{Z}}$ to ∞-categories with sifted colimits can be reduced to polynomial algebras. Animated rings should be thought as connective spectral commutative rings (i.e. E_∞-rings) with extra structure. Especially, after forgetting this extra structure, we can also define modules over animated rings (as modules over the underlying E_∞-ring). One important example of such a module is the cotangent complex. This module arises naturally if we want to define an analogue of the module of differentials as the module that represents the space of derivations.

The underlying E_∞-ring of an animated ring is a commutative algebra object in spectra. Thus, we can define homotopy groups of animated rings and automatically see (with the theory developed in [Lur17]) that we can associate to every animated ring A a $\mathbb{N}_0$-graded ring π_*A. Using this, we reduce definitions like smoothness of a morphism $A \to B \in \mathrm{AR}_{\mathbb{Z}}$ to smoothness of the ordinary rings $\pi_0 A \to \pi_0 B$ together with compatibility of the graded ring structure, i.e. $\pi_*A \otimes_{\pi_0 A} \pi_0 B \cong \pi_* B$. Analogously to the classical case, we have that for a smooth morphism of animated rings its cotangent complex (the module representing the space of derivations) is finite projective. We can upgrade this to an "if and only if" if we assume that on π_0 the ring homomorphism is finitely presented. This does not hold in the classical world, i.e. a ring homomorphism with a finite projective module of differentials, may not be smooth, e.g. non-smooth regular closed immersions.

Defining *derived stacks* is rather straightforward now. We set them as presheaves (of spaces) on $\mathrm{AR}_{\mathbb{Z}}$ which satisfy étale descent. One important class of examples are *affine derived schemes* which we set as representable presheaves on $\mathrm{AR}_{\mathbb{Z}}$. We can also define relative versions, where we replace $\mathbb{Z}$ with an animated ring. We will see that they naturally satisfy fpqc-descent. For affine derived schemes, it is easy to define properties by using their underlying animated rings. To do the same for derived stacks, we will need the notion of *n-geometric* morphisms. This notion is defined inductively, where we say a morphism $f\colon F \to G$ of derived stacks is (-1)-geometric, if the base change with an affine derived scheme is representable by an affine derived schemes. A (-1)-geometric morphism is smooth if it is so after base change to any affine. The morphism f is n-geometric if for any affine derived scheme $\mathrm{Spec}(A)$ with morphism $\mathrm{Spec}(A) \to G$ the base change $F \times_G \mathrm{Spec}(A)$ has a smooth $(n-1)$-geometric effective epimorphism by $\coprod \mathrm{Spec}(T_i)$, where an n-geometric morphism is smooth if after affine base change the induced maps of the atlas to the base is (-1)-geometric and smooth. For a good class[4] of properties $\mathbf{P}$ of affine derived schemes, e.g. smooth, flat,...[5], we can now say that a morphism of derived stacks has property $\mathbf{p} \in \mathbf{P}$ if it is n-geometric for some n and after base change with an affine derived scheme the atlas over the affine base has property $\mathbf{p}$. Since geometricity is defined by using smooth atlases, we are mostly interested in this property. One important aspect is that this property is strongly related to the cotangent complex.

Theorem 1.1 (4.77)**.** *Let $f\colon X \to Y$ be an n-geometric morphism of derived stacks. Then f is smooth if and only if $f_{|(\mathrm{Ring})}$ is locally of finite presentation and the cotangent complex of f exists, is perfect and has Tor-amplitude in $[-n-1, 0]$.*

The proof of this theorem uses a lot of homotopical algebraic geometric methods combining ideas of the homotopical world with ideas of the algebraic geometric world.

This theorem allows us to show that the stack of perfect complexes is locally geometric and locally of finite presentation.

Theorem 1.2 (5.14)**.** *The derived stack*

$$\begin{aligned}\mathrm{Perf}\colon \mathrm{AR}_R &\to \mathbb{S}\\ A &\mapsto (\mathrm{Mod}_A^{\mathrm{perf}})^{\simeq}\end{aligned}$$

is locally geometric and locally of finite presentation.

The idea is to write this stack as a filtered colimit of substacks $\mathrm{Perf}^{[a,b]}$, where we fix the Tor-amplitude of the perfect complexes. These are $(b-a+1)$-geometric stacks locally of finite presentation. To see this, we will use induction over the difference $b-a$ and cover it using morphisms of perfect modules with smaller Tor-amplitude by sending the morphism to its cofiber. Then we are basically finished by showing that

[4]With "good class" we mean stable under base change, composition, equivalences and smooth local on source and target.

[5]Note that the property étale is not smooth local on the source. We have to be careful if we want to define étale morphisms of n-geometric stacks.

the stack classifying morphisms of perfect modules with some given Tor-amplitude is geometric. An interesting substack is obtained by setting $a = b = 0$, then $\mathrm{Perf}^{[0,0]} \simeq \mathrm{Proj} \simeq \coprod_n \mathrm{BGL}_n$.

Derived F-zips

The second part of this thesis focuses on derived F-zips. To be more specific, we first define derived F-zips and then analyze the geometry of their moduli spaces. The definition of derived F-zips is influenced by the natural structure that arises on the de Rham hypercohomology for some proper smooth scheme morphism $X \to \mathrm{Spec}(A)$. We see for example that a the Hodge-filtration is just a functor $\mathbb{Z}^{\mathrm{op}} \to \mathcal{D}(A)^{\mathrm{perf}}$ that is bounded. This is what we call a *descending filtration*. With this notion, we set a *derived F-zip over $A \in \mathrm{AR}_{\mathbb{F}_p}$*, for some positive prime p, to be a tuple $(C^\bullet, D_\bullet, \phi, \varphi_\bullet)$, where M is a perfect module over A, $C^\bullet$ a bounded descending filtration, $D_\bullet$ a bounded ascending filtration, equivalences $\varphi_\bullet \colon (\mathrm{gr}^\bullet C)^{(1)} \xrightarrow{\sim} \mathrm{gr}^\bullet D$ and and equivalence $\phi \colon \mathrm{colim}_{\mathbb{Z}^{op}} C^\bullet \xrightarrow{\sim} \mathrm{colim}_{\mathbb{Z}} D_\bullet$. Varying A, this construction induces a derived stack (even a hypercomplete fpqc-sheaf). Our main goal is to show that this stack is locally geometric.

Theorem 1.3 (6.42)**.** *The derived stack*

$$\begin{aligned} F\text{-Zip} \colon \mathrm{AR}_{\mathbb{F}_p} &\to \mathbb{S}, \\ A &\mapsto \infty\text{-groupoid of derived } F\text{-zips over } A \end{aligned}$$

is locally geometric.

The idea of the proof is straightforward. We first look at the derived substack $F\text{-Zip}^{[a,b],S}$, for a finite subset $S \subseteq \mathbb{Z}$ and $a \leq b \in \mathbb{Z}$, classifying those derived F-zips $(C^\bullet, D_\bullet, \phi, \varphi_\bullet)$ where we fix the Tor-amplitude $[a, b]$ of all filtered pieces and $\mathrm{gr}^i C \simeq 0$ for $i \neq S$. In this way we only have to look at the stacks classifying two chains of morphisms of modules with fixed Tor-amplitude, that have a connecting equivalence at the last entry, such that the graded pieces are equivalent after Frobenius twist. Since perfect modules with fixed Tor-amplitude, morphisms of those and equivalences of those are geometric, we conclude the geometricity of $F\text{-Zip}^{[a,b],S}$.

We can also look at derived substacks $F\text{-Zip}^{\leq\tau}$, for a function $\tau \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}_0$ with finite support, classifying those derived F-zips $\underline{F} \coloneqq (C^\bullet, D_\bullet, \phi, \varphi_\bullet)$ where the fiberwise dimension of the $\pi_i(\mathrm{gr}^j C)$ are $\leq \tau(i, j)$. If we have equality and the $\pi_i(\mathrm{gr}^j C)$ are finite projective, we call $\underline{F}$ homotopy finite projective of type τ. By upper semi-continuity the dimension of fiberwise cohomology of perfect complexes, we see that $F\text{-Zip}^{\leq\tau}$ is an open substack of F-Zip and also geometric, as it is in fact open in some $F\text{-Zip}^{[a,b],S}$. Writing F-Zip as the filtered colimit of the $F\text{-Zip}^{\leq\tau}$, we conclude the theorem.

Since derived F-zips satisfy descent, we can glue this definition to any derived scheme S. There is also an ad hoc definition in the derived scheme case, but we can show that both definitions agree.

The definition is constructed in a way such that every proper smooth morphism $f \colon X \to S$ induces a derived F-zip $\underline{Rf_*\Omega^\bullet_{X/S}}$ over S.

Modification of filtrations and examples of derived F-zips

For the term *filtration* above, we do not enforce something like a monomorphism condition on the filtration. Even though it seems natural, it actually leads to another definition of derived F-zips, which we call *strong derived F-zips.* The difference of these two becomes apparent if we look at the corresponding spectral sequences.

Theorem 1.4 (6.71). *Let $f\colon X \to S$ be a smooth proper morphism of schemes. Let us consider the Hodge-de Rham spectral sequence*

$$E_1^{p,q} = R^q f_* \Omega^p_{X/S} \Rightarrow R^{p+q} f_* \Omega^\bullet_{X/S}.$$

Assume that all $R^i f_* \Omega^j_{X/S}$ are finite locally free. The derived F-zip $\underline{Rf_*\Omega^\bullet_{X/S}}$ is strong if and only if the Hodge-de Rham spectral sequence degenerates.

So, we do not expect the theory of strong derived F-zips to give us any new information if we want to consider geometric objects that do not induce classical F-zips. But, we can show that the derived stack of strong derived F-zips is open in the derived stack of derived F-zips. Further, looking at very specific types of strong derived F-zips, we can even make a connection to classical F-zips.

This connection can be generalized by looking at the full sub-∞-category of derived F-zips with degenerate spectral sequences[6] such that the graded pieces attached to the filtrations have finite projective homotopy groups of type τ, denoted by $\mathcal{X}^\tau$. It is not hard to see $\mathcal{X}^\tau$ is equivalent to the product of classical F-zips of type corresponding to the components of τ (see section 7 for more details).

We remark that we formulate all the results more generally for derived F-zips over arbitrary derived schemes of positive characteristic.

The above connection to classical F-zips also shows that in the case of K3-surfaces or proper smooth curves and abelian schemes the theory of strong derived F-zips gives no new information. In the K3-surfaces and proper smooth curve case, we can be more specific. Every derived F-zip of K3-type or proper smooth curve type is induced by a classical F-zip. This is, since in both cases there is only one cohomology group with a non-trivial filtration. This does not hold for abelian schemes (since they have a more complicated type), but as remarked earlier, the derived F-zip associated to an abelian scheme X/A is completely determined by the classical F-zips associated to $H^1_{\mathrm{dR}}(X/A)$ (since the Hodge-de Rham spectral sequence of abelian schemes is degenerate, the Hodge cohomologies are finite projective and we have $H^n_{\mathrm{dR}}(X/A) \cong \wedge^n H^1_{\mathrm{dR}}(X/A)$). Also one can look at the moduli stack of Enriques surfaces in characteristic 2. In this case the Hodge-de Rham spectral sequence does not degenerate in general. Hence, we cannot directly use the theory of F-zips by associating to an Enriques surface its de Rham cohomology but have to use derived F-zips for this approach. Using the upper semi-continuity of cohomology, we can see with the theory of derived F-zips that the

[6] For any animted ring A a functor $F\colon \mathrm{Fun}(\mathbb{Z}, \mathrm{Mod}_A)$ with $F(n) \simeq 0$ for $n \ll 0$ induces by [Lur17, Prop. 1.2.2.14] a spectral sequence of the form $E_1^{p,q} = \pi_{p+q}(\mathrm{gr}^p F) \Rightarrow \pi_{p+q}(\mathrm{colim}_{\mathbb{Z}} F)$. A derived F-zip over A comes equipped with two filtrations and thus induce two such spectral sequences, which we call the spectral sequences attached to the derived F-zip.

substacks classifying Enriques surfaces of type $\mathbb{Z}/2$ or μ_2 are open in the moduli of Enriques surfaces and the substack classifying Enriques surfaces of type α_2 is closed. These results give a new proof of the results of Liedtke [Lie15] who does not use the derived theory.

Derived F-zips with cup product

Let $f\colon X \to S$ be a proper smooth morphism of schemes in positive characteristic with geometrically connected fibers of fixed dimension n. Further, assume the Hodge-de Rham spectral sequence associated to f degenerates and the Hodge cohomologies are finite locally free. As in the classical case there is extra structure on the de Rham hypercohomology coming from the cup product, namely a perfect pairing. For classical F-zips this induces a G-zip structure on the F-zip associated to $H^n_{\mathrm{dR}}(X/S)$, for certain reductive groups over a field of characteristic $p > 0$. One could try to define a derived G-zip, for a reductive group G over a field k of characteristic $p > 0$, in such a way such that the cup product induces a derived $\widetilde{G}$-zip structure on the de Rham hypercohomology for some reductive group $\widetilde{G}$ over k. As explained in Section 9.1.2 the most obvious ways to generalize the theory of G-zips to derived G-zips are not quite right. The problem here is that we do not have a "good" way of defining derived group schemes. For example, we would like to have that the derived analogue of GL_n-torsors is given by perfect complexes of *Euler-characteristic* $\pm n$. But, as far as we know there is no such analogue.

Alternatively, we show that the symmetric monoidal category of classical F-zips over a scheme S in characteristic $p > 0$ is equivalent to the symmetric monoidal category of vector bundles over a certain algebraic stack $\mathfrak{X}_S$. The stack $\mathfrak{X}_S$ is given by pinching the projective line at 0 and ∞ via the Frobenius morphism and then taking the quotient with the induced $\mathbb{G}_{\mathrm{m},\mathbb{F}_p}$-action. The category of G-zips over S is then equivalent to the stack of G-torsors on $\mathfrak{X}_S$. To apply this construction to derived F-zips, we will show that perfect complexes over $\mathfrak{X}_S$ are precisely derived F-zips. But, we lack a definition of derived groups and torsors attaching extra structure to perfect complexes. So again, we did not follows this approach further.

For completion, we naively put the cup product structure into the definition of derived F-zips leading to the definition of dR-*zips*. This again is a sheaf and we can explicitly analyze the projection to derived F-zips.

Proposition 1.5 (9.7). *The induced morphism via forgetting the pairing*

$$p\colon \text{dR-Zip} \to F\text{-Zip}$$

is smooth and locally of finite presentation, in particular dR-Zip *is locally geometric and locally of finite presentation.*

Depending on the Tor-amplitude of the dR-zips, we can specify the properties of the above forgetful functor.

The derived world has another benefit. Usually, we can extend results for smooth objects to objects that are only *lci* (in fact to any animated algebra via left Kan extension). In the case of the de Rham hypercohomology, we know that its lci analogue is given by the derived de Rham complex (here lci is needed to assure perfectness of the cotangent complex). This seems like a good generalization of the de Rham hypercohomology since it comes equipped with two filtrations with Frobenius-equivalent graded pieces. But, one can show that these filtrations are not bounded in any way (see Section 9.1.1 for more details). So we would need a notion of derived F-zips with unbounded filtrations. But, then the obvious problem becomes the geometricity, since we would have to cover an infinite amount of information with the atlas, which is not clear at all - geometricity is a priori not preserved under arbitrary limits (and may not even be for cofiltered limits).

1.1 Structure of this paper

We start by summarizing [Lur17] (see Section 2). We try to show that spectral rings and modules behave in some sense like expected.

The next step is the introduction of derived commutative algebra, i.e. the theory of algebras over animated rings (see Section 3). We first define animated rings $\mathrm{AR}_{\mathbb{Z}}$, show that there is a relation to E_∞-rings and use this relation to define modules over animated rings. As a consequence, we can define the cotangent complex and define what properties of morphisms in $\mathrm{AR}_{\mathbb{Z}}$ are. We end this section by showing that the cotangent complex is highly related to smoothness of morphisms.

After talking about derived commutative algebra, we introduce the theory derived algebraic geometry (see Section 4). Mainly, we introduce the notion of derived stacks, geometricity of morphisms and derived schemes. We also talk about truncation of those and how it relates to classical algebraic geometry. Further, we again cover smoothness of such morphisms and how it relates to the cotangent complex.

We finish the summary on derived algebraic geometry by showing that the stack of perfect modules is locally geometric and locally of finite presentation (see Section 5).

Next, we talk about filtrations on the derived category and introduce derived F-zips (see Section 6). We show that the presheaf which assigns to an animated ring the ∞-category of derived F-zips is in fact a sheaf, so a derived stack, and even locally geometric. After the geometricity, we discuss some important substacks and try to generalize the notion of derived F-zips to derived schemes. We look at certain substacks that come naturally by looking at derived F-zips of certain type. Also, we look at the substack classifying those filtrations that are termwise monomorphisms. In particular, we show that this condition is under some assumptions equivalent to the degeneracy of the Hodge-de Rham spectral sequence.

We finish the study of derived F-zips by trying to connect classical F-zips with derived F-zips (see Section 7). We show that in the case of degenerating Hodge-de Rham spectral sequence there is no new information coming from derived F-zips. We lastly apply our theory to the moduli of Enriques surfaces (see Section 8).

We finish this paper by elaborating the problems that appeared while trying to generalize the theory of derived F-zips to case of proper lci morphisms and trying to define derived G-zips (see Section 9). For completion, we also naively equip derived F-zips with extra structure.

In the Appendix, we discuss the connection of classical G-zips and G-torsors on the pinched projective space modulo $\mathbb{G}_{\mathrm{m},\mathbb{F}_p}$-action $\mathfrak{X}$ and look at the perfect complexes over $\mathfrak{X}$.

1.2 Assumptions

All rings are commutative with one.

We work with the Zermelo-Frenkel axioms of set theory, with the axiom of choice and assume the existence of inaccessible regular cardinals.

Throughout this paper, we fix some uncountable inaccessible regular cardinal κ and the collection $U(\kappa)$ of all sets having cardinality $< \kappa$, which is a Grothendieck universe (and as a Grothendieck universe is uniquely determined by κ) and hence satisfies the usual axioms of set theory (see [Wil69]). When we talk about small, we mean $\mathcal{U}(\kappa)$-small. In the following, we will use some theorems, which assume smallness of the respective (∞-)categories. When needed, without further mentioning it, we assume that the corresponding (∞-)categories are contained in $\mathcal{U}(\kappa)$.

If we work with families of objects that are indexed by some object, we will assume, if not further mentioned, that the indexing object is a $\mathcal{U}(\kappa)$-small set.

1.3 Notation

We work in the setting of $(\infty, 1)$-categories (see[Lur09]). By abuse of notation for any 1-category C, we will always denote its nerve again with C, unless otherwise specified.

A *subcategory* $\mathcal{C}'$ of an ∞-category $\mathcal{C}$ is a simplicial subset $\mathcal{C}' \subseteq \mathcal{C}$ such that the inclusion is an inner fibration. In particular, any subcategory of an ∞-category is itself a ∞-category and we will not mention this fact.

- Δ denotes the simplex category (see [Lur21, 000A]), i.e. the category of finite non-empty linearly ordered sets, Δ_+ the category of (possibly empty) finite linearly ordered sets. We denote with Δ_s those finite linearly ordered sets whose morphisms are strictly increasing functions and with $\Delta_{s,+}$ those (possibly empty) finite linearly ordered sets whose morphisms are strictly increasing functions.

- With an ∞-category, we always mean an $(\infty, 1)$-category.

- $\mathbb{S}$ denotes the ∞-category of small spaces (also called ∞-groupoids or anima).

- Cat_∞ denotes the ∞-category of small ∞-categories.

- Sp denotes the ∞-category of spectra.

- For an E_∞-ring A, we denote the ∞-category of A-modules in spectra, i.e. $\mathrm{Mod}_A(\mathrm{Sp})$ in the notation of [Lur17], with Mod_A.

- For any ordered set $(S, \leq)$, we denote its corresponding ∞-category again with S, where the corresponding ∞-category of an ordered set is given by the nerve of $(S, \leq)$ seen as a 1-category (the objects are given by the elements of S and $\mathrm{Hom}_S(a, b) = *$ if and only if $a \leq b$ and otherwise empty).

- For any set S the ∞-category S^{disc} will denote the nerve of the set S seen as a discrete 1-category (the objects are given by the elements of S and $\mathrm{Hom}_S(a,a) = *$ for any $a \in S$ and otherwise empty).

- For any morphism $f\colon X \to Y$ in an ∞-category $\mathcal{C}$ with finite limits, we denote the functor from Δ_+ to $\mathcal{C}$ that is given by the Čech nerve of f (see [Lur09, §6.1.2]) if it exists by $\check{C}(Y/X)_\bullet$.

- Let $\mathcal{C}$ be an ∞-category with final object $*$. For morphisms $f\colon * \to X$ and $g\colon * \to X$, we denote the homotopy pullback $* \times_{f,X,g} *$ if it exists with $\Omega_{f,g}X$. If $\mathcal{C}$ has an initial object 0, then we denote the pullback $0 \times_X 0$ with ΩX.

- Let $f\colon X \to Y$ be a morphism in $\mathbb{S}$ and let $y \in Y$. We write $\mathrm{fib}_y(X \to Y)$ or $\mathrm{fib}_y(f)$ for the pullback $X \times_Y *$, where $*$ is the final object in $\mathbb{S}$ (up to homotopy) and the morphism $* \to Y$ is induced by the element y, which by abuse of notation, we also denote with y.

- For a morphism $f\colon M \to N$ in Mod_A, where A is some E_∞-ring, we set $\mathrm{fib}(f) = \mathrm{fib}(M \to N)$ (resp. $\mathrm{cofib}(f) = \mathrm{cofib}(M \to N)$) as the pullback (resp. pushout) of f with the essentially unique zero morphism $0 \to N$ (resp. $M \to 0$).

- When we say that a square diagram in an ∞-category $\mathcal{C}$ of the form

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

  is commutative, we always mean, that we can find a morphism $\Delta^1 \times \Delta^1 \to \mathcal{C}$ of ∞-categories that extends the above diagram.

Acknowledgement

First and foremost I want to thank my PhD advisor Torsten Wedhorn for his constant help and for all his patience while listening to my questions. He suggested this topic and encouraged me to learn a lot about higher algebra and derived algebraic geometry. I would also like to thank Anton Güthge who pointed out some mistakes in the earlier versions, Benjamin Antieu, Adeel Khan and Jonathan Weinberger for their help concerning derived algebraic geometry respectively higher topos theory, Christian Liedtke for discussions concerning Enriques surfaces and Timo Richarz for helpful discussions. Further, I would like to thank my family and Simone Steilberg for constant encouragement and patience, during my work on this thesis and my life through this pandemic. Lastly, I want to thank Rızacan Çiloğlu, Catrin Mair, Simone Steilberg and Thibaud van den Hove for proofreading my thesis.

2 Overview: Higher algebra

In this section, we want to summarize some important aspects of spectra, E_∞-rings and modules over those presented in [Lur17].

Our main interests are animated rings and modules over those. Animated rings are presented in Section 3, so we want to focus on modules. These will be defined over a monoidal ∞-category, in this case the ∞-category of spectra $\mathrm{Sp} \coloneqq \mathrm{Sp}(\mathbb{S})$. So, one should think of Sp as an ∞-category equipped with a tensor product and spectral modules as modules for this tensor product over some commutative ring, called E_∞-rings.

Let us start by recalling stable ∞-categories. An ∞-category is stable if it has a zero object, it admits fibers and cofibers and every cofiber sequence is a fiber sequence (see [Lur17, Def. 1.1.1.9]). This can be seen as an ∞-analogue of an abelian category. One very important feature of a stable ∞-category is that its homotopy category is automatically triangulated (see [Lur17, Thm. 1.1.2.14]). Stable ∞-categories have other nice stability properties, e.g. a square is a pullback if and only if it is a pushout and there exists finite limits and colimits (see [Lur17, Prop. 1.1.3.4]), but listing everything concerning stable ∞-categories would be to involved so we refer to [Lur17, §1].

Now let us come to the definition of the spectrum $\mathrm{Sp}(\mathcal{C})$ of a pointed ∞-category $\mathcal{C}$ with finite limits. One definition is obtained by setting $\mathrm{Sp}(\mathcal{C})$ as the ∞-category of excisive, reduced functors from[7] $\mathbb{S}_*^{\mathrm{fin}}$ to $\mathcal{C}$ (see [Lur17, Def. 1.2.4.8]). Alternatively, one obtains $\mathrm{Sp}(\mathcal{C})$ as the homotopy limit of the tower $\dots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}$ (see [Lur17, Rem. 1.4.2.25]). Both viewpoints are useful. An important property of $\mathrm{Sp}(\mathcal{C})$ is that it is a stable ∞-category and if $\mathcal{C}$ is presentable then so is $\mathrm{Sp}(\mathcal{C})$ (see [Lur17, Cor. 1.4.2.17, Prop. 1.4.4.4]). From now on, we assume $\mathcal{C}$ to be presentable. The first definition as functors allows us to define a functor

$$\Omega^\infty\colon \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$$

by evaluation on the zero sphere. In fact, $\mathcal{C}$ is stable if and only if Ω^∞ is an equivalence. Another property of Ω^∞ is that it admits a left adjoint Σ^∞ (see [Lur17, Prop 1.4.4.4]).

Let us set $\mathcal{C} = \mathbb{S}_*$, the ∞-category of pointed spaces and let us set $\mathrm{Sp} \coloneqq \mathrm{Sp}(\mathbb{S}_*)$. The second definition we gave allows us to identify the homotopy category hSp with the classical stable homotopy category[8] (see [Lur17, Rem. 1.4.3.2]). In fact, one can show that Sp is the ∞-category associated to the model category of symmetric spectra (see [Lur17, Ex. 4.1.8.6]). This allows us to define a monoidal structure on Sp using the monoidal structure on the underlying model category, given by the smash product.

[7] Here $\mathbb{S}_*^{\mathrm{fin}}$ denotes the smallest subcategory of $\mathbb{S}$ that contains the final object, is stable under finite colimits and consist of pointed objects, where pointed means objects $x \in \mathbb{S}$ with a morphism $* \to x$, (see [Lur17, Not. 1.4.2.5]).

[8] Symmetric spectra are certain sequences of Kan complexes $X_0, X_1, \dots$ with maps $\Sigma X_{n-1} \to X_n$. This category is equipped with a model structure (called the stable model structure) and is closed monoidal. One can also equip certain sequences of pointed topological spaces $X_0, X_1, \dots$ with maps $\Sigma X_{n-1} \to X_n$ with a model structure and endow it with a closed monoidal structure using the smash product. Both constructions are in fact Quillen equivalent (see [HSS00] for further information about symmetric spectra).

One important aspect of this monoidal structure is that its unit element is the sphere spectrum[9] and the tensor product preserves colimits in each variable (see [Lur17, Cor. 4.8.2.19]). This construction shows that for a spectrum object $X \in \mathrm{Sp}$ we have

$$\mathrm{Hom}_{\mathrm{Sp}}(S, X) \simeq \mathrm{Hom}_{\mathbb{S}}(\Delta^0, X) \simeq \Omega^\infty(X),$$

where S denotes the sphere spectrum. This is nothing new if we think about abelian groups for example, since $\mathbb{Z}$-module homomorphisms from $\mathbb{Z}$ to any abelian group are uniquely characterized by the elements of the group and since here $\mathrm{Hom}_{\mathrm{Sp}}(S, X)$ is a Kan-complex, we see that it is equivalent to the underlying space of the spectrum X. An important side remark is that the heart of spectra is naturally identified with (the nerve of) the category of abelian groups.

Using Ω^∞, stability of Sp and homotopy groups of Kan-complexes, we can define an accessible t-structure on Sp (see [Lur17, Prop 1.4.3.6]). In particular, we can define the homotopy groups of spectrum objects $X \in \mathrm{Sp}$, via $\pi_n X \simeq \pi_0 \Omega^\infty(X[-n])$ (see proof of [Lur17, Prop 1.4.3.6]) and if $n \geq 2$, then these are given by $\pi_n \Omega^\infty(X)$ (see [Lur17, Rem. 1.4.3.8]).

Before we can define modules, we start with E_∞-rings. We will not go into detail, since we will work with an analogue, namely animated rings. One should think of animated rings as E_∞-rings with an extra structure. This extra structure vanishes if we are in characteristic zero but gives us no relation except a conservative functor from animated rings to E_∞-rings in positive or mixed characteristic (see Proposition 3.5).

As stated above, one should think of Sp as an ∞-categorical analogue of abelian groups. To define commutative rings in this ∞-category one could use the theory of ∞-operads and describe E_∞-rings in terms of sections of the operad induced by the monoidal structure of Sp (see [Lur17, §2] for more information about ∞-operads and [Lur17, §3, 4] for the construction of rings using this approach). We will not describe how this is achieved but instead use a rectification argument, i.e. we set E_∞-rings as the ∞-category associated to the commutative algebra objects in the underlying model category of Sp. Both approaches are equivalent (see [Lur17, Thm. 4.5.4.7]) so we can think of E_∞-rings as certain commutative rings in the model category associated to Sp. Using the Eilenberg-Mac Lane spectrum one can see that for example ordinary commutative rings are discrete E_∞-rings. This construction is also vague, since it requires the ∞-categorical localization of cofibrant commutative algebra objects in the underlying model category. But contrary to ∞-operads, we think it gives a more classical feeling of commutative rings.

Now let us conclude this section with modules over E_∞-rings. Again [Lur17] deals with modules using ∞-operads (see [Lur17, §3, 4]). Analogous to the E_∞-ring case, we can apply a rectification statement to define modules using the monoidal structure on the underlying model category, again both constructions are equivalent (see [Lur17,

[9] We use the convention of [Lur17], where the sphere spectrum is the image of the final object $* \in \mathbb{S}$ by Σ^∞ (see [Lur17, §1.4.4] for more details).

Thm. 4.3.3.17]). For an E_∞-ring A, we will denote the ∞-category of spectral A-modules with Mod_A. By forgetting the module structure, we get a functor $\mathrm{Mod}_A \to \mathrm{Sp}$ (induced by the construction using ∞-operads, see [Lur17, Def. 3.3.3.8]). As in the classical case for abelian groups and modules, the forgetful functor is conservative (this follows from [Lur17, Cor. 4.2.3.2]). Further, if we restrict ourselves to connective[10] modules, then even the composition $\mathrm{Mod}_A^{\mathrm{cn}} \to \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathbb{S}$ is conservative (see [Lur17, Rem. 7.1.1.8]) This is an analogue to the fact that a morphism of classical modules is an isomorphism if and only if it is a bijection on the underlying sets. Another fact that is well known in the classical case is that, we have the following equivalence in $\mathbb{S}$

$$\mathrm{Hom}_{\mathrm{Mod}_A}(A, M) \simeq \mathrm{Hom}_{\mathrm{Sp}}(S, M) \simeq \Omega^\infty(M)$$

(see [Lur17, Cor. 4.2.4.7]).

As in the classical case, we can also define E_∞-algebras and modules over algebras by setting $E_\infty\text{-}\mathrm{Alg}_A \coloneqq \mathrm{CAlg}(\mathrm{Mod}_A)$, i.e. we endow A-modules with a monoidal structure and define A-algebras as commutative object in Mod_A (see [Lur17, 7.1.3.8]). Alternatively, we could look at the over category $E_\infty\text{-}\mathrm{Alg}_{A/}$ but both constructions are in fact equivalent (see [Lur17, Cor. 3.4.1.7]). We also have a forgetful functor from $E_\infty\text{-}\mathrm{Alg}_A$ to Sp, which under the identification $E_\infty\text{-}\mathrm{Alg}_A \simeq \mathrm{CAlg}(\mathrm{Mod}_A)$ factors through the forget functor $E_\infty\text{-}\mathrm{Alg}_A \to \mathrm{Mod}_A$ which is conservative (again follows from [Lur17, Cor. 4.2.3.2]). Further, the identification of $\pi_n R$ with $\pi_0 \mathrm{Hom}_{\mathrm{Sp}}(S[n], R)$ for an E_∞-ring R, where $\pi_n R$ is defined on the underlying spectrum of R, allows us to endow $\pi_* R \coloneqq \bigoplus_{n\in\mathbb{Z}} \pi_n R$ with a graded commutative ring structure (see [Lur17, 7 7.1.1, Rem. 7.1.1.6]). We can also endow $\pi_* M \coloneqq \bigoplus_{n\in\mathbb{Z}} \pi_n M$ for any R-module M, with a graded $\pi_* R$-module structure (see [Lur17, §7.1.1]).

The ∞-category of A-modules is also stable (see [Lur17, 7.1.1.5]) and has an accessible t-structure induced by the accessible t-structure on Sp. This t-structure allows us to identify the heart of Mod_A with the (nerve of the) ordinary category of $\pi_0 A$-modules (see [Lur17, Prop. 7.1.1.13]) (note that by the above $\pi_0 A$ is an ordinary commutative ring). This is analogous to E_∞-algebra case, where for a connective E_∞-ring A the discrete A-algebras are precisely the ordinary commutative $\pi_0 A$-algebras (see [Lur17, Prop. 7.1.3.18] (note that there is a typo in the statement)).

A key difference to connective E_∞-rings is that over ordinary (discrete) commutative rings R, the R-module spectra are not discrete R-modules but instead we have $\mathrm{Mod}_R \simeq \mathcal{D}(R)$ as symmetric monoidal ∞-categories, where $\mathcal{D}(R)$, denotes the derived ∞-category of R-modules[11] (see [Lur17, Thm. 7.1.2.13]). Let us also remark, that under this equivalence the homotopy groups of module spectra are isomorphic to the homology groups of the associated complex and since this is an equivalence of symmetric monoidal ∞-categories this isomorphism also respects the module structure (see Remark 3.20).

[10] Here an object c in an ∞-category $\mathcal{C}$ with a conservative functor $f\colon \mathcal{C} \to \mathcal{D}$ into a stable ∞-category with t-structure is connective, if $f(c)$ is connective, i.e. $\pi_i f(c) \cong 0$ for $i < 0$.

[11] The derived ∞-category of a Grothendieck abelian category $\mathcal{A}$ is the ∞-category associated to the model category of chain complexes $\mathrm{Ch}(\mathcal{A})$ (see [Lur17, Prop. 1.3.5.15.] and [Lur17, Prop. 1.3.5.3] for the model structure on chain complexes). The homotopy category $\mathrm{h}\mathcal{D}(\mathcal{A})$ is equivalent to the ordinary derived category $D(\mathcal{A})$ of $\mathcal{A}$.

3 Derived commutative algebra

In the following, R will be a ring.

In this section, we want to give a quick summary about animated rings, present the cotangent complex and analyse smooth morphisms between animated rings. Mainly, we show that these notions arise from our classical point of view and behave like one can expect.

3.1 Animated rings

In this section, we summarise important aspects of animated rings, for this we will follow [Lur18, §25].

By Poly_R we denote the category of polynomial R-algebras in finitely many variables. Then the category of R-algebras is naturally equivalent to the category of functors from $\mathrm{Poly}_R^{\mathrm{op}}$ to (sets) which preserve finite products[12]. Applying this construction to the ∞-categorical case, we obtain AR_R the ∞-category of animated R-algebras, i.e.

$$\mathrm{AR}_R \coloneqq \mathrm{Fun}_\pi(\mathrm{Poly}_R^{\mathrm{op}}, \mathbb{S}),$$

where the subscript π denotes the full subcategory of $\mathrm{Fun}(\mathrm{Poly}_R^{\mathrm{op}}, \mathbb{S})$, that preserve finite products. Alternatively, this ∞-category is obtained by freely adjoining sifted colimits to Poly_R (this is the meaning of [Lur09, Prop. 5.5.8.15] using that any element in AR_R can be obtained by a sifted colimit in Poly_R by [Lur09, Lem. 5.5.8.14, Cor. 5.5.8.17]).

For a cocomplete category C that is generated under colimits by its full subcategory of compact projective objects C^{sfp}, Cesnavicius-Scholze define the ∞-category *animation of C* in [CS21, §5.1] denoted by $\mathrm{Ani}(C)$. The ∞-category $\mathrm{Ani}(C)$ is the ∞-category freely generated under sifted colimits by C^{sfp}. In particular, with this definition, we see that $\mathrm{Ani}((R\text{-Alg})) \simeq \mathrm{AR}_R$. This process can also be applied to R-modules, which we will look at later in Section 3.2, and to abelian groups, where $\mathrm{Ani}((\mathrm{Ab}))$ recovers the ∞-category of simplicial abelian groups (see [CS21, §5.1] for more details). The animation of (Sets) recovers the ∞-category of ∞-groupoids, i.e. $\mathrm{Ani}((\mathrm{Sets})) \simeq \mathbb{S}$.

We have another description for animated R-algebras. Let $\mathbf{A}$ be the category of product preserving functors from $\mathrm{Poly}_R^{\mathrm{op}}$ to simplicial sets[13]. We obtain a model structure on $\mathbf{A}$ by the Quillen model structure on simplicial sets (see [Lur09, 5.5.9.1]) - this is often called the model category of *simplicial commutative R-algebras*. This model category is known to be a combinatorial, proper, simplicial model category (for more

[12] For a functor $F\colon \mathrm{Poly}_R^{\mathrm{op}} \to (\mathrm{Sets})$ that preserves finite products we can set $F(R[X])$ as the underlying ring of F, where the multiplication is induced by $R[T] \to R[T_1] \otimes_R R[T_2]$, $T \mapsto T_1T_2$ and the addition by $T \mapsto T_1 + T_2$. Conversely, for any R-algebra A, we can construct a contraviarant functor from Poly_R to (Sets) via $A \mapsto \mathrm{Hom}_{(R\text{-Alg})}(-, A)$. These constructions are inverse to each other.

[13] As for product preserving functors from $\mathrm{Poly}_R^{\mathrm{op}}$ to (Sets) it is not hard to see that a product preserving functor $F\colon \mathrm{Poly}_R^{\mathrm{op}} \to \mathbb{S}$ defines a simplicial commutative ring, via $F \mapsto F(R[X])$ (the face and degeneracy maps have to respect the ring structure by functoriality). In particular, in this way we can identify $\mathbf{A}$ with the category of simplicial commutative R-algebras.

details on these properties, we refer to [Qui67, Ch. II §4, §6]). The ∞-category associated to this model category (i.e. $\mathrm{N}^{\mathrm{hc}}(\mathbf{A}^\circ)$) is equivalent to AR_R, where $\mathbf{A}^\circ$ denotes the full subcategory consisting of fibrant/cofibrant objects (see [Lur09, Cor. 5.5.9.3]).

Definition 3.1. For a ring R, we define the ∞-category of *animated R-algebras*, denoted by AR_R as the ∞-category $\mathrm{Fun}_\pi(\mathrm{Poly}_R^{\mathrm{op}}, \mathbb{S})$. For an animated ring A, we define $\mathrm{AR}_A := (\mathrm{AR}_{\mathbb{Z}})_{A/}$ as the ∞-category of *animated A-algebras*.

Further, if $R = \mathbb{Z}$, we call an animated R-algebra an *animated ring*.

Remark 3.2. Note that for a ring R, we have $\mathrm{AR}_R \simeq (\mathrm{AR}_{\mathbb{Z}})_{R/}$ by [Lur18, Prop. 25.1.4.2].

Remark 3.3. Since AR_A is the over category of $\mathrm{AR}_{\mathbb{Z}}$ which is the ∞-category of a combinatorial model category (which is explained in the beginning), we see with [Lur09, Prop. 5.5.3.11, Prop. A.3.7.6] that AR_A is a presentable ∞-category.

The following theorem allows us to connect AR_R with $E_\infty\text{-}\mathrm{Alg}_R^{\mathrm{cn}}$. The idea is simple, as AR_R is generated by Poly_R under sifted colimits, any sifted colimit preserving functor is up to homotopy determined by its restriction to Poly_R. Since Poly_R lies fully in $E_\infty\text{-}\mathrm{Alg}_R^{\mathrm{cn}}$ (which has sifted colimits), we therefore get a functor $\theta\colon \mathrm{AR}_R \to E_\infty\text{-}\mathrm{Alg}_R^{\mathrm{cn}}$ corresponding to the inclusion $\mathrm{Poly}_R \hookrightarrow E_\infty\text{-}\mathrm{Alg}_R^{\mathrm{cn}}$. We can also use this philosophy to analyze functors by restricting them to Poly_R if they preserve sifted colimits.

Proposition 3.4. *Let $j\colon \mathrm{Poly}_R \to \mathrm{AR}_R$ denote the Yoneda-embedding. Then we have an equivalence of ∞-categories*

$$\mathrm{Fun}_{\mathrm{sift}}(\mathrm{AR}_R, E_\infty\text{-}\mathrm{Alg}_R^{\mathrm{cn}}) \to \mathrm{Fun}(\mathrm{Poly}_R, E_\infty\text{-}\mathrm{Alg}_R^{\mathrm{cn}}),$$

where the subscript sift *denotes the full subcategory of sifted colimit preserving functors.*

Proof. This follows from [Lur09, Prop. 5.5.8.15, Cor. 5.5.8.17] and note that AR_R has small colimits since it is presentable by Remark 3.3. ☐

Proposition 3.5. *The functor $\theta\colon \mathrm{AR}_R \to E_\infty\text{-}\mathrm{Alg}_R^{\mathrm{cn}}$ described above is conservative, has a left adjoint θ^L and a right adjoint θ^R.*

Proof. This is [Lur18, 25.1.2.2]. ☐

Remark 3.6. Let us take a closer look at the left adjoint of θ. We know that per definition $\mathbb{Z}[X]$ is a compact and projective object of $\mathrm{AR}_{\mathbb{Z}}$, so in particular the functor $\mathrm{Hom}_{\mathrm{AR}_{\mathbb{Z}}}(\mathbb{Z}[X], -)$ commutes with sifted colimits (see [Lur09, Prop. 5.5.8.25]) and since we can write any animated ring as a sifted colimit of polynomial rings, we see that for any animated ring A, we have $\mathrm{Hom}_{\mathrm{AR}_{\mathbb{Z}}}(\mathbb{Z}[X], A) \simeq \Omega^\infty\theta(A)$. We observe that for the free E_∞-$\mathbb{Z}$-algebra in one variable $\mathbb{Z}\{X\}$ the same equivalence holds, i.e. $\mathrm{Hom}_{E_\infty\text{-}\mathrm{Alg}_{\mathbb{Z}}^{\mathrm{cn}}}(\mathbb{Z}\{X\}, \theta(A)) \simeq \Omega^\infty\theta(A)$. Using the adjunction, we therefore see that $\theta^L(\mathbb{Z}\{X\}) \simeq \mathbb{Z}[X]$.

The functor θ allows us to view any $A \in \mathrm{AR}_R$ as a ring object in Sp. Thus we can associate fundamental groups to this object and also module objects in Sp.

Definition 3.7. Let $A \in \mathrm{AR}_R$. For any $i \in \mathbb{Z}$, we set $\pi_i(A) \coloneqq \pi_i(\theta(A))$ and we set $\mathrm{Mod}_A \coloneqq \mathrm{Mod}_{\theta(A)}$. We refer to elements of Mod_A as *A-modules*.

Recall from Section 2, that animated rings per definition have no negative homotopy groups and $\pi_* A$ is a graded ring.

Notation 3.8. We want to remark that we have the notion of truncation functors for animated rings (see [Lur18, §25.1.3]), denoted by $\tau_{\leq n}$ for $n \in \mathbb{N}_0$ and are induced by the truncations on the underlying E_∞-rings. For an animated ring A we denote $\tau_{\leq n} A$ with $A_{\leq n}$.

We denote with $(\mathrm{AR}_R)_{\leq n}$ the full subcategory of n-truncated animated R-algebras. The elements of $(\mathrm{AR}_R)_{\leq 0} \simeq (R\text{-Alg})$ are called *discrete*.

Remark 3.9. The inclusion of n-truncated animated R-algebras $(\mathrm{AR}_R)_{\leq n}$ into AR_R, for some $n \in \mathbb{N}_0$, has a left adjoint denoted by $\tau_{\leq n}$ (see [Lur18, Rem. 25.1.3.4]). Since per definition $\tau_{\leq 0} = \pi_0$, we see that passage to the underlying ring of an animated ring via π_0 preserves colimits.

We can view any connective E_∞-algebra over R as a connective R-module (more precisely $E_\infty\text{-}\mathrm{Alg}_R^{\mathrm{cn}} \simeq \mathrm{CAlg}(\mathrm{Mod}_R^{\mathrm{cn}})$). This induces a forgetful functor $E_\infty\text{-}\mathrm{Alg}_R^{\mathrm{cn}} \to \mathrm{Mod}_R^{\mathrm{cn}}$, which has a left adjoint (see [Lur17, Ex. 3.1.3.14]). Using the above left adjoint θ^L, we can associate an animated ring to any connective R-module M.

Definition 3.10. Let A be an animated ring. Let $M \in \mathrm{Mod}_A^{\mathrm{cn}}$ be a connective A-module. We denote the image of M under the left adjoint to the forgetful functor $E_\infty\text{-}\mathrm{Alg}_{\theta(A)}^{\mathrm{cn}} \to \mathrm{Mod}_A^{\mathrm{cn}}$ composed with θ^L by $\mathrm{Sym}_A(M)$ and call it the *symmetric animated A-algebra of M.*[14]

Further, in the following remark we want to explain that there are two possible ways to define homotopy groups on animated rings, rather naturally. But both notions are in fact equivalent (in the sense that the two notions produce isomorphic homotopy groups).

Remark 3.11. The homotopy groups of an animated ring can be defined alternatively via the following. We have a natural functor from rings to abelian groups and then to sets by forgetting the ring structure. This induces a functor from

$$F\colon \mathrm{AR}_{\mathbb{Z}} \to \mathrm{Ani}((\mathrm{Ab})) \to \mathrm{Ani}((\mathrm{Sets}))$$

(see [CS21, §5.1.4]). The animation of abelian groups is the ∞-category of simplicial abelian groups and the animation of sets is $\mathbb{S}$. Using this functor, we can also define the n-homotopy group of an animated ring A via $\pi_n F(A) \in \mathbb{S}$. This construction of the homotopy groups agrees with the construction of the homotopy groups via passage to spectra.

[14]By [Lur09, Prop. 5.2.5.1] the adjunction of θ and θ^L can be transfered to the adjunction of slice categories, i.e. θ induces an adjunction, which by abuse of notation we denote the same, between AR_A and $E_\infty\text{-}\mathrm{Alg}_{\theta(A)}^{\mathrm{cn}}$.

The reason for this is the following commutative diagram

$$\begin{tikzcd}
\mathrm{AR}_{\mathbb{Z}} \arrow[r, "\theta"] \arrow[d] \arrow[dd, bend right, "F"'] & E_\infty\text{-}\mathrm{Alg}^{\mathrm{cn}} \arrow[d] \\
\mathrm{Ani}((\mathrm{Ab})) \arrow[r] \arrow[d] & \mathcal{D}(\mathbb{Z})^{\mathrm{cn}} \arrow[dl, "\Omega^\infty"] \\
\mathbb{S} &
\end{tikzcd}$$

(as all of these functors commute with sifted colimits[15], we only need to check commutativity on polynomial $\mathbb{Z}$-algebras, which follows by construction).

We want to conclude this section by explaining localizations of animated rings.

Definition and Remark 3.12. Let $\mathcal{C}$ be a presentable ∞-category and let S be a set of morphisms in $\mathcal{C}$. Then, we say that an object $Z \in \mathcal{C}$ is *S-local* if for any morphism $f\colon X \to Y \in S$, we have that the morphism $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ induced by f is an equivalence. We say that a morphism $f\colon X \to Y$ in $\mathcal{C}$ is an *S-equivalence* if for any S-local object the morphism $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ induced by f is an equivalence (see [Lur09, Def. 5.5.4.1], note that this definition does not need presentability of $\mathcal{C}$ but as explained below presentability allows us to work with the full subcategory of S-local objects in a nice way).

The inclusion of the full subcategory $\mathcal{C}[S^{-1}]$ of S-local objects in $\mathcal{C}$ admits a left adjoint, which we call localization $\mathcal{C} \to \mathcal{C}[S^{-1}]$ (see [Lur09, Prop. 5.5.4.15]). The idea is to "complete" S by taking $\bar{S}$ as the set of S-equivalences in $\mathcal{C}$. Then $\mathcal{C}[S^{-1}]$ is the localization of $\mathcal{C}$ by $\bar{S}$, which is strongly saturated (see [Lur09, §5.5.4] for more details).

This is analogous to the classical localization, where even if we want to localize at one element of a ring, we have to automatically localize the multiplicative subset generated by the element.

Proposition and Definition 3.13 (Localization)**.** *Let A be an animated R-algebra and let $F \subseteq \pi_0(A) = \pi_0(\mathrm{Hom}_{\mathrm{Mod}_A}(A, A))$ be a subset. Then there exists $A[F^{-1}] \in \mathrm{AR}_A$, such that for all $B \in \mathrm{AR}_A$ the simplicial set $\mathrm{Hom}_{\mathrm{AR}_A}(A[F^{-1}], B)$ is nonempty if and only if the image of all $f \in F$ under $\pi_0(A) \to \pi_0(B)$ is invertible. Further if it is nonempty, then it is contractible.*

Proof. The proof is analogous to the proof of [TV08, Prop. 1.2.9.1], which treats the special case where F has only one element.

Let $\mathrm{Sym}\colon \mathrm{Mod}_A^{\mathrm{cn}} \to \mathrm{AR}_A$ be the left adjoint to the map $\mathrm{AR}_A \to E_\infty\text{-}\mathrm{Alg}_A^{\mathrm{cn}} \to \mathrm{Mod}_A^{\mathrm{cn}}$. Consider the set $S \coloneqq \{\mathrm{Sym}(f)\colon \mathrm{Sym}(A) \to \mathrm{Sym}(A) \mid f \in F\}$. We set $A[F^{-1}]$ as the image of A under the localization map $\mathrm{AR}_A \to \mathrm{AR}_A[S^{-1}]$, where $\mathrm{AR}_A[S^{-1}]$ denotes the full subcategory of AR_A of S-local objects (note that AR_A is presentable by Remark 3.3).

[15] For F and θ this follows from construction. That the forgetful functor from connective E_∞-rings to modules commutes with sifted colimits follows from the fact, that the tensor product on spectra commutes with sifted colimits (see [Lur17, 4.8.2.19, Cor. 3.2.3.2]), for Ω^∞ see [Lur17, Prop. 1.4.3.9].

An object $B \in \mathrm{AR}_A$ is S-local if and only if the induced map

$$f^*\colon \mathrm{Hom}_{\mathrm{Mod}_A}(A, B) \to \mathrm{Hom}_{\mathrm{Mod}_A}(A, B)$$

is an equivalence for all $f \in F$. Equivalently, f^* is an equivalence if and only if the multiplication by the image of f on $\pi_i B$ is an equivalence for all $f \in F$. Therefore, any A-algebra B is S-local if and only if the image of f under $\pi_0 A \to \pi_0 B$ is invertible for all $f \in F$.

Now assume that $\mathrm{Hom}_{\mathrm{AR}_A}(A[F^{-1}], B)$ is nonempty, then the morphism $\pi_0 A \to \pi_0 B$ factors through $\pi_0 A[F^{-1}]$, so every $f \in F$ has invertible image in $\pi_0 B$, since by definition $A[F^{-1}]$ is S-local. To see that if $\mathrm{Hom}_{\mathrm{AR}_A}(A[F^{-1}], B)$ is nonempty, then it is contractible, note that in this case B is S-local and we have

$$\mathrm{Hom}_{\mathrm{AR}_A}(A[F^{-1}], B) \simeq \mathrm{Hom}_{\mathrm{AR}_A}(A, B) \simeq *$$

by adjunction. $\square$

Remark 3.14. Note that in the proof of Proposition 3.13 if we have a subset $F \subseteq \pi_0 A$ and denote its generated multiplicative subset by S, then by [Lur09, Prop. 5.5.4.15] an animated A-algebra B is $\{\mathrm{Sym}(f)\colon \mathrm{Sym}(A) \to \mathrm{Sym}(A) \mid f \in F\}$-local if and only if it is $\{\mathrm{Sym}(s)\colon \mathrm{Sym}(A) \to \mathrm{Sym}(A) \mid s \in S\}$-local (note that AR_A is presentable by Remark 3.3).

Notation 3.15. Let A be an animated ring and $f \in \pi_0 A$. Then we define the localization by an element as $A[f^{-1}] := A[\{f\}^{-1}]$.

Remark 3.16. Let F be a subset of $\pi_0 A$ and let S be the multiplicative subset generated by F. By the universal property of the localization of rings, we know that $\pi_0 A[F^{-1}] \cong S^{-1}\pi_0 A$.

Now assume that F is given by a single element $f \in \pi_0 A$. After the characterization of étale morphisms via the cotangent complex, we will see that the $\pi_i A[f^{-1}] \cong (\pi_i A)_f$ for all $i \geq 0$ (see Lemma 4.65)

Definition 3.17. Let $A \to B$ be a morphism of animated rings. Then B is *locally of finite presentation* over A if it is compact as an animated A-algebra, i.e. the functor $\mathrm{Hom}_{\mathrm{AR}_A}(B, -)\colon \mathrm{AR}_A \to \mathbb{S}$ commutes with filtered colimits.

Remark 3.18. Our notion of "locally finite presentation" is *stronger* than the notion of "finitely presented" in the classical sense. What we mean is that if a morphism of animated rings $A \to B$ is locally of finite presentation, then the induced morphism of rings $\pi_0 A \to \pi_0 B$ is finitely presented. But the other way around is not true, as we will see that $A \to B$ is locally of finite presentation if and only if $\pi_0 A \to \pi_0 B$ is locally of finite presentation and its cotangent complex is perfect (see Proposition 3.57). An example of a finitely presented morphism with non-perfect cotangent complex is the non-lci morphism $\mathbb{F}_p \to \mathbb{F}_p[X, Y]/(X^2, XY, Y^2)$ (the non-perfectness follows from [Avr99, (1.3)]).

As open immersions are finitely presented in the classical world of algebraic geometry, we would expect a similar result in the derived world. This will get explicit later but first we would like to show that the fundamental example of an open immersion, the localization of a ring along an element, is locally of finite presentation.

Lemma 3.19. *Let A be an animated ring and $f \in \pi_0 A$. Then $A[f^{-1}]$ is locally of finite presentation over A.*

Proof. We have that A-algebra morphisms from the localization to any other A-algebra B are either empty or contractible, depending whether f is invertible in $\pi_0 B$. For any filtered system $(B_i)_{i\in I}$ of A-algebras, we have $\pi_0 \operatorname{colim}_{i\in I} B_i = \operatorname{colim}_{i\in I} \pi_0 B_i$. Therefore, we see that $\operatorname{Hom}_{\mathrm{AR}_R}(A[f^{-1}], \operatorname{colim}_{i\in I} B_i)$ is empty if f is not invertible in $\operatorname{colim}_{i\in I} \pi_0 B_i$ and if f is invertible in $\operatorname{colim}_{i\in I} \pi_0 B_i$, then $\operatorname{Hom}_{\mathrm{AR}_R}(A[f^{-1}], \operatorname{colim}_{i\in I} B_i)$ is contractible. Since $\pi_0 A[f^{-1}] \cong \pi_0(A)_f$ is locally of finite presentation as a $\pi_0 A$-algebra, we know that f is invertible in $\operatorname{colim} \pi_0 B_i$ if and only there is an $i' \in I$ such f is invertible in $\pi_0 B_{i'}$. So in particular, if there is such i', we have

$$\operatorname{Hom}_{\mathrm{AR}_R}(A[f^{-1}], \operatorname*{colim}_{i\in I} B_i) \simeq * \simeq \operatorname*{colim}_{i\in I} \operatorname{Hom}_{\mathrm{AR}_R}(A[f^{-1}], B_i)$$

and if there is no such i', we have

$$\operatorname{Hom}_{\mathrm{AR}_R}(A[f^{-1}], \operatorname*{colim}_{i\in I} B_i) = \emptyset = \operatorname*{colim}_{i\in I} \operatorname{Hom}_{\mathrm{AR}_R}(A[f^{-1}], B_i).$$

$\square$

3.2 Modules over animated rings

Let us recall some useful notions about modules over animated ring. In the following A will be an animated ring.

Remark 3.20. Before we start, let us remark that under the symmetric monoidal equivalence of stable ∞-categories $\mathrm{Mod}_R \simeq \mathcal{D}(R)$ explained in Section 2, for a ring R, the homotopy groups are isomorphic to the corresponding homology groups, this isomorphism respects the module structure (see [SS03, B.1]).

Remark 3.21. We want to make clear that throughout, we will work in *homological* notation. This is natural from the homotopy theory standpoint but differs from the algebraic geometry standpoint which uses *cohomological* notation. In particular, we will define notions such as "Tor-amplitude" homologically.

Definition and Remark 3.22. Let P be a connective A-module, then P is called *projective* if for all connective A-modules Q, we have $\operatorname{Ext}^1(P,Q) \cong 0$, where $\operatorname{Ext}^1(P,Q)$ is defined as $\pi_0 \operatorname{Hom}_{\mathrm{Mod}_A}(P, Q[1]) \cong \operatorname{Hom}_{h\,\mathrm{Mod}_A}(P, Q[1])$.[16]

Equivalently, P is projective if for all fiber sequences

$$M' \to M \to M'',$$

[16] Recall that the homotopy category of a stable ∞-categories is an additive category, so the expression $\operatorname{Ext}^1(P,Q) \cong 0$ makes sense.

where M, M', M'' are connective A-modules, the induced map $\mathrm{Ext}^0(P, M) \to \mathrm{Ext}^0(P, M'')$ is surjective (see [Lur17, Prop. 7.2.2.6]. This also shows equivalence with the definition given in [Lur17, Def. 7.2.2.4]).

We denote by $\mathrm{Proj}(A)$ the full subcategory of projective A-modules in Mod_A.

Remark 3.23. From the definition of projective modules it follows that if an A-module P is projective, then for any connective module Q we have $\mathrm{Ext}^i(P, Q) \cong 0$ for all $i \geq 0$. In fact, this condition is equivalent to the same condition with Q assumed to be discrete. This is also an equivalent definition of projective modules (see [Lur17, Prop. 7.2.2.6]).

Definition 3.24. A connective A-module M is called *flat*, if $\pi_0 M$ is a flat $\pi_0 A$-module and the natural morphism $\pi_i A \otimes_{\pi_0 A} \pi_0 M \to \pi_i M$ is an isomorphism.

The compatibility with the higher homotopy groups is important if we want to define for example *flat morphisms* of animated rings. The Tor-spectral sequence below then shows us that the homotopy groups are compatible with base change.

The following is a direct consequence of the definition of flatness.

Lemma 3.25. *Let P be an A-module.*

1. *If P is projective it is flat.*

2. *If P is flat, then it is projective if and only if $\pi_0 P$ is projective over $\pi_0 A$.*

Proof. See [Lur17, Lem. 7.2.2.14] and [Lur17, Prop. 7.2.2.18]. $\square$

We also have a homotopy equivalence relating projective modules over A and over $\pi_0 A$.

Proposition 3.26. *The base change with the natural map $A \to \pi_0 A$ induces an equivalence between $h\,\mathrm{Proj}(A)$ and the $h\,\mathrm{Proj}(\pi_0 A)$*[17].

Proof. This follows from [Lur17, Cor. 7.2.2.19]. $\square$

Definition 3.27. We call an A-module P *finite projective*, if it is projective and $\pi_0 P$ is finitely presented over $\pi_0 A$.

We can generalize this notion via the notion of perfectness.

Definition and Remark 3.28. An A-module P is called *perfect*, if it is a compact object of Mod_A. Equivalently, P is perfect if and only if there exists an A-module $P^\vee$ such that we have $\mathrm{Hom}_{\mathrm{Mod}_A}(P, -) \simeq \Omega^\infty(P^\vee \otimes_A -)$ (see [Lur17, Def. 7.2.4.1] and [Lur17, Prop. 7.2.4.2]).

Remark 3.29. If A is discrete, then we have $\mathrm{Mod}_A \simeq \mathcal{D}(\pi_0 A)$ as symmetric monoidal ∞-categories and a complex of A-modules is perfect in the our sense if and only if it is perfect in the classical sense (see [Sta19, 07LT]).

[17]Note, that since projective modules are flat, $h\,\mathrm{Proj}(\pi_0 A)$ is just the usual category of (classical) projective $\pi_0 A$-modules.

Remark 3.30 ([Lur17, Prop. 7.2.1.19])**.** Let us insert a quick remark about the Tor-spectral sequence associated to spectral modules. Let A be an E_∞-ring and $M, N \in \mathrm{Mod}_A$. Then $\pi_* M$ and $\pi_* N$ are graded $\pi_* A$-modules and we have a spectral sequence in graded abelian groups of the following form called the *Tor-spectral sequence*

$$E_2^{p,q} = \mathrm{Tor}_p^{\pi_* A}(\pi_* M, \pi_* N)_q \Rightarrow \pi_{p+q}(M \otimes_A N).$$

Here convergence is in the sense that $\pi_{p+q}(M \otimes_A N)$ has a filtration $F^\bullet$ such that $\mathrm{gr}_F^p \cong E_\infty^{p,q}$, there is a $k \leq 0$ such that $F^n \simeq 0$ for $n \leq k$ and $\mathrm{colim}_n F^n \simeq \pi_{p+q}(M \otimes_A N)$ (see [Lur17, Var. 7.2.1.15] for the construction of the Tor-group).

We also have the notion of a Tor-amplitude for A-modules. We will use the homological notation, since it is in line with the definitions given in homotopy theory.

Definition 3.31 ([AG14, Def. 2.11])**.** Let M be an A-module. Then we say that M has *Tor-amplitude (concentrated) in* $[a, b]$ for $a \leq b \in \mathbb{Z}$ if for all discrete A-modules N, we have

$$\pi_i(M \otimes_A N) = 0$$

for all $i \notin [a, b]$.

