Turbulent Poiseuille Flow with Wall Transpiration: Analytical Study and Direct Numerical Simulation

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Abstract. We present an analytical and direct numerical simulation (DNS) study to describe an incompressible, fully developed turbulent Poiseuille flow with wall transpiration i.e. uniform transverse velocity with constant flux on the wall. The DNS was conducted at $Re_\tau = 250$ for different relative transpiration velocities. The DNS data serve as a first test case of a new turbulent scaling law in the form of a logarithm. DNS data validates the new turbulent logarithmic scaling law derived from Lie symmetry theory of the infinite dimensional multi-point correlation equation and is principally different from the classical near-wall log-law. We will show that the DNS data agree with the new turbulent scaling law over practically the whole cross-section of the channel.

1. Introduction

A turbulent plane Poiseuille flow with wall transpiration serves as an interesting and important object of investigation. On the one hand, being homogeneous in streamwise and spanwise directions, it is among relatively simple near-wall flows. On the other hand, it has fundamental properties universal for flows with transpiration and longitudinal pressure gradient. Moreover, it is an example of channel flow with an asymptotic mean velocity profile and a peculiar shear stress distribution when the point of zero shear stress may not coincide with that of maximum mean velocity. The flow geometry is shown on the figures 1 (sketch) and 2(volume plot). Only one known experimental study of a pressure-driven Poiseuille flow with wall transpiration was performed by Zhapbasbayev & Isakhanova (1998) (see also Zhapbasbayev & Yershin (2003)). Experiments were carried out in an air duct at Reynolds numbers based on the bulk velocity and the channel half width 10400, 22400 and 34000 as well as the blowing velocity normalized by the bulk one, up to 0.01. Beside mean velocity also the Reynolds-stress-tensor components were measured. DNS of a turbulent Poiseuille flow was conducted by Sumitani & Kasagi (1995). The authors investigated Poiseuille flow at $Re_\tau = u_\tau h/\nu = 150$ and $v_0/u_\tau = 0.05$, where

$$u_\tau = \sqrt{\frac{h}{\rho}\frac{\partial p}{\partial y}\Big|_{y=0}} \tag{1}$$

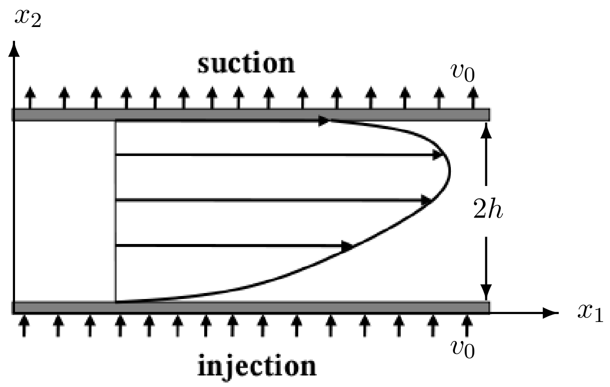

Figure 1. Sketch of Poiseuille-type flow with wall-transpiration.

is friction velocity calculated from the longitudinal mean pressure gradient, h is the channel half width and v_0 is the transpiration velocity.

For the same flow type, Nikitin & Pavel'ev (1998) generated results at $Re_\tau = 356, v_0/u_\tau = 0.112$ and $681.2, 0.118$, respectively. Vigdorovich, Hu & Coleman (2002) performed computation at $Re_\tau = 360$ and $v_0/u_\tau = 0.05$.
All the above mentioned DNS data were generated at low Reynolds numbers according to computational resources available ten years ago. However recently, Hoyas & Jiménez (2006) have studied a plane Poiseuille flow with impermeable walls for Reynolds numbers up to $Re_\tau = 2003$. Poiseuille flow without transpiration was studied analytically, by the method of matched asymptotic expansions at high Reynolds numbers by Yajnik (1970), Bush & Fendell (1972, 1973, 1974) and Lund & Bush (1980) with the use of some approximate particular closure hypotheses. In a series of papers Oberlack (2000, 2001); Oberlack & Rosteck (2010, 2011) the turbulent Poiseuille and related flows where studied using Lie symmetry theory investigating the infinite series of multi-point correlation equations and derived a variety of classical and new scaling laws. In the present paper DNS of the flow at Re_τ=250 and a wide range of the key parameter v_0 (transpiration velocity) was carried out to test a new theory based on Lie symmetries applied to the infinite set of multi-point correlation equations derived from the Navier-Stokes equations and to obtain the data on the microstructure of turbulent motion that cannot be provided by any present theory.

