

## Optical transmittance of photonic structures with linearly graded dielectric constituents<sup>1</sup>

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**Abstract.** The optical transmittance of one-dimensional photonic structures generated from lossless dielectric slabs with linear, or piecewise linear, profiles of their electric permittivity for linearly polarized, normally incident electromagnetic radiation is studied both analytically and numerically. Resorting to an analysis of the respective photonic modes, the dependence on frequency of the optical property addressed is first established for the singly graded slab and the doubly graded slab, i.e. for the proper dielectric constituents, and subsequently for the photonic structures derived from them. Considering the singly graded periodic structure, the transmittance reveals a series of photonic bands and gaps, exhibiting maxima less than unity, as for the singly graded slab, and tending to saturate in a periodic sequence of centro-symmetric columnar peaks, separated by gaps of identical widths, at large frequencies; an asset that recommends this type of photonic structure as ideal. Looking at the doubly graded periodic structure, the transmittance reveals two series of photonic bands and gaps, exhibiting maxima of full transparency, as for the doubly graded slab, or maxima of reduced transparency and tending to saturate in essentially constant values with zero gaps at large frequencies; a trait known for all frequencies from an optically homogeneous, non-dispersive medium of semi-infinite extent. The properties revealed for both the singly graded periodic structure and the doubly graded periodic structure offer unique possibilities regarding practical applications of such novel photonic composites, optical filters exploiting their transmittance or optical mirrors utilizing their reflectance only being the most obvious.

<sup>1</sup> This paper is dedicated to Professor A M Stoneham FRS on the occasion of his 70th birthday.

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**Contents**

<b>1. Introduction</b>	<b>2</b>
<b>2. Theory and results</b>	<b>3</b>
2.1. Singly graded slab . . . . .	4
2.2. Doubly graded slab . . . . .	7
2.3. Singly graded periodic structure . . . . .	10
2.4. Doubly graded periodic structure . . . . .	14
<b>3. Conclusions</b>	<b>18</b>
<b>Acknowledgment</b>	<b>19</b>
<b>Appendix. Stepwise constant periodic structure</b>	<b>19</b>
<b>References</b>	<b>22</b>

**1. Introduction**

Photonic crystals are regularly structured, synthetic composites made up of materials with different refractive indices. The terminology dates back to the year 1987, when three-dimensional dielectric structures of this kind were analyzed for the first time [1, 2]. One-dimensional periodic structures, however, had already been studied a century before [3]: multi-layer stacks of dielectric plates were then found to display spectral ranges of high optical reflectivity, now called stop bands or bandgaps [4]. Investigations of the effect of transparency bands on the spontaneous emission of light from atoms and molecules embedded in one-dimensional structures followed later on [5].

The two keynote works initially cited above sparked off a wealth of publications about photonic composites, and the literature tends to ever grow [6]–[18]. From a practical point of view, interest in such structures is due to their unique optical properties—selective transmission of electromagnetic waves in definite ranges of frequency perhaps being the most prominent of them—revealed when the structural period length is comparable with, or larger than, the vacuum wavelength of the irradiating electromagnetic field. These traits offer ready exploitation for modern photonics and optoelectronics applications that rely on, e.g. (i) fast optical switching, routing, filtering and forging spectrometers-on-a-chip as well as the availability of several laser types [12, 14] or (ii) techniques for preparing samples with desired characteristics by doping, infiltration or mechanical deformation [12, 14], [19]–[23].

Apart from employing conventional dielectrics, photonic crystals may be fabricated from materials prone to undergo order/disorder phase transitions, with performances that can be controlled by external parameters and fields. Foremost examples are structures involving ferroelectric [24, 25], ferromagnetic [26, 27] or superconducting [28, 29] constituents, which behave significantly different below and above the respective transition temperatures. In their low-temperature, ordered states, these materials are sensitive to applied electric or magnetic fields and also to mechanical stress; a fact likely to unfold additional and entirely new vistas for governing the optical properties of such composites.

Recently, graded materials distinguished by characteristics that change continuously over each layer have been suggested for use as building blocks in one-dimensional

photonic composites, with suitable space dependences of their electric permittivity or magnetic permeability, and hence their refractive index, ensuring significant bandgaps persist [30]. Whereas common multi-layer structures restrict themselves to regularly alternating homogeneous layers of different, but constant, thicknesses and refractive indices [31], graded multi-layer structures open up the additional possibility of shaping the refractive index's profile itself, and hence broaden the spectrum of reflection or transmission properties; these structures might also include homogeneous, alongside with graded, constituents, extending the optical diversity even more [32].

For single-layer structures displaying exponential, parabolic or linear profiles of the refractive index and linear profiles of the electric permittivity, the dispersion of electromagnetic waves as well as the optical transmittance and reflectance have been explored [31], [33]–[35]. Periodically modulated profiles of the electric permittivity have been considered on top, in which case selective optical transmittances were found, as with photonic crystals having two different kinds of spatially homogeneous constituents [36, 37]. For regular multi-layer stacks of exponentially index-graded plates combined with spatially homogeneous plates, the distribution of the electromagnetic field, the structure of the photonic bands and gaps as well as the dispersion of the optical reflectance have been obtained. It was established that, due to the insertion of the so-graded plates, the density and the ranges of frequency of the bandgaps rise (the locations of the gaps allowing control by adjustments of the refractive index) and the reflectance of these stacks augments [38].

Motivated by the opportunities for guiding electromagnetic radiation that the technique of grading provides, we here study the optical transmittance of one-dimensional photonic structures generated by lossless dielectric slabs with linear, or piecewise linear, profiles of their electric permittivity, aiming to examine whether intriguing features emerge for such structures too. Calling upon the convenient vector potential approach, in section 2 we introduce the photonic modes of singly and doubly graded dielectric slabs as well as those of periodic structures derived from them, assuming linearly polarized electromagnetic radiation at normal incidence. Applying these, we gain exact analytical representations of the transmittance and the structure of the photonic bands that cover the whole dispersion regime, and supplement them with simple asymptotic forms limited to large frequencies. Graphical illustrations serve to highlight the crucial points in each case. Finally, in section 3 we conclude by summarizing the insights and results obtained, suggesting potential applications of the novel photonic composites. A summary of mathematical details relating to periodic structures with stepwise constant electric permittivities quoted for comparison can be found in the [appendix](#).

## 2. Theory and results

Let us look at a singly graded dielectric slab of thickness  $d$  and, respectively, a doubly graded dielectric slab of thickness  $2d$  as constituents of one-dimensional periodic structures of semi-infinite extent, both spreading infinitely along the  $x$ - and  $y$ -directions of a Cartesian coordinate system  $x, y, z$ . We envisage that linearly polarized electromagnetic radiation propagating in the positive  $z$ -direction through the vacuum space  $z < 0$ , with the electric (magnetic) field oriented parallel to the  $x$ -axis and the magnetic (electric) field oriented parallel to the  $y$ -axis of this coordinate system, is normally incident on the surface  $z = 0$  of either type of slab, whose optical properties are supposed to be entirely determined by the principal components of the tensor of electric permittivity  $\varepsilon_\nu$  for  $\nu = x, y$ . Assuming a harmonic time dependence with arbitrary

frequency  $\omega \geq 0$ , the space-dependent parts of the vector potential in the Coulomb gauge  $\tilde{A}_v$ , i.e. the photonic modes, obey the master equation [29]

$$\frac{d^2 \tilde{A}_v}{dz^2} + k_0^2 \varepsilon_v \tilde{A}_v = 0, \quad (1)$$

identifying the wavenumber of electromagnetic radiation  $k_0 = \omega/c$ , where  $c$  means the vacuum speed of light. Solutions of equation (1) allow derivations of the electric and magnetic fields as well as the time-averaged flow of electromagnetic energy straightforwardly.

In the range  $0 \leq z \leq d$ , the principal components of the tensor of electric permittivity of the singly graded slab and the doubly graded slab shall be given by  $\varepsilon_v = \varepsilon_A(z)$ , where

$$\varepsilon_A(z) = \varepsilon_a + \left( \frac{\varepsilon_b - \varepsilon_a}{d} \right) (z - 0) \quad (2)$$

with constant dielectric parameters  $\varepsilon_a > 1$  and  $\varepsilon_b > 1$ , neglecting electromagnetic dispersion as well as dissipation and assuming in-plane crystalline isotropy. In the range  $d \leq z \leq 2d$ , the principal components of the tensor of electric permittivity of the doubly graded slab shall be given by  $\varepsilon_v = \varepsilon_B(z)$ , where

$$\varepsilon_B(z) = \varepsilon_b - \left( \frac{\varepsilon_b - \varepsilon_a}{d} \right) (z - d) \quad (3)$$

with the same dielectric parameters  $\varepsilon_a$  and  $\varepsilon_b$ , again neglecting electromagnetic dispersion as well as dissipation and assuming in-plane crystalline isotropy. On introducing the variables

$$\zeta_{A,B}(z) = -u^2 \varepsilon_{A,B}(z) \quad \text{with} \quad u = \text{sgn}(\varepsilon_b - \varepsilon_a) \left| \frac{k_0 d}{\varepsilon_a - \varepsilon_b} \right|^{1/3}, \quad (4)$$

which adopt the respective values  $\alpha = -u^2 \varepsilon_a$  for  $z = 0$  and  $\beta = -u^2 \varepsilon_b$  for  $z = d$ , equation (1) transforms into Airy's differential equations

$$\frac{d^2 \tilde{A}_v}{d\zeta_{A,B}^2} - \zeta_{A,B} \tilde{A}_v = 0 \quad (5)$$

solved by the Airy functions  $Ai$  and  $Bi$  of arguments  $\zeta_A(z)$  and  $\zeta_B(z)$  [39]. Investigations of the optical transmittance of the slabs or the periodic structures made up of them imply solving equation (1) for the photonic modes separately in  $z < 0$  and  $z \geq 0$  and joining the solutions together smoothly at  $z = 0$ ; a procedure that ensures continuity of the tangential components of both the electric and the magnetic field.