Lemma 3.32. *Let M be an A-module. Then M has Tor-amplitude in $[a, b]$ if and only if the ordinary complex $M \otimes_A \pi_0 A$ in $\mathcal{D}(\pi_0 A)$ has Tor-amplitude in $[a, b]$.*

Proof. Let $F\colon \mathrm{Mod}_{\pi_0 A} \to \mathrm{Mod}_A$ be the forgetful functor. This functor comes from the Cartesian fibration of [Lur09, Cor. 3.4.3.4]. In particular, we see that for a discrete $\pi_0 A$-module N the underlying spectra of N and $F(N)$ are equivalent. So, their homotopy groups are isomorphic, so $F(N)$ is discrete and up to equivalence determined by $\pi_0 F(N)$ (see [Lur17, Prop. 7.1.1.13]). Since $\pi_0 N$ determines N up to equivalence and $\pi_0 N \cong \pi_0 F(N)$, we see that the diagram

$$\begin{tikzcd} & N(\pi_0 A\text{-Mod}) \arrow[dl, hook', "i_1"'] \arrow[dr, hook, "i_2"] & \\ \mathrm{Mod}_{\pi_0 A} \arrow[rr, "F"] & & \mathrm{Mod}_A \end{tikzcd}$$

commutes on the level of elements up to equivalence. Therefore, we see that for any $N \in N(\pi_0 A\text{-Mod})$, we have

$$M \otimes_A i_2(N) \simeq M \otimes_A F(i_1(N)) \simeq M \otimes_A \pi_0 A \otimes_{\pi_0 A} i_1(N)$$

concluding the proof. $\square$

Lemma 3.33. *Let A be an animated R-algebra. Let P and Q be A-modules.*

1. *If P is perfect, then P has finite Tor-amplitude.*

2. *If B is an A-algebra and P has Tor-amplitude in $[a, b]$, then the B-module $P \otimes_A B$ has Tor-amplitude in $[a, b]$.*

3. *If P has Tor-amplitude in $[a,b]$ and Q has Tor-amplitude in $[c,d]$, then $P \otimes_A Q$ has Tor-amplitude in $[a+c, b+d]$.*

4. *If P, Q have Tor-amplitude in $[a,b]$, then for any morphism $f \colon P \to Q$ the fiber of f has Tor-amplitude in $[a-1,b]$ and the cofiber of f has Tor-amplitude in $[a, b+1]$.*

5. *If P is a perfect A-module with Tor-amplitude in $[0,b]$, with $0 \leq b$, then P is connective and $\pi_0 P \simeq \pi_0(P \otimes_A \pi_0 A)$.*

6. *P is perfect and has Tor-amplitude in $[a,a]$ if and only if P is equivalent to $M[a]$ for some finite projective A-module.*

7. *If P is perfect and has Tor-amplitude in $[a,b]$, then there exists a morphisms*

$$M[a] \to P$$

such that M is a finite projective A-module and the cofiber is perfect with Tor-amplitude in $[a+1, b]$.

Proof. Since modules over animated rings are defined as modules over their underlying E_∞-ring spectrum, this is [AG14, Prop. 2.13]. □

Let us conclude this section by specifically looking at connective modules over animated rings. These will be given by the animation of classical modules. This illustrates why we work with modules over spectra, as the animation of modules seems natural but produced only *connective* objects.

First let us consider the ∞-category $\mathrm{Mod}(\mathrm{Sp})$ of tuples (M, A), where A is an E_∞-R-algebra and M is an A-module. This ∞-category comes naturally with a cartesian fibration $\mathrm{Mod}(\mathrm{Sp}) \to E_\infty\text{-}\mathrm{Alg}_R$ (see [Lur17, §4.5] for more details).

Now we can define the ∞-category $\mathrm{AR}\text{-}\mathrm{Mod}_R \coloneqq \mathrm{Mod}(\mathrm{Sp}) \times_{\mathrm{CAlg}_R(\mathrm{Sp})} \mathrm{AR}_R$. Let us denote the full subcategory, consisting of objects $(M, A) \in \mathrm{AR}\text{-}\mathrm{Mod}_R$, where M is connective by $\mathrm{AR}\text{-}\mathrm{Mod}_R^{\mathrm{cn}}$. The next proposition shows that $\mathrm{AR}\text{-}\mathrm{Mod}_R^{\mathrm{cn}}$ is the animation of the category of tuples (A, M), where A is an R-algebra and M is an A-module.

Proposition 3.34. *Let $\mathcal{C} \subseteq \mathrm{AR}\text{-}\mathrm{Mod}_R^{\mathrm{cn}}$ be the full subcategory consisting of objects (M, A), where A is a polynomial R-algebra and M is a free A-module of finite rank. Let $\mathcal{E}$ be an ∞-category, which admits sifted colimits. Let us denote by $\mathrm{Fun}_{\mathrm{sift}}(\mathrm{AR}\text{-}\mathrm{Mod}_R^{\mathrm{cn}}, \mathcal{E})$ the full subcategory of $\mathrm{Fun}(\mathrm{AR}\text{-}\mathrm{Mod}_R^{\mathrm{cn}}, \mathcal{E})$ spanned by those functors, which preserve sifted colimits. Then the restriction functor*

$$\mathrm{Fun}_{\mathrm{sift}}(\mathrm{AR}\text{-}\mathrm{Mod}_R^{\mathrm{cn}}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

is an equivalence of ∞-categories.

Proof. This is [Lur18, Cor. 25.2.1.3], but since some references are broken, we recall the proof.

It suffices to show that $P_\Sigma(\mathcal{C}) \simeq \text{AR-Mod}_R^{\text{cn}}$, since then the proposition follows from [Lur09, Prop. 5.5.8.15], where $P_\Sigma(\mathcal{C})$ denotes those presheaves that preserve finite products.

The following is [Lur18, Prop. 25.2.1.2] (here the references are broken). Note that $\mathcal{C}$ consists of those objects in $\text{AR-Mod}_R^{\text{cn}}$ which are coproducts of finitely many copies of $C := (R[X], 0)$ and $D = (R, R)$. We especially see that C and D corepresent the functors $\text{AR-Mod}_R^{\text{cn}} \to \mathbb{S}$ given by

$$(A, M) \mapsto \Omega^\infty A^{\text{sp}}, \quad (A, M) \mapsto \Omega^\infty M$$

respectively. Since both functors preserve sifted colimits, the objects C and D are compact, projective and $\mathcal{C}$ consists of compact projective objects of $\text{AR-Mod}_R^{\text{cn}}$. It follows with [Lur09, Prop. 5.5.8.22], that the inclusion $\mathcal{C} \hookrightarrow \text{AR-Mod}_R^{\text{cn}}$ extends (see [Lur09, Prop. 5.5.8.15]) to a fully faithful functor $F\colon P_\Sigma(\mathcal{C}) \to \text{AR-Mod}_R^{\text{cn}}$, which commutes with sifted colimits. Since the inclusion preserves finite coproducts, we see that F preserves small colimits (see [Lur09, Prop. 5.5.8.15]) and therefore admits a right adjoint G, by the adjoint functor theorem (see [Lur09, Cor. 5.5.2.9]). To prove that F is an equivalence, it suffices to show that G is conservative.

To see this, note that since F is left adjoint and fully faithful, the unit map $\text{id} \to GF$ is an equivalence.

That G is conservative is clear, since the conservative functor

$$\text{AR-Mod}_R^{\text{cn}} \to \mathbb{S} \times \mathbb{S}, \quad (A, M) \mapsto (\Omega^\infty A^{\text{sp}}, \Omega^\infty M)$$

factors through G. $\square$

Notation 3.35. Again, for an animated ring A, we denote $\text{AR-Mod}_A := \text{AR-Mod} \times_{\text{Ani}} \text{AR}_A$ and $\text{AR-Mod}_A^{\text{cn}} := \text{AR-Mod}^{\text{cn}} \times_{\text{Ani}} \text{AR}_A$.

Remark 3.36. The above proposition also shows that the ∞-category of simplicial commutative R-modules, which is equivalent to $\text{Ani}(R\text{-Mod})$ (the animation of R-modules), is equivalent to the connective R-modules Mod_R^{cn}.

Lastly, we will insert a lemma which will show the uniqueness of the cotangent complex for derived stacks (see Definition 4.58).

Lemma 3.37. *Let $j\colon \text{Mod}_A \to \mathcal{P}(\text{Mod}_A^{\text{cn,op}})$ be the Yoneda embedding followed by restriction. Then for all $n \geq 0$ the restriction of j to the $(-n)$-connective objects is fully faithful.*

Proof. We will not prove this here and refer to [TV08, Prop. 1.2.11.3]. $\square$

3.3 The cotangent complex

We will define square zero extensions and the cotangent complex following [Lur18, §25].

Proposition 3.34 allows us to define square zero extensions. Namely, if we look at the functor $\mathcal{C} \to (\text{Ring}) \simeq (\text{AR}_R)_{\leq 0} \hookrightarrow \text{AR}_R$, where $\mathcal{C}$ is as in Proposition 3.34, given by $(M, A) \mapsto A \oplus M$, we see that it induces a functor $\text{AR-Mod}_R^{\text{cn}} \to \text{AR}_R$ commuting with sifted colimits.

Definition 3.38. For an $A \in \mathrm{AR}_R$ and a connective A-module M, we define the *square zero extension of A by M* as the image of (M, A) under the functor $\mathrm{AR}\text{-}\mathrm{Mod}_R^{\mathrm{cn}} \to \mathrm{AR}_R$ described above and denote the resulting animated R-algebra by $A \oplus M$.

Remark 3.39. Since the forgetful functor from $\mathrm{AR}_R \to \mathrm{Mod}_R$ preserves colimits, we see that the underlying module of $A \oplus M$, for some animated R-algebra A and a connective A-module M, is equivalent to the direct sum in Mod_R of A and M.

Remark 3.40. Let $A \in \mathrm{AR}_R$ and M be a connective A-module. In Mod_A we have (up to homotopy) unique maps $0 \to M \to 0$, these determine maps between animated rings $A \to A \oplus M \to A$. Thus, we can view $A \oplus M$ as an element of $(\mathrm{AR}_A)_{/A}$.

Since we have defined square zero extensions of an animated algebra by a connective module, we can now define the notion of a derivation.

Definition 3.41. Let $A \in \mathrm{AR}_R$ and $M \in \mathrm{Mod}_A^{\mathrm{cn}}$. The *space of R-linear derivations* $\mathrm{Der}_R(A, M)$ *of A into M* is defined as the mapping space $\mathrm{Hom}_{(\mathrm{AR}_R)_{/A}}(A, A \oplus M)$.

Definition 3.42. Let $A \in \mathrm{AR}_R$, $M \in \mathrm{Mod}_A^{\mathrm{cn}}$ and $d \in \mathrm{Der}_R(A, M)$. Then we define $A \oplus_d M$ as the pullback of $d\colon A \to A \oplus M$ and the trivial derivation s, i.e. we have a pullback diagram of the form

$$\begin{tikzcd} A \oplus_d M \arrow[r] \arrow[d] & A \arrow[d, "d"] \\ A \arrow[r, "s"] & A \oplus M. \end{tikzcd}$$

Next, we want to define the absolute cotangent complex associated to an aniamted ring A. This should be thought of as an ∞-analogue of the module of differentials. So, we will characterize it by a universal property.

Proposition and Definition 3.43. *Let $A \in \mathrm{AR}_R$. There is a connective A-module L_A and a derivation $\eta \in \mathrm{Der}_R(A, L_A)$ uniquely (up to equivalence) characterized by the property, that for every connective A-module M the map*

$$\mathrm{Hom}_{\mathrm{Mod}_A}(L_A, M) \to \mathrm{Der}_R(A, M)$$

induced by η is an equivalence.

We call the A-module $L_{A/R}$ the cotangent complex of A over R.

If $R = \mathbb{Z}$, then we write L_A and call it the absolute cotangent complex of A.

Proof. This is [Lur18, Prop. 25.3.1.5]. □

Remark 3.44. Let $A \to B$ be a morphism in AR_R. Then for any connective B-module M, we will see that we have a map $\mathrm{Der}_R(B, M) \to \mathrm{Der}_R(A, M)$, where we see M as an A-module via the forgetful functor. This follows from the following.

Note that by functoriality, we have a commutative diagram of the form

$$\begin{tikzcd} A \oplus M \arrow[r] \arrow[d] & A \arrow[d] \\ B \oplus M \arrow[r] & B. \end{tikzcd}$$

This induces a map $A \oplus M \to A \times_B (B \oplus M)$, which is an equivalence when passing to the underlying R-modules, since the underlying R-module of $B \oplus M$ is the direct sum of B and M. Therefore for an R-derivation $d\colon B \to B \oplus M$, we get the following diagram in AR_R with pullback squares

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \oplus M \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & B \oplus M \arrow[r] & B, \end{tikzcd}$$

where the composition of the horizontal arrows is the identity on A, respectively B. Therefore, an R-derivation of B induces an R-derivation of A.

Thus, per definition, we get a map

$$\mathrm{Hom}_B(L_{B/R}, M) \to \mathrm{Hom}_A(L_{A/R}, M) \simeq \mathrm{Hom}_B(L_{A/R} \otimes_A B, M).$$

Where the second map is induced by the adjunction of the forgetful functor and the tensor product. Now Lemma 3.37 induces a map $L_{A/R} \otimes_A B \to L_{B/A}$.

Definition 3.45. Let $A \to B$ be a morphism in AR_R. Then we define *the (relative) cotangent complex of B over A*, denoted by $L_{B/A}$, as the cofiber of the induced map $B \otimes_A L_{A/R} \to L_{B/R}$.

Remark 3.46. Let $A \to B$ be a morphism in AR_R. For every connective B-module M the definition of $L_{B/A}$ as the cofiber of $L_{A/R} \otimes_A B \to L_{B/R}$ induces an equivalence

$$\mathrm{Hom}_{\mathrm{Mod}_B}(L_{B/A}, M) \simeq \mathrm{fib}_{d_0}(\mathrm{Der}_R(B, M) \to \mathrm{Der}_R(A, M)),$$

where d_0 is the trivial R-derivation of A into M. Therefore, seeing $B \oplus M$ as an animated A-algebra, via the trivial derivation and the natural morphism $A \oplus M \to B \oplus M$, we have an equivalence

$$\mathrm{Hom}_{\mathrm{Mod}_B}(L_{B/A}, M) \xrightarrow{\sim} \mathrm{fib}_{\mathrm{id}_B}(\mathrm{Hom}_{\mathrm{AR}_A}(B, B \oplus M) \to \mathrm{Hom}_{\mathrm{AR}_A}(B, B)).$$

So the relative cotangent complex represents morphisms $B \to B \oplus M$, with an augmentation $B \oplus M \to B$, which are not only R-linear but in fact also A-linear, i.e. $\mathrm{Hom}_{\mathrm{Mod}_B}(L_{B/A}, M) \simeq \mathrm{Hom}_{(\mathrm{AR}_A)_{/B}}(B, B \oplus M)$.

Remark 3.47. We claim that the definition of the cotangent complex as the module representing derivations shows that for any pushout diagram of the form

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

in AR_R, we have $L_{B/A} \simeq L_{B'/A'} \otimes_{A'} B$.

Indeed, let M be a B-module. Using the arguments of Remark 3.44, we get a morphism

$$\mathrm{Hom}_{(\mathrm{AR}_A)_{/B}}(B, B \oplus M) \to \mathrm{Hom}_{(\mathrm{AR}_{A'})_{/B'}}(B', B' \oplus M).$$

Let $d \in \mathrm{Hom}_{(\mathrm{AR}_{A'})_{/B'}}(B', B' \oplus M)$, which is given by a diagram

$$B' \to B' \oplus M \to B'$$

such that the composition is homotopic to the identity on B'. By the universal property of the pushout, this induces an A-derivation of B into $B \oplus M$. Both constructions are inverse to each other using universal properties, so the morphism

$$\mathrm{Hom}_{(\mathrm{AR}_A)_{/B}}(B, B \oplus M) \to \mathrm{Hom}_{(\mathrm{AR}_{A'})_{/B'}}(B', B' \oplus M)$$

is an equivalence and using Remark 3.46, we are done.

Remark 3.48. We defined the cotangent complex of an animated R-algebra as the ∞-analogue of the Kähler-differentials. The same construction can be done for E_∞-algebras (see [Lur17, 7.3]). Thus, we could ask the question whether for a map of animated R-algebras $A \to B$ the relative cotangent complex $L_{B/A}$ is equivalent to the relative cotangent complex of the underlying map of E_∞-algebras $L^{\infty}_{B/A} := L_{\theta(B)/\theta(A)}$. In general, the answer is no, take for example $\mathbb{Z} \to \mathbb{Z}[X]$ (since this morphism in not formally smooth if we allow non-connective E_∞-rings, this follows from [TV08, Prop. 2.4.1.5]).

But we have an induced morphism $L^{\infty}_{A/B} \to L_{B/A}$ and passing to the homotopy groups, we see that the map $\pi_i L^{\infty}_{A/B} \to \pi_i L_{B/A}$ is an isomorphism if $i \leq 1$ and surjective for $i = 2$ (see [Lur18, 25.3.5.1], note that Lurie calls our cotangent complex *the algebraic cotangent complex*).

In characteristic 0 however we have that the map on homotopy groups is an isomorphism for all $i \in \mathbb{Z}$ (see [Lur18, 25.3.5.3]). This is not surprising, since in characteristic 0 we have an equivalence of animated rings and connective E_∞-rings.

Remark 3.49. Let R be a ring and A be an animated R-algebra. Let $B = \mathrm{Sym}_A(M)$ for some connective A-module M. We claim that $L_{B/A}$ is given by $M \otimes_A B$.

Indeed, for any connective B-module M', we have

$$\begin{aligned}&\mathrm{Hom}_{(\mathrm{AR}_A)_{/B}}(\mathrm{Sym}_A(M), A \oplus M') \\ &\qquad \simeq \mathrm{fib}_{\mathrm{id}_B}(\mathrm{Hom}_{\mathrm{AR}_A}(\mathrm{Sym}_A(M), B \oplus M') \to \mathrm{Hom}_{\mathrm{AR}_A}(\mathrm{Sym}_A(M), B)).\end{aligned}$$

Using the adjunction of the forget functor and Sym_A, we get a map $\iota\colon M \to B$ corresponding to the identity on B. Further, we have

$$\begin{aligned}&\mathrm{fib}_{\mathrm{id}_B}(\mathrm{Hom}_{\mathrm{AR}_A}(\mathrm{Sym}_A(M), B \oplus M') \to \mathrm{Hom}_{\mathrm{AR}_A}(\mathrm{Sym}_A(M), B)) \\ &\qquad \simeq \mathrm{fib}_{\iota}(\mathrm{Hom}_{\mathrm{Mod}_A}(M, B \oplus M') \to \mathrm{Hom}_{\mathrm{Mod}_A}(M, B)).\end{aligned}$$

Now the underlying module of $B \oplus M$ is the direct sum of B with M in Mod_A. Therefore, this fiber is equivalent to $\mathrm{Hom}_{\mathrm{Mod}_A}(M, M') \simeq \mathrm{Hom}_{\mathrm{Mod}_B}(M \otimes_A B, M')$. Hence, by the universal property the cotangent complex $L_{B/A}$ is given by $M \otimes_A B$.

Proposition 3.50. *Let A be an animated R-algebra and write A as the sifted colimit of polynomial R-algebras $P^\bullet$. Then we have $L_{A/R} \simeq \operatorname{colim} L_{P^\bullet/R} \otimes_{P^\bullet} A$.*

Proof. This is [Kha18, Lec. 5 Thm. 2.3]. For the convenience of the reader, we recall the proof.

Note that each P^i is equivalent to $\mathrm{Sym}(R^{n_i})$ for some $n_i \in \mathbb{N}$. Also by Remark 3.49, we have $\operatorname{colim} L_{P^\bullet/R} \otimes_{P^\bullet} A \simeq \operatorname{colim} R^\bullet \otimes_R P^\bullet \otimes_{P^\bullet} A \simeq \operatorname{colim} A^\bullet$, where $A^\bullet := R^\bullet \otimes_R A$. Thus, we have to show that

$$\lim \mathrm{Hom}_{\mathrm{Mod}_A}(A^\bullet, M) \simeq \lim \mathrm{fib}_{\iota_\bullet}(\mathrm{Hom}_{\mathrm{AR}_R}(P^\bullet, A \oplus M) \to \mathrm{Hom}_{\mathrm{AR}_R}(P^\bullet, A))$$

for all connective A-modules M, where $\iota_\bullet \colon P^\bullet \to A$ are the natural maps. But now we will see that both sides can be identified with $\lim \Omega^\infty(M)^\bullet$, so we conclude the equivalence.

Indeed, each A^{n_i} is free this is clear for the left hand side and for the right hand side note that $P^\bullet$ is given by symmetric algebras of free algebras and conclude using the above. □

Remark 3.51. Note that the proof of Proposition 3.50 shows that for a discrete ring R, the cotangent complex $L_{B/R}$ for some $B \in \mathrm{AR}_R$ agrees with the left Kan extension of $\Omega^1_{-/R} \colon \mathrm{Poly}_R \to \mathcal{D}(R)$ along the inclusion $\mathrm{Poly}_R \hookrightarrow \mathrm{AR}_R$. So in particular, our definition agrees with the classical definition in the discrete case[18].

Setting $R = \mathbb{Z}$, we get an analogous description of L_B.

We add a little lemma showing that the steps in the Postnikov towers are square-zero extensions. For E_∞-algebras this can be found in [Lur17, Cor. 7.4.1.28]. We will not prove this lemma, since it would be to involved. But a detailed (model categorical) proof can be found in the notes of Porta and Vezzosi [PV15].

Lemma 3.52. *Let A be an animated R-algebra. There exists a unique derivation $d \in \pi_0 \mathrm{Der}(A_{\leq n-1}, \pi_n(A)[n+1])$ such that the projection $A_{\leq n-1} \oplus_d \pi_n(A)[n+1] \to A_{\leq n-1}$ is equivalent to the natural morphism $A_{\leq n} \to A_{\leq n-1}$ (recall the notation from Definition 3.42).*

Proof. [TV08, Lem. 2.2.1.1]. □

3.4 Smooth and étale morphisms

In this section, we will follow [TV08]. In the reference Toën and Vezzosi deal with animated rings and derived algebraic geometry (in our sense) in the model categorical setting. Most definitions however are made in such a way, such that we can easily translate them to the ∞-categorical setting (for more explanation on how to go from model categories to ∞-categories, we recommend [Lur09] and [Lur17]).

Definition 3.53. A morphism $f \colon A \to B$ of animated R-algebras is called *flat (resp. faithfully flat, smooth, étale)* if the following two conditions are satisfied

[18]Let $L\Omega^1_{-/R}$ denote the left Kan extension of $\Omega^1_{-/R} \colon \mathrm{Poly}_R \to \mathcal{D}(R)$ along the inclusion $\mathrm{Poly}_R \hookrightarrow \mathrm{AR}_R$. Then $L\Omega^1_{-/R}$ factors through $\mathrm{AR}\text{-}\mathrm{Mod}^{\mathrm{cn}}_R$, since $\Omega^1_{A/R}$ has an A-module structure for some polynomial R-algebra A. Thus, we see that $L\Omega^1_{B/R}$ has a natural B-module structure and on polynomial algebras agrees with $L_{B/R}$ (note that this a priori proves the equivalence in $\mathcal{D}(R)$ and thus in Sp but the forget functor $\mathrm{AR}\text{-}\mathrm{Mod}^{\mathrm{cn}}_R \to \mathrm{Sp}$ is conservative by [Lur17, Cor. 4.2.3.2]).

(i) the induced ring homomorphism $\pi_0 f\colon \pi_0 A \to \pi_0 B$ is flat (resp. faithfully flat, smooth, étale), and

(ii) we have an isomorphism $\pi_* A \otimes_{\pi_0 A} \pi_0 B \to \pi_* B$ of graded rings.

Note that in the above definition $\pi_0 f$ is a morphism of commutative rings, so asking whether they are flat, étale or smooth is natural. The condition (ii) on the homotopy groups is a natural compatibility condition that assures that homotopy groups and base change commute in the sense that $\pi_n M \otimes_{\pi_0 A} \pi_0 B \cong \pi_n(M \otimes_A B)$ for an A-module M and a flat animated A-algebra B. (see [Lur17, Prop. 7.2.2.13]).

Let $f\colon A \to B$ be a homomorphism of rings. If f is smooth, we know that the module of differentials $\Omega_{B/A}$ is finite projective. The other direction is in general not correct, i.e. there are ring homomorphisms locally of finite presentation with finite projective module of differentials that are not smooth. One example is the projection $k[X] \to k[X]/(X) \cong k$ for a field k. Its module of differentials $\Omega_{k/k[X]}$ vanishes, so is in particular a finite dimensional k-vector space. But the projection is not smooth. One way to see this, is that the cotangent complex $L_{k/k[X]}$ is quasi-isomorphic to $(X)/(X^2)[-1]$[19] (see [Sta19, 07BU]). This condition on the cotangent complex comes rather naturally in derived algebraic geometry, as we can see the cotangent complex as the derived version of the module of differentials. In fact, we will make this explicit in Proposition 3.56 to see that a morphism of animated rings is smooth if and only if it is locally of finite presentation and its cotangent complex is finite projective.

But for technical reasons, we need to understand the cotangent complex of the natural maps $A_{\leq k} \to A_{\leq k-1}$. As these are given by square zero extensions by $\pi_k A$ (see Lemma 3.52) and isomorphisms on π_i for $i \leq k-1$, we will see that the cotangent complex is easier to understand.

Lemma 3.54. *Let A be an animated R-algebra and $k \geq 1$. Then there exists natural isomorphisms*

$$\pi_{k+1} L_{A_{\leq k-1}/A_{\leq k}} \cong \pi_k A,$$

and $\pi_i L_{A_{\leq k-1}/A_{\leq k}} \cong 0$ for $i \leq k$ (recall Notation 3.8 for the $A_{\leq *}$).

Proof. This is [TV08, Lem. 2.2.2.8] translated to ∞-categories.

We can also deduce this lemma using [Lur17] and [Lur18]. Note that the fiber of $A_{\leq k} \to A_{\leq k-1}$ as an A-module is given by $\pi_k A[k]$ and thus the cofiber is given by $\pi_k A[k+1]$. Now the first assertion follows from [Lur18, Rem. 25.3.6.5]. For the vanishing of the lower homotopy groups, note that for $i \leq k+2$, we have $\pi_i L^{\infty}_{A_{\leq k-1}/A_{\leq k}} \cong \pi_i L_{A_{\leq k-1}/A_{\leq k}}$ by [Lur18, Prop. 25.3.5.1], where $L^{\infty}_{A_{\leq k-1}/A_{\leq k}}$ is the cotangent complex associated to the underlying E_∞-algebra of $A_{\leq k-1} \to A_{\leq k}$ (see [Lur17, §7.3] for more details). Thus, the vanishing follows with [Lur17, Lem. 7.4.3.17]. $\square$

In the following proof, we want to compute

$$\mathrm{Tor}_*^{\pi_* A}(\pi_0 A, \pi_* M)$$

[19] As the projection $k[X] \to k$ is a regular immersion, we can use [Sta19, 08SJ].

for some $A \in \mathrm{AR}_R$ and an A-module M. The Tor-spectral sequence will be an important tool when calculating tensor products. But first lets talk about graded free resolutions, which compute the Tor-groups.

Remark 3.55 (Graded free resolutions)**.** Let $A \in \mathrm{AR}_R$ and fix an A-module M such that there is an $n \geq 0$ such that $\pi_k M = 0$ for all $k < n$. To compute the graded Tor group, we need a graded free resolution of $\pi_* M$, where a graded free resolution is an exact sequence $\cdots \to P_1 \to P_0 \to \pi_* M \to 0$ of graded $\pi_* A$-modules with P_i equal to the direct sum of shifts of $\pi_* A$ (see [Sta19, 09KK] for details on graded free resolutions). We can follow the existence of such a sequence inductively with [Sta19, 09KN]. Namely, set $M_0 \coloneqq \pi_* M$, then for any i, we can find a short exact sequence of the form $0 \to M_{i+1} \to P_i \to M_i \to 0$, such that P_i is the direct sum of shifts of $\pi_* A$ and is concentrated in degrees $\geq n$.

The P_i are constructed as follows, for any $m \in (M_i)_k$, we can find a $\pi_* A$-linear map $\pi_*(A)[k] \to M_i$ sending $1 \in \pi_*(A)[k]_k$ to m and $d(1)$ to $d(m)$. The module P_i is now defined as the direct sum over all non-zero degrees and corresponding elements. Therefore by construction P_i is a direct sum of shifts of $\pi_* A$ and further by induction, we see that if M_0 is concentrated in degrees $\geq n$, then for all $i \geq 0$ the module P_i is concentrated in degrees $\geq n$.

Proposition 3.56. *Let $f \colon A \to B$ be a morphism in AR_R.*

1. *The morphism f is smooth if and only if the B-module $L_{B/A}$ is finite projective and $\pi_0 B$ is of finite presentation over $\pi_0 A$.*

2. *The morphism f is étale if and only if $L_{B/A} \simeq 0$ and $\pi_0 B$ is of finite presentation over $\pi_0 A$.*

Proof. This is [TV08, Thm. 2.2.2.6] in the ∞-categorical setting. For the convenience of the reader, we recall the proof of the first assertion. The proof of the second assertion is left out and can be reconstructed following the proof of [TV08, Thm. 2.2.2.6].

Let f be a smooth morphism, then it is flat by assumption, thus $\pi_0 B = \pi_0 A \otimes_A B$ (by [Lur17, Prop. 7.2.2.15] $B \otimes_A \pi_0 A$ has to be discrete and since on π_0 it is an equivalence, we have the equivalence on the level of animated rings). In particular, we have

$$\pi_0 L_{B/A} = \pi_0(L_{B/A} \otimes_B \pi_0 B) = \pi_0 L_{\pi_0 B/\pi_0 A} = \Omega_{\pi_0 B/\pi_0 A}[0],$$

by compatibility of the cotangent complex with base change (see Remark 3.47). Since f is smooth, we see that $\pi_0(L_{B/A}) = \Omega_{\pi_0 B/\pi_0 A}[0]$ is finite projective over $\pi_0 B$. By Proposition 3.26, there is a projective B-module P with $P \otimes_B \pi_0 B \simeq \pi_0 L_{B/A}$ (thus P is in fact finite projective). Using the projectivity of P, we lift the natural projection $P \to \pi_0 L_{B/A}$ to a morphism $\phi \colon P \to L_{B/A}$ (see [Lur17, Prop. 7.2.2.6] and note that surjectivity of ϕ on π_0 implies that the fiber is connective). We want to show that ϕ is in fact an equivalence. For this it is enough to show $\mathrm{cofib}(\phi) \simeq 0$. By construction, it is clear that $\pi_0 \, \mathrm{cofib}(\phi) = 0$ and we will show by induction on n that $\pi_n \, \mathrm{cofib}(\phi) = 0$.

To see this, let $n > 0$, assume $\pi_k \, \mathrm{cofib}(\phi) = 0$ for $k < n$ and consider the following Tor spectral sequence (see Remark 3.30)

$$E_2^{p,q} = \mathrm{Tor}_p^{\pi_* B}(\pi_0 B, \pi_* \, \mathrm{cofib}(\phi))_q \Rightarrow \pi_{p+q}(\pi_0 B \otimes_B \mathrm{cofib}(\phi)) = 0.$$

To see that $\pi_0 B \otimes_B \operatorname{cofib}(\phi) \simeq 0$, note that $\phi \otimes \mathrm{id}_{\pi_0 B}$ is equivalent to the identity on $L_{\pi_0 B/\pi_0 A}$ and thus the cofiber of $\phi \otimes \mathrm{id}_{\pi_0 B}$ which is $\pi_0 B \otimes_B \operatorname{cofib}(\phi)$ vanishes. Let $P_\bullet$ be the graded free resolution of $\pi_* \operatorname{cofib}(\phi)$ constructed in Remark 3.55. Then $E_2^{p,q} = H^{-p}(\pi_0 B \otimes_{\pi_* B} P_\bullet)_q = 0$ for $q < n$, since it is a subgroup of a quotient of $(\pi_0 B \otimes_{\pi_* B} P_p)_q$, which itself is a quotient of $(P_p)_q$. Therefore, $\pi_n \operatorname{cofib}(\phi) \simeq H^0(\pi_0 B \otimes_{\pi_* B} P_\bullet)_n = E_2^{0,n} \simeq \pi_n(\pi_0 B \otimes_B \operatorname{cofib}(\phi)) \simeq 0$.

To see the first equivalence note that $H^0(\pi_0 B \otimes_{\pi_* B} P_\bullet) \simeq \pi_0 B \otimes_{\pi_* B} H^0(P_\bullet) \simeq \pi_0 B \otimes_{\pi_* B} \pi_* \operatorname{cofib}(\phi) \simeq \pi_* \operatorname{cofib}(\phi)$ (see [Sta19, 09LL] for the definition of the tensor product of dg-modules and note that taking the 0th-cohomology is just taking a cokernel which commutes with tensor products).

Now assume that $L_{B/A}$ is finite projective and $\pi_0 A \to \pi_0 B$ is of finite presentation. Consider the pushout square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \pi_0 A & \longrightarrow & C. \end{array}$$

We want to show that the natural morphism $C \to \pi_0 C \simeq \pi_0 B$ is an equivalence. For this assume there is a smallest integer $i > 0$ such that $\pi_i C \neq 0$. We get a fiber sequence

$$L_{C_{\leq i}/\pi_0 A} \otimes_{C_{\leq i}} \pi_0 C \to L_{\pi_0 C/\pi_0 A} \to L_{\pi_0 C/C_{\leq i}},$$

using Lemma 3.54 (actually its proof), we see that

$$\pi_i(L_{C/\pi_0 A} \otimes_C \pi_0 C) \simeq \pi_i(L_{C_{\leq i}/\pi_0 A} \otimes_{C_{\leq i}} \pi_0 C) \simeq \pi_{i+1}(L_{\pi_0 C/C_{\leq i}}) \simeq \pi_i(C).$$

But $L_{C/\pi_0 A} \otimes_C \pi_0 C$ is projective over $\pi_0 C$ and thus discrete which is a contradiction. Therefore, we see that $\pi_0 B \simeq \pi_0 A \otimes_A B$ and thus $L_{\pi_0 B/\pi_0 A} \simeq L_{B/A} \otimes_B \pi_0 B$ is discrete. Hence, with [Sta19, 07BU], we see that $\pi_0 A \to \pi_0 B$ is smooth.

It remains to show that the natural map $\phi\colon \pi_n A \otimes_{\pi_0 A} \pi_0 B \to \pi_n B$ is an equivalence. But this follows from $B \otimes_A \pi_0 A \simeq \pi_0 B$ and [Lur17, Thm. 7.2.2.15] (we have to test that $B \otimes_A M$ is discrete for all discrete A-modules M, but since $\pi_0 B$ is a flat $\pi_0 A$-module, we see that $B \otimes_A M \simeq B \otimes_A \pi_0 A \otimes_{\pi_0 A} M \simeq \pi_0 B \otimes_{\pi_0 A} M$ is discrete). $\square$

The following proposition is in a similar fashion to Proposition 3.56. Namely, we can characterize finitely presented morphisms by their cotangent complex and their behavior on the underlying discrete rings (i.e. on π_0).

Proposition 3.57. *Let $f\colon A \to B$ be a morphism in AR_R. Then f is locally of finite presentation if and only if the B-module $L_{B/A}$ is perfect and $\pi_0 B$ is of finite presentation over $\pi_0 A$*

Proof. We will not prove this and refer to [TV08, Prop. 2.2.2.4] or [Lur04, Prop. 3.2.18]. $\square$

Corollary 3.58. *Let $f\colon A \to B$ be a smooth morphism of animated R-algebras. Then f is locally of finite presentation.*

Proof. Combine Proposition 3.57 and 3.56. $\square$

One other important fact is that for an animated ring A, we can lift truncated étale maps $B \to \pi_0 A$ to étale maps $\widetilde{B} \to A$. The idea is to use Postnikov towers and the compatibility of cotangent complexes and truncations.

Proposition 3.59. *Let A be an animated R-algebra. Then the base change under the natural morphism $A \to \pi_0 A$ induces a equivalence ∞-categories of étale A-algebras and étale $\pi_0 A$-algebras.*

Proof. See [CS21, Prop. 5.2.3]. $\square$

Lastly, we can use Proposition 3.59 to show that any finite projective module P over an animated ring A is finite locally free.

Corollary 3.60. *Let A be an animated ring and let P be a finite projective module over A. Then there is a finite étale cover $(A \to A_i)_{i \in I}$, i.e. the A_i are étale A-algebras, where I is finite and $A \to \prod_{i \in I} A_i$ is faithfully flat, such that $P \otimes_A A_i$ is free of finite rank, i.e. $P \otimes_A A_i \simeq A_i^r$ for some $r \in \mathbb{N}$.*

Proof. Let $\mathrm{Proj}(A)$ denote the full subcategory of Mod_A of projective modules. We have an equivalence of categories $\mathrm{h}\,\mathrm{Proj}(A) \to \mathrm{h}\,\mathrm{Proj}(\pi_0 A) \simeq \mathrm{Proj}_{\pi_0 A}$ given by the tensor product (see [Lur17, Cor 7.2.2.19]). By definition, this restricts to an equivalence of finite projective modules. Since classical finite projective modules on $\pi_0 A$ are finite locally free, we know that there exists an open cover $(\widetilde{A}_i)$ of $\pi_0 A$ such that $\widetilde{A}_i \otimes_{\pi_0 A} \pi_0 A \otimes_A P$ is equivalent to some finite free $\widetilde{A}_i$-module. By Proposition 3.59, we can lift this open cover to an étale cover A_i of A (certainly $A \to \prod_{i \in I} A_i$ is étale and faithfully flatness can be checked on π_0). Since $A_i \otimes_A \pi_0 A = \widetilde{A}_i$, the equivalence of the categories involved shows the claim. $\square$

4 Derived algebraic geometry

For this section, we will closely follow [TV08, §2], [AG14] and the lecture notes of Adeel Khan [Kha18].

In [TV08] Toën and Vezzosi deal with derived algebraic geometry in the model categorical setting. This allows us to translate them easily to the ∞-categorical setting.

In [AG14] Antieu and Gepner deal with spectral algebraic geometry (in the ∞-categorical setting). The ideas are more or less the same as we use them and in some parts, we can transport proofs one-to-one. But it is important to note that there is no higher principle which concludes derived algebraic geometry (theory of certain sheaves on animated algebras) as a corollary of spectral algebraic geometry (theory of certain sheaves on E_∞-algebras). This is because there is no fully faithful embedding from animated algebras to E_∞-algebras in general. In characteristic 0 both ∞-categories are equivalent and thus the results from [TV08] and [AG14] agree. In characteristic $p > 0$ however, we have no such relation. Therefore the translation of [AG14] to our setting has to be treated with caution.

4.1 Affine derived schemes

In the following R will be a ring and A an animated R-algebra.

Let us define the étale and fpqc topology.

Proposition and Definition 4.1. *Let B an animated A-algebra.*

(a) There exists a Grothendieck topology on $\mathrm{AR}_A^{\mathrm{op}}$, called the fpqc-topology*, which can be described as follows: A sieve (see [Lur09, Def. 6.2.2.1]) $\mathcal{C} \subseteq (\mathrm{AR}_A^{op})_{/B} \simeq \mathrm{AR}_B^{op}$ is a covering sieve if and only if it contains a finite family $(B \to B_i)_{i\in I}$ for which the induced map $B \to \prod_{i\in I} B_i$ is faithfully flat.*

(b) There exists a Grothendieck topology on the full subcategory $(\mathrm{AR}_A^{\text{ét}})^{op}$ of étale A-algebras, called the étale-topology*, which can be described as follows: A sieve $\mathcal{C} \subseteq (\mathrm{AR}_A^{\text{ét}})^{\mathrm{op}}_{/B} \simeq (\mathrm{AR}_B^{\text{ét}})^{\mathrm{op}}$ is a covering sieve if and only if it contains a finite family $(B \to B_i)_{i\in I}$ for which the induced map $A \to \prod_{i\in I} B_i$ is faithfully flat (and étale, which is automatic).*

Proof. Let S be the collection of all faithfully flat (resp. faithfully flat and étale) morphisms in AR_R. It is enough to check that S satisfies the properties of [Lur18, Prop. A.3.2.1]. For this, we note that a morphism of animated rings is faithfully flat (resp. faithfully flat and étale) if and only if it is after passage to connective E_∞-rings. Since the functor $\theta\colon \mathrm{AR}_A \to E_\infty\text{-}\mathrm{Alg}^{\mathrm{cn}}_{\theta(A)}$ is conservative and commutes with limits and colimits (see Proposition 3.5), we see that S satisfies the properties of [Lur18, Prop. A.3.2.1] if and only if the collection of all faithfully flat (resp. faithfully flat and étale) morphisms in $E_\infty\text{-}\mathrm{Alg}^{\mathrm{cn}}_{\theta(A)}$ does so. But this follows from [Lur18, Prop. B.6.1.3] - see also [Lur18, Var. B.6.1.7] - (resp. [Lur18, Prop. B.6.2.1]). ☐

Definition and Remark 4.2. An *affine derived scheme over A* is a functor from AR_A to spaces (i.e. a presheaf on $\mathrm{AR}_A^{\mathrm{op}}$), which is equivalent to $\mathrm{Spec}(B) \coloneqq \mathrm{Hom}_{\mathrm{AR}_A}(B, -)$ for some $B \in \mathrm{AR}_A$.

Note that $\mathrm{Spec}(B)$ is an fpqc sheaf by [Lur18, D. 6.3.5] and Lemma 3.5

Definition 4.3. Let $\mathbf{P}$ be one of the following properties of a morphism animated rings: *flat, smooth, étale, locally of finite presentation.* We say that a morphisms of affine derived schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(C)$ has property $\mathbf{P}$ if the underlying homomorphism $C \to B$ has $\mathbf{P}$.

Remark 4.4. Let us remark that the above properties of morphisms of affine derived schemes are stable under equivalences, composition, pullbacks and are étale local on the source and target. This follows from classical theory (as found for example in [Sta19]) and Propositions 3.56 and 3.57, noting Proposition 3.26 and that (perfect) modules modules satisfy descent (see Remark 4.53).

Definition 4.5. For a discrete ring A, we set

$$\mathrm{Spec}(A)_{\mathrm{cl}} \coloneqq \mathrm{Hom}_{(\mathrm{Ring})}(A, -) \colon (\mathrm{Ring}) \to (\mathrm{Sets})$$

to be its underlying classical scheme. We will abuse notation and denote the underlying locally ringed space of $\mathrm{Spec}(A)_{\mathrm{cl}}$ the same.

Remark 4.6. The notation $(-)_{\mathrm{cl}}$ is introduced since even for a discrete ring A the corresponding derived stack $\mathrm{Spec}(A)$ is a sheaf with values in spaces. Thus, for a (possibly non discrete) animated ring B the space $\mathrm{Hom}_{\mathrm{Ani}}(A, B)$ need not to be discrete, e.g. $\mathrm{Hom}_{\mathrm{Ani}}(\mathbb{Z}[X], B) \simeq \Omega^\infty B$. But for example if we restrict ourself to discrete rings C, we have $\mathrm{Hom}_{\mathrm{Ani}}(A, C) \simeq \mathrm{Hom}_{(\mathrm{Ring})}(\pi_0 A, C)$ by adjunction, even when A is not discrete.

4.2 Geometric stacks

In this section we closely follow [AG14] and [TV08]. In [TV08] the notion of geometric stacks can be found in the context of model categories. Our notion agrees with the notions presented in [TV08] (note that they speak of n-representable morphisms rather than n-geometric) and we want to remark that the definition of geometric stacks in [AG14] is different from ours. The main point are the 0-geometric stacks. In our case, any scheme will be 1-geometric, whereas in [AG14] a qcqs scheme with non-affine diagonal will not be 1-geometric. Nevertheless, as the principal of the definition is analogous, the ideas presented in [AG14] often times agree with the ideas presented in [TV08].

Definition 4.7. Let A be an animated ring. A *derived stack over A* is a sheaf of spaces on $(\mathrm{AR}_A^{\text{ét}})^{\mathrm{op}}$. We denote the ∞-category of derived stacks over A by dSt_A. If $A = \mathbb{Z}$, we simply say derived stack and denote the ∞-category of derived stacks by dSt.

Remark 4.8. Let us remark, that this definition makes sense if we do not assume that AR_A for any animated ring A is small. The reason for this is that a priori the full subcategory of derived stacks in $\mathcal{P}((\mathrm{AR}_A^{\text{ét}})^{\mathrm{op}})$ is defined as the full subcategory of S-local objects, where S consists of monomorphisms $U \hookrightarrow \mathrm{Spec}(A)$ that come from a covering sieve in $(\mathrm{AR}_A^{\text{ét}})^{\mathrm{op}}$. But if we assume smallness of AR_A one sees that dSt is a topological localization of $\mathcal{P}((\mathrm{AR}_A^{\text{ét}})^{\mathrm{op}})$ (see [Lur09, Prop. 6.2.2.7]) which can come quite handy for example in Remark 4.50, where we use [Lur09, Prop. 5.5.4.2].

Definition 4.9 ([Lur09, §6.2.3]). Let A be an animated ring. A morphism $f\colon X \to Y$ in dSt_A is an *effective epimorphism* if the natural map $\operatorname{colim}_\Delta \check{C}(X/Y)_\bullet \to Y$ in $\mathcal{P}((\mathrm{AR}_A^{\text{ét}})^{\mathrm{op}})$ is an equivalence in dSt_A after sheafifcation[20].

Remark 4.10 (Effective epimorphisms). Let $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be a morphism of affine derived schemes. By [Lur09, Prop. 7.2.1.14] the map f is an effective epimorphism in dSt if and only if its 0-truncation, in the sense of [Lur09, §5.5.6], is an effective epimorphism. Up to homotopy $\tau_{\leq 0}\operatorname{Spec}(A)$ is given by the sheafification of $\pi_0 \operatorname{Hom}_{\mathrm{Ani}}(A, -)$ (see [Lur09, Prop. 5.5.6.28] together with Remark 4.8). Thus f is an effective epimorphism if and only if the sheafification of the induced map $\pi_0 f\colon \pi_0 \operatorname{Hom}(B, -) \to \pi_0 \operatorname{Hom}(A, -)$ is an epimorphism (note that a priori [Lur09, Prop. 7.2.1.14] tells us that $\pi_0 f^\sim$ needs to be an effective epimorphism but in a 1-topos every epimorphism is effective (see [MLM94, IV 7. Thm. 8])).

We will see in Remark 4.35 that the restriction of an affine derived scheme $\operatorname{Spec}(A)$ onto (Ring) preserves limits and colimits. In particular, any effective epimorphism of affine derived schemes $\operatorname{Spec}(A) \to \operatorname{Spec}(B)$ induces by adjunction an epimorphism of étale sheaves of sets $\operatorname{Hom}_{(\mathrm{Ring})}(\pi_0 A, -) \to \operatorname{Hom}_{(\mathrm{Ring})}(\pi_0 B, -)$, which in turn implies that the morphism of the underlying topological spaces of affine schemes, denoted by $|\operatorname{Spec}(\pi_0 A)_{\mathrm{cl}}| \to |\operatorname{Spec}(\pi_0 B)_{\mathrm{cl}}|$, is surjective. So for example if $B \to A$ is flat, then the effective surjectivity implies that it is in fact faithfully flat.

Definition 4.11 ([TV08, Def. 1.3.3.1]). We will define a geometric morphism inductively.

(1) A derived stack is (-1)-*geometric* or *affine* if it is equivalent to an affine derived scheme.

A morphism of derived stacks $X \to Y$ is (-1)-*geometric* or *affine* if for all affine schemes $\operatorname{Spec}(A)$ and all $\operatorname{Spec}(A) \to Y$ the base change $X \times_Y \operatorname{Spec}(A)$ is affine.

A (-1)-geometric morphism of derived stacks $X \to Y$ is *smooth* (resp. *étale*) if for all affine derived schemes $\operatorname{Spec}(A)$ and all morphisms $\operatorname{Spec}(A) \to Y$ the base change morphism $\operatorname{Spec}(B) \simeq X \times_Y \operatorname{Spec}(A) \to \operatorname{Spec}(A)$ corresponds to a smooth (resp. étale) morphism of animated rings.

Now let $n \geq 0$.

(2) An n-*atlas* of a derived stack X is a family $(\operatorname{Spec}(A_i) \to X)_{i \in I}$ of morphisms of derived stacks, such that

   (a) each $\operatorname{Spec}(A_i) \to X$ is $(n-1)$-geometric and smooth, and
   (b) the induced morphism $\coprod \operatorname{Spec}(A_i) \to X$ is an effective epimorphism.

If each of the morphisms $\operatorname{Spec}(A_i) \to X$ is étale, then we call the n-atlas *étale*.

A derived stack is called n-*geometric* (resp. n-*DM*), if

[20] We can describe dSt_A as a localization of $\mathcal{P}(\mathrm{AR}_A^{\mathrm{op}})$ (see [Lur09, §6.2.2]), so we get a functor $L\colon \mathcal{P}(\mathrm{AR}_A^{\mathrm{op}}) \to \mathrm{dSt}_A$ left adjoint to the inclusion, which we denote as sheafification.

(a) it has an n-atlas (resp. étale n-atlas), and

(b) the diagonal $X \xrightarrow{\Delta} X \times X$ is $(n-1)$-geometric.

(3) A morphism $X \to Y$ of derived stacks is called *n-geometric* (resp. *n-DM*) if for all affine derived scheme $\operatorname{Spec}(A)$ and all morphisms $\operatorname{Spec}(A) \to Y$ the base change $X \times_Y \operatorname{Spec}(A)$ is n-geometric (resp. n-DM).

An n-geometric morphism $X \to Y$ of derived stacks is called *smooth*, if for all affine derived scheme $\operatorname{Spec}(A)$ and all morphisms $\operatorname{Spec}(A) \to Y$ the base change $X \times_Y \operatorname{Spec}(A)$ has an n-atlas given by a family of affine derived schemes $(\operatorname{Spec}(A_i))_{i\in I}$, such that the induced morphism $A \to A_i$ is smooth.

An n-DM morphism $X \to Y$ of derived stacks is called *étale*, if for all affine derived scheme $\operatorname{Spec}(A)$ and all morphisms $\operatorname{Spec}(A) \to Y$ the base change $X \times_Y \operatorname{Spec}(A)$ has an étale n-atlas given by a family of affine derived schemes $(\operatorname{Spec}(A_i))_{i\in I}$, such that the induced morphism $A \to A_i$ is étale.

We call a morphism of derived stacks *geometric* (resp. *DM*) if it is n-geometric (resp. n-DM) for some $n \geq -1$.

Remark 4.12. From this definition one can see that an n-geometric morphism of derived stacks is automatically $(n+1)$-geometric.

Definition 4.13. Let $\mathbf{P}$ be a property of affine derived schemes that is stable under equivalences, pullbacks, compositions and is smooth-local on the source and target, then we say a morphism of derived stacks $X \to Y$ has $\mathbf{P}$ if it is geometric and for an affine $(n-1)$-atlas $(U_i)_{i\in I}$ of the pullback along an affine derived scheme $\operatorname{Spec}(B)$ the corresponding morphism $U_i \to \operatorname{Spec}(B)$ of affine schemes has $\mathbf{P}$.

Lemma 4.14. *The properties "locally of finite presentation", "flat" and "smooth" of morphisms of affine derived schemes satisfy the conditions of Definition 4.13.*

Proof. For flat and smooth this follows from the definition (note that on π_0 this follows from classical theory and since smooth covers are in particular flat and on π_0 faithfully flat, we see that the compatibility of the higher homotopy groups in the definition of smooth and flat morphisms is also clear).

For locally of finite presentation, we use Proposition 3.57 and the fact that on π_0 this follows from classical theory and that perfectness of modules can be checked fpqc-locally (we will see this in Remark 4.53). □

Our definition above differs from the theory developed in [TV08], where they assume the property to be étale-locally on the source and base. This would include the property *étale* but makes no sense for geometric stacks, since this would imply that a morphism of affine derived schemes is étale if and only if it is smooth locally étale but the next remark shows that this can not hold. We assume that there was a mixup in [TV08], since if one looks at later proofs where the condition étale is used one needs a stronger condition than geometric. This is not surprising, since this problem occurs even in the classical theory for Artin stacks. One can solve this for example by assuming that étale morphisms are always DM in the sense that after base change to an affine the resulting stack is actually a DM-stack, i.e. has an étale cover by schemes. We did this analogously but want to mention that this is only done for completion and is not used later on in any of the proofs.

Remark 4.15. We want to remark that the property *étale* is not smooth local on the base, since if it would be smooth local in our context, then it would be smooth local in classical theory of schemes which it is not.

Definition 4.16. A morphism of derived stacks is called *étale* if it is DM and étale.

Definition 4.17. A morphism of derived stacks $X \to Y$ is

1. an *open immersion* if it is a flat, locally of finitely presentation and a monomorphism, where flat is in the sense of Definition 4.13 and monomorphism means (-1)-truncated in the sense of [Lur09], i.e. the homotopy fibers of $X \to Y$ are either empty or contractible.

2. a *closed immersion* if it is affine and for any $\mathrm{Spec}(B) \to Y$ the corresponding morphism $X \times_Y \mathrm{Spec}(B) \simeq \mathrm{Spec}(C) \to \mathrm{Spec}(B)$ induces a surjection $\pi_0 B \to \pi_0 C$ of rings.

Definition 4.18. A morphism $f\colon X \to Y$ of derived stacks is a *locally closed immersion* if for all affine derived schemes $\mathrm{Spec}(A)$ and all morphisms $\mathrm{Spec}(A) \to Y$ the base change morphism $X \times_Y \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ factors as a closed immersion follows by an open immersion.

Remark 4.19. The definition of a closed immersion does not impose any monomorphism condition. This makes sense, since we will see that a monomorphism automatically has vanishing cotangent complex (see Lemma 4.64) and in particular, any closed immersion which is on t_0 of finite presentation and a monomorphism will be étale (this can proven analogously to [TV08, Cor. 2.2.5.6]). But as many naturally arising closed immersions are not étale, e.g. any regular immersion with non-vanishing cotangent complex, the above definition seems to be the one suited for the world of derived algebraic geometry.

Let us give an important example of an open immersion of derived stacks.

Lemma 4.20. *Let A be an animated R-algebra and let $f \in \pi_0 A$ be an element. The inclusion $j\colon \mathrm{Spec}(A[f^{-1}]) \hookrightarrow \mathrm{Spec}(A)$ is an open immersion.*

Proof. The proof is given in [Kha18, Lec. 3 Lem. 4.2]. But for the convenience of the reader, we recall the proof.

We have to check that j is a monomorphism, which is flat and locally of finite presentation. Locally of finite presentation follows from Lemma 3.19. Flatness, i.e. $\pi_0 A[f^{-1}] \otimes_{\pi_0 A} \pi_i A \simeq \pi_i A[f^{-1}]$ follows from $\pi_i(A[f^{-1}]) = \pi_i(A)[f^{-1}]$. To see that it is a monomorphism, we have to show that the homotopy fibers of $\mathrm{Hom}(A[f^{-1}], B) \to \mathrm{Hom}(A, B)$ for any $B \in \mathrm{AR}_R$ are either empty or contractible. But this follows from the general property of localization (see Lemma 3.13). □

Definition 4.21. A derived stack is called *separated* if the diagonal is a closed immersion.

Definition 4.22. A derived stack X is *quasi-compact* if there exists an n-atlas consisting of a single affine. A morphism $f\colon X \to Y$ of derived stacks is *quasi-compact* if for all affine derived schemes $\mathrm{Spec}(A)$ and all morphisms $\mathrm{Spec}(A) \to Y$ the base change $X \times_Y \mathrm{Spec}(A)$ is quasi-compact.

Remark 4.23. Since affine schemes are separated, we see that affine derived schemes are also separated (note that the diagonal of an affine scheme is representable and that we only have to check that the corresponding ring morphism on π_0 is surjective, which follows from classical theory).

Definition 4.24. A derived stack X is *locally geometric* if we can write X as the filtered colimit of geometric derived stacks X_i, with open immersions $X_i \hookrightarrow X$.

We say that locally geometric stack $X \simeq \mathrm{colim}_{i\in I} X_i$ is *locally of finite presentation* if each X_i is locally of finite presentation.

Definition and Proposition 4.25. *For a morphism of derived stacks $f\colon X \to Y$, we define $\mathrm{Im}(f)$ as an epi-mono factorisation $X \twoheadrightarrow \mathrm{Im}(f) \hookrightarrow Y$ of f (here "epi" means "effective epimorphism"). This factorisation is unique up to homotopy.*

Proof. The existence of such a factorisation follows from [Lur09, Ex. 5.2.8.16]. The uniqueness up to homotopy follows from [Lur09, 5.2.8.17] □

Remark 4.26. In the reference used in the above proof one shows that the image of a morphism $f\colon X \to Y$ is equivalent to the (-1)-truncation of f, which in turn is equivalent to the colimit of the Čech nerve of f (see [Lur09, Cor. 6.2.3.5] and note that per definition 0-connective morphisms are effective epimorphisms, see. [Lur09, Def. 6.5.1.10]).

The following lemmas are clear from the definitions and may seem unnecessary complicated but will enable us to give another definition of open immersion, which shows that we won't have to deal with geometricity of open immersions.

Lemma 4.27. *Let $\iota\colon U \hookrightarrow \mathrm{Spec}(A)$ be a monomorphism of derived stacks. Then U has an affine diagonal.*

Proof. Since ι is a monomorphism, we see that the diagonal of ι is an equivalence and hence we conclude. □

Lemma 4.28. *Let $(A_i)_{i\in I}$ be a family of animated B-algebras having the property $\boldsymbol{P}$, where $\boldsymbol{P}$ is as in Definition 4.13. Let U denote the image of the natural map $\coprod_{i\in I} \mathrm{Spec}(A_i) \to \mathrm{Spec}(B)$ and assume that the base change of $\mathrm{Spec}(A_i) \to U$ with any derived affine scheme and any morphism $\mathrm{Spec}(C) \to U$ is smooth (note that this makes sense by Lemma 4.27) and has property $\boldsymbol{P}$. Then the $\coprod_{i\in I} \mathrm{Spec}(A_i)$ is a 0-atlas for U, via the natural map and in particular $U \hookrightarrow \mathrm{Spec}(B)$ is 0-geometric and has property $\boldsymbol{P}$.*

Proof. This follows from the definitions. $\square$

Lemma 4.29. *Let $U \hookrightarrow X$ be a monomorphism of derived stacks and let $\boldsymbol{P}$ be a property as in Definition 4.13. Assume the base change with any* $\mathrm{Spec}(A) \to X$ *has a cover by a disjoint union of affine derived schemes over A such that the conditions of Lemma 4.28 are satisfied. Then $U \hookrightarrow X$ is* 0*-geometric and has property P.*

Proof. This follows from Lemma 4.28. $\square$

Lemma 4.30. *Let $\iota\colon U \hookrightarrow X$ be a geometric monomorphism of derived stacks. Then ι is* 0*-geometric.*

Proof. This follows from Lemma 4.29. $\square$

Remark 4.31. Lemma 4.30 implies for example that open immersions and locally closed immersions are automatically 0-geometric.

Lemma 4.32. *A morphism $U \to X$ of derived stacks is an open immersion if and only if for any* $\mathrm{Spec}(A) \to X$ *the base change is a monomorphism and there is an effective epimorphism* $\coprod_{i\in I} \mathrm{Spec}(A_i) \to \mathrm{Spec}(A) \times_X U$ *such that each* $\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$ *is an open immersion.*

Proof. This follows from Lemma 4.29. $\square$

We can also define derived versions of schemes with the notion of open immersions.

Definition 4.33. Let X be a derived stack. Then X is a *derived scheme* if it admits a cover $(\mathrm{Spec}(A_i) \hookrightarrow X)_{i\in I}$ such that each $\mathrm{Spec}(A_i) \hookrightarrow X$ is an open immersion (in particular X is 1-geometric).

Remark 4.34. If we have a morphism $\coprod_{i\in I} U_i \to X$ of derived stacks where each U_i is an open immersion, we sometimes write $\bigcup_{i\in I} U_i$ for its image.

If $X \to Y$ is a morphism of derived stacks, where the diagonal of Y is representable and X is a derived scheme, then the image of $X \to Y$ is a derived scheme.

If $\coprod_{i\in I} \mathrm{Spec}(A_i) \to X$ is a morphism of derived stacks, where X has representable diagonal and $\mathrm{Spec}(A_i)$ are affine open in X, then $\bigcup_{i\in I} \mathrm{Spec}(A_i)$ is an open substack of X.

Remark and Definition 4.35 (Truncation)**.** We give a quick summary of [TV08, §2.2.4].

Let $\mathbf{A}$ be the model category of simplicial commutative R-algebras, as explained in section 3.1. The inclusion $(R\text{-Alg}) \hookrightarrow \mathbf{A}$ has a left adjoint $\pi_0\colon \mathbf{A} \to (R\text{-Alg})$. This is a Quillen adjunction for the trivial model structure on $(R\text{-Alg})$. This induces an adjunction $\pi_0\colon \mathrm{AR}_R \rightleftarrows (R\text{-Alg})\colon i$. We therefore get adjunctions

$$i_!\colon \mathcal{P}((R\text{-Alg})^{\mathrm{op}}) \rightleftarrows \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})\colon i^*\colon \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}}) \rightleftarrows \mathcal{P}((R\text{-Alg})^{\mathrm{op}})\colon \pi_0^*$$

(here i^* (resp. π_0^*) is defined as the restriction of a presheaf and $\mathcal{P}(\mathcal{C})$ denotes the ∞-category of presheaves of spaces on $\mathcal{C}$ and $i_!$ is given by left Kan extension). The inclusion $(R\text{-Alg}) \hookrightarrow \mathbf{A}$ induces an equivalence to the discrete animated R-algebras and thus the restriction preserves i^* preserves sheaves and thus composing $i_!$ with the sheafification, we get the adjunction

$$i_!\colon \mathrm{Shv}_{\text{ét}}((R\text{-Alg})) \rightleftarrows \mathrm{Shv}_{\text{ét}}(\mathrm{AR}_R)\colon i^*\colon \mathrm{Shv}_{\text{ét}}(\mathrm{AR}_R) \rightleftarrows \mathrm{Shv}_{\text{ét}}((R\text{-Alg}))\colon \pi_0^*.$$

For convenience later on, we define $t_0 \coloneqq i^*$ and $\iota \coloneqq i_!$. Note, that by general theory of Kan extensions, the functor ι is indeed fully faithful (see [Lur09, §4.3.2]).

