13th European Turbulence Conference (ETC13) IOP Publishing
Journal of Physics: Conference Series **318** (2011) 042011 doi:10.1088/1742-6596/318/4/042011

Applications of the new symmetries of the multi-point correlation equations

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Abstract. We presently show that the infinite set of multi-point correlation equations, which are direct statistical consequences of the Navier-Stokes equations, admit a rather large set of Lie symmetry groups. Additional to the symmetries stemming from the Navier-Stokes equations a new scaling group and translational groups of the correlation vectors and all independent variables have been discovered. These new statistical groups have important consequences on our understanding of turbulent scaling laws. Exemplarily, we consider one of the key foundations of statistical turbulence theory, the universal law of the wall, and show that the log-law fundamentally relies on one of the new translational groups. Furthermore, we present rotating channel flows, where different rotational axes result in very different scaling laws.

1. Introduction

Turbulence represents an important field in hydrodynamics with applications in innumerable natural and technical systems. Examples for natural turbulent flows are atmospheric flow and oceanic current which to calculate is a crucial point in climate research. Only with the advent of super computers it became apparent that the Navier-Stokes equations provide a very good continuum mechanical model for turbulent flows, although practical flow problems at high Reynolds numbers cannot be calculated without invoking any additional assumptions.

In the most applications it is not at all necessary to know all the detailed fluctuations of velocity and pressure present in turbulent flows but for the most part statistical measures are sufficient. This represents the key idea of O. Reynolds who was the first to suggest a statistical description of turbulence. Due to the non-linearity of the Navier-Stokes equations an infinite set of statistical equations occurs.

The mathematical theory of Lie group analysis allows us a deeper insight into the statistical behavior of turbulence. This theory is developed to create special solutions of partial differential equations such as the Navier-Stokes equations or the resulting multi-point equations. At the end we will consider two different flow situations and determine the corresponding scaling laws.

This work presents the most important results from Oberlack & Rosteck (2010), Oberlack & Rosteck (2011), extended by new developments concerning the extension of the set of symmetries in Rosteck & Oberlack (2011) and applications to various rotating channel flows (see Oberlack & Rosteck (2011)).

2. Equations of statistical turbulence theory

2.1. The closure problem

Assuming an incompressible fluid under the assumption of a Newtonian material with constant density and viscosity, the given flow is described by the Navier-Stokes equations

$$\frac{\partial U_k}{\partial x_k} = 0\,, \tag{1}$$

$$\mathcal{M}_i(\boldsymbol{x}) = \frac{\partial U_i}{\partial t} + U_k \frac{\partial U_i}{\partial x_k} + \frac{\partial P}{\partial x_i} - \nu \frac{\partial^2 U_i}{\partial x_k \partial x_k} = 0 \;,\; i = 1,2,3\,, \tag{2}$$

where $t \in \mathbb{R}^+$, $\boldsymbol{x} \in \mathbb{R}^3$ and $\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{x}, t)$ represent time, position vector and instantaneous velocity vector. The pressure P^* already appears in its normalized form $P^* = \frac{P^*}{\rho}$, while the density ρ and the viscosity ν are positive constants.

After $\boldsymbol{U}$ and P are decomposed according to the Reynolds decomposition, i.e. $\boldsymbol{U} = \bar{\boldsymbol{U}} + \boldsymbol{u}$ and $P = \bar{P} + p$, we gain an averaged versions of the Navier-Stokes equations

$$\frac{\partial \bar{U}_k}{\partial x_k} = 0 \;, \tag{3}$$

$$\frac{\partial \bar{U}_i}{\partial t} + \bar{U}_k \frac{\partial \bar{U}_i}{\partial x_k} = -\frac{\partial \bar{P}}{\partial x_i} + \nu \frac{\partial^2 \bar{U}_i}{\partial x_k \partial x_k} - \frac{\partial \overline{u_i u_k}}{\partial x_k}\,,\; i = 1,2,3. \tag{4}$$

At this point we observe the well-known closure problem of turbulence since, compared to the original set of equations, the unknown Reynolds stress tensor $\overline{u_i u_k}$ appeared. However, rather different from the classical approach we will not proceed with deriving the Reynolds stress tensor transport equation which contains additional four unclosed tensors. Instead the multi-point correlation (MPC) approach is put forward out of two reasons. In every higher moment only one unclosed tensor appears and the multi-point correlation delivers additional information on the turbulence statistics such as length scale information.

For this we need the equations for the fluctuating quantities $\boldsymbol{u}$ and p which are derived by taking the differences between the averaged and the non-averaged equations, i.e. (1)−(3) and (2)−(4). The resulting fluctuation equations read

$$\frac{\partial u_k}{\partial x_k} = 0 \;, \tag{5}$$

$$\mathcal{N}_i(\boldsymbol{x}) = \frac{\bar{\mathrm{D}} u_i}{\bar{\mathrm{D}} t} + u_k \frac{\partial \bar{U}_i}{\partial x_k} - \frac{\partial \overline{u_i u_k}}{\partial x_k} + \frac{\partial u_i u_k}{\partial x_k} + \frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} = 0\,, \quad i = 1,2,3\,. \tag{6}$$

2.2. Multi-point correlation equations

The idea of two- and multi-point correlation equations in turbulence was presumably first established by Keller & Friedmann (1924). At that time the assumption was formulated that all correlation equations of orders higher than two may be neglected. This, however, was later refuted, so that all higher correlations have to be taken into account.