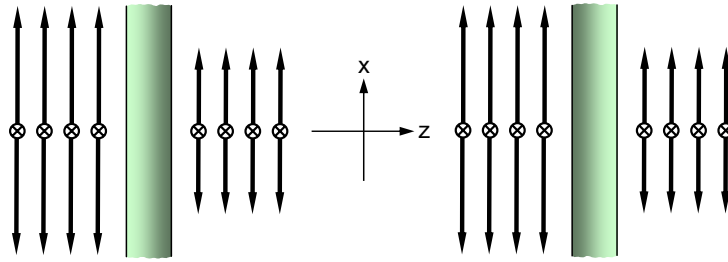
### 2.1. Singly graded slab

We first address the singly graded dielectric slab of thickness  $d$  faced with two vacua and excited by normally incident electromagnetic radiation, as figure 1 depicts. The photonic modes in the vacuum space  $z < 0$  are

$$\tilde{A}_v(z) = V_{1,\text{sg}}^{(s)} \exp(ik_0 z) + V_{2,\text{sg}}^{(s)} \exp(-ik_0 z) \quad (6)$$

with constants  $V_{1,\text{sg}}^{(s)}$  and  $V_{2,\text{sg}}^{(s)}$ . In the range  $0 \leq z \leq d$  occupied by the slab, they adopt the form

$$\tilde{A}_v(z) = A_{1,\text{sg}}^{(s)} Ai(\zeta_A(z)) + A_{2,\text{sg}}^{(s)} Bi(\zeta_A(z)) \quad (7)$$



**Figure 1.** Schematic view of the singly graded slab of thickness  $d$  excited by normally incident electromagnetic radiation with the electric (magnetic) field oriented parallel to the  $x$ -axis and the magnetic (electric) field oriented parallel to the  $y$ -axis of a Cartesian coordinate system  $x, y, z$ . The optical density of the slab is symbolized by the spatially varying colour depth for the dielectric parameters  $\varepsilon_a < \varepsilon_b$  (left) and, respectively,  $\varepsilon_a > \varepsilon_b$  (right).

owing to equations (2), (4) and (5), with constants  $A_{1,\text{sg}}^{(s)}$  and  $A_{2,\text{sg}}^{(s)}$ ; in the vacuum space  $z > d$ , they read

$$\tilde{A}_v(z) = V_{3,\text{sg}}^{(s)} \exp(ik_0z) \quad (8)$$

with a free amplitude  $V_{3,\text{sg}}^{(s)}$ .

Linking the photonic modes, equations (6)–(8), together smoothly at the surfaces  $z = 0$  and, respectively,  $z = d$  yields the constants

$$V_{1,\text{sg}}^{(s)} = \frac{\pi}{2} [M_{A-}(\alpha)N_{B-}(\beta) - M_{B-}(\alpha)N_{A-}(\beta)] V_{3,\text{sg}}^{(s)} \exp(ik_0d) \quad (9)$$

and

$$V_{2,\text{sg}}^{(s)} = \frac{\pi}{2} [M_{A+}(\alpha)N_{B+}(\beta) - M_{B+}(\alpha)N_{A+}(\beta)] V_{3,\text{sg}}^{(s)} \exp(ik_0d) \quad (10)$$

as well as

$$A_{1,\text{sg}}^{(s)} = \pi N_{B-}(\beta) V_{3,\text{sg}}^{(s)} \exp(ik_0d) \quad (11)$$

and

$$A_{2,\text{sg}}^{(s)} = -\pi N_{A-}(\beta) V_{3,\text{sg}}^{(s)} \exp(ik_0d) \quad (12)$$

in terms of  $V_{3,\text{sg}}^{(s)}$ , where

$$M_{A\pm}(\alpha) = Ai(\alpha) \pm \frac{i}{u} Ai'(\alpha) \quad (13)$$

and

$$M_{B\pm}(\alpha) = Bi(\alpha) \pm \frac{i}{u} Bi'(\alpha) \quad (14)$$

as well as

$$N_{A-}(\beta) = Ai'(\beta) - iuAi(\beta) \quad (15)$$

and

$$N_{B-}(\beta) = Bi'(\beta) - iuBi(\beta), \quad (16)$$

the primes symbolizing the derivatives of the Airy functions  $Ai'$  and  $Bi'$  for the respective arguments; a result that duplicates the findings of a previous analysis [40]. The transmittance  $T_{\text{sg}}^{(s)}$ , defined through the time-averaged flows of electromagnetic energy of the incident and transmitted waves using equations (6), (8) and (9), thus reads

$$T_{\text{sg}}^{(s)} = \frac{4/\pi^2}{|M_{A-}(\alpha)N_{B-}(\beta) - M_{B-}(\alpha)N_{A-}(\beta)|^2}. \quad (17)$$

In the steady-oscillations regime attained for  $\omega \gg \omega_{\text{sg}}$ , equation (17) takes on the asymptotic form

$$T_{\text{sg}}^{(s)} \cong \frac{1}{C_{1,\text{sg}}^2 \cos^2(\pi\omega/\omega_{\text{sg}}) + C_{2,\text{sg}}^2 \sin^2(\pi\omega/\omega_{\text{sg}})}, \quad (18)$$

where

$$C_{1,\text{sg}} = \frac{1}{2} \left( v + \frac{1}{v} \right) \quad (19)$$

with the modified ratio of dielectric parameters  $v = (\varepsilon_a/\varepsilon_b)^{1/4}$ , and

$$C_{2,\text{sg}} = \frac{1}{2} \left( n + \frac{1}{n} \right) \quad (20)$$

with the refractive index  $n = \varepsilon^{1/2}$  in terms of the average dielectric parameter  $\varepsilon = (\varepsilon_a\varepsilon_b)^{1/2}$ , apart from the characteristic frequency

$$\omega_{\text{sg}} = \frac{3\pi}{2} \left| \frac{\varepsilon_a - \varepsilon_b}{\varepsilon_a^{3/2} - \varepsilon_b^{3/2}} \right| \frac{c}{d}. \quad (21)$$

Equation (18) reveals maxima given by

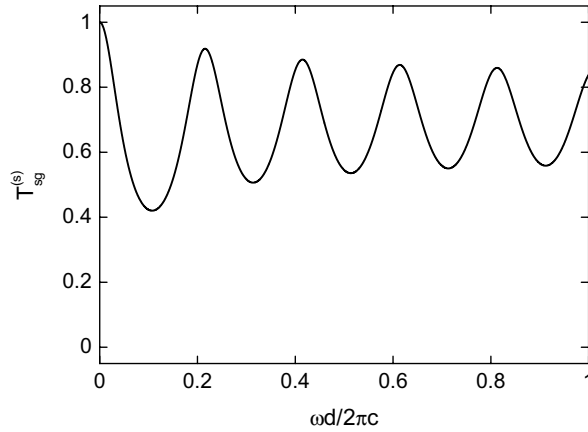
$$T_{\text{sg}}^{(s),\text{max}} \cong \frac{1}{C_{1,\text{sg}}^2} \quad \text{at} \quad \omega_j^{\text{max}} = j\omega_{\text{sg}}, \quad j \gg 1, \quad (22)$$

and minima given by

$$T_{\text{sg}}^{(s),\text{min}} \cong \frac{1}{C_{2,\text{sg}}^2} \quad \text{at} \quad \omega_j^{\text{min}} = \left( j - \frac{1}{2} \right) \omega_{\text{sg}}, \quad j \gg 1, \quad (23)$$

the coefficients  $C_{1,\text{sg}}$  and  $C_{2,\text{sg}}$  thus determining the finesse of the singly graded dielectric slab.

Figure 2 presents the variation with frequency of the transmittance of the singly graded dielectric slab, obtained from equation (17), for linearly polarized electromagnetic radiation with on-axis propagation, as illustrated in figure 1, taking the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  or, respectively,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$ . This shows that, because of the inversion symmetry of the radiation incidence regarding a positive (for  $\varepsilon_a < \varepsilon_b$ ) or negative (for  $\varepsilon_a > \varepsilon_b$ ) gradient of the electric permittivity of the slab, the transmittance in either case proves degenerate: starting from unity at zero frequency, quasi singly periodic oscillations for increasing frequency appear, with maxima delineated by a monotonically falling envelope and minima delineated by a monotonically rising envelope, tending to saturate in singly periodic oscillations at reduced transparency for large frequencies, as equation (18) predicts. Of course, periodic oscillations with a constant amplitude and full transparency at harmonically spaced resonant frequencies throughout would come to the fore, if the limit  $\varepsilon_b \rightarrow \varepsilon_a$  characterizing an optically homogeneous, non-dispersive slab were addressed.



