The dynamic stiffness of an air spring

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Abstract

The measurement of the dynamic stiffness of an air spring identifies a behaviour which until up now is not fully understood. Depending on whether the compression is isothermal or adiabatic the dynamic stiffness differs by a factor of 1.4 for a perfect diatomic gas. The frequency band in which the stiffness increase takes place is determined by the heat conduction from of the compressed air to the air spring wall. Since the heat transport is diffusive the change of stiffness happens to be in a surprisingly low frequency band ranging between 0.001 Hz and 0.1 Hz for a typical vehicle air spring. To understand this dynamic behaviour in detail, i.e. to find the temperature distribution within the spring, the energy equation must be solved using the momentum and mass balance simultaneously. This is done in an analytic manner by considering only small disturbances from the initial pressure, temperature, and density, when the air is at rest. The results show that an oscillating temperature boundary layer is formed in which the heat conduction takes places. With increasing dimensionless frequency, i.e. Peclet number, the boundary layer thickness increases and the stiffness becomes more and more close to its adiabatic value. In theory there is no need to use a heat transfer coefficient. Furthermore the theory serves as a way to determine the heat transfer coefficient. The dimensionless transfer coefficient, i.e. the Nusselt number, is useful when only the average temperature and pressure are of interest. This is usually the case when the air spring is considered as a connecting part between different masses in a dynamic system. It is found that the Nusselt number for the heat conduction inside the air spring is a constant ($Nu \approx 3.0$).

1 Boundary conditions for the simplified geometry

To have in space only a one dimensional problem the most simple geometric model of an air spring is used: two plane infinitely extending plates (with the initial separation distance h_0), one of which (the upper) is set into a harmonic oscillation perpendicular to its plane at frequency $f = \omega/(2\pi)$ and amplitude Δh (see figure 1). Since the plates are infinity large (or the lateral extension is much greater than h_0) there is only a velocity component υ in the normal y -direction.

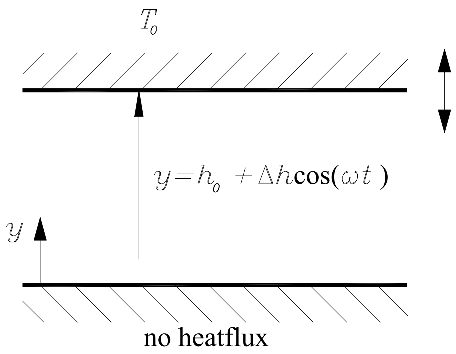


Figure 1: Model-geometry

The temperature of the upper plate is constant T_0, the lower plate is insulated. If there would be a line of symmetry at $y=0$, the problem would be equivalent. Hence the boundary conditions for the unknown velocity and temperature profile are:

$$\begin{aligned} &\upsilon(y=0)=0 \\ &\upsilon(y=h_0+\Delta h\cos(\omega t))=-\Delta h\omega\sin(\omega t) \\ &T(y=h_0+\Delta h\cos(\omega t))=T_0 \\ &\left(\frac{\partial T}{\partial y}\right)_{y=0}=0 \end{aligned} \tag{1}$$

The gas between the plates is considered to be calorically and thermally perfect (adiabatic exponent γ, ideal gas constant R). The dynamic viscosity η and the heat conductivity λ are constant. Like the gas velocity, the gas temperature T, density ρ, and pressure $p=\rho RT$ do depend only on the independent variables y and time t.

2 The dynamic stiffness as a function of the unknown heat transfer coefficient

In the first step the dynamic stiffness of the air spring is derived by considering only average values for temperature T, pressure p, and density ρ (note: the same notation is used, as for $T(y,t), p(y,t), \rho(y,t)$). Considering a gas volume reaching form $y=0$ to $y=h=h_0+\Delta h\cos(\omega t)$ the integral form of the conversation of mass and energy becomes

$$\begin{aligned} \dot{\rho}h+\rho\dot{h} &= 0, \\ \dot{p}h+\gamma p\dot{h}+(\gamma-1)\alpha(T-T_0) &= 0. \end{aligned} \tag{2}$$

