Reflective modular forms and vertex operator algebras

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Für meine Frau, Laura.

Zusammenfassung

In dieser Dissertation werden stark-rationale, holomorphe Vertex-Operator-Algebren und reflektive Modulformen untersucht. Wir beginnen damit, einer Vertex-Operator-Algebra mit zentraler Ladung $c = 24$ ihre Lie-Algebra der physikalischen Zustände zuzuordnen und studieren die zugehörige Lie-Klammer mit Hilfe von No-Ghost-Isomorphismen als bilineare Abbildung von Gewichtsräumen der Vertex-Operator-Algebra. Eine sorgfältige Analyse des No-Ghost-Theorems liefert Methoden, die eine explizite Beschreibung dieser Abbildungen durch Vertex-Algebra-Operationen ermöglicht. Im Anschluss zerlegen wir solche holomorphen Vertex-Operator-Algebren gemäß ihrer affinen Unterstruktur und zeigen, dass die zugehörigen Charakter vektorwertige Modulformen liefern. Hierfür werden Kowurzelgitter geeignet mit einfachen Strömen angereichert. Die Anhebung zu einem automorphen Produkt liefert die Produktseite der Nenneridentität der zugehörigen Lie-Algebra der physikalischen Zustände. Da dies eine verallgemeinerte Kac-Moody Algebra ist, folgt, dass das automorphe Produkt reflektiv ist. Schlussendlich studieren wir Gitter, welche reflektive Modulformen tragen. Dabei zeigen wir, dass es nur endlich viele solcher Gitter mit gerader Signatur, die skalierte hyperbolische Ebenen abspalten, gibt. Wir bestimmen explizite Schranken für die Stufe.

Abstract

In this thesis we mainly study strongly rational, holomorphic vertex operator algebras and reflective modular forms. First we associate the Lie algebra of physical states to a vertex operator algebra of central charge $c = 24$. We study the corresponding Lie bracket as a bilinear map between weight spaces of the vertex operator algebra. This makes use of no-ghost-isomorphisms. A careful analysis of the no-ghost theorem yields methods to evaluate those bilinear maps explicitly in terms of vertex algebra operations. Then we decompose such holomorphic vertex operator algebras according to their affine substructure and show that the corresponding characters are vector-valued modular forms for a coroot lattice, suitably enriched by simple currents. The associated automorphic product yields the product side of the denominator identity of the Lie algebra of physical states. Since this is a generalized Kac-Moody algebra it follows that this automorphic product is reflective. Finally we study lattices that admit a reflective modular form. We show, that there are just finitely many such lattices of even signature, which split rescaled hyperbolic planes. We determine explicit bounds for the levels.

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Contents

1 Introduction

Automorphic products, generalized Kac-Moody algebras and vertex operator algebras are three different but highly related objects in mathematics with appearances in algebraic geometry, number theory and conformal field theory. Vertex algebras were first introduced by Borcherds in [Bor86], as an attempt to formalize structures from conformal field theory. Later Frenkel, Lepowsky and Meurman slightly altered this definition and introduced vertex operator algebras in [FLM89]. Their goal was to axiomatize a suitable structure that is a $\mathbb{Z}$-graded representation $V^\natural$ of the monster group $\mathbb{M}$ such that each McKay-Thompson trace function

$$\mathrm{Tr}(g)(\tau) = \sum_{n\in\mathbb{Z}} \mathrm{tr}_{V_n^\natural}(g) q^{n-1} \tag{1}$$

defines a *Hauptmodul*, i.e. a biholomorphic map from some Riemann surface $\overline{\Gamma\backslash\mathbb{H}}$ of genus 0 to $\mathbb{P}^1(\mathbb{C})$. Here Γ is a suitable arithmetic subgroup of $\mathrm{SL}_2(\mathbb{R})$. The construction of such a representation $V^\natural$ was achieved in [FLM89] and it turned out, that this space is in fact a holomorphic vertex operator algebra of central charge $c = 24$. Furthermore Frenkel, Lepowsky and Meurman essentially proved the *Hauptmodul property* for a large subgroup of $\mathbb{M}$. Finally in [Bor92] Borcherds studied the monster Lie algebra $\mathfrak{m}^\natural$, which is a generalized Kac-Moody algebra, that can be derived from $V^\natural$ by use of a certain quantization process. Using the denominator identity and twisted versions of it he found recursion formulas for the coefficients of the McKay-Thompson trace functions that imply the Hauptmodul property for all elements of $\mathbb{M}$. Based on this work extensive research was done in automorphic forms, vertex operator algebras and generalized Kac-Moody algebras. A particular achievement, due to the work of many scientists, was the classification of all holomorphic vertex operator algebras V of central charge $c = 24$, with nontrivial weight-1 subspace V_1. In this thesis we want to make some contribution to this research. In the first section we to study the Lie bracket of Lie algebras like $\mathfrak{m}^\natural$ more carefully and derive an explicit expression in terms of vertex algebra operations of $V^\natural$. This answers a question raised by Borcherds in [Bor92]. In the second section we want to associate to each holomorphic vertex operator algebra of central charge $c = 24$, with nontrivial V_1, a suitable automorphic product which encodes the root structure of the corresponding Lie algebra of physical states. This automorphic product will turn out to be reflective, which means that its divisor is supported on roots of the corresponding lattice. Finally we want to describe some results concerning the classification problem of such reflective automorphic products.

Vertex operator algebras, generalized Kac-Moody algebras and automorphic products

Vertex operator algebras are in essence a $\mathbb{Z}$-graded vector space

$$V = \bigoplus_{n=0}^{\infty} V_n \tag{2}$$

with a complicated collection of multiplications. Those multiplications are defined by the *vertex operator* $Y(\cdot, z)$, which associates to each element $v \in V$ a formal operator-valued distribution

$$Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \in \mathrm{End}(V)[[z^{\pm 1}]], \tag{3}$$

that satisfies a collection of axioms. The most important axiom is the so called *locality axiom*, which yields relations between the *modes* v_n for different elements $v \in V$. A further crucial property of a vertex operator algebra is the existence of a *conformal structure*, which is a *Virasoro vector* $\omega \in V_2$ such that the corresponding modes defined by

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \tag{4}$$

generate a representation of the Virasoro algebra on V. This means that for some complex number $c \in \mathbb{C}$ we have

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n+m,0} c \mathrm{Id}_V. \tag{5}$$

This complex number $c \in \mathbb{C}$ will be called the *central charge* of V. We may introduce certain regularity assumptions and call a vertex operator algebra V *strongly rational* if they are satisfied. In this case V has a well-understood representation theory and the category of modules has a rich structure. In particular there exists a tensor product called the *fusion product* and denoted by $\boxtimes$. We call such a strongly rational vertex operator algebra V *holomorphic* if it is its only irreducible module. As mentioned above the moonshine module $V^\natural$ is an example of a holomorphic vertex operator algebra of central charge $c = 24$. Yet there are other interesting holomorphic vertex operator algebras of central charge $c = 24$ and a first important observation is, that the subspace V_1 of V carries the structure of a Lie algebra with the Lie bracket $[v, w] = v_0 w \in V_1$ for $v, w \in V_1$. Due to work of Schellekens in [Sch93] and Dong and Mason [DM04] it is known that V_1 satisfies precisely one of the following statements:

1. $V_1 = 0$.
2. V_1 is abelian of rank 24. In this case V is isomorphic to the lattice vertex operator algebra V_Λ, which correspond to the Leech lattice Λ.
3. V_1 is a semi-simple Lie algebra of rank 24 or less.

In [FLM89] Frenkel, Lepowsky and Meurman conjectured that in the case $V_1 = 0$ the vertex operator algebra V is isomorphic to the moonshine module $V^\natural$. This conjecture is therefore called the *FLM conjecture*. It is still open, however. Yet the case where V_1 is semi-simple is fully understood. In [Sch93] a complete classification of all possible semi-simple Lie algebras V_1 was obtained. It turns out that there are precisely 69 of such Lie algebras. This list of Lie algebras is called *Schellekens' list*. Schellekens subsequently conjectured that for each such Lie algebra there exists a unique holomorphic vertex operator algebra of central charge $c = 24$ that realizes this Lie algebra. This conjecture was proved in recent years by joint effort of many people. We may now discuss some important Lie algebras. Remember that a simple Lie algebra $\mathfrak{g}_0$ can be constructed by generators and relations. For every simple root α_i of $\mathfrak{g}_0$ one can find elements e_i, f_i and h_i in $\mathfrak{g}_0$ that satisfy certain relations depending on the Cartan matrix of $\mathfrak{g}_0$. One may relax the conditions on a matrix to be a Cartan matrix a bit to introduce *generalized Cartan matrices*. Those can now be used to define *Kac-Moody algebras* by generators and relations. These are infinite dimensional Lie algebras which still have almost all the properties of a simple Lie algebra. In particular they have a Cartan subalgebra, a root space decomposition and simple roots. There exists a well-understood theory of highest-weight modules for such Lie algebras and we have a denominator identity for them. A difference to simple Lie algebras is that there might be *imaginary roots*. Those are roots of non-positive length. Yet such an imaginary root can never be simple. An excellent standard textbook for this topic is [Kac90]. The theory of highest-weight modules of Kac-Moody algebras is of particular importance for the theory of vertex operator algebras. This is because suitable irreducible modules of Kac-Moody algebras carry the structure of a strongly rational vertex operator algebra, which are called *affine vertex operator algebras*. This type of vertex operator algebras plays an important role in Schellekens' work because the vertex operator subalgebra of V which is generated by V_1 is of this type. We discuss this in more detail later. In a series of papers [Bor86], [Bor88], [Bor91] and [Bor95b] Borcherds found a generalization of Kac-Moody algebras by relaxing the conditions on generalized Cartan matrices even further. Let I be some index set, then a symmetric matrix $A = (a_{ij})_{i,j \in I}$ is called *Borcherds-Cartan matrix* if it satisfies $a_{ij} \leq 0$ if $i \neq j$ and if $a_{ii} > 0$ implies $\frac{2a_{ij}}{a_{ii}} \in \mathbb{Z}$ for all $j \in I$. Given a Borcherds-Cartan matrix A, a *universal generalized Kac-Moody algebra* is a Lie algebra $\mathfrak{g}(A)$ with generators e_i, f_i and h_{ij} for $i, j \in I$ such that:

1. $\mathfrak{sl}_2$*-relations*: $[e_i, f_j] = h_{ij}$, $[h_{ij}, e_k] = \delta_{i,j} a_{ik} e_k$ and $[h_{ij}, f_k] = -\delta_{i,j} a_{ik} f_k$.
2. *Serre relations*: If $a_{ii} > 0$ then $\mathrm{ad}(e_i)^{1-2a_{ij}/a_{ii}}(e_j) = \mathrm{ad}(f_i)^{1-2a_{ij}/a_{ii}}(f_j) = 0$.

3. *Commutativity*: If $a_{ij} = 0$ then $[e_i, e_j] = [f_i, f_j] = 0$.

A *generalized Kac-Moody algebra* is any Lie algebra that is isomorphic to a quotient $\mathfrak{g}(A)/C$, where $\mathfrak{g}(A)$ is a universal generalized Kac-Moody algebra and C a subalgebra of its center. We usually assume that h_{ij} is contained in C if $i \neq j$ and put $h_i = h_{ii}$. This definition is in fact slightly more restrictive than the usual one but enough for our purpose. Similar to Kac-Moody algebras a generalized Kac-Moody algebra has a Cartan subalgebra and a root space decomposition. Yet a major difference is that there might be simple imaginary roots and they may not longer be linearly independent. Yet we can still introduce a *root lattice* R by the free $\mathbb{Z}$-span of all simple roots h_i. We may equip this lattice with a bilinear form by use of the Borcherds-Cartan matrix A by setting $(h_i, h_j) = a_{ij}$. We can introduce a *Weyl group* generated by reflections σ_r which correspond to real roots r as usual. Furthermore we have a meaningful representation theory and even have a *denominator identity* given by

$$e^{\rho} \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)} = \sum_{w \in W} \epsilon(w) w(S). \tag{6}$$

Here ρ is a *Weyl vector* and Δ^+ is the set of *positive* roots. S is a suitable sum of imaginary roots. One of the most famous generalized Kac-Moody algebras is certainly the monster Lie algebra $\mathfrak{m}^\natural$. We may project its root lattice into the Cartan subalgebra of $\mathfrak{m}^\natural$, which yields a lattice of rank 2. In fact this lattice is isomorphic to $\mathrm{II}_{1,1}$ and we may consider this as a root lattice as well. Applying the same projection to the denominator identity yields

$$p^{-1} \prod_{m>0, n \in \mathbb{Z}} (1 - p^m q^n)^{c(nm)} = j(p) - j(q), \tag{7}$$

where we have $p = e^{(1,0)}$ and $q = e^{(0,1)}$and the numbers $c(\cdot)$ are the Fourier coefficients of the J-function

$$J(\tau) = j(\tau) - 744 = \sum_{n=-1}^{\infty} c(n) q^n = q^{-1} + 0 + 196884q + \mathcal{O}(q^2). \tag{8}$$

We will come to the construction of the monster Lie algebra in more detail later. For now we focus on another remarkable observation. Clearly the expression in (7) does not just have meaning as a formal expression but defines a holomorphic function with certain modular properties, since j is a modular function. It is a general observation, that denominator identities of *interesting* generalized Kac-Moody algebras tend to have modular properties. By this we mean that their *product sides* define modular forms on suitable spaces. This may serve as a motivation for a more systematic construction of orthogonal modular forms introduced by Borcherds in [Bor95a] and [Bor98]. Let L be an even lattice of signature $(n, 2)$. We may set $V(\mathbb{C}) = L \otimes_{\mathbb{Z}} \mathbb{C}$ and introduce a space

$$\mathcal{K}(L) := \{[Z] \in \mathbb{P}(V(\mathbb{C})) : (Z, Z) = 0, (Z, \overline{Z}) < 0\}^+. \tag{9}$$

Here $\cdot^+$ means that we fix a choice out of two connected components of this space. It is furthermore natural to consider the *affine cone over* $\mathcal{K}(L)$, defined by

$$\tilde{\mathcal{K}}(L) := \{Z \in V(\mathbb{C}) \setminus \{0\} : [Z] \in \mathcal{K}(L)\}. \tag{10}$$

The *discriminant kernel* of $O(L)$ is the subgroup $\tilde{O}(L)$ of finite index of $O(L)$ which acts trivially in the discriminant $D(L) = L'/L$. We set $\Gamma(L) = \tilde{O}(L) \cap O^+(V)$. Take a subgroup $\Gamma \subset \Gamma(L)$ of finite index and a unitary character $\chi : \Gamma \to \mathbb{C}^*$. A meromorphic function $\Phi : \tilde{\mathcal{K}}(L) \to \mathbb{C}$ is called *modular form* of weight $k \in \mathbb{Z}$ for Γ and χ if for all $Z \in \tilde{\mathcal{K}}(L)$ we have

$$\Phi(MZ) = \chi(M)\Phi(Z) \; \forall M \in \Gamma \tag{11}$$

$$\Phi(tZ) = t^{-k}\Phi(Z) \; \forall t \in \mathbb{C}^*. \tag{12}$$

Such a function is an *orthogonal modular form* and we may furthermore call them *homogeneous*. Later we will need such modular forms for rational weight as well but we ignore this for now. This simplifies the discussion a bit. For a rational vector $v \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ with $(v, v) > 0$ we define the associated *rational quadratic divisor* by

$$\mathcal{K}_v(L) := \{[Z] \in \mathcal{K}(L) : (Z, v) = 0\}. \tag{13}$$

In the affine cone we denote the corresponding *rational quadratic divisor* of $v \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ with $(v, v) > 0$ by

$$v^{\perp} := \tilde{\mathcal{K}}_v(L) := \{Z \in \tilde{\mathcal{K}}(L) : [Z] \in \mathcal{K}_v(L)\}. \tag{14}$$

Now we want to discuss a construction for such modular forms. The Weil representation $\rho_{D(L)}$ of the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ is defined on the group algebra $\mathbb{C}[D(L)]$ of the discriminant form $D(L) = L'/L$. We may call a holomorphic function $f : \mathbb{H} \to \mathbb{C}[D(L)]$ a *vector-valued modular form* of weight $k \in \mathbb{Z}$ for ρ_D if it satisfies

$$f(m\tau) = \phi(\tau)^{2k} \rho_{D(L)}(m) f(\tau) \tag{15}$$

for every $(m, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$. Remember that $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix in $\mathrm{SL}_2(\mathbb{Z})$ and $\phi(\tau)$ is a holomorphic function such that $\phi^2(\tau) = c\tau + d$. A direct consequence of the invariance under $(T, 1) \in \mathrm{Mp}_2(\mathbb{Z})$ is that f has a Fourier expansion

$$f(\tau) = \sum_{\gamma \in D(L)} \sum_{n \in \mathbb{Z} - q(\gamma)} [f_\gamma](n) e(n\tau) \mathfrak{e}_\gamma. \tag{16}$$

We assume that all Fourier coefficients $[f_\gamma](n)$ with $n < 0$ are integers, i.e. $[f_\gamma](n) \in \mathbb{Z}$. Then there exists a, up to a constant factor, uniquely determined orthogonal modular form Φ_f which possesses a product expansion around some cusp and has a divisors which is a linear combination of rational quadratic divisors. More precisely for every primitive $v \in L$ with $v^2 > 0$ the rational quadratic divisor $v^{\perp}$ has vanishing order

$$\sum_{0 < x \in \mathbb{Q}, xv \in L'} [f_{xv+L}] \left(-x^2 \frac{v^2}{2} \right). \tag{17}$$

We call Φ_f the *automorphic product* associated with f. This is the main result of [Bor98]. We call a primitive element $v \in L$ a *root* of L, if its corresponding reflection, which is defined by

$$\sigma_v(x) = x - \frac{2(v, x)}{v^2} v \text{ for } x \in L \otimes \mathbb{Q}, \tag{18}$$

is contained in $O(L)$. We call an orthogonal modular form *reflective* if its divisor is a linear combination of rational quadratic divisors $v^{\perp}$ which correspond to roots v of L.

The Lie algebra of physical states

Let V be a strongly rational holomorphic vertex operator algebra of central charge $c = 24$. We consider the conformal lattice vertex algebra $V_{\mathrm{II}_{1,1}}$ associated to the *hyperbolic plane* $\mathrm{II}_{1,1}$. The tensor product $V \otimes V_{\mathrm{II}_{1,1}}$ carries the structure of a conformal vertex algebra of central charge $c = 26$ and we denote the Virasoro operators corresponding to the conformal structure by $L(k)$ for $k \in \mathbb{Z}$. We may introduce the *subspaces of physical states* for every $n \in \mathbb{Z}$ by

$$P^n = \{v \in V \otimes V_{\mathrm{II}_{1,1}} : L(m)v = 0 \text{ for all } m > 0 \text{ and } L(0)v = nv\}. \tag{19}$$

Of particular interest will be the quotient $P^1/L(-1)P^0$ because it turns out that for $x, y \in P^1$ the bracket, defined by $[x, y] = [x_0 y]$, equips this space with the structure of a Lie algebra. Since V is strongly rational it carries a unique invariant bilinear form $\langle \cdot, \cdot \rangle$ scaled such that $\langle \mathbf{1}, \mathbf{1} \rangle = 1$. This in turn induces an invariant bilinear form $(\cdot,\cdot)$ on the Lie algebra $P^1/L(-1)P^0$. We denote its kernel by $\ker(\cdot,\cdot)$ and introduce the Lie algebra

$$\mathfrak{g}(V) = (P^1/L(-1)P^0)/\ker(\cdot,\cdot). \tag{20}$$

This Lie algebra obviously carries an invariant non-degenerate bilinear form $(\cdot,\cdot)$ and will be called the *Lie algebra of physical states* associated with V. This construction is called the *old-covariant quantization* of V and has a few remarkable properties. First of all notice that the natural $\mathrm{II}_{1,1}$-grading of $V_{\mathrm{II}_{1,1}}$ induces a natural $\mathrm{II}_{1,1}$-Lie algebra grading of $\mathfrak{g}(V)$. Let us denote the corresponding weight spaces by $\mathfrak{g}(V)_\alpha$ for $\alpha \in \mathrm{II}_{1,1}$, then we have

$$\mathfrak{g}(V) = \bigoplus_{\alpha\in\mathrm{II}_{1,1}} \mathfrak{g}(V)_\alpha. \tag{21}$$

Similarly the group of automorphisms $\mathrm{Aut}(V)$ of V induces a natural group of automorphisms G acting on $\mathfrak{g}(V)$. Of course this group is in general just a subgroup of the group of Lie algebra automorphisms of $\mathfrak{g}(V)$. We may apply this construction to the moonshine module $V^\natural$ to obtain the monster Lie algebra $\mathfrak{m}^\natural = \mathfrak{g}(V^\natural)$ and its natural action of the monster group $\mathbb{M}$, which is induced from the group of automorphisms $\mathrm{Aut}(V^\natural) = \mathbb{M}$ of $V^\natural$. There exists a Lie algebra involution $\theta : \mathfrak{g}(V) \to \mathfrak{g}(V)$ which satisfies $\theta(\mathfrak{g}(V)_\alpha) = \mathfrak{g}(V)_{-\alpha}$ for every $r \in \mathrm{II}_{1,1}$ and which preserves the bilinear form $(\cdot,\cdot)$ and the group action of G. We may use this involution to introduce the *contravariant bilinear form*

$$(\cdot,\cdot)_0 = -(\cdot,\theta(\cdot)), \tag{22}$$

whose restriction to $\mathfrak{g}(V)_\alpha \times \mathfrak{g}(V)_\alpha$ is non-degenerate for every $\alpha \in \mathrm{II}_{1,1}$.

Theorem (no-ghost theorem)**.** *Let V be a strongly regular holomorphic vertex operator algebra of central charge $c_V = 24$ with a group of automorphisms G and let its associated Lie algebra of physical states be $\mathfrak{g}(V)$. Take $0 \neq \alpha \in \mathit{II}_{1,1}$. There exists a linear isomorphism*

$$\eta_\alpha : V_{1-\frac{\alpha^2}{2}} \to \mathfrak{g}(V)_\alpha \tag{23}$$

that preserves the group action of G and satisfies that for all $v, w \in V_{1-\frac{\alpha^2}{2}}$ we have

$$\langle v, w\rangle = (\eta_\alpha(v), \eta_\alpha(w))_0. \tag{24}$$

For $\alpha = 0$ there exists a linear isomorphism

$$\eta_0 : V_1 \oplus (\mathit{II}_{1,1} \otimes \mathbb{C}) \to \mathfrak{g}(V)_0 \tag{25}$$

that preserves the G-action and the corresponding bilinear forms as in the case $\alpha \neq 0$.

The no-ghost theorem is a remarkable result originally proved by physicists in [GT72] and discussed by mathematicians in [Fre85] and [Bor92]. If we assume that V has a suitable real structure $V_\mathbb{R}$ with a positive-definite bilinear form $\langle\cdot,\cdot\rangle$, the no-ghost theorem implies that the contravariant bilinear form $(\cdot,\cdot)_0$ is also positive-definite, when restricted to root spaces $\mathfrak{g}(V_\mathbb{R})_r$ with $r \neq 0$. Therefore the Lie algebra of physical states does not contain any *ghosts*, i.e. vectors of negative length. This justifies the name of the theorem. But this aspect does not really matter for us, therefore we don't make any positivity assumptions. In Borcherds' paper [Bor92] the no-ghost theorem is of particular importance because it allows the computation of the root multiplicities of roots of the monster Lie algebra $\mathfrak{m}^\natural$. This is because the *root grading* of $\mathfrak{m}^\natural$ is simply given by the $\mathrm{II}_{1,1}$-grading of $\mathfrak{m}^\natural$ and therefore $\alpha \in \mathrm{II}_{1,1}$ has multiplicity

$$\mathrm{mult}(\alpha) = \dim(\mathfrak{m}^\natural_\alpha) = \dim(V^\natural_{1-\alpha^2/2}). \tag{26}$$

Now it is possible to evaluate those multiplicities explicitly because the character $\mathrm{ch}_{V^\natural}$ is given by the J-function defined by (8). This information is necessary to obtain the denominator identity (7).

This together with suitable twisted denominator identities yield the recursion formulas that determine the McKay-Thompson trace formulas mentioned above. This finishes Borcherds' proof of the Hauptmodul property. Of course we can use the no-ghost isomorphisms η_α to define bilinear maps between subspaces of the vertex operator algebra V. More precisely for elements $\alpha, \beta \in \mathrm{II}_{1,1}$ we can define maps

$$\{\cdot,\cdot\}_{\alpha,\beta} : V_{1-\alpha^2/2} \times V_{1-\beta^2/2} \to V_{1-(\alpha+\beta)^2/2}, (v,w) \mapsto \eta^{-1}_{\alpha+\beta}([\eta_\alpha(v), \eta_\beta(w)]). \tag{27}$$

In section 15 of [Bor92] Borcherds asks for an explicit description of this bracket just in terms of vertex algebra operations of $V^\natural$. Clearly we can generalize this question to arbitrary holomorphic vertex operator algebra V of central charge $c = 24$. The first section of this thesis is concerned with an answer to this question. This is one of the main results of this thesis. In order to do this notice that in the construction of the no-ghost isomorphisms η_α there is some ambiguity, since those maps depend on certain choices of operators acting on $H(\alpha) = V \otimes V_{\mathrm{II}_{1,1},\alpha}$ for $\alpha \in \mathrm{II}_{1,1}$. By picking a standard basis $e, f \in \mathrm{II}_{1,1}$ with $e^2 = f^2 = 0$ and $(e,f) = 1$ it is possible to make these choices in a consistent way, however.

Theorem (Theorem 3.4.9). *Assume $\alpha, \beta, \alpha+\beta \notin e^\perp$. We may put $n = 1 - \frac{\alpha^2}{2}$ and $m = 1 - \frac{\beta^2}{2}$. For $v \in V_n$ and $w \in V_m$ we have*

$$\{v,w\}_{\alpha,\beta} = \tag{28}$$

$$\frac{\epsilon(\alpha,\beta)}{n!m!} \sum_{r=0}^{n} \sum_{s=0}^{m} \sum_{l_1,l_2=0}^{\infty} \sum_{k\in\mathbb{Z}} \sum_{\underline{m}\in B(l_1+l_2+k-(\alpha,\beta))} S_r[n]S_s[m]c(\underline{m})L(-\underline{m})(\mathfrak{i}^r_{l_1}(v)_k \mathfrak{j}^s_{l_2}(w)). \tag{29}$$

Of course we have to explain some notation in order to make sense of this result. Here ϵ is a choice of a 2-cocycle for the lattice vertex algebra $V_{\mathrm{II}_{1,1}}$. Furthermore the operators $\mathfrak{i}^r_l$ and $\mathfrak{j}^r_l$ are suitable linear combinations of products of Virasoro operators. By $\underline{m}$ we simply denote certain tuples of numbers. $L(-\underline{m})$ is then just $L(-m_1)\cdots L(-m_r)$ if $\underline{m} = (m_1, \cdots, m_r)$. Finally by expressions like $S_r[n]$ we denote the number $S_r(1,\cdots,n)$, where S_r is the r-th symmetric polynomial in n variables. We find that $\{v,w\}_{\alpha,\beta}$ can be expressed in terms of vertex algebra operations of V, so this is an answer to Borcherds question. Unfortunately this formula becomes rather complicated quickly and might therefore not be suitable for explicit computations.

Vertex operator algebras and reflective modular forms

In this section we study strongly rational holomorphic vertex operator algebras of central charge $c = 24$ and their connection with reflective automorphic products. We exclude the cases $V_1 = 0$ and V_1 abelian from this discussion. This work was strongly motivated by [CKS07]. Therein Creutzig, Klauer and Scheithauer study vertex operator algebras associated to particular entries of Schellekens' list. Namely those with $V_1 = A^r_{p-1,p}$ for $p = 2,3,5,7$ and $r = \frac{48}{(p-1)(p+1)}$. They prove that the corresponding Lie algebras of physical states are generalized Kac-Moody algebras and evaluate their denominator identities explicitly. As a consequence of the assumption that V is strongly rational we may equip V with an invariant bilinear form $\langle\cdot,\cdot\rangle$ which we scale such that $\langle \mathbf{1}, \mathbf{1}\rangle = -1$. Here by *invariant* we mean a certain invariance property with respect to the vertex operator $Y(\cdot,z)$. This is explained in more detail in the main section. Since $\mathfrak{g} := V_1$ is semi-simple we may decompose it into simple Lie algebras $\mathfrak{g}_i$ and we define numbers k_i by $\langle\cdot,\cdot\rangle|_{\mathfrak{g}_i\times\mathfrak{g}_i} = k_i(\cdot,\cdot)$. Here $(\cdot,\cdot)$ denotes the invariant bilinear form on $\mathfrak{g}_i$ which is scaled such that the highest root θ of $\mathfrak{g}_i$ satisfies $(\theta,\theta) = 2$. It is well-known that we have $k_i \in \mathbb{Z}_{>0}$ and we write

$$V_1 = \mathfrak{g} = \mathfrak{g}_{1,k_1} \oplus \cdots \oplus \mathfrak{g}_{r,k_r}. \tag{30}$$

The corresponding Cartan subalgebra will be denoted by $\mathfrak{h} = \mathfrak{h}_{1,k_1} \oplus \cdots \oplus \mathfrak{h}_{r,k_r}$. We denote the affine Kac-Moody algebra corresponding to $\mathfrak{g}_i$ by $\hat{\mathfrak{g}}_i$. The associated simple affine vertex operator algebra of level k_i is $L_{\hat{\mathfrak{g}}_i}(k_i\Lambda_0)$. Now the vertex operator subalgebra $V(\mathfrak{g}) \subset V$ generated by V_1 is isomorphic to

$$L_{\hat{\mathfrak{g}}_1}(k_1\Lambda_0) \otimes \cdots \otimes L_{\hat{\mathfrak{g}}_r}(k_r\Lambda_0). \tag{31}$$

This is a result of Dong and Mason, obtained in [DM06]. Furthermore by use of Proposition 4.1 in [DM04] we know that this isomorphism preserves the conformal structure. So we can consider V as a $V(\mathfrak{g})$-module. Yet since $V(\mathfrak{g})$ is also strongly rational, we know that V can be decomposed into finitely many irreducible $V(\mathfrak{g})$-modules. We obtain a decomposition

$$V = \bigoplus_{\Lambda^1,\cdots,\Lambda^r} m(\Lambda^1,\cdots,\Lambda^r) L_{\hat{\mathfrak{g}}_1}(\Lambda^1)\otimes\cdots\otimes L_{\hat{\mathfrak{g}}_r}(\Lambda^r). \tag{32}$$

Here the Λ^i are dominant integral weights of level k_i of $\hat{\mathfrak{g}}_i$, which parametrize the irreducible modules of the vertex operator algebra $L_{\hat{\mathfrak{g}}_i}(k_i\Lambda_0)$. We may use the coroots $\check{\alpha}_i$ as a basis of $\mathfrak{h}$, that spans a sublattice with bilinear form $\langle\cdot,\cdot\rangle$ that is isomorphic to

$$L = \check{Q}_1(k_1)\oplus\cdots\oplus\check{Q}_r(k_r). \tag{33}$$

Since the Cartan subalgebra $\mathfrak{h}$ acts semi-simply on V we may define a formal character

$$\chi_V(v,q) = \mathrm{Tr}_V\left(e^{2\pi i v_0} q^{L_0-1}\right). \tag{34}$$

Clearly we can identify $\mathfrak{h}$ with $L\otimes\mathbb{C}$ and as usual we write $q=e^{2\pi i\tau}$. Yet it turns out that this formal character defines a holomorphic function on $(L\otimes\mathbb{C})\times\mathbb{H}$ and as a consequence of Theorem 1.1 in [KM15] we obtain the following Proposition.

Proposition. *The character* $\chi_V : (L\otimes\mathbb{C})\times\mathbb{H}\to\mathbb{C}$ *is a nearly holomorphic Jacobi form of weight* 0 *and lattice index* L.

By this we mean that χ_V defines a holomorphic function and satisfies the relations

$$\chi_V\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right) = \exp\left(\pi i\frac{c(z,z)}{c\tau+d}\right)\chi_V(z,\tau) \text{ and} \tag{35}$$

$$\chi_V(z+\tau l+h,\tau) = \exp(-\pi i((l,l)\tau+2(l,h)))\chi_V(z,\tau), \tag{36}$$

for all $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$ and $l,h\in L$. This property was already used by Schellekens in [Sch93] to determine certain equations which yield strong restrictions on the structure of the Lie algebra $\mathfrak{g}$ and the decomposition (32). By solving them he proved his classification result. We can also define a vector-valued modular form f_V for the Weil representation ρ_D with $D=D(L)$ of weight $-\frac{\dim(\mathfrak{h})}{2}$ by use of the *theta decomposition* of χ_V. This is

$$\chi_V(z,\tau) = \sum_{\lambda\in D(L)} f_{V,\lambda}(\tau)\Theta^L_\lambda(z,\tau), \tag{37}$$

where $\Theta^L_\lambda(z,\tau)$ are the usual lattice theta functions associated with L. For a fixed lattice the functions χ_V and f_V encode the same information, so we may as well work with f_V and do so in the following. A *simple current* S of a vertex operator algebra V is an irreducible module of V such that each fusion product $S\boxtimes M$ with another irreducible module M is once again irreducible. For strongly rational simple affine vertex operator algebras $L_{\hat{\mathfrak{g}_0}}(k\Lambda_0)$, corresponding to a simple Lie algebra $\mathfrak{g}_0$, we can introduce a certain subset of the simple currents, the so called *cominimal simple currents*. Those are precisely the simple currents that correspond to cominimal weights of the Lie algebra $\mathfrak{g}_0$. We may consider the set S_V consisting of cominimal simple currents of $V(\mathfrak{g})$ that are contained in V. This just means that $S\in S_V$ if and only if $m(S)\neq 0$, where $m(S)$ is the multiplicity of the irreducible module S in the decomposition (32). In fact S_V is an abelian group under the fusion product, which acts naturally on the set of all irreducible modules of $V(\mathfrak{g})$, that are contained in V. The corresponding multiplicities are invariant under this action and therefore we can consider S_V-orbits of the decomposition (32). Since $V(\mathfrak{g})$ appears with multiplicity 1 it is clear that each cominimal simple current in V appears with multiplicity 1. It turns out that there is an isotropic subgroup G_V of the discriminant form $D(L)$ which is naturally isomorphic to S_V and will therefore be called the *subgroup of cominimal simple currents* of $D(L)$. We construct an even lattice by

$$H = \bigcup_{g \in G_V} g + L. \tag{38}$$

It turns out that the vector-valued modular form f_V satisfies that

1. for $\gamma \notin H' \subset L'$ we have $f_{V,\gamma} = 0$ and
2. for $\gamma \in L'$ and $g \in G_V$ we have $f_{V,g+\gamma} = f_{V,\gamma}$.

We may view f_V as a vector-valued modular form for the Weil representation of $D(H)$. The fact that f_V is essentially a character of V implies that V carries a natural H'-grading with respect to the semi-simple action of $\mathfrak{h}$. Now we can consider the Lie algebra of physical states $\mathfrak{g}(V)$ associated to V. The no-ghost theorem implies that we have a natural subspace $\mathcal{H} \subset \mathfrak{g}(V)$ which is isometric to $\mathfrak{h} \oplus (\mathrm{II}_{1,1} \otimes \mathbb{C})$ and preserves the semi-simple action of $\mathfrak{h}$ on V_n with respect to the no-ghost isomorphisms. This induces a natural $H' \oplus \mathrm{II}_{1,1}$-grading, i.e. we have

$$\mathfrak{g}(V) = \bigoplus_{\alpha \in H' \oplus \mathrm{II}_{1,1}} \mathfrak{g}_\alpha(V). \tag{39}$$

We obtain $\mathcal{H} = \mathfrak{g}_0(V)$ and find that this is a self-centralizing subalgebra of $\mathfrak{g}(V)$. This allows us to view (39) as a root space decomposition with respect to $\mathcal{H}$, which we therefore view as a *Cartan subalgebra* of $\mathfrak{g}(V)$. The no-ghost theorem can furthermore be used to check that the multiplicity of $0 \neq \alpha \in H' \oplus \mathrm{II}_{1,1}$ is given by

$$\mathrm{mult}(\alpha) = [f_{V,\alpha}]\left(-\frac{\alpha^2}{2}\right). \tag{40}$$

For $\alpha = 0$ the no-ghost theorem implies $\mathrm{mult}(0) = [f_{V,0}](0) + 2 = \dim(\mathfrak{h}) + 2$, which is of course just the dimension of the Cartan subalgebra $\mathcal{H}$ of $\mathfrak{g}(V)$. For the rest of the discussion we may assume that V has a real form with positive-definite structure. We make this assumption to prove the following theorem. Yet we conjecture that this assumption will turn out to be unnecessary.

Theorem (Theorem 4.5.6). *Assume that V has a real form with positive-definite structure. The Lie algebra $\mathfrak{g}(V)$ is a generalized Kac-Moody algebra with Cartan subalgebra $\mathcal{H}$ and root lattice $R = H' \oplus \mathrm{II}_{1,1}$.*

Since adding hyperbolic planes to a lattice does not change its discriminant form we may view f_V as a vector-valued modular form for the Weil representation corresponding to the lattice $H \oplus \mathrm{II}_{1,1} \oplus \mathrm{II}_{1,1}$ of weight $k = 1 - \frac{\dim(\mathfrak{h})+2}{2}$. Clearly its Fourier coefficients are integers, so we can consider the associated automorphic product Φ_V.

Theorem (Theorem 4.5.9). *Assume that V has a real form with positive-definite structure. The automorphic product Φ_V, associated to the vector-valued modular form f_V is holomorphic, strongly reflective and of singular weight.*

This is one of the main results of this thesis. Notice that we call a reflective modular form *strongly reflective* if the multiplicity of each rational quadratic divisor $r^\perp$, which corresponds to a root, has multiplicity 0 or 1. In the following we will give a brief sketch of the proof. The fact that each Fourier coefficient of f_V is a non-negative integer already implies that Φ_V has to be holomorphic. Because of $[f_{V,0}](0) = \dim(\mathfrak{h})$ we obtain that f_V is of singular weight. In order to prove that Φ_V is strongly reflective we have to make use of the fact that $\mathfrak{g}(V)$ is a generalized Kac-Moody algebra. Let $r \in H \oplus \mathrm{II}_{1,1}$, with $r^2 > 0$, be primitive such that Φ_V vanishes on $r^\perp$. Of course this implies that there exists a suitable $x \in \mathbb{Q}_{>0}$ such that $xr \in H' \oplus \mathrm{II}_{1,1}$ and xr is a root of $\mathfrak{g}(V)$. Because of $(xr)^2 > 0$ this is already a real root and so no scaling of xr can be another root of $\mathfrak{g}(V)$. This implies $\mathrm{mult}(r^\perp) = 1$. Since xr is a real root of $\mathfrak{g}(V)$ it is furthermore clear that the corresponding reflection $\sigma_{xr} = \sigma_r$ is contained in the Weyl group W. Yet we have $W \subset O(R) = O(H' \oplus \mathrm{II}_{1,1})$. This implies that the reflection σ_r is contained in $O(H \oplus \mathrm{II}_{1,1})$. Therefore Φ_V is strongly reflective.

The classification of reflective modular forms

Early in the study of reflective modular forms Gritsenko and Nikulin conjectured in [GN98] that up to scaling there are just finitely many lattices of signature $(n, 2)$ that admit a reflective modular form. This question was then studied extensively by Gritsenko and Nikulin themselves as well as several other authors. Namely Barnard in [Bar03], Dittmann in [Dit19], Scheithauer in [Sch06], Wang in [Wan19c], [Wan21], [Wan19a] and Ma in [Ma17], [Ma18]. In several of these papers full classifications where obtained in special cases. In the case of lattices with prime level $N = p$ a full classification of reflective lattices was achieved in [Wan19c]. Furthermore one of the main results of [Ma18] is that there are just finitely man reflective lattices that carry a reflective modular form with bounded slope. Here we say that a reflective modular form has *bounded slope* if the multiplicities in its divisor satisfy certain restrictions. Apart from these additional assumptions this answers the question of Gritsenko and Nikulin. But it seems to be complicated to derive sharp explicit bounds on the levels N of such reflective lattices by use of these results. Yet this was achieved in the case of lattices with squarefree level and some further assumptions in [Dit19].

Theorem ([Dit19, Theorem 1.1.]). *There are only finitely many even lattices L of signature $(n, 2)$, $n \geq 4$ and squarefree level N that split $II_{1,1} \oplus II_{1,1}(N)$ and carry a nonconstant reflective modular form.*

A crucial part of this statement is that the level N of such a reflective lattice has to solve an inequality induced by the *valence formula* for $\Gamma_0(N)$. This inequality can be solved explicitly and it is possible to give a table of all solutions. We aim to generalize this statement to lattices of even signature but arbitrary level N. Notice that lattices of squarefree level automatically have even signature therefore this does not have to be assumed. In the following let L be an even lattice of even signature $(n, 2)$ with $n \geq 4$ and level N that splits $\mathrm{II}_{1,1} \oplus \mathrm{II}_{1,1}(N)$. We may assume that $\Phi : \tilde{\mathcal{K}}(L) \to \mathbb{C}$ is a nonzero holomorphic reflective modular form of weight $k \in \mathbb{Z}$. As a consequence of *Bruiniers' converse theorem* we know that Φ has to be an automorphic product. See [Bru14] for details. Hence there exists a vector-valued modular form f for the Weil representation ρ_D with $D = D(L)$ of weight $k = 1 - \frac{n}{2}$ with suitable integrality condition on the Fourier coefficients such that $\Phi = \Phi_f$. A crucial first step will be to characterize reflective automorphic products Φ_f by properties of f.

Proposition (Prop. 5.1.2). *If the automorphic product Φ_f of f is reflective, then for every $\lambda \in D(L)$ the component f_λ of f satisfies: If f_λ has a pole at $i\infty$, then there exists a divisor $d|N$ such that $d\lambda = 0$ and $\frac{\lambda^2}{2} = \frac{1}{d} (\text{mod}\, \mathbb{Z})$. Furthermore there exists a number $c_\lambda \in \mathbb{C}$, such that the Fourier expansion of f_λ satisfies*

$$f_\lambda(\tau) = c_\lambda q^{-\frac{1}{d}} + \mathcal{O}(1). \tag{41}$$

For simplicity we may assume in the following discussion that f_0 has a pole of order 1 at $i\infty$. The next step is to associate to each such vector-valued modular form f a suitable nonzero nearly holomorphic modular form g_f for $\Gamma_0(N)$ of weight $k = 1 - \frac{n}{2}$ and some character, that satisfies strong bounds on the pole orders at cusps. More precisely we want a nonzero modular form $g_f \in M_k^!(\Gamma_0(N), \chi)$, such that the expansion g_s of g_f at the cusp s of $\Gamma_0(N)$ satisfies

$$g_s(\tau) \in \mathcal{O}\left(q^{-\frac{1}{t(s)}}\right). \tag{42}$$

Here by $t(s)$ we denote the width of the cusp s of $\Gamma_0(N)$. The existence of such a modular form then implies strong bounds on the level N of the lattice L, which can be made explicit. Before we discuss this in more detail we will introduce a construction for such a modular form g_f. In the special case of squarefree level N a construction for the modular form g_f was already obtained in [Bar03] and [Dit19]. Yet their method has no generalization to the case where N is not squarefree. We have to find such a modular form by other means. Roughly our result is that we can always construct such a modular form g_f by taking a suitable linear combination of components of f. We will sketch this in more detail. Assume first that the discriminant form D can be decomposed as $D = D_1 \oplus D_2$, then for $v = v_1 + v_2 \in D_1 \oplus D_2$ and $w_2 \in D_2$ we can define a pairing by

$$\langle v, w_2 \rangle = (v_2, w_2)v_1 \in D_1, \tag{43}$$

where $(\cdot, \cdot)$ is the usual sesquilinear form on $\mathbb{C}[D_2]$. Of course, if we have a holomorphic function $f : \mathbb{H} \to \mathbb{C}[D]$ the *reduction* $f'(\tau) = \langle f(\tau), w_2 \rangle$ is a holomorphic function $f' : \mathbb{H} \to \mathbb{C}[D_1]$ as well. Assume now that f is a vector-valued modular form for the Weil representation on D with character χ but possibly just for some subgroup $\Gamma_0(N_0)$ of $\mathrm{SL}_2(\mathbb{Z})$. Notice that we do not have to consider metaplectic covers here since we assume D to have even signature. Furthermore let the level of D be M and denote the levels of D_i by M_i for $i = 1, 2$. The numbers N_0, M_1 and M_2 shall be mutually coprime. In particular this implies $M = M_1 M_2$. In this setting it is possible to find a vector $w_2 \in \mathbb{C}[D_2]$, supported by isotropic elements of D_2, such that $f'(\tau) = \langle f(\tau), w_2 \rangle$ defines a vector-valued modular form for the Weil representation ρ_{D_1} of $\Gamma_0(N_0 M_2)$ for some character χ'. Notice that we may decompose $D(L)$ in its p-adic components by

$$D(L) = D_{p_1^{m_1}} \oplus \cdots \oplus D_{p_r^{m_r}} \tag{44}$$

if $N = p_1^{m_1} \cdots p_r^{m_r}$ is a prime decomposition. Now we may apply such reduction steps to the vector-valued modular form f, corresponding to the reflective automorphic product Φ_f, such that we obtain a nearly holomorphic modular form $g_f \in M_k^!(\Gamma_0(N), \chi)$. The task is to make sure that after each such step the new vector-valued modular form has *suitably bounded* pole orders and is still nonzero. It turns out that this can be done in a systematic way such that g_f satisfies bounds of the pole orders at cusps as in (42). Notice that this construction is just possible if each p-adic Jordan block D_{p^m} of $D(L)$ contains a nonzero isotropic element. We make sure that this is the case by assuming that L splits $\mathrm{II}_{1,1}(N)$. The assumption that f_0 has a pole at $i\infty$ is crucial to make sure that $g_f \neq 0$. Now we may apply the valence formula as given in Theorem 4.1.4 in [Ran08] to g_f and obtain an inequality

$$\frac{-k}{12} N \prod_{p|N} \left(1 + \frac{1}{p}\right) \leq \sum_{d|N} \phi\left(\gcd\left(d, \frac{N}{d}\right)\right). \tag{45}$$

Without too much effort we find that this inequality bounds k by 12, which in turn implies $n \leq 26$. We may furthermore deduce explicit bounds on N for each such k.

Theorem (Theorem 5.4.3)**.** *There are just finitely many even lattices L of even signature $(n, 2)$ with $n \geq 4$ and level N that split $\mathit{II}_{1,1} \oplus \mathit{II}_{1,1}(N)$ and carry a reflective automorphic product Φ_f such that f_0 has a pole of order 1 at $i\infty$. Moreover explicit bounds for the level N are given in Table 1.*

We have to explain what the entries in Table 1 mean. Take a solution N of the inequality (45) for some $k = -1, \cdots, -12$. Then the exponents m_i in the prime decomposition $N = p_1^{m_1} \cdots p_r^{m_r}$ are bounded by the corresponding entries in this table. We write $-$ instead of 0 because we view this as a trivial solution. Finally, we have to discuss a generalization of this method, which also works in the case where f_0 does not have a pole at $i\infty$. First, we observe that some component of f must have a pole at $i\infty$, since otherwise f would vanish. Around this observation we can now build a reduction method, which always yields a nonzero reduction f' for a nonzero modular form f. Yet this reduction f' will in general just be a modular form for $\Gamma_1(N)$ and the bounds of the pole orders, at the cusps of this group, are weaker as in (42). This method can now be used to construct a nearly holomorphic scalar-valued modular form g_f for $\Gamma_1(N)$, of weight k and some character χ, that satisfies

$$g_s(\tau) \in \mathcal{O}\left(q^{-\frac{B(N)}{t(s)}}\right), \tag{46}$$

for a certain *bound function* B. Here g_s is the expansion of g_f at the cusp s of $\Gamma_1(N)$ and $t(s)$ is the width of this cusp. As above, the valence formula implies an inequality, which can just be solved by finitely many integers N.

	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12
2	7	5	4	3	2	2	1	1	-	-	-	-
3	4	3	2	2	1	1	-	-	-	-	-	-
5	2	2	1	1	-	-	-	-	-	-	-	-
7	2	1	1	-	-	-	-	-	-	-	-	-
11	2	1	-	-	-	-	-	-	-	-	-	-
13	1	-	-	-	-	-	-	-	-	-	-	-
17	1	-	-	-	-	-	-	-	-	-	-	-
19	1	-	-	-	-	-	-	-	-	-	-	-
23	1	-	-	-	-	-	-	-	-	-	-	-

Table 1: Bounds on exponents in a prime factorization of N.

Theorem (Theorem 5.4.5). *There are just finitely many reflective even lattices L of even signature $(n,2)$ and level N that split $II_{1,1} \oplus II_{1,1}(N)$. Moreover we can give explicit bounds for the levels N.*

In this case we can again derive a table, similar to Table 1. Yet the obtained bounds will be much bigger. Mainly this is due to the fact that we work with $\Gamma_1(N)$ and the bound function B.

2 Preliminaries

In this section we discuss preliminary structures that are necessary for this thesis. We start with lattices and modular forms and later discuss generalized Kac-Moody algebras and vertex operator algebras. In particular we are going to discuss affine vertex operator algebras, which are very important for the later sections. Almost everything in this section is well-known and discussed elsewhere. We follow many of those sources, as indicated in the text.

2.1 Lattices and discriminant forms

In this subsection we start with lattices and discriminant forms for which important sources are [EH12], [Nik80] and [CS98]. A *lattice* L is a free $\mathbb{Z}$-module of finite rank d with a non-degenerate symmetric bilinear form $(\cdot,\cdot) : L \times L \to \mathbb{Z}$. Its *ambient space* V is the rational vector space $L \otimes_{\mathbb{Z}} \mathbb{Q}$ and we can naturally extend the bilinear form $(\cdot,\cdot)$ to V. By Sylvesters law of inertia the bilinear form $(\cdot,\cdot)$ can be diagonalized over $\mathbb{Q}$ and the number of positive eigenvalues r is independent of the particular diagonalization. Of course the number of negative eigenvalues $s = d - r$ is also determined and we call the pair (r, s) the *signature* of $(V, (\cdot,\cdot))$. The *signature* of L is simply the signature of its ambient space. The number $r - s$ will usually be called *signature* as well. For some free $\mathbb{Z}$-generators $e_1, \cdots, e_d$ of L we call the symmetric matrix $G = ((e_i, e_j))_{i,j}$ the *Gram matrix* of the lattice L with respect to those generators. The determinant $\det(G)$ is independent of the choice of generators and we call it the *discriminant* of L, denoted by $\mathrm{disc}(L)$. We call a lattice *integral* if $(x, y) \in \mathbb{Z}$ for all $x, y \in L$ and *even* if $(x, x) \in 2\mathbb{Z}$ for all $x \in L$. From now on we assume that any lattice L is even. For such a lattice L the *dual lattice* is

$$L' = \{v \in V : (v, x) \in \mathbb{Z}, \forall x \in L\}. \tag{47}$$

Clearly L' is also a lattice and we call $D := D(L) := L'/L$ its discriminant group. We easily observe that $|D(L)| = \mathrm{disc}(L)$. For $\lambda = [x] \in D(L)$ we set $q_D(\lambda) = q(x)(\mathrm{mod}\,\mathbb{Z})$, which is clearly a well-defined element in $\mathbb{Q}/\mathbb{Z}$. The *level of* L is the smallest positive integer N such that $Nq_D(\lambda) = 0$ for all $\lambda \in D$. An even lattice L will be called *unimodular* if $D(L) = 0$. A *discriminant form* is a finite abelian group D with a non-degenerate quadratic form $q_D : D \to \mathbb{Q}/\mathbb{Z}$, this is a map with

1. $q_D(a\lambda) = a^2 q_D(\lambda)$ for all $a \in \mathbb{Z}$ and $\lambda \in D$ and
2. $(\lambda, \mu) = q_D(\lambda + \mu) - q_D(\lambda) - q_D(\mu)$ is a non-degenerate bilinear form on $D \times D$.

For a rational number $x \in \mathbb{Q}$ we set

$$D(x) = \left\{ \gamma \in D : \frac{\gamma^2}{2} = x \ (\mathrm{mod}\ \mathbb{Z}) \right\}. \tag{48}$$

The discriminant group $D(L)$ of an even lattice L together with its associated quadratic form $q_{D(L)}$ is a discriminant form. An element $x \in L$ is called *isotropic* if $q(x) = 0$ and analogously we call an element $\lambda \in D$ of a discriminant form D *isotropic* if $q_D(\lambda) = 0$. Furthermore we call a subgroup $U \subset D$ *isotropic* if each element $\lambda \in U$ is isotropic. Assume there is an even lattice $L \subset H \subset L'$, then $U = H/L \subset D(L)$ defines an isotropic subgroup and conversely for every isotropic subgroup $U \subset D(L)$ the set

$$H_U = H = \bigcup_{\lambda \in U} \lambda + L \tag{49}$$

is an even lattice. This shows that even overlattices of L are uniquely parametrized by isotropic subgroups of $D(L)$. Another important invariant of an even lattice is its *genus*. The genus of an even lattice L is the set of all even lattices M of the same signature as L such that $L \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong M \otimes_{\mathbb{Z}} \mathbb{Z}_p$, where $\mathbb{Z}_p$ are the p-adic integers, for every prime p. An important result about the genus of even lattices is that two even lattices are in the same genus if and only if they have the same signature and their discriminant forms are isomorphic. See Corollary 1.9.4 in [Nik80] for this. We can now introduce a genus symbol $\mathrm{II}_{r,s}(D)$, which is the genus of all even lattices of signature (r, s) and discriminant form D. Like abelian groups discriminant forms can be decomposed into indecomposable components albeit not necessarily in a unique way. Following [CS98] and [Nik80] Scheithauer [Sch09] gave a nice overview of all possible non-trivial p-adic Jordan components. This is

1. For p an odd prime and $q = p^m > 1$ the non-trivial p-adic Jordan components of exponent q are $(q)^{\pm n}$ for $n \geq 1$. The indecomposable p-adic Jordan components of exponent q are $(q)^{\pm 1}$ and are generated by an element γ with $q\gamma = 0$ and $\frac{\gamma^2}{2} = \frac{a}{q} (\mathrm{mod}\ \mathbb{Z})$ for an integer a with $\left(\frac{a}{q}\right) = \pm 1$. Such components have level q and their p-excess is given by
$$p - \mathrm{excess}((q)^{\pm n}) = n(q-1) + 4k \ (\mathrm{mod}\ 8), \tag{50}$$
where $k = 1$ if q is not a square and the sign is -1 and $k = 0$ otherwise. We furthermore set $\gamma_p((q)^{\pm n}) = e(-(p - \mathrm{excess}((q)^{\pm n}))/8)$.

2. For $q = 2^m > 1$ the non-trivial even 2-adic Jordan components of exponent q are $(q)_{\mathrm{II}}^{\pm 2n}$ and the indecomposable even 2-adic Jordan components of exponent q are $(q)_{\mathrm{II}}^{\pm 2}$. They are generated by two elements $\lambda, \mu \in (q)_{\mathrm{II}}^{\pm 2}$ with $q\lambda = q\mu = 0$ and $(\lambda, \mu) = \frac{1}{q} (\mathrm{mod}\ \mathbb{Z})$ and $\frac{\gamma^2}{2} = \frac{\mu^2}{2} = 0 (\mathrm{mod}\ \mathbb{Z})$ for $(q)_{\mathrm{II}}^{+2}$ and $\frac{\gamma^2}{2} = \frac{\mu^2}{2} = \frac{1}{q} (\mathrm{mod}\ \mathbb{Z})$ for $(q)_{\mathrm{II}}^{-2}$. And their oddity is given by
$$\mathrm{oddity}((q)_{\mathrm{II}}^{\pm 2m}) = 4k \ (\mathrm{mod}\ 8), \tag{51}$$
where $k = 1$ if q is not a square and the sign is -1 and $k = 0$ otherwise.

3. For $q = 2^m > 1$ the non-trivial odd 2-adic Jordan components of exponent q are $(q)_t^{\pm n}$ with $n \geq 1$ and $t \in \mathbb{Z}/8\mathbb{Z}$. The indecomposable components are $(q)_t^{\pm 1}$ with $\left(\frac{t}{2}\right) = \pm 1$ and are generated by an element γ with $q\gamma = 0$ and $\frac{\gamma^2}{2} = \frac{t}{2q} (\mathrm{mod}\ \mathbb{Z})$. The level of those components is $2q$. And their oddity is given by
$$\mathrm{oddity}((q)_t^{\pm 2m}) = t + 4k \ (\mathrm{mod}\ 8), \tag{52}$$
where $k = 1$ if q is not a square and the sign is -1 and $k = 0$ otherwise. We furthermore set $\gamma_2((q)^{\pm 2n}) = e(\mathrm{oddity}((q)^{\pm 2n})/8)$.

To add two Jordan components of the same exponent q one has to multiply their signs, add their ranks and add their subscripts t if they have any. Every discriminant form can be constructed as a sum of the above components and is determined up to isomorphism by its decomposition in p-adic Jordan components. The signature $\mathrm{sign}(D) \in \mathbb{Z}/8\mathbb{Z}$ of a discriminant form is the signature of any even lattice L with $D = D(L)$. In Theorem 1.3.1 of [Nik80] Nikulin showed that two even lattices L_1 and L_2 have isomorphic discriminant forms if and only if there are even unimodular lattices S_1 and S_2, such that $L_1 \oplus S_1 \cong L_2 \oplus S_2$. Yet an unimodular lattice S of signature (r, s) satisfies $r - s \equiv 0 (\mathrm{mod}\ 8)$. Therefore the signature of a discriminant form is well-defined. One important property of the signature is the *oddity formula*
$$\mathrm{sign}(D) + \sum_{p \geq 3} p - \mathrm{excess(D)} = \mathrm{oddity}(D) \ (\mathrm{mod}\ 8). \tag{53}$$
Assume D is a discriminant form of level N and c any integer. Since c acts by multiplication as a group homomorphism on D we can consider its kernel D_c and its image D^c. D^c is the orthogonal complement of D_c in D. We furthermore define D^{c*} to be the set of elements $\alpha \in D$ that satisfy
$$c\frac{\gamma^2}{2} + (\alpha, \gamma) = 0 \ (\mathrm{mod}\ \mathbb{Z}) \tag{54}$$
for all $\gamma \in D_c$. In [Sch09] Scheithauer showed that D^{c*} is a coset of D^c, i.e. there is an element $x_c \in D$ such that $D^{c*} = x_c + D^c$. To describe x_c more precisely fix a Jordan decomposition of D and

assume $2^k||c$. If the 2-adic Jordan component of D of exponent 2^k is even we have $x_c = 0$. Otherwise we set $x_c = (2^{k-1}, \cdots, 2^{k-1})$. Furthermore in Proposition 2.2 of [Sch09] Scheithauer proved that for $\alpha = x_c + c\gamma \in D^{c*}$ the value of

$$c\frac{\gamma^2}{2} + (x_c, \gamma) \pmod{\mathbb{Z}} \tag{55}$$

does not depend on the choice of γ. We denote this number by $\frac{\alpha_c^2}{2} \pmod{\mathbb{Z}}$.

2.2 The Weil representation and modular forms

In this subsection we introduce the Weil representation and discuss vector-valued modular forms. A nice overview of this topic is given in [Bru04]. Further sources are [Bor98] and [Sch09]. The complex upper half plane will be denoted by $\mathbb{H}$ and for $\tau \in \mathbb{H}$ we write $\tau = x + iy$, with $x \in \mathbb{R}$ and $y \in \mathbb{R}_{>0}$. For any complex $z \in \mathbb{C}$ we put $e(z) = e^{2\pi i z}$ and denote by $\sqrt{z} = z^{1/2}$ the principal branch of the square root. This is $\arg(\sqrt{z}) \in]-\pi/2, \pi/2]$. In general for $b \in \mathbb{C}$ we write $z^b = e^{b\mathrm{Log}(z)}$, where $\mathrm{Log}(z)$ is the principal branch of the logarithm. For a matrix $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we write $j(m, \tau) = c\tau + d$ as usual. Taking $m \in \mathrm{SL}_2(\mathbb{R})$ we let ϕ be any holomorphic function on $\mathbb{H}$ with $\phi(\tau)^2 = j(m, \tau)$. The set of all possible pairs (m, ϕ) will be denoted by $\mathrm{Mp}_2(\mathbb{R})$ and can be turned into a group by

$$(m_1, \phi_1(\tau))(m_2, \phi_2(\tau)) = (m_1 m_2, \phi_1(A_2\tau)\phi_2(\tau)). \tag{56}$$

This group will be called the *metaplectic group* and is a double cover of $\mathrm{SL}_2(\mathbb{R})$, realized by the choice of a square root. There is an obvious covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ and we define $\mathrm{Mp}_2(\mathbb{Z})$ by the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under this map. The group $\mathrm{Mp}_2(\mathbb{Z})$ is well-known to be generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \tag{57}$$

$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right). \tag{58}$$

And with

$$Z = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right) \tag{59}$$

we have the equation

$$S^2 = (ST)^3 = Z. \tag{60}$$

Let D be a discriminant form with quadratic form $q_D = q$ and associated bilinear form $(\cdot, \cdot)$. We write $\mathfrak{e}_\gamma$ for the standard basis of the group ring $\mathbb{C}[D]$ with $\gamma \in D$. The standard scalar product on $\mathbb{C}[D]$, denoted by $(\cdot, \cdot)$, is linear in the first variable and anti-linear in the second and defined by $(\mathfrak{e}_\lambda, \mathfrak{e}_\mu) = \delta_{\lambda,\mu}$. On $\mathbb{C}[D]$ we can define the structure of a unitary representation ρ_D of $\mathrm{Mp}_2(\mathbb{Z})$ by

$$\rho_D(T)\mathfrak{e}_\gamma = e\left(-\frac{\gamma^2}{2}\right)\mathfrak{e}_\gamma \tag{61}$$

$$\rho_D(S)\mathfrak{e}_\gamma = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in D} e((\gamma, \beta))\mathfrak{e}_\beta. \tag{62}$$

The element Z acts as

$$\rho_D(Z)\mathfrak{e}_\gamma = e(\mathrm{sign}(D)/4)\mathfrak{e}_{-\gamma}. \tag{63}$$

The *dual Weil representation* ρ_D^* on $\mathbb{C}[D]$ is given by $\rho_D^*(m)\mathfrak{e}_\lambda = \overline{\rho_D(m)}\mathfrak{e}_\lambda$ for $m \in \mathrm{Mp}_2(\mathbb{Z})$. An important property of the Weil representation is that we have

$$\rho_{D_1 \oplus D_2} \cong \rho_{D_1} \otimes \rho_{D_2}. \tag{64}$$

We make use of this fact by introducing certain *partial pairings* between discriminant forms. Assume for simplicity that $D = D_1 \oplus D_2$ and $\text{level}(D_i) = M_i$ for $i = 1, 2$ with $M = M_1 M_2$ and $(M_1, M_2) = 1$. For $\lambda = \lambda_1 + \lambda_2 \in D_1 \oplus D_2 = D$ we identify $\mathfrak{e}_\lambda$ with $\mathfrak{e}_{\lambda_1} \otimes \mathfrak{e}_{\lambda_2}$ as usual and we define for $v \in \mathbb{C}[D_2]$

$$\langle \mathfrak{e}_\lambda, v\rangle = (\mathfrak{e}_{\lambda_2}, v)\mathfrak{e}_{\lambda_1} \in \mathbb{C}[D_1]. \tag{65}$$

We extend this by linearity to a partial pairing $\langle \cdot, \cdot \rangle : \mathbb{C}[D] \times \mathbb{C}[D_2] \to \mathbb{C}[D_1]$. Clearly we have for all $v \in \mathbb{C}[D_2]$ and $w \in \mathbb{C}[D]$ that

$$\rho_{D_1}(m)\langle w, v\rangle = \langle \rho_{D_1}(m) \otimes \text{Id}w, v\rangle \; \forall m \in \text{Mp}_2(\mathbb{Z}). \tag{66}$$

Let $f(\tau) = \sum_{\gamma \in D} f_\gamma(\tau)\mathfrak{e}_\gamma$ be a holomorphic function with values in $\mathbb{C}[D]$. For $(m, \phi) \in \text{Mp}_2(\mathbb{Z})$ we define a slash-operator acting on f by

$$(f|_k^{\rho_D}(m, \phi))(\tau) = \phi(\tau)^{-2k}\rho_D^{-1}(m)f(m\tau). \tag{67}$$

A holomorphic function $f : \mathbb{H} \to \mathbb{C}[D]$ will be called *modular form* of *weight* $k \in \frac{1}{2}\mathbb{Z}$ for the representation ρ_D if

$$(f|_k^{\rho_D}(m, \phi))(\tau) = f(\tau) \; \forall (m, \phi) \in \text{Mp}_2(\mathbb{Z}). \tag{68}$$

If f is a modular form and meromorphic at $i\infty$ we call it a *nearly holomorphic modular form.* If f is holomorphic at $i\infty$ we call it a *holomorphic modular form* and finally if it vanishes at $i\infty$ we call it a *cusp form.* It is well-known and easy to prove that, as a consequence of the invariance under T, the modular form f has a Fourier expansion

$$f(\tau) = \sum_{\gamma \in D} \sum_{n \in \mathbb{Z} - q(\gamma)} a_\gamma(n)e(n\tau)\mathfrak{e}_\gamma. \tag{69}$$

We furthermore call

$$\sum_{\gamma \in D} \sum_{n \in \mathbb{Z} - q(\gamma), n < 0} a_\gamma(n)e(n\tau)\mathfrak{e}_\gamma \tag{70}$$

the *principal part* of f. Assume that the discriminant form D is represented by an even lattice L of signature (n, m), i.e. $D = D(L)$. Let f be a modular form for the Weil representation on D. Then it is possible to reduce the lattice L to an even sublattice such that f can be reduced to a modular form for the discriminant form of this sublattice. Assume L has a primitive isotropic element $l \in L$. The lattice $L_l := l^\perp / l$ is integral and has signature $(n-1, m-1)$. Fix $\gamma \in L'$ such that $(\gamma, l) = 1$. We get a natural identification

$$L_l \cong L_{l,\gamma} := L \cap l^\perp \cap \gamma^\perp. \tag{71}$$

Following [Bru04, Section 2.1] we study some properties of $L'_{l,\gamma}/L_{l,\gamma}$. The divisor $\text{div}(l) \in \mathbb{Z}$ of l is defined by $(l, L) = \text{div}(l)\mathbb{Z}$. We can chose $\zeta \in L$ such that $(l, \zeta) = \text{div}(l)$. Such an element can be represented uniquely by

$$\zeta = \zeta_{l,\gamma} + \text{div}(l)\gamma + rl \tag{72}$$

with $\text{div}(l) \in \mathbb{Z} \in L'_{l,\gamma}$ and $r \in \mathbb{Q}$. One of its distinguished properties is $\frac{l}{\text{div}(l)} \in L'$. The Proposition 2.2 in[Bru04] tells us furthermore that

$$L = L_{l,\gamma} \oplus \zeta\mathbb{Z} \oplus l\mathbb{Z}. \tag{73}$$

Considering the sublattice

$$L'_0 = \{\lambda \in L' : (\lambda, l) \equiv 0 (\text{mod } \text{div}(l))\} \tag{74}$$

we can decompose $\lambda \in L'_0$ as $\lambda = \lambda_{l,\gamma} + y\gamma + xl$ with $x \in \mathbb{Q}$, $y \in \text{div}(l)\mathbb{Z}$ and $\lambda_{l,\gamma} \in L_{l,\gamma}$. This allows us to introduce an isomorphism of lattices by

$$p : L'_0 \to L'_{l,\gamma}, \lambda \mapsto \lambda_{l,\gamma} - \frac{(\lambda, l)}{\text{div}(l)}\zeta_{l,\gamma}. \tag{75}$$

One easily finds that $p(L) = L_{l,\gamma}$ such that we have a surjective map $p : L_0'/L \to L_{l,\gamma}'/L_{l,\gamma}$. Next we want to reduce the modular form f to a modular form $f_{L_{l,\gamma}}$ for the Weil representation of $D(L_{l,\gamma})$. We define a function $f_{L_{l,\gamma}} : \mathbb{H} \to \mathbb{C}[D(L_{l,\gamma})]$ by

$$f_{L_{l,\gamma}}(\tau) = \sum_{\beta \in L_{l,\gamma}'/L_{l,\gamma}} \sum_{\gamma \in L_0'/L, p(\gamma)=\beta} f_\gamma(\tau)\mathfrak{e}_\beta. \tag{76}$$

In Theorem 5.3. of [Bor98] it is proved that $f_{L_{l,\gamma}}$ is a nearly holomorphic modular form for the Weil representation on $D(L_{l,\gamma})$ of weight k if f is a nearly holomorphic modular form for the Weil representation on $D(L)$ of weight k. We call $f_{L_{l,\gamma}}$ the reduction of f to the lattice $L_{l,\gamma}$. Of course we have

$$\rho_D(Z^2)\mathfrak{e}_\gamma = e(\mathrm{sign}(D)/2)\mathfrak{e}_\gamma, \tag{77}$$

so if the signature of D is even we find that we can consider ρ_D as unitary representation of $\mathrm{SL}_2(\mathbb{Z})$. Since we are mainly interested in discriminant forms of even signature we specialize to this case for the rest of this subsection. Scheithauer showed in Theorem 4.7 of [Sch09] that the matrix $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ acts as

$$\rho_D(m)\mathfrak{e}_\gamma = \xi \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta \in D^{c*}} e(-a\beta_c^2/2)e(-b(\beta,\gamma))e(-bd(\gamma^2/2))\mathfrak{e}_{d\gamma+\beta} \tag{78}$$

on $\mathbb{C}[D]$, where ξ is a root of unity that depends on m and D. We will write $\xi = \xi_D(m)$ if we want to highlight this dependance. Take any congruence subgroup Γ of level N, i.e. $\Gamma(N) \subset \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and any character of finite order χ of Γ with $\Gamma(N) \subset \ker(\chi)$. We can define a vector valued modular form of weight $k \in \mathbb{Z}$ *for the congruence subgroup G with character χ* to be a holomorphic function $f : \mathbb{H} \to \mathbb{C}[D]$ that satisfies

$$(f|_k^{\rho_D} m)(\tau) = \chi(m)f(\tau) \;\forall m \in \Gamma. \tag{79}$$

Let s be any cusp of Γ and take $M_s \in \mathrm{SL}_2(\mathbb{Z})$ such that $M_s i\infty = s$. We can define f at the cusp s by $f_s(\tau) = (f|_k^{\rho_D} M_s)(\tau)$. We set

$$\Gamma_\infty^s = \Gamma_\infty \cap M_s^{-1}\Gamma M_s \tag{80}$$

and call the smallest positive integer $t(s)$ such that $\pm T^{t(s)} \in \Gamma_\infty^s$ the *width* of the cusp s. For simplicity we assume $-1 \in \Gamma$, then the width is the smallest positive integer such that $T^{t(s)} \in \Gamma_\infty^s$. We define $T_s \in \Gamma$ by $T^{t(s)} = M_s^{-1}T_sM_s$ and for any component $f_{s,\gamma}$ of f_s we obtain

$$f_{s,\gamma}(\tau + t(s)) = \chi(T_s)e(-t(s)\gamma^2/2)f_{s,\gamma}(\tau). \tag{81}$$

Since χ has finite order we get that $\chi(T_s)$ is a root of unity, i.e. we find $r_s \in \mathbb{Q}$ such that $\chi(T_s) = e(r_s)$. We put $g(\tau) = f_{s,\gamma}(t(s)\tau)$ and obtain $g(\tau+1) = \chi(T_s)e(-t(s)\gamma^2/2)g(\tau) = e(r_s - t(s)\gamma^2/2)g(\tau)$. Altogether we obtain a Fourier expansion of f at the cusp s by

$$f_s(\tau) = \sum_{\gamma \in D} \sum_{n \in \mathbb{Z}+(r_s - t(s)\gamma^2/2)} a_s(n)q_{t(s)}^n. \tag{82}$$

In [Bor00] the numbers r_s are evaluated in the special case of certain Dirichlet characters for $\Gamma = \Gamma_0(N)$. Next we specialize the discussion to $\Gamma = \Gamma_1(N)$. We may denote the level of the discriminant form D by M and assume $(N, M) = 1$. We furthermore assume that we can decompose D as $D = D_1 \oplus D_2$ with discriminant forms D_i of level $M_i = \mathrm{level}(D_i)$ for $i = 1, 2$. We assume $(M_1, M_2) = 1$ and obtain $M = M_1M_2$. For a fixed rational number $x \in \mathbb{Q}$ we embed $D(x)$ into D such that $v \in \mathbb{C}[D(x)]$ is just an element $v = \sum_{\lambda \in D} a_\lambda \mathfrak{e}_\lambda \in \mathbb{C}[D]$ with $a_\lambda = 0$ if $\lambda \notin D(x)$. Let $f : \mathbb{H} \to \mathbb{C}[D]$ be a modular form for $\Gamma_1(N)$ with character χ of $\Gamma_1(N)$ of finite order with $\Gamma(N) \subset \ker(\chi)$. For example for $x \in \mathbb{Q}$ with $Nx \in \mathbb{Z}$ we can define such a character by $\chi_x(M) = e(-bx)$ with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$. We define $f' : \mathbb{H} \to \mathbb{C}[D_1]$ by $f'(\tau) = \langle f(\tau), v \rangle$. Where $\langle \cdot, \cdot \rangle$ is the partial pairing defined by (65). Of course this is a holomorphic function.