**Figure 2.** Transmittance  $T_{sg}^{(s)}$  of the singly graded slab of thickness  $d$  as a function of the normalized frequency  $\omega d/2\pi c$  in the case of linearly polarized electromagnetic radiation due to on-axis propagation, for the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  or, respectively,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$ .

## 2.2. Doubly graded slab

We next consider the doubly graded dielectric slab of thickness  $2d$  faced with two vacua and excited by normally incident electromagnetic radiation, as figure 3 depicts. The photonic modes in the vacuum space  $z < 0$  are

$$\tilde{A}_v(z) = V_{1,dg}^{(s)} \exp(ik_0z) + V_{2,dg}^{(s)} \exp(-ik_0z) \quad (24)$$

with constants  $V_{1,dg}^{(s)}$  and  $V_{2,dg}^{(s)}$ . In the range  $0 \leq z \leq d$  of the slab, they adopt the form

$$\tilde{A}_v(z) = A_{1,dg}^{(s)} Ai(\zeta_A(z)) + A_{2,dg}^{(s)} Bi(\zeta_A(z)) \quad (25)$$

owing to equations (2), (4) and (5), with constants  $A_{1,dg}^{(s)}$  and  $A_{2,dg}^{(s)}$ ; in the range  $d \leq z \leq 2d$  of the slab, they adopt the form

$$\tilde{A}_v(z) = B_{1,dg}^{(s)} Ai(\zeta_B(z)) + B_{2,dg}^{(s)} Bi(\zeta_B(z)) \quad (26)$$

owing to equations (3), (4) and (5), with constants  $B_{1,dg}^{(s)}$  and  $B_{2,dg}^{(s)}$ , and in the vacuum space  $z > 2d$ , they read

$$\tilde{A}_v(z) = V_{3,dg}^{(s)} \exp(ik_0z) \quad (27)$$

with a free amplitude  $V_{3,dg}^{(s)}$ .

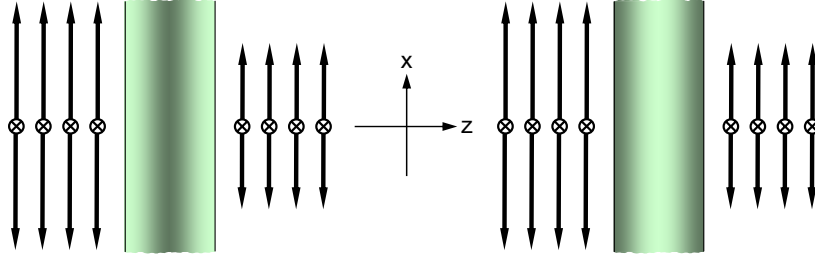
Linking the photonic modes, equations (24)–(27), together smoothly at the surface  $z = 0$ , the interface  $z = d$  and, respectively, the surface  $z = 2d$  yields the constants

$$V_{1,dg}^{(s)} = \frac{\pi^2}{2} [M_{A+}(\alpha) O_{BA-}(\alpha, \beta) + M_{B+}(\alpha) O_{AB-}(\alpha, \beta)] V_{3,dg}^{(s)} \exp(2ik_0d) \quad (28)$$

and

$$V_{2,dg}^{(s)} = \frac{\pi^2}{2} [M_{A-}(\alpha) O_{BA-}(\alpha, \beta) + M_{B-}(\alpha) O_{AB-}(\alpha, \beta)] V_{3,dg}^{(s)} \exp(2ik_0d) \quad (29)$$





**Figure 3.** Schematic view of the doubly graded slab of thickness  $2d$  excited by normally incident electromagnetic radiation with the electric (magnetic) field oriented parallel to the  $x$ -axis and the magnetic (electric) field oriented parallel to the  $y$ -axis of a Cartesian coordinate system  $x, y, z$ . The optical density of the slab is symbolized by the spatially varying colour depth for the dielectric parameters  $\varepsilon_a < \varepsilon_b$  (left) and, respectively,  $\varepsilon_a > \varepsilon_b$  (right).

as well as

$$A_{1,\text{dg}}^{(s)} = \pi^2 O_{BA-}(\alpha, \beta) V_{3,\text{dg}}^{(s)} \exp(2ik_0d) \quad (30)$$

and

$$A_{2,\text{dg}}^{(s)} = \pi^2 O_{AB-}(\alpha, \beta) V_{3,\text{dg}}^{(s)} \exp(2ik_0d) \quad (31)$$

as well as

$$B_{1,\text{dg}}^{(s)} = \pi N_{B-}(\alpha) V_{3,\text{dg}}^{(s)} \exp(2ik_0d) \quad (32)$$

and

$$B_{2,\text{dg}}^{(s)} = -\pi N_{A-}(\alpha) V_{3,\text{dg}}^{(s)} \exp(2ik_0d) \quad (33)$$

in terms of  $V_{3,\text{dg}}^{(s)}$ , referring to the definitions of equations (13)–(16), where

$$O_{AB-}(\alpha, \beta) = N_{A-}(\alpha) P_{BA+}(\beta) - 2N_{B-}(\alpha) Ai(\beta) Ai'(\beta) \quad (34)$$

and

$$O_{BA-}(\alpha, \beta) = N_{B-}(\alpha) P_{AB+}(\beta) - 2N_{A-}(\alpha) Bi(\beta) Bi'(\beta) \quad (35)$$

with

$$P_{AB+}(\beta) = Ai(\beta) Bi'(\beta) + Bi(\beta) Ai'(\beta) \quad (36)$$

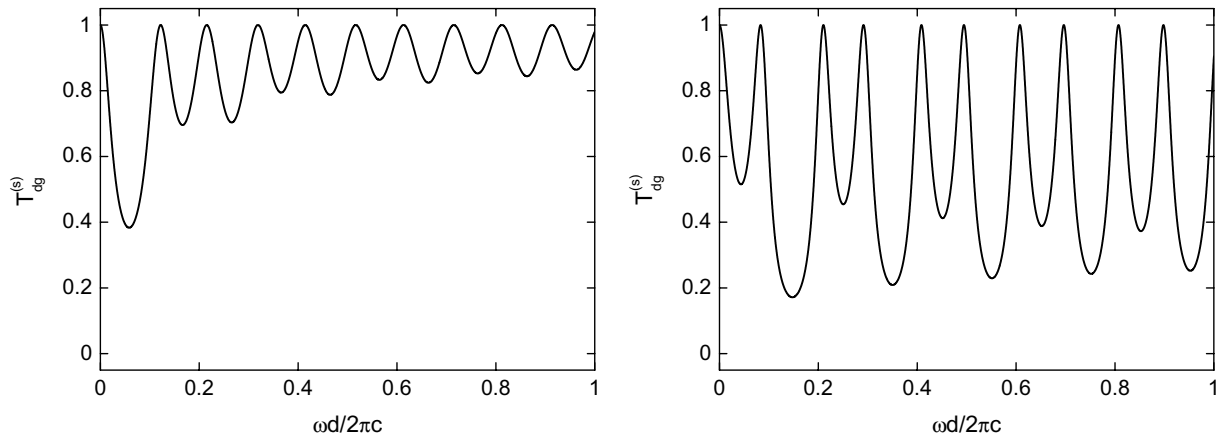
and

$$P_{BA+}(\beta) = Bi(\beta) Ai'(\beta) + Ai(\beta) Bi'(\beta), \quad (37)$$

the primes symbolizing the derivatives of the Airy functions  $Ai'$  and  $Bi'$  for the respective arguments. The transmittance  $T_{\text{dg}}^{(s)}$ , defined through the time-averaged flows of electromagnetic energy of the incident and transmitted waves using equations (24), (27) and (28), thus reads

$$T_{\text{dg}}^{(s)} = \frac{4/\pi^4}{|M_{A+}(\alpha) O_{BA-}(\alpha, \beta) + M_{B+}(\alpha) O_{AB-}(\alpha, \beta)|^2}. \quad (38)$$





**Figure 4.** Transmittance  $T_{\text{dg}}^{(s)}$  of the doubly graded slab of thickness  $2d$  as a function of the normalized frequency  $\omega d/2\pi c$  in the case of linearly polarized electromagnetic radiation due to on-axis propagation, for the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  (left) and, respectively,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$  (right).

In the steady-oscillations regime attained for  $\omega \gg \omega_{\text{dg}}$ , equation (38) takes on the asymptotic form

$$T_{\text{dg}}^{(s)} \cong \frac{1}{\cos^2(\pi\omega/\omega_{\text{dg}}) + C_{2,\text{dg}}^2 \sin^2(\pi\omega/\omega_{\text{dg}})}, \quad (39)$$

where

$$C_{2,\text{dg}} = \frac{1}{2} \left( n_a + \frac{1}{n_a} \right) \quad (40)$$

with the refractive index  $n_a = \varepsilon_a^{1/2}$ , apart from the characteristic frequency

$$\omega_{\text{dg}} = \frac{1}{2} \omega_{\text{sg}}. \quad (41)$$

Equation (39) reveals maxima given by

$$T_{\text{dg}}^{(s),\text{max}} = 1 \quad \text{at} \quad \omega_j^{\text{max}} = j\omega_{\text{dg}}, \quad j \gg 1, \quad (42)$$

and minima given by

$$T_{\text{dg}}^{(s),\text{min}} \cong \frac{1}{C_{2,\text{dg}}^2} \quad \text{at} \quad \omega_j^{\text{min}} = \left( j - \frac{1}{2} \right) \omega_{\text{dg}}, \quad j \gg 1, \quad (43)$$

the coefficient  $C_{2,\text{dg}}$  thus measuring the finesse of the doubly graded dielectric slab.