(For convenience $\partial/\partial t$ is noted here and in the following by a dot). The term $\alpha(T-T_0)$ represents the unknown heat flux from the air to the upper wall. Following Newton, the heat flux is proportional to the temperature difference:

$$q_y = \alpha(T - T_0). \tag{3}$$

In the next section the value of the Nussel number $Nu = \alpha h_0 / \lambda$ will be given. The nonlinear system (2) is linearised by using the ansatz

$$\begin{aligned}
&h = h_0 + \tilde{h} = h_0(1 + h^+ e^{i\omega t}),\\
&T = T_0 + \tilde{T} = T_0(1 + T^+ e^{i\omega t}),\\
&\rho = \rho_0 + \tilde{\rho} = \tilde{\rho}_0(1 + \rho^+ e^{i\omega t}),\\
&p = \rho_0 R T_0 + \tilde{\rho} R T_0 + \rho_0 R \tilde{T} = p_0 + \tilde{p} = \tilde{p}_0(1 + p^+ e^{i\omega t}).
\end{aligned} \tag{4}$$

The initial values are described by the index "0". They are assumed to be much greater than the perturbation quantities marked with a tilde. $h^+ = \Delta h / h_0 = \tilde{h} / h_0$ describes the dimensionless displacement plate distance, $T^+ = \tilde{T} / T_0$, $\rho^+ = \tilde{\rho} / \rho_0$, $p^+ = \tilde{p} / p_0$ the dimensionless perturbation values of temperature, pressure, and density.

Inserting (4) into (2) and neglecting all perturbation terms of higher order than one, we end up with:

$$\begin{aligned}
\rho^+ + h^+ &= 0,\\
p^+ + \gamma h^+ + \mathrm{i}\,(\gamma - 1)\frac{\alpha}{\omega\, p_0 h_0} T^+ &= 0,\\
p^+ &= \rho^+ + T^+.
\end{aligned} \tag{5}$$

If we choose $c_0 = p_0 / h_0$ as a reference stiffness at low frequencies, the dynamic stiffness of the device $c = \tilde{p} / \tilde{h}$ becomes

$$c^+ = \frac{c}{c_0} = \frac{\tilde{p}}{\tilde{h}} \frac{h_0}{p_0} = \frac{p^+}{h^+} = \frac{i\mathrm{Nu/Pe} - \gamma}{-i\gamma \mathrm{Nu/Pe} + 1}. \tag{6}$$

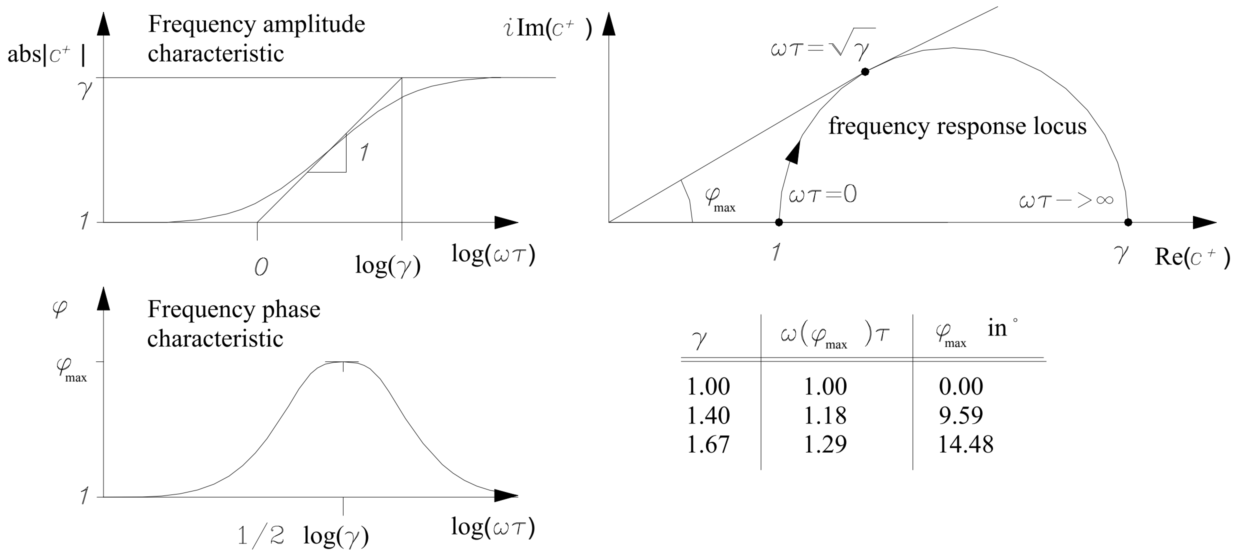
Other than the Nusselt number, the Peclet number is the most important dimensionless product. Here it is convenient to interpret the Peclet number as a dimensionless frequency.