For a derived scheme $X \in \mathrm{Shv}(\mathrm{AR}_R)^{\text{ét}}$, we denote its image under t_0 with X_{cl} and call it the *underlying classical scheme*. Note that $t_0(\mathrm{Spec}(A)) \simeq \mathrm{Spec}(\pi_0 A)_{\mathrm{cl}}$ and $\iota(\mathrm{Spec}(\pi_0 A)_{\mathrm{cl}}) \simeq \mathrm{Spec}(\pi_0 A)$ (by adjunction and the fact that any morphism from an animated ring to a discrete ring is characterized by the corresponding morphisms on discrete rings). Also by adjunction being an effective epimorphism is preserved under t_0. So if $\coprod_{i\in I} \mathrm{Spec}(A_i)$ is a Zariski atlas of X, we see that $\coprod_{i\in I} \mathrm{Spec}(\pi_0 A_i)_{\mathrm{cl}}$ is a cover of X_{cl} and thus X_{cl} has values in discrete spaces, i.e. sets (note that étale locally any morphism $\mathrm{Spec}(B) \to X_{\mathrm{cl}}$ factors through $\coprod_{i\in I} \mathrm{Spec}(\pi_0 A_i)_{\mathrm{cl}}$, in particular the points of X_{cl} can be computed by the points of its atlas, which are discrete). Hence, X_{cl} recovers the classical notion of a scheme.

Let us state a few interesting properties of t_0 and i.

1. The functor t_0 has a right and left adjoint (see above),
2. the functor t_0 preserves geometricity (here geometricity of sheaves in $(R\text{-Alg})$ is defined similarly to derived stacks, see [TV08, §2.1.1] for further information) and the properties flat, smooth and étale along geometric morphisms,
3. the functor ι preserves geometricity, homotopy pullbacks of n-geometric stacks along flat morphisms and sends flat (resp. smooth, étale) morphisms of n-geometric stacks to flat (resp. smooth, étale) morphisms of n-geometric stacks,
4. if $X \in \mathrm{Shv}((R\text{-Alg}))^{\text{ét}}$ is n-geometric and $X' \to \iota(X)$ is a flat morphism, then X' is the image of an n-geometric stack under ι.

A proof for these statements is given in [TV08, Prop. 2.2.4.4].

We list some properties of geometric morphisms of derived stacks.

Lemma 4.36. *Let $X \to Z$ and $Y \to Z$ be morphisms of derived stacks. If $X \to Z$ is n-geometric, then so is $X \times_Z Y \to Y$.*

Proof. This follows immediately from the definition. □

Lemma 4.37. *A morphism of derived stacks $X \to Y$ is n-geometric if and only if the base change under $\mathrm{Spec}(A) \to Y$ for any $A \in \mathrm{AR}_R$ is n-geometric.*

Proof. This follows immediately from the definitions. □

Lemma 4.38. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of derived stacks. If f and g are n-geometric, then so is $g \circ f$.*

Proof. The proof is straightforward using induction on n (see [TV08, Prop. 1.3.3.3 (3)]). $\square$

Proposition 4.39. *Let $f\colon X \to Y$ be a morphism of derived stacks. Assume X is n-geometric and the diagonal $Y \to Y \times Y$ is n-geometric. Then f is n-geometric.*

Proof. This is analogous to [AG14, Lem. 4.30].

Let $\coprod_{i\in I} \mathrm{Spec}(A_i) \twoheadrightarrow X$ be an n-atlas. Consider a morphism $\mathrm{Spec}(A) \to Y$, where A is an animated ring. Then we have a morphism $\coprod_{i\in I} \mathrm{Spec}(A_i) \times_Y \mathrm{Spec}(A) \to \coprod_{i\in I} \mathrm{Spec}(A_i \otimes A)$, which is n-geometric, since it is the base change of the diagonal under $\coprod_{i\in I} \mathrm{Spec}(A_i \otimes A)$. Therefore $\coprod_{i\in I} \mathrm{Spec}(A_i) \times_Y \mathrm{Spec}(A)$ has an n-atlas, say given by $(\mathrm{Spec}(B_j) \to \coprod_{i\in I} \mathrm{Spec}(A_i) \times_Y \mathrm{Spec}(A))_{j\in J}$ and by Lemma 4.38, we see that $(\mathrm{Spec}(B_j) \to X \times_Y \mathrm{Spec}(A))_{j\in J}$ is an n-atlas. This finishes the proof, but to make things clear, we finally get following diagram with pullback squares

$$\begin{tikzcd}
\coprod_{j\in J} \mathrm{Spec}(B_j) \arrow[d, two heads, "n\text{-atlas}"'] & & \coprod \mathrm{Spec}(A_i \otimes A) & \\
\coprod_{i\in I} \mathrm{Spec}(A_i) \times_Y \mathrm{Spec}(A) \arrow[urr] \arrow[rr, two heads, "n\text{-geom.}"] \arrow[d] & & X \times_Y \mathrm{Spec}(A) \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[d] \\
\coprod A_i \arrow[rr, two heads, "n\text{-atlas}"] & & X \arrow[r] & Y.
\end{tikzcd}$$

$\square$

Corollary 4.40. *Let X and Y be n-geometric stacks. Then any morphism $X \to Y$ is n-geometric.*

Proof. This follows immediately from the definitions and Proposition 4.39 $\square$

Proposition 4.41. *Let $X \to Y$ be an effective epimorphism of derived stacks and suppose that X and $X \times_Y X$ are n-geometric. Further, assume that the projections $X \times_Y X \to X$ are n-geometric and smooth. Then Y is an $(n+1)$-geometric stack. If in addition X is quasi-compact and $X \to Y$ is a quasi-compact morphism, then Y is quasi-compact. Finally if X is locally of finite presentation, then so is Y.*

Proof. This is analogous to [AG14, Lem. 4.29].

Let $\coprod_{i\in I} \mathrm{Spec}(A_i) \twoheadrightarrow X$ be an n-atlas. Consider the following diagram with pullback squares

$$\begin{tikzcd}
\mathrm{Spec}(A_i) \times_Y \mathrm{Spec}(A_j) \arrow[r] \arrow[d] & X \times_Y \mathrm{Spec}(A_j) \arrow[r] \arrow[d] & \mathrm{Spec}(A_j) \arrow[d] \\
\mathrm{Spec}(A_i) \times_Y X \arrow[r] \arrow[d] & X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(A_i) \arrow[r] & X \arrow[r, "f"] & Y.
\end{tikzcd}$$

It suffices to show that $\coprod_{i\in I}\operatorname{Spec}(A_i)\to X\to Y$ is an $(n+1)$-atlas. Since the projections $X\times_Y X\to X$ and $\operatorname{Spec}(A_i)\to X$ are n-geometric smooth, we will see that $\operatorname{Spec}(A_i)\to X\to Y$ is n-geometric smooth, proving our claim.

Indeed, let $\operatorname{Spec}(C)\to Y$ be a morphism from an affine derived scheme. Consider the base change $\coprod_{j\in I}\operatorname{Spec}(A_j)\times_Y\operatorname{Spec}(C)\twoheadrightarrow\operatorname{Spec}(C)$, which is an effective epimorphism. In particular, we can find an étale covering $\operatorname{Spec}(\widetilde{C})\to\operatorname{Spec}(C)$, which factors through $\coprod_{j\in I}\operatorname{Spec}(A_j)\times_Y\operatorname{Spec}(C)$. To show that $\operatorname{Spec}(A_i)\times_Y\operatorname{Spec}(C)$ has an n-atlas, it suffices to check that the base change with $\widetilde{C}$ has an n-atlas (see [TV08, Prop. 1.3.3.4]). Now let us look at the following diagram with pullback squares

$$
\begin{tikzcd}
Z \arrow[r] \arrow[d, "f'"] & W \arrow[r] \arrow[d] & \operatorname{Spec}(\widetilde{C}) \arrow[d, "f"] \\
\coprod_{j\in I}\operatorname{Spec}(A_i)\times_Y\operatorname{Spec}(A_j)\times_Y C \arrow[r, "h"] \arrow[d, two heads, "g'"] & \coprod_{j\in I} X\times_Y\operatorname{Spec}(A_j)\times_Y C \arrow[r, "l"] \arrow[d, two heads] & \coprod_{j\in I}\operatorname{Spec}(A_j)\times_Y C \arrow[d, two heads, "g"] \\
C\times_Y\operatorname{Spec}(A_i) \arrow[r] \arrow[d] & C\times_Y X \arrow[r] \arrow[d] & C \arrow[d] \\
\operatorname{Spec}(A_i) \arrow[r] & X \arrow[r] & Y.
\end{tikzcd}
$$

Since $g\circ f$ is affine étale effective epimorphism, we know that $g'\circ f'$ is affine étale effective epimorphism. Since the projections are n-geometric smooth and by the above the projection $\operatorname{Spec}(A_i)\times_Y\operatorname{Spec}(A_j)\to\operatorname{Spec}(A_i)$ is n-geometric smooth, we see that $l\circ h$ is n-geometric smooth. Therefore, Z has an n-atlas and since $g\circ f'$ is affine étale, we see that the n-atlas of Z gives an n-atlas of $\operatorname{Spec}(A_i)\times_Y\operatorname{Spec}(C)$.

The rest of the statement follows immediately by the definitions. □

We conclude this section with an important remark. This remark shows, for an animated ring A, how open subschemes of $\operatorname{Spec}(\pi_0 A)_{\mathrm{cl}}$ can be lifted to derived open subschemes of $\operatorname{Spec}(A)$. In particular, when we want to show that an inclusion of derived stacks is an open immersion, it suffices to show that it is an open immersion after applying t_0.

Remark 4.42 (Lifting opens along affines)**.** Let A be an animated ring. Assume we have an open subscheme $U\hookrightarrow\operatorname{Spec}(\pi_0 A)_{\mathrm{cl}}$ of an affine scheme. Let $(\operatorname{Spec}(\pi_0 A_{f_i})_{\mathrm{cl}}\to U)_{i\in I}$ be an affine open cover by basis elements. Certainly, we can lift this open cover to an open subscheme $V:=\operatorname{Im}(\coprod_{i\in I}\operatorname{Spec}(A[f_i^{-1}]))$, where the image is taken in $\operatorname{Spec}(A)$. Let B be a animated A-algebra with structure morphism $w\colon\operatorname{Spec}(B)\to\operatorname{Spec}(A)$. Then w factors through an $u\colon\operatorname{Spec}(B)\to V$, i.e. $u\in V(B)$, if and only if there is an étale cover $(B\to B_j)_{j\in J}$ such that for every j there is an i with $\pi_0 w_j(f_i)$ invertible, where w_j is the composition of w with the natural map $B\to B_j$.

To see this, assume we have a map $u\colon\operatorname{Spec}(B)\to V$ of derived A-schemes. Then base change with the affine open cover of V gives an affine open $\coprod_{i\in I}\operatorname{Spec}(B_i)$ cover of $\operatorname{Spec}(B)$ that maps to $\coprod_{i\in I}\operatorname{Spec}(A[f_i^{-1}])$ via projection (note that V is an open subscheme of an affine scheme and thus separated, so in particular the diagonal of V is affine). The projection $\coprod_{i\in I}\operatorname{Spec}(B_i)\to\coprod_{i\in I}\operatorname{Spec}(A[f_i^{-1}])$ is induced by the termwise projections (note that coproducts in ∞-topoi are universal). Thus, by the universal property of localization, we see that $\pi_0 w_i(f_i)$ is invertible in $\pi_0 B_i$.

Now assume there is an étale cover $(\operatorname{Spec}(B_j)\to\operatorname{Spec}(B))_{j\in J}$ such that for every j there is an i with $\operatorname{Spec}(B_j)\to\operatorname{Spec}(A[f_i^{-1}])$. In particular, we get a map $\operatorname{Spec}(B_j)\to\coprod_{i\in I}\operatorname{Spec}(A[f_i^{-1}])$ and by taking coproducts and fiber products, we get a map

$$\operatorname{Spec}(B)\simeq\operatorname{colim}_{\Delta}(\check{C}(\coprod_{j\in J}\operatorname{Spec}(B_j)/B)_\bullet)\to\coprod_{i\in I}\operatorname{Spec}(A[f_i^{-1}])\to V.$$

4.3 Quasi-coherent modules over derived stacks

In this section we will shortly look at quasi-coherent modules over derived stacks. We show that they behave as "expected". Namely, quasi-coherent modules over derived stacks still satisfy descent[21]. We also have pullback and pushforward functors that are adjoint to another. Further, we show that the ∞-category of quasi-coherent modules (in the derived sense) over a (classical) scheme X is equivalent to $\mathcal{D}_{\mathrm{qc}}(X)$.

We will closely follow [GR17, §I.3], [Lur04] and [Kha18] and generalize some results following their ideas.

Definition 4.43. Let X be a presheaf on $\mathrm{AR}_R^{\mathrm{op}}$, we define the ∞-*category of quasi-coherent modules over* X to be

$$\operatorname{QCoh}(X)\coloneqq\lim_{\operatorname{Spec}(A)\to X}\operatorname{Mod}_A.$$

An element $\mathcal{F}\in\operatorname{QCoh}(X)$ is called *quasi-coherent module over* X or $\mathcal{O}_X$*-module*. For any affine derived scheme $\operatorname{Spec}(A)$ and any morphism $f\colon\operatorname{Spec}(A)\to X$, we denote the image of a quasi-coherent module $\mathcal{F}$ under the projection $\operatorname{QCoh}(X)\to\operatorname{Mod}_A$, with $f^*\mathcal{F}$.

We define the ∞-category of *perfect quasi-coherent modules over* X to be

$$\operatorname{QCoh}_{\mathrm{perf}}(X)\coloneqq\lim_{\operatorname{Spec}(A)\to X}\operatorname{Mod}_A^{\mathrm{perf}}.$$

We say that a perfect quasi-coherent module $\mathcal{F}$ over X has *Tor-amplitude in* $[a,b]$ if for every derived affine scheme $\operatorname{Spec}(A)$ and any morphism $f\colon\operatorname{Spec}(A)\to X$ the A-module $f^*\mathcal{F}$ has Tor-amplitude in $[a,b]$.

Remark 4.44. We see that that by definition $\operatorname{QCoh}(-)$ (resp. $\operatorname{QCoh}_{\mathrm{perf}}(-)$) is a right Kan extension of $\operatorname{Mod}_-\colon\mathrm{AR}_R\to\mathrm{Cat}_\infty$ (resp. $\operatorname{Mod}_-^{\mathrm{perf}}\colon\mathrm{AR}_R\to\mathrm{Cat}_\infty$) onto $\mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})^{\mathrm{op}}$ along the Yoneda emebdding.

A priori Mod_- is a functor from animated rings to the ∞-category of not necessarily small ∞-categories. But for the purpose of this article if we talk about the right Kan extension along the Yoneda embedding to presheaves on $\mathrm{AR}_{\mathbb{Z}}$, we assume smallness of the module categories.

[21] We will make this explicit later on, as we did not define Grothendieck topologies on derived stacks.

Remark 4.45. Note that limits preserve monomorphisms in Cat_∞ (as this ∞-category has limits). Therefore, if $F, G\colon \mathrm{AR}_R \to \mathrm{Cat}_\infty$ are functors and $\alpha\colon F \to G$ is a natural transformation such that $\alpha(A)$ is a monomorphism for all $A \in \mathrm{AR}_R$, we see that for the induced morphism $R\alpha$ on the right Kan extensions RF resp. RG of F resp. G under the Yoneda embedding $\mathrm{AR}_R \hookrightarrow \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})^{\mathrm{op}}$ the evaluation on some $X \in \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})^{\mathrm{op}}$ yields a monomorphism $R\alpha(X)\colon RF(X) \hookrightarrow RG(X)$.

This can be applied to see that for example for all $\in \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})^{\mathrm{op}}$, we have that the natural morphism $\mathrm{QCoh}_{\mathrm{perf}}(X) \to \mathrm{QCoh}(X)$ is fully faithful and we can see $\mathrm{QCoh}_{\mathrm{perf}}(X)$ as a full subcategory of $\mathrm{QCoh}(X)$.

Lemma 4.46. *Let $\mathcal{C}$ be the limit of stable ∞-categories $\mathcal{C}_k$ indexed by some simplicial set K with finite limit preserving transition maps, then $\mathcal{C}$ is stable.*

Proof. Since $\mathcal{C}_k$ have finite limits, we know that $\mathcal{C}$ has finite limits. Then the spectrum of $\mathcal{C}$ is stable by [Lur17, Cor. 1.4.2.17], but by [Lur17, Rem. 1.4.2.25], we know that $\mathrm{Sp}(\mathcal{C})$ itself is a limit of the tower $\cdots \to \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*$. In particular, we have

$$\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Sp}(\lim_K \mathcal{C}_\bullet) \simeq \lim_K \mathrm{Sp}(\mathcal{C}_\bullet) \simeq \lim_K \mathcal{C}_\bullet \simeq \mathcal{C},$$

where we use [Lur17, Prop. 1.4.2.21] for the second to last equivalence. □

Remark 4.47. By Lemma 4.46, we know that for any $X \in \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})$ the ∞-category $\mathrm{QCoh}(X)$ is stable, since it is the limit of stable ∞-categories and the transition maps are given by base change (the base change functor preserves fiber sequences, as they are equivalently cofiber sequences, and finite products, that are equivalent to finite coproducts).

The following proposition is a generalization of [GR17, §I.3 Cor. 1.3.11] but we can follow the idea of the proof.

Proposition 4.48. *Let $\mathcal{C}$ be a presentable ∞-category and let $F\colon \mathrm{AR}_R \to \mathcal{C}$ be a (hypercomplete) sheaf with respect to the Grothendieck topology $\tau \in \{\mathrm{fpqc}, \text{étale}\}$ on AR_R. Let RF denote the right Kan extension of F along the Yoneda embedding $\mathrm{AR}_R \hookrightarrow \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})^{\mathrm{op}}$. Further let us denote the corresponding ∞-topos of (hypercomplete) τ-sheaves on AR_R with Shv_τ. Then for any diagram $p\colon K \to \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})$, where K is a simplicial set, and morphism $\mathrm{colim}_K X_k \to Y$ that becomes an equivalence in Shv_τ after sheafification[22], we have that the natural map $RF(Y) \to \lim_K RF(X_k)$ is an equivalence.*

Proof. First, let us note that since $\mathcal{C}$ is presentable, we can find a small subcategory $\mathcal{C}' \subseteq \mathcal{C}$ such that $\mathcal{C}$ is a localization of $\mathcal{P}(\mathcal{C}')$ (see [Lur09, Thm. 5.5.1.1]). In particular, the elements $RF(Y)$ and $\lim_K RF(X_k)$ may be regarded as functors from $\mathcal{C}'$ to $\mathbb{S}$ and the natural morphism is an equivalence if and only if it is an equivalence after composing with the evaluation for every $c \in \mathcal{C}'$ (see [Lur21, 01DK]). We note that the inclusion of $\mathcal{C}$ into $\mathcal{P}(\mathcal{C}')$ preserves limits and since the evaluation of a functor $G \in \mathcal{P}(\mathcal{C}')$ at c is

[22]Recall[20] that we can describe Shv_τ as a localization of $\mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})$ (as seen in the proof), so we get a functor $L\colon \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}}) \to \mathrm{Shv}_\tau$ left adjoint to the inclusion, which we call *sheafification*.

equivalent to $\mathrm{Hom}_{\mathcal{P}(\mathcal{C}')}(j(c), G)$, where $j\colon \mathcal{C}' \hookrightarrow \mathcal{P}(\mathcal{C}')$ denotes the Yoneda embedding (see [Lur09, Lem. 5.5.2.1]), we see that also the evaluation preserves limits. So it is enough to check that for every $c \in \mathcal{C}'$, the morphism $RF(Y)(c) \to \lim_{\Delta}(RF(X_k)(c))$ is an equivalence. In particular, we may replace F by $\mathrm{Hom}_{\mathcal{P}(\mathcal{C}')}(j(c), -) \circ F$ for any $c \in \mathcal{C}'$ and so without loss of generality, we may assume that $\mathcal{C} \simeq \mathbb{S}$.

We will first discuss the case of τ-sheaves. By definition of the ∞-category Shv_τ, we know that all τ-sheaves are S-local, where S is the collection of those monomorphism $U \hookrightarrow \mathrm{Spec}(A)$, where $A \in \mathrm{AR}_R$, such that it defines a τ-covering sieve (see [Lur09, §6.2.2] for details). This in particular defines a localization functor $L\colon \mathcal{P}(\mathrm{AR}_R) \to \mathcal{P}(\mathrm{AR}_R)$ with essential image given by Shv_τ. Using [Lur09, Prop. 5.5.4.2], we see that any equivalence in Shv_τ is local, i.e. any morphism $f\colon U \to V$ in $\mathcal{P}(\mathrm{AR}_R)$ such that Lf is an equivalence and any $Q \in \mathrm{Shv}_\tau$ we have that the natural morphism

$$\mathrm{Hom}_{\mathcal{P}(\mathrm{AR}_R)}(V, Q) \to \mathrm{Hom}_{\mathcal{P}(\mathrm{AR}_R)}(U, Q)$$

is an equivalence. In particular, in our situation, we have that

$$\mathrm{Hom}_{\mathcal{P}(\mathrm{AR}_R)}(Y, Q) \to \mathrm{Hom}_{\mathcal{P}(\mathrm{AR}_R)}(\operatorname*{colim}_K X_k, Q) \simeq \lim_K \mathrm{Hom}_{\mathcal{P}(\mathrm{AR}_R)}(X_k, Q)$$

is an equivalence (note that the colimit in the second Hom is taken in the ∞-category $\mathcal{P}(\mathrm{AR}_R)$, whereas for $Y \simeq \operatorname{colim}_K X_k$ colimit is taken in Shv_τ which do not agree in general since the inclusion $\mathrm{Shv}_\tau \hookrightarrow \mathcal{P}(\mathrm{AR}_R)$ does not preserve colimits in general).

Since F is a sheaf with respect to the topology τ, we therefore have

$$\mathrm{Hom}_{\mathcal{P}(\mathrm{AR}_R)}(Y, F) \simeq \lim_K \mathrm{Hom}_{\mathcal{P}(\mathrm{AR}_R)}(X_k, F).$$

Since we can write any presheaf on AR_R as a colimit of representable ones (see [Lur09, Lem. 5.1.5.3]) and the Yoneda lemma [Lur09, Lem. 5.5.2.1], we finally have the equivalence

$$RF(Y) \xrightarrow{\sim} \lim_K RF(X_k).$$

The case of hypercomplete τ-sheaves is completely analogous, noting that the ∞-topos of hypercomplete τ-sheaves can be realized as a localization of $\mathcal{P}(\mathrm{AR}_R)$ with respect to hypercovers (see [Lur09, Cor. 6.5.3.13]). $\square$

Definition 4.49. Let $\mathcal{C}$ be a presentable ∞-category and let τ be the fpqc or étale topology on AR_R. A functor $F\colon \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{C}$ is a *(hypercomplete) sheaf or satisfies τ-descent* if for any effective epimorphism $X \to Y$ (resp. a hypercover $X^\bullet \to Y$), we have

$$RF(Y) \simeq \lim_{\Delta} RF(\check{C}(X/Y)_\bullet) \text{ (resp. } RF(Y) \simeq \lim_{\Delta_s} RF(X^\bullet)).$$

Remark 4.50. In the setting of Proposition 4.48, we see that if F is a (hypercomplete) sheaf, then so is its right Kan extension RF.

Remark 4.51. An important example of a presentable ∞-category is the ∞-category Cat_∞ of small ∞-categories. Presentability of this ∞-category follows from the fact that it is the ∞-category of a combinatorial simplicial model category (marked simplicial sets with the model structure of [Lur09, Prop. 3.1.5.2]), which are precisely the presentable ∞-categories ([Lur09, Prop. A.3.7.6]).

Proposition 4.52. *Let $\mathcal{C}$ be a presentable ∞-category and let $F\colon \mathrm{AR}_R \to \mathcal{C}$ be an étale sheaf. Let RF denote a right Kan extension of F along the Yoneda embedding $\mathrm{AR}_R \hookrightarrow \mathcal{P}(\mathrm{AR}_R)^{\mathrm{op}}$. Then for any derived scheme X over R, the natural morphism*

$$RF(X) \to \lim_{\substack{U \hookrightarrow X \\ \text{affine open}}} F(U)$$

is an equivalence.

Proof. This lemma is a generalization of [Kha18, Lec. 1 Prop. 3.5] but can be proven the same. For the convenience of the reader, we give a proof.

Let X be derived scheme and $Y := \coprod_{i\in I} \operatorname{Spec}(A_i) \to X$ be a Zariski atlas. By Remark 4.50 we have

$$RF(X) \simeq \lim_{\Delta} RF(\check{C}(Y/X)_\bullet).$$

For any affine open $U \hookrightarrow X$ let $Y_U := \coprod_{i\in I} U \times_X \operatorname{Spec}(A_i) \to U$ be the induced cover on U. Thus the question reduces to showing the equivalence

$$RF(\check{C}(Y/X)_n) \to \lim_{\substack{U \hookrightarrow X \\ \text{affine open}}} RF(\check{C}(Y_U/U)_n),$$

for all $[n] \in \Delta$. By cofinality, we may replace X in the limit argument by $\check{C}(Y/X)_n$ for any n.

To see this, note that for every affine open $U \hookrightarrow \check{C}(X/Y)_n$, we get a morphism $U \to U \times_X U \simeq \check{C}(Y/X)_n \times_X U \times_{\check{C}(Y/X)_n} U \simeq \check{C}(Y_U/U)_n \times_{\check{C}(Y/X)_n} U \to \check{C}(Y_U/U)$. Thus

$$\lim_{\substack{U \hookrightarrow \check{C}(Y/X)_n \\ \text{affine open}}} RF(U) \simeq \lim_{\substack{U \hookrightarrow X \\ \text{affine open}}} RF(\check{C}(Y_U/U)_n).$$

Assume the pairwise intersection of the $\operatorname{Spec}(A_i)$ is affines, then X is affine and the question is trivial. Now assume the pairwise intersection is not affine then it is open in an affine scheme and thus separated. Thus they admit Zariski covers, where each of the pairwise intersection is affine. Repeating the whole process concludes the proof. $\square$

Remark 4.53. Let us remark that the functors $A \mapsto \mathrm{Mod}_A$ and $A \mapsto \mathrm{Mod}_A^{\mathrm{perf}}$ are hypercomplete sheaves for the fpqc topology. The first assertion follows by [Lur18, Cor. D.6.3.3]. The second assertion is clear since modules satisfy flat hyperdescent and since perfect modules are precisely the dualizable ones (see [Lur17, Prop. 7.2.2.4]), we can construct a dual fpqc locally (see [Lur17, Prop. 4.6.1.11]) - see the proof [AG14, Lem. 5.4] for a more detailed explanation.

Remark 4.54. Using the definition of the functors QCoh and $\mathrm{QCoh}_{\mathrm{perf}}$, we see with Remark 4.53 and Remark 4.50 that these functors satisfy descent in the sense that for any effective epimorphism of derived stacks $X \twoheadrightarrow Y$, we have

$$\mathrm{QCoh}(Y) \simeq \lim_{\Delta} \mathrm{QCoh}(\check{C}(X/Y)_\bullet) \text{ resp. } \mathrm{QCoh}_{\mathrm{perf}}(Y) \simeq \lim_{\Delta} \mathrm{QCoh}_{\mathrm{perf}}(\check{C}(X/Y)_\bullet).$$

Remark 4.55. Using Remark 4.44 and Proposition 4.52, we see that a quasi-coherent module over a derived scheme is given by a compatible family of modules $(\mathcal{F}_A)_A$ for every affine open $\mathrm{Spec}(A) \hookrightarrow X$.

Remark 4.56. Let us recall the derived direct and inverse image. For a morphism of animated rings $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, we get a forgetful functor $\mathrm{Mod}_B \to \mathrm{Mod}_A$ (this follows from [Lur17, 4.6.2.17]). This functor is right adjoint to the tensor product $B \otimes_A -$. We can globalize this to the case where we replace the domain by an arbitrary derived stack X. Namely, any quasi-coherent module $\mathcal{F}$ over X is determined by its underlying C-module $\iota^*\mathcal{F}$, for $\iota\colon \mathrm{Spec}(C) \to X$. Since C is naturally an A-algebra, we can forget the C-structure and view $\mathcal{F}$ as a limit in Mod_A. The tensor product with each such C also induces a functor from A-modules to quasi-coherent X-modules. We can also globalize this construction on the base for a geometric morphism of derived stacks, i.e. if $f\colon X \to S$ is a geometric morphism of derived stacks, we get an adjunction

$$f^*\colon \mathrm{QCoh}(S) \rightleftarrows \mathrm{QCoh}(X)\colon f_*,$$

note here that the right adjoint comes formally from the fact that the pullback by construction commutes with colimits. If one adds assumptions to f, then we can say more about the pushforward but we will not do this here and refer to [Lur04, §5.5] since it is not of interest for us.

If we work with classical schemes, we will write Lf^* and Rf_* to differentiate between the classical notions.

Proposition 4.57. *Let X be a scheme. Then we have an equivalence of ∞-categories $\mathcal{D}_{\mathrm{qc}}(X) \simeq \mathrm{QCoh}(X)$, where $\mathcal{D}_{\mathrm{qc}}(X)$ denoted the derived ∞-category of $\mathcal{O}_X$-modules with quasi-coherent cohomologies.*

Proof. This is shown in the spectral setting in [Lur18] and can be followed in our setting from Lurie's PhD thesis [Lur04]. For convenience of the reader we will show how to conclude this proposition as a consequence of both references.

A scheme X is per definition a locally ringed space $(X, \mathcal{O}_X)$. We let $X' \coloneqq \mathrm{Shv}_{(\mathrm{Sets})}(X)$ denote the Grothendieck topos associated to the small Zariski-site of X. The sheaf of rings $\mathcal{O}_X$ can be viewed as a ring object of X'. So in particular, the tuple $(X', \mathcal{O}_X)$ defines a locally ringed topos. We write $\mathcal{X}$ for the 1-localic ∞-topos assocaited to X' (which exists by [Lur09, Prop. 6.4.5.7]). As explained in [Lur18, Rem. 1.4.1.5], we can view $\mathcal{O}_X$ as a sheaf of connective 0-truncated E_∞-rings (which are just commutative rings) on $\mathcal{X}$, which we denote by $\mathcal{O}_{\mathcal{X}}$ and hence get a spectrally ringed space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, which is local (we refer to [Lur18, §I.1.1] for the definitions).

Since $\mathcal{O}_{\mathcal{X}}$ takes values in commutative rings, we can also view it as a sheaf on $\mathcal{X}$, with values in animated rings. Therefore $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ defines a spectral scheme resp. a derived scheme in the sense of [Lur18, §I.1] resp. [Lur04]. Note that also the definition of an $\mathcal{O}_{\mathcal{X}}$-module agrees in both references, i.e. in both references, we see an $\mathcal{O}_{\mathcal{X}}$-module as an $\mathcal{O}_{\mathcal{X}}$-module object in $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$, where $\mathcal{O}_{\mathcal{X}}$ is naturally seen as a sheaf with values in spectra.

By [Lur04, Thm. 4.6.5], we have an equivalence of $\mathrm{QCoh}(X)$ and ∞-categories of sheaf $\mathcal{O}_{\mathcal{X}}$-modules M on $\mathcal{X}$ such that

1. $\pi_i M$ (which is defined as the sheafification of the presheaf $V \mapsto \pi_i M(V)$) is a quasi-coherent sheaf on the underlying Deligne-Mumford stack of $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, and

2. the underlying sheaf of spaces of M is a hypersheaf (as explained above M can be seen as a sheaf on $\mathcal{X}$ with values in Sp, i.e. a limit preserving functor $M\colon \mathcal{X}^{\mathrm{op}} \to$ Sp, and composing with $\Omega^\infty\colon \mathrm{Sp} \to \mathbb{S}$ defines the underlying sheaf of spaces of M).

Using [Lur18, Prop. 2.2.6.1], we see that $\mathrm{QCoh}(X)$ is equivalent to the ∞-category of quasi-coherent sheaves on the spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Now [Lur18, Cor. 2.2.6.2] shows that indeed the derived ∞-category of $\mathcal{O}_X$-modules with quasi-coherent cohomology is equivalent to $\mathrm{QCoh}(X)$. □

4.4 The cotangent complex of a derived stack

This section is derived from [TV08, §1.4],[AG14, §4.2], [Kha18, Lecture 5].

In this section, we will globalize the results of Section 3.3 and 3.4 to geometric morphisms of derived stacks. For this, we will define a global version of the cotangent complex and list properties. Most importantly, we will show that any n-geometric morphism has a cotangent complex and that smooth morphisms are characterized by the cotangent complex. Further, we will use the results to show that geometric stacks are automatically hypercomplete sheaves for the étale topology.

We let R be a ring and assume every derived stack is a derived stack over R.

Let $f\colon X \to Y$ be morphism of derived stacks. Let $x\colon \mathrm{Spec}(A) \to X$ be an A-point, where A is an animated R-algebra. Let $M \in \mathrm{Mod}_A^{\mathrm{cn}}$ and let us look at the commutative square

$$\begin{tikzcd} X(A \oplus M) \arrow[r] \arrow[d] & X(A) \arrow[d, "f"] \\ Y(A \oplus M) \arrow[r] & Y(A), \end{tikzcd}$$

where the vertical arrow are given by the canonical projection $A \oplus M \to A$. We set the *dervations at the point x* as

$$\mathrm{Der}_x(X/Y, M) \coloneqq \mathrm{fib}_x(X(A \oplus M) \to X(A) \times_{Y(A)} Y(A \oplus M)),$$

where we see x as a point in the target via the natural map induced by $\mathrm{Spec}(A \oplus M) \to \mathrm{Spec}(A) \xrightarrow{x} X \xrightarrow{f} Y$.

Definition 4.58 ([TV08, Def. 1.4.1.5])**.** Let $f\colon X \to Y$ be a morphism of derived stacks. We say $L_{f,x} \in \mathrm{Mod}_A$ is a *cotangent complex for f at the point x*: $\mathrm{Spec}(A) \to X$, if it is $(-n)$-connective, for some $n \geq 0$ and for all $M \in \mathrm{Mod}_A^{\mathrm{cn}}$ there is a functorial equivalence

$$\mathrm{Hom}_{\mathrm{Mod}_A}(L_{f,x}, M) \simeq \mathrm{Der}_x(X/Y, M).$$

When such $L_{f,x}$ exists, we say f *admits a cotangent complex at the point x*. If there is no possibility of confusion, we also write $L_{X/Y,x}$ for $L_{f,x}$. We also write L_X if $Y \simeq \mathrm{Spec}(R)$.

Definition 4.59. Let $f\colon X \to Y$ be a morphism of derived stacks. We say that $L_f \in \mathrm{QCoh}(X)$ *is a cotangent complex for* f if for all points $x\colon \mathrm{Spec}(A) \to X$ the A-module x^*L_f is a cotangent complex for f at the point x.

If L_f exists, we say that f *admits a cotangent complex.* We will write $L_{f,x}$ instead of x^*L_f if f admits a cotangent complex.

If $Y \simeq \mathrm{Spec}(R)$ and L_f exists, we say that X admits an *absolute cotangent complex.*

Remark 4.60. By Lemma 3.37, the cotangent complex, for any morphism of derived stacks, is unique up to homotopy.

Remark 4.61. Note that any morphism of affine derived schemes $f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ admits a cotangent complex. For any point $x\colon \mathrm{Spec}(C) \to \mathrm{Spec}(B)$, we have $L_{\mathrm{Spec}(B)/\mathrm{Spec}(A),x} \coloneqq L_{B/A} \otimes_B C$.

Lemma 4.62. *Let $f\colon X \to Y$ be a morphism of derived stacks.*

1. *If X and Y admit absolute cotangent complexes, then f admits a cotangent complex and we have the following cofiber sequence for any point $x\colon \mathrm{Spec}(A) \to X$*

$$L_{Y,f\circ x} \to L_{X,x} \to L_{f,x}.$$

2. *If f admits a cotangent complex, then for any morphism of derived stacks $Z \to Y$ and any point $x\colon \mathrm{Spec}(A) \to X \times_Y Z$, we have*

$$L_{f,x} \simeq L_{X\times_Y Z/Z,x}.$$

3. *If for any morphism $x\colon \mathrm{Spec}(A) \to X$ the projection $\mathrm{pr}\colon X \times_{Y,f\circ x} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ admits a cotangent complex, then f admits a cotangent complex and further we have*

$$L_{f,x} \simeq L_{\mathrm{pr},x}.$$

4. *If for any point $x\colon \mathrm{Spec}(A) \to X$ the stack $X \times_{Y,f\circ x} \mathrm{Spec}(A)$ admits a cotangent complex, then f has a cotangent complex and we have*

$$L_{\mathrm{Spec}(A),\mathrm{id}_{\mathrm{Spec}(A)}} \to L_{X\times_Y A,x} \to L_{f,x}.$$

Proof. The proof in the model categorical case is given in [TV08, Lem. 1.4.1.16]. But these properties are straightforward to check.

Part *1* and *2* follow from the definitions. Part *3* follows from *2* and part *4* follows from *1*, *3* and Remark 4.61 □

Lemma 4.63. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a morphism of derived stacks. Assume Y/Z admits a cotangent complex, then X/Y admits a cotangent complex if and only if X/Z admits a cotangent complex. Further, we obtain a cofiber sequence*

$$f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y}$$

of quasi-coherent modules over X if the cotangent complexes exist.

Proof. This is stated in [Kha18, Lec. 5 Prop. 5.7]. But anyway we will give a proof.

Let us take a point $x\colon \operatorname{Spec}(A) \to X$. Then what we impose is that we have a cofiber sequence

$$L_{Y/Z,f\circ x} \to L_{X/Z,x} \to L_{X/Y,x}.$$

By Lemma 3.37 it is enough to show that for any connective A-module M the fiber of $\operatorname{Der}_x(X/Z, M) \to \operatorname{Der}_{f\circ x}(Y/Z, M)$ at the trivial derivation (given by $A \oplus M \to A \to X \to Y$) is given by $\operatorname{Der}_x(X/Y, M)$. But this is clear. □

Lemma 4.64. *Let $j\colon X \hookrightarrow Y$ be a monomorphism of derived stacks over A, then j admits a cotangent complex and $L_j \simeq 0$.*

Proof. The proof is the same as in [Kha18, Lec. 5 Prop. 5.9], but for the convenience of the reader, we recall the proof.

It suffices to show that at any point $x\colon \operatorname{Spec}(B) \to X$ the cotangent complex is zero, i.e. the space of derivations at x is contractible. Since j is a monomorphism, i.e. its fibers are (-1)-truncated (so either contractible or empty (see [Lur09, Def. 5.5.6.8])), we know that the canonical map

$$X(B \oplus M) \to X(B) \times_{Y(B)} Y(B \oplus M)$$

is also a monomorphism (note that any point in the fiber of the above morphism defines a point in the fiber of $X(B \oplus M) \to Y(B \oplus M)$, which per definition has either contractible or empty fibers). But the canonical map $u\colon \operatorname{Spec}(B\oplus M) \to \operatorname{Spec}(B) \to X$ defines a derivation, so the space of derivations is nonempty and thus contractible. □

Now we can easily see that the homotopy groups of the localization with respect to one element is given by the localization of the homotopy groups.

Lemma 4.65. *Let A be an animated ring and $f \in \pi_0 A = \pi_0 \operatorname{Hom}_{\operatorname{Mod}_A}(A, A)$. Then we have $\pi_i(A[f^{-1}]) \cong (\pi_i A)_f$ as $\pi_0 A$-modules.*

Proof. From Proposition 3.13 and Lemma 3.19 it follows that the map $\operatorname{Spec}(A[f^{-1}]) \hookrightarrow \operatorname{Spec}(A)$ is locally of finite presentation and a monomorphism. Since monomorphisms have a vanishing relative cotangent complex (see Lemma 4.64), we conclude with Proposition 3.56 that $A \to A[f^{-1}]$ is étale. Hence, we conclude using the definition of étale morphisms. □

Lemma 4.66. *Let $(A \to A_i)_{i\in I}$ be an étale covering in AR_R and let M be an A-module. Then the family induced by base change $(A\oplus M \to A_i \oplus (M \otimes_A A_i))_{i\in I}$ is an étale cover.*

Proof. We only need to show that $A_i \otimes_A (A \oplus M) \simeq A_i \oplus (M \otimes_A A_i)$, since étale covers are stable under base change. But by construction the functors $A_i \otimes_A (- \oplus M)$ and $(- \otimes_A A_i) \oplus (M \otimes_A A_i)$ from $\text{AR-Mod}_R^{\text{cn}}$ to AR_{A_i} commute with sifted colimits and thus we are reduced to classical commutative algebra, where it is clear. □

Lemma 4.67. *Let $f\colon X \to Y$ be a morphism of derived schemes. Then f admits a cotangent complex.*

Proof. A proof sketch is given in [Kha18, Lec. 5 Thm. 5.12], but since these are lecture notes, we recall the proof.

We may assume that $Y = \mathrm{Spec}(R)$ using Lemma 4.63.

By Proposition 4.52, we know that

$$\mathrm{QCoh}(X) \simeq \lim_{\substack{\mathrm{Spec}(B)\hookrightarrow X \\ \text{open immersion}}} \mathrm{Mod}_B\,.$$

For each of the open affines $\mathrm{Spec}(B)$ in X, we take $L_{\mathrm{Spec}(B)} := L_{B/\mathbb{Z}}$, viewed as a quasi-coherent sheaf on $\mathrm{Spec}(B)$. Since the cotangent complex is compatible with taking pullbacks, i.e. $L_B \otimes_B A \simeq L_A$ for a triangle

$$\begin{tikzcd}
\mathrm{Spec}(A) \arrow[d] \arrow[dr] & \\
\mathrm{Spec}(B) \arrow[r] & X,
\end{tikzcd}$$

where $\mathrm{Spec}(A), \mathrm{Spec}(B)$ are open in X, we see that this indeed defines an object in the limit, which we denote with L_X (for the compatibility, note that the relative cotangent complexes of monomorphisms vanish by Lemma 4.64). This defines a cotangent complex on X.

We have to show that L_X represents the space of derivations. Since we have glued the cotangent complex for affine opens, we will use a descent argument for arbitrary points. For this, we use that modules satisfy fpqc descent and that for any point $x\colon \mathrm{Spec}(B) \to X$, the base change with an affine open cover of X gives an affine open cover (B_i) of B. Hence, for a connective B-module M, it suffices to show that

$$\lim \mathrm{Hom}_{B_i}(L_{X,B_i}, M_i) \simeq \lim \mathrm{Der}_{B_i}(X/\mathbb{Z}, M_i),$$

where $M_i := M \otimes_B B_i$, which is clear termwise, since each B_i factors through some affine open A_i of X by construction (to write the derivations as a limit use the sheaf property of X, $\mathrm{Spec}(\mathbb{Z})$ and Lemma 4.66). Note that for each affine open the cotangent complex exists and we claim that $L_{X,B_i} \simeq L_{X,A_i} \otimes_{A_i} B_i \simeq L_{\mathrm{Spec}(A_i),A_i} \otimes_{A_i} B_i \simeq L_{\mathrm{Spec}(A_i),B_i}$, which concludes the lemma.

To see this, note that $\mathrm{Der}_{B_i}(A_i/X, M_i) \simeq 0$, since $\mathrm{Spec}(A_i) \hookrightarrow X$ is a monomorphism (see Lemma 4.64). Therefore $\mathrm{Der}_{B_i}(A_i/\mathbb{Z}, M_i) \simeq \mathrm{Der}_{B_i}(X/\mathbb{Z}, M_i)$ (since its fiber is $\mathrm{Der}_{B_i}(A_i/X, M_i)$), the same holds if we replace the point by A_i. Since the cotangent complexes at A_i exist, we get $L_{\mathrm{Spec}(A_i),A_i} \simeq L_{X,A_i}$ and after tensoring with B_i, we see that L_{X,B_i} is a cotangent complex for $X/\mathbb{Z}$ at B_i if and only if $\mathrm{Der}_{B_i}(A_i/\mathbb{Z}, M_i) \simeq \mathrm{Der}_{B_i}(X/\mathbb{Z}, M_i)$ but this we have seen above, i.e. we have equivalences

$$\mathrm{Hom}_{B_i}(L_{X,B_i}, M_i) \simeq \mathrm{Hom}_{B_i}(L_{\mathrm{Spec}(A_i),B_i}, M_i) \simeq \mathrm{Der}_{B_i}(A_i/\mathbb{Z}, M_i) \simeq \mathrm{Der}_{B_i}(X/\mathbb{Z}, M_i).$$

We remark that by this construction and commutativity of $\tau_{\geq 0}$ with limits, we have that L_X is connective. In particular, we have shown that L_X is a cotangent complex for $X/\mathbb{Z}$. □

Remark 4.68. Let us give another construction of a cotangent complex. Consider the functor $L_{-}\colon \mathrm{AR}_{\mathbb{Z}} \to \mathcal{D}(R)$ given by the usual cotangent complex seen as complex of abelian groups. We denote its right Kan extension along the inclusion $\mathrm{AR}_R \hookrightarrow \mathrm{dSch}^{\mathrm{op}}_{/\mathrm{Spec}(R)}$ with $\mathbb{R}L_{-}$. By the above proof, we see that for a derived scheme X, we have $L_X \simeq \mathbb{R}L_X$ in $\mathcal{D}(R)$. In particular, by stability of the derived ∞-category, the same holds for the relative cotangent complex.

Definition 4.69. We recall the notion of an obstruction theory for derived stacks respectively morphism of derived stacks (see [TV08, 1.4.2.1, 1.4.2.2]).

(i) A derived stack X is called *infinitesimally cartesian* or *inf-cartesian* if and only if for every animated R-algebra A, connective A-module M with $\pi_0 M = 0$ and derivation $d \in \mathrm{Der}(A, M)$ the pullback square

$$\begin{tikzcd} A \oplus_d M \arrow[r] \arrow[d] & A \arrow[d, "d"] \\ A \arrow[r, "s"] & A \oplus M, \end{tikzcd}$$

where s denotes the trivial derivation, induces a pullback square

$$\begin{tikzcd} X(A \oplus_d M) \arrow[r] \arrow[d] & X(A) \arrow[d, "d"] \\ X(A) \arrow[r, "s"] & X(A \oplus M). \end{tikzcd}$$

A morphism $f\colon X \to Y$ of derived stacks is called *infinitesimally cartesian* or *inf-cartesian* if and only if for every animated R-algebra A, connective A-module M with $\pi_0 M = 0$ and derivation $d \in \mathrm{Der}(A, M)$ we have a pullback square

$$\begin{tikzcd} X(A \oplus_d M) \arrow[r] \arrow[d] & Y(A \oplus_d M) \arrow[d] \\ X(A) \times_{X(A \oplus M)} X(A) \arrow[r] & Y(A) \times_{Y(A \oplus M)} Y(A). \end{tikzcd}$$

(ii) A derived stack X has an *obstruction theory* if and only if it has a cotangent complex and is infinitesimally cartesian.

A morphism of derived stacks $f\colon X \to Y$ has an *obstruction theory* if and only if it has a cotangent complex and is infinitesimally cartesian.

Definition 4.70 ([TV08, Def. 1.2.8.1])**.** Let $f\colon X \to Y$ be a morphism of derived stacks, we say f is *formally smooth* if for any $A \in \mathrm{AR}_R$, a connective A-module M with $\pi_0 M = 0$, and derivation $d \in \mathrm{Der}_R(A, M)$ the natural map

$$\pi_0 X(A \oplus_d M) \to \pi_0(X(A) \times_{Y(A)} Y(A \oplus_d M))$$

is surjective.

Lemma 4.71. *Let $f\colon X \to Y$ be a morphism of derived stacks.*

1. *If X and Y have an obstruction theory, then f has an obstruction theory.*

2. *If f has an obstruction theory, then for any morphism of derived stacks $Z \to Y$ the base change $Z \times_Y X \to Z$ has an obstruction theory.*

3. *If for any $A \in \mathrm{AR}_R$ and any morphism $\mathrm{Spec}(A) \to Y$ the base change $X \times_Y \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ has an obstruction theory, then f has an obstruction theory.*

Proof. This is [TV08, Lem. 1.4.2.3] but nevertheless we recall the proof.

The existence of the cotangent complex follows from Lemma 4.62.

Part *1* and *2* are clear by definition. For part *3* let B be an animated ring, M a connective B-module with $\pi_0 M = 0$ and $d \in \mathrm{Der}_R(B, M)$. We need to show that the diagram

$$\begin{tikzcd} X(A \oplus_d M) \arrow[r] \arrow[d] & Y(A \oplus_d M) \arrow[d] \\ X(A) \times_{X(A \oplus M)} X(A) \arrow[r] & Y(A) \times_{Y(A \oplus M)} Y(A) \end{tikzcd}$$

is a pullback diagram. Let $x \in Y(A \oplus_d M)$, we claim that it suffices to show that the induced morphism of the fibers of the two horizontal arrows at x is an equivalence.

Indeed, assume that we have a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & * \arrow[d, "x"] \\ X(A \oplus_d M) \arrow[r] \arrow[d] & Y(A \oplus_d M) \arrow[d] \\ X(A) \times_{X(A \oplus M)} X(A) \arrow[r] & Y(A) \times_{Y(A \oplus M)} Y(A) \end{tikzcd} \tag{4.4.1}$$

where the upper square and the outer square is a pullback. Let us also consider the pullback diagram

$$\begin{tikzcd} Z \arrow[r, "\alpha"] \arrow[d] & Y(A \oplus_d M) \arrow[d] \\ X(A) \times_{X(A \oplus M)} X(A) \arrow[r] & Y(A) \times_{Y(A \oplus M)} Y(A) \end{tikzcd}$$

(note that naturally $\mathrm{fib}_x(\alpha) \simeq F$ as the outer square of (4.4.1) is a pullback diagram). We have a naturally induced morphism of fiber sequences (i.e. a commutative diagram of the form)

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\simeq"] & X(A \oplus_d M) \arrow[r] \arrow[d] & Y(A \oplus_d M) \arrow[d, "\simeq"] \\ \mathrm{fib}_x(\alpha) \arrow[r] & Z \arrow[r] & Y(A \oplus_d M). \end{tikzcd}$$

The long exact homotopy sequence for fiber sequences in $\mathbb{S}$ now implies the claim.

That the upper square and the outer square of (4.4.1) are pullback diagrams follows from the fact that the pullback of f under the morphism corresponding to x has an obstruction theory. $\square$

The following technical lemma shows, how liftings along square zero extensions are linked to loops in the space of derivations. This is crucial, when dealing with formal smoothness of morphisms. Lifts of morphisms along square zero extensions are controlled by the cotangent complex which is, in some cases, easier to handle.

Lemma 4.72. *Let $f: X \to Y$ be a morphism of derived stacks, and assume f has an obstruction theory. Let $A \in \mathrm{AR}_R$, M be a connective A-module with $\pi_0 M = 0$ and $d \in \mathrm{Der}(A, M)$ a derivation. Let $x \in X(A) \times_{Y(A)} Y(A \oplus_d M)$ be a point and $L(x)$ the fiber of $X(A \oplus_d M) \to X(A) \times_{Y(A)} Y(A \oplus_d M)$ at x. There exists an element $\alpha(x) \in \pi_0 \mathrm{Hom}_{\mathrm{Mod}_A}(L_{f,x}, M)$ such that $L(x) \simeq \Omega_{\alpha(x),0} \mathrm{Hom}_{\mathrm{Mod}_A}(L_{f,x}, M)$, where we consider the pullback diagram*

$$\begin{tikzcd}
\Omega_{\alpha(x),0} \mathrm{Hom}_{\mathrm{Mod}_A}(L_{f,x}, M) \arrow[r] \arrow[d] & * \arrow[d, "\alpha(x)"] \\
* \arrow[r, "0"] & \mathrm{Hom}_{\mathrm{Mod}_A}(L_{f,x}, M).
\end{tikzcd}$$

Proof. This is [TV08, Prop. 1.4.2.6], but for the convenience of the reader we give a proof.

First, note that x corresponds to a diagram of the form

$$\begin{tikzcd}
\mathrm{Spec}(A) \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(A \oplus_d M) \arrow[r] & Y.
\end{tikzcd}$$

After composition with the natural maps, we get

$$\begin{tikzcd}
\mathrm{Spec}(A \oplus M) \arrow[r, "d"] \arrow[d, "s"] & \mathrm{Spec}(A) \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(A \oplus_d M) \arrow[r] & Y,
\end{tikzcd}$$

which gives a point $\alpha(x) \in \mathrm{Hom}_{\mathrm{dSt}_{A//Y}}(\mathrm{Spec}(A \oplus M), X) \simeq \mathrm{Der}_A(X/Y, M)$. Using that f is inf-cartesian, we get a pullback diagram

$$\begin{tikzcd}
L(x) \arrow[r] \arrow[d] & * \arrow[d, "\alpha(x)"] \\
* \arrow[r, "0"] & \mathrm{Der}_A(X/Y, M).
\end{tikzcd}$$

To see this, note the following commutative diagram with pullback squares.

$$\begin{tikzcd}
L(x) \arrow[r] \arrow[d] & * \arrow[d, "x"] & \\
X(A \oplus_d M) \arrow[r] \arrow[d] & X(A) \times_{Y(A)} Y(A \oplus_d M) \arrow[d, "\alpha"] & \\
* \arrow[r, "0"] \arrow[d] & \mathrm{Der}_A(X/Y, M) \arrow[r] \arrow[d] & * \arrow[d] \\
X(A) \times_{Y(A)} Y(A \oplus M) \arrow[r] \arrow[d, "\simeq"] & X(A \oplus M) \arrow[r] & X(A) \times_{Y(A)} Y(A \oplus M) \\
X(A) \times_{Y(A) \times_{Y(A \oplus M)} Y(A)} Y(A). & &
\end{tikzcd}$$

□

Lemma 4.73. *Any affine derived scheme $X \simeq \mathrm{Spec}(B)$ has an obstruction theory.*

Proof. Certainly, X has a cotangent complex by $L_{X,x} \simeq L_B \otimes_B A$, for $x\colon \mathrm{Spec}(A) \to X$. So we are left to show that X is infinitesimally cartesian. But this follows from compatability of the Hom functor with limits. □

Lemma 4.74. *Let $f\colon X \to Y$ be a morphism of affine derived schemes. If f is smooth, then it is formally smooth.*

Proof. This follows from [TV08, Prop. 2.2.5.1], but for convenience of the reader, we give a proof in our setting.

Let A be an animated R-algebra, M a connected C-module and $d \in \mathrm{Der}(A, M)$. Let $x \in \pi_0 X(A) \times_{Y(A)} Y(A \oplus_d M)$ be a point. We have to show that the fiber of

$$X(A \oplus_d M) \to X(A) \times_{Y(A)} Y(A \oplus_d M)$$

along x is nonempty. By Lemma 4.73 and 4.72, it suffices to show that $\pi_0 \operatorname{Hom}(L_{B/C} \otimes_B A, M)$ is contractible, where $X \simeq \mathrm{Spec}(B) \to \mathrm{Spec}(C) \simeq Y$. By Proposition 3.56 the A-module $L_{B/C} \otimes_B A$ is finite projective, so especially a retract of a free module (see [Lur17, Cor. 7.2.2.9]) and therefore $\pi_0 \operatorname{Hom}(L_{B/C} \otimes_B A, M)$ is a retract of a product of $\pi_0 M$, which is zero by hypothesis on M. □

Remark 4.75. We want to remark that Lemma 4.74 holds more generally. An n-geometric morphism of derived stacks is smooth if any only if after restriction to (Ring) via t_0 is locally of finite presentation and it is formally smooth. This is a bit technical but a proof of this is given for example in [TV08, Prop. 2.2.5.1].

The next proposition and corollary show, how smoothness of a geometric morphism is linked to its cotangent complex. This can be seen as a globalization of Proposition 3.56.

Proposition 4.76. *Let $f\colon X \to Y$ be an n-geometric morphism of derived stacks. Then f has an obstruction theory. Further if f is smooth, then f is formally smooth and L_f is perfect with Tor-amplitude in $[-n-1, 0]$.*

Proof. The proof of this lemma in the spectral setting is given in [AG14, Prop. 4.45]. The proof in the derived setting is analogous. But for the convenience of the reader, we give a proof.

We prove this lemma by induction over n. For the formal smoothness part we will first reduce to the case where Y is affine.

Indeed, let B be an animated R-algebra. Note that we have to show that for any point $x \in \pi_0(X(B) \times_{Y(B)} Y(B \oplus M))$ its fiber under $X(B \oplus_d M) \to X(B) \times_{Y(B)} Y(B \oplus_d M)$ is nonempty. The point x corresponds to a commutative diagram of the form

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(B \oplus_d M) \arrow[r] & Y. \end{tikzcd}$$

After base change, we get a diagram of the form

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] & X \times_Y \mathrm{Spec}(B \oplus_d M) \arrow[d] \\ \mathrm{Spec}(B \oplus_d M) \arrow[r, "\mathrm{id}"] & \mathrm{Spec}(B \oplus_d M) \end{tikzcd}$$

showing that, we can replace f by the projection $X \times_Y \mathrm{Spec}(B \oplus_d M) \to \mathrm{Spec}(B \oplus_d M)$, in particular, we can assume Y to be affine (this reduction is part of [TV08, Prop. 2.2.5.1]).

Further for the existence of an obstruction theory, we may assume without loss of generality that $Y \simeq \mathrm{Spec}(R)$ (see Lemma 4.71 and use that affine schemes have an obstruction theory by Lemma 4.74).

Let $n = -1$, then $X \simeq \mathrm{Spec}(A)$ and each A is a smooth R-algebra. In particular, we see with Lemma 4.73 and Proposition 3.56 that $L_{A/R}$ exists and is finite projective. The formal smoothness follows from Lemma 4.74.

Now assume $n \geq 0$ and let $p\colon U \simeq \coprod_{i \in I} \mathrm{Spec}(A_i) \to X$ be an n-atlas, where A_i are smooth R-algebras. Let B be a animated R-algebra, M be a connective B-module with $\pi_0 M = 0$ and $d \in \mathrm{Der}(B, M)$ a derivation.

Inf-cartesian. For this we will follow [TV08, Lem. 1.4.3.10].

By Lemma 4.66 any étale cover $B \to B'$ gives a cartesian square of the form

$$\begin{tikzcd} B' \oplus_d M \arrow[r] \arrow[d] & B' \arrow[d, "d"] \\ B' \arrow[r, "s"] & B' \oplus M, \end{tikzcd}$$

which covers the square induced by the derivation. So to check if $X(B \oplus_d M) \simeq X(B) \times_{X(B \oplus M)} X(B)$, we can pass to an étale cover of B. Therefore, we may assume that any image $x_1 \in X(B)$ of $x \in X(B) \times_{X(B\oplus M)} X(B)$ under the projection, lifts to a point in $u \in \mathrm{Spec}(A_i)(B)$, for some i. Next, we claim that the point x lifts to a point $y \in \mathrm{Spec}(A_i)(B) \times_{\mathrm{Spec}(A_i)(B\oplus M)} \mathrm{Spec}(A_i)(B)$.

To see this, consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(A_i)(B) \times_{\mathrm{Spec}(A_i)(B\oplus M)} \mathrm{Spec}(A_i)(B) \arrow[r, "f"] \arrow[d, "p"] & X(B) \times_{X(B\oplus M)} X(B) \arrow[d, "q"] \\
\mathrm{Spec}(A_i)(B) \arrow[r] & X(B).
\end{tikzcd}$$

Let $F(p)$ (resp. $F(q)$) denote the fiber of u (resp. x_1) under p (resp. q). We get a natural morphism $g\colon F(p) \to F(q)$. Moreover the fiber of f along x receives a natural morphism from $\mathrm{fib}_x(g)$. Therefore, to see that $\mathrm{fib}_x(f)$ is nonempty it is enough to show that $\mathrm{fib}_x(g)$ is nonempty. But now g is naturally identified, per definition, with the morphism $\Omega_{d',0}\,\mathrm{Der}_B(A_i, M) \to \Omega_{d',0}\,\mathrm{Der}_B(X, M)$, where d' is the derivation that is given by the image of u (note that $X(B) \to X(B \oplus M) \to X(B)$ is equivalent to the identity). Thus the fiber of g is given by $\Omega_{d',0}\,\mathrm{Der}_B(A_i/X, M)$, which is equivalent to $\Omega_{d',0}\,\mathrm{Hom}(L_{A_i/X,B}, M)$ by induction hypothesis. But now, again by induction hypothesis, we can find an étale cover of B such that $\pi_0\,\mathrm{Hom}(L_{A_i/X,B}, M) = 0$, since M is assumed to be connected, and therefore $\Omega_{d',0}\,\mathrm{Hom}(L_{A_i/X,B}, M)$ is nonempty.

Now consider the commutative digram

$$\begin{tikzcd}
\mathrm{Spec}(A_i)(B \oplus_d M) \arrow[r, "a"] \arrow[d] & \mathrm{Spec}(A_i)(B) \times_{\mathrm{Spec}(A_i)(B\oplus M)} \mathrm{Spec}(A_i)(B) \arrow[d] \\
X(B \oplus_d M) \arrow[r] & X(B) \times_{X(B\oplus M)} X(B).
\end{tikzcd}$$

By induction hypothesis this square is a pullback sqaure, further a is an equivalence by affineness. Since x lifts to a point in $y \in \mathrm{Spec}(A_i)(B) \times_{\mathrm{Spec}(A_i)(B\oplus M)} \mathrm{Spec}(A_i)(B)$, we see that the fiber at x is given by the fiber of a at y, which is nonempty and contractible (since affine schemes are inf-cartesian by Lemma 4.73).