Figure 4 presents the variation with frequency of the transmittance of the doubly graded dielectric slab, obtained from equation (38), for linearly polarized electromagnetic radiation with on-axis propagation, as illustrated in figure 3, taking the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  or, respectively,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$ . This shows that, because of the complementarity of the radiation incidence regarding a ‘convex’ (for  $\varepsilon_a < \varepsilon_b$ ) or ‘concave’ (for  $\varepsilon_a > \varepsilon_b$ ) profile of the electric permittivity of the slab, the transmittance in either case proves non-degenerate, having some traits in common, but also pointing to an essential difference: starting from unity at zero frequency, quasi doubly periodic oscillations for increasing frequency appear, with maxima

of full transparency at an infinite sequence of resonant frequencies and minima delineated by two associated envelopes both of which, if  $\varepsilon_a < \varepsilon_b$ , rise monotonically, whereas if  $\varepsilon_a > \varepsilon_b$ , one falls monotonically while the other rises monotonically, tending to saturate in unique singly periodic oscillations at a constant amplitude for large frequencies, as equation (39) predicts. It may be commented that the occurrence of maxima of full transmittance here is reminiscent of, e.g., the well-known phenomenon of excitation of resonant energy states of symmetric quantum wells [41]. As for the singly graded slab, periodic oscillations with a constant amplitude and full transparency at harmonically spaced resonant frequencies throughout would come to the fore, if the limit  $\varepsilon_b \rightarrow \varepsilon_a$  characterizing an optically homogeneous, non-dispersive slab were addressed.

### 2.3. Singly graded periodic structure

We now examine a one-dimensional dielectric photonic structure adjacent to a vacuum, generated by periodic continuation into the half-space  $z \geq 0$  of the singly graded dielectric slab of thickness  $d$  and excited by linearly polarized, normally incident electromagnetic radiation, as figure 5 depicts. Let us start by establishing the photonic modes. Thanks to Floquet's theorem [42], which introduces a wavenumber  $k_z$  confined to the first Brillouin zone  $-\pi/d \leq k_z \leq \pi/d$ , the following derivations may be restricted to a primitive unit cell. Thus, in the range  $0 \leq z \leq d$ , the solution of equation (1) is

$$\tilde{A}_v(z) = A_{1,\text{sg}}^{(\text{p})} Ai(\zeta_A(z)) + A_{2,\text{sg}}^{(\text{p})} Bi(\zeta_A(z)) \quad (44)$$

owing to equations (2), (4) and (5), with constants  $A_{1,\text{sg}}^{(\text{p})}$  and  $A_{2,\text{sg}}^{(\text{p})}$ , determined up to an arbitrary non-zero factor, to read

$$A_{1,\text{sg}}^{(\text{p})} = Bi(\beta) - Bi(\alpha) \exp(ik_z d) \quad (45)$$

and

$$A_{2,\text{sg}}^{(\text{p})} = Ai(\alpha) \exp(ik_z d) - Ai(\beta). \quad (46)$$

The dispersion of the photonic modes itself follows from the implicit equation

$$M_{AB-}(\alpha, \beta) - M_{BA-}(\beta, \alpha) = \frac{2}{\pi} \cos(k_z d), \quad (47)$$

which defines the relation  $\omega = \omega_j(k_z)$  for varying wavenumber  $k_z$ , the label  $j = 0, 1, 2, \dots$  distinguishing the photonic bands appropriate for the chosen direction of radiation incidence and type of polarization; it simultaneously implies the relation  $k_z = k_z(\omega_j)$  for varying frequency  $\omega_j$  of the photonic bands with label  $j = 0, 1, 2, \dots$  called for in the evaluation of the transmittance examined below. In equation (47),

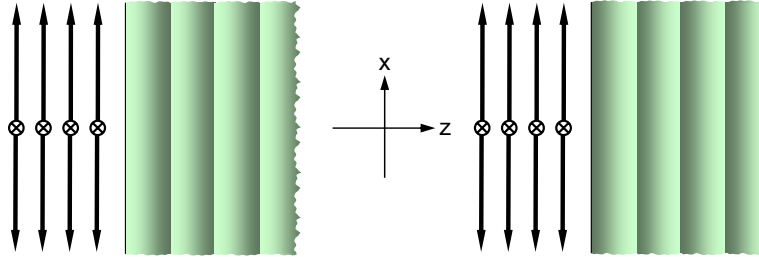
$$M_{AB-}(\alpha, \beta) = Ai(\alpha)Bi'(\beta) - Bi(\alpha)Ai'(\beta) \quad (48)$$

and

$$M_{BA-}(\beta, \alpha) = Bi(\beta)Ai'(\alpha) - Ai(\beta)Bi'(\alpha). \quad (49)$$

Asymptotic expansion of equation (47) for large frequencies  $\omega_j$ , when  $j \gg 1$ , yields the closed 'tight-binding' form

$$\omega_j(k_z) \cong \left[ j + \frac{1}{2} + (-1)^{j+1} \frac{1}{\pi} \arcsin \left( \frac{\cos(k_z d)}{C_{1,\text{sg}}} \right) \right] \omega_{\text{sg}} \quad (50)$$



**Figure 5.** Schematic view of the singly graded periodic structure of primitive translation  $d$  excited by normally incident electromagnetic radiation with the electric (magnetic) field oriented parallel to the  $x$ -axis and the magnetic (electric) field oriented parallel to the  $y$ -axis of a Cartesian coordinate system  $x, y, z$ . The optical density of the periodic structure is symbolized by the spatially varying colour depth for the dielectric parameters  $\varepsilon_a < \varepsilon_b$  (left) and, respectively,  $\varepsilon_a > \varepsilon_b$  (right).

with the quantity  $C_{1,\text{sg}}$  given by equation (19) and the inter-band spacing  $\omega_{\text{sg}}$  given by equation (21). This shows that the band half-width  $\Delta\omega_{\text{sg}}$  in turn amounts to

$$\Delta\omega_{\text{sg}} = \frac{1}{\pi} \text{arccsc}(C_{1,\text{sg}}) \omega_{\text{sg}}. \quad (51)$$

Figure 6 elucidates the dispersion of the photonic modes of the singly graded dielectric periodic structure obtained from equation (47) for normally incident, linearly polarized electromagnetic radiation, taking the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  or, equivalently,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$ . We note a quasi-periodic sequence of photonic bands and gaps controlling electromagnetic wave propagation along the structure, which tends to saturate in a periodic sequence of photonic bands for large frequencies, as equation (50) predicts.

The photonic modes in the vacuum space  $z < 0$  read

$$\tilde{A}_v(z) = V_{1,\text{sg}}^{(\text{p})} \exp(ik_0z) + V_{2,\text{sg}}^{(\text{p})} \exp(-ik_0z), \quad (52)$$

with amplitudes  $V_{1,\text{sg}}^{(\text{p})}$  and  $V_{2,\text{sg}}^{(\text{p})}$  given by

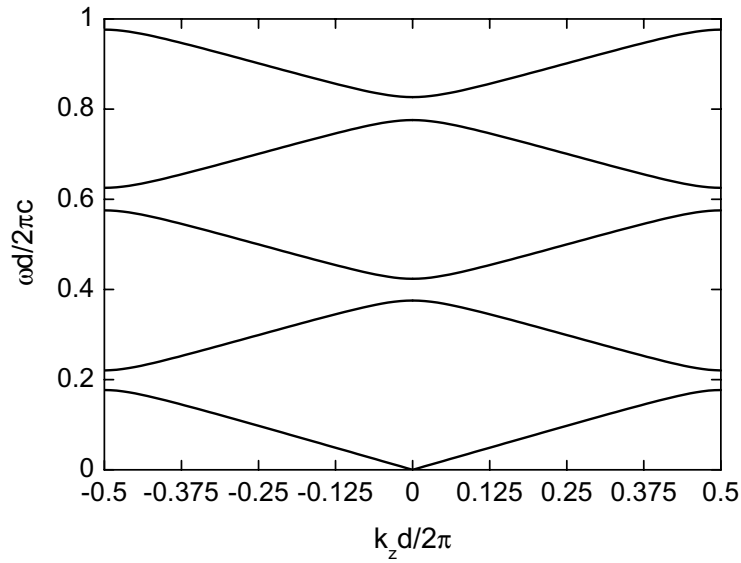
$$V_{1,\text{sg}}^{(\text{p})} = \frac{1}{2} \left[ M_{A+}(\alpha) A_{1,\text{sg}}^{(\text{p})} + M_{B+}(\alpha) A_{2,\text{sg}}^{(\text{p})} \right] \quad (53)$$

and

$$V_{2,\text{sg}}^{(\text{p})} = \frac{1}{2} \left[ M_{A-}(\alpha) A_{1,\text{sg}}^{(\text{p})} + M_{B-}(\alpha) A_{2,\text{sg}}^{(\text{p})} \right], \quad (54)$$

making recourse to the definitions of equations (13) and (14). The transmittance  $T_{\text{sg}}^{(\text{p})}$ , defined through the time-averaged flows of electromagnetic energy of the incident and transmitted waves using equations (44), (52) and (53), in the case of dielectric bands for  $0 \leq k_z \leq \pi/d$  or, respectively, air bands for  $-\pi/d \leq k_z < 0$ , with  $l = 0, 1, 2, \dots$ , thus reads

$$\left. \begin{array}{l} T_{2l,\text{sg}}^{(\text{p}+)} \\ T_{2l+1,\text{sg}}^{(\text{p}-)} \end{array} \right\} = \frac{4}{\pi} \left| \frac{\sin(k_z d)}{u} \right| \frac{|W_{AB-}(\alpha, \beta)|}{\left| M_{A+}(\alpha) A_{1,\text{sg}}^{(\text{p})} + M_{B+}(\alpha) A_{2,\text{sg}}^{(\text{p})} \right|^2}; \quad (55)$$