$$\mathrm{Nu} = \frac{\alpha h_0}{\lambda}, \qquad \mathrm{Pe} = \frac{\omega h_0^2}{\lambda / (\rho_0 c_p)}. \tag{7}$$

(c_p, c_v stands for the specific heat at constant pressure, volume respectively). The typical time of the phase change from isothermal to adiabatic is $\tau = h_0 c_p \rho_0 / \alpha$. Thus we can write (6) in the equivalent form

$$c^+ = \frac{1 + \mathrm{i}\ \omega\tau}{1 + \frac{1}{\gamma}\mathrm{i}\ \omega\tau}. \tag{8}$$

(in System control engineering equation (8) describes a phase lifting limb.) The bode diagram (dynamic stiffness and phase angle) for the air spring is shown in figure 2.



γ	$\omega(\varphi_{\max})\tau$	$\varphi_{\max}$ in $^\circ$
1.00	1.00	0.00
1.40	1.18	9.59
1.67	1.29	14.48

Figure 2: Bode diagram for the air spring

3 The unsteady temperature field

In the second step a way to determine the dynamic stiffness is needed without using the heat transfer coefficient. This can only be done by calculating the unsteady temperature field. This must solve the equations for conservation of mass, momentum, and energy in differential form:

$$
\begin{aligned}
\dot{\rho}+(\rho\upsilon)' &= 0\\
\rho\dot{\upsilon}+\rho\upsilon\upsilon' &= -p'+\hat{\eta}\upsilon''\\
c_v\rho\dot{T}+c_v\rho\upsilon T'-\frac{p}{\rho}\dot{\rho}-\frac{p}{\rho}\upsilon\rho' &= \lambda T''\\
p &= \rho RT.
\end{aligned}
\tag{9}
$$

(For convenience $\partial/\partial y$ is noted here and in the following by a dash). The viscosity $\hat{\eta}$ takes the pressure viscosity η_D into account:

$$
\hat{\eta} = \frac{4}{3}\eta+\eta_{\mathrm{D}} \;\rightarrow\; \hat{\mathrm{P}}\mathrm{r}=\frac{\hat{\eta}}{\rho_0 D}\;, D=\frac{\lambda}{c_p\rho}. \tag{10}
$$

D stands for heat conductivity number. According to Eucken in the case of vanishing pressure viscosity the Prandtl number is a function of γ

$$
\hat{\mathrm{P}}\mathrm{r} = \frac{4}{3}\left(\frac{4\gamma}{9\gamma-5}\right). \tag{11}
$$

A closed solution of the nonlinear problem (11) together with boundary conditions (1) is only feasible, if only small perturbations are considered:

$$\begin{aligned} \upsilon(y,t) &= \tilde{\upsilon}(y,t), \\ T(y,t) &= T_o + \tilde{T}(y,t), \\ \rho(y,t) &= \rho_o + \tilde{\rho}(y,t), \\ p(y,t) &= p_o + \tilde{p}(y,t). \end{aligned} \tag{12}$$

Inserting (12) into (9) and neglecting all perturbation terms of order higher than one, the result is:

$$\begin{aligned} \dot{\tilde{\rho}} + \rho_o \tilde{\upsilon}' &= 0, \\ \rho_o \dot{\tilde{\upsilon}} &= -\tilde{p}' + \hat{\eta}\tilde{\upsilon}'', \\ c_v \rho_o \dot{\tilde{T}} - RT_o \dot{\tilde{\rho}} &= \lambda \tilde{T}'', \\ \tilde{p} &= \tilde{\rho} RT_o + \rho_o R\tilde{T}, \quad p_o = \rho_o RT_o. \end{aligned} \tag{13}$$

In addition, the boundary conditions have to be linearised, by means of a Taylor expansion:

$$\begin{aligned} \tilde{\upsilon}(h_0) &= -\omega h \sin \omega t \quad , \quad \tilde{\upsilon}(0) = 0 \\ \tilde{T}(h_0) &= 0 \quad , \quad \tilde{T}'(0) = 0. \end{aligned} \tag{14}$$