2. Analysis

2.1. Statistical transport equation
It seems suitable to introduce the analysis by giving a short review of the pertinent features of the phenomenological description of turbulent channel flow along the walls with transpiration. Axial statistical transport equation has the following form:

$$v_0 \frac{\partial \bar{U}_1}{\partial x_2} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_1} + \nu \frac{\partial^2 \bar{U}_1}{\partial x_2^2} - \frac{\partial \overline{u_1' u_2'}}{\partial x_2} \tag{2}$$

where $v_0 = 0.003;\ 0.0164;\ 0.05$ for present DNS. The ordinary no-slip boundary conditions were imposed only on the u_1 and u_3 components at the walls, but a constant mean velocity was given to the wall-normal component (u_2). Due to friction at the walls, the flow constantly looses kinetic energy, and hence a forcing needs to be employed in order to keep the flow from decelerating. This was done by fixing mass-flow rate. Due to a pressure gradient the latter adapted at every time step in order to keep the bulk velocity u_b constant:

$$\frac{1}{2h}\int_{-h}^{h}\bar{U}_1(x_2)dx_2 = u_b \tag{3}$$

Mass transfer in and out of the system occurs normal to the porous surfaces only.

2.2. Lie symmetry analisys

One of the key objectives of the research is to further develop and validate an asymptotic and Lie symmetry group theory for turbulent Poiseuille flow with wall transpiration at high Reynolds numbers. Related to the former a friction law for Poiseuille flow with transpiration was derived in Vigdorovich & Oberlack (2008), which allows to describe the relation between the wall shear stress, the Reynolds number, and the transpiration velocity by a function of one variable. Further a velocity defect law was established, which generalizes the classical law for the core region in a channel with impermeable walls to the case of transpiration.

Subsequently we briefly describe the Lie symmetry and invariance structure of the present flow emloying the infinite set of multi-point correlation (MPC) equations for the velocity and pressure fluctuations $u(x,t)$ and $p(x,t)$

$$\begin{aligned} S_{i_{\{n+1\}}} = \frac{\partial R_{i_{\{n+1\}}}}{\partial t} + \sum_{l=1}^{n}\Bigg[\bar{U}_{k_{(l)}}(x_{(l)})\frac{\partial R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}} - \nu\frac{\partial^2 R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}} + R_{i_{\{n+1\}}[i_{(l)}\to k_{(l)}]}\frac{\partial \bar{U}_{i_{(l)}}}{\partial x_{k_{(l)}}} \\ - R_{i_{\{n\}}[i_{(l)}\to 0]}\frac{\partial \overline{u_{i_{(l)}}u_{k_{(l)}}}(x_{(l)})}{\partial x_{k_{(l)}}} + \frac{\partial P_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}} + \frac{\partial R_{i_{\{n+2\}}[i_{(n+1)}\to k_{(l)}]}[x_{(n+1)}\to x_{(l)}]}{\partial x_{k_{(l)}}}\Bigg], \end{aligned} \tag{4}$$

where n varies from 1 to ∞. In (4) the MPC tensor is defined as

$$R_{i_{\{n+1\}}} = \overline{u_{i_{(0)}}(x_{(0)})\cdot\ldots\cdot u_{i_{(n)}}(x_{(n)})}, \tag{5}$$

and the four variations of it needed in equations (4) are given in Oberlack (2000); Oberlack & Rosteck (2010). Adding the corresponding continuity equations and some side conditions for $R_{i_{\{n+1\}}}$ the whole set generates a complete statistical description of turbulence.