In order to write the MPC equations in a very compact form, we introduce the following notation. The multi-point velocity correlation tensor of order $n+1$ is defined as follows:

$$R_{i_{\{n+1\}}} = R_{i_{(0)} i_{(1)} \cdots i_{(n)}} = \overline{u_{i_{(0)}}(\boldsymbol{x}_{(0)}) \cdot \ldots \cdot u_{i_{(n)}}(\boldsymbol{x}_{(n)})} \;, \tag{7}$$

where the first index of the $\mathbf{R}$ tensor defines the tensor character of the term and the second index in braces denotes the order of the tensor. The curly brackets point out that not an index of a tensor but an enumeration is meant. Beginning the indices with 0 is an advantage when introducing a new coordinate system based on the Euclidean distance of two space points.

In order to set up the whole infinite set of equations some further notation is needed to understand the formulas describing the MPC equations. It happens that the list of indices is changed for one special point $x^{(l)}$, so that we write

$$R_{i_{\{n+1\}}[i_{(l)}\mapsto k_{(l)}]} = \overline{u_{i_{(0)}}(\boldsymbol{x}_{(0)})\cdot\ldots\cdot u_{i_{(l-1)}}(\boldsymbol{x}_{(l-1)})\cdot u_{k_{(l)}}(\boldsymbol{x}_{(l)})u_{i_{(l+1)}}(\boldsymbol{x}_{(l+1)})\cdot\ldots\cdot u_{i_{(n)}}(\boldsymbol{x}_{(n)})}\ . \tag{8}$$

If also the space point itself changes, e.g., we note

$$R_{i_{\{n+2\}}[i_{(n+1)}\mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)}\mapsto \boldsymbol{x}_{(l)}] = \overline{u_{i_{(0)}}(\boldsymbol{x}_{(0)})\cdot\ldots\cdot u_{i_{(n)}}(\boldsymbol{x}_{(n)})u_{k_{(l)}}(\boldsymbol{x}_{(l)})}\ , \tag{9}$$

If indices are missing e.g. between $i_{(l-1)}$ and $i_{(l+1)}$ we define

$$R_{i_{\{n\}}[i_{(l)}\mapsto\emptyset]} = \overline{u_{i_{(0)}}(\boldsymbol{x}_{(0)})\cdot\ldots\cdot u_{i_{(l-1)}}(\boldsymbol{x}_{(l-1)})u_{i_{(l+1)}}(\boldsymbol{x}_{(l+1)})\cdot\ldots\cdot u_{i_{(n)}}(\boldsymbol{x}_{(n)})}\,. \tag{10}$$

Finally, we introduce the velocity-pressure correlation

$$P_{i_{\{n\}}[l]} = \overline{U_{i_{(0)}}(\boldsymbol{x}_{(0)})\cdot\ldots\cdot u_{i_{(l-1)}}(\boldsymbol{x}_{(l-1)})p(\boldsymbol{x}_{(l)})u_{i_{(l+1)}}(\boldsymbol{x}_{(l+1)})\cdot\ldots\cdot u_{i_{(n)}}(\boldsymbol{x}_{(n)})}\,. \tag{11}$$

Afterwards, the equations determining the multi-point correlations can be formulated by the sum

$$\mathcal{T}_{i_{\{n+1\}}}(\boldsymbol{x}_{(0)},\ldots,\boldsymbol{x}_{(n)}) = \sum_{a=1}^{n}\overline{\mathcal{N}_{i_{(a)}}(\boldsymbol{x}_{(a)})\prod_{b=1,b\neq a}^{n} u_{i_{(b)}}(\boldsymbol{x}^{(b)})} = 0\,. \tag{12}$$

Inserting (6) and writing the resulting equation in a compact form we gain

$$\begin{aligned}\mathcal{T}_{i_{\{n+1\}}} = \frac{\partial R_{i_{\{n+1\}}}}{\partial t} + \sum_{l=0}^{n}\Bigg[\bar{U}_{k_{(l)}}(\boldsymbol{x}_{(l)})\frac{\partial R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}} + R_{i_{\{n+1\}}[i_{(l)}\mapsto k_{(l)}]}\frac{\partial \bar{U}_{i_{(l)}}(\boldsymbol{x}_{(l)})}{\partial x_{k_{(l)}}} + \frac{\partial P_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}}\\ -\nu\frac{\partial^2 R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}} - R_{i_{\{n\}}[i_{(l)}\mapsto\emptyset]}\frac{\partial \overline{u_{i_{(l)}}u_{k_{(l)}}}(\boldsymbol{x}_{(l)})}{\partial x_{k_{(l)}}} + \frac{\partial R_{i_{\{n+2\}}[i_{(n+1)}\mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)}\mapsto\boldsymbol{x}_{(l)}]}{\partial x_{k_{(l)}}}\Bigg] = 0\\ \text{for}\quad n = 1,\ldots,\infty\ .\end{aligned} \tag{13}$$

In the case $n = 2$ we have $R_{i_{\{1\}}[i_{(l)}\mapsto\emptyset]} = 0$. Further, the two-point correlation tensor $R_{i_{\{2\}}}$ has a close link to the Reynolds stress tensor, i.e. $\lim_{x_{(k)}\to x_{(l)}} R_{i_{\{2\}}}(\boldsymbol{x}_{(k)},\boldsymbol{x}_{(l)}) = \overline{u_{i_{(0)}}u_{i_{(1)}}}(\boldsymbol{x}_{(l)})$ with $k\neq l$ and arbitary $\boldsymbol{x}_{(k)}$ and $\boldsymbol{x}_{(l)}$ which is the key unclosed quantity in the Reynolds stress transport equation (4).