Since there is no boundary condition for the pressure and the density they are eliminated from (13)

$$\begin{aligned} \rho_0 \ddot{\tilde{\upsilon}} &= p_0 \tilde{\upsilon}'' - \rho_0 R \dot{\tilde{T}}' + \hat{\eta} \dot{\tilde{\upsilon}}'', \\ c_v \rho_o \dot{\tilde{T}} + p_o \tilde{\upsilon}' &= \lambda \tilde{T}''. \end{aligned} \tag{15}$$

To solve the system it is convinient to transforme it into the frequency range. The complex notation shall be used and only the physical meaning of the real part is valid:

$$\begin{aligned} \tilde{\upsilon}(y,t) &= \mathrm{i}\, h\omega\, e^{i\omega t} \varphi(y^+), \\ \tilde{T}(y,t) &= T_0 e^{i\omega t} \vartheta(y^+). \end{aligned} \tag{16}$$

Here $y^+ = y / h_0$ is the dimensionless coordinate, the dimensionless complex functions $\varphi(y^+)$, $\vartheta(y^+)$ are the (dimensionless complex amplitudes of) velocity and temperature respectively at the position y^+. With the ansatz (16) the system of partial differential equations (13) become a system of ordinary differential equation system in the unknown functions φ and ϑ:

$$\begin{aligned} h^+ \left(\kappa^2 + \mathrm{i}\,\gamma\ \mathrm{Pe}\ \hat{\mathrm{Pr}}\right)\varphi'' + \gamma\ \mathrm{Pe}^2 h^+ \varphi - \kappa^2 \vartheta' &= 0, \\ \gamma\vartheta'' - \mathrm{i}\ \mathrm{Pe}\ \ \vartheta - \mathrm{i}\,(\gamma - 1)\ \mathrm{Pe}\, h^+ \varphi' &= 0. \end{aligned} \tag{17}$$

The dimensionless plate distance $\kappa = ah_0 / D$ is built by the sonic velocity $a = \sqrt{\gamma RT_0}$ of the gas and the conductivity number D is introduced. From (14) it follows that the boundary conditions are transformed to

$$\begin{aligned} \varphi(1) = 1 \quad &, \quad \varphi(0) = 0, \\ \vartheta(1) = 0 \quad &, \quad \vartheta'(0) = 0. \end{aligned} \tag{18}$$

To solve the system (17), (18), the ansatz

$$\begin{pmatrix} \varphi \\ \vartheta \end{pmatrix} = \mathbf{r}\; e^{\mu\, y+} \tag{19}$$

is used. This leads to an eigenvalueproblem. The four eigenvalues and the accessory eigenvectors are

$$\mu_{1,2,3,4} = \pm\sqrt{\frac{1}{2}\frac{i\kappa^2 - (\gamma + \hat{\mathrm{P}}\mathrm{r})\ \mathrm{Pe} \pm \sqrt{\left\{\ \mathrm{i}\kappa^2 - (\gamma + \hat{\mathrm{P}}\mathrm{r})\ \mathrm{Pe}\right\}^2 + 4\mathrm{i}\kappa^2\mathrm{Pe} - 4\gamma\,\hat{\mathrm{P}}\mathrm{r}\,\mathrm{Pe}^2}}{\kappa^2 / \mathrm{Pe} + \mathrm{i}\gamma\,\hat{\mathrm{P}}\mathrm{r}}}, \tag{20}$$

$$r_{2j} = 1, \quad r_{1j} = \frac{\gamma\mu_j^2 - \mathrm{i\ Pe}}{\mathrm{i}\ (\gamma - 1)\mathrm{Pe}\ h^+ \mu_j}, \quad j = 1,..,4. \tag{21}$$

The linear combination of the four different solutions gives:

$$\begin{pmatrix} \varphi \\ \vartheta \end{pmatrix} = \sum_{j=1}^{4} a_j \begin{pmatrix} \dfrac{\gamma\mu_j^2 - \mathrm{i\ Pe}}{\mathrm{i}\ (\gamma - 1)\mathrm{Pe}\ h^+ \mu_j} \\ 1 \end{pmatrix} \exp\left(\mu_j y^+\right). \tag{22}$$

The unknown coefficients a_j are determined by the boundary conditions (18)

