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Orthogonal Eisenstein Series of Singular Weight

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Zusammenfassung

In dieser Doktorarbeit werden (nicht-)holomorphe orthogonale Eisensteinreihen mithilfe von Borcherds additiven Thetalift untersucht.

Dazu werden zunächst die Randkomponenten der orthogonalen Halbebene und ihrer Quotienten nach Kongruenzuntergruppen untersucht. Insbesondere der Fall von Primzahlstufe und quadratfreier Stufe wird behandelt.

Anschließend wird der additive Thetalift von nicht-holomorphen vektorwertigen Eisensteinreihen zur Weil-Darstellung eines Gitters der Signatur (b^+, b^-) betrachtet. Es wird die meromorphe Fortsetzung und Funktionalgleichung der Thetalifts gefolgert. Außerdem wird die Fourierentwicklung berechnet.

Im letzten Teil wird sich auf den Fall der Signatur $(2, l)$ spezialisiert und gezeigt, dass die additiven Thetalifts von nicht-holomorphen vektorwertigen Eisensteinreihen selbst nicht-holomorphe orthogonale Eisensteinreihen sind. Für diese ergibt sich damit ein neuer Beweis der meromorphen Fortsetzung und der Funktionalgleichung. Außerdem wird der Thetalift auf Injektivität und Surjektivität untersucht. Anschließend werden die holomorphen Eisensteinreihen betrachtet, indem die nicht-holomorphen Eisensteinreihen an speziellen Werten ausgewertet werden. Auch hier wird der Thetalift auf Injektivität und Surjektivität untersucht und letztendlich gezeigt, dass wenn das Gitter zwei hyperbolische Ebenen abspaltet, alle holomorphen Modulformen singulären Gewichts $\kappa = \frac{l}{2} - 1$, welche auf dem Rand linearkombinationen von Eisensteinreihen sind, als Thetalift geschrieben werden können.

Abstract

In this thesis we investigate (non-)holomorphic orthogonal Eisenstein series by using Borcherds' additive theta lift.

Therefore we start by looking at the boundary components of the orthogonal upper half-plane and its quotients by congruence subgroups. In particular we investigate the case of prime level and square-free level.

Afterwards we consider the additive theta lift of non-holomorphic vector-valued Eisenstein series with respect to the Weil representation of a lattice of signature (b^+, b^-). We will derive the meromorphic continuation and functional equation of the theta lifts. Moreover, we will calculate their Fourier expansion.

In the last part we will specialise to signature $(2, l)$ and show, that additive theta lifts of non-holomorphic vector-valued Eisenstein series are non-holomorphic orthogonal Eisenstein series. This yields a new proof of their meromorphic continuation and functional equation. Moreover, we will investigate if the theta lift is injective or surjective. Afterwards we consider the holomorphic Eisenstein series by evaluating the non-holomorphic Eisenstein series at special values. Again, we investigate, if the theta lift is injective or surjective and show, that if the lattice splits two hyperbolic planes, then all holomorphic modular forms of singular weight $\kappa = \frac{l}{2} - 1$, that are linear combinations of Eisenstein series on the boundary, can be written as theta lifts.

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Contents

Chapter 1

Introduction

Let L be an even lattice of signature $(2, l)$. There is an index 2 subgroup of the corresponding orthogonal group $\mathrm{O}^+(2, l) \subseteq \mathrm{O}(2, l)$ acting on the orthogonal upper half-plane $\mathbb{H}_l$. Similar to the case of elliptic modular forms, it is possible to define orthogonal modular forms, which can be seen to be global sections of a hermitian line bundle. For $l = 1$ we obtain the classical case of elliptic modular forms and for $l = 2$ we obtain Hilbert modular forms for real quadratic number fields.

An important problem is the construction of such modular forms, in particular for low weight. It turns out that there is a minimal positive weight in which non-zero holomorphic modular forms exist, which is given by $\kappa = \frac{l}{2} - 1$ for $l > 2$, see [Bun01]. The weight $\kappa = \frac{l}{2} - 1$ is called the singular weight, and modular forms of singular weight have many vanishing Fourier coefficients. In particular, there are no cusp forms of singular weight (for an analogous theory in the Siegel case see [Fre83], [Fre91], where it is shown that all holomorphic modular forms of singular weight are linear combinations of theta functions). In contrast to the symplectic case, there are no holomorphic theta functions for the orthogonal group (except for the exceptional isomorphisms between the orthogonal and symplectic group in low dimensions). Moreover, the usual constructions of holomorphic modular forms do not work immediately for low weight. For example the Eisenstein series and Poincaré series do not converge. On the other hand, using the celebrated multiplicative Borcherds lift of [Bor98, Theorem 13.3], examples of holomorphic modular forms of singular weight can be constructed. For results in this direction see [DHS15], [OS19], [Sch17]. Another method is to use the additive Borcherds lift of [Bor98, Theorem 14.3] using holomorphic modular forms of weight 0, i.e. invariant vectors, as input functions.

Our aim is to investigate Eisenstein series $\mathcal{E}_{\kappa,\lambda}(Z)$ of low weight, in particular of singular weight $\kappa = \frac{l}{2}-1$. Since for low weights these Eisenstein series do not converge, we have to consider non-holomorphic Eisenstein series $\mathcal{E}_{\kappa,\lambda}(Z,s)$, get a meromorphic continuation to all $s \in \mathbb{C}$ and hope that they yield holomorphic modular forms for special values of s. This will be done using the machinery of the additive Borcherds lift to obtain the Fourier expansion, the meromorphic continuation, a functional equation and to evaluate the Eisenstein series at special values of the spectral parameter s. The case of maximal lattices and weight $\kappa > \frac{l}{2} + 1$ was already considered by [Hir98] using another method.

We give a brief overview on the structure of this thesis. In the second chapter we will give the necessary preliminaries on lattices and vector-valued holomorphic modular forms. In particular, we will define the vector-valued non-holomorphic Eisenstein series and the Siegel theta function which are needed for the additive Borcherds lift.

In the third chapter we will introduce the orthogonal upper half-plane and holomorphic modular forms for the orthogonal group. Next we will introduce the hermitian structure and the weight κ Laplace operator. Moreover, we will recall certain growth estimates on Siegel domains which will be needed later.

In the fourth chapter we introduce the Baily-Borel boundary of the orthogonal upper half-plane and consider its geometry. In particular, we will calculate the corresponding parabolic subgroups in certain cases and introduce the Siegel operator which allows us to define boundary values of holomorphic modular forms.

In the fifth chapter we will consider the regularized theta lift of a general non-holomorphic vector-valued Eisenstein series and calculate its Fourier expansions. As a direct consequence we obtain a functional equation of the lift and we can study its poles. We finish the discussion by considering the case of Lorentzian lattices.

In the sixth and last chapter we specialize to the case of signature $(2,l)$ and show that the regularized theta lift of non-holomorphic vector-valued Eisenstein series are in fact non-holomorphic Eisenstein series for the orthogonal group. Together with the previous chapter this shows that they have a meromorphic continuation and a functional equation. Further one obtains their Fourier expansion. We will investigate the image and kernel of the regularized theta lift and show that it is surjective under certain assumptions (for example if the lattice splits two hyperbolic planes). Afterwards we will specialize to the harmonic points and study holomorphic Eisenstein series for the orthogonal group. In particular, we will construct holomorphic Eisenstein series of singular weight and show that under certain conditions all of them can be constructed using the regularized theta lift.

We will now go into more detail. Recall the metaplectic cover $\mathrm{Mp}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$. For an even lattice L there is a unitary representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$ which factors through a finite quotient. If L has signature $(2,l)$ with $l = 0 \bmod 2$, writing $\mathrm{Iso}(L'/L)$ for the set of isotropic elements of L'/L, we can consider the non-holomorphic Eisenstein series of weight k for $\beta \in \mathrm{Iso}(L'/L)$

$$E_{k,\beta}(\tau,s) = \frac{1}{2}\sum_{M\in\tilde{\Gamma}_\infty\backslash \mathrm{Mp}_2(\mathbb{Z})} y^s|_{k,L}M,$$

where $|_{k,L}$ is the Petersson slash operator of weight k with respect to ρ_L and $\tilde{\Gamma}_\infty$ is the stabilizer of ∞. One easily sees that $E_{k,-\beta} = (-1)^\kappa E_{k,\beta}$, where $\kappa = \frac{l}{2} - 1 + k$. As in the classical theory, they can be seen to be eigenfunctions of the weight k Laplace operator, have a functional equation and we can calculate their Fourier expansion (see [BK01], [BK03, Section 3], [DS05, Chapter 4], [KY10]).

Let $\mathrm{Iso}_0(L)$ be the set of primitive isotropic vectors in L. For $z \in \mathrm{Iso}_0(L)$ and a vector $z' \in L'$ with $(z,z') = 1$, consider the lattice $K = L \cap z^\perp \cap z'^\perp$. Let

$$\mathbb{H}_l^\pm = \{X + iY \in K \otimes \mathbb{C} \mid q(Y) > 0\}$$

and denote by $\mathbb{H}_l \subseteq \mathbb{H}_l^\pm$ one of its connected components. For an element $Z \in \mathbb{H}_l^\pm$ we write $Z_L = Z - (q(Z) + q(z'))z + z'$. This defines a biholomorphic map

$$\mathbb{H}_l^\pm \to \mathcal{K} = \{[Z_L] \in \mathbb{P}(V(\mathbb{C})) \mid (Z_L, Z_L) = 0, (Z_L, \overline{Z_L}) > 0\},$$

where $\mathbb{P}(V(\mathbb{C}))$ is the projective space of $V(\mathbb{C}) = L \otimes \mathbb{C}$. The connected component of $\mathcal{K}$ corresponding to $\mathbb{H}_l$ is denoted by $\mathcal{K}^+$ and we write $\mathrm{O}^+(V(\mathbb{R}))$ for the subgroup of $\mathrm{O}(V(\mathbb{R}))$ which preserves the components. For an appropriate subgroup $\Gamma \subseteq \mathrm{O}^+(V) := \mathrm{O}(V) \cap \mathrm{O}^+(V(\mathbb{R}))$ and a primitive isotropic vector $\lambda \in \mathrm{Iso}_0(L')$, let Γ_λ be the stabilizer of λ and N_λ be the order of λ in L'/L. Then the orthogonal Eisenstein series of weight κ corresponding to the cusp λ is defined by

$$\mathcal{E}_{\kappa,\lambda}(Z,s) = \sum_{\sigma\in\Gamma_\lambda\backslash\Gamma} q(Y)^s|_\kappa \sigma_\lambda\sigma,$$

where $\sigma_\lambda \in \mathrm{O}^+(V)$ with $\sigma_\lambda N_\lambda \lambda = z$ and one easily sees again that $\mathcal{E}_{\kappa,-\delta} = (-1)^\kappa \mathcal{E}_{\kappa,\delta}$. Of course, if κ is large enough (i.e. $\kappa > l$), the series converges absolutely for $s = 0$ and we obtain a holomorphic modular form of weight κ. But for smaller weight, in particular for singular weight $\kappa = \frac{l}{2} - 1$, we can not expect to obtain a holomorphic function at

$s = 0$. Nevertheless it turns out that these Eisenstein series are again Eigenfunctions of the weight κ Laplace operator and for $s = 0$ they are even harmonic.

The quotient space $\Gamma\backslash\mathbb{H}_l$ is usually not compact, but can be compactified using the theory of Baily-Borel by adding boundary components to $\mathbb{H}_l$ and extending the action of $\mathrm{O}^+(V(\mathbb{Q}))$ (see [BB66], [BJ06]). These boundary components are given by points corresponding to isotropic lines in L and by 1-dimensional boundary components isomorphic to the usual upper half-planes $\mathbb{H}$ corresponding to isotropic planes $I \subseteq L$. For such a 1-dimensional boundary component I and a holomorphic modular form $F : \mathbb{H}_l \to \mathbb{C}$, one can consider its boundary value $F|_I : \mathbb{H} \to \mathbb{C}$, which turns out to be a holomorphic modular form of the same weight with respect to an appropriate congruence subgroup of $\mathrm{SL}_2(\mathbb{Q})$. We will be interested in the holomorphic modular forms $F : \mathbb{H}_l \to \mathbb{C}$, such that for all 1-dimensional boundary components I, the restriction to the boundary $F|_I : \mathbb{H} \to \mathbb{C}$ is a linear combination of Eisenstein series. In particular, for κ large enough, the Eisenstein series $\mathcal{E}_{\kappa,\lambda}(Z) := \mathcal{E}_{\kappa,\lambda}(Z, 0)$ are linear combinations of Eisenstein series if restricted to the boundary. If a holomorphic modular form vanishes on the boundary, we call it a cusp form. For singular weight $\kappa = \frac{l}{2} - 1$, a holomorphic modular form is already fully determined by its boundary values and there are no cusp forms.

Now consider the natural map $\mathrm{O}^+(L) \to \mathrm{O}(L'/L)$ and denote by $\Gamma(L)$ its kernel. For $\lambda \in \mathrm{Iso}_0(L')$ we consider its image in L'/L. This yields a map $\Gamma(L)\backslash \mathrm{Iso}_0(L') \to \mathrm{Iso}(L'/L)$, which we will denote by π_L.

For an easier exposition we will assume throughout the introduction that L splits two hyperbolic planes over $\mathbb{Z}$ so that π_L is bijective. For $\delta \in L'/L$ isotropic, we can consider the orthogonal Eisenstein series

$$\mathcal{G}_{\kappa,\delta}(Z, s) = \mathcal{E}_{\kappa,\pi_L^{-1}(\delta)}(Z, s).$$

As in the vector-valued case we want to obtain a meromorphic continuation to all $s \in \mathbb{C}$ and a functional equation. We will do this by writing the orthogonal non-holomorphic Eisenstein series as a Borcherds lift of vector-valued non-holomorphic Eisenstein series. For the general theory of Eisenstein series see for example [Lan76].

Therefore, consider the Siegel theta function $\Theta_L(\tau, Z)$ of weight k (see Section 6.2) and define, following [Bor98], [Bru02], the additive Borcherds lift of the vector-valued Eisenstein series $E_{k,\beta}(\tau, s)$ by the regularized integral

$$\Phi_\beta(Z, s) := \int_{\mathrm{Mp}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}} \langle E_{k,\beta}(\tau, s), \Theta_L(\tau, Z)\rangle v^k \frac{\mathrm{d}u\mathrm{d}v}{v^2}.$$

Then we have the following

Theorem 1.0.1 (see Theorem 6.3.1 and Theorem 6.3.2)**.** *Let* $\kappa = \frac{l}{2} - 1 + k$. *We have*

$$\Phi_\beta(Z,s) = \frac{\Gamma(s+\kappa)}{(-2\pi i)^\kappa \pi^s} \sum_{\substack{\delta \in \mathrm{Iso}(L'/L) \\ \beta = k_{\delta\beta}\delta}} N_\delta^{2s+\kappa} \zeta_+^{k_{\delta\beta}}(2s+\kappa)\mathcal{G}_{\kappa,\delta}(Z,s),$$

where $\zeta_+^{k_{\delta\beta}}$ *are modified Riemann zeta functions and* N_δ *is the order of* δ*. Moreover, taking suitable linear combinations (with meromorphic functions in* s *as coefficients) one can construct every orthogonal Eisenstein series* $\mathcal{G}_{\kappa,\delta}(Z,s)$ *for* $\delta \in \mathrm{Iso}(L'/L)$*, i.e. the theta lift is bijective.*

This yields, in particular, a meromorphic continuation and the functional equation of the orthogonal Eisenstein series. Moreover, using the methods of [Bor98], one can calculate the Fourier expansion of orthogonal Eisenstein series.

Afterwards we specialize to the case $s = 0$ to obtain holomorphic Eisenstein series. For $k > 2$ this works immediately and we obtain

Theorem 1.0.2 (see Theorem 6.4.8)**.** *For* $k > 2$ *the lift* $\Phi_\beta(Z) := \Phi_\beta(Z,0)$ *is a holomorphic modular form which is a linear combination of Eisenstein series on the boundary. In particular, the Eisenstein series* $\mathcal{E}_{\kappa,\delta}(Z,0), \delta \in \mathrm{Iso}(L'/L)$ *are holomorphic modular forms whose value in a* 0*-dimensional cusp* $\lambda \in \Gamma(L)\backslash \mathrm{Iso}_0(L')$ *is* $1 + (-1)^\kappa$ *if* $\pi_L(\lambda) = \pm\delta$ *and vanishes otherwise. Moreover, up to addition of cusp forms, they span the space of holomorphic modular forms that are linear combinations of Eisenstein series on the boundary.*

For $k = 0$ it turns out that $\Phi_\beta(Z,s)$ is usually not holomorphic at $s = 0$. Using the functional equation relating the values at s with the values at $1-s$ we can investigate the behaviour at $s = 1$ instead, where $\Phi_\beta(Z,s)$ has a simple pole. We obtain

Theorem 1.0.3 (see Theorem 6.4.9)**.** *For* $k = 0$ *and thus* $\kappa = \frac{l}{2} - 1$*, the residue at* $s = 1$ *of* $\Phi_\beta(Z,s)$ *is a holomorphic modular form of singular weight. Moreover, every invariant vector for the Weil representation yields a linear combination* $\Phi_{\mathfrak{v}}$ *such that* $\Phi_{\mathfrak{v}}(Z)$ *is a holomorphic modular form. Again, they are linear combinations of Eisenstein series on the boundary.*

In fact, we will show that $\mathrm{res}_{s=1}\Phi_\beta(Z,s) = \Phi(Z, \mathrm{res}_{s=1} E_{0,\beta}(\tau,s))$. In contrast to $k > 2$, i.e. $\kappa \geq \frac{l}{2} + 1$, we do not have a surjectivity result at this point for $k = 0, \kappa = \frac{l}{2} - 1$.

Therefore, let now $F : \mathbb{H}_l \to \mathbb{C}$ be a modular form and consider the adjoint theta lift

$$\Phi^*(\tau, f) := \int_{\Gamma(L)\backslash\mathbb{H}_l} F(Z)\Theta_L(\tau, Z)q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}.$$

We will show that if F is harmonic and square-integrable, its lift $\Phi^*(\tau, F)$ is also harmonic (see Lemma 6.5.2). As a main theorem we obtain

Theorem 1.0.4 (see Theorem 6.5.3)**.** *If $F : \mathbb{H}_l \to \mathbb{C}$ is a holomorphic modular form of singular weight, then its theta lift $\Phi^*(\tau, f)$ is an invariant vector and we have*

$$\Phi^*(\tau, f) = \frac{\Gamma(l/2)}{2(2\pi)^{l/2}} \sum_{\substack{\gamma\in L'/L \\ q(\gamma)=0}} \sum_{\substack{\delta\in L'/L \\ q(\delta)=0 \\ \gamma=k_\delta\delta}} N_\delta^{l-\kappa}\zeta_+^{k_\delta}(l-\kappa)a_{F,\delta}(0)C(\delta)\mathfrak{e}_\gamma,$$

where $C(\delta)$ is a certain positive constant and $a_{F,\delta}(0)$ is the constant Fourier coefficient of F in the 0-dimensional cusp corresponding to δ.

As a corollary one obtains

Corollary 1.0.5 (see Corollary 6.5.8)**.** *Let $F : \mathbb{H}_l \to \mathbb{C}$ be a holomorphic modular form of singular weight κ. Then its theta lift $\Phi^*(\tau, F)$ vanishes if and only if F vanishes in every 0-dimensional cusp. In particular, using that Φ^* is adjoint to Φ one obtains an isomorphism between holomorphic modular forms that are linear combinations of Eisenstein series on the boundary and invariant vectors of the Weil representation.*

If L is a maximal lattice, then the space of invariant vectors is either 1-dimensional (if L is unimodular) or 0-dimensional (if L is not unimodular). Since maximal lattices of Witt rank 2 always split two hyperbolic planes over $\mathbb{Z}$, we obtain

Corollary 1.0.6 (see Corollary 6.5.9)**.** *Let $\kappa = \frac{l}{2} - 1$ be the singular weight. If L is a maximal lattice of Witt rank 2, then the space $\mathrm{M}_\kappa^{\partial\,\mathrm{Eis}}(\Gamma(L))$ is either 1-dimensional (if L is unimodular) or 0-dimensional (if L is not unimodular). Moreover, if $\kappa = 2, 4, 6, 8, 10, 14$, i.e. $l = 6, 10, 14, 18, 22, 30$, then we have $\mathrm{M}_\kappa(\Gamma(L)) = \mathrm{M}_\kappa^{\partial\,\mathrm{Eis}}(\Gamma(L))$ and the space of holomorphic modular forms of singular weight is either 0 or 1-dimensional depending on L being unimodular or not.*

If L does not split two hyperbolic planes over $\mathbb{Z}$, then we can still fully determine the image of the theta lift Φ (see Corollary 6.5.7).

Chapter 2

Preliminaries

2.1 Lattices

Throughout this section let R be a principal ideal domain, K its field of fractions and M a finitely generated free R-module.

Definition 2.1.1. A *quadratic form* on M is a map $q : M \to K$ which satisfies

(i) $q(rx) = r^2 q(x)$ for all $r \in R, x \in M$.

(ii) The map $(x, y) := q(x + y) - q(x) - q(y)$ is a symmetric bilinear form.

We say that q is *non-degenerate*, if for all $x \in M \setminus \{0\}$ there is $y \in M$ with $(x, y) \neq 0$. A non-zero vector $x \in M$ is called *isotropic* if $q(x) = 0$ and *anisotropic* otherwise.

Remark 2.1.2. Setting $x = y$ in (ii) yields $2q(x) = (x, x)$. In particular, if $2 \in R^\times$, then (ii) implies (i).

The pair (M, q) is called a *quadratic module* over R. If R is a field, we call it *quadratic space*. We say that $x, y \in M$ are *orthogonal* if $(x, y) = 0$.

Example 2.1.3. Let $b^+, b^- \in \mathbb{N}$ and $n = b^+ + b^-$. Then the map

$$(x_1, \ldots, x_n) \mapsto x_1^2 + \ldots + x_{b^+}^2 - x_{b^++1}^2 - \ldots - x_n^2$$

makes $\mathbb{R}^n$ into a quadratic space which we denote by $\mathbb{R}^{(b^+, b^-)}$.

Definition 2.1.4. Let $(M, q), (M', q')$ be quadratic modules over R. An R-linear map $\sigma : M \to M'$ is called *isometry*, if it is injective and satisfies $q'(\sigma(x)) = q(x)$ for all $x \in M$. The *orthogonal group* of M is defined by

$$\mathrm{O}(M) := \{\sigma \in \mathrm{Aut}(M) \mid \sigma \text{ isometry}\}.$$

We have the following well-known theorem

Theorem 2.1.5. *Let (V, q) be a quadratic space over $\mathbb{R}$ of dimension n. Then there are uniquely determined numbers b^+, b^- with $b^+ + b^- = n$ such that (V, q) is isometric to $\mathbb{R}^{(b^+,b^-)}$, i.e. there exists an isometry $\sigma : (V, q) \to \mathbb{R}^{(b^+,b^-)}$. The tuple (b^+, b^-) is called* signature *of (V, q).*

For a rational quadratic space (V, q) we define its *signature* to be the signature of $V(\mathbb{R}) := V \otimes_{\mathbb{Q}} \mathbb{R}$. We define the Witt-rank (or $\mathbb{Q}$-rank) to be the dimension of a maximal isotropic subspace of V.

Definition 2.1.6. A $\mathbb{Z}$*-lattice* (or just *lattice*) L is a finitely generated free $\mathbb{Z}$-module with a quadratic form with values in $\mathbb{Q}$. The *signature* of L is defined as the signature of $V(\mathbb{R}) = L \otimes_{\mathbb{Z}} \mathbb{R}$. We say that L is integral, if the bilinear form corresponding to the quadratic form only takes values in $\mathbb{Z}$. We say that L is even, if the quadratic form only takes values in $\mathbb{Z}$.

Example 2.1.7. Let $U = \mathbb{Z} \oplus \mathbb{Z}$ with basis e_1, e_2 such that $q(ae_1 + be_2) = ab$. Then U is an even lattice called *hyperbolic plane*.

For a lattice L and a natural number $N \in \mathbb{N}$ we define the scaled lattice $L(N)$ to be the same $\mathbb{Z}$-module with quadratic form $N \cdot q$. We call $x \in L \smallsetminus \{0\}$ *primitive*, if $\mathbb{Q}x \cap L = x\mathbb{Z}$. We write $\mathrm{Iso}_0(L)$ for the set of primitive isotropic elements. The *dual lattice* is defined by

$$L' := \{x \in V = L \otimes \mathbb{Q} \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L\}.$$

For an integral lattice L we have $L \subseteq L'$ and call L'/L the *discriminant group*. It is a finite abelian group.

From now on we will assume that L is an even lattice. The *level* of L is the smallest natural number $N \in \mathbb{N}$ such that $Nq(x) \in \mathbb{Z}$ for all $x \in L'$. Moreover, we define the *level* of $x \in L$ as the unique positive integer N_x with $(x, L) = N_x\mathbb{Z}$ (then $\frac{x}{N_x} \in L'$ is primitive as an element of L'). We have the following

Lemma 2.1.8. *Let L be an even lattice of level N and $z \in L$ primitive of level N_z. Then $N_z \mid N$.*

The quadratic form q induces a quadratic form $L'/L \to \mathbb{Q}/\mathbb{Z}$, which we also denote by q. Write $\mathrm{O}^+(V(\mathbb{R}))$ for the identity component of $\mathrm{O}(V(\mathbb{R}))$ and define $\mathrm{O}^+(L) := \mathrm{O}(L) \cap \mathrm{O}^+(V(\mathbb{R}))$. Since $\mathrm{O}(L)$ also acts on L', we obtain a map $\mathrm{O}^+(L) \to \mathrm{O}(L'/L)$. We write $\Gamma(L)$ for the kernel of this map. If $M \subseteq L$ is a sublattice, then $\Gamma(M) \subseteq \Gamma(L)$. A subgroup $\Gamma \subseteq \Gamma(L)$ is called congruence subgroup if it contains some $\Gamma(NL)$ for a natural number $N \in \mathbb{N}$. We call $\gamma \in L'/L$ *isotropic*, if $q(\gamma) = 0 \in \mathbb{Q}/\mathbb{Z}$ and we write $\mathrm{Iso}(L'/L)$ for the set of isotropic elements in L'/L.

Lemma 2.1.9. *Let L be an even lattice, $z, \zeta \in \mathrm{Iso}_0(L)$ of level $N_z = N_\zeta$ with $(z, \zeta) = N_z$. Then the scaled hyperbolic plane $U(N_z)$ spanned by z, ζ splits orthogonally from L.*

Proof. This is essentially [O'M63, 82:15]. Let W be the orthogonal complement of $U(N_z) \otimes \mathbb{Q}$. Then we have

$$L \otimes \mathbb{Q} = U(N_z) \otimes \mathbb{Q} \oplus W.$$

We obviously have

$$W \cap L \oplus U(N_z) \subseteq L.$$

We now show the converse inclusion. Therefore, let $x \in L$ and write $x = a + b$, $a \in W, b \in U(N_z) \otimes \mathbb{Q}$ as a decomposition in $L \otimes \mathbb{Q}$. For all $h \in U(N_z)$ we have

$$(x, h) = (a + b, h) = (a, h) + (b, h) = (b, h)$$

and thus

$$(b, U(N_z)) = (x, U(N_z)) \subseteq (x, z)\mathbb{Z} + (x, \zeta)\mathbb{Z} \subseteq N_z\mathbb{Z} + N_z\mathbb{Z} = N_z\mathbb{Z}.$$

Writing $b = b_1 z + b_2 \zeta$ yields

$$b_1 = \frac{(b, \zeta)}{N_z} \in \mathbb{Z} \quad \text{and} \quad b_2 = \frac{(b, z)}{N_z} \in \mathbb{Z}$$

and thus $b \in U(N_z)$. We obtain $a = x - b \in L$, i.e. $a \in W \cap L$. □

Corollary 2.1.10 ([Bru14, Lemma 5.1])**.** *Let L be an even lattice of level N and $z \in \mathrm{Iso}_0(L)$ of level $N_z = N$ or $N_z = 1$. Then L splits a scaled hyperbolic plane $U(N_z)$. In particular, if L is a lattice of prime level, then every primitive isotropic vector splits a scaled hyperbolic plane.*

Proof. If $N_z = 1$ let $z' \in L$ with $(z, z') = 1$. If $N_z = N$ let $z' \in L'$ with $(z, z') = 1$. Now define $\tilde{z}' = z' - N_z q(z') z / N_z$. Then $\zeta = N_z \tilde{z}' \in L$ is isotropic of level N_z and $(z, \zeta) = N_z$. □

We have a natural map

$$\pi_L : \Gamma(L) \backslash \operatorname{Iso}_0(L') \to \operatorname{Iso}(L'/L).$$

Lemma 2.1.11. *The map π_L is surjective if and only if L splits a hyperbolic plane.*

Proof. If π_L is surjective, then there is some primitive isotropic element in L of level 1, which yields a splitting of a hyperbolic plane. Conversely, assume

$$L = U \oplus L_1 = e_1\mathbb{Z} \oplus e_2\mathbb{Z} + L_1,$$

and let $\delta \in L'/L = L_1'/L_1$ be an isotropic element. Consider a preimage $\lambda \in L_1'$. Then $e_1 - q(\lambda)e_2 + \lambda \in L'$ is isotropic with image $\delta \in L'/L$. □

The following lemma goes back to Eichler and is called Eichler criterion.

Lemma 2.1.12 ([FH00, Lemma 4.4])**.** *If L splits two hyperbolic planes, then π_L is bijective.*

A useful theorem to decide if hyperbolic planes split is

Theorem 2.1.13 ([Nik79, Cor. 1.15.5])**.** *Let L be an even lattice of signature (m, n) with $m, n \geq 1$. If $m + n \geq 3 + l(L'/L)$, then L splits a hyperbolic plane. Here $l(L'/L)$ denotes the minimal number of generators for L'/L.*

2.2 Genus of a Lattice

Denote by $\mathbb{Z}_p$ the ring of p-adic integers.

Definition 2.2.1. Let L be an even lattice and p a prime. Then we obtain a *$\mathbb{Z}_p$-lattice* $L_p := L \otimes \mathbb{Z}_p$. We say that two lattices $L, \tilde{L}$ are in the same *genus*, if they have the same signature and $L_p \simeq \tilde{L}_p$ for all primes p.

Obviously, isomorphic lattices are in the same genus. The converse is not necessarily true. In every genus there are only finitely many isomorphisms classes. For indefinite lattices the number of isomorphism classes is often 1 as can be seen from the following

Theorem 2.2.2 ([CS99, 9.7, Theorem 21, p. 395])**.** *Let L be an indefinite lattice of dimension n and level N such that $\operatorname{gen}(L)$ contains more then one class. Then*

$$k^{\binom{n}{2}} \mid 4^{\lfloor \frac{n}{2} \rfloor} N^n$$

for some non-square $k = 0, 1 \bmod 4$.

This yields for square-free level

Corollary 2.2.3. *Let L be indefinite with square-free level N and dimension $n \geq 4$. Then $\operatorname{gen}(L)$ contains only one class.*

Proof. Let $k = 0, 1 \bmod 4$ and $2 \neq p \mid k$. Since $\binom{n}{2} > n$ for $n \geq 4$ the divisibility property can't hold for k (here we used that N is square free). If $2 \mid k$, then we already have $4 \mid k$, since $k = 0, 1 \bmod 4$. Since k is a non-square, there's either another prime $p \neq 2$ dividing k (see the first case) or $8 \mid k$. Since for $n \geq 3$ we have

$$3\binom{n}{2} > 2n \geq 2\left\lfloor \frac{n}{2} \right\rfloor + n$$

the divisibility property can't hold for $2 \mid k$ too (we used again that N is square-free). □

Another useful theorem is

Theorem 2.2.4 ([Nik79, Theorem 1.14.2])**.** *Let L be an even indefinite lattice satisfying $\operatorname{rank}(L) \geq l(L'_p/L_p) + 2$ for all primes p, then $\operatorname{gen}(L)$ contains only one class and the homomorphism $\rho : \mathrm{O}(L) \to \mathrm{O}(L'/L)$ is surjective. Here $l(L'_p/L_p)$ denotes the minimal number of generators of L'_p/L_p.*

Since the assumption is always true if L splits a hyperbolic plane, we obtain

Corollary 2.2.5. *Let L be an even indefinite lattice splitting a hyperbolic plane. Then $\operatorname{gen}(L)$ contains only one class and the homomorphism $\rho : \mathrm{O}(L) \to \mathrm{O}(L'/L)$ is surjective.*

For $\mathbb{Z}_p$-lattices we have the following

Theorem 2.2.6 ([O'M63, Thm. 92:3, Thm. 93:14])**.** *Let L, M, N be $\mathbb{Z}_p$-lattices with $L \oplus M \simeq L \oplus N$. If $p \neq 2$, then $M \simeq N$. If $p = 2$ and L is a scaled hyperbolic plane, then $M \simeq N$.*

As a corollary we obtain

Corollary 2.2.7. *Let $L \simeq U(N) \oplus L_1 \simeq U(N) \oplus L_2$ be a lattice. Then $\operatorname{gen}(L_1) = \operatorname{gen}(L_2)$.*

Theorem 2.2.8. *Let L be a lattice of square-free level N and let $z \in L$ be isotropic of level N_z. Then L splits a scaled hyperbolic plane $U(N_z)$ spanned by z and $N_z z'$ for some isotropic $z' \in L'$.*

Proof. According to [Dit18, Proposition 2.73] there exists an isotropic $z' \in L'$ such that $N_z z' \in L$ and $(z, z') = 1$. Let $K = z^\perp \cap z'^\perp \cap L$. Then $U(N_z) \oplus K \subseteq L$, where $U(N_z)$ is the hyperbolic plane spanned by $z, N_z z'$. According to [Bru02, Proposition 2.2] we have equality. □

2.3 Scalar-Valued Eisenstein Series

Let $\mathbb{H} := \{\tau = u + iv \in \mathbb{C} \mid v > 0\}$ be the usual upper half-plane. For $z \in \mathbb{C}$ we write $e(z) := e^{2\pi i z}$ and consider the usual action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

For $N \in \mathbb{N}$ define the principal congruence subgroup

$$\Gamma(N) := \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$$

and, more generally, for $N_1, N_2 \in \mathbb{N}, N = \operatorname{lcm}(N_1, N_2)$ we will also need the congruence subgroups

$$\Gamma(N_1, N_2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \bmod N, c \equiv 0 \bmod N_1, b \equiv 0 \bmod N_2 \right\}$$

and $\Gamma(N) = \Gamma(N, N), \Gamma_1(N) = \Gamma(N, 1)$. Following [DS05, Section 4.2] we define scalar-valued Eisenstein series for $\Gamma(N)$. Therefore, let $v = (c_v, d_v) \in (\mathbb{Z}/N\mathbb{Z})^2$ and $k \in \mathbb{Z}, k \geq 3$. Define the non-normalized Eisenstein series

$$G_k^v(\tau) = \sum_{(c,d) \equiv v \bmod N}{}' \frac{1}{(c\tau + d)^k},$$

where $\sum'$ means that we sum over non-zero pairs. Recall the *modified zeta function* $\zeta^{d_v}(s) = \sum'_{n\equiv d \bmod N} \frac{1}{n^s}$ and define the divisor sum

$$\sigma_s^v(n) := \sum_{\substack{m|n \\ \frac{n}{m}\equiv c_v \bmod N}} \operatorname{sgn}(m) m^s e\left(\frac{d_v m}{N}\right),$$

where the sum is over positive and negative divisors. We have

Theorem 2.3.1 ([DS05, Theorem 4.2.3])**.** *The Eisenstein series G_k^v are holomorphic modular forms of weight k with respect to $\Gamma(N)$, i.e. they are holomorphic, satisfy*

$$G_k^v(M\tau) = (c\tau + d)^k G_k^v(\tau)$$

for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(N)$ and the Fourier expansion of the Eisenstein series are given by

$$G_k^v(\tau) = \delta_{c_v,0}\zeta^{d_v}(k) + \frac{(-2\pi i)^k}{\Gamma(k)N^k} \sum_{n=1}^{\infty} \sigma_{k-1}^v(n) e\left(\frac{n\tau}{N}\right).$$

More generally, we will see Eisenstein series for $v = (c_v, d_v) \in \mathbb{Z}/N_1\mathbb{Z} \times \mathbb{Z}/N_2\mathbb{Z}$ with Fourier expansion

$$\tilde{G}_k^v(\tau) = \delta_{c_v,0}\zeta^{d_v}(k) + \frac{(-2\pi i)^k}{\Gamma(k)N_2^k} \sum_{n=1}^{\infty} \sigma_{k-1}^v(n) e\left(\frac{n\tau}{N_1}\right),$$

where

$$\sigma_s^v(n) := \sum_{\substack{m|n \\ \frac{n}{m}\equiv c_v \bmod N_1}} \operatorname{sgn}(m) m^s e\left(\frac{d_v m}{N_2}\right)$$

is a divisor sum over positive and negative divisors. We shortly show that they are in fact Eisenstein series of the previous type for $N = \operatorname{lcm}(N_1, N_2)$. Therefore, let $N_i N_i' = N$. Then we have

$$\begin{aligned} \zeta^{d_v}(k) &= \sum'_{n\equiv d_v \bmod N_2} \frac{1}{n^k} \\ &= N_2'^k \sum'_{n\equiv d_v \bmod N_2} \frac{1}{(N_2' n)^k} \\ &= N_2'^k \sum'_{n\equiv N_2' d_v \bmod N} \frac{1}{n^k} = N_2'^k \zeta^{N_2' d_v}(k), \end{aligned}$$

Moreover, a short calculation shows $\sigma_{k-1}^{(c_v,d_v)}(n) = \sigma_{k-1}^{(N_1' c_v, N_2' d_v)}(nN_1')$ (here, $N_1' c_v$ is meant to be $\bmod N$). Using that $\sigma_{k-1}^{(N_1' c_v, N_2' d_v)}(n)$ vanishes if $N_1' \nmid n$ we have

$$\tilde{G}_k^v(\tau) = \delta_{c_v,0}\zeta^{d_v}(k) + \frac{(-2\pi i)^k}{\Gamma(k)N_2^k} \sum_{n=1}^{\infty} \sigma_{k-1}^v(n) e\left(\frac{n\tau}{N_1}\right)$$

$$= N_2'^k \left(\delta_{c_v,0} \zeta^{N_2' d_v}(k) + \frac{(-2\pi i)^k}{\Gamma(k) N^k} \sum_{n=1}^{\infty} \sigma_{k-1}^{(N_1' c_v, N_2' d_v)}(n N_1') e\left(\frac{n N_1' \tau}{N} \right) \right)$$
$$= N_2'^k G_k^{(N_1' c_v, N_2' d_v)}(\tau)$$

and we define $G_k^v(\tau) = G_k^{(N_1' c_v, N_2' d_v)}(\tau) = N_2'^{-k} \tilde{G}_k^v(\tau)$.

Remark 2.3.2. For $k = 1, 2$ one can still use the Fourier expansion as the definition of the corresponding Eisenstein series G_k^v, but usually they will not be modular anymore and one needs a correction term to obtain modularity, see [DS05, Sections 4.6, 4.8].

2.4 Vector-Valued Modular Forms

We will now introduce the Weil representation and vector-valued modular forms.

Definition 2.4.1. We denote by $\mathrm{Mp}_2(\mathbb{R})$ the *metaplectic cover* of $\mathrm{SL}_2(\mathbb{R})$. It is realized as pairs (M, ϕ), where $M \in \mathrm{SL}_2(\mathbb{R})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic square root of $\tau \mapsto c\tau + d$. The product for $(M_1, \phi_1), (M_2, \phi_2) \in \mathrm{Mp}_2(\mathbb{R})$ is given by

$$(M_1, \phi_1(\tau))(M_2, \phi_2(\tau)) := (M_1 M_2, \phi_1(M_2 \tau) \phi_2(\tau)).$$

By $\mathrm{Mp}_2(\mathbb{Z})$ we denote the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map. It is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \quad \text{and} \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right).$$

We have the relation $S^2 = (ST)^3 = Z$, where $Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \right), i \right)$ is the standard generator of the center of $\mathrm{Mp}_2(\mathbb{Z})$. The assigment

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right)$$

defines a locally isomorphic embedding of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{Mp}_2(\mathbb{R})$. Furthermore, we will write $\Gamma_\infty := \{ \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right) \mid n \in \mathbb{Z} \} \subseteq \mathrm{SL}_2(\mathbb{Z})$ and $\tilde{\Gamma}_\infty := \{ (\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right), 1) \mid n \in \mathbb{Z} \} \subseteq \mathrm{Mp}_2(\mathbb{Z})$. For an even non-degenerate lattice L of signature (b^+, b^-) consider the group ring $\mathbb{C}[L'/L]$ with standard basis $(\mathfrak{e}_\gamma)_{\gamma \in L'/L}$. For $\mathfrak{v} = \sum_{\gamma \in L'/L} \mathfrak{v}_\gamma \mathfrak{e}_\gamma \in \mathbb{C}[L'/L]$ we write $\mathfrak{v}^* := \sum_{\gamma \in L'/L} \mathfrak{v}_\gamma \mathfrak{e}_{-\gamma}$. Moreover, we write $\langle \cdot, \cdot \rangle$ for the standard inner product on $\mathbb{C}[L'/L]$ which is anti-linear in the second variable. We denote the subspace of vectors that are supported on isotropic elements by $\mathrm{Iso}(\mathbb{C}[L'/L])$ and we introduce the notation $\mathfrak{e}_\gamma(\tau) := e(\tau) \mathfrak{e}_\gamma = e^{2\pi i \tau} \mathfrak{e}_\gamma$.