Existence. Let us look at the fiber L_f of $L_{f\circ p} \to L_p$. By induction hypothesis L_p and $L_{f\circ p}$ exist and if f is smooth both are perfect with Tor-amplitude in $[-n, 0]$ and $[0, 0]$ respectively. In particular if f is smooth then L_f is perfect and has Tor-amplitude in $[-n-1, 0]$. We have to show that L_f satisfies the universal property of the cotangent complex for f at any point. Let $x\colon \mathrm{Spec}(B) \to X$ be a morphism. We may assume, that x factors through p, since p is an effective epimorphism, so we can pass to an étale cover B which factors through U. Let $y\colon \mathrm{Spec}(B) \to U$ be such a factorisation. We get a map

$$F\colon \mathrm{Der}_y(U, N) \to \mathrm{Der}_x(X, N)$$

for any connective B-module N, which is surjective.

To see the surjectivity of F, note that by induction hypothesis p is formally smooth.

Further any element in $d \in \mathrm{Der}_x(X, N)$ corresponds to a diagram of the form

$$\begin{tikzcd} \mathrm{Spec}(B \oplus N) \arrow[r, "d"] & X \\ \mathrm{Spec}(B). \arrow[u] \arrow[ur, "x"'] & \end{tikzcd}$$

Using the factorization by U, we get an element in $U(B) \times_{X(B)} X(B \oplus N)$. By formal smoothness of p, we can lift this element to an element in $U(B \oplus N)$ (note that $B \times_{0,B\oplus N[1],0} B \simeq B \oplus N$). But by construction this element has to be a derivation of U at y.

The fiber of F along the derivation, which is induced by $\mathrm{Spec}(B \oplus N) \to \mathrm{Spec}(B) \xrightarrow{x} X$ is given by $\mathrm{Der}_y(U/X, N)$. Thus, we get a fiber sequence

$$\mathrm{Hom}_B(L_p, N) \to \mathrm{Hom}_B(L_{f\circ p}, N) \to \mathrm{Der}_x(X, N).$$

After delooping[23] and surjectivity of F, we see that $\mathrm{Der}_x(X, N)$ is the fiber of

$$B\,\mathrm{Hom}_B(L_p, N) \to B\,\mathrm{Hom}_B(L_{f\circ p}, N),$$

where the prefix "B" denotes the deloop, and therefore $\mathrm{Der}_x(X, N) \simeq \mathrm{Hom}_B(L_f, N)$.

To see this, note that the map from $\mathrm{Der}_x(X, N)$ to the fiber of $B\,\mathrm{Hom}_B(L_p, N) \to B\,\mathrm{Hom}_B(L_{f\circ p}, N)$ is by the five-lemma an equivalence on the homotopy groups.

Formal smoothness. This is part of [TV08, Prop. 2.2.5.1].

Assume f is smooth and $x\colon \mathrm{Spec}(B) \to X$ is a point. By the above f has an obstruction theory. Therefore by Lemma 4.72 it suffices to show that $\pi_0 \mathrm{Hom}(L_{f,x}, M)$ is contractible. But this follows from

$$\pi_0 \mathrm{Hom}(L_{f,x}, M) \simeq \pi_0(L_{f,x}^\vee \otimes_B M) \simeq \pi_0 L_{f,x}^\vee \otimes_{\pi_0 B} \pi_0 M \simeq 0,$$

by connectedness of M (note that the above construction of the relative cotangent complex implies that L_f is perfect in the smooth case and therefore dualizable and the dual is connective). $\square$

Corollary 4.77. *Let $f\colon X \to Y$ be an n-geometric morphism of derived stacks. Then f is smooth if and only if $t_0 f$ is locally of finite presentation and L_f exists, is perfect and has Tor-amplitude in $[-n-1, 0]$.*

Proof. The proof is the same as in the spectral setting presented in [AG14, Prop. 4.46] using Proposition 4.76. But for the convenience of the reader, we recall the proof.

We may assume that $Y \simeq \mathrm{Spec}(A)$ is affine (use Lemma 4.62). Let us fix an n-atlas $p\colon \coprod_{i\in I} \mathrm{Spec}(T_i) \to X$. The "only if" part is Proposition 4.76 and the fact that the T_i are smooth A-algebras by construction and thus are locally of finite presentation (see Proposition 3.56).

[23] For a pointed ∞-category $\mathcal{C}$ a deloop of an object $c \in \mathcal{C}$ is an object $c' \in \mathcal{C}$, such that $c \simeq \Omega c'$. For $\mathcal{C} = \mathbb{S}$ the ∞-category of spaces, there is a deloop for every object. This follows from the effectivity of groupoid objects in $\mathbb{S}$ (see [Lur09, Cor. 6.1.3.20]) (the map $x \to *$, where $x \in \mathbb{S}$, defines a simplicial object, which extends via the colimit to a Čech nerve).

For the "if" part assume the $\pi_0 T_i$ are locally of finite presentation over $\pi_0 B$, L_f exists, is perfect and has Tor-amplitude in $[-n-1,0]$. We have a cofiber sequence

$$p^* L_f \to L_{U/A} \to L_p,$$

where by construction L_f and L_p are perfect with Tor-amplitude in $[-n-1,0]$ and $[-n,0]$ respectively (for the existence and Tor-amplitude of L_p, we use Proposition 4.76). Therefore, $L_{U/A}$ is also perfect with Tor-amplitude in $[-n-1,0]$. But since U is the disjoint union of affines the Tor-amplitude of $L_{U/A}$ is concentrated in $[0,0]$ and thus $L_{U/A}$ is perfect and finite projective, which implies that $\mathrm{Spec}(T_i) \to \mathrm{Spec}(A)$ is smooth (see Lemma 3.56). □

Corollary 4.78. *Let $f\colon X \to Y$ be an n-geometric morphism of derived stacks locally of finite presentation. Then L_f is perfect.*

Proof. Per definition of perfect quasi-coherent modules over a derived stack, we have to check that for any point $x\colon \mathrm{Spec}(A) \to X$ the cotangent complex $L_{f,x}$ is a perfect A-module. By Lemma 4.62, we know that the cotangent complex of the projection $\mathrm{pr}\colon X \times_{Y, f\circ x} \mathrm{Spec}(A)\,\mathrm{Spec}(A)$ at the point induced by x is equivalent to $L_{f,x}$. So without loss of generality, we may assume that $Y \simeq \mathrm{Spec}(B)$ is affine.

Since f is n-geometric and locally of finite presentation, we know that there exists an n-atlas $(p_i\colon \mathrm{Spec}(A_i) \to X)_{i\in I}$ such that A_i are locally of finite presentation over A. Since perfect quasi-coherent modules satisfy fpqc descent (see Remark 4.54), we have that $L_f \in \mathrm{QCoh}_{\mathrm{perf}}(X)$ if and only if each $p_i^* L_f$ is perfect. But by Lemma 4.63, we have the following cofiber sequence

$$p_i^* L_f \to L_{\mathrm{Spec}(A_i)/\mathrm{Spec}(A)} \to L_{p_i}.$$

Since A_i is locally of finite presentation over A, we know by Proposition 3.57, that $L_{\mathrm{Spec}(A_i)/\mathrm{Spec}(A)}$ is perfect and since by definition p_i is smooth, we have with Proposition 4.76 that indeed $p_i^* L_f$ is perfect. □

The last part of this section is dedicated to show that a geometric derived stack X is automatically hypercomplete for the étale topology. The idea is to show that for any animated R-algebra A, we have $X(A) \simeq \lim_n X(A_{\leq n})$. Then we reduce to the case, where we look at n-truncated sheaves, which are always hypercomplete.

Lemma 4.79. *Let X be an n-geometric derived stack for some $n \geq -1$ and A an animated ring. Then the natural morphism $X \to \lim_n X \circ \tau_{\leq n}$ is an equivalence.*

Proof. This is analogous to [Lur04, Prop. 5.3.7].

We will do this by induction over n. This is certainly true for if X is affine. So assume that $n \geq 0$ and let $p\colon U := \coprod_{i\in I} \mathrm{Spec}(A_i) \to X$ be an n-atlas.

By definition $U \times_X U$ is $(n-1)$-geometric, this also holds for every successive fiber product, i.e. every element of the Čech nerve $\check{C}(U/X)_\bullet$ is $(n-1)$-geometric. Since p is an effective epimorphism, we have that the natural map $\mathrm{colim}_\Delta\, \check{C}(U/X)_\bullet \to X$ is an equivalence. By induction hypothesis, we have for every $[n] \in \Delta$ that the natural map

$\check{C}(U/X)_{[n]} \to \lim \check{C}(U/X)_{[n]} \circ \tau_{\leq n}$ is an equivalence. Thus, also its colimit under Δ is an equivalence, so we get an induced commutative diagram

$$\begin{tikzcd} \operatorname{colim}_\Delta \check{C}(U/X)_{[n]} \arrow[r] \arrow[d] & \operatorname{colim}_\Delta \lim_n \check{C}(U/X)_{[n]} \circ \tau_{\leq n} \arrow[d] \\ X \arrow[r] & \lim_n X \circ \tau_{\leq n}, \end{tikzcd}$$

where the top arrow and left vertical arrow are equivalences and the right vertical arrow is a monomorphism. Thus the bottom vertical arrow is an equivalence if $\lim_n U \circ \tau_{\leq n} \to \lim_n X \circ \tau_{\leq n}$ is an effective epimorphism. Let $x \in \lim_n X(A_{\leq n})$ and consider the projection onto $X(A_{\leq 0})$, denoted by x_0. Then we can find an étale cover $\widetilde{\pi_0 A}$ of $A_{\leq 0} \simeq \pi_0 A$ such that x_0 has a lift in $U(\widetilde{\pi_0 A})$. By Proposition 3.59 there is an étale cover $\widetilde{A}$ of A such that $\pi_0 \widetilde{A} \simeq \widetilde{\pi_0 A}$. In particular, we see that we can lift the image of x_0 in $X(\widetilde{A}_{\leq 0})$ under $U(\widetilde{A}_{\leq 0}) \to X(\widetilde{A}_{\leq 0})$. Now let x_n be the image of x in $X(A_{\leq n})$. We will show the result by induction. Assume the argument holds for $n-1$. In particular, let u_{n-1} be the lift of x_{n-1} under $U(\widetilde{A}_{\leq n-1}) \to X(\widetilde{A}_{\leq n-1})$. It is enough to prove that we can find a lift of (u_{n-1}, x_n) under $U(\widetilde{A}_{\leq n}) \to U(\widetilde{A}_{\leq n-1}) \times_{X(\widetilde{A}_{\leq n-1})} X(\widetilde{A}_{\leq n})$, since then for all $n \in \mathbb{N}_0$ there is a lift u_n of x_n compatible with the maps in the limit, i.e. we get an element $u \in \lim_n U(\widetilde{A}_{\leq n})$ that maps to the image of x in $\lim_n X(\widetilde{A}_{\leq n})$. But this follows from formal smoothness of $U \to X$ (see Proposition 4.76) and the fact that the map $A_{\leq n} \to A_{\leq n-1}$ is a square zero extension (see Lemma 3.52). $\square$

Lemma 4.80. *Let $X \to Y$ be an n-geometric morphism and A be a k-truncated animated R-algebra. Then $X(A) \to Y(A)$ is $(n+k+1)$-truncated.*

Proof. This is a consequence of Lemma 4.79 and analogous to [Lur04, Cor. 5.3.8].

We have to show that the fiber of $X(A) \to Y(A)$ is $(n+k+1)$-truncated. By Lemma 4.79 it suffices to show that for all $n \in \mathbb{N}_0$ the map $X(A_{\leq j}) \to X(A_{\leq j-1}) \times_{Y(A_{\leq n-1})} Y(A_{\leq j})$ is $(n+j+1)$-truncated, whenever $j \leq k$. But from Lemma 3.52 and Lemma 4.72 the fiber of the previous map is given by the loop of $\operatorname{Hom}_{\operatorname{Mod}_A}(L_{X/Y,A}, \pi_j A[j])$ which by adjunction is $(n+j+1)$-truncated, since by definition $L_{X/Y}[n]$ is connective[24]. $\square$

Lemma 4.81. *Let X be an n-geometric stack, then X is hypercomplete.*

Proof. This is a direct consequence of Lemma 4.79 and 4.80 and the fact that truncated ∞-topoi are automatically hypercomplete. This is analogous to [Lur04, Cor. 5.3.9] but anyway we will explain this.

By Lemma 4.79, we have $X \simeq X \circ \tau_{\leq n}$ and for any k-truncated animated ring A, we have that $X(A)$ is $(n+k+1)$-truncated (see Lemma 4.80), in particular $X \circ \tau_{\leq n}$ is hypercomplete (see [Lur09, Lem. 6.5.2.9]) and since the ∞-topos of hypercomplete sheaves has limits, we have that X is hypercomplete. $\square$

[24]Note that $\pi_{n+j+1+k} \operatorname{Hom}_{\operatorname{Mod}_A}(L_{X/Y,A}, \pi_j A[j-1]) \cong \pi_0 \operatorname{Hom}_{\operatorname{Mod}_A}(L_{X/Y,A}[n+j+1+k], \pi_j A[j+1])$ and since $L_{X/Y,A}[n+j+1+k] \in (\operatorname{Mod}_A)_{\geq j+1+k}$ and $\pi_j A[j+1] \in (\operatorname{Mod}_A)_{\leq -j+1}$, we see by definition of the t-structures that $\pi_{n+j+1+k} \operatorname{Hom}_{\operatorname{Mod}_A}(L_{X/Y,A}, \pi_j A[j+1]) \cong 0$ for $k \geq 1$.

5 The stack of perfect modules

In this section, we want to prove that the derived stack of perfect modules is locally geometric. This was already proven in [TV07] in the model categorical setting and in [AG14] in the spectral setting, but we recall the proof in its entirety in our setting.

We recall some lemmas needed for the proof, as they will become important later on when analyzing the substacks of derived F-zips.

Lemma 5.1. *Let A be a commutative ring and P be a perfect complex of A-modules and let $n \in \mathbb{N}_0$. Further, for $k \in \mathbb{Z}$ let $\beta_k\colon \operatorname{Spec}(A)_{\mathrm{cl}} \to \mathbb{N}_0$ be the function given by $s \mapsto \dim_{\kappa(s)} \pi_k(P \otimes_A \kappa(s))$. Then $\beta_k^{-1}([0,n])$ is quasi-compact open.*

Proof. By [Sta19, 0BDI] β_k is upper semi-continuous and locally constructible. As $\operatorname{Spec}(A)_{\mathrm{cl}}$ is affine it is quasi-compact quasi-separated and so we see that $\beta_k^{-1}([0,n])$ is quasi-compact open. $\square$

Remark 5.2. Let A be a commutative ring and P be a perfect complex of A-modules. Let $I \subseteq \mathbb{Z}$ be a finite subset and for $k \in \mathbb{Z}$ let β_k be as in Lemma 5.1 . Assume that $\beta_i \neq 0$ for $i \in I$ and zero everywhere else. Then using [Sta19, 0BCD,066N], we see that P has Tor-amplitude in $[\min(I), \max(I)]$.

Lemma 5.3. *Let A be a commutative ring and P be a perfect complex of A-modules. Then there exists a quasi-compact open subscheme $U \subseteq \operatorname{Spec}(A)_{\mathrm{cl}}$ with the following property,*

- *an affine scheme morphism $\operatorname{Spec}(B)_{\mathrm{cl}} \to \operatorname{Spec}(A)_{\mathrm{cl}}$ factors through U if and only if $P \otimes_A B \simeq 0$.*

Proof. Let β_k be as in Lemma 5.1. Then we set

$$U \coloneqq \bigcap_{k \in \mathbb{Z}} \beta_k^{-1}(\{0\}).$$

As P is perfect, so in particular has finite Tor-amplitude, this intersection has only finitely many pieces that are non equal to $\operatorname{Spec}(A)_{\mathrm{cl}}$. Therefore, U is a finite intersection of quasi-compact opens in an affine scheme (see Lemma 5.1) and thus quasi-compact open.

Now assume we have a morphism $\operatorname{Spec}(B)_{\mathrm{cl}} \to \operatorname{Spec}(A)_{\mathrm{cl}}$ such that $P \otimes_A B = 0$. Then certainly for any $b \in B$ and all $i \in \mathbb{Z}$ we have $\dim_{\kappa(b)} \pi_i(P \otimes_A B \otimes_B \kappa(b)) = 0$. Let $a \in \operatorname{Spec}(A)$ be the image of b. Then for all $i \in \mathbb{Z}$ we have the following equalities

$$\begin{aligned}
\dim_{\kappa(b)} \pi_i(P \otimes_A B \otimes_B \kappa(b)) &= \dim_{\kappa(b)} \pi_i(P \otimes_A \kappa(a) \otimes_{\kappa(a)} \kappa(b)) \\
&= \dim_{\kappa(b)} \pi_i(P \otimes_A \kappa(a)) \otimes_{\kappa(a)} \kappa(b) \\
&= \dim_{\kappa(a)} \pi_i(P \otimes_A \kappa(a)),
\end{aligned}$$

where we use flatness of field extensions in the second equality. Therefore, we see that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ factors through U.

For the other direction assume that $\mathrm{Spec}(B)_{\mathrm{cl}} \to \mathrm{Spec}(A)_{\mathrm{cl}}$ factors through U. Then for any $b \in \mathrm{Spec}(B)$ and any $i \in \mathbb{Z}$, we have that $\pi_i(P \otimes_A B \otimes_B \kappa(b)) = 0$. By Remark 5.2, we see that $P \otimes_A B$ is given by a finite projective module M concentrated in one degree. The fiberwise dimension of M is equal to 0 by assumption and thus by Nakayama $M = 0$. □

The next lemma shows that the vanishing locus of perfect complexes is quasi-compact open. This will be applied to the cofiber of morphisms of perfect complexes. In particular, the locus classifying equivalences between fixed perfect modules is therefore quasi-compact open.

Lemma 5.4. *Let $A \in \mathrm{AR}_R$ and P be a perfect A-module. Define the derived stack V_P via $V_P(B) =$ full sub-∞-category of $\mathrm{Hom}_{\mathrm{AR}_R}(A, B)$ consisting of morphisms $u\colon A \to B$ such that $P \otimes_{A,u} B \simeq 0$. This is a quasi-compact open substack of $\mathrm{Spec}(A)$.*

Proof. This is [TV07, Prop. 2.23] translated to our setting. But for the convenience of the reader, we give a proof.

Consider $Q \coloneqq P \otimes_A \pi_0 A$. Then Q is a perfect complex of $\pi_0 A$-modules. Lemma 5.3 shows that there is a quasi-compact open subscheme $U \subseteq \mathrm{Spec}(\pi_0 A)_{\mathrm{cl}}$, such that for any point $u\colon \mathrm{Spec}(R')_{\mathrm{cl}} \to \mathrm{Spec}(\pi_0 A)_{\mathrm{cl}}$, where R' is a commutative ring, the module $Q \otimes_{\pi_0 A} R'$ is isomorphic to 0 if and only if u factors through U.

Let $f_1, \dots, f_n \in \pi_0 A$, such that the $\pi_0 A_{f_i}$ covers U. Then $V \coloneqq \mathrm{Im}(\coprod_{i=1}^n \mathrm{Spec}(A[f_i^{-1}]))$, the image of $\coprod_{i=1}^n \mathrm{Spec}(A[f_i^{-1}]) \to \mathrm{Spec}(A)$, is equivalent to V_P.

Indeed, take a morphism $u\colon A \to B$ in AR_R. Then $u \in V(B)$ if and only if there exists an i such that $\pi_0 u(f_i)$ is étale locally invertible in $\pi_0 B$ (see Remark 4.42). This is equivalent to $P \otimes_A \pi_0 B \simeq 0$ by the choice of the f_i. This again is equivalent to $P \otimes_A B \simeq 0$.

To see this, assume the $P \otimes_A B \not\simeq 0$ and take the minimal $i \in \mathbb{Z}$ with $\pi_i(P \otimes_A B) \not\simeq 0$. Consider the Tor spectral sequence

$$E_2^{p,q} = \mathrm{Tor}_p^{\pi_* B}(\pi_*(P \otimes_A B), \pi_0 B)_q \Rightarrow \pi_{p+q}(P \otimes_A \pi_0 B).$$

By the definition of the graded tensor product, we see that $E_2^{0,q} = \pi_q(P \otimes_A B)$ for $q = i$ and 0 for $q < i$. As explained in Remark 3.55, we can choose a graded free resolution of $P \otimes_A B$ such that each term of the resolution concentrated in degrees $\geq i$. Therefore, the spectral sequence is concentrated in the quadrant where the left lower corner is at $p = 0$ and $q = i$. Thus, $\pi_i(P \otimes_A B) \cong E_2^{0,i} \cong \pi_i(P \otimes_A \pi_0 B) \cong 0$ contradicting the assumption.

The geometricity follows from Lemma 4.28. □

Next, we show that the stack classifying morphisms between perfect modules[25] is actually geometric and in good cases smooth. Since derived F-zips will come with two bounded perfect filtrations (i.e. finite chains of morphisms of perfect modules), this lemma is crucial for the geometricity of derived F-zips.

[25]Note that for two perfect A-modules P, Q over some animated ring A, we have $\mathrm{Hom}_{\mathrm{Mod}_B}(P \otimes_A B, Q \otimes_A B) \simeq \mathrm{Hom}_{\mathrm{Mod}_A}(P \otimes Q^\vee, B)$ as shown in the proof of Lemma 5.7.

Lemma 5.5. *Let A be an animated R-algebra. Let P be a perfect A-module with Tor-amplitude concentrated in $[a,b]$ with $a \leq 0$. Then the derived stack*

$$F_P^A \colon \mathrm{AR}_A \to \mathbb{S}$$
$$B \mapsto \mathrm{Hom}_{\mathrm{Mod}_A}(P, B)$$

is $(-a-1)$-geometric and locally of finite presentation over $\mathrm{Spec}(A)$[26]. *Further, the cotangent complex of F_P at a point $x\colon \mathrm{Spec}(B) \to F_P^A$ is given by*

$$L_{F_P,x} \simeq P \otimes_A B.$$

In particular, if $b \leq 0$, then F_P is smooth.

Proof. Before showing the geometricity, let us calculate the space of derivations of F_P and hence the cotangent complex.

Let $x\colon \mathrm{Spec}(B) \to F_P^A$ be a morphism of derived stacks corresponding to a morphism $f\colon P \to B$ in Mod_A and M be a connective B-module. We have that $\mathrm{Der}_x(F_P^A/A, M)$ is given by the fiber of $\mathrm{Hom}_A(P, B \oplus M) \to \mathrm{Hom}_A(P, B)$ at f. The underlying R-module of $B \oplus M$ is per construction the direct sum of the underlying R-module of B and of M. Therefore, any morphism $P \to B \oplus M$ is uniquely up to homotopy characterized by a morphism $P \to B$ and $P \to M$ and thus, we see that $\mathrm{Der}_x(F_P^A/A, M) \simeq \mathrm{Hom}_A(P, M) \simeq \mathrm{Hom}_A(P \otimes_A B, M)$. Hence, we have $L_{F_P,x} \simeq P \otimes_A B$.

Let us conclude the rest of the proof, which we will prove by induction on a.

If $a = 0$, then P is connective and $F_P^A \simeq \mathrm{Hom}_{\mathrm{AR}_A}(-, \mathrm{Sym}_A P)$ which has the desired properties.

Now assume $a < 0$. By Lemma 3.33 we have $P \simeq \mathrm{fib}(Q \to M[a+1])$, where Q has Tor-amplitude in $[a+1, b]$ and M is a finite projective A-module. Thus we get a fiber sequence

$$F_{M[a+1]}^A \to F_Q^A \to F_P^A \to F_{M[a]}.$$

By induction hypothesis F_Q is $(-a-2)$-geometric and locally of finite presentation. We will see that the map $p\colon F_Q^A \to F_P^A$ is an effective epimorphism.

Indeed, note that the above fiber sequence and projectivity of M imply that $\pi_0 p$ is surjective and thus p is an effective epimorphism (see Remark 4.10).

The diagonal $F_Q^A \times_{F_P^A} F_Q^A$ is given by $F_{Q \oplus_P Q}^A$[27], which will be $(-a-2)$-geometric with smooth projections to F_Q^A.

To see this, note that we have a fiber sequence $Q \to Q \oplus_P Q \to M[a+1]$ which has a retract. Thus the natural map $Q \oplus_P Q \to Q \oplus M[a+1]$ is an equivalence on the level of homotopy groups by the splitting lemma (the induced exact sequences are short exact, using the retract) and therefore $Q \oplus_P Q \simeq Q \oplus M[a+1]$. Hence, $F_{Q \oplus_P Q}^A \simeq F_Q^A \times F_{M[a+1]}^A$, which is the pullback of $(-a-2)$-geometric stack and thus itself geometric.

Also the projection to F_Q^A is smooth, because $F_{M[a+1]}^A$ is smooth (the smoothness of $F_{M[a+1]}^A$ follows since $L_{F_{M[a+1]}^A,x} \simeq M[a+1] \otimes_A B$ at a point $x\colon \mathrm{Spec}(B) \to F_{M[a+1]}^A$ and thus has Tor-amplitude in $[a+1, 0]$, which concludes (see Corollary 4.77)).

[26] Certainly, we can view F_P^A as a derived stack over R with a morphism to $\mathrm{Spec}(A)$. So for any animated R-algebra C that does not come with a morphism $A \to C$ the value of F_P^A is empty.

[27] Here $Q \oplus_P Q$ is defines as the pushout of the morphism $P \to Q$ with itself.

By Proposition 4.41, we see that $F_P^A \to \mathrm{Spec}(A)$ is a quasi-compact $(-a-1)$-geometric stack locally of finite presentation.

If $b \leq 0$, then $L_{F_P^A}$ is perfect with Tor-amplitude concentrated in degree $[a, 0]$ and therefore F_P^A is smooth by Corollary 4.77. □

Remark 5.6. A variant of Lemma 5.5 in the spectral setting can be found in [AG14, Thm. 5.2]. Alternatively, one can look at the proofs given in [TV07, Lem. 3.9] and [TV07, 3.12] to construct a proof in the model categorical setting.

Lemma 5.7. *The diagonal map* $\mathrm{Perf}^{[a,b]} \to \mathrm{Perf}^{[a,b]} \times_R \mathrm{Perf}^{[a,b]}$ *is* $(b-a)$*-geometric and locally of finite presentation.*

Proof. This is part of the proof of [AG14, Thm. 5.6] translated to our setting. For the convenience of the reader, we give a proof.

A morphism $\mathrm{Spec}(A) \to \mathrm{Perf}^{[a,b]} \times_R \mathrm{Perf}^{[a,b]}$ corresponds to two perfect modules P, Q with Tor-amplitude concentrated in $[a, b]$. The pullback under the diagonal classifies equivalences between P and Q. This is an open, 0-geometric substack of

$$\mathrm{Hom}_{\mathrm{Mod}_A}(P \otimes_A Q^\vee, -) \simeq \mathrm{Hom}_{\mathrm{Mod}_A}(P, Q \otimes_A -) \simeq \mathrm{Hom}_{\mathrm{Mod}_-}(P \otimes_A -, Q \otimes_A -),$$

(note that perfect modules are dualizable).

To see this, note that for any morphism $\mathrm{Spec}(B) \to \mathrm{Hom}_{\mathrm{Mod}_A}(P \otimes_A Q^\vee, -)$, given by a morphism $\varphi\colon P \otimes_A B \to Q \otimes_A B$, the stack $\mathrm{Equiv}(P, Q) \times_{\mathrm{Hom}_{\mathrm{Mod}_A}(P \otimes_A Q^\vee, -)} \mathrm{Spec}(B)$ classifies morphisms $u\colon B \to C$, where $\mathrm{cofib}\, \varphi \otimes_{B,u} C \simeq 0$, which is an open, 0-geometric substack of $\mathrm{Spec}(B)$ by Lemma 5.4.

Now $P \otimes Q^\vee$ is a perfect module of Tor-dimension $[a-b, b-a]$ (see Lemma 3.33) and thus Lemma 5.5 concludes the proof. □

Definition 5.8. Let $n \in \mathbb{N}$ and $A \in \mathrm{AR}_R$. We denote the ∞-category of finite projective A-modules of rank n with $\mathrm{BGL}_n(A)$.

Lemma 5.9. *Let* $n \in \mathbb{N}$*. The functor* $A \mapsto \mathrm{BGL}_n(A)$ *from* AR_R *to* Cat_∞ *satisfies fpqc descent.*

Proof. We already know that modules satisfy decent so it is enough to check that an A-module M is finite projective of rank n if it is after base change to an fpqc-cover $(A \to A_i)_{i \in I}$. Note that $\pi_0 A \to \pi_0 A_i$ is faithfully flat for every $i \in I$. Now assume that $M \otimes_A A_i$ is finite projective of rank n. Then it is in particular flat and we will show first that M is flat over A. By flatness, the natural map

$$\pi_j A_i \otimes_{\pi_0 A} \pi_0 M \cong \pi_j A_i \otimes_{\pi_0 A_i} \pi_0 M \otimes_{\pi_0 A} \pi_0 A_i \cong \pi_j A_i \otimes_{\pi_0 A_i} \pi_0(M \otimes_A A_i) \to \pi_j(M \otimes_A A_i)$$

is an equivalence. By flatness of $A \to A_i$, we have $\pi_j A_i \cong \pi_j A \otimes_{\pi_0 A} \pi_0 A_i$. Hence, we have $\pi_j A_i \otimes_{\pi_0 A} \pi_0 M \cong \pi_j A \otimes_{\pi_0 A} \pi_0 A_i \otimes_{\pi_0 A} \pi_0 M$. By faithfully flatness the map $\pi_j A \otimes_{\pi_0 A} \pi_0 M \to \pi_j M$ is an equivalence if and only if it so after base change to $\pi_0 A_i$ for all $i \in I$. But the above shows that this base change gives the map

$$\pi_j A \otimes_{\pi_0 A} \pi_0 A_i \otimes_{\pi_0 A} \pi_0 M \to \pi_j(M \otimes_A A_i)$$

and flatness of A_i over A shows that $\pi_j(M \otimes_A A_i) \cong \pi_j M \otimes_{\pi_0 A} A_i$ (see [Lur17, Prop. 7.2.2.13]) so indeed, $\pi_j A \otimes_{\pi_0 A} \pi_0 M \to \pi_j M$ is an equivalence. Therefore, M is flat over A.

Now a flat module is finite projective of rank n if it is so on π_0 (see Lemma 3.25 and use the definition of finite projectiveness) but this follows from classical faithfully flat descent. □

Remark 5.10. The inclusion of $\mathbb{S} \hookrightarrow \mathrm{Cat}_\infty$ is left adjoint to the functor $(-)^\simeq$ that passes to the largest Kan complex contained in an ∞-category (see [Lur09, Prop. 1.2.5.3]). Therefore if $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$ is a (hypercomplete) sheaf for some Grothendieck topology on $\mathcal{C}$ then also $F^\simeq := (-)^\simeq \circ F$ is one.

Definition 5.11. We define the derived stack classifying vector bundles as the stack

$$\begin{aligned} \mathrm{BGL}_{n,R}\colon \mathrm{AR}_R &\to \mathbb{S} \\ A &\mapsto \mathrm{BGL}_n(A)^\simeq. \end{aligned}$$

Further, we denote by $\mathrm{GL}_{n,R}$ the loop under the map $\mathrm{Spec}(R) \to \mathrm{BGL}_{n,R}$, which is given for an animated R-algebra A by $* \mapsto A^n$, i.e. we have the following pullback diagram in dSt_R

$$\begin{array}{ccc} \mathrm{GL}_{n,R} & \longrightarrow & \mathrm{Spec}(R) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & \mathrm{BGL}_{n,R} \end{array}$$

(note that for a commutative ring A, we have that $\mathrm{BGL}_{n,R}(A)$ is the groupoid of rank n vector bundles on A and thus $\mathrm{GL}_{n,R}(A)$ is indeed given by the points of the general linear group scheme of rank n).

Lemma 5.12. *Let* Proj_R *denote the derived stack classifying finite projective modules. Then* $\mathrm{Proj}_R \simeq \coprod_{n\in\mathbb{N}} \mathrm{BGL}_{n,R}$*, in particular* Proj_R *is 1-geometric and smooth.*

Further, $\mathrm{GL}_{n,R}$ *is an affine derived scheme and* Proj_R *has an affine diagonal.*

Proof. The proof is the same as [TV08, Cor. 1.3.7.12] but for the convenience of the reader, we give a sketch.

That $\mathrm{Proj}_R \simeq \coprod_{n\in\mathbb{N}} \mathrm{BGL}_{n,R}$, where $\mathrm{BGL}_{n,R}$ denotes the stack of finite projective modules of rank n, is clear. So it suffices to show that $\mathrm{BGL}_{R,n}$ is a 1-geometric smooth stack. It is enough to show that $\mathrm{GL}_{n,R} \to \mathrm{Spec}(R)$ is 0-geometric smooth. Then $\mathrm{GL}_{n,R} \to \mathrm{Spec}(R)$ defines a 0-Segal groupoid (see [TV08, Def. 1.3.4.1]) and $\mathrm{BGL}_{R,n}$ is 1-geometric (see [TV08, Prop. 1.3.4.2]). That $\mathrm{BGL}_{n,R}$ is smooth follows from the fact that the natural morphism $\mathrm{Spec}(R) \to \mathrm{BGL}_{n,R}$ gives a 1-atlas.

The claim about $\mathrm{GL}_{n,R}$ follows in the following way. The stack $\mathrm{GL}_{n,R}$ is equivalent to the stack classifying automorphisms of R^n, i.e. $\mathrm{GL}_{n,R}(A) \simeq \mathrm{Equiv}_A(A^n)$, for $A \in \mathrm{AR}_R$. But by Lemma 5.4 this is a 0-geometric open substack of $F_{R^{n^2}} \simeq \mathrm{Spec}(\mathrm{Sym}_R(R^{n^2}))$, which is a (-1)-geometric smooth stack.

Alternatively, one could follow [TV08, Prop. 1.3.7.10] and show directly that the inclusion $\iota\colon \mathrm{GL}_{n,R} \hookrightarrow F_{R^{n^2}}$ is representable and étale by showing that for any point

$x \in F_{R^{n^2}}(A)$, we have $\iota^{-1}(x) \simeq \mathrm{Spec}(A[\det(x)^{-1}])$. In particular, we see in this way that $\mathrm{GL}_{n,R}$ is representable by an affine derived scheme. This also shows that $\mathrm{BGL}_{n,R}$ has an affine diagonal, since $\mathrm{Spec}(R) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R) \to \mathrm{BGL}_{n,R} \times_{\mathrm{Spec}(R)} \mathrm{BGL}_{n,R}$ is a 1-atlas by the above and so, we have a pullback diagram of the form

$$\begin{tikzcd} \mathrm{GL}_{n,R} \arrow[r] \arrow[d] & \mathrm{Spec}(R) \simeq \mathrm{Spec}(R) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R) \arrow[d] \\ \mathrm{BGL}_{n,R} \arrow[r, "\Delta"] & \mathrm{BGL}_{n,R} \times_{\mathrm{Spec}(R)} \mathrm{BGL}_{n,R}, \end{tikzcd}$$

which shows that the diagonal is affine, since this can be tested after passing to a cover by affines (see [TV08, Lem. 1.3.2.8]). $\square$

Remark 5.13. For the proof of the next theorem, we want to remark some generalities about pullbacks of Kan complexes.

Let X, Y be Kan complexes, i.e. elements in $\mathbb{S}$, and assume we have a morphism $X \to \mathrm{Fun}(\partial\Delta^1, Y)$ in $\mathbb{S}$. We want to compute the following pullback in $\mathbb{S}$

$$\begin{tikzcd} W \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, Y) \arrow[d, "i"] \\ X \arrow[r] & \mathrm{Fun}(\partial\Delta^1, Y), \end{tikzcd}$$

where i is given by the restriction. In general this is a pullback in the ∞-categorical sense, which can also be computed on the level of model categories via the homotopy pullback (recall that $\mathbb{S}$ is the ∞-category associated to the model category of simplicial sets with the usual model structure (weak equivalences are given by weak equivalences of the underlying Kan complexes and fibrations are Kan fibrations)). We claim that the homotopy pullback of the underlying Kan complexes is equivalent in $\mathbb{S}$ to the ordinary pullback of simplicial sets if i is a Kan fibration (i.e. i is a fibration in the model category of simplicial sets and since each simplicial set involved is already a Kan complex they are per definition fibrant).

Indeed, this is a classical result and remarked in [Lur09, Rem. A.2.4.5] but we will shortly sketch the idea behind it. Let $\mathbf{A}$ be a combinatorial model category (e.g. the category of simplicial sets with the model structure explained above) and let I be the diagram category given by three objects $\{0, 1, 2\}$ together with morphisms $0 \to 2$, $1 \to 2$ and identities. One can attach the injective model structure onto the functor category $\mathrm{Fun}(I, \mathbf{A})$ by defining weak equivalences to be pointwise weak equivalences and defining cofibrations also pointwise (the fibration are then given by certain lifting properties, which we will not discuss on detail). The homotopy limit is defined as the right Quillen adjoint of the constant functor $\mathbf{A} \to \mathrm{Fun}(I, \mathbf{A})$, which is weakly equivalent to the the limit of a *fibrant* diagram in $\mathrm{Fun}(I, \mathbf{A})$, i.e. for a diagram $a \to c \leftarrow b$, the homotopy limit is defined as an object that is weakly equivalent to ordinary limit of $a' \to c' \leftarrow b'$, where $a' \to c'$ and $b' \to c'$ are fibrations, c' is fibrant and we have a commutative diagram of the form

$$\begin{tikzcd} a \arrow[r] \arrow[d] & c \arrow[d] & b \arrow[l] \arrow[d] \\ a' \arrow[r] & c' & b', \arrow[l] \end{tikzcd}$$

where the vertical arrows are weak equivalences (this is analogous to the theory of homotopy pushouts, which can be found in [Lur09, §A.2.4]). Now let I_R be the full subcategory of I generated by the elements 0 and 2 and define I_L as the full subcategory of I generated by the elements 1 and 2. The tuple (I_L, I_R) makes I into a Reedy category on I (see [Lur09, §A.2.9] for more on Reedy categories). As explained in [Lur09, Proposition A.2.9.19] there is a model structure on $\mathrm{Fun}(I, \mathbf{A})$ corresponding to the Reedy structure on I called the Reedy model structure. Important for us is that a diagram $a \to c \leftarrow b$ is fibrant for the Reedy model structure if a, b are fibrant and either $a \to c$ or $b \to c$ is a fibration. Further, as remarked in [Lur09, Rem. A.2.9.23] the Reedy model structure and the injective model structure are Quillen equivalent via the identity functor. Therefore, if we have a fibrant diagram with respect to the Reedy model structure, then the homotopy limit is per definition weak equivalent to the ordinary limit in $\mathbf{A}$. Since the homotopy limit in this case is the homotopy pullback, we are done.

But that i is a Kan fibration follows from [Lur21, Cor. 3.1.3.3] and hence we can compute the above pullback in the ∞-category $\mathbb{S}$ via the limit of the underlying simplicial sets.

Now assume that Y is an arbitrary ∞-category. We want to compute the following pullback in $\mathbb{S}$

$$\begin{tikzcd} W \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, Y)^{\simeq} \arrow[d, "i"] \\ X \arrow[r] & \mathrm{Fun}(\partial\Delta^1, Y)^{\simeq}, \end{tikzcd}$$

where i is naturally given by applying the functor $(-)^{\simeq}$ to the restriction. If i is a Kan fibration, then we can apply the argument above. In general it may not be clear if i is a Kan fibration. But in this case, we have that the natural morphism $F\colon \mathrm{Fun}(\Delta^1, Y) \to \mathrm{Fun}(\partial\Delta^1, Y)$ is an isofibration of ∞-categories (see [Lur21, 01F3]), meaning that it is an inner fibration on the level of homotopy categories, we have the following property: if $x \in h\,\mathrm{Fun}(\Delta^1, Y)$ and we have an isomorphism $u'\colon y \xrightarrow{\sim} F(x)$ then, there exists a $x' \in h\,\mathrm{Fun}(\Delta^1, Y)$ with an isomorphism $u\colon x' \to x$ such that $F(u) = u'$. In particular, [Lur21, Prop. 4.4.3.7] implies that i is a Kan fibration.

Theorem 5.14. *The derived stack*

$$\begin{aligned} \mathrm{Perf}_R\colon \mathrm{AR}_R &\to \mathbb{S} \\ A &\mapsto (\mathrm{Mod}_A^{\mathrm{perf}})^{\simeq} \end{aligned}$$

is locally geometric and locally of finite presentation.

To be more specific, we can write $\operatorname{Perf}_R = \operatorname{colim}_{a \le b} \operatorname{Perf}_R^{[a,b]}$*, where* $\operatorname{Perf}_R^{[a,b]}$ *is the moduli space consisting of perfect modules which have Tor-amplitude concentrated in degree* $[a,b]$ *and each* $\operatorname{Perf}_R^{[a,b]}$ *is* $(b-a+1)$*-geometric and locally of finite presentation and the inclusion* $\operatorname{Perf}_R^{[a,b]} \hookrightarrow \operatorname{Perf}_R$ *is a quasi-compact open immersion. If* $b - a \le 1$ *then* $\operatorname{Perf}_R^{[a,b]}$ *is in fact smooth.*

Proof. The proof in the model categorical setting can be found in [TV07, Prop. 3.7] and in the spectral setting in [AG14, Thm. 5.6]. The latter follows the former with few changes for readability. We will follow the proof presented in the latter using our setting.

We show that $\operatorname{Perf}^{[a,b]}$ is $n+1$-geometric, where $n = b - a$, by induction over n.

For $n = 0$, we are done, since then we have $\operatorname{Proj} \simeq \operatorname{Perf}^{[a,a]}$ (see Lemma 3.33), which is 1-geometric and locally of finite presentation by Lemma 5.12.

Now let $n > 0$ and assume $\operatorname{Perf}^{[a+1,b]}$ is n-geometric and locally of finite presentation. Let U be defined via the pullback diagram of derived stacks

$$\begin{tikzcd} U \arrow[r] \arrow[d, "p"'] & \operatorname{Fun}(\Delta^1, \operatorname{Mod}^{\mathrm{perf}})^{\simeq} \arrow[d] \\ \operatorname{Perf}^{[a+1,b]} \times_R \operatorname{Perf}^{[a+1,a+1]} \arrow[r] & \operatorname{Fun}(\partial\Delta^1, \operatorname{Mod}^{\mathrm{perf}})^{\simeq}. \end{tikzcd}$$

Let $\operatorname{Spec}(A) \to \operatorname{Perf}^{[a+1,b]} \times_R \operatorname{Perf}^{[a+1,a+1]}$ be given by (P,Q), where P is a perfect A-module of Tor-amplitude $[a+1,b]$ and Q is the $a+1$ shift of a finite projective A-module, then $p^*(P,Q)$ classifies morphisms between those, i.e. $p^*(P,Q) \simeq \operatorname{Spec}(\operatorname{Sym}(P \otimes_A Q^\vee))$ (note that $P \otimes_A Q^\vee$ is perfect and has Tor-amplitude in $[0, b-(a+1)]$ and thus is connective (see Lemma 3.33). Therefore p is (-1)-geometric and locally of finite presentation and with Lemma 5.7 and 4.38, we see that U is n-geometric and locally of finite presentation. Note that if $b - a \le 1$ then p is even smooth and using that $\operatorname{Perf}^{[a+1,a+1]}$ is smooth, we see that U is smooth.

By sending a morphism to its fiber, we get a morphism of derived stacks $q\colon U \to \operatorname{Perf}^{[a,b]}$. Using Proposition 4.39 with Lemma 5.7, we see that q is n-geometric, so it suffices to show that q is also smooth and an effective epimorphism.

That it is an effective epimorphism follows from Lemma 3.33. To check smoothness let $\operatorname{Spec}(A) \to \operatorname{Perf}^{[a,b]}$ be a morphism classified by a perfect A-module P with Tor-amplitude in $[a,b]$. Then $q^{-1}(P)(B)$, for some animated A-algebra B, consists of morphisms of perfect B-modules $f\colon Q \to M[a+1]$ such that $\operatorname{fib}(f) \simeq P \otimes_A B$, where Q has Tor-amplitude in $[a+1,b]$ and M is finite projective. Since locally every finite projective module is free of finite rank, we can decompose

$$q^{-1}(P) \simeq \coprod_m q^{-1}(P)_m,$$

where $q^{-1}(P)_m$ is the substack of $q^{-1}(P)$, where the classified morphisms have codomain given by the $a+1$ shift of free modules of rank m. The stack $q^{-1}(P)_m$ is equivalent to the stack classifying morphisms $A^m[a] \to P$, where the cofiber has Tor-amplitude in $[a+1,b]$, which is equivalent to $A^m[a] \to P$ beeing a surjection on π_a.

To see the equivalence of stacks let us look at $q^{-1}(P)_m(B)$. These are all morphisms $f\colon Q \to B^m[a+1]$ such that $\mathrm{fib}(f) \simeq P \otimes_A B$, again Q is a perfect B-module with Tor-amplitude in $[a+1,b]$. Since Mod_A is stable, we see that $P\otimes_A B \to Q \to B^m[a+1]$ is a fiber diagram if and only if it is a cofiber diagram and thus after shift we see that $q^{-1}(P)_m(B)$ consists of morphisms $g\colon B^m[a] \to P \otimes_A B$ such that its cofiber $\mathrm{cofib}(g)$ has Tor-amplitude in $[a+1,b]$. By Lemma 3.33 $\mathrm{cofib}(g)$ has Tor-amplitude in $[a,b]$. Since after tensoring $M^m[a] \to P \otimes_A M \to \mathrm{cofib}(g) \otimes_B M$, where M is a discrete $\pi_0 B$-module, we still have a cofiber sequence it is enough to check that $\pi_a(g \otimes \mathrm{id}_M)$ is a surjection. But since $\pi_a(P \otimes_A M) = \pi_a(P) \otimes_{\pi_0 A} M$ (use the degeneracy of the Tor-spectral sequence at $(0,a)$) and the ordinary tensor product of $\pi_0 A$-modules preserves surjections it is enough to check that $\pi_a g$ is surjective. And therefore $q^{-1}(P)_m(B)$ consists of morphisms $B^m[a] \to P \otimes_A B$, which are surjective on π_a (obviously any morphism $B^m[a] \to P \otimes_A B$ with cofiber having Tor-amplitude in $[a+1,b]$ is surjective on π_a).

By this characterization the stack $q^{-1}(P)_m$ is an open substack of $F^A_{(P^\vee)^m[a]}$ (see Lemma 5.5 for notation).

To see this, let $\mathrm{Spec}(B) \to F_{(P^\vee)^m[a]}$ be given by a morphism $\xi\colon B^m[a] \to P \otimes_A B$. Let Z be the pullback of $\mathrm{Spec}(B)$ along the inclusion $q^{-1}(P)_m \hookrightarrow F_{(P^\vee)^m[a]}$. In particular, for any animated A-algebra C, we have that $Z(C)$ consists of those morphisms $f\colon B \to C$, such that $\pi_a f^*\xi$ is surjective. Since $P\otimes_A B$ is perfect and has Tor-amplitude in $[a,b]$ its homotopy group $\pi_a(P\otimes_A B)$ is finitely presented (see [Lur17, Cor. 7.2.4.5]). Therefore, being surjective is an open condition on $\pi_0 B$ (see [GW10, Prop. 8.4]). Further refining by principal affine opens $D(f_i) \subseteq \mathrm{Spec}(\pi_0 B)$, we get an open substack $\bigcup \mathrm{Spec}(B[f_i^{-1}])$ of $\mathrm{Spec}(B)$. Now a morphism $u\colon B \to C$ is in $Z(C)$ if and only if étale locally there is an i such that $\pi_0 u(f_i)$ is invertible. Therefore, $Z \simeq \bigcup \mathrm{Spec}(B[f_i^{-1}])$.

Since $q^{-1}(P)_m$ is open in $F^A_{(P^\vee)^m[a]}$, which itself is smooth by Lemma 5.5, we see that $q^{-1}(P)$ is smooth over A, which concludes the proof.

Indeed, let P be a perfect A-module with Tor-amplitude in $[a,b]$. Then, by Lemma 3.33, we can find a cofiber sequence $M[a] \to P \to Q$, where Q is perfect of Tor-amplitude $[a+1,b]$ and M is finite projective. Analogous to the above, we see that P has Tor-amplitude in $[a+1,b]$.

For the open immersion part it suffices by induction to show that for all $a < b \in \mathbb{Z}$ the inclusion $\mathrm{Perf}_R^{[a+1,b]} \hookrightarrow \mathrm{Perf}_R^{[a,b]}$ is an open immersion. Let $A \in \mathrm{AR}_R$ and $\mathrm{Spec}(A) \to \mathrm{Perf}_R^{[a,b]}$ be a morphism classified by a perfect A-module P of Tor-amplitude $[a,b]$. By Lemma 3.33, we have a fiber sequence of A-modules $P \to M[a+1] \to Q$, where Q is perfect of Tor-amplitude in $[a+1,b]$ and M is finite projective. Now P has Tor-amplitude in $[a+1,b]$ if and only if $M \simeq 0$. But by Lemma 5.4, we see that the vanishing locus of M is a quasi-compact open in $\mathrm{Spec}(A)$, which concludes the proof. □

Corollary 5.15. *Let A be an animated R-algebra and let $\mathrm{Spec}(A) \to \mathrm{Perf}_R$ be a morphism given by a perfect A-module P. The cotangent complex $L_{\mathrm{Perf}_R,A}$ is perfect and if A/R is étale then the cotangent complex a that point is given by*

$$L_{\mathrm{Perf}_R,A} \simeq (P \otimes_A P^\vee)^\vee[-1].$$

Proof. This is analogous to [AG14, Cor. 5.9], but for the convenience of the reader we recall the proof.

The first assertion follows from Theorem 5.14 with Corollary 4.78.

For the second assertion let $\Omega_P \mathrm{Perf}_R$ denote the loop of Perf_R along $\mathrm{Spec}(A) \to \mathrm{Perf}_R$ classified by a perfect A-module P, i.e. the $\Omega_P \mathrm{Perf}_R \coloneqq \mathrm{Spec}(A) \times_{\mathrm{Perf}_R} \mathrm{Spec}(A)$. We know per definition of the cotangent complex that we have a pullback diagram

$$\begin{tikzcd} L_{\mathrm{Perf},A} \arrow[r] \arrow[d] & L_{A/R} \arrow[d] \\ L_{A/R} \arrow[r] & L_{\Omega_P \mathrm{Perf}_R,*}, \end{tikzcd}$$

where $*$ is the point corresponding to the canonical map $\mathrm{Spec}(A) \to \Omega_P \mathrm{Perf}_R$.

Let $T \in \mathrm{AR}_A$, then $\Omega_P \mathrm{Perf}_R(T) \simeq \mathrm{Equiv}_T(P \otimes_A T)$, where $\mathrm{Equiv}_T(P \otimes_A T)$ denotes the T-automorphisms of $P \otimes_A T$. In particular $\Omega_P \mathrm{Perf}$ is an open substack of

$$\mathrm{Hom}_A((P \otimes_A P^\vee)^\vee, -)$$

(after using adjunctions). By Lemma 5.5, we now have

$$L_{\Omega_P \mathrm{Perf}_R,*} \simeq (P \otimes_A P^\vee)^\vee.$$

Therefore, if A/R is étale, we have $\Sigma L_{\mathrm{Perf}_R,A} \simeq L_{\Omega_P \mathrm{Perf}_R,*}$, whe finishing the proof. $\square$

6 Derived F-zips

In the following we fix a prime p and an $\mathbb{F}_p$-algebra R. Starting from here it is important that we have chosen the ∞-category of animated rings for our study of derived algebraic geometry. The main reason is that we want to have a Frobenius in characteristic $p > 0$. For E_∞-rings it is not clear how to define a Frobenius morphism. But for animated rings we have naturally a Frobenius. Namely, if we see an animated ring A over $\mathbb{F}_p$ as a contravariant functor from $\mathrm{Poly}_{\mathbb{F}_p}$ to $\mathbb{S}$, then the Frobenius morphism induces a natural transformation of the animated ring to itself, which we denote by $\mathrm{Frob}\colon A \to A$. For any animated R-algebra A and any A-module M, we denote the base change of M under the Frobenius of A with $M^{(1)} := M \otimes_{A,\mathrm{Frob}} A$. If A is discrete, we can see an M-module via the equivalence $\mathrm{Mod}_A \simeq \mathcal{D}(A)$ as an element in the derived category and $M^{(1)} \simeq M \otimes^L_{A,\mathrm{Frob}} A$ (here we abuse notation and identify A with $\pi_0 A$ if A is discrete).

We want to define derived versions of F-zips presented in [MW04]. In the reference Moonen-Wedhorn define F-zips over schemes of characteristic $p > 0$ and analyze the corresponding classifying stack. One application is the F-zip associated to a scheme with degenerate Hodge-de Rham spectral sequence. Examples of those are abelian schemes and K3-surfaces. The degeneracy of the spectral sequence is used to get two filtrations (note that also the conjugate spectral sequence degenerates) on the i-th de Rham cohomology. Our goal is to eliminate the extra information given by the degeneracy of the spectral sequences. This information seems unnecessary, since the two spectral sequences are induced by filtrations on the de Rham hypercohomology and thus if we pass to the derived categories, we can use perfectness of the de Rham hypercohomology, the two filtrations and the Cartier-isomorphism to get derived F-zips, as explained in the following example.

Example 6.1. Let $f\colon X \to S$ be a proper smooth morphism of schemes, where S is an R-scheme. The complex $Rf_*\Omega^\bullet_{X/S}$ is perfect and commutes with arbitrary base change (see [Sta19, 0FM0]). The conjugate and Hodge filtrations on the de Rham complex induce functors $\mathbf{conj}\colon \mathbb{Z} \to \mathcal{D}(S)$ resp. $\mathbf{HDG}\colon \mathbb{Z}^{\mathrm{op}} \to \mathcal{D}(S)$ given by[28] $\mathbf{conj}(n) = Rf_*\tau_{\leq n}\Omega^\bullet_{X/S}$ resp. $\mathbf{HDG}(n) = Rf_*\sigma_{\geq n}\Omega^\bullet_{X/S}$ (recall that we see the ordered set $\mathbb{Z}$ as a 1-category (and thus via the Nerve functor as an ∞-category) where we have a unique map between $a, b \in \mathbb{Z}$ if and only if $a \leq b$). The associated colimits are naturally equivalent as we have

$$\operatorname*{colim}_{\mathbb{Z}} \mathbf{conj} \simeq Rf_*\Omega^\bullet_{X/S} \simeq \operatorname*{colim}_{\mathbb{Z}^{\mathrm{op}}} \mathbf{HDG}.$$

For $n \geq 0$, we have the following exact sequences of complexes of $f^{-1}\mathcal{O}_S$-modules

$$0 \longrightarrow \tau_{\leq n-1}\Omega^\bullet_{X/S} \longrightarrow \tau_{\leq n}\Omega^\bullet_{X/S} \longrightarrow \mathcal{H}^n(\Omega^\bullet_{X/S})[-n] \longrightarrow 0,$$

$$0 \longrightarrow \sigma_{\geq n+1}\Omega^\bullet_{X/S} \longrightarrow \sigma_{\geq n}\Omega^\bullet_{X/S} \longrightarrow \Omega^n_{X/S}[-n] \longrightarrow 0.$$

[28]Here $\tau_{\leq n}$ denotes the canonical truncation and $\sigma_{\geq n}$ the stupid truncation in the sense of [Sta19, 0118].

These induce fiber sequences in $\mathcal{D}(S)$ of the form

$$\mathbf{conj}(n-1) \longrightarrow \mathbf{conj}(n) \longrightarrow Rf_*\mathcal{H}^n(\Omega^\bullet_{X/S})[-n],$$

$$\mathbf{HDG}(n+1) \longrightarrow \mathbf{HDG}(n) \longrightarrow Rf_*\Omega^n_{X/S}[-n].$$

It makes sense to think of $Rf_*\mathcal{H}^n(\Omega^\bullet_{X/S})[-n]$ resp. $Rf_*\Omega^n_{X/S}[-n]$ as *"cokernels"* of the respective maps in the distinguished triangles (as they are the cofibers in the stable ∞-category $\mathcal{D}(S)$).

The notation of **conj** and **HDG** is chosen to indicate their influence on the classical conjugate and Hodge filtration. Using these functors, one can naturally associate converging spectral sequences (as explained for example in [Lur17, Def. 1.2.2.9, Prop. 1.2.2.14] or [Sta19, 0FM7] for the Hodge filtration) on $R^if_*\Omega^\bullet_{X/S}$. The filtration on the i-th cohomology of the colimit of **HDG** ($\simeq Rf_*\Omega^\bullet_{X/S}$), for example is given by

$$F^nR^if_*\Omega^\bullet_{X/S} := \operatorname{im}(H^i(\mathbf{HDG}(n)) \to R^if_*\Omega^\bullet_{X/S}).$$

The spectral sequence associated to the Hodge functor is given by

$$E_1^{p,q} = H^q(X, Rf_*\Omega^p_{X/S}) = H^{q+p}(X, Rf_*\Omega^p_{X/S}[-p]) \Rightarrow R^{p+q}f_*\Omega^\bullet_{X/S}.$$

Therefore, it seems reasonable to think of **conj** resp. **HDG** as an *ascending* resp. *descending filtration* (see Definition 6.2 below) with *graded pieces*

$$\operatorname{gr}^n \mathbf{conj} := Rf_*\mathcal{H}^n(\Omega^\bullet_{X/S})[-n] \;\text{ resp. }\; \operatorname{gr}^n \mathbf{HDG} := Rf_*\Omega^n_{X/S}[-n]$$

(see Definition 6.4 below).

The Cartier isomorphism gives an equivalence $Rf_*\mathcal{H}^n(\Omega^\bullet_{X/S}) \simeq Rf_*^{(1)}\Omega^n_{X^{(1)}/S}$. Again by [Sta19, 0FM0], $Rf_*\Omega^n_{X/S}$ commutes with arbitrary base change and therefore

$$(\operatorname{gr}^n \mathbf{HDG})^{(1)} \simeq (Rf_*\Omega^n_{X/S})^{(1)}[-n] \simeq Rf_*^{(1)}\Omega^n_{X^{(1)}/S}[-n] \simeq Rf_*\mathcal{H}^n(\Omega^\bullet_{X/S})[-n] \simeq \operatorname{gr}^n \mathbf{conj}.$$

We claim that **conj** and **HDG** take values in perfect complexes of $\mathcal{O}_S$-modules and their respective graded pieces are perfect.

Indeed, first note that we can check this Zariski locally, so we may assume that S is affine and in particular quasi-compact. Then for any $n \in \mathbb{Z}$ the complex $\operatorname{gr}^n \mathbf{HDG}$ is perfect and $Rf_*\Omega^\bullet_{X/S}$ are perfect and their formation commute with arbitrary base change (see [Sta19, 0FM0]). Since $\sigma_{\geq 0}\Omega^\bullet_{X/S} = \Omega^\bullet_{X/S}$, we see inductively using the distinguished triangles above that for all $n \in \mathbb{Z}$ the complex $\mathbf{HDG}(n)$ is perfect. Now certainly the base change of perfect complexes is perfect and therefore the Cartier isomorphism shows that the graded pieces of **conj** are also perfect. The quasi-compactness of S implies that there is an $n \in \mathbb{N}_0$ such that $\tau_{\leq n}\Omega^\bullet_{X/S} = \Omega^\bullet_{X/S}$ and thus again inductively with the distinguished triangles above, we see that $\mathbf{conj}(n)$ is perfect for all $n \in \mathbb{Z}$.

Note, that we heavily used that Zariski locally there is an $n \gg 0$ such that for any $k \leq 0$ and $j \geq 0$, we have $\mathbf{conj}(k-1) \simeq 0 \simeq \mathbf{HDG}(n+j)$ resp. $\mathbf{conj}(n+j) \simeq Rf_*\Omega^\bullet_{X/S} \simeq \mathbf{HDG}(k)$ and $\mathbf{conj}(n+j) \to \mathbf{conj}(n+j+1)$ resp. $\mathbf{HDG}(k+1) \to \mathbf{HDG}(k)$ is equivalent to the identity.

The example above gives us an idea for the definition of *derived F-zips* (see Definition 6.13). Namely, a derived F-zip should consist of *two filtrations* (one descending and one ascending) with perfect values that are locally determined by a finite chain of morphisms, i.e. functors

$$C^\bullet \in \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Perf}(S)) \text{ and } D_\bullet \in \mathrm{Fun}(\mathbb{Z}, \mathrm{Perf}(S)),$$

such that their colimits are equivalent and on affine opens are upto equivalence determined by their values on a finite ordered subset of $\mathbb{Z}$, together with *equivalences* $\varphi_\bullet$ of their graded pieces up to Frobenius twist, i.e. for $\mathrm{gr}^n C \coloneqq \mathrm{cofib}(C^{n+1} \to C^n)$ and $\mathrm{gr}^n D \coloneqq \mathrm{cofib}(D_{n-1} \to D_n)$ equivalences of the form

$$\varphi_n \coloneqq (\mathrm{gr}^n C)^{(1)} \xrightarrow{\sim} \mathrm{gr}^n D.$$

The ∞-category of derived F-zips should then be defined as the ∞-category of such triples $(C^\bullet, D_\bullet, \varphi_\bullet)$.

6.1 Filtrations

In the following A will denote an animated ring.

We will now define the notion of a *filtration* and *graded pieces* and look at properties of filtrations. These definitions are highly influenced by the work of Gwilliam-Pavlov [GP18] and Example 6.1.

Definition 6.2. An *ascending (resp. descending) filtration of A-modules* is an element $F \in \mathrm{Fun}(\mathbb{Z}, \mathrm{Mod}_A)$ (resp. $F \in \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Mod}_A)$).

We call an ascending (resp. descending) filtration F

(i) *right bounded* if there exists $i \in \mathbb{Z}$ such that the natural map $F(k) \to \mathrm{colim}_{\mathbb{Z}} F$ (resp. $F(k) \to \mathrm{colim}_{\mathbb{Z}^{\mathrm{op}}} F$) is an equivalence for all $i \leq k$ (resp. $i \geq k$),

(ii) *left bounded* if there exists $i \in \mathbb{Z}$ such that the natural map $0 \to F(k)$ is an equivalence for all $k \leq i$ (resp. $k \geq i$),

(iii) *bounded*, if it is left and right bounded,

(v) *perfect* if F takes values in $\mathrm{Mod}_A^{\mathrm{perf}}$,

(vi) *strong* if for all $i \leq j$ (resp. $j \leq i$), we have that $F(i) \to F(j)$ is a monomorphism.