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sum_{j=1}^{4} a_j \begin{pmatrix} \dfrac{\gamma\mu_j^2 - \mathrm{i\ Pe}}{\mathrm{i}\ (\gamma - 1)\mathrm{Pe}\ h^+ \mu_j} \\ 1 \end{pmatrix} \exp\left(\mu_j\right), \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \sum_{j=1}^{4} a_j \begin{pmatrix} \dfrac{\gamma\mu_j^2 - \mathrm{i\ Pe}}{\mathrm{i}\ (\gamma - 1)\mathrm{Pe}\ h^+ \mu_j} \\ \mu_j \end{pmatrix}. \tag{23}$$

Resultant are the now known velocity $\tilde{\upsilon}(y,t)$ and temperature $\tilde{T}(y,t)$ fields. The pressure and density obtained from (13) are:

$$\frac{\tilde{\rho}}{\rho_o} = \frac{1}{\gamma - 1}\left(\vartheta - \frac{\gamma}{\mathrm{i\ Pe}}\vartheta''\right)\exp(\mathrm{i}\ \omega t), \quad \frac{\tilde{p}}{p_o} = \frac{\tilde{T}}{T_0} + \frac{\tilde{\rho}}{\rho_0} = \left(\vartheta - h^+\varphi'\right)\exp(i\omega t). \tag{24}$$

For the dimensionless dynamic stiffness, only the pressure at the upper plate is of interest:

$$c^+ = \frac{c}{c_0} = \frac{\tilde{p}}{\tilde{h}}\frac{h_0}{p_0} = \frac{p^+}{h^+} = -\varphi'(1). \tag{25}$$

As expected c^+ does not depend on the dimensionless amplitude h^+ like it has to be in a linear case.

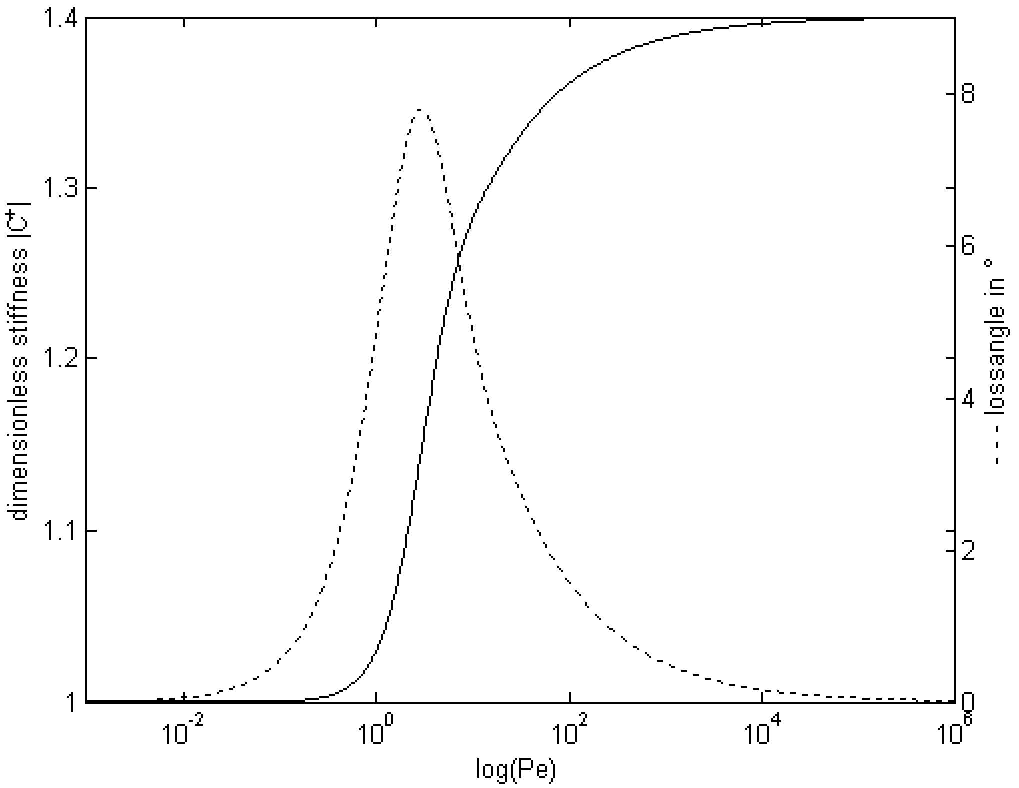


Figure 3: Bode diagram of the solution (25) for $\kappa = 10^8$, γ=1.4.