Definition 2.4.2. The *Weil representation* is the unitary representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$ defined by

$$\rho_L(T)\mathfrak{e}_\gamma := \mathfrak{e}_\gamma(q(\gamma)) \quad \text{and} \quad \rho_L(S)\mathfrak{e}_\gamma := \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{L'/L}} \sum_{\delta \in L'/L} \mathfrak{e}_\delta(-(\gamma,\delta)).$$

The Weil representation factors through a finite quotient of $\mathrm{Mp}_2(\mathbb{Z})$. Write $\mathrm{Inv}(\mathbb{C}[L'/L])$ for the subspace of vectors in $\mathbb{C}[L'/L]$ that are invariant under the Weil representation (see [Bie21], [Zem21] for results on the space of invariants and [ES17] for an algorithm to calculate this space).

Remark 2.4.3. If we take $-q$ as the quadratic form of L instead of q, the corresponding Weil representation is called *dual Weil representation*, which we denote by ρ_L^*.

A short calculation using orthogonality of characters shows $\rho_L(Z)\mathfrak{e}_\gamma = i^{b^- - b^+}\mathfrak{e}_{-\gamma}$. For $\beta, \gamma \in L'/L$ we define the coefficients

$$\rho_{\beta,\gamma}(M,\phi) := \langle \rho_L(M,\phi)\mathfrak{e}_\gamma, \mathfrak{e}_\beta \rangle \quad \text{and} \quad \rho^{-1}_{\beta,\gamma}(M,\phi) := \langle \rho_L^{-1}(M,\phi)\mathfrak{e}_\gamma, \mathfrak{e}_\beta \rangle.$$

Theorem 2.4.4 (Shintani's Formula, [Shi75, Proposition 1.6])**.** *For $M \in \mathrm{SL}_2(\mathbb{Z})$ the coefficient $\rho_{\beta,\gamma}(\tilde{M})$ is given by*

$$\sqrt{i}^{(b^- - b^+)(1-\mathrm{sgn}(d))} \delta_{\beta,a\gamma} e(abq(\beta))$$

if $c = 0$ and by

$$\frac{\sqrt{i}^{(b^- - b^+)\,\mathrm{sgn}(c)}}{|c|^{(b^- + b^+)/2}\sqrt{|L'/L|}} \sum_{r \in L/cL} e\left(\frac{a(\beta+r,\beta+r) - 2(\gamma,\beta+r) + d(\gamma,\gamma)}{2c}\right)$$

if $c \neq 0$.

The sum in Shintani's formula can be calculated, see [Sch09], [Str13, Theorem 1].

For a vector-valued function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ we write $f_\gamma : \mathbb{H} \to \mathbb{C}$ for its components, i.e. $f = \sum_{\gamma \in L'/L} f_\gamma \mathfrak{e}_\gamma$. For $k \in \frac{1}{2}\mathbb{Z}$ we define the *Petersson slash operator* $f \mapsto f|_{k,L}(M,\phi)$ by

$$(f|_{k,L}(M,\phi))(\tau) = \phi(\tau)^{-2k}\rho_L^{-1}(M,\phi)f(M\tau).$$

If $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is smooth and invariant under the action of T, i.e. $f|_{k,L}T = f$, then we have $f_\gamma(\tau+1) = e(q(\gamma))f_\gamma(\tau)$ and thus $e(-q(\gamma)\tau)f_\gamma(\tau)$ is a 1-periodic function with a Fourier expansion

$$e(-q(\gamma)\tau)f_\gamma(\tau) = \sum_{n=-\infty}^{\infty} b(\gamma,n,y)e(nu).$$

This yields a Fourier expansion

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z}+q(\gamma)} c(\gamma, n, y)\mathfrak{e}_\gamma(nu).$$

Definition 2.4.5. A function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is said to be a *modular form* of weight k with respect to the Weil representation if $f|_{k,L}(M,\phi) = f$ for all $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$. We call a modular form f a *holomorphic modular form* if f is holomorphic with Fourier expansion

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n \geq 0}} c(\gamma, n)\mathfrak{e}_\gamma(n\tau).$$

Obviously, non-trivial modular forms only exist for weights with $2k + b^- - b^+ = 0 \bmod 2$. Hence, if the dimension of L is even, there are only integral weights and the group $\mathrm{SL}_2(\mathbb{Z})$ instead of $\mathrm{Mp}_2(\mathbb{Z})$ suffices. The space of modular forms of weight 0 is given by the invariant vectors $\mathrm{Inv}(\mathbb{C}[L'/L])$.

We denote the *hyperbolic Laplace operator of weight* k by

$$\Delta_k = v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) - ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

It acts on smooth functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$ component-wise and commutes with the action of $\mathrm{Mp}_2(\mathbb{Z})$ in the sense that

$$\Delta_k(f|_{k,L}(M,\phi)) = (\Delta_k f)|_{k,L}(M,\phi)$$

for f smooth, $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$ and analogously for the dual Weil representation. It has a self-adjoint extension to the space of square-integrable modular forms of weight k with respect to the Petersson inner product

$$\langle f, g \rangle := \int_{\mathrm{Mp}_2(\mathbb{Z})\backslash\mathbb{H}} \langle f(\tau), g(\tau)\rangle v^k \frac{\mathrm{d}u\mathrm{d}v}{v^2}$$

for vector-valued square-integrable modular forms f, g of weight k. We will introduce two further differential operators, the *weight changing operators*. We define the *Maass raising operator* and the *Maass lowering operator* as

$$R_k = 2i\frac{\partial}{\partial \tau} + kv^{-1} \quad \text{and} \quad L_k = -2iv^2\frac{\partial}{\partial \overline{\tau}}.$$

The raising operator maps smooth modular forms of weight k to smooth modular forms of weight $k+2$. Similarly, the lowering operator maps smooth modular forms of weight

k to smooth modular forms of weight $k-2$. Moreover, they commute with the slash operator in the sense that

$$R_k(f|_{k,L}(M,\phi)) = (R_k f)|_{k+2,L}(M,\phi) \quad \text{and} \quad L_k(f|_{k,L}(M,\phi)) = (L_k f)|_{k-2,L}(M,\phi).$$

The Laplace operator can be expressed in terms of R_k and L_k by

$$\Delta_k = L_{k+2}R_k + k = R_{k-2}L_k.$$

Next we will discuss vector-valued modular forms for certain sublattices. Following [Bor98], let $z \in \mathrm{Iso}_0(L)$ and $z' \in L'$ with $(z,z') = 1$. Denote by N_z the level of z and consider the sublattice

$$K = L \cap z^{\perp} \cap z'^{\perp}.$$

Then K has signature (b^+-1, b^--1). For a vector $\lambda \in V(\mathbb{R})$ we write λ_K for its orthogonal projection to $K \otimes \mathbb{R}$, which is given by

$$\lambda_K = \lambda - (\lambda,z)z' + (\lambda,z')(z',z')z - (\lambda,z')z.$$

Let $\zeta \in L$ with $(z,\zeta) = N_z$. Then we have

$$L = K \oplus \mathbb{Z}\zeta + \mathbb{Z}z.$$

Consider the sublattice

$$L_0' = \{\lambda \in L' \mid (\lambda,z) \equiv 0 \bmod N_z\} \subseteq L'$$

and the projection

$$\pi : L_0' \to K', \quad \lambda \mapsto \pi(\lambda) = \lambda_K + \frac{(\lambda,z)}{N_z}\zeta_K.$$

This projection induces a surjective map $L_0'/L \to K'/K$ which we also denote by π and we have

$$L_0'/L = \{\lambda \in L'/L \mid (\lambda,z) \equiv 0 \bmod N_z\}.$$

We have the following connection between the Weil representations ρ_L and ρ_K.

Lemma 2.4.6. *Let $\gamma \in K'/K$. Then*

$$\rho_L(M) \sum_{m\in\mathbb{Z}/N_z\mathbb{Z}} \mathfrak{e}_{\gamma+\frac{mz}{N_z}} \left(-\frac{mn}{N_z}\right) = (\rho_K(M)\mathfrak{e}_\gamma) \sum_{m\in\mathbb{Z}/N_z\mathbb{Z}} \mathfrak{e}_{\frac{mz}{N_z}-ncz'} \left(-\frac{amn}{N_z} + q(z')acn^2\right).$$

Proof. Write $\lambda = \gamma + \frac{mz}{N_z}$. We first consider the case $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c = 0$. Then by Shintani's formula

$$\begin{aligned}
\rho_L(M)\mathfrak{e}_\lambda &= \sqrt{i}^{(b^- - b^+)(1-\operatorname{sgn}(d))} \sum_{\beta \in L'/L} \delta_{\beta, a\lambda} \mathfrak{e}_\beta(abq(\beta)) \\
&= \sqrt{i}^{(b^- - b^+)(1-\operatorname{sgn}(d))} \mathfrak{e}_{a\lambda}(abq(a\lambda)) \\
&= \sqrt{i}^{((b^- -1)-(b^+ -1))(1-\operatorname{sgn}(d))} \mathfrak{e}_{a\gamma}(abq(a\gamma)) \mathfrak{e}_{\frac{amz}{N_z}} = (\rho_K(M)\mathfrak{e}_\gamma)\mathfrak{e}_{\frac{amz}{N_z}}.
\end{aligned}$$

Multiplying by $e\left(-\frac{mn}{N_z}\right)$ and summing over $m \in \mathbb{Z}/N_z\mathbb{Z}$ yields the result. Now let $c \neq 0$. Again, Shintani's formula yields

$$\rho_L(M)\mathfrak{e}_\lambda = C_L(c) \sum_{\beta \in L'/L} \sum_{r \in L/cL} \mathfrak{e}_\beta \left(\frac{a(\beta + r, \beta + r) - 2(\lambda, \beta + r) + d(\lambda, \lambda)}{2c} \right),$$

where

$$C_L(c) = \frac{\sqrt{i}^{(b^- - b^+)\operatorname{sgn}(c)}}{|c|^{\frac{b^- + b^+}{2}} \sqrt{|L'/L|}} = \frac{\sqrt{i}^{((b^- -1)-(b^+ -1)\operatorname{sgn}(c)}}{|c|^{\frac{(b^- -1)+(b^+ -1)}{2}} \sqrt{|K'/K|}} \cdot \frac{1}{cN_z} = \frac{C_K(c)}{cN_z}.$$

Since $L = K \oplus \mathbb{Z}\zeta \oplus \mathbb{Z}z$ we can instead sum over $r + k\zeta + k'z, r \in K/cK, k, k' \in \mathbb{Z}/c\mathbb{Z}$ to obtain

$$\begin{aligned}
\rho_L(M)\mathfrak{e}_\lambda = C_L(c) &\sum_{\beta \in L'/L} \sum_{\substack{r \in K/cK \\ k \in \mathbb{Z}/c\mathbb{Z}}} e\left(\frac{a(\beta + r, \beta + r) - 2(\gamma, k\zeta) + d(\gamma, \gamma)}{2c} \right) \\
&\times e\left(\frac{2a(\beta + r, k\zeta) + a(k\zeta, k'\zeta) - 2(\gamma, k\zeta) - 2(\frac{mz}{N_z}, \beta + k\zeta)}{2c} \right) \\
&\times \sum_{k' \in \mathbb{Z}/c\mathbb{Z}} e\left(\frac{2a(\beta, k'z) + 2a(k\zeta, k'\zeta)}{2c} \right) \mathfrak{e}_\beta \\
= C_L(c) &\sum_{\beta \in L'/L} \sum_{\substack{r \in K/cK \\ k \in \mathbb{Z}/c\mathbb{Z}}} e\left(\frac{a(\beta + r, \beta + r) - 2(\gamma, k\zeta) + d(\gamma, \gamma)}{2c} \right) \\
&\times e\left(\frac{2a(\beta + r, k\zeta) + a(k\zeta, k'\zeta) - 2(\gamma, k\zeta) - 2(\frac{mz}{N_z}, \beta + k\zeta)}{2c} \right) \\
&\times \sum_{k' \in \mathbb{Z}/c\mathbb{Z}} e\left(k' \frac{a(\beta + k\zeta, k'z)}{2c} \right) \mathfrak{e}_\beta.
\end{aligned}$$

Now the last sum vanishes by orthogonality of characters unless

$$(\beta + k\zeta, z) = (\beta, z) + kN_z \equiv 0 \bmod c,$$

in which case it sums to c. This yields

$$\rho_L(M)\mathfrak{e}_\lambda = C_L(c)c \sum_{\beta \in L'/L} \sum_{\substack{r \in K/cK \\ k \in \mathbb{Z}/c\mathbb{Z} \\ c | kN_z + (\beta, z)}} e\left(\frac{a(\beta + r, \beta + r) - 2(\gamma, k\zeta) + d(\gamma, \gamma)}{2c} \right)$$

$$\times e\left(\frac{2a(\beta+r,k\zeta)+a(k\zeta,k'\zeta)-2(\gamma,k\zeta)-2(\frac{mz}{N_z},\beta+k\zeta)}{2c}\right)\mathfrak{e}_\beta.$$

We now multiply this by $e\left(-\frac{mn}{N_z}\right)$ and sum over $m\in\mathbb{Z}/N_z\mathbb{Z}$ to obtain

$$\begin{aligned}
&\rho_L(M)\sum_{m\in\mathbb{Z}/N_z\mathbb{Z}}\mathfrak{e}_{\gamma+\frac{mz}{N_z}}\left(-\frac{mn}{N_z}\right)\\
&=C_L(c)c\sum_{\beta\in L'/L}\sum_{\substack{r\in K/cK\\ k\in\mathbb{Z}/c\mathbb{Z}\\ c|kN_z+(\beta,z)}}e\left(\frac{a(\beta+r,\beta+r)-2(\gamma,k\zeta)+d(\gamma,\gamma)}{2c}\right)\\
&\times e\left(\frac{2a(\beta+r,k\zeta)+a(k\zeta,k'\zeta)-2(\gamma,k\zeta)}{2c}\right)\\
&\times\sum_{m\in\mathbb{Z}/N_z\mathbb{Z}}e\left(-\frac{(\frac{mz}{N_z},\beta+k\zeta+ncz')}{c}\right)\mathfrak{e}_\beta.
\end{aligned}$$

The latter sum again vanishes unless

$$(z,\beta+k\zeta+ncz')=(z,\beta+ncz')+kN_z\equiv 0 \bmod cN_z,$$

in which case it is equal to N_z. In particular, we have $(z,\beta+ncz')\equiv 0 \bmod N_z$ and thus $\beta+ncz'\in L_0'/L$. But every element in L_0'/L can be written as $\alpha+\frac{mz}{N_z}$ for some $\alpha\in K'/K$ and $m\in\mathbb{Z}/N_z\mathbb{Z}$, which shows that $(\beta+ncz',\frac{z}{N_z})=0$. But this implies $k\equiv 0 \mod c$. Hence we obtain

$$\begin{aligned}
&\rho_L(M)\sum_{m\in\mathbb{Z}/N_z\mathbb{Z}}\mathfrak{e}_{\gamma+\frac{mz}{N_z}}\left(-\frac{mn}{N_z}\right)\\
&=C_K(c)\sum_{\beta\in K'/K}\sum_{r\in K/cK}\mathfrak{e}_\beta\left(\frac{a(\beta+r,\beta+r)-2(\gamma,\beta+r)+d(\gamma,\gamma)}{2c}\right)\\
&\times\sum_{m\in\mathbb{Z}/N_z\mathbb{Z}}\mathfrak{e}_{\frac{mz}{N_z}-ncz'}\left(-\frac{amn}{N_z}+q(z')acn^2\right)\\
&=(\rho_K(M)\mathfrak{e}_\gamma)\sum_{m\in\mathbb{Z}/N_z\mathbb{Z}}\mathfrak{e}_{\frac{mz}{N_z}-ncz'}\left(-\frac{amn}{N_z}+q(z')acn^2\right),
\end{aligned}$$

which shows the claim. □

Now let $f:\mathbb{H}\to\mathbb{C}[L'/L]$ be a vector-valued function. Define

$$f^K(\tau,r,t):=\sum_{\gamma\in K'/K}f_\gamma^K(\tau,r,t)\mathfrak{e}_\gamma,$$

where

$$f_\gamma^K(\tau,r,t):=\sum_{\substack{\lambda\in L_0'/L\\ \pi(\lambda)=\gamma}}f_{\lambda+tz'}(\tau)e(-r(\lambda,z')+rtq(z')),$$

hence

$$f^K(\tau,r,t) = \sum_{\lambda\in L_0'/L} f_{\lambda+tz'}(\tau)\mathfrak{e}_{\pi(\lambda)}(-r(\lambda,z') + rtq(z')).$$

Then

Lemma 2.4.7 ([Bor98, Theorem 5.3])**.** *Assume f is modular of weight k with respect to the Weil representation. Then for $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$ with $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we have*

$$f^K(M\tau, ar+bt, cr+dt) = \phi(\tau)^{2k}\rho_K(M,\phi)f^K(\tau,r,t).$$

Moreover, we have the following

Lemma 2.4.8. *For vector-valued functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$, $g : \mathbb{H} \to \mathbb{C}[K'/K]$ we have*

$$\langle f(\tau), g(\tau)\sum_{m\in\mathbb{Z}/N\mathbb{Z}} \mathfrak{e}_{\frac{mz}{N}}\left(-\frac{mn}{N}\right)\rangle = \langle f^K(\tau,-n,0), g(\tau)\rangle.$$

Proof. This is a straight forward calculation. We have

$$\begin{aligned}\langle f(\tau), g(\tau)\sum_{m\in\mathbb{Z}/N\mathbb{Z}} \mathfrak{e}_{\frac{mz}{N}}\left(-\frac{mn}{N}\right)\rangle &= \sum_{\substack{\gamma\in K'/K\\ m\in\mathbb{Z}/N\mathbb{Z}}} f_{\gamma+\frac{mz}{N}}(\tau)\overline{g}_\gamma(\tau)e\left(n(\gamma+\frac{mz}{N},z')\right)\\ &= \sum_{\lambda\in L_0'/L} e(n(\lambda,z'))f_\lambda(\tau)\overline{g}_{\pi(\lambda)}(\tau)\\ &= \langle f^K(\tau,-n,0), g(\tau)\rangle,\end{aligned}$$

which shows the result. □

2.5 Vector-Valued Eisenstein Series

Let L be an even lattice of signature (b^+,b^-). Assume that $b^+ - b^-$ is even and let $k \in \mathbb{Z}$. Moreover, set $\kappa = \frac{b^- - b^+}{2} + k$. Let $\beta \in \mathrm{Iso}(L'/L)$ and define similar to [BK03] (they consider Eisenstein series with respect to the dual Weil representation ρ_L^*) the *vector-valued non-holomorphic Eisenstein series* of weight k by

$$E_{k,\beta}(\tau,s) = \frac{1}{2}\sum_{(M,\phi)\in\tilde{\Gamma}_\infty\backslash \mathrm{Mp}_2(\mathbb{Z})} (\mathrm{Im}(\tau)^s\mathfrak{e}_\beta)|_{k,L}(M,\phi).$$

For $\beta \in \mathrm{Iso}(L'/L)$ of order N_β and a character $\chi : (\mathbb{Z}/N_\beta\mathbb{Z})^\times \to \mathbb{C}^\times$ we define

$$E_{k,\beta,\chi}(\tau,s) := \sum_{n\in(\mathbb{Z}/N_\beta\mathbb{Z})^\times} \chi(n)E_{k,n\beta}(\tau,s)$$

More generally, for $\mathfrak{v} \in \mathrm{Iso}(\mathbb{C}[L'/L])$ we define

$$\begin{aligned} E_{k,\mathfrak{v}}(\tau, s) &= \frac{1}{2} \sum_{(M,\phi)\in\tilde{\Gamma}_\infty\backslash \mathrm{Mp}_2(\mathbb{Z})} (\mathrm{Im}(\tau)^s \mathfrak{v})|_{k,L}(M,\phi) \\ &= \sum_{\beta\in\mathrm{Iso}(L'/L)} \mathfrak{v}_\beta E_{k,\beta}(\tau, s). \end{aligned}$$

We have $E_{k,\mathfrak{v}^*} = (-1)^\kappa E_{k,\mathfrak{v}}$.

Lemma 2.5.1. *The Eisenstein series* $E_{k,\beta}$ *converges normally on* $\mathbb{H}$ *for* $\mathrm{Re}(s) > 1 - \frac{k}{2}$ *and defines a* $\mathrm{Mp}_2(\mathbb{Z})$*-invariant real analytic function, which is an Eigenfunction of the hyperbolic Laplace operator of weight* k *with Eigenvalue* $s(s+k-1)$.

Proof. Since the Weil representation is unitary, we have the majorant

$$\sum_{(M,\phi)\in\tilde{\Gamma}_\infty\backslash \mathrm{Mp}_2(\mathbb{Z})} \mathrm{Im}(M\tau)^{\mathrm{Re}(s)+\frac{k}{2}}.$$

The convergence now follows from the classical case. Moreover, a straight forward calculation yields

$$\Delta_k v^s = s(s+k-1)v^s$$

and since the Laplace operator commutes with the action of $\mathrm{Mp}_2(\mathbb{Z})$ we obtain

$$\Delta_k E_{k,\beta}(\tau, s) = s(s+k-1)\Delta_k E_{k,\beta}(\tau, s).$$

The $\mathrm{Mp}_2(\mathbb{Z})$-invariance is obvious. □

Remark 2.5.2. Since the Weil representation is trivial on $\Gamma(N)$, where N is the level of L, the Eisenstein series $E_{k,\beta}$ is a linear combination of $\mathrm{Mp}_2(\mathbb{Z})/\Gamma(N)$-translates of scalar-valued Eisenstein series for $\Gamma(N)$. In particular, we obtain a meromorphic continuation of $E_{k,\beta}(\tau, s)$ to all $s \in \mathbb{C}$.

Next we will calculate the Fourier expansion. Therefore, we have to introduce certain special functions. Recall the *Whittaker differential equation*

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(-\frac{1}{4} + \frac{\nu}{z} - \frac{\mu^2 - \frac{1}{4}}{z^2}\right) = 0.$$

One solution is given by the *Whittaker function* $W_{\nu,\mu}$, see [EMOT81, 6.9], [OLBC10, 13.14]. As $z \to 0$ we have

$$W_{\nu,\mu}(z) \sim \frac{\Gamma(2\mu)}{\Gamma(\mu - \nu + \frac{1}{2})} z^{-\mu+\frac{1}{2}},$$

for $\mu \geq \frac{1}{2}$ and real $y \to \infty$ we have

$$W_{\nu,\mu}(y) = e^{-\frac{y}{2}} y^{\nu}(1 + O(y^{-1})).$$

As an abbreviation we define for $s \in \mathbb{C}, y \in \mathbb{R} \smallsetminus \{0\}$

$$\mathcal{W}_s(y) = |y|^{-\frac{k}{2}} W_{\operatorname{sgn}(y)\frac{k}{2}, s-\frac{1}{2}}(|y|)$$

and the previous asymptotics yield

$$\mathcal{W}_s(y) = |y|^{-\frac{k}{2} + \operatorname{sgn}(y)\frac{k}{2}} e^{-\frac{|y|}{2}}(1 + O(y^{-1}))$$

for $y \in \mathbb{R}, y \to \infty$. Moreover, for $s = 0$ we have

$$\mathcal{W}_0(y) = e^{-\frac{y}{2}} \begin{cases} 1, & \text{if } y > 0, \\ \Gamma(1-k, |y|), & \text{if } y < 0, \end{cases}$$

where $\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} \mathrm{d}t$ is the incomplete Gamma function as in [OLBC10, 8.2].

Theorem 2.5.3 ([BK03])**.** *The Eisenstein series has the Fourier expansion*

$$E_{k,\beta}(\tau, s) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z} + q(\gamma)} c_{k,\beta}(\gamma, n, s, v) \mathfrak{e}_\gamma(nu),$$

where the coefficients $c_{k,\beta}(\gamma, n, s, v)$ are given by

$$\begin{cases} (\delta_{\beta,\gamma} + \delta_{-\beta,\gamma}) v^s + 2\pi v^{1-k-s} \frac{\Gamma(k+2s-1)}{\Gamma(k+s)\Gamma(s)} \sum_{c \in \mathbb{Z} \smallsetminus \{0\}} |2c|^{1-k-2s} H_c(\beta, 0, \gamma, 0), & \text{if } n = 0, \\ \frac{2^k \pi^{s+k} |n|^{s+k-1}}{\Gamma(s+k)} \mathcal{W}_s(4\pi n v) \sum'_{c \in \mathbb{Z}} |2c|^{1-k-2s} H_c(\beta, 0, \gamma, n), & \text{if } n > 0, \\ \frac{2^k \pi^{s+k} |n|^{s+k-1}}{\Gamma(s)} \mathcal{W}_s(4\pi n v) \sum'_{c \in \mathbb{Z}} |2c|^{1-k-2s} H_c(\beta, 0, \gamma, n), & \text{if } n < 0, \end{cases}$$

where $H_c(\beta, m, \gamma, n)$ denotes the generalized Kloosterman sum

$$H_c(\beta, m, \gamma, n) = \frac{e^{-\pi i \operatorname{sgn}(c)\frac{k}{2}}}{|c|} \sum_{\substack{d \ (c)^* \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z}) / \Gamma_\infty}} \rho_{\beta\gamma}^{-1} \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} e\left(\frac{ma + nd}{c}\right).$$

Remark 2.5.4. In [BK03] and [Wil19] they consider the Eisenstein series with respect to the dual Weil representation. The calculation is analogous except that one obtains a generalized Kloosterman sum with respect to the dual Weil representation. In [BK01], [Sch06], [Sch18] the Fourier expansion is calculated for holomorphic Eisenstein series, i.e. for $s = 0$ and $k > 2$. In particular, [Sch18] shows that the Fourier coefficients of the holomorphic Eisenstein series are in a certain cyclotomic field $\mathbb{Q}(\zeta_{\varphi(N_\beta)})$.

Proof. Let $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$. The Fourier coefficients are given by

$$c_{k,\beta}(\gamma, n, s, v) = \frac{1}{2}\int_0^1 \sum_{(M,\phi)\in\tilde{\Gamma}_\infty\backslash \mathrm{Mp}_2(\mathbb{Z})} \langle (\mathrm{Im}(\tau)^s \mathfrak{e}_\beta)|_{k,L}(M,\phi), \mathfrak{e}_\gamma(nu)\rangle \mathrm{d}u.$$

Splitting the sum into the sum over $\mathrm{id}, Z, Z^2, Z^3 \in \tilde{\Gamma}_\infty\backslash \mathrm{Mp}_2(\mathbb{Z})$ and the sum over the elements $((\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}), \phi) \in \tilde{\Gamma}_\infty\backslash \mathrm{Mp}_2(\mathbb{Z})$ with $c \neq 0$ and using the invariance of $\mathrm{Im}(\tau)^s\mathfrak{e}_\beta$ under the action of Z^2 yields for the coefficient $c_{k,\beta}(\gamma, n, s, v)$

$$\delta_{0,n}(\delta_{\beta,\gamma} + \delta_{-\beta,\gamma})v^s + v^s \sum_{\substack{c\neq 0 \\ (\begin{smallmatrix} a & b \\ c & d\end{smallmatrix})\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}} \rho^{-1}_{\beta\gamma}\widetilde{\begin{pmatrix} a & b \\ c & d\end{pmatrix}} \int_0^1 \frac{e(-nu)}{(c\tau + d)^{k+s}(c\overline{\tau} + d)^s}\mathrm{d}u$$

$$= \delta_{0,n}(\delta_{\beta,\gamma} + \delta_{-\beta,\gamma})v^s + v^s \sum_{\substack{c\neq 0 \\ (\begin{smallmatrix} a & b \\ c & d\end{smallmatrix})\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty}} \rho^{-1}_{\beta\gamma}\widetilde{\begin{pmatrix} a & b \\ c & d\end{pmatrix}} \int_{-\infty}^\infty \frac{e(-nu)}{(c\tau + d)^{k+s}(c\overline{\tau} + d)^s}\mathrm{d}u.$$

Using $\sqrt{c\tau + d} = \mathrm{sgn}(c)\sqrt{c}\sqrt{\tau + \frac{d}{c}}$ we obtain

$$\begin{aligned}\int_{-\infty}^\infty \frac{e(-nu)}{(c\tau + d)^{k+s}(c\overline{\tau} + d)^s}\mathrm{d}u &= |c|^{-k-2s}\,\mathrm{sgn}(c)^k \int_{-\infty}^\infty \frac{e(-nu)}{(\tau + \frac{d}{c})^{k+s}(\overline{\tau} + \frac{d}{c})^s}\mathrm{d}u \\ &= |c|^{-k-2s}\,\mathrm{sgn}(c)^k e\left(\frac{nd}{c}\right) \int_{-\infty}^\infty \frac{e(-nu)}{\tau^{k+s}\overline{\tau}^s}\mathrm{d}u\end{aligned}$$

and thus the Fourier coefficient is given by $\delta_{0,n}(\delta_{\beta,\gamma} + \delta_{-\beta,\gamma})v^s$ plus

$$\begin{aligned}& v^s \int_{-\infty}^\infty \frac{e(-nu)}{\tau^{k+s}\overline{\tau}^s}\mathrm{d}u \sum_{\substack{c\neq 0 \\ (\begin{smallmatrix} a & b \\ c & d\end{smallmatrix})\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty}} \rho^{-1}_{\beta\gamma}\widetilde{\begin{pmatrix} a & b \\ c & d\end{pmatrix}} |c|^{-k-2s}\,\mathrm{sgn}(c)^k e\left(\frac{nd}{c}\right) \\ &= v^s \int_{-\infty}^\infty \frac{e(-nu)}{\tau^{k+s}\overline{\tau}^s}\mathrm{d}u \sum_{c\in\mathbb{Z}}{}' |c|^{1-k-2s}\,\mathrm{sgn}(c)^k e^{\pi i\,\mathrm{sgn}(c)\frac{k}{2}} H_c(\beta, 0, \gamma, n) \\ &= v^s \int_{-\infty}^\infty \frac{e(-nu)}{\tau^{k+s}\overline{\tau}^s}\mathrm{d}u \sum_{c\in\mathbb{Z}}{}' |c|^{1-k-2s} H_c(\beta, 0, \gamma, n).\end{aligned}$$

For $n \neq 0$ the integral can be calculated using [EMOT54, 3.2(12)] and correcting the sign as pointed out in [BK03] and is given by

$$\begin{aligned}\int_{-\infty}^\infty \frac{e(-nu)}{\tau^{k+s}\overline{\tau}^s}\mathrm{d}u &= i^{-k}\int_{-\infty}^\infty \frac{e^{-2\pi i n u}}{(v - iu)^{k+s}(v + iu)^s}\mathrm{d}u \\ &= 2^k\pi^{s+k} i^{-k}|n|^{s+k-1}v^{-s}\mathcal{W}_s(4\pi n v)\begin{cases}\Gamma(k+s)^{-1}, & \text{if } n > 0, \\ \Gamma(s)^{-1}, & \text{if } n < 0.\end{cases}\end{aligned}$$

For $n = 0$ we have according to [OLBC10, 5.12.8], [Fre90, Chapter III, Lemma 4.4]

$$\int_{-\infty}^\infty \frac{1}{\tau^{k+s}\overline{\tau}^s}\mathrm{d}u = 2^{2-k-2s}\pi i^{-k}\frac{\Gamma(k + 2s - 1)}{\Gamma(k+s)\Gamma(s)}v^{1-k-2s}.$$

This shows the assertion. □

Remark 2.5.5. Since ρ_L factors through a finite group, the coefficients $\rho_{\beta,\gamma}$ are universally bounded and there exists $C > 0$ with $H_c(\beta, 0, \gamma, n) < C$ for all $\gamma \in L'/L, n \in \mathbb{Z} + q(\gamma), c \in \mathbb{Z} \smallsetminus \{0\}$. Thus the L-functions occurring in the Fourier expansion converge absolutely for $\mathrm{Re}(s) > 1 - \frac{k}{2}$.

We will frequently use the notation

$$\begin{aligned} c_{k,\beta}(\gamma, 0, s, v) &= (\delta_{\beta,\gamma} + \delta_{-\beta,\gamma})v^s + c_{k,\beta}(\gamma, 0, s)v^{1-k-s}, \\ c_{k,\beta}(\gamma, n, s, v) &= c_{k,\beta}(\gamma, n, s)\mathcal{W}_s(4\pi n v). \end{aligned}$$

The coefficients have the following asymptotics.

Lemma 2.5.6. *We have*

$$c_{k,\beta}(\gamma, n, s, v) = O(e^{-2\pi|n|})$$

for $v \geq 1$ as $n \to \pm\infty$.

Proof. The previous remark implies that the sum is bounded by a constant multiple of $\zeta(2s + k - 1)$. Thus the asymptotics for the Whittaker function yields

$$c_{k,\beta}(\gamma, n, s, v) \leq C|n|^{s+\frac{k}{2}(1+\mathrm{sgn}(n))-1} v^{-\frac{k}{2}+\mathrm{sgn}(n)\frac{k}{2}} e^{-2\pi|n|v} \leq \tilde{C}e^{-2\pi|n|}.$$

This shows the assertion. □

The Fourier expansion shows that the Eisenstein series $E_{k,\beta}$ span a space of dimension $|\mathrm{Iso}(L'/L)/\{\pm 1\}|$ if κ is even. If κ is odd, the Eisenstein series for $\beta \in \mathrm{Iso}(L'/L)$ with $\beta = -\beta$ vanish identically. Next we want to investigate the Eisenstein series at $s = 0$. Therefore, we set $E_{k,\beta}(\tau) := E_{k,\beta}(\tau, 0)$ and $c_{k,\beta}(\gamma, n) := c_{k,\beta}(\gamma, n, 0)$.

Proposition 2.5.7. *Let $k > 2$. Then $E_{k,\beta}(\tau)$ exists and is a vector-valued holomorphic modular form. The Fourier expansion is of the form*

$$\mathfrak{e}_\beta + (-1)^\kappa \mathfrak{e}_{-\beta} + \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n>0}} c_{k,\beta}(\gamma, b) e_\gamma(n\tau).$$

Proof. For $k > 2$ the defining series converges at $s = 0$. This implies all the assertions. It also follows from the fact that the vector-valued Eisenstein series are $\mathrm{Mp}_2(\mathbb{Z})/\Gamma(N)$-translates of the scalar valued ones. □

Proposition 2.5.8. *Let $k = 2$. Then there exists an invariant vector $\mathfrak{w} \in \operatorname{Inv}(\mathbb{C}[L'/L])$ such that*

$$E_{k,\beta}(\tau) = \mathfrak{w} v^{-1} + \mathfrak{e}_\beta + (-1)^\kappa \mathfrak{e}_{-\beta} + \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n>0}} c_{k,\beta}(\gamma, b) e_\gamma(n\tau)$$

Proof. By [Miy06, Theorem 7.2.12], [DS05, Section 4.6], [Wil19, Section 6] the Fourier expansion is given by

$$E_{2,\mathfrak{v}}(\tau) = \mathfrak{w} v^{-1} + \mathfrak{e}_\beta + (-1)^\kappa \mathfrak{e}_{-\beta} + \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n>0}} c_{k,\mathfrak{v}}(\gamma, n) \mathfrak{e}_\gamma(n\tau)$$

for some vector $\mathfrak{w} \in \mathbb{C}[L'/L]$. Applying the lowering operator L_k to $E_{2,\beta}(\tau)$ yields $\mathfrak{w}$, which therefore is an invariant vector. □

Proposition 2.5.9. *Let $k = 1$. Then there exists a vector $\mathfrak{w} \in \mathbb{C}[L'/L]$ such that*

$$E_{k,\beta}(\tau) = \mathfrak{w} + \mathfrak{e}_\beta + (-1)^\kappa \mathfrak{e}_{-\beta} + \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n>0}} c_{k,\beta}(\gamma, b) e_\gamma(n\tau)$$

Proof. Again, see [Miy06, Theorem 7.2.13], [DS05, Section 4.8], [Wil19, Section 4]. □

Proposition 2.5.10. *Let $k = 0$. Then the residue at $s = 1$ of the Eisenstein series $E_{k,\beta}(\tau, s)$ is an invariant vector. If $\mathfrak{v} \in \operatorname{Inv}(\mathbb{C}[L'/L])$ is an invariant vector, then $\mathfrak{v} = E_{k,\mathfrak{v}}(\tau)$.*

Proof. The first assertion follows from the corresponding result in the scalar-valued case, see [Iwa95]. For the second assertion observe that (for arbitrary weight k) for the suitable normalized scalar-valued Eisenstein series $E_k(\tau, s)$ we have

$$\begin{aligned} E_k(\tau, s)\mathfrak{v} &= \frac{1}{2} \sum_{(M,\phi) \in \tilde{\Gamma}_\infty \backslash \operatorname{Mp}_2(\mathbb{Z})} \operatorname{Im}(\tau)^s |_{k,L} M \mathfrak{v} \\ &= \frac{1}{2} \sum_{(M,\phi) \in \tilde{\Gamma}_\infty \backslash \operatorname{Mp}_2(\mathbb{Z})} \phi(\tau)^{-2k} \operatorname{Im}(M\tau)^s \rho_L^{-1}(M, \phi) \mathfrak{v} \\ &= \frac{1}{2} \sum_{(M,\phi) \in \tilde{\Gamma}_\infty \backslash \operatorname{Mp}_2(\mathbb{Z})} (\operatorname{Im}(\tau)^s \mathfrak{v})|_{k,L}(M, \phi) = E_{k,\mathfrak{v}}(\tau, s). \end{aligned}$$

Now the left-hand-side evaluates to $\mathfrak{v}$ at $s = 0$. □

According to [Hej83, Page 372] we have the functional equation

$$E_{k,\mathfrak{v}}(\tau, s) = \frac{1}{2} \sum_{\alpha \in \operatorname{Iso}(L'/L)} c_{k,\mathfrak{v}}(\alpha, 0, s) E_{k,\alpha}(\tau, 1 - k - s).$$

Next we will investigate the behaviour with respect to certain sublattices. As above, let $z \in \mathrm{Iso}_0(L)$ and $z' \in L'$ with $(z, z') = 1$. Recall that

$$E^K_{k,\beta}(\tau, s, r, t) = \sum_{\gamma \in K'/K} \sum_{\substack{\lambda \in L_0'/L \\ \pi(\lambda)=\gamma}} \langle E_{k,\beta}(\tau, s), \mathfrak{e}_{\lambda+tz'}\rangle \mathfrak{e}_\gamma(-r(\lambda, z') + rtq(z')).$$

Lemma 2.5.11. *If* $\beta \notin L_0'/L$ *we have* $E^K_{k,\beta}(\tau, s, 0, 0) = 0$ *and if* $\beta \in L_0'/L$ *we have* $E^K_{k,\beta}(\tau, s, 0, 0) = E_{k,\pi(\beta)}(\tau, s)$.

Proof. A straight forward calculation yields

$$\begin{aligned} E^K_{k,\beta}(\tau, s, 0, 0) &= \sum_{\gamma \in K'/K} \sum_{\substack{\lambda \in L_0'/L \\ \pi(\lambda)=\gamma}} \langle E_{k,\beta}(\tau, s), \mathfrak{e}_\lambda\rangle \mathfrak{e}_\gamma \\ &= \sum_{(M,\phi) \in \tilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \phi(\tau)^{-2k} \operatorname{Im}(M\tau)^s \sum_{\gamma \in K'/K} \sum_{\substack{\lambda \in L_0'/L \\ \pi(\lambda)=\gamma}} \langle \rho_L(M)^{-1} \mathfrak{e}_\beta, \mathfrak{e}_\lambda\rangle \mathfrak{e}_\gamma \\ &= \sum_{(M,\phi) \in \tilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \phi(\tau)^{-2k} \operatorname{Im}(M\tau)^s \sum_{\gamma \in K'/K} \langle \mathfrak{e}_\beta, \rho_L(M) \sum_{m \in \mathbb{Z}/N_z\mathbb{Z}} \mathfrak{e}_{\gamma + \frac{mz}{N_z}}\rangle \mathfrak{e}_\gamma. \end{aligned}$$

Now apply Lemma 2.4.6 to obtain

$$\sum_{(M,\phi) \in \tilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \phi(\tau)^{-2k} \operatorname{Im}(M\tau)^s \sum_{\gamma \in K'/K} \langle \mathfrak{e}_\beta, (\rho_K(M)\mathfrak{e}_\gamma) \sum_{m \in \mathbb{Z}/N_z\mathbb{Z}} \mathfrak{e}_{\frac{mz}{N_z}}\rangle \mathfrak{e}_\gamma$$

and Lemma 2.4.8 to get

$$\sum_{(M,\phi) \in \tilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \phi(\tau)^{-2k} \operatorname{Im}(M\tau)^s \sum_{\gamma \in K'/K} \langle \mathfrak{e}^K_\beta, \rho_K(M)\mathfrak{e}_\gamma\rangle \mathfrak{e}_\gamma.$$

Now $\mathfrak{e}^K_\beta = 0$ if $\beta \notin L_0'/L$ and $\mathfrak{e}^K_\beta = \mathfrak{e}_{\pi(\beta)}$ if $\beta \in L_0'/L$. Hence we obtain in the latter case

$$\begin{aligned} &\sum_{(M,\phi) \in \tilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \phi(\tau)^{-2k} \operatorname{Im}(M\tau)^s \sum_{\gamma \in K'/K} \langle \mathfrak{e}_{\pi(\beta)}, \rho_K(M)\mathfrak{e}_\gamma\rangle \mathfrak{e}_\gamma \\ &= \sum_{(M,\phi) \in \tilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \phi(\tau)^{-2k} \operatorname{Im}(M\tau)^s \sum_{\gamma \in K'/K} \langle \rho_K(M)^{-1}\mathfrak{e}_{\pi(\beta)}, \mathfrak{e}_\gamma\rangle \mathfrak{e}_\gamma \\ &= \sum_{(M,\phi) \in \tilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \phi(\tau)^{-2k} \operatorname{Im}(M\tau)^s \rho_K(M)^{-1}\mathfrak{e}_{\pi(\beta)} = E_{k,\pi(\beta)}(\tau, s). \end{aligned}$$

This shows the assertion for $\operatorname{Re}(s) \gg 0$ and by analytic continuation everywhere. □

Remark 2.5.12. There is also a more abstract proof. In the first case one observes that $E_{k,\beta}(\tau, s, 0, 0)$ is a square-integrable function with Eigenvalue $s(s+k-1)$. But the eigenvalues of Δ_k on square-integrable functions can only be in a certain subset of the real numbers, see [Hej76]. This implies that it must vanish everywhere by analytic continuation. In the second case one considers the difference of both sides and argues analogously.