**Figure 6.** Dispersion of the photonic modes of the singly graded periodic structure of primitive translation  $d$  in the case of linearly polarized electromagnetic radiation due to on-axis propagation, for the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  or, respectively,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$ : normalized frequency  $\omega d/2\pi c$  of the lowest five photonic bands as a function of the normalized wavenumber  $k_z d/2\pi$  within the first Brillouin zone.

likewise, this quantity, defined through the time-averaged flows of electromagnetic energy of the reflected and transmitted waves using equations (44), (52) and (54), in the case of dielectric bands for  $-\pi/d \leq k_z < 0$  or, respectively, air bands for  $0 \leq k_z \leq \pi/d$ , with  $l = 0, 1, 2, \dots$ , reads

$$\left. \begin{array}{l} T_{2l,sg}^{(p-)} \\ T_{2l+1,sg}^{(p+)} \end{array} \right\} = \frac{4}{\pi} \left| \frac{\sin(k_z d)}{u} \right| \frac{|W_{AB-}(\alpha, \beta)|}{\left| M_{A-}(\alpha) A_{1,sg}^{(p)} + M_{B-}(\alpha) A_{2,sg}^{(p)} \right|^2}. \quad (56)$$

Herein,

$$W_{AB-}(\alpha, \beta) = Ai(\alpha) Bi(\beta) - Bi(\alpha) Ai(\beta). \quad (57)$$

Both of these cases make use of the amplitudes  $A_{1,sg}^{(p)}$  and  $A_{2,sg}^{(p)}$  spelt out in equations (45) and (46) and require exploiting equation (47).

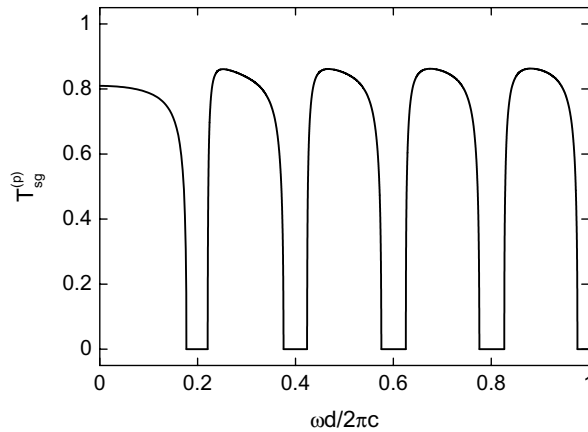
In the steady-state regime attained for  $\omega_j \gg \omega_{sg}$ , equations (55) and (56) take on the asymptotic form

$$T_{j,sg}^{(p)} \simeq \frac{2 |\operatorname{scn}(\pi \omega_j / \omega_{sg})|}{|\operatorname{scn}(\pi \omega_j / \omega_{sg})| + C_{2,sg} |\sin(\pi \omega_j / \omega_{sg})|} \quad (58)$$

$$\text{in } \left( j + \frac{1}{2} \right) \omega_{sg} - \Delta \omega_{sg} < \omega_j < \left( j + \frac{1}{2} \right) \omega_{sg} + \Delta \omega_{sg},$$

where

$$\operatorname{scn}(\pi \omega_j / \omega_{sg}) = \pm \left| 1 - C_{1,sg}^2 \cos^2(\pi \omega_j / \omega_{sg}) \right|^{1/2} \quad (59)$$



**Figure 7.** Transmittance  $T_{sg}^{(p)}$  of the singly graded periodic structure of primitive translation  $d$  as a function of the normalized frequency  $\omega d/2\pi c$  in the case of linearly polarized electromagnetic radiation due to on-axis propagation, for the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  or, respectively,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$ .

with the coefficients  $C_{1,sg}$  and  $C_{2,sg}$  defined in equations (19) and (20); they reveal maxima given by

$$T_{j,sg}^{(p),\max} \cong \frac{2}{1 + C_{2,sg}} \quad \text{at} \quad \omega_j^{\max} = \left(j + \frac{1}{2}\right) \omega_{sg}, \quad j \gg 1, \quad (60)$$

and minima given by

$$T_{j,sg}^{(p),\min} = 0$$

$$\text{in} \quad \left(j - \frac{1}{2}\right) \omega_{sg} + \Delta\omega_{sg} \leq \omega \leq \left(j + \frac{1}{2}\right) \omega_{sg} - \Delta\omega_{sg}, \quad j \gg 1. \quad (61)$$

Figure 7 presents the variation with frequency of the transmittance of the singly graded dielectric periodic structure, obtained from equations (55) and (56), for linearly polarized electromagnetic radiation with on-axis propagation, as illustrated in figure 5, taking the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  or, respectively,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$ . This shows that, because of the inversion symmetry of the radiation incidence regarding the succession of unidirectional, positive (for  $\varepsilon_a < \varepsilon_b$ ) or negative (for  $\varepsilon_a > \varepsilon_b$ ) gradients of the electric permittivity of the singly graded periodic structure, the transmittance in either situation (which bears witness to the sequence of photonic bands and gaps of figure 6) proves degenerate: starting from the value appropriate for an optically homogeneous, non-dispersive medium of semi-infinite extent, with average electric permittivity  $\varepsilon_0 = \varepsilon_a/2 + \varepsilon_b/2$  at zero frequency [29], a quasi-periodic sequence of biased columnar peaks for increased frequency appears, with maxima inside the photonic bands delineated by a constant, less than full, height, tending to saturate in a periodic sequence of centro-symmetric columnar peaks, separated by gaps of identical widths, for large frequencies, in keeping with the asymptotics disclosed by equation (58). Of course, a constant, less than maximum, transmittance for all frequencies would come to the fore, if the limit  $\varepsilon_b \rightarrow \varepsilon_a$  characterizing an optically homogeneous, non-dispersive medium of semi-infinite extent were addressed.

#### 2.4. Doubly graded periodic structure

We finally turn to a one-dimensional dielectric photonic structure adjacent to a vacuum, generated by periodic continuation into the half-space  $z \geq 0$  of the doubly graded dielectric slab of thickness  $2d$  and excited by linearly polarized, normally incident electromagnetic radiation, as figure 8 depicts. Let us start by establishing the photonic modes. Thanks to Floquet's theorem [42], which introduces a wavenumber  $k_z$  confined to the first Brillouin zone  $-\pi/2d \leq k_z \leq \pi/2d$ , the following derivations may be restricted to a primitive unit cell. Thus, in the range  $0 \leq z \leq d$ , the solution of equation (1) is

$$\tilde{A}_v(z) = A_{1,\text{dg}}^{(\text{p})} Ai(\zeta_A(z)) + A_{2,\text{dg}}^{(\text{p})} Bi(\zeta_A(z)) \quad (62)$$

owing to equations (2), (4) and (5), and in the range  $d \leq z \leq 2d$ , the solution of equation (1) is

$$\tilde{A}_v(z) = B_{1,\text{dg}}^{(\text{p})} Ai(\zeta_B(z)) + B_{2,\text{dg}}^{(\text{p})} Bi(\zeta_B(z)) \quad (63)$$

owing to equations (3), (4) and (5). The constants  $A_{1,\text{dg}}^{(\text{p})}$  and  $A_{2,\text{dg}}^{(\text{p})}$  as well as  $B_{1,\text{dg}}^{(\text{p})}$  and  $B_{2,\text{dg}}^{(\text{p})}$ , determined up to an arbitrary non-zero factor, read

$$A_{1,\text{dg}}^{(\text{p})} = Bi(\alpha) \left[ \frac{1}{\pi} \exp(2ik_z d) + P_{AB+}(\beta) \right] - 2Ai(\alpha)Bi(\beta)Bi'(\beta) \quad (64)$$

and

$$A_{2,\text{dg}}^{(\text{p})} = Ai(\alpha) \left[ P_{BA+}(\beta) - \frac{1}{\pi} \exp(2ik_z d) \right] - 2Bi(\alpha)Ai(\beta)Ai'(\beta) \quad (65)$$

as well as

$$B_{1,\text{dg}}^{(\text{p})} = Bi(\alpha) \left[ \frac{1}{\pi} + P_{AB+}(\beta) \exp(2ik_z d) \right] - 2Ai(\alpha)Bi(\beta)Bi'(\beta) \exp(2ik_z d) \quad (66)$$

and

$$B_{2,\text{dg}}^{(\text{p})} = Ai(\alpha) \left[ P_{BA+}(\beta) \exp(2ik_z d) - \frac{1}{\pi} \right] - 2Bi(\alpha)Ai(\beta)Ai'(\beta) \exp(2ik_z d). \quad (67)$$