Remark 6.3. The definition of a *strong filtration* seems natural, since for a discrete module M over a discrete ring A a filtration is usually defined as a filtered chain of submodules

$$\cdots \subseteq M_i \subseteq M_{i+1} \subseteq \cdots \subseteq M$$

(for we simplicity only consider ascending filtrations). But we can show that the Hodge filtration **HDG** of Example 6.1 is strong **if and only if** the Hodge-de Rham spectral sequence is degenerate (see Theorem 6.71). Since we are particularly interested in the cases where the Hodge-de Rham spectral sequence is non-degenerate, strong filtrations are not used in the definition of derived F-zips. Nevertheless, we include the discussion of strong filtrations since it seems natural to ask what happens if the filtrations are given by monomorphisms.

The ∞-category of A-modules is stable. Thinking of stable ∞-categories as analogs of abelian categories, we may think of cofibers as cokernels. This allows for a definition of graded pieces of a filtration, that was used in Example 6.1.

Definition 6.4. Let F be a ascending (resp. descending) filtration of A-modules. For any $i \in \mathbb{Z}$, we define the *i-th graded piece of F* as $\mathrm{gr}^i F \coloneqq \mathrm{cofib}(F(i-1) \to F(i))$ (resp. $\mathrm{gr}^i F \coloneqq \mathrm{cofib}(F(i+1) \to F(i))$).

Remark 6.5. By construction of the category $\mathrm{Fun}(\mathbb{Z}, \mathrm{Mod}_A)$ one sees that two filtrations F, G are equivalent if and only if there is a morphism $F \to G$ such that for all $n \in \mathbb{Z}$ the induced morphism $F(n) \to G(n)$ is an equivalence of A-modules. However, one can show that a morphism of bounded filtrations is an equivalence if and only if it induces an equivalence on the graded pieces (this is an easy consequence using induction or [GP18, Rem. 3.21]).

Remark 6.6. Note that for a perfect filtration F of A-modules, the graded pieces $\mathrm{gr}^i F$ are again perfect (since the ∞-category of perfect modules is per definition stable, see [Lur17, §7.2.4]).

Remark 6.7. We want to attach a monoidal structure to filtrations of A-modules (we will only consider ascending filtrations but the arguments work analogously for descending filtrations as explained in the end of the remark).

First, note that for any (symmetric) monoidal ∞-category $\mathcal{C}$ the ∞-category $\mathrm{Fun}(\mathbb{Z}, \mathcal{C})$ has two monoidal structures. The first one is simply given by termwise tensor product (see [Lur17, Rem. 2.1.3.4]), the other one is given by the Day convolution (see [Lur17, Ex. 2.2.6.17]). We will not use the monoidal structure given by termwise tensor product, since we want to consider bounded filtrations and for such we do not have a unit element with respect to the termwise tensor product. Having this in mind, will look closely into the monoidal structure induced by the Day convolution, which we will explain in the following.

For the Day convolution, we first need a (symmetric) monoidal structure on $\mathbb{Z}$. For this, we simply take $\mathbb{Z}$ with the usual addition, seen as a symmetric monoidal structure on $\mathbb{Z}$. Then the Day convolution of two elements $F, G \in \mathrm{Fun}(\mathbb{Z}, \mathrm{Mod}_A)$, denoted by $F \otimes G$ is given by the formula

$$(F \otimes G)(k) \simeq \operatorname*{colim}_{n+m \leq k} F(n) \otimes_A G(m),$$

where we take the colimit over the category of triples $(a, b, a+b \to k)$, where $a, b \in \mathbb{Z}$ and $a + b \to k$ is a morphism in $\mathbb{Z}$ (recall that this simply means $a + b \leq k$), the morphisms are given componentwise, i.e. a morphism

$$(a, b, a + b \to k) \to (a', b', a' + b' \to k')$$

is given by the relations $a \leq a'$, $b \leq b'$, and $k \leq k'$. A unit element for this tensor product is given by the bounded perfect filtration $A^{\text{triv}}_\bullet$ on A, where $A^{\text{triv}}_i \simeq A$ for $i \geq 0$ and 0 otherwise, the maps $A^{\text{triv}}_m \to A^{\text{triv}}_n$ for $0 \leq m \leq n$ are given by the identity and $A^{\text{triv}}_m \to A^{\text{triv}}_n$ for $m \leq n \leq 0$ are given by 0.

If we replace ascending filtrations with descending ones the relations above get opposed, i.e. we have unique morphisms $a \to b$ in $\mathbb{Z}^{\text{op}}$ if and only if $b \leq a$. Taking this to account we can dually define the Day convolution for descending filtrations similarly.

Notation 6.8. In the definition of derived F-zips we will have an ascending and a descending filtration. For clarity, we denote for an ascending filtration $F \in \text{Fun}(\mathbb{Z}, \text{Mod}_A)$ its values with $F_n := F(n)$ for any $n \in \mathbb{Z}$ and for a descending filtration $G \in \text{Fun}(\mathbb{Z}^{\text{op}}, \text{Mod}_A)$ its values with $G^n := G(n)$. We also denote the filtrations with $F_\bullet := F$ resp. $G^\bullet := G$.

For the gradings we omit the $\bullet$, i.e. $\text{gr}^i F := \text{gr}^i F_\bullet$ resp. $\text{gr}^i G := \text{gr}^i G^\bullet$

Remark 6.9. Let us visualize the Day convolution using an easy example. Let M, N be A-modules and let $C \to M$ and $D \to N$ be morphisms of A-modules. Now let us look at the filtrations $C_\bullet, D_\bullet$ given

$$C_\bullet \colon \ \dots \xrightarrow{0} 0 \xrightarrow{0} C \to M \xrightarrow{\text{id}} M \xrightarrow{\text{id}} \dots$$
$$D_\bullet \colon \ \dots \xrightarrow{0} 0 \xrightarrow{0} D \to M \xrightarrow{\text{id}} M \xrightarrow{\text{id}} \dots,$$

where we set $C_0 = C$ and $D_0 = D$. Then we have $(C_\bullet \otimes D_\bullet)_0 \simeq C \otimes_A D$ and the A-module $(C_\bullet \otimes D_\bullet)_1$ is given by the pushout of the following diagram

$$\begin{tikzcd}
 & C \otimes_A D \arrow[dl] \arrow[dr] & \\
M \otimes_A D & & C \otimes_A N.
\end{tikzcd}$$

The A-module $(C_\bullet \otimes D_\bullet)_2$ is given by the colimit of the diagram

$$\begin{tikzcd}
 & & C \otimes_A D \arrow[dl] \arrow[dd] \arrow[dr] & & \\
 & M \otimes_A D \arrow[dl, "\text{id}"'] \arrow[dr] & & C \otimes_A N \arrow[dl] \arrow[dr, "\text{id}"] & \\
M \otimes_A D & & M \otimes_A N & & C \otimes_A N,
\end{tikzcd}$$

in particular, we may forget about the top most module and only look at the colimit of the bottom zigzag (in the homotopy category). This diagram makes clear that the $(C_\bullet \otimes D_\bullet)_2 \simeq M \otimes_A N$. The same visualization works for higher degrees of the filtration $(C_\bullet \otimes D_\bullet)_\bullet$ and we will prove in the following proposition that the Day convolution descends to perfect bounded filtrations having this tree structure in mind.

Proposition 6.10. *The Day convolution on* $\mathrm{Fun}(\mathbb{Z}, \mathrm{Mod}_A)$ *descends to a symmetric monoidal structure on the full subcategory of perfect bounded ascending filtrations.*

The same holds perfect bounded descending filtrations.

Proof. That the unit element for the Day convolution is a bounded perfect filtration on an A-module is shown in Remark 6.7.

Let $C_\bullet$ and $D_\bullet$ be bounded ascending filtrations of A-modules. We claim that $(C_\bullet \otimes D_\bullet)_\bullet$ is a bounded ascending filtration.

That $(C_\bullet \otimes D_\bullet)_\bullet$ defines a left bounded filtration is clear. To see that it is also right bounded, fix some integers $k, k' \in \mathbb{Z}$ such that the natural morphisms

$$C_i \xrightarrow{\sim} \operatorname*{colim}_{\mathbb{Z}} C_\bullet \text{ and } D_j \xrightarrow{\sim} \operatorname*{colim}_{\mathbb{Z}} D_\bullet$$

are equivalences for all $i \geq k$ and $j \geq k'$. For simplicity, let us denote $M := \operatorname{colim}_{\mathbb{Z}} C_\bullet$ and $N := \operatorname{colim}_{\mathbb{Z}} D_\bullet$. First, note that the morphism $C_{i \leq i+1} \colon C_i \to C_{i+1}$ is an equivalence for all $i \geq k$. Using this equivalence, we may assume that $C_{i \leq i+1}$ is given by id_M (it is not hard to find and equivalence of filtrations), analogously we do the same for $D_\bullet$. Now let us look at $(C_\bullet \otimes D_\bullet)_{k+k'}$, we claim that this term is equivalent to $M \otimes N$. Indeed, $C_k \otimes_A D_{k'} \simeq M \otimes_A N$ by construction. Now let $(i, j) \in \mathbb{Z}^2$, such that $i + j \leq k + k'$ but $i > k$ or $j > k'$, so there is no morphism from $C_i \otimes D_j$ to $C_k \otimes_A D_{k'}$. Without loss of generality assume $i > k$ (in particular $j < k'$).

Let us visualize what we are going to do. Considering the zig-zag from Remark 6.9, we will look at the following diagram

$$\begin{tikzcd}
C_k \otimes_A D_j \arrow[d] \arrow[dr, "\mathrm{id}"] \arrow[ddddr, "f"] & & & \\
\vdots \arrow[d] & C_{k+1} \otimes_A D_j \arrow[dr, "\mathrm{id}"] & & \\
\vdots \arrow[d] & & \cdots \arrow[dr, "\mathrm{id}"] & \\
C_k \otimes_A D_{k'} \arrow[dr, "g"] & & & C_i \otimes_A D_j \arrow[dll, "h"] \\
& (C_\bullet \otimes D_\bullet)_{k+k'}. & &
\end{tikzcd}$$

By definition of colimits, we automatically get a homotopy between f and h and a homotopy between f and g. In particular, this diagram shows h is up to homotopy uniquely determined by f and g. But certainly the morphisms of the filtrations and g uniquely (up to homotopy) determine f. Using this and the universal property of colimits, we see that there exists a morphism $p \colon (C_\bullet \otimes D_\bullet)_{k+k'} \to C_k \otimes_A D_{k'}$ such that $\mathrm{id}_{C_k \otimes_A D_{k'}} \simeq p \circ g$. But $g \circ p$ induces a map $(C_\bullet \otimes D_\bullet)_{k+k'} \to (C_\bullet \otimes D_\bullet)_{k+k'}$ that is compatible with all transition maps in the colimit diagram and thus $g \circ p \simeq \mathrm{id}_{(C_\bullet \otimes D_\bullet)_{k+k'}}$. In other words $M \otimes_A N \simeq C_k \otimes_A D_{k'} \simeq (C_\bullet \otimes D_\bullet)_{k+k'}$.

The same argument shows that $(C_\bullet \otimes D_\bullet)_l \simeq M \otimes_A N$ and that the canonical maps $(C_\bullet \otimes D_\bullet)_l \to (C_\bullet \otimes D_\bullet)_{l+1}$ are homotopic to the identity for all $l \geq k + k'$. So for all $l \geq k + k'$, we have that the natural map

$$(C_\bullet \otimes D_\bullet)_l \xrightarrow{\sim} \operatorname*{colim}_{\mathbb{Z}} (C_\bullet \otimes D_\bullet)_\bullet \simeq M \otimes_A N$$

is an equivalence, i.e. $(C_\bullet \otimes D_\bullet)_\bullet$ is right bounded.

The above computations show that the since $C_\bullet$ and $D_\bullet$ are bounded, we have for any $k \in \mathbb{Z}$ that $(C_\bullet \otimes D_\bullet)_k$ is equivalent to the colimit taken over a finite filtered subset of $\mathbb{Z}$ (again seen as a category via nerve and morphisms uniqely given by relations). Since finite colimits of perfect modules are perfect, we see that $(C_\bullet \otimes D_\bullet)_\bullet$ is not only bounded but also perfect (note stable ∞-categories are closed under finite colimits, see [Lur17, Prop. 1.1.3.4]).

Combining everything above, we see that the Day convolution descends to bounded perfect filtrations and therefore gives us a symmetric monoidal structure on bounded perfect filtrations (see [Lur17, Prop. 2.2.1.1, Rem. 2.2.1.2]).

The proof for descending filtrations works analogously. □

Remark 6.11. For two filtrations $C_\bullet$ and $D_\bullet$ of A-modules, it is known that

$$\operatorname{gr}^k(C_\bullet \otimes D_\bullet) \simeq \bigoplus_{n \in \mathbb{Z}} \operatorname{gr}^n C \otimes_A \operatorname{gr}^{k-n} D$$

(see [BMS19, Lem. 5.2]).

Remark 6.12. Let us remark that the construction of the Day convolution can also be done for $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$, where $\mathcal{C}$ and $\mathcal{D}$ arbitrary symmetric monoidal ∞-categories (see [Lur17, Ex. 2.2.6.17]).

An interesting example for us occurs if $\mathcal{C} \simeq \mathbb{Z}^{\mathrm{disc}}$ (recall that this means the set $\mathbb{Z}$ as a discrete 1-category and thus an ∞-category via the Nerve functor), where we endow $\mathbb{Z}^{\mathrm{disc}}$ with a symmetric monoidal structure by addition, i.e. $a \otimes b := a + b$, and $\mathcal{D} \simeq \mathrm{Mod}_A^{\mathrm{perf}}$. Then for functors $F, G \in \operatorname{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathrm{Mod}_A^{\mathrm{perf}})$, we have

$$(F \otimes G)(k) \simeq \bigoplus_{n+m=k} F(n) \otimes_A G(m) \simeq \bigoplus_{n \in \mathbb{Z}} F(n) \otimes_A G(k-n).$$

This will become important, when constructing a symmetric monoidal structure on derived F-zips, since we have to take the morphisms between graded pieces to account and the behaviour of graded pieces of the tensor product of bounded perfect filtrations.

6.2 Derived F-zips over affine schemes

We are ready to define derived F-zips and we will do so by axiomatizing the structures occurring in Example 6.1. We will first restrict ourselves to the local case, i.e. we define derived F-zips over animated rings. The reason, besides simplicity, is that the theory of derived algebraic geometry was only developed for animated rings since we want a "nice" model category, such as the model category associated to animated rings. This is not a real issue since globalization of the results is achieved by considering right Kan extensions. There is also a direct way of defining derived F-zips for derived stacks but we will see that both constructions agree (see Remark 6.62).

Recall that we fixed an $\mathbb{F}_p$-algebra R in the beginning of this section.

Definition 6.13. Let A be an animated R-algebra. A *derived F-zip over A* is a tuple $(C^\bullet, D_\bullet, \phi, \varphi_\bullet)$ consisting of

- a descending bounded perfect filtration of A-modules $C^\bullet$,
- an ascending bounded perfect filtration of A-modules $D_\bullet$
- an equivalence $\phi\colon \operatorname{colim}_{\mathbb{Z}^{\mathrm{op}}} C^\bullet \simeq \operatorname{colim}_{\mathbb{Z}} D_\bullet$, and
- a family of equivalences $\varphi_k\colon (\operatorname{gr}^k C)^{(1)} \xrightarrow{\sim} \operatorname{gr}^k D$.

The ∞-category of F-zips over A, denoted by $F\text{-}\mathrm{Zip}_{\infty,R}(A)$, is defined as the full subcategory of

$$(\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Mod}_A^{\mathrm{perf}}) \times_{\mathrm{colim},\mathrm{Mod}_A,\mathrm{colim}} \mathrm{Fun}(\mathbb{Z}, \mathrm{Mod}_A^{\mathrm{perf}})) \\ \times_{((\operatorname{gr}^i_-)^{(1)},\operatorname{gr}^i_-)_{i\in\mathbb{Z}},\prod_{\mathbb{Z}}\mathrm{Fun}(\partial\Delta^1,\mathrm{Mod}_A^{\mathrm{perf}})} \prod_{\mathbb{Z}} \mathrm{Fun}(\Delta^1, \mathrm{Mod}_A^{\mathrm{perf}}),$$

consisting of derived F-zips over A.

For an animated R-algebra homomorphism $A \to A'$ we have an obvious base change functor $F\text{-}\mathrm{Zip}_{\infty,R}(A) \to F\text{-}\mathrm{Zip}_{\infty,R}(A')$ via the tensor product, where the filtrations are base changed component wise with induced morphisms.

Remark 6.14. In the above definition we have to fix the equivalence between the colimit of the ascending and descending filtration. This comes from the fact that we want to define derived F-zips as a full subcategory, as above. To be more specific let us look at a pullback diagram of ∞-categories

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & \mathcal{A} \arrow[d, "g"] \\ \mathcal{B} \arrow[r, "n"] & \mathcal{C}. \end{tikzcd}$$

A morphism $\Delta^0 \to \mathcal{D}$ is by definition the same as diagram as follows

$$\begin{tikzcd} \Delta^0 \arrow[r, "f"] \arrow[d, "m"] \arrow[dr, "h"] & \mathcal{A} \arrow[d, "g"] \\ \mathcal{B} \arrow[r, "n"] & \mathcal{C} \end{tikzcd}$$

with equivalences $g \circ f \simeq h$ and $n \circ m \simeq h$ in the ∞-category $\mathrm{Fun}(\Delta^0, \mathcal{C})$, i.e. 1-morphisms in $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\Delta^0, \mathcal{C}) \coloneqq \mathrm{Fun}(\Delta^0, \mathcal{C})^{\simeq}$ (resp. 2-morphisms in Cat_∞). Important here is that we have to fix the homotopy equivalences $g \circ f \simeq h$ and $n \circ m \simeq h$, i.e. they are an additional datum. So an object in $\mathcal{D}$ is the same as a tuple $(A, B, C) \in \mathcal{A} \times \mathcal{B} \times \mathcal{C}$ together with an equivalence $g(A) \simeq C$ and $n(B) \simeq C$. This is equivalent to giving a tuple (A, B, ϕ) of objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$ and an equivalence $\phi\colon g(A) \simeq n(B)$.

Remark 6.15. Let A be an animated R-algebra. The homotopy category of derived F-zips over A forgets the extra datum of the equivalence between the colimits. This follows from the fact the filtrations in the definition of a derived F-zips over A are bounded. So, any derived F-zip $(C^\bullet, D_\bullet, \phi, \varphi_\bullet)$ is isomorphic (not canonically) in $h\,F\text{-Zip}_\infty(S)$ to a derived F-zip, where the equivalence between the colimits is actually given by the identity. In particular, up to equivalence we may replace ϕ by identity and we will write $(C^\bullet, D_\bullet, \varphi_\bullet)$ in this case for a derived F-zip, when we work with derived F-zips up to homotopy.

Example 6.16. Let us come back to Example 6.1. Let $f\colon X \to \operatorname{Spec}(A)$ be a proper smooth morphism of schemes. Then the associated Hodge and conjugate filtration $\mathbf{HDG}$ and $\mathbf{conj}$ define a descending resp. ascending perfect bounded filtration of A-modules. We also have equivalences $\varphi_n\colon (\operatorname{gr}^n \mathbf{HDG})^{(1)} \xrightarrow{\sim} \operatorname{gr}^n \mathbf{conj}$ between the graded pieces (up to Frobenius twist), induced by the Cartier isomorphism. Therefore, we get a derived F-zip associated to the proper smooth map f of schemes

$$\underline{R\Gamma_{\mathrm{dR}}(X/A)} \coloneqq (\mathbf{HDG}^\bullet, \mathbf{conj}_\bullet, \varphi_\bullet)$$

(note that $\operatorname{colim}_{\mathbb{Z}^{\mathrm{op}}} \mathbf{HDG}^\bullet \simeq \operatorname{colim}_{\mathbb{Z}} \mathbf{conj}$ naturally by the identity).

Remark 6.17. Let us remark that for any $A \in \mathrm{AR}_R$ the ∞-category of F-zips over A is essentially small, even if we don't assume Mod_A to be small[29]. This is because the ∞-category Mod_A is compactly generated (see [Lur17, Prop. 7.2.4.2]) (thus accessible) and therefore the full subcategory of perfect objects is essentially small (see [Lur09, Prop. 5.4.2.2]). Hence for any small ∞-category K, the ∞-category $\operatorname{Fun}(K, \mathrm{Mod}_A^{\mathrm{perf}})$ is again essentially small (see [Lur09, 5.3.4.13, 5.4.4.3]). Finally, since $F\text{-Zip}_{\infty,R}(A)$ is a full subcategory of finite limits of those of the form above, we see that indeed $F\text{-Zip}_{\infty,R}(A)$ is essentially small (note that by [Lur09, Cor. 4.2.4.8] the ∞-category of small ∞-categories has small limits).

Lemma 6.18. *The ∞-category of derived F-zips over an animated R-algebra A is stable.*

Proof. We know that Mod_A and $\mathrm{Mod}_A^{\mathrm{perf}}$ are stable and thus also for any ∞-category $\mathcal{C}$ the ∞-category $\operatorname{Fun}(C, \mathrm{Mod}_A^{\mathrm{perf}})$ is stable. Since the limit of stable ∞-categories with finite limit preserving transition maps is stable (see Lemma 4.46) it is enough to show that $F\text{-Zip}_{\infty,R}(A)$ is a stable subcategory of

$$\Big(\operatorname{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Mod}_A^{\mathrm{perf}}) \times_{\operatorname{colim}, \mathrm{Mod}_A, \operatorname{colim}} \operatorname{Fun}(\mathbb{Z}, \mathrm{Mod}_A^{\mathrm{perf}})\Big) \\ \times_{((\operatorname{gr}^i_-)^{(1)}, \operatorname{gr}^i_-)_{i\in\mathbb{Z}}, \prod_{\mathbb{Z}} \operatorname{Fun}(\partial\Delta^1, \mathrm{Mod}_A^{\mathrm{perf}})} \prod_{\mathbb{Z}} \operatorname{Fun}(\Delta^1, \mathrm{Mod}_A^{\mathrm{perf}})$$

[29]We want to remark that we did not assume any smallness of the module categories explicitly and this remark shows that it is not needed in this section. But as Remark 4.54 shows, we need smallness of the module categories, for globalization purposes, i.e. when we want to extend derived F-zips to derived schemes via right Kan extension.

(note that $\prod_{\mathbb{Z}} \mathrm{Fun}(\Delta^1, \mathrm{Mod}_A^{\mathrm{perf}}) \simeq \mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathrm{Fun}(\Delta^1, \mathrm{Mod}_A^{\mathrm{perf}}))$ and filtered colimits preserve finite limits). For this, we have to show that the perfect bounded filtrations, equivalences between colimits of filtrations and equivalences between the graded pieces (up to Frobenius twist) are stable under shifts and cofibers.

That perfect bounded filtrations are stable under shift and cofibers follows immediately from the fact that limits and colimits of functors can be computed pointwise (see [Lur09, Cor. 5.1.2.3]). The same argument implies that equivalences between the graded pieces (up to Frobenius twist) are stable under cofibers and shifts. Since filtered colimits comute with shifts and cofibers, we also see that the equivalence between the colimits is preserved under those operations. □

In the following we want to construct a symmetric monoidal structure on derived F-zips. The idea is very simple. We know that derived F-zips are contained in a larger ∞-category (see Definition 6.13), let us denote this category with $\mathcal{C}$. This ∞-category $\mathcal{C}$ is constructed by limits of functor categories, that we can endow with the Day convolution. For the morphisms between graded pieces, we have to be bit careful but Remarks 6.11 and 6.12 show us that this will be not a problem. Since passing to the graded pieces and taking the colimit of a filtration are both monoidal functors, we see that indeed $\mathcal{C}$ is symmetric monoidal. Now we only need to show that the unit object of $\mathcal{C}$ is a derived F-zip, which follows immediately, the Day convolution of perfect bounded filtrations is bounded perfect (this is Proposition 6.10) and the induced morphism of graded pieces (up to Frobenius twist) is an equivalence, which is also immediate.

Proposition 6.19. *The ∞-category of derived F-zips over an animated R-algebra A admits a symmetric monoidal structure.*

Proof. We know that $\mathrm{Mod}_A^{\mathrm{perf}}$ admits a symmetric monoidal structure (see [Lur17, Rem. 2.2.1.2] and note that as an A-module A is perfect and the tensor product of perfect A-modules is again perfect). We now show how to construct a symmetric monoidal structure on derived F-zips.

The monoidal structure on the filtrations are given by the Day convolution (see Proposition 6.10). The monoidal structure on the equivalences of graded pieces is given in the following way.

We endow $\mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathrm{Mod}_A^{\mathrm{perf}})$ with the Day convolution (as explained in Remark 6.12), where we endow $\mathbb{Z}^{\mathrm{disc}}$ with a symmetric monoidal structure by usual addition. The unit object in $\mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathrm{Mod}_A^{\mathrm{perf}})$ is given by $A_{\mathrm{triv}}^{\mathrm{disc}}$, where $A_{\mathrm{triv}}^{\mathrm{disc}}(0) \simeq A$ and 0 otherwise. Now we endow the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathrm{Mod}_A^{\mathrm{perf}}))$ with the pointwise tensor product (see [Lur17, Rem. 2.1.3.4]) - we do exactly the same for $\mathrm{Fun}(\partial\Delta^1, \mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathrm{Mod}_A^{\mathrm{perf}}))$. Certainly, by this construction for a derived F-zip $(C^\bullet, D_\bullet, \varphi_\bullet)$ over A, the family $\varphi_\bullet$ defines an element in $\mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathrm{Mod}_A^{\mathrm{perf}}))$.

Now let us note that taking the colimit defines a symmetric monoidal functor from ascending (resp. descending) filtrations to Mod_A (as the tensor product of spectra commute with colimits in each variable, see Section 2). Also sending a filtration to its graded piece is symmetric monoidal by Remark 6.11. Therefore, we can attach a symmetric monoidal structure to

$$\begin{aligned}\mathcal{C}(A) \coloneqq (\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Mod}_A^{\mathrm{perf}}) \times_{\mathrm{colim}, \mathrm{Mod}_A, \mathrm{colim}} \mathrm{Fun}(\mathbb{Z}, \mathrm{Mod}_A^{\mathrm{perf}}))\\ \times_{\mathrm{Fun}(\partial\Delta^1, \mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathrm{Mod}_A^{\mathrm{perf}}))} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathrm{Mod}_A^{\mathrm{perf}}))\end{aligned}$$

where we use that the ∞-category of symmetric monoidal ∞-categories has limits (see[30] [Lur17, Rem. 2.4.2.6, Prop. 3.2.2.1]).

Since derived F-zips over A forms a full subcategory of $\mathcal{C}(A)$, it suffices to check that the unit element of $\mathcal{C}(A)$ is in $F\text{-}\mathrm{Zip}_{\infty,R}(A)$ and that it is closed under the tensor product (see [Lur17, Rem. 2.2.1.2]). But this follows from Proposition 6.10 and Remarks 6.11 and 6.12. Concretely, the unit element in $F\text{-}\mathrm{Zip}_{\infty,R}(A)$ is given by

$$\underline{\mathbb{1}_A} \coloneqq (A^{\bullet}_{\mathrm{triv}}, A_{\bullet}^{\mathrm{triv}}, \mathrm{id}_A, (\mathrm{id}_A)_0),$$

where $A_{\bullet}^{\mathrm{triv}}$ is defined as in Remark 6.7, $A^{\bullet}_{\mathrm{triv}}$ is defined dually , i.e is given by $A^n_{\mathrm{triv}} \simeq A$ for $n \leq 0$ zero elsewhere and the identity as transition maps, $(\mathrm{id}_A)_0$ denotes the family of morphisms $\varphi_\bullet$, where $\varphi_0 \simeq \mathrm{id}_A$ and 0 elsewhere.

Note that for $\varphi_\bullet, \vartheta_\bullet \in \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathrm{Mod}_A^{\mathrm{perf}}))$ which induce equivalences in $\mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathrm{Mod}_A^{\mathrm{perf}})$ their tensor product is still an equivalence, by the explicit description given in Remark 6.12. $\square$

Our next goal is to show that the functor that sends an animated R-algebra to the ∞-category of derived F-zips over it is locally geometric. For this we need that it is a hypercomplete sheaf for the étale topology. We will show that it is a hypercomplete sheaf even for the fpqc topology. Since every geometric derived stack is hypercomplete (see Lemma 4.81) the hypercompleteness condition - at least for the étale topology - is necessary.

Again the idea is very simple and follows the proof of descent for perfect modules seen in [AG14, Lem. 5.4]. We again embed F-zips into a larger category as in the proof of Proposition 6.19, which satisfies hyperdescent. Then we only need to check that the properties *bounded perfect* of a filtration and the property *equivalence* of a morphism between modules satisfy fpqc-hyperdescent. But since our cover is affine and perfectness is equivalent to dualizability both properties satisfy hyperdescent and we are done.

To see that the larger category satisfies descent one only needs that perfect filtrations satisfy descent which will follow from descent of perfect modules.

Lemma 6.20. *Let $F\colon \mathrm{AR}_R \to \mathrm{Cat}_\infty$ be a hypercomplete fpqc sheaf. Then for any ∞-category $\mathcal{C}$ the functor $\mathrm{Fun}(\mathcal{C}, F(-))\colon \mathrm{AR}_R \to \mathrm{Cat}_\infty$ is a hypercomplete fpqc sheaf.*

[30] The reference shows existence of limits in commutative algebra objects of symmetric monoidal ∞-categories. But using [Lur17, Prop. 4.1.7.10] we can endow the ∞-category of ∞-categories with the cartesian model structure (a concrete description of the associated ∞-operad is given in [Lur17, Notation 4.8.1.2]). The commutative algebra objects of the ∞-category of ∞-categories with this monoidal structure is then equivalent to the ∞-category of symmetric monoidal ∞-categories (defined for example in [Lur17, Var. 2.1.4.13]).

Proof. As $\mathrm{Fun}(\mathcal{C}, -)$ is right adjoint to the product it preserves limits. Since F is a hypercomplete sheaf, we see that indeed the natural morphism $\mathrm{Fun}(\mathcal{C}, F(A)) \xrightarrow{\sim} \lim_{\Delta_s} \mathrm{Fun}(\mathcal{C}, F(A^\bullet))$ is an equivalence for any fpqc hypercovering $A \to A^\bullet$. □

Definition and Proposition 6.21. *The functor*

$$\begin{aligned} F\text{-}\mathrm{Zip}_{\infty,R} \colon \mathrm{AR}_R &\to \mathrm{Cat}_\infty \\ A &\mapsto F\text{-}\mathrm{Zip}_{\infty,R}(A) \end{aligned}$$

is a hypercomplete sheaf for the fpqc topology.

Proof. In the following we will denote the functor $\mathrm{AR}_R \to \mathrm{Cat}_\infty,\ A \mapsto \mathrm{Mod}_A^{\mathrm{perf}}$ with $\mathrm{Mod}_{(-)}^{\mathrm{perf}}$ to avoid confusion with the notation of the stack of perfect modules over R.

Let $A \to A^\bullet$ be an fpqc hypercovering given by a functor $\Delta_{+,s} \to \mathrm{AR}_R$. We have to show that $F\text{-}\mathrm{Zip}_\infty(A) \to \lim_{\Delta_s} F\text{-}\mathrm{Zip}_{\infty,R}(A^\bullet)$ is an equivalence.

For convenience, we first set

$$\begin{aligned} \mathcal{C}(-) \coloneqq \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Mod}_-^{\mathrm{perf}}) &\times_{\mathrm{colim}, \mathrm{Mod}_-, \mathrm{colim}} \mathrm{Fun}(\mathbb{Z}, \mathrm{Mod}_-^{\mathrm{perf}}) \\ &\times_{\mathrm{Fun}(\partial\Delta^1, \mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathrm{Mod}_-^{\mathrm{perf}}))} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathrm{Mod}_-^{\mathrm{perf}})) \end{aligned}$$

For the fully faithfulness let us look at the following diagram

$$\begin{tikzcd} F\text{-}\mathrm{Zip}_{\infty,R}(A) \arrow[r] \arrow[d, hook] & \lim_{\Delta_s} F\text{-}\mathrm{Zip}_{\infty,R}(A^\bullet) \arrow[d, hook] \\ \mathcal{C}(A) \arrow[r] & \lim_{\Delta_s} \mathcal{C}(A^\bullet). \end{tikzcd}$$

By Lemma 6.20 and descent of (perfect) modules (see Remark 4.53), we see that $\mathcal{C}$ is a hypercomplete sheaf for the fpqc topology and thus the bottom horizontal arrow is an equivalence and thus the upper horizontal arrow is fully faithful.

For the essential surjectivity note that we have to check that a filtration is bounded if and only if it is fpqc hyperlocally. But this follows immediately from the definition of a hypercovering, since $A \to A_0$ has to be an fpqc-covering and thus if a filtration is bounded on the hypercovering it is certainly on A. Also we have to check that the induced morphism on the graded pieces (up to Frobenius twist) is an equivalence if and only if it is so fpqc hyperlocally but again this follows from descent of modules. □

Recall that the functor $(-)^{\simeq}$ that sends an ∞-category to its underlying Kan complex is right adjoint to the inclusion and thus preserves limits. In particular the hypercomplete sheaf $F\text{-}\mathrm{Zip}_{\infty,R}$ induces a derived stack. We want to show that this stack is locally geometric. To do so, we have to write it as a filtered colimit of geometric stacks. In the case of perfect modules, we restricted ourselves to perfect modules of fixed Tor-amplitude $\mathrm{Perf}_R^{[a,b]}$. To use the geometricity of $\mathrm{Perf}_R^{[a,b]}$ for our advantage, we fix the Tor-amplitude of the graded pieces associated to the descending filtration of a derived F-zip $(C^\bullet, D_\bullet, \phi, \varphi_\bullet)$. By boundedness of the filtrations this also fixes the Tor-amplitude of each C^i and D_i for all $i \in \mathbb{Z}$ (for D_i we use the equivalences given by $\varphi_\bullet$).

But this is still not enough for geometricity since we would need to cover filtrations that could get bigger and bigger. To solve this problem, we also fix a finite subset $S \subseteq \mathbb{Z}$, where the i-th graded piece vanishes for $i \notin S$. This is analogous to fixing the *type* (see Definition 6.28), which is done in the classical setting by [MW04]. This approach also works as seen later in Remark 6.41 but amounts to the same proof.

Definition 6.22. We define the *derived stack of F-zips*

$$\begin{aligned} F\text{-Zip}_R \colon \mathrm{AR}_R &\to \mathbb{S} \\ A &\mapsto F\text{-Zip}_{\infty,R}(A)^{\simeq}. \end{aligned}$$

For a finite subset $S \subseteq \mathbb{Z}$ and $a \leq b \in \mathbb{Z}$, we define $F\text{-Zip}_R^{[a,b],S}$ as the derived substack over $F\text{-Zip}_R$, where we restrict ourself to the F-zips $(C^\bullet, D_\bullet, \phi, \varphi_\bullet)$, such that $\mathrm{gr}^i C \simeq 0$ for $i \notin S$ and the Tor-amplitude of $\mathrm{gr}^i C$ is contained in $[a,b]$ for all $i \in S$ (note that both conditions can be tested locally and thus $F\text{-Zip}_R^{[a,b],S}$ indeed defines a derived substack).

Theorem 6.23. *Let $S \subseteq \mathbb{Z}$ be a finite subset and $a \leq b \in \mathbb{Z}$. The derived stack $F\text{-Zip}_R^{[a,b],S}$ is $(b-a+1)$-geometric and locally of finite presentation. Further, the functor $F\text{-Zip}_R^{[a,b],S} \to \mathrm{Perf}_R^{[a,b]}$ induced by $(C^\bullet, D_\bullet, \phi, \varphi_\bullet) \mapsto \mathrm{colim}_{\mathbb{Z}^{\mathrm{op}}} C^\bullet$ is locally of finite presentation.*

We want to remark that in fact F-Zip is locally geometric, which will be shown later on (see Theorem 6.42).

The idea of the proof is straightforward. We know that perfect modules with fixed Tor-amplitude, morphisms between those and stacks classifying equivalences between those are geometric. Since the filtrations have finite length, we can see them as finite chains between perfect modules with fixed Tor-amplitude, which is also geometric. The only thing left is to extend finite chains of perfect modules to functors from $\mathbb{Z}$ (resp. $\mathbb{Z}^{\mathrm{op}}$) to perfect modules with fixed degree where the graded pieces do not vanish. But this is also straightforward since the only thing left is to degenerate each vertex in the finite chain such that it sits in the right degree.

Proof of Theorem 6.23. Let k be the number of elements of S and let us index S in the following way $\{s_0 < \cdots < s_{k-1}\}$. Let us set $n \coloneqq b-a+1$. Consider the pullback square

$$\begin{tikzcd} V \arrow[r] \arrow[d, "p"] & \mathrm{Fun}(\Delta^1, \mathrm{Mod}^{\mathrm{perf}}_{(-)})^{\simeq} \arrow[d] \\ \mathrm{Perf}_R^{[a,b]} \times_R \mathrm{Perf}_R^{[a,b]} \arrow[r] & \mathrm{Fun}(\partial\Delta^1, \mathrm{Mod}^{\mathrm{perf}}_{(-)})^{\simeq} \end{tikzcd}$$

then V is an n-geometric stack locally of finite presentation, since itself classifies morphisms between perfect complexes with Tor amplitude in $[a,b]$ and thus the fiber of a point $\mathrm{Spec}(A) \to \mathrm{Perf}_R^{[a,b]} \times_R \mathrm{Perf}_R^{[a,b]}$, classified by perfect A-modules (P,Q) of Tor-amplitude in $[a,b]$, under p is given by $F_{P \otimes Q^\vee}$, which is $(n-2)$-geometric and locally of finite presentation by Lemma 5.5 (thus using that $\mathrm{Perf}_R^{[a,b]} \times_R \mathrm{Perf}_R^{[a,b]}$ is n-geometric (since $\mathrm{Perf}_R^{[a,b]}$ is n-geometric by Theorem 5.14) we see that V is n-geometric).

We will now glue copies of V together so that we can classify a chain of morphisms, and since F-zips have two filtrations, we will do this twice. Let us start with $\mathrm{codom}\colon V \to \mathrm{Perf}_R^{[a,b]}$, which sends a morphism to its codomain. This morphism is n-geometric and locally of finite presentation, since it is the composition of $V \to \mathrm{Perf}_R^{[a,b]} \times_R \mathrm{Perf}^{[a,b]}$ and $p_2\colon \mathrm{Perf}_R^{[a,b]} \times_R \mathrm{Perf}_R^{[a,b]} \to \mathrm{Perf}_R^{[a,b]}$, which both are n-geometric and locally of finite presentation (analogously the map $\mathrm{dom}\colon V \to \mathrm{Perf}^{[a,b]}$ which sends a morphism to its domain is n-geometric and locally of finite presentation). In particular $\widetilde{V_1} := V \times_{\mathrm{codom},\mathrm{Perf}_R^{[a,b]},\mathrm{codom}} V$ is n-geometric and locally of finite presentation.

The derived stack $\widetilde{V_1}$ classifies tuples of morphisms of perfect modules $(M \to M', N \to N')$, such M' is equivalent to N'. This is not an extra datum, as codom from V to $\mathrm{Perf}_R^{[a,b]}$ is pointwise a Kan fibration, so we can use Remark 5.13 to see that $\widetilde{V_1}$ is pointwise equivalent in $\mathbb{S}$ to the ordinary pullback of simplicial sets, where we do not need to keep track of the equivalence of M' and N'. Since we want to keep track of the equivalence between the colimits, we define V_1 via the pullback square

$$\begin{tikzcd} V_1 \arrow[r] \arrow[d, "\widetilde{p}"] & \mathrm{Fun}(\Delta^1, \mathrm{Perf}_R^{[a,b]}) \arrow[d] \\ \widetilde{V_1} \arrow[r, "(\mathrm{codom},\mathrm{codom})"] & \mathrm{Fun}(\partial\Delta^1, \mathrm{Perf}_R^{[a,b]}). \end{tikzcd}$$

Note that for any morphism $\mathrm{Spec}(A) \to \widetilde{V_1}$, classified by two morphisms $(M \to M', N \to N')$ of perfect A-modules of Tor-amplitude $[a,b]$, the fiber under $\widetilde{p}$ is given by the stack classifying equivalences between M' and N', which is open in the derived stack $\mathrm{Hom}_{\mathrm{Mod}_A}(M' \otimes_A (N')^\vee, -)$ over A by Lemma 5.4[31] and thus $(n-2)$-geometric and locally of finite presentation by Lemma 5.5. By our construction V_1 is n-geometric and classifies tuples (f, g, ψ), where f, g are morphisms of perfect modules and ψ is an equivalence between their codomains.

Let us set recursively

$$V_l := (V \times_R V) \times_{\mathrm{codom}\times\mathrm{codom},\mathrm{Perf}_R^{[a,b]}\times_R\mathrm{Perf}_R^{[a,b]},\mathrm{dom}\times\mathrm{dom}} V_{l-1},$$

for $l \geq 2$. Let us also set V_0 as the stack classifying equivalences between perfect modules (analogously defined as V). Here $\mathrm{dom}\times\mathrm{dom}\colon V_{l-1} \to \mathrm{Perf}_R^{[a,b]} \times_R \mathrm{Perf}_R^{[a,b]}$ is defined for $l > 2$ by projecting to $V \times_R V$ and then further projecting by $\mathrm{dom}\times\mathrm{dom}$ and for $l = 2$ it is directly given by $\mathrm{dom}\times\mathrm{dom}$ (note that we have two projections from V_1 to V). Using the same arguments as before, we see that V_l is n-geometric and locally of finite presentation for all $l \geq 0$. Now let us look at V_{k-1}. The stack V_{k-1} classifies two chains of length $k-1$ of morphisms of perfect modules with Tor-amplitude in $[a,b]$, with an equivalence of the ends of the two chains.

[31]This is seen in the proof of Lemma 5.7

Later in the proof, we will identify V_{k-1} with the stack that classifies perfect modules with Tor-amplitude in $[a,b]$ and two bounded filtrations with graded pieces of Tor-amplitude in $[a,b]$ (one descending, one ascending) with k non-trivial graded pieces.

We extend the chains by zero on the left by defining

$$\widetilde{V_{k-1}} \coloneqq V_{k-1} \times_{\mathrm{dom}\times\mathrm{dom},\mathrm{Perf}_R^{[a,b]}\times_R \mathrm{Perf}_R^{[a,b]},p_0\times p_0} W \times_R W,$$

where W is defined via the pullback

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "p_0"] & \mathrm{Fun}(\Delta^1, \mathrm{Mod}_{(-)}^{\mathrm{perf}})^{\simeq} \arrow[d] \\
\mathrm{Perf}_R^{[a,b]} \arrow[r, "M\mapsto (0,M)"] & \mathrm{Fun}(\partial\Delta^1, \mathrm{Mod}_{(-)}^{\mathrm{perf}})^{\simeq}.
\end{tikzcd}$$

Since 0 is the initial object in perfect modules, we see that $W \simeq \mathrm{Perf}_R^{[a,b]}$ and thus $\widetilde{V_{k-1}} \simeq V_{k-1}$, but this description will ease the connection to derived F-zips (we have to add zeroes to get left bounded filtrations and by working with $\widetilde{V_{k-1}}$ this is automatic if we degenerate the left most vertex). Note that we can identify an element in $\widetilde{V_{k-1}}(A)$ with two functors

$$C_\bullet, D_\bullet \colon S_- \coloneqq \{s_0 - 1 < s_0 < \cdots < s_{k-1}\} \to \mathrm{Mod}_A^{\mathrm{perf}}.$$

Sending $C_\bullet$ to the functor

$$\begin{aligned}
C^\bullet \colon S_+^{\mathrm{op}} \coloneqq \{s_0 < \cdots < s_{k-1} < s_{k-1}+1\}^{\mathrm{op}} &\to \mathrm{Mod}_A^{\mathrm{perf}} \\
s_i &\mapsto C_{s_{k-1-i}} \\
s_{k-1}+1 &\mapsto C_{s_0-1}
\end{aligned}$$

defines an obvious equivalence between the corresponding functor categories and so we may see an element $\widetilde{V_{k-1}}(A)$ as a tuple $(C^\bullet, D_\bullet, \psi)$ of a finite descending chain $C^\bullet$ of perfect A-modules, a finite ascending chain $D_\bullet$ of perfect A-modules and an equivalence of their respective colimits, i.e.

$$\psi \colon \operatorname*{colim}_{S_+^{\mathrm{op}}} C^\bullet \simeq \operatorname*{colim}_{S_-} D_\bullet.$$

With this identification, we get a morphism from $\widetilde{V_{k-1}}$ to the k-fold product of $\mathrm{Fun}(\partial\Delta^1, \mathrm{Perf})$, by sending a pair of chains to the graded pieces of the chains (resp. if we see them as filtrations, we send them to the non-trivial graded pieces), i.e. if the filtrations of an element in $\widetilde{V_{k-1}}(A)$, for some $A \in \mathrm{AR}_R$, is given by $(C^\bullet, D_\bullet, \psi)$, we take $((\mathrm{gr}^s C)^{(1)}, \mathrm{gr}^s D)_{s\in S}$ (here $\mathrm{gr}^s C$ is defined as the cofiber of $C^{s_{i+1}} \to C^{s_i}$, where s_i corresponds to s in the notation above, analogously for $\mathrm{gr}^s D$). With this let us look at the following pullback square

$$\begin{tikzcd}
\widetilde{V} \arrow[r] \arrow[d, "p"] & \prod_{s\in S} \mathrm{Fun}(\Delta^1, \mathrm{Perf}_R) \arrow[d] \\
\widetilde{V_{k-1}} \arrow[r, "((\mathrm{gr}^s(-))^{(1)}, \mathrm{gr}^s(-))_{s\in S}"] & \prod_{s\in S} \mathrm{Fun}(\partial\Delta^1, \mathrm{Perf}_R)
\end{tikzcd}$$

(note the difference to the previous squares: previously we considered morphisms in $\mathrm{Mod}_{-}^{\mathrm{perf}}$, whereas here we consider morphisms in the underlying Kan complex, so only invertible ones).

Now let $\mathrm{Spec}(A) \to \widetilde{V_k}$ be a morphism classified by a perfect A-module with Tor-amplitude in $[a,b]$ and a descending (resp. ascending) chain $C^\bullet$ (resp. $D_\bullet$). Then $p^{-1}(\mathrm{Spec}(A))$ is given by the stack

$$\prod_{s\in S} \mathrm{Equiv}((\mathrm{gr}^s C)^{(1)}, \mathrm{gr}^s D),$$

where $\mathrm{Equiv}((\mathrm{gr}^s C)^{(1)}, \mathrm{gr}^s D)$ denotes the stack classifying equivalences between $(\mathrm{gr}^s C)^{(1)}$ and $\mathrm{gr}^s D$. This stack is n-geometric and locally of finite presentation (since each term is n-geometric and locally of finite presentation)[32] and therefore, we see that $\widetilde{V}$ is n-geometric and locally of finite presentation. By construction, we also have that the morphism

$$\widetilde{V} \to \mathrm{Perf}_R^{[a,b]}$$

given by sending one of the chains to its colimit is n-geometric and locally of finite presentation (as it is given by projections down to $\mathrm{Perf}_R^{[a,b]}$ which are all n-geometric and locally of finite presentation).

We can naturally define a functor $F\colon \widetilde{V} \to F\text{-}\mathrm{Zip}_R^{[a,b],S}$ by extending the filtrations via identity to get non-trivial graded pieces at the points of S and the isomorphisms of the graded pieces by the essentially unique zero morphism.

To be more specific we will show how to give F as a functor of simplicial sets. The extension by identity will just be degeneration of simplicial sets. Let $\sigma \in \widetilde{V}(A)$ be an m-simplex. Then σ is given by functors

$$D\colon S_- \times \Delta^m \to \mathrm{Mod}_A^{\mathrm{perf}}, \quad C\colon S_+^{\mathrm{op}} \times \Delta^m \to \mathrm{Mod}_A^{\mathrm{perf}}$$

with

$$\psi\colon C_{|\{s_0\}\times\Delta^m} \simeq D_{|\{s_{k-1}\}\times\Delta^m}, \quad C_{|\{s_{k-1}+1\}\times\Delta^m} \simeq 0 \simeq D_{|\{s_0-1\}\times\Delta^m},$$

and $(\varphi_s)_{s\in S}$, where $\varphi_s\colon \Delta^1 \times \Delta^m \to \mathrm{Perf}_R(A)$ is a functor between simplicial sets, such that

$$\varphi_{s|\Delta^{\{0\}}\times\Delta^{\{i\}}} \simeq (\mathrm{gr}^s C)^{(1)} \text{ and } \varphi_{s|\Delta^{\{1\}}\times\Delta^{\{i\}}} \simeq \mathrm{gr}^s D.$$

We will show how to define an extension of D, i.e. a functor $\bar{D}\colon \mathbb{Z}\times\Delta^m \to \mathrm{Mod}_A^{\mathrm{perf}}$, that restricted to S_- is equivalent to D.

Let (α,σ) be an l-simplex in $\mathbb{Z}\times\Delta^m$, note that α is given by a sequence of integers $\alpha_0 \leq \dots \leq \alpha_{l-1}$. We are going to count the number of α_i that are between two vertices in S_- and degenerate our finite chain for this amount. This can be thought of adding the right amount of identities between vertices such that the resulting filtration has non-vanishing graded pieces precisely at all $s \in S$.

[32] Again, $\mathrm{Equiv}((\mathrm{gr}^s C)^{(1)}, \mathrm{gr}^s D)$ is open in the derived stack $\mathrm{Hom}_{\mathrm{Mod}_A}((\mathrm{gr}^s C)^{(1)} \otimes (\mathrm{gr}^s D)^\vee, -)$ over A by lemma 5.4 and $\mathrm{Hom}_{\mathrm{Mod}_A}((\mathrm{gr}^s C)^{(1)} \otimes (\mathrm{gr}^s D)^\vee, -) \to \mathrm{Spec}(A)$ is $(n-2)$-geometric by Lemma 5.5.

Let us note the degeneracy maps of S_- with $\lambda_\bullet$ (here we see the finite ordered set S_- as an ∞-category and set $\lambda_{-\infty}$ as the degeneracy map corresponding to $s_0 - 1$). For $s_d \in S_-$ define sets $A_d \coloneqq \{\alpha_j \mid s_d \leq \alpha_j \leq s_{d+1}\}$, where we set $s_k = \infty$ and $A_{-\infty} \coloneqq \{\alpha_j \mid \alpha_j \leq s_0 - 1\}$. Let us further set $n_i = \# A_i$. We consider the l-simplex in S_- of the form

$$\bar{\alpha} \coloneqq \lambda_{k-1}^{n_{k-1}-1} \circ \cdots \circ \lambda_{-\infty}^{n_{-\infty}-1} \langle s_j \mid A_j \neq \emptyset \rangle,$$

where $\lambda_i^{-1} \coloneqq \mathrm{id}$. Thus, we can set $\bar{D}(\alpha, \sigma) \coloneqq D(\bar{\alpha}, \sigma)$. Similarly, this can be done to extend C, ψ and $(\varphi_s)_s$ which defines the functor F.

We also get a projection $P\colon F\text{-}\mathrm{Zip}_R^{[a,b],S} \to \widetilde{V}$ via restricting the filtrations to $\{s_0 - 1 \leq s_0 \leq \cdots \leq s_{k-1}\} \simeq S_-$ resp. $\{s_0 \leq \cdots \leq s_{k-1} \leq s_{k-1} + 1\} \simeq S_+$ and the equivalences to $\mathrm{Fun}(\Delta^1, \mathrm{Fun}(S^{\mathrm{disc}}, \mathrm{Perf}_R(A)))$. Since by this construction $P \circ F$ is the identity, we see that F is in fact a monomorphism and since it is an effective epimorphism (by Remark 6.15 F is pointwise essentially surjective on the level of homotopy categories), this shows that F is an equivalence of derived stacks. □

Corollary 6.24. *Let $S \subseteq \mathbb{Z}$ be a finite subset and $a \leq b \in \mathbb{Z}$. The derived stack $F\text{-}\mathrm{Zip}_R^{[a,b],S}$ has a perfect cotangent complex*

Proof. This follows from Theorem 6.23 and Corollary 4.78.

□

6.3 On some substacks of *F*-Zip

In this section, we want to define the *type* of a derived F-zip and look at the derived substacks classified by the type. We do the same with those derived F-zips where the underlying module has some fixed Euler-characteristic. These derived substacks will be open (resp. locally closed) and we will use these to write the derived stack of derived F-zips as a filtered colimit of open derived substacks.

We do this in the spirit of the classical theory of F-zips over a scheme S. There the type of a classical F-zip $(M, C^\bullet, D_\bullet, \varphi_\bullet)$ is a function from S to functions $\mathbb{Z} \to \mathbb{N}_0$ with finite support that assigns for a point $s \in S$ the function $k \mapsto \dim_{\kappa(s)}(\mathrm{gr}_C^k M \otimes_{\mathcal{O}_S} \kappa(s))$ (one uses that the graded pieces of a classical F-zip are finite projective and thus the dimension on the fibers is locally constant with respect to s).

Since we are working with complexes we have to modify the definition of the type. To be more specific, we will look at fiberwise dimensions of all cohomologies at once. For a perfect complex P over S the assignment $s \in S \mapsto H^i(P \otimes_{\mathcal{O}_S} \kappa(s))$ defines an upper semi-continuous function. Since derived F-zips have only finitely many nonzero graded pieces, we will use this result to analyze the geometry of the derived substacks classifying derived F-zips with certain type.

Definition and Remark 6.25. Let A be an animated ring and let P be a perfect A-module. We define the function

$$\begin{aligned} \beta_P\colon \mathrm{Spec}(\pi_0 A)_{\mathrm{cl}} &\to \mathbb{N}_0^{\mathbb{Z}} \\ a &\mapsto (\dim_{\kappa(a)} \pi_i(P \otimes_A \kappa(a)))_{i \in \mathbb{Z}}. \end{aligned}$$

Since $P \otimes_A \pi_0 A$ is perfect and thus has bounded Tor-amplitude, we see that β_P takes values in functions $\mathbb{Z} \to \mathbb{N}_0$ with finite support and $\beta_P^{-1}(([0,k_i])_{i\in\mathbb{Z}})$ is open and quasi-compact for any $(k_i)_i \in \mathbb{N}_0^{\mathbb{Z}}$ (see Lemma 5.1).

Let $I \subseteq \mathbb{Z}$ be a finite subset. Assume that $\mathrm{Supp}(\beta_P(a)) \subseteq I$ for all $a \in \mathrm{Spec}(\pi_0 A)_{\mathrm{cl}}$. Recall from Remark 5.2 that this implies that P has Tor-amplitude in $[\min(I), \max(I)]$. Further, as explained in the proof of Lemma 5.3, the Lemma of Nakayama implies that if β_P constant with value equal to $(0)_{i\in\mathbb{Z}}$, then $P \simeq 0$.

The following definition will be used to ease notation.

Definition 6.26. Let $f\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be a function. We say f has *finite support* if the induced function $\mathbb{Z} \times \mathbb{Z} \to \mathbb{N}_0$ given by $(n,m) \mapsto f(n)_m$ has finite support.

Remark 6.27. A function $f\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ has finite support if the sets

$$\{n \in \mathbb{Z} \mid f(n) \neq (0)_{i\in\mathbb{Z}}\}, \quad \{k \in \mathbb{Z} \mid f(n)_k \neq 0\}$$

are finite.

Definition and Remark 6.28. Let A be an animated R-algebra and $\underline{F} := (C^\bullet, D_\bullet, \phi, \varphi_\bullet)$ be a derived F-zip over A. Consider the function

$$\beta_{\underline{F}}\colon a \mapsto (k \mapsto \beta_{\mathrm{gr}^k C}(a))$$

from $\mathrm{Spec}(\pi_0 A)_{\mathrm{cl}}$ to functions with finite support.

1. The function $\beta_{\underline{F}}$ is called *type of the derived F-zip $\underline{F}$*.
2. Let $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be a function with finite support. We say that $\underline{F}$ *has type* $\leq \tau$ if for all $a \in \mathrm{Spec}(\pi_0 A)_{\mathrm{cl}}$ and all $n, m \in \mathbb{Z}$, we have $\beta_{\underline{F}}(a) \leq \tau$.[33]

Further, for any $a \in \mathrm{Spec}(\pi_0 A)_{\mathrm{cl}}$ there exists a quasi-compact open neighbourhood U_a of a (resp. locally closed subset V_a containing a) and a function $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ with finite support, such that $\beta_{\underline{F}|U_a} \leq \tau$ in the sense above (resp. $\beta_{\underline{F}|V_a}$ is constant an equal to τ) (this follows from Lemma 5.1).

Remark 6.29. Let us come back to our example of a proper smooth morphism $f\colon X \to \mathrm{Spec}(A)$. Let $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be a function with finite support. Then the derived F-zip $\underline{R\Gamma_{\mathrm{dR}}(X/A)}$ of Example 6.16 has type $\leq \tau$ if the Hodge numbers

$$\dim_{\kappa(a)} H^{-j-i}(X_{\kappa(a)}, \Omega^i_{X_{\kappa(a)}/\kappa(a)}) \leq \tau(i)_j$$

for all $i, j \in \mathbb{Z}$ and $a \in \mathrm{Spec}(A)$ (note that the minus signs in the Hodge-numbers appears since we use *homological* notation).

[33] Again, we view $\beta_P(a)$ and τ as functions from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{N}_0$ and define the inequality pointwise.

Definition and Remark 6.30. Let A be an animated R-algebra and $\underline{F} \coloneqq (C^\bullet, D_\bullet, \phi, \varphi_\bullet)$ be a derived F-zip over A. Let us look at the function

$$\begin{aligned}\chi_k(\underline{F})\colon \operatorname{Spec}(\pi_0 A)_{\mathrm{cl}} &\to \mathbb{Z}\\ s &\mapsto \chi(\operatorname{gr}^k C \otimes_A \kappa(s)).\end{aligned}$$

This is a locally constant function (see [Sta19, 0B9T]). Since the filtrations on derived F-zips are bounded, we also know that the function $\chi_{\underline{F}}\colon a \mapsto (k \mapsto \chi_k(\underline{F})(a))$ is also locally constant as a map from $\operatorname{Spec}(\pi_0 A)_{\mathrm{cl}}$ to functions $\mathbb{Z} \to \mathbb{Z}$ with finite support. We call $\chi_{\underline{F}}$ the *Euler-characteristic* of $\underline{F}$.

If $\tau\colon \mathbb{Z} \to \mathbb{Z}$ is a function with finite support, we say $\underline{F}$ *has Euler-characteristic* τ if for $\chi_{\underline{F}}$ is constant with value τ.

The reason behind the following definition gets clear later on. One problem will be that we cannot classify F-zips of fixed type, since the type is only "upper semi-continuous" (we do not explicitly define this notion but hope that the idea is clear from the previous definitions and remarks). This can be resolved when we assume that the homotopies of the graded pieces are finite projective.

Definition 6.31. Let A be an animated ring.

1. Let $M \in \operatorname{Mod}_A$ be a perfect A-module. Fix a map $r\colon \mathbb{Z} \to \mathbb{N}_0$. We call M *homotopy finite projective of rank* r, if for all schemes S and scheme morphism $f\colon S \to \operatorname{Spec}(\pi_0 A)_{\mathrm{cl}}$ the $\mathcal{O}_S$-module $\pi_i(f^*(M \otimes_A \pi_0 A))$ is finite locally free of rank r_i.
2. Let $\underline{F} \coloneqq (C^\bullet, D_\bullet, \phi, \varphi_\bullet)$ be a derived F-zip over A and $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be a function with finite support. We say that $\underline{F}$ is *homotopy finite projective of type* τ if for all $i \in \mathbb{Z}$, we have that $\operatorname{gr}^i C$ is homotopy finite projective of rank $\tau(i)$.

Remark 6.32. By the above definition if a derived F-zip is homotopy finite projective of type τ, for τ like above, then its ascending filtration has type τ.

Definition 6.33. Let A be an animated R-algebra. Let $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be a function with finite support and finite value.

(i) We set $F\text{-}\mathrm{Zip}^{\leq\tau}_{\infty,R}(A)$ to be the subcategory of $F\text{-}\mathrm{Zip}_{\infty,R}(A)$ consisting of those derived F-zips with type $\leq \tau$.

(ii) We set $F\text{-}\mathrm{Zip}^{\tau}_{\infty,R}(A)$ to be the subcategory of $F\text{-}\mathrm{Zip}_{\infty,R}(A)$, of those derived F-zips that are homotopy finite projective of type τ.

The associated functors denoted by $F\text{-}\mathrm{Zip}^{\leq\tau}_{\infty,R}$ and $F\text{-}\mathrm{Zip}^{\tau}_{\infty,R}$ and the associated functors AR_R to $\mathbb{S}$ are denoted by $F\text{-}\mathrm{Zip}^{\leq\tau}_{R}$ respectively $F\text{-}\mathrm{Zip}^{\tau}_{R}$.