2.6 Siegel Theta Function

Let p be a polynomial on $\mathbb{R}^{(b^+,b^-)}$ which is homogeneous of degree κ^+ in the positive definite variables and homogeneous of degree κ^- in the negative definite variables. For an isometry $\nu : L \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}^{(b^+,b^-)}$ we write ν^+ and ν^- for the inverse image of $\mathbb{R}^{(b^+,0)}$ and $\mathbb{R}^{(0,b^-)}$. For an element $\lambda \in L \otimes_{\mathbb{Z}} \mathbb{R}$ we write $\lambda_{\nu^\pm}$ for the projection of λ onto $\nu^\pm$. The positive definite *majorant* associated to ν is then given by $q_\nu(\lambda) = q(\lambda_{\nu^+}) - q(\lambda_{\nu^-})$. For $\gamma \in L'/L, \tau \in \mathbb{H}$ and an isometry $\nu : L \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}^{(b^+,b^-)}$ we define the *Siegel theta function*

$$\begin{aligned}\theta_\gamma(\tau,\alpha,\beta,\nu,p) := v^{\frac{b^-}{2}+\kappa^-} \sum_{\lambda\in\gamma+L} &\exp\left(\frac{\Delta}{8\pi v}\right)(p)(\nu(\lambda+\beta)) \\ &\times e(\tau q((\lambda+\beta)_{\nu^+}) + \overline{\tau} q((\lambda+\beta)_{\nu^-}) - (\lambda+\beta/2,\alpha)),\end{aligned}$$

where Δ is the usual *Laplace operator* on $\mathbb{R}^{b^++b^-}$ and $\alpha, \beta \in L \otimes \mathbb{R}$. Moreover, we define

$$\begin{aligned}\Theta_L(\tau,\alpha,\beta,\nu,p) &:= \sum_{\gamma\in L'/L} \theta_\gamma(\tau,\alpha,\beta,\nu,p)\mathfrak{e}_\gamma \\ &= v^{\frac{b^-}{2}+\kappa^-} \sum_{\lambda\in L'} \exp\left(\frac{\Delta}{8\pi v}\right)(p)(\nu(\lambda+\beta)) \\ &\quad \times \mathfrak{e}_\lambda(\tau q((\lambda+\beta)_{\nu^+}) + \overline{\tau} q((\lambda+\beta)_{\nu^-}) - (\lambda+\beta/2,\alpha)).\end{aligned}$$

For $\alpha = \beta = 0$ we write

$$\begin{aligned}\theta_\gamma(\tau,\nu,p) &:= v^{\frac{b^-}{2}+\kappa^-} \sum_{\lambda\in\gamma+L} \exp\left(-\frac{\Delta}{8\pi v}\right)(p)(\nu(\lambda))\mathfrak{e}_\lambda(\tau q(\lambda_{\nu^+}) + \overline{\tau} q(\lambda_{\nu^-})) \\ &= v^{\frac{b^-}{2}+\kappa^-} \sum_{\lambda\in\gamma+L} \exp\left(-\frac{\Delta}{8\pi v}\right)(p)(\nu(\lambda))\mathfrak{e}_\lambda(ivq_\nu(\lambda) + uq(\lambda))\end{aligned}$$

and

$$\begin{aligned}\Theta_L(\tau,\nu,p) &:= \sum_{\gamma\in L'/L} \theta_\gamma(\tau,\nu,p)\mathfrak{e}_\gamma \\ &= v^{\frac{b^-}{2}+\kappa^-} \sum_{\lambda\in L'} \exp\left(-\frac{\Delta}{8\pi v}\right)(p)(\nu(\lambda))e(\tau q(\lambda_{\nu^+}) + \overline{\tau} q(\lambda_{\nu^-})) \\ &= v^{\frac{b^-}{2}+\kappa^-} \sum_{\lambda\in L'} \exp\left(-\frac{\Delta}{8\pi v}\right)(p)(\nu(\lambda))e(ivq_\nu(\lambda) + uq(\lambda)).\end{aligned}$$

This is the theta function in [Bor98] multiplied with $v^{\frac{b^-}{2}+\kappa^-}$. Using Poisson summation one obtains

Theorem 2.6.1 ([Bor98, Theorem 4.1]). *For $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z}), M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ we have*

$$\Theta_L(M\tau, a\alpha + b\beta, c\alpha + d\beta, \nu, p) = \phi(\tau)^{b^++2\kappa^+-b^--2\kappa^-}\rho_L(M,\phi)\Theta_L(\tau,\alpha,\beta,\nu,p).$$

In particular, for $\alpha = \beta = 0$, the theta function $\Theta_L(\tau,\nu,p)$ has weight $\frac{b^+}{2} + \kappa^+ - \frac{b^-}{2} - \kappa^-$.

Example 2.6.2. Let p be a polynomial that is invariant under $\mathrm{O}(b^+) \times \mathrm{O}(b^-)$. Then $\exp\left(\frac{\Delta}{8\pi v}\right)(p)(\nu(\lambda+\beta))$ only depends on ν^+ (respectively ν^-). Hence the theta function can be seen to be a function on the *Grassmannian*

$$\mathrm{Gr}^+(L \otimes_{\mathbb{Z}} \mathbb{R}) := \{\nu \subseteq L \otimes_{\mathbb{Z}} \mathbb{R} \mid \dim(\nu) = b^+, q|_\nu > 0\}.$$

Example 2.6.3. A particularly nice example is, if p is harmonic with respect to Δ, i.e. if $\Delta p = 0$. Then the theta function is given by

$$\Theta_L(\tau,\nu,p) = v^{\frac{b^-}{2}+\kappa^-} \sum_{\lambda \in L'} p(\nu(\lambda)) \mathfrak{e}_\lambda(ivq_\nu(\lambda) + uq(\lambda)).$$

Borcherds shows in [Bor98] that the theta function can be written as a Poincaré series. We will indicate the construction here. Therefore, let $z \in \mathrm{Iso}_0(L)$ and $z' \in L'$ with $(z,z') = 1$. Recall the notation from above. For an isometry $\nu : L \otimes \mathbb{R} \to \mathbb{R}^{(b^+,b^-)}$ we write $\omega^\pm$ for the orthogonal complement of $z_{\nu^\pm}$ in $\nu^\pm$. This yields a decomposition

$$L \otimes \mathbb{R} = \omega^+ \oplus \mathbb{R} z_{\nu^+} \oplus \omega^- \oplus \mathbb{R} z_{\nu^-}$$

and for $\lambda \in L \otimes \mathbb{R}$ we write $\lambda_{\omega^\pm}$ for the corresponding projections of λ onto $\omega^\pm$. Additionally the map

$$\omega : L \otimes \mathbb{R} \to \mathbb{R}^{(b^+,b^-)}, \lambda \mapsto \nu(\lambda_{\omega^+} + \lambda_{\omega^-})$$

can be seen to be an isometry $K \otimes \mathbb{R} \to \mathbb{R}^{(b^+-1,b^--1)}$ by restriction. For a homogeneous polynomial p of degree (κ^+,κ^-) on $\mathbb{R}^{(b^+,b^-)}$ we now define the homogeneous polynomials p_{ω,h^+,h^-} of degree $(\kappa^+ - h^+, \kappa^- - h^-)$ by

$$p(\nu(\lambda)) = \sum_{h^+,h^-} (\lambda, z_{\nu^+})^{h^+} (\lambda, z_{\nu^-})^{h^-} p_{\omega,h^+,h^-}(\omega(\lambda)).$$

We have the following

Theorem 2.6.4 ([Bor98, Theorem 5.2])**.** *Let* $\mu = -z' + \frac{z_{\nu^+}}{2z_{\nu^+}^2} + \frac{z_{\nu^-}}{2z_{\nu^-}^2} \in L \otimes \mathbb{R}$. *Then*

$$\begin{aligned}\theta_{\gamma+L}(\tau,\nu,p) = \frac{1}{\sqrt{2z_{\nu^+}^2}} &\sum_{\substack{c,d \in \mathbb{Z} \\ c \equiv (\gamma,z) \bmod N_z}} \sum_{h,h^+,h^-} \frac{h!\left(\frac{z_{\nu^+}}{\pi}\right)^h}{(-2i)^{h^++h^-}} \binom{h^+}{h}\binom{h^-}{h} \\ &\times \frac{(c\overline{\tau}+d)^{h^+-h}(c\tau+d)^{h^--h}}{v^{h^+-h}} e\left(-\frac{|c\tau+d|^2}{4ivz_{\nu^+}^2} - (\gamma,z')d + q(z')cd\right) \\ &\times \theta_{K+\pi(\gamma-cz')}(\tau, d\mu_K, -c\mu_K, \omega, p_{\omega,h^+,h^-}),\end{aligned}$$

where we used the notation $z_{\nu^+}^2 = (z_{\nu^+}, z_{\nu^+})$.

We will now rewrite this more explicitly as a Poincaré series.

Theorem 2.6.5. *We have*

$$\Theta_L(\tau,\nu,p) = \frac{1}{\sqrt{2z_{\nu^+}^2}}\sum_h h!\left(\frac{z_{\nu^+}^2}{4\pi}\right)^h \Theta_K(\tau,\omega,p_{\omega,h,h})\sum_{m\in\mathbb{Z}/N_z\mathbb{Z}}\mathfrak{e}_{\frac{mz}{N_z}}$$
$$+\frac{1}{\sqrt{2z_{\nu^+}^2}}\sum_{M\in\Gamma_\infty\backslash\Gamma}\sum_{h,h^+,h^-}\frac{h!\left(-\frac{z_{\nu^+}^2}{\pi}\right)^h}{(-2i)^{h^++h^-}}\binom{h^+}{h}\binom{h^-}{h}\sum_{n=1}^{\infty}n^{h^++h^--2h}$$
$$\times\frac{(c\tau+d)^{\frac{b^-}{2}+\kappa^--\frac{b^+}{2}-\kappa^+}}{\mathrm{Im}(M\tau)^{h^+-h}}\exp\left(-\frac{\pi n^2}{2\,\mathrm{Im}(M\tau)z_{\nu^+}^2}\right)$$
$$\times\rho_L(M)^{-1}\left(\Theta_K(M\tau,n\mu_K,0,\omega,p_{\omega,h^+,h^-})\sum_{m\in\mathbb{Z}/N\mathbb{Z}}\mathfrak{e}_{\frac{mz}{N}}\left(-\frac{mn}{N}\right)\right).$$

Proof. We have

$$\Theta_L(\tau,\nu,p) = \frac{1}{\sqrt{2z_{\nu^+}^2}}\sum_{c,d\in\mathbb{Z}}\sum_{\substack{\gamma\in L'/L\\ c\equiv(\gamma,z)\bmod N_z}}\sum_{h,h^+,h^-}\frac{h!\left(-\frac{z_{\nu^+}^2}{\pi}\right)^h}{(-2i)^{h^++h^-}}\binom{h^+}{h}\binom{h^-}{h}$$
$$\times\frac{(c\overline{\tau}+d)^{h^+-h}(c\tau+d)^{h^--h}}{v^{h^+-h}}e\left(-\frac{|c\tau+d|^2}{4ivz_{\nu^+}^2}-(\gamma,z')d+q(z')cd\right)$$
$$\times\theta_{K+\pi(\gamma-cz')}(\tau,d\mu_K,-c\mu_K,\omega,p_{\omega,h^+,h^-})\mathfrak{e}_\gamma.$$

We make the change $\gamma\mapsto\gamma+cz'$ and sum over coprime c,d to see that $\Theta_L(\tau,\nu,p)$ is given by

$$\frac{1}{\sqrt{2z_{\nu^+}^2}}\sum_h\frac{h!\left(-\frac{z_{\nu^+}^2}{\pi}\right)^h}{(-2i)^{2h}}\sum_{\gamma\in L_0'/L}\theta_{K+\pi(\gamma)}(\tau,\omega,p_{\omega,h,h})\mathfrak{e}_\gamma$$
$$+\frac{1}{\sqrt{2z_{\nu^+}^2}}\sum_{(c,d)=1}\sum_{h,h^+,h^-}\frac{h!\left(-\frac{z_{\nu^+}^2}{\pi}\right)^h}{(-2i)^{h^++h^-}}\binom{h^+}{h}\binom{h^-}{h}\sum_{n=1}^{\infty}n^{h^++h^--2h}$$
$$\times\frac{(c\overline{\tau}+d)^{h^+-h}(c\tau+d)^{h^--h}}{v^{h^+-h}}e\left(-\frac{n^2|c\tau+d|^2}{4ivz_{\nu^+}^2}\right)$$
$$\times\sum_{\gamma\in L_0'/L}\theta_{K+\pi(\gamma)}(\tau,nd\mu_K,-nc\mu_K,\omega,p_{\omega,h^+,h^-})\mathfrak{e}_{\gamma+ncz'}\left(-(\gamma,z')nd-q(z')n^2cd\right).$$

The elements $\gamma\in L_0'/L$ are represented by $\gamma+\frac{mz}{N_z}$ for $\gamma\in K'/K, m\in\mathbb{Z}/N_z\mathbb{Z}$. Hence we obtain using the transformation formula for the theta function

$$\sum_{\gamma\in L_0'/L}\theta_{K+\pi(\gamma)}(\tau,nd\mu_K,-nc\mu_K,\omega,p_{\omega,h^+,h^-})\mathfrak{e}_{\gamma+ncz'}\left(-(\gamma,z')nd-q(z')n^2cd\right)$$

$$= \sum_{\gamma\in K'/K} \theta_{K+\gamma}(\tau, nd\mu_K, -nc\mu_K, \omega, p_{\omega,h^+,h^-})\mathfrak{e}_\gamma \sum_{m\in\mathbb{Z}/N_z\mathbb{Z}} \mathfrak{e}_{\frac{mz}{N_z}+ncz'}\left(-\frac{mnd}{N_z} - q(z')n^2cd\right)$$

$$= \Theta_K(\tau, nd\mu_K, -nc\mu_K, \omega, p_{\omega,h^+,h^-}) \sum_{m\in\mathbb{Z}/N_z\mathbb{Z}} \mathfrak{e}_{\frac{mz}{N_z}+ncz'}\left(-\frac{mnd}{N_z} - q(z')n^2cd\right)$$

$$= (c\tau+d)^{-\frac{b^+-1}{2}-\kappa^++h^++\frac{b^--1}{2}+\kappa^--h^-}$$

$$\times \left(\rho_K^{-1}(M)\Theta_K(M\tau, n\mu_K, 0, \omega, p_{\omega,h^+,h^-})\right) \sum_{m\in\mathbb{Z}/N_z\mathbb{Z}} \mathfrak{e}_{\frac{mz}{N_z}+ncz'}\left(-\frac{mnd}{N_z} - q(z')n^2cd\right)$$

$$= (c\tau+d)^{-\frac{b^+-1}{2}-\kappa^++h^++\frac{b^--1}{2}+\kappa^--h^-}$$

$$\times \rho_L(M)^{-1}\left(\Theta_K(M\tau, n\mu_K, 0, \omega, p_{\omega,h^+,h^-}) \sum_{m\in\mathbb{Z}/N_z\mathbb{Z}} \mathfrak{e}_{\frac{mz}{N_z}}\left(-\frac{mn}{N_z}\right)\right),$$

where $M = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. This yields

$$\Theta_L(\tau,\nu,p) = \frac{1}{\sqrt{2z_{\nu^+}^2}} \sum_h h!\left(-\frac{z_{\nu^+}^2}{4\pi}\right)^h \Theta_K(\tau,\omega,p_{\omega,h,h}) \sum_{m\in\mathbb{Z}/N_z\mathbb{Z}} \mathfrak{e}_{\frac{mz}{N_z}}$$

$$+ \frac{1}{\sqrt{2z_{\nu^+}^2}} \sum_{M\in\Gamma_\infty\backslash\Gamma} \sum_{h,h^+,h^-} \frac{h!\left(-\frac{z_{\nu^+}^2}{\pi}\right)^h}{(-2i)^{h^++h^-}} \binom{h^+}{h}\binom{h^-}{h} \sum_{n=1}^\infty n^{h^++h^--2h}$$

$$\times \frac{(c\tau+d)^{\frac{b^-}{2}+\kappa^--\frac{b^+}{2}-\kappa^+}}{\mathrm{Im}(M\tau)^{h^+-h}} e\left(-\frac{n^2}{4i\,\mathrm{Im}(M\tau)z_{\nu^+}^2}\right)$$

$$\times \rho_L(M)^{-1}\left(\Theta_K(M\tau, n\mu_K, 0, \omega, p_{\omega,h^+,h^-}) \sum_{m\in\mathbb{Z}/N_z\mathbb{Z}} \mathfrak{e}_{\frac{mz}{N_z}}\left(-\frac{mn}{N_z}\right)\right).$$

□

Chapter 3

Orthogonal Upper Half Plane

3.1 Orthogonal Upper Half Plane

Let L be an even lattice of signature (b^+, b^-). Let $V = L \otimes_{\mathbb{Z}} \mathbb{Q}, V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$. Then $G := \mathrm{O}(V(\mathbb{R})) \simeq \mathrm{O}(b^+, b^-)$ and $K = \mathrm{O}(b^+) \times \mathrm{O}(b^-)$ is a maximal compact subgroup. The Riemannian symmetric space

$$G/K \simeq \mathrm{O}(b^+, b^-)/\,\mathrm{O}(b^+) \times \mathrm{O}(b^-) \simeq \mathrm{SO}^+(b^+, b^-)/\,\mathrm{SO}(b^+) \times \mathrm{SO}(b^-),$$

where $\mathrm{SO}^+(b^+, b^-)$ denotes the identity component of the special orthogonal group, is hermitian if and only if $b^+ = 2$ or $b^- = 2$ (since this is equivalent to K containing $\mathrm{SO}(2)$ in its centralizer). We will be mainly interested in this case. We will give different models of the symmetric space G/K.

Grassmannian Model

We consider the Grassmannian of b^+-dimensional subspaces of $V(\mathbb{R})$ on which the quadratic form is positive definite, i.e.

$$\mathrm{Gr}^+(V(\mathbb{R})) := \{\nu \subseteq V(\mathbb{R}) \mid \dim(\nu) = b^+, q|_\nu > 0\}.$$

We have an obvious action of G on $\mathrm{Gr}^+(V(\mathbb{R}))$ and if we fix some $\nu_0 \in \mathrm{Gr}^+(V(\mathbb{R}))$, then its stabilizer K is isomorphic to $\mathrm{O}(b^+) \times \mathrm{O}(b^-)$. By Witt's theorem the action of G is transitive and hence we have $\mathrm{Gr}^+(V(\mathbb{R})) \simeq G/K$. This description is called *Grassmannian* *model*. The disadvantage is that we do not see the complex structure in signature $(2,l)$. We want to mention that the assignment $\nu \to q_\nu$, where q_ν is the majorant associated to ν, defines an isomorphism from $\mathrm{Gr}^+(V(\mathbb{R}))$ to the space of Hermite's minimal majorants of q.

Projective Model

Assume now that $(b^+, b^-) = (2,l)$. Consider the projective space $\mathbb{P}(V(\mathbb{C}))$, where $V(\mathbb{C}) := V(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. This is a complex manifold of dimension $l+1$. Let

$$\mathcal{N} := \{[Z] \in \mathbb{P}(V(\mathbb{C})) \mid (Z,Z) = 0\}$$

and

$$\mathcal{K} := \{[Z] \in \mathcal{N} \mid (Z,\overline{Z}) > 0\}.$$

Observe that the relations in $\mathcal{N}$ and $\mathcal{K}$ are in fact independend of the choice of representative and hence $\mathcal{N}$ and $\mathcal{K}$ are well defined. Moreover, $\mathcal{N}$ is a closed submanifold of $\mathbb{P}(V(\mathbb{C}))$ of dimension l and $\mathcal{K}$ is an open subset of $\mathcal{N}$. It has two connected components interchanged by $Z \mapsto \overline{Z}$, one of which we denote by $\mathcal{K}^+$. Let $\mathrm{O}^+(V(\mathbb{R})) \subseteq \mathrm{O}(V(\mathbb{R}))$ fixing $\mathcal{K}^+$. We have the following

Lemma 3.1.1 ([BGHZ08, Lemma 2.17])**.** *The assignment* $[Z] = [X+iY] \mapsto \mathbb{R}X + \mathbb{R}Y$ *defines a real analytic isomorphism* $\mathcal{K}^+ \to \mathrm{Gr}^+(V(\mathbb{R}))$*. In particular, the Grassmannian* $\mathrm{Gr}^+(V(\mathbb{R}))$ *has a complex structure.*

Proof. A short calculation shows that for $Z = X + iY \in V(\mathbb{C})$ with $[Z] \in \mathcal{K}$ we have

$$\begin{aligned}(Z,Z) &= (X,X) - (Y,Y) + 2i(X,Y) = 0,\\ (Z,\overline{Z}) &= (X,X) + (Y,Y) > 0,\end{aligned}$$

which is equivalent to $X \perp Y$ and $(X,X) = (Y,Y) > 0$, i.e. X,Y span a 2-dimensional positive definite plane and hence $\mathbb{R}X + \mathbb{R}Y \in \mathrm{Gr}^+(V)$. Conversely, let $\nu \in \mathrm{Gr}^+(V)$ and choose a basis X,Y with $X \perp Y, (X,X) = (Y,Y)$. Then $Z = X + iY \in \mathcal{K}$ and by taking a suitable orientation of X,Y we obtain $Z = X + iY \in \mathcal{K}^+$. □

The complex manifold $\mathcal{K}^+$ is called the *projective model*. Write $\tilde{\mathcal{K}}^+$ for the preimage of $\mathcal{K}^+$ under the projection $V(\mathbb{C}) \to \mathbb{P}(V(\mathbb{C}))$.

Tube Domain Model

Again assume that L has signature $(2, l)$ and let $z \in V$ be isotropic, $z' \in V$ with $(z, z') = 1$. Let $\tilde{z} = z' - q(z')z$ and $K = L \cap z^\perp \cap z'^\perp$. Define

$$\mathbb{H}_l^\pm := \{Z = X + iY \in K \otimes_{\mathbb{Z}} \mathbb{C} \mid q(Y) > 0\}.$$

For $Z \in \mathbb{H}_l$ define

$$Z_L := Z - q(Z)z + \tilde{z},$$

hence

$$X_L = X + (q(Y) - q(X))z + \tilde{z}, \quad Y_L = Y - (X, Y)z.$$

Then we have

Lemma 3.1.2 ([BGHZ08, Lemma 2.18])**.** *The assignment*

$$\psi : \mathbb{H}_l^\pm \to \mathcal{K}, \quad Z \mapsto [Z_L]$$

defines a biholomorphic map.

Proof. One easily checks that $X_L \perp Y_L$ and $(X_L, X_L) = (Y_L, Y_L)$, hence $[Z_L] \in \mathcal{K}$. Conversely let $[Z_L] = [X_L + iY_L] \in \mathcal{K}$. Then X_L, Y_L span a 2-dimensional positive definite subspace. Hence we have $(Z_L, z) \neq 0$ and we can choose a representative of $[Z_L]$ of the form $Z_L = Z + bz + \tilde{z}$. The assumption $q(Z_L) = 0$ implies $b = -q(Z)$ and $(Z_L, \overline{Z_L}) > 0$ implies $q(Y) > 0$. □

The component of $\mathbb{H}_l^\pm$ that is mapped to $\mathcal{K}^+$ is denoted by $\mathbb{H}_l$ and we call it *orthogonal upper half-plane*.

Remark 3.1.3. By setting $i\mathcal{C} = \mathbb{H}_l \cap i(K \otimes_{\mathbb{Z}} \mathbb{R})$ we see that $\mathbb{H}_l = K \otimes_{\mathbb{Z}} \mathbb{R} + i\mathcal{C}$ is a *tube domain* and $\mathcal{C}$ is a connected component of

$$\mathcal{C}^\pm := \{Y \in K \otimes_{\mathbb{Z}} \mathbb{R} | q(Y) > 0\}.$$

The biholomorphic map $\psi : \mathbb{H}_l \to \mathcal{K}^+$ now induces an action of $\mathrm{O}^+(V(\mathbb{R}))$ on $\mathbb{H}_l$. The action is given in the following way. Let $Z \in \mathbb{H}_l$ and $\sigma \in \mathrm{O}^+(V(\mathbb{R}))$. Then

$$\sigma(Z_L) = Z' - az + b\tilde{z}$$

for some $a, b \in \mathbb{R}$ and $Z' \in K \otimes \mathbb{R}$ and thus $\sigma Z = b^{-1}Z'$ with $b = (\sigma(Z_L), z)$.

Definition 3.1.4. The function

$$j : \mathrm{O}^+(V(\mathbb{R})) \times \mathbb{H}_l \to \mathbb{C}^\times, \quad j(\sigma, Z) := (\sigma(Z_L), z)$$

is called *factor of automorphy*.

Remark 3.1.5. We have

$$j(\sigma, Z)(\sigma Z)_L = \sigma(Z_L)$$

and the cocycle relation

$$j(\sigma_1\sigma_2, Z) = j(\sigma_1, \sigma_2 Z) j(\sigma_2, Z).$$

The following lemma is a generalisation of the classical formula $\mathrm{Im}(M\tau) = \frac{\mathrm{Im}(\tau)}{|j(M,\tau)|^2}$.

Lemma 3.1.6 ([Bru02, Lemma 3.20])**.** *For $Z \in \mathbb{H}_l$ and $\sigma \in \mathrm{O}^+(V(\mathbb{R}))$ we have*

$$q(\mathrm{Im}(\sigma Z)) = \frac{q(\mathrm{Im}(Z))}{|j(\sigma, Z)|^2}.$$

Proof. Using $4q(\mathrm{Im}(Z)) = (Z_L, \overline{Z}_L)$ we have

$$\begin{aligned} q(\mathrm{Im}(\sigma Z)) &= \frac{1}{4}(\sigma Z, \overline{\sigma Z}) \\ &= \frac{1}{4}\left(\frac{\sigma(Z_L)}{(\sigma(Z_L), z)}, \frac{\overline{\sigma(Z_L)}}{(\overline{\sigma(Z_L)}, z)}\right) \\ &= \frac{1}{4|j(\sigma, Z)|^2}(Z_L, \overline{Z}_L) = \frac{q(\mathrm{Im}(Z))}{|j(\sigma, Z)|^2}. \end{aligned}$$

□

3.2 Orthogonal Modular Forms

As above let L be an even lattice of signature $(2, l)$, $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$, $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$ and $\Gamma \subseteq \Gamma(L)$ a congruence subgroup.

Definition 3.2.1. A function $F : \tilde{\mathcal{K}}^+ \to \mathbb{C}$ is called *modular form* of weight $\kappa \in \mathbb{Z}$ with respect to Γ if it satisfies

(i) $F(tZ) = t^{-\kappa} F(Z)$ for all $t \in \mathbb{C}^\times$.

(ii) $F(\sigma Z) = F(Z)$ for all $\sigma \in \Gamma$.

Again, let $z \in \mathrm{Iso}_0(L), z' \in L', (z, z') = 1$. Then we have

Lemma 3.2.2. *For a modular form $F : \tilde{\mathcal{K}}^+ \to \mathbb{C}$ of weight $\kappa \in \mathbb{Z}$ with respect to Γ define*

$$F_z : \mathbb{H}_l \to \mathbb{C}, \quad F_z(Z) := F(Z_L) = F(Z - q(Z)z + \tilde{z}).$$

Then F_z satisfies

$$F_z(\sigma Z) = j(\sigma, Z)^\kappa F_z(Z)$$

for all $\sigma \in \Gamma$ and we have a bijective correspondence between modular forms and functions with this transformation property.

Proof. We have

$$F_z(\sigma Z) = F((\sigma Z)_L) = F(j(\sigma, Z)^{-1}\sigma(Z_L)) = j(\sigma, Z)^\kappa F(Z_L) = j(\sigma, Z)^\kappa F_z(Z).$$

Conversely, let $F_z : \mathbb{H}_l \to \mathbb{C}$ be a function with the above transformation property. Every element in $\tilde{\mathcal{K}}^+$ can be uniquely written as tZ_L for some $t \in \mathbb{C}^\times, Z \in \mathbb{H}_l$. Now

$$F(tZ_L) := t^{-\kappa} F_z(Z)$$

defines a modular form of weight $\kappa \in \mathbb{Z}$ with respect to Γ. □

For $F_z : \mathbb{H}_l \to \mathbb{C}$ and $\sigma \in \mathrm{O}^+(V)$ we define the weight κ slash operator $F_z|_\kappa \sigma$ by

$$(F_z|_\kappa \sigma)(Z) := j(\sigma, Z)^{-\kappa} F_z(\sigma Z).$$

Then the modular forms of weight κ on $\mathbb{H}_l$ are exactly the functions that are invariant under the slash operator $|_\kappa \sigma$ for all $\sigma \in \Gamma$.

Definition 3.2.3. Let $u \in V(\mathbb{R})$ be an isotropic vector and $v \in V(\mathbb{R})$ orthogonal to u. The *Eichler transformation* is defined by

$$E((u, v)(\lambda) = \lambda - (\lambda, u)v + (\lambda, v)u - q(v)(\lambda, u)u$$

for $\lambda \in V(\mathbb{R})$. It is an element of $\mathrm{O}^+(V(\mathbb{R}))$ and if $u, v \in L$ it is contained in the discriminant kernel $\Gamma(L)$.

Remark 3.2.4. We have

$$E(u,v_1)E(u,v_2) = E(u,v_1+v_2)$$

and if we write $K = L \cap z^{\perp} \cap z'^{\perp}, W = K \otimes \mathbb{Q}$, then for $v \in W, Z \in \mathbb{H}_l$ we have

$$E(z,-v)Z = Z + v.$$

In particular, the translations by elements in K are in the discriminant kernel $\Gamma(L)$.

If F_z is a modular form that is holomorphic on $\mathbb{H}_l$ with respect to $\Gamma(L)$, it is invariant under translations of K and thus has a Fourier expansion of the form

$$F_z(Z) = \sum_{\lambda \in K'} a_z(\lambda) e(\lambda, Z),$$

where $e(\lambda, Z) := e((\lambda, Z)) = e^{2\pi i(\lambda, Z)}$. For general congruence subgroups Γ one has to consider appropriate sublattices $K_0 \subseteq K$.

Definition 3.2.5. We say that a modular form F is a *holomorphic modular form* if F is holomorphic and for all choices of z, z' we have that $a_z(\lambda) \neq 0$ implies $\lambda \in \overline{\mathcal{C}}$. We write $\mathrm{M}_\kappa(\Gamma)$ for the space of holomorphic modular forms. We call a holomorphic modular form F of weight κ for some congruence subgroup Γ a *cusp form* if $a_z(\lambda) \neq 0$ implies $\lambda \in \mathcal{C}$ for all choices of z, z'. We write $\mathrm{S}_\kappa(\Gamma)$ for the space of cusp forms. We call a holomorphic modular form F *singular*, if for all choices of z, z' we have that $a_z(\lambda) \neq 0$ implies $q(\lambda) = 0$.

It turns out that every modular form that is holomorphic on $\mathbb{H}_l$ is already a holomorphic modular form if $l \geq 3$.

Proposition 3.2.6 (Koecher Principle, [Bru02, Proposition 4.15]). *Let $l \geq 3$ and $F : \mathbb{H}_l \to \mathbb{C}$ be a holomorphic function satisfying*

(i) $F(Z+k) = F(Z)$ *for* $k \in K$,

(ii) $F(\sigma Z) = F(Z)$ *for* $\sigma \in \Gamma(K)$, *where* $\Gamma(K)$ *is considered as a subgroup of* $\Gamma(L)$.

Then F has a Fourier expansion of the form

$$\sum_{\lambda \in K' \cap \overline{\mathcal{C}}} a(\lambda) e(\lambda, Z).$$

In particular, if F is a modular form that is holomorphic on $\mathbb{H}_l$, then it is already a holomorphic modular form.

Remark 3.2.7. In fact it suffices that l is larger then the Witt-rank of L (which is automatically fulfilled for $l \geq 3$).

We want to mention another result that plays an important role for us.

Theorem 3.2.8 (Singular Weights, [Bun01, Satz 3.1.19])**.** *Assume that L has Witt-rank 2 (which is automatically true for $l \geq 5$). Let $F : \mathbb{H}_l \to \mathbb{C}$ be a holomorphic modular form of weight $\kappa \in \mathbb{Z}$ with respect to $\Gamma(L)$. Then $\kappa \geq \frac{l}{2} - 1$. If f has weight $\kappa = \frac{l}{2} - 1$, so called* singular weight*, then f is a singular. In particular, there are no cusp forms of singular weight, i.e. $\mathrm{S}_\kappa(\Gamma(L)) = \{0\}$. Moreover, every singular modular form has singular weight.*

3.3 Hermitian Geometry for Signature $(2, l)$

As mentioned earlier, the complex manifold $\mathbb{H}_l$ carries the structure of a hermitian symmetric space. We will make this more precise now. Again, let $z \in \mathrm{Iso}_0(L)$ and $z' \in L'$ with $(z, z') = 1$. Write $W = z^\perp \cap z'^\perp \subseteq L \otimes \mathbb{Q}$ and let $b_1, \ldots, b_l$ be a basis of $W(\mathbb{R}) = W \otimes \mathbb{R}$ such that

$$q(y_1 b_1 + \ldots + y_l b_l) = y_1^2 - y_2^2 - \ldots - y_l^2.$$

If $Z = z_1 b_1 + \ldots + z_l b_l \in W(\mathbb{C})$, we write $Z = (z_1, \ldots, z_l)$ and similarly for $X, Y \in W(\mathbb{R})$. The connected component of $\mathbb{H}_l^\pm$ can be chosen such that $\mathbb{H}_l = W(\mathbb{R}) + i\mathcal{C}$ with

$$\mathcal{C} = \{Y \in W(\mathbb{R}) | q(Y) > 0, y_1 > 0\}.$$

The $(1,1)$-form

$$\omega = -\frac{i}{2}\partial\overline{\partial}\log(q(Y)))$$

is invariant under $\mathrm{O}^+(V(\mathbb{R}))$ by Lemma 3.1.6. Write

$$\omega = -\frac{i}{2}\partial\overline{\partial}\log(q(Y))) = \frac{i}{2}\sum_{i,j} h_{ij}(Y)\mathrm{d}z_i \wedge \mathrm{d}\overline{z}_j,$$

where $h(Z) = h(Y) = (h_{ij}(Y))_{1\leq i,j\leq l}$ is the associated hermitian form given by

$$h(Y) = \frac{1}{q(Y)^2}\begin{pmatrix} y_1^2 & -y_1y_2 & -y_1y_3 & \cdots & -y_1y_l \\ -y_1y_2 & y_2^2 & y_2y_3 & \cdots & y_2y_l \\ -y_1y_3 & y_2y_3 & y_3^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & y_{l-1}y_l \\ -y_1y_l & y_2y_l & \cdots & y_{l-1}y_l & y_l^2 \end{pmatrix} + \frac{1}{2q(Y)}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

and its inverse is given by

$$h^{-1}(Y) = 4\begin{pmatrix} y_1^2 & y_1y_2 & y_1y_3 & \cdots & y_1y_l \\ y_1y_2 & y_2^2 & y_2y_3 & \cdots & y_2y_l \\ y_1y_3 & y_2y_3 & y_3^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & y_{l-1}y_l \\ y_1y_l & y_2y_l & \cdots & y_{l-1}y_l & y_l^2 \end{pmatrix} + 2q(Y)\begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

We show that the hermitian form is positive definite. Since $\mathrm{O}^+(V(\mathbb{R}))$ acts transitively, it is sufficient to check this on the special point $(i, 0, \ldots, 0)$. In this case we have

$$h(Y) = \frac{1}{2}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

which is obviously positive definite. Since ω is obviously closed, it defines an $\mathrm{O}^+(V(\mathbb{R}))$-invariant Kähler metric on $\mathbb{H}_l$. The determinant of $h(Y)$ is given by $\det(h) = \frac{1}{2^l q(Y)^l} = \frac{1}{Y^{2l}}$ and hence the volume form of the underlying Riemannian metric g is given by

$$\begin{aligned} \omega_g &= \frac{1}{(2iY^2)^l} \mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \ldots \wedge \mathrm{d}z_l \wedge \mathrm{d}\overline{z}_l \\ &= \frac{1}{Y^{2l}} \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}x_l \wedge \mathrm{d}y_l. \end{aligned}$$

We write

$$\widehat{\mathrm{d}z_i} = \mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \ldots \wedge \mathrm{d}\overline{z_{i-1}} \wedge \mathrm{d}\overline{z}_i \wedge \ldots \wedge \mathrm{d}z_l \wedge \mathrm{d}\overline{z}_l$$

(i.e. the $\mathrm{d}z_i$ term is missing) and similarly for $\widehat{\mathrm{d}\overline{z}_i}$. Then we have

$$\mathrm{d}z_i \wedge \widehat{\mathrm{d}z_i} = (4iq(Y))^l \omega_g, \quad \mathrm{d}\overline{z}_i \wedge \widehat{\mathrm{d}\overline{z}_i} = -(4iq(Y))^l \omega_g.$$

The Hodge-$\overline{*}$-operator is defined by the equality

$$\alpha \wedge \overline{*}\beta = \langle \alpha, \beta \rangle \omega_g,$$

and thus we have

$$\overline{*}\mathrm{d}\overline{z}_i = -\frac{1}{2}\sum_{j=1}^{l} \frac{h^{ji}(Y)}{(4iq(Y))^l} \widehat{\mathrm{d}\overline{z}_j}.$$

Now let Γ be a torsion-free congruence subgroup. Then the quotient $\Gamma\backslash\mathbb{H}_l$ is a hermitian manifold, where the metric comes from the hermitian metric of $\mathbb{H}_l$. Modular forms of weight κ are the global sections of a hermitian line bundle $\mathcal{L}_\kappa$ on $\Gamma\backslash\mathbb{H}_l$, where the hermitian metric is given by the Petersson metric $F(Z)\overline{G(Z)}q(Y)^\kappa$. If $U \subseteq \Gamma\backslash\mathbb{H}_l$ is open, the sections $\mathcal{L}_\kappa(U)$ are the holomorphic functions $F : \pi^{-1}(U) \to \mathbb{C}$ satisfying the transformation law of modular forms of weight κ. Here $\pi : \mathbb{H}_l \to \Gamma\backslash\mathbb{H}_l$ is the projection. The dual bundle $\mathcal{L}_\kappa^*$ of $\mathcal{L}_\kappa$ can be identified using the hermitian metric with the line bundle $\mathcal{L}_{-\kappa}$ of modular forms of weight $-\kappa$ and the mapping $F(Z) \mapsto q(Y)^\kappa\overline{F(Z)}$ defines an anti-linear bundle isomorphism. This yields a Hodge-$\overline{*}$-operator

$$\overline{*}_\kappa : \mathcal{A}^{p,q}(\Gamma\backslash\mathbb{H}_l, \mathcal{L}_\kappa) \to \mathcal{A}^{n-p,n-q}(\Gamma\backslash\mathbb{H}_l, \mathcal{L}_{-\kappa}), \quad \overline{*}_\kappa(\phi \otimes F) := (\overline{*}\phi) \otimes (q(Y)^\kappa\overline{F(Z)}).$$

The weight κ Laplace operator is then defined by

$$\Omega_\kappa = \overline{*}_{-\kappa}\overline{\partial}\overline{*}_\kappa\overline{\partial} + \overline{\partial}\overline{*}_{-\kappa}\overline{\partial}\overline{*}_\kappa.$$

In particular, on 0-form we have

$$\Omega_\kappa = \overline{*}_{-\kappa}\overline{\partial}\overline{*}_\kappa\overline{\partial}.$$

Theorem 3.3.1. *For a modular form F of weight κ the weight κ Laplace operator is given by*

$$\begin{aligned}\Omega_\kappa F(Z) &= \frac{q(Y)^{l-\kappa}}{2}\sum_{j=1}^{l}\sum_{i=1}^{l}\frac{\partial}{\partial z_j}\left(h^{ji}(Y)q(Y)^{\kappa-l}\frac{\partial F(Z)}{\partial \overline{z}_i}\right)\\ &= 2\sum_{j=1}^{l}\sum_{i=1}^{l}y_iy_j\frac{\partial^2 F(Z)}{\partial z_j\partial\overline{z}_i} - q(Y)\left(\frac{\partial^2 F(Z)}{\partial z_1\partial\overline{z}_1} - \sum_{i=2}^{l}\frac{\partial^2 F(Z)}{\partial z_i\partial\overline{z}_i}\right) - i\kappa\sum_{i=1}^{l}y_i\frac{\partial F(Z)}{\partial\overline{z}_i}.\end{aligned}$$