In the process of searching Lie groups for the MPC equations (13) the non-linearity implies a main disadvantage. This is the reason that we used another set of multi-point equations based on the instantaneous velocities, so that the corresponding multi-point correlation is defined by $H_{i_{\{n+1\}}} = \overline{U_{i_{(0)}}(\boldsymbol{x}_{(0)})\cdot\ldots\cdot U_{i_{(n)}}(\boldsymbol{x}_{(n)})}$. The resulting equations form a linear set of equations equivalent to (13) which is much more suited for the analysis. The symmetries, found for the **H** approach, can be directly transferred to the presently employed **R**-**P** approach.

From equation (5) a continuity equation for the multi-point equations

$$\frac{\partial R_{i_{\{n+1\}}[i_{(a)}\mapsto k_{(a)}]}}{\partial x_{k_{(a)}}} = 0\quad,\quad\frac{\partial P_{i_{\{n\}}[k][i_{(l)}\mapsto m_{(l)}]}}{\partial x_{m_{(l)}}} = 0\quad\text{for}\quad a,k,l = 0,\ldots,n\ \text{and}\ k\neq l \tag{14}$$

can be derived. Further there exist side conditions, e.g. $R_{i_{(0)}i_{(1)}}(\boldsymbol{x}_{(0)},\boldsymbol{x}_{(1)}) = R_{i_{(1)}i_{(0)}}(\boldsymbol{x}_{(1)},\boldsymbol{x}_{(0)})$, which result from the definition of the MPC (7).

3. Symmetries of statistical transport equations

After recalling the Lie symmetries of the Euler and Navier-Stokes equations we will transfer these symmetries to its corresponding ones for the MPC equations. Then we will present some additional symmetries of the MPC equations which are not reflected in the original Euler and Navier-Stokes equations. Although the additional symmetries are found for the **H**-approach, we will show only the transfered ones in the **R** notation out of reasons of brevity (see Oberlack & Rosteck (2010)).

In order to appreciate the analysis on Lie symmetries below we will define its basic concepts. Suppose the system of partial differential equations under investigation is given by

$$\boldsymbol{F}(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{z}^{(1)}, \boldsymbol{z}^{(2)}, \dots) = 0, \tag{15}$$

where $\boldsymbol{y}$ and $\boldsymbol{z}$ are the independent and the dependent variables respectively and $\boldsymbol{z}^{(n)}$ refers to all n^{th}-order derivatives of any component of $\boldsymbol{z}$ with respect to any component of $\boldsymbol{y}$.

A transformation $\boldsymbol{y} = \boldsymbol{\phi}(\boldsymbol{y}^*, \boldsymbol{z}^*)$ and $\boldsymbol{z} = \boldsymbol{\psi}(\boldsymbol{y}^*, \boldsymbol{z}^*)$ is called a *symmetry transformation* of the equation (15) if the equivalence

$$\boldsymbol{F}(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{z}^{(1)}, \dots) = 0 \quad \Leftrightarrow \quad \boldsymbol{F}(\boldsymbol{y}^*, \boldsymbol{z}^*, \boldsymbol{z}^{*(1)}, \dots) = 0 \tag{16}$$

holds, i.e. the given transformation substituted into (15) does not change the form of equation (15) if written in the new variables $\boldsymbol{y}^*$ and $\boldsymbol{z}^*$.

A second concept which will be heavily relied on is that of an *invariant*. It refers to quantities that do not change structure under a given symmetry i. e. $I(\boldsymbol{y}, \boldsymbol{z}) = I(\boldsymbol{\phi}(\boldsymbol{y}^*, \boldsymbol{z}^*), \boldsymbol{\psi}(\boldsymbol{y}^*, \boldsymbol{z}^*)) = I(\boldsymbol{y}^*, \boldsymbol{z}^*)$.

The final concept in this context is that of an *invariant solution*. The invariants may be taken as new dependent and independent variables which in turn leads to a reduction of the number of the independent variables. The self-similarity solution well-known in mechanics corresponds to invariant solutions with certain scaling properties.

3.1. Symmetries of the Euler and Navier-Stokes equations

The Euler equations, i.e. equation (1) and (2) with $\nu = 0$ admit a ten-parameter symmetry group,

$$\begin{aligned}
T_1 &: \ t^* = t + a_1, \ \boldsymbol{x}^* = \boldsymbol{x}, \ \boldsymbol{U}^* = \boldsymbol{U}, \ P^* = P, \\
T_2 &: \ t^* = t, \ \boldsymbol{x}^* = \mathrm{e}^{a_2}\boldsymbol{x}, \ \boldsymbol{U}^* = \mathrm{e}^{a_2}\boldsymbol{U}, \ P^* = \mathrm{e}^{2a_2}P, \\
T_3 &: \ t^* = \mathrm{e}^{a_3}t, \ \boldsymbol{x}^* = \boldsymbol{x}, \ \boldsymbol{U}^* = \mathrm{e}^{-a_3}\boldsymbol{U}, \ P^* = \mathrm{e}^{-2a_3}P, \\
T_4 - T_6 &: \ t^* = t, \ \boldsymbol{x}^* = \mathbf{a}\cdot\boldsymbol{x}, \ \boldsymbol{U}^* = \mathbf{a}\cdot\boldsymbol{U}, \ P^* = P, \\
T_7 - T_9 &: \ t^* = t, \ \boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{f}(t), \ \boldsymbol{U}^* = \boldsymbol{U} + \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}t}, \ P^* = P - \boldsymbol{x}\cdot\frac{\mathrm{d}^2\boldsymbol{f}}{\mathrm{d}t^2}, \\
T_{10} &: \ t^* = t, \ \boldsymbol{x}^* = \boldsymbol{x}, \ \boldsymbol{U}^* = \boldsymbol{U}, \ P^* = P + f_4(t) \ ,
\end{aligned} \tag{17}$$