Proposition 6.34. *The functors $F\text{-}\mathrm{Zip}^{\leq\tau}_{\infty,R}$ and $F\text{-}\mathrm{Zip}^{\tau}_{\infty,R}$ are hypercomplete fpqc sheaves. In particular, $F\text{-}\mathrm{Zip}^{\leq\tau}_{R}$ and $F\text{-}\mathrm{Zip}^{\tau}_{R}$ are derived substacks of $F\text{-}\mathrm{Zip}_{R}$.*

Proof. Using the arguments as in the proof of Proposition 6.21, we only have to show that if $A \to \widetilde{A}$ is faithfully flat, a derived F-zip over A has type $\leq \tau$ respectively is homotopy finite projective of type τ if and only if it has so after base change to $\widetilde{A}$. But by faithfully flatness, we know that $\operatorname{Spec}(\pi_0\widetilde{A})_{\mathrm{cl}} \to \operatorname{Spec}(\pi_0 A)_{\mathrm{cl}}$ is faithfully flat. Now the definitions involved easily show the claim noting that for A-module M and any commutative diagram of the form

$$\begin{tikzcd}
\kappa(a') \arrow[r] \arrow[d] & \operatorname{Spec}(\pi_0\widetilde{A}) \arrow[d] \\
\kappa(a) \arrow[r] & \operatorname{Spec}(\pi_0 A),
\end{tikzcd}$$

where $\kappa(a)$ is the residue field of a point $a \in \operatorname{Spec}(\pi_0 A)$ and $\kappa(a')$ is the residue field of a lift $a' \in \operatorname{Spec}(\pi_0\widetilde{A})$ of a (exists by faithfully flatness), we have

$$\begin{aligned}
\dim_{\kappa(a')} \pi_i(M \otimes_A \widetilde{A} \otimes_{\widetilde{A}} \kappa(a')) &= \dim_{\kappa(a')} \pi_i \left(M \otimes_A \kappa(a) \otimes_{\kappa(a)} \kappa(a')\right) \\
&= \dim_{\kappa(a')} \pi_i \left(M \otimes_A \kappa(a)\right) \otimes_{\kappa(a)} \kappa(a') \\
&= \dim_{\kappa(a)} \pi_i \left(M \otimes_A \kappa(a)\right),
\end{aligned}$$

where we use flatness of field extensions for the second equality. □

Definition 6.35. Let A be an animated R-algebra. Let $\tau\colon \mathbb{Z} \to \mathbb{Z}$ be a locally constant function with finite support. We set $F\text{-Zip}^{\chi=\tau}_{\infty,R}(A)$ to be the subcategory of $F\text{-Zip}_{\infty,R}(A)$ classifying those derived F-zips $\underline{F}$ with $\chi_{\underline{F}} = \tau$.

The associated functor is denoted by $F\text{-Zip}^{\chi=\tau}_{\infty,R}$ and the associated presheaf from AR_R to $\mathbb{S}$ is denoted by $F\text{-Zip}^{\chi=\tau}_{R}$.

Lemma 6.36. *The functor $F\text{-Zip}^{\chi=\tau}_{\infty,R}$ is a hypercomplete fpqc sheaf. In particular, the functor $F\text{-Zip}^{\chi=\tau}_{R}$ is a derived substack of $F\text{-Zip}_R$.*

Proof. The proof is completely analogous to the proof of Proposition 6.34. □

In the following, we want to show that the inclusion of the derived substacks $F\text{-Zip}^{\leq\tau}$, $F\text{-Zip}^{\tau}$ and $F\text{-Zip}^{\chi=\tau}$ into $F\text{-Zip}$ are in fact geometric. To show geometricity of $F\text{-Zip}^{\tau}$ we will need a proposition from the upcoming book of Görtz-Wedhorn. This proposition in particular shows the reason behind the definition of *homotopy finite projectiveness*. The finite projectiveness of the homotopy groups is needed to have some geometric structure if we fix the type.

Lemma 6.37 ([GW])**.** *Let S be a scheme, let E be a perfect complex in $D(S)$ of Tor-amplitude $[a,b]$, and let $I \subseteq [a,b]$ be an interval containing a or b. Fix a map $r\colon I \to \mathbb{N}_0$, $i \mapsto r_i$. Then there exists a unique locally closed subscheme $j\colon Z = Z_r \hookrightarrow S$ such that a morphism $f\colon T \to S$ factors through Z if and only if for all morphisms $g\colon T' \to T$ the $\mathcal{O}_{T'}$-module $\pi_i(L(f \circ g)^*E)$ finite locally free of rank r_i for all $i \in I$. Moreover,*

(1) *the immersion $j\colon Z \hookrightarrow S$ is of finite presentation,*

(2) *as a set one has*

$$Z = \{s \in S \mid \dim_{\kappa(s)} \pi_i(E \otimes^L_{\mathcal{O}_S} \kappa(s)) = r_i \text{ for all } i \in I\},$$

(3) *if* $f\colon T \to S$ *factors as* $T \xrightarrow{\bar{f}} Z \xrightarrow{j} S$, *then* $\pi_i(Lf^*E \otimes^L_{\mathcal{O}_T} \mathcal{G}) = \bar{f}^*\pi_i(j^*E) \otimes_{\mathcal{O}_T} \mathcal{G}$ *for all* $i \in I$ *and for all quasi-coherent* $\mathcal{O}_T$*-modules* $\mathcal{G}$.

Proof. See Lemma C.2. $\square$

We are going to show that derived F-zips of certain type are classified by open (resp. locally closed) substacks of $F\text{-Zip}_R$. We have seen in Remark 4.42 that an open immersion $U \hookrightarrow \operatorname{Spec}(\pi_0 A)$ of R-algebras, where $A \in \mathrm{AR}_R$, can be lifted to an open immersion $\widetilde{U} \hookrightarrow \operatorname{Spec}(A)$. Also any morphism $\operatorname{Spec}(T) \to \operatorname{Spec}(A)$ factors through $\widetilde{U}$ if étale locally $\operatorname{Spec}(\pi_0 T) \to \operatorname{Spec}(\pi_0 A)$ factors through U.

We could now try to do the same for closed immersions. So, for a closed immersion, let's say induced by an element $a \in \pi_0 A$, $\operatorname{Spec}(\pi_0 A/(a)) \to \operatorname{Spec}(\pi_0 A)$, we get a closed immersion $\operatorname{Spec}(A/\!\!/(a)) \to \operatorname{Spec}(A)$, where $A/\!\!/(a)$ is the derived quotient[34]. But it is not clear that a morphism $\operatorname{Spec}(T) \to \operatorname{Spec}(A)$ factors through $\operatorname{Spec}(A/\!\!/(a))$ if and only étale locally on π_0 it does. In particular, in the following proposition we cannot show that $F\text{-Zip}^\tau \hookrightarrow F\text{-Zip}^{\le\tau}$ is a locally closed immersion, but only after restricting the functors to R-algebras (we can even show that it is open and closed).

Remark 6.38. In the next Proposition, we will analyze the geometric structure of the inclusion $i\colon F\text{-Zip}_R^{\le\tau} \hookrightarrow F\text{-Zip}_R$. For this, we want to understand the pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname{Spec}(A) \arrow[d] \\ F\text{-Zip}_R^{\le\tau} \arrow[r, "i"] & F\text{-Zip}_R, \end{tikzcd}$$

where $\operatorname{Spec}(A) \to F\text{-Zip}_R$ is given by some derived F-zip $\underline{F} \coloneqq (C^\bullet, D_\bullet, \phi, \varphi_\bullet)$. As explained in Remark 6.15, we can find an equivalence between $\underline{F}$ and a derived F-zip $\underline{G} \coloneqq (C'^\bullet, D'_\bullet, \phi', \varphi'_\bullet)$, where ϕ' is given by the identity on $\operatorname{colim}_{\mathbb{Z}^{\mathrm{op}}} C'^\bullet$. This equivalence between $\underline{F}$ and $\underline{G}$ defines homotopies

$$\begin{tikzcd} \operatorname{Spec}(A) \arrow[r, "\mathrm{id}_{\operatorname{Spec}(A)}"] \arrow[dr, "\underline{G}"'] & \operatorname{Spec}(A) \arrow[d, "\underline{F}"] \\ & F\text{-Zip}_R, \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}(A) \arrow[r, "\mathrm{id}_{\operatorname{Spec}(A)}"] \arrow[dr, "\underline{F}"'] & \operatorname{Spec}(A) \arrow[d, "\underline{G}"] \\ & F\text{-Zip}_R. \end{tikzcd}$$

[34]The derived quotient is defined in the following way. An element in $a \in \pi_0 A = \pi_0 \Omega^\infty A$ gives rise to an element $f \in \operatorname{Hom}(\mathbb{Z}[X], A) \simeq \Omega^\infty A$. Thus we can define the derived quotient as

$$A/\!\!/(a) \coloneqq A \otimes_{f, \mathbb{Z}[X], X \mapsto 0} \mathbb{Z}.$$

So in particular, the pullbacks $F\text{-Zip}_R^{\leq\tau} \times_{i,F\text{-Zip}_R,\underline{F}} \operatorname{Spec}(A)$ and $F\text{-Zip}_R^{\leq\tau} \times_{i,F\text{-Zip}_R,\underline{G}} \operatorname{Spec}(A)$ are equivalent. Thus, for geometric properties of the inclusion i, we may only work with derived F-zips of the form $\underline{G}$ (i.e. those derived F-zips where the equivalence between the colimits is given via the identity).

The same reasoning works also for other substacks.

Proposition 6.39. *Let $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be a function with finite support. Then the inclusion $i\colon F\text{-Zip}_R^{\leq\tau} \hookrightarrow F\text{-Zip}_R$ is a quasi-compact open immersion (in particular 0-geometric by Proposition 4.30).*

Let further $p\colon F\text{-Zip}_R^{\tau} \hookrightarrow F\text{-Zip}_R^{\leq\tau}$ denote the inclusion. Then t_0p[35] is a closed immersion locally of finite presentation.

Proof. Let $\operatorname{Spec}(A) \to F\text{-Zip}_R$ be a morphism of derived stacks with $A \in \mathrm{AR}_{\mathbb{F}_p}$ classified by a derived F-zip $\underline{F} = (C^\bullet, D_\bullet, \varphi_\bullet)$[36]. Then a morphism $f\colon \operatorname{Spec}(T) \to \operatorname{Spec}(A)$ factors through $i^{-1}(\underline{F}) \coloneqq \operatorname{Spec}(A) \times_{F\text{-Zip}_R} F\text{-Zip}^{\leq\tau}$ if and only if $f^*\underline{F}$ has type $\leq \tau$. By Lemma 5.1, we know that there is a quasi-compact open subscheme $\widetilde{U}$ of $\operatorname{Spec}(\pi_0 A)$ classifying those points of $\operatorname{Spec}(\pi_0 A)$, where $\underline{F}$ has type $\leq \tau$ (note that the filtrations are bounded and that perfect complexes have only finitely many non-zero homotopy groups). We claim that f factors over $i^{-1}(\underline{F})$ if and only if it factors over the lift U of $\widetilde{U}$ constructed in Remark 4.42, i.e. if we write $\widetilde{U} = \bigcup_{i=0}^n \operatorname{Spec}(\pi_0 A_{f_i})$, we define U as the image of $\coprod_{i=0}^n \operatorname{Spec}(A[f_i^{-1}]) \to \operatorname{Spec}(A)$ (note that this construction implies that U is quasi-compact).

Indeed, it is clear that if f factors through U, then it certainly factors through $i^{-1}(\underline{F})$. Now assume f factors through $i^{-1}(\underline{F})$. In particular, we have

$$\dim_{\kappa(t)} \pi_i(f^* \operatorname{gr}^j C \otimes_T \kappa(t)) \leq \tau(j)_i$$

for all $t \in \operatorname{Spec}(\pi_0 T)$ and $i, j \in \mathbb{Z}$. But we have the following equalities

$$\begin{aligned}
\dim_{\kappa(t)} \pi_i(f^* \operatorname{gr}^j C \otimes_T \kappa(t)) &= \dim_{\kappa(t)} \pi_i(\operatorname{gr}^j C \otimes_A \kappa(t)) \\
&= \dim_{\kappa(t)} \pi_i \left(\operatorname{gr}^j C \otimes_A \kappa(\pi_0 f(t)) \otimes_{\kappa(\pi_0 f(t))} \kappa(t)\right) \\
&= \dim_{\kappa(t)} \pi_i \left(\operatorname{gr}^j C \otimes_A \kappa(\pi_0 f(t))\right) \otimes_{\kappa(\pi_0 f(t))} \kappa(t) \\
&= \dim_{\kappa(\pi_0 f(t))} \pi_i \left(\operatorname{gr}^j C \otimes_A \kappa(\pi_0 f(t))\right),
\end{aligned}$$

where we use flatness of $\kappa(f(t)) \to \kappa(t)$ in the fourth equality. This shows that $\pi_0 f$ factors through $\widetilde{U}$. Let us write $\widetilde{U} = \bigcup_{j\in J} \operatorname{Spec}((\pi_0 A)_{f_j})_{\mathrm{cl}}$ as a finite union of principal affine opens in $\operatorname{Spec}(\pi_0 A)$. Then f factors though $U = \bigcup_{j\in J} \operatorname{Spec}(A[f_j^{-1}])$ if and only if there is an étale cover $(T \to T_k)_{k\in I}$ such that $\operatorname{Spec}(\pi_0 T_i)$ factors through some $\operatorname{Spec}((\pi_0 A)_{f_j})$ (see Remark 4.42). But this is clear since, the base change of $\coprod_{j\in J} \operatorname{Spec}((\pi_0 A)_{f_j}) \to \widetilde{U}$ to $\operatorname{Spec}(\pi_0 T)$ gives an étale cover of $\operatorname{Spec}(\pi_0 T)$, which can be lifted to an étale cover of $\operatorname{Spec}(T)$ (use Proposition 3.59 and note that faithfully flatness can be checked on π_0), where this property holds per definition.

[35] Recall that $t_0 p$ is the restriction of p to R-algebras by the inclusion $(R\text{-Alg}) \hookrightarrow \mathrm{AR}_R$.

[36] Keeping Remark 6.38 in mind, we do not need to keep track of the equivalence connecting the colimits of the ascending and descending filtrations.

For $t_0\, F\text{-Zip}_R^{\tau}$, let $\operatorname{Spec}(A) \to t_0\, F\text{-Zip}_R^{\leq \tau}$ classified by a derived F-zip $\underline{F}$ over an R-algebra A. Then a morphism $f\colon \operatorname{Spec}(T) \to \operatorname{Spec}(A)$ factors through the projection $t_0 p^{-1}(A) \to \operatorname{Spec}(A)$ if and only if $f^*\underline{F}$ is homotopy finite projective of type τ. By Lemma 6.37 and finiteness of the filtrations, we can find a locally closed subscheme Z of $\operatorname{Spec}(A)$, where $Z \hookrightarrow \operatorname{Spec}(A)$ is finitely presented such that f factors through Z if and only if $f^*\underline{F}$ is homotopy finite projective of type τ. So in particular, $Z \simeq t_0 p^{-1}(A)$.

What is left to show is that Z is also closed in $\operatorname{Spec}(A)$. Using the upper semi-continuity of the betti numbers (see Lemma 5.1), we see that the image of Z in $\operatorname{Spec}(A)$ is a closed subset and therefore, the immersion $Z \hookrightarrow \operatorname{Spec}(A)$ is in fact closed (see [Sta19, 01IQ]). $\square$

Proposition 6.40. *Let $\tau\colon \mathbb{Z} \to \mathbb{Z}$ be a function with finite support. The inclusion $i\colon F\text{-Zip}_R^{\chi=\tau} \hookrightarrow F\text{-Zip}_R$ is an open immersion of derived stacks and further, $t_0 i$ is an open and closed immersion.*

Proof. This completely analogous to proof of 6.39 with Remark 6.75. But nevertheless, we give a proof for completion.

Let $\operatorname{Spec}(A) \to F\text{-Zip}$ be a morphism of derived stacks with $A \in \mathrm{AR}_{\mathbb{F}_p}$ classified by a derived F-zip $\underline{F} = (C^\bullet, D_\bullet, \varphi_\bullet)$. Then a morphism $f\colon \operatorname{Spec}(T) \to \operatorname{Spec}(A)$ factors through $i^{-1}(\underline{F})$ if and only if $\chi(f^*\underline{F}) = \tau$. Since χ is locally constant, we know that we can find an open and closed subscheme $\widetilde{U}$ of $\operatorname{Spec}(\pi_0 A)$ classifying those points of $\operatorname{Spec}(\pi_0 A)$, where $\underline{F}$ has Euler-characteristic τ (note that the filtrations are bounded). Let U be the lift of $\widetilde{U}$ on $\operatorname{Spec}(A)$ constructed in Remark 4.42. Then by construction U is open in $\operatorname{Spec}(A)$. We claim that f factors through $i^{-1}(\underline{F})$ if and only if it factors through U.

Indeed if f factors through U then certainly it also factors through $i^{-1}(\underline{F})$. Now assume f factors through $i^{-1}(\underline{F})$. In particular, we have $\chi(f^* \operatorname{gr}_C^k M \otimes_T \kappa(t)) = \tau(k)$ for all $t \in \operatorname{Spec}(\pi_0 T)$ and $k \in \mathbb{Z}$. But analogous to the proof of Proposition 6.39, we have
$$\chi(f^* \operatorname{gr}_C^k M \otimes_T \kappa(t)) = \chi(f^* \operatorname{gr}_C^k M \otimes_A \kappa(\pi_0 f(t))).$$
This shows that $\pi_0 f$ factors through $\widetilde{U}$. Now, this factorization can be lifted to a factorization of f through U (again the argumentation is the same as in the proof of Proposition 6.39).

That $t_0 i$ is open and closed follows immediately from the above. $\square$

Remark 6.41. By the above $F\text{-Zip}_R^{\leq\tau} \hookrightarrow F\text{-Zip}_R$ is a 0-geometric open immersion. In fact, we can see that $\operatorname{colim}_\tau F\text{-Zip}_R^{\leq\tau} \simeq F\text{-Zip}_R$, where τ runs through the functions $\mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ with finite support. This colimit is filtered, as we can view τ as a function $\mathbb{Z} \times \mathbb{Z} \to \mathbb{N}_0$ and get a pointwise oder.

For the equivalence, note that any F-zip $\underline{F}$ has finitely many graded pieces, which all have finite Tor-amplitude, so we can find a function $\sigma\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ with finite support, such that $\underline{F}$ has type $\leq \sigma$, i.e. $\underline{F} \in F\text{-Zip}^{\leq\sigma}$.

Further, the inclusion $F\text{-Zip}_R^{\leq\tau} \hookrightarrow F\text{-Zip}_R$ has to factor through some $F\text{-Zip}_R^{[a,b],S}$, since the filtrations are bounded and by Remark 6.25. By the same arguments as in the proof of 6.39, we see that $F\text{-Zip}_R^{\leq\tau} \hookrightarrow F\text{-Zip}_R^{[a,b],S}$ is quasi-compact open immersion, which implies, that $F\text{-Zip}_R^{\leq\tau}$ itself is $(b-a+1)$-geometric and locally of finite presentation. In particular, we can write $F\text{-Zip}_R$ as the filtered colimit of geometric derived open substacks

$$F\text{-Zip}_R \simeq \varinjlim_{\tau} F\text{-Zip}_R^{\leq\tau}\,.$$

Theorem 6.42. *The derived stack $F\text{-Zip}_R$ is locally geometric and locally of finite presentation.*

Proof. This follows from Remark 6.41. □

6.4 Strong derived F-zips over affine schemes

In the following we want to look at derived F-zips, where the underlying ascending and descending filtration are strong. Since a morphism between modules is a monomorphism if and only if the diagonal is an equivalence it is not hard to see with Lemma 5.4 that restricting to those derived F-zips where the underlying filtrations are strong gives an open derived substack of F-Zip. We can also use this open derived substack to easily embed the stack of classical F-zips into the derived version.

Definition 6.43. Let A be an animated R-algebra. We set the full sub-∞-category $sF\text{-Zip}_{\infty,R}(A)$ of $F\text{-Zip}_{\infty,R}(A)$ as those derived F-zips $(C^\bullet, D_\bullet, \phi, \varphi)$, where the filtrations $C^\bullet$, $D_\bullet$ are in fact strong filtrations. An element in $sF\text{-Zip}_{\infty,R}(A)$ is called *strong derived F-zip over A*.

Remark 6.44. The base change of strong derived F-zips is again strong. We only need to check that monomorphisms of A-modules are compatible with base change, for some animated ring A. But a morphism of A-modules $M \to N$ is a monomorphism if and only if the diagonal $M \to M \times_N M$ is an equivalence[37]. Since every pullback square in a stable ∞-category is a pushout square (see [Lur17, Prop. 1.1.3.4]), we see that the tensor product of A-modules commutes with pullbacks. Therefore, the pullback induces a functor $A \mapsto sF\text{-Zip}_{\infty,R}(A)$ from AR_R to Cat_∞.

Remark 6.45. Again, this may seem like a useful definition but as it turns out the strongness condition is very strong (see Theorem 6.71). This is because in triangulated categories monomorphisms are automatically *split*. So a strong filtration is automatically determined by its underlying graded pieces, which indicates that the corresponding spectral sequence should degenerate. But the reason behind the derived F-zips is precisely the study of those filtrations with non degenerate spectral sequences.

Definition and Proposition 6.46. *The presheaf*

$$\begin{aligned} sF\text{-Zip}_{\infty,R}\colon \mathrm{AR}_R &\to \mathrm{Cat}_\infty \\ A &\mapsto sF\text{-Zip}_{\infty,R}(A) \end{aligned}$$

is a hypercomplete sheaf for the fpqc topology.

[37] To see this note that Mod_A has finite limits (see [Lur17, Cor. 4.2.3.3]), for any animated ring A, and thus a morphism of A-modules $M \to N$ is a monomorphism if and only if the diagonal $M \to M \times_N M$ is an equivalence (see [Lur09, Lem 5.5.6.15]).

Proof. This is completely analogous to the proof of Proposition 6.21 using the fact that a morphism of modules is a monomorphism if and only if it is after passage to an fpqc cover (since a morphism of modules is a monomorphism if and only if the diagonal is an equivalence[37]). □

Definition 6.47. We define the *derived stack of strong F-zips* as

$$\begin{aligned} sF\text{-}\mathrm{Zip}_R\colon \mathrm{AR}_R &\to \mathbb{S} \\ A &\mapsto sF\text{-}\mathrm{Zip}_{\infty,R}(A)^{\simeq}. \end{aligned}$$

Similarly to Definition 6.22, we define for any finite set $S \in \mathbb{Z}$ and $a \leq b \in \mathbb{Z}$ the derived substack $sF\text{-}\mathrm{Zip}_R^{[a,b],S}$ of $sF\text{-}\mathrm{Zip}_R$.

Proposition 6.48. *Let $S \in \mathbb{Z}$ be a finite subset and $a \leq b \in \mathbb{Z}$. The inclusion of derived stacks $sF\text{-}\mathrm{Zip}_R^{[a,b],S} \hookrightarrow F\text{-}\mathrm{Zip}_R^{[a,b],S}$ is a quasi-compact open immersion. In particular, $sF\text{-}\mathrm{Zip}_R^{[a,b],S}$ is geometric.*

Proof. This follows directly from Theorem 6.23 and the fact that the stack classifying monomorphisms between perfect complexes is open.

To see this let V denote the derived stack classifying morphisms between perfect modules. Let $\iota\colon U \hookrightarrow V$ be the substack classifying those morphisms that are monomorphisms. We claim that ι is a quasi-compact open immersion. Indeed, let $\mathrm{Spec}(A) \to V$ be a morphism of derived stacks, where A is an animated ring, given by a morphism $f\colon M \to N$ of perfect A-modules. Now f is a monomorphism if and only if the diagonal $\Delta_f\colon M \to M \times_N M$ is an equivalence. But the stack classifying those morphisms $A \to B$ such that Δ_f is an equivalence is quasi-compact open by Lemma 5.4, as Δ_f is an equivalence if and only if $\mathrm{cofib}(\Delta_f) \simeq 0$.

To finish the proof note that we can extend the quasi-compact openness condition to derived F-zips since our filtrations are bounded. □

Definition and Remark 6.49. For a function $\sigma\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$, we can define

$$sF\text{-}\mathrm{Zip}_R^{\leq\sigma} \coloneqq sF\text{-}\mathrm{Zip}_R \times_{F\text{-}\mathrm{Zip}_R} F\text{-}\mathrm{Zip}_R^{\leq\sigma} \text{ and } sF\text{-}\mathrm{Zip}_R^{\sigma} \coloneqq sF\text{-}\mathrm{Zip}_R \times_{F\text{-}\mathrm{Zip}_R} F\text{-}\mathrm{Zip}_R^{\sigma}.$$

We also see, that we can write $sF\text{-}\mathrm{Zip}_R \simeq \mathrm{colim}_\tau\, sF\text{-}\mathrm{Zip}_R^{\leq\tau}$ as a filtered colimit, where the colimit is over functions $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ with finite support, with the ordering as in Remark 6.34. Again the $sF\text{-}\mathrm{Zip}_R^{\leq\tau}$ are open in $sF\text{-}\mathrm{Zip}_R$ and $t_0\, sF\text{-}\mathrm{Zip}_R^{\tau}$ are closed in $t_0\, sF\text{-}\mathrm{Zip}_R^{\leq\tau}$.

Proposition 6.50. *The derived stack $sF\text{-}\mathrm{Zip}_R$ is locally geometric and locally of finite presentation.*

Proof. This follows from Remark 6.49. □

In the next proposition, we want to show that the stack of classical F-zips lies quasi-compact open in $t_0\, sF\text{-}\mathrm{Zip}_R$. To do so, we fix some type of a classical F-zip, lets say

σ (as the classical F-zips is the disjoint union of such, this is enough). Then we only look at strong derived F-zips, where the graded pieces are all finite projective modules sitting in one degree. This can be also be achieved by fixing a type τ of the strong derived F-zips (see Remark 6.25). But if we choose τ nicely in relation to σ, we see that strong derived F-zips of type τ are precisely classical F-zips of type σ. Since we only work with derived F-zips corresponding to finite projective modules, we thus see the quasi-compact openness of classical F-zips in derived ones.

Lemma 6.51. *Let $\sigma\colon \mathbb{Z} \to \mathbb{N}_0$ be a function with finite support. Let us set $\tau^\sigma\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ to be the function given by $k \mapsto \tau^\sigma(k)_0 = \sigma(k)$ and $\tau^\sigma(k)_j = 0$ for $j \neq 0$. Let $\operatorname{cl} F\text{-}\mathrm{Zip}_R$ denote the classical stack of F-zips. Then the inclusion $\operatorname{cl} F\text{-}\mathrm{Zip}_R^\sigma \hookrightarrow t_0\, sF\text{-}\mathrm{Zip}_R$ is a quasi-compact open and closed immersion.*

Proof. We know that $t_0\, sF\text{-}\mathrm{Zip}_R^{\leq \tau^\sigma}$ is open in $t_0\, sF\text{-}\mathrm{Zip}_R$ but by our construction for an element $\underline{F} = (C^\bullet, D_\bullet, \varphi_\bullet)$ given by $\operatorname{Spec}(A) \to t_0\, sF\text{-}\mathrm{Zip}_R^{\leq \tau^\sigma}$, we have that $\operatorname{gr}^k C$ is a finite projective module sitting in degree 0. In particular, the function $\beta_{\operatorname{gr}^k C}$ is locally constant. Therefore, even $t_0\, sF\text{-}\mathrm{Zip}_R^{\tau^\sigma}$ is quasi-compact open and closed in $t_0\, sF\text{-}\mathrm{Zip}_R$. Now we have an equivalence $\operatorname{cl} F\text{-}\mathrm{Zip}_R^\sigma \simeq t_0\, sF\text{-}\mathrm{Zip}_R^{\tau^\sigma}$ concluding the proof. $\square$

6.5 Globalization

6.5.1 Globalization of derived stacks

In the following, we want to look at derived F-zips over derived schemes. We also want to look at some properties of the corresponding sheaf. Important here is that derived stacks take *affine derived schemes* as parameters and not *derived schemes*. Let us show how to fix this.

Let R be a ring. We start by extending derived stacks $X\colon \mathrm{AR}_R \to \mathbb{S}$ via right Kan extension to a presheaf $\mathbb{R}X\colon \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})^{\mathrm{op}} \to \mathbb{S}$. Using Remark 4.50, we see that $\mathbb{R}X$ is in fact an étale sheaf. In particular, since every derived scheme has an open cover by affines, $\mathbb{R}X_{|\,\mathrm{dSch}}$ is uniquely determined by $\mathbb{R}X_{|\mathrm{AR}_R} \simeq X$.

In the case of derived F-zips, we could finish this section with the arguments above. But we can also define derived F-zips over derived schemes analogous to Definition 6.22. We will see that this definition agrees with the definition given by right Kan extension.

6.5.2 Filtrations over derived schemes

We want to globalize the construction of derived F-zips. We could do this by right Kan extension but also give a direct definition by globalizing filtrations and defining it analogously to Definition 6.22. In fact both definitions will agree (see Lemma 6.62).

When working with derived schemes, we always assume that our module categories are small, i.e. for any animated ring A, we assume that Mod_A is small. This is, as we want to use Proposition 4.48 to see that quasi-coherent modules satisfy descent.

Let us look at the functor from AR_R to Cat_∞ given by $A \mapsto \mathrm{Fun}(\mathbb{Z}, \mathrm{Mod}_A)$. By Lemma 6.20, we know that this functor satisfies fpqc-hyperdescent. So its right Kan extension to derived schemes will still be an fpqc sheaf (see Remark 4.54). Since $\mathrm{Fun}(\mathbb{Z}, -)$ commutes with limits (which was used in the proof of Lemma 6.20), we immediately see that $\mathbb{R}\,\mathrm{Fun}(\mathbb{Z}, \mathrm{Mod}_-)(S) \simeq \mathrm{Fun}(\mathbb{Z}, \mathrm{QCoh}(S))$ for any derived scheme S.

We have that perfectness of a filtration, boundedness and strongness can be checked fpqc hyperlocally (which was used in the proof of Proposition 6.21 and Proposition 6.46). So in particular using the same arguments as in Proposition 6.21, we see that bounded (resp. perfect or strong) filtrations also satisfy fpqc-hyperdescent and we can right Kan extend to derived schemes and get a sheaf denoted by

$$\mathrm{Fun}^b(\mathbb{Z}, \mathrm{QCoh}(-)) \text{ (resp. } \mathrm{Fun}_{\mathrm{perf}}(\mathbb{Z}, \mathrm{QCoh}(-)) \text{ or } \mathrm{Fun}^s(\mathbb{Z}, \mathrm{QCoh}(-))).$$

By Proposition 4.52, we see that for a derived scheme S an element in $\mathrm{Fun}^b(\mathbb{Z}, \mathrm{QCoh}(-))$ (resp. $\mathrm{Fun}_{\mathrm{perf}}(\mathbb{Z}, \mathrm{QCoh}(-))$ or $\mathrm{Fun}^s(\mathbb{Z}, \mathrm{QCoh}(-))$) is given by a functor $F \in \mathrm{Fun}(\mathbb{Z}, \mathrm{QCoh}(S))$ such that for any affine open $\iota\colon \mathrm{Spec}(A) \hookrightarrow S$ the ascending filtration $\iota^* F$ is bounded (resp. perfect or strong).

Also we have that a functor $F \in \mathrm{Fun}(\mathbb{Z}, \mathrm{QCoh}(S))$ is in $\mathrm{Fun}^b(\mathbb{Z}, \mathrm{QCoh}(-))$ (resp. $\mathrm{Fun}_{\mathrm{perf}}(\mathbb{Z}, \mathrm{QCoh}(-))$ or $\mathrm{Fun}^s(\mathbb{Z}, \mathrm{QCoh}(-))$) if and only if there is a flat atlas $(\mathrm{Spec}(A_i) \xrightarrow{p_i} S)_{i \in I}$ such that $p_i^* F$ is bounded (resp. perfect or strong).

Let us note the above stays true if we replace $\mathbb{Z}$ with $\mathbb{Z}^{op}$ (or in general with any ∞-category but we do not need this).

Definition 6.52. Let S be a derived scheme. An *ascending (resp. descending) filtration of quasi-coherent modules over* S is an element $F \in \mathrm{Fun}(\mathbb{Z}, \mathrm{QCoh}(S))$ (resp. $F \in \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{QCoh}(S))$).

We say that F is

- (i) *locally bounded* if F lies in $\mathrm{Fun}^b(\mathbb{Z}, \mathrm{QCoh}(S))$ (resp. $\mathrm{Fun}^b(\mathbb{Z}^{\mathrm{op}}, \mathrm{QCoh}(S))$),
- (ii) *perfect* if F lies in $\mathrm{Fun}_{\mathrm{perf}}(\mathbb{Z}, \mathrm{QCoh}(S))$ (resp. $\mathrm{Fun}_{\mathrm{perf}}(\mathbb{Z}^{\mathrm{op}}, \mathrm{QCoh}(S))$),
- (iii) *strong* if F lies in $\mathrm{Fun}^s(\mathbb{Z}, \mathrm{QCoh}(S))$ (resp. $\mathrm{Fun}^s(\mathbb{Z}^{\mathrm{op}}, \mathrm{QCoh}(S))$).

Remark 6.53. Note that by Remark 4.45 for any $X \in \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})$ the ∞-categories $\mathrm{Fun}_{\mathrm{perf}}(\mathbb{Z}, \mathrm{QCoh}(S))$, $\mathrm{Fun}^b(\mathbb{Z}, \mathrm{QCoh}(S))$ and $\mathrm{Fun}^s(\mathbb{Z}, \mathrm{QCoh}(S))$ can be seen as full sub ∞-categories of $\mathrm{Fun}(\mathbb{Z}, \mathrm{QCoh}(S))$ (the same holds with $\mathbb{Z}$ replaced by $\mathbb{Z}^{\mathrm{op}}$)

Lemma 6.54. *Let S be a derived scheme and F an ascending (resp. descending) strong filtration of $\mathcal{O}_S$-modules and let $i \in \mathbb{Z}$. Then for all $i \in \mathbb{Z}$ the morphism $F(i) \to F(i+1)$ (resp. $F(i) \to F(i-1)$) is a monomorphism.*

Proof. Note that $\mathrm{Fun}(\mathbb{Z}, \mathrm{QCoh}(S))$ has finite products by Remark 4.47 and the fact that limits can be computed pointwise.

Now we are done, since it suffices to show that the diagonal of $F(i) \to F(i+1)$ is an equivalence (see [Lur09, Lem 5.5.6.15]), which can be checked Zariski locally, where it is true by definition.

For descending filtrations the arguments are analogous. □

Notation 6.55. Let S be a derived scheme and let $\mathcal{F}$ be an $\mathcal{O}_S$-module. If F is an descending filtration on $\mathcal{F}$, we write $F^k := F(k)$, for $k \in \mathbb{Z}$ and $F^\bullet := F$. If G is an ascending filtration on $\mathcal{F}$, we write $G_\bullet$ for G and denote its points by G_k.

Definition 6.56. Let S be a derived scheme and let $\mathcal{F}$ be an $\mathcal{O}_S$-module. Let F be a ascending (resp. descending) filtration on $\mathcal{F}$. For any $i \in \mathbb{Z}$, we define the *i-th graded piece of F* as $\operatorname{gr}^i F := \operatorname{cofib}(F(i-1) \to F(i))$ (resp. $\operatorname{gr}^i F := \operatorname{cofib}(F(i+1) \to F(i))$).

6.5.3 Derived F-zips over schemes

Before defining derived F-zips for derived schemes, we first have to make sense of the Frobenius of derived schemes. Classically the Frobenius on a scheme X is equivalent to the morphism given by composition $X \to X^{(1)} \to X$ of the relative Frobenius and the natural map. The points of $X^{(1)}$ are given by restriction along the Frobenius. This can be used to define the Frobenius for derived schemes (even for derived stacks) as in the following.

Remark and Definition 6.57. Let X be a derived scheme over R. For an animated R-algebra A we have an R-morphism $\operatorname{Spec}(A) \to X^{(1)} := X \times_{\operatorname{Spec}(R),\operatorname{Frob}_R} \operatorname{Spec}(R)$ if and only if there is a morphism $\operatorname{Spec}(A_{\operatorname{Frob}_R}) \to X$, where $A_{\operatorname{Frob}_R}$ is the restriction of A along the Frobenius[38], i.e. $X^{(1)}(A) \simeq X(A_{\operatorname{Frob}_R})$. Also we have an R-algebra map $\operatorname{Frob}_A \colon A \to A_{\operatorname{Frob}_R}$. Thus $X(\operatorname{Frob}_A)$ induces a map $F_{X/S} \colon X \to X^{(1)}$, which we call *relative Frobenius of X*. The composition with the projection gives $F_X \colon X \to X^{(1)} \to X$ called the *Frobenius of X*.

Remark 6.58. Let S be a classical scheme. As the (nerve of the) category of schemes lies fully in the ∞-category of derived schemes, we see by construction that F_S agrees with the classical Frobenius morphism. This can be tested on points given by R-algebras, where it holds per definition.

The definition above also agrees with the definition of the Frobenius on animated R-algebras, as we have $\mathrm{AR}_R \simeq \operatorname{Fun}_\pi(\operatorname{Poly}_R^{\mathrm{op}}, \mathbb{S})$ (recall that the Frobenius is induced by the Frobenius on polynomial R-algebras).

Moreover, this argument shows that the Frobenius morphism on derived schemes defined above is equivalent to the morphism induced by right Kan extension of the Frobenius on animated rings.

Definition 6.59. Let S be a derived scheme over R. A *derived F-zip over S* is a tuple $(C^\bullet, D_\bullet, \phi, \varphi_\bullet)$ consisting of

- a descending locally bounded perfect filtration $C^\bullet$ of quasi-coherent modules over S,
- an ascending locally bounded perfect filtration $D_\bullet$ of quasi-coherent modules over S,

[38]Using the Frobenius on R, we can restrict any R-algebra A along the Froebnius $\operatorname{Frob}_R \colon R \to R$, i.e. $A_{\operatorname{Frob}_R}$ is the animated R-algebra obtained by composing the natural morphism $R \to A$ with Frob_R.

- an equivalence $\phi\colon \operatorname{colim}_{\mathbb{Z}^{\mathrm{op}}} C^\bullet \simeq \operatorname{colim}_{\mathbb{Z}} D_\bullet$, and
- a family of equivalences $\varphi_k\colon F_S^* \operatorname{gr}^k C \xrightarrow{\sim} \operatorname{gr}^k D$.

The ∞-category of F-zips over S, i.e. the full subcategory of

$$\begin{aligned}(\mathrm{Fun}_{\mathrm{perf}}(\mathbb{Z}^{\mathrm{op}}, \mathrm{QCoh}(S)) \times_{\mathrm{colim},\mathrm{QCoh}(S),\mathrm{colim}} \mathrm{Fun}_{\mathrm{perf}}(\mathbb{Z}, \mathrm{QCoh}(S)))\\ \times_{\prod_{\mathbb{Z}} \mathrm{Fun}(\partial\Delta^1, \mathrm{QCoh}(S))} \prod_{\mathbb{Z}} \mathrm{Fun}(\Delta^1, \mathrm{QCoh}(S))\end{aligned}$$

consisting of F-zips, is denoted by $F\text{-}\mathrm{Zip}_{\infty,R}(S)$.

For a morphism $S' \to S$ of derived schemes over R we have an obvious base change functor $F\text{-}\mathrm{Zip}_{\infty,R}(S) \to F\text{-}\mathrm{Zip}_{\infty,R}(S')$ via the pullback.

Remark 6.60. By definition for any affine derived scheme $\mathrm{Spec}(A)$, the ∞-category $F\text{-}\mathrm{Zip}_{\infty,R}(\mathrm{Spec}(A))$ indeed recovers Definition 6.22.

Also, as in Remark 6.15, on affine schemes, up to equivalence we may assume that the equivalence between the colimits of the filtrations of a derived F-zip is given by identity.

Remark 6.61. Note that if we have a locally bounded perfect filtration $C^\bullet$ over some derived scheme S, we have that its colimit in $\mathrm{QCoh}(S)$ is actually perfect. This can be checked Zariski locally, where the filtrations actually become bounded (note that the colimit is filtered) and thus the colimit can be taken over a finite subcategory of $\mathbb{Z}$ and thus is perfect.

The next lemma shows that the ∞-category of derived F-zips satisfies fpqc descent and that we can extend the definition of derived F-zips to arbitrary derived (pre-)stacks.

Lemma 6.62. *Let $\mathbb{R}\, F\text{-}\mathrm{Zip}_{\infty,R}$ be the the right Kan extension of $F\text{-}\mathrm{Zip}_{\infty,R}\colon \mathrm{AR}_R \to \mathbb{S}$ along the Yoneda embedding $\mathrm{AR}_R \hookrightarrow \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})^{\mathrm{op}}$. Then for any derived scheme S over R, we have that the natural morphism induced by base change*

$$F\text{-}\mathrm{Zip}_{\infty,R}(S) \to \mathbb{R}\, F\text{-}\mathrm{Zip}_{\infty,R}(S)$$

is an equivalence.

Proof. Affine locally on S the assertion is certainly true. So it is enough to show that for an affine open cover $(\mathrm{Spec}(A_i) \hookrightarrow S)_{i\in I}$, we have

$$F\text{-}\mathrm{Zip}_{\infty,R}(S) \simeq \lim_{\Delta} F\text{-}\mathrm{Zip}_{\infty,R}(\check{C}(\coprod \mathrm{Spec}(A_i)/S)_\bullet).$$

This is again completely analogous to the proof in the affine case (see Proposition 6.21), where we embedded derived F-zips into a larger category that satisfied descent. Here we have to use that

$$\begin{aligned}S \mapsto \mathrm{Fun}_{\mathrm{perf}}(\mathbb{Z}^{\mathrm{op}}, \mathrm{QCoh}(-)) \times_{\mathrm{colim},\mathrm{QCoh}(-),\mathrm{colim}} \mathrm{Fun}_{\mathrm{perf}}(\mathbb{Z}, \mathrm{QCoh}(S))\\ \times_{\prod_{\mathbb{Z}} \mathrm{Fun}(\partial\Delta^1, \mathrm{QCoh}(-))} \prod_{\mathbb{Z}} \mathrm{Fun}(\Delta^1, \mathrm{QCoh}(S))\end{aligned}$$

is given by the limit of right Kan extensions (see discussion in Section 6.5.2) of sheaves and Remark 4.50. $\square$

The above lemma allows us to globalize the derived stack of derived F-zips and gives us a direct description of the points of it.

Definition and Remark 6.63. We set

$$F\text{-Zip}_R\colon \mathcal{P}(\mathrm{AR}_R)^{\mathrm{op}} \to \mathbb{S}$$

as the right Kan extension of $F\text{-Zip}\colon \mathrm{AR}_R \to \mathbb{S}$ along the inclusion $\mathrm{Ani} \hookrightarrow \mathcal{P}(\mathrm{AR}_R)^{\mathrm{op}}$.

By Remark 4.50, we see that F-Zip is a hypercomplete fpqc-sheaf.

Further, we have for any derived R-scheme S that $F\text{-Zip}_R(S) \simeq F\text{-Zip}_{\infty,R}(S)^{\simeq}$ by Lemma 6.62 as $(-)^{\simeq}$ commutes with limits.

Example 6.64. Let us globalize Example 6.16. Let $f\colon X \to S$ be a proper smooth morphism of schemes. Again, the associated Hodge and conjugate filtration $\mathbf{HDG}$ and $\mathbf{conj}$ define a descending resp. ascending perfect bounded filtration of quasi-coherent modules over S. We also have equivalences $\varphi_n\colon F_S^* \operatorname{gr}^n \mathbf{HDG} \xrightarrow{\sim} \operatorname{gr}^n \mathbf{conj}$ between the graded pieces (up to Frobenius twist), induced by the Cartier isomorphism. Therefore, we get a derived F-zip associated to the proper smooth map f of schemes

$$\underline{Rf_*\Omega^\bullet_{X/S}} := (\mathbf{HDG}^\bullet, \mathbf{conj}_\bullet, \varphi_\bullet).$$

Example 6.65. The above construction works analogously for log smooth scheme morphisms (i.e. schemes with a fine log structure as explained in [Kat89]).

If $f\colon X \to S$ is a proper log smooth morphism of Cartier type (note that f is per definition integral and thus flat, see [Kat89, Cor. 4.5]), then $\Omega^1_{X/S}$ (the sheaf of log differentials) is locally free of finite rank (see [Kat89, Prop. 3.10]) and because f is proper and flat the associated Hodge filtrations $\mathbf{HDG}_{\log}$ are perfect (use the distinguished triangle associated to the stupid truncation and conclude via induction and the fact that f is proper, locally of finite presentation and flat, see [Sta19, 0B91] - this is analogous to the proof of [Sta19, 0FM0]). We also have a Cartier isomorphism in this setting (see [Kat89, Thm. 4.12]) (this implies, using the distinguished triangles for the conjugate filtration $\mathbf{conj}_{\log}$ and induction, that the conjugate filtration is perfect) and so we have equivalences $\varphi_n\colon F_S^* \operatorname{gr}^n \mathbf{HDG}_{\log} \xrightarrow{\sim} \operatorname{gr}^n \mathbf{conj}^{\log}$ thus analogous to the above, we can attach the structure of a derived F-zip to f via, again,

$$(\mathbf{HDG}^\bullet_{\log}, \mathbf{conj}^{\log}_\bullet, \varphi_\bullet)$$

Let us consider the notion of strong F-zips. The condition that the filtration is given by monomorphisms seems very natural, but as Theorem 6.71 will show, in this case we can not expect a generalization from classical theory. Especially, the following lemma shows that perfect complexes with finite projective cohomologies are particularly easy to handle.

Lemma 6.66. *Let A be a ring and let P be a perfect complex over A such that for all $i \in \mathbb{Z}$ the A-module $\pi_i(P)$ is finite projective. Then there exists a quasi-isomorphism $P \xrightarrow{\sim} \bigoplus_{n\in\mathbb{Z}} \pi_n(P)[n]$.*

Proof. Since P is perfect, we may assume that there exists $a \leq b \in \mathbb{Z}$ such that P has Tor-amplitude in $[a, b]$. we further assume that P is represented by the complex of finite projective A-modules

$$\cdots \to 0 \to P_b \xrightarrow{\partial_b} P_{b-1} \xrightarrow{\partial_{b-1}} \ldots \xrightarrow{\partial_{a+1}} P_a \to 0 \to \cdots.$$

Let us define a new complex $P^{\leq a}$ given by

$$\cdots \to 0 \to P_b \xrightarrow{\partial_b} P_{b-1} \xrightarrow{\partial_{b-1}} \ldots \xrightarrow{\partial_{a+1}} \mathrm{im}(\partial_{a+1}) \to 0 \to \cdots.$$

We get a short exact sequences of complexes

$$0 \to P^{\leq a} \to P \to \pi_a(P)[a] \to 0.$$

Since $\pi_a(P)$ is projective, this induces a section $\pi_a(P) \to P_a$ and we can extend this to a morphism $\pi_a(P)[a] \to P$ which induces a section of $P \to \pi_a(P)[a]$. Also, this induces a retraction of $P^{\leq a} \to P$ and s in particular, $P \simeq P^{\leq a} \oplus \pi_a(P)[a]$. Now we claim that $P^{\leq a}$ is perfect and has Tor-amplitude in $[a+1, b]$ concluding the proof by induction over the Tor-amplitude of P.

Indeed, first note that $P^{\leq a}$ is equivalent to the complex

$$\cdots \to 0 \to P_b \xrightarrow{\partial_b} P_{b-1} \xrightarrow{\partial_{b-1}} \ldots \xrightarrow{\partial_a} P_{a+1} \to 0 \to \cdots$$

which is by construction a complex of finite projective modules concentrated in degrees $[a+1, b]$, i.e a perfect complex of Tor-amplitude in $[a+1, b]$. $\square$

We can use this lemma to see that a morphism of perfect complexes with finite projective is a monomorphism if and only if it is so on the cohomologies. This is clear since if the induced map on the cohomologies is injective, then the long exact homotopy sequences corresponding to a fiber sequence consists of short exact sequences. The projectiveness gives us retractions on the level of cohomology groups and thus a retraction on the whole complex.

Lemma 6.67. *Let A be a ring and let P, Q be a perfect complexes over A. Further assume we have a morphism $f\colon P \to Q$ such that for all $i \in \mathbb{Z}$ the A-modules $\pi_i P$, $\pi_i Q$ and $\pi_i \operatorname{cofib}(f)$ are finite projective. Then f is a monomorphism if and only if $\pi_i f$ is injective for all $i \in \mathbb{Z}$ (or equivalently[39] the morphism $\pi_i Q \to \pi_i \operatorname{cofib}(f)$ is surjective for all $i \in \mathbb{Z}$).*

Proof. The *only if* direction is clear[40].

For the *if* direction let $\pi_i f$ be injective for all $i \in \mathbb{Z}$. We may assume by Lemma 6.66 that $P \xrightarrow{\sim} \bigoplus_{n \in \mathbb{Z}} \pi_n P[n]$ and $Q \xrightarrow{\sim} \bigoplus_{n \in \mathbb{Z}} \pi_n Q[n]$. It is enough to find retractions g_i of $\pi_i f$ since then this induces a retraction of f implying that f is a monomorphism.

[39] This follows from long exact homotopy sequence induced by the fiber sequence $P \to Q \to \operatorname{cofib}(f)$.

[40] Let $P \xrightarrow{f} Q \xrightarrow{g} \operatorname{cofib}(f)$ be a cofiber sequence of A-modules and let $h\colon \operatorname{cofib}(f)[-1] \to P$ be the naturally induced morphism. By construction, we have $f \circ h \simeq 0$ and as f is a monomorphism this implies $h \simeq 0$ and so indeed $\ker(\pi_i f) = \mathrm{im}(\pi_{i+1} h) = 0$.

Since $\pi_i f$ is injective for all $i \in \mathbb{Z}$, we get short exact sequences

$$0 \to \pi_i P \xrightarrow{\pi_i f} \pi_i Q \to \pi_i \operatorname{cofib}(f) \to 0. \tag{6.5.1}$$

As $\pi_i \operatorname{cofib}(f)$ is projective, we see that the short exact sequence (6.5.1) is split, giving us the retractions of $\pi_i f$. $\square$

We want to show that the Hodge-spectral sequence associated to a proper smooth morphism degenerates if the Hodge-filtration is strong. For this we want to reduce to the Artinian case. Before that, we will reduce to the noetherian case. This is standard for proper smooth morphisms and the finite locally freeness of the E_1-page of the spectral sequence can also be reduced to the noetherian case, as in the proof of [Kat72, (2.3.2)]. We will worry about the strongness of the filtration. The idea is to use Lemma 6.67, so that we only need to worry about the condition that a surjective morphism of finite projective modules can be descended along projective limits of schemes. But as we only work with perfect bounded filtrations this will be a consequence of induction.

Proposition 6.68. *Let $f\colon X \to S$ be a proper smooth morphism of schemes, with $S = \operatorname{Spec}(A)$ affine. Assume that all $R^i f_* \Omega^j_{X/S}$ are finite locally free and that the Hodge-filtration* **HDG** *is strong. Then there exists an affine noetherian subring $A' \subseteq A$, a proper smooth morphism of schemes $f'\colon X' \to S' \coloneqq \operatorname{Spec}(A')$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & S \arrow[d, "g"] \\ X' \arrow[r, "f'"] & S' \end{tikzcd}$$

is cartesian, where g corresponds to the inclusion $A' \subseteq A$. Further, $R^i f'_* \Omega^j_{X/S}$ is finite locally free for all $i, j \in \mathbb{Z}$ and the Hodge-filtration associated to f' is strong.

Proof. The existence of a subring $A_0 \subseteq A$ and a proper smooth A_0-scheme X_0 such that the base change of $f_0\colon X_0 \to \operatorname{Spec}(A_0)$ under $\operatorname{Spec}(A) \to \operatorname{Spec}(A_0)$ is equal to f is standard[41]. As in the proof of [Kat72, (2.3.2)], we can find a noetherian subring $A_0 \subseteq A' \subseteq A$ such that $R^i f'_* \Omega^k_{X'/S'}$ commutes with base change and is finite locally free for all $i, j \in \mathbb{Z}$, where $f'\colon X' \to S' \coloneqq \operatorname{Spec}(A')$ is the base change of f_0. Let us also note that for all $i, k \in \mathbb{Z}$ the A-module $R^i f_* \sigma_{\geq k} \Omega^\bullet_{X/S}$ is finite locally free[42].

What is left to show is that after possibly enlarging A' the Hodge-filtration associated to f' is strong. The idea is to show that $R^i f'_* \sigma_{\geq k} \Omega^\bullet_{X'/S'}$ is finite projective and the

[41] Use [GW10, Thm. 10.69] to find an affine noetherian $\widetilde{S} = \operatorname{Spec}(\widetilde{A})$ and morphisms $S \to \widetilde{S}$ and $\widetilde{X} \to \widetilde{S}$ such that the induced base change morphism is isomorphic to f. Then write S as a projective limit of affine $\widetilde{S}$-schemes of finite type by adjoining variables to $\widetilde{A}$ and conclude with [Gro66, (8.10.5)] and [Gro67, (17.7.8)].

[42] Indeed, as the Hodge-filtration associated to f is strong, we have for all $i, k \in \mathbb{Z}$ short exact sequences of the form $0 \to R^i f_* \sigma_{\geq k+1} \Omega^\bullet_{X/S} \to R^i f_* \sigma_{\geq k} \Omega^\bullet_{X/S} \to R^{i-k} f_* \Omega^k_{X/S} \to 0$. Since the Hodge-filtration is bounded and $R^{i-k} f_* \Omega^k_{X/S}$ is finite locally free for all $k, i \in \mathbb{Z}$, we see inductively that $R^i f_* \sigma_{\geq k} \Omega^\bullet_{X/S}$ is finite locally free.

induced maps $R^if'_*\sigma_{\geq k+1}\Omega^\bullet_{X'/S'} \to R^if'_*\sigma_{\geq k}\Omega^\bullet_{X'/S'}$ are injective for all $i,k \in \mathbb{Z}$, so we can use Lemma 6.67. We will show both simultaneously by descending induction over k.

Let us first look for any $k \in \mathbb{Z}$ at the fiber sequence

$$Rf'_*\sigma_{\geq k+1}\Omega^\bullet_{X'/S'} \to Rf'_*\sigma_{\geq k}\Omega^\bullet_{X'/S'} \xrightarrow{q_k} Rf'_*\Omega^k_{X'/S'}[-k]. \tag{6.5.2}$$

Since $\Omega^\bullet_{X'/S'}$ is concentrated in finite degrees, we will use induction to see that for all $i,k \in \mathbb{Z}$ the formation of $R^if'_*\sigma_{\geq k}\Omega^\bullet_{X'/S'}$ commutes with arbitrary base change in the sense that for any A'-algebra B, the natural base change morphism

$$R^if'_*\sigma_{\geq k}\Omega^\bullet_{X'/S'} \otimes_{A'} B \to H^i(Rf'_*\sigma_{\geq k}\Omega_{X'/S'} \otimes^L_{A'} B)$$

is an equivalence. This will imply that after possibly enlarging A', that $R^if'_*\sigma_{\geq k}\Omega^\bullet_{X'/S'}$ are finite locally free (again with [Gro66, (11.2.6.1)]). It is important, that the Hodge-filtration is bounded and since $Rf'_*\sigma_{\geq k}\Omega^\bullet_{X'/S'}$ is perfect there are only finitely many non-trivial cohomology groups, so we can indeed use [Gro66, (11.2.6.1)].

Indeed, let fix some $k \in \mathbb{Z}$, then the fiber sequence (6.5.2) induces the following long exact sequence for $i \in \mathbb{Z}$

$$\begin{aligned} R^{i-1-k}f'_*\Omega^k_{X'/S'} \to R^if'_*\sigma_{\geq k+1}\Omega^\bullet_{X'/S'} \to R^if'_*\sigma_{\geq k}\Omega^\bullet_{X'/S'} &\xrightarrow{H^i(q_k)} R^{i-k}f'_*\Omega^k_{X'/S'} \\ &\to R^{i+1}f'_*\sigma_{\geq k+1}\Omega^\bullet_{X'/S'}. \end{aligned}$$

Let us assume that the formation of $R^if'_*\sigma_{\geq k+1}\Omega^\bullet_{X'/S'}$ commutes with arbitrary base change for all $i \in \mathbb{Z}$. After possibly enlarging A', we may assume that $R^if'_*\sigma_{\geq k+1}\Omega^\bullet_{X'/S'}$ is finite locally free for all $i \in \mathbb{Z}$ (again this follows from [Gro66, (11.2.6.1)]). We claim that $H^i(q_k)$ is surjective for all $i,k \in \mathbb{Z}$. This is equivalent to the fact that the morphism

$$R^{i-k}f'_*\Omega^k_{X'/S'} \to R^{i+1}f'_*\sigma_{\geq k+1}\Omega^\bullet_{X'/S'}$$

is zero for all $i \in \mathbb{Z}$. Since both domain and target are finite locally free and their formation commutes with base change, we see that after base change to $S = \mathrm{Spec}(A)$ this holds. But the morphism $A' \to A$ is given by inclusion of rings. So if after base change to A the morphism is zero it had to be zero before. Therefore, we have short exact sequences

$$0 \to R^if'_*\sigma_{\geq k+1}\Omega^\bullet_{X'/S'} \to R^if'_*\sigma_{\geq k}\Omega^\bullet_{X'/S'} \xrightarrow{H^i(q_k)} R^{i-k}f'_*\Omega^k_{X'/S'} \to 0 \tag{6.5.3}$$

for all $i \in \mathbb{Z}$. This first of all shows that $R^if'_*\sigma_{\geq k}\Omega^\bullet_{X'/S'}$ is finite projective for all $i \in \mathbb{Z}$. Not only that, the short exact sequence (6.5.3) stays exact, after base change to any A'-algebra B (see [Sta19, 00HL]). So the natural base change maps induces the following commutative diagram with exact columns

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
R^i f'_* \sigma_{\geq k+1} \Omega^\bullet_{X'/S'} \otimes_{A'} B \arrow[r, "\sim"] \arrow[d] & H^i(Rf'_* \sigma_{\geq k+1} \Omega^\bullet_{X'/S'} \otimes^L_{A'} B) \arrow[d] \\
R^i f'_* \sigma_{\geq k} \Omega^\bullet_{X'/S'} \otimes_{A'} B \arrow[r] \arrow[d] & H^i(Rf'_* \sigma_{\geq k} \Omega^\bullet_{X'/S'} \otimes^L_{A'} B) \arrow[d] \\
R^{i-k} f'_* \Omega^k_{X'/S'} \otimes_{A'} B \arrow[r, "\sim"] \arrow[d] & H^{i-k}(Rf'_* \Omega^k_{X'/S'} \otimes^L_{A'} B) \arrow[d] \\
0 & 0.
\end{tikzcd}$$

Thus, the five lemma shows that the formation of $R^i f'_* \sigma_{\geq k} \Omega^\bullet_{X'/S'}$ commutes with arbitrary base change concluding the induction.

Now, as the sequence (6.5.3) is short exact, we can use Lemma 6.67 to see that indeed the Hodge-filtration associated to f' is strong.

□

Remark 6.69. Let us remark that in the setting of Proposition 6.68, after possibly enlarging A', the $R^i f'_* \sigma_{\geq k} \Omega^\bullet_{X'/S'}$ are finite locally free for all $i, k \in \mathbb{Z}$ and their formation commute with arbitrary base change. This follows from the proof of Proposition 6.68.

On the other side, we can show that if the Hodge-de Rham spectral sequence degenerates, then the Hodge-filtration is strong. Again Lemma 6.67 will be crucial. Also, we do not need the particular form of the Hodge filtration and thus we will show generally that a bounded perfect filtration with degenerate associated spectral sequence is automatically strong. Later on in Section 7, we will use this result to show that the derived F-zips associated to a proper smooth morphism with degenerative Hodge-de Rham spectral sequence and finite projective E_1-terms is completely determined by the underlying classical F-zips. So, for example in the abelian scheme case, the theory of derived F-zips gives us no new information and recovers the classical theory by passing to the cohomologies of the filtration (in a suitable sense as explained in Section 7).

Proposition 6.70. *Let A be a ring and $C^\bullet$ be a descending bounded perfect filtration of A-modules. Assume that $\pi_i \operatorname{gr}^k C$ are finite projective for all $i, k \in \mathbb{Z}$ and that the spectral sequence associated to $C^\bullet$*

$$E_1^{p,q} = \pi_{p+q} \operatorname{gr}^p C \Rightarrow \pi_{p+q} \operatorname*{colim}_{\mathbb{Z}^{\mathrm{op}}} C^\bullet$$

degenerates. Then the filtration $C^\bullet$ is strong and the statement stays true if we replace $C^\bullet$ by an ascending filtration.

Proof. For convenience, let us set $M := \operatorname{colim}_{\mathbb{Z}^{\mathrm{op}}} C^\bullet$.

It is enough to show[43]

- if for any $k \in \mathbb{Z}$ the natural map $C^k \to M$ is a monomorphism, then the map $C^{k+1} \to C^k$ is a monomorphism.

Also, we will see that changing a descending filtration to an ascending one only changes the indices in the following proof and thus works similarly.

So let us fix some $k \in \mathbb{Z}$ and assume that $C^k \to M$ is a monomorphism. The degeneracy of the spectral sequence implies that $E_1^{k,q} = \pi_{k+q} \operatorname{gr}^k C$ is naturally isomorphic to $E_\infty^{k,q}$ for any $q \in \mathbb{Z}$. By construction of the spectral sequence (see [Lur17, §1.2.2]), we see that $E_\infty^{k,q} \cong \operatorname{im}(h_q)/\operatorname{im}(g_q)$, where $h_q \colon \pi_{k+q}C^k \to \pi_{k+q}M$ and $g_q \colon \pi_{k+q}C^{k+1} \to \pi_{k+q}M$. As $C^k \to M$ is a monomorphism, we see that h_q is injective and therefore $\operatorname{im}(h_q) \cong \pi_{k+q}C^k$. As the filtration on M induced by the spectral sequence is bounded and the graded pieces are finite projective by degeneracy, we see that $\pi_{k+q}C^k$ is finite projective (since it is isomorphic to a filtered piece by $\operatorname{im}(h_q) \cong \pi_{k+q}C^k$). We claim that the morphism $f_q \colon \pi_{k+q}C^k \to \pi_{k+q} \operatorname{gr}^k C$ induced by the fiber sequence

$$C^{k+1} \to C^k \to \operatorname{gr}^k C,$$

is surjective for all $q \in \mathbb{Z}$.