Proof. We have

$$\begin{aligned}\overline{*}_\kappa\overline{\partial}F(Z) &= \sum_{i=1}^{l}q(Y)^k\frac{\partial\overline{F(Z)}}{\partial z_i}\overline{*}\mathrm{d}\overline{z}_i\\ &= -\frac{1}{2(4i)^l}\sum_{j=1}^{l}\sum_{i=1}^{l}h^{ji}(Y)q(Y)^{\kappa-l}\frac{\partial\overline{F(Z)}}{\partial z_i}\widehat{\mathrm{d}\overline{z}_j}.\end{aligned}$$

Applying $\overline{*}_{-\kappa}\overline{\partial}$ yields

$$\begin{aligned}\Omega_\kappa F(Z) &= -\frac{1}{2(4i)^l}\sum_{j=1}^{l}\sum_{i=1}^{l}\overline{*}_{-\kappa}\overline{\partial}\left(h^{ji}(Y)q(Y)^{\kappa-l}\frac{\partial\overline{F(Z)}}{\partial z_i}\widehat{\mathrm{d}\overline{z}_j}\right)\\ &= -\frac{1}{2(4i)^l}\sum_{j=1}^{l}\sum_{i=1}^{l}\overline{*}_{-\kappa}\frac{\partial}{\partial\overline{z}_j}\left(h^{ji}(Y)q(Y)^{\kappa-l}\frac{\partial\overline{F(Z)}}{\partial z_i}\right)\mathrm{d}\overline{z}_j\wedge\widehat{\mathrm{d}\overline{z}_j}\\ &= \frac{q(Y)^l}{2}\sum_{j=1}^{l}\sum_{i=1}^{l}\overline{*}_{-\kappa}\frac{\partial}{\partial\overline{z}_j}\left(h^{ji}(Y)q(Y)^{\kappa-l}\frac{\partial\overline{F(Z)}}{\partial z_i}\right)\omega_g\end{aligned}$$

$$= \frac{q(Y)^{l-\kappa}}{2} \sum_{j=1}^{l} \sum_{i=1}^{l} \frac{\partial}{\partial z_j} \left(h^{ji}(Y) q(Y)^{\kappa-l} \frac{\partial F(Z)}{\partial \overline{z}_i} \right) \omega_g$$
$$= \frac{1}{2} \sum_{j=1}^{l} \sum_{i=1}^{l} h^{ij}(Y) \frac{\partial^2 F(Z)}{\partial z_j \partial \overline{z}_i} + \frac{q(Y)^{l-\kappa}}{2} \sum_{j=1}^{l} \sum_{i=1}^{l} \left(\frac{\partial}{\partial z_j} h^{ij}(Y) q(Y)^{\kappa-l} \right) \frac{\partial F(Z)}{\partial \overline{z}_i}.$$

A short calculation yields

$$\frac{\partial q(Y)}{\partial y_1} = 2y_1, \quad \frac{\partial q(Y)}{\partial y_j} = -2y_j, j > 1, \quad \frac{\partial h^{ij}(Y)}{\partial y_1} = 4y_i.$$

Thus, for $i = 1$ we have

$$\sum_{j=1}^{l} \frac{\partial}{\partial z_j} h^{ij}(Y) q(Y)^{\kappa-l}$$
$$= -\frac{i}{2} \sum_{j=1}^{l} \left(q(Y)^{\kappa-l} \frac{\partial h^{ij}(Y)}{\partial y_j} + (\kappa - l) h^{ij}(Y) q(Y)^{\kappa-l-1} \frac{\partial q(Y)}{\partial y_j} \right)$$
$$= -\frac{i}{2} \left(4q(Y)^{\kappa-l} l y_1 + 2(\kappa - l) q(Y)^{\kappa-l-1} \left((4y_1^2 - 2q(Y)) y_1 - 4y_1 \sum_{j=2}^{l} y_j^2 \right) \right)$$
$$= -\frac{i}{2} \left(4q(Y)^{\kappa-l} l y_1 + 2y_1(\kappa - l) q(Y)^{\kappa-l-1} \left(4y_1^2 - 2q(Y) + 4 \sum_{j=2}^{l} y_j^2 \right) \right)$$
$$= -2i\kappa q(Y)^{\kappa-l} y_1$$

and similarly for $i > 1$

$$\sum_{j=1}^{l} \frac{\partial}{\partial z_j} h^{ij}(Y) q(Y)^{\kappa-l}$$
$$= -\frac{i}{2} \left(4q(Y)^{\kappa-l} l y_i + 2(\kappa - l) q(Y)^{\kappa-l-1} \left(4y_i y_1^2 - 4y_i \sum_{j=2}^{l} y_j^2 - 2q(Y) y_i \right) \right)$$
$$= -2i\kappa q(Y)^{\kappa-l} y_i$$

and hence

$$\Omega_\kappa F(Z) = \frac{1}{2} \sum_{j=1}^{l} \sum_{i=1}^{l} h^{ij}(Y) \frac{\partial^2 F(Z)}{\partial z_j \partial \overline{z}_i} - i\kappa \sum_{i=1}^{l} y_i \frac{\partial F(Z)}{\partial \overline{z}_i}$$
$$= 2 \sum_{j=1}^{l} \sum_{i=1}^{l} y_i y_j \frac{\partial^2 F(Z)}{\partial z_j \partial \overline{z}_i} - q(Y) \left(\frac{\partial^2 F(Z)}{\partial z_1 \partial \overline{z}_1} - \sum_{i=2}^{l} \frac{\partial^2 F(Z)}{\partial z_i \partial \overline{z}_i} \right) - i\kappa \sum_{i=1}^{l} y_i \frac{\partial F(Z)}{\partial \overline{z}_i}.$$

□

Remark 3.3.2. There is an ad hoc definition of the weight (m, n) Laplace operator given by [Zem17] which coincides with $4\Omega_\kappa$ for $m = \kappa, n = 0$. The weight κ Laplace operator commutes with the weight κ slash operator and satisfies

$$\Omega_\kappa q(Y)^s = s(s + \kappa - l/2) q(Y)^s.$$

There are also weight raising and weight lowering operators in [Zem17] defined by

$$R_\kappa := \frac{\partial^2}{\partial z_1^2} - \sum_{i=2}^{l} \frac{\partial^2}{\partial z_i^2} - \frac{i(\kappa+1-\frac{l}{2})}{q(Y)} \sum_{i=1}^{l} y_i \frac{\partial}{\partial z_i} - \frac{\kappa(\kappa+1-\frac{l}{2})}{2q(Y)},$$
$$L_\kappa := q(Y)^2 \frac{\partial^2}{\partial \overline{z}_1^2} - q(Y)^2 \sum_{i=2}^{l} \frac{\partial^2}{\partial \overline{z}_i^2} + iq(Y)\left(1-\frac{l}{2}\right) \sum_{i=1}^{l} y_i \frac{\partial}{\partial \overline{z}_i}.$$

They commute with the slash operators in the sense that

$$(R_\kappa F)|_{\kappa+2}\sigma = R_\kappa(F|_\kappa \sigma)$$
$$(L_\kappa F)|_{\kappa-2}\sigma = L_\kappa(F|_\kappa \sigma)$$

for every twice differentiable function $F : \mathbb{H}_l \to \mathbb{C}$ and every $\sigma \in \mathrm{O}^+(V)$. Moreover, a short calculation shows

$$R_\kappa q(Y)^s = (s+\kappa)\left(\frac{l}{2}-1-\kappa-s\right) q(Y)^{s-1},$$
$$L_\kappa q(Y)^s = s\left(\frac{l}{2}-1-s\right) q(Y)^{s+1}.$$

Proposition 3.3.3 ([Che73, Section 3, Example (B)]). *Let M be a complete connected hermitian manifold and let E be an hermitian vector bundle with corresponding Laplace operator $\Delta_E = \overline{\star}_{E^*}\overline{\partial}\overline{\star}_E\overline{\partial}$. If u, v are smooth square integrable sections of E such that $\Delta_E u, \Delta_E v$ are also square integrable. Then we have*

$$(\Delta_E u, v) = (u, \Delta_E, v).$$

Remark 3.3.4. The hermitian inner product for global sections F, G of $\mathcal{L}_\kappa$ is now given by

$$\int_{\Gamma\backslash\mathbb{H}_l} F(Z)\overline{G(Z)} q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l},$$

if the integral exists. We will see in the next section that this is for example the case if $\kappa = \frac{l}{2} - 1$ and F, G are holomorphic modular forms or for arbitrary weight if one of them is a cusp form. By Proposition 3.3.3 we have

$$\int_{\Gamma\backslash\mathbb{H}_l} \Omega_\kappa F(Z)\overline{G(Z)} q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l} = \int_{\Gamma\backslash\mathbb{H}_l} F(Z)\overline{\Omega_\kappa G(Z)} q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}$$

if F, G are square-integrable global sections of $\mathcal{L}_\kappa$.

3.4 Siegel Domains and Growth Estimates

We will now derive certain growth conditions for square-integrability of modular form. Therefore, let $z \in \mathrm{Iso}_0(L), z' \in L', (z, z') = 1, K = z^\perp \cap z'^\perp \cap L$ and $d \in \mathrm{Iso}_0(K), d' \in K'$

with $(d, d') = 1$. We write $D = K \cap d^\perp \cap d'^\perp$. Moreover, we choose the component $\mathbb{H}_l$ such that we have $(Y, d) > 0$ for $X + iY \in \mathbb{H}_l$. For $Z = X + iY \in \mathbb{H}_l$ we write $Z = z_1 \tilde{d} + z_2 d + Z_D$, where $\tilde{d} = d' - q(d')d$.

Definition 3.4.1. For $t > 0$ we define the *Siegel domain* $\mathcal{S}_t$ as the set of $Z = X + iY \in \mathbb{H}_l$ satisfying

$$\begin{aligned} x_1^2 + x_2^2 + |q(X_D)| &< t^2, \\ 1/t &< y_1, \\ y_1^2 &< t^2 q(Y), \\ |q(Y_D)| &< t^2 y_1^2. \end{aligned}$$

The set of $Y, q(Y) > 0, (Y, d) > 0$ satisfying the last three inequalities is denoted by $\mathcal{R}_t$.

We have the following

Proposition 3.4.2 ([Bru02, Proposition 4.10])**.** *Let $\Gamma \subseteq \Gamma(L)$ be a subgroup of finite index.*

(i) For any $t > 0$ and any $g \in \mathrm{O}^+(V)$ the set

$$\{\sigma \in \Gamma \mid \sigma g \mathcal{S}_t \cap \mathcal{S}_t \neq \emptyset\}$$

is finite.

(ii) There exists a $t > 0$ and finitely many $g_1, \dots, g_n \in \mathrm{O}^+(V)$ such that for

$$\mathcal{S} = g_1 \mathcal{S}_t \cup \ldots \cup g_n \mathcal{S}_t$$

we have $\Gamma \mathcal{S} = \mathbb{H}_l$.

The invariant volume element on $\mathbb{H}_l$ is given by

$$\frac{\mathrm{d}X \mathrm{d}Y}{q(Y)^l}.$$

The previous proposition means, in particular, that for a Γ-invariant measurable function $F : \mathbb{H}_l \to \mathbb{C}$ we have $F \in L^p(\mathbb{H}_l/\Gamma)$ if and only if $\int_{\mathcal{S}} |F(Z)|^p \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l} < \infty$. This again is the case if for every choice of z, z', d, d' and $t > 0$ the integral $\int_{\mathcal{S}_t} |F(Z)|^p \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}$ is finite (see [Bru02, Lemma 4.16-4.18]). We will need the following estimates.

Lemma 3.4.3 ([Bru02, Lemma 4.13]). *Let $t > 0$. Then there exists $\varepsilon > 0$ such that for all $\lambda = (\lambda_1, \lambda_2, \lambda_D) \in K'$ and $Y \in \mathcal{R}_t$ we have*

$$\frac{(\lambda, Y)^2}{Y^2} - q(\lambda) \geq \varepsilon(y_2^2\lambda_1^2/2 + y_1^2\lambda_2^2/2 + q(Y)q(\lambda_D))/Y^2.$$

Using the inequality

$$q(Y) < y_1y_2$$

this yields

$$\frac{(\lambda, Y)^2}{Y^2} - q(\lambda) > \varepsilon(y_2/y_1\lambda_1^2 + y_1/y_2\lambda_2^2 + q(\lambda_D)).$$

Lemma 3.4.4. *Suppose $2p < l$. Then*

$$\int_{\mathcal{S}_t} q(Y)^p \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l} < \infty.$$

Together with the Koecher principle this shows

Corollary 3.4.5. *If F is a holomorphic modular form of singular weight $\kappa = \frac{l}{2} - 1$, then F is a square-integrable section of the corresponding line bundle $\mathcal{L}_\kappa$. Moreover, every cusp form is square-integrable.*

Chapter 4

Boundary of the Orthogonal Upper Half Plane

4.1 Boundary and Siegel Operator

Consider the projective model $\mathcal{K}^+ \subseteq \mathcal{K} \subseteq \mathcal{N}$. Then we can compactify $\mathcal{K}$ naively by taking its closure inside $\mathcal{N}$, i.e. the boundary is

$$\partial\mathcal{K}^+ = \{[Z] \in \mathcal{N} \mid (Z, \overline{Z}) = 0\}.$$

This means that for $[Z] \in \partial\mathcal{K}$ we have $(X, X) = (Y, Y) = 0$. Hence X, Y span an isotropic subspace of $V(\mathbb{R})$, which can be either 1-dimensional or 2-dimensional. Conversely, let $\langle z \rangle \subseteq V(\mathbb{R})$ be an isotropic line with generator z. Take two sequences $x_n, y_n \in V(\mathbb{R})$ with positive norm which are orthogonal and converge to z. Then $[x_n + iy_n] \in \mathcal{K}^+$ with limit $[(1 + i)z] = [z] \in \partial\mathcal{K}^+$ for $n \to \infty$. Hence every isotropic line in $V(\mathbb{R})$ represents a boundary point of $\mathcal{K}^+$ in $\mathcal{N}$.

Definition 4.1.1. A boundary point of $\mathcal{K}^+$ of the form $[z] \in \mathcal{N}$ which is represented by a real isotropic line is called *special boundary point*. A set consisting of one special boundary point is called *zero-dimensional boundary component*. A non-special boundary point is called *generic boundary point*.

Now let $I \subseteq V(\mathbb{R})$ be an isotropic plane and consider the set of all boundary points which can be represented by elements of $I \otimes \mathbb{C}$.

Definition 4.1.2. For an isotropic plane $I \subseteq V(\mathbb{R})$ the set of all generic boundary points which can be represented by an element of $I \otimes \mathbb{C}$ is called *one-dimensional boundary component* attached to I. By a *boundary component* we mean a zero-dimensional or a one-dimensional boundary component.

Lemma 4.1.3 ([BF01, Remark and Definition 2.1])**.** *The one-dimensional boundary components are isomorphic to usual upper half-planes. Moreover, there is a bijective correspondence between boundary components and non-zero isotropic subspaces of* $V(\mathbb{R})$.

Proof. Let $I \subseteq V(\mathbb{R})$ be an isotropic plane. Take a basis z, d of I and consider $\tilde{z}, \tilde{d}$ isotropic such that $(z, \tilde{z}) = (d, \tilde{d}) = 1$ and all other products vanish. We will use the shorthand notation (z_1, z_2, z_3, z_4) for $z_1\tilde{z} + z_2 z + z_3\tilde{d} + z_4 d$. Then the elements of I have the form $(0, z_2, 0, z_4)$. Assume that this is not the multiple of a real point, i.e. it is not a special boundary points. Then $z_2 \neq 0 \neq z_4$ and we can normalize it such that $z_4 = 1$, i.e. we have a point of the form $(0, \tau, 0, 1)$ for some $\tau \in \mathbb{C} \setminus \mathbb{R}$. making suitable choices of the basis and $\mathcal{K}^+$ we can assume that $(1, 1, i, i) \in \mathcal{K}^+$. Then we have an embedding

$$\mathbb{H} \times \mathbb{H} \to \mathcal{K}^+, \quad (\tau_1, \tau_2) \mapsto (1, -\tau_1\tau_2, \tau_1, \tau_2).$$

In the projective space we have

$$\lim_{t\to\infty}[1, -\tau it, \tau, it] = \lim_{t\to\infty}\left[\frac{1}{it}, -\tau, \frac{\tau}{it}, 1\right] = [0, -\tau, 0, 1].$$

Thus the point $[0, -\tau, 0, 1]$ is in the boundary of $\mathcal{K}^+$ if and only if $v = \operatorname{Im}(\tau) > 0$. The set of all boundary points represented by elements in $I \otimes \mathbb{C}$ can be identified with $\mathbb{H} \cup \mathbb{R} \cup \infty$. In particular, if we let $\tau = iv$ with $v \to \infty$, we obtain the special boundary point $[z]$. □

In particular, this means that every boundary component is a hermitian symmetric domain of lower dimension.

Definition 4.1.4. A boundary component is called *rational boundary component* if the corresponding isotropic subspace is defined over $\mathbb{Q}$. Write $\mathbb{H}_l^*$ for the union of $\mathbb{H}_l \simeq \mathcal{K}^+$ together with all rational boundary components. Then the rational orthogonal group $\mathrm{O}^+(V) = \mathrm{O}^+(V(\mathbb{R})) \cap \mathrm{O}(V)$ acts on $\mathbb{H}_l^*$.

By the theory of Baily-Borel (see [BB66], [BJ06]), there is a topology on $\mathbb{H}_l^*$ such that for congruence subgroups $\Gamma \subseteq \mathrm{O}^+(V)$ the quotients $X_\Gamma = \mathbb{H}_l^*/\Gamma$ are normal compact complex analytic spaces. In fact, using holomorphic modular forms of suitably high weight one can show that X_Γ can be embedded into projective space, which gives it, using Chow's theorem, the structure of a normal projective algebraic variety. The next theorem describes the topology on $\mathbb{H}_l^*$. Therefore, let $\overline{\mathcal{S}_t}$ be the closure of $\mathcal{S}_t \subseteq \mathbb{H}_l^*$ with respect to the topology induced by the usual topology of $\mathcal{N}$.

Theorem 4.1.5 ([BB66, Section 4]). *There exists a unique topology, called the* Satake topology*, on $\mathbb{H}_l^*$ which has the following properties:*

1. *The set $\mathbb{H}_l$ is open in $\mathbb{H}_l^*$ and the induced topology is the usual one.*
2. *The induced topology on $\overline{\mathcal{S}_t}$ coming from the Satake topology and from the usual topology on $\mathcal{N}$ agree.*
3. *The rational orthogonal group $\mathrm{O}^+(V)$ acts as a group of topological automorphisms on $\mathbb{H}_l^*$.*
4. *Let $\Gamma \subseteq \mathrm{O}^+(V)$ be a subgroup which is commensurable with $\mathrm{O}^+(L)$ for some lattice $L \subseteq V$. Then every $a \in \mathbb{H}_l^*$ admits a fundamental system of neighbourhoods U which are invariant under the stabilizer Γ_a and such that*

   $$U/\Gamma_a \to \mathbb{H}_l^*/\Gamma$$

   is an open embedding.

Let $I = \langle z \rangle \subseteq V$ be an isotropic line and $z \in \mathrm{Iso}_0(L), z' \in L', (z, z') = 1$. Consider the corresponding orthogonal upper half-plane $\mathbb{H}_l$. Then we have

$$\lim_{t\to\infty}[(tZ)_L] = \lim_{t\to\infty}[tZ - t^2 q(Z)z + \tilde{z}] = [z] \in \partial\mathcal{K}^+$$

and hence for a modular form $F : \tilde{\mathcal{K}}^+ \to \mathbb{C}$ of weight κ with respect to Γ we define the value in the cusp z as

$$F(z) := \lim_{t\to\infty} F_z(tZ) = \lim_{t\to\infty} F((tZ)_L)$$

if it exists. If we multiply z by a non-zero constant $t \in \mathbb{Q}$, then the value is multiplied by t^κ. Assume that F_z has a Fourier expansion of the form

$$F_z(Z) = \sum_{\lambda \in K_0' \cap \overline{\mathcal{C}}} a_z(\lambda) e(\lambda, Z).$$

Then the value in the cusp exists and we have

$$F(z) = a_z(0).$$

We will usually normalize this value by assuming $z \in L$ is primitive isotropic so that the value in the cusp $I = \langle z \rangle$ is well-defined up to multiplication by $(-1)^\kappa$. A straight forward calculation shows that

$$\lim_{t\to\infty} F_{\sigma^{-1}z}(tZ) = \lim_{t\to\infty} F_z|_\kappa \sigma^{-1}(tZ).$$

Let $I \subseteq V$ be an isotropic plane and, as above, $z \in \mathrm{Iso}_0(L), z' \in L', (z,z') = 1, K = z^\perp \cap z'^\perp \cap L, d \in \mathrm{Iso}_0(K), d' \in K', (d,d') = 1$ with $I = \langle z, d\rangle$. Let $D = d^\perp \cap d'^\perp \cap K$ and let $d_3, \dots, d_l$ be a basis of $D \otimes_{\mathbb{Z}} \mathbb{Q}$. Recall the orthogonal upper half-plane

$$\mathbb{H}_l = K \otimes_{\mathbb{Z}} \mathbb{R} + i\mathcal{C} = \{Z = X + iY \in K \otimes_{\mathbb{Z}} \mathbb{C} \mid q(Y) > 0, (Y,d) > 0\}$$

and we write $Z = z_1\tilde{d} + z_2 d + Z_D$, where $\tilde{d} = d' - q(d')d$. For $\tau \in \mathbb{H}$ and $t \in \mathbb{R}_{>0}$ we have $\tau\tilde{d} + itd \in \mathbb{H}_l$, which corresponds to $[\tilde{z} - \tau itz + \tau\tilde{d} + itd] \in \mathcal{K}^+$. For a modular form $F : \tilde{\mathcal{K}}^+ \to \mathbb{C}$ we define the *Siegel operator* corresponding to the boundary component I as (if it exists)

$$F|_I : \mathbb{H} \to \mathbb{C}, \quad \tau \mapsto F|_I(\tau) := \lim_{t\to\infty} F_z(\tau\tilde{d} + itd).$$

This depends of course on the choice of the basis. For $\lambda = (\lambda_1, \lambda_2, \lambda_U) \in W(\mathbb{R}) = K \otimes_{\mathbb{Z}} \mathbb{R}$ we have

$$(\lambda, \tau\tilde{d} + itd) = it\lambda_1 + \tau\lambda_2.$$

Assume that F_z has a Fourier expansion of the form

$$F_z(Z) = \sum_{\lambda \in K_0' \cap \overline{\mathcal{C}}} a_z(\lambda) e(\lambda, Z).$$

Then the Siegel operator exists and we have

$$\begin{aligned}
F|_I(\tau) &= \lim_{t\to\infty} \sum_{(\lambda_1,\lambda_2,\lambda_D)\in K_0'\cap\overline{\mathcal{C}}} a_z(\lambda_1,\lambda_2,\lambda_D) e(\lambda_1\tilde{d} + \lambda_2 d + \lambda_D, \tau d' + itd) \\
&= \lim_{t\to\infty} \sum_{(\lambda_1,\lambda_2,\lambda_D)\in K_0'\cap\overline{\mathcal{C}}} a_z(\lambda_1,\lambda_2,\lambda_D) \exp(-2\pi t\lambda_1) e(\lambda_2\tau) \\
&= \sum_{\substack{(0,\lambda_2,0)\in K_0' \\ \lambda_2 \geq 0}} a_z(0,\lambda_2,0) e(\lambda_2\tau).
\end{aligned}$$

Now let

$$P_I(\mathbb{R}) = \{\sigma \in \mathrm{O}^+(V(\mathbb{R})) \mid \sigma I = I\}$$

be the stabilizer of I and consider the normal subgroup

$$Z_I(\mathbb{R}) = \{\sigma \in \mathrm{O}^+(V(\mathbb{R})) \mid \sigma x = x \text{ for all } x \in I\}.$$

Then $P_I(\mathbb{R})$ is a parabolic subgroup of $\mathrm{O}^+(V(\mathbb{R}))$. The quotient $P_I(\mathbb{R})/Z_I(\mathbb{R})$ acts on the boundary component I and can in fact be identified with its automorphism group $\mathrm{SL}_2(\mathbb{R})/\{\pm\mathrm{id}\}$ (see [AMRT10, Theorem 3.10] for the general case, we will calculate this stabilizer in the next section). The image of Γ in $\mathrm{SL}_2(\mathbb{Q})/\{\pm\mathrm{id}\}$, is an arithmetic subgroup and $F|_I$ is then a modular form of weight κ with respect to this arithmetic subgroup.

Remark 4.1.6. A modular form $F : \tilde{\mathcal{K}}^+ \to \mathbb{C}$ that is holomorphic is a holomorphic modular form if and only if $F|_I$ exists for every boundary component. It is a cusp form if and only if it vanishes at every boundary component.

Definition 4.1.7. We say that a holomorphic modular form F is a linear combination of Eisenstein series on the boundary if $F|_I$ is a linear combination of Eisenstein series for all 1-dimensional boundary components I. We write $\mathrm{M}_\kappa^{\partial\,\mathrm{Eis}}(\Gamma)$ for the space of holomorphic modular forms that are linear combinations of Eisenstein series on the boundary. In particular, we have $\mathrm{S}_\kappa(\Gamma) \subseteq \mathrm{M}_\kappa^{\partial\,\mathrm{Eis}}(\Gamma)$.

Remark 4.1.8. If a holomorphic modular form F is a linear combination of Eisenstein series on the boundary, then its restrictions to the boundary are fully determined by the values in the 0-dimensional cusps, i.e. the constant Fourier coefficients. If F is a holomorphic modular form of singular weight, then it is fully determined by its restrictions to the boundary.

Definition 4.1.9. For $\Gamma = \Gamma(L)$ recall the map

$$\pi_L : \Gamma(L)\backslash \mathrm{Iso}_0(L') \to \mathrm{Iso}(L'/L)$$

and that every element in $\Gamma(L)\backslash \mathrm{Iso}_0(L')$ corresponds to a 0-dimensional cusp of $\Gamma(L)\backslash\mathbb{H}_l$. We write $\mathrm{M}_\kappa^\pi(\Gamma(L))$ for the subspace of $\mathrm{M}_\kappa^{\partial\,\mathrm{Eis}}(\Gamma(L))$ that consists of holomorphic modular forms whose values in the 0-dimensional cusps only depend on their image in L'/L. In particular, if $\delta = -\delta \in L'/L$, the value in the cusps corresponding to δ vanish if κ is odd. We have $\mathrm{S}_\kappa(\Gamma(L)) \subseteq \mathrm{M}_\kappa^\pi(\Gamma(L))$ and if π is injective we have $\mathrm{M}_\kappa^\pi(\Gamma(L)) = \mathrm{M}_\kappa^{\partial\,\mathrm{Eis}}(\Gamma(L))$.

4.2 Parabolic Subgroups

We will now calculate the parabolic subgroups $P_I(\mathbb{R})$ and their intersections with $\mathrm{O}^+(L)$ and $\Gamma(L)$. First assume that I is a line and choose a basis $e_1, e_2, d_1, \ldots d_n$ of V such that $I = \langle e_1\rangle$ and the bilinear form has matrix representation

$$B = \left({}_{1}{}^{1} \right) \perp B'.$$

Write $K = L \cap \langle d_1, \dots, d_n \rangle$. Then according to [AD15, Section 3.3] the connected component of the parabolic subgroup has the decomposition

$$P_I^+(\mathbb{R}) = (\mathbb{R}_{>0} \cdot \mathrm{O}^+(K \otimes_{\mathbb{Z}} \mathbb{R})) \ltimes K \otimes_{\mathbb{Z}} \mathbb{R},$$

where we have the identifications

$$\mathbb{R}_{>0} \to \mathrm{O}(V(\mathbb{R})), \quad t \mapsto \begin{pmatrix} t & & & & \\ & t^{-1} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$K \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{O}(V(\mathbb{R})), \quad x \mapsto \begin{pmatrix} 1 & -q(x) & -(x, d_1) & \dots & -(x, d_n) \\ & 1 & & & \\ & x_1 & 1 & & \\ & \vdots & & \ddots & \\ & x_n & & & 1 \end{pmatrix},$$

where $x = \sum_{j=1}^n x_j d_j$. Let $\mathbb{H}_l$ be the tube domain realization corresponding to e_1, e_2. We shortly calculate the action of $P_I^+(\mathbb{R})$ on $\mathbb{H}_l$. We obviously have $j(\sigma, Z) = 1$ for all $\sigma \in P_I^+(\mathbb{R}), Z \in \mathbb{H}_l$. Moreover, $\mathrm{O}^+(W(\mathbb{R}))$ and $\mathbb{R}_{>0}$ act by multiplication and $W(\mathbb{R})$ acts by translation.

Write

$$P_I^+(\mathrm{O}^+(L)) := P_I^+(\mathbb{R}) \cap \mathrm{O}^+(L) \quad \text{and} \quad P_I^+(\Gamma(L)) := P_I^+(\mathbb{R}) \cap \Gamma(L).$$

Assume that L has a decomposition

$$L = U(M_I) \oplus K = e_1 \oplus M_I e_2 \oplus K.$$

Let $M \in P_I^+(\mathrm{O}^+(L))$ with decomposition corresponding to $t \in \mathbb{R}_{>0}, x \in W(\mathbb{R}), M' \in \mathrm{O}^+(W(\mathbb{R}))$. Then M acts on $e_1, M_I e_2$ and $\lambda \in K$ as

$$M_I e_2 \mapsto -t M_I q(x) e_1 + t^{-1} M_I e_2 + M_I M' x, \quad \lambda \mapsto -(x, \lambda) e_1 + M' \lambda.$$

This shows $t = 1, M' \in \mathrm{O}^+(K)$ and $(x, \lambda) \in \mathbb{Z}$ for all $\lambda \in K$, i.e. $x \in K'$. Moreover, one sees $M_I x \in K$ and $M_I q(x) \in \mathbb{Z}$, which yields

$$M \in P_I^+(\mathrm{O}^+(L)) = \mathrm{O}^+(K) \ltimes \left(K' \cap \frac{1}{M_I} K \right).$$

Similarly, if $M \in P_I^+(\Gamma(L))$ we have $M' \in \Gamma(K)$ and $x \in K$, hence

$$P_I^+(\Gamma(L)) = \Gamma(K) \ltimes K.$$

Now let $I \subseteq V$ be an isotropic plane. We choose a basis $e_1, \dots, e_4, d_1, \dots, d_{n-2}$ of V such that $I = \langle e_1, e_2 \rangle$ and the bilinear form has matrix representation

$$B = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \perp B'.$$

We write again $D = K \cap \langle d_1, \dots, d_{n-2} \rangle$. Then again following [AD15, Section 3.3], the connected component of the parabolic subgroup has the decomposition

$$P_I^+(\mathbb{R}) = (\mathrm{SL}_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cdot \mathrm{O}(D \otimes_{\mathbb{Z}} \mathbb{R})) \ltimes (D \otimes_{\mathbb{Z}} \mathbb{R} \times D \otimes_{\mathbb{Z}} \mathbb{R} \times \mathbb{R}),$$

where we have the identifications

$$\mathrm{SL}_2(\mathbb{R}) \to \mathrm{O}(V(\mathbb{R})), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & -b & & & & & \\ -c & d & & & & & \\ & & a & b & & & \\ & & c & d & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

$$\mathbb{R}_{>0} \to \mathrm{O}(V(\mathbb{R})), \quad t \mapsto \begin{pmatrix} t & & & & & & \\ & t & & & & & \\ & & t^{-1} & & & & \\ & & & t^{-1} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

$$D \otimes_{\mathbb{Z}} \mathbb{R} \times D \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{O}(V(\mathbb{R})),$$

$$(x, y) \mapsto \begin{pmatrix} 1 & & -(x,y) & -q(y) & -(y,d_1) & \dots & -(y,d_{n-2}) \\ & 1 & -q(x) & 0 & -(x,d_1) & \dots & -(x,d_{n-2}) \\ & & 1 & & & & \\ & & & 1 & & & \\ & & x_1 & y_1 & 1 & & \\ & & \vdots & \vdots & & \ddots & \\ & & x_{n-2} & y_{n-2} & & & 1 \end{pmatrix},$$

$$\mathbb{R} \to \mathrm{O}(V(\mathbb{R})), \quad z \mapsto \begin{pmatrix} 1 & & -z & & & & \\ & 1 & & z & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

where $x = \sum_{j=1}^{n-2} x_j d_j, y = \sum_{j=1}^{n-2} y_j d_j$. We will calculate the action on the boundary component corresponding to I and the factor of automorphy. In the previous notation we have $e_1 = z, e_4 = z', e_2 = d, e_3 = d'$. Recall that the boundary is given by points $\lim_{t\to\infty}(\tau e_3 + ite_2)_L$ for $\tau \in \mathbb{H}$. The image in the projective model is

$$(\tau e_3 + ite_2)_L = e_4 - \tau ite_1 + \tau e_3 + ite_2$$

and $\mathrm{SL}_2(\mathbb{R})$ is the only part of $P_I^+(\mathbb{R})$ that acts non-trivially on the boundary component via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau e_3 + ite_2)_L = -(a\tau + b)ite_1 + it(c\tau + d)e_2 + (a\tau + b)e_3 + (c\tau + d)e_4.$$

Hence we have $j(\sigma, \tau e_3 + ite_2) = (c\tau + d)$, if $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with the above embedding into $P_I^+(\mathbb{R})$ and in $\mathbb{H}_l$ we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau e_3 + ite_2) = ite_2 + \sigma\tau e_3,$$

where $\sigma\tau$ is the usual action on $\mathbb{H}$.

Now assume that L has an orthogonal decomposition

$$L = U(M_I) \oplus U(N_I) \oplus D = e_1 \oplus N_I e_4 \oplus e_2 \oplus M_I e_3 \oplus D.$$

Again, let $M \in P_I^+(\mathrm{O}^+(L))$ with decomposition corresponding to $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}), t \in \mathbb{R}^+, x, y \in D \otimes_{\mathbb{Z}} \mathbb{R}, z \in \mathbb{R}$ and $M' \in \mathrm{O}(D \otimes_{\mathbb{Z}} \mathbb{R})$. Calculating the action yields for $\lambda \in D$

$$\begin{aligned} e_1 \mapsto & ate_1 - cte_2, \quad e_2 \mapsto -bte_1 + dte_2 \\ M_I e_3 \mapsto & -t(a(M_I z + M_I(x,y)) - bM_I q(x))e_1 \\ & + t(c(M_I z + M_I(x,y)) - dM_I q(x))e_2 \\ & + at^{-1} M_I e_3 + ct^{-1} M_I e_4 + M_I M' x \\ N_I e_4 \mapsto & -t(aN_I q(y) + bN_I z)e_1 + t(cN_I q(y) + dN_I z)e_2 \end{aligned}$$

$$+ bt^{-1}N_I e_3 + dt^{-1}N_I e_4 + N_I M' y,$$
$$\lambda \mapsto -t(ay - bx, \lambda)e_1 + t(cy - dx, \lambda)e_2 + M'\lambda.$$

The action of $M \in P_I^+(\mathrm{O}^+(L))$ on e_1, e_2 shows that $t\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an integral matrix with determinant $t^2 \in \mathbb{Z}$. Similarly, the action on $M_I e_3, N_I e_4$ shows that $t^{-1}\begin{pmatrix} a & \frac{N_I}{M_I}b \\ \frac{M_I}{N_I}c & d \end{pmatrix}$ is an integral matrix with determinant $t^{-2} \in \mathbb{Z}$. This shows that $t = 1$ and $\frac{M_I}{N_I}c, \frac{N_I}{M_I}b \in \mathbb{Z}$, in particular, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0\left(\frac{N_I}{M_I}, \frac{M_I}{N_I}\right) \subseteq \mathrm{SL}_2(\mathbb{Z})$, where $\Gamma_0\left(\frac{N_I}{M_I}, \frac{M_I}{N_I}\right)$ consists of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $\frac{M_I}{N_I}c, \frac{N_I}{M_I}b \in \mathbb{Z}$. The action of M on $\lambda \in D$ shows that $M' \in \mathrm{O}(D)$ and $ay - bx, cy - dx \in D'$. But this implies $x, y \in D'$. Now the action on $M_I e_3, N_I e_4$ shows that $M_I x \in D, N_I y \in D$. The action on $M_I e_3$ yields

$$a(M_I z + (M_I x, y)) - bM_I q(x), c(M_I z + (M_I x, y)) - dM_I q(x) \in \mathbb{Z},$$

hence $M_I z + (M_I x, y), M_I q(x) \in \mathbb{Z}$ and since $M_I x \in D$ and $y \in D'$ we obtain $M_I z \in \mathbb{Z}$. Similar, using the action on $N_I e_4$ we obtain $N_I z \in \mathbb{Z}$ and thus $P_I^+(\mathrm{O}^+(L))$ is given by

$$\left(\Gamma_0\left(\frac{N_I}{M_I}, \frac{M_I}{N_I}\right) \cdot \mathrm{O}(D)\right) \ltimes \left(\left(D' \cap \frac{1}{M_I}D\right) \times \left(D' \cap \frac{1}{N_I}D\right) \times \frac{1}{\gcd(M_I, N_I)}\mathbb{Z}\right),$$

where $\Gamma_0\left(\frac{M_I}{N_I}, \frac{N_I}{M_I}\right)$ consists of matrices as above. The only part of $P_I^+(\mathrm{O}^+(L))$ that acts non-trivially on the boundary component corresponding to I is given by $\Gamma_0\left(\frac{N_I}{M_I}, \frac{M_I}{N_I}\right)$. For the discriminant kernel we directly see that $M' \in \Gamma(D), x, y \in D$ and $z \in \mathbb{Z}$.

$$P_I^+(\Gamma(L)) = (\Gamma(M_I, N_I) \cdot \Gamma(D)) \ltimes (D \times D \times \mathbb{Z}).$$

Since $\Gamma(M_I, N_I)$ is the only part of $P_I^+(\Gamma(L))$ that acts non-trivially on the boundary component corresponding to I, the boundary component of $\Gamma(L)\backslash\mathbb{H}_l$ corresponding to I is isomorphic to $\Gamma(M_I, N_I)\backslash\mathbb{H}$.

4.3 Boundary Components

In this section we will examine the boundary and obtain generalizations of the results in [AD15] to non-maximal lattices.

Lemma 4.3.1. *Let V be a rational quadratic space with* Witt-rank *at least 2, i.e. the dimension of a (and by Witt's theorem any) maximal isotropic subspace is at least 2. Moreover, let $e_1, e_2 \in V \setminus \{0\}$ be isotropic vectors. Then there are two isotropic planes I_1, I_2 with non-trivial intersection and $e_1 \in I_1, e_2 \in I_2$.*

Proof. Since the Witt-rank is greater or equal to 2 there is an isotropic plane $I_1 \subseteq V$ containing e_1. Assume $e_2 \notin I_1$ (otherwise we take $I_1 = I_2$). Then $\dim(e_2^\perp) = \dim(V) - 1$ and thus the intersection $I_1 \cap e_2^\perp$ is non-empty and $I_2 = \langle I_1 \cap e_2^\perp, e_2 \rangle$ is an isotropic plane containing e_2. □

From now on we assume that L is an even lattice of signature $(2, l)$ and $V = L \otimes \mathbb{Q}$.

Corollary 4.3.2. *Assume that V has Witt-rank 2. Then the boundary of the corresponding orthogonal upper half-plane is connected.*

Proof. The boundary corresponds to rational isotropic subspaces and for two boundary components I, I' we have $I \subseteq \overline{I'}$ if and only if the corresponding isotropic subspace of I is a subspace of I'. Since these isotropic subspaces are non-trivially connected by the previous lemma we obtain the result. □

Let $I \subseteq L$ an isotropic plane such that we have an orthogonal splitting

$$L = U(M_I) \oplus U(N_I) \oplus D_I = (\mathbb{Z}e_1 \oplus \mathbb{Z}e_2) \oplus (\mathbb{Z}e_3 \oplus \mathbb{Z}e_4) \oplus D_I$$

with $I = \mathbb{Z}e_1 \oplus \mathbb{Z}e_3$.

Lemma 4.3.3. *Assume for an isotropic plane I we have two orthogonal splittings*

$$L = U(M_I) \oplus U(N_I) \oplus D_I = U(M_I') \oplus U(N_I') \oplus D_I'$$

as above. Then $D_I \simeq D_I'$.