where a_1-a_3 are independent group-parameters, $\mathbf{a}$ denotes a constant rotation matrix with the properties $\mathbf{a}\cdot\mathbf{a}^{\mathsf{T}} = \mathbf{a}^{\mathsf{T}}\cdot\mathbf{a} = \mathbf{I}$ and $|\mathbf{a}| = 1$. Moreover $\boldsymbol{f}(t) = (f_1(t), f_2(t), f_3(t))^{\mathsf{T}}$ with twice differentiable functions f_1-f_3 and $f_4(t)$ may have arbitrary time dependence.

The physical meaning of T_1 is a time translation while T_4-T_6 designate rotation invariance. The symmetries T_7-T_9 comprise translational invariance in space for constant f_1-f_3 as well as the classical Galilei group if f_1-f_3 are linear in time. In its rather general form T_7-T_9 and T_{10} are direct consequences of an incompressible flow and do not have a counterpart in the case of compressible flows. The complete record of all point-symmetries (17) was first published by Pukhnachev (1972).

Invoking a formal transfer from Euler to the Navier-Stokes equations symmetry properties change and a recombination of the two scaling symmetries T_2 and T_3 is observed

$$T_{NaSt}: \quad t^* = \mathrm{e}^{2a_4} t, \;\; \boldsymbol{x}^* = \mathrm{e}^{a_4}\boldsymbol{x}, \;\; \boldsymbol{U}^* = \mathrm{e}^{-a_4}\boldsymbol{U}, \;\; P^* = \mathrm{e}^{-2a_4}P, \tag{18}$$

while the remaining groups stay unaltered.

It should be noted that additional symmetries exist for dimensional restricted cases such as plane or axisymmetric flows (see Andreev & Rodionov, 1988; Cantwell, 1978).

3.2. Symmetries of the MPC implied by Euler and Navier-Stokes symmetries

A first set of symmetries can be directly derived from the symmetries of the Euler equations. Applying the Reynolds decompostion to these symmetries we easily show that

$$\begin{aligned}
\bar{T}_1 &: t^* = t + a_1,\, \boldsymbol{x}^* = \boldsymbol{x},\, \boldsymbol{r}^*_{(l)} = \boldsymbol{r}_{(l)},\, \bar{\boldsymbol{U}}^* = \bar{\boldsymbol{U}},\, \bar{P}^* = \bar{P},\, \mathbf{R}^*_{\{n\}} = \mathbf{R}_{\{n\}},\, \mathbf{P}^*_{\{n\}} = \mathbf{P}_{\{n\}},\\
\bar{T}_2 &: t^* = t,\;\; \boldsymbol{x}^* = \mathrm{e}^{a_2}\boldsymbol{x},\;\; \boldsymbol{r}^*_{(l)} = \mathrm{e}^{a_2}\boldsymbol{r}_{(l)},\;\; \bar{\boldsymbol{U}}^* = \mathrm{e}^{a_2}\bar{\boldsymbol{U}},\\
&\quad \bar{P}^* = \mathrm{e}^{2a_2}\bar{P},\;\; \mathbf{R}^*_{\{n\}} = \mathrm{e}^{na_2}\mathbf{R}_{\{n\}}, \mathbf{P}^*_{\{n\}} = \mathrm{e}^{(n+2)a_2}\mathbf{P}_{\{n\}},\\
\bar{T}_3 &: t^* = \mathrm{e}^{a_3}t,\;\; \boldsymbol{x}^* = \boldsymbol{x},\;\; \boldsymbol{r}^*_{(l)} = \boldsymbol{r}_{(l)},\;\; \bar{\boldsymbol{U}}^* = \mathrm{e}^{-a_3}\bar{\boldsymbol{U}},\\
&\quad \bar{P}^* = \mathrm{e}^{-2a_3}\bar{P},\;\; \mathbf{R}^*_{\{n\}} = \mathrm{e}^{-na_3}\mathbf{R}_{\{n\}},\;\; \mathbf{P}^*_{\{n\}} = \mathrm{e}^{-(n+2)a_3}\mathbf{P}_{\{n\}},\\
\bar{T}_4\text{–}\bar{T}_6 &: t^* = t,\;\; \boldsymbol{x}^* = \mathbf{a}\cdot\boldsymbol{x},\;\; \boldsymbol{r}^*_{(l)} = \boldsymbol{r}_{(l)},\;\; \bar{\boldsymbol{U}}^* = \mathbf{a}\cdot\bar{\boldsymbol{U}},\;\; \bar{P}^* = \bar{P},\\
&\quad \mathbf{R}^*_{\{n\}} = \mathbf{A}_{\{n\}} \otimes \mathbf{R}_{\{n\}},\;\; \mathbf{P}^*_{\{n\}} = \mathbf{A}_{\{n\}} \otimes \mathbf{P}_{\{n\}},\\
\bar{T}_7\text{–}\bar{T}_9 &: t^* = t,\;\; \boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{f}(t),\;\; \boldsymbol{r}^*_{(l)} = \boldsymbol{r}_{(l)},\;\; \bar{\boldsymbol{U}}^* = \bar{\boldsymbol{U}} + \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}t},\\
&\quad \bar{P}^* = \bar{P} - \boldsymbol{x}\cdot\frac{\mathrm{d}^2\boldsymbol{f}}{\mathrm{d}t^2},\;\; \mathbf{R}^*_{\{n\}} = \mathbf{R}_{\{n\}},\;\; \mathbf{P}^*_{\{n\}} = \mathbf{P}_{\{n\}},\\
\bar{T}_{10} &: t^* = t,\, \boldsymbol{x}^* = \boldsymbol{x},\, \boldsymbol{r}^*_{(l)} = \boldsymbol{r}_{(l)},\, \bar{\boldsymbol{U}}^* = \bar{\boldsymbol{U}},\, \bar{P}^* = \bar{P} + f_4(t),\, \mathbf{R}^*_{\{n\}} = \mathbf{R}_{\{n\}},\, \mathbf{P}^*_{\{n\}} = \mathbf{P}_{\{n\}},
\end{aligned} \tag{19}$$