For the present flow we have the boundary condition (BC) $\bar{U}_i(x_1, x_2 = \pm 1, x_3) = (0, v_0, 0)^T$ at the wall. For the transverse mean flow this implies that $\bar{U}_2(x_2) = const$ and together with the BC we obtain $\bar{U}_2(x_2) = v_0$ which leads to a simple and direct coupling of the BC and the MPC equation.

For the general invariance condition, i.e. requirement for a turbulent scaling law of plane shear flows, and including the new statistical symmetries found in Oberlack & Rosteck (2010, 2011) we find

$$\frac{\mathrm{d}x_2}{k_1x_2+k_x} = \frac{\mathrm{d}r_{(k)}}{k_1r_{(k)}} = \frac{\mathrm{d}\bar{U}_1}{(k_1-k_2+k_a)\bar{U}_1+l_1} = \frac{\mathrm{d}R_{(11)}(x_2,\underline{r})}{I(x_2,\underline{r})} = \cdots \quad . \tag{6}$$

Presently we have include the constraint $k_1 - k_2 + k_a = 0$ from the BC $\bar{U}_2(x_2) = v_0$. Imposing this onto (6) leads to new scaling laws particularly including a new logarithmic mean flow scaling law

$$\frac{\bar{U}_1}{v_0} = \gamma_1 \ln\left(\frac{x_2}{h}\right) + \gamma_2 \tag{7}$$

valid in the core region of a turbulent channel flow and presumed that v_0 is sufficiently large. For brevity scaling laws for higher order moments have been omitted. A first clear hint towards the validity of the new log-law may be taken from figure 3. Most important, and other than the classical near-wall log-law, we have the mean velocity normalized on the transpiration velocity.

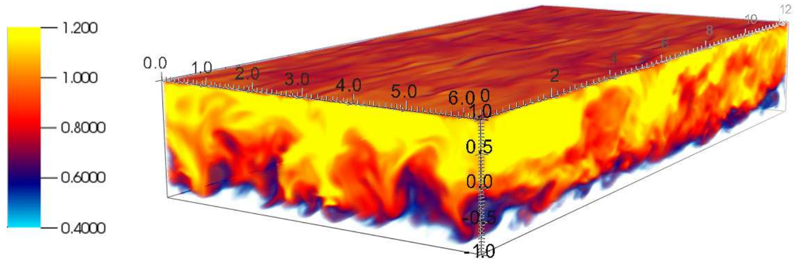

Figure 2. 3D instantaneous velocity field of the channel flow with transpiration. Injection side on the bottom, suction side on the top.

3. DNS of the turbulent channel flow

3.1. General information about the code

The numerical code used in the simulations was originally developed at School of Aeronautics, Technical University of Madrid (Hoyas & Jiménez, 2008; del Álamo & Jiménez, 2003). Various DNS of a turbulent channel flow have been performed in a computational box: $L_{x_1} = 4\pi h$, $L_{x_2} = 2$ and $L_{x_3} = 2\pi h$ (fig. 2), at a Reynolds number Re_τ=250 based on the friction velocity u_τ. All quantities in the code were normalized by half width h of the channel and bulk velocity. The number of collocation points used in the present simulations is $N_{x_1} = 768$, $N_{x_2} = 251$, $N_{x_3} = 256$. The spatial discretization uses dealised Fourier expansions in x_1 and x_3 directions and a seven-point compact finite differences in x_2, with fourth-order consistency and extended spectral-like resolution. The temporal discretization is third-order semi-implicit Runge-Kutta.

3.2. Validation of the code

Some DNS of the fully developed turbulent Poiseuille flow with uniform wall injection and suction have been conducted, one of which was carried out by Sumitani & Kasagi (1995). We verified our code using their results (fig. 3,4) comparing mean velocity profiles $\bar{U}_1$ and Reynolds-stresses $\overline{u'_i u'_j}$. From Sumitani & Kasagi (1995) we adopted both the friction Reynolds number of Re_τ=150 and a constant mean transverse velocity $v_0/u_\tau = 0.05$.