The dispersion of the photonic modes itself follows from the implicit equation

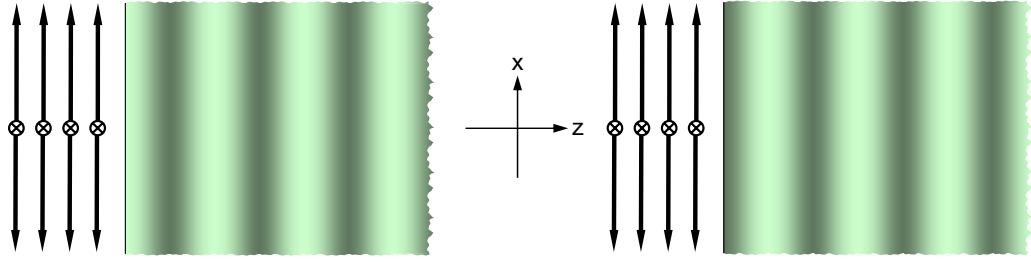
$$Q_{AABB}(\alpha, \beta) + P_{AB+}(\alpha)P_{BA+}(\beta) + Q_{BBAA}(\alpha, \beta) = \frac{1}{\pi^2} \cos(2k_z d), \quad (68)$$

which defines the relation  $\omega = \omega_j(k_z)$  for varying wavenumber  $k_z$ , the label  $j = 0, 1, 2, \dots$  distinguishing the photonic bands appropriate for the chosen direction of radiation incidence and type of polarization; it simultaneously implies the relation  $k_z = k_z(\omega_j)$  for varying frequency  $\omega_j$  of the photonic bands with label  $j = 0, 1, 2, \dots$  called for in the evaluation of the transmittance examined below. In equation (68),

$$Q_{AABB}(\alpha, \beta) = -2Ai(\alpha)Ai'(\alpha)Bi(\beta)Bi'(\beta), \quad (69)$$

apart from the definitions of equations (36), (37), and

$$Q_{BBAA}(\alpha, \beta) = -2Bi(\alpha)Bi'(\alpha)Ai(\beta)Ai'(\beta). \quad (70)$$



**Figure 8.** Schematic view of the doubly graded periodic structure of primitive translation  $2d$  excited by normally incident electromagnetic radiation with the electric (magnetic) field oriented parallel to the  $x$ -axis and the magnetic (electric) field oriented parallel to the  $y$ -axis of a Cartesian coordinate system  $x, y, z$ . The optical density of the periodic structure is symbolized by the spatially varying colour depth for the dielectric parameters  $\varepsilon_a < \varepsilon_b$  (left) and, respectively,  $\varepsilon_a > \varepsilon_b$  (right).

Asymptotic expansion of equation (68) for large frequencies  $\omega_j$ , when  $j \gg 1$ , yields the ‘empty-lattice’ form

$$\omega_j(k_z) \cong \left[ j + \frac{1}{2} + (-1)^{j+1} \left( \frac{1}{2} - \frac{|2k_z d|}{\pi} \right) \right] \omega_{\text{dg}} \quad (71)$$

with the inter-band spacing  $\omega_{\text{dg}}$  given by equation (41). This shows that the band half-width  $\Delta\omega_{\text{dg}}$  in turn amounts to

$$\Delta\omega_{\text{dg}} = \frac{1}{2} \omega_{\text{dg}}, \quad (72)$$

indicating that bandgaps disappear.

Figure 9 elucidates the dispersion of the photonic modes of the doubly graded dielectric periodic structure obtained from equation (68) for normally incident, linearly polarized electromagnetic radiation, taking the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  or, equivalently,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$ . We note a quasi-periodic sequence of photonic bands and gaps controlling electromagnetic wave propagation along the structure, which tends to saturate in a periodic sequence of photonic bands for large frequencies, as equation (71) predicts. Because of band splitting and bandgaps ceasing to exist, as equation (72) states, there are twice as many bands in the Brillouin zone of halved extent compared to the singly graded periodic structure examined above.

The photonic modes in the vacuum space  $z < 0$  read

$$\tilde{A}_v(z) = V_{1,\text{dg}}^{(\text{p})} \exp(ik_0 z) + V_{2,\text{dg}}^{(\text{p})} \exp(-ik_0 z), \quad (73)$$

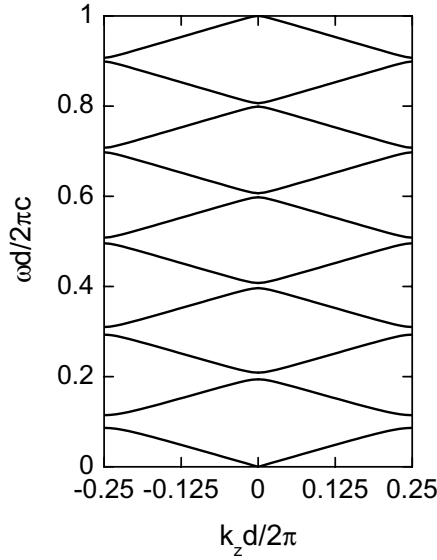
with amplitudes  $V_{1,\text{dg}}^{(\text{p})}$  and  $V_{2,\text{dg}}^{(\text{p})}$  given by

$$V_{1,\text{dg}}^{(\text{p})} = \frac{1}{2} \left[ M_{A+}(\alpha) A_{1,\text{dg}}^{(\text{p})} + M_{B+}(\alpha) A_{2,\text{dg}}^{(\text{p})} \right] \quad (74)$$

and

$$V_{2,\text{dg}}^{(\text{p})} = \frac{1}{2} \left[ M_{A-}(\alpha) A_{1,\text{dg}}^{(\text{p})} + M_{B-}(\alpha) A_{2,\text{dg}}^{(\text{p})} \right], \quad (75)$$





**Figure 9.** Dispersion of the photonic modes of the doubly graded periodic structure of primitive translation  $2d$  in the case of linearly polarized electromagnetic radiation due to on-axis propagation, for the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  or, respectively,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$ : normalized frequency  $\omega d/2\pi c$  of the lowest ten photonic bands as a function of the normalized wavenumber  $k_z d/2\pi$  within the first Brillouin zone.

making recourse to the definitions of equations (13) and (14). The transmittance  $T_{\text{dg}}^{(\text{p})}$ , defined through the time-averaged flows of electromagnetic energy of the incident and transmitted waves using equations (62), (73) and (74), in the case of dielectric bands for  $0 \leq k_z \leq \pi/2d$  or, respectively, air bands for  $-\pi/2d \leq k_z < 0$ , with  $l = 0, 1, 2, \dots$ , thus reads

$$\left. \begin{array}{l} T_{2l,\text{dg}}^{(\text{p}+)} \\ T_{2l+1,\text{dg}}^{(\text{p}-)} \end{array} \right\} = \frac{8}{\pi^2} \left| \frac{\sin(2k_z d)}{u} \right| \frac{|M_{AB-}(\alpha, \beta) W_{AB-}(\alpha, \beta)|}{|M_{A+}(\alpha) A_{1,\text{dg}}^{(\text{p})} + M_{B+}(\alpha) A_{2,\text{dg}}^{(\text{p})}|^2}; \quad (76)$$

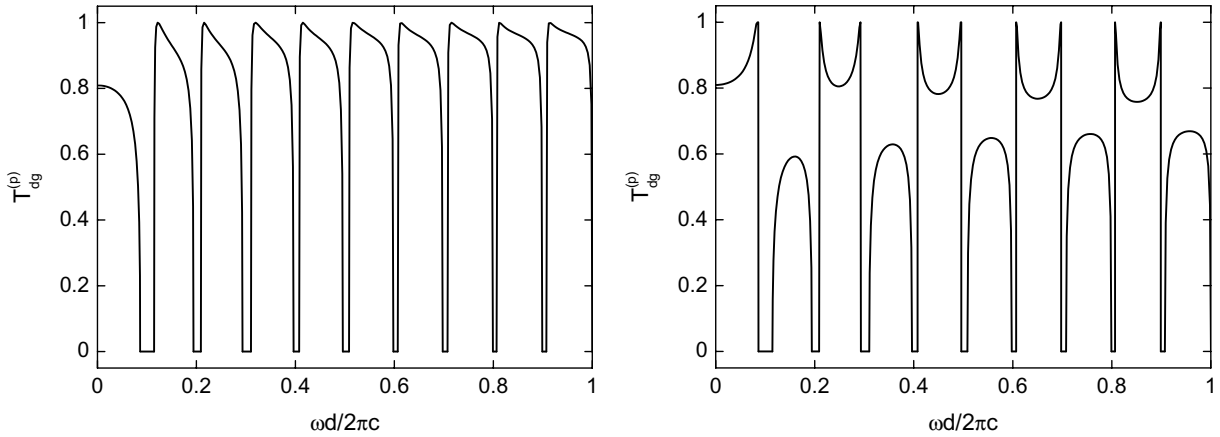
likewise, this quantity, defined through the time-averaged flows of electromagnetic energy of the reflected and transmitted waves using equations (62), (73) and (75), in the case of dielectric bands for  $-\pi/2d \leq k_z < 0$  or, respectively, air bands for  $0 \leq k_z \leq \pi/2d$ , with  $l = 0, 1, 2, \dots$ , reads

$$\left. \begin{array}{l} T_{2l,\text{dg}}^{(\text{p}-)} \\ T_{2l+1,\text{dg}}^{(\text{p}+)} \end{array} \right\} = \frac{8}{\pi^2} \left| \frac{\sin(2k_z d)}{u} \right| \frac{|M_{AB-}(\alpha, \beta) W_{AB-}(\alpha, \beta)|}{|M_{A-}(\alpha) A_{1,\text{dg}}^{(\text{p})} + M_{B-}(\alpha) A_{2,\text{dg}}^{(\text{p})}|^2}. \quad (77)$$

Herein, the definitions of equations (48) and (57) apply. Both of these cases make use of the amplitudes  $A_{1,\text{dg}}^{(\text{p})}$  and  $A_{2,\text{dg}}^{(\text{p})}$  spelt out in equations (64) and (65) and require exploiting equation (68).