are symmetries of the MPC equations (13), where all function and parameter definitions are adopted from 3.1 and $\mathbf{A}$ is a concatenation of rotation matrices as $A_{i_{(0)}j_{(0)}i_{(1)}j_{(1)}\cdots i_{(n)}j_{(n)}} = a_{i_{(0)}j_{(0)}}a_{i_{(1)}j_{(1)}}\ldots a_{i_{(n)}j_{(n)}}$. Formulating the MPC equations in relative coordinates the vector $\boldsymbol{r} = \boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}$ is introduced (see Oberlack & Rosteck (2010) and all given symmetries are formulated in relativ coordinates.

3.3. Statistical symmetries of the MPC equations

It is clear that a general symmetry analysis of the MPC equations could result in further symmetries, which have no correspondance in the Navier-Stokes equations so that these symmetries will be called *statistical symmetries*.

In Oberlack & Rosteck (2010) we showed already the first new symmetries, which can be found for the MPC equations. Considering this work the basis of this analysis does not lie on the MPC equations in the $\mathbf{R}$ formulation, but the previous mentioned $\mathbf{H}$ formalism is needed.

This entire new set of symmetries for the **R**-**P**-system

$$\bar{T}'_1 : t^* = t,\, \boldsymbol{x}^* = \boldsymbol{x},\, \boldsymbol{r}^*_{(l)} = \boldsymbol{r}_{(l)} + \boldsymbol{a}_{(l)},\, \bar{\boldsymbol{U}}^* = \bar{\boldsymbol{U}},\, \bar{P}^* = \bar{P},\, \mathbf{R}^*_{\{n\}} = \mathbf{R}_{\{n\}},\, \mathbf{P}^*_{\{n\}} = \mathbf{P}_{\{n\}}, \cdots \tag{20}$$

$$\begin{aligned}
\bar{T}'_{2_{\{1\}}} &: t^* = t,\;\; \boldsymbol{x}^* = \boldsymbol{x},\;\; \boldsymbol{r}^*_{(l)} = \boldsymbol{r}_{(l)},\;\; \bar{U}^*_{i_{(0)}} = \bar{U}_{i_{(0)}} + C_{i_{(0)}},\\
&\quad R^*_{i_{(0)}i_{(1)}} = R_{i_{(0)}i_{(1)}} + \bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}} - \left(\bar{U}_{i_{(0)}} + C_{i_{(0)}}\right)\left(\bar{U}_{i_{(1)}} + C_{i_{(1)}}\right),\;\; \cdots
\end{aligned} \tag{21}$$

$$\bar{T}'_{2_{\{2\}}}: t^* = t,\ \boldsymbol{x}^* = \boldsymbol{x},\ \boldsymbol{r}^*_{(l)} = \boldsymbol{r}_{(l)},\ \bar{U}^*_{i_{(0)}} = \bar{U}_{i_{(0)}},\ R^*_{i_{(0)}i_{(1)}} = R_{i_{(0)}i_{(1)}} + C_{i_{(0)}i_{(1)}},\ \cdots \tag{22}$$

$$\bar{T}'_s: t^* = t,\ \boldsymbol{x}^* = \boldsymbol{x},\ \boldsymbol{r}^*_{(l)} = \boldsymbol{r}_{(l)},\ \bar{U}^*_{i_{(0)}} = \mathrm{e}^{a_s}\bar{U}_{i_{(0)}},$$
$$R^*_{i_{(0)}i_{(1)}} = \mathrm{e}^{a_s}\left[R_{i_{(0)}i_{(1)}} + (1-\mathrm{e}^{a_s})\,\bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}}\right],\ \cdots, \tag{23}$$

can be seperated in three sets of symmetries. Note that this group is not related to the classical translation group in usual $\boldsymbol{x}$-space (here $T_7 - T_9$ in equation (17) with $\boldsymbol{f} = const.$).

A translation of the relative coordinates (20) occurs, where $\boldsymbol{a}_{(l)}$ represents the related set of group parameters. The second set of statistical symmetries was in fact already partially identified in Oberlack (2000), however, falsely taken for the Galilean group, where $C_{i_{(0)}}$, $D_{i_{(0)}}$, $C_{i_{(1)}}$, ... refer to group parameters. In this representation only the translation for $R_{i_{\{n\}}}$, $n = 1, 2$, is mentioned, although for arbitary n a respective symmetry can be formulated. The third statistical group (23) that has been identified denotes simple scaling of all MPC tensors.