Proposition 2.2.1. *Take a vector valued modular form f of weight k for ρ_D for $\Gamma_1(N)$ with character χ. For $x \in \mathbb{Q}$ and any $v \in \mathbb{C}[D_2(x)]$ the holomorphic function f', defined by $f'(\tau) = \langle f(\tau), v\rangle$ is a vector valued modular form for ρ_{D_1} for $\Gamma_1(NM_2)$ with character $\chi\chi_x$. Let s be a cusp of $\Gamma_1(NM_2)$ and $M_s \in SL_2(\mathbb{Z})$ with $M_s i\infty = s$. We can consider s as a cusp of $\Gamma_1(N)$ and write $f_s = f|_k M_s$. Then $f'_s = f'|_k M_s$ is given by*

$$f'_s(\tau) = \sum_{\lambda_1 \in D_1} \left(\sum_{\lambda_2 \in D_2} (\rho_{D_2}(M_s)\mathfrak{e}_{\lambda_2}, v)(f_s)_{\lambda_1+\lambda_2}(\tau) \right) \mathfrak{e}_{\lambda_1}. \tag{83}$$

Proof. A direct consequence of (78) is that

$$\rho_{D_2}(m)\mathfrak{e}_{\lambda_2} = \chi_{D_2}(m)\chi_{\lambda_2^2/2}(m)\mathfrak{e}_{d\lambda_2} = \chi_{\lambda_2^2/2}(m)\mathfrak{e}_{\lambda_2}, \tag{84}$$

for all $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(M_2)$. Notice that we have $\chi_{D_2}(m) = 1$ and $d\lambda_2 = \lambda_2$, because of $a \equiv d \equiv 1 (\mathrm{mod}\ M_2)$. Using this we can compute for $m \in \Gamma_1(NM_1)$ that

$$\begin{aligned}
(f'|_k^{\rho_{D_1}} m)(\tau) &= j(m,\tau)^{-k}\rho_{D_1}^{-1}(m) f'(m\tau) && (85)\\
&= \langle j(m,\tau)^{-k}\rho_{D_1}^{-1}(m) \otimes \mathrm{Id} f(m\tau), v\rangle && (86)\\
&= \langle j(m,\tau)^{-k}\mathrm{Id} \otimes \rho_{D_2}(m)\rho_D^{-1}(m) f(m\tau), v\rangle && (87)\\
&= \langle \chi(m)\mathrm{Id} \otimes \rho_{D_2}(m) f(\tau), v\rangle && (88)\\
&= \sum_{\lambda \in D} \chi(m)\langle \mathrm{Id} \otimes \rho_{D_2}(m) f_{\lambda_1+\lambda_2}(\tau)\mathfrak{e}_\lambda, v\rangle && (89)\\
&= \sum_{\lambda_1 \in D_1} \sum_{\lambda_2 \in D_2(x)} \chi(m)\chi_x(m)(\mathfrak{e}_{\lambda_2}, v) f_{\lambda_1+\lambda_2}(\tau)\mathfrak{e}_{\lambda_1} && (90)\\
&= \chi(m)\chi_x(m) f'(\tau). && (91)
\end{aligned}$$

So f' is indeed a vector valued modular form as stated in the proposition. The statement about f'_s follows by a similar argument. □

An important special case is $v = \sum_{\lambda \in D_2(0)} a_\lambda \mathfrak{e}_\lambda \in \mathbb{C}[D_2(0)]$, i.e. v is supported by isotropic elements of D_2. If v furthermore satisfies $a_{d\lambda} = a_\lambda$ for all $\lambda \in D_2(0)$ and $d \in (\mathbb{Z}/M_2\mathbb{Z})^*$ we call it *invariant* under $(\mathbb{Z}/M_2\mathbb{Z})^*$.

Proposition 2.2.2. *Take a vector valued modular form f of weight k for ρ_D for $\Gamma_0(N)$ with Dedekind character χ. For any $v \in \mathbb{C}[D_2(0)]$ invariant under $(\mathbb{Z}/M_2\mathbb{Z})^*$ the holomorphic function f', defined by $f'(\tau) = \langle f(\tau), v\rangle$ is a vector valued modular form for ρ_{D_1} for $\Gamma_0(NM_2)$ with character $\chi\chi_{D_2}$. Let s be a cusp of $\Gamma_0(NM_2)$ and $M_s \in SL_2(\mathbb{Z})$ with $M_s i\infty = s$. We can consider s as a cusp of $\Gamma_0(N)$ and write $f_s = f|_k M_s$. Then $f'_s = f'|_k M_s$ is given by*

$$f'_s(\tau) = \sum_{\lambda_1 \in D_1} \left(\sum_{\lambda_2 \in D_2} (\rho_{D_2}(M_s)\mathfrak{e}_{\lambda_2}, v)(f_s)_{\lambda_1+\lambda_2}(\tau) \right) \mathfrak{e}_{\lambda_1}. \tag{92}$$

Proof. This proof makes use of

$$\rho_{D_2}(m)\mathfrak{e}_{\lambda_2} = \chi_{D_2}(m)\mathfrak{e}_{d\lambda_2} \tag{93}$$

for any $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M_2)$ which is a consequence of (78) as well. The rest of the reasoning is similar to the proof of Proposition 2.2.1. □

2.3 Jacobi forms of lattice index

In this subsection we introduce Jacobi forms of lattice index, which are closely related to vector-valued modular forms for the Weil representation. A classical introduction is [EZ13]. Further sources are [CG13] and [GN] and there are well-written expositions in [Wan19b] and [Moc19]. The book [CS17] gives another short introduction. We especially follow [Wan19b] closely. We assume that L is an even positive-definite lattice. As usual we denote is bilinear form by $(\cdot,\cdot)$ and its dual lattice by L'. To the lattice L we can associate its *Heisenberg group*

$$H(L\otimes\mathbb{R}) := \{[x,y;r] : x,y\in L\otimes\mathbb{R}, r\in\mathbb{R}\}, \tag{94}$$

equipped with the group structure defined by

$$[x_1,y_1;r_1][x_2,y_2;r_2] = [x_1+x_2, y_1+y_2; r_1+r_2+1/2((x_1,y_2)-(x_2,y_1))]. \tag{95}$$

The *integral Heisenberg group* is given by the subgroup

$$H(L) := \{[x,y;r] : x,y\in L, r+1/2(x,y)\in\mathbb{Z}\}. \tag{96}$$

For $m=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{R})$ we set

$$m[x,y;r] := [dx-cy, ay-bx : r] = [(x,y)m^{-1};r]\in H(L\otimes\mathbb{R}). \tag{97}$$

This defines an action of $\mathrm{SL}_2(\mathbb{R})$ on $H(L\otimes\mathbb{R})$ and we can consider the semi-direct product $\mathrm{SL}_2(\mathbb{R})\rtimes H(L\otimes\mathbb{R})$ with the group structure

$$(m_1,h_1)(m_2,h_2) = (m_1m_2, (m_2^{-1}h_1)h_2). \tag{98}$$

We denote this group by $\Gamma^J(L\otimes\mathbb{R})$ and call it the *Jacobi group*. The *integral Jacobi group* is given by the subgroup

$$\Gamma^J(L) := \mathrm{SL}_2(\mathbb{Z})\rtimes H(L). \tag{99}$$

Let ϕ be a holomorphic function on $\mathbb{H}\times(L\otimes\mathbb{C})$. We can define slash-operators $|_k m$ and $|_k h$ for $k\in\mathbb{Z}$, $m=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{R})$ and $h=[x,y;r]\in H(L\otimes\mathbb{R})$ by

$$(\phi|_k m)(\tau,z) = j(m,\tau)^{-k}\exp\left(-\pi i\frac{c(z,z)}{j(m,\tau)}\right)\phi\left(m\tau,\frac{z}{j(m,\tau)}\right) \tag{100}$$

$$(\phi|_k h)(\tau,z) = \exp\left(\pi i((x,x)\tau+2(x,z)+(x,y)+2r)\right)\phi(\tau,z+x\tau+y). \tag{101}$$

By use of $(m,h)=(m,[0,0;0])(\mathrm{Id},h)$ we can set

$$(\phi|_k(m,h))(\tau,z) := (\phi|_k(m,[0,0;0]))(\tau,z)(\phi|_k(\mathrm{Id},h))(\tau,z). \tag{102}$$

This defines an action of the Jacobi group $\Gamma^J(L\otimes\mathbb{R})$ on the space of such functions. We can introduce such slash-operators for half-integers $k\in\frac{1}{2}\mathbb{Z}$ as well. But one has to introduce suitable multiplier systems or replace the Jacobi group by a suitable double cover. Since we are not in need of this we just refer to the literature. See [CS17], [Wan19b] and [GN]. A holomorphic function ϕ on $\mathbb{H}\times(L\otimes\mathbb{C})$ will be called *weakly holomorphic Jacobi form of weight k and index L* if it satisfies

$$(\phi|_k(m,h))(\tau,z) = \phi(\tau,z)\ \forall(m,h)\in H(L) \tag{103}$$

and admits for some $n_1\in\mathbb{Z}$ a Fourier expansion like

$$\phi(\tau,z) = \sum_{n>n_1, l\in L'} f(n,l)e^{2\pi i(n\tau+(l,z))}. \tag{104}$$

If a weakly holomorphic Jacobi form ϕ satisfies $f(n,l) = 0$ for all $n < 0$ and $l \in L'$ we call it a *weak Jacobi form*. We put $N(n,l) := 2n - (l,l)$ and call ϕ a *holomorphic Jacobi form* if $f(n,l) = 0$ unless $N(n,l) \geq 0$. Finally we call ϕ a *cusp form* if $f(n,l) = 0$ unless $N(n,l) > 0$. The number $N(n,l)$ is called the *hyperbolic norm* of the Fourier coefficient $f(n,l)$ and we call

$$\sum_{n>n_1, l\in L' N(n,l)<0} f(n,l)e^{2\pi i(n\tau+(l,z))} \tag{105}$$

the *singular part* of ϕ. The corresponding spaces of Jacobi forms will be denoted by $J_{k,L}^{w.h.}$, $J_{k,L}^{w}$, $J_{k,L}$ and $J_{k,L}^{c}$. Given a positive-definite lattice L we can define the *Jacobi theta series of* L for $\lambda \in D(L)$ by

$$\Theta_\lambda^L(\tau, z) = \sum_{l\in\lambda+L} e^{\pi i((l,l)\tau+2(l,z))}. \tag{106}$$

It is well known that those functions converge locally uniformly on $\mathbb{H} \times (L \otimes \mathbb{C})$ and therefore define holomorphic functions on this space. The function $\Theta^L(\tau, z) = \sum_{\lambda\in D(L)} \Theta_\lambda^L(\tau, z)\mathfrak{e}_\lambda$ has many remarkable properties. For example $\Theta^L(\tau, 0)$ defines a vector valued modular form of weight $\frac{\mathrm{rank}(L)}{2}$ for the Weil representation $\rho_{D(L)}$ and the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$. This is a special case of Theorem 4.1. in [Bor98]. For fixed $\tau \in \mathbb{H}$ the functions $\Theta_\lambda^L(\tau, z)$ are linearly independent and any weakly holomorphic Jacobi form $\phi \in J_{k,L}^{w.h.}$ can be decomposed as

$$\phi(\tau, z) = \sum_{\lambda\in D(L)} f_\lambda(\tau)\Theta_\lambda^L(\tau, z). \tag{107}$$

We call this the *theta decomposition of* ϕ. See in particular the discussion in section 1 of [CG13] and the literature cited therein. In fact the functions $f_\lambda(\tau)$ define a nearly holomorphic modular form $f_\phi(\tau) = \sum_{\lambda\in D(L)} f_\lambda(\tau)\mathfrak{e}_\lambda$ of weight $k + \frac{\mathrm{rank}(L)}{2} \in \frac{1}{2}\mathbb{Z}$ for the Weil representation $\rho_{D(L)}$ for the full metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$.

Theorem 2.3.1 (theta decomposition)**.** *Let L be an even positive-definite lattice. For a weakly holomorphic Jacobi form of weight $k \in \mathbb{Z}$ and index L the map*

$$\phi \mapsto f_\phi \tag{108}$$

defines an isomorphism from the space of weakly holomorphic modular forms $J_{k,L}^{w.h.}$ of weight k and index L to the space of nearly holomorphic modular forms $M^!_{k+\frac{rank(L)}{2}}\left(\rho_{D(L)}\right)$ of weight $k + \frac{rank(L)}{2}$ for $\rho_{D(L)}$. Restricted to the subspace of holomorphic (resp. cusp) Jacobi forms this induces an isomorphism to the space of holomorphic (resp. cusp) vector valued modular forms.

For further details about the theta decomposition see [Wan19b] and [Gri12]. Especially useful is [Moc19] and the literature therein. Notice in particular that the space of Jacobi forms $J_{k,L}^{w.h.}$ depends on a particular even positive-definite lattice, whereas the space of modular forms $\mathrm{M}^!_{k+\frac{\mathrm{rank}(L)}{2}}\left(\rho_{D(L)}\right)$ just depends on the discriminant form $D(L)$ of this lattice.

2.4 Orthogonal modular forms

Orthogonal modular forms are meromorphic functions defined on suitable domains, associated to the orthogonal group of some lattice, that have some invariance property under a suitable group action. Due to their importance in algebraic geometry, number theory and representation theory those functions are studied extensively in the literature. Here we give a brief introduction to this topic and discuss *automorphic products*, a particularly important class of orthogonal modular forms. The fundamental references are [Bor95a] and [Bor98]. A further important exposition is [Bru04]. A slightly different approach towards automorphic products was taken in [GN]. The following discussion is based on

[Wan19b], [Bru04] and [Bor98]. A *hermitian symmetric space* H is a connected hermitian manifold such that each point $p \in H$ is an isolated fixed point of an involutive holomorphic isometry. A hermitian symmetric space will be called *irreducible* if its only possible decomposition into a product of hermitian symmetric spaces consists of itself. We can distinguish three different types of hermitian symmetric domains. Those types are the *euclidean type*, the *compact type* and the *non-compact type*. For us just the non-compact type matters. We call a hermitian symmetric space, that can be written as a product of irreducible hermitian symmetric spaces of non-compact type, a *hermitian symmetric domain*. Let H be a hermitian symmetric domain and fix any point $p \in H$, which will be called *base point*. We denote the group of isometries of H by $\mathrm{I}(H)$. The following theorem is crucial in the understanding of hermitian symmetric domains.

Theorem 2.4.1. *Let H be a hermitian symmetric domain and p its base point.*

1. *$I(H)$ carries a natural Lie group structure.*
2. *The subgroup $K_p := \{g \in I(H) : g(p) = p\}$ is compact in $I(H)$ and the map*
$$I(H)^\circ/K_p \to H, gK_p \mapsto g(p) \tag{109}$$
is an isomorphism of real C^∞-manifolds.
3. *The quotient $I(H)^\circ/K_p$ can be canonically equipped with the structure of a complex manifold such that this isomorphism is holomorphic.*
4. *Conversely for a pair of a connected Lie group G and a closed subgroup K the quotient G/K can be equipped with the structure of a hermitian symmetric domain if certain condition are satisfied.*

The statement of this theorem is a combination of Theorem IV 3.3. in [Hel01] and Proposition VIII 4.2. in [Hel01]. This book is in fact a general reference for hermitian symmetric spaces. We refer the reader to it for any further details. Using this statement we can consider a hermitian symmetric domain H as a certain quotient of Lie groups G/K. We can now think of *modular forms* as meromorphic functions on G/H that have certain invariance properties under a suitable discrete subgroup $\Gamma \subset G$. In the following we are mainly interested in *orthogonal modular forms*. Those are modular forms defined on hermitian symmetric domains of the form $O(n+2)/O(n) \times O(2)$. Before this we introduce a more explicit description of such hermitian symmetric domains. Let L be an even lattice of signature $(n, 2)$ with $n \geq 4$. We denote its real and complex ambient spaces by $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ and $V(\mathbb{C}) = V \otimes_{\mathbb{R}} \mathbb{C}$ respectively. We set

$$\mathcal{K}(L) := \{[Z] \in \mathbb{P}(V(\mathbb{C})) : (Z, Z) = 0, (Z, \overline{Z}) < 0\}^+. \tag{110}$$

Here $\cdot^+$ means that we fix a choice out of two connected components of this space. The orthogonal group $O(V)$ has a natural subgroup $O^+(V)$ of index 2 which preserves $\mathcal{K}(L)$. In fact for a choice of base point $p \in \mathcal{K}(L)$ the space $\mathcal{K}(L)$ carries the structure of a hermitian symmetric domain that represents the quotient $\mathcal{D}_L = O^+(V)/\mathrm{Stab}_p(O^+(V))$. We call $\mathcal{K}(L)$ the *projective model* of the hermitian symmetric domain $\mathcal{D}_L$. Notice that there are further possibilities to describe the quotient $\mathcal{D}_L$ in a more explicit way. For example there are the *Grassmannian models*. See [Bru04] and [Bor98] for this. We stick to the projective model and introduce the *affine cone over* $\mathcal{K}(L)$ by

$$\tilde{\mathcal{K}}(L) := \{Z \in V(\mathbb{C}) \smallsetminus \{0\} : [Z] \in \mathcal{K}(L)\}. \tag{111}$$

For a rational vector $v \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ with $(v, v) > 0$ we define the associated *rational quadratic divisor* by

$$\mathcal{K}_v(L) := \{[Z] \in \mathcal{K}(L) : (Z, v) = 0\}. \tag{112}$$

In the affine cone we denote the corresponding rational quadratic divisor of $v \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ with $(v, v) > 0$ by

$$v^\perp := \tilde{\mathcal{K}}_v(L) := \{Z \in \tilde{\mathcal{K}}(L) : [Z] \in \mathcal{K}_v(L)\}. \tag{113}$$

The *discriminant kernel* of $O(L)$ is the subgroup $\tilde{O}(L)$ of finite index of $O(L)$ which acts trivially in the discriminant $D(L) = L'/L$. We set $\Gamma(L) = \tilde{O}(L) \cap O^+(V)$. Take a subgroup $\Gamma \subset \Gamma(L)$ of finite index and a unitary character $\chi : \Gamma \to \mathbb{C}^*$. A meromorphic function $\Phi : \tilde{\mathcal{K}}(L) \to \mathbb{C}$ is called *modular form* of weight $k \in \mathbb{Z}$ for Γ and χ if for all $Z \in \tilde{\mathcal{K}}(L)$ we have

$$\Phi(MZ) = \chi(M)\Phi(Z)\ \forall M \in \Gamma \tag{114}$$

$$\Phi(tZ) = t^{-k}\Phi(Z)\ \forall t \in \mathbb{C}^*. \tag{115}$$

In the following we will call such a modular form *homogeneous*. This is motivated by equation (115). The *modular variety of orthogonal type* is given by the quotient $\Gamma\backslash\mathcal{K}(L)$. It is well-known to carry the structure of a complex space. By the theory of Baily and Borel [BB66] we know that this space has a compactification, the so called *Baily-Borel compactification* $\overline{\Gamma\backslash\mathcal{K}(L)}^{BB}$. More precisely they showed that $\Gamma\backslash\mathcal{K}(L)$ can be extended to a set $\overline{\Gamma\backslash\mathcal{K}(L)}^{BB}$ by certain *boundary components* of dimension 0 and 1, such that this set has a natural structure of a complex space that extends the structure of $\Gamma\backslash\mathcal{K}(L)$. In fact $\overline{\Gamma\backslash\mathcal{K}(L)}^{BB}$ even carries the structure of a projective variety. We call those boundary components the *cusps* of $\mathcal{K}(L)$ or $\Gamma\backslash\mathcal{K}(L)$ respectively. A 0-dimensional cusp of $\mathcal{K}(L)$ can be represented by a primitive isotropic vector $l \in L$. In the following we discuss what it means to *expand* a modular form Φ at such a cusp. We consider the lattice L_l and fix $\gamma \in L'$ such that $(l, \gamma) = 1$. Following the discussion on the previous section we introduce the lattices $L_{l,\gamma} = L \cap l^\perp \cap \gamma^\perp$ and L'_0 and use all other notations from there. We assume furthermore that $L_{l,\gamma}$ contains an isotropic element. Using $L \otimes \mathbb{C} = L_{l,\gamma} \otimes \mathbb{C} \oplus \mathbb{C}\gamma \oplus \mathbb{C}l$ we can write $Z_L = Z + a\gamma + bl$ for $Z_L \in L \otimes \mathbb{C}$ and $Z \in L_{l,\gamma} \otimes \mathbb{C}$. The following elaboration is based on Section 3.2 in [Bru04]. In the affine cone $\tilde{\mathcal{K}}(L)$ we define subsets

$$\tilde{\mathcal{K}}(L)_l := \{Z_L \in \tilde{\mathcal{K}}(L) : (Z_L, l) = 1\}. \tag{116}$$

Notice that a vector $Z_L = X_L + iY_L \in V(\mathbb{Z})$ satisfies $Z_L^2 = 0$ and $(Z_L, \overline{Z_L}) < 0$ if and only if $X_L \perp Y_L$ and $X_L^2 = Y_L^2 < 0$. So X_L and Y_L span a 2-dimensional negative definite subspace in V. A primitive isotropic vector l must therefore satisfy $(Z_L, l) \neq 0$ if $[Z_L] \in \mathcal{K}(L)$. We can use this to express $\tilde{\mathcal{K}}(L)$ in a disjoint union

$$\tilde{\mathcal{K}}(L) = \bigcup_{\alpha \in \mathbb{C}^*} \tilde{\mathcal{K}}(L)_{\alpha l} \tag{117}$$

for any primitive isotropic vector $l \in L$. Furthermore the set

$$\{Z = X + iY \in L_{l,\gamma} \otimes \mathbb{C} : (Y, Y) < 0\}. \tag{118}$$

has two connected components and one of them gets mapped into $\mathcal{K}(L)$ by

$$Z \mapsto [Z_L] = [Z + \gamma + (-q(Z) - q(\gamma))l]. \tag{119}$$

We denote this connected component by $\mathbb{H}_l$. We can write

$$\mathbb{H}_l = L_{l,\gamma} \otimes \mathbb{R} + iC^+. \tag{120}$$

Here C^+ is called the *positive cone* and defined to be the connected component of $\{Y \in L_{l,\gamma} \otimes \mathbb{R} : (Y, Y) < 0\}$ which is contained in $\mathbb{H}_l$. By $\mathrm{pr}_l(Z) = Z + \gamma + (-q(Z) - q(\gamma))l$ we can define a map $\mathbb{H}_l \to \tilde{\mathcal{K}}(L)_l$ which is biholomorphic. Of course the map $\tilde{\mathcal{K}}(L)_l \to \mathcal{K}(L)$ defined by $Z_L \mapsto [Z_L]$ is also biholomorphic, so we get a composition of biholomorphic maps

$$\mathbb{H}_l \xrightarrow{\mathrm{pr}_l} \tilde{\mathcal{K}}(L)_l \xrightarrow{[\cdot]} \mathcal{K}(L). \tag{121}$$

The space $\mathbb{H}_l$ will be called the *tube domain* of $\mathcal{K}(L)$ corresponding to l. Of course the space $\mathbb{H}_l$ and the projection pr_l depend on the choice of $\gamma \in L'$ but we keep this implicit. For $\lambda \in L \otimes \mathbb{Q}$ with positive norm we can introduce subsets $\lambda^\perp$ of $\mathbb{H}_l$ similar to (113). Assume $\lambda = \lambda_{l,\gamma} + a\gamma + bl$ for $\lambda_{l,\gamma} \in L_{l,\gamma} \otimes \mathbb{Q}$ and $a, b \in \mathbb{Q}$. We put

$$\lambda^\perp = \{Z \in \mathbb{H}_l : (\mathrm{pr}_l(Z), \lambda) = 0\} \subset \mathbb{H}_l. \tag{122}$$

and for $\beta \in L'/L$ and $m \in \mathbb{Z} - \frac{\beta^2}{2}$ we set

$$H(\beta, m) = \bigcup_{\lambda \in \beta + L, q(\lambda) = m} \lambda^\perp \subset \mathbb{H}_l. \tag{123}$$

Notice that the subsets $\lambda^\perp$ and $H(\beta, m)$ have codimension 1 within $\mathbb{H}_l$. We can furthermore introduce a divisor on $\mathbb{H}_l$

$$\sum_{\lambda \in \beta + L, q(\lambda) = m} \lambda^\perp \tag{124}$$

which is supported by $H(\beta, m)$ and called a *Heegner divisor.* We will call the connected components of

$$\mathbb{H}_l \setminus H(\beta, m) \tag{125}$$

the *Weyl chambers* of index (β, m). To each Weyl chamber W we can associate a *Weyl vector* $\rho_{\beta,m}(W)$ in $L_{l,\gamma} \otimes \mathbb{R}$ but we refer to the literature for the definition. See Definition 3.5. in [Bru04]. Due to the decomposition in (117) we find a unique number $J_{l,\gamma}(g, Z) \in \mathbb{C}^*$ and an element $g \cdot Z \in \mathbb{H}_l$ such that

$$g\mathrm{pr}_l(Z) = J_{l,\gamma}(g, Z)\mathrm{pr}_l(g \cdot Z) \tag{126}$$

for any $g \in O^+(L)$ and $Z \in \mathbb{H}_l$. This defines an action of $O^+(L)$ on $\mathbb{H}_l$. For a meromorphic function $F : \tilde{\mathcal{K}}(L) \to \mathbb{C}$ we can define a meromorphic function $F_l : \mathbb{H}_l \to \mathbb{C}$ by $F_l = F \circ \mathrm{pr}_l$. Assume that Φ is a meromorphic modular form of weight k for a finite index subgroup $\Gamma \subset \Gamma(l)$ and a character $\chi : \Gamma \to \mathbb{C}^*$. Then Φ_l satisfies

$$\Phi_l|_k g(Z) := J_{l,\gamma}(g, Z)^{-k}\Phi_l(g \cdot Z) = \chi(g)\Phi_l(Z)\ \forall Z \in \mathbb{H}_l. \tag{127}$$

We call Φ_l the *expansion of* Φ *at the cusp* l. Of course a modular form Φ is uniquely determined by its expansion Φ_l at the cusp l. So far we just worked with integral weights $k \in \mathbb{Z}$. For now we assume that k is a rational number, i.e. $k \in \mathbb{Q}$. This discussion relies on section 3.3 in [Bru04]. We fix a choice of holomorphic logarithm $\mathrm{Log}(j(g, Z))$ for each $g \in O^+(V)$ and $Z \in \mathbb{H}_l$ such that we can define

$$j(g, Z)^k = e^{k\mathrm{Log}(j(g,Z))}. \tag{128}$$

For every rational $k \in \mathbb{Q}$ there exists a map ω_k from $O^+(V) \times O^+(V)$ to the set of roots of unity of order bounded by the denominator of k such that

$$j(g_1g_2, Z)^k = \omega_k(g_1, g_2)j(g_1, g_2Z)^k g(g_2, Z)^k. \tag{129}$$

Notice that ω_k just depends on $k(\mathrm{mod}\ \mathbb{Z})$. Let $\Gamma \subset O^+(V)$ be a subgroup as above. A *multiplier system* of weight k of Γ is a map

$$\chi : \Gamma \to S^1 = \{t \in \mathbb{C} : |t| = 1\} \tag{130}$$

that satisfies

$$\chi(g_1g_2) = \omega_k(g_1, g_2)\chi(g_1)\chi(g_2). \tag{131}$$

for all $g_1, g_2 \in \Gamma$. If k is an integer the definition of a multiplier system reduces to a character of Γ. So far we we just considered modular forms Φ defined on $\tilde{\mathcal{K}}(L)$.

Definition 2.4.2. Take $k \in \mathbb{Q}$. Let $\Gamma \subset \Gamma(L)$ be a subgroup of finite index and χ a multiplier system of weight k. A meromorphic function $\Phi : \mathbb{H}_l \to \mathbb{C}$ is called *meromorphic modular form* of weight r and multiplier system χ with respect to Γ, if

$$\Phi(gZ) = \chi(g)j(g, Z)^k\Phi(Z) \tag{132}$$

for all $g \in \Gamma$. If Φ is holomorphic on $\mathbb{H}_l$ we call it a *holomorphic modular form.*

This is Definition 3.18. in [Bru04]. In the following we will work with this definition of modular forms, since it makes it easier to consider modular forms of rational weight $k \in \mathbb{Q}$. If Φ is a *homogeneous* modular form of integer weight $k \in \mathbb{Z}$ with some character χ for a group Γ, then its expansion Φ_l at the cusp l is a modular form in the sense of Definition 2.4.2. Let f be a nearly holomorphic modular form of weight $k = 1 - \frac{n}{2}$ for the Weil representation $\rho_{D(L)}$. We assume that the Fourier coefficients $[f_\gamma](n)$ contributing to the principal part, i.e. with $n < 0$, satisfy $[f_\gamma](n) \in \mathbb{Z}$. The *Weyl chambers* with respect to f are the connected components of

$$\mathbb{H}_l \setminus \bigcup_{\beta \in L_0'/L} \bigcup_{m<0, [f_\beta](m) \neq 0} H(\beta, m). \tag{133}$$

For each such Weyl chamber W and each $\beta \in L_0'/L$ and $m \in \mathbb{Z} - \frac{\beta^2}{2}$ we find a Weyl chamber $W_{\beta,m}$ of index (β, m) such that $W \subset W_{\beta,m}$. We can define the *Weyl vector* $\rho_{W,f}$ attached to W and f by

$$\rho_{W,f} = \frac{1}{2} \sum_{\beta \in L_0'/L} \sum_{m \in \mathbb{Z} - \frac{\beta^2}{2}, m<0} [f_\beta](m) \rho_{\beta,m}(W_{\beta,m}). \tag{134}$$

For any Weyl chamber W with respect to f and every $\lambda \in L_{l,\gamma}'$ we write $(\lambda, W) < 0$ if $(\lambda, w) < 0$ for each $w \in W$.

Theorem 2.4.3 (Borcherds 1998)**.** *Let L be an even lattice of signature $(n, 2)$ for $n \geq 3$. Take a primitive isotropic vector $l \in L$ and some $\gamma \in L'$ with $(l, \gamma) = 1$. As above we consider the space $\mathbb{H}_l$. Let f be a nearly holomorphic modular form of weight $k = 1 - \frac{n}{2}$ for the Weil representation $\rho_{D(L)}$. We assume that the Fourier coefficients $[f_\gamma](n)$ contributing to the principal part, i.e. with $n < 0$, satisfy $[f_\gamma](n) \in \mathbb{Z}$. Then there exists a meromorphic function $\Phi_f : \mathbb{H}_l \to \mathbb{C}$ with the following properties:*

1. *The function Φ_f is a meromorphic modular form of rational weight $\frac{[f_0](0)}{2}$ for the group $\Gamma(L)$ and some multiplier system χ of finite order. If $[f_0](0)$ is an even integer than χ is just a character.*
2. *The only zeros or poles of Φ_f are on rational quadratic divisors $\lambda^\perp$ for $\lambda \in L$ with $(\lambda, \lambda) > 0$ are zeros or poles of vanishing order*

$$\sum_{0<x \in \mathbb{Q}, x\lambda \in L'} [f_{x\lambda+L}] \left(-x^2 \frac{\lambda^2}{2} \right). \tag{135}$$

3. *For each Weyl chamber $W \subset \mathbb{H}_l$ with respect to f the modular form Φ_f has an infinite product expansion converging in a neighborhood of the cusp l. Up to a constant this product is given by*

$$e((Z, \rho_{W,f})) \prod_{\lambda \in L_{l,\gamma}', (\lambda, W)<0} \prod_{\delta \in L_0'/L, p(\delta) = \lambda + L_{l,\gamma}} (1 - e((\delta, \gamma) + (Z, \lambda)))^{[f_\delta](-\lambda^2/2)}. \tag{136}$$

Here $\rho_{W,f}$ is the Weyl vector attached to W and f.

This is Theorem 13.3 in [Bor98]. The precise formulation is Theorem 3.22 in [Bru04]. A meromorphic modular form Φ_f will be called an *automorphic product* of the nearly holomorphic modular form f. A necessary condition for a meromorphic modular form to be an automorphic product is of course that its divisor is linear combination of rational quadratic divisors. Yet for suitable lattices this property is also sufficient. See in particular the work of Bruinier in [Bru04] and [Bru14].

Theorem 2.4.4 (Bruiniers converse theorem)**.** *Let L be an even lattice of signature $(n, 2)$ with $n \geq 4$ such that there is a positive definite lattice and a positive integer m with $L = K \oplus II_{1,1} \oplus II_{1,1}(m)$. Then every modular form for the discriminant kernel $\Gamma(L)$ whose divisor is a linear combination of rational quadratic divisors is a multiple of a Borcherds product of a nearly holomorphic modular form for the Weil representation.*

In the following we specialize the discussion to automorphic products where we assume that the divisor has special properties. Let L be an even lattice of signature $(n, 2)$. Remember that the *divisor* of $r \in L$ is the positive integer $\mathrm{div}(r)$ given by $(r, L) = \mathrm{div}(r)\mathbb{Z}$. The element $r \in L$ is called *primitive* if $\mathbb{Q}r \cap L = \mathbb{Z}r$. It is easy to see that an element $r \in L$ is primitive if and only if there is an element $v_r \in L'$ such that $(r, v_r) = 1$. Notice that this does not depend on the assumption that L is an even lattice. We can characterize primitive elements by that in any rational lattice with non-degenerate bilinear form such that $(L')' = L$.

Lemma 2.4.5. *For every $r \in L \setminus \{0\}$ we set $r^* := \frac{1}{div(r)} r \in L'$. This element has the following properties:*

1. *The element $r^* \in L'$ is primitive and we have $\mathbb{Q}r \cap L' = \mathbb{Z}r^*$.*
2. *For every $r \in L \setminus \{0\}$ and every $q \in \mathbb{Q}$ with $qr \in L$ we have $div(qr) = q div(r)$ and $(qr)^* = r^*$.*
3. *For every $r \in L \setminus \{0\}$ we have $ord([r^*]) | div(r)$ and $ord([r^*]) = div(r)$ if and only if r is primitive in L.*
4. *For a primitive element $r^* \in L'$ we can set $r_0 = ord([r^*])r^*$ and this element is primitive in L.*

Proof. We have $(r, L) = \mathrm{div}(r)\mathbb{Z}$. This implies that $(r^*, L) = \mathbb{Z}$ and therefore r^* is primitive in L'. Obviously we have $\mathbb{Q}r \cap L' = \mathbb{Z}r^*$. Assume $q \in \mathbb{Q}$ satisfies $qr \in L$, then we have

$$\mathrm{div}(qr)\mathbb{Z} = (qr, L) = q\mathrm{div}(r)\mathbb{Z}. \tag{137}$$

This implies $\mathrm{div}(qr) = q\mathrm{div}(r)$ and $(qr)^* = r^*$ is a direct consequence. Because of $\mathrm{div}(r)r^* \in L$ we obviously have $\mathrm{ord}([r^*])|\mathrm{div}(r)$. If r is primitive then $\mathrm{ord}([r^*])r^* = \frac{\mathrm{ord}([r^*])}{\mathrm{div}(r)} r \in L$ implies $\frac{\mathrm{ord}([r^*])}{\mathrm{div}(r)} \in \mathbb{Z}$, so we get $\mathrm{ord}([r^*]) = \mathrm{div}(r)$. Now we assume that $\mathrm{ord}([r^*]) = \mathrm{div}(r)$ and $qr \in L$ for some $q \in \mathbb{Q}$. We have seen above that $(qr)^* = r^*$ and $\mathrm{ord}([(r)^*]) = \mathrm{ord}([(qr)^*])|\mathrm{div}(qr)$. But we also have $\mathrm{div}(qr) = q\mathrm{div}(r) = q\mathrm{ord}([(r)^*])$ and this implies $q \in \mathbb{Z}$. Therefore $r \in L$ is primitive. Let now $r^* \in L'$ be any primitive element of L'. Of course we have $r_0 := \mathrm{ord}([r^*])r^* \in L$. Since r^* is primitive we find $v_{r^*} \in L$ with $(r^*, v_{r^*}) = 1$. This implies

$$\mathrm{div}(r_0)\mathbb{Z} = (r_0, L) = \mathrm{ord}([r^*])(r^*, L) = \mathrm{ord}([r^*])\mathbb{Z}. \tag{138}$$

We obtain $\mathrm{div}(r_0) = \mathrm{ord}([r^*])$ and this implies that r_0 is primitive. □

For a vector $r \in L'$ with $r^2 \neq 0$, the corresponding *reflection* $\sigma_r : L \otimes \mathbb{Q} \to L \otimes \mathbb{Q}$ is defined by

$$\sigma_r(x) = x - \frac{2(r, x)}{(r, r)} r \text{ for } x \in L \otimes \mathbb{Q}. \tag{139}$$

Definition 2.4.6. A primitive vector $r \in L$ of positive norm will be called a *root* or *reflective* if σ_r is an automorphism of L.

Clearly a primitive vector $r \in L$ is a root if and only if $\frac{2r}{r^2} \in L'$. A rational quadratic divisor $r^\perp$ is called *reflective* if r is a root of L. For simplicity we will call a reflective rational quadratic divisor $r^\perp$ a *reflective divisor.*

Lemma 2.4.7. *Let L be an even non-degenerate lattice and $r \in L$ a primitive element. The following statements are equivalent:*

1. *The vector r is a root in L.*
2. *We have $\frac{2}{(r^*)^2} r^* \in L$.*
3. *The number $\frac{2}{(r^*)^2}$ is an integer and we have $ord([r^*])|\frac{2}{(r^*)^2}$.*

Proof. We start with 1. $\Rightarrow$ 2.: Since r^* is primitive we find $v_{r^*} \in L$ with $(v_{r^*}, r^*) = 1$, so we get $\frac{2}{(r^*)^2} r^* = v_{r^*} - \sigma_r(v_{r^*}) \in L$. Next we show 2. $\Rightarrow$ 3.: Since r^* is primitive we find an element $v_{r^*} \in L$ such that $(r^*, v_{r^*}) = 1$. We obtain that $\frac{2}{(r^*)^2} = \left(\frac{2r^*}{(r^*)^2}, v_{r^*}\right)$ must be an integer. The statement is a direct consequence. Finally we discuss 3. $\Rightarrow$ 1.: The statement directly implies that $\sigma_r(x) \in L$ for all $x \in L$. So σ_r is an automorphism of L. Since r is primitive it is also a root. □

For the following discussion we fix a primitive isotropic vector $l \in L$ and $\gamma \in L'$ with $(l, \gamma) = 1$. We may assume that $L_{l,\gamma}$ contains at least one non-trivial isotropic vector as well.

Definition 2.4.8. Let L be an even lattice of signature $(n, 2)$ and $k \in \mathbb{Q}$. A non-constant meromorphic modular form $\Phi : \mathbb{H}_l \to \mathbb{C}$ of weight k for the finite index subgroup $\Gamma \subset \Gamma(L)$ is called *reflective* if its divisor is a linear combination of reflective rational quadratic divisors. We call the modular form Φ *strongly reflective* if it is holomorphic and the multiplicity of each reflective quadratic divisor $r^\perp$ is either 0 or 1.

Of particular interest for us will be *reflective automorphic products*, i.e. automorphic products that are reflective modular forms. We will furthermore say that a reflective modular form is *reflective with a* 2*-root* if there is a root $r \in L$ with $r^2 = 2$ and $\mathrm{div}(r) = 1$, such that this reflective modular form satisfies $\mathrm{mult}(r^\perp) \neq 0$.

2.5 Kac-Moody algebras and their representation theory

Semi-simple Lie algebras are well-known to have a realization by generators and relations. Essentially one associates to each simple root α_i, of such a Lie algebra, some generators e_i, f_i and h_i which satisfy a collection of relations determined by the Cartan matrix of the Lie algebra. A particular feature of those Cartan matrices is that they are positive-definite. Conversely, some properties of those matrices can be used to define such Cartan matrices in an abstract way. The construction by generators and relations now yields precisely the semi-simple Lie algebras. It turns out that this construction still gives interesting Lie algebras, if the conditions on those Cartan matrices are loosened in a certain way. In particular, if we omit the positive-definiteness we obtain *Kac-Moody algebras*. The corresponding matrices are then called *generalized Cartan matrices*. Usually they are infinite dimensional but besides that, they have properties similar to those of semi-simple Lie algebras. For example they have a sensible root structure and a well understood theory of highest-weight modules. A very important subclass of such Kac-Moody algebras is given by the *affine Kac-Moody algebras*. Their most intriguing property is, that we don't have to construct them by generators and relations. This is because there is a construction for affine Kac-Moody algebras by extension of simple Lie algebras. In this subsection we give a brief introduction to those Lie algebras and discuss this construction in detail. Excellent introductory literature about Kac-Moody algebras is [Kac90], [KP84], [KW88] and [KMPS90]. Since this topic is important in string theory and conformal field theory there is also excellent literature from physics, namely [Fuc95] and [GO86]. For an introduction to semi-simple Lie algebras see [Hum12]. The following elaboration is based on all those sources. In particular [Kac90]. For the sake of simplicity we focus on *untwisted* affine Kac-Moody algebras. Let $\mathfrak{g}$ be a simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and a root system $\overline{\Delta}$. We fix a choice of simple roots

$$\overline{\Pi} = \{\alpha_1, \cdots, \alpha_n\} \subset \mathfrak{h} \tag{140}$$

and denote the corresponding set of coroots by

$$\check{\overline{\Pi}} = \{\check{\alpha}_1, \cdots, \check{\alpha}_n\} \subset \mathfrak{h}^*. \tag{141}$$

We introduce the *Cartan matrix* $\overline{A} = (a_{i,j})_{1 \leq i,j \leq n}$ of $\mathfrak{g}$ by $a_{i,j} = \langle \alpha_i, \check{\alpha}_j \rangle$ as usual. Here by $\langle \cdot, \cdot \rangle$ we denote the usual pairing. The *Killing form* of $\mathfrak{g}$ will be denoted by $(\cdot, \cdot)_K$ and the *normalized invariant bilinear form* $(\cdot, \cdot)$ is the Killing form rescaled such that $(\theta, \theta) = 2$. Here we denote by θ the highest root of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple there can just be two root lengths and the highest root θ is always a long root. Under the identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ induced by $(\cdot,\cdot)$ we can identify $\check{\alpha}_i$ with $2\alpha_i/(\alpha_i,\alpha_i)$. The corresponding *fundamental weights* $\overline{\Lambda}_i \in \mathfrak{h}^*$ are defined by

$$\overline{\Lambda}_i(\alpha_j) = \delta_{i,j}. \tag{142}$$

For such a simple Lie algebra $\mathfrak{g}$ the corresponding *untwisted affine Kac-Moody algebra* is

$$\hat{\mathfrak{g}} = \mathbb{C}[t,t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d. \tag{143}$$

Here K is a central element in d is a derivative such that the Lie bracket $[\cdot,\cdot]$ of $\hat{\mathfrak{g}}$ is defined by

$$[t^m \otimes x, t^n \otimes y] = t^{n+m} \otimes [x,y] + m\delta_{n+m,0}(x,y)K \tag{144}$$

$$[d, t^n \otimes y] = nt^n \otimes y. \tag{145}$$

We introduce the commutative subalgebra

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d \tag{146}$$

of $\hat{\mathfrak{g}}$. We call $\hat{\mathfrak{h}}$ the *affine Cartan subalgebra* or simply *Cartan subalgebra* of $\hat{\mathfrak{g}}$. By extending $\lambda \in \mathfrak{h}^*$ trivially to $\hat{\mathfrak{h}}^*$ we get $\mathfrak{h}^* \subset \hat{\mathfrak{h}}^*$. We define elements $\Lambda_0, \delta \in \hat{\mathfrak{h}}^*$ by $\Lambda_0(\mathfrak{h} \oplus \mathbb{C}d) = 0$, $\Lambda_0(K) = 1$ $\delta(\mathfrak{h} \oplus \mathbb{C}K) = 0$ and $\delta(d) = 1$. It is clear that we obtain

$$\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta. \tag{147}$$

We define a projection $\bar{\cdot} : \hat{\mathfrak{h}}^* \to \mathfrak{h}^*$ by $\bar{\cdot}|_{\mathfrak{h}^*} = \mathrm{id}|_{\mathfrak{h}^*}$ and $\bar{\cdot}|_{\mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta} = 0$. By $\alpha_0 = \delta - \theta$ we can define a simple root and it is well known that

$$\Pi = \{\alpha_0, \alpha_1, \cdots, \alpha_n\} \tag{148}$$

is a set of simple roots of $\hat{\mathfrak{g}}$. Furthermore by $\check{\alpha}_0 = K - \check{\theta}$ we can define a simple coroot such that

$$\check{\Pi} = \{\check{\alpha}_0, \check{\alpha}_1, \cdots, \check{\alpha}_n\} \tag{149}$$

is the corresponding set of coroots of $\hat{\mathfrak{g}}$. As in the finite case we can define a matrix $A = (a_{i,j})_{0 \leq i,j \leq n}$ by $a_{i,j} = \langle \alpha_i, \check{\alpha}_j \rangle$. This is the so called *generalized Cartan matrix* of $\hat{\mathfrak{g}}$. Of course we can reproduce the finite Cartan matrix $\overline{A}$ by deleting the 0-row and 0-column. Following §6.1 in [Kac90] we can associate to each simple root α_i certain numerical labels a_i and $\check{a}_i$, the so called *Coxeter labels* and *dual Coxeter labels.* Using those labels we introduce the numbers

$$h = \sum_{i=0}^{n} a_i \text{ and} \tag{150}$$

$$\check{h} = \sum_{i=0}^{n} \check{a}_i, \tag{151}$$

which we call *Coxeter number* and *dual Coxeter number.* In §6.1 of [Kac90] we can find a table with those numbers for all simple untwisted affine Kac-Moody algebras. Using the fact that $\check{a}_0 = 1$ we can write

$$\delta = \sum_{i=0}^{n} a_i \alpha_i \in Q \text{ and} \tag{152}$$

$$K = \sum_{i=0}^{n} \check{a}_i \check{\alpha}_i \in \check{Q}, \tag{153}$$

where Q and $\check{Q}$ are the *root lattice* and *coroot lattice* of $\hat{\mathfrak{g}}$, i.e. the $\mathbb{Z}$-span of the roots and coroots respectively. Another important property of the Coxeter labels is $\check{a}_i | a_i$ and the fact that for some symmetric matrix B we have

$$A = \text{diag}(a_0/\check{a}_0, \cdots, a_n/\check{a}_n)B. \tag{154}$$

Next we need to extend the normalized bilinear form $(\cdot, \cdot)$ to $\hat{\mathfrak{g}}$. We do this by

$$(t^m \otimes x, t^n \otimes y) = \delta_{n,m}(x, y), \tag{155}$$

$$(t^m \otimes x, K) = (t^m \otimes x, d) = 0, \tag{156}$$

$$(K, K) = (d, d) = 0 \text{ and} \tag{157}$$

$$(K, d) = a_0. \tag{158}$$

This bilinear form induces a identification map $\nu : \hat{\mathfrak{h}} \to \hat{\mathfrak{h}}^*$ that satisfies

$$\check{a}_i \nu(\check{\alpha}_i) = a_i \alpha_i, \tag{159}$$

$$\nu(K) = \delta \text{ and} \tag{160}$$

$$\nu(d) = a_0 \Lambda_0. \tag{161}$$

The induced normalized bilinear form $(\cdot, \cdot)$ on $\hat{\mathfrak{h}}^*$ satisfies

$$(\alpha_i, \alpha_j) = (\check{a}_i/a_i) a_{ij} \ \forall i, j = 0, \cdots, n \tag{162}$$

$$(\alpha_i, \Lambda_0) = 0 \ \forall i = 1, \cdots, n \tag{163}$$

$$(\alpha_0, \Lambda_0) = a_0^{-1} \text{ and} \tag{164}$$

$$(\Lambda_0, \Lambda_0) = 0. \tag{165}$$

For $\lambda \in \hat{\mathfrak{h}}^*$ we can now write

$$\lambda = \overline{\lambda} + \langle \lambda, K \rangle \Lambda_0 + (\lambda, \Lambda_0)\delta. \tag{166}$$

The *Weyl vector* $\rho \in \hat{\mathfrak{h}}^*$, defined by $\langle \rho, \check{\alpha}_i \rangle = 1$ and $\langle \rho, d \rangle = 0$ satisfies

$$\rho = \overline{\rho} + \check{h}\Lambda_0. \tag{167}$$

There is a very interesting connection between the dimension $\dim(\mathfrak{g})$ and the length of the Weyl vector. More precisely we have the *strange formula of Freudenthal-de Vries*, which is

$$\frac{|\overline{\rho}|^2}{2\check{h}} = \frac{\dim(\mathfrak{g})}{24}. \tag{168}$$

For every simple root $\alpha_i \in \Pi$ we define a reflection r_i acting on $\hat{\mathfrak{h}}^*$ by

$$r_i(\lambda) = \lambda - \langle \lambda, \check{\alpha}_i \rangle \alpha_i. \tag{169}$$

The group generated by all r_i is called *affine Weyl group* and denoted by W. Take $w = r_{i_1} \cdots r_{i_s} \in W$. We call such an expression *reduced* if w can't be written as a product of reflections r_i, corresponding to simple roots, with less then s factors. In this case we call s the *length of* w and denote it by $\epsilon(w)$. We denote the set of all roots of $\hat{\mathfrak{g}}$ by Δ. Of course we can construct real forms $\hat{\mathfrak{h}}_{\mathbb{R}}$ and $\hat{\mathfrak{h}}^*_{\mathbb{R}}$ of $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^*$ simply by considering the $\mathbb{R}$-span of the simple roots or coroots respectively. The *fundamental Weyl chamber* is

$$C = \{h \in \hat{\mathfrak{h}}_{\mathbb{R}} : \langle \alpha_i, h \rangle \geq 0, \text{ for } i = 0, \cdots, n\}. \tag{170}$$

For $w \in W$ the sets $w(C)$ are called *Weyl chambers*. Of course we can introduce the *dual fundamental Weyl chamber* by

$$\check{C} = \{\lambda \in \hat{\mathfrak{h}}^*_{\mathbb{R}} : \langle \check{\alpha}_i, \lambda \rangle \geq 0, \text{ for } i = 0, \cdots, n\} \tag{171}$$

and the *dual Weyl chambers* by $w(\check{C})$. A root $\alpha \in \Delta$ will be called *real* if there is $w \in W$ such that $w(\alpha)$ is a simple root. We denote the set of real roots by Δ^{re} and the set of positive real roots by Δ^{re}_+.

Proposition 2.5.1 ([Kac90, §5.1]). *Let α be a real root of an untwisted affine Kac-Moody algebra $\hat{\mathfrak{g}}$. Then we have:*

1. *mult$(\alpha) = 1$.*
2. *$k\alpha$ is a root if and only if $k = \pm 1$.*
3. *If $\beta \in \Delta$ then there exist nonnegative integers p and q related by the equation $p - q = \langle \beta, \check{\alpha} \rangle$, such that $\beta + k\alpha \in \Delta \cup \{0\}$ if and only if $-p \leq k \leq q$, $k \in \mathbb{Z}$.*
4. *For the bilinear form defined above we have $(\alpha, \alpha) > 0$ and if $\alpha = \sum k_i \alpha_i$, then $k_i(\alpha_i, \alpha_i) \in (\alpha, \alpha)\mathbb{Z}$.*

A root $\alpha \in \Delta$ which is not real will be called *imaginary* and we denote the set of imaginary roots by Δ^{im} and the set of positive imaginary roots by Δ^{im}_+. We have $\Delta^{im} = \Delta^{im}_+ \cup (-\Delta^{im}_+)$.

Proposition 2.5.2 ([Kac90, §5.2]). *Imaginary roots have the following properties:*

1. *The set Δ^{im}_+ is W-invariant.*
2. *For every $\alpha \in \Delta^{im}_+$ there exits a unique root $\beta \in -\check{C}$ that is W-equivalent to α.*
3. *The root $\alpha \in \Delta$ is imaginary if and only if $(\alpha, \alpha) \leq 0$.*

Next we want to express the roots Δ of $\hat{\mathfrak{g}}$ in terms of roots $\overline{\Delta}$ of $\mathfrak{g}$. Following §6.3 in [Kac90] we get

$$\Delta^{im} = \{\pm\delta, \pm 2\delta, \pm 3\delta, \cdots\} \text{ and } \Delta^{im}_+ = \{\delta, 2\delta, 3\delta, \cdots\} \tag{172}$$

for the imaginary roots. The real roots are given by

$$\Delta^{re} = \{\alpha + n\delta : \alpha \in \overline{\Delta}, n \in \mathbb{Z}\} \tag{173}$$

and the positive real roots are

$$\Delta^{re}_+ = \{\alpha + n\delta \in \Delta^{re} : \alpha \in \overline{\Delta}, n > 0\} \cup \overline{\Delta}_+. \tag{174}$$

We can define subset $\hat{\mathfrak{n}}_\pm$ of $\hat{\mathfrak{g}}$ by

$$\hat{\mathfrak{n}}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \hat{\mathfrak{g}}_\alpha. \tag{175}$$

Of course we have the property

$$\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{g}} \oplus \hat{\mathfrak{n}}_+ \tag{176}$$

and obtain

$$U(\hat{\mathfrak{g}}) = U(\hat{\mathfrak{n}}_-) \otimes U(\hat{\mathfrak{g}}) \otimes U(\hat{\mathfrak{n}}_+). \tag{177}$$

We denote the Weyl group of $\mathfrak{g}$ by $\overline{W}$ and discuss its relation with W. See §6.5 in [Kac90] for this and further details. For $\alpha \in \mathfrak{h}^*$ we introduce an endomorphism t_α on $\hat{\mathfrak{h}}^*$ by

$$t_\alpha(\lambda) = \lambda + \langle \lambda, K \rangle \alpha - ((\lambda, \alpha) + \frac{1}{2}|\alpha|^2 \langle \lambda, K \rangle)\delta. \tag{178}$$

For $\alpha, \beta \in \mathfrak{h}^*$ and $w \in \overline{W}$ we have

$$t_\alpha t_\beta = t_{\alpha+\beta} \text{ and } t_{w(\alpha)} = w t_\alpha w^{-1}. \tag{179}$$

We can introduce a sublattice M of $\mathfrak{h}^*_{\mathbb{R}}$ by $M = \nu\left(\check{\overline{Q}}\right) \subset \overline{Q}$. Of course this is just the coroot lattice of $\mathfrak{g}$ embedded into $\mathfrak{h}^*_{\mathbb{R}}$. By (178) we can define a faithful action of the lattice M on $\hat{\mathfrak{h}}^*$. Considered as a subgroup of GL$(\hat{\mathfrak{h}}^*)$ we denote the corresponding group by T and call it the *group of translations*. The finite Weyl group $\overline{W}$ and the group of translations can both be considered as subgroups of W.

Proposition 2.5.3. *The Weyl group W is a semidirect product $W = \overline{W} \ltimes T$.*

As usual we can introduce *fundamental weights* Λ_i of $\hat{\mathfrak{g}}$ by $\Lambda_i(\alpha_j) = \delta_{i,j}$ and their relation to the fundamental weights $\overline{\Lambda}_i$ of $\mathfrak{g}$ is

$$\Lambda_i = \overline{\Lambda}_i + \check{a}_i\Lambda_0. \tag{180}$$

We can write $\Lambda \in \hat{\mathfrak{h}}^*$ as

$$\Lambda = \sum_{i=0}^{n} n_i\Lambda_i + c\delta \tag{181}$$

with *labels* $n_i = \langle \Lambda, \check{\alpha}_i \rangle$ and $c = \langle \Lambda, d \rangle$. The number $\langle \Lambda, K \rangle$ will be called the *level* of Λ. In the following we introduce some basic representation theory for affine Kac-Moody algebras. This is highly influenced by sections §9 to §12 in [Kac90]. Consult this for any further details. We consider a $\hat{\mathfrak{g}}$-module V which is $\hat{\mathfrak{h}}$-diagonalizable, such that its weight spaces V_λ are finite-dimensional. For $\lambda \in \hat{\mathfrak{h}}^*$ we introduce $D(\lambda) = \{\mu \in \hat{\mathfrak{h}}^* : \mu \leq \lambda\}$. Assume furthermore that there are finitely many $\lambda_1, \cdots, \lambda_s \in \hat{\mathfrak{h}}^*$ such that

$$P(V) \subset \bigcup_{i=1}^{s} D(\lambda_i). \tag{182}$$

We define the *category* $\mathcal{O}$ of $\hat{\mathfrak{g}}$ by taking all $\hat{\mathfrak{g}}$-modules V with the previously mentioned properties as objects. The morphisms are just the homomorphisms of $\hat{\mathfrak{g}}$-modules. A $\hat{\mathfrak{g}}$-module V in category $\mathcal{O}$ is called *highest-weight module* of *highest weight* $\Lambda \in P(V)$ if there is a nonzero vector $v_\Lambda \in V$ such that

$$\hat{\mathfrak{n}}_+ v_\Lambda = 0 \text{ and } hv_\Lambda = \Lambda(h)v_\Lambda \text{ for all } h \in \hat{\mathfrak{h}} \text{ and} \tag{183}$$

$$U(\hat{\mathfrak{g}})v_\Lambda = V. \tag{184}$$

A highest-weight module $M(\Lambda)$ of highest weight $\Lambda \in \hat{\mathfrak{h}}$ is called *Verma module of highest-weight* Λ if every $\hat{\mathfrak{g}}$-module of highest-weight Λ is a quotient of $M(\Lambda)$.

Proposition 2.5.4 ([Kac90, Prop. 9.2.])**.** *Take $\Lambda \in \hat{\mathfrak{h}}$.*

1. *There exists a unique Verma module $M(\Lambda)$ up to isomorphism.*
2. *Viewed as a $U(\hat{\mathfrak{n}}_-)$-module $M(\Lambda)$ is free of rank 1 and generated by a highest-weight vector.*
3. *$M(\Lambda)$ contains a unique proper maximal submodule $M'(\Lambda)$.*

As a consequence of this proposition there is a unique irreducible highest-weight module $L(\Lambda) = M(\Lambda)/M'(\Lambda)$ of highest-weight Λ. Furthermore it is well-known that any irreducible module in category $\mathcal{O}$ is of this form for some highest-weight $\Lambda \in \hat{\mathfrak{h}}$. For an object V in the category $\mathcal{O}$ we call a vector $v \in V_\lambda$ *primitive* if there is a submodule $U \subset V$ such that $v \notin U$ and $\hat{\mathfrak{n}}_+ v \subset U$. The module V is generated by its primitive vectors as a $\hat{\mathfrak{g}}$-module. If a weight $\lambda \in P(V)$ admits a primitive weight vector we call it *primitive* as well. Next we consider formal sums of the form

$$\sum_{\lambda \in \hat{\mathfrak{h}}^*} c(\lambda)e^\lambda, \tag{185}$$

where $c(\lambda) \in \mathbb{C}$ with the property that $c(\lambda) = 0$ for λ not contained in the union of finitely many $D(\mu)$. We denote the set of all such formal sums by $\mathcal{E}$. By extension of $e^\lambda e^\mu = e^{\lambda+\mu}$ we can turn this set into a associative commutative algebra over $\mathbb{C}$. See §9.7 in [Kac90] for details. In particular we can define an action of the Weyl group W on $\mathcal{E}$ by $w(e^\lambda) = e^{w(\lambda)}$. For a module V in category $\mathcal{O}$ we write $\text{mult}_V(\lambda)$ for $\dim(V_\lambda)$ so that we can introduce the *formal character of* V by

$$\text{ch}(V) = \sum_{\lambda \in P(V)} \text{mult}_V(\lambda)e^\lambda \in \mathcal{E}. \tag{186}$$

The character of a module V is one of its most important invariants. We can define *Kostants partition function* K by

$$\prod_{\alpha\in\Delta_+}(1-e(\alpha))^{-\text{mult}(\alpha)} = \sum_{\beta\in\hat{\mathfrak{h}}^*} K(\beta)e(\beta). \tag{187}$$

Then the multiplicities of the weights of the Verma module $M(\Lambda)$ are given by

$$\text{mult}_{M(\Lambda)}(\lambda) = K(\Lambda-\lambda). \tag{188}$$

Usually a module V in category $\mathcal{O}$ does not possess a composition series but there are related structures, such that we can still define a *multiplicity* $[V : L(\mu)]$ of $L(\mu)$ in V. The multiplicity of $L(\mu)$ in V is nonzero if and only if μ is a primitive weight of V. Using those multiplicities we can express the character $\text{ch}(V)$ as

$$\text{ch}(V) = \sum_{\lambda\in\hat{\mathfrak{h}}^*}[V : L(\lambda)]\text{ch}(L(\lambda)). \tag{189}$$

See sections §9.6 and §9.7 [Kac90] for details about this. The study of the characters $\text{ch}(L(\lambda))$ is an important task in the representation theory of Kac-Moody algebras. Following §10 in [Kac90] we introduce

$$P = \{h\in\hat{\mathfrak{h}^*} : \langle\lambda,\check{\alpha}_i\rangle\in\mathbb{Z} \text{ for } i = 0,\cdots,n\}, \tag{190}$$
$$P_+ = \{h\in P : \langle\lambda,\check{\alpha}_i\rangle \geq 0 \text{ for } i = 0,\cdots,n\} \text{ and} \tag{191}$$
$$P_{++} = \{h\in P : \langle\lambda,\check{\alpha}_i\rangle > 0 \text{ for } i = 0,\cdots,n\}. \tag{192}$$

We call P the *weight lattice* of $\hat{\mathfrak{g}}$ and its elements are called *integral weights*. Elements in P_+ are called *dominant weights* and elements in P_{++} are called *regular dominant weights*. We call an irreducible module $L(\Lambda)$ *integrable* if $\Lambda\in P_+$. See §3 and §10 in [Kac90] for details and an alternative approach. Integrable modules have many remarkable properties. See in particular § 10 in [Kac90]. We discuss a few of those properties in the following. Starting with the fact that for $\Lambda\in P_+$ we have

$$\text{mult}_{L(\Lambda)}(\lambda) = \text{mult}_{L(\Lambda)}(w(\lambda)) \text{ for } w\in W. \tag{193}$$

So the multiplicities of the weights $\lambda\in P(L(\Lambda))$ are invariant under the Weyl group W. Another way to express this fact is

$$w(\text{ch}(L(\Lambda))) = \text{ch}(L(\Lambda)). \tag{194}$$

In the following we mainly work with irreducible integrable modules $L(\Lambda)$, therefore we simplify notation by $\text{mult}_\Lambda(\alpha)$ for $\text{mult}_{L(\Lambda)}(\alpha)$, ch_Λ for $\text{ch}(L(\Lambda))$ and $P(\Lambda)$ for $P(L(\Lambda))$.