Indeed, recall from [Lur17, §1.2.2] that

$$E_r^{p,q} := \operatorname{im}(\pi_{p+q} \operatorname{cofib}(C^{p+r} \to C^p) \to \pi_{p+q} \operatorname{cofib}(C^{p+1} \to C^{p-r+1})).$$

As the spectral sequence is degenerate, the proof of [Lur17, Prop. 1.2.2.7 (2)] shows that for any $r \geq 2$, we get a commutative diagram

$$\begin{tikzcd}
\pi_{k+q} \operatorname{cofib}(C^{k+r} \to C^k) \arrow[r] \arrow[dr, "\phi_r"'] & \pi_{k+q} \operatorname{cofib}(C^{k+1} \to C^{k-r+1}) \\
 & E_{r-1}^{k,q}, \arrow[u, "\vartheta_r"']
\end{tikzcd}$$

where ϕ_r is surjective and ϑ_r is injective. The morphism ϕ_r is induced by the natural morphism $\operatorname{cofib}(C^{k+r} \to C^k) \to \operatorname{cofib}(C^{k+1} \to C^{k-r+1})$. Since $C^\bullet$ is bounded, we see that for r large enough, we have $\operatorname{cofib}(C^{k+r} \to C^k) \simeq C^k$. Thus, we get a morphism $\alpha_q \colon \pi_{k+q}C^k \to E_1^{k,q} = \pi_{k+p} \operatorname{gr}^k C$ that is surjective using the ϕ_r and the degeneracy. But by construction α_q is induced by the natural map $C^k \to \operatorname{gr}^k C$ and hence α_q agrees with f_q showing the desired surjectivity.

By surjectivity of the f_q, we have for all $q \in \mathbb{Z}$ short exact sequences of the form

$$0 \to \pi_{k+q}C^{k+1} \to \pi_{k+q}C^k \to \pi_{k+q} \operatorname{gr}^k C \to 0,$$

showing that $\pi_{k+q}C^{k+1}$ is finite projective for all $q \in \mathbb{Z}$. Therefore, we can conclude the proof with Lemma 6.67. $\square$

[43] As the $C^\bullet$ is bounded, we have that there is exists a n large enough such that $C^n \to M$ is an equivalence and thus also a monomorphism. Hence, induction indeed concludes that $C^\bullet$ is strong.

Combining all the arguments before, we get the connection between the degeneracy of the Hodge-de Rham spectral sequence and strongness of the Hodge-filtration.

Theorem 6.71. *Let $f\colon X \to S$ be a smooth proper morphism of schemes. Let us consider the Hodge-de Rham spectral sequence*

$$E_1^{p,q} = R^q f_*\Omega^p_{X/S} \Rightarrow R^{p+q} f_*\Omega^\bullet_{X/S}.$$

Assume that all $R^i f_*\Omega^j_{X/S}$ are finite locally free. The derived F-zip $\underline{Rf_*\Omega^\bullet_{X/S}}$ of Example 6.64 is strong (in the sense that **HDG** *and* **conj** *are strong) if and only if the Hodge-de Rham spectral sequence degenerates.*

Proof. The *if* part follows from Proposition 6.70.

For the *only if* part let us first assume that $S = \operatorname{Spec}(A)$ is the spectrum of a local Artin ring, then we can check this via comparing lengths of the limit term and the E_1 terms of the spectral sequence. But in this case this is clear, since we get a short exact sequence

$$0 \to H^i(X, \sigma_{\geq n+1}\Omega^\bullet_{X/S}) \to H^i(X, \sigma_{\geq n}\Omega^\bullet_{X/S}) \to H^{i-n}(X, \Omega^n_{X/S}) \to 0$$

for all $n \geq 0$ and $i \in \mathbb{Z}$ (by strongness of our filtration). This implies that

$$\operatorname{length}_A H^i(X, \sigma_{\geq n}\Omega^\bullet_{X/S}) = \operatorname{length}_A H^i(X, \sigma_{\geq n+1}\Omega^\bullet_{X/S}) + \operatorname{length}_A H^{i-n}(X, \Omega^n_{X/S}),$$

for all $n \geq 0$ and $i \in \mathbb{Z}$. As $H^i_{\mathrm{dR}}(X/S) = H^i(X, \sigma_{\geq 0}\Omega^\bullet_{X/S})$, this implies inductively that

$$\operatorname{length}_A H^i_{\mathrm{dR}}(X/S) = \sum_{n\geq 0} \operatorname{length}_A H^{i-n}(X, \Omega^n_{X/S}) = \sum_{p+q=i} \operatorname{length}_A H^q(X, \Omega^p_{X/S})$$

Thus, we get for all $i \in \mathbb{Z}$ the equation

$$\sum_{p+q=i} \operatorname{length}_A E_\infty^{p,q} = \operatorname{length}_A H^i_{\mathrm{dR}}(X/S) = \sum_{p+q=i} \operatorname{length}_A H^q(X, \Omega^p_{X/S}),$$

so in particular in this case the Hodge-de Rham spectral sequence degenerates.

Now let us show how to reduce to the case, where S is the spectrum of a local Artin ring (we do this analogous to the proof of [Kat72, (2.3.2)]). First of all the question is local, since cohomology commutes with arbitrary flat base change, so we may assume that $S = \operatorname{Spec}(A)$ is affine. Using Proposition 6.68 we may assume that S is noetherian. Localizing further, we can even assume that S is given by the spectrum of a local noetherian ring (as cohomologies commute with flat base change and filtered colimits are exact). By faithfully flatness of completion (see [Sta19, 00MC]) we can assume that S is the spectrum of a complete noetherian local ring $(A, \mathfrak{m})$ and therefore if the Hodge spectral sequence degenerates for all $S_n = \operatorname{Spec}(A/\mathfrak{m}^n)$, then it degenerates on the limit of all $A/\mathfrak{m}^n$, namely A, since finite modules over complete rings are complete (see [Sta19, 00MA] and note that the $R^i f_*\Omega^j_{X/S}$ are finite projective and therefore their formation commute with arbitrary base change). Hence, we may assume that A is a local Artin ring, which we already discussed in the beginning.

$\square$

We conclude this section by defining the substacks corresponding to a function $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$. Again, we could do this by right Kan extension but as before there is also an ad hoc definition that agrees with the one given by right Kan extension.

The following definitions are globalizations of the definitions given in section 6.3.

Definition 6.72. Let S be a derived scheme and let $\mathcal{F} \in \mathrm{Perf}(S)$. We define the function

$$\begin{aligned} \beta_{\mathcal{F}}\colon S_{\mathrm{cl}} &\to \mathbb{N}_0^{\mathbb{Z}} \\ s &\mapsto (\dim_{\kappa(s)} \pi_i(\mathrm{cl}^*\mathcal{F} \otimes^L_{\mathcal{O}_{S_{\mathrm{cl}}}} \kappa(s)))_{i\in\mathbb{Z}}, \end{aligned}$$

where $\mathrm{cl}\colon S_{\mathrm{cl}} \to S$ is the natural morphism.

This function is locally upper semi-continuous in the sense that for every $s \in S_{\mathrm{cl}}$ there is a neighbourhood U_s such that for any family $(k_i)_i \in \mathbb{N}_0^{\mathbb{Z}}$ the set $\beta^{-1}_{\mathcal{F}|U_s}(([0,k_i])_i)$ is open (see [Sta19, 0BDI]).

Remark 6.73. Note that in the above definition, we implicitly assume that $\mathrm{cl}^*\mathcal{F}$ is a perfect complex of quasi-coherent $\mathcal{O}_{S_{\mathrm{cl}}}$-module. This makes sense as $\mathrm{cl}^*\mathcal{F}$ is in $\mathcal{D}_{\mathrm{qc}}(S_{\mathrm{cl}})$ by Proposition 4.57 and by definition perfect.

Definition and Remark 6.74. Let S be a derived scheme. Let $\underline{F} \coloneqq (C^\bullet, D_\bullet, \phi, \varphi_\bullet)$ be a derived F-zip over S. Consider the function

$$\beta_{\underline{F}}\colon s \mapsto (k \mapsto \beta_{\mathrm{gr}^k C}(s))$$

from S_{cl} to functions $\mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$.

1. The function $\beta_{\underline{F}}$ is called *type of the derived F-zip $\underline{F}$*.
2. Let $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be a function with finite support. We say that $\underline{F}$ *has type* $\leq \tau$ if for all $s \in S_{\mathrm{cl}}$, we have $\beta_{\underline{F}}(s) \leq \tau$ (again the relation is given pointwise as functions $\mathbb{Z} \times \mathbb{Z} \to \mathbb{N}_0$).

Further, for any $s \in S_{\mathrm{cl}}$ there exists a quasi-compact open (resp. locally closed) neighbourhood U_s of s and a function $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ with finite support, such that $\beta_{\underline{F}|U_s} \leq \tau$ in the sense above (resp. $\beta_{\underline{F}|U_s}$ is constant an equal to τ) (this follows from Lemma 5.1).

Definition and Remark 6.75. Let S be a derived scheme. Let $\underline{F} \coloneqq (C^\bullet, D_\bullet, \phi, \varphi_\bullet)$ be a derived F-zip over S. Let us look at the function

$$\begin{aligned} \chi_k(\underline{F})\colon S_{\mathrm{cl}} &\to \mathbb{Z} \\ s &\mapsto \chi(\mathrm{cl}^* \mathrm{gr}^k C \otimes_{\mathcal{O}_{S_{\mathrm{cl}}}} \kappa(s)), \end{aligned}$$

where $\mathrm{cl}\colon S_{\mathrm{cl}} \to S$ is the natural morphism This is a locally constant function (see [Sta19, 0B9T]). Since the filtrations on derived F-zips are locally bounded, we also know that the function $\chi_{\underline{F}}\colon s \mapsto (k \mapsto \chi_k(\underline{F})(s))$ is also locally constant as a map from S_{cl} to functions $\mathbb{Z} \to \mathbb{Z}$ with finite support. We call $\chi_{\underline{F}}$ the *Euler-characteristic* of $\underline{F}$.

If $\tau\colon \mathbb{Z} \to \mathbb{Z}$ is a function with finite support, we say $\underline{F}$ *has Euler-characteristic* τ if for $\chi_{\underline{F}}$ is constant with value τ.

Definition 6.76. Let S be a derived scheme.

1. Let $M \in \operatorname{Perf}(S)$ be a perfect module over S. Fix a map $r\colon \mathbb{Z} \to \mathbb{N}_0$. We call M *homotopy finite locally free of rank* r, if for each scheme $f\colon X \to S_{\mathrm{cl}}$ the $\mathcal{O}_X$-module $\pi_i(f^*\mathrm{cl}^* M)$ is finite locally free of rank r_i, where we say that it is finite locally free of rank 0 if it is isomorphic to 0.

2. Let $\underline{F} \coloneqq (C^\bullet, D_\bullet, \varphi_\bullet)$ be a derived F-zip over S and $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be a function with finite support. We say that $\underline{F}$ is *homotopy finite locally free of type* τ if for all $i \in \mathbb{Z}$, we have that $\operatorname{gr}^i C$ is homotopy finite locally free of rank $\tau(i)$.

Definition 6.77. Let $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be function with finite support and let S be a derived R-scheme. We set $F\text{-}\mathrm{Zip}^{\leq\tau}_{\infty,R}(S)$ (resp. $F\text{-}\mathrm{Zip}^{\tau}_{\infty,R}(S)$, $F\text{-}\mathrm{Zip}^{\chi=\tau}_{\infty,R}(S)$) as the full subcategory of derived F-zips over S of type $\leq \tau$ (resp. homotopy finite locally free of type τ, of Euler-characteristic τ).

Lemma 6.78. *Let $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be function with finite support. Let $\mathbb{R}\, F\text{-}\mathrm{Zip}^{\leq\tau}_{\infty,R}$ (resp. $\mathbb{R}\, F\text{-}\mathrm{Zip}^{\tau}_{\infty,R}$, $\mathbb{R}\, F\text{-}\mathrm{Zip}^{\chi=\tau}_{\infty,R}$) be the the right Kan extension of $F\text{-}\mathrm{Zip}^{\leq\tau}_{\infty,R}$ (resp. $F\text{-}\mathrm{Zip}^{\tau}_{\infty,R}$, $F\text{-}\mathrm{Zip}^{\chi=\tau}_{\infty,R}$) along the Yoneda embedding $\mathrm{AR}_R \hookrightarrow \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})^{\mathrm{op}}$. Then for any derived scheme S over R, we have that the natural morphism induced by base change*

$$
\begin{aligned}
&F\text{-}\mathrm{Zip}^{\leq\tau}_{\infty,R}(S) \to \mathbb{R}\, F\text{-}\mathrm{Zip}^{\leq\tau}_{\infty,R}(S)\\
&(\text{resp. } F\text{-}\mathrm{Zip}^{\tau}_{\infty,R}(S) \to \mathbb{R}\, F\text{-}\mathrm{Zip}^{\tau}_{\infty,R}(S),\ F\text{-}\mathrm{Zip}^{\chi=\tau}_{\infty,R}(S) \to \mathbb{R}\, F\text{-}\mathrm{Zip}^{\chi=\tau}_{\infty,R}(S))
\end{aligned}
$$

is an equivalence.

Proof. This is completely analogous to the proof of Lemma 6.62. We only need to varify that the properties "has type $\leq \tau$", "is homotopy finite locally free of type τ" and "has Euler characteristic τ" can be checked on an affine open cover of S but this is clear. $\square$

Definition and Remark 6.79. Let $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be function with finite support and let S be a derived R-scheme. We set

$$F\text{-}\mathrm{Zip}^{\leq\tau}_{R}\colon \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})^{\mathrm{op}} \to \mathbb{S}$$

(resp. $F\text{-}\mathrm{Zip}^{\tau}_{R}$, $F\text{-}\mathrm{Zip}^{\chi=\tau}_{R}$) as the right Kan extension of $F\text{-}\mathrm{Zip}^{\leq\tau}_{R}$ (resp. $F\text{-}\mathrm{Zip}^{\tau}_{R}$, $F\text{-}\mathrm{Zip}^{\chi=\tau}_{R}$) along the Yoneda embedding $\mathrm{AR}_R \hookrightarrow \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})^{\mathrm{op}}$.

By Remark 4.50, these define fpqc sheaves and define subsheaves of $F\text{-}\mathrm{Zip}_R$.

7 Connection to classical theory

Again, in the following R will be an $\mathbb{F}_p$-algebra.

7.1 Derived F-zips with degenerating spectral sequences

In Lemma 6.51, we showed that classical F-zips can be included into the theory of derived F-zips. But what if the homotopy groups of associated to the graded pieces of a derived F-zip are finite locally free and the associated spectral sequences degenerate (see Definition 7.1). Then we would expect that we have a functor $\pi_n \colon \mathcal{X} \to \mathrm{cl}\, F\text{-Zip}$, where $\mathcal{X}$ is a suitable substack of $t_0\, F\text{-Zip}$, given by sending the underlying module to its n-th homotopy group and looking at the associated filtrations.

In fact, we will show that there is even more. For a smooth proper scheme morphism with degenerating Hodge-de Rham spectral sequence and finite locally free cohomologies, we get an F-zip. Important here is that the graded pieces and de Rham cohomology are finite locally free. For filtrations in our sense, we also get spectral sequences, i.e. for some R-algebra A and a bounded perfect filtrations $C^\bullet \in \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{D}(A))$ and $D_\bullet \in \mathrm{Fun}(\mathbb{Z}, \mathcal{D}(A))$, we have a spectral sequence

$$E_1^{p,q} = \pi_{p+q}(\mathrm{gr}^p\, C) \Rightarrow \pi_{p+q} \operatorname*{colim}_{\mathbb{Z}^{\mathrm{op}}} C^\bullet \text{ and } E_1^{p,q} = \pi_{p+q}(\mathrm{gr}^p\, D) \Rightarrow \pi_{p+q} \operatorname*{colim}_{\mathbb{Z}} D_\bullet$$

(see [Lur17, Prop. 1.2.2.14]). If we assume that the $\pi_{p+q}(\mathrm{gr}^p\, C)$ are finite projective and that the above spectral sequences are degenerate, we can associate for any $n \in \mathbb{Z}$ a classical F-zip to a derived F-zip $\underline{F} := (C^\bullet, D_\bullet, \phi, \varphi_\bullet)$ via

$$\underline{F} \mapsto \pi_n \underline{F} := (\pi_n(\operatorname*{colim}_{\mathbb{Z}^{\mathrm{op}}} C^\bullet), \widetilde{C}^\bullet, \widetilde{D}_\bullet, \pi_n \varphi_\bullet),$$

where $\widetilde{C}^\bullet$ resp. $\widetilde{D}_\bullet$ is the filtration associated to the spectral sequences induced by $C^\bullet$ resp. $D_\bullet$. Let us verify that $(\pi_n(\mathrm{colim}_{\mathbb{Z}^{\mathrm{op}}} C^\bullet), \widetilde{C}^\bullet, \widetilde{D}_\bullet, \pi_n \varphi_\bullet)$ is a classical F-zip.

For convenience let us set $M := \pi_n(\mathrm{colim}_{\mathbb{Z}^{\mathrm{op}}} C^\bullet)$. By definition, we also have $\pi_n \,\mathrm{colim}_{\mathbb{Z}}\, D \cong M$. First of all note that both $\widetilde{C}^\bullet$ and $\widetilde{D}_\bullet$ are finite and their graded pieces are by degeneracy of the spectral sequences equivalent to

$$\mathrm{gr}^i_{\widetilde{C}}\, M = \pi_n(\mathrm{gr}^i\, C), \quad \mathrm{gr}^i_{\widetilde{D}}\, M = \pi_n(\mathrm{gr}^i\, D).$$

By homotopy finite projectiveness, we have that all graded pieces of $\widetilde{C}^\bullet$ resp. $\widetilde{D}_\bullet$ are finite projective[44]. Since the filtrations are bounded, we see that the pieces of the filtrations are also finite projective and thus also M. The only thing left to see is that $\pi_n \varphi_i$ induce isomorphisms $(\mathrm{gr}^i_{\widetilde{C}}\, M)^{(1)} \xrightarrow{\sim} \mathrm{gr}^i_{\widetilde{D}}\, M$. But this again follows from the degeneracy of the spectral sequences (resp. the description above induced by the degeneracy) and the fact that homotopy finite projectiveness implies compatibility with base change (along Frobenius) by Lemma 6.37 (iii).

Further, if the derived F-zip $\underline{F}$ is homotopy finite projective of some type τ, then $\pi_n \underline{F}$ has type $\tau_n \colon k \mapsto \tau(k)_n$.

[44] Note that homotopy finite projectiveness of $\pi_n\, \mathrm{gr}^i\, C$ implies compatibility with base change (along Frobenius) by Lemma 6.37 (iii) and so $\pi_n\, \mathrm{gr}^i\, D \cong \pi_n(\mathrm{gr}^i\, C^{(1)}) \cong \pi_n(\mathrm{gr}^i\, C)^{(1)}$ is finite projective.

Moreover, using the arguments in the proof of Theorem 6.71, we see that a derived F-zip homotopy finite projective of some type with degenerating spectral sequences associated to the filtrations (as above) is automatically strong. Let us make everything, we said above more precise.

Definition 7.1. Let A be an R-algebra. A derived F-zip $(C^\bullet, D_\bullet, \phi, \varphi_\bullet)$ is called *degenerate* if the spectral sequences

$$E_1^{p,q} = \pi_{p+q}(\mathrm{gr}^p C) \Rightarrow \pi_{p+q} \operatorname*{colim}_{\mathbb{Z}^{\mathrm{op}}} C^\bullet \text{ and } E_1^{p,q} = \pi_{p+q}(\mathrm{gr}^p D) \Rightarrow \pi_{p+q} \operatorname*{colim}_{\mathbb{Z}} D_\bullet$$

associated to the filtrations (see [Lur17, Prop. 1.2.2.14]) degenerate.

Lemma 7.2. *Let A be an R-algebra and let $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be a function with finite support. If a derived F-zip $\underline{F}$ over A is homotopy finite projective of type τ and degenerate, then $\underline{F}$ is strong.*

Proof. This follows from Proposition 6.70. □

Proposition 7.3. *Let A be an R-algebra and let $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be a function with finite support. Further, let $\mathcal{X}^\tau_{\infty,R}(A) \subseteq F\text{-}\mathrm{Zip}^\tau_{\infty,R}(A)$ denote the full subcategory of those derived F-zips that are homotopy finite projective of type τ and degenerate. Then $A \mapsto \mathcal{X}^\tau_{\infty,R}(A)$ defines a hypercomplete sheaf for the fpqc-topology.*

Moreover, let $\mathcal{X}^\tau_R$ denote the associated derived stack. Then the inclusion

$$i\colon \mathcal{X}^\tau_R \hookrightarrow t_0\, F\text{-}\mathrm{Zip}^\tau_R$$

is a closed immersion.

Proof. Analogous to Proposition 6.46 it suffices to check that the transition maps of the spectral sequences are zero if and only if they are zero fpqc locally, but as these are maps between discrete modules, we see that this is certainly an fpqc local property.

Let $\mathrm{Spec}(A) \to t_0\, F\text{-}\mathrm{Zip}^\tau_R$ be given a derived F-Zip $\underline{F}$ that is homotopy finite projective of type τ. This in particular implies that the formation of the homologies of the graded pieces commute with arbitrary base change by Lemma 6.37 (iii). Now a morphism $f\colon \mathrm{Spec}(T) \to \mathrm{Spec}(A)$ factors through $\mathcal{X}^\tau_R \times_{t_0\, F\text{-}\mathrm{Zip}^\tau_R} \mathrm{Spec}(A)$ if and only if the spectral sequence associated to the filtrations of $f^*\underline{F}$ degenerate. Again this is equivalent to the differentials of the spectral sequences being zero. By commutativity of the homologies with base change and the fact that being zero for a morphism of finite projective modules is a closed property (see [GW10, Prop. 8.4 (2)]), we see that i is in fact a closed immersion. □

Remark 7.4. Let A be a R-algebra. Let $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be a function with finite support and let $n \in \mathbb{Z}$. As explained in the beginning of this section, we get a map of derived stacks

$$\pi_n\colon \mathcal{X}^\tau_R \to \mathrm{cl}\, F\text{-}\mathrm{Zip}^{\tau_n}_R$$

that is induced by $\underline{F} \mapsto \pi_n \underline{F}$, where $\tau_n\colon \mathbb{Z} \to \mathbb{N}_0$ is given by the function $k \mapsto \tau(k)_n$. Also by Lemma 7.2, we see that the inclusion $\mathcal{X}^\tau_R \hookrightarrow t_0\, F\text{-}\mathrm{Zip}^\tau_R$ factors through the open derived substack $t_0\, sF\text{-}\mathrm{Zip}^\tau_R$.

We can also inlcude $\operatorname{cl} F\text{-}\mathrm{Zip}_R^{\tau_n}$ into $\mathcal{X}_R^\tau$ by considering the following functor

$$(-)[n]\colon \underline{M} \coloneqq (M, C^\bullet, D_\bullet, \varphi_\bullet) \mapsto \underline{M}[n] \coloneqq (C_n^\bullet, D_\bullet^n, \mathrm{id}_M^n, \varphi_\bullet^n),$$

where $C_n^k \coloneqq C^k[n]$, $D_k^n \coloneqq D_k[n]$, $\mathrm{id}_M^n \coloneqq \mathrm{id}_M[n]$ and $\varphi_k^n \coloneqq \varphi_k[n]$ (this is just the n-shift of $\underline{M}[0]$ in $F\text{-}\mathrm{Zip}_R(A)$). Thus, π_n defines a section of $(-)[n]$.

We see that the morphism given

$$\prod_{n\in\mathbb{Z}} \operatorname{cl} F\text{-}\mathrm{Zip}_R^{\tau_n} \hookrightarrow \mathcal{X}_R^\tau, \quad (\underline{M}_n)_{n\in\mathbb{Z}} \mapsto \bigoplus_{n\in\mathbb{Z}} \underline{M}_n[n]$$

is a monomorphism, as it has a section given by $\prod_{n\in\mathbb{Z}} \pi_n$.

Lemma 7.5. *Let $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be a function with finite support and finitely many values. Then the monomorphism $\prod_{n\in\mathbb{Z}} \operatorname{cl} F\text{-}\mathrm{Zip}_R^{\tau_n} \hookrightarrow \mathcal{X}_R^\tau$ defined in Remark 7.4 is an equivalence of derived stacks.*

Proof. We have to show that the map $\prod_{n\in\mathbb{Z}} \operatorname{cl} F\text{-}\mathrm{Zip}_R^{\tau_n} \hookrightarrow \mathcal{X}_R^\tau$ is an effective epimorphism.

It is enough to show that for an R-algebra A every $\underline{F} \in \mathcal{X}_R^\tau(A)$ is equivalent to $\bigoplus_{n\in\mathbb{Z}} \underline{M}_n[n]$ for some F-zips $\underline{M}_n \in \operatorname{cl} F\text{-}\mathrm{Zip}_R^{\tau_n}(A)$.

Let us set $\underline{F} \simeq (C^\bullet, D_\bullet, \varphi_\bullet)$ (see Remark 6.15). We can assume that for every $k \in \mathbb{Z}$, we have $C^k \simeq \bigoplus_{n\in\mathbb{Z}} C_n^k[n]$, $D_k \simeq \bigoplus_{n\in\mathbb{Z}} D_k^n[n]$ by Lemma 6.66. Let $\pi_n\underline{F} = (M, \widetilde{C}^\bullet, \widetilde{D}_\bullet, \pi_n\varphi_\bullet)$ as in the beginning of this section. By strongness of $\underline{F}$ and construction of $\widetilde{C}^\bullet$ resp. $\widetilde{D}_\bullet$, we see that $\widetilde{C}^k = C_n^k$ resp. $\widetilde{D}_k = D_k^n$. Thus, we immediately see $\underline{F} \simeq \bigoplus_{n\in\mathbb{Z}} \pi_n\underline{F}[n]$. $\square$

Lemma 7.6. *Let us fix some $n \in \mathbb{Z}$. Further, let $\sigma\colon \mathbb{Z} \to \mathbb{N}_0$ be a function with finite support and $\tau_n^\sigma\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be given by $k \mapsto \tau_n^\sigma(k)_n = \sigma(k)$ and $k \mapsto \tau_n^\sigma(k)_m = 0$ for $m \neq n$. Then the inclusion $\mathcal{X}_R^\sigma \hookrightarrow t_0\, F\text{-}\mathrm{Zip}_R^\sigma$ is quasi-compact open.*

Proof. By Lemma 7.5, we have an equivalence $\mathcal{X}_R^\sigma \simeq \operatorname{cl} F\text{-}\mathrm{Zip}_R^{\tau_n^\sigma}$ which is quasi-compact open in $t_0\, F\text{-}\mathrm{Zip}_R^\sigma$ by Lemma 6.51 and Proposition 6.48 (note that Lemma 6.51 assumes $n = 0$ but the proof for arbitrary n works similarly). $\square$

The results in this section show us that for morphisms with degenerating Hodge-de Rham spectral sequence there is no new information coming from the theory of derived F-zips. This for example will also show that in the case for abelian schemes, where the F-zips associated to its de Rham cohomology are already determined by its H^1_{dR}, the derived F-zip is also determined by H^1_{dR} (see Section 7.2.1).

In the next section, we want to discuss some classical examples, like curves and K3-surface. Analogous to the abelian scheme case, we can use Lemma 7.5 to determine the associated derived F-zips by their classical counterparts. We will not do this, as this is completely analogous but focus on derived F-zips with type given by the types associated to proper smooth curves and K3 surfaces. As the type $\underline{Rf_*\Omega_{X/S}}$ in these cases will have a certain form, we will see that any strong derived F-zip with the same type is equivalent to a derived F-zip coming from a classical one.

7.2 Classical examples

We want to look at derived F-zips associated to abelian schemes, proper smooth curves and K3-surfaces, and explicitely show that we do not get anything new from the theory of derived F-zips.

7.2.1 Abelian schemes

Let $X \to S$ be an abelian scheme of relative dimension n. A classical result is that

$$H^i_{\mathrm{dR}}(X/S) = \wedge^i H^1_{\mathrm{dR}}(X/S),$$

$H^1_{\mathrm{dR}}(X/S)$ is locally free of rank $2n$ (and thus also the $H^i_{\mathrm{dR}}(X/S)$ are finite locally free), the $R^j f_* \Omega^j_{X/S}$ are finite locally free and the Hodge-de Rham spectral sequence degenerates (see [BBM82, Prop. 2.5.2]).

We can even go further and say that the underlying F-zips of an abelian scheme is characterized by the underling F-zip associated H^1_{dR}, i.e.

$$\underline{H}^i_{\mathrm{dR}}(X/S) = \wedge^i \underline{H}^1_{\mathrm{dR}}(X/S)$$

(see [PWZ15, Ex. 9.9]).

Therefore Lemma 7.5 (or rather its proof) implies the following.

Proposition 7.7. *Let $f\colon X \to S$ be an abelian scheme of relative dimension n. The derived F-zip $\underline{R\Gamma_{\mathrm{dR}}(X/S)}$ is equivalent to the derived F-zip $\bigoplus_{k=0}^{2n} \wedge^k \underline{H}^1_{\mathrm{dR}}(X/S)[k]$ (see Remark 7.4 for the notation).*

Proof. See the discussion above. $\square$

7.2.2 Proper smooth curves

Let C be a proper smooth connected curve of genus g over an algebraically closed field k of characteristic $p > 0$. The de Rham complex consists of two terms. By the degeneracy of the spectral sequence, we know that

$$H^n_{\mathrm{dR}}(C/k) \cong \begin{cases} \Gamma(C, \mathcal{O}_C) = k & \text{if } n = 0, \\ H^1(C, \mathcal{O}_C) \oplus \Gamma(C, \Omega^1_{C/k}) & \text{if } n = 1, \\ H^1(C, \Omega^1_{C/k}) = k & \text{if } n = 2, \\ 0 & \text{else.} \end{cases}$$

Further, we know that $g = \dim_k H^1(C, \mathcal{O}_C) = \dim \Gamma(C, \Omega^1_{C/k})$. Therefore, the de Rham hypercohomology of C is a perfect complex of Tor-amplitude in $[-2, 0]$ and the filtrations are of the form

$$\begin{aligned} \mathbf{HDG}^\bullet\colon \cdots \to 0 \to \mathbf{HDG}^1 \to R\Gamma_{\mathrm{dR}}(C/k) \to R\Gamma_{\mathrm{dR}}(C/k) \to \dots \\ \mathbf{conj}_\bullet\colon \cdots \to 0 \to \mathbf{conj}_0 \to R\Gamma_{\mathrm{dR}}(C/k) \to R\Gamma_{\mathrm{dR}}(C/k) \to \dots \end{aligned}$$

The graded pieces are given by

$$\mathrm{gr}^0\,\mathbf{HDG} \simeq R\Gamma(C,\mathcal{O}_C) \text{ and } \mathrm{gr}^1\,\mathbf{HDG} \simeq R\Gamma(C,\Omega^1_{C/k})[-1]$$

which are perfect complexes of Tor-amplitude in $[-1,0]$ and $[-2,-1]$ (in homological notation).

Let $\mathcal{M}$ denote the moduli stack of smooth proper curves $X \to S$ (see [Sta19, 0DMJ]). The map

$$R\Gamma_{\mathrm{dR}}\colon X/S \mapsto \underline{R\Gamma_{\mathrm{dR}}(X/S)}$$

from $\mathcal{M}$ to $t_0\,F\text{-Zip}^{[-2,0],\{-1,0\}}$ gives a decomposition $\mathcal{M} = \coprod_g \mathcal{M}_g$ into open and closed substacks classifying smooth proper curves of genus g. This follows from the fact that the Euler-characteristic of the graded pieces of the de Rham hypercohomology are determined by the genus and from Proposition 6.40.

Consider the function $\sigma\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ defined as $\sigma(0)_0 = \sigma(1)_{-2} = 1$ and $\sigma(0)_{-1} = \sigma(1)_{-1} = g$ and otherwise zero, for some $g \in \mathbb{N}$. An example of a strong derived F-zip homotopy finite projective of type σ is $\underline{R\Gamma_{\mathrm{dR}}(C/k)}$ by the above (strongness follows from Theorem 6.71). As the de Rham cohomologies are finite projective and the Hodge-spectral sequence degenerates, we see that locally $\underline{R\Gamma_{\mathrm{dR}}(C/k)}$ is determined by the graded pieces of the Hodge and conjugate filtration (see Lemma 6.66). This allows us to construct $\underline{R\Gamma_{\mathrm{dR}}(C/k)}$ from the classical F-zip $\underline{H}^1_{\mathrm{dR}}(C/k)$. But not only that, since σ is not too complicated (there is only one non-trivial homotopy with non-trivial filtration) it seems reasonable that any derived F-zip of type σ is equivalent to one that is induced by a classical F-zip (see below for more details).

First let us show how to extend a classical F-zip of type $\tau\colon \mathbb{Z} \to \mathbb{N}_0,\ k \mapsto \sigma(k)_{-1}$ to a derived F-zip of type σ.

Construction 7.8. Let A be an $\mathbb{F}_p$-algebra. Recall that the natural morphism $A \to A^{(1)}$ is an isomorphism of rings. Further, let τ be as above.

Let $\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ be a classical F-zip over A of type τ. We set $M^+ \coloneqq A[0] \oplus M[-1] \oplus A[-2]$, $C_+ \coloneqq C^1[-1] \oplus A[-2]$ and $D^+ \coloneqq D_0[-1] \oplus A[0]$ as complexes in $\mathcal{D}(A)$. This defines a descending filtration $C_+^\bullet\colon C_+ \to M^+$, where M^+ is in degree 0 and an ascending filtration $D_\bullet^+\colon D^+ \to M^+$, where M^+ is in degree 1, of A-modules. We also get natural equivalences between the graded pieces of the filtrations up to Frobenius twist induced by $A^{(1)} \xrightarrow{\sim} A$ and $\varphi_\bullet$, denoted by $\varphi_\bullet^+$. We define a new derived F-zip over A via

$$\underline{M}^+ \coloneqq (C_+^\bullet, D_\bullet^+, \varphi_\bullet^+).$$

The idea of the above construction is to take a classical F-zip and extend it by a trivial F-zip into homotopical direction. So in the above construction $\underline{M}^+$ is a classical F-zip shifted to degree -1 (homological) and then we add a trivial F-zip via the direct sum to the homotopical degree 0 and -2. All the information of $\underline{M}^+$ as a derived F-zip lies in homotpical degree -1. In particular, we can recover derived F-zips with type as $\underline{M}^+$ from classical F-zips.

Proposition 7.9. *Let σ and τ be defined as above. Then for an $\mathbb{F}_p$-algebra A the map*

$$\begin{aligned}\alpha\colon \operatorname{cl} F\text{-}\mathrm{Zip}^{\tau}_R(A) &\to sF\text{-}\mathrm{Zip}^{\sigma}_R(A)\\ \underline{M} &\mapsto \underline{M}^+\end{aligned}$$

induces an effective epimorphism $\operatorname{cl} F\text{-}\mathrm{Zip}^{\tau}_R \to sF\text{-}\mathrm{Zip}^{\sigma}_R$ *of derived stacks.*

Proof. Let A be an $\mathbb{F}_p$-algebra. Consider a derived F-zip $\underline{F} = (C^\bullet_F, D^F_\bullet, \varphi^F_\bullet)$ over A that is homotopy finite projective of type σ over A. We claim that there is a classical M-zip $\underline{M} = (M, C^\bullet, D_\bullet, \varphi)$ of type τ that induces an equivalence $\underline{M}^+ \xrightarrow{\sim} \underline{F}$.

We can apply Lemma 6.66 to the filtrations and graded pieces of $\underline{F}$ and may only work with perfect complexes over A that have vanishing differentials, i.e. direct sums of shifts of finite free modules (note that we use that the filtrations are finite).

Using the explicit type of $\underline{F}$, we get a long exact homotopy sequence

$$0 \to (C^0_F)_0 \to A \xrightarrow{\partial} (C^1_F)_{-1} \to (C^0_F)_{-1} \to A^g \xrightarrow{\partial'} (C^1_F)_{-2} \to (C^0_F)_{-2} \to 0,$$

and using the ascending filtration, we get the long exact homotopy sequence

$$0 \to (D^F_0)_0 \to (D^F_1)_0 \to 0 \to (D^F_0)_{-1} \to (D^F_1)_{-1} \to (A^g)^{(1)} \to 0 \to (D^F_1)_{-2} \to A^{(1)} \to 0.$$

The strongness of our filtrations show that ∂, ∂' are zero and we see that we can set

$$\begin{aligned}&M = F_{-1},\\ &C^\bullet\colon 0 = C^2 \subseteq (C^1_F)_{-1} \subseteq F_{-1} = C^0,\\ &D_\bullet\colon 0 = D_{-1} \subseteq (D^F_0)_{-1} \subseteq F_{-1} = D_1, \text{ and}\\ &\varphi_0 = \pi_{-1}\varphi^F_0,\ \varphi_1 = \pi_{-1}\varphi^F_1.\end{aligned}$$

The acyclicity of the complexes involved give us an equivalence $\underline{M}^+ \xrightarrow{\sim} \underline{F}$. ☐

7.2.3 K3-surfaces

Let X be a K3-surface over a field k. It is well known, that the Hodge-de Rham spectral sequence of X/k degenerates and that the Hodge-numbers are given as follows $h^{0,0} = h^{0,2} = h^{2,0} = h^{2,2} = 1$, $h^{1,1} = 20$ and otherwise zero. This in particular gives us the type of the derived F-zip associated to a K3-surface over an arbitrary scheme in positive characteristic (which is written out in the following remark).

Using this, we will show as in the case of proper smooth curves that every derived F-zip of the same type as in the K3-surface case is equivalent to one which comes from the classical F-zip of type associated to H^2_{dR} of a K3-surface.

Remark 7.10. Let $\sigma\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be the function given by $\sigma(0)_0 = \sigma(0)_{-2} = \sigma(2)_{-2} = \sigma(2)_{-4} = 1$, $\sigma(1)_{-2} = 20$ and otherwise zero.

Let $\underline{F} \coloneqq (C^\bullet, D_\bullet, \varphi_\bullet) \in t_0\, F\text{-}\mathrm{Zip}^{\sigma}(A)$. We may assume by Lemma 6.66 that every perfect complex associated to $\underline{F}$ has vanishing differentials, i.e. the filtrations and graded pieces are perfect complexes of A-modules with vanishing differentials. So, we can write

$$C^\bullet \simeq \bigoplus_{n\in\mathbb{Z}} (C^\bullet)_n[n] \text{ and } D_\bullet \simeq \bigoplus_{n\in\mathbb{Z}} (D_\bullet)_n[n].$$

By this $C^2 \simeq \pi_{-2}\operatorname{gr}^2_C[-2] \oplus \pi_{-4}\operatorname{gr}^2_C[-4]$ and we have long exact homotopy sequences

$$\begin{aligned} 0 \to (C^2)_{-2} \to &(C^1)_{-2} \to \pi_{-2}\operatorname{gr}^1_C \to 0 \to (C^2)_{-3} \\ &\to 0 \to 0 \to (C^2)_{-4} \to (C^1)_{-4} \to 0, \end{aligned}$$

$$\begin{aligned} 0 \to (C^0)_0 \to \pi_0\operatorname{gr}^0_C \to 0 \to (C^0)_{-1} \to 0 \to (C^1)_{-2} \to (C^0)_{-2} \to \pi_{-2}\operatorname{gr}^0_C \\ \to 0 \to (C^0)_{-3} \to 0 \to (C^1)_{-4} \to (C^0)_{-4} \to 0. \end{aligned}$$

Further, we have $D_0 \simeq \pi_0\operatorname{gr}^0_D[0] \oplus \pi_{-2}\operatorname{gr}^0_D[-2]$ and long exact homotopy sequences

$$0 \to (D_0)_0 \to (D_1)_0 \to 0 \to 0 \to (D_1)_{-1} \to 0 \to (D_0)_{-2} \to (D_1)_{-2} \to \pi_{-2}\operatorname{gr}^1_D \to 0,$$

$$\begin{aligned} 0 \to (D_1)_0 \to (D_2)_0 \to 0 \to 0 \to (D_2)_{-1} \to 0 \to (D_1)_{-2} \to (D_2)_{-2} \\ \to \pi_{-2}\operatorname{gr}^2_D \to 0 \to (D_2)_{-3} \to 0 \to 0 \to (D_2)_{-4} \to \pi_{-4}\operatorname{gr}^2_D \to 0. \end{aligned}$$

In particular, we have that $\underline{F}$ is a strong derived F-zip as the homotopies are finite locally free, which allows us to construct sections.

Corollary 7.11. *Let X/S be a K3-surface, then the Hodge-de Rham spectral sequence associated to X/S degenerates.*

Proof. Combine Remark 7.10 and Theorem 6.71. □

Lemma 7.12. *Let $\sigma\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be the function given by some*

$$\sigma(0)_0, \sigma(0)_{-2}, \sigma(1)_{-2}, \sigma(2)_{-2}, \sigma(2)_{-4} \in \mathbb{N}_0$$

and otherwise zero. Then the inclusion

$$sF\text{-Zip}^\sigma \hookrightarrow F\text{-Zip}^\sigma$$

is an equivalence.

Proof. We only have to check that it is an effective epimorphism, which can be checked locally. Now the argumentation as in Remark 7.10 concludes the proof. □

Again, as in the proper smooth curve case, we want to construct a derived F-zip of type σ out of a classical F-zip and show that all derived F-zips of type σ are given by those.

In particular, the derived F-zip associated to a K3-surface will carry no additional information besides the classical F-zip attached to its H^2_{dR}.

Construction 7.13. Let $\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ be a classical F-zip over A of type τ, where $\tau(2) = \tau(0) = 1$ and $\tau(1) = 20$ and otherwise zero. We set $M^+ \coloneqq A[0] \oplus M[-2] \oplus A[-4]$, $C^2_+ \coloneqq C^2[-2] \oplus A[-4]$, $C^1_+ = C^1_+[-2] \oplus A[-4]$, $D^+_0 \coloneqq D_0[-2] \oplus A[-4]$ and $D^+_1 = A[0] \oplus D_1[-2]$. This defines a descending filtration $C^\bullet_+ \colon C^2_+ \to C^1_+ \to M^+$ and an ascending filtration $D^+_\bullet \colon D^+_0 \to D^+_1 \to M^+$. We also get natural equivalences between the graded pieces of the filtrations up to Frobenius twist induced by $A^{(1)} \xrightarrow{\sim} A$ and $\varphi_\bullet$, denoted by $\varphi^+_\bullet$. We define a new derived F-zip over A via

$$\underline{M}^+ \coloneqq (C^\bullet_+, D^+_\bullet, \varphi^+_\bullet).$$

Proposition 7.14. *let τ be as in Construction 7.13 and let σ be as in Remark 7.10. Then for an $\mathbb{F}_p$-algebra A the map*

$$\begin{aligned} \alpha \colon \mathrm{cl}\, F\text{-}\mathrm{Zip}^\tau_R(A) &\to F\text{-}\mathrm{Zip}^\sigma_R(A) \\ \underline{M} &\mapsto \underline{M}^+ \end{aligned}$$

induces an effective epimorphism $\mathrm{cl}\, F\text{-}\mathrm{Zip}^\tau_R \to t_0\, F\text{-}\mathrm{Zip}^\sigma_R$ *of derived stacks.*

Proof. Using that a derived F-zip of type σ is automatically strong (see Lemma 7.12), we see that the proof is analogous to the proof of Proposition 7.9 with Construction 7.13. □

8 Application to Enriques surfaces

One of the main reasons behind the theory of derived F-zips is to extend the theory of F-zips such that we can use it on geometric objects that have non-degenerate Hodge-de Rham spectral sequence. One example of such geometric objects are Enriques surfaces in characteristic $= 2$. Here, we have three types of Enriques surfaces: $\mathbb{Z}/2\mathbb{Z}$, μ_2 and α_2. The Enriques surfaces of type α_2 are in particular interest for us since they have non-degenerate Hodge-de Rham spectral sequences (for the other types the spectral sequences degenerate). One can show (see [Lie15]) that the moduli stack $\mathcal{M}$ of Enriques surfaces has three substacks $\mathcal{M}_{\alpha_2}$, $\mathcal{M}_{\mathbb{Z}/2\mathbb{Z}}$ and $\mathcal{M}_{\mu_2}$ that classify precisely these three types and that they are closed and open resp. open in $\mathcal{M}$. We will come to the same result using the theory of derived F-zips (see Proposition 8.9) since the substacks corresponding to the types of Enriques surfaces can be classified by their corresponding type of the derived F-zip associated to the de Rham hypercohomology.

8.1 Overview

We will shortly recall the definition of Enriques surfaces and some properties. We use the upcoming book of Cossec, Dolgachev and Liedtke as a reference (see [CDL21]). For this fix an algebraically closed field k of characteristic $p > 0$.

Definition 8.1. An Enriques surface is a proper smooth surface over k with Kodaira dimension 0 and $b_2(X) \coloneqq \dim_{\mathbb{Q}_\ell} H^2_{\text{ét}}(X, \mathbb{Q}_\ell) = 10$, where $\ell \neq p$ is a prime.

Proposition 8.2. *Let S be an Enriques surface over k. If the characteristic is $p = 2$, then the group scheme of divisor classes which are numerically equivalent to* 0*, denoted by* $\mathrm{Pic}^\tau_{S/k}$ *is either* $\mathbb{Z}/2, \mu_2$ *or* α_2*. In characteristic* > 2*, we have* $\mathrm{Pic}^\tau_{S/k} \cong \mathbb{Z}/2$.

Definition 8.3. Let S be an Enriques surface over k and assume $p = 2$, then we call S *classical* (resp. *singular* or *supersingular*) or of *type* $\mathbb{Z}/2\mathbb{Z}$ *(resp.* μ_2 *or* α_2*)* if $\mathrm{Pic}^\tau_{S/k}$ is isomorphic to $\mathbb{Z}/2$ (resp. μ_2 or α_2).

Proposition 8.4. *Let S be an Enriques surface over k. The associated Hodge-de Rham spectral sequence degenerates if and only if S is not supersingular.*

Proof. This is [CDL21, Cor 1.4.15] but let us recall the arguments (note that there is a typo in the reference, as they compute the crystalline cohomology and conclude the de Rham cohomology by the universal coefficient formula, which implies the numbers in Table (8.1.2)).

In [CDL21, §1.4 Table 1.2, Table 1.3] they give the exact Hodge-numbers and dimensions of the de Rham cohomology, which in particular implies the result about degeneracy.

Let us be a bit more precise and recall the important numbers. Let $h^{i,j}$ denote the k-dimension of $H^j(S, \Omega^i_{S/k})$ and h^i_{dR} the dimension of $H^i(S, \Omega^\bullet_{S/k})$. Then we have the following table linking the type of S with the Hodge-numbers.

(8.1.1)

$\mathrm{Pic}^{\tau}_{S/k}$	$h^{0,0}$	$h^{1,0}$	$h^{0,1}$	$h^{0,2}$	$h^{1,1}$	$h^{2,0}$
μ_2	1	0	1	1	10	1
$\mathbb{Z}/2$	1	1	0	0	12	0
α_2	1	1	1	1	12	1

The dimension of the de Rham cohomology is given as follows (this does not depend on the type of the Enriques surfaces).

(8.1.2)

h^0_{dR}	h^1_{dR}	h^2_{dR}	h^3_{dR}	h^4_{dR}
1	1	12	0	1

By Serre duality this table is enough to conclude (non-)degeneracy. □

We denote by $(\mathrm{Pic}^{\tau}_{S/k})^D$ the Cartier dual of $\mathrm{Pic}^{\tau}_{S/k}$. Note that $\alpha_2^D = \alpha_2$, $\mathbb{Z}/2^D = \mu_2$ and $\mu_2^D = \mathbb{Z}/2$.

Proposition 8.5. *Let S be an Enriques surface over k. There exists a non-trivial $(\mathrm{Pic}^{\tau}_{S/k})^D$-torsor*

$$\pi\colon X \to S$$

In particular, π is finite flat of degree 2. Note that if $p \neq 2$ or S is of type μ_2, then π is étale.

Proof. See [CDL21, Thm. 1.3.1]. □

Definition 8.6. A finite flat map $X \to S$ of degree 2 is called K3-*cover.*

Proposition 8.7. *Let S be an Enriques surface over k. Let $\pi\colon X \to S$ be a* K3*-cover. Then X is integral Gorenstein, satisfying*

$$H^1(X, \mathcal{O}_X) = 0, \qquad \omega_X \cong \mathcal{O}_X.$$

Further, we have

1. *if $p \neq 2$ or S is of type μ_2, then X is a smooth* K3*-surface, and*
2. *if $p = 2$ and S is of type $\mathbb{Z}/2$ or α_2, then X is not a smooth surface.*

Proof. See [CDL21, Prop. 1.3.3]. □

Definition 8.8. Let S be a $\mathbb{F}_p$-scheme. An *Enriques surface X over S* is a proper smooth morphism of algebraic spaces $f\colon X \to S$ such that the geometric fibers of f are Enriques surfaces.

8.2 Derived F-zips associated to Enriques surfaces

In the following every scheme will be in characteristic 2.

We let $\mathcal{M}$ denote that stack classifying Enriques surfaces with "nice" polarization, i.e. the functor that sends an $\mathbb{F}_2$-scheme S to the groupoid of pairs $(X/S, \mathcal{L})$ consisting of Enriques surfaces $X \to S$ with "nice" line bundle $\mathcal{L}$ on X - the term "nice" means a polarization such that $\mathcal{M}$ defines an Artin stack, examples of such classifying stacks are given in [CDL21, Thm. 5.11.6] and [Lie15, §5] (we only need that $\mathcal{M}$ is an Artin stack and are not interested in the polarization itself and as there are many different such polarizations such that $\mathcal{M}$ is an Artin stack, we omit the explicit description). By our previous constructions, we get a morphism

$$p\colon \mathcal{M} \to t_0\, F\text{-Zip}_S, \quad (X/S, \mathcal{L}) \mapsto \underline{Rf_*\Omega^\bullet_{X/S}}.$$

The Hodge-numbers and dimension of the de Rham cohmology for Enriques surfaces over algebraically closed fields, define types for the underlying F-zip (see Tables 8.1.1 8.1.2). We denote those by $\tau_{\mathbb{Z}/2}, \tau_{\mu_2}, \tau_{\alpha_2}$[45] for the types defined by the Hodge-numbers of $\mathbb{Z}/2, \mu_2, \alpha_2$-Enriques surfaces respectively.

We denote the corresponding loci with $\mathcal{M}_{\mathbb{Z}/2}$ and $\mathcal{M}_{\mu_2}$, i.e. these denote the substacks classifying Enriques surfaces of type $\mathbb{Z}/2$ (resp. μ_2). We denote the substack of α_2 Enrique surfaces $f\colon X \to S$ such that $R^i f_*\Omega^j_{X/S}$ is finite locally free for all $i, j \in \mathbb{Z}$ with $\mathcal{M}_{\alpha_2}$[46]. With these definitions, we see that

$$p^{-1}(t_0\, F\text{-Zip}^{\leq \tau_{\alpha_2}}) \coloneqq \mathcal{M} \times_{t_0\, F\text{-Zip}_S} t_0\, F\text{-Zip}^{\leq \tau_{\alpha_2}} \simeq \mathcal{M}$$

and we will see in the following that the substacks $\mathcal{M}_{\mathbb{Z}/2}$ and $\mathcal{M}_{\mu_2}$ are open in $\mathcal{M}$ and $\mathcal{M}_{\alpha_2}$ is locally closed in $\mathcal{M}$.

Proposition 8.9. *The substacks $\mathcal{M}_{\mathbb{Z}/2}$ and $\mathcal{M}_{\mu_2}$ are open algebraic substacks and $\mathcal{M}_{\alpha_2}$ is a closed algebraic substack of $\mathcal{M}$ locally of finite presentation.*

Proof. Let us look at $\mathcal{M}_{\mathbb{Z}/2} \simeq p^{-1}(t_0\, F\text{-Zip}^{\leq \tau_{\mathbb{Z}/2}})$. We claim that this is an open substack of $\mathcal{M}$. Since $\mathcal{M}$ is an Artin stack, we know that it is a 1-geometric 1-truncated derived stack in our sense. In particular, since the base change of open immersions are open immersion (i.e. flat, locally finitely presented monomorphisms), we know with Remark 4.35 and Proposition 6.39 that $\mathcal{M}_{\mathbb{Z}/2} \hookrightarrow \mathcal{M}$ is a flat, locally finitely presented monomorphism and in particular $\mathcal{M}_{\mathbb{Z}/2}$ is 1-geometric. Since $\mathcal{M}_{\mathbb{Z}/2} \hookrightarrow \mathcal{M}$ is a monomorphism (i.e. (-1)-truncated), we see that $\mathcal{M}_{\mathbb{Z}/2}$ is 1-truncated (see [Lur09, Lem. 5.5.6.14]). In fact, we claim that this shows that $\mathcal{M}_{\mathbb{Z}/2}$ is an algebraic stack.

To see this note that since $\mathcal{M}_{\mathbb{Z}/2} \hookrightarrow \mathcal{M}$ is a monomorphism, the diagonal of $\mathcal{M}_{\mathbb{Z}/2}$ is representable by an algebraic space. Further, we claim that 1-geometricity of $\mathcal{M}_{\mathbb{Z}/2}$ implies that we have a smooth atlas by a coproduct of affine schemes, so a scheme.

[45] Recall that the types are given by $\tau_*(i)_j \coloneqq h^{i,-j-i}$ (the hodge numbers of the corresponding types). The table (8.1.1) shows that $\tau_{\mu_2} \leq \tau_{\alpha_2}$ and $\tau_{\mathbb{Z}/2} \leq \tau_{\alpha_2}$ and no relation between τ_{μ_2} and $\tau_{\mathbb{Z}/2}$.

[46] Note that if $f\colon X \to S$ is a locally noetherian reduced Enriques surface, we can use [Gro63, Prop. (7.8.4)] to see that $R^i f_*\Omega^j_{X/S}$ is finite locally free for all $i, j \in \mathbb{Z}$.

To see the last part, let us look at a smooth 0-atlas $q\colon \coprod \operatorname{Spec}(A_i) \twoheadrightarrow \mathcal{M}_{\mathbb{Z}/2}$. We have to check that this is smooth in the classical sense. For that consider the base change with an affine scheme $\operatorname{Spec}(B) \to \mathcal{M}_{\mathbb{Z}/2}$, denoted by X. This is an algebraic space and by geometricity has a smooth cover $\coprod \operatorname{Spec}(B_i)$ by some smooth B-algebras B_i such that each $g\colon \operatorname{Spec}(B_i) \to X$ is affine. So, we have a diagram of the following form with cartesian square

$$\begin{tikzcd}
\coprod \operatorname{Spec}(B_i) \arrow[r, "g"] & X \arrow[r, "f"] \arrow[d] & \operatorname{Spec}(B) \arrow[d] \\
& \coprod \operatorname{Spec}(A_i) \arrow[r, "q"] & \mathcal{M}_{\mathbb{Z}/2}.
\end{tikzcd}$$

As g is smooth and surjective and $f \circ g$ is smooth, we know by descent that f is smooth (as the property "smooth" is smooth local on the source for algebraic spaces, see [Sta19, 06F2]). Certainly f is also surjective, as it is the base change of and effective epimorphism. Therefore, by definition we see that p is smooth and surjective.

The same argumentation works if we replace $\mathcal{M}_{\mathbb{Z}/2}$ with $\mathcal{M}_{\mu_2}$.

For the supersingular locus, we note that the inclusion

$$t_0\, F\text{-}\mathrm{Zip}^{\tau_{\alpha_2}} \hookrightarrow t_0\, F\text{-}\mathrm{Zip}^{\tau_{\leq \alpha_2}}$$

is a closed immersion locally of finite presentation (again by Proposition 6.39). So, analogous to the above, we see that $\mathcal{M}_{\alpha_2}$ is an algebraic substack $\mathcal{M}$ such that $\mathcal{M}_{\alpha_2} \hookrightarrow \mathcal{M}$ is a closed immersion of algebraic stacks locally of finite presentation. $\square$

9 Derived F-zips with cup product

Here we discuss two possible generalizations of derived F-zips. Firstly, we could try to extend the theory of derived F-zips in such a way such that we can attach a derived F-zip to an lci morphism. Secondly, we could extend the theory of derived F-zips to the theory of derived G-zips as in [PWZ15], for a reductive group G and hope that the extra structure on the de Rham hypercohomology given by the cup product (see Section 9.2) endows it with a G-zip structure.

We will discuss both cases and show that the naive way of extending derived F-zips does not work in both cases. But for completion, we will look at derived F-zips with some extra structure that is given by a perfect pairing. Again, we can not connect this to the theory of G-zips but this is just a very naive approach we want to discuss.

9.1 Problems

Now let us discuss the problems that occurred when trying to generalize the theory.

9.1.1 Derived F-zips for lci morphisms

We could have defined derived F-zips not over animated rings but over usual commutative rings in positive characteristic. One benefit of the animation process is that simplicially every commutative ring can be approximated by smooth rings. One often uses this to generalize theories that work in the smooth case to the non-smooth case. To define derived F-zips, we looked at the de Rham hypercohomology of a smooth proper scheme. So to define a theory of derived F-zips that works for non-smooth schemes, we would need a non-smooth analogue of the de Rham hypercohomology. The most natural generalization comes from looking at the de Rham complex as a functor from smooth $\mathbb{F}_p$-algebras and looking at its left Kan extension to animated $\mathbb{F}_p$-algebras. This is done for example in [Bha12b],[Bha12a] or [Ill71] and is called the derived de Rham complex, denoted by $\mathrm{dR}_{X/R}$[47] for a scheme X over some ring R of positive characteristic. Let us state some facts about the derived de Rham complex that can also be found in the mentioned articles by Bhatt or in the book by Illusie.

In the smooth case this gives the usual de Rham hypercohomology $R\Gamma_{\mathrm{dR}}(X/R)$. The derived de Rham complex comes naturally with two filtrations, one is the conjugate filtration and one is the Hodge filtration. Both come from the conjugate respectively the Hodge filtration on the de Rham complex by extending via left Kan extension. The graded pieces are given by

$$\mathrm{gr}^i_{\mathrm{conj}}\,\mathrm{dR}_{X/R} \simeq \wedge^i L_{X^{(1)}/R}, \quad \mathrm{gr}^i_{\mathrm{HDG}}\,\mathrm{dR}_{X/R} \simeq \wedge^i L_{X/R},$$

in particular they are isomorphic upto Frobenius twist.

Even though it seems natural, it is not clear that the Hodge filtration is complete, i.e. $\varprojlim_i \mathrm{HDG}(i) \simeq 0$. Further, the filtrations may not be finite in any way. This holds

[47] We define $\mathrm{dR}_{-/R}$ as the left Kan extension of the functor $P \mapsto \Omega^\bullet_{P/R}$ along the inclusion $\mathrm{Poly}_R \hookrightarrow \mathrm{AR}_R$. Then, we denote with $\mathbb{R}\,\mathrm{dR}_{-/R}$ the right Kan extension of $\mathrm{dR}_{-/R}$ along the Yoneda embedding $\mathrm{AR}_R \hookrightarrow \mathcal{P}(\mathrm{AR}_R^{\mathrm{op}})^{\mathrm{op}}$ and set $\mathrm{dR}_{X/R} \coloneqq \mathbb{R}\,\mathrm{dR}_{-/R}(X)$.

more or less for any variety with isolated lci singularities. One very generic example is $A \coloneqq k[\varepsilon]/(\varepsilon^p)$ for some field k of characteristic $p > 0$. One can show that $\wedge^n L_{A/k}$ is not quasi-isomorphic to 0 for any $n \in \mathbb{N}_0$ (see [Bha12c, Rem. 2.2]). This obstruction comes from the fact that in the lci case $L_{A/k}$ is a complex concentrated in two degrees and after base change to k one can see that it is given by the direct sum of exterior powers and shifts of k (see proof of [Bha12c, Lem. 2.1]). Now the exterior power of the shift of a module can be computed by its free divided power, which will not vanish even for higher powers (see [Lur18, Prop. 25.2.4.2]). To avoid problems, we could define derived F-zips using non-complete filtrations but the problem here is actually the unboundedness of the filtrations. There is no need for an n-atlas if we allow infinite filtrations since we would need to cover finitely many data at once, which renders this approach a priori useless.

9.1.2 Derived G-zips

The theory of G-zips, for a connected reductive group G over a field of characteristic $p > 0$, endows the theory of F-zips with extra structure related to the group. The motivation behind this is that we have a cup product on the de Rham cohomologies of smooth proper maps, where the relative Hodge-de Rham spectral sequence degenerates. In even degrees this endows the F-zip associated to the de Rham cohomology with a twisted symmetric structure and in odd degrees with a twisted symplectic structure. All of this can be found in [PWZ15].

There are 3 equivalent approaches to the theory of G-zips. The first one is to first identify the stack of F-zips over a scheme S with the stack of vector bundles on some quotient stack $\mathfrak{X}$, where the quotient stack is defined via the following recipe. We take $\mathbb{P}^1$ and pinch the point at ∞ and the 0 point up to Frobenius twist together. Now we let $\mathbb{G}_m$ act on the affine line around 0 in degree 1 and on the affine line around ∞ in degree -1. Let us make precise what happens here. Vector bundles on $[\mathbb{A}^1/\mathbb{G}_m]$ are finitely filtered vector bundles, where depending on the action of $\mathbb{G}_m$, in our case multiplication with an element in $\mathbb{G}_m$ resp. with the inverse, we get an increasing resp. decreasing filtration (see Appendix A for further details). Thus, a vector bundle on $[\mathbb{P}^1/\mathbb{G}_\mathrm{m}]$ gives a vector bundle with an ascending and a descending filtration. The pullback to the 0 resp. ∞ gives us the graded pieces. Gluing 0 and ∞ together along the Frobenius, we see that a vector bundle on $\mathfrak{X}$ gives us an F-zip. Now a G-zip is just a G-torsor over $\mathfrak{X}$ (see Theorem A.5).

Secondly one can realize G-zips as exact fiber functors from finite G-representations to F-zips. Using that F-zips are the same as finite dimensional vector bundles over the above quotient stack $\mathfrak{X}$ and using Tannaka duality, one sees that this description and the first one agree.

Lastly there is a description of G-zips as a quotient stack $[G/E]$. We spare the details for the reader and refer to [PWZ15], where also the equivalence with the second description can be found.