In Rosteck & Oberlack (2011) we proved that (21) or (22) can be generalized, so that e.g. $C_{i_{(0)}}$ is a function of time and then a derivative of $C_{i_{(0)}}$ appears also for $\mathbf{P}^*_{\{n\}}$. Also another symmetry working on the pressure-velocity correlations can be identified. The basis reason to find these symmetries was that the whole set of given symmetries must form a Lie algebra. Its concrete form is omitted at this point because it is not needed for the further considerations.

4. Turbulent scaling laws

Using Lie theory to analyse the MPC equations (13) was first done only with the classical symmetries resulting from the Euler equations Oberlack (2000). Already these symmetries allowed a wide class of possible scaling laws for special flow characteristica.

First hints towards a considerably extended set of symmetries for the MPC equations in the form (14) or (13) may e.g. be taken from Oberlack (2000) and Khujadze & Oberlack (2004), where new statistical symmetries were necessary to calculate the appearing scaling law. Its importance was not observed therein - rather it was stated that they may be mathematical artifacts of the averaging process and probably physically irrelevant. The set of new symmetries was first presented and its key importance for turbulence recognized in Oberlack & Rosteck (2010) and later extended in Rosteck & Oberlack (2011).

The first scaling law calculated with the extended set of symmetries was the exponential decay produced by a fractal grid (see Khujadze & Oberlack (2004); Oberlack & Rosteck (2010)). The resulting scaling law shows a perfect agreement to the data by Seoud & Vassilicos (2007). This very positive result motivates to consider some other flows. Here we will deal later with the log law and rotating channel flows.

The following scaling laws will be derived on a pure mathematical way, where the basis is given by the MPC equations (13). Combining the symmetries of these equations invariant solution can be formed, while the number of used symmetries is limited by the boundary conditions (see Bluman *et al.* (2009)).

4.1. Stationary wall-bounded turbulent shear flows

Due to their eminent practical importance wall-bounded shear flows are by far the most intensively investigated turbulent flows thereby employing a vast number of numerical, experimental and modeling approaches and this, in fact, for more than a century.

From all the theoretical approaches the universal law of the wall is the most widely cited and also accepted approach with its essential ingredient being the logarithmic law of the wall. Though a variety of different approaches have been put forward for its derivation neither of them have employed the full multi-point equations, which are the basis for statistical turbulence, nor do they solve an equation that is related to the Navier-Stokes equations.

Reasonable for this case is a transformation of the points $\boldsymbol{x}_{i_{(0)}} \cdots \boldsymbol{x}_{i_{(n)}}$ into a coordinate system of related coordinates, meaning $\boldsymbol{x}_{(0)}$, $\boldsymbol{r}_{(1)} = \boldsymbol{x}_{(1)} - \boldsymbol{x}_{(0)}$, $\cdots$ $\boldsymbol{r}_{(n+1)} = \boldsymbol{x}_{(n+1)} - \boldsymbol{x}_{(0)}$. In this case equation (13) reduces for $n = 2$ in relative coordinates to

$$\begin{aligned}\mathcal{T}_{i_{\{2\}}} &= \frac{\bar{\mathrm{D}} R_{ij}}{\bar{\mathrm{D}} t} + R_{kj} \frac{\partial \bar{U}_i(\boldsymbol{x},t)}{\partial x_k} + R_{ik} \left.\frac{\partial \bar{U}_j(\boldsymbol{x},t)}{\partial x_k}\right|_{\boldsymbol{x}+\boldsymbol{r}} + \left[\bar{U}_k\left(\boldsymbol{x}+\boldsymbol{r},t\right) - \bar{U}_k\left(\boldsymbol{x},t\right)\right] \frac{\partial R_{ij}}{\partial r_k} \\ &+ \frac{\partial \overline{pu}_j}{\partial x_i} - \frac{\partial \overline{pu}_j}{\partial r_i} + \frac{\partial \overline{u_i p}}{\partial r_j} - \nu \left[\frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - 2\frac{\partial^2 R_{ij}}{\partial x_k \partial r_k} + 2\frac{\partial^2 R_{ij}}{\partial r_k \partial r_k}\right] \\ &+ \frac{\partial R_{(ik)j}}{\partial x_k} - \frac{\partial}{\partial r_k}\left[R_{(ik)j} - R_{i(jk)}\right] = 0 \ . \end{aligned} \tag{24}$$

In addition, the continuity equations, the side condition and, of course, also the symmetries transform to the new coordinate system.

Already in Oberlack (1995) it was observed that in the limit of high Reynolds numbers and $|\boldsymbol{r}| \gg \eta_K$ the logarithmic wall law allows for a self-similar solution of the two-point correlation equation (24).

Within this subsection we exclusively examine wall-parallel turbulent flows only depending on the wall-normal coordinate x_2. Further, we only explicitly write the two-point correlation R_{ij} though all results are also valid for all higher order correlations. This finally yields $\bar{U}_1 = \bar{U}_1(x_2)$, $R_{ij} = R_{ij}(x_2, \boldsymbol{r})$, $\ldots$.

With these geometrical assumptions we identify a reduced set of groups, where the two scaling groups $\bar{T}_2$ and $\bar{T}_3$ and the translation invariance form $\bar{T}_7 - \bar{T}_9$ in x_2-direction in (19) remain. Additionally, it is necessary to use the above-mentioned statistical symmetries, especially the translational group in correlation space (20), the translational group (21) for $\bar{U}_1$, the translational group (22) for R_{ij} and finally the scaling group (23) Applying these statistical symmetries is essential for calculating the following scaling laws.