Theorem 2.5.5 (character formula)**.** *Take $\Lambda\in P_+$. Let $L(\Lambda)$ be the corresponding irreducible integrable module of $\hat{\mathfrak{g}}$. Then we have*

$$ch_\Lambda = \frac{\sum_{w\in W}\epsilon(w)e(w(\Lambda+\rho)-\rho)}{\prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{mult_\Lambda(\alpha)}}. \tag{195}$$

A famous special case of this theorem is $\Lambda = 0$. In this case we get $\text{ch}_\Lambda = \text{ch}_0 = 1e^0 = 1$ so that we obtain the *denominator identity*

$$\prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{\text{mult}_\Lambda(\alpha)} = \sum_{w\in W}\epsilon(w)e(w(\rho)-\rho). \tag{196}$$

Another important property is the *multiplicity formula*

$$\text{mult}_\Lambda(\lambda) = \sum_{w\in W}\epsilon(w)K(w(\Lambda+\rho)-(\lambda+\rho)). \tag{197}$$

A direct consequence of the multiplicity formula is

$$\text{mult}_{\Lambda+a\delta}(\lambda+a\delta) = \text{mult}_\Lambda(\lambda)\ \forall a\in\mathbb{C}. \tag{198}$$

So far we considered formal characters ch_V, i.e. formal expansions in a certain $\mathbb{C}$-algebra $\mathcal{E}$. Now we focus on *analytic* properties of functions induced by such characters. Notice therefore that e^{λ} induces a holomorphic function on $\hat{\mathfrak{h}}$ by $h \mapsto e^{\langle \lambda, h\rangle}$. For any module V in category $\mathcal{O}$ we define a function

$$h \mapsto \mathrm{ch}_V(h) = \sum_{\lambda \in P(V)} \mathrm{mult}_V(\lambda) e^{\langle \lambda, h\rangle}. \tag{199}$$

We define the set $Y(V)$ to be the subset of $\hat{\mathfrak{h}}$ on which this series converges absolutely. We define sets

$$Y = \{h \in \hat{\mathfrak{h}} : \sum_{\alpha \in \Delta_+} \mathrm{mult}(\alpha) |e^{-\langle \alpha, h\rangle}| < \infty\} \text{ and} \tag{200}$$

$$Y_N = \{h \in \hat{\mathfrak{h}} : \mathrm{Re}(\langle \alpha_i, h\rangle) > N \text{ for } i = 0, \cdots, n\}. \tag{201}$$

For A highest-weight module V we have that $Y(V)$ is a convex set and

$$Y \cap Y_0 \subset Y(V). \tag{202}$$

See Lemma 10.6 in [Kac90] for this. We go back to the special case $V = L(\Lambda)$ with $\Lambda \in P_+$. An explicit description of $Y(L(\Lambda))$ is given in §11.10 of [Kac90]. Assume that $\langle \Lambda, \hat{\alpha}_i\rangle \neq 0$ for some i, then the region of absolute convergence of ch_λ is

$$Y(L(\Lambda)) = \{h \in \hat{\mathfrak{h}} : \sum_{\alpha \in \Delta_+} \mathrm{mult}(\alpha) |e^{-\langle \alpha, h\rangle}| < \infty\} = \{h \in \hat{\mathfrak{h}} : \mathrm{Re}(\langle \delta, h\rangle) > 0\}. \tag{203}$$

In particular we see that the region of absolute convergence $Y(L(\Lambda))$ does not depend on the choice of $\Lambda \in P_+$. Following Proposition 10.6 in [Kac90] we find that ch_Λ defines a holomorphic function on $\mathrm{Int}(Y(L(\Lambda)))$. We call this holomorphic function the *character* of $L(\Lambda)$ and denote it by ch_Λ. Furthermore the series $\sum_{w \in W} l(w) e^{w(\Lambda+\rho)}$ converges absolutely on $\mathrm{Int}(Y(\Lambda))$ as well. We need to understand more about the weights $P(\Lambda)$ of an irreducible module $L(\Lambda)$ with $\Lambda \in P_+$.

Proposition 2.5.6 ([Kac90, Prop. 11.4])**.** *Let $\Lambda \in P_+$ and $\lambda, \mu \in P(\Lambda)$. Then*

1. $(\Lambda, \Lambda) - (\lambda, \mu) \geq 0$ *and equality holds if and only if* $\lambda = \mu \in W\Lambda$.
2. $|\Lambda + \rho|^2 - |\lambda + \rho|^2 \geq 0$ *and equality holds if and only if* $\lambda = \Lambda$.

If the highest-weight Λ of some highest-weight module V has level $\langle \Lambda, K\rangle$ then we say that also the module V has level $\langle \Lambda, K\rangle$. For $\Lambda \in P_+$ we obtain

$$k := \langle \Lambda, K\rangle = \sum_{i=0}^{n} \check{a}_i \langle \Lambda, \check{\alpha}_i\rangle \in \mathbb{Z}_{\geq 0}. \tag{204}$$

On $L(\Lambda)_\lambda$ the element K has to act as multiplication by $\langle \lambda, K\rangle$ but obviously also by $\langle \Lambda, K\rangle$, since its kernel is a submodule. We obtain $\langle \Lambda, K\rangle = \langle \lambda, K\rangle$ for all $\lambda \in P(\Lambda)$. Note that we have

$$P = \sum_{i=0}^{n} \mathbb{Z}\Lambda_i + \mathbb{C}\delta \text{ and} \tag{205}$$

$$P_+ = \sum_{i=0}^{n} \mathbb{Z}_+\Lambda_i + \mathbb{C}\delta. \tag{206}$$

We set furthermore

$$P^k = \{\lambda \in P : \langle \lambda, K\rangle = k\} \text{ and } P^k_+ = P^k \cap P_+. \tag{207}$$

Proposition 2.5.7 ([Kac90, Prop. 12.5.])**.** *Take $\Lambda \in P^k_+$. Then $L(\Lambda)$ satisfies:*

1. $P(\Lambda) = W\{\lambda \in P_+ : \lambda \leq \Lambda\}$.

2. $P(\Lambda) = (\Lambda + Q) \cap$ *convex hull of* $W\Lambda$.

3. *If* $\lambda, \mu \in P(\Lambda)$ *and* μ *lies in the convex hull of* $W\lambda$*, then* $\mathit{mult}_\Lambda(\mu) \geq \mathit{mult}_\Lambda(\lambda)$.

4. $P(\Lambda)$ *lies in the paraboloid* $\{\lambda \in \hat{\mathfrak{h}}^*_{\mathbb{R}} : |\overline{\lambda}|^2 + 2k(\lambda, \Lambda_0) \leq |\Lambda|^2; \langle \lambda, K\rangle = k\}$ *and the intersection of* $P(\Lambda)$ *with the boundary of this paraboloid is* $W\Lambda$.

5. *For* $\lambda \in P(\Lambda)$ *the set of* $t \in \mathbb{Z}$ *such that* $\lambda - t\delta \in P(\Lambda)$ *is an interval* $[-p, +\infty)$ *with* $p \geq 0$ *and* $t \mapsto \mathit{mult}_\Lambda(\lambda - t\delta)$ *is a nondecreasing function on this interval. Moreover, if* $x \in \mathfrak{g}_{-\delta}$, $x \neq 0$ *then the map* $x : L(\Delta)_{\lambda - t\delta} \to L(\Lambda)_{\lambda-(t+1)\delta}$ *is injective.*

6. *Set* $\mathfrak{n}_-^{(\delta)} = \bigoplus_{n>0} \hat{\mathfrak{g}}_{-n\delta}$ *then* $L(\Lambda)$ *is a free* $U\left(\mathfrak{n}_-^{(\delta)}\right)$*-module.*

We call a weight $\lambda \in P(\Lambda)$ *maximal* if $\lambda + \delta \notin P(\Lambda)$. The set of maximal weights of $L(\Lambda)$ will be denoted by $\max(\Lambda)$ and we have a disjoint union

$$P(\Lambda) = \bigcup_{\lambda \in \max(\Lambda)} \{\lambda - n\delta : n \in \mathbb{Z}_+\}. \tag{208}$$

Now we follow §12.7 in [Kac90] by introducing the *modular anomaly* of Λ by

$$m_\Lambda = \frac{|\Lambda + \rho|^2}{2(k + \check{h})} - \frac{|\rho|^2}{2\check{h}}. \tag{209}$$

For $\lambda \in \hat{\mathfrak{h}}^*$ we set

$$m_{\Lambda,\lambda} = m_\Lambda - \frac{|\lambda|^2}{2k}. \tag{210}$$

Now we can introduce the *string function* of $\lambda \in \hat{\mathfrak{h}}^*$ by

$$c^\Lambda_\lambda = e^{-m_{\Lambda,\lambda}\delta} \sum_{n \in \mathbb{C}} \mathrm{mult}_\Lambda(\lambda - n\delta) e^{-n\delta}. \tag{211}$$

The string functions converge absolutely to holomorphic functions on $Y(L(\Lambda))$ and have a some remarkable properties. Assume therefore that $\Lambda \in P^k_+$ and $\lambda \in P(\Lambda)$. Of course we have $\langle \Lambda, d\rangle = \langle \lambda, d\rangle (\mathrm{mod}\, \mathbb{Z})$. As a consequence we can rewrite (211) as

$$c^\Lambda_\lambda = e^{-m_{\Lambda,\lambda}\delta} \sum_{n \in \langle \lambda - \Lambda, d\rangle + \mathbb{Z}} \mathrm{mult}_\Lambda(\lambda - n\delta) e^{-n\delta} \tag{212}$$

for any $\lambda \in \hat{\mathfrak{h}}^*$. Using this we find that the string function c^Λ_λ just depends on $\lambda (\mathrm{mod}\, \mathbb{C}\delta)$ and using (198) we find that it also just depends on $\Lambda (\mathrm{mod}\, \mathbb{C}\delta)$. This together with invariance under the Weyl group $W = \overline{W} \ltimes T$ in (213) we obtain

$$c^\Lambda_{w(\lambda)+k\gamma+a\delta} = c^\Lambda_\lambda \text{ for } w \in \overline{W}, \gamma \in M, a \in \mathbb{C} \text{ and} \tag{213}$$

$$c^{\Lambda+a\delta}_\lambda = c^\Lambda_\lambda \text{ for } a \in \mathbb{C}. \tag{214}$$

For each $\lambda \in \hat{\mathfrak{h}}^*$ with $\mathrm{level}(\lambda) = k > 0$ we can furthermore introduce its associated *theta series*

$$\Theta_\lambda = e^{-\frac{|\lambda|^2}{2k}\delta} \sum_{t \in T} e^{t(\lambda)} = e^{k\Lambda_0} \sum_{t \in M + k^{-1}\overline{\lambda}} e^{-\frac{1}{2}k|\gamma|^2\delta + k\gamma}. \tag{215}$$

A crucial property of those theta series is that we have

$$\Theta_{\lambda + k\alpha + a\delta} = \Theta_\lambda \; \forall \alpha \in M, a \in \mathbb{C}. \tag{216}$$

We can now introduce the *normalized character* $\chi_\Lambda = e^{-m_\Lambda\delta}\mathrm{ch}_\Lambda$. The normalized character just depends on the class $\Lambda(\text{mod } \mathbb{C}\delta)$. This allows us to consider the weight lattice P (mod $\mathbb{C}\delta$). we make use of this in the *theta decomposition*

$$\chi_\Lambda = \sum_{\lambda\in P^k \text{ mod } (kM+\delta\mathbb{C})} c_\lambda^\Lambda \Theta_\lambda. \tag{217}$$

Following §13.2 in [Kac90] we chose an orthonormal basis $v_1, \cdots, v_n$ of $\mathfrak{h}_\mathbb{R}$ and introduce coordinates of $\hat{\mathfrak{h}}$ by

$$v = 2\pi i \left(\sum_{i=1}^n z_i v_i - \tau d + uK\right). \tag{218}$$

As usual set $z = \sum_{i=1}^n z_i v_i \in \mathfrak{h}$, such that we can write (z,τ,u) for $v \in \hat{\mathfrak{h}}$. The space of absolute convergence of ch_Λ can now be parametrized as

$$Y = \{(z,\tau,u) : z \in \mathfrak{h}, t, u \in \mathbb{C}, \mathrm{Im}(\tau) > 0\}. \tag{219}$$

Since ch_Λ, Θ_λ and c_λ^Λ converge to holomorphic functions on Y we can write $\mathrm{ch}_\Lambda = \mathrm{ch}_\Lambda(z,\tau,u)$, $\Theta_\lambda = \Theta_\lambda(z,\tau,u)$ and $c_\lambda^\Lambda = c_\lambda^\Lambda(\tau)$. It is easy to see that c_λ^Λ does not depend on z and u. As usual we write q for $e^{2\pi i\tau}$ and obtain

$$c_\lambda^\Lambda(\tau) = q^{m_{\Lambda,\lambda}} \sum_{n\in\mathbb{C}} \mathrm{mult}_\Lambda(\lambda - n\delta) q^n. \tag{220}$$

The theta series Θ_λ can now be written in more classical terms as well. We have

$$\Theta_\lambda(z,\tau,u) = e^{2\pi iku} \sum_{\gamma\in M+k^{-1}\overline{\lambda}} q^{k\gamma^2/2} e^{2\pi ik(\gamma,z)}. \tag{221}$$

In fact those functions are not just holomorphic but satisfy remarkable *modularity properties.*

Theorem 2.5.8 ([Kac90, Thm. 13.8.]). *Let $\hat{\mathfrak{g}}$ be an affine untwisted Kac-Moody algebra as above. Take $\Lambda \in P_+^k$ for some $k \geq 0$.*

1. *We have*

$$\chi_\Lambda\left(\frac{z}{\tau}, -\frac{1}{\tau}, u - \frac{(z,z)}{2\tau}\right) = \sum_{\Lambda'\in P_+^k (\text{mod}\, \mathbb{C}\delta)} \mathcal{S}_{\Lambda,\Lambda'} \chi_{\Lambda'}(z,\tau,u) \tag{222}$$

   and

$$\chi_\Lambda\left(z, \tau+1, u\right) = e^{2\pi i m_{\overline{\Lambda}}} \chi_\Lambda(z,\tau,u). \tag{223}$$

   Where the matrix $\mathcal{S}$ is given by

$$\mathcal{S}_{\Lambda,\Lambda'} = i^{|\overline{\Delta}_+|} |M'/(k+\check{h})M|^{-1/2} \sum_{w\in\overline{W}} \epsilon(w) e^{-\frac{2\pi i(\overline{\Lambda}+\overline{\rho}, w(\overline{\Lambda'}+\overline{\rho}))}{k+\check{h}}}. \tag{224}$$

2. *The linear span of the normalized characters χ_Λ for $\Lambda \in P_+^k(\text{mod}\, \mathbb{C}\delta)$ is invariant under the action of $SL_2(\mathbb{Z})$ defined by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z,\tau,u) = f\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}, u - \frac{c(z,z)}{2(c\tau+d)}\right). \tag{225}$$

The matrix $\mathcal{S} = \mathcal{S}_k = (\mathcal{S}_{\Lambda,\Lambda'})_{\Lambda,\Lambda'\in P_+^k(\text{mod}\,\mathbb{C}\delta)}$ is unitary and symmetric. Of course the theta functions Θ_λ and the string functions c_λ^Λ have similar transformation properties.

Theorem 2.5.9 ([Kac90, Thm. 13.5.]). *The theta function Θ_λ satisfies*

$$\Theta_\lambda\left(\frac{z}{\tau}, -\frac{1}{\tau}, u - \frac{(z,z)}{2\tau}\right) = (-i\tau)^{n/2}|M'/kM|^{-1/2} \sum_{\mu \in P^k (\mathit{mod}\, kM + \mathbb{C}\delta)} e^{-\frac{2\pi i}{k}(\overline{\lambda}, \overline{\mu})}\Theta_\mu(z,\tau,u) \quad (226)$$

$$\Theta_\lambda(z, \tau+1, u) = e^{\pi i |\lambda|^2/k}\Theta_\lambda(z,\tau,u). \quad (227)$$

Now we may come to the transformation properties of the string functions.

Theorem 2.5.10. *Let $\hat{\mathfrak{g}}$ be an affine untwisted Kac-Moody algebra of rank $n+1$ as above. Take $\Lambda \in P_+^k$ for some $k \geq 0$ and $\lambda \in \hat{\mathfrak{h}}^*$. Then we have*

$$c_\lambda^\Lambda\left(-\frac{1}{\tau}\right) = |M'/kM|^{-1/2}(-i\tau)^{-n/2} \sum_{\Lambda' \in P_+^k (\mathit{mod}\, \mathbb{C}\delta), \lambda' \in P^k (\mathit{mod}\, kM + \mathbb{C}\delta)} \mathcal{S}_{\Lambda,\Lambda'} e^{2\pi i \frac{(\overline{\lambda}, \overline{\lambda'})}{k}} c_{\lambda'}^{\Lambda'}(\tau) \quad (228)$$

$$c_\lambda^\Lambda(\tau+1) = e^{2\pi i (m_{\Lambda,\lambda} + \langle \lambda - \Lambda, d\rangle)} c_\lambda^\Lambda(\tau). \quad (229)$$

Finally we discuss an important theorem about a certain subset of fundamental weights. See § 13.11 in [Kac90] for details of this. We can label the simple roots by the set $\{0, \cdots, n\}$ as usual. An important subset of this index set is given by

$$J = \{0 \leq j \leq n : a_j = 1\}, \quad (230)$$

where a_j is, as usual, the Coxeter label of the simple root α_i. Remember that $a_j = 1$ implies $\check{a}_j = 1$ and therefore we have $\alpha_j = \check{\alpha}_j$ for all $j \in J$.

Theorem 2.5.11 ([Kac90, Thm. 13.11.]). *For $\Lambda \in P_+^k$ we have*

$$2k(\Lambda, \rho) \geq \check{h}(\Lambda, \Lambda) \quad (231)$$

and the equality holds if and only if $\Lambda = k\Lambda_j (\mathit{mod}\, \mathbb{C}\delta)$ with $j \in J$.

Later we will be in need to find bounds on pole orders of string functions at $i\infty$. The following proposition will be helpful in this.

Proposition 2.5.12 ([Kac90, Prop. 13.11.]). *Let $\Lambda \in P_+^k$, $k > 0$ and $\lambda \in P(\Lambda)$. Then we have*

$$m_{\Lambda,\lambda} \geq -\frac{|\rho|^2}{2\check{h}} \frac{k}{k+\check{h}} \quad (232)$$

with equality if and only if $\Lambda = k\Lambda_j (\mathit{mod}\, \mathbb{C}\delta)$ with $j \in J$ and $\lambda = w(\Lambda)$ for some $w \in W$.

2.6 Generalized Kac-Moody algebras

In the beginning of the previous section we explained that we can generalize semi-simple Lie algebras by weakening certain assumptions on Cartan matrices that yield Lie algebras by a construction with generators and relations. In this section we discuss a further generalization of generalized Cartan matrices, which we will call *Borcherds-Cartan matrices.* By use of generators and relations we can associate to each Borcherds-Cartan matrix a *generalized Kac-Moody algebra.* Those are Lie algebras which generalize both, semi-simple Lie algebras and Kac-Moody algebras. For generalized Kac-Moody algebras over the real numbers $\mathbb{R}$ see [Bor88], [Bor91], [Bor95b] and [Jur98]. A further short introduction is given in section §11.13 of [Kac90]. For generalized Kac-Moody algebras over the complex numbers a good discussion is given in [Car16]. We will work with generalized Kac-Moody algebras over both, fields in the following. Since mostly it does not matter, if we work over $\mathbb{R}$ or $\mathbb{C}$ we will keep the field implicit. The following exposition is based on all those sources. Let I be a countable index set and $A = (a_{ij})_{i,j \in I}$ a real matrix. We call A a *Borcherds-Cartan matrix* if the following properties are satisfied:

1. A is symmetric, this means $a_{ij} = a_{ji}$ for all $i, j \in I$.
2. $a_{ij} \leq 0$ if $i \neq j$.
3. $a_{ii} > 0$ implies that $\frac{2a_{ij}}{a_{ii}} \in \mathbb{Z}$ for all $j \in I$.

Let A be a Borcherds-Cartan matrix with index set I. The *universal generalized Kac-Moody algebra* $\mathfrak{g}(A)$ associated with A is the complex Lie algebra with generators e_i, f_i and h_{ij} for $i, j \in I$ and relations for $i, j, k, l \in I$:

1. $\mathfrak{sl}_2$*-relations*: $[e_i, f_j] = h_{ij}$, $[h_{ij}, e_k] = \delta_{i,j} a_{ik} e_k$ and $[h_{ij}, f_k] = -\delta_{i,j} a_{ik} f_k$.
2. *Serre relations*: If $a_{ii} > 0$ then $\mathrm{ad}(e_i)^{1-2a_{ij}/a_{ii}}(e_j) = \mathrm{ad}(f_i)^{1-2a_{ij}/a_{ii}}(f_j) = 0$.
3. *Commutativity*: If $a_{ij} = 0$ then $[e_i, e_j] = [f_i, f_j] = 0$.

Following the arguments in section 3 in [Jur98] we can introduce subalgebras $\mathfrak{n}_+$, $\mathfrak{n}_-$ and $\mathfrak{h}(A)$ of $\mathfrak{g}(A)$ generated by all e_i, all f_i and h_{ij} for $i, j \in I$ respectively. They yield a direct sum decomposition

$$\mathfrak{g}(A) = \mathfrak{n}_+ \oplus \mathfrak{h}(A) \oplus \mathfrak{n}_-. \tag{233}$$

We can introduce an invariant symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}(A)$ which is uniquely determined by $(e_i, f_j) = \delta_{i,j}$. This bilinear form satisfies $(h_i, h_j) = a_{ij}$, where we write $h_i = h_{ii}$ for all $i \in I$. The element h_{ij} vanishes unless the i-th and the j-th row of A are equal, i.e. $a_{ik} = a_{jk}$ for all $k \in I$. Furthermore elements h_{ij} lie in the kernel $\ker(\cdot, \cdot)$ and are central in $\mathfrak{g}(A)$. The center $\mathfrak{c}$ of $\mathfrak{g}(A)$ is contained in $\mathfrak{h}(A)$. For a family $(n_i)_{i \in I}$ of positive integers with $n_i = n_j$ if $h_{ij} \neq 0$ we can introduce a $\mathbb{Z}$-grading of $\mathfrak{g}(A)$ by $\deg(e_i) = -\deg(f_i) = n_i \; \forall i \in I$ and denote the corresponding spaces by $\mathfrak{g}(A)_n$ for $n \in \mathbb{Z}$. We obtain a decomposition

$$\mathfrak{g}(A) = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}(A)_n \tag{234}$$

with $\mathfrak{g}(A)_0 = \mathfrak{h}(A)$ and $G_n \perp G_m$ unless $n + m = 0$. Notice that the spaces G_n might not have finite dimension. A Lie algebra $\mathfrak{g}$ is called *generalized Kac-Moody algebra* if there is a generalized Cartan matrix A such that for a subspace $C \subset \mathfrak{c}$ of the center of $\mathfrak{g}(A)$ the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{g}(A)/C$. Notice that this is a slightly more restrictive definition than the one usually given. In [Bor95b] and [Car16] the definition of a generalized Kac-Moody algebra also contains all extensions of such Lie algebras by suitable commutative Lie algebras of outer derivations. We consider such extensions as well but will not call them a generalized Kac-Moody algebra in the following. Of course a generalized Kac-Moody algebra $\mathfrak{g}$ has generators e_i, f_i and h_{ij} for $i \in I$ which are just the classes of e_i, f_i and h_{ij} in the quotient $\mathfrak{g}(A)/C$. We denote the span of all h_{ij} by $\mathfrak{h}$ and observe $\mathfrak{h} = \mathfrak{h}(A)/C$. Yet notice that since C is contained in the center $\mathfrak{c}$ it is also contained in the kernel of the bilinear form $(\cdot, \cdot)$. This implies that the generalized Kac-Moody algebra $\mathfrak{g}$ carries a natural invariant bilinear form which satisfies $(e_i, f_j) = \delta_{i,j}$ and $(h_i, h_j) = a_{ij}$. The $\mathbb{Z}$-grading given in (234) induces a natural $\mathbb{Z}$-grading of $\mathfrak{g}$. We denote the corresponding weight spaces by $\mathfrak{g}_n$. A central element $c \in \mathfrak{g}$ is contained in $\mathfrak{h}$ and satisfies $(c, h_{i,j}) = 0$. This implies that $c \in \ker(\cdot, \cdot)$. So the center of $\mathfrak{g}$ is contained in the kernel of the bilinear form $(\cdot, \cdot)$. If the bilinear form $(\cdot, \cdot)$ of $\mathfrak{g}$ is non-degenerate we find that $\mathfrak{g}$ has trivial center. This implies $\mathfrak{g} = \mathfrak{g}(A)/\mathfrak{c}$, where $\mathfrak{c}$ is the center of $\mathfrak{g}(A)$ and not just a subset of the center. See Theorem 3.1. in [Jur98] and the discussion afterwards. We introduce derivations $d_i : \mathfrak{g} \to \mathfrak{g}$ by

$$d_i(e_j) = -d_i(f_j) = \delta_{i,j} \text{ for all } i, j \in I. \tag{235}$$

They form a commutative Lie algebra $\mathfrak{d}$ with the obvious commutator bracket and we introduce the extension

$$\mathfrak{g}^e = \mathfrak{g} \ltimes \mathfrak{d} = (\mathfrak{g}(A)/C) \ltimes \mathfrak{d}. \tag{236}$$

The extended Lie algebra $\mathfrak{g}^e$ has a *Cartan subalgebra* $\mathfrak{h}^e$ given by

$$\mathfrak{h}^e = \mathfrak{h} \ltimes \mathfrak{d}. \tag{237}$$

We can now consider the dual space $(\mathfrak{h}^e)^* = \mathfrak{h}^* \oplus \mathfrak{d}^*$ and introduce for $\alpha \in (\mathfrak{h}^e)^*$ spaces

$$\mathfrak{g}'_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}^e\}. \tag{238}$$

It is easy to see that we have $\mathfrak{h} = \mathfrak{g}'_0$. As usual we call $\alpha \in (\mathfrak{h}^e)^*$ a *root* if $\alpha \neq 0$ and $\mathfrak{g}'_\alpha \neq 0$. If α is root then we call $\mathfrak{g}'_\alpha$ its *root space*. The dimension $\dim(\mathfrak{g}'_\alpha)$ will also be called the *multiplicity* of α and denoted by $\mathrm{mult}(\alpha)$ for any $\alpha \in Q$. For $i \in I$ we define a root α_i by $[h, e_i] = \alpha_i(h)e_i$ for every $h \in \mathfrak{h}^e$. Those roots are called *simple roots*. This is similar to the construction of the root spaces of Kac-Moody algebras. The only difference is that we don't have to work with the extended Cartan subalgebra $\mathfrak{h}^e$ in this case. The reason for this is that without this extension the simple roots α_i might not be linearly independent. Since $\mathfrak{n}_+$ is spanned by elements of the form $x(i_1, \cdots, i_r) = [e_{i_1}, [e_{i_2}, \cdots [e_{i_{r-1}}, e_{i_r}] \cdots]]$ it is clear that $\mathfrak{n}_+$ can be decomposed into root spaces $\mathfrak{g}'_\alpha$, where α can be written as $\alpha = \sum_{i \in I} n_i \alpha_i$ with $n_i \in \mathbb{Z}_{\geq 0}$. Such roots will be called *positive* and we denote the set of all positive roots by Δ_+. The eigenvalue of d_i for the eigenvector $x(i_1, \cdots, i_r)$ counts how often i appears in the tuple $(i_1, \cdots, i_r)$ therefore it is clear that $\mathfrak{g}'_{\alpha_i}$ is spanned by e_i. This implies that the root spaces of simple roots are 1-dimensional. By replacing e_i with f_i we get a similar decomposition for $\mathfrak{n}_-$ where each root α satisfies $\alpha = \sum_{i \in I} n_i \alpha_i$ with $n_i \in \mathbb{Z}_{\leq 0}$. Those roots will be called *negative* and we denote the set of all negative roots by Δ_-. Using this we get

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}'_\alpha. \tag{239}$$

We denote the set of roots of $\mathfrak{g}$ by Δ. This set satisfies $\Delta = \Delta_+ \cup \Delta_-$. See [Kac90], section 2 of [Jur98] and [Bor88] for details. Altogether we obtain the *abstract root space decomposition* of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}'_\alpha. \tag{240}$$

The $\mathbb{Z}$-span of all simple roots will be denoted by Q and is called the *root lattice* and we denote its real ambient space by $R_\mathbb{R}$ and its complex ambient space by R. Of course R is a subspace of $(\mathfrak{h}^e)^*$. Since the simple roots are linearly independent they form a basis of R and we can equip both Q, $R_\mathbb{R}$ and R with a symmetric bilinear form $(\cdot, \cdot)$ by

$$(\alpha_i, \alpha_j) = a_{ij} \ \forall i, j \in I. \tag{241}$$

Obviously the values of $(\cdot, \cdot)$ over $R_\mathbb{R}$ are contained in $\mathbb{R}$, since the Borcherds-Cartan matrix A just has real entries. We call a root $\alpha \in \Delta$ *real* if $(\alpha, \alpha) > 0$ and *imaginary* otherwise. The real simple roots are precisely the simple roots α_i with $a_{ii} > 0$. We will write I_0 for the subset of I defined by $a_{ii} > 0$. We can define a *Weyl vector* $\rho \in R^*_\mathbb{R}$ by

$$\langle \rho, \alpha_i \rangle = \frac{\alpha_i^2}{2} \ \forall i \in I. \tag{242}$$

This vector is clearly well-defined since the simple root are linearly independent. If possible we identify the Weyl vector ρ with a vector $\rho \in R_\mathbb{R}$ that satisfies $(\rho, \alpha_i) = \alpha_i^2/2$ for all $i \in I$. In this case we call a fixed choice of such a vector *Weyl vector* as well. We denote the set of simple roots by Π. For the set of real simple roots we write Π^{re} and for the set of imaginary simple roots we write Π^{im}. Analogously we write Δ^{re} for the set of all real roots and Δ^{im} for the set of all imaginary roots. We can introduce the *Weyl chamber* by

$$C = \{x \in R_\mathbb{R} : (x, \alpha_i) \leq 0, \ \forall i \in I_0\}. \tag{243}$$

For a real simple root α_i we define a linear isometry $r_i : R \to R$ by

$$r_i(\lambda) = \lambda - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i. \tag{244}$$

We call this map the *reflection* associated with the real simple root $\alpha_i \in \Pi^{re}$. Clearly such reflections preserve the root lattice Q and satisfy $r_i(\alpha_i) = -\alpha_i$. The group W generated by all r_i for $i \in I_0$ is called the *Weyl group* of $\mathfrak{g}$. Naturally we have $W \subset \mathrm{GL}(R)$. For a detailed discussion of the Weyl group see section 2 in [Jur96], section §11.13. in [Kac90] and [Bor88].

Proposition 2.6.1 ([Bor88, Prop. 2.1.]). *Every positive root $\alpha \in Q$ is conjugate under the Weyl group W to a simple real root or a positive root in the Weyl chamber C.*

Since the Weyl group is a group of isometries it is clear that $\alpha \in \Delta_+$ is conjugate to a real simple root if and only if α is a real root. The following proposition is based on Proposition 2.3. in [Jur96].

Proposition 2.6.2. *The Weyl group W and the roots Δ have the following properties:*

1. *A real root $\alpha \in \Delta^{re}$ satisfies dim$(\mathfrak{g}'_\alpha) = 1$ and is conjugate under the Weyl group to a real simple root. Furthermore α is a sum of real simple roots.*
2. *A imaginary root $\alpha \in \Delta^{im}$ is conjugate to an element in the Weyl chamber C.*
3. *The element $w \in W$ preserves the dimension of the root spaces, i.e. we have dim$(\mathfrak{g}'_{w(\alpha)}) =$ dim$(\mathfrak{g}'_\alpha)$ for all $\alpha \in Q$.*
4. *We have those identities:*

$$W\Delta^{re} = \Delta^{re} \tag{245}$$
$$W\Delta^{im} = \Delta^{im} \tag{246}$$
$$\Delta^{re} = -\Delta^{re} \tag{247}$$
$$\Delta^{im} = -\Delta^{im} \tag{248}$$
$$W(\Delta^{im} \cap \Delta_+) = \Delta^{im} \cap \Delta_+ \tag{249}$$

For simplicity in the following we assume that the Weyl vector ρ can indeed be considered as an element in R. See section 4 in [Bor88] for details. We can consider the usual formal exponential expressions e^μ for any $\mu \in R$ with the property that we have $e^{\mu+\lambda} = e^\mu e^\lambda$ for $\lambda, \mu \in R$. There is a natural action of the Weyl group W on such expressions and we can consider infinite formal sums over such expressions. Notice that the product of two such sums might not be well-defined but we consider their product if it is well-defined. This is similar to the case of Kac-Moody algebras. Let s be any sum of elements in Π^{im}. We set $\epsilon(s) = (-1)^m$ if s is the sum of m distinct pairwise perpendicular elements and $\epsilon(s) = 0$ otherwise. Then we set $S = e^\rho \sum_s \epsilon(s) e^{-s}$, where this sum runs over all sums of elements in Π^{im}. A fundamental result about generalized Kac-Moody algebras is that they have a *denominator identity*

$$e^\rho \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)} = \sum_{w \in W} \epsilon(w) w(S). \tag{250}$$

This result is due to Borcherds. See [Bor88] and furthermore the discussion in [Kac90] and [Jur98]. The denominator identity is of fundamental importance in the theory of generalized Kac-Moody algebras. It also played a crucial role in Borcherds' proof of the moonshine conjecture. We now introduce our first non-trivial example of a generalized Kac-Moody algebra. See [Bor92] and [Jur98] for this discussion. We consider the J-function, i.e. the nearly holomorphic $\mathrm{SL}_2(\mathbb{Z})$-invariant modular function whose Fourier expansion is

$$J(\tau) = j(\tau) - 744 = \sum_{n=-1}^{\infty} c(n) q^n = q^{-1} + 0 + 196884q + \mathcal{O}(q^2). \tag{251}$$

Let $A^\natural$ be the symmetric matrix with blocks indexed by $-1, 1, 2, 3, \cdots$ where the block (i, j) has entries $-(i+j)$ and size $c(i) \times c(j)$. This is

$$A^\natural = \left(\begin{array}{c|ccc|ccc|c} 2 & 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots \\ \hline 0 & -1 & \cdots & -1 & -2 & \cdots & -2 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & -1 & \cdots & -1 & -2 & \cdots & -2 & \cdots \\ \hline -1 & -2 & \cdots & -2 & -3 & \cdots & -3 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\ -1 & -2 & \cdots & -2 & -3 & \cdots & -3 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right). \tag{252}$$

Clearly this matrix satisfies the axioms of a Bocherds-Cartan matrix, therefore we can construct an associated universal generalized Kac-Moody algebra $\mathfrak{g}(A^\natural)$.

Definition 2.6.3. The *monster Lie algebra* $\mathfrak{m}^\natural$ is the generalized Kac-Moody algebra $\mathfrak{g}(A^\natural)/\mathfrak{c}$, where $\mathfrak{c}$ is the center of $\mathfrak{g}(A^\natural)$.

The monster Lie algebra is the most famous generalized Kac-Moody algebra because of its importance in Borcherds' proof of the monstrous moonshine conjecture. Notice that his definition of the monster Lie algebra is very different from ours. In fact one of the most remarkable properties of this Lie algebra is that it has a natural action of the *Conway-Norton monster group* $\mathbb{M}$. Clearly our definition is not useful to recognize such a $\mathbb{M}$-action. This action comes naturally by Borcherds' definition, however. We will discuss this later. Of course it is clear that this definition is not useful to see if a given Lie algebra is a generalized Kac-Moody algebra or not. Yet in [Bor91] and [Bor95b] Borcherds found a characterization for generalized Kac-Moody algebras which can be used to do this.

Theorem 2.6.4 ([Bor91, Thm. 1])**.** *A real Lie algebra $\mathfrak{g}$ that satisfies the following condition is a generalized Kac-Moody algebra.*

1. *$\mathfrak{g}$ can be $\mathbb{Z}$-graded as $\mathfrak{g} = \oplus_{n\in\mathbb{Z}}\mathfrak{g}_n$, with $\mathfrak{g}_n$ finite dimensional for $n \neq 0$.*
2. *$\mathfrak{g}$ has an involution ω which maps $\mathfrak{g}_n$ to $\mathfrak{g}_{-n}$ and acts as -1 on $\mathfrak{g}_0$.*
3. *$\mathfrak{g}$ has a symmetric invariant bilinear form $(\cdot,\cdot)$ which is preserved by ω and such that $\mathfrak{g}_m$ and $\mathfrak{g}_n$ are orthogonal unless $m = -n$.*
4. *If $g \in \mathfrak{g}_n$, $g \neq 0$ and $n \neq 0$, then $(g, \omega(g)) > 0$.*

Usually it turn out to be hard to construct the involution ω explicitly. Therefore it is often hard to use this characterization. In [Bor95b] Borcherds gave a more flexible characterization, which also has a complex version. This is given in [Car16]. Yet in the following we will just make use of the version given above. For the following discussion we will make a few additional assumptions on the structure of the generalized Kac-Moody algebra $\mathfrak{g}$ under consideration. We introduce subspaces

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\} \tag{253}$$

for every $\alpha \in \mathfrak{h}^*$ and call such an $\alpha \neq 0$ a *root* if $\mathfrak{g}_\alpha \neq 0$. Assume that the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is self-centralizing and finite-dimensional. This implies that we have $\mathfrak{h} = \mathfrak{g}_0$. Furthermore we assume that $\mathfrak{g}$ has a *root space decomposition*

$$\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha \tag{254}$$

and that all root spaces $\mathfrak{g}_\alpha$ are finite-dimensional. Clearly the root space decomposition is closely related to the abstract root space decomposition defined in (240). Since $\mathfrak{h} \subset \mathfrak{h}^e$ we can consider restrictions $\alpha'|_{\mathfrak{h}^e} = \alpha \in \mathfrak{h}^*$ for any $\alpha' \in \Delta$ and we obtain

$$\mathfrak{g}_\alpha = \bigoplus_{\alpha'|_{\mathfrak{h}^e}=\alpha} \mathfrak{g}'_{\alpha'}. \tag{255}$$

This sum runs over all roots $\alpha' \in \Delta$ with $\alpha'|_{\mathfrak{h}^e} = \alpha \in \mathfrak{h}^*$. Furthermore we assume that the invariant bilinear form $(\cdot,\cdot)$ of $\mathfrak{g}$ is non-degenerate. As a consequence all generators h_{ij} in $\mathfrak{h}$ vanish for $i \neq j$ and we can identify $\mathfrak{h}$ with its dual $\mathfrak{h}^*$ by this bilinear form. In particular every root $\alpha \in \mathfrak{h}^*$ can be identified with an element $h \in \mathfrak{h}$ which we call a *root* as well. We define an isometry between the ambient space of the root lattice and the Cartan subalgebra $p : R \to \mathfrak{h}$ by $\alpha_i \mapsto h_i$. This map usually has a large kernel, since the generators h_i are not linearly independent. Regardless it is common to call the generators *simple roots* and call the sublattice $p(Q)$ of $\mathfrak{h}$ the *root lattice*. We call $h \in \mathfrak{h}$ a *root* if its preimage $p^{-1}(h)$ contains a root of Δ. Under this map an element $\lambda \in R$ gets mapped to the element $h = p(\lambda)$ which satisfies

$$\lambda|_{\mathfrak{h}}(h') = (h, h') \; \forall h' \in \mathfrak{h}. \tag{256}$$

So $h \in \mathfrak{h}$ is a root if and only if it corresponds to a root in the root space decomposition (254) under the identification of $\mathfrak{h}$ with $\mathfrak{h}^*$. Let $\alpha \in \Delta^{re}$ be a real root then it is easy the see that $h = p(\alpha)$ has exactly one root in its preimage $p^{-1}(h)$, which is α of course. Yet several different imaginary roots $\alpha \in \Delta$ can get mapped to the same element $h \in \mathfrak{h}$ under p. This might lead to some ambiguity since we also called the number $\dim(\mathfrak{g}_\alpha)$ the multiplicity of the root α. Usually all roots in the preimage $p^{-1}(h_i)$ of a simple root h_i will be simple roots in Q, but this does not have to be the case. If the Weyl vector ρ defines an element in R we can clearly map it into $\mathfrak{h}$ and call its image *Weyl vector* as well. For $i \in I_0$ we can define an isometry $r'_i : \mathfrak{h} \to \mathfrak{h}$ by

$$r'_i(h) = h - \frac{2(h, h_i)}{(h_i, h_i)} h_i \tag{257}$$

and this induces a natural action of the Weyl group W on $\mathfrak{h}$ which commutes with p. See the discussion in section 4 of [Bor92] for more details of this. In the following we apply those notational changes to the root system of the monster Lie algebra $\mathfrak{m}^\natural$. It turns out that the root lattice $p(Q)$ is isomorphic to the lattice $\mathrm{II}_{1,1}$ and under this identification the positive simple roots in $\mathrm{II}_{1,1}$ are precisely the elements $\binom{1}{n}$ with $n = -1, 1, 2, 3, \cdots$. Here $\binom{1}{-1}$ is the only positive real simple root, so all $\binom{1}{n}$ for $n = 1, 2, 3, \cdots$ are imaginary. Under this map all imaginary simple root α_i, corresponding to a column in the j-th column of blocks in the matrix $A^\natural$, get mapped to the simple root $\binom{1}{j}$. Therefore this imaginary simple root has multiplicity $c(j)$. See Theorem 7.2. in [Bor92]. A generalized Kac-Moody algebra $\mathfrak{g}$ is clearly determined if its simple roots and their scalar products are given, since this determines the Borcherds-Cartan matrix. Yet for a fixed Cartan subalgebra $\mathfrak{h}$ with bilinear form $(\cdot,\cdot)$ a generalized Kac-Moody algbera $\mathfrak{g}$ is also determined, up to isomorphism, by its root multiplicities.

Proposition 2.6.5 ([Car16, Lemma 3.4.4.]). *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be generalized Kac-Moody algebras with finite-dimensional Cartan $\mathfrak{h}_1$ and $\mathfrak{h}_2$, which are self centralizing such that all root spaces in the decomposition (254) are finite-dimensional. Given an isometric isomorphism $f : \mathfrak{h}_1 \to \mathfrak{h}_2$, then this map can be extended to a Lie algebra isomorphism $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ if and only if the root multiplicities are identical under the isometry.*

2.7 Vertex operator algebras

In this subsection we give a brief introduction to vertex operator algebras and their most important properties. There is extensive literature on this subject to which we refer the reader for any details. For example there is [FBZ04], [LL04], [Kac98] and [FHLS93]. A recent and well-written introduction is also given in [Möl16]. The following discussion is highly influenced by those sources. Before we can come to the definition of vertex operator algebras we have to discuss some formal calculus. We closely follow [FBZ04] on this. Fix a complex vector space V. For formal variables $z_1, \cdots, z_n$ we can introduce the space

$$V[[z_1^{\pm 1}, \cdots, z_n^{\pm 1}]] = \left\{ A(z_1, \cdots, z_n) = \sum_{i_1, \cdots, i_n \in \mathbb{Z}} A_{i_q, \cdots, i_n} z_1^{i_1} \cdots z_n^{i_n} : A_{i_1, \cdots, i_n} \in V \right\} \tag{258}$$

of *formal distributions*. Assume for a moment that V is not just a complex vector space but a $\mathbb{C}$-algebra. For formal variables $w_1, \cdots, w_m$ and a corresponding formal distribution $B(w_1, \cdots, w_m)$ we can consider a product $A(z_1, \cdots, z_n)B(w_1, \cdots, w_m)$ as an element in $V[[z_1^{\pm 1}, \cdots, z_n^{\pm 1}, w_1^{\pm 1}, \cdots, w_m^{\pm 1}]]$ but in general it does not make sense to consider a product of two elements in $V[[z_1^{\pm 1}, \cdots, z_n^{\pm 1}]]$ since the coefficients might not be well-defined. One of the most important formal distributions is the *formal δ-function*, defined by

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n \in \mathbb{C}[[z^{\pm 1}]]. \tag{259}$$

A fundamental property of the δ-function is that

$$f(z)\delta(z) = f(1)\delta(z) \text{ for all } f(z) \in \mathbb{C}[z^{\pm 1}]. \tag{260}$$

Another basic but important relation for formal variables x and y is

$$x^{-1}\delta\left(\frac{y}{x}\right) = y^{-1}\delta\left(\frac{y}{x}\right) = y^{-1}\delta\left(\frac{x}{y}\right). \tag{261}$$

See section 2.1 in [LL04] for this. We know introduce the space of *formal Laurent series* with values in V by

$$V((z)) = \left\{ A(z) = \sum_{n=N}^{\infty} A_n z^n : A_n \in V \text{ and } n \in \mathbb{Z} \right\}. \tag{262}$$

If V is a $\mathbb{C}$-algbera than so is $V((z))$ since products $A(z)B(z)$ make sense for formal Laurent series $A(z), B(z) \in V((z))$. For formal variables v and w we can now consider spaces $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$. Here $\mathbb{C}((z))((w))$ is the space of formal Laurent series in the variable w with values in $\mathbb{C}((z))$ and $\mathbb{C}((w))((z))$ is the space of formal Laurent series in the variable z with values in $\mathbb{C}((w))$. Those spaces are not equal of course, for example we have $\sum_{n=0}^{\infty} z^{-n} w^n \in \mathbb{C}((z))((w))$ but also $\sum_{n=0}^{\infty} z^{-n} w^n \notin \mathbb{C}((w))((z))$. Yet we have the crucial

$$\mathbb{C}((z))((w)) \cap \mathbb{C}((w))((z)) = \mathbb{C}[[z, w]][z^{-1}, w^{-1}]. \tag{263}$$

The field of fraction of $\mathbb{C}[[z, w]]$ will be denote by $\mathbb{C}((z, w))$. Following [LL04] we set $S = \{w, z, w \pm z\}$ and define the subalgebra $\mathbb{C}((z, w))_S$ of $\mathbb{C}((z, w))$ generated by $z^{\pm 1}$, $w^{\pm 1}$ and $(z \pm w)^{-1}$. We can define embeddings $\iota_{z,w}\mathbb{C}((z, w))_S \to \mathbb{C}((z))((w))$ and $\iota_{w,z}\mathbb{C}((z, w))_S \to \mathbb{C}((w))((z))$ by defining them on the generators of $\mathbb{C}((z, w))_S$. The definition on $z^{\pm 1}$ and $w^{\pm 1}$ should be obvious and on $(z \pm w)^{-1}$ we define

$$\iota_{z,w}((z \pm w)^{-1}) = \sum_{n=0}^{\infty} (\mp 1)^n z^{-n-1} w^n \in \mathbb{C}((z))((w)) \text{ and} \tag{264}$$

$$\iota_{z,w}((z \pm w)^{-1}) = -\sum_{n=0}^{\infty} (\mp 1)^n w^{-n-1} z^n \in \mathbb{C}((w))((z)). \tag{265}$$

For an element $v \in \mathbb{C}[[z, w]][z^{-1}, w^{-1}]$ of course we have $\iota_{z,w}(v) = \iota_{w,z}(v)$. Notice furthermore that in $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$ we have

$$\delta\left(\frac{z}{w}\right) = \iota_{z,w}((z - w)^{-1}) - \iota_{w,z}((z - w)^{-1}). \tag{266}$$

We also need certain expressions in three variables z_0, z_1 and z_2. Namely we need to consider expressions like $z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)$. We will as usual expand those expressions as in

$$\iota_{z_1,z_2}\left(\delta\left(\frac{z_1-z_2}{z_0}\right)\right)=\sum_{m\in\mathbb{Z}_{\geq 0},n\in\mathbb{Z}}(-1)^m\binom{n}{m}z_0^{-n}z_1^{n-m}z_2^m. \tag{267}$$

This is common in the literature and so the embedding ι_{z_1,z_2} is usually omitted. See [FHLS93] or [LL04] for this. Basic properties of those expressions are

$$\iota_{z_2,z_0}z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)=\iota_{z_1,z_0}z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) \text{ and} \tag{268}$$

$$\iota_{z_1,z_2}z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)-\iota_{z_2,z_1}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)=\iota_{z_1,z_0}z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right). \tag{269}$$

The following treatment of fields and their locality is based on [FBZ04].

Definition 2.7.1. Let V be a complex vector space. A *field* is a formal distribution

$$A(z)=\sum_{j\in\mathbb{Z}}A_jz^{-j}\in \mathrm{End}(V)[[z^{-1}]] \tag{270}$$

such that we have for any $v\in V$ that $A_jv=0$ for j large enough. This just means $A(z)v\in V((z))$.

Assume now that the vector space V is $\mathbb{Z}_{\geq 0}$-graded with homogeneous subsets $V_n=\{v\in V : \deg(v)=n\}$. We call a field A *homogeneous* of conformal dimension $\Delta\in\mathbb{Z}$ if the modes A_j are homogeneous of degree $-j+\Delta$. For fields $A(z)$ and $B(z)$ we have the make sense out of the composition $A(z)B(w)$ in the following. Take $v\in V$ and $\rho\in V^*$ then we can consider the usual pairing $\langle\rho, A(z)B(w)v\rangle\in\mathbb{C}[[z^{\pm1},w^{\pm1}]]$. Using the properties of a field we observe $\langle\rho, A(z)B(w)v\rangle\in\mathbb{C}((z))((w))$ and $\langle\rho, B(w)A(z)v\rangle\in\mathbb{C}((w))((z))$.

Definition 2.7.2. Two fields $A(z)$ and $B(w)$ are *local* with respect to each other if for every $v\in V$ and $\rho\in V^*$ there is an element

$$f_{v,\rho}\in\mathbb{C}[[z,w]][z^{-1},w^{-1},(z-w)^{-1}] \tag{271}$$

such that $\langle\rho, A(z)B(w)v\rangle=\iota_{z,w}(f_{v,\rho})$ and $\langle\rho, B(w)A(z)v\rangle=\iota_{w,z}(f_{v,\rho})$. In other words

$$\langle\rho, A(z)B(w)v\rangle \text{ and } \langle\rho, B(w)A(z)v\rangle \tag{272}$$

are just different expansion of the element $f_{v,\rho}$.

Furthermore we have that two fields $A(z)$ and $B(w)$ are local if and only if there is an integer $N\in\mathbb{Z}_{\geq 0}$ such that

$$(z-w)^N[A(z),B(w)]=0 \tag{273}$$

holds in $\mathrm{End}(V)[[z^{\pm1},w^{\pm1}]]$. For further details about this formal calculus the reader may consult the literature indicated above. For a field $A(z)$ we set

$$A_+(z)=\sum_{n\geq 0}A_nz^n \text{ and } A_-(z)=\sum_{n\leq 0}A_nz^n. \tag{274}$$

Given two field $A(z)$ and $B(w)$ we want to make sense out of the expression $A(z)B(w)|_{z=w}$. This is usually not possible but following [FBZ04] we can introduce the *normally ordered product* of the fields $A(z)$ and $B(w)$ by

$$:A(z)B(w):=A_+(z)B(w)+B(w)A_-(z) \tag{275}$$

$$=\sum_{n\in\mathbb{Z}}\left(\sum_{m<0}A_mB_nz^{-m-1}+\sum_{m\geq 0}B_nA_mz^{-m-1}\right)w^{-n-1}. \tag{276}$$

The normally ordered product of two fields has now basically all properties we want from a product of fields. This means that the specialization $: A(z)B(w) : |_{z=w}$ makes sense and we denote it by $: A(z)B(z) :$. Furthermore For any $v \in V$ and $\rho \in V^*$ we get

$$\langle \rho, : A(z)B(w) : v \rangle \in \mathbb{C}[[z,w]][z^{-1}, w^{-1}]. \tag{277}$$

For three fields $A(z)$, $B(z)$ and $C(z)$ we can define

$$: A(z)B(z)C(z) :=: A(z)(: B(z)C(z) :) : \tag{278}$$

and extend this definition to any finite number of fields in the obvious way.

Definition 2.7.3 (Vertex algebras)**.** A *vertex algebra* is a collection of data:

1. A $\mathbb{Z}_+$-graded complex vector space
$$V = \bigoplus_{n=0}^{\infty} V_n \tag{279}$$
which we call *space of states.*

2. A vector $\mathbf{1} \in V_0$ which we call *vacuum vector.*

3. A *translation operator* $T : V \to V$ of degree 1.

4. A linear operator $Y(\cdot, z) : V \to \mathrm{End}(V)[[z^{\pm 1}]]$ which maps each $A \in V_n$ to a field
$$Y(A,z) = \sum_{m \in \mathbb{Z}} A_m z^{-m-1} \tag{280}$$
of conformal dimension n, i.e. $\deg(A_m) = -m + n - 1$.

This data satisfies the following properties:

1. We have $Y(\mathbf{1}, z) = \mathrm{Id}_V$ and for any $A \in V$ we have
$$Y(A,z)\mathbf{1} \in V[[z]] \tag{281}$$
such that we can evaluate $Y(A,z)\mathbf{1}|_{z=0} = A$. This is the *vacuum axiom.*

2. For any $A \in V$ we have
$$[T, Y(A,z)] = \partial_z Y(A,z) \tag{282}$$
and $T\mathbf{1} = 0$. We call this the *translation axiom.*

3. For $A, B \in V$ we demand that $Y(A,z)$ and $Y(B,z)$ are local with respect to each other. This is the *locality axiom.*

The locality axiom is the most important property of a vertex algebra. There are several ways to state it. Given all other axioms we could replace it by the *Borcherds identity* which states that for any $l, k, m \in \mathbb{Z}$ and any $A, B \in V$ we have

$$\sum_{j=0}^{\infty} \binom{l}{j} ((-1)^j A_{m+l-j} B_{k+j} - (-1)^{j+l} B_{k+l-j} A_{m+j}) = \sum_{n=0}^{\infty} \binom{m}{n} (A_{n+l}B)_{m+k-n}. \tag{283}$$

Yet another equivalent formulation is the *Jacobi identity* which states that for any $A, B \in V$ we have

$$\iota_{z_1,z_2} z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y(A,z_1)Y(B,z_2) - \iota_{z_2,z_1} z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(B,z_2)Y(A,z_1)$$
$$= \iota_{z_1,z_0} z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(A,z_0)B, z_2). \tag{284}$$

See [FHLS93] and [LL04] for details of the Jacobi identity. For computations the Borcherds identity will turn out to be the most useful formulation of locality axiom. Another useful formula, which is a direct consequence of the Borcherds identity is the *commutator formula*. It states that for $A, B \in V$ we have

$$[A_m, B_k] = \sum_{n \geq 0} \binom{m}{n} (A_n B)_{m+k-n}. \tag{285}$$

We can introduce the *Virasoro Lie algebra* or just the *Virasoro algebra of central charge* c to be the Lie algebra Vir spanned by L_n for $n \in \mathbb{Z}$ and a central element C over $\mathbb{C}$ with relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n+m,0}C. \tag{286}$$

We will call a module M of the Virasoro algebra Vir a *Virasoro module of central charge* $c \in \mathbb{C}$ if the central element C acts as $c\mathrm{Id}_M$. Such a module Let V be a vertex algebra and denote its homogeneous subspaces by V_n for $n \in \mathbb{Z}_{\geq 0}$ as above. We call a vector $\omega \in V_2$ a *conformal vector of central charge* $c \in \mathbb{C}$ of V if the modes L_n defined by

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \tag{287}$$

and the element $C = c\mathrm{Id}_V$ induce the structure of a Virasoro module of central charge $c \in \mathbb{C}$ on V. This just means that we have for $n, m \in \mathbb{Z}$ that

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n+m,0}c\mathrm{Id}_V. \tag{288}$$

Furthermore we demand that $L_0 v = nv$ for $v \in V_n$ and $L_{-1} = T$. This means in particular that V can be decomposed into L_0-eigenspaces of the Virasoro algebra and this eigenspace decomposition coincides with the $\mathbb{Z}_{\geq 0}$-grading of the vertex algebra.

Definition 2.7.4. A pair (V, ω) consisting of a vertex algebra V and a conformal vector of central charge $c \in \mathbb{C}$ such that $\dim(V_n) < \infty$ for all $n \in \mathbb{Z}$ and $\dim(V_n) = 0$ for n sufficiently negative will be called *vertex operator algebra*. We call ω the *Virasoro vector* of this vertex operator algebra and c its *central charge*.

Of course we can introduce the usual substructures like *vertex operator subalgebras*, *ideals* and *homomorphisms* of vertex operator algebras. Clearly we have those structures for vertex algebras as well. See chapter 1 in [FBZ04] and chapter 3.9 in [LL04] for details of this. Vertex algebras are quite sophisticated objects, therefore it is hard to construct explicit examples. Yet there is a construction using *generators and relations* that can be used to build many examples. In the following we discuss this construction briefly following [FBZ04]. Given a $\mathbb{Z}_{\geq 0}$-graded complex vector space V, a non-trivial vector $\mathbf{1} \in V_0$ and an endomorphism T of V of degree 1 . Assume that S is a countable ordered set and $\{a^\alpha\}_{\alpha \in S}$ a collection of homogeneous vectors in V. Furthermore we assume that for each $\alpha \in S$ we are given a field

$$a^\alpha(z) = \sum_{n \in \mathbb{Z}} a_n^\alpha z^{-n-1} \in \mathrm{End}(V)[[z^{\pm 1}]] \tag{289}$$

such that the following conditions hold:

1. For all $\alpha \in S$ we have $a^\alpha(z)\mathbf{1} = a^\alpha + z(\cdots)$.
2. $T\mathbf{1} = 0$ and $[T, a^\alpha(z)] = \partial_z a^\alpha(z)$.
3. The filed $a^\alpha(z)$ are mutually local.
4. V is spanned by

$$a_{j_1}^{\alpha_1} \cdots a_{j_m}^{\alpha_m}\mathbf{1} \text{ for } j_i < 0. \tag{290}$$

Theorem 2.7.5 (Reconstruction theorem). *Under the above assumptions the assignment*

$$Y(a^{\alpha_1}_{j_1}\cdots a^{\alpha_m}_{j_m}\mathbf{1},z)=\frac{1}{(-j_1-1)!\cdots(-j_m-1)!}:\partial_z^{-j_1-1}a^{\alpha_1}(z)\cdots\partial_z^{-j_m-1}a^{\alpha_m}(z): \tag{291}$$

defines a vertex algebra structure on V. Moreover this is the unique vertex algebra structure on V with the property that $Y(a^\alpha,z)=a^\alpha(z)$.

For this theorem and further details see section 3.6. in [FBZ04]. Above we introduced the Virasoro algebra Vir. We set $Vir_0=\mathbb{C}L_0\oplus\mathbb{C}C$ and $Vir_n=\mathbb{C}L_{-n}$ such that we get the structure of a $\mathbb{Z}$-graded Lie algebra

$$Vir=\bigoplus_{n\in\mathbb{Z}}Vir_n. \tag{292}$$

This is just the ad(L_0)-eigenspace decomposition. Following Remark 6.1.1 in [LL04] we note that instead of a gradation by *degree* this is a gradation by *weight*. We define a few subsets of Vir by

$$Vir_{\leq n}=\bigoplus_{k\leq n}Vir_k \text{ and } Vir_{\geq n}=\bigoplus_{k\geq n}Vir_k. \tag{293}$$

We furthermore introduce $Vir_-=Vir_{\leq -1}$ and $Vir_+=Vir_{\geq 1}$. For any complex number $c\in\mathbb{C}$ we introduce a 1-dimensional representation $\mathbb{C}_c$ of $Vir_{\leq 1}$ by letting C act as multiplication by c and L_n acts trivially for $n\leq 1$. The induced representation of Vir is given by

$$V(c,0)=U(Vir)\otimes_{U(Vir_{\leq 1})}\mathbb{C}_c. \tag{294}$$

We denote the element $1\otimes 1$ by v_c. By use of the Poincaré-Birkhoff-Witt theorem we find that $V(c,0)$ has a basis

$$L(-n_1)\cdots L(-n_r)v_c \tag{295}$$

for $n_1\geq n_2\geq\cdots\geq n_r\geq 2$. Of course this space has a natural $\mathbb{Z}_{\geq 0}$-grading given by the eigenspaces of L_0. This is

$$V(c,0)_n=\text{span}_{\mathbb{C}}(\{L(-n_1)\cdots L(-n_r)v_c : n_1+\cdots+n_r=n\}). \tag{296}$$

We can introduce an element $\omega=L_{-2}v_c$ and a field

$$Y(\omega,z)=T(z)=\sum_{n\in\mathbb{Z}}L_nz^{-n-2}\in\text{End}(V(c,0))[[z^{\pm 1}]]. \tag{297}$$

Furthermore we set $\mathbf{1}=v_c$ and $T=L_{-1}$. We observe that this data satisfies the conditions given above. Therefore by use of the reconstruction theorem $V(c,0)$ can be equipped with a unique vertex algebra structure such that $Y(\omega,z)=T(z)$. Finally we notice that the vector $\omega\in V(c,0)_2$ satisfies all the properties of a conformal vector of central charge c of the Vertex algebra $V(c,0)$. As a consequence the vertex algebra $V(c,0)$ is a vertex operator algebra of central charge c with Virasoro vector ω. We call this vertex operator algebra the *Virasoro vertex operator algebra of central charge* c. This is our first non-trivial example of a vertex operator algebra. Assume now that V is a vertex operator algebra of central charge $c\in\mathbb{C}$ with Virasoro vector ω. Consider the smallest vertex operator subalgebra $U\subset V$. Of course this is the vertex operator subalgebra generated by the Virasoro vector ω and isomorphic to $V(c,0)$. We find that any vertex operator algebra of central charge c has $V(c,0)$ as a vertex operator subalgebra. Vetex algebras have furthermore an interesting representation theory which we are going to discuss now. This is a huge field therefore we have to refer to the literature for details again. See [FHLS93] and [LL04] for the basics. We follow their approach closely. Let V be a vertex algebra.

Definition 2.7.6. A V*-module* is a vector space W equipped with a linear map

$$Y_W(\cdot,z):V\to\text{End}(W)[[z^{\pm 1}]] \tag{298}$$

such that *all the defining properties of vertex algebra that make sense hold*. This means that $Y_W(v,z)$ is a field in End$(W)[[z^{\pm 1}]]$ for every $v\in V$ and furthermore:

1. We have the *vacuum property* $Y_W(\mathbf{1}, z) = \mathrm{Id}_W$ and

2. we have the *Jacobi identity*, i.e. for any $A, B \in V$ we have

$$\iota_{z_1,z_2} z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_W(A, z_1)Y_W(B, z_2) - \iota_{z_2,z_1} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_W(B, z_2)Y_W(A, z_1)$$
$$= \iota_{z_1,z_0} z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_W(Y(A, z_0)B, z_2). \quad (299)$$

For any $v \in V$ we call $Y_W(v, z)$ the vertex operator on W associated with $v \in V$. Of course we can always consider V itself as a V-module. We call it the *adjoint module*. Assume now that V is a vertex operator algebra with Virasoro vector $\omega \in V_2$. Let W be a V-module, where we consider V as a vertex algebra. Then we can define operators $L_n \in \mathrm{End}(W)$ for $n \in \mathbb{Z}$ by

$$Y_W(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \quad (300)$$

Proposition 2.7.7 ([LL04, Prop. 4.1.5])**.** *Let V be a vertex operator algebra and W a V-module viewed as a vertex algebra then the endomorphisms L_n satisfy:*

$$[L_{-1}, Y_W(v, z)] = Y_W(L_{-1}v, z) = \partial_z Y_W(v, z) \text{ for } v \in V. \quad (301)$$

Furthermore for $n, m \in \mathbb{Z}$ we have

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n+m,0} c Id_W. \quad (302)$$

This shows that we can consider a V-module W as a Virasoro algebra module of central charge c.

Definition 2.7.8. Let V be a vertex operator algebra. A *V-module* is a module W of V viewed as a vertex algebra such that

$$W = \amalg_{h\in\mathbb{C}} W_h, \quad (303)$$

where we have

$$W_h = \{w \in W : L_0 w = hw\}. \quad (304)$$

Furthermore we assume that the subspaces W_h of weight h vectors are of finite dimension and $W_h = 0$ if $\mathrm{Re}(h)$ is sufficiently negative.

Sometimes in the literature it is assumed that the V-module W is graded by $\mathbb{Q}$. For example this is assumed in [FHLS93]. For us this does not make a huge difference since all modules we will need in the following have a $\mathbb{Q}$-grading. We call a subspace $U \subset W$ a *V-submodule of W* if it satisfies $Y_W(v, z)u \in U((z))$ for all $v \in V$ and $u \in U$. A V-module W will be called irreducible if has no non-trivial submodules. A vertex operator algebra V will be called *simple* if its adjoint module is irreducible. We can easily introduce the usual notions like *V-module homomorphisms,V-module endomorphisms* and *V-module isomorphisms* and so on. See section 4.5 in [LL04] for details. Given a homogeneous $v \in V$. Then the mode v_n acts with weight $\mathrm{wt}(v_n) = \mathrm{wt}(v) - n - 1$ on W. As a consequence for any $\alpha \in \mathbb{C}$ the space

$$W_{[\alpha]} = \bigoplus_{h\in\alpha+\mathbb{Z}} W_h \quad (305)$$

is a V-submodule of W. Notice that if W is irreducible there is a $\alpha \in \mathbb{C}$ such that $W = W_{[\alpha]}$. We denote the minimal element $h \in \alpha + \mathbb{Z}$ such that $W_h \neq 0$ by $\rho(W)$ and call it the *conformal weight of W*. Assume that we have vertex operator algebras V_1 and V_2 of central charge $c_1, c_2 \in \mathbb{C}$ respectively. Furthermore assume that we are given modules W_1 and W_2 of V_1 and V_2. The tensor product $V_1 \otimes_{\mathbb{C}} V_2$ carries a natural structure of a vertex operator algebra. We define the grading by

$\deg(v_1 \otimes v_2) = \deg(v_1) + \deg(v_2)$ and introduce a vacuum vector $\mathbf{1} = \mathbf{1}_1 \otimes \mathbf{1}_2$, where $\mathbf{1}_i$ is the vacuum vector of V_i for $i = 1, 2$. We can furthermore introduce a translation by $T = T_1 \otimes \mathrm{Id}_2 + \mathrm{Id}_1 \otimes T_2$ and finally a vertex operator

$$Y(A_1 \otimes A_2, z) = Y(A_1, z) \otimes Y(A_2, z). \tag{306}$$

So far this defines the structure of a vertex algebra on $V_1 \otimes V_2$, which we call the *tensor product of* V_1 *and* V_2. By

$$\omega = \omega_1 \otimes \mathbf{1}_2 \otimes \mathbf{1}_1 \otimes \omega_2 \tag{307}$$

we can define a Virasoro vector for $V_1 \otimes V_2$ such that this tensor product carries the structure of a vertex operator algebra of central charge $c = c_1 + c_2$. A direct consequence of (306) is

$$(A_1 \otimes A_2)_k = \sum_{n+m+1=k} A_{1,n} \otimes A_{2,m}. \tag{308}$$

Clearly this operator makes sense since applied to any vector in $V_1 \otimes V_2$ just finitely many summand can be non-zero. The tensor product $W_1 \otimes_{\mathbb{C}} W_2$ can now be equipped with the structure of a $V_1 \otimes V_2$-module by

$$Y_{W_1 \otimes_{\mathbb{C}} W_2}(v_1 \otimes v_2, z) = Y_{W_1}(v_1, z) \otimes Y_{W_2}(v_2, z). \tag{309}$$

See section 1.5 [FBZ04], section 4.6 in [LL04] and [FHLS93] for details and proofs for those statements. Of course we can extend those tensor products to any finite number of vertex operator algebras and modules. Since we consider all vertex algebras over the complex numbers we usually omit the subscript $\cdot_{\mathbb{C}}$ in the tensor product $\otimes_{\mathbb{C}}$. This tensor product should not be confused with the *fusion product* $M_1 \boxtimes_V M_2$ of two V-modules M_1 and M_2. Before we can introduce this, we need a few further construction involving modules. We mainly follow [FHLS93] in this. Therefore we might simply assume that the V-modules are $\mathbb{Q}$-graded, such that we meet the definition of V-modules therein. So we assume that the V-module W has a grading

$$W = \bigoplus_{n \in \mathbb{Q}} W_n. \tag{310}$$

Its graded dual is

$$W' = \bigoplus_{n \in \mathbb{Q}} W_n^*. \tag{311}$$

The *adjoint vertex operator* Y' is the linear map

$$Y' : V \to \mathrm{End}(W')[[z^{\pm 1}]], v \mapsto Y'(v, z) = \sum_{n \in \mathbb{Z}} v'_n z^{-n-1}, \tag{312}$$

where we have $v'_n \in \mathrm{End}(W')$ as usual, which is characterized by the condition

$$\langle Y'(v, z)w', w\rangle = \langle w', Y(e^{zL_1}(-z^{-2})^{L_0} v, z^{-1})w\rangle \tag{313}$$

for all $v \in V$, $w' \in W'$ and $w \in W$. We may equip W with the obvious $\mathbb{Q}$-grading given by $W'_n = W_n^*$.