In our context the first and second approach seem to be the natural ones. The first approach seems to be a bit tricky, since we would need to show that there is a quotient stack, such that that perfect complexes over this stack gives us the derived F-zips. Naturally one could take $\mathfrak{X}$ as the desired stack and as explained in Appendix B the perfect complexes on $\mathfrak{X}$ recover derived F-zips. But we still lack a good notion of derived groups and torsors attaching extra structure to perfect complexes.

For the second approach, we would need a replacement for finite G-representations. Looking at the works of Iwanari and Bhatt on derived Tannaka duality, it seems natural to replace $\mathrm{Rep}(G_{\mathbb{F}_p})$ with $\mathrm{Perf}(\mathrm{B}G_{\mathbb{F}_p})$. But even though natural, it will turn out to be not the right approach. The problem here is $\mathrm{B}G$. It is the classifying stack associated to a classical group scheme. Thus looking at exact fiber functors and Tannaka duality (which we do not have) one could argue that it should give us the classical theory of G-zips embedded into the derived setting. This is not the same as a derived analogue. For example, $\mathbb{G}_m$-zips are the same as F-zips of rank 1. We would expect derived $\mathbb{G}_m$-zips to be derived F-zips of of Euler characteristic ± 1. But we will see, that this is not completely true for exact fiber functors from $\mathrm{Perf}(\mathrm{B}\mathbb{G}_{\mathrm{m},\mathbb{F}_p})$ to $F\text{-}\mathrm{Zip}(A)$ for some $\mathbb{F}_p$-algebra A. Instead of derived F-zips of Euler characteristic ± 1 they give us derived F-zips, where the cohomologies are finite locally free, meaning that they give us (more or less) the classical theory.

To make everything we wrote precise, let F be an exact monoidal functor from $\mathrm{Perf}(\mathrm{B}\mathbb{G}_{\mathrm{m},\mathbb{F}_p})$ to $F\text{-}\mathrm{Zip}(A)$ for $\mathbb{F}_p$-algebra A. Since over a field any complex is quasi-isomorphic to a complex with zero differentials, we see that the descent condition (induced by the Bar resolution of $\mathrm{B}\mathbb{G}_{\mathrm{m},\mathbb{F}_p}$) for a complex in $\mathrm{Perf}(\mathrm{B}\mathbb{G}_{\mathrm{m},\mathbb{F}_p})$ is a condition on the cohomologies of the complex. With this, we see that $E \in \mathrm{Perf}(\mathrm{B}\mathbb{G}_{\mathrm{m},\mathbb{F}_p})$ is equivalent to a finite direct sum of $E_i[i]$, where $i \in \mathbb{Z}$ and E_i are finite projective graded modules. Since F is exact, we see that F is already determined, up to equivalence, by its image on vector bundles on $\mathrm{B}\mathbb{G}_{\mathrm{m}}$, i.e. finite $\mathbb{G}_{\mathrm{m}}$-representations, seen as complexes concentrated in degree 0. But finite $\mathbb{G}_{\mathrm{m}}$-representations are generated under the tensor product by the standard representation, hence F is already determined, up to equivalence, by its image of $\mathbb{F}_p$, seen as a graded vector space in degree 1. This is certainly an invertible element in $\mathrm{Perf}(\mathrm{B}\mathbb{G}_{\mathrm{m},\mathbb{F}_p})$ and thus $F(\mathbb{F}_p)$ must also be invertible. Since the monoidal structure on $F\text{-}\mathrm{Zip}(A)$ is given componentwise, we see that the underlying module of the F-zip has to be invertible. But invertible perfect complexes over A are locally shifts of line bundles and globally given by the direct sum of shifts of finite projective modules (see [Sta19, 0FNT]). This is too much and would give us the classical theory of F-zips after passing to the cohomology.

9.2 Extra structure coming from geometry

In this section, we naively put extra structure on derived F-zips by looking at the extra structure on the de Rham hypercohomology coming from the cup product, namely a perfect pairing on the underlying module of a derived F-zip.

In the following we fix a ring R of characteristic $p > 0$.

Definition 9.1. Let A be an animated ring. Let M and N be perfect A-modules. A *perfect pairing of M and N* is a morphism $M \otimes N \to A$ such that the induced morphism $M \to N^\vee$ is an equivalence.

Remark 9.2. Let A be an animated ring. Note that any equivalence between perfect A-modules M, N of the form $M \to N^\vee$ induces a perfect pairing $M \otimes N \to A$ by adjunction. So giving a perfect pairing $M \otimes N \to A$ is equivalent to giving an equivalence $M \to N^\vee$.

Definition 9.3. Let A be an animated ring. We define the ∞-category of perfect pairings PP_A over A as the full subcategory of X, where X is given by the pullback diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Mod}_A^{\mathrm{perf}} \times \mathrm{Mod}_A^{\mathrm{perf}} \arrow[d, "(M,N)\mapsto(M,N^\vee)"] \\ \mathrm{Fun}(\Delta^1, \mathrm{Mod}_A^{\mathrm{perf}}) \arrow[r, "r"] & \mathrm{Fun}(\partial\Delta^1, \mathrm{Mod}_A^{\mathrm{perf}}), \end{tikzcd}$$

of those morphisms $M \to N^\vee$ that are an equivalence (note that r is given by restriction and by [Lur21, 01F3] is an isofibration of simplicial sets. Thus X is equivalent in Cat_∞ to the usual pullback of simplicial sets[48] - so indeed X classifies morphisms of A-modules $M \to N^\vee$).

Definition 9.4. Let A be an animated $\mathbb{F}_p$-algebra, $a \leq b \in \mathbb{Z}$ and $S \subset \mathbb{Z}$ be a finite subset. Let $\mathrm{dR\text{-}Zip}_\infty^{[a,b],S}(A)$ denote the ∞-category

$$F\text{-}\mathrm{Zip}_\infty^{[a,b],S}(A) \times_{(\mathrm{colim}[b-a],(\mathrm{colim})^\vee),\mathrm{Mod}_A^{\mathrm{perf}} \times \mathrm{Mod}_A^{\mathrm{perf}}} \mathrm{PP}_A,$$

i.e. the ∞-category consisting of tuples $(\underline{F}, \psi)$, where $\underline{F} := (C^\bullet, D_\bullet, \phi, \varphi_\bullet)$ is a derived F-zip with $M := \mathrm{colim}_{\mathbb{Z}^{\mathrm{op}}} C$ and a perfect pairing $\psi \colon M \otimes M \to \mathbb{1}[a-b]$.[49] We set

$$\mathrm{dR\text{-}Zip}_\infty(A) := \operatorname*{colim}_{\substack{a \leq b, \\ S \subseteq \mathbb{Z} \text{ finite}}} \mathrm{dR\text{-}Zip}_\infty^{[a,b],S}(A)$$

and call its elements dR-*zips over* A.

Next we want to show that for any proper smooth morphism $f \colon X \to S$ of schemes, we can attach a dR-zip structure to the derived F-zip $\underline{Rf_*\Omega^\bullet_{X/S}}$. This structure comes naturally from the cup product.

Lemma 9.5. *Let A be a ring and X be a proper smooth scheme over A of relative dimension n. The de Rham hypercohomology $R\Gamma(X, \Omega^\bullet_{X/A})$ has Tor-amplitude in $[-2n, 0]$.*

Proof. Indeed, first of all, we claim that $\dim_{\kappa(a)} \pi_i(X_{\kappa(a)}, \Omega^\bullet_{X_a/\kappa(a)})$ is zero for all $a \in \mathrm{Spec}(A)$ if $i \notin [-2n, 0]$.

[48] To check that X is equivalent to the usual pullback of simplicial sets, we may use Yoneda and check that for any ∞-category $\mathcal{C}$ the ∞-groupoid $\mathrm{Fun}(\mathcal{C}, X)^\simeq$ is given by the usual pullback of simplicial sets, but this follows from Remark 5.13

[49] Again, as explained in the definition of PP_A, we have that the pullback diagram defining $\mathrm{dR\text{-}Zip}_\infty^{[a,b],S}(A)$ is equivalent in Cat_∞ to the ordinary pullback of simlicial sets, as the projection from PP_A to $\mathrm{Mod}_A \times \mathrm{Mod}_A$ is an isofibration (since isofibrations are stable under pullbacks of simplicial sets by [Lur21, 01H4]).

This follows from Grothendieck vanishing (see [Har77, Thm. 2.7]) in the following way. The perfect complex $R\Gamma(X_{\kappa(a)}, \Omega^k_{X_{\kappa(a)}/\kappa(a)})[-k]$ has by Grothendieck vanishing nonzero homotopies in degrees $-n-k, \ldots, -k$.

Now, the distinguished triangle associated to the stupid truncation

$$R\Gamma(X_{\kappa(a)}, \sigma_{\geq k+1}\Omega^\bullet_{X_{\kappa(a)}/\kappa(a)}) \to R\Gamma(X_{\kappa(a)}, \sigma_{\geq k}\Omega^\bullet_{X_{\kappa(a)}/\kappa(a)}) \to R\Gamma(X_{\kappa(a)}, \Omega^k_{X_{\kappa(a)}/\kappa(a)})[-k]$$

shows by induction that $\dim_{\kappa(a)} \pi_i(X_{\kappa(a)}, \Omega^\bullet_{X_a/\kappa(a)})$ is nonzero if and only if $i \in [-2n, 0]$.

Further, it suffices to check Zariski locally on $\mathrm{Spec}(A)$ that $R\Gamma(X, \Omega^\bullet_{X/A})$ has Tor-amplitude in $[-2n, 0]$. Any point $a \in \mathrm{Spec}(A)$ has an affine open neighbourhood $U = \mathrm{Spec}(A_f)$ such that $R\Gamma(X, \Omega^\bullet_{X/A})_{|U}$ is by [Sta19, 0BCD] equivalent to a complex of the form

$$\cdots \to 0 \to A_f^{d_0} \to A_f^{d_{-1}} \to \cdots \to A_f^{d_{-2n}} \to 0 \to \cdots,$$

where $A_f^{d_i}$ sits in homological degree $-i$ and

$$d_i := \dim_{\kappa(a)} \pi_i(R\Gamma(X, \Omega^\bullet_{X/A}) \otimes^L_A \kappa(a)) = \dim_{\kappa(a)} \pi_i(X_{\kappa(a)}, \Omega^\bullet_{X_a/\kappa(a)})$$

(the last equality follows as the formation of the de Rham hypercohomology commutes with arbitrary base change, see [Sta19, 0FM0]). □

Example 9.6. Let A be a ring and X be a proper smooth scheme over A with nonempty fibers of equidimension n. By Lemma 9.5 $R\Gamma(X, \Omega^\bullet_{X/A})$ has Tor amplitude in $[-2n, 0]$. Further, the de Rham hypercohomology admits a perfect pairing

$$R\Gamma(X, \Omega^\bullet_{X/A}) \otimes^L_A R\Gamma(X, \Omega^\bullet_{X/A})[2n] \to A$$

(see [Sta19, 0G8K]). Hence, this induces a dR-zip structure on $\underline{R\Gamma_{\mathrm{dR}}(X/A)}$.

Proposition 9.7. *The functor*

$$\begin{aligned} \text{dR-Zip}_{\infty,R}\colon \mathrm{AR}_R &\to \mathrm{Cat}_\infty \\ A &\mapsto \text{dR-Zip}_\infty(A) \end{aligned}$$

defines a hypercomplete sheaf for the fpqc topology. We denote the associated derived stack with dR-Zip_R.

Let $S \subseteq \mathbb{Z}$ be a finite subset, $a \leq b \in \mathbb{Z}$ and set $n := b-a$, then the induced morphism

$$p^{[a,b],S}\colon \text{dR-Zip}_R^{[a,b],S} \to F\text{-Zip}_R^{[a,b],S}$$

is $2n$-geometric and smooth. Further, $\text{dR-Zip}_R^{[a,b],S}$ *is $2n$-geometric if $n \geq 1$ (resp. 1-geometric if $n = 0$) and locally of finite presentation .*

Proof. Fix a finite subset $S \subseteq \mathbb{Z}$, $a \leq b \in \mathbb{Z}$ and set $n := b-a$. For a derived F-zip $\underline{F} = (C^\bullet, D_\bullet, \phi, \varphi_\bullet)$, let us set $M_{\underline{F}} := \mathrm{colim}_{\mathbb{Z}^{\mathrm{op}}} C^\bullet$. Let us look at the following pullback square

$$\begin{tikzcd} X \arrow[rr] \arrow[d] & & \mathrm{Fun}(\Delta^1, \mathrm{Perf}) \arrow[d] \\ F\text{-Zip}^{[a,b],S}_{\infty,R} \arrow[rr, "\underline{F} \mapsto (M_{\underline{F}}, M^\vee_{\underline{F}}[-n])"] & & \mathrm{Fun}(\partial\Delta^1, \mathrm{Perf}). \end{tikzcd}$$

Noting that a perfect pairing of $M_{\underline{F}}$ and $M_{\underline{F}}^{\vee}[-n]$ is the same as an equivalence $M_{\underline{F}} \xrightarrow{\sim} M_{\underline{F}}^{\vee}[-n]$, we see that $\text{dR-Zip}_{\infty}^{[a,b],S} \simeq X$. In particular, $\text{dR-Zip}_{\infty,R}^{[a,b],S}$ satisfies fpqc hyperdescent.

Finally let $p^{[a,b],S}\colon \text{dR-Zip}_R^{[a,b],S} \to F\text{-Zip}_R^{[a,b],S}$ denote the induced morphism of derived stacks. For a derived F-zip $\underline{F}$ over some animated $\mathbb{F}_p$-algebra A, we have that $(p^{[a,b],S})^{-1}(\underline{F}) \simeq \operatorname{Equiv}(M_{\underline{F}}, M_{\underline{F}}^{\vee}[-n])$, which is $2n$-geometric and smooth, as $M_{\underline{F}} \otimes_A M_{\underline{F}}^{\vee}[-n]$ has Tor-amplitude in $[-2n, 0]$ (see Lemma 5.4 and 5.5). Now the assertions on p follow immediately per definition and by Theorem 6.23, we get the results on $\text{dR-Zip}_R^{[a,b],S}$. $\square$

Corollary 9.8. *The derived stack* dR-Zip_R *is locally geometric and locally of finite presentation.*

Proof. Let $\tau\colon \mathbb{Z} \to \mathbb{N}_0^{\mathbb{Z}}$ be a function with finite support. We know that the inclusion $F\text{-Zip}_R^{\leq\tau} \hookrightarrow F\text{-Zip}_R$ is a quasi-compact open immersion and factors as a geometric morphism through $F\text{-Zip}_R^{[a,b],S}$ for some finite subset $S \subseteq \mathbb{Z}$ and $a \leq b \in \mathbb{Z}$ (see Remark 6.41). In particular, we see that the pullback of $F\text{-Zip}_R^{\leq\tau}$ along $p^{[a,b],S}$, denoted by $\text{dR-Zip}^{\leq\tau}$, is again geometric by Proposition 9.7 and Theorem 6.23 and quasi-compact open in dR-Zip_R. Since $F\text{-Zip}_R \simeq \operatorname{colim}_\tau F\text{-Zip}_R^{\leq}$, we see that $\text{dR-Zip}_R \simeq \operatorname{colim}_\tau \text{dR-Zip}_R^{\leq\tau}$ and so is locally geometric.

That dR-Zip_R is locally of finite presentation follows analogously from the fact that $F\text{-Zip}_R^{\leq\tau}$ is locally of finite presentation. $\square$

Appendix

A Comparison of G-zips and G-torsors

Throughout, we let k be a field of characteristic $p > 0$.

Let us first construct the quotient stack that will classify F-zips. Let S be a k-scheme. Consider the two closed subschemes $\{0\}$ and $\{\infty\}$ inside $\mathbb{P}^1$ that are naturally isomorphic. Let φ denote the isomorphism of $\{\infty\}$ and $\{0\}$ composed with the Frobenius. We know by [Fer03, Thm. 7.1] that the following pushout exists in the category of $\mathbb{F}_p$-schemes

$$\begin{tikzcd} \{\infty\} \amalg \{0\} \arrow[r, "{(\mathrm{id},\varphi)}"] \arrow[d, hook, "\iota"] & \{\infty\} \arrow[d] \\ \mathbb{P}^1_S \arrow[r] & X_S \end{tikzcd}$$

(note that the Frobenius is integral and thus the morphism from the coproduct is integral and obviously the inclusion of the two points is a closed immersion). The new space X_S is the $\mathbb{P}^1_S$ where we pinch the point 0 and ∞ together. We have a $\mathbb{G}_{\mathrm{m},S}$-action on $\mathbb{P}^1_S$, where we act on the affine line around 0 via multiplication and on the affine line around ∞ by multiplication with the inverse. Since $\mathbb{G}_{\mathrm{m},S} \times_S \mathbb{P}^1_S \cong \mathbb{G}_{\mathrm{m},\mathbb{F}_p} \times_{\mathbb{F}_p} \mathbb{P}^1_S$, we see that the $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{P}^1_S$ as $\mathbb{F}_p$-schemes is the same as the action as S-schemes. Therefore, we get an induced $\mathbb{G}_{\mathrm{m},\mathbb{F}_p}$-action on X.

Let us note that we have the following pushout diagram

(A.1.1)
$$\begin{tikzcd} \mathbb{G}_{\mathrm{m},\mathbb{F}_p} \times_{\mathbb{F}_p} \{\infty\} \amalg \{0\} \arrow[r, "{\mathrm{id} \times (\mathrm{id},\varphi)}"] \arrow[d, hook, "\mathrm{id} \times \iota"] & \mathbb{G}_{\mathrm{m},\mathbb{F}_p} \times_{\mathbb{F}_p} \{\infty\} \arrow[d] \\ \mathbb{G}_{\mathrm{m},\mathbb{F}_p} \times_{\mathbb{F}_p} \mathbb{P}^1_S \arrow[r] & \mathbb{G}_{\mathrm{m},\mathbb{F}_p} \times_{\mathbb{F}_p} X_S \end{tikzcd}$$

which is induced by the $\mathbb{G}_{\mathrm{m}}$-actions. The following diagram with commutative squares shows that indeed $\mathbb{G}_{\mathrm{m},\mathbb{F}_p} \times_{\mathbb{F}_p} X_S$ fulfills the universal property of the pushout above

$$\begin{tikzcd} & \{\infty\} \amalg \{0\} \arrow[r, "{(\mathrm{id},\varphi)}"] \arrow[d, "s"] & \{\infty\} \arrow[d, "s"] \\ \{\infty\} \amalg \{0\} \arrow[r, "s"] \arrow[d, "\iota"] & \mathbb{G}_{\mathrm{m},\mathbb{F}_p} \times_{\mathbb{F}_p} \{\infty\} \amalg \{0\} \arrow[r, "{\mathrm{id} \times (\mathrm{id},\varphi)}"] \arrow[d, "\mathrm{id} \times \iota"] & \mathbb{G}_{\mathrm{m},\mathbb{F}_p} \times_{\mathbb{F}_p} \{\infty\} \\ \mathbb{P}^1_S \arrow[r, "s"] & \mathbb{G}_{\mathrm{m},\mathbb{F}_p} \times_{\mathbb{F}_p} \mathbb{P}^1_S & \end{tikzcd}$$

where the morphism s denotes the natural 0-section.

Now let us set $\mathfrak{X}_S$ as the quotient stack $[X_S/\mathbb{G}_{\mathrm{m},\mathbb{F}_p}]$. We claim that $\mathfrak{X}_S$ is an Artin-stack, which follows from the following lemma and [Sta19, 04TK].

Lemma A.1. *The group action of $\mathbb{G}_{\mathrm{m},\mathbb{F}_p}$ on X_S is smooth.*

Proof. Since the action of $\mathbb{G}_{\mathrm{m},\mathbb{F}_p}$ on X_S is induced by the pushout diagram A.1.1, we see with[50] [Sta19, 08KQ] that the $\mathbb{G}_{\mathrm{m},\mathbb{F}_p}$-action is smooth if and only if the $\mathbb{G}_{\mathrm{m},\mathbb{F}_p}$-action on $\{\infty\}$ and $\mathbb{P}^1_S$ is smooth. The smoothness of the former action is clear. For the latter it is enough to see that the $\mathbb{G}_{\mathrm{m},S}$-action on $\mathbb{A}^1_S$ by multiplication of degree 1 is smooth (the degree -1 case is completely similar).

Indeed, the question is local, so we may assume that $S = \mathrm{Spec}(R)$ is affine. Then the ring map corresponding to the $\mathbb{G}_{\mathrm{m},S}$-action is equivalently given by

$$R[X] \to R[X][Y_1, Y_2, Z]/(Y_1Y_2 - 1, ZY_1 - X).$$

The Jacobi-matrix corresponding to this map is given by

$$\begin{pmatrix} Y_2 & Y_1 & 0 \\ Z & 0 & Y_1 \end{pmatrix}.$$

For any point in $\mathfrak{p} \in \mathrm{Spec}(R[X][Y_1, Y_2, Z]/(Y_1Y_2 - 1, ZY_1 - X))$ we have that this matrix has full rank, as $Y_1Y_2 - 1 \in \mathfrak{p}$. Therefore, we see that indeed the $\mathbb{G}_{\mathrm{m},S}$-action $\mathbb{A}^1_S$ via multiplication of degree 1 is smooth of relative dimension 1 concluding the proof. $\square$

We will start by showing that vector bundles over $\mathfrak{X}$ are the same as F-zips. Then the comparison of G-zips and G-torsors on $\mathfrak{X}$ is a Tannaka duality like statement.

Theorem A.2. *Let S be a k-scheme and let $n \in \mathbb{N}$. There is an equivalence*

$$\mathrm{VB}_n(\mathfrak{X}_S) \simeq \mathrm{cl}\, F\text{-}\mathrm{Zip}^n(S)$$

of categories.

Proof. A vector bundle on $\mathfrak{X}_S$ is by descent a $\mathbb{G}_{\mathrm{m},\mathbb{F}_p}$-equivariant vector bundle on X_S. Since X_S is the coequalizer of $\{0\}$, where one of the morphisms is given by the identity and the other sends $\{0\}$ to $\{\infty\}$ and twist by Frobenius, we see that a vector bundle on X is given by a finite locally free $\mathbb{P}^1_S$-modules with an isomorphism after pullback to $\{\infty\}$ respectively $\{0\}$ with Frobenius twist (see [TT16, Cor. 6.5]). A vector bundle of rank n on $\mathbb{P}^1_S$ is given by a tuple (V, W) of vector bundles of rank n on $\mathbb{A}^1_S$ such that $V_{|D(0)} \cong W_{|D(\infty)}$. Let (V, W) be such a tuple. Further assume that we have an isomorphism $W^{(1)}_{|\{\infty\}} \to V_{|\{0\}}$ this defines a vector bundle on X_S. The $\mathbb{G}_{\mathrm{m},\mathbb{F}_p}$-equivariance induces a grading on V and W (this is for example explained in [Sta19, 03LE]). Since $\mathbb{G}_{\mathrm{m},\mathbb{F}_p}$ acts on the affine line around $\{0\}$ by multiplying in degree 1 and on the affine line around $\{\infty\}$ by multiplying in degree -1, we see that per definition of the grading the corresponding endomorphisms of V and W seen as k-vector spaces are morphisms of graded vector spaces in degree 1 resp. -1. This construction gives an ascending chain

$$\cdots \to V_{i-1} \to V_i \to V_{i+1} \to \ldots$$

[50] Locally the pushout of schemes is given by the fiber product of the rings corresponding to affine opens, as one sees in the construction (see for example the proof of the existence given in [Sta19, 0E25]). So in particular, we can apply the reference.

and a descending chain

$$\cdots \to W^{i+1} \to W^i \to W^{i-1} \to \dots .$$

The pullback to $\{0\}$ gives the direct sum of graded pieces and considering the Frobenius twist, we see that this isomorphism is the same as the datum of a family of isomorphisms $\varphi\colon (\mathrm{gr}^i_W)^{(1)} \to \mathrm{gr}^i_V$ for each i, also since the pullback to $\{0\}$ also has to be a vector bundle, we see that each graded piece has to be finite locally free (since the direct sum of all has to be finite locally free of rank n). This in particular shows that the filtrations stabilize, i.e. there only finitely many nonzero graded pieces. The pullback of these filtrations to $D(0)$ resp. $D(\infty)$ gives the underlying module of the filtrations, which have to be isomorphic, i.e. $V_\bullet$ and $W^\bullet$ define an ascending resp. descending filtration on the same module.

Putting all these data together, we see that the category of vector bundles on $\mathfrak{X}_S$ of rank n is equivalent to the category of F-zips over S of rank n.

□

Remark A.3. The above proposition shows in particular, that the functor

$$\mathrm{VB}(\mathfrak{X})\colon S \mapsto \mathrm{VB}(\mathfrak{X}_S)$$

is a sheaf for the fppf topology and even and Artin stack.

Corollary A.4. *There is an equivalence of Artin-stacks*

$$\mathrm{VB}(\mathfrak{X}) \simeq \mathrm{cl}\, F\text{-}\mathrm{Zip}_k\,.$$

Further, for any scheme S, we have a monoidal equivalence of symmetric monoidal categories $\mathrm{VB}(\mathfrak{X}_S) \simeq \mathrm{cl}\, F\text{-}\mathrm{Zip}_k(S)$

Proof. This is clear with Theorem A.2. Note that the proof of Theorem A.2 shows this equivalence respects the symmetric monoidal structures. □

Corollary A.5. *Let G be a linear algebraic group over k such that its identity component is reductive. We have an isomorphism of Artin-stacks*

$$G\text{-}\mathrm{Tors}(\mathfrak{X}) \cong G\text{-}\mathrm{zip}_k.$$

Proof. Using Theorem A.2, we see with [PWZ15, Thm. 7.13] that it is enough to show that G-Tors($\mathfrak{X}$) is equivalent to the stack of fiber functors from the symmetric monoidal category of G representations to the symmetric monoidal category $\mathrm{VB}(\mathfrak{X})$, which we denote by $\mathrm{Hom}^{\otimes}(\mathrm{Rep}(G), \mathrm{VB}(\mathfrak{X}))$.[51]

Let S be a k-scheme and $\mathfrak{X}_S^\bullet$ denote the Čech nerve of $\mathfrak{X}_S$ (which by construction is termwise given by a scheme). Since BG is a sheaf for the fppf topology, we get by

[51]The associated pre-stack is given by the assignment $S \mapsto \mathrm{Hom}(\mathrm{Rep}(G), \mathrm{VB}(\mathfrak{X}_S))$. That this is a stack, follows from [SR72, III.3.2.1.2]. In general, for any symmetric monoidal categories A and B, we denote with $\mathrm{Hom}^{\otimes}(A, B)$ the groupoid of fiber functors from A to B.

definition $G\text{-Tors}(\mathfrak{X}_S) = \mathrm{Hom}_{P((\mathrm{Sch}/k))}(\mathfrak{X}_S, \mathrm{B}G) = \lim_{n\in\mathbb{N}} \mathrm{Hom}_{P((\mathrm{Sch}/k))}(\mathfrak{X}_S^n, \mathrm{B}G)$, by Tannaka duality (see [Zie15, Thm. 2.3]) we have

$$\lim_{n\in\mathbb{N}} \mathrm{Hom}_{P((\mathrm{Sch}/k))}(\mathfrak{X}_S^n, \mathrm{B}G) = \lim_{n\in\mathbb{N}} \mathrm{Hom}^{\otimes}(\mathrm{Rep}(G), \mathrm{VB}(\mathfrak{X}_S^n))$$

(here we embed the categories involved, which are $(2,1)$-categories, naturally into the world of ∞-categories (for example via the Duskin-Nerve which is explained in [Lur21, §I.2.3]), then the limit above is the usual limit in the ∞-categorical sense). Since the pre-stack of fiber functors $\mathrm{Hom}^{\otimes}(\mathrm{Rep}(G), \mathrm{VB}(\mathfrak{X}))$ satisfies fpqc decent (see [SR72, III.3.2.1.2], infact one uses this reference together with [Zie15, Thm. 2.1] to see that the stack of fiber functors defines a gerbe, which is needed in [Zie15, Thm. 2.3]), we have

$$\lim_{n\in\mathbb{N}} \mathrm{Hom}^{\otimes}(\mathrm{Rep}(G), \mathrm{VB}(\mathfrak{X}_S^n)) = \mathrm{Hom}^{\otimes}(\mathrm{Rep}(G), \mathrm{VB}(\mathfrak{X}_S))$$

concluding the proof. $\square$

B Perfect complexes on the pinched projective space

In this section, we want to understand the perfect complexes on the pinched projective space, i.e. the ∞-category $\mathrm{Perf}(\mathfrak{X}_S)$ for any scheme S (See Appendix A for the notation).

As explained in [HL13, Prop. 4.1.1] quasi-coherent sheaves on $\left[\mathbb{A}^1_S/\,\mathbb{G}_{\mathrm{m},S}\right]$ are the same as $\mathbb{Z}$-indexed diagrams of quasi-coherent $\mathcal{O}_S$-modules, so a chain of morphisms of $\mathcal{O}_S$-modules

$$\dots \to \mathcal{F}_i \to \mathcal{F}_{i+1} \to \mathcal{F}_{i+2} \to \dots$$

(for vector bundles we showed the computation behind it in Theorem A.2). Equivalently the category of quasi-coherent sheaves on $\left[\mathbb{A}^1_S/\,\mathbb{G}_{\mathrm{m},S}\right]$ is given by the category of graded $\mathcal{O}_S$-modules together with an endomorphism of degree 1 (this endomorphism is induced by multiplication with X). This gives a description of the category of chain complexes and so

$$D_{\mathrm{qc}}(\left[\mathbb{A}^1_S/\,\mathbb{G}_{\mathrm{m},S}\right]) \simeq \mathrm{Fun}(\mathbb{Z}, D_{\mathrm{qc}}(S))$$

Now let us endow the abelian category of chain complexes of quasi-coherent modules over $\left[\mathbb{A}^1_S/\,\mathbb{G}_{\mathrm{m},S}\right]$ with the usual model structure and $\mathrm{Fun}(\mathbb{Z}, \mathrm{Ch}(\mathrm{QCoh}(S)))$ with the pointwise model structure. The natural identification of the categories explained above induces a Quillen equivalence and therefore an equivalence of ∞-categories

$$\mathcal{D}_{\mathrm{qc}}(\left[\mathbb{A}^1_S/\,\mathbb{G}_{\mathrm{m},S}\right]) \simeq \mathrm{Fun}(\mathbb{Z}, \mathcal{D}_{\mathrm{qc}}(S)).$$

In fact, this equivalence can be upgraded naturally to a symmetric monoidal equivalence.

We will compute $\mathrm{Perf}(\mathfrak{X}_S)$ via descent. For this, we want to understand the pullback of perfect complexes along $\left[\{0\}/\,\mathbb{G}_{\mathrm{m},\mathbb{F}_p}\right] \to \left[\mathbb{A}^1_S/\,\mathbb{G}_{\mathrm{m},\mathbb{F}_p}\right]$. We will need the following proposition to ease computation.

Proposition B.1. *Let X be a scheme and $i\colon Z \hookrightarrow X$ be a closed immersion. Then the pushforward $i_*\colon \mathcal{D}_{\mathrm{qc}}(Z) \to \mathcal{D}_{\mathrm{qc}}(X)$ is fully faithful.*

Proof. Passing to an affine open cover of X, we may assume that $X = \operatorname{Spec}(A)$ is affine (and hence $Z = \operatorname{Spec}(A/I)$ is also affine). As a fully faithful functor in Cat_∞ is equivalently a monomorphism, we see that a limit of fully faithful functors is fully faithful (since Cat_∞ has small limits, as it is presentable, see Remark 4.51). Further, since $\mathcal{D}(A)$ is right-complete (see [Lur17, Prop. 1.3.5.21]), we may write it as a limit of $\mathcal{D}(A)_{\geq n}$ and replace $\mathcal{D}(A)$ and $\mathcal{D}(A/I)$ by $\mathcal{D}(A)^+$ and $\mathcal{D}(A/I)^+$ respectively (the full subcategories of *right* bounded objects).

Using [TT90, Lem. 1.9.5], we see that for any $M \in \mathcal{D}(A/I)^+$, we have that i_*M is quasi-isomorphic to a complex $F^\bullet$, where F^i is an A-module that is annihilated by I. Let us denote the full subcategory spanned by the essential image of i_* by $\mathcal{D}_I(A)^+$. Let $M, N \in \mathcal{D}(A/I)^+$ and let us look at $\operatorname{Hom}_{\mathcal{D}_I(A)^+}(i_*M, i_*N)$. Let $J^\bullet$ be a fibrant resolution of i_*N, i.e. an injective chain complex $J^\bullet$ of A-modules that are annihilated by I and a quasi-isomorphsim $J^\bullet \xrightarrow{\sim} i_*N$. Then, we have

$$\operatorname{Hom}_{\mathcal{D}_I(A)^+}(i_*M, i_*N) \simeq \operatorname{Hom}_{\mathcal{D}(A)^+}(i_*M, i_*N) \simeq \operatorname{Hom}_{\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^+(A))}(i_*M, J^\bullet).$$

Since per definition the restriction $(A/I\text{-Mod}) \to (A\text{-Mod})$ is fully faithful with essential image given by the A-modules that are annihilated by I, we see that the same holds for the restriction $\iota\colon \mathrm{Ch}^+(A/I) \to \mathrm{Ch}^+(A)$. The fully faithfulness stays true if we attach a dg-structure on chain complexes. So, we may view ι as a monomorphism of dg-categories. The category of dg-categories admits a combinatorial model structure such that the dg-nerve functor N_{dg} from dg-categories to simplicial sets with the Joyal model structure becomes right Quillen (see [Lur17, Prop. 1.3.1.19, 1.3.1.20]). Thus, N_{dg} induces a limit preserving functor from the ∞-category of dg-categories to Cat_∞. In particular, $\mathrm{N}_{\mathrm{dg}}(\iota)$ is a monomorphism. Therefore, we can write $J^\bullet \simeq \iota(\widetilde{N})$, where $\widetilde{N} \in \mathrm{Ch}^+(A/I)$ is an injective resolution of N and we have equivalences

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{D}_I(A)^+}(i_*M, i_*N) &\simeq \operatorname{Hom}_{\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^+(A))}(\iota(M), J^\bullet) \\
&\simeq \operatorname{Hom}_{\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^+(A))}(\iota(M), \iota(\widetilde{N})) \\
&\simeq \operatorname{Hom}_{\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^+(A/I))}(M, N) \simeq \operatorname{Hom}_{\mathcal{D}(A/I)^+}(M, N).
\end{aligned}$$

Hence, i_* is fully faithful as desired. □

Since we are interested in morphisms between algebraic stacks, let us show how to generalize this result to fit into our picture.

Corollary B.2. *Let S be a scheme. Further, let G be a group scheme over S and let X, Z be S-schemes with a G-action denoted by a_X and a_Z respectively. Let $i\colon Z \hookrightarrow X$ be a G-equivariant closed immersion such that the diagram*

$$\begin{tikzcd}
G \times_S Z \arrow[r, "\mathrm{id} \times i"] \arrow[d, "a_Z"] & G \times_S X \arrow[d, "a_X"] \\
Z \arrow[r, hook, "i"] & X
\end{tikzcd}$$

is cartesian. Then the restriction functor $i_*\colon \mathcal{D}_{\mathrm{qc}}(Z) \to \mathcal{D}_{\mathrm{qc}}(X)$ is fully faithful.

Proof. Using the Barr-resolution of X and Z, we get a commutative diagram like follows

$$\begin{tikzcd}
\dots \arrow[r, shift left=1.5] \arrow[r, shift left=0.5] \arrow[r, shift right=0.5] \arrow[d] & G \times_S Z \arrow[r, shift left, "p"] \arrow[r, shift right, "a_Z"'] \arrow[d, "\mathrm{id} \times i"] & Z \arrow[d, "i"] \\
\dots \arrow[r, shift left=1.5] \arrow[r, shift left=0.5] \arrow[r, shift right=0.5] & G \times_S X \arrow[r, shift left, "p"] \arrow[r, shift right, "a_X"'] & X,
\end{tikzcd}$$

where the vertical arrows are all closed immersion. As monomorphism of ∞-categories are preserved by limits, we conclude with Proposition B.1. $\square$

Proposition B.3. *Let $\mathbb{G}_{\mathrm{m},S}$ act on $\mathbb{A}^1_S$ by multiplication of degree 1 (resp. -1) and trivially on the closed subscheme $\{0\} \subseteq \mathbb{A}^1_S$. We denote with $f\colon \left[\{0\}/\,\mathbb{G}_{\mathrm{m},S}\right] \to \left[\mathbb{A}^1_S/\,\mathbb{G}_{\mathrm{m},S}\right]$ the naturally induced morphism. Further, let $M \in \mathcal{D}_{\mathrm{qc}}(\left[\mathbb{A}^1_S/\,\mathbb{G}_{\mathrm{m},S}\right])$. Then we have*

$$Lf^*M \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i M,$$

where we consider M as an element in $\mathrm{Fun}(\mathbb{Z}, \mathcal{D}_{\mathrm{qc}}(S))$.

Proof. By Corollary B.2 it is enough to show that after restricting to $\mathcal{D}_{\mathrm{qc}}(\left[\mathbb{A}^1_S/\,\mathbb{G}_{\mathrm{m},S}\right])$, we have $Lf^*M \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i M$. In particular, to compute Lf^*, we may resolve $\mathcal{O}_S$ as a K-flat complex in $\mathcal{D}_{\mathrm{qc}}(\left[\mathbb{A}^1_S/\,\mathbb{G}_{\mathrm{m},S}\right])$, which is straightforward as we are going to see.

We will give a proof in the case that $\mathbb{G}_{\mathrm{m},S}$ acts by multiplication of degree 1. The degree -1 case is completely analogous and we will note the places where the proof changes.

Important for us is that as explained above a quasi-coherent $\mathbb{G}_{\mathrm{m},S}$-equivariant $\mathcal{O}_{\mathbb{A}^1_S}$-module $\mathcal{F}$ is equivalently a graded $\mathcal{O}_S$-module $\mathcal{F} = \bigoplus_{i\in\mathbb{Z}} \mathcal{F}^i$ together with an endomorphism $\mathcal{F} \to \mathcal{F}$ of degree 1 (resp. -1) that is induced by multiplication with X.

The category of quasi-coherent modules over $\left[\{0\}/\,\mathbb{G}_{\mathrm{m},S}\right]$ is analogously equivalent to the category of quasi-coherent graded $\mathcal{O}_S$-modules. As f is equivariant we get a pullback functor f^* from the category of cochain complexes of graded $\mathcal{O}_S$-modules with endomorphism of degree 1 (resp. -1) to the category of cochain complexes of graded $\mathcal{O}_S$-modules. Let us write M as $(M_\bullet, \partial_\bullet) = (\bigoplus_{i\in\mathbb{Z}} M^i_\bullet, \bigoplus_{i\in\mathbb{Z}} \partial^i_\bullet)$ (a chain complex of graded $\mathcal{O}_S$-modules) together with an endomorphism $X\colon M \to M$ that is induced by multiplication with X. The complex f^*M is given by $M \otimes_{\mathcal{O}_{\mathbb{A}^1_S}} \mathcal{O}_S$, where we identify $\mathcal{O}_S$ with $\mathcal{O}_S[X]/(X)$ (which endows $\mathcal{O}_S$ with a trivial grading and degree 1 (resp. -1) endomorphism given by 0).

To compute Lf^*M, it is enough to find a cochain complex $P \in \mathcal{D}_{\mathrm{qc}}(\left[\mathbb{A}^1_S/\,\mathbb{G}_{\mathrm{m},S}\right])$ with a quasi-isomorphism $P \xrightarrow{\sim} \mathcal{O}_S$ in $\mathcal{D}_{\mathrm{qc}}(\left[\mathbb{A}^1_S/\,\mathbb{G}_{\mathrm{m},S}\right])$, such that the functor $P \otimes_{\mathcal{O}_{\mathbb{A}^1_S}} -$ in the category of cochain complexes of $\mathbb{G}_{\mathrm{m},S}$-equivariant $\mathcal{O}_{\mathbb{A}^1_S}$-modules is exact. Then Lf^*M is equivalent to $M \otimes_{\mathcal{O}_{\mathbb{A}^1_S}} P$. We claim that P is naturally given by the Koszul complex of $\mathcal{O}_S$.

Indeed, a flat resolution of $\mathcal{O}_S$ is given by the complex $P^\bullet$ that has zero everywhere except in degrees -1 and 0, where it is given by the morphism

$$\mathcal{O}_S[X] \xrightarrow{\cdot X} \mathcal{O}_S[X].$$

Now $\mathcal{O}_S[X]$ is endowed with the obvious grading together with an endomorphism of degree 1 (resp. -1) by multiplication with X. Then $P^\bullet$ becomes an element in $\mathcal{D}_{\mathrm{qc}}([\mathbb{A}^1_S/\mathbb{G}_{\mathrm{m},S}])$ by shifting the grading of P^{-1} by -1 (resp. 1). Note that the functor $P^\bullet \otimes_{\mathcal{O}_{\mathbb{A}^1_S}} -$ is exact and therefore we have

$$Lf^*M \simeq P^\bullet \otimes_{\mathcal{O}_{\mathbb{A}^1_S}} M.$$

Now let us explicitly compute Lf^*M. By definition of the tensor product of chain complexes, we see with the above that $M \otimes_{\mathcal{O}_{\mathbb{A}^1_S}} P^\bullet$ is equivalent to the complex

$$(M_{n+1} \oplus M_n, \iota_n)_{n\in\mathbb{Z}},$$

where the ι_n differentials are given by

$$\begin{pmatrix} \partial_{n+1} & 0 \\ (-1)^n X & \partial_n \end{pmatrix}.$$

The induced grading is given by $(M_{n+1} \oplus M_n)^i = M^{i-1}_{n+1} \oplus M^i_n$ (resp. $(M_{n+1} \oplus M_n)^i = M^{i+1}_{n+1} \oplus M^i_n$).

Now let us analyze the graded pieces $\mathrm{gr}^i M$. As explained above, we can also consider

$$\dots \to (M^{i-1}_\bullet, \partial^{-1}_\bullet) \xrightarrow{\cdot X} (M^i_\bullet, \partial^i_\bullet) \to \dots$$

as a filtration in $\mathcal{D}_{\mathrm{qc}}(S)$. Let us calculate $\mathrm{cofib}((M^{i-1}_\bullet, \partial^{i-1}_\bullet) \xrightarrow{\cdot X} (M^i_\bullet, \partial^i_\bullet))$. We can do so, by calculating the cone of multiplication with X, which is given by

$$(M^{i-1}_{n+1} \oplus M^i_n)_{n\in\mathbb{Z}}, \iota^i_n),$$

where the differentials ι^i_n are, up to equivalence given by

$$\begin{pmatrix} \partial^{i-1}_{n+1} & 0 \\ (-1)^n X & \partial^i_n \end{pmatrix}$$

(For $\mathrm{cofib}((M^{i+1}_\bullet, \partial^{i-1}_\bullet) \xrightarrow{\cdot X} (M^i_\bullet, \partial^i_\bullet))$ this is analogous just changes the indices).

Finally, these constructions imply that $Lf^*M \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i M$ in $\mathcal{D}_{\mathrm{qc}}([\mathbb{A}^1_S/\mathbb{G}_{\mathrm{m},S}])$. $\square$

We are finally ready to compute the perfect complexes on $\mathfrak{X}_S$ for any scheme S. We will describe it as a full subcategory of

$$\begin{aligned}\mathcal{C}(S) \coloneqq (\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{D}_{\mathrm{qc}}(S)) \times_{\mathrm{colim}, \mathcal{D}_{\mathrm{qc}}(S), \mathrm{colim}} \mathrm{Fun}(\mathbb{Z}, \mathcal{D}_{\mathrm{qc}}(S))) \\ \times_{(\bigoplus (\mathrm{gr}^i)^{(1)}, \bigoplus \mathrm{gr}^i), \mathcal{D}_{\mathrm{qc}}(S) \times \mathcal{D}_{\mathrm{qc}}(S), \mathrm{dom} \times \mathrm{codom}} \mathrm{Fun}(\Delta^1, \mathcal{D}_{\mathrm{qc}}(S)).\end{aligned}$$

Theorem B.4. *For any scheme S the ∞-category $\mathrm{Perf}(\mathfrak{X}_S)$ is equivalent to the full subcategory in $\mathcal{C}(S)$ of tuples $(C^\bullet, D_\bullet, \phi, \varphi)$, where*

- $C^\bullet \in \mathrm{Fun}(\mathbb{Z}^{op}, \mathcal{D}_{\mathrm{qc}}(S))$*, such that each* $\mathrm{gr}^i C$ *and* $\bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i C$ *are perfect,*
- $D_\bullet \in \mathrm{Fun}(\mathbb{Z}, \mathcal{D}_{\mathrm{qc}}(S))$*, such that each* $\mathrm{gr}^i D$ *and* $\bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i D$ *are perfect,*
- $\phi\colon \mathrm{colim}_{\mathbb{Z}^{\mathrm{op}}} C^\bullet \xrightarrow{\sim} \mathrm{colim}_{\mathbb{Z}} D^\bullet$ *is an equivalence, and*
- *and* $\varphi\colon \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i C^{(1)} \xrightarrow{\sim} \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i D$ *is an equivalence.*

Proof. By Proposition 4.48, we can use the Barr-resolution of $\mathfrak{X}_S$ to see that

(B.1.1) $$\mathrm{Perf}(\mathfrak{X}_S) \simeq \lim(\mathrm{Perf}(X_S) \rightrightarrows \mathrm{Perf}(X_S \times_{\mathbb{F}_p} \mathbb{G}_{\mathrm{m},\mathbb{F}_p}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots).$$

We can again use Proposition 4.48 to see that we have a limit diagram of the form

$$\begin{array}{ccc} \mathrm{Perf}(X_S) & \longrightarrow & \mathrm{Perf}(\mathbb{P}^1_S) \\ \downarrow & & \downarrow \\ \mathrm{Perf}(\{\infty\}) & \longrightarrow & \mathrm{Perf}(\{\infty\}) \times \mathrm{Perf}(\{0\}). \end{array}$$

Since the $\mathbb{G}_{\mathrm{m},\mathbb{F}_p}$-action on X_S is induced by the pushout of the $\mathbb{G}_{\mathrm{m},\mathbb{F}_p}$-actions on $\mathbb{P}^1_S$, $\{\infty\}$ and $\{\infty\} \amalg \{0\}$ (see (A.1.1)), we see that an object $X \in \mathrm{Perf}(\mathfrak{X}_S)$ corresponds to a tuple (M, N, φ), where

- $M \in \mathrm{Perf}(\big[\mathbb{P}^1_S/\,\mathbb{G}_{\mathrm{m},\mathbb{F}_p}\big])$,
- $N \in \mathrm{Perf}(\big[\{\infty\}/\,\mathbb{G}_{\mathrm{m},\mathbb{F}_p}\big])$, and
- φ is an equivalence of the images of M and N in
$$\mathrm{Perf}(\big[\{\infty\}/\,\mathbb{G}_{\mathrm{m},\mathbb{F}_p}\big]) \times \mathrm{Perf}(\big[\{0\}/\,\mathbb{G}_{\mathrm{m},\mathbb{F}_p}\big]).$$

Using the standard cover of $\mathbb{P}^1_S$ by affine lines and the discussion in the beginning, we see that $M \in \mathrm{Perf}(\big[\mathbb{P}^1_S/\,\mathbb{G}_{\mathrm{m},\mathbb{F}_p}\big])$ is equivalently given by a tuple $(C^\bullet, D_\bullet, \phi)$, where $C^\bullet \in \mathrm{Fun}(\mathbb{Z}^{op}, \mathcal{D}_{\mathrm{qc}}(S))$ is perfect, $D_\bullet \in \mathrm{Fun}(\mathbb{Z}, \mathcal{D}_{\mathrm{qc}}(S))$ is perfect and an equivalence $\phi\colon \mathrm{colim}_{\mathbb{Z}^{\mathrm{op}}} C^\bullet \xrightarrow{\sim} \mathrm{colim}_{\mathbb{Z}} D^\bullet$.[52]

Further, (as explained in the proof of Proposition B.3) $\mathrm{Perf}(\big[\{\infty\}/\,\mathbb{G}_{\mathrm{m},\mathbb{F}_p}\big])$ consists of perfect chain complexes of graded $\mathcal{O}_S$-modules. Also, we have seen in Proposition B.3 that the image of $(C^\bullet, D_\bullet)$ in $\mathrm{Perf}(\big[\{\infty\}/\,\mathbb{G}_{\mathrm{m},\mathbb{F}_p}\big]) \times \mathrm{Perf}(\big[\{0\}/\,\mathbb{G}_{\mathrm{m},\mathbb{F}_p}\big])$ is equivalent to $(\bigoplus_i \mathrm{gr}^i C, \bigoplus_i \mathrm{gr}^i D)$.

Lastly, we want to remark that by [GP18, Prop. 2.45] an element in $\mathrm{Fun}(\mathbb{Z}^{op}, \mathcal{D}_{\mathrm{qc}}(S))$ (resp. $\mathrm{Fun}(\mathbb{Z}, \mathcal{D}_{\mathrm{qc}}(S))$) is perfect if and only if each graded piece perfect.

Putting all of this together, we get the desired description of $\mathrm{Perf}(\mathfrak{X}_S)$ as a full subcategory of $\mathcal{C}(S)$. $\square$

[52] By construction $\mathbb{P}^1_S$ is the pushout of the maps $\mathrm{Spec}(\mathcal{O}_S[X, X^{-1}]) \xrightarrow{x \mapsto x^{-1}} \mathrm{Spec}(\mathcal{O}_S[X])$ and $\mathrm{Spec}(\mathcal{O}_S[X, X^{-1}]) \xrightarrow{x \mapsto x} \mathrm{Spec}(\mathcal{O}_S[X])$, where the maps are given on T-valued points. So again, the description of $\mathrm{Perf}(\mathbb{P}^1_S)$ follows from Proposition 4.48 and the fact that the derived pullback along open immersions is given by the usual pullback.

It is clear by construction that F-Zip(S) is a full subcategory of $\mathrm{Perf}(\mathfrak{X}_S)$. But let us show that we have an equivalence. This will follow immediately if we can show that the filtrations associated to an element in $\mathrm{Perf}(\mathfrak{X}_S)$ are locally bounded.

Lemma B.5. *Let S be a scheme and let $F \in \mathrm{Fun}(\mathbb{Z}, \mathcal{D}_{\mathrm{qc}}(S))$ be an ascending filtration such that $\mathrm{gr}^i F$ and $\bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i F$ are perfect $\mathcal{O}_S$-modules. Then F is locally bounded and perfect.*

The assertion stays true if we replace $\mathbb{Z}$ by $\mathbb{Z}^{\mathrm{op}}$.

Proof. As this is a local question, we may assume that $S = \mathrm{Spec}(A)$ is affine. Fiberwise the question is clear, since a perfect complex over a field is quasi-isomorphic to a finite direct sum of finite dimensional vector spaces sitting in one degree. For every point $s \in S$, we can find an open neighbourhood U_s around s such that only finitely many $\mathrm{gr}^i F$ are nonzero (see [Sta19, 0BCD]). As S is quasi-compact, we conclude the lemma[53]. $\square$

Corollary B.6. *Let R be an $\mathbb{F}_p$-algebra and S an R-scheme. Then we have*

$$F\text{-}\mathrm{Zip}_R(S) \simeq \mathrm{Perf}(\mathfrak{X}_S).$$

Proof. This follows immediately combining Theorem B.4, Lemma B.5 and that finite direct sums in $\mathcal{D}_{\mathrm{qc}}(S)$ are the same as finite products as it is stable[54]. $\square$

C Miscellaneous from Algebraic Geometry II [GW]

This section is dedicated for propositions of the upcoming book of Görtz-Wedhorn that are needed in this thesis. Everything is this section is taken out of [GW] made available for us by the authors and no originality is claimed. We state an prove some of the results of [GW] for completion, as it is not publicly available.

Lemma C.1 ([GW])**.** *Let R be a ring and let $d\colon M \to N$ be an R-linear map of finitely generated proejctive R-modules. Then the following assertions are equivalent.*

(i) $\mathrm{Coker}(d)$ *is a finitely generated projective R-module.*

(ii) $\mathrm{Im}(d)$ *is a direct summand of N and for every R-module Q one has* $\mathrm{Im}(d) \otimes_R Q = \mathrm{Im}(d \otimes \mathrm{id}_Q)$.

(iii) $\mathrm{Im}(d)$ *is a finitely generated projective R-module and for every $s \in \mathrm{Spec}(R)$ one has* $\mathrm{Im}(d) \otimes_R \kappa(s) = \mathrm{Im}(d \otimes \mathrm{id}_{\kappa(s)})$.

(vi) $\mathrm{Ker}(d)$ *is a direct summand of M and for every R-module Q one has* $\mathrm{Ker}(d) \otimes_R Q = \mathrm{Ker}(d \otimes \mathrm{id}_Q)$.

[53] Inductively any bounded filtration with perfect graded pieces is perfect.

[54] The equivalence between finite direct sums and products shows that for an element $(C^\bullet, D_\bullet, \phi, \varphi) \in \mathrm{Perf}(\mathfrak{X}_S)$ the equivalence $\varphi\colon \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i C^{(1)} \xrightarrow{\sim} \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i D$ is equivalently given by equivalences $\varphi_i\colon \mathrm{gr}^i C^{(1)} \xrightarrow{\sim} \mathrm{gr}^i D$.

(v) *For every* $s \in \operatorname{Spec}(R)$ *one has* $\operatorname{Ker}(d) \otimes_R \kappa(s) = \operatorname{Ker}(d \otimes \mathrm{id}_{\kappa(s)})$.

Proof. If (i) holds, the exact sequence $0 \to \operatorname{Im}(d) \to N \to \operatorname{Coker}(d) \to 0$ splits and stays exact after tensoring with any R-module Q (see [Sta19, 00HL]). As $\operatorname{Coker}(d \otimes \mathrm{id}_Q) = \operatorname{Coker}(d) \otimes_R Q$, this implies (ii). Clearly (ii) $\Rightarrow$ (iii). If $\operatorname{Im}(d)$ is projective, then the exact sequence $0 \to \operatorname{Ker}(d) \to M \to \operatorname{Im}(d) \to 0$ show with the previous argument that (ii) implies (iv) and (iii) implies (v). Clearly, (iv) implies (v). It remains to show that (v) implies (i).

We will show this by recalling another lemma of the upcoming book of Görtz-Wedhorn [GW]. Namely, for $s \in \operatorname{Spec}(R)$ the map $\operatorname{Ker}(d) \otimes_R \kappa(s) \to \operatorname{Ker}(d \otimes \mathrm{id}_{\kappa(s)})$ is surjective if and only if there is an $f \in R$ with $s \in D(f)$ such that $\operatorname{Coker}(d)_f$ is finite projective.

Assume that $\operatorname{Ker}(d) \otimes_R \kappa(s) \to \operatorname{Ker}(d \otimes \mathrm{id}_{\kappa(s)})$ is surjective. As $\operatorname{Coker}(d)$ is of finite presentation, it suffices to show that the localization of $\operatorname{Coker}(d)$ in s is a free module (see [GW10, Prop. 7.27]). Therefore, we may assume that R is a local ring and s is the closed point in $\operatorname{Spec}(R)$. Then $\operatorname{Ker}(d) \to \operatorname{Ker}(d \otimes \mathrm{id}_{\kappa(s)})$ is surjective and M, N are free of ranks m, n, say. Set $\kappa := \kappa(s)$ and $r := \operatorname{rk}(d \otimes \mathrm{id}_\kappa)$. We will show that $\operatorname{Im}(d)$ is a direct summand of N of rank r. Let $\bar{d}\colon M/\operatorname{Ker}(d) \to N$ be the induced injective map. It suffices to show that $\bar{d} \otimes \mathrm{id}_\kappa$ is injective (see [GW10, Prop. 8.10]).

Choose $x_1, \dots, x_r \in \operatorname{Ker}(d)$ the map to a basis of $\operatorname{Ker}(d \otimes \mathrm{id}_\kappa)$. We also choose $x_{r+1}, \dots, x_m$ in M whose image in $M \otimes_R \kappa$ yields a basis of $M \otimes_R \kappa/\operatorname{Ker}(d \otimes \mathrm{id}_\kappa)$. Then $x_1, \dots, x_m$ generate M by Nakayama's lemma and hence form a basis because M if free of rank $m = \dim_\kappa(M \otimes_R \kappa)$. As $x_1, \dots, x_r \in \operatorname{Ker}(d)$, for $i > r$, the images $\bar{x}_i$ of x_i in $M/\operatorname{Ker}(d)$ generate this module. Now $\bar{d} \otimes \mathrm{id}_\kappa$ maps $\bar{x}_i \otimes 1$ to $d \otimes \mathrm{id}_\kappa(x_i \otimes 1)$ for $i > r$ and these elements are linearly independent. This shows that $\bar{d} \otimes \mathrm{id}_\kappa$ is injective.

If $\operatorname{Im}(d)$ has a complement L in N, then $\operatorname{Coker}(d) \cong L$ is finite projective. This shows the necessity of the above assertion and the implication (v) to (i).

The sufficiency of the above assertion follows from the equivalence of the assertions in the lemma applied to the map $d \otimes \mathrm{id}_{R_f}$. □

Lemma C.2 ([GW]). *Let S be a scheme, let E be a perfect complex in $D(S)$ of Tor-amplitude $[a,b]$, and let $I \subseteq [a,b]$ be an interval containing a or b. Fix a map $r\colon I \to \mathbb{N}_0$, $i \mapsto r_i$. Then there exists a unique locally closed subscheme $j\colon Z = Z_r \hookrightarrow S$ such that a morphism $f\colon T \to S$ factors through Z if and only if for all morphisms $g\colon T' \to T$ the $\mathcal{O}_{T'}$-module $\pi_i(L(f \circ g)^*E)$ finite locally free of rank r_i for all $i \in I$. Moreover,*

(1) *the immersion $j\colon Z \hookrightarrow S$ is of finite presentation,*

(2) *as a set one has*

$$Z = \{s \in S \mid \dim_{\kappa(s)} \pi_i(E \otimes^L_{\mathcal{O}_S} \kappa(s)) = r_i \textit{ for all } i \in I\},$$

(3) *if $f\colon T \to S$ factors as $T \xrightarrow{\bar{f}} Z \xrightarrow{j} S$, then $\pi_i(Lf^*E \otimes^L_{\mathcal{O}_T} \mathcal{G}) = \bar{f}^*\pi_i(j^*E) \otimes_{\mathcal{O}_T} \mathcal{G}$ for all $i \in I$ and for all quasi-coherent $\mathcal{O}_T$-modules $\mathcal{G}$.*

Proof. Uniqueness follows from the fact that Z is characterised by a universal property. Thus, we may work locally and assume that $S = \operatorname{Spec}(R)$ is affine. In particular, we may assume that E is represented by a complex of finite free R-modules concentrated in degree $[a,b]$. Let us denote the rank of each E_i with n_i and the differentials $d_a\colon E_a \to E_{a-1}$.

The interval I is either of the form $[a,b']$ or $[a',b]$ for some $a \leq b'$ resp. $a' \leq b$. First consider $I = [a',b]$. The condition that $\pi_b(E) = \operatorname{Ker}(d_b)$ is locally free and that its formation commutes with base change is equivalent to $\operatorname{Coker}(d_b)$ beeing locally free (see Lemma C.1). In this case, $\pi_b(E)$ has rank r_a if and only if $\operatorname{Coker}(d_b)$ has rank $t_b \coloneqq n_{b-1}-n_b+r_b$. Furthermore, in this case $\operatorname{Im}(d_b)$ is a direct summand of E_{b-1}, it is locally free of rank $n_{b-1}-t_b = n_b - r_b$ and its formation commutes with arbitrary base change (again by Lemma C.1). Therefore, we can apply Lemma C.1 to $E_{b-1}/\operatorname{Im}(d_b) \to E_{b-2}$ and see that $\pi_{b-1}(E)$ is finite locally free of rank r_{b-1} and its formation commutes with arbitrary base change, and hence it is a direct summand of $E_{b-1}/\operatorname{Im}(d_b)$ if and only if $\operatorname{Coker}(d_{b-1})$ is locally free of rank $t_{b-1} \coloneqq (n_{b-2}-n_{b-1}+n_b)+(r_{b-1}-r_b)$. Proceeding by induction on sees that for $i \leq b$ the R-module $\pi_i(E)$ is locally free of rank r_i and that its formation commutes with arbitrary base change if and only if $\operatorname{Coker}(d_i)$ is locally free of rank

$$t_i \coloneqq \sum_{j=-1}^{b-i}(-1)^{j+1}n_{i-j} + \sum_{j=0}^{b-i}(-1)^j r_{i-j}.$$

In fact Lemma C.1 shows that in this case the formation of $\operatorname{Ker}(d_i)$ and of $\operatorname{Im}(d_{i-1})$ also commutes with tensoring by any R-module Q. Hence,

$$\pi_i(E \otimes_R Q) = \pi_i(E) \otimes_R Q \tag{C.1.1}$$

for any R-module Q.

There is a subscheme Z_i such that $f\colon T \to S$ factors through Z_i if and only if $f^*\operatorname{Coker}(d_i)$ is locally free of rank t_i (see [GW10, Thm. 11.17]). As the formation of $\operatorname{Coker}(d_i)$, we can take $Z = \bigcap_{i\in I} Z_i$ the scheme theoretic intersection.

To construct Z in the case that $I = [a,b']$, one proceed similarly.

As $\operatorname{Coker}(d_i)$ is of finite presentation, all immersions $Z_i \to S$ are of finite presentation (see remark after [GW10, Thm. 11.17]). Hence, $Z \to S$ is of finite presentation.

A point $s \in S$ is contained in Z if and only if $\pi_i(E \otimes_{\mathcal{O}_S} \kappa(s))$ is of rank r_i and its formation commutes with arbitrary base change $T \to \operatorname{Spec}(\kappa(s))$. But this base change is flat and the second condition holds automatically, which shows (2).

The last assertion holds by construction of Z and (C.1.1). ∎

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