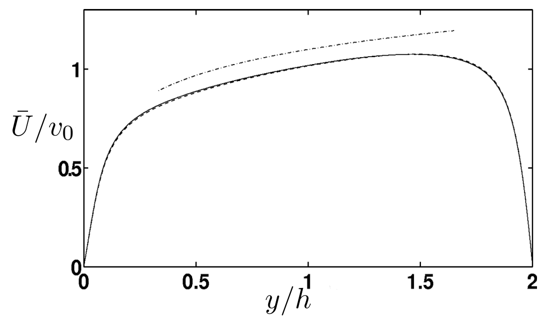

Figure 3. Comparison of mean velocity profiles of our DNS (solid line) to results Sumitani & Kasagi (1995) (dotted line). The dash-dotted line is the new log-law (for better visibility the curve is shifted up)

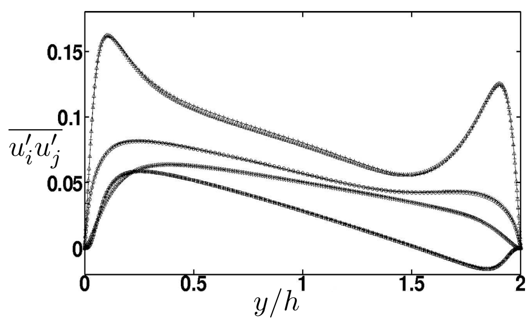

Figure 4. Comparison of Reynolds-stresses of our DNS to results Sumitani & Kasagi (1995). From the top to the bottom: $\overline{u'_1 u'_1}, \overline{u'_3 u'_3}, \overline{u'_2 u'_2}, \overline{u'_1 u'_2}$.

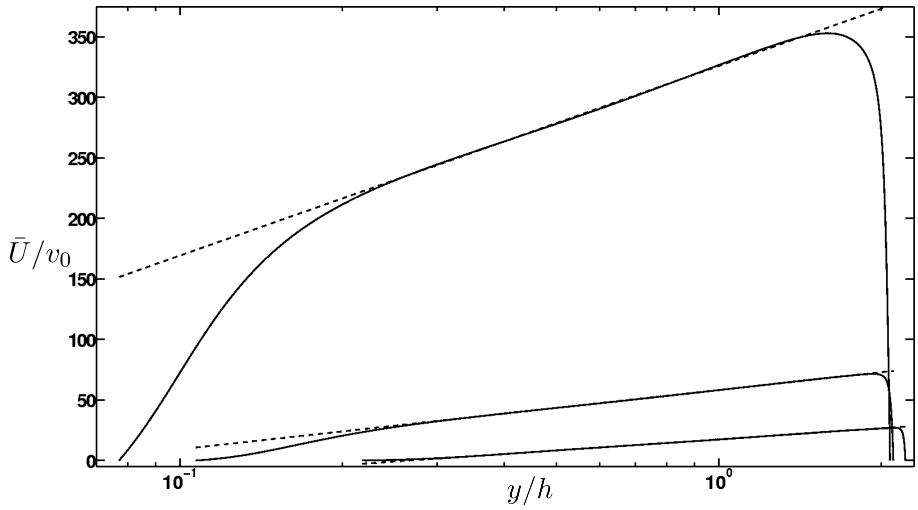

Figure 5. Comparison of the new log-lows (dashed line) with corresponding mean velocity profiles (solid line) obtained from DNS. Cases with different transpiration velocities, from top to the bottom $v_o/u_\tau = 0.05$, $v_o/u_\tau = 0.16$, $v_o/u_\tau = 0.26$.

4. Results

It has been widely believed that the mean velocity profile in the intermediate region in wall-bounded turbulent flow is adequately described by the von Kármán-Prandtl universal logarithmic law of the wall. In section 2 of the paper we presented a new logarithmic scaling law for the mean velocity profile. The main new feature of the law is its dependence on the transpiration velocity v_0. We performed a series of DNS for different values of transpiration velocity at fixed Re_τ=250. Analysis of the new log-law and mean velocity profiles gave us different overlap regions depending on transpiration velocity. For moderately large transverse velocity numbers ($v_0/u_\tau = 0.16$) new law is valid not only in the core region of the flow but up to 85% of the entire channel width. For the smallest v_0 we have chosen ($v_0/u_\tau = 0.05$) log region is 65% (fig. 5).