Theorem 2.7.9 ([FHLS93, Thm. 5.2.1.])**.** *The pair* (W', Y') *carries the structure of a* V*-module.*

We call the V-module W' the V*-module contragredient to* W or simply the *contragredient module of* W. The contragredient module W' of W has a few remarkable properties. We discuss them now following section 5.3 in [FHLS93].

Proposition 2.7.10 ([FHLS93, Prop. 5.3.1.])**.** *There are natural identifications between the contragredient module of the contragredient* $(W')'$ *and* W *and between the double-adjoint operators* $Y''(\cdot, z)$ *and* $Y(\cdot, z)$.

Notice furthermore that W is irreducible if and only if W' is irreducible. The existence of a nondegenerate bilinear form $(\cdot,\cdot)$ on W such that $W_n \perp W_m$ for $n \neq m$ is equivalent to linear isomorphism $\phi : W \to W'$ that preserves the gradation by

$$\langle \phi(w_1), w_2 \rangle = (w_1, w_2). \tag{314}$$

Such a linear isomorphism is a V-module isomorphism if and only if we have

$$(Y(v,z)w_1, w_2) = (w_1, Y(e^{zL_1}(-z^{-2})^{L_0}v, z^{-1})w_2) \tag{315}$$

for all $v \in V$ and $w_1, w_2 \in W$. We call such a bilinear form *invariant*. So this tells us that a V-module can be equipped with an invariant bilinear form if and only if it is isomorphic to its contragredient module. For elements $v, w \in V$ we can use (315) and the fact that $\mathrm{Res}_z(z^{-1}Y(v,z)\mathbf{1}) = v$ to prove

$$(v,w)\mathbf{1} = \mathrm{Res}_z \left(z^{-1} Y(e^{zL_1}(-z^{-2})^{L_0}v, z^{-1})w \right). \tag{316}$$

We call a vertex operator algebra V *self-contragredient* if its adjoint module is isomorphic to its contragredient module. We get that a vertex operator algebra can be equipped with an invariant bilinear form if and only if it is self-contragredient.

Proposition 2.7.11 ([FHLS93, Prop. 5.3.6.]). *Let V be a self-contragredient vertex operator algebra equipped with the invariant bilinear form $(\cdot,\cdot)$ then this bilinear form is symmetric.*

Usually we assume self-contragredience for a vertex operator algebra, since an invariant symmetric bilinear form is a strong useful structure to work with. But this is usually not enough. We need to assume certain further *regularity properties* to get a better understanding of the V-modules. What we call a V-module is usually called *ordinary V-module* in the literature. This helps to distinguish them from certain more general types of modules which are *weak modules* and *admissible modules*. A *weak module M* of a vertex operator algebra that satisfies all properties of a V-module except that we don't assume that it carries a grading. A weak module W is called *admissible* if it carries a $\mathbb{Z}_{\geq 0}$-grading

$$W = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} W(n) \tag{317}$$

such that for any $r, m \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$ and $a \in V_r$ we have

$$a_m W(n) \subset W(r + n - m - 1). \tag{318}$$

Of course any ordinary module is admissible, therefore we get

$$\{\text{ordinary } V\text{-modules}\} \subset \{\text{admissible } V\text{-modules}\} \subset \{\text{weak } V\text{-modules}\}. \tag{319}$$

See [DLM97] for the details of those definitions. We call an admissible module *simple* if its only $\mathbb{Z}_{\geq 0}$-graded submodules are 0 and W. A vertex operator algebra V is called *rational* if every admissible V-module is a direct sum of simple admissible V-modules. Following Remark 2.4 [DLM97] we note that if V is rational then every simple admissible module is ordinary and it has only finitely many simple inequivalent modules. Those simple admissible modules are of course irreducible as ordinary modules and we denote the set of all those irreducible ordinary modules up to isomorphism by $\mathrm{Irr}(V)$. A vertex operator algebra is called *regular* if each weak module is a direct sum of irreducible ordinary modules. Of course every regular vertex operator algebra is also rational. One important property of the tensor product is that it preserves regularity, i.e. if V_1 and V_2 are regular vertex operator algebras then is $V_1 \otimes V_2$ regular as well. Next we define the space

$$C_2(V) = \mathrm{span}_{\mathbb{C}}(\{v_{-2}w : v, w \in V\}). \tag{320}$$

We call a vertex operator algebra *C_2-cofinite* if the space $C_2(V)$ has finite codimension. Finally we call a vertex operator algebra of *CFT-type* if $V_0 = \mathbb{C}\mathbf{1}$. This condition is also called *Zhus cofiniteness condition*. Now we can clarify the relations between these conditions. In [Li99] it is proved that every regular vertex operator algebra is C_2-cofinite.

Theorem 2.7.12 ([ABD04, Thm. 4.5]). *A vertex operator algebra V of CFT-type is regular if and only if it is C_2-cofinite and rational.*

We are mainly interest in vertex operator algebras that are rational and C_2-cofinite.

Theorem 2.7.13 ([DLM00, Thm. 11.3]). *Take a rational and C_2-cofinite vertex operator algebra V. Then the central charge of V is rational and each irreducible module has rational conformal weight.*

A *strongly rational* vertex operator algebra is a vertex operator algebra that is self-contragredient, irreducible, regular and of CFT-type. A strongly rational vertex operator algebra is of course C_2-cofinite and rational by Theorem 2.7.12. In the following we discuss the most important properties of their modules. We follow section 5.4. in [FHLS93] for the introduction of *intertwining operators*.

Definition 2.7.14 ([FHLS93, Def. 5.4.1.]). Let V be a vertex operator algebra and let (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) be three V-modules. An *intertwining operator of type* $\binom{W_3}{W_1\, W_2}$ is a linear map

$$W_1 \to \operatorname{Hom}(W_2, W_3)\{z\}, z \mapsto \mathcal{Y}(w, z) = \sum_{n\in\mathbb{C}} w_n z^{-n-1} \tag{321}$$

such that *all properties of a module action that make sense hold.* That is, for $v \in V$, $w_1 \in W_1$ and $w_2 \in W_2$

$$(w_1)_n w_2 = 0 \text{ for } n \text{ sufficiently large} \tag{322}$$

and the following *Jacobi identity* holds for the operators $Y(v, \cdot)$, $\mathcal{Y}(w_1, \cdot)$ acting on the element w_2:

$$\begin{aligned}\iota_{z_1,z_2} z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_3(v, z_1)\mathcal{Y}(w_1, z_2)w_2 - \iota_{z_2,z_1} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) \mathcal{Y}(w_1, z_2)Y_2(v, z_1)w_2 \\ = \iota_{z_1,z_0} z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) \mathcal{Y}(Y_1(v, z_0)w_1, z_2)w_2.\end{aligned} \tag{323}$$

Furthermore we assume the *translation axiom*

$$\partial_z \mathcal{Y}(w_1, z) = \mathcal{Y}(L_{-1}w_1, z). \tag{324}$$

Notice that all appearing expressions are algebraically meaningful. In [FHLS93] the definition is just for $\mathbb{Q}$-graded modules but we use the more general version since we introduced modules that way. This does not make a difference for strongly rational vertex operator algebras, however. Notice that it is common to denote an intertwining operator by $\mathcal{Y}_{W_1\, W_2}^{W_3}$. We denote the space of intertwining operators of type $\binom{W_3}{W_1\, W_2}$ by $I\binom{W_3}{W_1\, W_2}$. The *fusion rules* are the numbers

$$N_{W_1\, W_2}^{W_3} = \dim I \begin{pmatrix} W_3 \\ W_1\ W_2 \end{pmatrix}. \tag{325}$$

Of course we already have seen examples of intertwining operators because the vertex operator Y is an intertwining operator of type $\binom{V}{V\, V}$ and the vertex operator Y_W is an intertwining operator of type $\binom{W}{V\, W}$ for any V-module W. In general the fusion rules might be ∞, but this is not the case for strongly rational vertex operator algebras. For any intertwining operator $\mathcal{Y} \in I\binom{W_3}{W_1\, W_2}$ we can introduce an *adjoint operator* $\mathcal{Y}'$ by

$$\langle \mathcal{Y}'(w_1, z)w_3', w_2\rangle = \langle w_3', \mathcal{Y}(e^{zL_1}(-z^{-2})^{L_0}w_1, z^{-1})w_2\rangle \tag{326}$$

for $w_i \in W_i$ and $w_i' \in W_i'$ and $i = 1, 2, 3$.

Theorem 2.7.15 ([FHLS93, Thm. 5.5.1.]). *The adjoint operator $\mathcal{Y}'$ of an intertwining operator $\mathcal{Y}$ of type $I\binom{W_3}{W_1\, W_2}$ is an intertwining operator of type $I\binom{W_2'}{W_1\, W_3'}$.*

We can of course consider the *double-adjoint operator* $\mathcal{Y}''$ and by Proposition 5.5.2. in [FHLS93] we know that this operator equals the intertwining operator $\mathcal{Y}$ such that the correspondence $\mathcal{Y} \mapsto \mathcal{Y}'$ defines a linear isomorphism from $I\binom{W_3}{W_1\,W_2}$ to $I\binom{W_2'}{W_1\,W_3'}$. As a consequence we get

$$N^{W_3}_{W_1\,W_2} = N^{W_2'}_{W_1\,W_3'}. \tag{327}$$

By use of *skew-symmetry* we can associate an operator $\mathcal{Y}^* : W_2 \otimes W_1 \to W_3[[z^{\pm 1}]]$ to any $\mathcal{Y} \in I\binom{W_3}{W_1\,W_2}$ by

$$\mathcal{Y}^*(w_2, z)w_1 = e^{zL_{-1}}\mathcal{Y}(w_1, -z)w_2 \tag{328}$$

for any $w_1 \in W_1$ and $w_2 \in W_2$. $\mathcal{Y}^*$ is an intertwining operator of type $I\binom{W_3}{W_2\,W_1}$ and satisfies $\mathcal{Y}^{**} = \mathcal{Y}$ therefore we obtain a linear isomorphism by $\mathcal{Y} \mapsto \mathcal{Y}^*$ from $I\binom{W_3}{W_1\,W_2}$ to $I\binom{W_3}{W_2\,W_1}$. We obtain

$$N^{W_3}_{W_1\,W_2} = N^{W_3}_{W_2\,W_1}. \tag{329}$$

For simplicity we can write N_{ijk} for $N^{W_k'}_{W_i\,W_j}$ then we get

$$N_{ijk} = N_{\sigma(i)\sigma(j)\sigma(k)} \tag{330}$$

for any permutation σ of $\{i, j, k\}$. This properties is called the *S_3-symmetry of the fusion rules.* See section 5.5. in [FHLS93] for details.

Definition 2.7.16 ([Li98, Def. 3.1])**.** Let V be a vertex operator algebra and W_1 and W_2 be V-modules. A *fusion product* of the ordered pair (W_1, W_2) is a pair consisting of a V-module $W_1 \boxtimes_V W_2$ and an intertwining operator $\mathcal{F}(\cdot, z)$ of type $\binom{W_1 \boxtimes_V W_2}{W_1\,W_2}$ such that for any V-module U and any intertwining operator $\mathcal{Y}$ of type $\binom{U}{W_1\,W_2}$ there is a unique V-module homomorphism $\phi : W_1 \boxtimes_V W_2 \to U$ such that $\mathcal{Y}(\cdot, z) = \phi \circ \mathcal{F}(\cdot, z)$.

By the usual argument we can check that the a fusion product $W_1 \boxtimes_V W_2$ is uniquely determined up to a unique isomorphism if it exists. Of course it is not at all clear that such a fusion product exits. The theory of fusion products is vast and complicated therefore we just discuss the principal statements.

Theorem 2.7.17 ([Li98, Thm. 3.20])**.** *Assume that V is a rational vertex operator algebra and (W_1, W_2) a ordered pair of V-modules, then a fusion product $W_1 \boxtimes_V W_2$ exists.*

In the following we assume that V is a strongly rational vertex operator algebra. For V-modules W_1, W_2 and W_3 we have

$$\mathrm{Hom}_V(W_1 \boxtimes_V W_2, W_3) \cong I\begin{pmatrix} W_3 \\ W_1\,W_2 \end{pmatrix}, \tag{331}$$

therefore we have $\dim \mathrm{Hom}_V(W_1 \boxtimes_V W_2, W_3) = N^{W_3}_{W_1\,W_2}$. This isomorphism maps each homomorphism $\phi \in \mathrm{Hom}_V(W_1 \boxtimes_V W_2, W_3)$ to $\phi \circ \mathcal{F}(\cdot, z)$, where $\mathcal{F}$ is the intertwining operator corresponding to the fusion product. A few further remarkable properties are the *commutativity* and the *associativity* of the fusion product. Notice that those properties just hold for the underlying V-module structure, however. We have

$$W_1 \boxtimes_V W_2 \cong W_2 \boxtimes_V W_1 \tag{332}$$

and

$$W_1 \boxtimes_V (W_2 \boxtimes_V W_3) \cong (W_1 \boxtimes_V W_2) \boxtimes_V W_3, \tag{333}$$

where both isomorphisms are isomorphisms of V-modules. We introduce the *fusion algebra* of V

$$\mathcal{V}(V) = \bigoplus_{W \in \mathrm{Irr}(V)} \mathbb{C}W. \tag{334}$$

The algebra structure is defined by linear extension of

$$W_1 \boxtimes_V W_2 = \sum_{W_3 \in \mathrm{Irr}(V)} N_{W_1\,W_2}^{W_3} W_3. \tag{335}$$

The sum on the right is isomorphic to $W_1 \boxtimes_V W_2$ as a V-module because of Schurs lemma and the fact that $\dim(\mathrm{Hom}_V(W_1 \boxtimes_V W_2, W_3)) = N_{W_1\,W_2}^{W_3}$. Since we assume that V is strongly rational we know that V is simple. Therefore it is contained in $\mathrm{Irr}(V)$. Following Remark 3.5 in [Li98] we know that $V \boxtimes_V W = W$ for any V-module W, so we get that V acts as a unit on the fusion algebra $\mathcal{V}(V)$. We call a V-module U a *simple current*, if its fusion product with any irreducible module is irreducible again. As a first consequence we get that a simple current has to be U irreducible since V is irreducible therefore we get that $U = V \boxtimes_V U$ has to be irreducible. We denote the subset of simple currents of $\mathrm{Irr}(V)$ by $S(V)$ and notice that this set is closed under the fusion product. This is a simple consequence of the associativity.

Proposition 2.7.18 ([LY08, Cor. 1]). *Assume that V is a strongly rational vertex operator algebra. Then the following holds.*

1. *Every simple current V-module is irreducible.*
2. *An irreducible V-module U is a simple current if and only if $U \boxtimes_V U' = V$.*
3. *The set of simple currents $S(V)$ forms a multiplicative abelian group in $\mathcal{V}(V)$ under the fusion product.*

Recall that U' denotes the contragredient V-module of U. In the following we introduce characters of modules and their relation to the fusion algebra. For any $v \in V$ that is homogeneous we write $\sigma(v) = v_{\deg(v)-1}$ such that this operator has degree 0. We can then introduce the *formal graded trace* of W by

$$\mathrm{Tr}_W(v,q) = q^{\rho(W)-\frac{c}{24}} \sum_{n=0}^{\infty} \mathrm{tr}_{W_{\rho(W)+n}}(\sigma(v))q^n. \tag{336}$$

The formal graded trace can be extended by linearity to all of V. Zhu studied those traces in his landmark paper [Zhu96] and showed that those traces in fact converge for $q = e^{2\pi i\tau}$ to holomorphic function on the upper half plane $\mathbb{H}$. Furthermore Zhu introduced a certain deformation of the grading of vertex operator algebra which we will call *Zhus second grading*. We denote the weight spaces with relation to this grading by $V_{[n]}$ such that we have

$$V = \bigoplus_{n=0}^{\infty} V_{[n]} \tag{337}$$

and we denote the corresponding degree operator by $\deg[\cdot]$ to separate it from the degree operator $\deg(\cdot)$ that correspond to the usual grading.

Theorem 2.7.19 ([Zhu96]). *Let V be a strongly rational vertex operator algebra of central charge c and $W \in Irr(V)$. Then:*

1. *Let $q = e^{2\pi i\tau}$. Then the formal sum $Tr_W(v,\tau)$ converges to a holomorphic function on the complex upper half plane $\mathbb{H}$.*
2. *Let $v \in V_{[k]}$ be of degree k in Zhus second grading. Then there is a representation*

$$\rho_V : SL_2(\mathbb{Z}) \to GL(\mathcal{V}(V)) \tag{338}$$

of $SL_2(\mathbb{Z})$ on the fusion algebra $\mathcal{V}(V)$ such that for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ we have

$$Tr_W(v,\gamma\tau) = (c\tau+d)^k \sum_{M \in Irr(V)} \rho(\gamma)_{W,M} Tr_M(v,\tau). \tag{339}$$

The precise version of this theorem is from [vEMS17] and can also be found in [Möl16]. Since the functions $\mathrm{Tr}_W(v,\tau)$ are linear independent this representation is uniquely determined. We call it *Zhus representation*. The specialization

$$\mathrm{ch}_W(\tau) = \mathrm{Tr}_W(\mathbf{1},\tau) = q^{\rho(W)-\frac{c}{24}} \sum_{n=0}^{\infty} \dim(W_{\rho(W)+n}) q^n \tag{340}$$

is called the *character of* W. Since $\mathrm{SL}_2(\mathbb{Z})$ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ the elements $\mathcal{S} = \rho_V(S)$ and $\mathcal{T} = \rho_V(T)$ are of particular importance. The *Verlinde formula* is a relation between the matrix $\mathcal{S}$ and the fusion rules. The particular version for vertex operator algebras is due to Huang and one of the deepest and most important results in the entire field. See [Hua08]. Once again we state the version given in [vEMS17].

Theorem 2.7.20. *Let V be a strongly rational vertex operator algebra. Then*

1. *The matrix $\mathcal{S}$ is symmetric and $\mathcal{S}^2$ is the permutation matrix sending W to its contragredient module W'. Moreover $\mathcal{S}_{V,W} \neq 0$ for $W \in Irr(V)$.*

2. *The fusion rules satisfy the Verlinde formula*

$$N_{M\,N}^{W} = \sum_{U \in Irr(V)} \frac{\mathcal{S}_{M,U}\mathcal{S}_{N,U}\mathcal{S}_{W',U}}{\mathcal{S}_{V,U}}. \tag{341}$$

The *translation matrix* $\mathcal{T}$ satisfies

$$\mathcal{T}_{W,M} = \delta_{W,M}\mathcal{T}_W = \delta_{W,M} e^{2\pi i(\rho(W)-\frac{c}{24})}. \tag{342}$$

In the following we introduce *quantum dimensions* and discuss further properties of the $\mathcal{S}$-matrix. This follows [DJX13]. Let V be a strongly rational vertex operator algebra and M a V-module. The *quantum dimension* of M is given by

$$\mathrm{qdim}_V(M) = \lim_{y\to 0} \frac{\mathrm{ch}_M(iy)}{\mathrm{ch}_V(iy)} \tag{343}$$

Clearly this definition can be extended to more general vertex operator algebras V, but then it is not clear if such a quantum dimension exists for any V-module W. We assume that V is strongly rational and has irreducible modules W_i for some index set $i \in I$ with $0 \in I$ and $V = W_0$. Then all quantum dimensions exist if the conformal weights $\rho(W_i)$ are positive for all $i \in I \setminus \{0\}$. Furthermore all quantum dimensions are positive numbers. See Lemma 4.2. in [DJX13]. We can introduce the *global dimension of* V by

$$\mathrm{glob}(V) = \sum_{i\in I} \mathrm{qdim}_V(W_i)^2. \tag{344}$$

Proposition 2.7.21 ([DJX13]). *We assume that V is a strongly rational vertex operator algebra with irreducible modules W_i for $i \in I$. We assume $0 \in I$ and write $V = W_0$. Then the quantum dimensions have the following properties:*

1. *$qdim_V(W_i) = \frac{\mathcal{S}_{W_i,V}}{\mathcal{S}_{V,V}}$.*

2. *$qdim_V(W_i \boxtimes W_j) = qdim_V(W_i) qdim_V(W_j)$.*

3. *$glob(V) = \frac{1}{\mathcal{S}_{V,V}^2}$.*

4. *$qdim_V(W_i) = 1$ if and only if W_i is a simple current.*

Finally we need to discuss some further relations between the fusion rules, the $\mathcal{S}$-matrix and the quantum dimensions.

Proposition 2.7.22. *Let V be a strongly rational vertex operator algebra with irreducible modules W_i for $i \in I$ and $V = W_0$. Then we have*

$$\mathcal{S}_{W_i,W_k}\mathcal{S}_{W_j,W_k} = \mathcal{S}_{V,V}\, qdim_V(W_k) \sum_{i\in I} N_{W_i\,W_j}^{W_l} \mathcal{S}_{W_l,W_k} \tag{345}$$

and furthermore we have

$$\mathcal{S}_{W_i,W_j} = \mathcal{S}_{V,V} \sum_{i\in I} N_{W_i\,W_j}^{W_k}\, qdim_V(W_k) \mathcal{T}_{W_i}\mathcal{T}_{W_j}\mathcal{T}_{W_k}^{-1}\mathcal{T}_V^{-1}. \tag{346}$$

Proof. This is an explicit computation that makes use of the Verlinde formula. □

2.8 Affine vertex operator algebras

Affine vertex operator algebras are vertex operator algebras, which are build on integrable highest-weight modules of certain affine Kac-Moody algebras. They are of particular importance in physics, since they represent *Wess-Zumino-Witten models* in conformal field theory. An original source is [FZ92]. Further expositions are given in [FBZ04], [LL04] and [Kac98]. For a perspective more related to physics [Fuc95] and [FMS12] are a good source. This section is based on the entire discussion in section 2.5 and we keep all notations from there. In particular we assume again that $\mathfrak{g}$ is a simple Lie algebra. As usual we denote the corresponding affine Kac-Moody algebra by $\hat{\mathfrak{g}}$ and consider its subalgebra

$$\mathfrak{g}' = \mathbb{C}[t,t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K. \tag{347}$$

This space is of course equipped with the restricted Lie algebra structure of $\hat{\mathfrak{g}}$. We can introduce a *degree* by $\deg(x(n)) = -n$ and set $\mathfrak{g}'_- = \mathfrak{g} \otimes t\mathbb{C}[t]$, $\mathfrak{g}'_+ = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$ and $\mathfrak{g}'_0 = \mathfrak{g} \otimes t^0 \oplus \mathbb{C}K$. Notice that $\mathfrak{g}'_-$ is the subspace of positive degree and that $\mathfrak{g}'_+$ is the subspace of negative degree. We put $\mathfrak{b}' = \mathfrak{g}'_0 \oplus \mathfrak{g}'_-$ and denote the 1-dimensional representation of $\mathfrak{b}'$ on which K acts as $k \in \mathbb{C}$ and $\mathfrak{g}'_-$ acts trivially by $\mathbb{C}_k$. We introduce the *vacuum representation of level k* of $\mathfrak{g}'$ as

$$V_k(\mathfrak{g}', 0) = U(\mathfrak{g}') \otimes_{U(\mathfrak{b}')} \mathbb{C}_k. \tag{348}$$

More general for any $\mu \in \mathfrak{h}^*$ we can consider the irreducible module $L(\mu)$ of highest-weight μ of $\mathfrak{g}$. We can introduce a $\mathfrak{g}'$-module $V_k(\mathfrak{g}', \mu)$ by

$$V_k(\mathfrak{g}', \mu) = U(\mathfrak{g}') \otimes_{U(\mathfrak{b}')} L(\mu), \tag{349}$$

where we let K act as k on $L(\mu)$. We denote the maximal proper $\mathfrak{g}'$-submodule of $V_k(\mathfrak{g}', \mu)$ by $J_k(\mathfrak{g}', \mu)$ and introduce the irreducible $\mathfrak{g}'$-module

$$L_k(\mathfrak{g}', \mu) = V_k(\mathfrak{g}', \mu)/J_k(\mathfrak{g}', \mu). \tag{350}$$

Proposition 2.8.1 ([FZ92, Prop. 2.1.1.]). *$L_k(\mathfrak{g}', \mu)$ is the unique $\mathfrak{g}'$-module satisfying the following properties.*

1. *$L_k(\mathfrak{g}', \mu)$ is irreducible as a $\mathfrak{g}'$-module.*

2. *The central element K acts as kId on $L_k(\mathfrak{g}', \mu)$.*

3. *$V_\mu = \{a \in L_k(\mathfrak{g}', \mu) : \mathfrak{g}'_- a = 0\}$ is the irreducible $\mathfrak{g}$-module with highest-weight μ.*

We denote the vector $1 \otimes 1 \in V_k(\mathfrak{g}', 0)$ by v_k. Let J^a for $a = 1, \cdots, d$ with $d = \dim(\mathfrak{g})$ be a basis of $\mathfrak{g}$. Of course we can span $V_k(\mathfrak{g}', 0)$ by elements of the form

$$J^{a_1}(-n_1) \cdots J^{a_r}(-n_r)v_k, \tag{351}$$

where we have $a_i = 1, \cdots, d$ and $n_i > 0$ for all $i = 1, \cdots, r$. We introduce a degree by

$$\deg(J^{a_1}(-n_1) \cdots J^{a_r}(-n_r)v_k) = n_1 + \cdots + n_r. \tag{352}$$

Finally for every $J^a = J^a(-1)v_k$ we introduce fields

$$Y(J^a(-1)v_k, z) = J^a(z) = \sum_{n\in\mathbb{Z}} J^a(n)z^{-n-1} \in \mathrm{End}(V_k(\mathfrak{g}', 0))[[z^{\pm 1}]]. \tag{353}$$

Using the reconstruction theorem we get that $V_k(\mathfrak{g}', 0)$ is a vertex algebra. In the following we assume $k+\check{h} \neq 0$. For our choice of a basis J^a for $a = 1, \cdots, d$ we denote its dual basis by J_a for $a = 1, \cdots, d$, i.e. we have

$$(J^a, J_b) = \delta_{a,b} \tag{354}$$

for all $a, b = 1, \cdots, d$. The *Sugawara vector* is

$$\omega = \frac{1}{2(k+\check{h})} \sum_{a=1}^{d} J_a(-1)J^a(-1)v_k. \tag{355}$$

In fact the Sugawara vector is a conformal vector of central charge

$$c(\mathfrak{g}, k) = \frac{k\dim(\mathfrak{g})}{k+\check{h}}. \tag{356}$$

Equipped with the Sugawara vector as Virasoro vector $V_k(\mathfrak{g}', 0)$ has a natural structure of a vertex operator algebra of central charge $c(\mathfrak{g}, k)$. See [FBZ04], [LL04] and [FZ92] for this construction. Let $I \subset V_k(\mathfrak{g}', 0)$ be a vertex operator algebra ideal. Then $V_k(\mathfrak{g}', 0)/I$ is a vertex algebra as well. If we have $\mathbf{1} \notin I$ and $\omega \notin I$ then $V_k(\mathfrak{g}', 0)/I$ is a vertex operator algebra of central charge $c(\mathfrak{g}, k)$. Let now $J_k(\mathfrak{g}', 0)$ be the maximal proper submodule of $V_k(\mathfrak{g}', 0)$ for $k \neq 0$ we have $\mathbf{1} \notin J_k(\mathfrak{g}', 0)$ as well as $\omega \notin J_k(\mathfrak{g}', 0)$, therefore $L_k(\mathfrak{g}', 0) = V_k(\mathfrak{g}', 0)/J_k(\mathfrak{g}', 0)$ is a vertex operator algebra of central charge $c(\mathfrak{g}, k)$. Since $L_k(\mathfrak{g}', 0)$ is an irreducible $\mathfrak{g}'$-module it is a simple vertex operator algebra.

Theorem 2.8.2 ([FZ92, Thm. 3.1.3.]). *Let k be a positive integer, then the vertex operator algebra $L_k(\mathfrak{g}', 0)$ is rational. Let μ run through all integrable weights such that $(\mu, \theta) \leq k$, then $L_k(\mathfrak{g}', \mu)$ provides a complete list of irreducible $L_k(\mathfrak{g}', 0)$-modules.*

In order to get all regularity properties we need for the vertex operator algebra $L_k(\mathfrak{g}', 0)$, we come to the following statement.

Theorem 2.8.3 ([DLM97, Thm. 3.7.]). *Let k be a positive integer, then the vertex operator algebra $L_k(\mathfrak{g}', 0)$ is regular.*

From now on we assume that k is a positive integer. Since $L_k(\mathfrak{g}', 0)$ is clearly of CFT-type it has to be C_2-cofinite by Theorem 2.7.12.

Theorem 2.8.4. *Let k be a positive integer, then the vertex operator algebra $L_k(\mathfrak{g}', 0)$ is strongly rational.*

For every integral $\lambda \in \mathfrak{h}^*$ with $(\lambda, \theta) \leq k$ we set $\Lambda = k\Lambda_0 + \lambda \in \hat{\mathfrak{h}}^*$ and introduce the *vacuum anomaly*

$$h_\Lambda = \frac{(\Lambda + 2\rho, \Lambda)}{2(k+\check{h})} = \frac{(\lambda + 2\overline{\rho}, \lambda)}{2(k+\check{h})}. \tag{357}$$

The vacuum anomaly h_Λ is exactly the conformal weight of the module $L_k(\mathfrak{g}', \lambda)$. See Corollary 12.8. in [Kac90]. We can define an action of $d \in \hat{\mathfrak{g}}$ on $L_k(\mathfrak{g}', \lambda)$ by

$$d \mapsto h_\Lambda \mathrm{Id} - L(0). \tag{358}$$

It turns out that this extends to a natural action of $\hat{\mathfrak{g}}$ on $L_k(\mathfrak{g}', \lambda)$ under which we have a natural $\hat{\mathfrak{g}}$-module isomorphism from $L_k(\mathfrak{g}', \lambda)$ to $L(\Lambda)$. Using this we can naturally identify the vertex operator algebra $L_k(\mathfrak{g}', 0)$ with $L(k\Lambda_0)$. Furthermore, the set of irreducible modules $L_k(\mathfrak{g}', \lambda)$ for integrable $\lambda \in \mathfrak{h}^*$ with $(\lambda, \theta) \leq k$ can be identified with $L(\Lambda)$ for $\Lambda \in P_+^k$ and $\Lambda(d) = 0$ along the map $\lambda \mapsto \Lambda = k\Lambda_0 + \lambda$. See section §12.8 in [Kac90] and section 6.6 in [LL04] for details of this construction. From now on we will work with this realization of the simple affine vertex operator algebras. Clearly the characters $\mathrm{ch}_{L(\Lambda)}(\tau)$ are given by $\chi_\Lambda(0, \tau, 0)$. The $\mathcal{S}$-matrix, defined by Zhus theorem 2.7.19 is given explicitly by (224). In chapter 2.5 we usually worked with classes $[\Lambda] \in P_+^k(\mathrm{mod}\ \mathbb{C}\delta)$. Since each such class has a unique representant which satisfies $\Lambda(d) = 0$ we will usually identify the class with this representant. In this sense we can parametrize irreducible $L(k\Lambda_0)$-modules by classes in $P_+^k(\mathrm{mod}\ \mathbb{C}\delta)$.

2.9 The moonshine module and Schellekens' list

Holomorphic vertex operator algebras are vertex operator algebras with, up to isomorphism, just one irreducible module. Their classification is of particular importance because of its relations to the classification of even and unimodular lattices. Much progress has been made in this field and many people contributed to this. We will give a brief introduction to this topic but focus on the case of central charge $c = 24$. The most basic construction of holomorphic vertex operator algebras makes use of even, unimodular and positive-definite lattices L. We briefly introduce this well-known construction in the following. See [LL04], [Kac98] and [FBZ04] for details. A nice exposition is also given in [Möl16]. We set $\mathfrak{h} = L \otimes_\mathbb{Z} \mathbb{C}$ and consider this as an abelian Lie algebra with invariant bilinear form. We introduce an affine Lie algebra $\hat{\mathfrak{h}}$, the *Heisenberg algebra*, by

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K \tag{359}$$

where K is a central element in this Lie algebra and we have

$$[x \otimes t^n, y \otimes t^m] = (x, y) n \delta_{n+m,0} K \tag{360}$$

for $n, m \in \mathbb{Z}$ and $x, y \in \mathfrak{h}$. For simplicity we will usually write $x(n)$ for $x \otimes t^n$. We introduce subalgebras of $\hat{\mathfrak{h}}$ by $\hat{\mathfrak{h}}_+ = \mathfrak{h} \otimes t\mathbb{C}[t]$, $\hat{\mathfrak{h}}_- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]$ and $\hat{\mathfrak{h}}_0 = \mathfrak{h} \otimes t^0 \oplus \mathbb{C}K$. We finally set $\mathfrak{b} = \hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_0$. Of course we can introduce *degrees* by $\deg(x(n)) = -n$ and consider the usual grading by degree here. For $l \in \mathbb{C}$ let $\mathbb{C}_l$ be the 1-dimensional module of $\mathfrak{b}$ on which we have $Kv = lv$ and $h(n)v = 0$ for all $v \in \mathbb{C}$, $h \in \mathfrak{h}$ and $n \geq 0$. We call l the *level* of $\mathbb{C}_l$. We introduce the $\hat{\mathfrak{h}}$-module

$$V_{\hat{\mathfrak{h}}}(l, 0) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{b}})} \mathbb{C}_l \tag{361}$$

of *level* l. We write $\mathbf{1} = 1 \otimes 1$ in this space. Let $h_1, \cdots h_d$ be a basis of $\mathfrak{h}$, i.e. we have $d = \mathrm{rank}(L)$, then it is clear that $V_{\hat{\mathfrak{h}}}(l, 0)$ is spanned by elements

$$x_k(-n_k) \cdots x_1(-n_1)\mathbf{1} \tag{362}$$

Where we have $x_i \in \{h_1, \cdots h_d\}$ and $n_i > 0$ for all $i = 1, \cdots, k$. We introduce a $\mathbb{Z}_{\geq 0}$-grading of $V_{\hat{\mathfrak{h}}}(l, 0)$ by

$$\deg(x_k(-n_k) \cdots x_1(-n_1)\mathbf{1}) = n_k + \cdots + n_1. \tag{363}$$

Using this we obtain as $\mathbb{Z}_{\geq 0}$-graded vector spaces that

$$V_{\hat{\mathfrak{h}}}(l, 0) = U(\hat{\mathfrak{h}}_-) = S(\hat{\mathfrak{h}}_-). \tag{364}$$

For every $h \in \mathfrak{h}$ we can now introduce a field

$$Y(h(-1)\mathbf{1}, z) = h(z) = \sum_{n\in\mathbb{Z}} h(n)z^{-n-1} \in \mathrm{End}(V_{\hat{\mathfrak{h}}}(l,0))[[z^{\pm 1}]]. \tag{365}$$

By use of the reconstruction theorem we find that this defines a vertex algebra structure on $V_{\hat{\mathfrak{h}}}(l,0)$ which we call the *Heisenberg vertex algebra of level* l. We denote the dual basis of $h_1, \cdots, h_d$ by $h^1, \cdots, h^d$, then we find that

$$\omega = \frac{1}{2l}\sum_{i=1}^{d} h^i(-1)h_i(-1)\mathbf{1} \tag{366}$$

is a conformal vector such that we can turn $V_{\hat{\mathfrak{h}}}(l,0)$ into a conformal vertex operator algebra of central charge d. We call it the the *Heisenberg vertex operator algebra of level* l. It is well-known that there exists a 2-cocycle $\epsilon : L \times L \to \{\pm 1\}$ such that $\epsilon(\alpha,\alpha) = (-1)^{\frac{(\alpha,\alpha)}{2}}$ and $\epsilon(\alpha,\beta)\epsilon(\beta,\alpha) = (-1)^{(\alpha,\beta)}$. This is just one possible choice for this 2-cocycle. See [Kac98] and [LL04] for details. We consider the *twisted group algebra* $\mathbb{C}_\epsilon[L]$ defined by the complex span of e^α for $\alpha \in L$ and multiplication $e^\alpha e^\beta = \epsilon(\alpha,\beta)e^{\alpha+\beta}$. We can consider the vector space

$$V_L = V_{\hat{\mathfrak{h}}}(1,0) \otimes \mathbb{C}_\epsilon[L]. \tag{367}$$

By keeping all previous notations form the Heisenberg vertex operator algebra and $\alpha \in L$ we find that this space is spanned by the

$$x_k(-n_k)\cdots x_1(-n_1)\mathbf{1} \otimes e^\alpha. \tag{368}$$

Clearly this defines a natural L-grading of V_L and for $\alpha \in L$ we denote the corresponding weight space by $(V_L)_\alpha$. We can furthermore introduce a $\mathbb{Z}_{\geq 0}$-grading of V_L by

$$\deg(x_k(-n_k)\cdots x_1(-n_1)\mathbf{1} \otimes e^\alpha) = n_k + \cdots + n_1 + \frac{(\alpha,\alpha)}{2}. \tag{369}$$

We introduce an action of $\hat{\mathfrak{h}}$ on $\mathbb{C}_\epsilon[L]$ by $h(n)e^\alpha = Ke^\alpha = 0$ for all $n \neq 0$ and

$$h(0)e^\alpha = (h,\alpha)e^\alpha \tag{370}$$

for all $h \in \mathfrak{h}$. We extend this action to V_L along the tensor product in the obvious way. For $\alpha = 0$ we can introduce fields

$$Y(h(-1)\mathbf{1} \otimes e^\alpha, z) = h(z) = \sum_{n\in\mathbb{Z}} h(n)z^{-n-1}. \tag{371}$$

For $\alpha \neq 0$ we set

$$Y(\mathbf{1} \otimes e^0, z) = e^\alpha z^\alpha \mathrm{exp}\left(\sum_{n=0}^{\infty} \alpha(-n)\frac{z^n}{n}\right) \exp\left(\sum_{n=0}^{\infty} \alpha(n)\frac{z^{-n}}{-n}\right) \tag{372}$$

where z^α acts on V_L as $z^{\alpha(0)}$. More precisely for $h \in \mathfrak{h}$ we define z^h by $z^h v = z^{(h,\beta)}v$ for $v \in (V_L)_\beta$. We can furthermore define a vacuum vector of V_L by $\mathbf{1} \otimes e^0$ and abuse notation by denoting it by $\mathbf{1}$ as well. By use of the reconstruction theorem we can introduce a vertex algebra structure on V_L. This is the *lattice vertex algebra* associated to L. Let again $h_1, \cdots, h_d$ be a basis of $\mathfrak{h}$ and $h^1, \cdots, h^d$ its dual basis, then

$$\omega = \frac{1}{2}\sum_{i=1}^{d} h^i(-1)h_i(-1)\mathbf{1} \tag{373}$$

is a conformal vector of central charge $c = \mathrm{rank}(L)$. The vertex algebra V_L, equipped with this conformal vector as Virasoro vector, is called the *conformal lattice vertex algebra* of central charge $c = \mathrm{rank}(L)$ associated with L. In general this is not quite a vertex operator algebra since the restrictions on the dimensions of the weight spaces might not be satisfied. If the lattice L is positive-definite then V_L is in fact a strongly rational vertex operator algebra. See [Don93], [DLM97] and [DLM00] for details of this. A direct and easy computation reveals that the Virasoro operators L_k are given by

$$L_k = \frac{1}{2}\sum_{i=1}^{d}\sum_{n+m=k} : h^i(n)h_i(m) : . \tag{374}$$

Let L be an even positive-definite lattice. In [Don93] it is proved that for $\lambda \in L'$ the space

$$V_{\lambda+L} = V_{\hat{\mathfrak{h}}}(1,0) \otimes \mathbb{C}_\epsilon[\lambda + L] \tag{375}$$

can be equipped with the structure of an irreducible V_L-module that up to isomorphism just depends on $[\lambda] \in L'/L$. Furthermore each irreducible V_L-module is of this form. In particular we have $V_{0+L} \cong V_L$. See Theorem 3.1. in [Don93]. Another interesting property is that the fusion product of two irreducible V_L-module is given by

$$V_{\lambda+L} \boxtimes_{V_L} V_{\mu+L} = V_{\lambda+\mu+L}. \tag{376}$$

As a consequence we find that every irreducible V_L-module is a simple current and the fusion algebra $\mathcal{V}(V_L)$ is isomorphic to $\mathbb{C}[L'/L]$.

Definition 2.9.1. A strongly rational vertex operator algebra is called *holomorphic* if each of its irreducible modules is isomorphic to its adjoint module.

As a consequence of Zhus theorem 2.7.19 we know that the character ch_V of a holomorphic vertex operator algebra V of central charge c has to be a modular function for $\mathrm{SL}_2(\mathbb{Z})$ for some character $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}^*$. It can be shown that $\rho(S) = 1$, therefore the equation $(ST)^3 = S^2$ in $\mathrm{SL}_2(\mathbb{Z})$ implies that $\rho(T)^3 = 1$. Yet we know that $\rho(T) = e^{-2\pi i\frac{c}{24}}$ so we obtain $8|c$. This is a fundamental result about about holomorphic vertex operator algebras. An even, unimodular, positive-definite lattice L always satisfies $8|\mathrm{rank}(L)$. A proof for this can be given by similar arguments or simply observe that the associated vertex operator algebra V_L is holomorphic and has central charge $c = \mathrm{rank}(L)$. So for every even, unimodular, positive-definite lattice L we can construct a holomorphic vertex operator algebra.

Theorem 2.9.2 ([DM04, Thm. 1 and Thm. 2])**.** *Let V be a holomorphic vertex operator algebra of central charge c.*

1. *Assume $c = 8$. Then V is isomorphic to V_{E_8}, the holomorphic vertex operator algebra associated to the lattice E_8.*
2. *Assume $c = 16$. Then V is isomorphic to V_L, where L is one of the two even, unimodular, positive-definite lattices of rank 16.*

For central charge $c \geq 32$ there is no reasonable classification of holomorphic vertex operator algebras in sight. One of the reasons for this is that there is not even a classification for even, unimodular, positive-definite lattices L that satisfy $\mathrm{rank}(L) \geq 32$. For lattices of rank 24 we have *Niemeiers classification*, however.

Theorem 2.9.3 (Niemeier)**.** *Up to isomorphism there are precisely 24 even unimodular positive-definite lattices of rank 24 and they are uniquely determined by their root lattices.*

For each Niemeier lattice L we can consider the corresponding lattice vertex operator algebra V_L. In particular we can do this for the *Leech lattice* Λ, the unique Niemeier lattice without roots. Here, a *root* is simply an element $v \in \Lambda$ with length $v^2 = 2$. The associated vertex operator algebra is the V_Λ *Leech lattice vertex operator algebra* of central charge 24. Yet there are more holomorphic vertex operator algebras of central charge $c = 24$ then just the 24 lattice vertex operator algebras. The most famous example is the *moonshine module* $V^\natural$. This vertex operator algebra was first constructed in [FLM89]. Essentially they took the Leech lattice vertex operator algebra V_Λ and considered the *orbifold* of a certain automorphism $\sigma \in \mathrm{Aut}(V_\Lambda)$ of order 2. By this construction one extends the fixed-point vertex operator algebra V_Λ^σ by an irreducible module of it such that the corresponding space can be equipped with the structure of a holomorphic vertex operator algebra. This is a rather complicated construction and all of [FLM89] is dedicated to it. A few of the remarkable properties of the moonshine module are that we have $V_1^\natural = 0$ and

$$\mathrm{ch}_{V^\natural}(\tau) = j(\tau) = q^{-1} + 0 + 196884q + \mathcal{O}(q^2). \tag{377}$$

Yet the most important property of $V^\natural$ is that its automorphism group $\mathrm{Aut}(V^\natural)$ is isomorphic to the *Conway-Norton monster group* $\mathbb{M}$. This is precisely the $\mathbb{M}$-representation that appears in Borcherds' proof of the moonshine conjecture. We may now introduce a fundamental invariant of a holomorphic vertex operator algebra V of central charge $c = 24$. This is its weight-1 space V_1, equipped with a Lie algebra structure by $[v, w] = v_0 w$ for $v, w \in V_1$.

Theorem 2.9.4 ([DM04, Thm. 3]). *Let V be a holomorphic vertex operator algebras of central charge $c = 24$. Then the Lie algebra V_1 is reductive and exactly one of the following:*

1. $V_1 = 0$.
2. *V_1 is abelian in which case V is isomorphic to the Leech lattice vertex operator algebra V_Λ.*
3. *V_1 is a semi-simple Lie algebra of rank* 24. *In this case V is isomorphic to a Niemeier lattice vertex operator algebra.*
4. *V_1 is a semi-simple Lie algebra and its rank is less than* 24.

In the following we will focus on the case, where V_1 is semi-simple. The symmetric, non-degenerate, invariant bilinear form of V, which is unique up to scaling, will be denoted $\langle \cdot, \cdot \rangle$. We assume it normalized such that $\langle \mathbf{1}, \mathbf{1} \rangle = -1$. We decompose V_1 into simple Lie algebras $\mathfrak{g}_i$ and we define numbers k_i by $\langle \cdot, \cdot \rangle|_{\mathfrak{g}_i \times \mathfrak{g}_i} = k_i(\cdot, \cdot)$. In fact we have $k_i \in \mathbb{Z}_{>0}$. See [DM06]. We denote this decomposition by

$$V_1 = \mathfrak{g} = \mathfrak{g}_{1,k_1} \oplus \cdots \oplus \mathfrak{g}_{r,k_r}. \tag{378}$$

The corresponding Cartan subalgebra will be denoted by

$$\mathfrak{h} = \mathfrak{h}_{1,k_1} \oplus \cdots \oplus \mathfrak{h}_{r,k_r}. \tag{379}$$

The vertex operator subalgebra $V(\mathfrak{g}) = \langle V_1 \rangle$ generated by V_1 turns out to be isomorphic to

$$L_{\hat{\mathfrak{g}}_1}(k_1\Lambda_0) \otimes \cdots \otimes L_{\hat{\mathfrak{g}}_r}(k_r\Lambda_0). \tag{380}$$

See [DM06] for this. By use of Proposition 4.1 in [DM04] we know that their Virasoro vectors are equal. This allows us to consider V as a $V(\mathfrak{g})$-module. Since $V(\mathfrak{g})$ is strongly rational we know that V can be decomposed into finitely many irreducible $V(\mathfrak{g})$-modules. We obtain

$$V = \bigoplus_{\Lambda^1, \cdots, \Lambda^r} m(\Lambda^1, \cdots, \Lambda^r) L_{\hat{\mathfrak{g}}_1}(\Lambda^1) \otimes \cdots \otimes L_{\hat{\mathfrak{g}}_r}(\Lambda^r). \tag{381}$$

This decomposition was first studied by Schellekens in [Sch93], where he classified all possible Lie algebra structures of V_1 and all induced decompositions (381).

Theorem 2.9.5 (Schellekens' list, [vEMS17, Thm. 6.4.]). *Let V be a strongly rational holomorphic vertex operator algebra of central charge $c = 24$. Then either $V_1 = 0$ or $dim(V_1) = 24$ and V is isomorphic to the Leech lattice vertex operator algebra or V_1 is isomorphic to one of* 69 *semi-simple Lie algebras described in Table 1 of [Sch93].*

We refer to this list of Lie algebras as *Schellekens' list* in the following. This result was reproved in [vEMS17] by similar arguments and in [vELMS21] by use of the *very strange formula*. The last approach does not reproduce the $V(\mathfrak{g})$-module decomposition, however. Of course it is not at all clear that for every Lie algebra $\mathfrak{g}$ in Schellekens' list there is a holomorphic vertex operator algebra V of central charge $c = 24$ with $V_1 = \mathfrak{g}$. This problem was addressed in recent years by a community of people and their joint effort led to the following theorem.

Theorem 2.9.6. *For every Lie algebra $\mathfrak{g} \neq 0$ in Schellekens' list there exists a unique holomorphic vertex operator algebra V of central charge $c = 24$ with $V_1 = \mathfrak{g}$.*

The *cyclic orbifold construction* developed in [Möl16] and [vEMS17] was an important contribution to this question. For every nontrivial Lie algebra in Schellekens' list it can be used to construct a holomorphic vertex operator algebra of central charge $c = 24$ that yields this Lie algebra as its weight-1 space. See in particular [MS19]. An excellent overview on this topic is given in [LS19] and the literature cited therein. Yet the case $V_1 = 0$ is still wide-open. In [FLM89] Frenkel, Lepowsky and Meurman, conjectured that a holomorphic vertex operator algebra V of central charge $c = 24$ with $V_1 = 0$ is isomorphic to the moonshine module $V^\natural$. This conjecture is also called the *FLM conjecture*.

3 The Lie algebra of physical states and the no-ghost theorem

Motivated by the *old covariant quantization* in physics we can associate to a vertex operator algebra a certain *Lie algebra of physical states*. This was famously used by Borcherds to construct the *monster Lie algebra*, which is the Lie algebra of physical states of the moonshine module $V^\natural$. An important tool of his proof of the monstrous moonshine conjecture was the famous no-ghost theorem from string theory. Using it, one can identify root spaces of the Lie algebra of physical states with certain subspaces of the corresponding vertex operator algebra. This makes it possible to determine the root multiplicities in terms of Fourier coefficients of the character of the vertex operator algebra. Furthermore those identifications can be used to lift the Lie bracket to bilinear maps on suitable weight spaces of the vertex operator algebra. In [Bor92] Borcherds asked for an explicit description of such maps in terms of vertex algebra operations. The main result of this section is an explicit answer to this question. First we discuss Borcherds' sketch of proof of the no-ghost theorem and restate most of the results in terms of an important operator E. This operator will turn out to be crucial to describe the identifications, induced by the no-ghost theorem. Now we can use this to evaluate the Lie bracket explicitly in terms of vertex operator operations.

3.1 The old covariant quantization and the no-ghost theorem

In this section we introduce the old covariant quantization of certain modules of the Virasoro Lie algebra. Based on this structure we discuss the famous no-ghost theorem form string theory. The first proof of the no-ghost theorem was given in [GT72]. A related but slightly different approach was taken in [Fre85]. This was highly influential for the following discussion. We follow this source in particular in the proof of the no-ghost theorem. The most prominent application of the no-ghost theorem was in Borcherds proof of the *Hauptmodul part* of the monstrous moonshine conjecture. See [Bor92]. Therein a sketch of proof for the no-ghost theorem is given as well. It follows the approach in the previously mentioned sources. As usual we denote the Virasoro Lie algebra by Vir. Let V be a complex Vir-module with the following properties:

1. V is of central charge $c = 24$, i.e. the central element $C \in Vir$ acts as multiplication by 24 on V.

2. V is equipped with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ such that L_{-n} is the adjoint operator of L_n. We call this property *Virasoro-invariance.*

3. The operator L_0 defines by $V_n = \{v \in V : L_0 v = nv\}$ a weight grading

$$V = \bigoplus_{n=0}^{\infty} V_n \tag{382}$$

   and we assume that each weight space V_n is finite-dimensional.

4. the subspace V_0 has dimension 1 and is spanned by an element $\mathbf{1}$, which we call *vacuum.* We furthermore assume $\langle \mathbf{1}, \mathbf{1} \rangle = 1$ and $L_n \mathbf{1} = 0$ for all $n \geq -1$.

Definition 3.1.1. We call a Vir-module V, which satisfies the previously discussed properties, a *lowest-weight* module of central charge $c = 24$ of the Virasoro Lie algebra Vir.

Let V be a lowest weight module of central charge $c_V = 24$ of Vir. We easily check that $V_n \perp V_m$ unless $n = m$. We consider the conformal lattice vertex algebra $V_{\mathrm{II}_{1,1}}$ associated to the Lorentzian lattice $\mathrm{II}_{1,1}$ in rank 2 which has central charge $c_{\mathrm{II}_{1,1}} = 2$. Where the conformal structure of $V_{\mathrm{II}_{1,1}}$ is given as usual by (373). As in section 2.9 we write $\mathfrak{h} = \mathrm{II}_{1,1} \otimes \mathbb{C}$. Then this conformal vertex algebra has a natural $\mathrm{II}_{1,1}$-grading, given by

$$V_{\mathrm{II}_{1,1},\alpha} = S(\hat{\mathfrak{h}}_-) \otimes e^\alpha \tag{383}$$

for $\alpha \in \mathrm{II}_{1,1}$. The Vir-module $V \otimes V_{\mathrm{II}_{1,1}}$ has central charge $c = c_V + c_{\mathrm{II}_{1,1}} = 26$. The vertex algebra $V_{\mathrm{II}_{1,1}}$ carries a symmetric non-degenerate bilinear form $(\cdot,\cdot)_{\mathrm{II}_{1,1}}$ which we assume normalized such that $(\mathbf{1},\mathbf{1})_{\mathrm{II}_{1,1}} = -1$. Together with the bilinear form $\langle\cdot,\cdot\rangle$ of V this induces a natural symmetric non-degenerate bilinear form $(\cdot,\cdot)$ on $V \otimes V_{\mathrm{II}_{1,1}}$ by

$$(\cdot,\cdot) = \langle\cdot,\cdot\rangle \otimes (\cdot,\cdot)_{\mathrm{II}_{1,1}}. \tag{384}$$

Since in the following we will act with the Virasoro Lie algebra Vir on V, $V_{\mathrm{II}_{1,1}}$ and on $V \otimes V_{II_{1,1}}$ we will denote the Virasoro operators that act on $V \otimes V_{II_{1,1}}$ by $L(n)$ for $n \in \mathbb{Z}$. This helps to avoid misunderstandings. In particular the adjoint operator of $L(n)$ is $L(-n)$ for all $n \in \mathbb{Z}$, i.e. we have for all $v, w \in V \otimes V_{II_{1,1}}$ and $n \in \mathbb{Z}$ that

$$(L(n)v, w) = (v, L(-n)w). \tag{385}$$

The Virasoro operators that act on $V \otimes V_{\mathrm{II}_{1,1}}$ will also be denoted by $L(n)$, yet for the Virasoro operators acting on V we write L_n. The *space of primary states* of $V \otimes V_{\mathrm{II}_{1,1}}$ is given by

$$P = \{v \in V \otimes V_{\mathrm{II}_{1,1}} : L(k)v = 0\ \forall k > 0\}. \tag{386}$$

The operator $L(0)$ induces a natural weight grading of P and we write

$$P^n = \{v \in P : L(0)v = nv\}. \tag{387}$$

We furthermore have a natural $\mathrm{II}_{1,1}$-grading of $V \otimes V_{\mathrm{II}_{1,1}}$, simply given by subspaces $V \otimes V_{\mathrm{II}_{1,1},\alpha}$ for $\alpha \in \mathrm{II}_{1,1}$. In the following we are going to denote this space by $H(\alpha)$. For each $\alpha \in \mathrm{II}_{1,1}$ we introduce subspaces $P(\alpha) = P \cap V \otimes V_{\mathrm{II}_{1,1},\alpha}$ and $P^n(\alpha) = P^n \cap P(\alpha)$ and notice that we have

$$P = \bigoplus_{\alpha \in \mathrm{II}_{1,1}} P(\alpha) \text{ and} \tag{388}$$

$$P^n = \bigoplus_{\alpha \in \mathrm{II}_{1,1}} P^n(\alpha). \tag{389}$$

This is due to the fact that each weight-α space $V_{\mathrm{II}_{1,1},\alpha}$ of $V_{\mathrm{II}_{1,1}}$ is closed under the Virasoro Lie algebra. The kernel $\ker(\cdot,\cdot)$ of the bilinear form on P^1 is homogeneous with respect to the $\mathrm{II}_{1,1}$-grading of P^1 and its $\mathrm{II}_{1,1}$-grade spaces are given by

$$\ker(\cdot,\cdot)_\alpha = \{v \in P^1(\alpha) : v \perp P^1(-\alpha)\}. \tag{390}$$

Definition 3.1.2. On the conformal vertex algebra $V_{\mathrm{II}_{1,1}}$ we can define a linear involutive map θ by

$$\theta(e^\alpha) = \epsilon(\alpha,-\alpha)(-1)^{(\alpha,\alpha)/2}e^{-\alpha} \text{ and } \theta(h(n)) = -h(n) \tag{391}$$

for all $\alpha \in \mathrm{II}_{1,1}$, $h \in \mathrm{II}_{1,1} \otimes \mathbb{C}$ and $n \in \mathbb{Z}$. Clearly this extends to an involution on $V \otimes V_{\mathrm{II}_{1,1}}$ by $\theta = \mathrm{Id} \otimes \theta$. We can use it to define a non-degenerate bilinear form on $V \otimes V_{\mathrm{II}_{1,1}}$ by

$$(\cdot,\cdot)_0 = -(\cdot,\theta(\cdot)). \tag{392}$$

We call $(\cdot,\cdot)_0$ the symmetric *contravariant bilinear form* on $V \otimes V_{\mathrm{II}_{1,1}}$.

The involution θ clearly commutes with the Virasoro operators, since it defines an automorphism of the conformal vertex algebra structure of $V_{\mathrm{II}_{1,1}}$ and acts as the identity on V. By a direct computation we can check that θ preserves the bilinear form $(\cdot,\cdot)_{\mathrm{II}_{1,1}}$, i.e. for $v, w \in V_{\mathrm{II}_{1,1}}$ we have

$$(\theta(v), \theta(w))_{\mathrm{II}_{1,1}} = (v, w)_{\mathrm{II}_{1,1}}. \tag{393}$$

We observe directly that a similar equation holds for $(\cdot,\cdot)$ on $V \otimes V_{\mathrm{II}_{1,1}}$. Altogether we find that θ preserves the subspaces P and P^n but maps $H(\alpha)$, $P(\alpha)$, $P^n(\alpha)$ and $\ker(\cdot,\cdot)_\alpha$ to $H(-\alpha)$, $P(-\alpha)$, $P^n(-\alpha)$ and $\ker(\cdot,\cdot)_{-\alpha}$. We furthermore have

$$\ker(\cdot,\cdot)_\alpha = \ker\left((\cdot,\cdot)_0|_{P^1(\alpha)\times P^1(\alpha)}\right). \tag{394}$$

The advantage of the contravariant bilinear form $(\cdot,\cdot)_0$ over $(\cdot,\cdot)$ lies in the fact, that we can restrict it to spaces $H(\alpha)$, $P(\alpha)$ and $P^n(\alpha)$ and obtain a *meaningful* Virasoro-invariant bilinear form. In particular the restriction of $(\cdot,\cdot)_0$ to $H(\alpha)$ is non-degenerate.

Proposition 3.1.3. *The $\mathit{II}_{1,1}$-grading of $V_{\mathit{II}_{1,1}}$ naturally induces a $\mathit{II}_{1,1}$-grading of the space $P^1/\mathit{ker}(\cdot,\cdot)$, i.e. we have*

$$P^1/\mathit{ker}(\cdot,\cdot) = \bigoplus_{\alpha\in \mathit{II}_{1,1}} P^1(\alpha)/\mathit{ker}(\cdot,\cdot)_\alpha. \tag{395}$$

The involution θ on $V\otimes V_{\mathit{II}_{1,1}}$ induces a natural linear involutive map

$$\theta : P^1/\mathit{ker}(\cdot,\cdot) \to P^1/\mathit{ker}(\cdot,\cdot), \tag{396}$$

which maps $P^1(\alpha)/\mathit{ker}(\cdot,\cdot)_\alpha$ to $P^1(-\alpha)/\mathit{ker}(\cdot,\cdot)_{-\alpha}$ for every $\alpha \in \mathit{II}_{1,1}$. The induced symmetric non-degenerate contravariant bilinear form

$$(\cdot,\cdot)_0 = -(\cdot,\theta(\cdot)) \tag{397}$$

is just the induced bilinear form of $(\cdot,\cdot)_0$ on $V\otimes V_{\mathit{II}_{1,1}}$.

Let G be a group of lowest weight Vir-module automorphisms that acts on V. This means that every element of g acts as a linear automorphism on V which commutes with all Virasoro operators L_n and C and preserves the bilinear form $(\cdot,\cdot)$ and the vacuum $\mathbf{1}$. Clearly since G commutes with L_0 it also preserves the corresponding L_0-grading. We will call such a group G a *group of symmetries* of the lowest-weight Vir-module V of central charge $c_V = 24$. We extend the action of G to $V\otimes V_{\mathrm{II}_{1,1}}$ by letting it act trivially on $V_{\mathrm{II}_{1,1}}$. This action commutes with the Virasoro operators $L(n)$ and C acting on the space $V\otimes V_{\mathrm{II}_{1,1}}$ and preserves the bilinear form. This implies that it preserves the spaces P, P^n, $P(\alpha)$, $P^n(\alpha)$ and $\ker(\cdot,\cdot)_\alpha$ and therefore defines a natural action G-action on the quotient $P^1/\ker(\cdot,\cdot)$ which preserves its $\mathrm{II}_{1,1}$-grading.

Definition 3.1.4. Let V be a lowest-weight Vir-module V of central charge $c_V = 24$ with a group of symmetries G. We introduce the space

$$\mathfrak{g}(V) = P^1/\ker(\cdot,\cdot) \tag{398}$$

equipped with a natural non-degenerate bilinear form $(\cdot,\cdot)$, an involution θ and a contravariant bilinear form $(\cdot,\cdot)_0$. Furthermore a $\mathrm{II}_{1,1}$-grading as in (395) and a *group of symmetries* G and call it the *spaces of physical states* of V.

In the remainder of this section we are going to study the structure of the space of physical states in more detail.

Theorem 3.1.5 (no-ghost theorem)**.** *Let V be a lowest-weight Vir-module V of central charge $c_V = 24$ with a group of symmetries G and let its space of physical states be $\mathfrak{g}(V)$. Take $\alpha\in \mathit{II}_{1,1}$. The subspace $\mathfrak{g}(V)_\alpha$ of $\mathfrak{g}(V)$ is naturally isomorphic to $V_{1-\frac{\alpha^2}{2}}$, as a G-module with G-invariant bilinear form if $\alpha\neq 0$ and to $V_1\oplus(\mathit{II}_{1,1}\otimes\mathbb{C})$ if $\alpha = 0$.*

This formulation of the no-ghost theorem is from [Bor92], where everything is done over the real numbers $\mathbb{R}$, however. First the no-ghost theorem was proved in [GT72]. If we assume that the Vir-module V is a real Vir-module with positive-definite bilinear form then it can be proved that the space of physical states can naturally equipped with a bilinear form $(\cdot,\cdot)_0$ whose restriction to $\mathfrak{g}(V)_r$ with $r \neq 0$ is positive-definite. This implies that the space $\mathfrak{g}(V)_r$ does not contain any *ghosts*, which are vectors of negative norm. This aspect of the no-ghost theorem does not matter for us in the current section and we don't assume any such real and positive-definite structure for the Vir-module V in the following. We will just work with non-degenerate bilinear forms over the complex numbers $\mathbb{C}$. The following discussion mainly follows [Fre85] and [Bor92]. By similar methods the no-ghost theorem was also proved in [Jur98]. We start with the case $0 \neq \alpha \in \mathrm{II}_{1,1}$. We fix an element $w_\alpha \in \mathrm{II}_{1,1}$ such that $(w_\alpha, w_\alpha) = 0$ and $(w_\alpha, \alpha) = 1$. In the following discussion we will make constant use of (309). For $n \in \mathbb{Z}$ we introduce operators

$$K_\alpha(n) = (\mathbf{1} \otimes w_\alpha(-1)\mathbf{1})(n) = \mathrm{Id} \otimes w_\alpha(n). \tag{399}$$

Notice that in the literature it is not common to indicate the element $\alpha \in \mathrm{II}_{1,1}$ with a subscript $(\cdot)_\alpha$ but we do this because later we will have to work with several element in $\mathrm{II}_{1,1}$ at the same time. Before we discuss its properties we state some useful standard relations that hold in $V_{\mathrm{II}_{1,1}}$. For $\alpha, \beta \in \mathfrak{h}$ and $n, m \in \mathbb{Z}$ we have

$$[\alpha(m), \beta(n)] = (\alpha, \beta) m \delta_{n+m,0} \text{ and} \tag{400}$$

$$[L(m), \alpha(n)] = -n\alpha(n+m). \tag{401}$$

See (8.6.42) and (8.7.13) in [FLM89]. With those relation we easily check on $V \otimes V_{\mathrm{II}_{1,1}}$ that for $n, m \in \mathbb{Z}$ we have

$$[K_\alpha(m), K_\alpha(n)] = 0 \text{ and} \tag{402}$$

$$[L(m), K_\alpha(n)] = -nK_\alpha(m+n). \tag{403}$$

Clearly for $v \in H(\alpha)$ and tuples $\lambda = (\lambda_1, \cdots, \lambda_n)$ and $\mu = (\mu_1, \cdots, \mu_m)$ with $\lambda_i, \mu_j \in \mathbb{N}$ we can set

$$v_{\lambda,\mu} = L(-1)^{\lambda_1} \cdots L(-n)^{\lambda_n} K_\alpha(-1)^{\mu_1} \cdots K_\alpha(-m)^{\mu_m} v. \tag{404}$$

Now we define a few important subspaces of $H(\alpha)$ by

- $P(\alpha)$ to be the space $P \cap H(\alpha)$ and we have $P^1(\alpha) = P^1 \cap P(\alpha)$.
- $K(\alpha)$ is the subspace of $H(\alpha)$ annihilated by all $K_\alpha(n)$ for $n > 0$.
- $T(\alpha)$ is the intersection $P(\alpha) \cap K(\alpha)$ and $T^1(\alpha)$ is the intersection $P^1(\alpha) \cap K(\alpha)$.
- $G(\alpha)$ is the span of all $t_{\lambda,\mu} := L(-1)^{\lambda_1} \cdots L(-n)^{\lambda_n} K_\alpha(-1)^{\mu_1} \cdots K_\alpha(-m)^{\mu_m} t$ for $\lambda_i, \mu_j \in \mathbb{N}$ with $\sum_i \lambda_i + \sum_j \mu_j > 0$ and $t \in T(\alpha)$.
- $K'(\alpha)$ is the subspace of $G(\alpha)$ generate by all $t_{0,\mu}$ with $\mu \neq 0$.
- $S(\alpha)$ is the space of spurious vectors in $H(\alpha)$, those are all vectors spanned by $t_{\lambda,\mu}$ with $\lambda \neq 0$.
- N is the radical of the bilinear form in $P(\alpha)$, this is $P(\alpha) \cap S(\alpha)$. N^1 is given by $P^1(\alpha) \cap S(\alpha)$.
- Ve^α is the subspace, defined by $V \otimes e^\alpha$.