Proof. Let e_i and f_i be the corresponding basis vectors of the scaled hyperbolic planes. The identity map on L induces an isomorphism on $I^\perp$

$$\phi : \mathbb{Z}e_1 \oplus \mathbb{Z}e_3 \oplus D_I \to \mathbb{Z}f_1 \oplus \mathbb{Z}f_3 \oplus D_I'.$$

Since I is isotropic, the projection $\pi : \mathbb{Z}f_1 \oplus \mathbb{Z}f_3 \oplus D_I' \to D_I'$ preserves the quadratic form and so does

$$\pi \circ \phi : \mathbb{Z}e_1 \oplus \mathbb{Z}e_3 \oplus D_I \to D_I'.$$

Then $\pi \circ \phi$ is surjective with kernel $I = \mathbb{Z}e_1 \oplus \mathbb{Z}e_3$, since D_I is definite. Therefore, the restriction to D_I is an isomorphism. □

Using

$$U(M_I) \oplus U(N_I) \simeq U(\gcd(M_I, N_I)) \oplus U(\operatorname{lcm}(M_I, N_I))$$

we immediately obtain

Corollary 4.3.4. *Let I, I' be isotropic planes with splittings as above. Then I' lies in the $\mathrm{O}^+(L)$-orbit of I if and only if $D_I \simeq D_{I'}$ and*

$$U(M_I) \oplus U(N_I) \simeq U(M_{I'}) \oplus U(N_{I'})$$

or equivalently if $\gcd(M_I, N_I) = \gcd(M_{I'}, N_{I'})$ *and* $\operatorname{lcm}(M_I, N_I) = \operatorname{lcm}(M_{I'}, N_{I'})$.

This classifies all $\mathrm{O}^+(L)$-orbits of isotropic planes whenever every isotropic plane yields a splitting as above. So we assume now that every isotropic plane $I \subseteq L$ comes with a tuple (M_I, N_I, D_I) with a negative definite lattice D_I and $M_I \mid N_I$ such that

$$L = U(M_I) \oplus U(N_I) \oplus D_I.$$

For $D \in \operatorname{gen}(D_I)$ we obviously have

$$\tilde{L} = U(M_I) \oplus U(N_I) \oplus D \in \operatorname{gen}(L).$$

We have the following converse of this observation

Theorem 4.3.5. *Assume that we have a splitting*

$$L = U(M) \oplus U(N) \oplus D.$$

Then

$$\tilde{L} = U(M) \oplus U(N) \oplus \tilde{D} \in \operatorname{gen}(L)$$

if and only if $\tilde{D} \in \operatorname{gen}(D)$.

Proof. For the converse direction assume

$$\tilde{L} = U(M) \oplus U(N) \oplus \tilde{D} \in \operatorname{gen}(L).$$

Then by definition

$$(U(M) \oplus U(N) \oplus \tilde{D}) \otimes \mathbb{Z}_p = (U(M) \oplus U(N) \oplus D) \otimes \mathbb{Z}_p$$

for all primes p. Since we can cancel scaled hyperbolic planes by Theorem 2.2.6, we obtain $D \otimes \mathbb{Z}_p = \tilde{D} \otimes \mathbb{Z}_p$ for all primes p and thus $\tilde{D} \in \operatorname{gen}(D)$. □

As a corollary we obtain

Corollary 4.3.6. *Assume that* $\mathrm{gen}(L)$ *contains only one class and let* $I \subsetneq L$ *be an isotropic plane with corresponding tuple* (D_I, M_I, N_I) *such that*

$$L = U(M_I) \oplus U(N_I) \oplus D_I.$$

Then

$$L = U(M_I) \oplus U(N_I) \oplus D$$

if and only if $D \in \mathrm{gen}(D_I)$.

This essentially completes the analysis for $\mathrm{O}^+(L)$. The number of $\Gamma(L)$-orbits in an $\mathrm{O}^+(L)$-orbit of an isotropic plane is now given by

$$\frac{[\mathrm{O}^+(L) : \Gamma(L)]}{[P_I(\mathrm{O}^+(L)) : P_I(\Gamma(L))]} = \frac{\rho(\mathrm{O}^+(L))}{\rho(P_I(\mathrm{O}^+(L)))},$$

where ρ is the map $\mathrm{O}^+(L) \to \mathrm{O}(L'/L)$. Recall that according to Theorem 2.2.4 this map is surjective if $l \geq l(L_p'/L_p)$ for all primes p, which is, in particular, satisfied if L splits a hyperbolic plane. More general results for the surjectivity can be found in [MM, Chapter 8, Section 5, 7]. For $M_I = N_I = 1$ we have $|\rho(P_I(\mathrm{O}^+(L)))| = |\rho(\mathrm{O}(D_I))|$, see [AD15, Corollary 5.2.7]. More generally, [AD15, Theorem 5.4.2] calculates these numbers if L is a scaled maximal lattice.

Next we consider the 0-dimensional cusps.

Lemma 4.3.7. *Let* $e_1, e_2 \in L'$ *be primitive isotropic of order* N *in* L'/L *such that we have corresponding splittings*

$$L = U(N) + L_1 = U(N) + L_2.$$

Then $\mathrm{gen}(L_1) = \mathrm{gen}(L_2)$. *In particular, if* $\mathrm{gen}(L_1)$ *consists only of one class, the cusps corresponding to* e_1 *and* e_2 *are* $\mathrm{O}^+(L)$*-equivalent.*

Observe that if the level of L is square-free, Corollary 2.2.3 yields

Corollary 4.3.8. *Assume that* L *has square-free level* N*. Then two cusps corresponding to primitive isotropic vectors* $e_1, e_2 \in L$ *are* $\mathrm{O}^+(L)$*-equivalent if and only if they have the same level.*

Moreover, by Eichler's criterion, see Lemma 2.1.12, we have

Corollary 4.3.9. *Assume that L splits two hyperbolic planes. Then the number of 0-dimensional cusps with respect to $\Gamma(L)$ is given by $|\mathrm{Iso}(L'/L)/\{\pm 1\}|$.*

Example 4.3.10. Assume that L has prime level p. Then every isotropic vector $z \in L$ has either level 1 or level p and yields a splitting of a scaled hyperbolic plane. Hence there are only three possible cases that can occur for an isotropic plane I. The corresponding tuples are $(1, 1, D_{(1,1)}), (1, p, D_{(1,p)}), (p, p, D_{(p,p)})$. Using the decomposition of the parabolic subgroup of the last section can be used to calculate the number of boundary components. So the boundary consists of points (corresponding to the orbits of primitive isotropic vectors) and they are connected by 1-dimensional boundary components isomorphic to $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}, \Gamma_1(p)\backslash\mathbb{H}$ and $\Gamma(p)\backslash\mathbb{H}$. If we assume that L splits two hyperbolic planes, then there is exactly one level 1 cusp and $|\mathrm{Iso}_0(L'/L)/\{\pm 1\}| - 1$ cusps of level p. At the level 1 cusp there is a bunch of projective lines intersecting exactly at the level 1 cusp. Moreover, there are 1-dimensional boundary components isomorphic to $\Gamma_1(p)\backslash\mathbb{H}$ connecting the level 1 and level p cusps and there are 1-dimensional boundary components isomorphic to $\Gamma(p)\backslash\mathbb{H}$ connecting the level p cusps. The number of 1-dimensional boundary components can be calculated using the structure of the parabolics.

For square-free level we see analogously that the boundary components with respect to $\Gamma(L)$ are isomorphic to $\Gamma(M_I, N_I)\backslash\mathbb{H}$ for $M_I \mid N_I \mid N$ and one can calculate the number of boundary components similar to the prime level case.

Chapter 5

Lifting Vector-Valued Eisenstein Series

5.1 Regularized Theta Lift

Let L be an even lattice of signature (b^+, b^-) and p a polynomial on $\mathbb{R}^{(b^+,b^-)}$ which is homogeneous of degree κ^+ in the first b^+ variables and homogeneous of degree κ^- in the last b^- variables. Denote by $\Theta_L(\tau, \nu, p)$ the corresponding theta series. Let $k = \frac{b^+}{2} + \kappa^+ - \frac{b^-}{2} - \kappa^-$. Write $\kappa = \kappa^+ - \kappa^-$ and assume $b^+ - b^- \equiv 0 \bmod 2$. For $\beta \in \mathrm{Iso}(L'/L)$ recall the vector-valued non-holomorphic Eisenstein series $E_{k,\beta}(\tau, s)$ of weight k for the Weil representation. Then

$$\langle E_{k,\beta}(\tau, s), \Theta_L(\tau, \nu, p)\rangle v^k$$

is invariant under $\mathrm{Mp}_2(\mathbb{Z})$ in τ. Hence we define

$$\Phi_\beta(\nu, p, s, t) = \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle E_{k,\beta}(\tau, s), \Theta_L(\tau, \nu, p)\rangle v^{k-t} \frac{\mathrm{d}u\mathrm{d}v}{v^2},$$

where

$$\mathcal{F}_T = \{\tau = u + iv \in \mathcal{F} \mid v \leq T\}$$

and set

$$\Phi_\beta(\nu, p, s) = \mathrm{CT}_{t=0}\, \Phi_\beta(\nu, p, s, t),$$

where $\mathrm{CT}_{t=0}$ denotes the constant term in the Laurent expansion at $t = 0$ (of the meromorphic continuation). For $\mathfrak{v} \in \mathrm{Iso}(\mathbb{C}[L'/L])$ we define $\Phi_{\mathfrak{v}}$ analogously. By linearity of the theta lift, it often suffices to only consider Φ_β for $\beta \in \mathrm{Iso}(L'/L)$.

Theorem 5.1.1. *For an isometry $\nu : L \otimes \mathbb{R} \to \mathbb{R}^{b^+,b^-}$ the regularized theta integral converges and defines a holomorphic function for $\mathrm{Re}(t) > \mathrm{Re}(s) > 1$ which has a meromorphic continuation in t to $t = 0$ and to all $s \in \mathbb{C}$. The possible poles come from the poles* *of $E_{k,\beta}(\tau,s)$ except for finitely many poles at $s = 1 - \frac{b^+}{2} + j, \frac{b^+}{2} - j, j \in \mathbb{N}_{>0}$, which do not occur if p is harmonic of positive degree. Moreover, the functional equation of $E_{k,\beta}(\tau,s)$ yields a functional equation for the lift $\Phi_\beta(\nu,p,s)$.*

Proof. We have

$$\begin{aligned}\Phi_\beta(\nu,p,s,t) &= \lim_{T\to\infty}\int_{\mathcal{F}_T}\langle E_{k,\beta}(\tau,s),\Theta_L(\tau,\nu,p)\rangle v^{k-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}\\ &= \int_{\mathcal{F}_1}\langle E_{k,\beta}(\tau,s),\Theta_L(\tau,\nu,p)\rangle v^{k-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}\\ &+ \int_{v=1}^{\infty}\int_{u=0}^{1}\langle E_{k,\beta}(\tau,s),\Theta_L(\tau,\nu,p)\rangle v^{k-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}.\end{aligned}$$

Since $\mathcal{F}_1$ is compact and the integrand is holomorphic in s and t, the first integral converges and defines a holomorphic function for all $s,t\in\mathbb{C}$, where $E_{k,\beta}$ has no pole. Therefore, it is sufficient to consider the second summand, i.e.

$$\varphi(\nu,p,s,t) = \int_{v=1}^{\infty}\int_{u=0}^{1}\langle E_{k,\beta}(\tau,s),\Theta_L(\tau,\nu,p)\rangle v^{k-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}.$$

Inserting the Fourier coefficients of $E_{k,\beta}$ and Θ_L yields

$$\begin{aligned}\varphi(\nu,p,s,t) &= \int_{v=1}^{\infty}\int_{u=0}^{1}\sum_{\lambda\in L'}\sum_{n\in\mathbb{Z}+q(\lambda)} c_{k,\beta}(\lambda,n,s,v)\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(\nu(\lambda))\\ &\times\exp(-2\pi v q_\nu(\lambda))e\left((n-q(\lambda))u\right)v^{\frac{b^+}{2}+\kappa^+-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}\\ &= \int_{v=1}^{\infty}\int_{u=0}^{1}\sum_{\lambda\in L'}\sum_{n\in\mathbb{Z}} c_{k,\beta}(\lambda,n+q(\lambda),s,v)\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(\nu(\lambda))\\ &\times\exp(-2\pi v q_\nu(\lambda))e\left(nu\right)v^{\frac{b^+}{2}+\kappa^+-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}.\end{aligned}$$

Carrying out the integration over u yields

$$\begin{aligned}&\int_{v=1}^{\infty}\sum_{\lambda\in L'} c_{k,\beta}(\lambda,q(\lambda),s,v)\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(\nu(\lambda))\exp(-2\pi v q_\nu(\lambda))v^{\frac{b^+}{2}+\kappa^+-t}\frac{\mathrm{d}v}{v^2}\\ &= \int_{v=1}^{\infty} c_{k,\beta}(0,0,s,v)\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(0)v^{\frac{b^+}{2}+\kappa^+-t}\frac{\mathrm{d}v}{v^2}\\ &+ \int_{v=1}^{\infty}\sum_{\substack{\lambda\in L'\setminus\{0\}\\ q(\lambda)=0}} c_{k,\beta}(\lambda,0,s,v)\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(\nu(\lambda))\exp(-2\pi v q_\nu(\lambda))v^{\frac{b^+}{2}+\kappa^+-t}\frac{\mathrm{d}v}{v^2}\\ &+ \int_{v=1}^{\infty}\sum_{\substack{\lambda\in L'\setminus\{0\}\\ q(\lambda)\neq 0}} c_{k,\beta}(\lambda,q(\lambda),s,v)\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(\nu(\lambda))\exp(-2\pi v q_\nu(\lambda))v^{\frac{b^+}{2}+\kappa^+-t}\frac{\mathrm{d}v}{v^2}.\end{aligned}$$

Inserting

$$c_{k,\beta}(0,0,s,v) = 2\delta_{\beta,0}y^s + c_{k,\beta}(0,0,s)y^{1-s-k}$$

yields for $\varphi(\nu,p,s,t)$

$$2\delta_{\beta,0}\int_{v=1}^{\infty} c_{k,\beta}(0,0,s,v)\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(0)v^{\frac{b^+}{2}+\kappa^++s-2-t}\mathrm{d}v$$

$$+\,c_{k,\beta}(0,0,s)\int_{v=1}^{\infty}\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(0)v^{\frac{b^+}{2}+\kappa^+-s-k-1-t}\mathrm{d}v$$

$$+\int_{v=1}^{\infty}\sum_{\substack{\lambda\in\pm\beta+L\smallsetminus\{0\}\\ q(\lambda)=0}}\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(\nu(\lambda))\exp(-2\pi v q_\nu(\lambda))v^{\frac{b^+}{2}+\kappa^++s-2-t}\mathrm{d}v$$

$$+\int_{v=1}^{\infty}\sum_{\substack{\lambda\in L'\smallsetminus\{0\}\\ q(\lambda)=0}}c_{k,\beta}(\lambda,0,s)\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(\nu(\lambda))\exp(-2\pi v q_\nu(\lambda))v^{\frac{b^+}{2}+\kappa^+-s-k-1-t}\mathrm{d}v$$

$$+\int_{v=1}^{\infty}\sum_{\substack{\lambda\in L'\smallsetminus\{0\}\\ q(\lambda)\neq 0}}c_{k,\beta}(\lambda,q(\lambda),s,v)\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(\nu(\lambda))\exp(-2\pi v q_\nu(\lambda))v^{\frac{b^+}{2}+\kappa^+-2-t}\mathrm{d}v.$$

The first integral converges for $\mathrm{Re}(s) < 1-\frac{b^+}{2}+\mathrm{Re}(t)$ and is equal to

$$\sum_{j=0}^{\infty}\frac{\Delta^j\overline{p}(0)}{(-8\pi)^j(t-s-\frac{b^+}{2}+1+j)j!}.$$

Similarly, the second integral converges for $\mathrm{Re}(s) > \frac{b^+}{2} - Re(t)$ and is equal to

$$\sum_{j=0}^{\infty}\frac{\Delta^j\overline{p}(0)}{(-8\pi)^j(s-\frac{b^+}{2}+t+j)j!}.$$

Both are finite sums and define meromorphic functions in s and t. The possible poles match the poles described in the theorem. The sum in the third integral is a subseries of a theta function attached to the positive definite majorant q_ν and thus the integral converges for all $s,t\in\mathbb{C}$. The convergence of the last two integrals follows from the asymptotics

$$c_{k,\beta}(\gamma,n,s,v) = O(e^{-2\pi|n|})$$

for $v\geq 1$ as $n\to\pm\infty$. □

5.2 Unfolding Against $E_{k,\beta}(\tau,s)$

We will now calculate the theta lift $\Phi_\beta(\nu,p,s)$ by unfolding against the Eisenstein series.

Theorem 5.2.1. *The theta lift ist given by*

$$\Phi_\beta(\nu,p,s) = 2\sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=0\\ \lambda\neq 0}}\sum_{j=0}^{\infty}\frac{\Delta^j\overline{p}(\nu(\lambda))}{(-8\pi)^j j!}\frac{\Gamma(s+\frac{b^+}{2}+\kappa^+-1-j)}{(2\pi q_\nu(\lambda))^{s+\frac{b^+}{2}+\kappa^+-1-j}}.$$

The series converges for $\mathrm{Re}(s) \gg 0$. *We can rewrite this to*

$$\Phi_\beta(\nu,p,s) = 2\sum_{\lambda\in\mathrm{Iso}_0(L')} \zeta_+^{k_{\lambda\beta}}(2s+b^+ +\kappa-2)\sum_{j=0}^{\infty}\frac{\Delta^j\overline{p}(\nu(\lambda))}{(-8\pi)^j j!}\frac{\Gamma(s+\frac{b^+}{2}+\kappa^+-1-j)}{(2\pi q_\nu(\lambda))^{s+\frac{b^+}{2}+\kappa^+-1-j}},$$

where for $\lambda\in\mathrm{Iso}_0(L')$ *we let* $k_{\lambda\beta}\in\mathbb{Z}/N_\lambda\mathbb{Z}$ *with* $k_{\lambda\beta}\lambda\in\beta+L$ *and the summands with* $\beta+L\cap\mathbb{Z}\lambda=\varnothing$ *are meant to be zero.*

Proof. Using the theta transformation formula we have that $\Phi_\beta(\nu,p,s,t)$ is given by

$$\begin{aligned}
&\lim_{T\to\infty}\int_{\mathcal{F}_T}\langle E_{k,\beta}(\tau,s),\Theta_L(\tau,\nu,p)\rangle v^{k-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}\\
&=\frac{1}{2}\lim_{T\to\infty}\int_{\mathcal{F}_T}\sum_{(M,\phi)\in\tilde{\Gamma}_\infty\backslash\mathrm{Mp}_2(\mathbb{Z})}\langle(\mathfrak{e}_\beta v^s)\mid_k(M,\phi),\Theta_L(\tau,\nu,p)\rangle v^{k-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}\\
&=\frac{1}{2}\lim_{T\to\infty}\int_{\mathcal{F}_T}\sum_{(M,\phi)\in\tilde{\Gamma}_\infty\backslash\mathrm{Mp}_2(\mathbb{Z})}\langle\phi(\tau)^{-k}\rho_L(M,\phi)^{-1}\mathfrak{e}_\beta\,\mathrm{Im}(M\tau)^s,\Theta_L(\tau,\nu,p)\rangle v^{k-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}\\
&=\frac{1}{2}\lim_{T\to\infty}\int_{\mathcal{F}_T}\sum_{(M,\phi)\in\tilde{\Gamma}_\infty\backslash\mathrm{Mp}_2(\mathbb{Z})}\langle\mathfrak{e}_\beta\,\mathrm{Im}(M\tau)^s,\rho_L(M,\phi)\Theta_L(\tau,\nu,p)\rangle\phi(\tau)^{-k}v^{k-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}\\
&=\lim_{T\to\infty}\int_{\mathcal{F}_T}\sum_{M\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}\langle\mathfrak{e}_\beta,\Theta_L(M\tau,\nu,p)\rangle\,\mathrm{Im}(M\tau)^{k+s}v^{-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}\\
&=\lim_{T\to\infty}\int_{\mathcal{F}_T}\sum_{M\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}\overline{\theta_\beta(M\tau,\nu,p)}\,\mathrm{Im}(M\tau)^{k+s}v^{-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}\\
&=2\lim_{T\to\infty}\int_{\mathcal{F}_T}\overline{\theta_\beta(\tau,\nu,p)}v^{k+s-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}\\
&+\lim_{T\to\infty}\int_{\mathcal{F}_T}\sum_{\substack{M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})\\c\neq0}}\overline{\theta_\beta(M\tau,\nu,p)}\,\mathrm{Im}(M\tau)^{k+s}v^{-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}.
\end{aligned}$$

By the growth of the theta function, the integral

$$\int_{\mathcal{G}}\theta_\beta(\tau,\nu,p)v^{k+s}\frac{\mathrm{d}u\mathrm{d}v}{v^2},$$

where

$$\mathcal{G}=\{\tau=u+iv\in\mathbb{H}\mid |u|\le\frac{1}{2},|\tau|\le1\},$$

converges absolutely for $\mathrm{Re}(s)\gg0$. Thus we can set $t=0$, take the limit $T\to\infty$ and unfold the second summand to obtain for the constant term at $t=0$

$$\begin{aligned}
&\lim_{T\to\infty}\int_{\mathcal{F}_T}\sum_{\substack{M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})\\c\neq0}}\overline{\theta_\beta(M\tau,\nu,p)}\,\mathrm{Im}(M\tau)^{k+s}v^{-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}\\
&=\sum_{\substack{M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})\\c\neq0}}\int_{\mathcal{F}}\overline{\theta_\beta(M\tau,\nu,p)}\,\mathrm{Im}(M\tau)^{k+s}v^{-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}
\end{aligned}$$

$$= 2\int_{\mathcal{G}} \overline{\theta_\beta(\tau,\nu,s)} v^{k+s} \frac{\mathrm{d}u\mathrm{d}v}{v^2}.$$

In the first summand we cut off $\mathcal{F}_1$ and add it to the second integral to obtain

$$2\,\mathrm{CT}_{t=0}\int_{v=1}^{\infty}\int_{u=0}^{1} \overline{\theta_\beta(\tau,\nu,s)} v^{k+s-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2} + 2\int_{v=0}^{1}\int_{u=0}^{1} \overline{\theta_\beta(\tau,\nu,s)} v^{k+s}\frac{\mathrm{d}u\mathrm{d}v}{v^2}.$$

Now we insert the Fourier expansion of

$$\theta_\beta(\tau,\nu,p) = v^{\frac{b^-}{2}+\kappa^-}\sum_{\lambda\in\beta+L}\exp\left(-\frac{\Delta}{8\pi v}\right)(p)(\nu(\lambda))\exp(-2\pi v q_\nu(\lambda))e(q(\lambda)u)$$

which leads, by evaluating the integral over u and using $q(\beta)\in\mathbb{Z}$, to

$$\begin{aligned}
&2\,\mathrm{CT}_{t=0}\int_{v=1}^{\infty}\int_{u=0}^{1}\sum_{\lambda\in\beta+L}\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(\nu(\lambda))\\
&\exp(-2\pi v q_\nu(\lambda))e(-q(\lambda)u)v^{s+\frac{b^+}{2}+\kappa^+-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}\\
&+2\int_{v=0}^{1}\int_{u=0}^{1}\sum_{\lambda\in\beta+L}\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(\nu(\lambda))\\
&\exp(-2\pi v q_\nu(\lambda))e(-q(\lambda)u)v^{s+\frac{b^+}{2}+\kappa^+}\frac{\mathrm{d}u\mathrm{d}v}{v^2}\\
&=2\,\mathrm{CT}_{t=0}\int_{v=1}^{\infty}\sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=0}}\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(\nu(\lambda))\exp(-2\pi v q_\nu(\lambda))v^{s+\frac{b^+}{2}+\kappa^+-t}\frac{\mathrm{d}v}{v^2}\\
&+2\int_{v=0}^{1}\sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=0}}\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(\nu(\lambda))\exp(-2\pi v q_\nu(\lambda))v^{s+\frac{b^+}{2}+\kappa^+}\frac{\mathrm{d}v}{v^2}.
\end{aligned}$$

Observe that for the $\lambda = 0$ term we have

$$\begin{aligned}
&2\,\mathrm{CT}_{t=0}\int_{v=1}^{\infty}\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(0)v^{s+\frac{b^+}{2}+\kappa^+-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}\\
&=-2\int_{v=0}^{1}\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(0)v^{s+\frac{b^+}{2}+\kappa^+}\frac{\mathrm{d}v}{v^2}
\end{aligned}$$

and the rest of the terms converge for $t = 0$. Hence we obtain

$$\begin{aligned}
\Phi_\beta(\nu,p,s) &= 2\sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=0\\ \lambda\neq 0}}\int_{v=0}^{\infty}\exp\left(-\frac{\Delta}{8\pi v}\right)(\overline{p})(\nu(\lambda))\exp(-2\pi v q_\nu(\lambda))v^{s+\frac{b^+}{2}+\kappa^+}\frac{\mathrm{d}v}{v^2}\\
&= 2\sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=0\\ \lambda\neq 0}}\sum_{j=0}^{\infty}\frac{\Delta^j\overline{p}(\nu(\lambda))}{(-8\pi)^j j!}\int_{v=0}^{\infty}\exp(-2\pi v q_\nu(\lambda))v^{s+\frac{b^+}{2}+\kappa^+-j}\frac{\mathrm{d}v}{v^2}\\
&= 2\sum_{\substack{\lambda\in\beta+L\\ q(\lambda)=0\\ \lambda\neq 0}}\sum_{j=0}^{\infty}\frac{\Delta^j\overline{p}(\nu(\lambda))}{(-8\pi)^j j!}\frac{\Gamma(s+\frac{b^+}{2}+\kappa^+-1-j)}{(2\pi q_\nu(\lambda))^{s+\frac{b^+}{2}+\kappa^+-1-j}}.
\end{aligned}$$

For $\lambda \in \mathrm{Iso}_0(L')$ let $k_{\lambda\beta} \in \mathbb{Z}/N_\lambda\mathbb{Z}$ with $k_{\lambda\beta}\lambda \in \beta + L$. Then we have

$$\begin{aligned}\Phi_\beta(\nu, p, s) &= 2 \sum_{\lambda \in \mathrm{Iso}_0(L')} \sum_{\substack{n=1 \\ n \equiv k_{\lambda\beta}}}^{\infty} \sum_{j=0}^{\infty} \frac{\Delta^j \overline{p}(\nu(n\lambda))}{(-8\pi)^j j!} \frac{\Gamma(s + \frac{b^+}{2} + \kappa^+ - 1 - j)}{(2\pi q_\nu(n\lambda))^{s + \frac{b^+}{2} + \kappa^+ - 1 - j}} \\ &= 2 \sum_{\lambda \in \mathrm{Iso}_0(L')} \sum_{\substack{n=1 \\ n \equiv k_{\lambda\beta}}}^{\infty} \sum_{j=0}^{\infty} \frac{\Delta^j \overline{p}(\nu(\lambda))}{(-8\pi)^j j!} \frac{\Gamma(s + \frac{b^+}{2} + \kappa^+ - 1 - j)}{(2\pi q_\nu(\lambda))^{s + \frac{b^+}{2} + \kappa^+ - 1 - j}} \frac{1}{n^{2s + b^+ + \kappa^+ - \kappa^- - 2}} \\ &= 2 \sum_{\lambda \in \mathrm{Iso}_0(L')} \zeta_+^{k_{\lambda\beta}}(2s + b^+ + \kappa - 2) \sum_{j=0}^{\infty} \frac{\Delta^j \overline{p}(\nu(\lambda))}{(-8\pi)^j j!} \frac{\Gamma(s + \frac{b^+}{2} + \kappa^+ - 1 - j)}{(2\pi q_\nu(\lambda))^{s + \frac{b^+}{2} + \kappa^+ - 1 - j}},\end{aligned}$$

where the summands with $\beta + L \cap \mathbb{Z}\lambda = \emptyset$ are meant to be zero.

$\square$

5.3 Unfolding Against $\Theta(\tau, \nu, p)$

Next we calculate the Fourier expansion of the theta lift by unfolding against the theta series.

Theorem 5.3.1 ([Bor98, Theorem 7.1])**.** *The function $\Phi_\beta(\nu, p, s)$ is equal to the constant term at $t = 0$ of*

$$\begin{aligned}&\frac{1}{\sqrt{2}|z_{\nu^+}|} \sum_{h=0}^{\infty} h! \left(\frac{z_{\nu^+}^2}{4\pi}\right)^h \Phi_\beta^K(w, p_{w,h,h}, s) + \frac{\sqrt{2}}{|z_{\nu^+}|} \sum_{h,h^+,h^-} \frac{h! \left(-\frac{z_{\nu^+}^2}{\pi}\right)^h}{(2i)^{h^+ + h^-}} \binom{h^+}{h} \binom{h^-}{h} \\ &\cdot \sum_{\lambda \in K'} \sum_{\substack{\delta \in L_0'/L \\ \pi(\delta) = \lambda + K}} \sum_{j=0}^{\infty} \frac{(-\Delta)^j (\overline{p}_{w,h^+,h^-})(w(\lambda))}{(8\pi)^j j!} \sum_{n=1}^{\infty} n^{h^+ + h^- - 2h} e(n((\lambda, \mu_K) + (\delta, z'))) \\ &\cdot \int_{v=0}^{\infty} \exp\left(-\frac{\pi n^2}{2v z_{\nu^+}^2} - 2\pi v q_w(\lambda)\right) c_{k,\beta}(\delta, q(\lambda), s, v) v^{\frac{b^+ - 5}{2} + \kappa^+ + h - h^+ - h^- - j - t} \mathrm{d}v,\end{aligned}$$

where $\Phi_\beta^K(w, p_{w,h,h}, s)$ denotes the constant term at $t = 0$ of the regularized theta integral for $E_{k,\beta}^K(\tau, s, 0, 0)$

$$\lim_{T \to \infty} \int_{\mathcal{F}_T} \langle E_{k,\beta}^K(\tau, s, 0, 0), \Theta_K(\tau, w, p_{w,h,h}) \rangle v^{\frac{b^+ - 1}{2} + \kappa^+ - h - t} \frac{\mathrm{d}x \mathrm{d}v}{v^2},$$

which by Lemma 2.5.11 vanishes for $\beta \notin L_0'/L$ and is given by the theta lift of $E_{k,\pi(\beta)}(\tau, s)$ for $\beta \in L_0'/L$.

Proof. One uses the expansion of Theorem 2.6.5 and unfolds with respect to this Poincaré series. This yields

$$\frac{1}{\sqrt{2}|z_{\nu^+}|}\sum_{h=0}^{\infty} h!\left(\frac{z_{\nu^+}^2}{4\pi}\right)^h \lim_{T\to\infty}\int_{\mathcal{F}_T}\langle E_{k,\beta}(\tau,s),\Theta_K(\tau,\omega,p_{\omega,h,h})\sum_{m\in\mathbb{Z}/N_z\mathbb{Z}}\mathfrak{e}_{\frac{mz}{N_z}}\rangle v^{k-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}$$

$$+\frac{1}{\sqrt{2}|z_{\nu^+}|}\sum_{h,h^+,h^-}\frac{h!\left(-\frac{z_{\nu^+}^2}{\pi}\right)^h}{(2i)^{h^++h^-}}\binom{h^+}{h}\binom{h^-}{h}$$

$$\times\lim_{T\to\infty}\int_{\mathcal{F}_T}\sum_{M\in\Gamma_\infty\backslash\Gamma}\sum_{n=1}^{\infty} n^{h^++h^--2h}\frac{(c\overline{\tau}+d)^{-k}}{\mathrm{Im}(M\tau)^{h^+-h}}\exp\left(-\frac{\pi n^2}{2\,\mathrm{Im}(M\tau)z_{\nu^+}^2}\right)$$

$$\times\langle E_{k,\beta}(\tau,s),\rho_L(M)^{-1}\left(\Theta_K(M\tau,n\mu_K,0,\omega,p_{\omega,h^+,h^-})\sum_{m\in\mathbb{Z}/N\mathbb{Z}}\mathfrak{e}_{\frac{mz}{N}}\left(-\frac{mn}{N}\right)\right)\rangle v^{k-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}.$$

Applying Lemma 2.4.8 now shows that the first summand has the correct form. For the second summand one uses the transformation formula of $E_{k,\beta}(\tau,s)$, together with Lemma 2.4.8 to obtain

$$\frac{1}{\sqrt{2}|z_{\nu^+}|}\sum_{h,h^+,h^-}\frac{h!\left(-\frac{z_{\nu^+}^2}{\pi}\right)^h}{(-2i)^{h^++h^-}}\binom{h^+}{h}\binom{h^-}{h}$$

$$\times\lim_{T\to\infty}\int_{\mathcal{F}_T}\sum_{M\in\Gamma_\infty\backslash\Gamma}\sum_{n=1}^{\infty} n^{h^++h^--2h}\exp\left(-\frac{\pi n^2}{2\,\mathrm{Im}(M\tau)z_{\nu^+}^2}\right)\mathrm{Im}(M\tau)^{k-h^++h}$$

$$\times\langle E_{k,\beta}^K(M\tau,s,-n,0),\Theta_K(M\tau,n\mu_K,0,\omega,p_{\omega,h^+,h^-})\rangle v^{-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}.$$

The usual unfolding trick now leads to

$$\frac{\sqrt{2}}{|z_{\nu^+}|}\sum_{h,h^+,h^-}\frac{h!\left(-\frac{z_{\nu^+}^2}{\pi}\right)^h}{(-2i)^{h^++h^-}}\binom{h^+}{h}\binom{h^-}{h}\int_{u=0}^{1}\int_{v=0}^{\infty}\sum_{n=1}^{\infty} n^{h^++h^--2h}\exp\left(-\frac{\pi n^2}{2vz_{\nu^+}^2}\right)$$

$$\times\langle E_{k,\beta}^K(\tau,s,-n,0),\Theta_K(\tau,n\mu_K,0,\omega,p_{\omega,h^+,h^-})\rangle v^{k-h^++h-t}\frac{\mathrm{d}u\mathrm{d}v}{v^2}.$$

It is justified by the polynomial growth of the Eisenstein series at every cusp. We will only consider the integral now. We plug in the Fourier expansion

$$E_{k,\beta}^K(\tau,s,-n,0)=\sum_{\gamma\in K'/K}\sum_{\substack{\delta\in L_0'/L\\ \pi(\delta)=\gamma}}\sum_{r\in\mathbb{Z}+q(\delta)}c_{k,\beta}(\delta,r,s,v)\mathfrak{e}_\gamma(ru-n(\delta,z'))$$

and

$$\Theta_K(\tau,n\mu_K,w,p_{w,h^+,h^-})$$
$$=v^{\frac{b^--1}{2}+\kappa^--h^-}\sum_{\lambda\in K'}\exp\left(\frac{\Delta}{8\pi v}\right)(p_{w,h^+,h^-})(w(\lambda))\mathfrak{e}_\lambda(q(\lambda)u+ivq_w(\lambda)-n(\lambda,\mu_K))$$

and carry out the integration over u to obtain

$$\int_{v=0}^{\infty} \sum_{\lambda\in K'} \sum_{\substack{\delta\in L_0'/L\\ \pi(\delta)=\lambda+K}} \exp\left(\frac{\Delta}{8\pi v}\right)(\overline{p}_{w,h^+,h^-})(w(\lambda)) \sum_{n=1}^{\infty} n^{h^++h^--2h} \exp\left(-\frac{\pi n^2}{2vz_{\nu^+}^2}\right)$$

$$\times c_{k,\beta}(\delta, q(\lambda), s, v) e(ivq_w(\lambda) + n(\delta, z') + n(\lambda, \mu_K)) v^{\frac{b^+-1}{2}+\kappa^+-h^+-h^-+h-t} \frac{\mathrm{d}v}{v^2}$$

$$= \sum_{\lambda\in K'} \sum_{\substack{\delta\in L_0'/L\\ \pi(\delta)=\lambda+K}} \sum_{j=0}^{\infty} \frac{\Delta^j(\overline{p}_{w,h^+,h^-})(w(\lambda))}{j!(8\pi)^j} \sum_{n=1}^{\infty} n^{h^++h^--2h} e(n(\delta, z') + n(\lambda, \mu_K))$$

$$\times \int_{v=0}^{\infty} c_{k,\beta}(\delta, q(\lambda), s, v) \exp\left(-\frac{\pi n^2}{2vz_{\nu^+}^2} - 2\pi v q_w(\lambda)\right) v^{\frac{b^+-1}{2}+\kappa^+-h^+-h^-+h-j-t} \frac{\mathrm{d}v}{v^2}.$$

This shows the result. □

The integral can be calculated in certain cases.

Lemma 5.3.2 ([Bor98, Lemma 7.3])**.** *For* $\lambda = 0$ *the integral is equal to*

$$(\delta_{\beta,\delta} + (-1)^{\kappa}\delta_{-\beta,\delta}) \left(\frac{\pi n^2}{2z_{\nu^+}^2}\right)^{s+\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t}$$

$$\times \Gamma\left(-s - \frac{b^+-3}{2} - \kappa^+ - h + h^+ + h^- + j + t\right)$$

$$+ c_{k,\beta}(\delta, 0, s) \left(\frac{\pi n^2}{2z_{\nu^+}^2}\right)^{\frac{b^+-1}{2}+\kappa^++h-h^+-h^--j-k-s-t}$$

$$\times \Gamma\left(-\frac{b^+-1}{2} - \kappa^+ - h + h^+ + h^- + j + k + s + t\right).$$

Moreover, the term with $\lambda = 0$ *is then given by*

$$\sum_{j=0}^{\infty} \frac{(-\Delta)^j(\overline{p}_{w,h^+,h^-})(0)}{(8\pi)^j j!} \left(\frac{\pi}{2z_{\nu^+}^2}\right)^{\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t}$$

$$\sum_{b,c\in\mathbb{Z}/N_z\mathbb{Z}} e\left(\frac{bc}{N_z}\right)\left(\Gamma\left(-s - \frac{b^+-3}{2} - \kappa^+ - h + h^+ + h^- + j + t\right)\left(\frac{\pi}{2z_{\nu^+}^2}\right)^{s}\right.$$

$$\times (\delta_{\beta,\frac{bz}{N_z}} + (-1)^{\kappa}\delta_{-\beta,\frac{bz}{N_z}})\zeta_+^c(-2s - b^+ + 3 - 2\kappa^+ + h^+ + h^- + 2j + 2t)$$

$$+ \Gamma\left(-\frac{b^+-1}{2} - \kappa^+ - h + h^+ + h^- + j + k + s + t\right)\left(\frac{\pi}{2z_{\nu^+}^2}\right)^{1-k-s}$$

$$\left.\times c_{k,\beta}\left(\frac{bz}{N_z}, 0, s\right)\zeta_+^c(-b^+ + 1 - 2\kappa^+ + h^+ + h^- + 2j + 2k + 2s + 2t)\right).$$

Proof. Inserting the Fourier coefficients yields

$$(\delta_{\beta,\delta} + (-1)^{\kappa}\delta_{-\beta,\delta}) \int_{v=0}^{\infty} \exp\left(-\frac{\pi n^2}{2vz_{\nu^+}^2}\right) v^{s+\frac{b^+-5}{2}+\kappa^++h-h^+-h^--j-t} \mathrm{d}v$$

$$+ c_{k,\beta}(\delta, 0, s) \int_{v=0}^{\infty} \exp\left(-\frac{\pi n^2}{2vz_{\nu^+}^2}\right) v^{\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-k-s-t} \mathrm{d}v.$$

Using the integral for the gamma function

$$\int_{v=0}^{\infty} \exp\left(-\frac{\alpha}{v}\right) v^{\beta} \mathrm{d}v = \alpha^{\beta+1}\Gamma(-\beta-1)$$

for $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) < -1$ with $\alpha = \frac{\pi n^2}{2z_{\nu^+}^2}$ we obtain

$$\begin{aligned}
&(\delta_{\beta,\delta} + (-1)^{\kappa}\delta_{-\beta,\delta}) \left(\frac{\pi n^2}{2z_{\nu^+}^2}\right)^{s+\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t} \\
&\times \Gamma\left(-s - \frac{b^+-3}{2} - \kappa^+ - h + h^+ + h^- + j + t\right) \\
&+ c_{k,\beta}(\delta, 0, s) \left(\frac{\pi n^2}{2z_{\nu^+}^2}\right)^{\frac{b^+-1}{2}+\kappa^++h-h^+-h^--j-k-s-t} \\
&\times \Gamma\left(-\frac{b^+-1}{2} - \kappa^+ - h + h^+ + h^- + j + k + s + t\right).
\end{aligned}$$

This yields for the summand with $\lambda = 0$

$$\begin{aligned}
&\sum_{\substack{\delta\in L_0'/L \\ \pi(\delta)=0+K}} \sum_{j=0}^{\infty} \frac{(-\Delta)^j(\overline{p}_{w,h^+,h^-})(0)}{(8\pi)^j j!} \sum_{n=1}^{\infty} n^{h^++h^--2h} e(n(\delta, z')) \\
&\times \Bigg((\delta_{\beta,\delta} + (-1)^{\kappa}\delta_{-\beta,\delta}) \left(\frac{\pi n^2}{2z_{\nu^+}^2}\right)^{s+\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t} \\
&\times \Gamma\left(-s - \frac{b^+-3}{2} - \kappa^+ - h + h^+ + h^- + j + t\right) \\
&+ c_{k,\beta}(\delta, 0, s) \left(\frac{\pi n^2}{2z_{\nu^+}^2}\right)^{\frac{b^+-1}{2}+\kappa^++h-h^+-h^--j-k-s-t} \\
&\times \Gamma\left(-\frac{b^+-1}{2} - \kappa^+ - h + h^+ + h^- + j + k + s + t\right)\Bigg).
\end{aligned}$$

Using that a set of representatives for $\delta \in L_0'/L$ with $p(\delta) = 0 + K$ is given by $\frac{bz}{N_z}$ where b runs through a set of representatives modulo N_z yields

$$\begin{aligned}
&\sum_{j=0}^{\infty} \frac{(-\Delta)^j(\overline{p}_{w,h^+,h^-})(0)}{(8\pi)^j j!} \Gamma\left(-s - \frac{b^+-3}{2} - \kappa^+ - h + h^+ + h^- + j + t\right) \\
&\times \left(\frac{\pi}{2z_{\nu^+}^2}\right)^{s+\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t} \sum_{b\in\mathbb{Z}/N_z\mathbb{Z}} (\delta_{\beta,\frac{bz}{N_z}} + (-1)^{\kappa}\delta_{-\beta,\frac{bz}{N_z}}) \\
&\times \sum_{n=1}^{\infty} e\left(\frac{nb}{N_z}\right) n^{2s+b^+-3+2\kappa^+-h^+-h^--2j-2t}
\end{aligned}$$

$$+\sum_{j=0}^{\infty}\frac{(-\Delta)^j(\overline{p}_{w,h^+,h^-})(0)}{(8\pi)^j j!}\Gamma\left(-\frac{b^+-1}{2}-\kappa^+-h+h^++h^-+j+k+s+t\right)$$
$$\times\left(\frac{\pi}{2z_{\nu^+}^2}\right)^{\frac{b^+-1}{2}+\kappa^++h-h^+-h^--j-k-s-t}\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}}c_{k,\beta}\left(\frac{bz}{N_z},0,s\right)$$
$$\times\sum_{n=1}^{\infty}e\left(\frac{nb}{N_z}\right)n^{b^+-1+2\kappa^+-h^+-h^--2j-2k-2s-2t}.$$

This shows the result. □

Lemma 5.3.3 ([Bor98, Lemma 7.2]). *For $\lambda \neq 0$ with $q(\lambda) = 0$ the integral is equal to*

$$2(\delta_{\beta,\delta}+(-1)^{\kappa}\delta_{-\beta,\delta})\left(\frac{n}{2|\lambda_{\nu^+}||z_{\nu^+}|}\right)^{s+\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t}$$
$$\times K_{s+\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t}\left(2\pi n\frac{|\lambda_{\nu^+}|}{|z_{\nu^+}|}\right)$$
$$+2c_{k,\beta}(\delta,0,s)\left(\frac{n}{2|\lambda_{\nu^+}||z_{\nu^+}|}\right)^{\frac{b^+-1}{2}+\kappa^++h-h^+-h^--j-k-s-t}$$
$$\times K_{\frac{b^+-1}{2}+\kappa^++h-h^+-h^--j-k-s-t}\left(2\pi n\frac{|\lambda_{\nu^+}|}{|z_{\nu^+}|}\right),$$

where $K_\nu(z)$ is the K-Bessel function as in [OLBC10, 10.25], [EMOT81, 6.9.1]. In particular, by the fast decay of the K-Bessel function, the sum over n defines a meromorphic function in s, where the poles come from the coefficients $c_{k,\beta}(\delta,0,s)$.