In the steady-state regime attained for  $\omega_j \gg \omega_{\text{dg}}$ , equations (76) and (77) take on the asymptotic form

$$T_{j,\text{dg}}^{(\text{p})} \cong \frac{2}{1 + C_{2,\text{dg}}} \quad \text{in} \quad j\omega_{\text{dg}} < \omega_j < (j+1)\omega_{\text{dg}} \quad (78)$$



**Figure 10.** Transmittance  $T_{\text{dg}}^{(\text{p})}$  of the doubly graded periodic structure of primitive translation  $2d$  as a function of the normalized frequency  $\omega d/2\pi c$  in the case of linearly polarized electromagnetic radiation due to on-axis propagation, for the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  (left) and, respectively,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$  (right).

with the quantity  $C_{2,\text{dg}}$  defined in equation (40); they reveal maxima given by

$$T_{j,\text{dg}}^{(\text{p}),\text{max}} = 1 \quad \text{at} \quad \omega_j^{\text{max}} = j\omega_{\text{dg}+}, \quad j \gg 1, \quad (79)$$

and minima given by

$$T_{j,\text{dg}}^{(\text{p}),\text{min}} = 0 \quad \text{at} \quad \omega_j^{\text{min}} = j\omega_{\text{dg}-}, \quad j \gg 1. \quad (80)$$

Figure 10 presents the variation with frequency of the transmittance of the doubly graded dielectric periodic structure, obtained from equations (76) and (77), for linearly polarized electromagnetic radiation with on-axis propagation, as illustrated in figure 8, taking the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  or, respectively,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$ . This shows that, because of the complementarity of the radiation incidence regarding the succession of ‘convex’ (for  $\varepsilon_a < \varepsilon_b$ ) or ‘concave’ (for  $\varepsilon_a > \varepsilon_b$ ) profiles of the electric permittivity of the doubly graded periodic structure, the transmittance in either situation (which bears witness to the sequence of photonic bands and gaps of figure 9) proves non-degenerate, having some traits in common, but also pointing to essential differences: starting from the value appropriate for an optically homogeneous, non-dispersive medium of semi-infinite extent, with average electric permittivity  $\varepsilon_0 = \varepsilon_a/2 + \varepsilon_b/2$  at zero frequency [29], if  $\varepsilon_a < \varepsilon_b$ , a quasi singly periodic sequence of biased columnar peaks for increasing frequency appears, with resonant maxima close to the lower edges of the photonic bands, whereas if  $\varepsilon_a > \varepsilon_b$ , a quasi doubly periodic sequence of dented resonant peaks and biased columnar peaks for increasing frequency appears, with minima or, respectively, maxima inside the photonic bands delineated by monotonically descending or, respectively, ascending heights; these tend to saturate in sequences of peaks of rectangular shape, i.e. in constant, higher (for  $\varepsilon_a < \varepsilon_b$ ) or, respectively, lower (for  $\varepsilon_a > \varepsilon_b$ ), but less than full, transmittances appropriate for an optically homogeneous structure of semi-infinite extent,

with electric permittivity  $\varepsilon_\infty = \varepsilon_a$  for large frequencies [29] (apart from resonances at an infinite sequence of evenly spaced frequencies associated with the edges of the gapless photonic bands), in keeping with the asymptotics disclosed by equation (78), where the doubly graded periodic structure loses its significance. It may be commented that the occurrence of maxima of full transmittance here is reminiscent of, e.g., the well-known phenomenon of resonant tunneling through periodic molecular structures [43]. As for the singly graded periodic structure studied above, a constant, less than full transmittance for all frequencies would come to the fore, if the limit  $\varepsilon_b \rightarrow \varepsilon_a$  characterizing an optically homogeneous, non-dispersive medium of semi-infinite extent were addressed.

### 3. Conclusions

By means of a vector potential approach, we have explored, both analytically and numerically, the optical transmittance of one-dimensional photonic structures in the form of periodically arranged, lossless dielectric slabs with linear, or piecewise linear, profiles of their electric permittivity, assuming linearly polarized electromagnetic radiation at normal incidence. Resorting to an analysis of the respective photonic modes, the dependence on frequency (or, equivalently, the variation with the characteristic structural length) of the optical property addressed has first been established for the singly graded slab and the doubly graded slab, i.e. for the proper dielectric constituents, and subsequently for the photonic structures derived from them.

Considering the singly graded periodic structure, with its sawtooth profile of the electric permittivity, the following key results came to the fore: the transmittance reveals a series of photonic bands and gaps, which proves degenerate as regards positive or negative gradients of the electric permittivity, exhibiting maxima less than unity, as for the singly graded slab, and tending to saturate in a periodic sequence of centro-symmetric columnar peaks, separated by gaps of identical widths, at large frequencies (cf figure 7); an asset that recommends this type of photonic structure as ideal. The occurrence of regularity of the transmittance is particularly notable, since it contrasts fundamentally with the behaviour found in a conventional multi-layer stack of dielectric slabs having spatially uniform permittivities  $\varepsilon_a$  and  $\varepsilon_b$ . Such a stepwise constant periodic structure may be related to the singly graded one considered here by replacing the linear segments of the profile of the electric permittivity with equilateral steps from  $\varepsilon_a$  to  $\varepsilon_b$ . This confirms that, in general, no single periodicity in the band structure of the transmittance of a conventional, twofold multi-layer stack exists (cf figure A.1).

Looking at the doubly graded periodic structure, with its triangular profiles of the electric permittivity, the following key results came to the fore: the transmittance reveals two series of photonic bands and gaps, which prove non-degenerate as regards the succession of ‘convex’ or ‘concave’ profiles of the electric permittivity, exhibiting maxima of full transparency, as for the doubly graded slab, or maxima of reduced transparency and tending to saturate in essentially constant values with zero gaps at large frequencies (cf figure 10); a trait known for all frequencies from an optically homogeneous, non-dispersive medium of semi-infinite extent. The occurrence of uniformity of the transmittance is particularly notable, since it contrasts fundamentally with the behaviour found in a conventional multi-layer stack of dielectric slabs having spatially uniform permittivities  $\varepsilon_a$  and  $\varepsilon_b$ . Such a stepwise constant periodic structure may be related to the doubly graded one considered here by replacing the linear segments of the profile of the electric permittivity with equilateral steps from  $\varepsilon_a$  to  $\varepsilon_b$  and from  $\varepsilon_b$  to  $\varepsilon_a$ .

This confirms that, in general, no uniformity in the band structures of the transmittance of a conventional, threefold multi-layer stack exists (cf figure A.2).

The properties revealed for both the singly graded periodic structure and the doubly graded periodic structure with linear, or piecewise linear, profiles of the electric permittivity offer unique possibilities regarding practical applications of such novel photonic composites, optical filters exploiting their transmittance or optical mirrors utilizing their reflectance (which here is just the transmittance's complement) only being the most obvious. The simple asymptotic forms derived, with their succinct exposition of the dependences on the various geometrical and material parameters involved, can serve as convenient guides to actual designs thereby.

### Acknowledgment

Valuable discussions with Y A Genenko are gratefully acknowledged.