Figure 3 shows a typical bode-diagram of solution (25) for $\kappa = 10^8$ in the relevant Peclet-range. The phase lifting behaviour, already inspected in section 2, is qualitatively present. The isothermal and adiabatic / isentropic limiting values in the sum of the admittance function are confirmed.

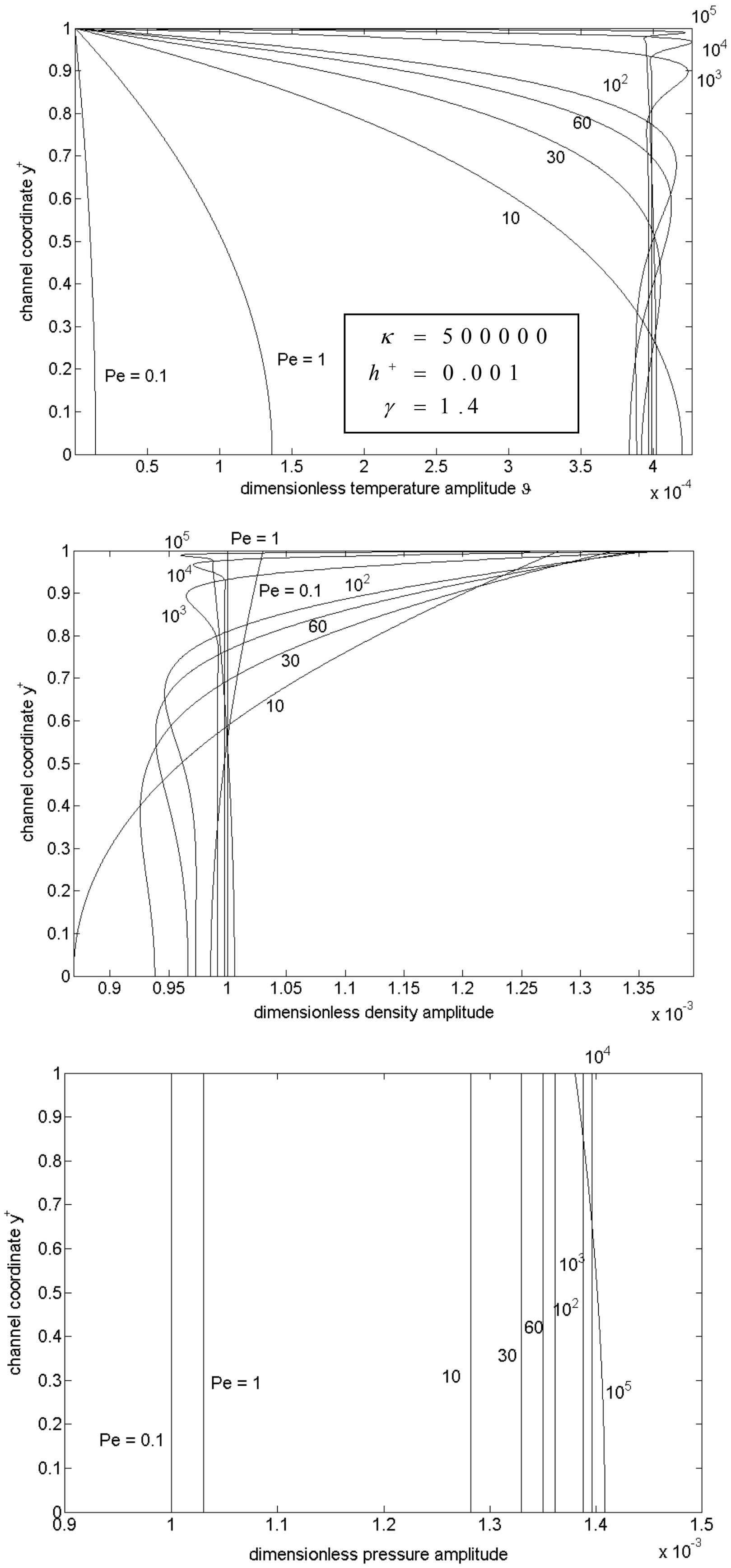
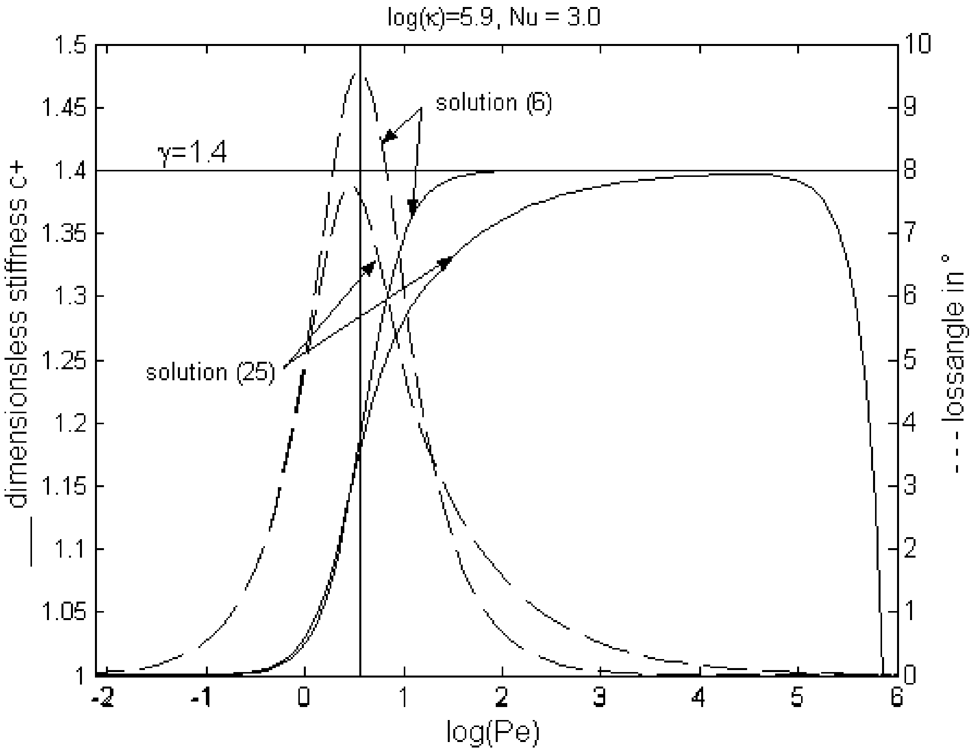