In the following we need a more detailed discussion of the properties of those subspaces of $H(\alpha)$.

Lemma 3.1.6 ([Bor92, Lemma 5.1/5.2]). *We have*

1. *For a fixed vector $t \in T(\alpha)$ of nonzero norm the set of all $t_{\lambda,\mu}$ is linearly independent.*
2. *For an orthogonal basis t_k of $T(\alpha)$ the set $(t_k)_{\lambda,\mu}$ defines a basis of $H(\alpha)$.*
3. *We obtain $H(\alpha) = T(\alpha) \oplus G(\alpha)$, $K(\alpha) = T(\alpha) \oplus K'(\alpha)$ and $H(\alpha) = K(\alpha) \oplus S(\alpha)$.*

4. *We also have $K(\alpha) = Ve^{\alpha} \oplus K'(\alpha)$ and $G(\alpha) = K'(\alpha) \oplus S(\alpha)$.*

5. *The restriction of the bilinear form $(\cdot,\cdot)_0$ to $T(\alpha)$ is non-degenerate and $K'(\alpha)$ is the kernel of the bilinear form $(\cdot,\cdot)_0$ on $K(\alpha)$.*

Proof. Proves for the statements in this lemma are given in [Bor92] and [Jur98]. □

We define a generating series

$$K_{\alpha}(z) = zY(\mathbf{1} \otimes w_{\alpha}(-1)\mathbf{1}, z) = \sum_{n\in\mathbb{Z}} K_{\alpha}(n) z^{-n} = \mathrm{Id} z^0 + K_0(z) \tag{405}$$

on $H(\alpha)$. This makes sense because of $K_{\alpha}(0) = \mathrm{Id}$. It can be checked that its formal inverse $K(z)^{-1}$ is well-defined on $H(\alpha)$, more precisely we have formally

$$K_{\alpha}(z)^{-1} = (1 + K_0(z))^{-1} = 1 - K_0(z) + K_0(z)^2 - K_0(z)^3 + - \cdots . \tag{406}$$

This is possible because for any fixed $v \in H(\alpha)$ we have $K_0^m(z)v = 0$ for $m \in \mathbb{N}$ sufficiently large. Using this we can define operators $D_{\alpha}(n)$ on $H(\alpha)$ by

$$D_{\alpha}(n) = \int_C K_{\alpha}(z)^{-1} z^n \frac{dz}{z}. \tag{407}$$

Here by $\int_C \cdot dz$ we denote a formal contour integral over some suitable contour C around $z = 0$. To perform explicit computations it is useful to have another description of the operators $D_{\alpha}(n)$. We define

$$D_{\alpha}(n,m) := \sum_{k_1+\cdots+k_m=n, k_1,\cdots,k_m\neq 0} K_{\alpha}(k_1)\cdots K_{\alpha}(k_m), \tag{408}$$

and $D_{\alpha}(n,0) := \delta_{n,0}$. Those operators are just the modes of the field $K_0^m(z)$, more precisely we have

$$K_0^m(z) = \sum_{n\in\mathbb{Z}\setminus\{0\}} D_{\alpha}(n,m) z^{-n}. \tag{409}$$

We use this to obtain

$$D_{\alpha}(n) = \sum_{m=0}^{\infty} (-1)^m D_{\alpha}(n,m). \tag{410}$$

By a direct computation we can see that just finitely many summands in (408) can act nontrivially on any element in $v \in H(\alpha)$ and for m sufficiently large we have $D_{\alpha}(n,m)v = 0$. Of course this is the same computation that shows $K_0^m(z)v = 0$ fur m sufficiently large. This implies that $D_{\alpha}(n,m)$ and $D_{\alpha}(n)$ are well-defined operators on $H(\alpha)$ and we could use (408) and (410) as definitions.

Lemma 3.1.7. *For integers $k, n \in \mathbb{Z}$ and $m \in \mathbb{N}$ we have*

1. *$[L(k), D_{\alpha}(n,m)] = mkD_{\alpha}(k+n, m-1) - (k+n-mk)D_{\alpha}(k+n,m)$ and*

2. *$[L(k), D_{\alpha}(n)] = -(2k+n)D_{\alpha}(k+n)$.*

Proof. We prove those statements by direct computation. We have

$$\begin{aligned}
[L(k), D_{\alpha}(n,m)] &= \left[L(k), \sum_{k_1+\cdots+k_m=n, k_1,\cdots,k_m\neq 0} K_{\alpha}(k_1)\cdots K_{\alpha}(k_m) \right] && (411)\\
&= mk \sum_{k_1+\cdots+k_{m-1}=n+k, k_1,\cdots,k_{m-1}\neq 0} K_{\alpha}(k_1)\cdots K_{\alpha}(k_{m-1}) && (412)\\
&\quad - (k+n-mk) \sum_{k_1+\cdots+k_m=n+k, k_1,\cdots,k_m\neq 0} K_{\alpha}(k_1)\cdots K_{\alpha}(k_m) && (413)\\
&= mkD_{\alpha}(n+k, m-1) - (n+k-mk)D_{\alpha}(n+k,m) && (414)
\end{aligned}$$

Using this result we obtain

$$[L(k), D_\alpha(n)] = \sum_{m=0}^{\infty} (-1)^m [L(k), D_\alpha(n,m)] \tag{415}$$

$$= \sum_{m=0}^{\infty} (-1)^m \left(mkD_\alpha(k+n, m-1) - (k+n-mk)D_\alpha(k+n,m)\right) \tag{416}$$

$$= \sum_{m=0}^{\infty} (-1)^m \left(-(n+2k)\right) D_\alpha(n+k,m) \tag{417}$$

$$= -(2k+n)D_\alpha(n+k). \tag{418}$$

Now the lemma is proved. □

Definition 3.1.8. For every $\alpha \in \mathrm{II}_{1,1}$ we introduce the operator E_α on $H(\alpha)$ by

$$E_\alpha = (D_\alpha(0)-1)(L(0)-1) + \sum_{n=1}^{\infty} (D_\alpha(-n)L(n) + L(-n)D_\alpha(n)). \tag{419}$$

The operator E_α is of fundamental importance for the ongoing discussion. Therefore we are in need of explicit formulas for the commutator of E_α with $L(n)$, $K_\alpha(n)$ and $D_\alpha(n)$. We start with the easiest, which is

$$[K_\alpha(n), E_\alpha] = -nK_\alpha(n) \; \forall n \in \mathbb{Z}. \tag{420}$$

This equation is easy to prove by use of (402).

Proposition 3.1.9. *On the space $H(\alpha)$ the operator E_α satisfies for all $n \in \mathbb{Z}$ that*

$$[L(n), E_\alpha] = -nL(n) + \frac{n^3-n}{12}(c-26)D_\alpha(n). \tag{421}$$

Proof. By a lengthy but direct computation similar to the proof of Lemma 3.1.7 we obtain

$$\left[L(n), \sum_{k=1}^{\infty}(D_\alpha(-k)L(k) + L(-k)D_\alpha(k))\right] = \tag{422}$$

$$\frac{n^3-n}{12}(c-26)D_\alpha(n) - 2nD_\alpha(n) + 2nD_\alpha(n)L(0) - nD_\alpha(0)L(n) \tag{423}$$

and

$$[L(n), (D_\alpha(0)-1)(L(0)-1)] = -2nD_\alpha(n)L(0) + nD_\alpha(0)L(n) + 2nD_\alpha(n) - nL(n). \tag{424}$$

Both equations together imply the statement of the proposition. □

Of course during the entire section we assumed that $c = 26$. So the statement of Proposition 3.1.9 really just is

$$[L(n), E_\alpha] = -nL(n). \tag{425}$$

The reason why we did not state the proposition like that is to make clear that it is this commutator formula where the property $c = 26$ becomes crucial.

Lemma 3.1.10. *For any $\alpha \in II_{1,1}$ and all $k_i, n \in \mathbb{Z}$ and $m \in \mathbb{N}$ we have*

$$[K_\alpha(k_1)\cdots K_\alpha(k_r), E_\alpha] = -(k_1+\cdots+k_r)K_\alpha(k_1)\cdots K_\alpha(k_r) \tag{426}$$

$$[D_\alpha(n,m), E_\alpha] = -nD_\alpha(n,m) \tag{427}$$

$$[D_\alpha(n), E_\alpha] = -nD_\alpha(n) \tag{428}$$

Proof. The first equation can be proved by induction over r. The second and third equation are direct consequences of this. □

In summary we showed that for $X(n) = K_\alpha(n), D_\alpha(n), L(n)$ we have $[X(n), E_\alpha] = -nX(n)$ for all $n \in \mathbb{Z}$.

Proposition 3.1.11. *The operator E_α has eigenvectors*

$$E_\alpha t_{\lambda,\mu} = -(\lambda_1 + 2\lambda_2 + \cdots + n\lambda_n + \mu_1 + 2\mu_2 + \cdots + m\mu_m)t_{\lambda,\mu}. \tag{429}$$

We fix an orthogonal basis t_k of $T(\alpha)$ and consider the eigenvectors $(t_k)_{\lambda,\mu}$ of E_k which satisfy:

1. *The eigenvalues of E_α are all nonpositive integer.*
2. *The space $H(\alpha)$ is spanned by the eigenvectors $(t_k)_{\lambda,\mu}$ of E_α.*
3. *The space $G(\alpha)$ is spanned by the eigenvectors $(t_k)_{\lambda,\mu}$ of E_α with $\lambda \neq 0$ or $\mu \neq 0$.*
4. *The space $S(\alpha)$ is spanned by the eigenvectors $(t_k)_{\lambda,\mu}$ of E_α with $\lambda \neq 0$.*
5. *The space $K(\alpha)$ is spanned by the eigenvectors $(t_k)_{0,\mu}$ of E_α.*
6. *The space $K'(\alpha)$ is spanned by the eigenvectors $(t_k)_{0,\mu}$ of E_α with $\mu \neq 0$.*
7. *The space $T(\alpha)$ is spanned by the eigenvectors $(t_k)_{0,0}$ of E_α.*

Proof. The equation (429) is a direct computation that makes use of $E_\alpha t = 0$ for $t \in T(\alpha)$. Which follows from $D_\alpha(0)|_{H(\alpha)} = \text{id}$. The rest of the statement is just a reformulation of Lemma 3.1.6 in terms of the operator E_α. □

Definition 3.1.12. The decomposition of $H(\alpha)$ into eigenspaces of E_α defines projections into those spaces. We denote the projection into $T(\alpha)$, the eigenspace corresponding to the eigenvalue 0, by

$$\mathcal{P} : H(\alpha) \to T(\alpha). \tag{430}$$

Since we furthermore have a direct sum $K(\alpha) = Ve^\alpha \oplus K'(\alpha)$ and therefore $H(\alpha) = Ve^\alpha \oplus K'(\alpha) \oplus S(\alpha)$ we also get an induced projection

$$\mathcal{P}' : H(\alpha) \to Ve^\alpha. \tag{431}$$

Lemma 3.1.13 ([Bor92, Lemma 5.3]). *The restriction $\mathcal{P}|_{Ve^\alpha}$ of $\mathcal{P}$ to Ve^α defines a linear isomorphism from Ve^α to $T(\alpha)$ and its inverse is given by $\mathcal{P}'|_{T(\alpha)}$.*

Proof. The restriction $\mathcal{P}|_{Ve^\alpha}$ defines a linear map from Ve^α to $T(\alpha)$. Clearly $\mathcal{P}'|_{T(\alpha)}$ defines a linear map from $T(\alpha)$ to Ve^α. Since their compositions just yield the identity maps on the corresponding spaces the statement is proved. □

Lemma 3.1.14 ([Bor92, Lemma 5.5]). *$P^1(\alpha)$ is the direct sum of $T^1(\alpha)$ and N^1.*

Proof. We have to show that each $p \in P^1$ can be written as $t + n$ for unique elements $t \in T^1(\alpha)$ and $n \in N^1$. We know that we can write $p = k + s$ for unique $k \in K^1(\alpha)$ and $s \in S^1(\alpha)$. Because $S(\alpha)$ is spanned by elements $t_{\lambda,\mu}$ with $\lambda \neq 0$ and $t \in T(\alpha)$ it is clear that the operator E_α preserves spurious states. This means $E_\alpha s \in S(\alpha)$ for each $s \in S(\alpha)$. For $p \in P^1(\alpha)$ we get

$$E_\alpha p = \sum_{m=1}^{\infty} L(-m)D_\alpha(m)p \in S(\alpha). \tag{432}$$

Together this implies $E_\alpha k = E_\alpha p - E_\alpha s \in S(\alpha)$. But because E_α preserves $K(\alpha)$ this yields $E_\alpha k = 0$ since we have $S(\alpha) \cap K(\alpha) = \{0\}$. This proves $k \in T^1(\alpha)$ and we get $s = p - k \in P^1(\alpha)$ since $T^1(\alpha) \subset P^1(\alpha)$. Finally observe $s \in N^1 = P^1(\alpha) \cap S^1(\alpha)$. □

In [Bor92] Borcherds gives a different proof for this lemma. In both proofs it is crucial that the Virasoro algebra acts with central charge $c = 26$. We need this here because otherwise the operator E_α would not satisfy equation (429).

Proposition 3.1.15. *For every $0 \neq \alpha \in II_{1,1}$ we can define a linear isomorphism*

$$\eta_\alpha : V_{1-\alpha^2/2} \to \mathfrak{g}(V)_\alpha = P^1(\alpha)/N^1,\ v \mapsto [\mathcal{P}(v \otimes e^\alpha)], \tag{433}$$

which preserves the G-invariant bilinear forms, i.e. for $v, w \in V_{1-\alpha^2/2}$ we have

$$\langle v, w\rangle = (v \otimes e^\alpha, w \otimes e^\alpha)_0 = ([\mathcal{P}(v \otimes e^\alpha)], [\mathcal{P}(w \otimes e^\alpha)])_0 = (\eta_\alpha(v), \eta_\alpha(w))_0. \tag{434}$$

Furthermore this linear isomorphism preserves the group action of G, i.e. for $g \in G$ and $v \in V_{1-\alpha^2/2}$ we have

$$g\eta_\alpha(v) = \eta_\alpha(gv). \tag{435}$$

Proof. Since the linear isomorphism $(P)|_{Ve^\alpha}$ preserves the $L(0)$-grading it is clear it defines a linear isomorphism from $V_{1-\alpha^2/2}e^\alpha$ to $T^1(\alpha)$. Using Lemma 3.1.14 we find that each class $P^1(\alpha)/N^1$ contains a unique representant in $T^1(\alpha)$, therefore the map η_α defines a linear isomorphism. For the invariance of the bilinear form we evaluate

$$(v \otimes e^\alpha, w \otimes e^\alpha)_0 = -\langle v, w\rangle(e^\alpha, \theta(e^\alpha)) = -(v, w)(\mathbf{1}, \mathbf{1})_{\mathrm{II}_{1,1}} = \langle v, w\rangle. \tag{436}$$

The equation (434) is a direct consequence of this. For $g \in G$ and $v \otimes e^\alpha = t + k$ with $t \in T^1(\alpha)$ and $k \in N^1$ we have

$$(gv) \otimes e^\alpha = gt + gk = gt (\mathrm{mod}\ N^1), \tag{437}$$

since the group action of G preserves $T^1(\alpha)$ and N^1. This is a consequence of the fact that G commutes with the Virasoro operators and preserves the bilinear form. □

Proposition 3.1.16. *We have*

$$\mathfrak{g}(V)_0 = V_1 \otimes e^0 \oplus II_{1,1} \otimes \mathbb{C}, \tag{438}$$

and the map

$$\eta_0 : V_1 \oplus (II_{1,1} \otimes \mathbb{C}) \to \mathfrak{g}(V)_0,\ v + \alpha \mapsto v \otimes e^0 + \mathbf{1} \otimes \alpha(-1)e^0 \tag{439}$$

defines a linear isomorphism that preserves the bilinear forms and the group action of G.

Proof. First we notice that we have

$$H^1(0) = V_1 \otimes e^0 \oplus V_0 \otimes S(\hat{\mathfrak{h}}_-)_1 \otimes e^0 = P^1(0). \tag{440}$$

The first equation is clear and for the second we have to check that each element of the form $v \otimes e^0 + \mathbf{1} \otimes h(-1)e^0$ for $v \in V_1$ and $h \in \mathrm{II}_{1,1} \otimes \mathbb{C}$ vanishes under all $L(n)$ for $n > 0$. Clearly we have $L_1 v = c\mathbf{1}$ for some constant $c \in \mathbb{C}$. We get

$$c = \langle L_1 v, \mathbf{1}\rangle = \langle v, L_{-1}\mathbf{1}\rangle = 0. \tag{441}$$

For each $h \in \mathrm{II}_{1,1} \otimes \mathbb{C}$ we can check by use of (374) that $L(1)h(-1)e^0 = 0$. Since the restriction of the bilinear form $(\cdot, \cdot)_0$ to this space $P^1(0)$ is clearly non-degenerate we obtain $\mathfrak{g}_0(V) = P^1(0)$. Therefore the map η_0 defines a linear isomorphism that preserves the group action of G. Next we have to consider the bilinear forms. For $v, w \in V_1$ we have

$$\begin{aligned}(v \otimes e^0, w \otimes e^0)_0 &= \langle v, w\rangle(e^0, e^0)_0 && (442)\\ &= -\langle v, w\rangle(e^0, \theta(e^0))_{\mathrm{II}_{1,1}} && (443)\\ &= -\langle v, w\rangle(e^0, e^0)_{\mathrm{II}_{1,1}} && (444)\\ &= \langle v, w\rangle. && (445)\end{aligned}$$

For $h_1, h_2 \in \mathrm{II}_{1,1} \otimes \mathbb{C}$ we obtain

$$\begin{aligned}(\mathbf{1} \otimes h_1(-1)e^0, \mathbf{1} \otimes h_2(-1)e^0)_0 &= \langle \mathbf{1}, \mathbf{1} \rangle (h_1(-1)e^0, h_2(-1)e^0)_0 && (446)\\ &= -\langle \mathbf{1}, \mathbf{1} \rangle (h_1(-1)e^0, \theta(h_2(-1)e^0))_{\mathrm{II}_{1,1}} && (447)\\ &= \langle \mathbf{1}, \mathbf{1} \rangle (h_1(-1)e^0, h_2(-1)e^0)_{\mathrm{II}_{1,1}} && (448)\\ &= -\langle \mathbf{1}, \mathbf{1} \rangle (e^0, [h_1(1), h_2(-1)]e^0)_{\mathrm{II}_{1,1}} && (449)\\ &= -\langle \mathbf{1}, \mathbf{1} \rangle (h_1, h_2)(e^0, e^0)_{\mathrm{II}_{1,1}} && (450)\\ &= (h_1, h_2). && (451)\end{aligned}$$

Because of $v \otimes e^0 \perp \mathbf{1} \otimes h(-1)e^0$ for $v \in V_1$ and $h \in \mathrm{II}_{1,1} \otimes \mathbb{C}$ the statement is proved. □

Clearly the combination of Proposition 3.1.15 and Proposition 3.1.16 just yields the no-ghost theorem. Notice that we can use the operator E_α to describe the projection $\mathcal{P} : H(\alpha) \to T(\alpha)$ in a more explicit way. For $x \in H(\alpha)$ we have

$$\mathcal{P}x = \frac{1}{d!} \prod_{i=1}^{d} (i + E_\alpha)x \tag{452}$$

where $d \in \mathbb{N}$ is sufficiently large such that $L(k)x = 0$ for all $k \geq d$. For $v \in V_n$ we obtain

$$\mathcal{P}v \otimes e^\alpha = \frac{1}{n!} \prod_{i=1}^{n} (i + E_\alpha)v \otimes e^\alpha = \sum_{i=0}^{n} S_i(1, \cdots, n)E_\alpha^i v \otimes e^\alpha, \tag{453}$$

where S_k are the usual symmetric polynomials in n variables. For short we denote the number $S_r(1, \cdots, n)$ by $S_r[n]$.

3.2 The Lie algebra of physical states

In the previous section we assumed that V is a lowest-weight Vir-module of central charge $c_V = 24$ and studied its space of physical states $\mathfrak{g}(V)$. In this section we specialize the discussion to the case where V is a vertex operator algebra of central charge $c = 24$. We assume furthermore that V is self-contragredient, i.e. that it carries a non-degenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$. This invariant bilinear form shall be normalized such that $\langle \mathbf{1}, \mathbf{1} \rangle = 1$. Furthermore we assume that the vacuum $\mathbf{1}$ spans the space V_0 over $\mathbb{C}$. Those assumptions turn V into a lowest-weight Vir-module of central charge $c_V = 24$ in the sense of Definition 3.1.1. Notice that we assume all those properties throughout the entire section. Clearly this turns $V \otimes V_{\mathrm{II}_{1,1}}$ into a conformal vertex algebra of central charge $c = 26$.

Proposition 3.2.1. *Let V be a vertex operator algebra of central charge $c = 24$ which satisfies all assumptions made above. The space*

$$\mathfrak{g}'(V) = P^1 / L(-1)P^0 \tag{454}$$

can be equipped with the structure of a Lie algebra by use of the bracket

$$[\cdot, \cdot] : \mathfrak{g}'(V) \times \mathfrak{g}'(V) \to \mathfrak{g}'(V), \; ([x], [y]) \mapsto [x_0 y]. \tag{455}$$

The subspace $L(-1)P^0$ is furthermore contained in ker$(\cdot, \cdot)$ and therefore the invariant bilinear form $(\cdot, \cdot)$ of P^1 can be restricted to the quotient $\mathfrak{g}'(V)$ and induces a bilinear form which is invariant under the Lie algebra structure, i.e.

$$([u, v], w) + (v, [u, w]) = 0 \tag{456}$$

for all $u, v, w \in \mathfrak{g}'(V)$. The $II_{1,1}$-grading of $V_{II_{1,1}}$ induces a natural $II_{1,1}$-Lie algebra grading of $\mathfrak{g}'(V)$, i.e. we have

$$\mathfrak{g}'(V) = \bigoplus_{\alpha \in II_{1,1}} \mathfrak{g}'_\alpha(V) \tag{457}$$

and

$$[\mathfrak{g}'_\alpha(V), \mathfrak{g}'_\beta(V)] \subset \mathfrak{g}'_{\alpha+\beta}(V). \tag{458}$$

Proof. The fact that this bracket defines a Lie algebra structure is a standard computation that makes use of the Borcherds identity of the vertex algebra $V \otimes V_{\mathrm{II}_{1,1}}$. Since the adjoint of $L(-1)$ is $L(1)$ and P^1 vanishes under $L(1)$ it is clear that $L(-1)P^0$ is contained in the kernel of the bilinear form. ☐

Since we prefer a space with a non-degenerate bilinear form we can consider the quotient

$$\mathfrak{g}(V) = \mathfrak{g}'(V)/\mathrm{ker}(\cdot,\cdot). \tag{459}$$

Clearly this quotient preserves the Lie algebra structure and the $\mathrm{II}_{1,1}$-grading. Furthermore we have

$$\mathfrak{g}(V) = \mathfrak{g}'(V)/\mathrm{ker}(\cdot,\cdot) = P^1/\mathrm{ker}(\cdot,\cdot), \tag{460}$$

such that the underlying vector space of the Lie algebra $\mathfrak{g}(V)$ coincides with the spaces of physical states of V. The bilinear form $(\cdot,\cdot)$ and the $\mathrm{II}_{1,1}$-grading coincide under this identification as well therefore we can equip the space of physical states of V with a natural Lie algebra structure. Assume that the group of symmetries G of V is a group of vertex operator algebra automorphisms then the induced group of symmetries of $\mathfrak{g}(V)$ is a group of Lie algebra automorphisms that preserves the invariant bilinear form and the $\mathrm{II}_{1,1}$-Lie algebra grading.

Definition 3.2.2. Let V be a self-contragredient vertex operator algebra of central charge $c_V = 24$ and assume that the corresponding bilinear form $\langle\cdot,\cdot\rangle$ is scaled such that $\langle\mathbf{1},\mathbf{1}\rangle = 1$. Furthermore we assume V_0 to be spanned by $\mathbf{1}$. Assume that G is a group of vertex operator algebra automorphisms of V. The *Lie algebra of physical states* $\mathfrak{g}(V)$ of V with symmetry group G is the Lie algebra structure on the spaces of physical states of V induced by the Lie algebra structure of $\mathfrak{g}'(V)$ together with its invariant bilinear form $(\cdot,\cdot)$ and the Lie algebra grading

$$\mathfrak{g}(V) = \bigoplus_{\alpha\in\mathrm{II}_{1,1}} \mathfrak{g}_\alpha(V). \tag{461}$$

In Definition 3.1.2 we introduced an involution θ on $V \otimes V_{\mathrm{II}_{1,1}}$ which is clearly an involution of conformal vertex algebras since it acts as the identity on V. This implies that θ induces a Lie algebra involution θ on $\mathfrak{g}(V)$ that preserves the bilinear form the commutes with the group of symmetries G and satisfies

$$\theta(\mathfrak{g}_\alpha(V)) = \mathfrak{g}_{-\alpha}(V) \tag{462}$$

for all $\alpha \in \mathrm{II}_{1,1}$. This allows us to introduce a *contravariant bilinear form*

$$(\cdot,\cdot)_0 = -(\cdot,\theta(\cdot)) \tag{463}$$

on $\mathfrak{g}(V)$ such that its restriction to each $\mathrm{II}_{1,1}$-grade space $\mathfrak{g}_\alpha(V)$ is non-degenerate. Notice that the contravariant bilinear form is not invariant in general, since we have

$$(a,[x,b])_0 = ([a,\theta(x)],b)_0 \tag{464}$$

for $a, b, x \in \mathfrak{g}(V)$. We can apply the no-ghost theorem 3.1.5 to study the $\mathrm{II}_{1,1}$-grade spaces $\mathfrak{g}_\alpha(V)$ for $\alpha \in \mathrm{II}_{1,1}$. For every $\alpha \in \mathrm{II}_{1,1} \setminus \{0\}$ this yields a linear isomorphism $\eta_\alpha : V_{1-\alpha^2/2} \to \mathfrak{g}(V)_\alpha$ that commutes with the group action of G and satisfies for every $v, w \in V_{1-\alpha^2/2}$ that

$$\langle v, w\rangle = (\eta_\alpha(v), \eta_\alpha(w))_0. \tag{465}$$

Clearly we obtain for every $\alpha \in \mathrm{II}_{1,1} \setminus \{0\}$ that

$$\dim(\mathfrak{g}_\alpha(V)) = \dim(V_{1-\alpha^2/2}). \tag{466}$$

We can now discuss the case $\alpha = 0$. The Proposition 3.1.16 induces a linear isomorphism $\eta_0 : V_1 \oplus (\mathrm{II}_{1,1} \otimes \mathbb{C}) \to \mathfrak{g}(V)_0$ which for all $v, w \in V_1$ and $\alpha, \beta \in \mathrm{II}_{1,1} \otimes \mathbb{C}$ satisfies

$$(\eta_0(v+\alpha), \eta_0(w+\beta))_0 = \langle v, w \rangle + (\alpha, \beta). \tag{467}$$

Yet we are not just interested in the value of the contravariant bilinear form $(\cdot,\cdot)_0$ but also in the values of invariant bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}_0(V)$. For all $v, w \in V_1$ and $\alpha, \beta \in \mathrm{II}_{1,1} \otimes \mathbb{C}$ we have

$$(\eta_0(v+\alpha), \eta_0(w+\beta)) = -\langle v, w \rangle + (\alpha, \beta). \tag{468}$$

This is a direct consequence of $(e^0, e^0)_{\mathrm{II}_{1,1}} = -1$. So far we discussed how we can transport the group action and the bilinear forms from V to $\mathfrak{g}(V)$. Yet the most interesting structure of $\mathfrak{g}(V)$ is its Lie algebra structure and we can use the no-ghost isomorphisms to transport the Lie bracket to V.

Definition 3.2.3. For $\alpha, \beta \in \mathrm{II}_{1,1} \setminus \{0\}$ we define maps

$$\{\cdot,\cdot\}_{\alpha,\beta} : V_{1-\frac{\alpha^2}{2}} \times V_{1-\frac{\beta^2}{2}} \to V_{1-\frac{(\alpha+\beta)^2}{2}},\ (v, w) \mapsto \eta^{-1}_{\alpha+\beta}([\eta_\alpha(v), \eta_\beta(w)]). \tag{469}$$

More explicitly for $v \in V_{1-\alpha^2}$ and $w \in V_{1-\beta^2/2}$ we have

$$\{v, w\}_{\alpha,\beta} = \mathcal{P}'((\mathcal{P}v \otimes e^\alpha)_0(\mathcal{P}w \otimes e^\beta)). \tag{470}$$

Borcherds asked in section 15 of [Bor92] for an explicit description of the map $\{\cdot,\cdot\}_{\alpha,\beta}$ in term of vertex algebra operations on V in the special case of the moonshine module $V = V^\natural$. In the remainder of this section we want to derive such an explicit description.

3.3 Schur polynomials and the lattice vertex algebra $V_{\mathrm{II}_{1,1}}$

In this section we discuss the lattice vertex algebra $V_{\mathrm{II}_{1,1}}$ in more detail. In particular we introduce the corresponding *Schur polynomials*. We aim to evaluate the projection $\mathcal{P}'$ explicitly for certain elements of spaces of the form $H(\beta)$ for $\beta \in \mathrm{II}_{1,1} \setminus \{0\}$. As usual we fix an element $w_\beta \in \mathrm{II}_{1,1}$ such that $(\beta, w_\beta) = 1$ and $(w_\beta, w_\beta) = 0$ and introduce the corresponding operators $K_\beta(n)$ as in section 3.1.

Definition 3.3.1. For $\alpha \in \mathrm{II}_{1,1}$ we define the *Schur polynomials* $S_k^+(\alpha)$ in vertex operators of $V_{\mathrm{II}_{1,1}}$ by

$$E(e^\alpha, z)^+ := \exp\left(\sum_{m=1}^{\infty} \alpha(-m)\frac{z^m}{m}\right) = \sum_{k=0}^{\infty} S_k^+(\alpha) z^k. \tag{471}$$

By comparing coefficients of the formal derivation of equation (471) we get

$$k S_k^+(\alpha) = \sum_{m=1}^{k} S_{k-m}^+(\alpha)\alpha(-m). \tag{472}$$

In the following we fix $\alpha, \beta \in \mathrm{II}_{1,1}$ and chose w_β as usual. We define numbers

$$A(m) := -(\alpha, w_\beta)(1-(\alpha, w_\beta))^{m-1} \tag{473}$$

and check by direct computation that in the case $(\alpha, w_\beta) \neq 0$ they satisfy the equation

$$\sum_{i=1}^{m-1} A(i) = -\left(\frac{1}{(\alpha, w_\beta)} A(m) + 1\right). \tag{474}$$

In fact we can check, that this equation determines the numbers $A(m)$ already.

Lemma 3.3.2. *For a non-negative integer k and a non-negative integer m we have*

$$[L(-k), S^+_m(\alpha)] = \sum_{h=1}^{m} \alpha(-k-h) S^+_{m-h}(\alpha). \tag{475}$$

Proof. We prove this by induction over m. The statement is clearly valid for $m=0$ and $m=1$. Assume now that the statement holds for all $m < m_0$ for some m_0. We get

$$[L(-k), S^+_{m_0}(\alpha)] \tag{476}$$
$$= \frac{1}{m_0}\left[L(-k), \sum_{i=1}^{m_0} S^+_{m_0-i}(\alpha)\alpha(-i)\right] \tag{477}$$
$$= \frac{1}{m_0}\sum_{i=1}^{m_0}\Big(L(-k)S^+_{m_0-i}(\alpha)\alpha(-i) - S^+_{m_0-i}(\alpha)\alpha(-i)L(-k)\Big) \tag{478}$$
$$= \frac{1}{m_0}\sum_{i=1}^{m_0}\Big(([L(-k),\alpha(-i)] + \alpha(-i)L(-k))S^+_{m_0-i}(\alpha) - \alpha(-i)S^+_{m_0-i}(\alpha)L(-k)\Big) \tag{479}$$
$$= \frac{1}{m_0}\sum_{i=1}^{m_0}\Big((i\alpha(-(k+i))S^+_{m_0-i}(\alpha) + \alpha(-i)[L(-k), S^+_{m_0-i}(\alpha)]\Big) \tag{480}$$
$$= \frac{1}{m_0}\left(\sum_{i=1}^{m_0} i\alpha(-(k+i))S^+_{m_0-i}(\alpha) + \sum_{i=1}^{m_0}\alpha(-i)\sum_{j=1}^{m_0-i}\alpha(-k-j)S^+_{m_0-i-j}(\alpha)\right) \tag{481}$$
$$= \frac{1}{m_0}\left(\sum_{i=1}^{m_0} i\alpha(-(k+i))S^+_{m_0-i}(\alpha) + \sum_{j=1}^{m_0}\alpha(-(k+j))(m_0-j)S^+_{m_0-j}(\alpha)\right) \tag{482}$$
$$= \sum_{i=1}^{m_0}\alpha(-k-i)S^+_{m_0-i}(\alpha) \tag{483}$$

Each step in those equations is easy to verify therefore the statement is proved. □

Notice that we have to commute elements of the form $\alpha(n)$ and $\alpha(m)$ to obtain equation (482). The assumption that k is non-negative makes sure that all the commutators, that appear, vanish. Otherwise the statement would look a bit more complicated. In order to determine the projection $\mathcal{P}'$ of expressions like $v \otimes kS^+_k(\alpha)e^\beta$ we need two lemmas which help us to determine such projections recursively.

Lemma 3.3.3. *For $k \in \mathbb{N}$ and $v \in V$ we have*

$$v \otimes kS^+_k(\alpha)e^\beta = -\sum_{m=1}^{k} A(m)v \otimes L(-m)S^+_{k-m}(\alpha)e^\beta (\textit{mod}\, G(\beta)). \tag{484}$$

Proof. The elements β and w_β clearly form a basis of $\mathrm{II}_{1,1}\otimes\mathbb{C}$ and we have $\alpha = (\alpha, w_\beta)\beta + cw_\beta$ for some constant $c \in \mathbb{C}$. Furthermore the dual basis of (β, w_β) is given by $(w_\beta, \beta - \beta^2 w_\beta)$ and we use this to compute

$$L(-m)e^\beta = \beta(-m)e^\beta + w_\beta(-1)(\cdots)e^\beta + \cdots + w_\beta(-m)(\cdots)e^\beta. \tag{485}$$

If $(\alpha, w_\beta) = 0$ we obtain $\alpha = cw_\beta$ and this implies

$$v \otimes kS^+_k(\alpha)e^\beta = 0(\mathrm{mod}\, G(\beta)). \tag{486}$$

Because of $A(m) = 0$ for all $m = 1, \cdots, k$ the statement is proved in this case. Form now on we assume $(\alpha, w_\beta) \neq 0$. This allows us to compute

$$v \otimes S^+_{k-m}(\alpha)L(-m)e^\beta = v \otimes S^+_{k-m}(\alpha)(\beta(-m)e^\beta + w_\beta(-1)(\cdots)e^\beta + \cdots + w_\beta(-m)e^\beta) \quad (487)$$

$$= v \otimes S^+_{k-m}(\alpha)\beta(-m)e^\beta + K_\beta(-1)(\cdots) + \cdots + K_\beta(-m)(\cdots) \quad (488)$$

$$= v \otimes S^+_{k-m}(\alpha)\beta(-m)e^\beta (\text{mod } G(\beta)) \quad (489)$$

$$= \frac{1}{(\alpha, w_\beta)} v \otimes S^+_{k-m}(\alpha)\alpha(-m)e^\beta (\text{mod } G(\beta)). \quad (490)$$

Now we can compute

$$\sum_{m=1}^{k} A(m)v \otimes L(-m)S^+_{k-m}(\alpha)e^\beta \quad (491)$$

$$= \sum_{m=1}^{k} A(m)v \otimes \left(S^+_{k-m}(\alpha)L(-m)e^\beta + \sum_{h=1}^{k-m} \alpha(-m-h)S^+_{k-m-h}(\alpha)e^\beta \right) \quad (492)$$

$$= -\sum_{m=1}^{k} v \otimes S^+_{k-m}(\alpha)\alpha(-m)e^\beta + \sum_{m=1}^{k} \left(\frac{1}{(\alpha, w_\beta)} A(m) + 1 \right) v \otimes S^+_{k-m}(\alpha)\alpha(-m)e^\beta \quad (493)$$

$$+ \sum_{m=1}^{k} \sum_{h=1}^{k-m} A(m)v \otimes \alpha(-m-h)S^+_{k-m-h}(\alpha)e^\beta \ (\text{mod } G(\beta)) \quad (494)$$

$$= -\sum_{m=1}^{k} v \otimes S^+_{k-m}(\alpha)\alpha(-m)e^\beta + \sum_{m=1}^{k} \left(\frac{1}{(\alpha, w_\beta)} A(m) + 1 \right) v \otimes S^+_{k-m}(\alpha)\alpha(-m)e^\beta \quad (495)$$

$$+ \sum_{i=2}^{k} \left(\sum_{j=1}^{i-1} A(j) \right) v \otimes \alpha(-i)S^+_{k-i}(\alpha)e^\beta \ (\text{mod } G(\beta)) \quad (496)$$

$$= -\sum_{m=1}^{k} v \otimes S^+_{k-m}(\alpha)\alpha(-m)e^\beta = -v \otimes kS^+_k(\alpha)e^\beta \quad (497)$$

In this computation we used the change of indices $i = m + h$ and $j = m$ and the fact

$$\left(\frac{1}{(\alpha, w_\beta)} A(1) + 1 \right) = 0. \quad (498)$$

This is precisely the statement of the lemma. ☐

Lemma 3.3.4. *For $k \in \mathbb{N}$ and $v \in V$ we have*

$$kP'(v \otimes S^+_k(\alpha)e^\beta) = \sum_{m=1}^{k} A(m)P'(L_{-m}v \otimes S^+_{k-m}(\alpha)e^\beta). \quad (499)$$

Proof. We have $\sum_{m=1}^{k} A(m)L(-m)(v \otimes S^+_{k-m}(\alpha)e^\beta) = 0$ (mod $G(\beta)$). We immediately get

$$\sum_{m=1}^{k} A(m)L_{-m}v \otimes S^+_{k-m}(\alpha)e^\beta = -\sum_{m=1}^{k} A(m)v \otimes L(-m)S^+_{k-m}(\alpha)e^\beta \quad (500)$$

$$= kv \otimes S^+_k(\alpha)e^\beta (\text{mod } G(\beta)). \quad (501)$$

Since the kernel of P' is precisely $G(\beta)$ we are done. ☐

Before we can come to the main result of this section we have to introduce some notation. In the following we will write $\mathbb{N}_+$ for the set of positive integers. We define sets

$$B^j(k) := \{\underline{m} = (m_1, \cdots, m_j) \in \mathbb{N}_+^j : m_1 + \cdots + m_j = k\}, \tag{502}$$

$B^0(k) := \emptyset$ and $B^0(0) := \{\underline{0} = (0)\}$ for $k \in \mathbb{N}_+$ and $j \in \mathbb{N}_+$. Clearly we have $B^j(k) = \emptyset$ for $j > k$. We use those sets to define

$$B(k) = \bigcup_{j=0}^{k} B^j(k). \tag{503}$$

For later use we define the numbers $p(k, j)$ to be the cardinality of $B^j(k)$ and polynomials

$$d_k = \sum_{m=0}^{k} p(k, m) T^m. \tag{504}$$

We have numbers $c(\underline{m}) = \prod_{i=1}^{j} \frac{A(m_i)}{m_i + \cdots + m_k}$ and $c(\underline{0}) = 1$. We also write $L(-\underline{m}) = L_{-m_j} \cdots L_{-m_1}$ and $L(-\underline{0}) = \mathrm{Id}$. All this prepares us to state and prove the main result of this subsection.

Proposition 3.3.5. *For $v \in V$ and $k \in \mathbb{N}$ we have*

$$P'(v \otimes S_k^+(\alpha) e^\beta) = \sum_{j=0}^{k} \sum_{\underline{m} \in B^j(k)} c(\underline{m}) L(-\underline{m}) v = \sum_{\underline{m} \in B(k)} c(\underline{m}) L(-\underline{m}) v. \tag{505}$$

Proof. We prove this by induction. The case $k = 0$ is clear. Notice that we have

$$B^{j+1}(k) = \{(m, \underline{m}) : m = 1, \cdots, k-1, \underline{m} \in B^j(k-m)\} \tag{506}$$

$$= \{(m, \underline{m}) : m = 1, \cdots, k-j, \underline{m} \in B^j(k-m)\}. \tag{507}$$

Therefore for any function X, defined on the sets $B^j(k)$ we have

$$\sum_{m=1}^{k-1} \sum_{j=1}^{k-m} \sum_{\underline{m} \in B^j(k-m)} X((m, \underline{m})) = \sum_{j=1}^{k-1} \sum_{m=1}^{k-1} \sum_{\underline{m} \in B^j(k-m)} X((m, \underline{m})) = \sum_{j=2}^{k} \sum_{\underline{m}' \in B^j(k)} X(\underline{m}'). \tag{508}$$

Taking into account that $\frac{A(m)}{k} c(\underline{m}) = c(\underline{m}')$ and $L(-\underline{m}') = L(-\underline{m}) L_{-m}$ for $\underline{m}' = (m, \underline{m})$ we can carry out the induction using (508) and get

$$P'(v \otimes S_k^+(\alpha) e^\beta) \tag{509}$$

$$= \frac{A(k)}{k} L_{-k} v + \sum_{m=1}^{k-1} \frac{A(m)}{k} P'(L_{-m} v \otimes S_{k-m}^+(\alpha) e^\beta) \tag{510}$$

$$= \frac{A(k)}{k} L_{-k} v + \sum_{m=1}^{k-1} \frac{A(m)}{k} \sum_{j=1}^{k-m} \sum_{\underline{m} \in B^j(k-m)} c(\underline{m}) L(-\underline{m}) L_{-m} v \tag{511}$$

$$= \frac{A(k)}{k} L_{-k} v + \sum_{m=1}^{k-1} \sum_{j=1}^{k-m} \sum_{\underline{m} \in B^j(k-m)} \frac{A(m)}{k} c(\underline{m}) L(-\underline{m}) L_{-m} v \tag{512}$$

$$= \frac{A(k)}{k} L_{-k} v + \sum_{j=2}^{k} \sum_{\underline{m}' \in B^{j+1}(k)} c(\underline{m}') L(-\underline{m}') v \tag{513}$$

$$= \sum_{j=0}^{k} \sum_{\underline{m}' \in B^j(k)} c(\underline{m}') L(-\underline{m}') v \tag{514}$$

$$= \sum_{\underline{m}' \in B(k)} c(\underline{m}') L(-\underline{m}') v \tag{515}$$

This is precisely the statement of the proposition if we substitute $\underline{m}'$ with $\underline{m}$. □

3.4 The Lie bracket of the Lie algebra of physical states

In this section we come to one of the main results of this thesis, an explicit formula for the bracket (470). In order to prove this we have to evaluate the term $(\mathcal{P}v \otimes e^{\alpha})_0(\mathcal{P}w \otimes e^{\beta})(\operatorname{mod} G(\alpha+\beta))$ in a suitable way, such that we can compute its projection under $\mathcal{P}'$ explicitly. Since those projections depend on a choice of an element $w_\alpha \in \mathrm{II}_{1,1}$ with $(\alpha, w_\alpha) = 1$ and $w_\alpha^2 = 0$ for every $\alpha \in \mathrm{II}_{1,1} \setminus \{0\}$ there is some ambiguity in those projections. Yet it is possible to make these choices in a *compatible* way, such that the corresponding operators $K_\alpha(n)$ are closely related.

Lemma 3.4.1. *The lattice $\mathrm{II}_{1,1}$ contains elements $e, f \in \mathrm{II}_{1,1}$ such that*

1. $e^2 = f^2 = 0$ *and* $(e, f) = 1$ *and*
2. *for all* $\alpha \in \mathrm{II}_{1,1}$ *with* $\alpha^2 \neq 0$ *we have* $(\alpha, e) \neq 0$ *and* $(\alpha, f) \neq 0$.

Each pair e, f with this properties spans $\mathrm{II}_{1,1} \otimes \mathbb{C}$ and $v = ae + bf$ satisfies $v^2 = 0$ if and only of $a = 0$ or $b = 0$.

Proof. The lattice $\mathrm{II}_{1,1}$ can be given explicitly by $\mathbb{Z}e \oplus \mathbb{Z}f$ with $e^2 = f^2 = 0$ and $(e, f) = 1$. Clearly e, f is a pair with the properties in the statement. Assume now e, f be any such pair. This pair spans $\mathrm{II}_{1,1} \otimes \mathbb{C}$ because otherwise we would have $e = cf$ for some $c \in \mathbb{C}$ but this clearly contradicts $(e, f) = 1$. A direct consequence of $v = ae + bf$ is $v^2 = ab$. This implies $v^2 = 0$ if and only if $a = 0$ or $b = 0$. □

Of course the statement of this lemma is obvious and the proof is easy. We just state it to collect all the statements by which we characterize such a basis e, f.

Definition 3.4.2. We fix a pair of elements e, f in $\mathrm{II}_{1,1}$ as in Lemma 3.4.1. For $\alpha \in \mathrm{II}_{1,1}$ with $\alpha^2 \neq 0$ and $\alpha = nf \in \mathrm{II}_{1,1}$ we set

$$w_\alpha = \frac{1}{(w, \alpha)} e \tag{516}$$

and for $\alpha = ne \in \mathrm{II}_{1,1}$ we set

$$w_\alpha = \frac{1}{(w', \alpha)} f. \tag{517}$$

From now on we chose a pair e, f as in Definition 3.4.2 and keep it fixed. For every $0 \neq \alpha \in \mathrm{II}_{1,1}$ we define w_α as in this definition. Clearly if $\alpha, \beta \in \mathrm{II}_{1,1}$ satisfy $\alpha, \beta \notin e^\perp$ we get for all $n \in \mathbb{Z}$ that

$$K_\alpha(n) = (\beta, w_\alpha) K_\beta(n) = \frac{1}{(\alpha, w_\beta)} K_\beta(n). \tag{518}$$

Definition 3.4.3. For $k \in \mathbb{N}_+$ and $j = 0, \cdots, k-1$ we define polynomials $p_j^k \in \mathbb{Z}[T]$ by

1. $p_0^k = T^k$ for $j = 0$,
2. $p_j^k = T p_j^{k-1} + p_{j-1}^{k-1}$ for $j = 1, \cdots, k-2$ and
3. $p_{k-1}^k = kT$ for $j = k-1$.

In the following we will evaluate the polynomials p_j^k at special places, which are usually integer. Of course for $m \in \mathbb{Z}$ we denote the corresponding value by $p_j^k(m)$.

Lemma 3.4.4. *Take $\alpha \in \mathrm{II}_{1,1}$. For $k \in \mathbb{N}$ and $m \in \mathbb{Z}$ we have*

$$[D_\alpha(m), E_\alpha^k] = -\sum_{j=0}^{k-1} p_j^k(m) D_\alpha(m) E_\alpha^j. \tag{519}$$

Proof. We prove this by induction. For $k = 0$ the statement is clear and for $k = 1$ the statement is given in Lemma 3.1.10. Assuming the statement for k we get

$$[E_\alpha^{k+1}, D_\alpha(m)] = E_\alpha[E_\alpha^k, D_\alpha(m)] + [E_\alpha, D_\alpha(m)]E_\alpha^k \tag{520}$$

$$= \sum_{j=0}^{k-1} p_j^k(m) E_\alpha D_\alpha(m) E_\alpha^j + p_0^1(m) D_\alpha(m) E_\alpha^k \tag{521}$$

$$= \sum_{j=0}^{k-1} p_j^k(m)(m D_\alpha(m) E_\alpha^j + D_\alpha(m) E_\alpha^{j+1}) \tag{522}$$

$$+ p_0^1(m) D_\alpha(m) E_\alpha^k \tag{523}$$

$$= p_0^k(m)(m) D_\alpha(m) E_\alpha^0 \tag{524}$$

$$+ \sum_{j=1}^{k-1} \left[p_j^k(m) m D_\alpha(m) E_\alpha^j + p_{j-1}^k(m) D_\alpha(m) E_\alpha^j \right] \tag{525}$$

$$+ p_{k-1}^k(m) D_\alpha(m) E_\alpha^k + p_0^1(m) D(m) E_\alpha^k \tag{526}$$

$$= p_0^k(m)(m) D_\alpha(m) E_\alpha^0 \tag{527}$$

$$+ \sum_{j=1}^{k-1} \left(p_j^k(m) m + p_{j-1}^k(m) \right) D_\alpha(m) E_\alpha^j \tag{528}$$

$$+ (p_{k-1}^k(m) + p_0^1(m)) D_\alpha(m) E_\alpha^k \tag{529}$$

$$= \sum_{j=0}^{k} p_j^{k+1}(m) D_\alpha(m) E_\alpha^j. \tag{530}$$

This proves the statement. □

Now we will start to evaluate the maps $\{\cdot,\cdot\}_{\alpha,\beta}$. Remember the polynomials d_k, defined in (504). As explained above for a number $x \in \mathbb{C}$ we denote the evaluation of the polynomial d_k at x by $d_k(x)$.

Lemma 3.4.5. *Assume $\alpha, \beta, \alpha+\beta \notin e^\perp$. For $x \in K(\alpha)$, $y \in K(\beta)$ and $n \in \mathbb{N}_+$, $k \in \mathbb{Z}$ we have*

$$(D_\alpha(-n)x)_k y = (-1)^n d_n((w_\alpha, \beta)) x_{k-n} y \ (\textit{mod}\ G(\alpha+\beta)) \ \textit{and} \tag{531}$$

$$x_k(D_\beta(-n)y) = d_n((\alpha, w_\beta)) x_{k-n} y \ (\textit{mod}\ G(\alpha+\beta)). \tag{532}$$

Proof. Because of $\alpha, \beta, \alpha+\beta \notin e^\perp$ we have that $K_\alpha(j)y = (w_\alpha, \beta) K_\beta(j) y$ and $K_\alpha(j)y = (w_\alpha, \alpha+\beta) K_{\alpha+\beta}(j) y'$ for $y \in K(\beta)$ and $y' \in K(\alpha+\beta)$ for all $j \in \mathbb{Z}$. See (518) and the corresponding discussion. Using this and the Borcherds identity (283) we get

$$(K_\alpha(-n)x)_k y = \sum_{j=0}^{\infty} (-1)^j \binom{-n}{j} (K_\alpha(-n-j)(x_{k+j} y) - (-1)^n x_{k-n-j}(K_\alpha(j) y)) \tag{533}$$

$$= -(-1)^n (w_\alpha, \beta) x_{k-n} y + K_{\alpha+\beta}(-n)(\cdots) + K_{\alpha+\beta}(-n-1)(\cdots) + \cdots \tag{534}$$

$$= -(-1)^n (w_\alpha, \beta) x_{k-n} y \ (\text{mod}\ G(\alpha+\beta)). \tag{535}$$

Using this we obtain

$$(D(-n,m)x)_k y = \sum_{n_1+\cdots+n_m = n, n_1, \cdots, n_m > 0} (K_\alpha(-n_1) \cdots K_\alpha(-n_m) x)_k y \tag{536}$$

$$= \sum_{n_1+\cdots+n_m = n, n_1, \cdots, n_m > 0} (-1)^{m+n} (w_\alpha, \beta)^m x_{k-n} y \ (\text{mod}\ G(\alpha+\beta)) \tag{537}$$

$$= (-1)^{m+n} (w_\alpha, \beta)^m p(n,m) x_{k-n} y \ (\text{mod}\ G(\alpha+\beta)) \tag{538}$$

Here we used that $K_\alpha(n)x$ vanish for all positive n. Taking the sum over m yields the stated result for $D(n)$. The second equation can be proved analogously. □

Definition 3.4.6. For fixed $\alpha, \beta \notin e^{\perp}$ with $\alpha + \beta \notin e^{\perp}$ we denote the numbers $d_n((w_\alpha, \beta))$ and $d_n((\alpha, w_\beta))$ by $d(n)$ and $d'(n)$ respectively. For $m \in \mathbb{N}$ we set

$$c_1^k(m) := (-1)^m d(m) p_0^{k-1}(-m) \tag{539}$$

and for $(m_j, \cdots, m_1) \in \mathbb{N}^j$ we recursively define

$$c_j^k(m_j, \cdots, m_1) := (-1)^{m_1} d(m_1) \sum_{i=j}^{k} p_{i-1}^{k-1}(-m_1) c_{j-1}^{i-1}(m_j, \cdots, m_2). \tag{540}$$

Furthermore we define

$$\tilde{c}_1^k(m) := d'(m) p_0^{k-1}(-m) \tag{541}$$

and again we define recursively

$$\tilde{c}_j^k(m_j, \cdots, m_1) := d'(m_1) \sum_{i=j}^{k} p_{i-1}^{k-1}(-m_1) \tilde{c}_{j-1}^{i-1}(m_j, \cdots, m_2). \tag{542}$$

Explicit evaluation of the recursive definition of the maps c_j^k in (540) gives

$$c_j^k(m_j, \cdots, m_1) = (-1)^{m_1 + \cdots + m_j} d(m_1) \cdots d(m_j) \tag{543}$$

$$\sum_{i_1=j}^{k} \sum_{i_2=j-1}^{i_1-1} \cdots \sum_{i_{j-1}=1}^{i_{j-2}-1} p_{i_1-1}^{k-1}(-m_1) p_{i_2-1}^{i_1-2}(-m_2) \cdots p_0^{i_{j-1}-2}(-m_j). \tag{544}$$

A similar formula can be derived for $\tilde{c}_j^k(m_j, \cdots, m_1)$.

Lemma 3.4.7. *Assume $\alpha, \beta, \alpha + \beta \notin e^{\perp}$. For $k \in \mathbb{N}_+$, $h \in \mathbb{Z}$ and $v, w \in V$ we have for all $x \in K(\alpha)$ and $y \in K(\beta)$ that*

$$(E_\alpha^k v \otimes e^\alpha)_h y = \sum_{i=1}^{k} \sum_{\underline{m} \in \mathbb{N}_+^i} c_i^k(\underline{m})((L(\underline{m})v) \otimes e^\alpha)_{h-|\underline{m}|} y \ (\mathit{mod}\, G(\alpha + \beta)) \text{ and} \tag{545}$$

$$x_h(E_\beta^k w \otimes e^\beta) = \sum_{i=1}^{k} \sum_{\underline{m} \in \mathbb{N}_+^i} \tilde{c}_i^k(\underline{m}) x_{h-|\underline{m}|}((L(\underline{m})w) \otimes e^\beta) \ (\mathit{mod}\, G(\alpha + \beta)). \tag{546}$$

Proof. We prove this statement by induction. Let us assume $k = 1$. We evaluate

$$(E_\alpha v \otimes e^\alpha)_h y \tag{547}$$

$$= \sum_{n=1}^{\infty} (D_\alpha(-n)(L_n v \otimes e^\alpha))_h \, y \tag{548}$$

$$= \sum_{n=1}^{\infty} (-1)^n d_n (L_n v \otimes e^\alpha)_{h-n} y (\mathrm{mod}\, G(\alpha + \beta)) \tag{549}$$

Here we made use of Lemma 3.4.5. This proves the case $k = 1$. Now we are ready to do the induction step by assuming the statement for some k and proving it for $k+1$. During this computation we denote

E_α simply by E. We obtain

$$(E^{k+1}v \otimes e^\alpha)_h y \tag{550}$$

$$= \sum_{m=1}^{\infty} (E^k D(-m) L_m v \otimes e^\alpha)_h y \tag{551}$$

$$= \sum_{m=1}^{\infty} \left(\left\{ D(-m)E^k + \sum_{j=0}^{k-1} p_j^k(-m) D(-m) E^j \right\} L_m v \otimes e^\alpha \right)_h y \tag{552}$$

$$= \sum_{m=1}^{\infty} (-1)^m p_0^k(-m) d(m) (L_m v \otimes e^\alpha)_h y \pmod{G(\alpha+\beta)} \tag{553}$$

$$+ \sum_{m=1}^{\infty} \sum_{j=1}^{k} p_j^k(-m) \left(D(-m) E^j L_m v \otimes e^\alpha \right)_h y \tag{554}$$

$$= \sum_{m=1}^{\infty} (-1)^m p_0^k(-m) d(m) (L_m v \otimes e^\alpha)_h y \tag{555}$$

$$+ \sum_{m=1}^{\infty} \sum_{j=1}^{k} (-1)^m p_j^k(-m) d(m) \left(E^j L_m v \otimes e^\alpha \right)_{h-m} y \pmod{G(\alpha+\beta)} \tag{556}$$

$$= \sum_{m=1}^{\infty} (-1)^m p_0^k(-m) d(m) (L_m v \otimes e^\alpha)_h y \tag{557}$$

$$+ \sum_{m=1}^{\infty} \sum_{j=1}^{k} (-1)^m p_j^k(-m) d(m) \sum_{i=1}^{j} \sum_{\underline{m} \in \mathbb{N}_+^i} c_i^j(\underline{m}) \left(L(\underline{m}) L_m v \otimes e^\alpha \right)_{h-|\underline{m}|-m} y \pmod{G(\alpha+\beta)} \tag{558}$$

$$= \sum_{m=1}^{\infty} (-1)^m p_0^k(-m) d(m) (L_m v \otimes e^\alpha)_h y \tag{559}$$

$$+ \sum_{j=1}^{k} \sum_{i=1}^{j} \sum_{m=1}^{\infty} \sum_{\underline{m} \in \mathbb{N}_+^i} (-1)^m p_j^k(-m) d(m) c_i^j(\underline{m}) \left(L(\underline{m}) L_m v \otimes e^\alpha \right)_{h-|\underline{m}|-m} y \pmod{G(\alpha+\beta)} \tag{560}$$

$$= \sum_{m=1}^{\infty} (-1)^m p_0^k(-m) d(m) (L_m v \otimes e^\alpha)_h y \tag{561}$$

$$+ \sum_{i=2}^{k+1} \sum_{j=i}^{k+1} \sum_{\underline{m'} \in \mathbb{N}_+^i} (-1)^{m_1} p_{j-1}^k(-m_1) d(m_1) c_{i-1}^{j-1}(\underline{m}) \left(L(\underline{m'}) v \otimes e^\alpha \right)_{h-|\underline{m'}|} y \pmod{G(\alpha+\beta)} \tag{562}$$

$$= \sum_{m=1}^{\infty} (-1)^m p_0^k(-m) d(m) (L_m v \otimes e^\alpha)_h y \tag{563}$$

$$+ \sum_{i=2}^{k+1} \sum_{\underline{m'} \in \mathbb{N}_+^i} \sum_{j=i}^{k+1} \left\{ (-1)^m p_{j-1}^k(-m) d(m) c_{i-1}^{j-1}(\underline{m}) \right\} \left(L(\underline{m'}) v \otimes e^\alpha \right)_{h-|\underline{m'}|} y \pmod{G(\alpha+\beta)} \tag{564}$$

$$= \sum_{i=1}^{k+1} \sum_{\underline{m} \in \mathbb{N}_+^i} c_i^{k+1}(\underline{m}) ((L(\underline{m}) v) \otimes e^\alpha)_{h-|\underline{m}|} y \pmod{G(\alpha+\beta)} \tag{565}$$

We used the previous results of this section and changed the indices several times. The second equation follows analogously. □

Definition 3.4.8. For $k \in \mathbb{N}_+$ and $n \in \mathbb{N}$ we set

$$\mathfrak{i}_n^k = \sum_{i=1}^{k} \sum_{\underline{m} \in B^i(n)} c_i^k(\underline{m}) L(\underline{m}), \tag{566}$$

$$\mathfrak{j}_n^k = \sum_{i=1}^{k} \sum_{\underline{m} \in B^i(n)} \tilde{c}_i^k(\underline{m}) L(\underline{m}) \text{ and} \tag{567}$$

$$\mathfrak{i}_n^0 = \mathfrak{j}_n^0 = \delta_{0,n}\mathrm{Id}. \tag{568}$$

As usual for $v, w \in V$ we denote the evaluation of $\mathfrak{i}_n^k$ and $\mathfrak{j}_n^k$ at these vectors by $\mathfrak{i}_n^k(v)$ and $\mathfrak{j}_n^k(w)$ respectively. Assume once again that $\alpha, \beta, \alpha + \beta \notin e^\perp$. For $k \in \mathbb{N}$, $h \in \mathbb{Z}$, $v, w \in V$, $x \in K(\alpha)$ and $y \in K(\beta)$ we have

$$(E_\alpha^k v \otimes e^\alpha)_h y = \sum_{n=0}^{\infty} (\mathfrak{i}_n^k(v) \otimes e^\alpha)_{h-n} y \text{ (mod } G(\alpha+\beta)) \text{ and} \tag{569}$$

$$x_h(E_\beta^k w \otimes e^\beta) = \sum_{n=0}^{\infty} x_{h-n}(\mathfrak{j}_n^k(w) \otimes e^\beta) \text{ (mod } G(\alpha+\beta)). \tag{570}$$

This is clearly just a reformulation of the statement of Lemma 3.4.7 by use of the operators $\mathfrak{i}_n^k$ and $\mathfrak{j}_n^k$. We are now ready to come to the first main result of this thesis. This is the explicit evaluation of the maps $\{\cdot,\cdot\}_{\alpha,\beta}$ in terms of vertex algebra operations. In the special case $V = V^\natural$, where we have $\mathfrak{g}(V^\natural) = \mathfrak{m}^\natural$, this answers question (4) in section 15 of [Bor92].

Theorem 3.4.9. *Assume $\alpha, \beta, \alpha+\beta \notin e^\perp$. We may put $n = 1 - \frac{\alpha^2}{2}$ and $m = 1 - \frac{\beta^2}{2}$. For $v \in V_n$ and $w \in V_m$ we have*

$$\{v, w\}_{\alpha,\beta} = \tag{571}$$

$$\frac{\epsilon(\alpha,\beta)}{n!m!} \sum_{r=0}^{n} \sum_{s=0}^{m} \sum_{l_1,l_2=0}^{\infty} \sum_{k \in \mathbb{Z}} \sum_{\underline{m} \in B(l_1+l_2+k-(\alpha,\beta))} S_r[n] S_s[m] c(\underline{m}) L(-\underline{m}) (\mathfrak{i}_{l_1}^r(v)_k \mathfrak{j}_{l_2}^s(w)). \tag{572}$$

Proof. For elements $v \in V_n$ and $w \in V_m$ we may use (453) to obtain

$$(\mathcal{P}v \otimes e^\alpha)_0(\mathcal{P}w \otimes e^\beta) = \frac{1}{n!m!} \sum_{r=0}^{n} \sum_{s=0}^{m} S_r[n] S_s[m] E_\alpha^s(v \otimes e^\alpha)_0 E_\beta^r(w \otimes e^\beta). \tag{573}$$

Remember that it is enough to evaluate this expression modulo $G(\alpha+\beta)$, since this is precisely the kernel of $\mathcal{P}'$. As a first step in this direction we may use (569) and (570) to obtain

$$E_\alpha^s(v \otimes e^\alpha)_0 E_\beta^r(w \otimes e^\beta) = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} (\mathfrak{i}_{l_1}^r(v) \otimes e^\alpha)_{-l_1-l_2} (\mathfrak{j}_{l_2}^s(w) \otimes e^\beta) (\text{mod } G(\alpha+\beta)). \tag{574}$$

Using (308) and (471) we compute

$$(\mathfrak{i}_{l_1}^r(v) \otimes e^\alpha)_{-l_1-l_2} (\mathfrak{j}_{l_2}^s(w) \otimes e^\beta) = \epsilon(\alpha,\beta) \sum_{k \in \mathbb{Z}} \mathfrak{i}_{l_1}^r(v)_k \mathfrak{j}_{l_2}^s(w) \otimes S^+_{k+l_1+l_2-(\alpha,\beta)}(\alpha) e^{\alpha+\beta}. \tag{575}$$

Now we are in a decent spot because we can use Proposition 3.3.5 to compute the projection $\mathcal{P}'$ of expressions like this. Putting all those formulas together the statement follows. ☐

4 From vertex operator algebras to reflective modular forms

In [Sch93] Schellekens studied the Lie algebra structure of the weight-1 subspaces V_1 of holomorphic vertex operator algebras V of central charge $c = 24$. His remarkable result was that there are just 71 possibilities for this Lie algebra. Furthermore he determined in each case the corresponding decomposition of V into affine modules. Subsequently he conjectured that for each such Lie algebra there exists a unique holomorphic vertex operator algebra with central charge $c = 24$, that realizes this Lie algebra. For the cases $V_1 \neq 0$ this conjecture is no proved by work of several authors. In [CKS07] Creutzig, Klauer and Scheithauer studied holomorphic vertex operator algebras of central charge $c = 24$ with $V_1 = A^r_{p-1,p}$ for $p = 2, 3, 5, 7$ and $r = \frac{48}{(p-1)(p+1)}$ in more detail. Using the explicitly known decomposition in affine modules of V they extend the *coroot lattice* of V_1 by a suitable isotropic subgroup of simple currents. Then they show that this lattice is the dual lattice of the root lattice of the Lie algebra of physical states $\mathfrak{g}(V)$, which turns out to be a generalized Kac-Moody algebra. Finally they find that the automorphic product associated to a *character* of V yields precisely the denominator identity of this Lie algebra and is reflective. In this section we essentially want to repeat this program for a holomorphic vertex operator algebra V of central charge $c = 24$ with semi-simple Lie algebra V_1. The main difference to [CKS07] is that we don't use any explicitly given decomposition of V into affine modules. As a consequence we will have to replace several explicit computations in [CKS07] by abstract arguments. This makes our result independent of Schellekens' list.

4.1 Cominimal simple currents of affine vertex operator algebras

Let $\mathfrak{g}$ be a simple Lie algebra and take $k \in \mathbb{Z}_{>0}$. In this subsection we give an introduction to *cominimal* simple currents of the simple affine vertex operator algebra $L_{\hat{\mathfrak{g}}}(k\Lambda_0)$. These are essentially simple currents, whose action on the irreducible modules is induced from an *outer automorphism* of the extended Dynkin diagram of $\hat{\mathfrak{g}}$. We furthermore discuss a well-known but important symmetry of the $\mathcal{S}$-matrix of $L_{\hat{\mathfrak{g}}}(k\Lambda_0)$ under those simple currents. This will be an important tool to show that the group of cominimal simple currents can be viewed as a subgroup of the discriminant form $D(M(k))$, where as usual M is a coroot lattice of $\mathfrak{g}$. For simplicity we will usually just write $L(k\Lambda_0)$ for $L_{\hat{\mathfrak{g}}}(k\Lambda_0)$ and use all notations from the chapters 2.5 and 2.8. Most of the content in this section is well-known. See in particular [KP84], [KW88] and [CKS07]. See also the fifth section in [Fuc95]. The best source for the previously mentioned symmetry of the $\mathcal{S}$-matrix of $L_{\hat{\mathfrak{g}}}(k\Lambda_0)$ under the action of the cominimal simple currents is [FMS12]. The irreducible modules of $L(k\Lambda_0)$ are parametrized by $\Lambda \in P^k_+$ with $\Lambda(d) = 0$. Usually we will write $P^k_+(\operatorname{mod} \mathbb{C}\delta)$ for the set of such elements. This keeps implicit that we always chose the unique representant which satisfies $\Lambda(d) = 0$. In (230) we introduced a set J with $j \in J$ if and only if the corresponding Coxeter label satisfies $a_j = 1$.