Proof. Again we insert the Fourier coefficients to obtain

$$(\delta_{\beta,\delta}+(-1)^{\kappa}\delta_{-\beta,\delta})\int_{v=0}^{\infty}\exp\left(-\frac{\pi n^2}{2vz_{\nu^+}^2}-2\pi v q_w(\lambda)\right)v^{s+\frac{b^+-5}{2}+\kappa^++h-h^+-h^--j-t}\mathrm{d}v$$
$$+c_{k,\beta}(\delta,0,s)\int_{v=0}^{\infty}\exp\left(-\frac{\pi n^2}{2vz_{\nu^+}^2}-2\pi v q_w(\lambda)\right)v^{\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-k-s-t}\mathrm{d}v.$$

We use the formula [EMOT54, p. 313, 6.3(17)]

$$\int_{v=0}^{\infty}\exp\left(-\alpha v-\frac{\beta}{v}\right)v^{\gamma}\mathrm{d}v = 2\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}(\gamma+1)}K_{\gamma+1}(2\sqrt{\alpha\beta})$$

for $\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0$ with $\alpha = 2\pi q_w(\lambda), \beta = \frac{\pi n^2}{2z_{\nu^+}^2}$. This yields

$$2(\delta_{\beta,\delta}+(-1)^{\kappa}\delta_{-\beta,\delta})\left(\frac{n^2}{4q_w(\lambda)z_{\nu^+}^2}\right)^{\frac{1}{2}(s+\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t)}$$
$$\cdot K_{s+\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t}\left(2\pi n\sqrt{\frac{q_w(\lambda)}{z_{\nu^+}^2}}\right)$$

$$+ 2c_{k,\beta}(\delta,0,s)\left(\frac{n^2}{4q_w(\lambda)z_{\nu^+}^2}\right)^{\frac{1}{2}(\frac{b^+-1}{2}+\kappa^++h-h^+-h^--j-k-s-t)}$$
$$\cdot K_{\frac{b^+-1}{2}+\kappa^++h-h^+-h^--j-k-s-t}\left(2\pi n\sqrt{\frac{q_w(\lambda)}{z_{\nu^+}^2}}\right).$$

Using $q_w(\lambda) = \lambda_{w^+}^2$ if $q(\lambda) = 0$ we can rewrite this to

$$2(\delta_{\beta,\delta} + (-1)^\kappa\delta_{-\beta,\delta})\left(\frac{n}{2|\lambda_{w^+}||z_{\nu^+}|}\right)^{s+\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t}$$
$$\times K_{s+\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t}\left(2\pi n\frac{|\lambda_{w^+}|}{|z_{\nu^+}|}\right)$$
$$+ 2c_{k,\beta}(\delta,0,s)\left(\frac{n}{2|\lambda_{w^+}||z_{\nu^+}|}\right)^{\frac{b^+-1}{2}+\kappa^++h-h^+-h^--j-k-s-t}$$
$$\times K_{\frac{b^+-1}{2}+\kappa^++h-h^+-h^--j-k-s-t}\left(2\pi n\frac{|\lambda_{w^+}|}{|z_{\nu^+}|}\right).$$

□

Lemma 5.3.4. *For $q(\lambda) > 0$ and $s = 0$ we have*

$$\int_{v=0}^{\infty}\exp\left(-\frac{\pi n^2}{2vz_{\nu^+}^2} - 2\pi v q_w(\lambda)\right)c_{k,\beta}(\delta,q(\lambda),0,v)v^{\frac{b^+-5}{2}+\kappa^++h-h^+-h^--j-t}\mathrm{d}v$$
$$= 2c_{k,\beta}(\delta,q(\lambda),0)\left(\frac{n}{2|\lambda_{w^+}||z_{\nu^+}|}\right)^{\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t} K_{\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t}\left(2\pi n\frac{|\lambda_{w^+}|}{|z_{\nu^+}|}\right)$$

Proof. We have

$$\mathcal{W}_0(v) = e^{-\frac{v}{2}}\cdot\begin{cases}1 & \text{if } v > 0,\\ \Gamma(1-k,-v) & \text{if } v < 0.\end{cases}$$

Since

$$c_{k,\beta}(\delta,q(\lambda),s,v) = c_{k,\beta}(\delta,q(\lambda),s)\mathcal{W}_s(4\pi q(\lambda)v)$$

we have to evaluate

$$\int_{v=0}^{\infty}\exp\left(-\frac{\pi n^2}{2vz_{\nu^+}^2} - 2\pi v\lambda_{w^+}^2\right)v^{\frac{b^+-5}{2}+\kappa^++h-h^+-h^--j-t}\mathrm{d}v.$$

We use again the formula [EMOT54, p. 313, 6.3(17)]

$$\int_{v=0}^{\infty}\exp\left(-\alpha v - \frac{\beta}{v}\right)v^\gamma\mathrm{d}v = 2\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}(\gamma+1)}K_{\gamma+1}(2\sqrt{\alpha\beta})$$

with $\alpha = 2\pi\lambda_{w^+}^2, \beta = \frac{\pi n^2}{2z_{\nu^+}^2}, \gamma = \frac{b^+-5}{2} + \kappa^+ + h - h^+ - h^- - j - t$ to obtain

$$\int_{v=0}^{\infty} \exp\left(-\frac{\pi n^2}{2vz_{\nu^+}^2} - 2\pi v\lambda_{w^+}^2\right) v^{\frac{b^+-5}{2}+\kappa^++h-h^+-h^--j-t} \mathrm{d}v$$
$$= 2\left(\frac{n}{2|\lambda_{w^+}||z_{\nu^+}|}\right)^{\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t} K_{\frac{b^+-3}{2}+\kappa^++h-h^+-h^--j-t}\left(2\pi n\frac{|\lambda_{w^+}|}{|z_{\nu^+}|}\right).$$

□

For $q(\lambda) < 0$ one obtains the special function $\mathcal{V}.(\cdot,\cdot)$ defined in [Bru02, Equation 3.25], but we will not need this here.

5.4 Lorentzian Lattices

We shortly investigate the theta lift for lattices of signature $(1, l-1), l > 2$. We assume that $p(x_1) = x_1^\kappa$ with $\kappa = \kappa^+ = k + \frac{l}{2} - 1, \kappa^- = 0$. Let $d \in \mathrm{Iso}_0(L)$. The set of vectors $\nu_1 \in V(\mathbb{R})$ with norm 1 has two components, one given by

$$V_1 = \{\nu_1 \in V(\mathbb{R}) \mid \nu_1^2 = 1, (d, \nu_1) > 0\}.$$

Via the mapping $\nu_1 \mapsto \mathbb{R}\nu_1$, we can identify V_1 with the Grassmannian $\mathrm{Gr}^+(V(\mathbb{R}))$, the set of positive definite 1-dimensional subspaces. For an element $\nu_1 \in V_1$, the map $a\nu_1 \mapsto a$ defines an isometry $\mathbb{R}\nu_1 \to \mathbb{R}^{(1,0)}$ and by abuse of notation we will identify ν_1 with every isometry which is equal to this isometry on $\mathbb{R}\nu_1$. For $\lambda \in V(\mathbb{R})$ we write λ_{ν_1} and $\lambda_{\nu_1^\perp}$ for the projection onto $\mathbb{R}\nu_1$ and $(\mathbb{R}\nu_1)^\perp$. The quadratic form then decomposes into $q(\lambda) = q(\lambda_{\nu_1}) + q(\lambda_{\nu_1^\perp})$ and we write $q_{\nu_1}(\lambda) = q(\lambda_{\nu_1}) - q(\lambda_{\nu_1^\perp})$ for the positive definite majorant. We now have

$$d_{\nu_1} = (d, \nu_1)\nu_1,$$
$$|d_{\nu_1}| = (d, \nu_1),$$
$$\mu = -d' + \frac{(d,\nu_1)\nu_1}{2(d,\nu_1)^2} - \frac{d - (d,\nu_1)\nu_1}{2(d,\nu_1)^2} = \frac{\nu_1}{(d,\nu_1)} - d' - \frac{d}{2(d,\nu_1)^2}$$

and since the polynomial p only depends on the positive definite variable we can consider $\Theta_L(\tau, \nu_1)$ and $\Phi_\beta(\nu_1, s)$ as a function on V_1. We now have

Corollary 5.4.1. *The Fourier expansion of $\Phi_\beta(\nu_1, s)$ is given by*

$$\frac{\sqrt{2}}{(2i)^\kappa (d,\nu_1)} \sum_{\lambda \in K'} \sum_{\substack{\delta \in L_0'/L \\ \pi(\delta) = \lambda + K}} \sum_{n=1}^{\infty} n^\kappa e\left(n\left(\frac{(\lambda,\nu_1)}{(d,\nu_1)} + (\delta, d')\right)\right)$$

$$\times \int_{v=0}^{\infty} \exp\left(-\frac{\pi n^2}{2v(d,\nu_1)^2} + 2\pi v q(\lambda)\right) c_{k,\beta}(\delta, q(\lambda), s, v) v^{-2-t} \mathrm{d}v.$$

Proof. We have $p_{w,h^+,h^-} = 0$ except for $p_{w,\kappa,0} = 1$. In particular, $p_{w,h,h} = 0$, since $\kappa > 0$, and therefore the first summand in Theorem 5.3.1 vanishes. Since K is negative definite, we have $q(\lambda) = -q_w(\lambda)$ for $\lambda \in K'$. Now using Theorem 5.3.1 with the observation that

$$(\lambda, \mu_K) = \frac{(\lambda, \nu_1)}{(d, \nu_1)}$$

for $\lambda \in K'$ we obtain the result. □

Corollary 5.4.2. *The constant Fourier coefficient in the Fourier expansion of $\Phi_\beta(\nu_1, s)$ is equal to*

$$\sum_{b,c \in \mathbb{Z}/N_d\mathbb{Z}} e\left(\frac{bc}{N_d}\right)\left(\Gamma(1-s)\left(\frac{2(d,\nu_1)^2}{\pi}\right)^{1-s} (\delta_{\beta,\frac{bd}{N_d}} + (-1)^\kappa \delta_{-\beta,\frac{bd}{N_d}})\zeta_+^c(2(1-s)-\kappa)\right.$$
$$\left. + \Gamma(k+s)\left(\frac{2(d,\nu_1)^2}{\pi}\right)^{k+s} c_{k,\beta}\left(\frac{bd}{N_d}, 0, s\right)\zeta_+^c(2(k+s)-\kappa)\right).$$

Corollary 5.4.3. *We have $\Phi_\beta(\nu_1) = \Phi_\beta(\nu_1, 0) = 0$ for $k > 0, \kappa \geq 2$, except if $k = 2$ and $\kappa = 3$.*

Proof. Since $k > 0$ we have $c_{k,\beta}(\delta, n, 0) = 0$ for $n < 0$ and since K is negative definite only the constant Fourier coefficient does not vanish. Hence we have

$$\Phi_\beta(\nu_1) = \sum_{b,c \in \mathbb{Z}/N_d\mathbb{Z}} e\left(\frac{bc}{N_d}\right)\left(\frac{2(d,\nu_1)^2}{\pi}(\delta_{\beta,\frac{bd}{N_d}} + (-1)^\kappa \delta_{-\beta,\frac{bd}{N_d}})\zeta_+^c(2-\kappa)\right.$$
$$\left. + \Gamma(k)\left(\frac{2(d,\nu_1)^2}{\pi}\right)^{k} c_{k,\beta}\left(\frac{bd}{N_d}, 0, 0\right)\zeta_+^c(2k-\kappa)\right).$$

The first summand can now be rewritten to

$$\frac{2(d,\nu_1)^2}{\pi} \sum_{b,c \in \mathbb{Z}/N_d\mathbb{Z}} e\left(\frac{bc}{N_d}\right)(\delta_{\beta,\frac{bd}{N_d}} + (-1)^\kappa \delta_{-\beta,\frac{bd}{N_d}})\zeta_+^c(2-\kappa).$$

Since

$$\zeta_+^c(2-\kappa) + (-1)^\kappa \zeta_+^{-c}(2-\kappa) = 0,$$

the first summand vanishes. For the second summand observe that $c_{k,\beta}(\delta, 0, 0)$ vanishes except for $k = 1, 2$. For $k = 1$ we again have $2k - \kappa \leq 0$ and using $c_{k,\beta}(-\delta, 0, 0) = (-1)^\kappa c_{k,\beta}(\delta, 0, 0)$ we see the vanishing of the sum. For $k = 2$ we have $2k - \kappa \leq 0$ except for $\kappa = 3$. □

Of course, this reproduces the result of [Bor98, Theorem 10.3] with $E_{k,\beta}(\tau)$ as the input function.

Lemma 5.4.4. *Let* $k = 0, \kappa = \frac{l}{2} - 1 > 1$*, i.e.* $l > 4$*. Then we have* $\operatorname{res}_{s=1} \Phi_\beta(\nu_1, s) = 0$*. For* $\kappa = 1$*, i.e.* $l = 4$*, it is a constant.*

Proof. One first observes that the non-zero Fourier coefficients are holomorphic in $s = 1$. Then one sees analogously to the previous proof that the constant Fourier coefficients are also holomorphic in $s = 1$ for $\kappa > 1$. For $\kappa = 1$ the second summand which is independend of ν_1 has a simple pole at $s = 1$ coming from $c_{k,\beta}\left(\frac{bd}{N_d}, 0, s\right)$. □

Again, this reproduces the result of [Bor98, Theorem 10.3] with $\operatorname{res}_{s=1} E_{0,\beta}(\tau, s)$ as input function.

Chapter 6

Orthogonal Eisenstein Series as Theta Lifts

6.1 Orthogonal Eisenstein Series

Let L be an even lattice of signature $(2,l)$ and let $\kappa \in \mathbb{Z}$. Let $\lambda \in \mathrm{Iso}_0(L')$ and recall the tube domain representation $\mathbb{H}_l$ corresponding to a fixed cusp z. Denote by $\Gamma(L)_\lambda \subseteq \Gamma(L)$ the stabilizer of λ in $\Gamma(L)$, write N_λ for the order of λ in L'/L and write $\sigma_\lambda \in \mathrm{O}^+(V)$ for an element satisfying $\sigma_\lambda N_\lambda \lambda = z$. Then $q(Y)^s|_\kappa \sigma_\lambda$ has weight κ with respect to $\Gamma(L)_\lambda$. Hence we define the non-holomorphic Eisenstein series

$$
\begin{aligned}
\mathcal{E}_{\kappa,\lambda}(Z,s) &:= \sum_{\sigma \in \Gamma(L)_\lambda \backslash \Gamma(L)} q(Y)^s|_\kappa \sigma_\lambda \sigma \\
&= \sum_{\sigma \in \Gamma(L)_\lambda \backslash \Gamma(L)} j(\sigma_\lambda \sigma, Z)^{-\kappa} \left(\frac{q(Y)}{|j(\sigma_\lambda \sigma, Z)|^2} \right)^s \\
&= \sum_{\sigma \in \Gamma(L)_\lambda \backslash \Gamma(L)} (N_\lambda \lambda, \sigma(Z_L))^{-\kappa} \left(\frac{q(Y)}{|(N_\lambda \lambda, \sigma(Z_L))|^2} \right)^s
\end{aligned}
$$

for $Z \in \mathbb{H}_l$ and $\mathrm{Re}(s) \gg 0$. The Eisenstein series does not depend on the choice of σ_λ. We have $\Gamma(L)_{-\lambda} = \Gamma(L)_\lambda$ and $\mathcal{E}_{\kappa,-\lambda}(Z,s) = (-1)^\kappa \mathcal{E}_{\kappa,\lambda}(Z,s)$.

Lemma 6.1.1. *The Eisenstein series $\mathcal{E}_{\kappa,\lambda}(Z,s)$ converges for $\mathrm{Re}(s) > \frac{l-\kappa}{2}$ and defines a modular form of weight κ with respect to $\Gamma(L)$. In particular, for $\kappa > l$, the Eisenstein series $\mathcal{E}_{\kappa,\lambda}(Z) := \mathcal{E}_{\kappa,\lambda}(Z,0)$ is a holomorphic modular form.*

Proof. We have

$$\sum_{\sigma\in\Gamma(L)_\lambda\backslash\Gamma(L)} \left| \frac{1}{(N_\lambda\lambda, \sigma(Z_L))^\kappa} \left(\frac{q(Y)}{|(N_\lambda\lambda, \sigma(Z_L))|^2} \right)^s \right| = q(Y)^{\mathrm{Re}(s)} \sum_{\sigma\in\Gamma_\lambda\backslash\Gamma} |(\sigma^{-1}(N_\lambda\lambda), Z_L)|^{-2\,\mathrm{Re}(s)-\kappa}$$
$$\leq q(Y)^{\mathrm{Re}(s)} \sum_{\lambda\in\mathrm{Iso}(L)} |(\lambda, Z_L)|^{-2\,\mathrm{Re}(s)-\kappa}.$$

Using

$$2q_Z(\lambda) = 4q(\lambda_Z) = 2(\lambda_Z, \lambda_Z) = \frac{(\lambda, X_L)^2 + (\lambda, Y_L)^2}{q(Y)} = \frac{|(\lambda, Z)|^2}{q(Y)}$$

for $q(\lambda) = 0$, we obtain

$$q(Y)^{\mathrm{Re}(s)} \sum_{\lambda\in\mathrm{Iso}(L)} |(\lambda, Z_L)|^{-2\,\mathrm{Re}(s)-\kappa} = 2^{-s-\frac{\kappa}{2}} q(Y)^{-\frac{\kappa}{2}} \sum_{\lambda\in\mathrm{Iso}(L)} q_Z(\lambda)^{-\mathrm{Re}(s)-\frac{\kappa}{2}}.$$

For varying Z in a compact subset there is a positive definite quadratic form $\tilde{q}$ with $\tilde{q}(\lambda) \leq q_Z(\lambda)$ for all $\lambda \in L$. By the next lemma the Eisenstein series converges uniformly on compact subsets of $\mathbb{H}_l$ for $\mathrm{Re}(s) > \frac{l-\kappa}{2}$. □

Lemma 6.1.2. *Let L be an even lattice of signature (b^+, b^-), $b^- \in 2\mathbb{Z}$ with quadratic form q and let $\tilde{q}$ be a positive definite quadratic form on the underlying module of L. Then the series*

$$\sum_{\lambda\in\mathrm{Iso}(L)} \tilde{q}(\lambda)^{-s}$$

converges for $\mathrm{Re}(s) > \frac{b^+ + b^-}{2} - 1$.

Proof. By going to appropriate finite index sublattices, we can assume $L = L^+ \oplus L^-$ for some positive definite and negative definite lattices L^+, L^- of rank b^+, b^-. Now let

$$q_{L^+,L^-}(\lambda) = q(\lambda^+) - q(\lambda^-)$$

be the positive definite majorant corresponding to the above decomposition. Since all norms on $\mathbb{R}^n$ are equivalent, there is a constant $C > 0$ with

$$C q_{L^+,L^-}(\lambda) \leq \tilde{q}(\lambda).$$

Now we have

$$\sum_{\lambda\in\mathrm{Iso}(L)} \tilde{q}(\lambda)^{-s} \ll \sum_{\substack{\lambda^\pm\in L^\pm \\ q(\lambda^+)=-q(\lambda^-)}} (q(\lambda^+) - q(\lambda^-))^{-s}$$
$$= 2^{-s} \sum_{\lambda^+\in L^+} q(\lambda^+)^{-s} |\{\lambda^- \in L^- \mid -q(\lambda^-) = q(\lambda^+)\}|.$$

Since

$$|\{\lambda^- \in L^- \mid -q(\lambda^-) = q(\lambda^+)\}| \ll q(\lambda^+)^{\frac{b^-}{2}-1}$$

for even b^- (using theta functions), we obtain

$$\sum_{\lambda\in\mathrm{Iso}(L)} \tilde{q}(\lambda)^{-s} \ll \sum_{\lambda^+\in L^+} q(\lambda^+)^{-s+\frac{b^-}{2}-1}.$$

Now the last series is just the Epstein zeta function attached to L^+ which converges for $\mathrm{Re}(s) > \frac{b^+ + b^-}{2} - 1$. □

Lemma 6.1.3. *The Eisenstein series $\mathcal{E}_{\kappa,\lambda}(Z,s)$ is orthogonal to cusp forms, i.e. we have*

$$\int_{\Gamma(L)\backslash\mathbb{H}_l} \mathcal{E}_{\kappa,\lambda}(Z,s)\overline{F(Z)}q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l} = 0$$

for all cusp forms F and $\mathrm{Re}(s) \gg 0$.

Proof. The integral exists for $\mathrm{Re}(s) \gg 0$ since $\mathcal{E}_{\kappa,\lambda}(Z,s)$ is bounded by $q(Y)^s$ on every Siegel domain for $\mathrm{Re}(s) \gg 0$ and cusp forms decay exponentially. Assume now without loss of generality that $N_\lambda \lambda = z$. We have

$$\begin{aligned}
&\int_{\Gamma(L)\backslash\mathbb{H}_l} \mathcal{E}_{\kappa,z}(Z,s)\overline{F(Z)}q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\\
&= \sum_{\sigma\in\Gamma(L)_z\backslash\Gamma(L)} \int_{\Gamma(L)\backslash\mathbb{H}_l} (z,\sigma(Z_L))^{-\kappa}\left(\frac{q(Y)}{|(z,\sigma(Z_L))|^2}\right)^s \overline{F(Z)}q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\\
&= \int_{\Gamma(L)_z\backslash\mathbb{H}_l} \overline{F(Z)}q(Y)^{s+\kappa}\frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}.
\end{aligned}$$

Now plug in the Fourier expansion of f to obtain

$$\sum_{\substack{\lambda\in K\\ \lambda\in\mathcal{C}}} \overline{a_z(\lambda)} \int_{\Gamma(L)_z\backslash\mathbb{H}_l} \overline{e(\lambda,Z)}q(Y)^{s+\kappa}\frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}.$$

Now the inner integral vanishes since $\lambda \neq 0$ and

$$\begin{aligned}
&\int_{\Gamma(L)_z\backslash\mathbb{H}_l} \overline{e(\lambda,Z)}q(Y)^{s+\kappa}\frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\\
&= \int_{\Gamma(L)_z\backslash\mathbb{H}_l} \overline{e(\lambda,E(z,X_0)Z)}q(\mathrm{Im}(E(z,X_0)Z))^{s+\kappa}\frac{\mathrm{d}X\mathrm{d}Y}{q(\mathrm{Im}(E(z,X_0)Z))^l}\\
&= \overline{e(\lambda,X_0)} \int_{\Gamma(L)_z\backslash\mathbb{H}_l} \overline{e(\lambda,Z)}q(Y)^{s+\kappa}\frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}
\end{aligned}$$

for all $X_0 \in W(\mathbb{R})$. □

Recall the map

$$\pi_L : \Gamma(L)\backslash \operatorname{Iso}_0(L') \to \operatorname{Iso}(L'/L).$$

For $\delta \in \operatorname{Iso}(L'/L)$ we let

$$\mathcal{G}_{\kappa,\delta} := \sum_{\lambda\in\pi_L^{-1}(\delta)} \mathcal{E}_{\kappa,\lambda},$$

in particular, $\mathcal{G}_{\kappa,\delta} = 0$ if the preimage is empty and if $\delta = -\delta$ for odd κ. More generally, for $\mathfrak{w} \in \operatorname{Iso}(\mathbb{C}[L'/L])$ we let

$$\mathcal{G}_{\kappa,\mathfrak{w}} := \sum_{\delta\in\operatorname{Iso}(L'/L)} \mathfrak{w}_\delta \mathcal{G}_{\kappa,\delta}.$$

For $\delta \in \operatorname{Iso}(L'/L)$ of order N_δ and χ a Dirichlet character of modulus N_δ we define

$$\mathcal{G}_{\kappa,\delta,\chi} := \sum_{m\in(\mathbb{Z}/N_\delta\mathbb{Z})^\times} \chi(m)\mathcal{G}_{\kappa,m\delta}.$$

By orthogonality of characters, the space generated by $\mathcal{G}_{\kappa,\delta}$ and the space generated by $\mathcal{G}_{\kappa,\delta,\chi}$ coincide. We have

$$\Omega_\kappa \mathcal{E}_{\kappa,\lambda}(Z,s) = s\left(s+\kappa-\frac{l}{2}\right)\mathcal{E}_{\kappa,\lambda}(Z,s),$$

i.e. the harmonic points of $\mathcal{E}_{\kappa,\lambda}$ are $s = 0$ and $s = \frac{l}{2} - \kappa$. Moreover, we have

$$\begin{aligned}
R_\kappa \mathcal{E}_{\kappa,\lambda}(Z,s) &= (s+\kappa)\left(\frac{l}{2}-1-\kappa-s\right)\mathcal{E}_{\kappa+2,\lambda}(Z,s-1),\\
L_\kappa \mathcal{E}_{\kappa,\lambda}(Z,s) &= s\left(\frac{l}{2}-1-s\right)\mathcal{E}_{\kappa-2,\lambda}(Z,s+1).
\end{aligned}$$

6.2 Theta Functions

As before, let L be an even lattice of signature $(2,l)$ and let $\kappa \in \mathbb{Z}$. Consider the polynomial $p(x_1,x_2) := (x_1 + ix_2)^\kappa$ on $\mathbb{R}^{(2,l)}$ and recall the identification $\mathcal{K}^+ \to \operatorname{Gr}^+(V(\mathbb{R}))$, $[Z] = [X+iY] \mapsto \mathbb{R}X + \mathbb{R}Y$. For $Z = X + iY \in \tilde{\mathcal{K}}^+$ consider the isometry

$$\nu_Z : \mathbb{R}X + \mathbb{R}Y \to \mathbb{R}^{(2,0)}, \quad aX + bY \mapsto |Y|\begin{pmatrix} a \\ b \end{pmatrix}.$$

By abuse of notation we will also write ν_Z for any isometry which equals ν_Z on $\mathbb{R}X + \mathbb{R}Y$. For $\lambda \in V(\mathbb{R})$ we write λ_Z and $\lambda_{Z^\perp}$ for the projection of λ onto $\mathbb{R}X + \mathbb{R}Y$ and its orthogonal complement. The corresponding positive definite majorant is denoted by q_Z. The map

$$\lambda \mapsto p(\nu_Z(\lambda)) = \frac{(\lambda, Z)^\kappa}{|Y|^\kappa}$$

is then well-defined although ν_Z is only well-defined on $\mathbb{R}X+\mathbb{R}Y$, since p only depends on the positive definite variables. Hence we define

$$\Theta_L(\tau,Z) := \frac{i^\kappa}{2|Y|^\kappa}\Theta(\tau,\nu_Z,p) = \frac{v^{\frac{l}{2}}}{2(-2i)^\kappa}\sum_{\lambda\in L'}\frac{(\lambda,Z)^\kappa}{q(Y)^\kappa}\mathfrak{e}_\lambda(\tau q(\lambda_Z)+\overline{\tau}q(\lambda_{Z^\perp}))$$

which is modular of weight $k=1-\frac{l}{2}+\kappa$ in $\tau=u+iv\in\mathbb{H}$ and weight κ in $Z=X+iY\in\tilde{\mathcal{K}}^+$. Let $z\in\mathrm{Iso}_0(L)$ of level N_z and $z'\in L'$ with $(z,z')=1$ write $K=L\cap z^\perp\cap z'^\perp$. Further, let $d\in\mathrm{Iso}_0(K)$ of level N_d and $d'\in K'$ with $(d,d')=1$ and $D=K\cap d^\perp\cap d'^\perp$. Moreover, we let $\tilde{z}=z'-q(z')z, \tilde{d}=d'-q(d')d$. Recall the corresponding upper half-plane model $\mathbb{H}_l$ so that we can view Θ_L as a modular form on $\mathbb{H}_l$. For $Z\in\mathbb{H}_l$ we use the obvious notion for $\nu_Z,\lambda_Z,\ldots$ etc. and we write Z_L for the corresponding element in $\tilde{\mathcal{K}}^+$. We then have for $Z=X+iY\in\mathbb{H}_l$.

$$\begin{aligned}
Z_L &= Z+\tilde{z}-q(Z)z\\
X_L &= X+\tilde{z}+(q(Y)-q(X))z\\
Y_L &= Y-(X,Y)z\\
z_Z &= \frac{(z,X_L)}{X_L^2}X_L+\frac{(z,Y_L)}{Y_L^2}Y_L = \frac{1}{Y_L^2}X_L\\
|z_Z| &= \frac{1}{|Y|}\\
\mu_K &= X\\
\omega^+ &= \mathbb{R}Y\\
\lambda_{\omega^+} &= \frac{(\lambda,Y)}{Y^2}Y \quad \text{for } \lambda\in K'\\
|\lambda_{\omega^+}| &= \frac{|(\lambda,Y)|}{|Y|}.
\end{aligned}$$

We will use these frequently. For $\lambda\in K\otimes\mathbb{R}$ we have

$$\begin{aligned}
p(\nu_Z(\lambda)) &= \left(\frac{(\lambda,X+iY)}{|Y|}\right)^\kappa\\
&= \sum_h(\lambda,X)^h\binom{\kappa}{h}i^{\kappa-h}|Y|^{-\kappa}(\lambda,Y)^{\kappa-h}\\
&= \sum_h(\lambda,z_Z)^h\binom{\kappa}{h}i^{\kappa-h}|Y|^{2h-\kappa}(\lambda,Y)^{\kappa-h},
\end{aligned}$$

hence

$$p_{\omega,h}(\omega(\lambda)) := p_{\omega,h,0}(\omega(\lambda)) = \binom{\kappa}{h}i^{\kappa-h}|Y|^h(\lambda,Y/|Y|)^{\kappa-h},$$

i.e. $p_{\omega,h}(x_2) = \binom{\kappa}{h} i^{\kappa-h}|Y|x_2^{\kappa-h}$ and $p_{\omega,h^+,h^-} = 0$ otherwise and we will write $p_{Y,h}$ instead of $p_{\omega,h}$. We expand $\Theta_L(\tau,Z)$ with respect to z,z',K to obtain

$$\frac{1}{2\sqrt{2}i^\kappa|Y|^{\kappa-1}}\Theta_K(\tau,Y/|Y|,p_{Y,0})\sum_{m_z\in\mathbb{Z}/N_z\mathbb{Z}}\mathfrak{e}_{\frac{m_z z}{N_z}}$$
$$+\frac{1}{2\sqrt{2}i^\kappa|Y|^{\kappa-1}}\sum_{M\in\Gamma_\infty\backslash\Gamma}\sum_{h=0}^{\kappa}\sum_{n=1}^{\infty}n^h\frac{(c\tau+d)^{\frac{l}{2}-1-\kappa}}{(-2i)^h\operatorname{Im}(M\tau)^h}\exp\left(-\frac{q(Y)n^2\pi}{\operatorname{Im}(M\tau)}\right)$$
$$\times\rho_L^{-1}(M)\left(\Theta_K(M\tau,nX,0,Y/|Y|,p_{Y,h})\sum_{m_z\in\mathbb{Z}/N_z\mathbb{Z}}\mathfrak{e}_{\frac{m_z z}{N_z}}\left(-\frac{m_z n}{N_z}\right)\right),$$

where we have used the identification of Section 5.4. Next we expand $\Theta_K(\tau,Y,p_{Y,0})$ in the first summand with respect to d,d',D to obtain

$$\Theta_L(\tau,Z)=\delta_{\kappa,0}\frac{q(Y)}{2y_1}\Theta_D(\tau)\sum_{\substack{m_d\in\mathbb{Z}/N_d\mathbb{Z}\\ m_z\in\mathbb{Z}/N_z\mathbb{Z}}}\mathfrak{e}_{\frac{m_d d}{N_d}+\frac{m_z z}{N_z}}$$
$$+\frac{q(Y)}{|Y|^\kappa 2^{\kappa+1}y_1}\sum_{M\in\Gamma_\infty\backslash\Gamma}\sum_{n=1}^{\infty}n^\kappa\frac{(c\tau+d)^{\frac{l}{2}-1-\kappa}}{\operatorname{Im}(M\tau)^\kappa}\exp\left(-\frac{q(Y)\pi n^2}{\operatorname{Im}(M\tau)y_1^2}\right)$$
$$\times\rho_L(M)^{-1}\left(\Theta_D(M\tau,nY_D/(|Y|y_1),0)\sum_{\substack{m_d\in\mathbb{Z}/N_d\mathbb{Z}\\ m_z\in\mathbb{Z}/N_z\mathbb{Z}}}\mathfrak{e}_{\frac{m_d d}{N_d}+\frac{m_z z}{N_z}}\left(-\frac{m_d n}{N_d}\right)\right)$$
$$+\frac{i^\kappa}{2\sqrt{2}|Y|^{\kappa-1}}\sum_{M\in\Gamma_\infty\backslash\Gamma}\sum_{h=0}^{\kappa}\sum_{n=1}^{\infty}n^h\frac{(c\tau+d)^{\frac{l}{2}-1-\kappa}}{(-2i)^h\operatorname{Im}(M\tau)^h}\exp\left(-\frac{q(Y)n^2\pi}{\operatorname{Im}(M\tau)}\right)$$
$$\times\rho_L^{-1}(M)\left(\Theta_K(M\tau,nX,0,Y/|Y|,p_{Y,h})\sum_{m_z\in\mathbb{Z}/N_z Z}\mathfrak{e}_{\frac{m_z z}{N_z}}\left(-\frac{m_z n}{N_z}\right)\right).$$

We want to get bounds for $\Theta_L(\tau,Z)$ and $\Omega_k\overline{\Theta_L(\tau,Z)}$, where Ω_κ is the weight κ Laplace operator in Z. According to [Zem15, Proposition 2.5] we have

$$\Omega_\kappa\overline{\Theta_L(\tau,Z)}=\overline{\Delta_k\Theta_L(\tau,Z)},$$

where Δ_k is the weight $k=\kappa-\frac{l}{2}+1$ Laplace operator in τ. Let

$$f(\tau,Z,n):=\frac{\sqrt{q(Y)\pi}n}{v^\kappa y_1}\exp\left(-\frac{q(Y)\pi n^2}{vy_1^2}\right)\Theta_D(\tau,nY_D/(|Y|y_1),0)$$

and

$$g_h(\tau,Z,n):=|Y|^{1-\kappa}v^{-h}\exp\left(-\frac{q(Y)n^2\pi}{v}\right)\Theta_K(\tau,nX,0,Y/|Y|,p_{Y,h})$$

so that $\Theta_L(\tau,Z)$ is given by

$$\delta_{\kappa,0}\frac{q(Y)}{2y_1}\Theta_D(\tau)\sum_{\substack{m_d\in\mathbb{Z}/N_d\mathbb{Z}\\ m_z\in\mathbb{Z}/N_z\mathbb{Z}}}\mathfrak{e}_{\frac{m_d d}{N_d}+\frac{m_z z}{N_z}}$$

$$+\frac{|Y|^{1-\kappa}}{2^{\kappa+1}\sqrt{2}}\sum_{M\in\Gamma_\infty\backslash\Gamma}\sum_{n=1}^{\infty}n^{\kappa-1}\left(f(\tau,Z,n)\sum_{\substack{m_d\in\mathbb{Z}/N_d\mathbb{Z}\\ m_z\in\mathbb{Z}/N_z\mathbb{Z}}}\mathfrak{e}_{\frac{m_d d}{N_d}+\frac{m_z z}{N_z}}\left(-\frac{m_d n}{N_d}\right)\right)\Bigg|_{k,L}M$$

$$+\frac{i^\kappa}{2\sqrt{2}}\sum_{M\in\Gamma_\infty\backslash\Gamma}\sum_{h=0}^{\kappa}\sum_{n=1}^{\infty}\frac{n^h}{(-2i)^h}\left(g_h(\tau,Z,n)\sum_{m_z\in\mathbb{Z}/N_z Z}\mathfrak{e}_{\frac{m_z z}{N_z}}\left(-\frac{m_z n}{N_z}\right)\right)\Bigg|_{k,L}M.$$

Since the Laplace operator commutes with the slash operator, we only have to find sufficient bounds for $f,\Delta_k f$ and $g_h,\Delta_k g_h$. We will need the following elementary lemma.

Lemma 6.2.1. *Let* $a>0, b, c\geq 0, n\in\mathbb{N}$.

(i) The function $x^n\exp(-ax^2)$ *has a maximum given by* $\left(\frac{n}{2ae}\right)^{\frac{n}{2}}$.

(ii) The function $x^n\exp(-ax)$ *has a maximum on* $x\geq 0$ *given by* $\left(\frac{n}{ae}\right)^n$.

(iii) The function $bx+\frac{c}{x}$ *has a minimum on* $x>0$ *given by* $2\sqrt{bc}$

We start with a bound for

$$f(\tau,Z,n)=\frac{\sqrt{q(Y)}n}{v^\kappa y_1}\exp\left(-\frac{q(Y)\pi n^2}{vy_1^2}\right)\Theta_D(\tau,nY_D/(|Y|y_1),0).$$

Observe that the absolute value of f and $\Delta_k f$ can be bounded by finite sums of the form

$$v^{-i}(yn)^j\exp\left(-\frac{(yn)^2}{v}\right)\sum_{\lambda\in D'}q(\lambda)^l\exp(-2\pi vq(\lambda)).$$

We have

Lemma 6.2.2. *Let*

$$f_{i,j,l}(v,y,n):=v^{-i}(yn)^j\exp\left(-\frac{3(yn)^2}{v}\right)\sum_{\lambda\in D'}q(\lambda)^l\exp(-2\pi vq(\lambda)).$$

Assume that $v\in\mathbb{R}_{>0}, y>C>0$ *and* $n\geq 1$. *Then we have the bound*

$$f_{i,j,l}(v,y,n)\ll v^{-i+j}\exp\left(-\frac{C^2n^2}{3v}\right),$$

where the implied constant is independent of v,y,n. *In particular, we obtain with* $y^2=\frac{q(Y)\pi}{y_1^2}, C^2=\frac{\pi}{t^2}$

$$f(\tau,Z,n)\ll v^{s_1}\exp\left(-\frac{\pi n^2}{3vt^2}\right)$$

and

$$\Delta_k f(\tau,Z,n)\ll v^{s_2}\exp\left(-\frac{\pi n^2}{3vt^2}\right)$$

for some $s_1,s_2\in\mathbb{R}$.