Applying Lie theory (see Bluman *et al.* (2009)), the set of remaining symmetries can be formed into an invariance condition of the MPC equations

$$\frac{\mathrm{d}x_2}{k_2 x_2 + k_{x_2}} = \frac{\mathrm{d}r_{[k]}}{k_2 r_{[k]} + k_{r_{[k]}}} = \frac{\mathrm{d}\bar{U}_1}{(k_2 - k_3 + k_s)\bar{U}_1 + k_{\bar{U}_1}} = \frac{\mathrm{d}R_{[ij]}}{\xi_{R_{[ij]}}} = \cdots \ , \tag{25}$$

$$\begin{aligned}\xi_{R_{ij}} &= (2k_2 - 2k_3 + k_s)R_{ij} - \big(k_s \bar{U}_1(x_2)\bar{U}_1(x_2 + r_2) + \\ & k_{\bar{U}_1}(\bar{U}_1(x_2) + \bar{U}_1(x_2 + r_2))\big)\,\delta_{i1}\delta_{j1} + k_{R_{ij}}\ ,\end{aligned} \tag{26}$$

where no summation is implied by the indices in square brackets and instead a concatenation is implied where the indices are consecutively assigned its values. For brevity explicit dependencies on the independent variables are only given where there is an unambiguity. In general, any set of parameters k_i generates a set of invariants which are in fact invariant solutions.

In fact, with a distinct combinations of parameters k_2, k_3 and k_s a multitude of flows may be described where here we first focus on the log-law. We may keep in mind that $\bar{U}_1$ exclusively depends on x_2 and not on $\boldsymbol{r}$.

Considering the classical case of the logarithmic wall law the reason of the appearing symmetry breaking can be found by revisiting the key idea of von Kármán. He assumed that close to the wall the wall-friction velocity u_τ is the only parameter determining the flow, while a symmetry breaking of the form $k_2 - k_3 + k_s = 0$ (see Oberlack & Rosteck, 2010) follows.

Under this assumption (25) leads to the classical dimensionless logarithmic wall law

$$u^+ = \frac{1}{\kappa}\ln(x_2^+ + A^+) + C \ , \tag{27}$$

where the constants κ, A^+ and C depend on k_2, k_{x_2}, $k_{\bar{U}_1}$ and an integration constant. Moreover, the two-point correlation can be derived by (25), formulated in dimensionless form and reduced to Reynolds stresses by taking the limit $r \to 0$ so that we gain

$$R_{ij} = (x_2^+ + A^+)^\gamma D_{ij} + B_{ij}\,, \quad ij \neq 11\,, \tag{28}$$

$$R_{11} = D_{11}(x_2^+ + A^+)^\gamma - \frac{1}{\kappa^2}\ln^2(x_2^+ + A^+) - 2\frac{1}{\kappa}C\ln(x_2^+ + A^+) + B_{11}\,, \tag{29}$$

where the new constants γ, D_{ij} and B_{ij} are combinations of the k_α appearing in equation (25). It is remarkable to note that γ is the same constant in all higher moments, so that the main behavior of these scaling laws only depends on a reduced set of parameters.

4.2. Rotating channel

The second application to be focussed on is the rotating channel flow, where different rotational axes will be considered (see also Oberlack & Rosteck (2011)).

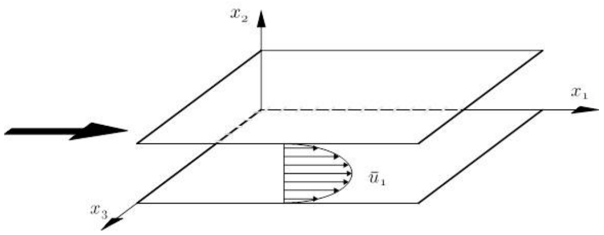

Figure 1. Flow geometry of the pressure driven channel flow.

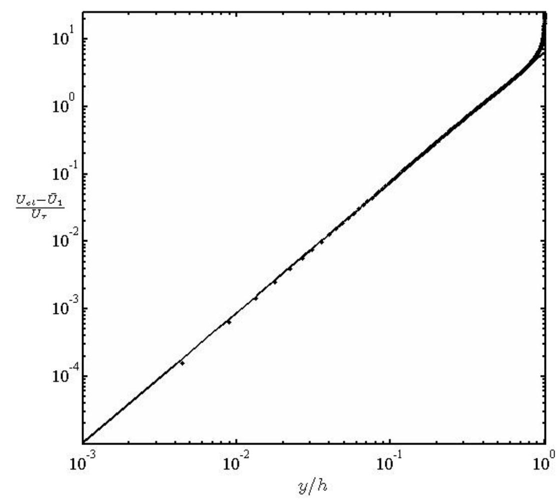

Figure 2. Comparison of the scaling law (−) in (32) with the DNS data ($\cdots$) of Hoyas & Jimenez (2006) at $Re_\tau = 2003$.

Using our symmetry analysis in order to gain scaling laws, the calculated symmetries have to be transformed into the coordinate system of a rotating frame. Then the invariant system can be developed and for each rotational axis the symmetries used in this case must be determined.

This leads to a rather complex and involved form of the operator (25) so details have to be omitted and only results for the mean flow will be given.

The certainly easiest case represents the rotation around the x_3 direction, so that only Ω_3 is non-zero. This was already analysed in Oberlack (2001) and the scaling law

$$\bar{U}_1(x_2) = \alpha_{rot}\Omega_3 x_2 + \bar{U}_{cl} \tag{30}$$

was formulated, where $\bar{U}_{cl}$ is the averaged velocity in the centreline. In this case the classical symmetries i.e. scaling in space and the Galilei invariance are used and extended by the action of the new scaling symmetry (23) and the translation of the velocities (21).