Definition 4.1.1. An irreducible module $L(\Lambda)$ of $L(k\Lambda_0)$ will be called *cominimal simple current* if $\Lambda = k\Lambda_j$ for some $j \in J$.

Following Proposition 5.1. in [DLM96] it is clear that each cominimal simple current is a simple current of $L(k\Lambda_0)$. In fact it turns out that almost all simple currents of affine vertex operator algebras of $L(k\Lambda_0)$ are cominimal simple currents. More precisely, the only simple affine vertex operator algebra which has a simple current that is not cominimal is $L_{\hat{E}_8}(2\Lambda_0)$. See for example [Fuc95]. Yet in the following we will just work with cominimal simple currents. Using the Table Aff 1 in [Kac90] we get the following list of all cominimal simple currents:

$$A_{n,k} : k\Lambda_0, \cdots, k\Lambda_n \tag{576}$$
$$B_{n,k} : k\Lambda_0, k\Lambda_1 \tag{577}$$
$$C_{n,k} : k\Lambda_0, k\Lambda_n \tag{578}$$
$$D_{n,k} : k\Lambda_0, k\Lambda_1, k\Lambda_{n-1}, k\Lambda_n \tag{579}$$
$$E_{6,k} : k\Lambda_0, k\Lambda_1, k\Lambda_5 \tag{580}$$
$$E_{7,k} : k\Lambda_0, k\Lambda_6 \tag{581}$$
$$E_{8,k} : k\Lambda_0 \tag{582}$$
$$F_{4,k} : k\Lambda_0 \tag{583}$$
$$G_{2,k} : k\Lambda_0 \tag{584}$$

In Theorem 5.10 of [DLM96] it is proved that the direct sum of all cominimal simple currents of an affine vertex operator algebra $L(k\Lambda_0)$ can be equipped with the structure of an abelian intertwining algebra. In the following we want to study the fusion product of a cominimal simple current with any other irreducible module of $L(k\Lambda_0)$. The coroot lattice $\overline{\check{Q}}$ of $\mathfrak{g}$ is contained in $\mathfrak{h}$ but in section 2.5 we introduced the map $\nu : \mathfrak{h} \to \mathfrak{h}^*$ which maps $\overline{\check{Q}}$ to the lattice M with which we identify the coroot lattice $\overline{\check{Q}}$ in the following. We therefore write M instead of $\check{Q}$ and consider it as a subset of $\mathfrak{h}^*$. The dual lattice M' of M is just the weight lattice $\overline{P}$, i.e. the $\mathbb{Z}$-span of the fundamental weights $\overline{\Lambda_j}$ of $\mathfrak{g}$. As in section 2.5 we indicate by a symbol $\bar{\cdot}$ that the object $\cdot$ corresponds to a simple Lie algebra $\mathfrak{g}$ and not to an affine Kac-Moody algebra $\hat{\mathfrak{g}}$. Clearly for a non-negative integer $k \in \mathbb{Z}_{>0}$ we have

$$P^k = k\Lambda_0 + \overline{P} + \mathbb{C}\delta. \tag{585}$$

This allows us to identify an element $\Lambda \in P^k \pmod{\mathbb{C}\delta}$ with an element $\overline{\Lambda}$ in $\overline{P}$. We consider the lattice $M(k)$ whose dual lattice is given by $\frac{1}{k}\overline{P}(k)$. We define a map

$$\iota : \overline{P} \to \frac{1}{k}\overline{P}(k),\ \overline{\Lambda} \mapsto \frac{1}{k}\overline{\Lambda}. \tag{586}$$

which is an isomorphism of free $\mathbb{Z}$-modules. Clearly we can extend this map to P^k simply my mapping $\Lambda \in P^k$ to $\iota(\overline{\Lambda})$. We denote this map by ι as well. We can check that for $\overline{\lambda_1}, \overline{\lambda_2} \in \overline{P}$ this map satisfies

$$(\iota(\overline{\lambda_1}), \iota(\overline{\lambda_2}))_{M(k)'} = \frac{(\overline{\lambda_1}, \overline{\lambda_2})}{k}. \tag{587}$$

In the last equation we equipped the bilinear form $(\cdot,\cdot)_{M(k)'}$ of $M(k)'$ with a subscript $\cdot_{M(k)'}$ to remind us that this bilinear form has to be considered rescaled by k. In fact ι induces an isomorphism of discriminant forms

$$\iota : \overline{P}/kM \to \frac{1}{k}\overline{P}(k)/M(k) = D(M(k)). \tag{588}$$

Notice that the quadratic form of $\overline{P}/kM$ is given by

$$q_{\overline{P}/kM}([\overline{\lambda}]) = \frac{(\overline{\lambda}, \overline{\lambda})}{2k} (\text{mod}\ \mathbb{Z}), \tag{589}$$

for elements $\overline{\lambda} \in \overline{P}$. A special property of the cominimal simple currents is that for every $i \in J$ we have

$$h(k\Lambda_i) = \frac{(k\Lambda_i + 2\rho, k\Lambda_i)}{2(k+\check{h})} = \frac{(k\Lambda_i, k\Lambda_i)}{2(k+\check{h})} + \frac{k(2\rho, k\Lambda_i)}{2(k+\check{h})k} \tag{590}$$
$$= \frac{k(k\Lambda_i, k\Lambda_i)}{2(k+\check{h})k} + \frac{\check{h}(\Lambda_i, \Lambda_i)}{2(k+\check{h})k} = \frac{(k\Lambda_i, k\Lambda_i)}{2k} = k\frac{(\Lambda_i, \Lambda_i)}{2} \tag{591}$$
$$= \frac{(\iota(k\Lambda_i), \iota(k\Lambda_i))_{M(k)'}}{2}. \tag{592}$$

In this computation we make use of Theorem 2.5.11. We can use ι to define a map by

$$e_k : P^k \to D(M(k)),\ \Lambda \mapsto [\iota(\Lambda)] = \left[\frac{1}{k}\overline{\Lambda}\right]. \tag{593}$$

Proposition 4.1.2. *For every positive integer $k \in \mathbb{Z}_{>0}$ the map $e_k|_{P^k_+(mod\, \mathbb{C}\delta)}$ is injective.*

Proof. We take $\Lambda^1, \Lambda^2 \in P^k_+$ with $\Lambda^1(d) = \Lambda^2(d) = 0$ and have for $j = 1, 2$ that

$$\Lambda^j = \sum_{i=0}^{n} n_i^j \Lambda_i \tag{594}$$

with $n_i^j \geq 0$ and

$$k = n_0^j + \sum_{i=1}^{n} \check{a}_i n_i^j. \tag{595}$$

Assume now that we have $e_k(\Lambda^1) = e_k(\Lambda^2)$. This implies

$$\frac{1}{k}\overline{\Lambda^1} - \frac{1}{k}\overline{\Lambda^2} = \frac{1}{k}\sum_{i=1}^{n}(n_j^1 - n_j^2)\overline{\Lambda_i} = \sum_{i=1}^{n} r_i \check{\alpha}_i \in \check{\overline{Q}}. \tag{596}$$

By pairing with $\check{\alpha}_j$ this yields

$$n_j^1 - n_j^2 = k\sum_{i=1}^{n} r_i(\check{\alpha}_i, \check{\alpha}_j) \in k\mathbb{Z}. \tag{597}$$

Clearly we have $-k \leq n_j^1 - n_j^2 \leq k$ for all $j = 1, \cdots, n$. So we get $n_j^1 - n_j^2 = -k, 0, k$ for all j. If we have $n_j^1 - n_j^2 = 0$ for all j the proof is done so lets assume that there exists j_0 with $n_{j_0}^1 - n_{j_0}^2 = k$, i.e. we have $n_{j_0}^1 = k$ and $n_{j_0}^2 = 0$. This is just possible if $\check{a}_{j_0} = 1$ and $n_l^1 = 0$ for all $l \neq j_0$. Now $n_l^1 - n_l^2$ either vanishes for all $l \neq j_0$ or there is a unique l_0 with $n_{l_0}^1 - n_{l_0}^2 = -k$ in which case we get $n_{l_0}^2 = k$. Then we must have $\check{a}_{l_0} = 1$ and $n_l^2 = 0$ for all $l \neq 0$. Altogether we find that the difference $\frac{1}{k}\overline{\Lambda^1} - \frac{1}{k}\overline{\Lambda^2}$ is either of the form $\overline{\Lambda_l}$ for l with $\check{a}_l = 1$ or of the form $\overline{\Lambda_i} - \overline{\Lambda_j}$ for i and j with $\check{a}_i = \check{a}_j = 1$. A case-by-case inspection shows that such elements can't be contained in $\check{\overline{Q}}$ unless $l = 0$ or $i = j$. ☐

The Proposition 4.1.2 allows us to view the set of irreducible modules of $L(k\Lambda_0)$ as a subset of the discriminant form $D(M(k))$, which we will do in the following. Next we introduce a certain group of *outer symmetries* of $\hat{\mathfrak{g}}$. This exposition is based on [KP84] and [KW88]. We denote the *coweight lattice* of $\mathfrak{g}$ by $\check{\overline{P}}$. This is the lattice in $\mathfrak{h}$ which is spanned by the coweights $\check{\overline{\Lambda}}_i$, defined by

$$\langle \check{\overline{\Lambda}}_i, \alpha_j \rangle = \delta_{i,j}. \tag{598}$$

This lattice can be mapped into $\mathfrak{h}^*$ by ν and we denote its image by $L_0 := \nu(\check{\overline{P}})$. The corresponding group of translations is

$$T_0 = \{t_\alpha : \alpha \in L_0\}. \tag{599}$$

Here t_α is just the translation defined by (178). We consider the group $W_0 := T_0 \ltimes \overline{W}$. The Weyl group $W = T \ltimes \overline{W}$ is obviously a subgroup of W_0, because of $T \subset T_0$. As in section §6.6 of [Kac90] and section 4.8 of [KP84] we can introduce an *affine action* of W_0 on $\mathfrak{h}^*_{\mathbb{R}}$ by

$$\text{af}(w) = w\ \forall w \in \overline{W} \text{ and} \tag{600}$$

$$\text{af}(t_\alpha)(\lambda) = \lambda + \alpha\ \forall \alpha \in L_0, \lambda \in \overline{\mathfrak{h}}^*_{\mathbb{R}}. \tag{601}$$

For $\lambda \in \hat{\mathfrak{h}}^*_{\mathbb{R}}$ with $\langle \lambda, K \rangle = 1$ we have

$$\overline{w(\lambda)} = \text{af}(w)(\overline{\lambda}). \tag{602}$$

An important subgroup of W_0 is given by

$$W_0^+ = \{\sigma \in W_0 : \sigma\Delta_+ = \Delta_+\}. \tag{603}$$

An important property is $W \cap W_0^+ = \{1\}$. Let $\mathrm{Aut}(\check{\Pi})$ be the group of symmetries of the Dynkin diagram of $\hat{\mathfrak{g}}$. This group acts naturally on P by $\sigma\Lambda_i = \Lambda_{\sigma(i)}$ for $\sigma \in \mathrm{Aut}(\check{\Pi})$. It furthermore turns out that W_0^+ acts as a group of permutations on the set of fundamental weights. More precisely for every $w \in W_0^+$ there exists an element $\sigma_w \in \mathrm{Aut}(\check{\Pi})$ such that $w(\Lambda_i) = \Lambda_{\sigma_w(i)}(\mathrm{mod}\,\mathbb{C}\delta)$ for all fundamental weights Λ_i. The map $w \mapsto \sigma_w$ defines a natural embedding of W_0^+ into $\mathrm{Aut}(\check{\Pi})$ and it maps W_0^+ onto the unique normal subgroup of $\mathrm{Aut}(\check{\Pi})$ which is isomorphic to the abelian group $\overline{P}/\overline{Q}$. See Proposition 1.3 in [KW88] for details. We clearly have $a_i = a_{\sigma_w(i)}$ and $\check{a}_i = \check{a}_{\sigma_w(i)}$ for all $i = 0, \cdots, n$. Therefore the action of W_0^+ preserves $P_+(\mathrm{mod}\,\mathbb{C}\delta)$ and $P_+^k(\mathrm{mod}\,\mathbb{C}\delta)$. This can be considered as an action on the set of irreducible modules of the vertex operator algebra $L(k\Lambda_0)$. Those permutations induce furthermore a simply transitive action of W_0^+ on the set J and this action can be described equivalently by the action of $\mathrm{af}(W_0^+)$ on the set

$$\mathcal{J} = \{\overline{\Lambda_j} : j \in J\}. \tag{604}$$

See Proposition 4.27 in [KP84] for a proof. We obtain a natural bijection

$$W_0^+ \to \mathcal{J},\ w \mapsto \mathrm{af}(w)0. \tag{605}$$

We denote the preimage of $\overline{\Lambda_i}$ for $i \in J$ under this map by w_i. Using that for $w = t_\alpha\overline{w} \in W_0^+$ we clearly have $\mathrm{af}(w)0 = \mathrm{af}(t_\alpha\overline{w})0 = \alpha$ we get that w_i can be expressed as $t_{\overline{\Lambda_i}}\overline{w_i}$ for a uniquely determined element $\overline{w_i} \in \overline{W}$. This shows that we can associate to each cominimal simple current $S = L(k\Lambda_i)$ with $i \in J$ a unique element $w_i \in W_0^+$. We will denote it also by $w_S \in W_0^+$ and the corresponding permutation will be denoted by σ_S. The following lemma is based on Proposition 5.1 in [CKS07] and the proofs are almost identical.

Lemma 4.1.3. *Let $\sigma_S \in W_0^+$ be the symmetry of the Dynkin diagram of $\hat{\mathfrak{g}}$, corresponding to a cominimal simple current S of $L(k\Lambda_0)$, then we have*

$$c^{\sigma_S(\Lambda)}_{\sigma_S(\lambda)}(\tau) = c^\Lambda_\lambda(\tau) \text{ for all } \Lambda \in \hat{P}^k_+, \lambda \in \hat{\mathfrak{h}}^*. \tag{606}$$

Proof. First we extend $\sigma_S \in W_0^+ \subset \mathrm{GL}(\hat{\mathfrak{h}}^*)$ to an automorphism of $\hat{\mathfrak{g}}$. Now we can define a *new $\hat{\mathfrak{g}}$-module* structure on $L_{\mathfrak{g}}(\sigma_S(\Lambda))$ by acting with σ_S on $\hat{\mathfrak{g}}$ first. Identifying $\hat{\mathfrak{g}}$ with itself along σ_S we find that $L_{\mathfrak{g}}(\sigma_S(\Lambda))$ and $L_{\mathfrak{g}}(\Lambda)$ can be identified as well. Essentially we just relabeled the simple roots along the symmetry σ_S. The statement is a direct consequence of this. □

The $\mathcal{S}$-matrix of the vertex operator algebra $L(k\Lambda_0)$ has a very important transformation property under this symmetry, more precisely for $\Lambda, M \in P_+^k(\mathrm{mod}\,\mathbb{C}\delta)$ and any $j \in J$ we have

$$\mathcal{S}_{w_j(\Lambda),M} = e^{-2\pi i(\overline{\Lambda_j},\overline{M})}\mathcal{S}_{\Lambda,M} = e^{-2\pi i(e_k(k\Lambda_j),e_k(M))_{D(M(k))}}\mathcal{S}_{\Lambda,M}. \tag{607}$$

See equation (14.255) in [FMS12] and the proof therein for the first equation. The second equation is clear. We denote the weight which corresponds to the fusion product of the cominimal simple current S with an irreducible module $L(\Lambda)$ by $S \boxtimes \Lambda$.

Proposition 4.1.4. *Take $\Lambda \in P_+^k(\mathrm{mod}\,\mathbb{C}\delta)$ and $j \in J$. The fusion product of the cominimal simple current $S = L(k\Lambda_j)$ with the irreducible module $L(\Lambda)$ is given by*

$$L(k\Lambda_j) \boxtimes_{L(k\Lambda_0)} L(\Lambda) \cong L(w_j(\Lambda)). \tag{608}$$

Proof. Take any $M \in P_+^k$ with $M(d) = 0$, such that $L(M)$ is an irreducible module of $L(k\Lambda_0)$. Using (345) we obtain

$$\mathcal{S}_{k\Lambda_j,M}\mathcal{S}_{\Lambda,M} = \mathcal{S}_{k\Lambda_0,M}\mathcal{S}_{k\Lambda_j\boxtimes\Lambda,M}. \tag{609}$$

Using (607) we get

$$\mathcal{S}_{k\Lambda_j,M}\mathcal{S}_{\Lambda,M} = \mathcal{S}_{k\Lambda_0,M}\mathcal{S}_{w_j(\Lambda),M}. \tag{610}$$

Because of $\mathcal{S}_{L(k\Lambda_0),L(M)} \neq 0$ we obtain for all M that

$$\mathcal{S}_{k\Lambda_j\boxtimes\Lambda,M} = \mathcal{S}_{w_j(\Lambda),M}. \tag{611}$$

Since the matrix $\mathcal{S}$ is invertible it can't have two common rows. Therefore $L(k\Lambda_j) \boxtimes_{L(k\Lambda_0)} L(\Lambda)$ and $L(w_j(\Lambda))$ must correspond to the same row. This implies the statement of the proposition. □

In Proposition 2.7.18 we saw that the set of simple currents of a strongly rational vertex operator algebra carries the structure of an abelian group. Clearly the fusion product also equips the set of cominimal simple currents $\mathcal{C}(k\Lambda_0)$ with an abelian group structure.

Corollary 4.1.5. *The map $A : W_0^+ \to \mathcal{C}(k\Lambda_0),\ w_s \mapsto s$ is an isomorphism of abelian groups.*

Proof. This map is clearly bijective since the map defined in (605) is bijective. Take cominimal simple currents $S_1 = L(k\Lambda_i)$ and $S_2 = L(k\Lambda_j)$ in $\mathcal{C}(k\Lambda_0)$ with $i, j \in J$. By use of Proposition 4.1.4 we get

$$S_1 \boxtimes S_2 = L(k\Lambda_i) \boxtimes L(k\Lambda_j) = w_i(k\Lambda_j) = w_i w_j(k\Lambda_0). \tag{612}$$

This clearly implies the statement. □

We can use (346) to obtain for $i \in J$ and $\Lambda \in P_+^k$ that

$$\frac{\mathcal{S}_{k\Lambda_i,\Lambda}}{\mathcal{S}_{k\Lambda_0,k\Lambda_0}} = \mathrm{qdim}(L(\Lambda))\mathcal{T}_{k\Lambda_i}\mathcal{T}_\Lambda\mathcal{T}^{-1}_{k\Lambda_i\boxtimes\Lambda}\mathcal{T}^{-1}_{k\Lambda_0}. \tag{613}$$

This directly implies

$$\frac{\mathcal{S}_{k\Lambda_i,\Lambda}}{\mathcal{S}_{k\Lambda_0,\Lambda}} = e^{-2\pi i(h(k\Lambda_i\boxtimes\Lambda)-h(k\Lambda_i)-h(\Lambda))} \tag{614}$$

and we get by use of (607) that

$$(e_k(k\Lambda_i), e_k(M))_{D(M(k))} = h(k\Lambda_i \boxtimes \Lambda) - h(k\Lambda_i) - h(\Lambda) (\mathrm{mod}\ \mathbb{Z}). \tag{615}$$

The finite Weyl group $\overline{W}$ preserves the coroot lattice $\check{\overline{Q}}$ and acts therefore on the discriminant form $D(M(k))$. For $i \in J$ and a fundamental reflection r_j we have

$$r_j(\overline{\Lambda_i}) = \overline{\Lambda_i} - \langle\overline{\Lambda_i}, \check{\alpha}_j\rangle\alpha_j = \overline{\Lambda_i} - \delta_{i,j}\check{\alpha}_j. \tag{616}$$

This implies that the images of the simple currents $e_k(k\Lambda_i) = [\overline{\Lambda_i}]$ for $i \in J$ are fixed-points of the action of $\overline{W}$ on $D(M(k))$. Using this we observe that for all $i, j \in J$ we have

$$e_k(w_i(k\Lambda_j)) = [\mathrm{af}(w_i)(\overline{\Lambda_j})] = [\overline{\Lambda_i} + \overline{\Lambda_j}]. \tag{617}$$

As a consequence we obtain that the restriction of e_k to the set of cominimal simple currents $\mathcal{C}(k\Lambda_0)$ defines an embedding of abelian groups into $D(M(k))$. Of course we use Proposition 4.1.2 to see that this map defines an embedding, i.e. is injective.

Definition 4.1.6. We denote the subgroup $e_k(\mathcal{C}(k\Lambda_0))$ of $D(M(k))$ by G and call it the *group of cominimal simple currents* in $D(M(k))$.

The affine action of the group W_0 on $\mathfrak{h}_\mathbb{R}^*$ can clearly by used to define an action of W_0 on $D(M(k))$ by

$$w[\lambda] = [\mathrm{af}(w)\lambda] \tag{618}$$

for $w \in W_0$ and $\lambda \in M(k)'$. Take $\Lambda \in P_+^k(\mathrm{mod}\ \mathbb{C}\delta)$, then we have for $w_i \in W_0^+$ that

$$w_i e_k(\Lambda) = e_k(k\Lambda_i \boxtimes \Lambda) = [\overline{w_i}(\overline{\Lambda_i} + \iota(\Lambda))]. \tag{619}$$

This can be checked by direct computation.

4.2 The affine substructure of a vertex operator algebra and Jacobi forms

In this subsection we consider a holomorphic vertex operator algebra V of central charge $c = 24$ such that the Lie algebra $\mathfrak{g} = V_1$ is semi-simple. We may decompose V as a module under the affine vertex operator algebra $V(\mathfrak{g})$, generated by V_1. It will turn out that a suitable character χ_V of V is a Jacobi form of lattice index and that the components of the corresponding vector-valued modular form f_V have nice properties at $i\infty$. We use all the notations and assumptions made in section 2.9. In particular we decompose $\mathfrak{g}$ into simple Lie algebras $\mathfrak{g}_i$ and find positive integers k_i such that the restriction of the invariant bilinear form $\langle \cdot, \cdot \rangle$ of V, which is scaled such that $\langle \mathbf{1}, \mathbf{1} \rangle = -1$, satisfies

$$\langle \cdot, \cdot \rangle|_{\mathfrak{g}_i \times \mathfrak{g}_i} = k_i (\cdot, \cdot)_i. \tag{620}$$

Here $(\cdot, \cdot)_i$ is the invariant bilinear form of $\mathfrak{g}_i$, which is scaled such that long roots have length 2. We denote this decomposition as in (378) and we denote the vertex operator subalgebra generated by V_1 as usual by $V(\mathfrak{g})$. It is strongly rational and isomorphic to the tensor product of the affine vertex operator algebras of level k_i generated by the $\mathfrak{g}_i$. This is (380). We get the usual finite decomposition (381) which is

$$V = \bigoplus_{\Lambda^1, \cdots, \Lambda^r} m(\Lambda^1, \cdots, \Lambda^r) L_{\hat{\mathfrak{g}}_1}(\Lambda^1) \otimes \cdots \otimes L_{\hat{\mathfrak{g}}_r}(\Lambda^r). \tag{621}$$

Here Λ^i runs through all of $P_+^{k_i} (\text{mod } \mathbb{C}\delta)$, i.e. the set of all dominant weights of level k_i of $\hat{\mathfrak{g}}_i$. We denote the set of all modules appearing in this decomposition of V by M_V, i.e. we have

$$M_V = \{(\Lambda^1, \cdots, \Lambda^r) : m(\Lambda^1, \cdots, \Lambda^r) \neq 0\}. \tag{622}$$

In the following we will mostly just write $\Lambda = (\Lambda^1, \cdots, \Lambda^r)$ for an irreducible module but we also write $L(\Lambda)$ occasionally to make clear that we consider modules and not just weights. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ can be decomposed as

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r, \tag{623}$$

where $\mathfrak{h}_i$ is the Cartan subalgebra of $\mathfrak{g}_i$ and within each $\mathfrak{h}_i$ we denote the coroots of $\mathfrak{g}_i$ by $\check{\alpha}_{i,j}$ with $1 \leq j \leq \dim(\mathfrak{h}_i) = d_i$. We order those coroots in an obvious way and denote them by $\check{\alpha}_l$ for $1 \leq l \leq \dim(\mathfrak{h}) = d$ if we don't want to make use of the fact that they are contained in a particular subspace $\mathfrak{h}_i$. Within $\mathfrak{h}$ we span a lattice L by the coroots $\check{\alpha}_l$ with bilinear form $\langle \cdot, \cdot \rangle$. Clearly this lattice satisfies

$$L = \check{Q}_1(k_1) \oplus \cdots \oplus \check{Q}_r(k_r), \tag{624}$$

where $\check{Q}_i$ is the coroot lattice of $\mathfrak{g}_i$. We call the lattice L the *scaled coroot lattice* of $\mathfrak{g}$ or simply the *coroot lattice* if we want to keep the scaling implicit. This lattice is clearly even and positive-definite. As usual we map the coroot lattice $\check{Q}_i$ into $\mathfrak{h}_i^*$ and denote its image by M_i, such that we can identify L with $M_1(k_1) \oplus \cdots \oplus M_r(k_r)$. We can introduce maps

$$\iota_i : \overline{P_i} \to \frac{1}{k_i} \overline{P_i}(k_i) \tag{625}$$

as in (586) and combine them to a map $\iota : \overline{P} \to L'$. Of course we can extend ι by linearity to a $\mathbb{C}$-linear map $\iota : \mathfrak{h}^* \to L' \otimes \mathbb{C}$. Previously we used the bilinear forms $(\cdot, \cdot)_i$ to identify $\mathfrak{h}_i^*$ with $\mathfrak{h}_i$. In this sense we may view $\overline{P}_i$ as a sublattice of $\mathfrak{h}_i$ and may consider $L' \otimes \mathbb{C}$ as $\mathfrak{h}$ equipped with the bilinear form $\langle \cdot, \cdot \rangle$. Then $\iota : \mathfrak{h}^* \to L' \otimes \mathbb{C} = \mathfrak{h}$ is just the usual identification of $\mathfrak{h}^*$ with $\mathfrak{h}$ by use of $\langle \cdot, \cdot \rangle$. Yet it is clear that ι does not define an isometry for the usual scalar product $(\cdot, \cdot)_i$ of $\overline{P}_i$. Yet for elements $\overline{\Lambda_1} = \sum_i \overline{\Lambda_1^i} \in \overline{P}$ and $\overline{\Lambda_2} = \sum_i \overline{\Lambda_2^i} \in \overline{P}$, with $\overline{\Lambda_1^i}, \overline{\Lambda_2^i} \in \overline{P_i}$, we have

$$\frac{(\overline{\Lambda_1^1}, \overline{\Lambda_2^1})_1}{k_1} + \cdots + \frac{(\overline{\Lambda_1^r}, \overline{\Lambda_2^r})_r}{k_r} = (\iota(\overline{\Lambda_1}), \iota(\overline{\Lambda_2}))_{L'}. \tag{626}$$

This induces furthermore an isomorphism of discriminant forms

$$\iota : \overline{P_1}/k_1M_1 \oplus \cdots \oplus \overline{P_r}/k_rM_r \to D(L). \tag{627}$$

In general we will work with all structures and notations introduced in the sections 2.5, 2.8 and 4.1. We will indicate the corresponding simple component $\mathfrak{g}_i$ by an index $\cdot_i$ or $\cdot^i$ and omit such indices for structures corresponding to $\mathfrak{g}$. We can furthermore define a map

$$e = e_{k_1} \oplus \cdots \oplus e_{k_r} : P^{k_1} \oplus \cdots \oplus P^{k_r} \to D(L) \tag{628}$$

and use it to embed the set of irreducible modules $\mathrm{Irr}(V(\mathfrak{g}))$ into $D(L)$ as a set. This is possible since $e|_{\mathrm{Irr}(V(\mathfrak{g}))}$ is injective, which is a consequence of Proposition 4.1.2. We will call a simple current of $V(\mathfrak{g})$ a *cominimal simple current* if it is a tensor product of cominimal simple currents of each component $L_{\hat{\mathfrak{g}}_i}(k_i\Lambda_0)$. We denote the set of all cominimal simple currents of $V(\mathfrak{g})$ by $\mathcal{C}(V(\mathfrak{g}))$ and observe that the fusion product defines an abelian group structure on this set. Clearly we just have

$$\mathcal{C}(V(\mathfrak{g})) = \mathcal{C}(k_1\Lambda_0) \oplus \cdots \oplus \mathcal{C}(k_r\Lambda_0) \tag{629}$$

as abelian groups. Of course all the structures introduced in section 4.1 can be naturally extended to this group. For a cominimal simple current $S \in \mathcal{C}(V(\mathfrak{g}))$ and an arbitrary irreducible module $\Lambda \in \mathrm{Irr}(V)(\mathfrak{g})$ we can use (615) to get

$$(e(S), e(\Lambda))_{D(L)} = h(S \boxtimes \Lambda) - h(S) - h(\Lambda) (\mathrm{mod}\, \mathbb{Z}). \tag{630}$$

We will denote the set of all cominimal simple currents, which are contained in V, by S_V. This just means $S_V = \mathcal{C}(V(\mathfrak{g})) \cap M_V$.

Definition 4.2.1. We denote the subgroup $e(\mathcal{C}(V(\mathfrak{g})))$ of $D(L)$ by G and call it the *group of cominimal simple currents* of $D(L)$. Furthermore we set $G_V = G \cap e(M_V)$. This is the subset of cominimal simple currents contained in V. Of course we have $e(S_V) = G_V$.

We decompose any vector $v \in \mathfrak{h}$ as $v = v^1 + \cdots + v^r$ with $v^i \in \mathfrak{h}_i$ and we introduce variables $z^i \in \mathbb{C}^{d_i}$ by

$$v^i = \sum_{j=1}^{d_i} z^i_j \check{\alpha}_{ij}. \tag{631}$$

The vector $v \in \mathfrak{h}$ will be considered as an element in $L \otimes \mathbb{C}$ under the identification with the variables $z = z^1 + \cdots + z^r \in \mathbb{C}^d$ for this basis. Yet for $v \in \mathfrak{h}$ and a formal $q = e^{2\pi i \tau}$ we introduce a formal character

$$\chi_V(v, q) = \mathrm{Tr}_V \left(e^{2\pi i v_0} q^{L_0 - 1}\right). \tag{632}$$

The trace of $e^{2\pi i v_0} q^{L_0}$ on $L_{\hat{\mathfrak{g}}_1}(\Lambda^1) \otimes \cdots \otimes L_{\hat{\mathfrak{g}}_r}(\Lambda^r)$ clearly satisfies

$$\mathrm{Tr}_{L_{\hat{\mathfrak{g}}_1}(\Lambda^1)\otimes\cdots\otimes L_{\hat{\mathfrak{g}}_r}(\Lambda^r)} \left(e^{2\pi i v_0} q^{L_0}\right) \tag{633}$$

$$= \mathrm{Tr}_{L_{\hat{\mathfrak{g}}_1}(\Lambda^1)} \left(e^{2\pi i v^1_0} q^{L_0}\right) \cdots \mathrm{Tr}_{L_{\hat{\mathfrak{g}}_r}(\Lambda^r)} \left(e^{2\pi i v^r_0} q^{L_0}\right) \tag{634}$$

$$= \mathrm{Tr}_{L_{\hat{\mathfrak{g}}_1}(\Lambda^1)} \left(e^{2\pi i v^1_0} e^{2\pi i \tau (h(\Lambda^1)\mathrm{Id} - d)}\right) \cdots \mathrm{Tr}_{L_{\hat{\mathfrak{g}}_r}(\Lambda^r)} \left(e^{2\pi i v^r_0} e^{2\pi i \tau (h(\Lambda^r)\mathrm{Id} - d)}\right) \tag{635}$$

$$= q^{h(\Lambda^1)+\cdots+h(\Lambda^r)} \mathrm{Tr}_{L_{\hat{\mathfrak{g}}_1}(\Lambda^1)} \left(e^{2\pi i (v^1_0 - \tau d)}\right) \cdots \mathrm{Tr}_{L_{\hat{\mathfrak{g}}_r}(\Lambda^r)} \left(e^{2\pi i (v^r_0 - \tau d)}\right) \tag{636}$$

$$= q^{m_{\Lambda^1}+\cdots+m_{\Lambda^r}+\frac{1}{24}(c(k_1)+\cdots+c(k_r))} \mathrm{Tr}_{L_{\hat{\mathfrak{g}}_1}(\Lambda^1)} \left(e^{2\pi i (v^1_0 - \tau d)}\right) \cdots \mathrm{Tr}_{L_{\hat{\mathfrak{g}}_r}(\Lambda^r)} \left(e^{2\pi i (v^r_0 - \tau d)}\right) \tag{637}$$

$$= q \chi_{\Lambda^1}(z^1, \tau, 0) \cdots \chi_{\Lambda^r}(z^r, \tau, 0) \tag{638}$$

Here $c(k_i)$ is the central charge of $L_{\hat{\mathfrak{g}}_i}(k_i\Lambda_0)$ therefore we have $c(k_1) + \cdots + c(k_r) = 24$. And we made use of $h(\Lambda^i) = m_{\Lambda^i} + \frac{1}{24} c(k_i)$. See equation (12.8.12) in [Kac90] for this. As a direct consequence we get

$$\chi_V(v, q) = \sum_{\Lambda^1,\cdots,\Lambda^r} m(\Lambda^1, \cdots, \Lambda^r) \chi_{\Lambda^1}(z^1, \tau, 0) \cdots \chi_{\Lambda^r}(z^r, \tau, 0). \tag{639}$$

Notice that this was so far a computation of formal series. Yet in section 2.5 we saw that the normalized characters $\chi_{\Lambda^i}(z^i, \tau, 0)$ define holomorphic functions on $\mathfrak{h} \times \mathbb{H}$. Therefore the formal character $\chi_V(v, q)$ defines a holomorphic function on $(L \otimes \mathbb{C}) \times \mathbb{H}$ which we denote $\chi_V(z, \tau)$ with the previously discussed identification of $v \in \mathfrak{h}$ and $z \in L \otimes \mathbb{C}$. We call this function the *character* of V and consider the equation (639) as an equation of holomorphic functions instead of an equation of formal expressions. By $z = 0$ we clearly just obtain Zhus character $\mathrm{ch}_V(\tau)$ of V. Characters like χ_V are also called *Jacobi trace functions* and were studied extensively by Krauel and Mason in [KM15]. An obvious application of Theorem 1.1. in their paper yields the relations

$$\chi_V\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = \exp\left(\pi i \frac{c(z, z)}{c\tau + d}\right) \chi_V(z, \tau) \tag{640}$$

$$\chi_V(z + \tau l + h, \tau) = \exp(-\pi i((l, l)\tau + 2(l, h)))\chi_V(z, \tau), \tag{641}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $l, h \in L$. Clearly those relations induce the invariance of χ_V under the integral Jacobi group $\Gamma^J(L)$, defined in (99).

Proposition 4.2.2. *The character* $\chi_V : (L \otimes \mathbb{C}) \times \mathbb{H} \to \mathbb{C}$ *is a nearly holomorphic Jacobi form of weight* 0 *and lattice index* L.

Using Theorem 2.3.1 we know that there exists a vector-valued modular form f_V of weight $-\frac{\dim(\mathfrak{h})}{2}$ for the Weil representation on $\mathbb{C}[D(L)]$ of the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$. This modular form is uniquely determined by the theta decomposition

$$\chi_V(z, \tau) = \sum_{\lambda \in D(L)} f_{V,\lambda}(\tau)\Theta^L_\lambda(z, \tau), \tag{642}$$

where $f_{V,\lambda}(\tau)$ are simply the components of f_V. This means that we have

$$f_V(\tau) = \sum_{\lambda \in D(L)} f_{V,\lambda}(\tau)\mathfrak{e}_\lambda. \tag{643}$$

We call this vector-valued modular form the *vector-valued modular form associated with* V. In the following we want to determine a more explicit description of the components of this modular form. In (221) we described a theta series Θ_λ for every $\lambda \in \hat{\mathfrak{h}}_i^*$. For $\lambda = k_i\Lambda_0 + k_i\overline{\lambda_i} \in P_i^{k_i}$ with $\overline{\lambda_i} \in \frac{1}{k_i}\overline{P_i}$ we can express this theta series with the usual lattice theta series $\Theta^{M_i(k_i)}_{e_{k_i}(\lambda)}$ which was defined in (106). More precisely we have

$$\Theta_\lambda(z^i, \tau, 0) = \Theta^{M_i(k_i)}_{e_{k_i}(k_i\Lambda_0 + k_i\overline{\lambda_i})}(z^i, \tau) = \Theta^{M_i(k_i)}_{[\lambda_i]}(z^i, \tau). \tag{644}$$

Usually we will denote a representant of the class $\lambda_i \in D(M_i(k_i))$ by $k_i\Lambda_0 + k_i\lambda_i \in P^{k_i}$. We will just do this if the exact choice of the representant does not matter. The theta decomposition (217) of the normalized character χ_{Λ^i} for $\Lambda^i \in P_+^{k_i}$ can now be written as

$$\chi_{\Lambda^i}(z^i, \tau) = \sum_{\lambda_i \in D(M_i(k_i))} c^{\Lambda^i}_{k_i\Lambda_0 + k_i\lambda_i}(\tau)\Theta^{M_i(k_i)}_{\lambda_i}(z^i, \tau). \tag{645}$$

We decompose elements $\lambda \in D(L)$ as $\lambda = \lambda_1 + \cdots + \lambda_r$ with $\lambda_i \in D(M_i(k_i))$ and compute

$$\chi_V(z,\tau) = \sum_{\Lambda^1,\cdots,\Lambda^r} m(\Lambda^1,\cdots,\Lambda^r)\chi_{\Lambda^1}(z^1,\tau)\cdots\chi_{\Lambda^r}(z^r,\tau) \tag{646}$$

$$= \sum_{\Lambda^1,\cdots,\Lambda^r} m(\Lambda^1,\cdots,\Lambda^r)\prod_{i=1}^r \sum_{\lambda_i\in D(M_i(k_i))} c^{\Lambda^i}_{k_i\Lambda_0+k_i\lambda_i}(\tau)\Theta^{M_i(k_i)}_{\lambda_i}(z^i,\tau) \tag{647}$$

$$= \sum_{\lambda\in D(L)}\sum_{\Lambda^1,\cdots,\Lambda^r} m(\Lambda^1,\cdots,\Lambda^r)\prod_{i=1}^r c^{\Lambda^i}_{k_i\Lambda_0+k_i\lambda_i}(\tau)\Theta^{M_i(k_i)}_{\lambda_i}(z^i,\tau) \tag{648}$$

$$= \sum_{\lambda\in D(L)}\sum_{\Lambda^1,\cdots,\Lambda^r} m(\Lambda^1,\cdots,\Lambda^r)\prod_{i=1}^r c^{\Lambda^i}_{k_i\Lambda_0+k_i\lambda_i}(\tau)\Theta^{L}_{\lambda}(z,\tau). \tag{649}$$

Since the components $f_{V,\lambda}$ of f_V for $\lambda \in D(L)$ are uniquely determined by the theta decomposition (642) it is clear that we have

$$f_{V,\lambda}(\tau) = \sum_{\Lambda^1,\cdots,\Lambda^r} m(\Lambda^1,\cdots,\Lambda^r)\prod_{i=1}^r c^{\Lambda^i}_{k_i\Lambda_0+k_i\lambda_i}(\tau). \tag{650}$$

For $\lambda = \lambda_1 + \cdots + \lambda_r \in D(L)$ and $\Lambda = (\Lambda^1,\cdots,\Lambda^r) \in \mathrm{Irr}(V(\mathfrak{g}))$ we set

$$c^{\Lambda}_{\lambda}(\tau) = \prod_{i=1}^r c^{\Lambda^i}_{k_i\Lambda_0+k_i\lambda_i}(\tau) \tag{651}$$

and $m(\Lambda) = m(\Lambda^1,\cdots,\Lambda^r)$, such that we can write

$$f_{V,\lambda}(\tau) = \sum_{\Lambda\in M_V} m(\Lambda)c^{\Lambda}_{\lambda}(\tau). \tag{652}$$

Proposition 4.2.3 (Schellekens' equation). *Let $\mathfrak{g}_i$ be any simple component of the Lie algebra V_1, k_i its level and $\check{h}_i$ its dual Coxeter number, then we have*

$$24\check{h}_i = k_i(\mathit{dim}(V_1) - 24). \tag{653}$$

See [Sch93], [vEMS17] and [DM04] for this. The equation (653) is very important in Schellekens' classification of the weight-1 spaces of holomorphic vertex operator algebras of central charge $c = 24$, because it drastically reduces the number of possible Lie algebras. More precisely there are just 221 Lie algebras that solve this equation. They are given in a table in [vELMS21]. We fix a $\lambda \in D(L)$. If $c^{\Lambda^i}_{k_i\Lambda_0+k_i\lambda_i} \neq 0$ then there exists a unique element $\lambda_i' \in k_i\Lambda_0^i + k_i\lambda_i + \mathbb{C}\delta$ which is a maximal weight of Λ^i. We get

$$c^{\Lambda^i}_{k_i\Lambda_0+k_i\lambda_i}(\tau) = c^{\Lambda^i}_{\lambda_i'}(\tau) = q^{m_{\Lambda^i,\lambda_i'}}\sum_{n=0}^{\infty}\mathrm{mult}_{L(\Lambda^i)}(\lambda_i' - n\delta)q^n. \tag{654}$$

Using the equations (168) and (653) and Proposition 2.5.12 we get an upper bound on the pole order of $c^{\Lambda}_{\lambda}(\tau)$ by

$$\mathrm{ord}_{i\infty}\left(\prod_{i=1}^r c^{\Lambda^i}_{k_i\Lambda_0+k_i\lambda_i}(\tau)\right) = -\sum_{i=1}^r m_{\Lambda^i,\lambda_i'} \leq \sum_{i=1}^r \frac{|\rho_i|^2}{2\check{h}_i}\frac{k_i}{k_i+\check{h}_i} = \sum_{i=1}^r \frac{\dim(\mathfrak{g}_i)}{24}\frac{24}{\dim(\mathfrak{g})} = 1. \tag{655}$$

Clearly equality holds if and only if $-m_{\Lambda^i,\lambda_i'} = \frac{|\rho_i|^2}{2\check{h}_i}\frac{k_i}{k_i+\check{h}_i}$ for all $i = 1,\cdots,r$. By use of Proposition 2.5.12 we find that this is possible if and only if each Λ^i satisfies $\Lambda^i = k_i\Lambda^i_j$ for some j with $a_j = 1$ and $k_i\Lambda_0 + k_i\lambda_i = w(k_i\Lambda^i_j)$ for some element $w \in W_i$, which is the affine Weyl group of $\hat{\mathfrak{g}}_i$. Since f_λ is a linear combination of such string functions it is clear that for any $\lambda \in D(L)$ we have

$$\mathrm{ord}_{i\infty}(f_\lambda) \leq 1. \tag{656}$$

For $\alpha \in \mathfrak{h}^*$ we can define

$$V_\alpha = \{v \in V : h_0 v = \alpha(h) v \,\forall h \in \mathfrak{h}\}. \tag{657}$$

Since h_0 preserves the L_0-grading it is clear that we have $(V_\alpha)_n = (V_n)_\alpha$ for all $n \in \mathbb{Z}$. Clearly if we have $(V_\alpha)_n \neq 0$ for some n and α, then α must be contained in the weight lattice, i.e. $\alpha \in \overline{P}$. The character of V_α is defined by

$$\mathrm{ch}(V_\alpha)(\tau) = \mathrm{Tr}_{V_\alpha}\left(q^{L_0 - \frac{c}{24}}\right) = \sum_{n=0}^{\infty} \dim((V_\alpha)_n) q^{n-1}. \tag{658}$$

We take some $\Lambda \in M_V$ and study $L(\Lambda)_\alpha$. We decompose $\alpha = \overline{\alpha}_1 + \cdots + \overline{\alpha}_r$, with $\overline{\alpha}_i \in \mathfrak{h}_i^*$ and set $\alpha_i = k_i \Lambda_0 + \overline{\alpha}_i$. The space $L(\Lambda^i)_{\overline{\alpha_i}}$ clearly has character

$$\mathrm{ch}_{L(\Lambda^i)_{\overline{\alpha_i}}}(\tau) = \mathrm{Tr}_{L(\Lambda^i)_{\overline{\alpha_i}}}\left(q^{L_0 - \frac{c(k_i)}{24}}\right) = q^{h(\Lambda^i) - \frac{c(k_i)}{24}} \sum_{n \in \mathbb{C}} \mathrm{mult}_{\Lambda^i}(\alpha_i - n\delta) q^n = q^{\frac{(\alpha_i, \alpha_i)_i}{2k_i}} c_{\alpha_i}^{\Lambda^i}(\tau). \tag{659}$$

Using the fact that we have

$$L(\Lambda)_\alpha = L(\Lambda^1)_{\overline{\alpha_1}} \otimes \cdots \otimes L(\Lambda^r)_{\overline{\alpha_r}} \tag{660}$$

we obtain

$$\mathrm{ch}_{L(\Lambda)_\alpha}(\tau) = q^{\frac{(\alpha_1,\alpha_1)_1}{2k_1} + \ldots + \frac{(\alpha_r,\alpha_r)_r}{2k_r}} c_{\alpha_1}^{\Lambda^1}(\tau) \cdots c_{\alpha_r}^{\Lambda^r}(\tau). \tag{661}$$

Now we can compute

$$\begin{aligned} \mathrm{ch}_{V_\alpha}(\tau) &= \sum_{\Lambda \in M} m(\Lambda) \mathrm{ch}_{L(\Lambda)_{\overline{\alpha}}}(\tau) & (662)\\ &= q^{\frac{(\alpha_1,\alpha_1)_1}{2k_1} + \ldots + \frac{(\alpha_r,\alpha_r)_r}{2k_r}} \sum_{\Lambda \in M} m(\Lambda) c_{\alpha_1}^{\Lambda^1}(\tau) \cdots c_{\alpha_r}^{\Lambda^r}(\tau) & (663)\\ &= q^{\frac{(\iota(\alpha),\iota(\alpha))}{2}} f_{\iota(\alpha)}(\tau). & (664) \end{aligned}$$

Of course the subspace V_0 of V has the form

$$V_0 = \mathbb{C}\mathbf{1} \oplus \mathfrak{h} \oplus \cdots, \tag{665}$$

this is a direct consequence of the structure of the root space decomposition of $V_1 = \mathfrak{g}$. We obtain

$$f_0(\tau) = \mathrm{ch}_{V_0}(\tau) = q^{-1} + \dim(\mathfrak{h}) + \mathcal{O}(q). \tag{666}$$

Proposition 4.2.4. *The vector-valued modular form f_V for the Weil representation $\rho_{D(L)}$, defined by the theta decomposition of the character $\chi_V(z,\tau)$, has weight $-\frac{\dim(\mathfrak{h})}{2}$ and satisfies*

$$\mathrm{ord}_{i\infty}(f_{V,\lambda}) \leq 1 \tag{667}$$

for every component at $\lambda \in D(L)$. This bound is reached if and only if $\lambda \in G_V$, i.e. λ is a class of a cominimal simple current contained in V. The Fourier expansion of $f_0(\tau)$ is given by

$$f_0(\tau) = q^{-1} + \dim(\mathfrak{h}) + \mathcal{O}(q). \tag{668}$$

Proof. Most of the statement was already discussed above. The only statement that remains to be checked is that $\mathrm{ord}_{i\infty}(f_\lambda) = 1$ if and only if $\lambda \in G_V$. If we have $\lambda \in G_V$ and $\Lambda \in M_V$ with $e(\Lambda) = \lambda$, then the summand $c_\lambda^\Lambda(\tau)$ must have a pole of order 1 at $i\infty$, because each component λ_i satisfies

$$m_{\Lambda^i, k_i\Lambda_0 + k_i\lambda_i'} = -\frac{|\rho_i|^2}{2\check{h}_i} \frac{k_i}{k_i + \check{h}_i}, \tag{669}$$

where $\lambda_i' \in k_i\Lambda_0^i + k_i\lambda_i + \mathbb{C}\delta$ is the unique representant of the class λ that is a maximal weight. This implies that $\mathrm{ord}_{i\infty}(f_\lambda) = 1$. Conversely $\mathrm{ord}_{i\infty}(f_\lambda) = 1$ implies that there is a $\Lambda \in M_V$ such that $\mathrm{ord}_{i\infty}(c_\lambda^\Lambda) = 1$. But this implies equation (669) for all components Λ^i and λ_i of Λ and λ. By Theorem 2.5.12 we get $\Lambda^i = k_i \Lambda_{j_i}^i$ for some $j_i \in J_i$ and we find $w^i \in W_i$ with $\lambda_i = w^i(k_i\Lambda_{j_i}^i)$. Here by W_i we mean the affine Weyl group of $\hat{\mathfrak{g}}_i$. We easily check that $e_{k_i}(w^i(k_i\Lambda_{j_i}^i)) = e(k_i\Lambda_{j_i}^i)$. So we obtain $\lambda = e(k_1\Lambda_{j_1}^1 + \cdots + k_n\Lambda_{j_n}^n) \in G_V$. □

4.3 Modular invariants and cominimal simple current extensions

A *modular invariant* is a quadratic matrix M with non-negative integral entries that satisfies a certain invariance property under an action of $\mathrm{SL}_2(\mathbb{Z})$. It turns out that every extension of a vertex operator algebra V_0 defines such a modular invariant and therefore it is an important invariant of such an extension. In general it is interesting but complicated to classify all modular invariants corresponding to such a representation of $\mathrm{SL}_2(\mathbb{Z})$. Yet there are a few special cases where a classification is known. See for example [DL15] and the literature cited therein for a recent discussion. In this section we consider the modular invariant corresponding to the extension $V(\mathfrak{g}) \subset V$, where V is a holomorphic vertex operator of central charge $c = 24$ and $V(\mathfrak{g})$ the vertex operator subalgebra generated by V_1. We obtain some invariance properties under the action of simple currents for those modular invariants and finally prove that the extension of $V(\mathfrak{g})$ by all cominimal simple currents contained in V is a *simple current extension* of $V(\mathfrak{g})$. Most of the arguments in this section can be found in [Sch93] as well. The following definition is taken from [DL15].

Definition 4.3.1. Let V_0 be a strongly rational vertex operator algebra with irreducible modules $W_0^0 = V_0, W_1^0 \cdots, W_p^0$. The corresponding representation of $\mathrm{SL}_2(\mathbb{Z})$, defined by Theorem 2.7.19 will be denoted ρ_{V_0} and as usual we set $\mathcal{T} = \rho_{V_0}(T)$ and $\mathcal{S} = \rho_{V_0}(S)$. Then a $(p+1) \times (p+1)$-matrix $\mathcal{M}$ is called a *modular invariant of* V_0 if

1. Every entry of $\mathcal{M}$ is a non-negative integer,
2. $\mathcal{M}_{00} = 1$ and
3. $\mathcal{T}\mathcal{M} = \mathcal{M}\mathcal{T}$ and $\mathcal{S}\mathcal{M} = \mathcal{M}\mathcal{S}$.

Let V be a strongly rational extension of V_0 and assume that its irreducible modules are given by $W_0 = V, W_1, \cdots, W_q$. Each W_i can be decomposed into V_0-modules with multiplicities $\mathrm{mult}_{W_i}(W_j^0)$. The formal graded traces $\mathrm{Tr}_{W_i}(v,\tau)$ satisfies therefore

$$\mathrm{Tr}_{W_i}(v,\tau) = \sum_{i=0}^{p} \mathrm{mult}_{W_i}(W_j^0)\mathrm{Tr}_{W_j^0}(v,\tau). \tag{670}$$

For $i, j = 0, \cdots p$ we define a matrix M_{ij} by

$$\mathcal{Z}(v_1, v_2, \tau_1, \tau) = \sum_{l=0}^{q} \mathrm{Tr}_{W_l}(v_1,\tau_1)\overline{\mathrm{Tr}_{W_l}(v_2,\tau_2)} \tag{671}$$

$$= \sum_{i,j=0}^{p} \mathcal{M}_{ij}\mathrm{Tr}_{W_i^0}(v_1,\tau_1)\overline{\mathrm{Tr}_{W_j^0}(v_2,\tau_2)}. \tag{672}$$

Clearly this matrix is well-defined and uniquely determined since the formal traces of V_0 are linearly independent. A direct consequence of Zhus theorem 2.7.19 is that $\mathcal{M}$ is a modular invariant of V_0. We furthermore have

$$\mathcal{M}_{ij} = \sum_{l=0}^{q} \mathrm{mult}_{W_l}(W_i^0)\mathrm{mult}_{W_l}(W_j^0). \tag{673}$$

Not every modular invariant of V_0 comes from an extension V but it is still reasonable to try to classify them. See for example [DL15]. Let now V be a holomorphic vertex operator algebra of central charge $c = 24$ such that V_1 is a semi-simple Lie algebra. As we have seen in the previous chapter V is an extension of the strongly regular vertex operator algebra $V(\mathfrak{g})$ and it decomposes as (381).

Definition 4.3.2. Let V be a a holomorphic vertex operator algebra of central charge $c = 24$ such that V_1 is a semi-simple Lie algebra. We call the modular invariant of $V(\mathfrak{g})$ corresponding to the extension $V(\mathfrak{g}) \subset V$ the *modular invariant of* V and we denote it by $\mathcal{M}_V$.

We denote the matrix entries of the modular invariant of V by $\mathcal{M}_V(\Lambda, \Lambda')$ and find

$$\mathcal{M}_V(\Lambda, \Lambda') = m(\Lambda)m(\Lambda'), \tag{674}$$

where Λ and Λ' parametrize the $V(\mathfrak{g})$-modules as usual.

Proposition 4.3.3. *The modular invariant $\mathcal{M}_V$ of V satisfies $\rho_{V(\mathfrak{g})}(\gamma)\mathcal{M}_V = \mathcal{M}_V$ for all $\gamma \in SL_2(\mathbb{Z})$.*

Proof. Clearly the statement is equivalent to

$$m(\Lambda) = \sum_{\Lambda'} m(\Lambda')\rho_{V(\mathfrak{g})}(\gamma)_{\Lambda,\Lambda'}, \tag{675}$$

for all $V(\mathfrak{g})$-modules $L(\Lambda)$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. We are going to prove this statement in the following. We can equip V as well as $V(\mathfrak{g})$ with Zhus second grading, which we indicate by subscripts $(\cdot)_{[k]}$. Take $v \in V(\mathfrak{g})_{[k]} \subset V_{[k]}$. As a consequence of Theorem 2.7.19 we get that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\mathrm{Tr}_V(v, \gamma\tau) = (c\tau + d)^k \mathrm{Tr}_V(v, \tau) = (c\tau + d)^k \sum_{\Lambda \in \mathrm{Irr}(V(\mathfrak{g}))} m(\Lambda)\mathrm{Tr}_\Lambda(v, \tau). \tag{676}$$

Since we can apply the same theorem to the trace function of $V(\mathfrak{g})$ as well we obtain

$$\begin{aligned}
\mathrm{Tr}_V(v, \gamma\tau) &= \sum_{\Lambda \in \mathrm{Irr}(V(\mathfrak{g}))} m(\Lambda)\mathrm{Tr}_\Lambda(v, \gamma\tau) && (677)\\
&= (c\tau + d)^k \sum_{\Lambda \in \mathrm{Irr}(V(\mathfrak{g}))} m(\Lambda) \sum_{\Lambda' \in \mathrm{Irr}(V(\mathfrak{g}))} \rho_{V(\mathfrak{g})}(\gamma)_{\Lambda,\Lambda'}\mathrm{Tr}_{\Lambda'}(v, \tau) && (678)\\
&= (c\tau + d)^k \sum_{\Lambda' \in \mathrm{Irr}(V(\mathfrak{g}))} \left[\sum_{\Lambda \in \mathrm{Irr}(V(\mathfrak{g}))} m(\Lambda)\rho_{V(\mathfrak{g})}(\gamma)_{\Lambda,\Lambda'} \right] \mathrm{Tr}_{\Lambda'}(v, \tau) && (679)
\end{aligned}$$

Since the trace functions $\mathrm{Tr}_\Lambda(v, \tau)$ of $V(\mathfrak{g})$ are linearly independent as functions of $v \in V(\mathfrak{g})_{[k]}$ for all $k \in \mathbb{Z}_{\geq 0}$ and $\tau \in \mathbb{H}$ we can compare coefficients. ☐

Lemma 4.3.4. *Take a cominimal simple current $S \in S_V$ contained in V and an irreducible module $\Lambda \in M_V$ contained in V. Then we have $h(\Lambda) = 0(mod\,\mathbb{Z})$ and $h(S \boxtimes \Lambda) = 0(mod\,\mathbb{Z})$.*

Proof. By use of Proposition 4.3.3 we obtain with $\gamma = T$ that

$$m(\Lambda) = e^{2\pi i (h(\Lambda))} m(\Lambda) \tag{680}$$

for all $\Lambda \in \mathrm{Irr}(V(\mathfrak{g}))$. Clearly $m(\Lambda) \neq 0$ implies $h(\Lambda) = 0(\mathrm{mod}\ \mathbb{Z})$. If $m(S \boxtimes \Lambda) \neq 0$ we are done. So we assume that $m(S \boxtimes \Lambda) = 0$. This is clearly just possible if the vertex operator Y_V of V restricted to $S \otimes \Lambda$ vanishes, i.e. if it yields the trivial intertwining operator. Yet if the simple current S os contained in V its contragredient module S^{-1} has to be contained in V as well, since otherwise V can't be self-contragredient. Because of $S \boxtimes S^{-1} = V(\mathfrak{g})$ we get that $Y(v_S, z)v_{S^{-1}} \in V(\mathfrak{g})((z))$ for all $v_S \in S$ and $v_{S^{-1}} \in S^{-1}$. The corresponding intertwining operator can't vanish, because otherwise the bilinear form of V would be degenerate. Clearly the modes of some elements defined by $Y(v_S, z)v_{S^{-1}}$ have to act nontrivially on Λ because they contain in particular the vacuum $\mathbf{1}$. If $S^{-1} \boxtimes \Lambda$ is not contained in V the Jacobi identity implies a contradiction, because its right-hand side would have to vanish for all $v_S \in S$, $v_{S^{-1}} \in S^{-1}$ and $w \in \Lambda$. We obtain that $S^{-1} \boxtimes \Lambda$ must be contained in V. This implies $h(S^{-1} \boxtimes \Lambda) = 0(\mathrm{mod}\ \mathbb{Z})$. Now we evaluate

$$\begin{aligned}
m(\Lambda) &= m(S \boxtimes (S^{-1} \boxtimes \Lambda)) && (681)\\
&= \sum_{\Lambda'} m(\Lambda')e^{-2\pi i h(S \boxtimes \Lambda')} S_{\Lambda', S^{-1} \boxtimes \Lambda} && (682)\\
&= \sum_{\Lambda'} m(\Lambda')e^{-2\pi i h(S \boxtimes \Lambda')} e^{-2\pi i h(S^{-1} \boxtimes \Lambda')} S_{\Lambda',\Lambda}. && (683)
\end{aligned}$$

Here we made use of the fact that $h(S) = 0(\text{mod } \mathbb{Z})$ and $h(\Lambda') = 0(\text{mod } \mathbb{Z})$ if $m(\Lambda') \neq 0$. This clearly implies

$$m(\Lambda) = e^{-2\pi i(h(S\boxtimes\Lambda)+h(S^{-1}\boxtimes\Lambda))}m(\Lambda), \tag{684}$$

since $\mathcal{S}$ is invertible. Together $m(\Lambda) \neq 0$ and $h(S^{-1} \boxtimes \Lambda) = 0(\text{mod } \mathbb{Z})$ imply that we have $h(S \boxtimes \Lambda) = 0(\text{mod } \mathbb{Z})$. ☐

One of the crucial observations of Schellekens in [Sch93] was, that the multiplicity of a module in (381) is invariant under the fusion product with a simple current $S \in S_V$. This is the theorem in Section 3 of [Sch93].

Proposition 4.3.5 ([Sch93, Section 3]). *The set S_V of cominimal simple currents contained in V forms an abelian group under the fusion rules. This group has a natural action on the set M_V and the multiplicities $m(\cdot)$ are invariant under this action, i.e. for any $S \in S_V$ and $\Lambda \in M_V$ we have*

$$m(S \boxtimes \Lambda) = m(\Lambda). \tag{685}$$

Proof. Take a cominimal simple current $S \in S_V$ and any irreducible module $\Lambda \in \text{Irr}(V(\mathfrak{g}))$. We obtain

$$\begin{aligned} m(S \boxtimes \Lambda) &= \sum_{\Lambda'} m(\Lambda')S_{\Lambda',S\boxtimes\Lambda} && (686)\\ &= \sum_{\Lambda'} m(\Lambda')e^{-2\pi i(h(S\boxtimes\Lambda')-h(S)-h(\Lambda'))}S_{\Lambda',\Lambda} && (687)\\ &= \sum_{\Lambda'} m(\Lambda')S_{\Lambda',\Lambda} && (688)\\ &= m(\Lambda). && (689)\end{aligned}$$

Notice that we made use of Lemma 4.3.4 and the fact that S is contained in V. This equation implies that $S \boxtimes \Lambda$ is contained in V if and only Λ is contained in V. In particular the set S_V is closed under the fusion product and therefore an abelian group. ☐

In Definition 4.2.1 we introduced the subset $G_V \subset D(L)$ of simple current contained in V. A direct consequence of the fact that S_V is an abelian group under the fusion product is, that G_V is a subgroup of $D(L)$. Furthermore we clearly have $m(k\Lambda_0) = 1$ since the weight-0 space V_0 of V has dimension 1. This implies $m(S) = 1$ for all cominimal simple currents S contained in V.

Theorem 4.3.6. *The restriction of the vertex operator of V to*

$$V_{sc} = V(\mathfrak{g})_{S_V} = \bigoplus_{S\in S_V} L(S) \tag{690}$$

defines the structure of a S_V-graded simple current extension vertex operator algebra of $V(\mathfrak{g})$.

Proof. Since the cominimal simple currents S_V are closed under the fusion product the space V_{sc} is closed under the restriction of the vertex operator of V. This defines the structure of a vertex operator algebra which extends $V(\mathfrak{g})$ because Y satisfies the Jacobi identity in V already. Finally we have to show that V_{sc} is simple. Assume $J \subset V_{sc}$ to be an ideal that contains a nontrivial element $v \in J$. Of course v can be written as a linear combination elements $v_S \in L(S)$ for some $S \in S_V$. Since the bilinear form $(\cdot,\cdot)$ of V is non-degenerate we find an element $w \in V$ with $(v, w) \neq 0$. Clearly we can find such an element in $w \in V_{sc}$ since we have $L(S) \perp W$ for every irreducible module W of $V(\mathfrak{g})$ except $L(S^{-1})$. Now we use (316) to generate the element $(v, w)\mathbf{1}$ in J. Now we can obviously generate all of V_{sc} within J and obtain $J = V_{sc}$. We see that V_{sc} is simple. This show that V_{sc} satisfies all axioms of a simple current extension vertex operator algebra as given in [Yam04]. ☐

4.4 The simple current glue group and its lattice extension

In this subsection we introduce the *simple current glue group* G_V and construct a lattice extension H of L by use of this group. We furthermore show that the vector-valued modular form f_V, which is associated to V, may be considered as a vector-valued modular form for the Weil representation $\rho_{D(H)}$ on the discriminant form of H. A similar construction was used in [CKS07] as well. Yet we have to replace some explicit computations by abstract arguments in the proofs of some properties of f_V.

Lemma 4.4.1. *The subgroup G_V of $D(L)$ is isotropic and we have $e(M_V) \subset G_V^{\perp}$.*

Proof. This statement is an obvious consequence of (630) and Lemma 4.3.4. □

Following the discussion of (49) we may use an isotropic subgroup of the discriminant form of an even non-degenerate lattice to *glue* it together and obtain a lattice, which is still even and has the same rank and dimension.

Definition 4.4.2. We call the group G_V the *simple current glue group* and we denote the even and positive-definite lattice generated by G_V and L by

$$H := (G_V, L) = \bigcup_{g \in G_V} g + L. \tag{691}$$

The lattice H is clearly even since G_V is isotropic. We have $H' = (L, G_V^{\perp})$ and $H'/H = G_V^{\perp}/G_V$ as abelian groups. Since we also have $e(M_V) \subset H'$ we can associate to each element in M_V a class in H'/H. Clearly such a class can contain more then one element of M_V. Just observe that each simple current of M_V is contained in the class [0].

Proposition 4.4.3. *The vector-valued modular form f_V, defined by the theta correspondence of the character χ_V satisfies:*

1. *For $\lambda \in D(L)$ with $\gamma \notin G_V^{\perp}$ we have $f_{V,\lambda} = 0$.*
2. *We have $f_{V,g+\lambda} = f_{V,\lambda}$ for all $g \in G_V$ and $\lambda \in D(L)$.*

Proof. Using formula (652) we observe that $f_\lambda \neq 0$ implies that there is an element $\Lambda = (\Lambda^1, \cdots, \Lambda^r) \in M_V$ such that that $c^{\Lambda}_{\lambda} \neq 0$. We make use of (651) and find that for every $i = 1, \cdots, r$ we have $c^{\Lambda^i}_{k_i\Lambda_0 + k_i\lambda_i} \neq 0$. For simplicity we may set $\tilde{\lambda}_i = k_i\Lambda_0 + k_i\lambda_i$. Since $c^{\Lambda^i}_{\tilde{\lambda}_i} \neq 0$ is just possible if $\tilde{\lambda}_i$ is a weight of the module $L_{\hat{\mathfrak{g}}_i}(\Lambda^i)$ it is clear that we find integers r_j such that $\Lambda^i - \tilde{\lambda}_i = \sum_{j=0}^{n_i} r_j \alpha^i_j$. Take the cominimal weight $k_i\Lambda^i_l$ for the Lie algebra $\hat{\mathfrak{g}}_i$, i.e. we have $l \in J_i$. Here by J_i we denote the set of indices corresponding to $\hat{\mathfrak{g}}_i$ such that the Coxeter label a_j satisfies $a_j = 1$ for all $j \in J_i$. In particular $L_{\hat{\mathfrak{g}}_i}(k_i\Lambda^i_l)$ is a cominimal simple current of $L_{\hat{\mathfrak{g}}_i}(k_i\Lambda_0)$. As a consequence of $a_l = 1$ we get $\alpha^i_l = \check{\alpha}^i_l$. Using this we find

$$(\Lambda^i - \tilde{\lambda}_i, k_i\Lambda^i_l) = \left(\sum_{j=0}^{n_i} r_j \alpha^i_j, k_i\Lambda^i_l \right) = \sum_{j=0}^{n_i} r_j \left(\alpha^i_j, k_i\Lambda^i_l \right) = k_i \sum_{j=0}^{n_i} r_j \delta_{j,l} \equiv 0 (\mathrm{mod}\, k_i\mathbb{Z}) \tag{692}$$

A direct consequence of this is that we must have $(e_i(\Lambda^i) - \lambda_i, e_i(S)) = 0(\mathrm{mod}\, \mathbb{Z})$ for all cominimal simple currents S of $L_{\hat{\mathfrak{g}}_i}(k_i\Lambda_0)$. Putting this together we obtain $(e(\Lambda) - \lambda, g) = 0(\mathrm{mod}\, \mathbb{Z})$ for every $g \in G$. Remember that G is the subgroup of cominimal simple currents of $D(L)$. Those are all cominimal simple currents of $V(\mathfrak{g})$ and not just those contained in V. Yet due to Lemma 4.4.1 we know that we have $(e(\Lambda), g) = 0(\mathrm{mod}\, \mathbb{Z})$ for all $g \in G_V$. Altogether we find that $f_\lambda \neq 0$ implies $(\lambda, g) = 0(\mathrm{mod}\, \mathbb{Z})$ for all $g \in G_V$. This is the first statement of the proposition. Take a cominimal simple current $S \in S_V$, which is contained in V. We may assume that S is given by $(k_1\Lambda^1_{l_1}, \cdots, k_r\Lambda^r_{l_r})$ and we set $g = e(S) \in G_V$. As usual we have a corresponding symmetry $\sigma_S \in W_0^+$ and we assume that σ_S consists component-wise of symmetries $\sigma_i \in W_{0,i}^+$. Here $W_{0,i}^+$ is simply the group of outer symmetries corresponding to the component $\mathfrak{g}_i$. We may now compute

$$f_{V,g+\lambda}(\tau) = \sum_{\Lambda \in M} m(\Lambda) c^{\Lambda}_{g+\lambda}(\tau) \tag{693}$$

$$= \sum_{\Lambda \in M} m(\Lambda) \prod_{i=1}^{r} c^{\Lambda^i}_{k_i\Lambda_0 + k_i\overline{\Lambda^i_{l_i}} + k_i\lambda_i}(\tau) \tag{694}$$

$$= \sum_{\Lambda \in M} m(\sigma_S(\Lambda)) \prod_{i=1}^{r} c^{\sigma_i(\Lambda^i)}_{\sigma_i(k_i\Lambda_0 + k_i\lambda_i)}(\tau) \tag{695}$$

$$= \sum_{\Lambda \in M} m(\Lambda) c^{\Lambda}_{\lambda}(\tau) = f_\lambda(\tau). \tag{696}$$

We may now indicate in more detail how the individual steps in this computation work. Remember that σ_i is given by $w_i t_{\overline{\Lambda^i_{l_i}}}$ for w_i a suitable element in the finite Weyl group of $\mathfrak{g}_i$. We discussed this in section 4.1. Remember furthermore that the string functions are invariant under the action of the Weyl group. This implies (695). For the next step we made use of Lemma 4.1.3 and the fact that the multiplicities $m(\Lambda)$ just depend on the orbit of the group S_V of simple currents contained in V. This proves the statement of the proposition. □

This properties of f_V will turn out to be crucial in the following. In the next proposition we study vector-valued modular forms with such properties and their relation to Jacobi forms. Notice that in the next proposition the lattice L is an arbitrary positive-definite even lattice and G is any isotropic subgroup in its discriminant group.