Proof. We split up the exponential term and use Lemma 6.2.1 to obtain

$$(yn)^j \exp\left(-\frac{(yn)^2}{v}\right) = (yn)^j \exp\left(-\frac{(yn)^2}{3v}\right)\exp\left(-\frac{(yn)^2}{3v}\right)\exp\left(-\frac{(yn)^2}{3v}\right)$$
$$\ll v^j \exp\left(-\frac{(yn)^2}{3v}\right)\exp\left(-\frac{C^2}{3v}\right).$$

This yields

$$\begin{aligned}
&f_{i,j,l}(v,y,n)\\
&\ll v^{-i+j}\exp\left(-\frac{(yn)^2}{3v}\right)\sum_{\lambda\in D'} q(\lambda)^l \exp\left(-2\pi v q(\lambda) - \frac{C^2}{3v}\right)\\
&\ll v^{-i+j}\exp\left(-\frac{(yn)^2}{3v}\right)\sum_{\lambda\in D'} q(\lambda)^l \exp\left(-2C\sqrt{2\pi q(\lambda)}\right) \ll \exp\left(-\frac{(yn)^2}{3v}\right)
\end{aligned}$$

where we have used Lemma 6.2.1 again. □

Next consider

$$g_h(\tau, Z, n) = |Y|^{1-\kappa} v^{-h} \exp\left(-\frac{q(Y)n^2\pi}{v}\right)\Theta_K(\tau, nX, 0, Y/|Y|, p_{Y,h}).$$

The theta function is given by $v^{\frac{l-1}{2}}$ times

$$\begin{aligned}
&\sum_{\lambda\in K'} \exp\left(-\frac{\Delta}{8\pi v}\right)(p_{Y,h})(\omega_Y(\lambda))\mathfrak{e}_\lambda\left(uq(\lambda) + iv\left(\frac{(\lambda,Y)^2}{Y^2} - q(\lambda)\right) - (\lambda, nX)\right)\\
&= \sum_{\lambda\in K'} \exp\left(-\frac{\Delta}{8\pi v}\right)(p_{Y,h})(\omega_Y(\lambda))\mathfrak{e}_\lambda\left(uq(\lambda) - (\lambda, nX)\right)\exp\left(-2\pi v\left(\frac{(\lambda,Y)^2}{Y^2} - q(\lambda)\right)\right).
\end{aligned}$$

The terms

$$\exp\left(-\frac{\Delta}{8\pi v}\right)(p_{Y,h})(\omega_Y(\lambda))$$

are given by a finite sum of constants times terms of the form

$$v^{-j}|Y|^h(\lambda, Y/|Y|)^{\kappa-h-2j}.$$

Again, we see that the absolute value of g_h and $\Delta_k g_h$ can be bounded by a finite sum of terms of the form

$$|Y|^h v^{-i}\exp\left(-\frac{q(Y)n^2\pi}{v}\right)\sum_{\lambda\in K'}\left(\frac{(\lambda,Y)^2}{Y^2}\right)^j q(\lambda)^l \exp\left(-2\pi v\left(\frac{(\lambda,Y)^2}{Y^2} - q(\lambda)\right)\right).$$

Lemma 6.2.3. *Let $g_{h,i,j,l}(v,Y,n)$ denote the function*

$$|Y|^h v^{-i}\exp\left(-\frac{q(Y)n^2\pi}{v}\right)\sum_{\lambda\in K'}\left(\frac{(\lambda,Y)^2}{Y^2}\right)^j q(\lambda)^l \exp\left(-2\pi v\left(\frac{(\lambda,Y)^2}{Y^2} - q(\lambda)\right)\right)$$

for $v \in \mathbb{R}_{>0}, Y \in \mathcal{R}_t$ *and* $n \geq 1$. *Then*

$$g_{h,i,j,l}(v,Y,n) \ll v^{\frac{h}{2}-i-j} \exp\left(-\frac{q(Y)n^2\pi}{4v}\right),$$

where the implied constant is independent of v, Y, n. *In particular, we have*

$$g_h(\tau, Z, n) \ll v^{s_1} \exp\left(-\frac{q(Y)n^2\pi}{4v}\right)$$

and

$$\Delta_k g_h(\tau, Z, n) \ll v^{s_2} \exp\left(-\frac{q(Y)n^2\pi}{4v}\right)$$

for some $s_1, s_2 \in \mathbb{R}$ *independent of* v, Y, n, h.

Proof. We have

$$\left(\frac{(\lambda, Y)^2}{Y^2}\right)^j \leq (y_2/y_1\lambda_1^2 + y_1/y_2\lambda_2^2 + q(\lambda_D))^j$$

and by Lemma 3.4.3 the exponential term can be bounded by

$$\exp\left(-2\pi\varepsilon v\left(y_2/y_1\lambda_1^2 + y_1/y_2\lambda_2^2 + q(\lambda_D)\right)\right)$$

on $\mathcal{R}_t$, which yields the bound

$$\begin{aligned}
&g_{h,i,j,l}(v,Y,n)\\
&\leq |Y|^h v^{-i} \exp\left(-\frac{q(Y)n^2\pi}{v}\right) \sum_{\lambda \in K'} q(\lambda)^l \exp\left(-\pi\varepsilon v\left(y_2/y_1\lambda_1^2 + y_1/y_2\lambda_2^2 + q(\lambda_D)\right)\right)\\
&\times (y_2/y_1\lambda_1^2 + y_1/y_2\lambda_2^2 + q(\lambda_D))^j \exp\left(-\pi\varepsilon v\left(y_2/y_1\lambda_1^2 + y_1/y_2\lambda_2^2 + q(\lambda_D)\right)\right).
\end{aligned}$$

Again we split up the exponential term Lemma 6.2.1 to obtain

$$\begin{aligned}
|Y|^h \exp\left(-\frac{q(Y)n^2\pi}{v}\right) &= |Y|^h \exp\left(-\frac{q(Y)n^2\pi}{4v}\right) \exp\left(-\frac{3q(Y)n^2\pi}{4v}\right)\\
&\ll v^{\frac{h}{2}} \exp\left(-\frac{3q(Y)n^2\pi}{4v}\right).
\end{aligned}$$

We obtain

$$\begin{aligned}
&g_{h,i,j,l}(v,Y,n)\\
&\ll v^{-i-j+\frac{h}{2}} \exp\left(-\frac{3q(Y)n^2\pi}{4v}\right) \sum_{\lambda \in K'} q(\lambda)^l \exp\left(-\pi\varepsilon v\left(y_2/y_1\lambda_1^2 + y_1/y_2\lambda_2^2 + q(\lambda_D)\right)\right),
\end{aligned}$$

where we have used Lemma 6.2.1 again. Now use $y_1 > t^{-1}, y_2/y_1 > t^{-2}, q(Y) > \frac{y_1 y_2}{1+t^4}$ and $q(Y) > t^{-4}$ on every Siegel domain $\mathcal{S}_t$ to obtain

$$g_{h,i,j,l}(v,Y,n)$$

$$
\begin{aligned}
&\ll v^{-i-j+\frac{h}{2}} \exp\left(-\frac{q(Y)n^2\pi}{4v}\right) \\
&\sum_{\lambda\in K'} q(\lambda)^l \exp\left(-\pi\varepsilon v\left(\frac{y_2\lambda_1^2}{y_1}+q(\lambda_D)\right)-\frac{q(Y)n^2\pi}{4v}-2\pi\varepsilon v\frac{y_1\lambda_2^2}{y_2}-\frac{q(Y)n^2\pi}{4v}\right) \\
&\leq v^{-i-j+\frac{h}{2}} \exp\left(-\frac{q(Y)n^2\pi}{4v}\right) \\
&\sum_{\lambda\in K'} q(\lambda)^l \exp\left(-\pi\varepsilon v\left(\frac{\lambda_1^2}{t^2}+q(\lambda_D)\right)-\frac{\pi}{4t^4 v}-2\pi\varepsilon v\frac{y_1\lambda_2^2}{y_2}-\frac{y_1y_2\pi}{4v(1+t^4)}\right) \\
&\ll v^{-i-j+\frac{h}{2}} \exp\left(-\frac{q(Y)n^2\pi}{4v}\right)\sum_{\lambda\in K'} q(\lambda)^l \exp\left(-\pi\sqrt{\varepsilon}\left(|\lambda_1|+t|\lambda_D|+\frac{\sqrt{2}|\lambda_2|}{t\sqrt{1+t^4}}\right)\right) \\
&\ll v^{-i-j+\frac{h}{2}} \exp\left(-\frac{q(Y)n^2\pi}{4v}\right).
\end{aligned}
$$

□

Proposition 6.2.4. *Let $C \in \mathbb{R}_{>0}$. The theta function $\Theta_L(\tau, Z)$ is bounded by a constant times*

$$v^s|Y|^{1-\kappa}\left(1+\delta_{\kappa,0}\frac{|Y|}{y_1}\right)$$

for all $Z \in \mathcal{S}_t$ and $\tau \in \mathbb{H}$ with $\mathrm{Im}(\tau) > C$ and some $s \in \mathbb{R}$. The constant only depends on the constant C. Similarly, the function $\Omega_\kappa\overline{\Theta_L} = \overline{\Delta_k\Theta_L}$ is bounded by

$$v^s|Y|^{1-\kappa}$$

for all $Z \in \mathcal{S}_t$ and $\tau \in \mathbb{H}$ with $\mathrm{Im}(\tau) > C$ and some $s \in \mathbb{R}$.

Proof. This is now a direct consequence of the previous two Lemmas together with the fact that the Laplace operator Δ_k commutes with the slash operator $|_{k,L}$. □

Corollary 6.2.5. *Both, the theta function Θ_L and $\Omega_\kappa\overline{\Theta_L}$ are square-integrable for $l \geq 3$ and $\kappa = \frac{l}{2} - 1 + k > 0$.*

Proof. The square is bounded by

$$q(Y)^{1-\kappa}$$

and hence we have to show that

$$\int_{\mathcal{S}_t} q(Y)\frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l} < \infty.$$

By Lemma 3.4.4 this is the case if $l \geq 3$. □

6.3 Orthogonal Eisenstein Series as Theta Lifts

Write $\Phi_\beta(Z,s) := \frac{i^\kappa}{2|Y|^\kappa}\Phi_\beta(\nu_Z,p,s)$. Then $\Phi_\beta(Z,s)$ and more generally $\Phi_{\mathfrak{v}}(Z,s)$ is modular with respect to $\Gamma(L)$. The functional equation for $E_{k,\mathfrak{v}}(\tau,s)$ yields a functional equation for the lift $\Phi_{\mathfrak{v}}(Z,s)$. To be precise, we have

$$\Phi_{\mathfrak{v}}(Z,s) = \frac{1}{2}\sum_{\alpha\in\mathrm{Iso}(L'/L)} c_{k,\mathfrak{v}}(\alpha,0,s)\Phi_\alpha(\tau,1-k-s).$$

Theorem 6.3.1. *The theta lift is equal to*

$$\begin{aligned}\Phi_\beta(Z,s) &= \frac{\Gamma(s+\kappa)}{(-2\pi i)^\kappa\pi^s}\sum_{\lambda\in\Gamma(L)\backslash\mathrm{Iso}_0(L)} N_\lambda^{2s+\kappa}\zeta_+^{k_{\lambda\beta}}(2s+\kappa)\mathcal{E}_{\kappa,\lambda}(Z,s)\\ &= \frac{\Gamma(s+\kappa)}{(-2\pi i)^\kappa\pi^s}\sum_{\delta\in\mathrm{Iso}(L'/L)} N_\delta^{2s+\kappa}\zeta_+^{k_{\delta\beta}}(2s+\kappa)\mathcal{G}_{\kappa,\delta}(Z,s),\end{aligned}$$

where $k_{\delta\beta}\in\mathbb{Z}/N_\delta\mathbb{Z}$ with $\beta = k_{\delta\beta}\delta$ (and the corresponding summands vanish if such a $k_{\delta\beta}$ does not exist), N_λ is the level of λ and N_δ is the order of δ.

Proof. By Theorem 5.2.1 we have

$$\Phi_\beta(Z,s) = \frac{\Gamma(s+\kappa)}{(-2i)^\kappa}\sum_{\lambda\in\mathrm{Iso}_0(L')}\zeta_+^{k_{\lambda\beta}}(2s+\kappa)\frac{(\lambda,\overline{Z_L})^\kappa}{q(Y)^\kappa}\frac{1}{(2\pi q_Z(\lambda))^{s+\kappa}},$$

where the summands with $\beta + L\cap\mathbb{Z}\lambda = \varnothing$ are meant to be zero. For $q(\lambda)=0$ we have

$$2q_Z(\lambda) = 4q(\lambda_Z) = \frac{|(\lambda,Z_L)|^2}{q(Y)}$$

and thus

$$\begin{aligned}\Phi_\beta(Z,s) &= \frac{\Gamma(s+\kappa)}{(-2\pi i)^\kappa\pi^s}\sum_{\lambda\in\mathrm{Iso}_0(L')}\frac{\zeta_+^{k_{\lambda\beta}}(2s+\kappa)}{(\lambda,Z_L)^\kappa}\left(\frac{q(Y)}{|(\lambda,Z_L)|^2}\right)^s\\ &= \frac{\Gamma(s+\kappa)}{(-2\pi i)^\kappa\pi^s}\sum_{\lambda\in\Gamma(L)\backslash\mathrm{Iso}_0(L')}\sum_{\sigma\in\Gamma(L)_\lambda\backslash\Gamma(L)}\frac{\zeta_+^{k_{\sigma(\lambda)\beta}}(2s+\kappa)}{(\sigma(\lambda),Z_L)^\kappa}\left(\frac{q(Y)}{|(\sigma(\lambda),Z_L)|^2}\right)^s.\end{aligned}$$

Since we have $k_{\sigma(\lambda)\beta} = k_{\lambda\beta}$ and $\sigma(\beta+L) = \beta+L$ for $\sigma\in\Gamma(L)$, this yields for $\Phi_\beta(Z,s)$

$$\begin{aligned}&\frac{\Gamma(s+\kappa)}{(-2\pi i)^\kappa\pi^s}\sum_{\lambda\in\Gamma(L)\backslash\mathrm{Iso}_0(L')}\zeta_+^{k_{\lambda\beta}}(2s+\kappa)\sum_{\sigma\in\Gamma(L)_\lambda\backslash\Gamma(L)}\frac{1}{(\sigma(\lambda),Z_L)^\kappa}\left(\frac{q(Y)}{|(\sigma(\lambda),Z_L)|^2}\right)^s\\ &= \frac{\Gamma(s+\kappa)}{(-2\pi i)^\kappa\pi^s}\sum_{\lambda\in\Gamma(L)\backslash\mathrm{Iso}_0(L)} N_\lambda^{2s+\kappa}\zeta_+^{k_{\lambda\beta}}(2s+\kappa)\mathcal{E}_{\kappa,\lambda}(Z,s)\\ &= \frac{\Gamma(s+\kappa)}{(-2\pi i)^\kappa\pi^s}\sum_{\delta\in\mathrm{Iso}(L'/L)} N_\delta^{2s+\kappa}\zeta_+^{k_{\delta\beta}}(2s+\kappa)\mathcal{G}_{\kappa,\delta}(Z,s)\end{aligned}$$

□

In particular, we obtain a map from $\mathrm{Iso}(\mathbb{C}[L'/L])$ to the space of non-holomorphic Eisenstein series sending $\mathfrak{v} \in \mathrm{Iso}(\mathbb{C}[L'/L])$ to $\Phi_{\mathfrak{v}}$.

Theorem 6.3.2. *The theta lifts $\Phi_\beta(Z,s)$ generate the space of $\mathcal{G}_{\kappa,\delta,\chi}$ and hence the space of all Eisenstein series $\mathcal{G}_{\kappa,\delta}$ for $\Gamma(L)$. In particular, if π_L is injective, then the theta lift is surjective onto non-holomorphic Eisenstein series and if π_L is surjective, the theta lift is injective on non-holomorphic Eisenstein series.*

Proof. We first resort the sum. Instead of summing over $\delta \in L'/L$ with $q(\delta) = 0$ and $\beta = k_{\delta\beta}\delta$, we sum over all cyclic isotropic subgroups containing β and then the generators of the subgroup. This yields

$$\begin{aligned}
\Phi_\beta(Z,s) &= \frac{\Gamma(s+\kappa)}{(-2i)^\kappa \pi^{s+\kappa}} \sum_{\substack{\beta\in H \\ H \text{ isotropic}}} \sum_{\langle\delta\rangle = H} N_\delta^{2s+\kappa} \zeta_+^{k_{\delta\beta}} (2s+\kappa)\mathcal{G}_{\kappa,\delta}(Z,s) \\
&= \frac{\Gamma(s+\kappa)}{(-2\pi i)^\kappa \pi^{s}} \sum_{\substack{\beta\in H = \langle\delta\rangle \\ H \text{ isotropic}}} N_\delta^{2s+\kappa} \sum_{n\in(\mathbb{Z}/N_\delta\mathbb{Z})^\times} \zeta_+^{k_{n\delta\beta}} (2s+\kappa)\mathcal{G}_{\kappa,n\delta}(Z,s) \\
&= \frac{\Gamma(s+\kappa)}{(-2\pi i)^\kappa \pi^{s}} \sum_{\substack{\beta\in H = \langle\delta\rangle \\ H \text{ isotropic}}} N_\delta^{2s+\kappa} \sum_{n\in(\mathbb{Z}/N_\delta\mathbb{Z})^\times} \zeta_+^{n^* k_{\delta\beta}} (2s+\kappa)\mathcal{G}_{\kappa,n\delta}(Z,s),
\end{aligned}$$

where $n^* n = 1 \bmod N_\delta$. Let χ be a Dirichlet character of modulus N_β and recall

$$E_{k,\beta,\chi} := \sum_{m\in(\mathbb{Z}/N_\beta\mathbb{Z})^\times} \chi(m) E_{k,m\beta}.$$

Then the corresponding lift $\Phi_{\beta,\chi}(Z,s)$ is given by

$$\frac{\Gamma(s+\kappa)}{(-2\pi i)^\kappa \pi^{s}} \sum_{m\in(\mathbb{Z}/N_\beta\mathbb{Z})^\times} \chi(m) \sum_{\substack{m\beta\in H = \langle\delta\rangle \\ H \text{ isotropic}}} N_\delta^{2s+\kappa} \sum_{n\in(\mathbb{Z}/N_\delta\mathbb{Z})^\times} \zeta_+^{n^* k_{\delta m\beta}} (2s+\kappa)\mathcal{G}_{\kappa,n\delta}(Z,s).$$

Now for $m \in (\mathbb{Z}/N_\beta\mathbb{Z})^\times$ we have $m\beta \in H$ if and only if $\beta \in H$. Moreover, we have $n^* k_{\delta m\beta} = k_{\delta n^* m\beta}$ and thus $\Phi_{\beta,\chi}(Z,s)$ is equal to

$$\begin{aligned}
&\frac{\Gamma(s+\kappa)}{(-2\pi i)^\kappa \pi^{s}} \sum_{\substack{\beta\in H = \langle\delta\rangle \\ H \text{ isotropic}}} N_\delta^{2s+\kappa} \sum_{n\in(\mathbb{Z}/N_\delta\mathbb{Z})^\times} \mathcal{G}_{\kappa,n\delta}(Z,s) \sum_{m\in(\mathbb{Z}/N_\beta\mathbb{Z})^\times} \chi(m) \zeta_+^{n^* k_{\delta m\beta}} (2s+\kappa) \\
&= \frac{\Gamma(s+\kappa)}{(-2\pi i)^\kappa \pi^{s}} \sum_{\substack{\beta\in H = \langle\delta\rangle \\ H \text{ isotropic}}} N_\delta^{2s+\kappa} \sum_{n\in(\mathbb{Z}/N_\delta\mathbb{Z})^\times} \chi(n)\mathcal{G}_{\kappa,n\delta}(Z,s) \sum_{m\in(\mathbb{Z}/N_\beta\mathbb{Z})^\times} \chi(m) \zeta_+^{k_{\delta m\beta}} (2s+\kappa) \\
&= \frac{\Gamma(s+\kappa)}{(-2\pi i)^\kappa \pi^{s}} \sum_{\substack{\beta\in H = \langle\delta\rangle \\ H \text{ isotropic}}} N_\delta^{2s+\kappa} \sum_{m\in(\mathbb{Z}/N_\beta\mathbb{Z})^\times} \chi(m) \zeta_+^{k_{\delta m\beta}} (2s+\kappa)\mathcal{G}_{\kappa,\delta,\chi}(Z,s).
\end{aligned}$$

Observe that if $\langle\beta\rangle$ is a maximal cyclic isotropic subgroup, then

$$\begin{aligned}\Phi_{\beta,\chi}(Z,s) &= \frac{\Gamma(s+\kappa)N_\beta^{2s+\kappa}}{(-2\pi i)^\kappa\pi^s}\sum_{m\in(\mathbb{Z}/N_\beta\mathbb{Z})^\times}\chi(m)\zeta_+^m(2s+\kappa)\mathcal{G}_{\kappa,\beta,\chi}(Z,s)\\ &= \frac{\Gamma(s+\kappa)N_\beta^{2s+\kappa}}{(-2\pi i)^\kappa\pi^s}L(2s+\kappa,\chi)\mathcal{G}_{\kappa,\beta,\chi}(Z,s).\end{aligned}$$

And inductive argument now shows the result. □

An immediate consequence is

Corollary 6.3.3. *The non-holomorphic Eisenstein series $\mathcal{G}_{\kappa,\delta}(Z,s)$ for $\delta\in\mathrm{Iso}(L'/L)$ have functional equations coming from the functional equations of $E_{k,\beta}(\tau,s)$ for $\beta\in\mathrm{Iso}(L'/L)$.*

The results of Section 5.3 yield the Fourier expansion

Theorem 6.3.4. *Let $z\in\mathrm{Iso}_0(L)$ of level N_z and let $z'\in L'$ with $(z,z')=1$. The theta lift $\Phi_\beta(Z,s)$ has the Fourier expansion in the cusp z given by*

$$\frac{i^\kappa}{2\sqrt{2}|Y|^{\kappa-1}}\Phi_\beta^K\left(\frac{Y}{|Y|},s\right)+\sum_{\lambda\in K'}b_\beta(\lambda,Y,s)e(\lambda,X),$$

where

$$\Phi_\beta^K\left(\frac{Y}{|Y|},s\right)=\begin{cases}0 & \text{if } (\beta,z)\neq 0 \bmod N_z,\\ \Phi_{\pi(\beta)}^K\left(\frac{Y}{|Y|},(ix_2)^\kappa,s\right) & \text{if } (\beta,z)=0 \bmod N_z,\end{cases}$$

and the Fourier coefficients are given by

$$\begin{aligned}b_\beta(0,Y,s)=\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}}\Bigg(&\frac{\Gamma(s+\kappa)N_z^{2s+\kappa}}{(-2\pi i)^\kappa\pi^s}q(Y)^s(\delta_{\beta,\frac{bz}{N_z}}+(-1)^\kappa\delta_{-\beta,\frac{bz}{N_z}})\zeta_+^b(2s+\kappa)\\ &+\frac{\Gamma(1-s-k+\kappa)N_z^{2-2s-2k+\kappa}}{(-2\pi i)^\kappa\pi^{1-s-k}}q(Y)^{1-s-k}c_{k,\beta}\left(\frac{bz}{N_z},0,s\right)\zeta_+^b(2-2s-2k+\kappa)\Bigg),\end{aligned}$$

for $q(\lambda)=0,\lambda\neq0$ the coefficient $b_\beta(\lambda,Y,s)$ is given by

$$\begin{aligned}&\frac{2|(\lambda,Y)|^{\frac12}}{2^\kappa}\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}}\left(\frac{q(Y)^s}{|(\lambda,Y)|^s}e\left(-\frac{(\lambda,\zeta)}{N_z}\right)\right.\\ &\times\sum_{n|\lambda}n^{2s-1+\kappa}(\delta_{\beta,\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z}}+(-1)^\kappa\delta_{-\beta,\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z}})e\left(\frac{nb}{N_z}\right)\\ &\times\sum_{h=0}^\infty\sum_{j=0}^\infty\frac{(-1)^j}{(4\pi|(\lambda,Y)|)^jj!}\binom{\kappa}{h}\frac{(\kappa-h)!}{(\kappa-h-2j)!}\left(\frac{(\lambda,Y)}{|(\lambda,Y)|}\right)^{\kappa-h}K_{s-\frac12+\kappa-h-j}(2\pi|(\lambda,Y)|)\end{aligned}$$

$$+\frac{q(Y)^{1-s-k}}{|(\lambda,Y)|^{1-s-k}}e\left(-\frac{(\lambda,\zeta)}{N_z}\right)\sum_{n|\lambda}n^{1-2s-2k+\kappa}c_{k,\beta}\left(\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z},0,s\right)e\left(\frac{nb}{N_z}\right)$$
$$\times\sum_{h=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{(4\pi|(\lambda,Y)|)^j j!}\binom{\kappa}{h}\frac{(\kappa-h)!}{(\kappa-h-2j)!}\left(\frac{(\lambda,Y)}{|(\lambda,Y)|}\right)^{\kappa-h}K_{\frac{1}{2}-s-k+\kappa-h-j}(2\pi|(\lambda,Y)|)\Bigg)$$

and for $q(\lambda)\neq 0$ the coefficient $b_\beta(\lambda,Y,s)$ is given by

$$\frac{1}{\sqrt{2}|Y|^{\kappa-1}}\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}}\sum_{h=0}^{\infty}(2i)^{-h}\sum_{j=0}^{\infty}\frac{(-1)^j i^h}{(8\pi)^j j!}\binom{\kappa}{h}\frac{(\kappa-h)!}{(\kappa-h-2j)!}|Y|^h\left(\frac{(\lambda,Y)}{|Y|}\right)^{\kappa-h-2j}$$
$$\times e\left(-\frac{(\lambda,\zeta)}{N_z}\right)\sum_{n|\lambda}n^{2j-\kappa+2h}e\left(\frac{nb}{N_z}\right)c_{k,\beta}\left(\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z},\frac{q(\lambda)}{n^2},s\right)$$
$$\times\int_0^\infty\exp\left(-\frac{\pi n^2}{2vz_Z^2}-\frac{2\pi v q_w(\lambda)}{n^2}\right)\mathcal{W}_s\left(4\pi\frac{q(\lambda)}{n^2}v\right)v^{-\frac{3}{2}+\kappa-h-j}\mathrm{d}v.$$

Proof. The terms for $h\neq 0\neq h^-$ in Theorem 5.3.1 do not exist. Hence we write h instead of h^+ and $p_{w,h}=p_{w,h,0}$ to obtain

$$\frac{i^\kappa|Y|^{1-\kappa}}{2\sqrt{2}}\Phi_\beta^K(w,p_{w,0},s)+\frac{i^\kappa|Y|^{1-\kappa}}{\sqrt{2}}\sum_h\frac{1}{(2i)^h}$$
$$\times\sum_{\lambda\in K'}\sum_{\substack{\delta\in L_0'/L\\ \pi(\delta)=\lambda+K}}\sum_{j=0}^{\infty}\frac{(-\Delta)^j(\overline{p}_{w,h})(w(\lambda))}{(8\pi)^j j!}\sum_{n=1}^{\infty}n^h e(n((\lambda,X)+(\delta,z')))$$
$$\times\int_{v=0}^{\infty}\exp\left(-\frac{\pi n^2 q(Y)}{v}-2\pi v q_w(\lambda)\right)c_{k,\beta}(\delta,q(\lambda),s,v)v^{-\frac{3}{2}+\kappa-h-j-t}\mathrm{d}v.$$

We have

$$\begin{aligned}p(\nu(\lambda))&=\left(\frac{(\lambda,X+iY)}{|Y|}\right)^\kappa\\&=\sum_h(\lambda,X)^h\binom{\kappa}{h}i^{\kappa-h}|Y|^{-\kappa}(\lambda,Y)^{\kappa-h}\\&=\sum_h(\lambda,z_{\nu^+})^h\binom{\kappa}{h}i^{\kappa-h}|Y|^{2h-\kappa}(\lambda,Y)^{\kappa-h},\end{aligned}$$

hence

$$p_{w,h}(w(\lambda))=\binom{\kappa}{h}i^{\kappa-h}|Y|^h(\lambda,Y/|Y|)^{\kappa-h}$$

and

$$(-\Delta)^j\overline{p}_{w,h}(w(\lambda))=(-1)^j i^{h-\kappa}\binom{\kappa}{h}\frac{(\kappa-h)!}{(\kappa-h-2j)!}|Y|^h(\lambda,Y/|Y|)^{\kappa-h-2j}.$$

The term for $\lambda=0$ is now given by

$$\frac{i^\kappa|Y|^{1-\kappa}}{\sqrt{2}}\sum_h\frac{1}{(2i)^h}\sum_{j=0}^{\infty}\frac{(-\Delta)^j(\overline{p}_{w,h})(0)}{(8\pi)^j j!}(\pi q(Y))^{\kappa-h-j-\frac{1}{2}}$$

$$\times \sum_{b,c\in\mathbb{Z}/N_z\mathbb{Z}} e\left(\frac{bc}{N_z}\right)\Bigg(\Gamma\left(\frac{1}{2}-s-\kappa+h+j\right)(\pi q(Y))^s$$
$$\times (\delta_{\beta,\frac{bz}{N_z}} + (-1)^\kappa \delta_{-\beta,\frac{bz}{N_z}})\zeta_+^c(-2s+1-2\kappa+h+2j)$$
$$+\Gamma\left(-\frac{1}{2}-\kappa+h+j+k+s\right)(\pi q(Y))^{1-k-s}$$
$$\times c_{k,\beta}\left(\frac{bz}{N_z},0,s\right)\zeta_+^c(-1-2\kappa+h+2j+2k+2s)\Bigg).$$

Obviously

$$(-\Delta)^j(\overline{p}_{w,\kappa-2j})(0) = \binom{\kappa}{2j}(2j)!|Y|^{\kappa-2j}$$

and $(-\Delta)^j(\overline{p}_{w,h})(0) = 0$ for $\kappa - 2j \neq h$. Hence we can rewrite this to

$$\frac{1}{\sqrt{\pi}2^\kappa}\sum_{j=0}^\infty (-1)^j\binom{\kappa}{2j}\frac{(2j)!}{4^j j!}$$
$$\times \sum_{b,c\in\mathbb{Z}/N_z\mathbb{Z}} e\left(\frac{bc}{N_z}\right)\Bigg(\Gamma\left(\frac{1}{2}-s-j\right)(\pi q(Y))^s(\delta_{\beta,\frac{bz}{N_z}} + (-1)^\kappa\delta_{-\beta,\frac{bz}{N_z}})\zeta_+^c(1-2s-\kappa)$$
$$+\Gamma\left(s-\frac{1}{2}-j+k\right)(\pi q(Y))^{1-k-s}c_{k,\beta}\left(\frac{bz}{N_z},0,s\right)\zeta_+^c(2s-1-\kappa+2k)\Bigg).$$

Using the duplication formula

$$\frac{(2j)!}{4^j j!} = \frac{\Gamma(j+\frac{1}{2})}{\sqrt{\pi}}$$

and Lemma A.1 we obtain

$$\sum_{b,c\in\mathbb{Z}/N_z\mathbb{Z}} e\left(\frac{bc}{N_z}\right)\Bigg(\frac{\Gamma(1-2s-\kappa)}{\Gamma(1-s-\kappa)}4^s\pi^s q(Y)^s(\delta_{\beta,\frac{bz}{N_z}} + (-1)^\kappa\delta_{-\beta,\frac{bz}{N_z}})\zeta_+^c(1-2s-\kappa)$$
$$+\frac{\Gamma(2s+2k-1-\kappa)}{\Gamma(s+k-\kappa)}4^{1-s-k}\pi^{1-s-k}q(Y)^{1-s-k}c_{k,\beta}\left(\frac{bz}{N_z},0,s\right)\zeta_+^c(2s+2k-1-\kappa)\Bigg).$$

Applying Lemma A.5 yields

$$\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}}\Bigg(\frac{N_z^{2s+\kappa}\Gamma(s+\kappa)}{(-2\pi i)^\kappa\pi^s}q(Y)^s(\delta_{\beta,\frac{bz}{N_z}} + (-1)^\kappa\delta_{-\beta,\frac{bz}{N_z}})\zeta_+^b(2s+\kappa)$$
$$+\frac{N_z^{2-2s-2k+\kappa}\Gamma(1-s-k+\kappa)}{(-2\pi i)^\kappa\pi^{1-s-k}}q(Y)^{1-s-k}c_{k,\beta}\left(\frac{bz}{N_z},0,s\right)\zeta_+^b(2-2s-2k+\kappa)\Bigg).$$

Now, the terms for $q(\lambda) = 0, \lambda \neq 0$ are given by

$$\frac{i^\kappa|Y|^{1-\kappa}}{\sqrt{2}}\sum_h\frac{1}{(2i)^h}\sum_{\substack{\delta\in L_0'/L\\ \pi(\delta)=\lambda+K}}\sum_{j=0}^\infty\frac{(-\Delta)^j(\overline{p}_{w,h})(w(\lambda))}{(8\pi)^j j!}\sum_{n=1}^\infty n^h e(n((\lambda,X)+(\delta,z')))$$

$$\times \int_{v=0}^{\infty} \exp\left(-\frac{\pi n^2 q(Y)}{v} - 2\pi v q_w(\lambda)\right) c_{k,\beta}(\delta, q(\lambda), s, v) v^{-\frac{3}{2}+\kappa-h-j-t} \mathrm{d}v$$

and the integral is

$$2(\delta_{\beta,\delta} + (-1)^\kappa \delta_{-\beta,\delta}) \left(\frac{nq(Y)}{|(\lambda, Y)|}\right)^{s-\frac{1}{2}+\kappa-h-j-t} K_{s-\frac{1}{2}+\kappa-h-j-t}(2\pi n|(\lambda, Y)|)$$
$$+ 2c_{k,\beta}(\delta, 0, s) \left(\frac{nq(Y)}{|(\lambda, Y)|}\right)^{\frac{1}{2}+\kappa-h-j-k-s-t} K_{\frac{1}{2}+\kappa-h-j-k-s-t}(2\pi n|(\lambda, Y)|).$$

Now plug in the definition of $(-\Delta)^j(\overline{p}_{w,h})(w(\lambda))$ and reorder the sum as a divisor sum to obtain for $t = 0$

$$2^{-\kappa} \sum_h \sum_{j=0}^{\infty} \frac{(-1)^j}{(4\pi|(\lambda, Y)|)^j j!} \binom{\kappa}{h} \frac{(\kappa-h)!}{(\kappa-h)!} \left(\frac{(\lambda, Y)}{|(\lambda, Y)|}\right)^{\kappa-h} \sum_{n|\lambda}^{\infty} \sum_{\substack{\delta \in L_0'/L \\ \pi(\delta)=\frac{\lambda}{n}+K}} e((\lambda, X) + (n\delta, z'))$$
$$\times \Bigg(2(\delta_{\beta,\delta} + (-1)^\kappa \delta_{-\beta,\delta}) n^{2s-1+\kappa} |(\lambda, Y)|^{\frac{1}{2}-s} q(Y)^s K_{s-\frac{1}{2}+\kappa-h-j}(2\pi|(\lambda, Y)|)$$
$$+ 2c_{k,\beta}(\delta, 0, s) n^{1-\kappa-2k-2s} |(\lambda, Y)|^{s+k-\frac{1}{2}} q(Y)^{1-k-s} K_{\frac{1}{2}+\kappa-h-j-k-s}(2\pi|(\lambda, Y)|) \Bigg).$$

Now $\delta \in L_0'/L$ with $\pi(\delta) = \frac{\lambda}{n} + K$ are given by $\frac{\lambda}{n} - \frac{(\lambda,\zeta)}{nN_z} z + \frac{b}{N_z} z$, where b runs through $\mathbb{Z}/N_z\mathbb{Z}$. This shows the result. □

Write

$$\varphi_\beta\left(\frac{z}{N_z}, s\right) := \frac{\Gamma(1-s-k+\kappa) N_z^{2-2s-2k+\kappa}}{(-2\pi i)^\kappa \pi^{1-s-k}} \sum_{b \in \mathbb{Z}/N_z\mathbb{Z}} c_{k,\beta}\left(\frac{bz}{N_z}, 0, s\right) \zeta_+^b(2-2s-2k+\kappa)$$

for the Fourier coefficient of $q(Y)^{1-k-s}$. Then, as we would expect, the functional equation has the form

$$\Phi_\beta(Z, s) = \frac{1}{2} \sum_{\alpha \in \mathrm{Iso}(L'/L)} \varphi_\beta(\alpha, s) \mathcal{G}_{\kappa,\alpha}(Z, 1-k-s).$$

6.4 Theta Lifts at Harmonic Points

We will now consider the Theta lifts at their harmonic points, i.e. $s = 0$ (and for $k = 0$ at $s = 1$). We set $\Phi_\beta(Z) = \Phi_\beta(Z, 0)$ and $b_\beta(\lambda, Y) = b_\beta(\lambda, Y, 0)$. We can calculate the Fourier coefficient for $q(\lambda) > 0$.

Lemma 6.4.1. *For $q(\lambda) > 0$ and $s = 0$ the coefficient $b_\beta(\lambda, Y)$ is given by*

$$\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}} e\left(-\frac{(\lambda,\zeta)}{N_z}\right)\sum_{n|\lambda} n^{\kappa-1} e\left(\frac{nb}{N_z}\right) c_{k,\beta}\left(\frac{\lambda}{n} - \frac{(\lambda,\zeta)}{nN_z}z + \frac{bz}{N_z}, \frac{q(\lambda)}{n^2}, 0\right) e(\lambda, iY),$$

if $(\lambda, Y) > 0$ and $b_\beta(\lambda, Y, 0) = 0$ if $(\lambda, Y) < 0$. For $q(\lambda) = 0, \lambda \neq 0$ the coefficient $b_\beta(\lambda, Y)$ is given by

$$\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}} \Bigg(e\left(-\frac{(\lambda,\zeta)}{N_z}\right)\sum_{n|\lambda} n^{\kappa-1} e\left(\frac{nb}{N_z}\right) (\delta_{\beta,\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z}} + (-1)^\kappa \delta_{-\beta,\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z}}) e(\lambda, iY)$$
$$+ 2|(\lambda,Y)|^{\frac{1}{2}} \frac{q(Y)^{1-k}}{|(\lambda,Y)|^{1-k}} e\left(-\frac{(\lambda,\zeta)}{N_z}\right)\sum_{n|\lambda} n^{1-2k+\kappa} c_{k,\beta}\left(\frac{\lambda}{n} - \frac{(\lambda,\zeta)}{nN_z}z + \frac{bz}{N_z}, 0, 0\right) e\left(\frac{nb}{N_z}\right)$$
$$\times \sum_{h=0}^\infty \sum_{j=0}^\infty \frac{(-1)^j}{(4\pi|(\lambda,Y)|)^j j!}\binom{\kappa}{h}\frac{(\kappa-h)!}{(\kappa-h-2j)!}\left(\frac{(\lambda,Y)}{|(\lambda,Y)|}\right)^{\kappa-h} K_{\frac{1}{2}-k+\kappa-h-j}(2\pi(\lambda,Y))\Bigg)$$

if $(\lambda, Y) > 0$ and by

$$2|(\lambda,Y)|^{\frac{1}{2}} \frac{q(Y)^{1-k}}{|(\lambda,Y)|^{1-k}} \sum_{b\in\mathbb{Z}/N_z\mathbb{Z}} e\left(-\frac{(\lambda,\zeta)}{N_z}\right)$$
$$\times \sum_{n|\lambda} n^{1-2k+\kappa} c_{k,\beta}\left(\frac{\lambda}{n} - \frac{(\lambda,\zeta)}{nN_z}z + \frac{bz}{N_z}, 0, 0\right) e\left(\frac{nb}{N_z}\right)$$
$$\times \sum_{h=0}^\infty \sum_{j=0}^\infty \frac{(-1)^j}{(4\pi|(\lambda,Y)|)^j j!}\binom{\kappa}{h}\frac{(\kappa-h)!}{(\kappa-h-2j)!}\left(\frac{(\lambda,Y)}{|(\lambda,Y)|}\right)^{\kappa-h} K_{\frac{1}{2}-k+\kappa-h-j}(2\pi|(\lambda,Y)|)$$

if $(\lambda, Y) < 0$.