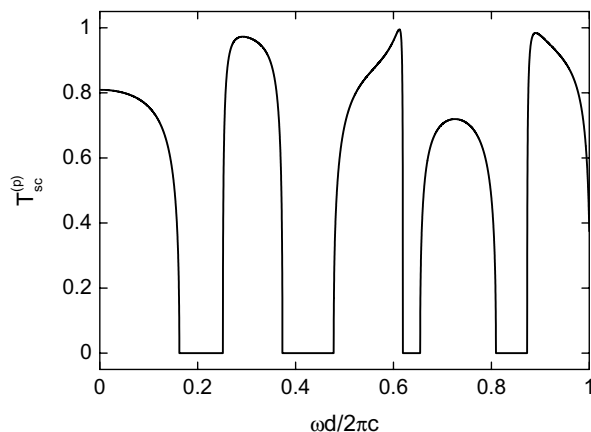
### Appendix. Stepwise constant periodic structure

We here address a one-dimensional dielectric photonic structure adjacent to a vacuum, generated by periodic continuation into the half-space  $z \geq 0$  of a twofold or, possibly, threefold stack of dielectric slabs of total thickness  $2s$  spreading infinitely along the  $x$ - and  $y$ -directions of the Cartesian coordinate system  $x, y, z$ . As in the main body of this work, we envisage that linearly polarized electromagnetic radiation propagating in the positive  $z$ -direction through the vacuum space  $z < 0$ , with the electric (magnetic) field oriented parallel to the  $x$ -axis and the magnetic (electric) field oriented parallel to the  $y$ -axis of the said coordinate system, is normally incident on the surface  $z = 0$  of the stack. Each of its constituent slabs, with relative thicknesses characterized by the structural fraction  $q$  for  $0 < q \leq 1$ , shall be entirely delineated by constant values of the principal components of the respective tensors of electric permittivity  $\varepsilon_\nu = \varepsilon_a > 1$  or  $\varepsilon_\nu = \varepsilon_b > 1$  for  $\nu = x, y$ , neglecting electromagnetic dispersion as well as dissipation and assuming in-plane crystalline isotropy. Specifically, in the range  $0 \leq z < qs$ , the electric permittivity shall be  $\varepsilon_\nu = \varepsilon_a$ ; in the range  $qs \leq z < (q+1)s$ , the electric permittivity shall be  $\varepsilon_\nu = \varepsilon_b$ ; in the range  $(q+1)s \leq z < 2s$ , the electric permittivity shall again be  $\varepsilon_\nu = \varepsilon_a$ . The following compilation of results draws heavily on previous analysis [29, 31].

Introducing the wavenumbers  $k_a = k_0 n_a$  and  $k_b = k_0 n_b$  linked to the refractive indices of the dielectric constituents,  $n_a = \varepsilon_a^{1/2}$  and  $n_b = \varepsilon_b^{1/2}$ , the dispersion of the photonic modes obeys the implicit equation

$$\cos(k_a s) \cos(k_b s) - \frac{1}{2} \left( \frac{n_a}{n_b} + \frac{n_b}{n_a} \right) \sin(k_b s) \sin(k_a s) = \cos(2k_z s), \quad (\text{A.1})$$

which defines the relation  $\omega = \omega_j(k_z)$  for varying wavenumber  $k_z$  confined to the first Brillouin zone  $-\pi/2s \leq k_z \leq \pi/2s$ , the label  $j = 0, 1, 2, \dots$  distinguishing the photonic bands appropriate for the chosen direction of radiation incidence and type of polarization; it simultaneously implies the relation  $k_z = k_z(\omega_j)$  for varying frequency  $\omega_j$  of the photonic bands with label  $j = 0, 1, 2, \dots$  called for in the evaluation of the transmittance examined below.



**Figure A.1.** Transmittance  $T_{sc}^{(p)}$  of a stepwise constant periodic structure with the characteristics  $s = d/2$  and  $q = 1$  as a function of the normalized frequency  $\omega d/2\pi c$  in the case of linearly polarized electromagnetic radiation due to on-axis propagation, for the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  or, respectively,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$ .

The transmittance  $T_{sc}^{(p)}$ , defined through the time-averaged flows of electromagnetic energy of the incident and transmitted waves, in the case of dielectric bands for  $0 \leq k_z \leq \pi/2s$  or, respectively, air bands for  $-\pi/2s \leq k_z < 0$ , with  $l = 0, 1, 2, \dots$ , reads

$$\left. \begin{array}{l} T_{2l,sc}^{(p+)} \\ T_{2l+1,sc}^{(p-)} \end{array} \right\} = 4n_a \frac{|A_{1,sc}^{(p)}|^2 - |A_{2,sc}^{(p)}|^2}{|(1+n_a)A_{1,sc}^{(p)} + (1-n_a)A_{2,sc}^{(p)}|^2}; \quad (\text{A.2})$$

likewise, this quantity, defined through the time-averaged flows of electromagnetic energy of the reflected and transmitted waves, in the case of dielectric bands for  $-\pi/2s \leq k_z < 0$  or, respectively, air bands for  $0 \leq k_z \leq \pi/2s$ , with  $l = 0, 1, 2, \dots$ , reads

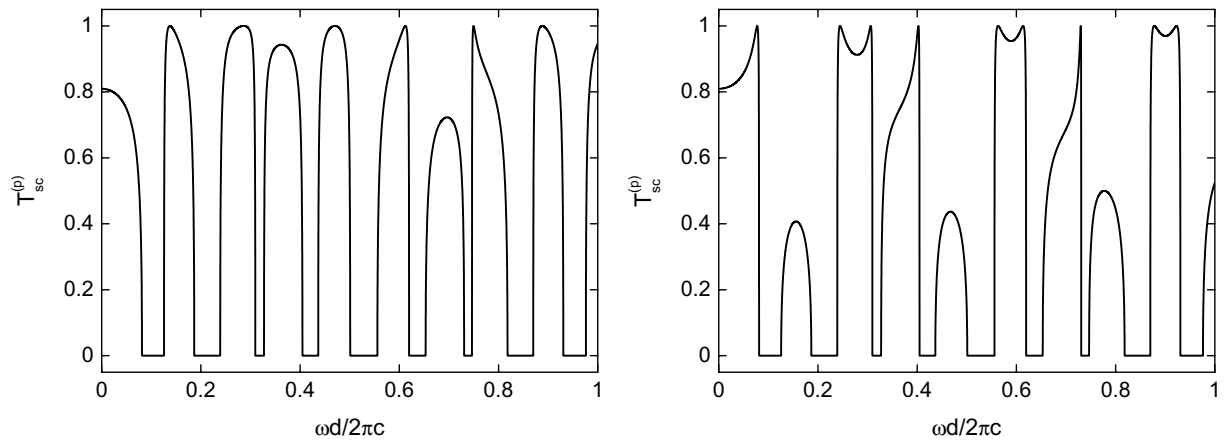
$$\left. \begin{array}{l} T_{2l,sc}^{(p-)} \\ T_{2l+1,sc}^{(p+)} \end{array} \right\} = 4n_a \frac{|A_{2,sc}^{(p)}|^2 - |A_{1,sc}^{(p)}|^2}{|(1-n_a)A_{1,sc}^{(p)} + (1+n_a)A_{2,sc}^{(p)}|^2}. \quad (\text{A.3})$$

Both of these cases make use of amplitudes  $A_{1,sc}^{(p)}$  and  $A_{2,sc}^{(p)}$  given by

$$A_{1,sc}^{(p)} = \frac{1}{2} \left\{ \left(1 + \frac{n_b}{n_a}\right) B_{1,sc}^{(p)} \exp[-iq(k_a - k_b)s] + \left(1 - \frac{n_b}{n_a}\right) B_{2,sc}^{(p)} \exp[-iq(k_a + k_b)s] \right\} \quad (\text{A.4})$$

and

$$A_{2,sc}^{(p)} = \frac{1}{2} \left\{ \left(1 - \frac{n_b}{n_a}\right) B_{1,sc}^{(p)} \exp[iq(k_a + k_b)s] + \left(1 + \frac{n_b}{n_a}\right) B_{2,sc}^{(p)} \exp[iq(k_a - k_b)s] \right\}, \quad (\text{A.5})$$



**Figure A.2.** Transmittance  $T_{sc}^{(p)}$  of a stepwise constant periodic structure with the characteristics  $s = d$  and  $q = 1/2$  as a function of the normalized frequency  $\omega d/2\pi c$  in the case of linearly polarized electromagnetic radiation due to on-axis propagation, for the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  (left) and, respectively,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$  (right).

with

$$B_{1,sc}^{(p)} = 2 \exp[-i(q+1)k_b s] - \left(1 + \frac{n_b}{n_a}\right) \exp[i(2k_z + k_a - qk_b)s] \\ - \left(1 - \frac{n_b}{n_a}\right) \exp[i(2k_z - k_a - qk_b)s] \quad (\text{A.6})$$

and

$$B_{2,sc}^{(p)} = -2 \exp[i(q+1)k_b s] + \left(1 + \frac{n_b}{n_a}\right) \exp[i(2k_z - k_a + qk_b)s] \\ + \left(1 - \frac{n_b}{n_a}\right) \exp[i(2k_z + k_a + qk_b)s], \quad (\text{A.7})$$

and require exploiting equation (A.1).

Figures A.1 and A.2 present the variation with frequency of the transmittance of a regular twofold stack of dielectric slabs of period length  $d$  (characterized by  $s = d/2$  and  $q = 1$ ) and, respectively, a regular threefold stack of dielectric slabs of period length  $2d$  (characterized by  $s = d$  and  $q = 1/2$ ), obtained from equations (A.2) and (A.3), for linearly polarized electromagnetic radiation with on-axis propagation, taking the dielectric parameters  $\varepsilon_a = 2$  and  $\varepsilon_b = 11$  or, respectively,  $\varepsilon_a = 11$  and  $\varepsilon_b = 2$ . This shows that, because of the inversion symmetry of the radiation incidence regarding the succession of positive (for  $\varepsilon_a < \varepsilon_b$ ) or negative (for  $\varepsilon_a > \varepsilon_b$ ) steps of the electric permittivity of the twofold stack, the transmittance in either situation proves degenerate, whereas because of the complementarity of the radiation incidence regarding the succession of positive/negative (for  $\varepsilon_a < \varepsilon_b$ ) or negative/positive (for  $\varepsilon_a > \varepsilon_b$ ) steps of the electric permittivity of the threefold stack, the transmittance in either situation proves non-degenerate.

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