Proposition 4.4.4. *Take an even positive-definite lattice L. Let J be a weakly holomorphic Jacobi form of weight k and index L, i.e. $J \in J^{w.h.}_{k,L}$. Let $\tilde{f}$ be the vector-valued modular form for the Weil representation $\rho_{D(L)}$, defined by the theta-decomposition of J. For an isotropic subgroup $G \subset D(L)$ we can extend the lattice L to an even positive-definite lattice $H := (L, G)$. The vector-valued modular form $\tilde{f}$ has the properties*

1. *For $\gamma \in D(L)$ with $\gamma \notin G^\perp$ we have $\tilde{f}_\gamma = 0$ and*
2. *We have $\tilde{f}_{g+\gamma} = \tilde{f}_\gamma$ for all $g \in G$ and $\gamma \in D(L)$,*

if and only if J is a weakly holomorphic Jacobi form of weight k and index H. In this case the theta-decomposition of J for the lattice H induces a vector-valued modular form f for the Weil representation $\rho_{D(H)}$ and it satisfies

$$f_{[\gamma]_N} = \tilde{f}_{[\gamma]_L} \text{ for all } \gamma \in H^\sharp. \tag{697}$$

Proof. First we assume that the vector-valued modular form $\tilde{f}$ has the properties in the statement. Take $g_1, g_2 \in G$ and $r \in \mathbb{Q}$ such that $r + \frac{1}{2}(g_1, g_2) \equiv 0 (\mathrm{mod}\, \mathbb{Z})$ and set $h = [g_1, g_2; r]$. We have to show that $J(\tau, z)|_{k,t}[h] = J(\tau, z)$. First we check for $\mu \in H' \subset L'$ and $l \in \mu + L$ that we have

$$\exp[\pi i((g_1, g_1)\tau + 2(g_1, z) + (g_1, g_2) + 2r + (l, l)\tau + 2(l, z + g_1\tau + g_2))] \tag{698}$$

$$= \exp[\pi i((g_1 + l, g_1 + l)\tau + 2(g_1 + l, z) + 2(l, g_2))] \tag{699}$$

$$= \exp[\pi i((g_1 + l, g_1 + l)\tau + 2(g_1 + l, z))] \tag{700}$$

Notice that $\mu \in N'$ is necessary for (700) the previous equation would hold for $\mu \in L'$ as well. We go on observing that

$$\Theta^H_\mu(\tau, z) = \sum_{g \in G} \Theta^L_{g+\gamma}(\tau, z). \tag{701}$$

Using those facts we can compute

$$J(\tau,z)|_{k,t}[h] = \sum_{\gamma\in L'/L} \exp[\pi i(\cdots)]\Theta^L_\gamma(\tau,z+g_1\tau+g_2)\tilde{f}_\gamma(\tau) \tag{702}$$

$$= \sum_{\gamma\in H'/L} \exp[\pi i((g_1+l,g_1+l)\tau+2(g_1+l,z))]\Theta^L_\gamma(\tau,z+g_1\tau+g_2)\tilde{f}_\gamma(\tau) \tag{703}$$

$$= \sum_{\gamma\in H'/L} \Theta^L_{g_1+\gamma}(\tau,z)\tilde{f}_\gamma(\tau) \tag{704}$$

$$= \sum_{\gamma\in H'/L} \Theta^L_{\gamma}(\tau,z)\tilde{f}_{\gamma-g_1}(\tau) \tag{705}$$

$$= \sum_{\gamma\in H'/L} \Theta^L_{\gamma}(\tau,z)\tilde{f}_{\gamma}(\tau) = \sum_{\gamma\in L'/L} \Theta^L_{\gamma}(\tau,z)\tilde{f}_{\gamma}(\tau) = J(\tau,z). \tag{706}$$

In the first equation of this computation we replaced (698) by $\exp[\pi i(\cdots)]$. This show that J is a Jacobi form of lattice index H. We consider its theta decomposition now. This is

$$J(\tau,z) = \sum_{\gamma\in H'/L} \Theta^L_{\gamma}(\tau,z)\tilde{f}_{\gamma}(\tau) \tag{707}$$

$$= \sum_{\gamma\in H'/H}\sum_{g\in G} \Theta^L_{\gamma+g}(\tau,z)\tilde{f}_{\gamma+g}(\tau) \tag{708}$$

$$= \sum_{\gamma\in H'/H}\left(\sum_{g\in G} \Theta^L_{\gamma+g}(\tau,z)\right)\tilde{f}_{\gamma}(\tau) \tag{709}$$

$$= \sum_{\gamma\in H'/H} \Theta^H_{\gamma}(\tau,z)\tilde{f}_{\gamma}(\tau). \tag{710}$$

Now we proof the other direction of the statement. We assume that J is a Jacobi form of lattice index H. The corresponding vector-valued modular form for the Weil representation ρ_H will be denoted f. Considering J as a Jacobi form of lattice index L we can associate a vector-valued modular form f' for the Weil representation ρ_L to it. Now we lift f to a vector-valued modular form $\tilde{f}$ of ρ_L by $\tilde{f}_\lambda = f_{\lambda+H}$ if $\lambda\in H'$ and $\tilde{f}_\lambda = 0$ if $\lambda\notin H'$. Now we compute

$$J(\tau,z) = \sum_{\gamma\in H'/H} \Theta^H_{\gamma}(\tau,z)f'_{\gamma}(\tau) \tag{711}$$

$$= \sum_{\gamma\in H'/H}\sum_{g\in G} \Theta^L_{g+\gamma}(\tau,z)f'_{\gamma}(\tau) \tag{712}$$

$$= \sum_{\gamma\in H'/L} \Theta^L_{\gamma}(\tau,z)\tilde{f}_{\gamma}(\tau) \tag{713}$$

$$= \sum_{\gamma\in L'/L} \Theta^L_{\gamma}(\tau,z)\tilde{f}_{\gamma}(\tau) \tag{714}$$

Since the theta decomposition of a Jacobi form defines a unique vector-valued modular form for the Weil representation we get $\tilde{f} = f'$. This implies that f has the properties of the proposition. □

Of course we can apply this proposition to f_V and find that χ_V is a nearly holomorphic Jacobi form of weight 0 of lattice index H.

Proposition 4.4.5. *The character χ_V is a weakly holomorphic Jacobi form of weight 0 and lattice index H and the corresponding vector-valued modular form f_V satisfies*

$$f_{V,\lambda} = \sum_{\Lambda\in M} m(\Lambda)c^\Lambda_\lambda(\tau) \tag{715}$$

for all $\lambda\in D(H)$. Furthermore the only component which has a pole of order 1 at $i\infty$ is the component f_0 or in other words if $f_{V,\lambda}$ with $\lambda\neq 0$ has pole at $i\infty$, then its pole order is strictly less than 1.

Proof. This statement is essentially a consequence of the previous proposition. Regardless we will give some details. Instead of f_V we denote the vector-valued modular forms corresponding to χ_V by f^L or f^H, depending on the lattice in consideration. The first statement is a consequence of the previous proposition and the formula for the components follows from $f^H_{[v]_H} = f^L_{[v]_L}$ for $v \in H'$ and (652). Assume that $f^H_{[v]_H}$ has pole of order 1 at $i\infty$ for some $v \in H'$. Then $f^L_{[v]_L}$ has to have a pole of order 1 at $i\infty$ as well. Yet in Proposition 4.2.4 we saw that f^L_γ has a pole of order 1 at $i\infty$ if and only if $\gamma = [v]_L$ is the class of a simple current contained in S_V. But then we already have $[v]_H = 0$. □

In the following we will always consider f_V as a vector-valued modular form of weight $-\frac{\dim(\mathfrak{h})}{2}$ for the Weil representation $\rho_{D(H)}$ of the group $\mathrm{Mp}_2(\mathbb{Z})$. A direct consequence of (715) is that all Fourier coefficients of f_V are non-negative integers and we have

$$f_{V,0}(\tau) = q^{-1} + \dim(\mathfrak{h}) + \mathcal{O}(q). \tag{716}$$

Of course the discriminant form $D(H)$ may be represented by the lattice $H_2 = H \oplus \mathrm{II}_{1,1} \oplus \mathrm{II}_{1,1}$ as well. This shows that f_V satisfies all necessary conditions of Theorem 2.4.3, such that we can associate an automorphic product Φ_V to f_V.

4.5 The Lie algebra of physical states and its root lattice

In this subsection we show that the automorphic product Φ_V, associated to f_V, is a reflective modular form. In order to do this we have to show that the Lie algebra of physical states $\mathfrak{g}(V)$ is a generalized Kac-Moody algebra. This is one of the main results of this thesis. For simplicity we may assume, that V has a real form with a suitable positive-definite structure. The reflectivity of Φ_V will be a consequence of the fact, that a product expansion of it describes the root structure of the Lie algebra $\mathfrak{g}(V)$ explicitly. More precisely this product is the *product side* of the denominator identitiy of this generalized Kac-Moody algebra. Special cases of this result were obtained in [CKS07] already, by use of Schellekens' list. In this section we make use of all the notations from previous sections. In particular we obtain the decomposition (381) and construct the lattices L and H by use of the usual non-degenerate bilinear form $\langle\cdot,\cdot\rangle$. This is the unique invariant bilinear form of V such that the vacuum $\mathbf{1}$ satisfies $\langle\mathbf{1},\mathbf{1}\rangle = -1$. Yet in the following we will also have to work with the invariant bilinear form

$$\langle\cdot,\cdot\rangle' = -\langle\cdot,\cdot\rangle. \tag{717}$$

This is because, as explained in section 3.1 and section 3.2, the vertex operator algebra V equipped with the invariant bilinear form $\langle\cdot,\cdot\rangle'$ is a lowest weight Vir-module of central charge $c_V = 24$ in the sense of Definition 3.1.1. For this we have to make use of $\langle\mathbf{1},\mathbf{1}\rangle' = 1$. Now we can associate to V its Lie algebra of physical states $\mathfrak{g}(V)$ as in Definition 3.2.2. We keep all notations and structures introduced in section 3.1 expect that we write $\langle\cdot,\cdot\rangle'$ for the appropriate bilinear form on V. In particular we equip $\mathfrak{g}(V)$ with an invariant bilinear form $(\cdot,\cdot)$, a contravariant bilinear form $(\cdot,\cdot)_0$, a $\mathrm{II}_{1,1}$-grading

$$\mathfrak{g}(V) = \bigoplus_{r\in\mathrm{II}_{1,1}} \mathfrak{g}_r(V) \tag{718}$$

and some group of symmetries G. We can take G to be the automorphism group $\mathrm{Aut}(V)$ of V but this does not matter in the following. We may fix vectors $e, f \in \mathrm{II}_{1,1}$ as in Lemma 3.4.1 such that we can associate to each $r \in \mathrm{II}_{1,1}$ a vector $w_r \in \mathrm{II}_{1,1}$ as in Definition 3.4.2. As usual we denote the induced no-ghost isomorphisms by η_r. For $r \in \mathrm{II}_{1,1}\setminus\{0\}$ and $v, w \in V_{1-r^2/2}$ this means

$$(\eta_r(v), \eta_r(w))_0 = \langle v, w\rangle' = -\langle v, w\rangle. \tag{719}$$

In the case $r = 0$ we have the no ghost isomorphism

$$\eta_0 : V_1 \oplus \mathrm{II}_{1,1} \to \mathfrak{g}(V)_0,\ v + \alpha \mapsto v \otimes e^0 + \mathbf{1} \otimes \alpha(-1) \otimes e^0. \tag{720}$$

Using (468) we find that for all $v, w \in V_1$ and $\alpha, \beta \in \mathrm{II}_{1,1} \otimes \mathbb{C}$ the invariant bilinear form on $\mathfrak{g}_0(V)$ satisfies

$$(\eta_0(v+\alpha), \eta_0(w+\beta)) = -\langle v, w\rangle' + (\alpha, \beta) = \langle v, w\rangle + (\alpha, \beta). \quad (721)$$

All this is just Proposition 3.1.16. In particular this allows us to embed the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} = V_1$ into $\mathfrak{g}(V)_0$.

Lemma 4.5.1. *For every $r \in II_{1,1}$ the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ acts as $\eta_0(\mathfrak{h})$ on $\mathfrak{g}_r(V)$ and on $V_{1-r^2/2}$ with the usual action of V_1. The no-ghost isomorphism η_r preserves this action, i.e. we have for every $h \in \mathfrak{h}$ and $v \in V_{1-r^2/2}$ that*

$$\eta_r(h_0 v) = [\eta_0(h), \eta_r(v)]. \quad (722)$$

Proof. The action of $\mathfrak{h}$ decomposes each V_n into weight spaces $(V_\alpha)_n$ for some $\alpha \in \mathfrak{h}^*$. Clearly we just have $\alpha \in \overline{P}$ but this does not matter yet. We may assume $v \in (V_\alpha)_{1-r^2/2}$ and obtain

$$h_0 v = \alpha(h) v \;\forall h \in \mathfrak{h}. \quad (723)$$

Now we have to evaluate $[\eta_0(h), \eta_r(v)]$. Notice that $(\eta_0(h))_0$ is just the operator $h_0 \otimes \mathrm{Id}$ on $V \otimes V_{\mathrm{II}_{1,1}}$. Since for $r = 0$ the statement is clear we may assume $r \neq 0$ and set $n = 1 - r^2/2$. Then the class $\eta_r(v)$ has a unique representant $t_v \in T^1(r)$ given by

$$t_v = \mathcal{P}(v \otimes e^r) = \sum_{i=0}^{n} S_i[n] E_r^i (v \otimes e^r) \in T^1(r). \quad (724)$$

On the space $K^1(r)$ the operator E_r acts as

$$\sum_{n=1}^{\infty} D_r(n) L(n) \quad (725)$$

and therefore commutes with $h_0 \otimes \mathrm{Id}$ since h_0 commutes with each L_n on V. Clearly $h_0 \otimes \mathrm{Id}$ acts by multiplication with $\alpha(h)$ on $v \otimes e^r$ so we get

$$(h_0 \otimes \mathrm{Id}) t_v = \sum_{i=0}^{n} S_i[n] E_r^i ((h_0 \otimes \mathrm{Id}) v \otimes e^r) = \alpha(h) t_v. \quad (726)$$

We obtain $[\eta_0(h), \eta_r(v)] = \alpha(h)\eta_r(v)$ and the statement is proved. □

As usual we denote the real span of the simple coroots of $\mathfrak{g}$ by $\mathfrak{h}_\mathbb{R}$ and consider it as a real form of $\mathfrak{h}$. In the following we identify $\mathfrak{h}_\mathbb{R}$ with $\mathfrak{h}_\mathbb{R}^*$ and $\mathfrak{h}$ with $\mathfrak{h}^*$ by use of the bilinear form $\langle \cdot, \cdot \rangle$. Notice that this is slightly different from the identification we used in previous sections. There we identified the Cartan subalgebras $\mathfrak{h}_i$, corresponding to simple components of $\mathfrak{g}$, with their dual $\mathfrak{h}_i^*$ by use of a bilinear form $(\cdot, \cdot)_i$ normalized such that the longest root of $\mathfrak{g}_i$ has length 2. Using this identification we obtain for $\alpha \in \overline{P} \subset \mathfrak{h}^*$ that $h \in \mathfrak{h}$ and $x \in V_\alpha$ satisfy

$$h_0 x = \alpha(h) x = \langle \iota(\alpha), h \rangle x, \quad (727)$$

where $\iota : \overline{P} \to L'$ is given componentwise as in (625). In this sense we view the lattice L' simply as the $\mathbb{Z}$-span of the dual basis of $\mathfrak{h}$ of the basis of simple coroots $\check{\overline{\alpha}}_i$ of $\mathfrak{h}$ with respect to $\langle \cdot, \cdot \rangle$. In each component $\mathfrak{h}_i$ this just means

$$\langle \iota\left(\overline{\Lambda_j^i}\right), \check{\overline{\alpha_k}} \rangle = \delta_{j,k} \quad (728)$$

and L' is the $\mathbb{Z}$-span of all $\iota\left(\overline{\Lambda_j^i}\right)$ running through all components $\mathfrak{h}_i$ of the Cartan subalgebra $\mathfrak{h}$, equipped with the bilinear form $\langle \cdot, \cdot \rangle$. We introduce a subspace of $\mathfrak{g}(V)$ by

$$\mathcal{H} = \eta_0 \left(\mathfrak{h} \oplus \mathrm{II}_{1,1} \otimes \mathbb{C} \right) \quad (729)$$

and we can view L' as a sublattice of $\mathcal{H}$ by embedding along η_0. Notice that we equip $\mathcal{H}$ with a bilinear form by restriction of the invariant non-degenerate bilinear form $(\cdot, \cdot)$ of the Lie algebra of physical states $\mathfrak{g}(V)$. We denote this bilinear form on $\mathcal{H}$ by $(\cdot, \cdot)_\mathcal{H}$ in the following.

Lemma 4.5.2. *We identify the image of $L' \oplus II_{1,1}$ under η_0 with itself, i.e. we denote it by $L' \oplus II_{1,1}$. It is a lattice of full rank within $\mathcal{H}$ and satisfies*

$$\mathcal{H} = (L' \oplus II_{1,1}) \otimes \mathbb{C} \tag{730}$$

as isometric vector spaces.

Proof. Because of $\mathfrak{h} = L' \otimes \mathbb{C}$ it is clear that the lattice $L' \oplus \mathrm{II}_{1,1}$ is of full rank. The lattice is equipped with the bilinear form $\langle \cdot, \cdot \rangle$ and the space $\mathcal{H}$ is equipped with the bilinear form $(\cdot, \cdot)_{\mathcal{H}}$. So equation (721) shows that η_0 induces an isometry. $\square$

For each $\overline{\alpha} \in L'$ and $r \in \mathrm{II}_{1,1}$ we define a subspace of $\mathfrak{g}_r(V)$ by

$$\mathfrak{g}_r(V)_{\overline{\alpha}} = \{x \in \mathfrak{g}_r(V) : [\eta_0(h), x] = \langle \overline{\alpha}, h \rangle x, \ \forall h \in \mathfrak{h}\} \tag{731}$$

Since each L_0-weight space V_n of V can be decomposed as $V_n = \bigoplus_{\alpha \in \overline{P}} (V_\alpha)_n$ the Lemma 4.5.1 can be used to obtain a decomposition

$$\mathfrak{g}_r(V) = \bigoplus_{\overline{\alpha} \in L'} \mathfrak{g}_r(V)_{\overline{\alpha}}. \tag{732}$$

For $\alpha = \overline{\alpha} + r \in L' \oplus \mathrm{II}_{1,1}$ we introduce eigenspaces

$$\mathfrak{g}_\alpha(V) = \mathfrak{g}_r(V)_{\overline{\alpha}} \tag{733}$$

which clearly satisfy

$$\mathfrak{g}_\alpha(V) = \{x \in \mathfrak{g}(V) : [h, x] = (h, \alpha)_{\mathcal{H}} x \ \forall h \in \mathcal{H}\}. \tag{734}$$

This implies a decomposition into $\mathcal{H}$-eigenspaces of $\mathfrak{g}(V)$ by

$$\mathfrak{g}(V) = \bigoplus_{\alpha \in L' \oplus \mathrm{II}_{1,1}} \mathfrak{g}_\alpha(V). \tag{735}$$

In the following we will just work with this decomposition therefore an expression like $\mathfrak{g}_\alpha(V)$ always corresponds to a space of the form $\mathfrak{g}_r(V)_{\overline{\alpha}}$. In particular by $\mathfrak{g}_0(V)$ we mean the weight space corresponding to $0 \in L' \oplus \mathrm{II}_{1,1}$ which is $\mathfrak{g}_0(V)_0$. This decomposition has the usual properties of a root space decomposition, in particular we have

$$[\mathfrak{g}_\alpha(V), \mathfrak{g}_\beta(V)] \subset \mathfrak{g}_{\alpha+\beta}(V). \tag{736}$$

This is a direct consequence of the Jacobi identity of the Lie algebra $\mathfrak{g}(V)$.

Definition 4.5.3. We call $\alpha \in L' \oplus \mathrm{II}_{1,1}$ a *root* of $\mathfrak{g}(V)$ if $\mathfrak{g}_\alpha(V) \neq 0$ and $\alpha \neq 0$. The $\mathbb{Z}$-span R of all roots of $\mathfrak{g}(V)$ is a sublattice of $L' \oplus \mathrm{II}_{1,1}$ and will be called the *root lattice* of $\mathfrak{g}(V)$.

So far we have seen that f_V is a nearly holomorphic vector-valued modular form of weight $k = -\frac{\dim(\mathfrak{h})}{2}$ for the Weil representation of the discriminant form $D(H)$. As explained above already, we may view f_V also as a vector-valued modular form for lattices like $H_1 = H \oplus \mathrm{II}_{1,1}$ and $H_2 = H \oplus \mathrm{II}_{1,1} \oplus \mathrm{II}_{1,1}$ of signature $(n+1, 1)$ and $(n+2, 2)$, respectively. This is obvious because those lattices represent the same discriminant form. We will switch between these lattices constantly in the following. In the next proposition we will view f_V once again as a modular form for the lattice L or more precisely $L \oplus \mathrm{II}_{1,1}$.

Proposition 4.5.4. *Let V be a holomorphic vertex operator algebra of central charge $c = 24$ such that V_1 is a semisimple Lie algebra. The subspace $\mathcal{H}$ of the Lie algebra of physical states $\mathfrak{g}(V)$ is equal to $\mathfrak{g}_0(V)$ and a self-centralizing subalgebra which has a natural totally real form given by*

$$\mathcal{H}_{\mathbb{R}} = \eta_0 (\mathfrak{h}_{\mathbb{R}} \oplus II_{1,1} \otimes \mathbb{R}) = (L' \oplus II_{1,1}) \otimes \mathbb{R}, \tag{737}$$

i.e. all roots act with real eigenvalues. For $\alpha \in L' \oplus II_{1,1}$ the dimension of the corresponding root space $\mathfrak{g}_\alpha(V)$ is given by

$$dim\,(\mathfrak{g}_\alpha(V)) = f_{V,\alpha}\left(-\frac{\alpha^2}{2}\right) \text{ for } \alpha \in L' \oplus II_{1,1} \setminus \{0\} \text{ and} \tag{738}$$

$$dim\,(\mathfrak{g}_0(V)) = dim\,(\mathfrak{h}) + 2. \tag{739}$$

As a consequence the root lattice R of the Lie algebra of physical states $\mathfrak{g}(V)$ is contained in the lattice $H' \oplus II_{1,1} \subset L' \oplus II_{1,1}$, which implies that we have a decomposition

$$\mathfrak{g}(V) = \bigoplus_{\alpha \in H' \oplus II_{1,1}} \mathfrak{g}_\alpha(V). \tag{740}$$

Proof. The subset $\mathcal{H}$ is clearly contained in $\mathfrak{g}_0(V)$. Conversely we must have $\mathfrak{g}_0(V) \subset \mathcal{H}$, since we have $\mathfrak{h} = \mathfrak{g}_0$ for the semi-simple Lie algebra $\mathfrak{g} = V_1$. This directly implies that $\mathcal{H}$ is self-centralizing. In order to see that $\mathcal{H}_\mathbb{R}$ is a totally real form we have to check that $(\cdot,\cdot)_\mathcal{H}$ is real-valued over $\mathcal{H}_\mathbb{R}$ and that all roots of $\mathfrak{g}(V)$ are contained in $\mathcal{H}_\mathbb{R}$. Both is clear since $(\cdot,\cdot)_\mathcal{H}$ is real-valued over $L' \oplus \mathrm{II}_{1,1}$ and all roots are contained in this lattice. Equation (739) is a direct consequence of this discussion. We take $\alpha = \overline{\alpha} + r \in L' \oplus \mathrm{II}_{1,1}$ and some $\alpha' \in \mathfrak{h}^*$ with $\iota(\alpha') = \overline{\alpha}$.

$$\mathfrak{g}_\alpha(V) = \mathfrak{g}_r(V)_{\overline{\alpha}} = \eta_r((V_{\alpha'})_{1-r^2/2}) \tag{741}$$

as a consequence of Lemma 4.5.1. Using the equation (664) we obtain

$$\mathrm{ch}_{V_{\alpha'}}(\tau) = q^{\frac{\overline{\alpha}^2}{2}} f_{V,\overline{\alpha}}(\tau). \tag{742}$$

The equation (738) is a direct consequence of this. Using Proposition 4.4.3 we know that $f_{V,\alpha} = 0$ unless $\alpha \in H' \oplus \mathrm{II}_{1,1}$, so it is clear that the root lattice R of $\mathfrak{g}(V)$ is contained in $H' \oplus \mathrm{II}_{1,1}$. Altogether we have proved the statements. □

Remember that the root lattice R of $\mathfrak{g}(V)$ is the sublattice of $H' \oplus \mathrm{II}_{1,1}$ spanned by all roots $\alpha \in H' \oplus \mathrm{II}_{1,1}$ of $\mathfrak{g}(V)$, i.e. all elements that satisfy $\mathfrak{g}_\alpha(V) \neq 0$. In the following we will determine this lattice explicitly.

Proposition 4.5.5. *The root lattice R of $\mathfrak{g}(V)$ is given by $R = H' \oplus II_{1,1}$.*

Proof. In this proof we consider f_V as a vector-valued modular form for the Weil representation of $D(H_1)$ and write f instead of f_V for simplicity. Of course we have $R \subset H' \oplus \mathrm{II}_{1,1}$. First we have to show that $H \oplus \mathrm{II}_{1,1} \subset R$. Take $r \in \mathrm{II}_{1,1}$ with $r^2 = 0$. We obtain $[f_r]\left(-\frac{r^2}{2}\right) = [f_0](0) = \dim(\mathfrak{h}) \neq 0$. Therefore r is contained in R. Since $\mathrm{II}_{1,1}$ can be spanned by such elements we have $\mathrm{II}_{1,1} \subset R$. Take $\alpha \in H$. We can clearly fin $l \in \mathrm{II}_{1,1}$ such that $[f_{\alpha+l}]\left(-\frac{(\alpha+l)^2}{2}\right) = [f_0]\left(-\frac{(\alpha+l)^2}{2}\right) \neq 0$. We obtain $\alpha + l \in R$. But this implies

$$H \oplus \mathrm{II}_{1,1} \subset R \subset H' \oplus \mathrm{II}_{1,1}. \tag{743}$$

This allows us to few $R/(H \oplus \mathrm{II}_{1,1})$ as a subset of the discriminant form $D(H)$. Clearly it is enough to prove $H' \oplus \mathrm{II}_{1,1} \subset R$, but this is equivalent to $R' \subset H \oplus \mathrm{II}_{1,1}$. Therefore we have to show that $r \in R'$ satisfies $r \in H \oplus \mathrm{II}_{1,1}$. Observe first that if $\lambda \notin R$ then $f_\lambda = 0$, since otherwise we could choose $l \in \mathrm{II}_{1,1}$ with $[f_{\lambda+l}]\left(-\frac{(\lambda+l)^2}{2}\right) \neq 0$ but then we would have $\lambda \in R$. A contradiction. Now we compute

$$f_r(S\tau) = \frac{e(\text{sign}(D(H))/8)}{\sqrt{|D(H)|}}\tau^k \sum_{\beta\in D(H)} e((r,\beta))f_\beta(\tau) \tag{744}$$

$$= \frac{e(\text{sign}(D(H))/8)}{\sqrt{|D(H)|}}\tau^k \sum_{\beta\in R/(H\oplus \mathrm{II}_{1,1})} e((r,\beta))f_\beta(\tau) \tag{745}$$

$$= \frac{e(\text{sign}(D(H))/8)}{\sqrt{|D(H)|}}\tau^k \sum_{\beta\in R/(H\oplus \mathrm{II}_{1,1})} f_\beta(\tau) \tag{746}$$

$$= f_0(S\tau). \tag{747}$$

We used $f_\beta = 0$ if $\beta \notin R$ and $(r,\beta) \in \mathbb{Z}$ for all $\beta \in R$. This implies $f_r = f_0$. Because of $f_r(\tau) = f_0(\tau) = q^{-1} + \mathcal{O}(1)$ we obtain $[r] = 0$ by Theorem 4.4.5 but this just shows $r \in H \oplus \mathrm{II}_{1,1}$. □

So far we discussed some properties of the Lie algebra of physical states $\mathfrak{g}(V)$, in particular its root space decomposition. Next we need to achieve a more structural result similar to the observation in [CKS07] that the Lie algebras of physical states associated to the vertex operator algebra studied therein are generalized Kac-Moody algebras. In order to prove this we make some additional assumptions about the structure of the vertex operator algebra V. We say that V has a *real and positive-definite structure* if V satisfies the following properties:

1. V has a real form $V_\mathbb{R}$ such that $V_{\mathbb{R},1}$ is given by the split real form of V_1.
2. We furthermore assume that $V_\mathbb{R}$ has an involutive, isometric vertex operator algebra automorphism $\theta_\mathbb{R}$, such that $\langle v, \theta_\mathbb{R}(v)\rangle' > 0$ for all $v \in V_\mathbb{R} \setminus \{0\}$.

In the following we assume, that V has a real and positive-definite structure. Of course making an assumption like this is rather unpleasant. This is because in practice it is usually hard to verify, that a vertex operator algebra has such a structure. Yet we make this assumption because it drastically simplifies the proof of the next theorem.

Theorem 4.5.6. *Assume that V has a real and positive-definite structure. The Lie algebra $\mathfrak{g}(V)$ is a generalized Kac-Moody algebra with Cartan subalgebra $\mathcal{H}$ and root lattice $R = H' \oplus II_{1,1}$.*

Proof. We can apply the old covariant quantization to the real form $V_\mathbb{R}$ such that we obtain a corresponding Lie algebra of physical states $\mathfrak{g}(V_\mathbb{R})$. We may now check, that $\mathfrak{g}(V_\mathbb{R})$ satisfies the condition in Theorem 2.6.4. Since we assume that $V_{\mathbb{R},1}$ is the split real form of V_1 it contains its Cartan, i.e. the real span of its simple roots. We denote it by $\mathfrak{h}_\mathbb{R}$, as above. This, together with the previously obtain results over the complex numbers, yields a decomposition

$$V_\mathbb{R} = \bigoplus_{\alpha\in H'} V_{\mathbb{R},\alpha}. \tag{748}$$

Of course here we made use of the identification of $\mathfrak{h}_\mathbb{R}^*$ with $\mathfrak{h}_\mathbb{R}$, as explained above. This induces a Lie algebra grading of $\mathfrak{g}(V_\mathbb{R})$, given by

$$\mathfrak{g}(V_\mathbb{R}) = \bigoplus_{\alpha\in H'\oplus \mathrm{II}_{1,1}} \mathfrak{g}_\alpha(V_\mathbb{R}) \tag{749}$$

and we have $\mathfrak{g}_0(V_\mathbb{R}) = \mathcal{H}_\mathbb{R}$. Of course $\mathcal{H}_\mathbb{R}$ is a Cartan subalgebra of $\mathfrak{g}(V_\mathbb{R})$ and has a natural semisimple action on each root space $\mathfrak{g}_\alpha(V_\mathbb{R})$. It is given by the bilinear form $(\cdot,\cdot)_{\mathcal{H}_\mathbb{R}}$ of $\mathcal{H}_\mathbb{R}$. Remember that on the lattice vertex algebra $V_{\mathrm{II}_{1,1}}$ we have an involution θ which maps $V_{\mathrm{II}_{1,1},r}$ to $V_{\mathrm{II}_{1,1},-r}$ and acts as -1 on $V_{\mathrm{II}_{1,1},0}$. Of course, here we really work with the usual real form of $V_{\mathrm{II}_{1,1}}$. The tensor product map $\theta_\mathbb{R} \otimes \theta$ now induces a natural involution ω on the Lie algebra $\mathfrak{g}(V_\mathbb{R})$. It maps $\mathfrak{g}_\alpha(V_\mathbb{R})$ to $\mathfrak{g}_{-\alpha}(V_\mathbb{R})$ and acts as -1 on $\mathfrak{g}_0(V_\mathbb{R})$. Both statements are a direct consequence of the fact that $\theta_\mathbb{R}$ maps $V_{\mathbb{R},\alpha}$ to $V_{\mathbb{R},-\alpha}$

for $\alpha \in H'$. Take a root $\alpha \in H' \oplus \mathrm{II}_{1,1}$ and $x, y \in \mathfrak{g}_\alpha(V)$. By use of the no-ghost theorem 3.1.5 we find that we have $-(x, \omega(y)) = \langle v, \theta_{\mathbb{R}}(w)\rangle'$ for suitable elements $v, w \in V_{\mathbb{R}}$ with $\eta_\alpha(v) = x$ and $\eta_\alpha(w) = y$. The symmetric invariant bilinear form $-(\cdot,\cdot)$ preserves ω, we have $\mathfrak{g}_\alpha(V_{\mathbb{R}}) \perp \mathfrak{g}_\beta(V_{\mathbb{R}})$ unless $\alpha + \beta = 0$ and $-(x, \omega(x)) > 0$ for all $x \in \mathfrak{g}_\alpha(V_{\mathbb{R}})$ with $\alpha \neq 0$ and $x \neq 0$. So the last thing we have to check is the existence of a suitable $\mathbb{Z}$-grading. We may choose a vector $h' \in (H' \oplus \mathrm{II}_{1,1}) \otimes \mathbb{Q}$, which is *regular*, i.e. we have $(h', \alpha) \neq 0$ for all $\alpha \in H' \oplus \mathrm{II}_{1,1} \setminus \{0\}$. Rescaled suitably we may assume $(h', \alpha) \in \mathbb{Z}$ and for every $n \in \mathbb{Z}$ we set

$$\mathfrak{g}_n = \bigoplus_{(\alpha, h') = n} \mathfrak{g}_\alpha(V_{\mathbb{R}}). \tag{750}$$

Of course this defines a $\mathbb{Z}$-grading of $\mathfrak{g}(V_{\mathbb{R}})$. We have $\mathfrak{g}_0 = \mathfrak{g}_0(V_{\mathbb{R}})$ and each $\mathfrak{g}_n$ is clearly finite dimensional, since each $\mathfrak{g}_\alpha(V_{\mathbb{R}})$ is finite dimensional. The rest of the statement is clear and so we observe, that $\mathfrak{g}(V_{\mathbb{R}})$ is a generalized Kac-Moody algebra with root lattice $R = H' \oplus \mathrm{II}_{1,1}$. Of course its complexification $\mathfrak{g}(V)$ is then a generalized Kac-Moody algebra as well with root lattice $R = H' \oplus \mathrm{II}_{1,1}$. □

The proof of this theorem is similar to Borcherds' proof of Theorem 6.2 in [Bor92]. We expect it to be possible to prove this theorem without the assumption, that V has a real and positive-definite structure. In order to do this one would have to use Lemma 3.4.2. in [Car16], which is a characterization of complex generalized Kac-Moody algebras. The only difficulty is the verification of the 6th condition in this lemma. Unfortunately, this seems to be hard. Yet it is an interesting question for further research. Above we introduced the lattice H_2 of signature $(n+2, 2)$. We fix a basis e, f, e_2, f_2 of $\mathrm{II}_{1,1} \oplus \mathrm{II}_{1,1}$ with the usual properties, i.e. $e^2 = f^2 = e_2^2 = f_2^2 = 0$ and $(e, f) = (e_2, f_2) = 1$. Clearly we also assume $(e, e_2) = (e, f_2) = 0$ and $(f, e_2) = (f, f_2) = 0$. This allows us to write every $h_2 \in H_2$ as

$$h_2 = h + ae + bf + ce_2 + df_2 \tag{751}$$

for $h \in H$ and $a, b, c, d \in \mathbb{Z}$. We are going to identify the lattices H_1 and H with the sublattices of H_2 defined by $c = d = 0$ and $a = b = c = d = 0$ in equation (751) respectively. We may set $l = e_2$ and $\gamma = f_2$ to construct the usual tube domain $\mathbb{H}_l$ of the space $\mathcal{K}(H_2)$. In particular we obtain $(H_2)_{l,\gamma} = H_1$. Since we already discussed that we can view f_V as a modular form for the Weil representation on $D(H_2)$ of weight $k = 1 - \frac{n+2}{2}$ and the Fourier coefficients of the components of f_V are non-negative integers all assumptions of Theorem 2.4.3 are satisfied.

Definition 4.5.7. We associate to f_V its automorphic product $\Phi_V : \mathbb{H}_l \to \mathbb{C}$. This automorphic form is holomorphic and of singular weight, since the Fourier coefficients of f_V are non-negative integers and we have $[f_{V,0}](0) = \dim(\mathfrak{h}) = n$.

In the following we just consider the automorphic product Φ_V, associated with f_V. We understand all multiplicities mult$(r^\perp)$ of rational quadratic divisors $r^\perp$ with respect to this orthogonal modular form.

Lemma 4.5.8. *Let $r^* \in H' \oplus II_{1,1} \oplus II_{1,1}$ be primitive. Then there exists a primitive $\lambda^* \in H' \oplus II_{1,1}$ with $[r^*] = [\lambda^*]$ and $(r^*)^2 = (\lambda^*)^2$. We set $r = ord([r^*])r^*$ and $\lambda = ord([\lambda^*])\lambda^*$. They are both primitive and we have mult$(r^\perp)$ =mult$(\lambda^\perp)$ and r is a root in $H \oplus 2II_{1,1}$ if and only if λ is a root in $H \oplus II_{1,1}$.*

Proof. Assume we have $r^* = h + l_1 + l_2$ for $h \in H'$ and l_1, l_2 elements in the corresponding copies of $II_{1,1}$. Choose a primitive $l \in \mathrm{II}_{1,1}$ with $l^2 = l_1^2 + l_2^2$ and set $\lambda^* = h + l$. This is a primitive element in $H' \oplus \mathrm{II}_{1,1}$ and we obviously get $[r^*] = [\lambda^*]$ and $(r^*)^2 = (\lambda^*)^2$. The multiplicities are equal since we have $\left[f_{V,[n\lambda^*]}\right](-n^2(\lambda^*)^2/2) = \left[f_{V,[nr^*]}\right](-n^2(r^*)^2/2)$ for all positive integers n. For the last statement we use Lemma 2.4.7 and apply it to both r^* and λ^*. □

We are now ready to prove one of the main results of this thesis. Remember that we assume that V has a real and positive-definite structure. Otherwise we would not know that $\mathfrak{g}(V)$ is a generalized Kac-Moody algebra with root lattice $R = H' \oplus \mathrm{II}_{1,1}$.

Theorem 4.5.9. *The automorphic product Φ_V, associated to the vector-valued modular form f_V, is strongly reflective and of singular weight.*

Proof. The fact that we have $[f_{V,0}](0) = \dim(\mathfrak{h})$ directly implies, that Φ_V is of singular weight. Since each Fourier coefficient of f_V is a non-negative integer is is clear that Φ_V is holomorphic. Assume that $r \in H_2$ is a primitive vector such that $\mathrm{mult}(r^\perp) \neq 0$. We have to show that r is a root of H_2 and $\mathrm{mult}(r^\perp) = 1$. As usual we set $r^* = \frac{1}{\mathrm{div}(r)} r$. This is a primitive element in H_2'. Using Lemma 4.5.8 we can replace r^* by $\lambda^* \in H' \oplus \mathrm{II}_{1,1}$ and have to show that λ is a root of $H \oplus \mathrm{II}_{1,1}$. We have

$$\mathrm{mult}(\lambda^\perp) = \sum_{n=1}^{\infty} [f_{n\lambda^*}] \left(-n^2 \frac{(\lambda^*)^2}{2} \right) \tag{752}$$

and let n_0 be the minimal integer such that $[f_{n_0\lambda^*}] \left(-n_0^2 \frac{(\lambda^*)^2}{2} \right) \neq 0$. But this number is the multiplicity of the real root $\alpha = n_0\lambda^*$ of the generalized Kac-Moody algebra $\mathfrak{g}(V)$. It is a well-know fact of the theory of generalized Kac-Moody algebras that real roots have multiplicity 1 and $k\alpha$ must have multiplicity 0 unless $k = -1, 0, 1$. So we get

$$\mathrm{mult}(\lambda^\perp) = [f_{n_0\lambda^*}] \left(-n_0^2 \frac{(\lambda^*)^2}{2} \right) = 1. \tag{753}$$

Since α is a root of $\mathfrak{g}(V)$ we furthermore get $\sigma_\lambda = \sigma_\alpha \in W \subset O(R) = O(H' \oplus \mathrm{II}_{1,1})$, where W is the Weyl group of $\mathfrak{g}(V)$. As a consequence we get $\frac{2}{(\lambda^*)^2}\lambda^* \in (H' \oplus \mathrm{II}_{1,1})' = H \oplus \mathrm{II}_{1,1}$ but this implies that λ is a root of $H \oplus \mathrm{II}_{1,1}$. This is just an application of Lemma 2.4.7. □

5 Classification of reflective automorphic products

Reflective modular forms are holomorphic modular forms for the orthogonal group of some lattice, such that their divisor is a linear combination of reflective rational quadratic divisors. These modular forms play an important role in different areas of mathematics. In particular in algebraic geometry, number theory, the theory of reflection groups and conformal field theory. In conformal field theory, or more particular the theory of vertex operator algebras, their importance comes form the fact that denominator identities of interesting generalized Kac-Moody algebras tend to define reflective modular forms. Classification results of reflective modular forms therefore have strong consequences for all of those field. This is an active area of research and there have been several recent contributions. In the first section we sketch some of these results as a preparation for our own research work. Then we will discuss a new reduction method, which can be used to reduce a vector-valued modular form f with *small pole orders* to a scalar-valued, nearly holomorphic modular form g for a suitable congruence subgroup with character. This scalar-valued modular form will still have small pole orders at cusps of this congruence subgroup. Finally the *valence formula* implies strong bounds on the levels of the possible congruence groups that admit such modular forms g. Since vector-valued modular forms f, that correspond to reflective automorphic products Φ_f, have *small pole orders*, we can apply this method in this case. We obtain strong restrictions on the level of a lattice that can carry a reflective automorphic product. Together with known bounds on the rank of such lattices this implies, that there are just finitely many reflective lattices, which split certain hyperbolic planes.

5.1 Reflective modular forms

In this section we want to introduce some recent results in the classification problem of reflective modular forms. Important starting points are [Bor00] and the PhD thesis [Bar03]. In the case of squarefree level contributions were made by Scheithauer in [Sch06] and [Sch17]. Those results were extended by Dittmann in [Dit19] and [Dit18]. Furthermore Wang contributed several classification results in [Wan19c], [Wan21], [Wan19a] and [Wan19b]. In particular he obtained a full classification of lattices of prime level, that carry a reflective modular form. Wangs work is partly based on methods that rely on Jacobi forms instead of vector-valued modular forms, as it is done in the work of Scheithauer and Dittmann. A more geometric approach was taken by Ma in [Ma17] and [Ma18]. A detailed overview of all recent results was given in [Wan19b]. In the following we will reduce the discussion to the case of lattices, which carry not just a reflective modular form but a reflective automorphic product. Yet for many lattices this does not make a difference since we have Bruiniers converse theorem 2.4.4. All the papers above are an important source for the following discussion as well as a motivation for our own research. Throughout this section we let L be an even lattice of signature $(n, 2)$. If an even lattice has odd signature, then its level is divisible by 4. So, if we assume the level of the lattice L to be squarefree, it is obviously of even signature. This simplifies the discussion because the structure of the discriminant form $D(L)$ gets less complicated. We will not assume that the level of L is squarefree. Yet we will assume, that its signature is even. Let Φ be a holomorphic reflective automorphic product of weight $k_0 \in \mathbb{Z}$. We put $k = 1 - \frac{n}{2}$ and assume that $f \in M_k^!\left(\rho_{D(L)}\right)$ is a vector-valued modular form that lifts to Φ, i.e. we have $\Phi = \Phi_f$. Since the lattice L is of even signature we will view f as a vector-valued modular form for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ instead of its metaplectic cover $\mathrm{Mp}_2(\mathbb{Z})$. The reflectiveness of an automorphic product Φ_f can be characterized by its corresponding vector-valued modular form.

Proposition 5.1.1 ([Dit19, Prop. 4.3.]). *Suppose L has squarefree level and splits $II_{1,1}$. Then the automorphic product Φ_f is reflective if and only if the corresponding vector-valued modular form f satisfies:*

1. *If $\gamma \in D(L)$ has order m and corresponds to roots, then the Fourier coefficients of f_γ at $i\infty$ is*

$$f_\gamma(\tau) = c_{\gamma,-1/m} q^{-1/m} + \mathcal{O}(1) \tag{754}$$

with $c_{\gamma,-1/m} \geq 0$ and

2. *f_γ is holomorphic at $i\infty$ for all other $\gamma \in D(L)$.*

Moreover, Φ is strongly reflective if and only if all $c_{\gamma,-1/m}$ are at most 1.

Since we do not make the assumption that the level of L is squarefree we need a more general characterization.

Proposition 5.1.2. *We assume that L is an even lattice of signature $(n, 2)$ with $n \geq 3$ that splits $II_{1,1}$. Let f be a nearly holomorphic modular form of weight $k = 1 - \frac{n}{2}$ for the Weil representation $\rho_{D(L)}$. We assume furthermore that the Fourier coefficients of f_λ, which contribute to the principal part, are integers. If the automorphic product Φ_f of f is reflective then for $\lambda \in D(L)$ the components f_λ of f satisfy: If f_λ has a pole at $i\infty$, then there exists a divisor $d|N$ such that $d\lambda = 0$ and $\frac{\lambda^2}{2} = \frac{1}{d}(\mathrm{mod}\, \mathbb{Z})$. Furthermore there exists a number $c_\lambda \in \mathbb{C}$, such that the Fourier expansion of f_λ satisfies*

$$f_\lambda(\tau) = c_\lambda q^{-\frac{1}{d}} + \mathcal{O}(1). \tag{755}$$

If furthermore Φ_f is reflective with 2-roots then we have $[f_0](-1) \neq 0$.

Proof. We assume that Φ_f is reflective. We fix $\lambda \in D(L)$ and assume that there exists an $x > 0$ such that $[f_\lambda](-x) \neq 0$. Of course we have $\frac{\lambda^2}{2} \equiv x(\mathrm{mod}\, \mathbb{Z})$. Let n_0 be the maximal integer such that $[f_{n_0\lambda}](-n_0^2 x) \neq 0$. We fix a representant $\lambda \in L'$ for the class $\lambda \in D(L)$ and choose a primitive $l \in \mathrm{II}_{1,1}$ such that $\mu^* := n_0\lambda + l$ is primitive in L' and satisfies $(\mu^*)^2 = 2n_0^2 x$. As above we set $\mu = \mathrm{ord}([\mu^*])\mu^*$. This is a primitive element in L and we get

$$\mathrm{mult}(\mu^\perp) = \sum_{n=1}^{\infty} [f_{n\mu^*}] \left(-n^2 \frac{(\mu^*)^2}{2}\right) = \sum_{n=1}^{\infty} [f_{nn_0\lambda}] \left(-n^2 n_0^2 x\right) = [f_{n_0\lambda}](-n_0^2 x) \neq 0, \tag{756}$$

by the maximality of n_0. Since Φ_f is reflective we get that μ is a root of L. By use of Lemma 2.4.7 we get a $d|N$ with $d = \frac{2}{(\mu^*)^2}$ and $d\mu^* \in L$. We obtain $x = \frac{1}{n_0^2 d}$ and $n_0 d\lambda \in L$. Clearly we have $n_0^2 d\lambda \in L$ and $n_0^2 d|N$ as well. A direct consequence of this is that $x \leq 1$ and this implies that there can be at most one number $x > 0$ with the property that $[f_\lambda](-x) \neq 0$. Altogether we have proved that f_λ has the stated property. Assume now that Φ_f is reflective with 2-root, then we find $l \in L$ such that $l^2 = 2$ and $\mathrm{div}(l) = 1$ with $\mathrm{mult}(l^\perp) \neq 0$. Clearly we have $l^* = l$ and therefore we get

$$0 \neq \mathrm{mult}(l^\perp) = \sum_{n=1}^{\infty} [f_{nl^*}] \left(-n^2 \frac{(l^*)^2}{2}\right) = \sum_{n=1}^{\infty} [f_0] \left(-n^2\right) = [f_0](-1). \tag{757}$$

This is the statement we wanted to show. □

The proof of this proposition is quite similar to the proof given by Dittmann in [Dit19], which clearly was an important source of ideas.

Definition 5.1.3. We say that an even lattice of even signature $(n, 2)$ is *reflective* if there exists a nonzero reflective modular form Φ on $\tilde{\mathcal{K}}(L)$. In this case we also say that L *admits a reflective modular form.* If there exists even a nonzero reflective automorphic product on $\tilde{\mathcal{K}}(L)$ we say that L *admits a reflective automorphic product.*

This definition allows us to restate the question of the classification of reflective modular forms in terms of a classification of reflective lattices. We gain some flexibility by this since it allows us to manipulate a reflective modular form in certain ways. For example if a lattice admits a reflective modular form it also admits a *symmetric* reflective modular form, which is a reflective automorphic form invariant under $O(D(L))$. The reason for this is that we can always symmetrize a reflective modular form without losing its reflectiveness.

Proposition 5.1.4. *Let L be an even lattice of even signature $(n, 2)$ with $n \geq 4$ that splits $II_{1,1}$. If L admits a reflective automorphic product its signature is bounded by $n \leq 26$.*

Proof. We assume that L is an even lattice as in the statement that admits a reflective automorphic product Φ which correspond to a vector-valued modular form f for the Weil representation on $\mathbb{C}[D(L)]$ of weight $k = 1 - \frac{n}{2}$ for the group $\mathrm{SL}_2(\mathbb{Z})$. A direct consequence of equation (78) is that the component f_0 of f is a nearly holomorphic modular form of weight k for the group $\Gamma_0(N)$ with character $\chi_{D(L)}$. Here N is the level of the discriminant form $D(L)$ and $\chi_{D(L)}$ is the associated character. So we have $f_0 \in M_k^!(\Gamma_0(N), \chi_{D(L)})$. Let s be a cusp of $\Gamma_0(N)$, then the expansion $f_{0,s}$ of f_0 at s is clearly just a linear combination of components f_λ of f for $\lambda \in D(L)$. Using Proposition 5.1.2 we find that the vanishing orders of f_0 at cusps have to be bounded by

$$-1 \leq \mathrm{ord}_s(f_0) = \mathrm{ord}_{i\infty}(f_{0,s}). \tag{758}$$

Notice that if Φ is nonzero its weight $k_0 = \frac{[f_0](0)}{2}$ has to be nonzero as well. This implies $f_0 \neq 0$ and so the valence formula as given in Theorem 4.1.4 in [Ran08] yields the inequality

$$\sum_{s \in \Gamma_0(N) \backslash \mathbb{P}^1(\mathbb{Q})} [\mathrm{PSL}_2(\mathbb{Z})_s : \mathrm{P}\Gamma_0(N)_s] \, \mathrm{ord}_s(f_0) \leq \frac{k}{12} \left[\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{P}\Gamma_0(N)\right]. \tag{759}$$

Using the bounds on $\mathrm{ord}_s(f_0)$ we find

$$\frac{-k}{12} \left[\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{P}\Gamma_0(N)\right] \leq \sum_{s \in \Gamma_0(N) \backslash \mathbb{P}^1(\mathbb{Q})} [\mathrm{PSL}_2(\mathbb{Z})_s : \mathrm{P}\Gamma_0(N)_s] \,, \tag{760}$$

which directly implies $-k \leq 12$ but this is equivalent to $n \leq 26$. □

The statement of this proposition a well-known and can also be proved by the observation, that $\Delta f_0 \in M_{12+k}\left(\Gamma_0(N), \chi_D\right)$. Here Δ is, as usual, the modular discriminant. Yet this proof clearly demonstrates the strength of equation (759). In the following we will write $t(s)$ for the *width* of the cusp s, i.e. $t(s) = [\mathrm{PSL}_2(\mathbb{Z})_s : \mathrm{P}\Gamma_0(N)_s]$ and we put $\epsilon_0(N) = [\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{P}\Gamma_0(N)]$. So for every $0 \neq g \in M_k^!(\Gamma_0(N), \chi_{D(L)})$ we have the inequality

$$\sum_{s \in \Gamma_0(N) \backslash \mathbb{P}^1(\mathbb{Q})} t(s) \mathrm{ord}_s(g) \leq \frac{k}{12} \epsilon_0(N). \tag{761}$$

In order to restrict the set of lattices that admit a reflective automorphic product we will have to find bounds on the level N of those lattices. This is done in the case of lattices of squarefree level N that split $\mathrm{II}_{1,1} \oplus \mathrm{II}_{1,1}(N)$ in [Dit19]. To do this one has to find a suitable nearly holomorphic modular form $g \in M_k^!(\Gamma_0(N), \chi_{D(L)})$ that satisfies bounds on the vanishing order that are even stronger than those given in (758). For this we may replace the vector-valued modular form by its symmetrization

$$\mathrm{sym}(f) = \frac{1}{|\mathrm{Aut}(D(L))|} \sum_{\sigma \in \mathrm{Aut}(D(L))} \sigma(f). \tag{762}$$

This is a *symmetric* vector-valued modular for the Weil representation of $\mathbb{C}[D(L)]$. i.e. a vector-valued modular form that is invariant under the group $\mathrm{Aut}(D(L))$. As a consequence of Corollary 5.5 in [Sch15] one finds a nearly holomorphic modular form $g \in M_k^!(\Gamma_0(N), \chi_{D(L)})$ that satisfies

$$f(\tau) = F_{\Gamma_0(N), g, 0}(\tau) = \sum_{M \in \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z})} g|_k M(\tau) \rho_{D(L)}(M^{-1}) \mathfrak{e}_0. \tag{763}$$

The modular form g satisfies the bounds of the vanishing orders we seek.

Lemma 5.1.5 ([Dit19, Lemma 4.4.])**.** *Let L be an even lattice of signature $(n,2)$ and squarefree level N that splits $\mathit{II}_{1,1} \oplus \mathit{II}_{1,1}(N)$, then the modular form $g \in M_k^!(\Gamma_0(N), \chi_{D(L)})$ associated to a reflective automorphic product, as above, satisfies $g_s \in \mathcal{O}\left(q^{-\frac{1}{t(s)}}\right)$ for every cusp s of $\Gamma_0(N)$.*

Putting this modular form into (761) yields an inequality that just can be solved by finitely many natural numbers $N \in \mathbb{Z}_{\geq 0}$.

Theorem 5.1.6 ([Dit19, Theorem 1.1.]). *There are only finitely many even lattices L of signature $(n, 2)$, $n \geq 4$ and squarefree level N that split $II_{1,1} \oplus II_{1,1}(N)$ and carry a nonconstant reflective modular form.*

One of the aims of this section is to generalize this theorem to the case of lattices with even signature but possibly non-squarefree level. This will be a further principal result of this thesis. In this proof the only difficulty is to associate to each vector-valued modular from f, which corresponds to a reflective automorphic product Φ_f, a suitable scalar-valued modular form $g \in M_k^!(\Gamma_0(N), \chi_{D(L)})$ which satisfies strong bounds of pole orders at cusps. The equation (761) will then imply a bounds in the corresponding level N. Since N is also the level of the corresponding lattice the theorem follows.

5.2 A partial reduction to $\Gamma_0(N)$

In this section we introduce a method which allows us to reduce a vector-valued modular form f for a Weil representation on a discriminant form to another vector-valued modular form f' for a Weil representation on some smaller discriminant form. An important property of this method is that if the components of the modular form f have *small pole orders*, then the reduced modular form f' will have the same property. The idea is essentially to consider suitable linear combinations of the components of f, such that all poles at cusps, that don't satisfy suitable bounds, cancel each other. During the whole section we consider a discriminant form D of level M with even signature and take an even lattice L of signature $(n, 2)$ with discriminant form $D(L) = D$. Let $f : \mathbb{H} \to \mathbb{C}[D]$ be a vector-valued modular form for the Weil representation of the group $\Gamma_0(N)$ of weight $k = 1 - \frac{n}{2}$ for some Dirichlet character χ of $\Gamma_0(N)$. We assume that the component f_0 of f has a pole of order 1 at $i\infty$, i.e. we have

$$f_0(\tau) = cq^{-1} + \mathcal{O}(1) \tag{764}$$

for some $c \in \mathbb{C}^*$. Later, we will allow $c = 0$ but here we make this assumption because it drastically simplifies the discussion. Following Borcherds in section 8 of [Bor00] we find that a full set of representatives of the cusps of $\Gamma_0(N)$ is given by $\frac{a}{c} \in \mathbb{Q}$ for $c|N$, $c > 0$ and $0 < a \leq \left(c, \frac{N}{c}\right)$ with $(a, c) = 1$. Usually we will work with those representatives unless stated otherwise. The width of the cusp $s = \frac{a}{c}$ is given by $t(s) = \frac{N}{(c^2, N)}$ and for each such cusp $s = \frac{a}{c}$ we pick and fix a matrix $M_s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $\frac{a}{c} = M_s i\infty$.

Definition 5.2.1. Assume $N \in \mathbb{Z}_{\geq 0}$ satisfies $(N, M) = 1$. Let $f : \mathbb{H} \to \mathbb{C}[D]$ be a nearly holomorphic vector-valued modular form for the Weil representation of $\Gamma_0(N)$ with character χ. We say that f has *small pole orders* if f satisfies that for every cusp s of $\Gamma_0(N)$ the Fourier expansion f_s at the cusp s of f has the following property: If $(f_s)_\lambda(\tau)$ has a pole at $i\infty$, there exists a divisor $d_\lambda|M$ and an integer $x \in \mathbb{Z}$ with $(x, d_\lambda) = 1$ such that $d_\lambda \lambda = 0$, $\frac{\lambda^2}{2} = \frac{x}{d_\lambda} (\mathrm{mod}\ \mathbb{Z})$ and

$$(f_s)_\lambda \in \mathcal{O}\left(q^{-\frac{1}{t(s)d_\lambda}}\right). \tag{765}$$

We assume that our vector-valued modular form f has small pole orders in the sense of this definition. The discriminant D has level M and we take a prime p and an integer $m \in \mathbb{Z}_{\geq 0}$ such that $p^m || M$. Let us assume $M = M'p^m$ for some M' such that $(M', p^m) = 1$. We can decompose D as

$$D = D_{M'} \oplus D_{p^m}. \tag{766}$$

First we assume that p is an odd prime, then we obtain a Jordan decomposition of D_{p^m} by

$$D_{p^m} \sim (p)^{\epsilon_1 n_1} \cdots (p^m)^{\epsilon_m n_m} \tag{767}$$

with $\epsilon_i = \pm 1$ and $n_i \in \mathbb{Z}_{\leq 0}$ for $i = 1, \cdots, m$. In particular we have $n_m \neq 0$ because otherwise we would have $p^m \nmid N$. We may take a vector

$$v = \sum_{\mu \in D_{p^m}(0)} a_\mu \mathfrak{e}_\mu \in \mathbb{C}[D_{p^m}(0)]. \tag{768}$$

which satisfies $a_0 \neq 0$ and is invariant under $(\mathbb{Z}/p^m\mathbb{Z})^*$. As a consequence of Proposition 2.2.2 we obtain that $f'(\tau) = \langle f(\tau), v \rangle$ is a vector-valued modular form for the Weil representation on $\mathbb{C}[D_{M'}]$ for the group $\Gamma_0(Np^m)$ of weight k with character $\chi\chi_{D_{p^m}}$. Since f has small pole orders the only component f_λ corresponding to an isotropic element $\lambda \in D$ that can have a pole at $i\infty$ is f_0. This is because if f_λ has a pole at $i\infty$ for some isotropic $\lambda \in D$ there has to be a divisor $d_\lambda | N$ with all the properties discussed above. We directly find $d_\lambda = 1$, however. Now we get $\lambda = d_\lambda \lambda = 0$. A direct consequence is that we have

$$f_0'(\tau) = \overline{a_0} f_0(\tau) + \mathcal{O}(1) = \overline{a_0} v q^{-1} + \mathcal{O}(1). \tag{769}$$

So f_0' has a pole of order 1 at $i\infty$ as well. In the following we want to find a vector v as above with $a_0 \neq 0$ such that f' has small pole orders as well. If this is possible or not will depend on the structure of the discriminant form.

Definition 5.2.2. We call the p-adic Jordan block D_{p^m} of D *regular* if it contains a nontrivial isotropic element. Otherwise we will call it *irregular*. Furthermore we will call a discriminant form D *regular* if each p-adic Jordan block of it is regular.

For odd prime numbers p it is easy to classify all regular p-adic Jordan blocks. Assume that D_{p^m} is a discriminant form of level p^m with a p-adic Jordan decomposition as in (767). Assume that $m \geq 2$ than we may pick an element $\gamma \in (p^m)^{\epsilon_m n_m}$ of multiplicity $\mathrm{mult}(\gamma) = 1$. Clearly $p^{m-1}\gamma$ is nontrivial and isotropic, therefore D_{p^m} is regular if $m \geq 2$. Assume $m = 1$, then there exists an explicit formula to compute the number of isotropic elements. See Proposition 3.2 in [Sch06].

Lemma 5.2.3. *Let p be an odd prime. A p-adic Jordan block D_{p^m} is irregular if and only if its p-adic Jordan symbol is $(p)^{\pm 1}$ or $(p)^{-\left(\frac{-1}{p}\right)2}$.*

In the following discussion we may assume that D_{p^m} is regular. We set some terminology. Let $\frac{a}{c} \in \mathbb{Q}$ be representant of a cusp s of $\Gamma_0(Np^m)$ with $c | Np^m$, $0 < a < \left(c, \frac{Np^m}{c}\right)$ and $(a, c) = 1$. Clearly this fraction also represents a cusp of $\Gamma_0(N)$ which we will call s as well. We will denote the width of s with respect to $\Gamma_0(N)$ by $t_N(s)$ and with respect to $\Gamma_0(Np^m)$ by $t_{Np^m}(s)$. As proved in Proposition 2.2.2 the components of the expansion at the cusp s of f' is given by

$$(f_s')_{\lambda_1}(\tau) = \sum_{\lambda_2 \in D_2} (\rho_{D_2}(M_s)\mathfrak{e}_{\lambda_2}, v)(f_s)_{\lambda_1 + \lambda_2}(\tau). \tag{770}$$

Lemma 5.2.4. *Assume that $(f_s)_{\lambda_1 + \lambda_2}$ has a pole at $i\infty$ for some $\lambda_1 \in D_{M'}$ and $\lambda_2 \in D_{p^m}$. Then we find $d_1 | M'$ and $d_2 | p^m$ with $d_1\lambda_1 = 0$ and $d_2\lambda_2 = 0$ and some $x, y \in \mathbb{Z}$ with $(x, d_1) = (y, d_2) = 1$ such that $\frac{\lambda_1^2}{2} = \frac{x}{d_1} (\mathrm{mod}\, \mathbb{Z})$ and $\frac{\lambda_2^2}{2} = \frac{y}{d_2} (\mathrm{mod}\, \mathbb{Z})$.*

Proof. Since f has small pole orders we find a divisor $d_0 | M = M'p^m$ with $d_0(\lambda_1 + \lambda_2) = 0$ and some $x_0 \in \mathbb{Z}$ with $(x_0, d_0) = 1$ and $\frac{(\lambda_1+\lambda_2)^2}{2} = \frac{x_0}{d_0} (\mathrm{mod}\, \mathbb{Z})$. We set $d_1 = (d_0, M')$ and $d_2 = (d_0, p^m)$ and find $d_1\lambda_1 = 0$ and $d_2\lambda_2 = 0$. Obviously we find $x', y' \in \mathbb{Z}$ such that $\frac{(\lambda_1)^2}{2} = \frac{x'}{M'} (\mathrm{mod}\, \mathbb{Z})$ and $\frac{(\lambda_2)^2}{2} = \frac{y'}{p^m} (\mathrm{mod}\, \mathbb{Z})$. Because of $(M', p^m) = 1$ we must have $d\frac{(\lambda_1)^2}{2} = \frac{dx'}{M'} = \frac{d_1 d_2 x'}{M'} = 0 (\mathrm{mod}\, \mathbb{Z})$. Since we have $(d_2, M') = 1$ as well this is just possible if already $\frac{d_1 x'}{M'} = 0 (\mathrm{mod}\, \mathbb{Z})$. So we find $x \in \mathbb{Z}$ such that $\frac{x'}{M'} = \frac{x}{d_1}$, i.e. $\frac{(\lambda_1)^2}{2} = \frac{x}{d_1} (\mathrm{mod}\, \mathbb{Z})$. Analogously we obtain $y \in \mathbb{Z}$ such that $\frac{(\lambda_2)^2}{2} = \frac{y}{d_2} (\mathrm{mod}\, \mathbb{Z})$. We get $\frac{x_0}{d_1 d_2} = \frac{x_0}{d_0} = \frac{x}{d_1} + \frac{y}{d_2} (\mathrm{mod}\, \mathbb{Z})$. But because of $(x, d_1 d_2) = 1$ we must have $(x, d_1) = 1$ as well. Analogously we obtain $(y, d_2) = 1$. □

Since we fix a decomposition (767) we can always decompose an $\lambda \in D_{p^m}$ as

$$\lambda = \lambda_1 + \cdots + \lambda_m \tag{771}$$

with $\lambda_i \in (p^i)^{\epsilon_i n_i}$.