5. Conclusion

A new scaling law for a turbulent channel flow with wall transpiration obtained from the Lie group theory was validated using DNS. A rather extended region of validity in the core of the channel and the dependence on the transpiration velocity are key properties of the new log-law. Simulations at higher Reynolds numbers are required and are presently run for a further validation of the presented results.

References

Bush, W. & Fendell, F. 1972 Asymptotic analysis of turbulent channel and boundary layer flow. *J. Fluid Mech.* **56** (4), pp. 657-681.

Bush, W. & Fendell, F. 1973 Asymptotic analysis of turbulent channel flow for mean turbulent energy closures. *J. Fluid Mech.* **16** (8), pp. 1189-1197.

Bush, W. & Fendell, F. 1974 Asymptotic analysis of turbulent channel flow for mixing length theory. *SIAM J. Appl. Math.* **26** (2), pp. 413-427.

del Álamo, J. C. & Jiménez, J. 2003 Spectra of the very large anisotropic scales in turbulent channels. *Phys. of Fluids* **15**(6), pp. L41-L44.

Hoyas, S. & Jiménez, J. 2006 Scaling of velocity fluctuations in turbulent channels up to $Re_\tau = 2000$. *Phys. of Fluids* **18**, 011702-1-3.

Hoyas, S. & Jiménez, J. 2008 Reynolds number dependence of the Reynolds-stress budgets in channels. *Phys. of Fluids* **20**, 101511-1–101511-8.

Lund, K. & Bush, W. 1980 Asymptotic analysis of plane turbulent Couette-Poiseuille flows. *J. Fluid Mech.* **96** (1), pp. 81-104.

Nikitin, N. & Pavel'ev, A 1998 Turbulent flow in channels with permeable walls. The results of DNS and the three-parameter-model. *Fluid Dynamics* **33**,(6), 413.

Oberlack, M. 2000 Symmetrie, Invarianz und Selbstähnlichkeit in der Turbulenz. *Habilitation thesis* RWTH Aachen.

Oberlack, M. 2001 A unified approach for symmetries in plane parallel turbulent shear flows. *J. Fluid Mech.* **427**, 299-328.

Oberlack, M. & Rosteck A. 2010 New statistical symmetries of the multi-point equations and its importance for turbulent scaling laws. *Discrete Contin. Dyn. Syst., Ser. S* **3**, 451-471.

Oberlack, M. & Rosteck A. 2011 Lie algebra of the symmetries of the multi-point equations in statistical turbulence theory. *J. Nonlinear Math. Phys.* **18**, pp. 251-264.

Sumitani, Y. & Kasagi, N. 1995 Direct numerical simulations of turbulent transport with uniform wall injection and suction. *AIAA Journal*,**33**(7), 1220-1228.

Vigdorovich, I. I., Hu, Z. W. & Coleman, G. N. 2002 Analytical study and DNS of turbulent Couette-Poiseuille flow with uniform transpiration *Advances in Turbulence IX, Proceedings of the Ninth European Turbulence Conference*, (ed. Castro, I. P., Hancock, P. E.). *CIMNE, Barcelona* pp. 867-870.

Vigdorovich, I. & Oberlack, M. 2008 Analytical study of turbulent Poiseuille flow with wall transpiration. *Phys. Fluids* **20**, 055102-1–055102-9.

Yajnik, K. 1970 Asymptotic theory of turbulent shear flows. *J. Fluid Mech.* **42** (2), pp. 411-427.

Zhapbasbayev, U. & Isakhanova, G., 1998 Developed turbulent flow in a plane channel with simultaneous injection through one porous wall and suction through the other. *J. Appl. Mech. Tech. Phys.* **39**, 53.

Zhapbasbayev, U. & Yershin, Sh., 2003 Developed turbulent flow in a channel with mass exchange through porous walls. *Proceedings of the Fourth International Symposium on Turbulence, Heat and Mass Transfer.* (ed. Hanjalic K. et. al.) *Begell Hous, Inc.* pp. 173-180.