Figure 4: Amplitude field of the temperature, density and pressure

Figure 4 shows the amplitude field of the temperature, density and pressure at different Peclet-numbers. For Peclet-values lower than 0.1 there is only a small change of temperature $|\vartheta(y^+)|$ over the channel height. For increasing Peclet-values (frequencies) the formation of a boundary layer becomes visible. For a Peclet-number of 1000, the heat conduction is restricted to a boundary layer thickness of 20% of the plate distance.



Fgure 5: Bode diagram; comparison of the solutions (6) and (25); determination of the Nusselt-number

By comparing the dynamic stiffness versus frequency of model (6) and (25) a Nusselt-relation can be found. The Nussel-number is a constant of approx. 3.0. Figure 5 shows a very good correlation of both models at Peclet numbers up to one. As long as $\kappa >> \mathrm{Pe}$ is valid, both models show the same asymptotes for high and low Peclet numbers.

If the Peclet number is of the order of the dimensionless panel distance, model (25) shows a behaviour which can not be explained by the homogeneous model (6): the result (25) shows a drop of the dynamic stiffness when the Peclet number comes close to the dimensionless plate distance κ (see figure 5). This can be understood by looking at the pressure profiles in figure 4 at high Peclet-numbers. The bending of the pressure profile shows the initial formation of a standing pressure wave between the walls. This isentropic limiting case can be studied by neglecting the friction term in the momentum equation and replacing the energy equation by the isentropic relation $p\rho^{-\gamma} = const$. Doing so, it follows

$$c^+ = -\gamma \frac{X_s}{\tan(X_s)}, \text{ with } X_s = \frac{Pe}{\kappa}. \tag{26}$$

The dynamic stiffness for

$$X_s = \frac{Pe}{\kappa} = n\frac{\pi}{2}, \quad \text{n=0,2,4...} \tag{27}$$

becomes singular, i. e. rigid, and for

$$X_s = \frac{Pe}{\kappa} = n\frac{\pi}{2}, \quad \text{n=1,3,5...} \tag{28}$$

ideal soft. As has been said: standing waves are forming. In the rigid case a velocity knot is formed near the upper wall. This is only compatible with the boundary condition at this plate if the velocity in the field becomes singular. In the isentropic case, the complex velocity amplitude becomes

$$\varphi(\mathrm{y}^+) = \frac{\sin(X_s y^+)}{\sin(X_s)} \tag{29}$$