Lemma 5.2.5. *Let p be an odd prime and D_{p^m} a p-adic Jordan block with a decomposition as in (767). Let $\lambda \in D_{p^m}$ be an element such that there exists a divisor $d|p^m$ and an integer $x \in \mathbb{Z}$ with $(d, x) = 1$ such that $d\lambda = 0$ and $\frac{\lambda^2}{2} = \frac{x}{d}(\mathrm{mod}\,\mathbb{Z})$. Assume $d = p^i$, then we have $\lambda = 0$ or the multiplicity of λ_i in D_{p^m} is 1. In this case the multiplicity of λ in D_{p^m} is 1 as well.*

Proof. Of course we have $\lambda = 0$ if and only if $d = 1$. So we can assume $\lambda \neq 0$ and $d = p^i$ with $1 \leq i \leq m$. Notice that $(p^j)^{\epsilon_j n_j}$ is just $(\mathbb{Z}/p^j\mathbb{Z})^{n_j}$ as an abelian group. So it is clear that for $i+1 \leq j \leq m$ we find λ'_j in $(p^j)^{\epsilon_j n_j}$ with

$$\lambda = \lambda_1 + \cdots + \lambda_i + p\lambda'_{i+1} + \cdots + p^{m-i}\lambda'_m. \tag{772}$$

For every $j = 1, \cdots, m$ we find $a_j \in \mathbb{Z}$ such that $\frac{(\lambda_j)^2}{2} = \frac{a_j}{p^j}(\mathrm{mod}\,\mathbb{Z})$ if $1 \leq j \leq i$ and $\frac{(\lambda'_j)^2}{2} = \frac{a_j}{p^j}(\mathrm{mod}\,\mathbb{Z})$ for $i+1 \leq j \leq m$. Putting this together we obtain

$$\frac{x}{p^i} = \frac{\lambda^2}{2} = \frac{a_1}{p} + \cdots + \frac{a_i}{p^i} + p^2\frac{a_{i+1}}{p^{i+1}} + \cdots + p^{2(m-i)}\frac{a_m}{p^m}(\mathrm{mod}\,\mathbb{Z}). \tag{773}$$

By multiplication with p^{i-1} we obtain $\frac{x}{p} = \frac{a_i}{p}(\mathrm{mod}\,\mathbb{Z})$. Since we have $(x, p) = 1$ and therefore $\frac{x}{p} \neq 0(\mathrm{mod}\,\mathbb{Z})$ we obtain $\frac{a_i}{p} \neq 0(\mathrm{mod}\,\mathbb{Z})$ and $(a_i, p) = 1$. In particular we must have $\lambda_i \neq 0$. We have to show that multiplicity of λ_i is 1. Assume this statement to be wrong, then we find $\lambda'_i \in (p^i)^{\epsilon_i n_i}$ such that $\lambda_i = p\lambda'_i$. We can furthermore pick an integer $a'_i \in \mathbb{Z}$ such that $\frac{(\lambda')_i^2}{2} = \frac{a'_i}{p^i}(\mathrm{mod}\,\mathbb{Z})$. Yet as a consequence we get $\frac{a_i}{p^i} = \frac{a'_i}{p^{i-2}}(\mathrm{mod}\,\mathbb{Z})$, which is clearly not possible because of $(p, a_i) = 1$. But then the multiplicity of λ_i has to be 1. This implies that the multiplicity of λ is 1 as well. □

We may introduce a subset of $D_{p^m}(0)$ by

$$I_p = \left\{\gamma \in D_{p^m} : \mathrm{mult}(\gamma) \geq p^{m-1} \text{ and } \frac{\gamma^2}{2} = 0(\mathrm{mod}\,\mathbb{Z})\right\} \subset (p^m)^{\epsilon_m n_m}. \tag{774}$$

Clearly if $\lambda \in I_p$ then we have for the decomposition $\lambda = \lambda_1 + \cdots + \lambda_m$ that $\lambda_i = 0$ if $i < m$, i.e. $\lambda = \lambda_m$. We check furthermore that if $m > 1$ the condition $\frac{\lambda^2}{2} = 0(\mathrm{mod}\,\mathbb{Z})$ is true for all $\lambda \in D_{p^m}$ with $\mathrm{mult}(\lambda) \geq p^{m-1}$. Furthermore we set $\mathring{I}_p = I_p \setminus \{0\}$ and make an Ansatz

$$v = \sum_{\mu \in I_p} a_\mu \mathfrak{e}_\mu = a_0\mathfrak{e}_0 + \sum_{\mu \in \mathring{I}_p} a_\mu \mathfrak{e}_\mu \tag{775}$$

with the assumption that $a_0 \neq 0$ and $a_{d\mu} = a_\mu$ for all $d \in (\mathbb{Z}/p^m\mathbb{Z})^*$ and $\mu \in I_p$. Let U be any subset of D_{p^m}. We denote its *characteristic function* by δ_U, i.e. for $\nu \in D_p^m$ we have $\delta_U(\nu) = 1$ if $\nu \in U$ and $\delta_U(\nu) = 0$ otherwise.

Lemma 5.2.6. *Let s be a cusp of $\Gamma_0(Np^m)$ represented by $\frac{a}{c}$ with $c|Np^m$, $c > 0$ and $0 < a < \left(c, \frac{Np^m}{c}\right)$ with $(a, c) = 1$. Take a matrix $M_s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $s = M_s i\infty$. Furthermore let $\lambda_2 \in D_{p^m}$ be an element such that there are $d|p^m$ and $x \in \mathbb{Z}$ with $(x, d) = 1$, $d\lambda_2 = 0$ and $\frac{\lambda_2^2}{2} = \frac{x}{d}(\mathrm{mod}\,\mathbb{Z})$. If the cusp s satisfies $p|c$ we have*

$$\left(\rho_{D_{p^m}}(M_s)\mathfrak{e}_{\lambda_2}, v\right) = \xi_{D_{p^m}}(M_s)\delta_{\lambda_2,0}\frac{\sqrt{|(D_{p^m})_{c_{p^m}}|}}{\sqrt{|D_{p^m}|}}\sum_{\mu \in I_p}\delta_{D_{p^m}^{c_{p^m}}}(\mu)\overline{a_\mu}e(-a(\mu)_c^2/2)). \tag{776}$$

If the cusp s satisfies $p \nmid c$ we have

$$\left(\rho_{D_{p^m}}(M_s)\mathfrak{e}_{\lambda_2}, v\right) = \xi_{D_{p^m}}(M_s)\frac{1}{\sqrt{|D_{p^m}|}} e(-dc_N^{-1}(\lambda_2^2/2)) \sum_{\mu \in I_p} \overline{a_\mu} e(c_N^{-1}(\mu, \lambda_2)). \tag{777}$$

Proof. We fix a cusp s of $\Gamma_0(Np^m)$ and a rational number $\frac{a}{c} \in \mathbb{Q}$ which represents this cusp with the properties $c|Np^m$, $c > 0$ and $0 < a \leq \left(c, \frac{Np^m}{c}\right)$ with $(a, c) = 1$. We pick $b, d \in \mathbb{Z}$ such that $M_s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Clearly we have $s = \frac{a}{c} = M_s i\infty$. We decompose c as $c = c_N c_{p^m}$ with $c_N = (c, N)$ and $c_{p^m} = (c, p^m)$. Take $\mu \in I_p$. We start by evaluating $\left(\rho_{D_{p^m}}(M_s)\mathfrak{e}_{\lambda_2}, \mathfrak{e}_\mu\right)$. Since p is an odd prime we obtain $D_{p^m}^{c*} = D_{p^m}^{c} = D_{p^m}^{c_{p^m}}$. So the elements in $D_{p^m}^{c*}$ are just the elements $\gamma \in D_{p^m}$ which satisfy $\mathrm{mult}(\gamma) \geq c_{p^m}$. We fix an element $\lambda_2 \in D_{p^m}$ with the properties as in the statement. By use of (78) we obtain

$$\begin{aligned}
&(\rho_{D_{p^m}}(M)\mathfrak{e}_{\lambda_2}, e_\mu) && (778)\\
&= \xi_{D_{p^m}}(M_s)\frac{\sqrt{|(D_{p^m})_{c_{p^m}}|}}{\sqrt{|D_{p^m}|}} \sum_{\beta \in D_{p^m}^{c_p}} e(-a\beta_c^2/2 - b(\beta, \lambda_2) - bd(\lambda_2^2/2))(d\lambda_2 + \beta, \mu) && (779)\\
&= \delta_{D_{p^m}^{c_{p^m}}}(\mu - d\lambda_2)\xi_{D_{p^m}}(M_s)\frac{\sqrt{|(D_{p^m})_{c_{p^m}}|}}{\sqrt{|D_{p^m}|}} e(-a(\mu - d\lambda_2)_c^2/2 - b(\mu - d\lambda_2, \lambda_2) - bd(\lambda_2^2/2)) && (780)
\end{aligned}$$

Assume $p|c^{p^m}$ in this case we may pick an integer d^{-1} such that $d^{-1}d \equiv 1 (\mathrm{mod}\, p^m)$. So clearly if $\beta = \mu - d\lambda_2 \in D_{p^m}^{c_{p^m}}$ we obtain that $\lambda_2 = d^{-1}(\mu - \beta)$ has multiplicity $\mathrm{mult}(\beta) \geq p$. But this already implies $\lambda_2 = 0$ by Lemma 5.2.5. In this case we get

$$\begin{aligned}
&(\rho_{D_{p^m}}(M)\mathfrak{e}_0, e_\mu) && (781)\\
&= \delta_{D_{p^m}^{c_{p^m}}}(\mu)\xi_{D_{p^m}}(M_s)\frac{\sqrt{|(D_{p^m})_{c_{p^m}}|}}{\sqrt{|D_{p^m}|}} e(-a(\mu)_c^2/2)). && (782)
\end{aligned}$$

So altogether we obtain

$$\left(\rho_{D_{p^m}}(M_s)\mathfrak{e}_{\lambda_2}, v\right) = \xi_{D_{p^m}}(M_s)\delta_{\lambda_2,0}\frac{\sqrt{|(D_{p^m})_{c_{p^m}}|}}{\sqrt{|D_{p^m}|}} \sum_{\mu \in I_p} \delta_{D_{p^m}^{c_{p^m}}}(\mu)\overline{a_\mu} e(-a(\mu)_c^2/2)). \tag{783}$$

Let us now assume that $p \nmid c$, i.e. we have $c_{p^m} = 1$. We pick $c_N^{-1} \in \mathbb{Z}$ such that $c_N^{-1}c_N = 1(\mathrm{mod}\, p^m)$. Similar to the reasoning above we obtain

$$\begin{aligned}
&(\rho_{D_{p^m}}(M)\mathfrak{e}_{\lambda_2}, e_\mu) && (784)\\
&= \xi_{D_{p^m}}(M_s)\frac{1}{\sqrt{|D_{p^m}|}} \sum_{\beta \in D_{p^m}} e(-a\beta_c^2/2 - b(\beta, \lambda_2) - bd(\lambda_2^2/2))\mu_{d\lambda_2+\beta,\mu} && (785)\\
&= \xi_{D_{p^m}}(M_s)\frac{1}{\sqrt{|D_{p^m}|}} e(-ac_N^{-1}(\mu - d\lambda_2)^2/2 - b(\mu - d\lambda_2, \lambda_2) - bd(\lambda_2^2/2)) && (786)\\
&= \xi_{D_{p^m}}(M_s)\frac{1}{\sqrt{|D_{p^m}|}} e(c_N^{-1}(\mu, \lambda_2) - dc_N^{-1}(\lambda_2^2/2)) && (787)
\end{aligned}$$

So we obtain

$$\left(\rho_{D_{p^m}}(M_s)\mathfrak{e}_{\lambda_2}, v\right) = \xi_{D_{p^m}}(M_s)\frac{1}{\sqrt{|D_{p^m}|}}e(-dc_N^{-1}(\lambda_2^2/2))\sum_{\mu\in I_p}\overline{a_\mu}e(c_N^{-1}(\mu,\lambda_2)). \tag{788}$$

This proves the statement. □

Proposition 5.2.7. *Let p be an odd prime and assume that the p-adic Jordan block D_{p^m} is regular. Then there exists a solution v for the Ansatz (775) such that $f' = \langle f, v\rangle$ is a vector-valued modular form for the Weil representation on $\mathbb{C}[D_{M'}]$ for the group $\Gamma_0(Np^m)$ of weight k and character $\chi\chi_{D_{p^m}}$ with small pole orders and f_0' has a pole at $i\infty$ of order 1.*

Proof. Clearly any solution of the Ansatz (775) yields a vector-valued modular form for the Weil representation on $\mathbb{C}[D_{M'}]$ for the group $\Gamma_0(Np^m)$ of weight k and character $\chi\chi_{D_{p^m}}$. By use of (769) it is furthermore clear that f_0' has a pole of order 1 at $i\infty$. If $(f_s')_{\lambda_1}$ has a pole at $i\infty$ for some cusp s of $\Gamma_0(Np^m)$ and $\lambda_1 \in D_{M'}$ then Lemma 5.2.4 provides us with a divisor $d_1|M'$ and an integer $x \in \mathbb{Z}$ with $(d_1, x) = 1$, $d_1\lambda_1 = 0$ and $\frac{\lambda_1^2}{2} = \frac{x}{d_1}(\mathrm{mod}\ \mathbb{Z})$. It remains to be checked that each component $(f_s')_{\lambda_1}$ satisfies

$$(f_s')_{\lambda_1}(\tau) = \mathcal{O}\left(q^{-\frac{1}{t_{Np^m}(s)d_{\lambda_1}}}\right). \tag{789}$$

Since we assume f to have small pole orders we know that if $(f_s)_{\lambda_1+\lambda_2}$ has a pole at $i\infty$ we have

$$(f_s)_{\lambda_1+\lambda_2}(\tau) = \mathcal{O}\left(q^{-\frac{1}{t_N(s)d_{\lambda_1}d_{\lambda_2}}}\right). \tag{790}$$

Clearly we have $t_{Np^m}(s) = t_N(s)t_{p^m}(s)$ and therefore we just have to find a solution v for the Ansatz above such that for every cusp s of $\Gamma_0(Np^m)$ and every λ_2 we have

$$d_{\lambda_2} < t_{p^m}(s) \Rightarrow \left(\rho_{D_{p^m}}(M_s)\mathfrak{e}_{\lambda_2}, v\right) = 0. \tag{791}$$

We write $c_{p^m} = p^i||c$. We may start with the case that $p|c$, i.e. $1 \le i$. The width $t_{p^i}(s)$ is given by $t_{p^i}(s) = \frac{p^m}{(p^{2i},p^m)}$. Since there is nothing to prove if $t_{p^i}(s) = 1$ we may assume $2 \le 2i < m$. As a consequence of Lemma 5.2.6 we may assume $\lambda_2 = 0$, since otherwise the statement is clear. Take any $\mu \in I_p$ and observe $i < \frac{m}{2} < m-1$. It is clear that we have $\delta_{D_{p^m}^{c_{p^m}}}(\mu) = 1$. So we are left with the evaluation of $\frac{\mu_c^2}{2}(\mathrm{mod}\ \mathbb{Z})$. As usual we may pick an integer $c_N^{-1} \in \mathbb{Z}$ such that $c_Nc_N^{-1} = 1(\mathrm{mod}\ p^m)$ and we obtain $\frac{\mu_c^2}{2} = c_N^{-1}\frac{\mu_{p^i}^2}{2}(\mathrm{mod}\ \mathbb{Z})$. We clearly find $\mu' \in (p^m)^{\epsilon_m n_m}$ with $\mu = p^{m-1}\mu'$ and some integer $y \in \mathbb{Z}$ $\frac{(\mu')^2}{2} = \frac{y}{p^m}(\mathrm{mod}\ \mathbb{Z})$. So we obtain

$$\frac{\mu_{p^i}^2}{2} = p^i\frac{(p^{m-i-1}\mu')^2}{2} = p^ip^{2(m-i-1)}\frac{y}{p^m} = p^{m-2-i}y = 0(\mathrm{mod}\ \mathbb{Z}). \tag{792}$$

Putting things together we obtain for every cusp s as above that

$$\left(\rho_{D_{p^m}}(M_s)\mathfrak{e}_0, v\right) = \xi_{D_{p^m}}(M_s)\frac{\sqrt{|(D_{p^m})_{c_{p^m}}|}}{\sqrt{|D_{p^m}|}}\sum_{\mu\in I_p}\overline{a_\mu}. \tag{793}$$

Let now $s = \frac{a}{c}$ be a cusp such that $p \nmid c$, i.e. $c_p = 1$ and $t_{p^m}(s) = p^m$. We furthermore assume $m \ge 2$. Let $\lambda_2 \in D_{p^m}$ be as above, so there exists a divisor $d_{\lambda_2}|p^m$ and $y \in \mathbb{Z}$ such that $(d_{\lambda_2}, y) = 1$, $d_{\lambda_2}\lambda_2 = 0$ and $\frac{\lambda_2^2}{2} = \frac{y}{d_{\lambda_2}}(\mathrm{mod}\ \mathbb{Z})$. We may assume $d_{\lambda_2} = p^j$ and $j \le m-1$, since otherwise we would not have $d_{\lambda_2} < t_{p^m}(s)$. Similar to the decomposition in (772) we can find an element $\lambda_m' \in (p^m)^{\epsilon_m n_m}$ such that λ_2 can be decomposed as

$$\lambda_2 = (\lambda_2)_1 + \cdots + (\lambda_2)_{m-1} + p\lambda_m' \tag{794}$$

with $(\lambda_2)_i \in (p^i)^{\epsilon_i n_i}$. Using this decomposition it is clear that we have $(\lambda_2, \mu) = 0(\text{mod } \mathbb{Z})$. So far this just works for $m \geq 2$. But in the case $M = 1$ we get $t_p(s) = p$ in this situation so the only choice for λ_2, such that $d_{\lambda_2} < t_p(s)$, is $\lambda_2 = 0$. But then we clearly have $(\lambda_2, \mu) = (0, \mu) = 0(\text{mod } \mathbb{Z})$. So for every cusp s with $p \nmid c$ and every λ_2 with $d_{\lambda_2} < t_{p^m}(s)$ we get

$$\left(\rho_{D_{p^m}}(M_s)\mathfrak{e}_{\lambda_2}, v\right) = \xi_{D_{p^m}}(M_s)\frac{1}{\sqrt{|D_{p^m}|}}e(-dc_N^{-1}(\lambda_2^2/2)) \sum_{\mu \in I_p} \overline{a_\mu}. \tag{795}$$

We find that in order to satisfy the condition in (791) a solution for the Ansatz (775) just has to solve

$$\sum_{\mu \in I_p} \overline{a_\mu} = 0. \tag{796}$$

Notice that we have $|\mathring{I}_p| \neq 0$ since D_{p^m} is regular. So we can easily solve this for example we can put $a_0 = 1$ and $a_\mu = -\frac{1}{|\mathring{I}_p|}$. This solution is invariant under $(\mathbb{Z}/p^m\mathbb{Z})^*$ and satisfies therefore all the properties we want. □

So far we assumed that p is an odd prime. In the following discussion we assume that $p = 2$. This discussion is similar to the discussion for odd primes p. Yet some details are different and this is why we treat this case separately. We assume that the level M of D is of the form $M = M'2^m$ with $(M', 2^m) = 1$. For simplicity we assume furthermore that the 2-adic Jordan decomposition of D_{2^m} is of the form

$$D_{2^m} \sim (2)_{t_1}^{\epsilon_1 n_1} \cdots (2^{m-1})_{t_{m-1}}^{\epsilon_{m-1} n_{m-1}} (2^m)_{\mathrm{II}}^{\epsilon_m n_m}. \tag{797}$$

For $i = 1, \cdots, m-1$ we may have $t_i \in \mathbb{Z}/8\mathbb{Z}$ or simply $t_i = \mathrm{II}$, if the i-th component is even. Yet we demand that the 2-adic Jordan component of maximal exponent 2^m is even. Or in other words we assume that $n_m \neq 0$. If the component of exponent 2^i is even we have $2|n_i$ of course. Once again we put

$$I_2 = \left\{\gamma \in D_{2^m} : \text{mult}(\gamma) \geq 2^{m-1} \text{ and } \frac{\gamma^2}{2} = 0(\text{mod } \mathbb{Z})\right\} \subset (2^m)_{\mathrm{II}}^{\epsilon_m n_m} \tag{798}$$

and we set $\mathring{I}_2 = I_2 \setminus \{0\}$. As a consequence of Proposition 3.1 in [Sch06] we obtain the following lemma.

Lemma 5.2.8. *Let D_{2^m} be a 2-adic Jordan block of the form as in (797). Then D_{2^m} is irregular if and only if its 2-adic Jordan symbol is given by $(2)_{\mathit{II}}^{-2}$.*

As above we make an Ansatz

$$v = \sum_{\mu \in I_2} a_\mu \mathfrak{e}_\mu = a_0 \mathfrak{e}_0 + \sum_{\mu \in \mathring{I}_2} a_\mu \mathfrak{e}_\mu \tag{799}$$

with the assumption that $a_0 \neq 0$ and $a_{d\mu} = a_\mu$ for all $d \in (\mathbb{Z}/2^m\mathbb{Z})^*$ and $\mu \in I_2$. We find that $f'(\tau) = \langle f(\tau), v\rangle$ is a vector-valued modular form for the Weil representation on $\mathbb{C}[D_{M'}]$ for the group $\Gamma_0(N2^m)$ of weight k with character $\chi\chi_{D_{2^m}}$. This is a consequence of Proposition 2.2.2. This implies that we are once again left with the evaluation of the pole orders of the components of f' at cusps, i.e. we have to show that f' has small pole orders. Assume $(f'_s)_{\lambda_1}$ has a pole at $i\infty$ using (770) and Lemma 5.2.4 we find $d_{\lambda_1}|M'$ and $x \in \mathbb{Z}$ with $(x, d_{\lambda_1}) = 1$, $d_{\lambda_1}\lambda_1 = 0$ and $\frac{\lambda_1^2}{2} = \frac{x}{d_{\lambda_1}}(\text{mod } \mathbb{Z})$. Similarly for every $\lambda_2 \in D_{2^m}$ such that $(f_s)_{\lambda_1+\lambda_2}$ has a pole at $i\infty$ we find $d_{\lambda_2}|2^m$ and $y \in \mathbb{Z}$ with $(y, d_{\lambda_2}) = 1$, $d_{\lambda_2}\lambda_2 = 0$ and $\frac{\lambda_2^2}{2} = \frac{y}{d_{\lambda_2}}(\text{mod } \mathbb{Z})$. In order to prove that f' has small pole orders it is once again enough to show that for each pole s of $\Gamma_0(N2^m)$ and d_{λ_2} we have

$$d_{\lambda_2} < t_{2^m}(s) \Rightarrow (\rho_{D_{2^m}}(M_s)\mathfrak{e}_{\lambda_2}, v) = 0. \tag{800}$$

In the evaluation of $\left(\rho_{D_{p^m}}(M_s)\mathfrak{e}_{\lambda_2}, v\right)$ for odd primes p the Lemma 5.2.5 was clearly quite useful. We made use of it in the proof of Lemma 5.2.6. For the prime number $p = 2$ we argue in a slightly different manner, since this lemma does not directly generalize to this case. We represent the cusp s of $\Gamma_0(N2^m)$ as usual by a fraction $\frac{a}{c} \in \mathbb{Q}$ with the usual properties. We decompose $c = c_N c_{2^m}$ as above and assume $c_{2^m} = 2^i$. In the case of $t_{2^m}(s) = 1$ the condition (800) is trivial so we may assume $2i < m$ in the following. Let U be any subset of D_{2^m}. We can introduce a *characteristic function* δ_U as above.

Lemma 5.2.9. *Let s be a cusp of $\Gamma_0(N2^m)$ represented by $\frac{a}{c}$ with $c|N2^m$, $c > 0$ and $0 < a < \left(c, \frac{N2^m}{c}\right)$ with $(a, c) = 1$. Take a matrix $M_s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $s = M_s i\infty$. We assume $2^i||c$. Furthermore let $\lambda_2 \in D_{2^m}$ be an element such that there are $d_{\lambda_2}|2^m$ and $x \in \mathbb{Z}$ with $(x, d_{\lambda_2}) = 1$, $d_{\lambda_2}\lambda_2 = 0$ and $\frac{\lambda_2^2}{2} = \frac{x}{d_{\lambda_2}} (\mathrm{mod}\, \mathbb{Z})$. We furthermore assume that $d_{\lambda_2} < t_{2^m}(s)$. In the case $1 \leq i < 2m$ we have*

$$(\rho_{D_{2^m}}(M_s)\mathfrak{e}_{\lambda_2}, v) = \xi_{D_{2^m}}(M_s)\frac{\sqrt{|(D_{2^m})_c|}}{\sqrt{|D_{2^m}|}}\delta_{D_2^{c*}}(\lambda_2)e\left(-ad^2\frac{(\lambda_2)_c^2}{2} + bd\frac{\lambda_2^2}{2}\right)\sum_{\mu \in I_2}\overline{a_\mu}. \tag{801}$$

In the case $i = 0$ we have

$$\left(\rho_{D_{p^m}}(M_s)\mathfrak{e}_{\lambda_2}, v\right) = \frac{1}{\sqrt{|D_{2^m}|}}e(-dc_N^{-1}(\lambda_2^2/2))\sum_{\mu \in I_2}\overline{a_\mu}. \tag{802}$$

Proof. We start with the case $i = 0$. Clearly we have $D_{2^m}^{c*} = D_{2^m}$. For $\mu \in I_2$ we have

$$(\rho_{D_{2^m}}(M)\mathfrak{e}_{\lambda_2}, e_\mu) \tag{803}$$
$$= \xi_{D_{2^m}}(M_s)\frac{1}{\sqrt{|D_{2^m}|}}\sum_{\beta \in D_{2^m}} e(-a\beta_c^2/2 - b(\beta, \lambda_2) - bd(\lambda_2^2/2))\mu_{d\lambda_2+\beta,\mu} \tag{804}$$
$$= \xi_{D_{2^m}}(M_s)\frac{1}{\sqrt{|D_{2^m}|}}e(-ac_N^{-1}(\mu - d\lambda_2)^2/2 - b(\mu - d\lambda_2, \lambda_2) - bd(\lambda_2^2/2)) \tag{805}$$
$$= \xi_{D_{2^m}}(M_s)\frac{1}{\sqrt{|D_{2^m}|}}e(c_N^{-1}(\mu, \lambda_2) - dc_N^{-1}(\lambda_2^2/2)) \tag{806}$$

So we obtain

$$(\rho_{D_{2^m}}(M)\mathfrak{e}_{\lambda_2}, v) \tag{807}$$
$$= \frac{1}{\sqrt{|D_{2^m}|}}e(-dc_N^{-1}(\lambda_2^2/2))\sum_{\mu \in I_2}\overline{a_\mu}e(c_N^{-1}(\mu, \lambda_2)) \tag{808}$$
$$= \frac{1}{\sqrt{|D_{2^m}|}}e(-dc_N^{-1}(\lambda_2^2/2))\sum_{\mu \in I_2}\overline{a_\mu}. \tag{809}$$

Notice that the last equation makes use of $(\mu, \lambda_2) = 0(\mathrm{mod}\, \mathbb{Z})$. This is obvious for $\mu = 0$. We consider the case $\mu \neq 0$. For $m > 1$ we have $\mu = 2^{m-1}\mu'$ and we have a decomposition

$$\lambda_2 = (\lambda_2)_1 + \cdots + (\lambda_2)_{m-1} + 2\lambda'_m \tag{810}$$

simply because we have $d_{\lambda_2}\lambda_2 = 0$ and $d_{\lambda_2} \leq 2^{m-1} < 2^m = t_{2^m}(s)$. In the case $m = 1$ we obtain $d_{\lambda_2} = 1$ so we have $\lambda_2 = 0$. So $(\mu, \lambda_2) = 0(\mathrm{mod}\, \mathbb{Z})$ is clear as well. This is of course the same argument as in the discussion of odd primes p. We consider the case $1 \leq i < \frac{m}{2}$. For $\mu \in I_2$ we have

$$(\rho_{D_{2^m}}(M)\mathfrak{e}_{\lambda_2}, e_\mu) \tag{811}$$
$$= \xi_{D_{2^m}}(M_s)\frac{\sqrt{|(D_{2^m})_c|}}{\sqrt{|D_{2^m}|}} \sum_{\beta \in D_{2^m}^{c*}} e\left(-a\frac{\beta_c^2}{2} - b(\beta,\lambda_2) - bd\frac{\lambda_2^2}{2}\right)(\mathfrak{e}_{d\lambda_2+\beta}, \mathfrak{e}_\mu) \tag{812}$$
$$= \xi_{D_{2^m}}(M_s)\frac{\sqrt{|(D_{2^m})_c|}}{\sqrt{|D_{2^m}|}}\delta_{D_2^{c*}}(\mu - d\lambda_2)e\left(-a\frac{(\mu-d\lambda_2)_c^2}{2} - b(\mu - d\lambda_2, \lambda_2) - bd\frac{\lambda_2^2}{2}\right) \tag{813}$$
$$= \xi_{D_{2^m}}(M_s)\frac{\sqrt{|(D_{2^m})_c|}}{\sqrt{|D_{2^m}|}}\delta_{D_2^{c*}}(\lambda_2)e\left(-ad^2\frac{(\lambda_2)_c^2}{2} + bd\frac{\lambda_2^2}{2}\right) \tag{814}$$

The equation (814) needs some further explanation. Because of $i < \frac{m}{2} \leq m-1$ we have for every $\mu \in I_2$ that $\mu - d\lambda_2 \in D_2^{c*}$ if and only if $\lambda_2 \in D_2^{c*}$. Using this it is easy to prove that $\frac{(\mu - d\lambda_2)_c^2}{2} = d^2\frac{(\lambda_2)_c^2}{2}(\text{mod }\mathbb{Z})$. We furthermore have $(\mu,\lambda_2) = 0(\text{mod }\mathbb{Z})$ as above. Applying this result to v we obtain

$$(\rho_{D_{2^m}}(M)\mathfrak{e}_{\lambda_2}, v) = \xi_{D_{2^m}}(M_s)\frac{\sqrt{|(D_{2^m})_c|}}{\sqrt{|D_{2^m}|}}\delta_{D_2^{c*}}(\lambda_2)e\left(-ad^2\frac{(\lambda_2)_c^2}{2} + bd\frac{\lambda_2^2}{2}\right)\sum_{\mu\in I_2}\overline{a_\mu}. \tag{815}$$

This is exactly the statement of the lemma. □

Proposition 5.2.10. *Assume that the* 2*-adic Jordan block* D_{2^m} *is regular and of the form as in (797). Then there exists a solution* v *for the Ansatz (775) such that* $f' = \langle f, v\rangle$ *is a vector-valued modular form for the Weil representation on* $\mathbb{C}[D_{M'}]$ *for the group* $\Gamma_0(N2^m)$ *of weight* k *and character* $\chi\chi_{D_{2^m}}$ *with small pole orders and* f_0' *has a pole at* $i\infty$ *of order* 1.

Proof. We may pick a solution v for the Ansatz (799) such that $a_0 \neq 0$ and $a_{d\mu} = a_\mu$ for $d \in (\mathbb{Z}/2^m\mathbb{Z})^*$ and $\mu \in I_2$. As explained above Proposition 2.2.2 implies that $f' = \langle f, v\rangle$ is a vector-valued modular form for the Weil representation on $\mathbb{C}[D_{M'}]$ for the group $\Gamma_0(N2^m)$ of weight k and character $\chi\chi_{D_{2^m}}$. Because of $a_0 \neq 0$ we know that f_0' has a pole of order 1 at $i\infty$. The Lemma 5.2.9 implies that f' has small pole order if $\sum_{\mu\in I_2}\overline{a_\mu} = 0$. W can easily pick a v such that this is true as well. For example we can set $a_0 = 1$ and $a_\mu = -\frac{1}{|I_2|}$. □

5.3 A partial reduction to $\Gamma_1(N)$

The reduction method of the previous section has the disadvantage that the reduction f' of a vector-valued modular form f of small pole order might vanish if the component f_0 of f does not have a pole of order 1 at $i\infty$. In this section we relax the condition that f has small pole order suitably by use of a certain *bound function* B. We then say that f has *pole orders bounded by* B. Next we introduce an alternative reduction method, which yields a reduction f', that has pole orders bounded by B, if f has this property. Furthermore this reduction f' is nonzero, if f is nonzero. This does not longer depend on a pole of f_0 at $i\infty$. Even though the principal idea is still the same, we have to work with the group $\Gamma_1(N)$ instead of $\Gamma_0(N)$. This is because we can no longer work just with isotropic elements and need more flexibility. Let $N \in \mathbb{Z}_{\geq 0}$ be a non-negative integer with prime decomposition

$$N = p_1^{r_1}\cdots p_n^{r_n}. \tag{816}$$

We have the usual *floor*-function $\lfloor x\rfloor$ which maps $x \in \mathbb{R}$ to the largest integer n such that $n \leq x$. Furthermore we have the *ceil*-function $\lceil x\rceil$ which maps $x \in \mathbb{R}$ to the minimal integer n such that $n \geq x$. For a prime power p^r we define a *bound function* B by $B(p^r) := p^{\lceil\frac{r}{2}\rceil}$ for $r \geq 2$ and $B(p^r) = 1$ if $r = 0, 1$. We may extend this function to all integers by $B(nm) = B(n)B(m)$ in the case $(n,m) = 1$. For the natural number N from above we obtain

$$B(N) = B(p_1^{r_1})\cdots B(p_n^{r_n}). \tag{817}$$

The basic assumptions in this section are still similar to those in the previous section. We assume that L is an even lattice of signature $(n, 2)$ in the genus corresponding to the discriminant form D, i.e. we have $D = D(L)$. Let the level of D be M. Throughout this section we consider a nonzero vector-valued modular form $f : \mathbb{H} \to \mathbb{C}[D]$ for the Weil representation of the group $\Gamma_1(N)$ of weight $k = 1 - \frac{n}{2}$ for some character χ of $\Gamma_1(N)$ with $(N, M) = 1$.

Definition 5.3.1. Assume $N \in \mathbb{Z}_{\geq 0}$ satisfies $(N, M) = 1$. Let $f : \mathbb{H} \to \mathbb{C}[D]$ be a nearly holomorphic vector-valued modular form of weight k for the Weil representation of $\Gamma_1(N)$ with character χ. We say that f has *pole orders bounded by* B if it satisfies that for every cusp s of $\Gamma_1(N)$ the Fourier expansion f_s at the cusp s of f has the following property: If $(f_s)_\lambda(\tau)$ has a pole at $i\infty$ then there exists a divisor $d_\lambda | M$ and an integer $x \in \mathbb{Z}$ with $(x, d_\lambda) = 1$ such that $d_\lambda \lambda = 0$, $\frac{\lambda^2}{2} = \frac{x}{d_\lambda} (\text{mod } \mathbb{Z})$ and

$$(f_s)_\lambda \in \mathcal{O}\left(q^{-\frac{B(N)}{t_N(s) d_\lambda}}\right). \tag{818}$$

This definition reasonably generalizes what we called modular forms of small pole orders in the last section. Another way to state the condition (818) is

$$t_N(s) d_\lambda \text{ord}_{i\infty}\left((f_s)_\lambda\right) \leq B(N). \tag{819}$$

We will usually work with this inequality in the following. We assume that our modular form f has pole orders bounded by B. Let p be a prime such that there exists $m \in \mathbb{Z}_{\geq 0}$, $m \geq 1$ with $p^m || M$. We write $M = M' p^m$. For any rational number $x \in \mathbb{Q}$ and any vector $v \in \mathbb{C}[D_{p^m}(x)]$ the holomorphic function $f'(\tau) = \langle f(\tau), v \rangle$ is a vector-valued modular form for the Weil representation $\rho_{D_{M'}}$ of weight k for the group $\Gamma_1(Np^m)$ with character $\chi\chi_x$. Now we want to find such a vector $v \in \mathbb{C}[D_{p^m}(x)]$ such that F' has pole orders bounded by B. The following proposition is the main result of this subsection.

Proposition 5.3.2. *Let D be a regular discriminant form of level M and assume that the 2-adic Jordan component $D_{2^{m_2}}$ of D, with $2^{m_2} || M$, is of the form (797). Assume $N \in \mathbb{Z}_{\geq 0}$ satisfies $(N, M) = 1$ and let $f : \mathbb{H} \to \mathbb{C}[D]$ be a nonzero vector-valued modular form of weight k for the Weil representation of $\Gamma_1(N)$ with character χ. Assume f has pole orders bounded by B. Let p be a prime such that there exists an integer $m \geq 1$ with $p^m || M$. We write $M = M' p^m$. Then there exists $x \in \mathbb{Q}$ and a vector $v \in D_{p^m}(x)$ such that $f'(\tau) = \langle f(\tau), v \rangle$ is a nonzero vector-valued modular form of weight k for the Weil representation $\rho_{D_{M'}}$ of $\Gamma_1(Np^m)$ with character $\chi\chi_x$ which has pole orders bounded by B.*

Notice first that there has to exist a cusp s of $\Gamma_1(N)$ and an element $\beta \in D$ such that $(f_s)_\beta$ has a pole at $i\infty$. This is because it is easy to see that each component of f_β of f has to be a nearly holomorphic modular form for the group $\Gamma_1(NM)$ of weight k for some character χ'. Yet we have $k < 0$ therefore the valence formula implies $f_\beta = 0$ if f_β does not have a a pole at some cusp of $\Gamma_1(NM)$. Yet the expansion at such a cusp s is a linear combination of components of f_s. So if for all cusps s of $\Gamma_0(NM)$ and $\beta \in D$ the functions $(f_s)_\beta$ are holomorphic at $i\infty$ then we get $f_\beta = 0$ for each $\beta \in D$. This is clearly not possible if $f \neq 0$. Assume $(f_s)_\beta$ has a cusp at $i\infty$. By assumption f has pole orders bounded by B, so we find $d_\beta | M$ and $x \in \mathbb{Z}$ with $(x, d_\beta) = 1$, $d_\beta \beta = 0$ and $\frac{\beta^2}{2} = \frac{x}{d_\beta} (\text{mod } \mathbb{Z})$. We decompose β as $\beta = \beta' + \beta_p$ with $\beta' \in D_{M'}$ and $\beta_p \in D_{p^m}$ and we write d_β as $d_\beta = d'_\beta d_{\beta,p}$ with $d_{\beta'} = (M' d_\beta)$ and $d_{\beta_p} = (d_\beta, p^m)$. As usual we have $d_{\beta'} \beta' = 0$ and $d_{\beta_p} \beta_p = 0$ and find $x_1, x_2 \in \mathbb{Z}$ with $(d_{\beta'}, x_1) = (d_{\beta_p}, x_2) = 1$ such that $\frac{(\beta')^2}{2} = \frac{x_1}{d_{\beta'}} (\text{mod } \mathbb{Z})$ and $\frac{(\beta_p)^2}{2} = \frac{x_2}{d_{\beta_p}} (\text{mod } \mathbb{Z})$. This is just Lemma 5.2.4. There exists a cusp s_0 of $\Gamma_1(N)$ and some $\delta \in D$ such that $(f_{s_0})_\delta$ has a pole at $i\infty$ and d_{δ_p} is *minimal* in the sense that if for any other cusp s of $\Gamma_1(N)$ and any other $\beta \in D$ the function $(f_s)_\beta$ has a pole at $i\infty$, then we have $d_{\delta_p} | d_{\beta_p}$. We fix an element $\delta \in D$ such that d_{δ_p} is minimal and for $d_{\delta_p} = p^i$ we call i *minimal* as well. Let s be any cusp of $\Gamma_1(N)$. Then there exists a rational number $\frac{a}{c}$ with $(a, c) = 1$, $p^m | c$ and $\frac{a}{c} \in \Gamma_1(N)s$, i.e. $\frac{a}{c}$ represents the cusp s. We chose $M_s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ with $\frac{a}{c} = M_s i\infty$. Of course the fraction $\frac{a}{c}$ represents a cusp of $\Gamma_1(Np^m)$ as well. We denote this cusp by s' and we set $M_{s'} = M_s$. Clearly different cusps of $\Gamma_1(N_p^m)$ may corresponding to the same cusp of $\Gamma_1(N)$ but by s' we always denote the choice induced by this construction.

Lemma 5.3.3. *Chose a cusp s_0 of $\Gamma_1(N)$ and $\delta \in D$ such that $(f_{s_0})_\delta$ has a pole at $i\infty$ and d_{δ_p} is minimal. As explained above we represent s_0 by a rational number $\frac{a_0}{c_0}$ with $p^m|c_0$ and fix $M_{s_0} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in SL_2(\mathbb{Z})$. We denote the cusp of $\Gamma_1(Np^m)$ represented by $\frac{a_0}{c_0}$ by s_0' and pick $M_{s_0'} = M_{s_0}$. We put $\mu = d_0\delta_p \in D_{p^r}$ and $v_0 = e^\mu \in \mathbb{C}[D_{p^m}(x)]$ with $x \in \mathbb{Q}$ such that $x = \frac{\mu^2}{2} (\mathrm{mod}\, \mathbb{Z})$. Then the function $f'(\tau) = \langle f(\tau), v_0 \rangle$ is a vector-valued modular form of weight k for the Weil representation $\rho_{D_{M'}}$ of $\Gamma_1(Np^m)$ with character $\chi\chi_x$ and we have*

$$(f'_{s_0'})_{\delta'}(\tau) = \xi_{D_{p^m}}(M_{s_0}) e\left(-b_0 d_0 \frac{\delta_p^2}{2}\right) (f_{s_0})_\delta(\tau). \tag{820}$$

This implies that $(f'_{s_0'})_{\delta'}$ has a pole at $i\infty$.

Proof. This proof is a direct computation that can easily carried out by use of Proposition 2.2.2. □

The statement of this lemma should be viewed as an analog of equation (764). In the following we will extend the vector v_0 such that f' still has some pole but also has pole orders bounded by B. As above we start with the case of odd primes p.

Lemma 5.3.4. *We assume $m = 1$ and that D_p is regular, then there exists a number $x \in \mathbb{Q}$ and a vector $v \in \mathbb{C}[D_p(x)]$ such that $f'(\tau) = \langle f(\tau), v \rangle$ is a nonzero vector-valued modular form of weight k for the Weil representation $\rho_{D_{M'}}$ of $\Gamma_1(Np^m)$, with character $\chi\chi_x$, which has pole orders bounded by B.*

Proof. Chose a cusp s_0 of $\Gamma_1(N)$ and $\delta \in D$ such that $(f_{s_0})_\delta$ has a pole at $i\infty$ and d_{δ_p} is minimal. Again we represent this cusp by a fraction $\frac{a}{c}$ as in Lemma 5.3.3 and chose $M_{s_0} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ accordingly. We write $d_{\delta_p} = p^i$ we either have $i = 0$ or $i = 1$. Let us assume $i = 1$ first. This is the trivial case, since for every cusp $\tilde{s}$ of $\Gamma_1(Np)$ and every $\beta \in D$ such that $(f_{\tilde{s}})_\beta$ has a pole at $i\infty$ we have

$$t_{Np}(\tilde{s}) d_{\beta'} \mathrm{ord}_{i\infty}((f_{\tilde{s}})_\beta) \leq t_N(\tilde{s}) d_{\beta'} p \mathrm{ord}_{i\infty}((f_{\tilde{s}})_\beta) \leq B(N) = B(Np). \tag{821}$$

Of course we view $\tilde{s}$ as a cusp of $\Gamma_1(N)$ if we consider $f_{\tilde{s}}$. We may pick v as in Lemma 5.3.3 to obtain a function f' with all the properties of the statement. We consider the case i=0. Let $\gamma \in D_p$ be a nonzero isotropic vector. Clearly γ has multiplicity 1 in this case. This implies that $(f_s)_{\beta'+\gamma}$ has to be holomorphic at $i\infty$ for all cusps s of $\Gamma_1(N)$ and all $\beta' \in D_{M'}$ because of $1\gamma \neq 0$. If we put $v = e^0 - e^{d\gamma}$ we obtain

$$\left(f'_{s_0'}\right)_{\delta'}(\tau) = \xi_{D_p}(M_{s_0}) \left((f_{s_0})_{\delta'+0}(\tau) - (f_{s_0})_{\delta'+\gamma}(\tau)\right), \tag{822}$$

which has a pole at $i\infty$ since $(f_{s_0})_{\delta'+0}$ has a pole at $i\infty$. This implies that f' is nonzero. We have to show that F' has pole order bounded by B. For cusps s of $\Gamma_1(Np)$ that satisfy $t_p(s) = 1$ the statement is once again trivial. This is because of $t_{Np}(s) = t_N(s)$ in this case. We may assume $t_p(s) = p$. Of course we have

$$(f'_s)_{\lambda'}(\tau) = \sum_{\lambda_p \in D_p} (\rho_{D_p}(M_s)\mathfrak{e}_{\lambda_p}, v)(f_s)_{\lambda'+\lambda_p}(\tau). \tag{823}$$

Since for poles coming from summands $(f_s)_{\lambda'+\lambda_p}(\tau)$ with $d_{\lambda_p} = p$ the inequality (819) is obviously satisfied we just have to show that $(\rho_{D_p}(M_s)\mathfrak{e}_0, v) = 0$. Since $t_p(s) = p$ implies $p \nmid c$ we pick an integer $c^{-1} \in \mathbb{Z}$ with $c^{-1}c = 1 (\mathrm{mod}\, p)$ and compute

$$(\rho_{D_p}(M_s)\mathfrak{e}_0, v) = \xi_{D_p}(M_s) \frac{1}{\sqrt{|D_p|}} \left[e\left(-ac^{-1}\frac{0^2}{2}\right) - e\left(-ac^{-1}\frac{(d\gamma)^2}{2}\right) \right] = 0. \tag{824}$$

Now we have checked that f' has pole orders bounded by B. □

From now on we assume that $m \geq 2$. Once again pick a cusp s_0 of $\Gamma_1(N)$ and $\delta \in D$ such that $(f_{s_0})_\delta$ has a pole at $i\infty$ and $d_{\delta_p} = p^i$ is minimal. A direct consequence of this is that for every cusp s of $\Gamma_1(Np^m)$ and every $\lambda' \in D_{M'}$, such that $(f'_s)_{\lambda'}$ has a pole at $i\infty$, we have

$$\mathrm{ord}_{i\infty}\left((f'_s)_{\lambda'}\right) \leq \frac{B(N)}{t_N(s)d_{\lambda'}p^i}. \tag{825}$$

We will make frequent use of this inequality. Let us assume $\lfloor \frac{m}{2} \rfloor \leq i$ for now. As a direct consequence of (825) we obtain that for every cusp s of $\Gamma_1(Np^m)$ and every $\lambda' \in D_{M'}$ we have for any choice of $v \in \mathbb{C}[D_{p^m}(x)]$ for $x \in \mathbb{Q}$ that

$$\mathrm{ord}_{i\infty}\left((F'_s)_{\lambda'}\right) \leq \frac{B(N)p^{\lceil \frac{m}{2} \rceil}}{t_N(s)d_{\lambda'}p^m} \leq \frac{B(Np^m)}{t_{Np^m}(s)d_{\lambda'}}. \tag{826}$$

In other words if $\lfloor \frac{m}{2} \rfloor \leq i$ we may pick $x \in \mathbb{Q}$ and $v \in \mathbb{C}[D_{p^m}(x)]$ as in Lemma 5.3.3. From now on we assume $i < \lfloor \frac{m}{2} \rfloor$. Since we have $p^m || M$ it is clear that D_{p^m} has a Jordan decomposition as in (767) with $n_m \neq 0$. We take an element $\gamma' \in (p^m)^{\epsilon_m n_m}$ of multiplicity 1 and put $\gamma = p^{m-i-1}\gamma'$. The assumption $i < \lfloor \frac{m}{2} \rfloor$ implies $0 \leq m - 2i - 2$ and we obtain that γ is isotropic and $(\gamma, \delta_p) = 0$. We fix a cusp s_0 and $\delta \in D$ such that $(f_{s_0})_\delta$ has a pole at $i\infty$ and d_{δ_p} is minimal as in Lemma 5.3.3. We use all notations form there and set $\mu_1 = d_0\delta_p$ and $\mu_2 = d_0\delta_p + d_0\gamma$. We chose $x \in \mathbb{Q}$ such that $x = \frac{\mu_1^2}{2} = \frac{\mu_2^2}{2} \pmod{\mathbb{Z}}$. Now we can introduce

$$v = e^{\mu_1} - e^{\mu_2} \in \mathbb{C}[D_{p^m}(x)]. \tag{827}$$

We introduce a vector-valued modular form f' by $f'(\tau) = \langle f(\tau), v \rangle$ of weight k for the Weil representation $\rho_{D_{M'}}$ of $\Gamma_1(Np^m)$ with character $\chi\chi_x$. Now we evaluate

$$(f'_{s'_0})_{\delta'}(\tau) = \xi_{D_{p^m}}(M_{s_0})e(-b_0d_0x)\left((f_{s_0})_{\delta'+\delta_p}(\tau) - (f_{s_0})_{\delta'+\gamma}(\tau)\right), \tag{828}$$

by a computation similar to the proof of Lemma 5.3.3. Since $(f_{s_0})_{\delta'+\gamma}(\tau)$ can't have a pole at $i\infty$ it is clear that $(f'_{s'_0})_{\delta'}(\tau)$ must have a pole at $i\infty$. This implies $f' \neq 0$. We have to show that f has pole orders bounded by B. We start with a cusp s of $\Gamma_1(Np^m)$ and assume that s is represented by a rational number $\frac{a}{c} \in \mathbb{Q}$ with $(a, c) = 1$ and pick $M_s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $\frac{a}{c} = M_s i\infty$ as usual. We set $c_N = (c, N)$ and $c_{p^m} = (c, p^m) = p^h$ for some $0 \leq h \leq m$.

Lemma 5.3.5. *Let s be a cusp of $\Gamma_1(Np^m)$ such that h satisfies $m - 2i - 1 \leq h$. Then for every $\lambda' \in D_{M'}$ such that $(f'_s)_{\lambda'}$ has a pole at $i\infty$ we have*

$$t_{Np^m}(s)d_{\lambda'}\mathit{ord}_{i\infty}((f'_s)_{\lambda'}) \leq B(Np^m). \tag{829}$$

Proof. Of course we have $t_{p^m}(s) \leq \frac{p^m}{p^h}$ and $m - 2i - 1 \leq h$ implies $t_{p^m}(s) \leq \frac{p^m}{p^h} \leq p^{2i+1}$. Using (825) we obtain

$$t_{Np^m}(s)d_{\lambda'}\mathrm{ord}_{i\infty}((f'_s)_{\lambda'}) \leq t_{Np^m}(s)d_{\lambda'}\frac{B(N)}{t_N(s)d_{\lambda'}p^i} \leq B(N)p^{i+1} \leq B(Np^m). \tag{830}$$

For the last inequality we made use of the assumption $i < \lfloor \frac{m}{2} \rfloor$. □

Of course the proof of this lemma is trivial in the sense that we just work with the obvious bound (825). Therefore this result would hold for any choice of vector v not just the one made in (827).

Lemma 5.3.6. *Let s be a cusp of $\Gamma_1(Np^m)$ represented by $\frac{a}{c}$ as above. We decompose c as $c = c_N c_{p^m}$ with $c_{p^m} = p^h$ and pick $M_s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathit{SL}_2(\mathbb{Z})$. We assume $h \leq m - 2i - 2$. For a fixed element*

$\mu \in D_{p^m}$ we have $\beta_1 = \mu_1 - d\mu \in D^c_{p^r}$ if and only if we have $\beta_2 = \mu_2 - d\mu \in D^c_{p^r}$. In the case $\beta_1 \notin D^c_{p^r}$ we find

$$(\rho_{D_{p^m}}(M_s)\mathfrak{e}_\mu, v) = 0 \tag{831}$$

and in case $\beta_1 \in D^c_{p^r}$ we obtain

$$(\rho_{D_{p^m}}(M_s)\mathfrak{e}_\mu, v) = \xi_{D_{p^m}}(M_s) e\left(-bd\frac{\mu^2}{2}\right)\frac{\sqrt{|(D_{p^m})_c|}}{\sqrt{|(D_{p^m})|}} * \tag{832}$$

$$\left[e\left(-ac_N^{-1}\frac{(\beta_1)^2_{c_{p^m}}}{2}\right)e(-b(\beta_1,\mu)) - e\left(-ac_N^{-1}\frac{(\beta_2)^2_{c_{p^m}}}{2}\right)e(-b(\beta_2,\mu))\right] \tag{833}$$

Proof. Notice that we have $h < h+i+1 \le m-i-1$, so we find $d\gamma = p^{m-i-1}d\gamma' \in D^c_{p^m}$. If $\beta_1 = \mu_1 - d\mu \in D^c_{p^r}$, then clearly we have $\beta_2 = \beta_1 + d\gamma = \mu_2 - d\mu \in D^c_{p^r}$ as well. If we have $\beta_2 \in D^c_{p^r}$ we find $\beta_1 = \beta_2 - d\gamma \in D^c_{p^r}$. The equations (831) and (832) are now direct consequences of (78). □

Lemma 5.3.7. *We still assume $i < \lfloor \frac{m}{2} \rfloor$. Let s be a cusp of $\Gamma_1(Np^m)$ such that h satisfies $h \le m-2i-2$. Then for every $\lambda' \in D_{M'}$ such that $(f'_s)_{\lambda'}$ has a pole at $i\infty$ we have*

$$t_{Np^m}(s) d_{\lambda'} \mathit{ord}_{i\infty}((F'_s)_{\lambda'}) \le B(Np^m). \tag{834}$$

Proof. The function $(f'_s)_{\lambda'}$ is as usual given by

$$(f'_s)_{\lambda'}(\tau) = \sum_{\mu \in D_{p^m}} (\rho_{D_{p^m}}(M_s)\mathfrak{e}_\mu, v)(f_s)_{\lambda'+\mu}(\tau). \tag{835}$$

Assuming that $(f_s)_{\lambda_1+\mu}$ has a pole at $i\infty$ we find $d_\mu | p^m$ and $y \in \mathbb{Z}$ with $(y, d_\mu) = 1$, $d_\mu \mu = 0$ and $\frac{\mu^2}{2} = \frac{y}{d_\mu} (\mathrm{mod}\ \mathbb{Z})$. We define $0 \le l \le m$ by $d_\mu = p^l$. A first property of l is $i \le l$ because i is minimal. It is enough to check that we have

$$d_\mu p^{\lceil \frac{r}{2} \rceil} < t_{p^m}(s) \Rightarrow \left(\rho_{D_{p^m}}(M_s)\mathfrak{e}_\mu, v\right) = 0. \tag{836}$$

We start the discussion with the case $h = 0$, i.e. we have $p \nmid c$. In this case we have $t_{p^m}(s) = p^m$. If $m - i - 1 < l$ in 836 is clearly not satisfied, therefore we may assume $l \le m - i - 1$ in the following. Using all notations from Lemma 5.3.6 we obtain $D^c_{p^m} = D_{p^m}$ and

$$\frac{\beta_2^2}{2} = \frac{\beta_1^2}{2} - d(\mu, \gamma)(\mathrm{mod}\ \mathbb{Z}). \tag{837}$$

Using this and (832) it is now clear that we have $(\rho_{D_p}(M_s)\mathfrak{e}_\mu, v) = 0$ if we have $(\mu, \gamma) = 0(\mathrm{mod}\ \mathbb{Z})$. This is because the expression inside of the brackets $[\cdots]$ in (832) vanishes. Because of $p^l \mu = 0$ we can find a decomposition of μ as in (772), i.e. we find suitable elements $\mu_i \in (p^i)^{\epsilon_i n_i}$ in the decomposition of D_{p^m} such that

$$\mu = \mu_1 + \cdots + \mu_l + p\mu_{l+1} + \cdots + p^{m-l}\mu_m. \tag{838}$$

We use this decomposition to obtain

$$(\mu, \gamma) = p^{m-l}(\mu_r, \gamma) = p^{(m-i-1)-l}p^m(\mu_r, \gamma') = 0(\mathrm{mod}\ \mathbb{Z}. \tag{839}$$

This finishes the case $h = 0$. We now assume $1 \le h \le m-2i-2$. A first consequence of this is that the entry d in the matrix M_s has to satisfy $p \nmid d$ because of $(c, d) = 1$. Above we explained that we have $i \le l$ let us assume $i < l$ first. Of course we have $\mathrm{mult}((\mu_1)_l) \ge p$, simply because $\mathrm{mult}((\delta_p)_l) \ge p$. Assume now there exists $\beta_1 \in D^c_{p^m}$ such that $\mu_1 = d\mu + \beta_1$, then we can decompose this component-wise along the decomposition (767) as usual and obtain $(\mu_1)_j = d\mu_j + (\beta_1)_j$ for every $j = 1, \cdots, m$.

For $j = l$ we obtain in particular $\mu_l = d^{-1}((\mu_1)_l - (\beta_1)_l)$, where as usual $d^{-1} \in \mathbb{Z}$ is any integer such that $dd^{-1} = 1(\text{mod } p^m)$. Such an integer exists because of $p \nmid d$. Yet because of $\text{mult}((\beta_1)_l) \geq p$ and $\text{mult}((\delta_p)_l) \geq p$ we must have $\text{mult}(\mu_l) \geq p$ as well. Yet this is a contradiction with the fact that $\text{mult}(\mu_l) = 1$ by Lemma 5.2.5. We find that in the case $i < l$ there can not exist $\beta_1 \in D^c_{p^m}$ such that $\mu_1 = d\mu + \beta_1$. So the Lemma 5.3.6 implies $(\rho_{D_{p^m}}(M_s)\mathfrak{e}_\mu, v) = 0$. We are left with the case $l = i$. In the following we may assume that $\mu_1 = d\mu + \beta_1$ for some $\beta_1 \in D^c_{p^m}$ since otherwise the statement is once again clear. Once again by Lemma 5.3.6 it is enough to prove that we have $(\beta_1, \mu) = (\beta_2, \mu)(\text{mod } \mathbb{Z})$ and $\frac{(\beta_1)^2_{c_{p^m}}}{2} = \frac{(\beta_2)^2_{c_{p^m}}}{2}(\text{mod } \mathbb{Z})$ in order to obtain $(\rho_{D_{p^m}}(M_s)\mathfrak{e}_\mu, v) = 0$. Using $c_{p^m} = p^h$ and the assumption $h \leq m - 2i - 2$ we can compute this explicitly. □

By putting together all previous results we have now proved Proposition 5.3.2 in the special case of odd primes p. We are left with the case $p = 2$. Yet the assumption that the 2-adic Jordan block $D_{2^{m_2}}$ has a decomposition as in (797) makes sure that this discussion is analogous to the case of odd primes.

5.4 Classification of reflective automorphic products on regular lattices

In this section we want to prove one of the main results of this thesis. We assume that L is an even lattice of even signature $(n, 2)$ with $n \geq 4$ and level N that splits $\text{II}_{1,1} \oplus \text{II}_{1,1}(N)$. Our goal is to show, that just finitely many lattices L, with those properties, are reflective. A first observation is that L satisfies all assumptions of Bruiniers converse theorem 2.4.4. This implies that every orthogonal modular form for the discriminant kernel $\Gamma(L)$ on $\mathcal{K}(L)$, with rational quadratic divisor, is already an automorphic product. In particular the lattice L is reflective if and only if it admits a reflective automorphic product Φ_f. Here f is a suitable vector-valued modular form. We may now apply the previously introduced reduction methods to obtain a scalar-valued modular form g_f for some congruence subgroup, which satisfies certain bounds for the pole order at any cusp of this congruence group. By use of the valence formula we will find bounds for the level N of the lattice L. We start with the assumption that L is reflective with 2-root. This means, that there exists a nonzero reflective automorphic product Φ_f on $\mathcal{K}(L)$ that has a 2-root.

Lemma 5.4.1. *Let p be a prime number and $m \in \mathbb{Z}_{\geq 0}$ an integer such that $p^m || N$, then the p-adic Jordan component of $D = D(L)$ is regular and if $p = 2$ the 2-adic Jordan component D_{2^m} is of the form as in (797).*

Proof. By assumption L splits $\text{II}_{1,1}(N)$ which we may span by e and f with $e^2 = f^2 = 0$ and $(e, f) = N$. Because of $\text{II}_{1,1}(N)' = \frac{1}{N}\text{II}_{1,1}(N)$ we have $\frac{1}{N}e \in \text{II}_{1,1}(N)'$. So for every p-adic component we can construct an isotropic element of maximal order in $D(L)$. This already implies the statement. □

Proposition 5.4.2. *Let L be an even lattice of even signature $(n, 2)$ with $n \geq 4$ and level N that splits $II_{1,1} \oplus II_{1,1}(N)$ and is reflective with a 2-root. Then with $k = 1 - \frac{n}{2}$ the level N satisfies the inequality*

$$\frac{-k}{12} N \prod_{p|N} \left(1 + \frac{1}{p}\right) \leq \sum_{d|N} \phi\left(gcd\left(d, \frac{N}{d}\right)\right). \tag{840}$$

Let N be a solution of this inequality then the maximal exponent with which a prime number can appear in the prime decomposition of N is given in Table 2.

Proof. Let L be a lattice with all the properties as in the statement. So there exists an automorphic product Φ_f that is reflective with 2-root and corresponds to a vector-valued modular form f for the Weil representation of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D(L)]$ of weight k. Using Proposition 5.1.2 we find that f has small pole orders and f_0 has a pole of order 1 at $i\infty$. As a consequence of Lemma 5.4.1 each p-adic Jordan component of $D(L)$ is regular and therefore we can use the Propositions 5.2.7 and 5.2.10 to reduce f step-by-step to vector-valued modular forms for smaller groups and smaller discriminant forms until we are left with a scalar-valued nearly holomorphic modular form g_f of weight k for the group $\Gamma_0(N)$

and character $\chi_{D(L)}$ which has small pole orders. Notice that each p-adic component D_{p^m} of $D(L)$ has even signature, simply because $D(L)$ has even signature. For odd primes p this is clear and for the case $p = 2$ we make use of the oddity formula (53). For a cusp s of $\Gamma_0(N)$ the expansion g_s of g_f at s satisfies $g_s \in \mathcal{O}\left(q^{-\frac{1}{t_N(s)}}\right)$. Since g_f has a pole at $i\infty$ we clearly have $g_f \neq 0$ and so we can apply the valence formula and obtain that g_f solves the inequality (761). We directly obtain

$$\frac{-k}{12}\epsilon_0(N) \leq \epsilon_\infty(N), \tag{841}$$

where $\epsilon_\infty(N)$ is the number of cusps of $\Gamma_0(N)$. Standard expressions for $\epsilon_0(N)$ and $\epsilon_\infty(N)$, given in [DS06], yield the inequality. In order to obtain Table 2 just notice that if N is a solution of inequality (840), then so is each divisor of it. Using this fact it is enough to determine for each prime p a maximal exponent $n_p(-k)$ such that $p^{n_p(-k)}$ solves the inequality. This can easily be done by a computer. □

In Table 2 we write $-$ instead of 0 because 1 is a trivial solution for the inequality (840) if $-k \leq 12$. Clearly many combinations of products of powers of primes that can be build by use of Table 2 will not

	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12
2	7	5	4	3	2	2	1	1	-	-	-	-
3	4	3	2	2	1	1	-	-	-	-	-	-
5	2	2	1	1	-	-	-	-	-	-	-	-
7	2	1	1	-	-	-	-	-	-	-	-	-
11	2	1	-	-	-	-	-	-	-	-	-	-
13	1	-	-	-	-	-	-	-	-	-	-	-
17	1	-	-	-	-	-	-	-	-	-	-	-
19	1	-	-	-	-	-	-	-	-	-	-	-
23	1	-	-	-	-	-	-	-	-	-	-	-

Table 2: Bounds on exponents in a prime factorization of N.

solve inequality (840). Yet it is easy to check which combinations do solve the inequality and which do not. This can now be used to give more explicit upper bounds for the levels N for any signature $(n, 2)$ of the lattice L.

Theorem 5.4.3. *There are just finitely many even lattices L of even signature $(n, 2)$ with $n \geq 4$ and level N that split $II_{1,1} \oplus II_{1,1}(N)$ and are reflective with a 2-root. Moreover, explicit bounds for the levels N are given in Table 2.*

Proof. Using the Propositions 5.1.4 and 5.4.2 we find that the rank $n+2$ of L is bounded by 28 and the level N is bounded by the entries in Table 2. Since there are only finitely many lattices with fixed rank and level the statement is proved. □

Now we focus on the case, where f_0 does not necessarily have a pole at $i\infty$. So, we will have to work with the partial reduction for $\Gamma_1(N)$.

Proposition 5.4.4. *Assume that L is an even lattice of even signature $(n, 2)$ with $n \geq 4$ and level N that splits $II_{1,1} \oplus II_{1,1}(N)$ and is reflective. Then with $k = 1 - \frac{n}{2}$ the level N satisfies the inequality*

$$\frac{-k}{12}N^2 \prod_{p|N}\left(1 - \frac{1}{p^2}\right) \leq B(N) \sum_{d|N} \phi(d)\phi\left(\frac{N}{d}\right). \tag{842}$$

Let N be a solution of this inequality then the maximal exponent with which a prime number can appear in the prime decomposition of N is given in Table 3.

Proof. Let L be a lattice with all the properties as in the statement. So there exists a nonzero automorphic product Φ_f that is reflective and corresponds to a nonzero vector-valued modular form f for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D(L)]$ of weight k. Using Proposition 5.1.2, we find that f has pole orders bounded by B. As a consequence of Lemma 5.4.1 we can use the Proposition 5.3.2 to reduce f step-by-step to vector-valued modular forms for smaller groups and smaller discriminant forms until we are left with a scalar-valued nearly holomorphic modular form g_f of weight k for the group $\Gamma_1(N)$ and some character χ, which has pole order bounded by B. So for a cusp s of $\Gamma_1(N)$ the expansion g_s of g_f at s satisfies $g_s \in \mathcal{O}\left(q^{-\frac{B(N)}{t_N(s)}}\right)$. Since g_f is nonzero we can apply the valence formula and obtain that g_f solves the inequality (761). We directly obtain

$$\frac{-k}{12}\epsilon_1(N) \leq B(N)\epsilon_\infty(N) \tag{843}$$

Here $\epsilon_\infty(N)$ is the number of cusps of $\Gamma_1(N)$ and $\epsilon_1(N)$ is $[\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{P}\Gamma_1(N)]$ by definition. By use of standard expressions for $\epsilon_1(N)$ and $\epsilon_\infty(N)$, given in [DS06], we obtain the inequality (842). Obtaining Table 3 amounts to compute explicit bounds for the solutions of the inequality (842). This is slightly less straightforward but still similar to the previous case in Proposition 5.4.2. □

The inequality (842) has much more solutions than (840). We can observe this directly by comparing Table 2 with Table 3. This is due to the fact that we have to work with the partial reduction for $\Gamma_1(N)$ and the bound function B. Notice furthermore that in the special case of squarefree level N a similar table was given in [Dit19]. We are now ready to come to the main theorem of this section, which is

	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12
2	13	9	9	7	7	5	5	5	3	3	3	3
3	7	7	5	5	3	3	3	3	3	3	-	-
5	5	5	3	3	3	3	3	-	-	-	-	-
7	5	3	3	3	3	-	-	-	-	-	-	-
11	3	3	3	-	-	-	-	-	-	-	-	-
13	3	3	3	-	-	-	-	-	-	-	-	-
17	3	3	-	-	-	-	-	-	-	-	-	-
19	3	3	-	-	-	-	-	-	-	-	-	-
23	3	-	-	-	-	-	-	-	-	-	-	-
29	3	-	-	-	-	-	-	-	-	-	-	-
31	3	-	-	-	-	-	-	-	-	-	-	-
37	3	-	-	-	-	-	-	-	-	-	-	-
41	3	-	-	-	-	-	-	-	-	-	-	-
43	3	-	-	-	-	-	-	-	-	-	-	-

Table 3: Bounds on exponents in a prime factorization of N.

one of the principal results of this thesis.

Theorem 5.4.5. *There are just finitely many reflective even lattices L of even signature $(n, 2)$ and level N that split $I\!I_{1,1} \oplus I\!I_{1,1}(N)$. Moreover, explicit bounds for the levels N are given in Table 3.*

The proof of this theorem is similar to the proof of Theorem 5.4.3. Essentially, we just replace the Proposition 5.4.2 by Proposition 5.4.4 in the reasoning.

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