Next, assuming a rotation axis lying along the x_2 direction, two velocity components $\bar{U}_1$ and $\bar{U}_3$ have to be taken into consideration since the Coriolis force induces a cross flow. Again, both averaged velocities may only depend on x_2. Different to the first case is that one additional symmetry appears, namely translation in time i.e. $\bar{T}'_1$ in (20). From this we derive the new Ω_2 depending scaling laws

$$\bar{U}_1 = \left(\frac{y}{h}\right)^b \left[a_1 \cos\left(cRo_2 \cdot \ln \frac{y}{h}\right) + a_2 \sin\left(cRo_2 \cdot \ln \frac{y}{h}\right)\right] + d_1(Ro_2)$$
$$\bar{U}_3 = \left(\frac{y}{h}\right)^b \left[a_1 \sin\left(cRo_2 \cdot \ln \frac{y}{h}\right) - a_2 \cos\left(cRo_2 \cdot \ln \frac{y}{h}\right)\right] + d_2(Ro_2)\,. \tag{31}$$

A simplification of these equation can be done assuming a non-rotating channel flow. In this case we obtain the core region in defect scaling

$$\frac{U_{cl} - \bar{U}_1}{u_\tau} = a\left(\frac{y}{h}\right)^b\,, \tag{32}$$

where U_{cl} is the velocity at the center of the channel and u_τ is the friction velocity. A comparison to DNS data of Hoyas & Jimenez (2006) at $Re_\tau = 2003$ to the scaling law in Figure 2 shows an almost perfect agreement, here the parameters are fitted to $a = 6.43$ and $b = 1.93$.

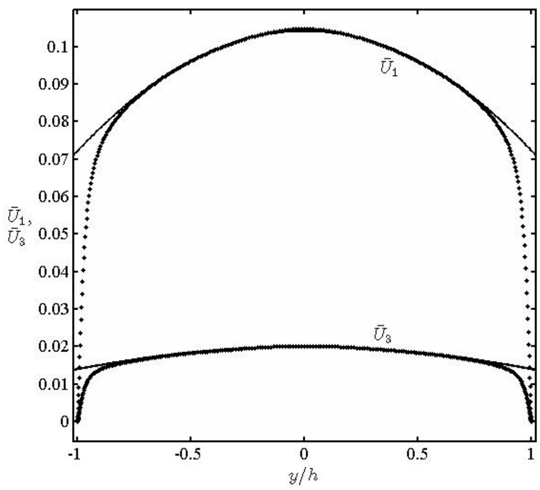

Figure 3. Comparison of the scaling law (−) in (31) with the DNS data (···) of Mehdizadeh & Oberlack (2010) at $Ro_2 = 0.011$.

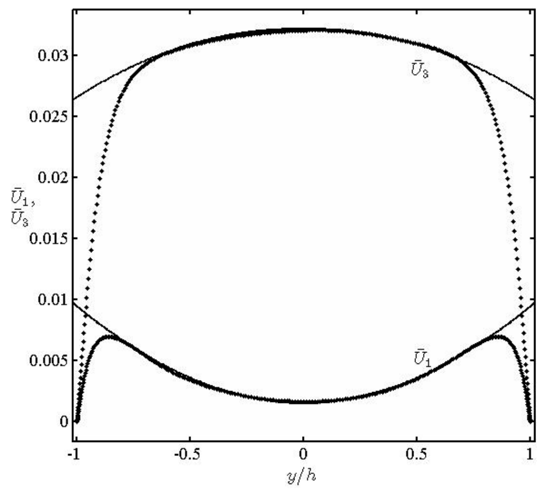

Figure 4. Comparison of the scaling law (−) in (31) with the DNS data (···) of Mehdizadeh & Oberlack (2010) at $Ro_2 = 0.18$.

Finally we consider the rotating case and compare the DNS data of Mehdizadeh & Oberlack (2010) at $Re_\tau = 360$ with the scaling law (31). Results are depicted for two different rotation numbers $Ro_2 = \frac{2\Omega_2 h}{u_{\tau 0}}$ in the figures 3 and 4 exhibiting an excellent fit in the center of the channel for all cases. $u_{\tau 0}$ refers to the friction velocity of the non-rotating case. It is to note from all the DNS data sets in Mehdizadeh & Oberlack (2010) we find that with an increasing Ω_2 the magnitudes of $\bar{U}_1$ and $\bar{U}_3$ switch position since with increasing rotation rates $\bar{U}_1$ is suppressed while $\bar{U}_3$ increases up to a certain point and decreases again though to a smaller extend compared to $\bar{U}_1$. This behavior is exactly described by the scaling law (31).

5. Summary and outlook

Within the present contribution it was shown that the admitted symmetry groups of the infinite set of multi-point correlation equations are considerably extended by three classes of groups compared to those originally stemming from the Euler and the Navier-Stokes equations. In fact, it was demonstrated that it is exactly these symmetries which are essentially needed to validate certain classical scaling laws such as the log-law from first principles and also to derive a large set of new scaling laws.

In spite of the very impressive results which give a much deeper understanding on turbulence statistics there are still some key open questions to be answered. Still, we cannot assure that all symetries are found, so that there is a development to find a algorithmic way to determine all symmetries. Furthermore, the appearing group parameters given in the scaling laws have certain decisive values which are to be determined. A method is necessary in order to gain them without fitting. Finally, we clearly observe that certain scaling laws such as the log law only cover certain regions of a turbulent flow and are usually embedded within other layers of turbulence. The matching of turbulent scaling laws is still an open question.

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