Proof. Recall that for $q(\lambda) > 0$ the coefficient $b_\beta(\lambda, Y, s)$ is given by

$$\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}} \Bigg(\frac{1}{\sqrt{2}|Y|^{\kappa-1}} \sum_{h=0}^\infty (2i)^{-h} \sum_{j=0}^\infty \frac{(-1)^j i^h}{(8\pi)^j j!}\binom{\kappa}{h}\frac{(\kappa-h)!}{(\kappa-h-2j)!}|Y|^h \left(\frac{(\lambda,Y)}{|Y|}\right)^{\kappa-h-2j}$$
$$\times e\left(-\frac{(\lambda,\zeta)}{N_z}\right)\sum_{n|\lambda} n^{2j-\kappa+2h} e\left(\frac{nb}{N_z}\right) c_{k,\beta}\left(\frac{\lambda}{n} - \frac{(\lambda,\zeta)}{nN_z}z + \frac{bz}{N_z}, \frac{q(\lambda)}{n^2}, s\right)$$
$$\times \int_0^\infty \exp\left(-\frac{\pi n^2}{2vz_Z^2} - \frac{2\pi v q_w(\lambda)}{n^2}\right)\mathcal{W}_s\left(4\pi\frac{q(\lambda)}{n^2}v\right) v^{-\frac{3}{2}+\kappa-h-j}\mathrm{d}v\Bigg).$$

We now plug in $s = 0$. We have

$$\int_0^\infty \exp\left(-\frac{\pi n^2 q(Y)}{v} - \frac{2\pi v q_w(\lambda)}{n^2}\right)\mathcal{W}_0\left(4\pi\frac{q(\lambda)}{n^2}v\right) v^{-\frac{3}{2}+\kappa-h-j}\mathrm{d}v$$
$$= 2\left(\frac{n^2 q(Y)}{|(\lambda,Y)|}\right)^{-\frac{1}{2}+\kappa-h-j} K_{-\frac{1}{2}+\kappa-h-j}\left(2\pi|(\lambda,Y)|\right).$$

This yields for $b_\beta(\lambda, Y)$

$$2^{1-\kappa}|(\lambda,Y)|^{\frac{1}{2}}\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}} e\left(-\frac{(\lambda,\zeta)}{N_z}\right)\sum_{n|\lambda} n^{\kappa-1} e\left(\frac{nb}{N_z}\right) c_{k,\beta}\left(\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z},\frac{q(\lambda)}{n^2},s\right)$$
$$\times\sum_{h=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{(4\pi|(\lambda,Y)|)^j j!}\binom{\kappa}{h}\frac{(\kappa-h)!}{(\kappa-h-2j)!}\left(\frac{(\lambda,Y)}{|(\lambda,Y)|}\right)^{\kappa-h} K_{-\frac{1}{2}+\kappa-h-j}\left(2\pi|(\lambda,Y)|\right).$$

Now use Lemma A.3. The case $q(\lambda) = 0, \lambda \neq 0$ is exactly the same, see also [Bor98, Theorem 14.3]. □

Definition 6.4.2. Define the holomorphic part of the theta lift $\Phi^+_\beta(Z)$ to be

$$\frac{\Gamma(\kappa)N_z^\kappa}{(-2\pi i)^\kappa}\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}}\delta_{\beta,\frac{bz}{N_z}}\zeta^b(\kappa)$$
$$+\sum_{\substack{\lambda\in K'\\ q(\lambda)=0\\ (\lambda,Y)>0}}\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}}\sum_{n|\lambda} n^{\kappa-1}e\left(\frac{nb}{N_z}\right)(\delta_{\beta,\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z}}+(-1)^\kappa\delta_{-\beta,\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z}})e\left(\left(\lambda, Z-\frac{\zeta}{N_z}\right)\right)$$
$$+\sum_{\substack{\lambda\in K'\\ q(\lambda)>0\\ (\lambda,Y)>0}}\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}}\sum_{n|\lambda} n^{\kappa-1}e\left(\frac{nb}{N_z}\right)c_{k,\beta}\left(\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z},\frac{q(\lambda)}{n^2},0\right)e\left(\left(\lambda, Z-\frac{\zeta}{N_z}\right)\right)$$

and the non-holomorphic part $\Phi^-_\beta(Z) = \Phi_\beta(Z) - \Phi^+_\beta(Z)$, which is given by

$$\frac{i^\kappa}{2\sqrt{2}|Y|^{\kappa-1}}\Phi^K_\beta\left(\frac{Y}{|Y|}\right)+\sum_{\substack{\lambda\in K'\\ q(\lambda)<0}} b_\beta(\lambda,Y)e(\lambda,X)$$
$$+\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}}\frac{\Gamma(1-k+\kappa)N_z^{2-2k+\kappa}}{(-2\pi i)^\kappa\pi^{1-k}}q(Y)^{1-k}c_{k,\beta}\left(\frac{bz}{N_z},0,0\right)\zeta^b_+(2-2k+\kappa)$$
$$+\sum_{\substack{\lambda\in K'\\ \lambda\neq 0\\ q(\lambda)=0}} 2|(\lambda,Y)|^{\frac{1}{2}}\frac{q(Y)^{1-k}}{|(\lambda,Y)|^{1-k}}e\left(-\frac{(\lambda,\zeta)}{N_z}\right)$$
$$\times\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}} e\left(\frac{nb}{N_z}\right)\sum_{n|\lambda} n^{1-2k+\kappa}c_{k,\beta}\left(\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z},0,0\right)$$
$$\times\sum_{h=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{(4\pi|(\lambda,Y)|)^j j!}\binom{\kappa}{h}\frac{(\kappa-h)!}{(\kappa-h-2j)!}\left(\frac{(\lambda,Y)}{|(\lambda,Y)|}\right)^{\kappa-h}$$
$$\times K_{\frac{1}{2}-k+\kappa-h-j}(2\pi|(\lambda,Y)|)e(\lambda,X).$$

Remark 6.4.3. Assume that $E_{k,\mathfrak{v}}(\tau) := E_{k,\mathfrak{v}}(\tau,0)$ is holomorphic. Then $\Phi^-_{\mathfrak{v}}(Z)$ vanishes identically and hence $\Phi_{\mathfrak{v}}(Z) = \Phi^+_{\mathfrak{v}}(Z)$ is a holomorphic modular form. See also [Bor98, Theorem 14.3]. Write $\mathrm{M}^\Phi_\kappa(\Gamma(L))$ for the space of holomorphic modular forms of weight κ that are given as a theta lift $\Phi_{\mathfrak{v}}(Z)$ for some $\mathfrak{v}\in\mathrm{Iso}(\mathbb{C}[L'/L])$.

Definition 6.4.4. Define the holomorphic boundary part of the theta lift $\Phi_\beta^{\partial+}(Z)$ to be

$$\frac{\Gamma(\kappa)N_z^\kappa}{(-2\pi i)^\kappa}\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}}\delta_{\beta,\frac{bz}{N_z}}\zeta^b(\kappa)$$
$$+\sum_{\substack{\lambda\in K'\\ q(\lambda)=0\\ (\lambda,Y)>0}}\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}}\sum_{n|\lambda}n^{\kappa-1}e\left(\frac{nb}{N_z}\right)(\delta_{\beta,\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z}}+(-1)^\kappa\delta_{-\beta,\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z}})e\left(\lambda,Z-\frac{\zeta}{N_z}\right).$$

Theorem 6.4.5. *We have*

$$\Phi_\beta^{\partial+}(Z)=\frac{\Gamma(\kappa)N_z^\kappa}{(-2\pi i)^\kappa}\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}}\delta_{\beta,\frac{bz}{N_z}}\zeta^b(\kappa)$$
$$+\sum_{\substack{\lambda\in\mathrm{Iso}_0(K)\\ (\lambda,Y)>0}}\sum_{\substack{b\in\mathbb{Z}/N_z\mathbb{Z}\\ c\in\mathbb{Z}/N_\lambda\mathbb{Z}}}\delta_{\beta,\frac{c\lambda}{N_\lambda}-\frac{c(\lambda,\zeta)}{N_\lambda N_z}z+\frac{bz}{N}}\sum_{m=1}^\infty\sigma_{\kappa-1}^{c,b}(m)e\left(\frac{m(\lambda,Z-\frac{\zeta_K}{N_z})}{N_\lambda}\right).$$

In particular, for an isotropic plane $I\subseteq L\otimes\mathbb{Q}$ with $I=\langle z,d\rangle$, we have, writing $\beta=\frac{c_\beta d}{N_d}-\frac{c_\beta(d,\zeta)z}{N_dN_z}+\frac{b_\beta z}{N_z}$, (if such a decomposition exists it is unique and if it does not exist then $\Phi_\beta^{\partial+}|_I$ vanishes identically)

$$\Phi_\beta^{\partial+}|_I(\tau)=\frac{\Gamma(\kappa)N_z^\kappa}{(-2\pi i)^\kappa}\delta_{c_\beta}\zeta^{b_\beta}(\kappa)+\sum_{m=1}^\infty\sigma_{\kappa-1}^{c_\beta,b_\beta}(m)e\left(\frac{m(\tau-(d,\frac{\zeta_K}{N_z}))}{N_d}\right),$$

which is (the holomorphic part of) an Eisenstein series on the boundary component I. Observe that the constant term only depends on the image of $\frac{z}{N_z}$ in L'/L.

Proof. For $\lambda\in K$ primitive let N_λ be its level. Then we can rewrite the second summand as

$$\sum_{\substack{\lambda\in K'\\ q(\lambda)=0\\ (\lambda,Y)>0}}\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}}\sum_{n|\lambda}n^{\kappa-1}e\left(\frac{nb}{N_z}\right)$$
$$\times(\delta_{\beta,\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z}}+(-1)^\kappa\delta_{-\beta,\frac{\lambda}{n}-\frac{(\lambda,\zeta)}{nN_z}z+\frac{bz}{N_z}})e\left(\lambda,Z-\frac{\zeta}{N_z}\right)$$
$$=\sum_{\substack{\lambda\in\mathrm{Iso}_0(K)\\ (\lambda,Y)>0}}\sum_{m=1}^\infty\sum_{b\in\mathbb{Z}/N_z\mathbb{Z}}\sum_{n|m}n^{\kappa-1}e\left(\frac{nb}{N_z}\right)$$
$$\times(\delta_{\beta,\frac{m\lambda}{nN_\lambda}-\frac{(m\lambda,\zeta)}{nN_\lambda N_z}z+\frac{bz}{N_z}}+(-1)^\kappa\delta_{-\beta,\frac{m\lambda}{nN_\lambda}-\frac{(m\lambda,\zeta)}{nN_\lambda N_z}z+\frac{bz}{N_z}})e\left(\frac{m(\lambda,Z-\frac{\zeta_K}{N_z})}{N_\lambda}\right)$$
$$=\sum_{\substack{\lambda\in\mathrm{Iso}_0(K)\\ (\lambda,Y)>0}}\sum_{\substack{b\in\mathbb{Z}/N_z\mathbb{Z}\\ c\in\mathbb{Z}/N_\lambda\mathbb{Z}}}(\delta_{\beta,\frac{c\lambda}{N_\lambda}-\frac{c(\lambda,\zeta)}{N_\lambda N}z+\frac{bz}{N_z}}+(-1)^\kappa\delta_{-\beta,\frac{c\lambda}{N_\lambda}-\frac{c(\lambda,\zeta)}{N_\lambda N_z}z+\frac{bz}{N_z}})$$

$$\times \sum_{m=1}^{\infty} \sum_{\substack{n|m \\ \frac{m}{n} \equiv c \bmod N_\lambda}} n^{\kappa-1} e\left(\frac{nb}{N_z}\right) e\left(\frac{m(\lambda, Z - \frac{\zeta_K}{N_z})}{N_\lambda}\right)$$

Summing over positive and negative divisors in the divisor sum we can rewrite this as

$$\Phi_\beta^{\partial+}(Z) = \frac{\Gamma(\kappa) N_z^\kappa}{(-2\pi i)^\kappa} \sum_{b \in \mathbb{Z}/N_z\mathbb{Z}} \delta_{\beta, \frac{bz}{N_z}} \zeta^b(\kappa)$$
$$+ \sum_{\substack{\lambda \in \mathrm{Iso}_0(K) \\ (\lambda, Y) > 0}} \sum_{\substack{b \in \mathbb{Z}/N_z\mathbb{Z} \\ c \in \mathbb{Z}/N_\lambda\mathbb{Z}}} \delta_{\beta, \frac{c\lambda}{N_\lambda} - \frac{c(\lambda,\zeta)}{N_\lambda N_z} z + \frac{bz}{N_z}} \sum_{m=1}^{\infty} \sigma_{\kappa-1}^{c,b}(m) e\left(\frac{m(\lambda, Z - \frac{\zeta_K}{N_z})}{N_\lambda}\right).$$

Now let $I \subseteq L \otimes \mathbb{Q}$ be an isotropic plane with $I = \langle z, d \rangle, d \in \mathrm{Iso}_0(K)$ and consider the Siegel operator corresponding to this plane. Then $\Phi_\beta^{\partial+}|_I(\tau)$ is given by

$$\frac{\Gamma(\kappa) N_z^\kappa}{(-2\pi i)^\kappa} \sum_{b \in \mathbb{Z}/N_z\mathbb{Z}} \delta_{\beta, \frac{bz}{N_z}} \zeta^b(\kappa) + \sum_{\substack{b \in \mathbb{Z}/N_z\mathbb{Z} \\ c \in \mathbb{Z}/N_d\mathbb{Z}}} \delta_{\beta, \frac{cd}{N_d} - \frac{c(d,\zeta)}{N_d N_z} z + \frac{bz}{N_z}} \sum_{m=1}^{\infty} \sigma_{\kappa-1}^{c,b}(m) e\left(\frac{m(\tau - (d, \frac{\zeta_K}{N_z}))}{N_d}\right).$$

Writing $\beta = \frac{c_\beta d}{N_d} - \frac{c_\beta (d,\zeta) z}{N_d N_z} + \frac{b_\beta z}{N_z}$ (if such a decomposition exists it is unique and if it does not exist then $\Phi_\beta^{\partial+}|_I$ vanishes identically), we obtain

$$\begin{aligned}\Phi_\beta^{\partial+}|_I(\tau) &= \frac{\Gamma(\kappa) N_z^\kappa}{(-2\pi i)^\kappa} \delta_{c_\beta} \zeta^{b_\beta}(\kappa) + \sum_{m=1}^{\infty} \sigma_{\kappa-1}^{c_\beta, b_\beta}(m) e\left(\frac{m(\tau - (d, \frac{\zeta_K}{N_z}))}{N_d}\right) \\ &= \frac{\Gamma(\kappa) N_z^\kappa}{(-2\pi i)^\kappa} G_\kappa^{(c_\beta, b_\beta)}\left(\tau - \left(d, \frac{\zeta_K}{N_z}\right)\right),\end{aligned}$$

which shows the result □

Theorem 6.4.6. *Assume that the map π_L is surjective. Then the theta lift is injective.*

Proof. Assume that there is some $\mathfrak{v} \in \mathrm{Iso}(\mathbb{C}[L'/L])$ with $\Phi_\mathfrak{v}(Z) = 0$. Then, in particular, $\Phi_\mathfrak{v}^{\partial+}(Z) = 0$ for every cusp z, i.e.

$$\sum_{b \in \mathbb{Z}/N_z\mathbb{Z}} \mathfrak{v}_{\frac{bz}{N_z}} \zeta^b(\kappa) = 0$$

for all cusps z. We want to show that $E_{k,\mathfrak{v}}(\tau, 0) = 0$, which is equivalent to $\mathfrak{v} + (-1)^\kappa \mathfrak{v}^* = 0$. Therefore, assume there is some $\delta \in \mathrm{Iso}(L'/L)$ with $\mathfrak{v}_\delta \neq -(-1)^\kappa \mathfrak{v}_{-\delta}$. By surjectivity of π_L there is a cusp z corresponding to δ. Choose such δ with minimal order. Then by assumption the value in the cusp z is

$$\sum_{b \in (\mathbb{Z}/N_z\mathbb{Z})^\times} \mathfrak{v}_{b\delta} \zeta^b(\kappa) = 0.$$

But of course this is also true for the cusps corresponding to $c\delta$ for $c \in (\mathbb{Z}/N_z\mathbb{Z})^\times$, i.e. we have

$$\sum_{b\in(\mathbb{Z}/N_z\mathbb{Z})^\times} \mathfrak{v}_{bc\delta}\zeta^{b}(\kappa) = \sum_{b\in(\mathbb{Z}/N_z\mathbb{Z})^\times} \mathfrak{v}_{b\delta}\zeta^{bc^*}(\kappa) = 0.$$

Rewrite this using $\zeta^{bc^*}(\kappa) = \zeta_+^{bc^*}(\kappa) + (-1)^\kappa \zeta_+^{-bc^*}(\kappa)$ to obtain

$$\sum_{b\in(\mathbb{Z}/N_z\mathbb{Z})^\times} (\mathfrak{v}_{b\delta} + (-1)^\kappa \mathfrak{v}_{-b\delta})\zeta_+^{bc^*}(\kappa) = 0.$$

For a character $\chi : (\mathbb{Z}/N_z\mathbb{Z})^\times \to \mathbb{C}^\times$ consider

$$\begin{aligned} 0 &= \sum_{c\in(\mathbb{Z}/N_z\mathbb{Z})^\times} \chi(c) \sum_{b\in(\mathbb{Z}/N_z\mathbb{Z})^\times} (\mathfrak{v}_{b\delta} + (-1)^\kappa \mathfrak{v}_{-b\delta})\zeta_+^{bc^*}(\kappa) \\ &= \sum_{b\in(\mathbb{Z}/N_z\mathbb{Z})^\times} \chi(b)(\mathfrak{v}_{b\delta} + (-1)^\kappa \mathfrak{v}_{-b\delta}) \sum_{c\in(\mathbb{Z}/N_z\mathbb{Z})^\times} \chi(c)\zeta_+^{c^*}(\kappa) \\ &= L(\overline{\chi}, \kappa) \sum_{b\in(\mathbb{Z}/N_z\mathbb{Z})^\times} \chi(b)(\mathfrak{v}_{b\delta} + (-1)^\kappa \mathfrak{v}_{-b\delta}). \end{aligned}$$

Now $L(\overline{\chi}, \kappa) \neq 0$ and thus

$$\sum_{b\in(\mathbb{Z}/N_z\mathbb{Z})^\times} \chi(b)(\mathfrak{v}_{b\delta} + (-1)^\kappa \mathfrak{v}_{-b\delta}) = 0$$

for all Dirichlet characters χ. But that means $\mathfrak{v}_{b\delta} + (-1)^\kappa \mathfrak{v}_{-b\delta} = 0$ for all $b \in (\mathbb{Z}/N_z\mathbb{Z})^\times$ contradicting the assumption $\mathfrak{v}_\delta \neq -(-1)^\kappa \mathfrak{v}_{-\delta}$. □

Theorem 6.4.7. *For κ even and every $\delta \in L'/L$ isotropic there is a theta lift $\Phi_\mathfrak{v}$ for some $\mathfrak{v} \in \mathrm{Iso}(\mathbb{C}[L'/L])$ such that the holomorphic part vanishes in all cusps except for the cusps corresponding to $\pm\delta$. For κ odd this is true if $\delta \neq -\delta$.*

Proof. This is essentially Theorem 6.3.2. If δ generates a maximal cyclic isotropic subgroup, then the holomorphic part of $\Phi_{c\delta}, c \in (\mathbb{Z}/N_\delta\mathbb{Z})^\times$ vanishes in every cusp except for the cusps corresponding to the generators of $\langle\delta\rangle$. In the cusps corresponding to $b\delta$ for $b \in (\mathbb{Z}/N_\delta\mathbb{Z})^\times$ the value is given by

$$\frac{\Gamma(\kappa)N_\delta^\kappa}{(-2\pi i)^\kappa}\zeta^{cb^*}(\kappa),$$

where $b^*b = 1 \in (\mathbb{Z}/N_\delta\mathbb{Z})^\times$. Now consider the linear combination

$$\frac{(-2\pi i)^\kappa}{2\Gamma(\kappa)N_\delta^\kappa} \sum_\chi \frac{1}{L(\chi, \kappa)} \sum_{c\in(\mathbb{Z}/N_\delta\mathbb{Z})^\times} \chi(c)\Phi_{c\delta},$$

whose value in the cusp $b\delta$ for $b \in (\mathbb{Z}/N_\delta\mathbb{Z})^\times$ is given by

$$\frac{1}{2}\sum_\chi \frac{1}{L(\chi, \kappa)} \sum_{c\in(\mathbb{Z}/N_\delta\mathbb{Z})^\times} \chi(c)\zeta^{cb^*}(\kappa)$$

$$= \frac{1}{2}\sum_{\chi}\frac{\chi(b)}{L(\chi,\kappa)}\sum_{c\in(\mathbb{Z}/N_\delta\mathbb{Z})^\times}\chi(c)\zeta^c(\kappa) = \frac{1}{2}\sum_{\chi}\left(\chi(b)+(-1)^\kappa\chi(-b)\right),$$

i.e. the value in the cusps vanishes except for the cusps corresponding to δ, where the value is 1. Now do induction over the maximal length of chains of cyclic isotropic subgroups containing δ. □

This means, in particular, that for $F \in \mathrm{M}^\pi_\kappa(\Gamma(L))$ there is a theta lift $\Phi_\mathfrak{v}$ for some $\mathfrak{v} \in \mathrm{Iso}(\mathbb{C}[L'/L])$ whose holomorphic boundary part is given by the boundary part of F (observe that for κ odd the values in cusps corresponding to $\delta \in \mathrm{Iso}(L'/L)$ with $\delta = -\delta$ must be zero). We will show that in this case $\Phi_\mathfrak{v}$ is already holomorphic and the difference $\Phi_\mathfrak{v} - F$ is a cusp form, i.e. we have $\mathrm{M}^\pi_\kappa(\Gamma(L)) = \mathrm{S}_\kappa(\Gamma(L)) + \mathrm{M}^\Phi_\kappa(\Gamma(L))$.

Theorem 6.4.8. *Let $k > 2$ and thus $\kappa > \frac{l}{2}+1 > 1$. Then $\Phi_\beta(Z) = \Phi^+_\beta(Z)$ is a holomorphic modular form of weight κ which is an Eisenstein series on the boundary. In particular, we have $\mathrm{M}^\pi_\kappa(\Gamma(L)) = \mathrm{S}_\kappa(\Gamma(L)) + \mathrm{M}^\Phi_\kappa(\Gamma(L))$ in this case.*

Proof. Using that the coefficients $c_{k,\beta}(\gamma,n,0)$ vanish for $n \leq 0$ one obtains the result using Theorem 6.3.4 and Lemma 6.4.1. Of course, this reproduces the result of [Bor98, Theorem 14.3]. □

Using Theorem 6.3.2 we obtain the Fourier expansion of the Eisenstein series $\mathcal{G}_{\kappa,\mathfrak{v}}(Z)$ for an invariant vector $\mathfrak{v} \in \mathrm{Iso}(\mathbb{C}[L'/L])$ and that the Eisenstein series $\mathcal{G}_{\kappa,\mathfrak{v}}(Z) = \mathcal{G}_{\kappa,\mathfrak{v}}(Z,0)$ for $\mathfrak{v} \in \mathrm{Iso}(\mathbb{C}[L'/L])$ are holomorphic if $\kappa > \frac{l}{2}+1$. Moreover, if

$$\pi_L : \Gamma(L)\backslash \mathrm{Iso}_0(L') \to \mathrm{Iso}(L'/L)$$

is injective, then we obtain all holomorphic orthogonal Eisenstein series as a lift of vector-valued Eisenstein series.

If $k = 0$, the Eisenstein series $E_{k,\mathfrak{v}}(\tau) = E_{k,\mathfrak{v}}(\tau,0)$ are usually not holomorphic in τ. Hence we can not expect that $\mathcal{G}_{\kappa,\mathfrak{v}}(Z) = \mathcal{G}_{\kappa,\mathfrak{v}}(Z,0)$ is holomorphic. Since $\mathrm{res}_{s=1} E_{k,\mathfrak{v}}(\tau,s)$ is always an invariant vector we would expect $\mathrm{res}_{s=1}\mathcal{G}_{\kappa,\mathfrak{v}}(Z,s)$ to be a holomorphic modular form. In fact, this is true by the following theorem, which was proven in my master thesis.

Theorem 6.4.9. *Let $k = 0, \kappa = \frac{l}{2} - 1 > 1$, i.e. $l > 4$. In the cusp z we have the expansion*

$$\mathrm{res}_{s=1}\Phi_\beta(Z,s) = \Phi(Z, \mathrm{res}_{s=1} E_{0,\beta}(\cdot,s))$$

$$= \frac{\Gamma(\kappa)N_z^\kappa}{(-2\pi i)^\kappa} \sum_{b\in\mathbb{Z}/N_z\mathbb{Z}} \mathrm{res}_{s=1}\, c_{k,\beta}\left(\frac{bz}{N_z}, 0, s\right)\zeta^b(\kappa)$$

$$+ \sum_{\substack{\lambda\in K' \\ q(\lambda)=0 \\ (\lambda,Y)>0}} e\left(-\frac{(\lambda,\zeta)}{N_z}\right) \sum_{b\in\mathbb{Z}/N_z\mathbb{Z}} \sum_{n|\lambda} n^{\kappa-1} e\left(\frac{nb}{N_z}\right) \mathrm{res}_{s=1}\, c_{k,\beta}\left(\frac{\lambda}{n} - \frac{(\lambda,\zeta)}{nN_z}z + \frac{bz}{N_z}, 0, s\right) e(\lambda, Z).$$

In particular, every invariant vector yields a holomorphic modular form of singular weight which is a linear combination of Eisenstein series on the boundary. For $\kappa = 1$, i.e. $l = 4$, it is the same expression plus the constant $\frac{i}{2\sqrt{2}}\, \mathrm{res}_{s=1}\, \Phi_\beta^K\left(\frac{Y}{|Y|}, s\right)$.

Proof. One observes that the terms with $q(\lambda) \neq 0$ are holomorphic in $s = 1$ and hence their residue vanishes (this follows from the corresponding result for vector-valued Eisenstein series). The calculation for the other Fourier coefficients is analogous to the case for $s = 0$. See also [Bor98, Theorem 14.3]. □

The question if this yields all holomorphic modular forms of singular weight which are linear combinations of Eisenstein series on the boundary will be answered later. We want to mention that we can also construct these in a different way. For an invariant vector $\mathfrak{v} \in \mathrm{Inv}(\mathbb{C}[L'/L])$ we have

$$E_k(\tau, s)\mathfrak{v} = \sum_{\beta\in\mathrm{Iso}(L'/L)} \mathfrak{v}_\beta E_{k,\beta}(\tau, s),$$

where $E_k(\tau, s)$ is the usual suitably normalized scalar-valued Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$. Now the left-hand-side is holomorphic in $s = 0$ and equal to a multiple of $\mathfrak{v}$ and the lift of $E_k(\tau, s)\mathfrak{v}$ is holomorphic in $s = 0$ and yields a holomorphic modular form for $s = 0$ as in the case $k > 2$.

Recall the definition of $\mathrm{M}_\kappa^\pi(\Gamma(L))$ in Definition 4.1.9 and $\mathrm{M}_\kappa^\Phi(\Gamma(L))$ in Remark 6.4.3. For $k = 2, l > 2$, we have

Theorem 6.4.10. *We have* $\mathrm{M}_\kappa^\pi(\Gamma(L)) = \mathrm{S}_\kappa(\Gamma(L)) + \mathrm{M}_\kappa^\Phi(\Gamma(L))$.

Proof. For $F \in \mathrm{M}_\kappa^\pi(\Gamma(L))$ consider the corresponding linear combination of theta lifts such that the holomorphic boundary part of $G(Z) = F(Z) - \Phi(Z)$ vanishes. Now one sees that $G(Z)$ is bounded by $q(Y)^{-1}$ and hence is square-integrable and harmonic. Therefore, it must be holomorphic and since the boundary part vanishes it is a cusp form. □

6.5 Lifting Holomorphic Orthogonal Modular Forms

Let L be an even lattice of signature $(2,l)$. For $z \in \mathrm{Iso}_0(L)$ and $z' \in L'$ with $(z,z') = 1$ write $K_z = L \cap z^\perp \cap z'^\perp$. Let $b_1, \ldots, b_l$ be a basis of $K_z \otimes \mathbb{R}$ with $b_1 \perp \langle b_2, \ldots, b_l \rangle$ and $q(b_1) > 0$. If $Z = z_1 b_1 + z_2 b_2 + z_3 b_3 + \ldots + z_l b_l \in K_z \otimes \mathbb{C}$, we write $Z = (z_1, \ldots, z_l)$ and similarly $X = (x_1, \ldots, x_l), Y = (y_1, \ldots, y_l)$ if $Z = X + iY$ with $X, Y \in K_z \otimes \mathbb{R}$. Denote by $\mathbb{H}_l = K_z \otimes \mathbb{R} + i\mathcal{C}$ the corresponding tube domain model, where

$$\mathcal{C} = \{Y = (y_1, \ldots, y_l) \in K_z \otimes \mathbb{R} \mid y_1 > 0, q(Y) > 0\}.$$

For $\lambda \in \mathrm{Iso}_0(L')$ let N_λ be the order of λ in L'/L and let $\sigma_\lambda \in \mathrm{O}^+(V)$ with $\sigma_\lambda N_\lambda \lambda = z$ and set $\lambda' = \sigma_\lambda^{-1} z'$. Define

$$K_\lambda = \lambda^\perp \cap \lambda'^\perp \cap L = L \cap \sigma_\lambda^{-1}(K \otimes \mathbb{R}).$$

Then

$$\sigma_\lambda \Gamma(L)_\lambda \sigma_\lambda^{-1} = \Gamma(\sigma_\lambda L)_z \supseteq (\sigma_\lambda K_{\sigma_\lambda^{-1} z}) \rtimes \Gamma(\sigma_\lambda K_{\sigma_\lambda^{-1} z}),$$

where $\sigma_\lambda K_{\sigma_\lambda^{-1} z}$ acts via translation and $\Gamma(\sigma_\lambda K_{\sigma_\lambda^{-1} z})$ via multiplication on $\mathbb{H}_l = K_z \otimes \mathbb{R} + i\mathcal{C}$. A fundamental domain is given by $\mathcal{F} = \mathcal{F}_1 + i\mathcal{F}_2$, where $\mathcal{F}_1$ is a fundamental domain of the action $\sigma_\lambda K_{\sigma_\lambda^{-1} z}$ on $K_z \otimes \mathbb{R}$ and $\mathcal{F}_2$ is a fundamental domain of the action $\Gamma(\sigma_\lambda K_{\sigma_\lambda^{-1} z})$ on $\mathcal{C}$. Moreover, recall the map

$$\pi_L : \Gamma(L) \backslash \mathrm{Iso}_0(L') \to \mathrm{Iso}(L'/L).$$

Let $F : \mathbb{H}_l \to \mathbb{C}$ be a modular form of weight κ. We define its theta lift to be (if it exists)

$$\Phi^*(\tau, F) := \int_{\Gamma(L)\backslash \mathbb{H}_l} F(Z) \Theta_L(\tau, Z) q(Y)^\kappa \frac{\mathrm{d}X \mathrm{d}Y}{q(Y)^l}.$$

Proposition 6.2.4 and Lemma 3.4.4 show that if F is a holomorphic modular form of singular weight $\kappa = \frac{l}{2} - 1$ then the theta lift exists (in fact, the lift exists for $\kappa < l - 1$ and even for arbitrary weight if F is a cusp form).

Lemma 6.5.1. *Let $F : \mathbb{H}_l \to \mathbb{C}$ be a holomorphic modular form of singular weight $\kappa = \frac{l}{2} - 1$ and $\mathfrak{v} \in \mathrm{Inv}(\mathbb{C}[L'/L])$ an invariant vector. Then*

$$\langle \Phi^*(\tau, F), \mathfrak{v} \rangle = \langle F, \Phi(Z, \mathfrak{v}) \rangle,$$

where the left hand side denotes the Petersson inner product on invariant vectors and the right hand side denotes the Petersson inner product on holomorphic modular forms of singular weight. In particular, the theta lifts are adjoint to eachother.

Proof. We have

$$\begin{aligned}
\langle \Phi^*(\tau, F), \mathfrak{v}\rangle &= \int_{\mathrm{Mp}_2(\mathbb{Z})\backslash\mathbb{H}} \sum_{\gamma\in L'/L} \int_{\Gamma(L)\backslash\mathbb{H}_l} F(Z)\Theta_\gamma(\tau, Z) q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\overline{\mathfrak{v}_\gamma(\tau)}\frac{\mathrm{d}x\mathrm{d}y}{y^2}\\
&= \int_{\Gamma(L)\backslash\mathbb{H}_l} F(Z) \int_{\mathrm{Mp}_2(\mathbb{Z})\backslash\mathbb{H}} \overline{\sum_{\gamma\in L'/L} \mathfrak{v}_\gamma(\tau)\overline{\Theta_\gamma(\tau, Z)}}\frac{\mathrm{d}x\mathrm{d}y}{y^2} q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\\
&= \int_{\Gamma(L)\backslash\mathbb{H}_l} F(Z)\overline{\Phi(Z, \mathfrak{v})} q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\\
&= \langle F, \Phi(Z, \mathfrak{v})\rangle.
\end{aligned}$$

□

One of our main tools will be

Lemma 6.5.2. *Let* $l \geq 4, \kappa = \frac{l}{2} - 1$. *If* F *and* $\Omega_\kappa F$ *are square-integrable, then the theta lift exists and we have* $\Phi^*(\tau, \Omega_\kappa F) = \Delta_0 \Phi^*(\tau, F)$. *In particular, if* F *is holomorphic, then* $\Phi^*(\tau, F)$ *exists and is harmonic.*

Proof. Let $\Gamma \subseteq \Gamma(L)$ be a finite index torsion-free subgroup. Then $\Gamma\backslash\mathbb{H}_l$ is a complete connected hermitian manifold. By Corollary 6.2.5, Θ_L and $\Omega_\kappa\overline{\Theta_L}$ are square-integrable and using the assumption on F and $\Omega_\kappa F$ we can apply Proposition 3.3.3 to square-integrable sections of the hermitian line bundle of modular forms of weight κ. This yields, using $\Omega_\kappa\overline{\Theta_L} = \overline{\Delta_0\Theta_L}$,

$$\begin{aligned}
\Delta_0\Phi^*(\tau, F) &= \Delta_0 \int_{\Gamma(L)\backslash\mathbb{H}_l} F(Z)\Theta_L(\tau, Z) q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\\
&= [\Gamma(L) : \Gamma]\Delta_0 \int_{\Gamma\backslash\mathbb{H}_l} F(Z)\Theta_L(\tau, Z) q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\\
&= [\Gamma(L) : \Gamma] \int_{\Gamma\backslash\mathbb{H}_l} F(Z)(\Delta_0\Theta_L(\tau, Z)) q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\\
&= [\Gamma(L) : \Gamma] \int_{\Gamma\backslash\mathbb{H}_l} F(Z)(\overline{\Omega_\kappa\overline{\Theta_L(\tau, Z)}}) q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\\
&= [\Gamma(L) : \Gamma] \int_{\Gamma\backslash\mathbb{H}_l} (\Omega_\kappa F(Z))\Theta_L(\tau, Z) q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\\
&= \int_{\Gamma(L)\backslash\mathbb{H}_l} (\Omega_\kappa F(Z))\Theta_L(\tau, Z) q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l} = \Phi^*(\tau, \Omega_\kappa F).
\end{aligned}$$

The other assertions follow immediately. □

Theorem 6.5.3. *Let* F *be a holomorphic modular form of singular weight* $\kappa = \frac{l}{2} - 1 > 0$, *i.e.* $l > 2$. *Then its theta lift*

$$\Phi^*(\tau, F) = \int_{\Gamma(L)\backslash\mathbb{H}_l} F(Z)\Theta_L(\tau, Z) q(Y)^\kappa \frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}$$

is an invariant vector given by

$$\frac{\Gamma(l/2)}{2(2\pi)^{l/2}}\sum_{\gamma\in\mathrm{Iso}(L'/L)}\sum_{\substack{\delta\in\mathrm{Iso}(L'/L)\\ \gamma=k_\delta\delta}} N_\delta^{l-\kappa}\zeta_+^{k_\delta}(l-\kappa)\sum_{\lambda\in\pi^{-1}(\delta)} a_{F,\lambda}(0)C(\lambda)\mathfrak{e}_\gamma,$$

where $C(\lambda)>0$ *are positive constants depending on* λ.

Proof. First observe that the integral converges since holomorphic modular forms of singular weight are square integrable by Corollary 3.4.5. We have

$$\begin{aligned}\Phi^*(\tau,F) &= \int_{\Gamma(L)\backslash\mathbb{H}_l} F(Z)\Theta_L(\tau,Z)q(Y)^\kappa\frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\\ &= \frac{v^{\frac{l}{2}}}{2}\int_{\Gamma(L)\backslash\mathbb{H}_l} F(Z)\sum_{\lambda\in L'}\frac{(\lambda,Z_L)^\kappa}{q(Y)^\kappa}\exp(-2\pi v q_Z(\lambda))\mathfrak{e}_\lambda(uq(\lambda))q(Y)^\kappa\frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\\ &= \frac{v^{\frac{l}{2}}}{2}\sum_{\lambda\in L'}\int_{\Gamma(L)\backslash\mathbb{H}_l} F(Z)(\lambda,Z_L)^\kappa\exp(-2\pi v q_Z(\lambda))\frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\mathfrak{e}_\lambda(uq(\lambda))\end{aligned}$$

and remark that the lift grows polynomially by Proposition 6.2.4. Moreover, by Lemma 6.5.2 it is harmonic of weight 0 and thus its growth comes from the constant Fourier coefficient (this follows from its Fourier expansion, see for example [Iwa95, Proposition 1.5]). The constant Fourier coefficients are then given by (the $\lambda=0$ term vanishes since $\kappa>0$)

$$\begin{aligned}&\frac{v^{\frac{l}{2}}}{2}\sum_{\substack{\lambda\in\Gamma(L)\backslash L'\\ q(\lambda)=0}}\int_{\Gamma(L)_\lambda\backslash\mathbb{H}_l} F(Z)(\lambda,Z_L)^\kappa\exp(-2\pi v q_Z(\lambda))\frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\mathfrak{e}_\lambda\\ &= \frac{v^{\frac{l}{2}}}{2}\sum_{\lambda\in\Gamma(L)\backslash\mathrm{Iso}_0(L')}\sum_{m=1}^\infty m^\kappa\int_{\Gamma(L)_\lambda\backslash\mathbb{H}_l} F(Z)(\lambda,Z_L)^\kappa\exp(-2\pi v m^2 q_{Z_L}(\lambda))\frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\mathfrak{e}_{m\lambda}.\end{aligned}$$

As above let $\sigma_\lambda\in\mathrm{O}^+(V)$ such that $\sigma_\lambda N_\lambda\lambda=z$, where N_λ is the order of λ in L'/L, and write $\lambda'=\sigma_\lambda^{-1}(z')$. Then we can rewrite the integral to (observe that $(z,Z_L)=1$ and $q_Z(z)=1/q(Y)$)

$$N_\lambda^{-\kappa}\int_{\Gamma(\sigma_\lambda L)_z\backslash\mathbb{H}_l}(F|_\kappa\sigma_\lambda)(Z)\exp(-2\pi v(m/N_\lambda)^2/q(Y))\frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}.$$

and hence, using the Fourier expansion of $F|_\kappa\sigma_\lambda$

$$\begin{aligned}&N_\lambda^{-\kappa}\sum_{\delta\in\sigma_\lambda K_\lambda'} a_{F,\lambda}(\delta)\int_{\Gamma(\sigma_\lambda L)_z\backslash\mathbb{H}_l} e(\delta,Z)\exp(-2\pi v(m/N_\lambda)^2/q(Y))\frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\\ &= N_\lambda^{-\kappa}[\Gamma(L)_\lambda : K_\lambda\rtimes\Gamma(K_\lambda)]\sum_{\delta\in\sigma_\lambda K_\lambda'} a_{F,\lambda}(\delta)\\ &\times\int_{\sigma_\lambda K_\lambda\rtimes\Gamma(\sigma_\lambda K_\lambda)\backslash\mathbb{H}_l} e(\delta,Z)\exp(-2\pi v(m/N_\lambda)^2/q(Y))\frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}\end{aligned}$$

$$
\begin{aligned}
&= N_\lambda^{-\kappa}[\Gamma(L)_\lambda : K_\lambda \rtimes \Gamma(K_\lambda)] \sum_{\delta\in\sigma_\lambda K'_\lambda} a_{F,\lambda}(\delta) \\
&\times \int_{\sigma_\lambda K_\lambda \backslash K_z\otimes\mathbb{R}} e(\delta, X)\mathrm{d}X \int_{\Gamma(\sigma_\lambda K_\lambda)\backslash\mathcal{C}} e(\delta, iY)\exp(-2\pi v(m/N_\lambda)^2/q(Y))\frac{\mathrm{d}Y}{q(Y)^l} \\
&= a_{F,\lambda}(0)\operatorname{vol}(K_\lambda) N_\lambda^{-\kappa}[\Gamma(L)_\lambda : K_\lambda \rtimes \Gamma(K_\lambda)] \\
&\times \int_{\Gamma(\sigma_\lambda L)_z\backslash\mathcal{C}} \exp(-2\pi v(m/N_\lambda)^2/q(Y))\frac{\mathrm{d}X\mathrm{d}Y}{q(Y)^l}.
\end{aligned}
$$

Consider the diffeomorphism

$$
\varphi : [0,\infty)\times\mathbb{R}^{l-1} \to \mathcal{C}, \quad (r, y_2, \dots, y_l) \mapsto \sqrt{r}(\sqrt{1 - q(0, y_2, \dots, y_l)}, y_2, \dots, y_l)
$$

with inverse

$$
\varphi^{-1} : \mathcal{C} \to [0,\infty)\times\mathbb{R}^{l-1}, \quad Y \mapsto (q(Y), y_2/\sqrt{q(Y)}, \dots, y_l/\sqrt{q(Y)}).
$$

Then we have for integrable $f : \mathcal{C} \to \mathbb{C}$

$$
\begin{aligned}
\int_{\mathcal{C}} f(Y)\mathrm{d}Y &= \int_0^\infty \int_{\mathbb{R}^{l-1}} f(\varphi(r, y_2, \dots, y_l))|\det(\varphi'(r, y_1, \dots, y_l))|\mathrm{d}y_2 \dots \mathrm{d}y_l \mathrm{d}r \\
&= \int_{\mathbb{R}^{l-1}} \int_0^\infty r^{\frac{l}{2}} f(\varphi(r, y_2, \dots, y_l))\frac{\mathrm{d}r}{r}|\det(\varphi'(1, y_1, \dots, y_l))|\mathrm{d}y_2 \dots \mathrm{d}y_l.
\end{aligned}
$$

Here $f(Y) = g(q(Y))$ for $g(r) = \exp(2\pi v(m/N_\lambda)^2/r)r^{-l}$ and thus

$$
f(\varphi(r, y_2, \dots, y_l)) = g(r) = \exp(2\pi v(m/N_\lambda)^2/r)r^{-l}.
$$

Hence we obtain

$$
\begin{aligned}
&a_{F,\lambda}(0)\operatorname{vol}(K_\lambda) N_\lambda^{-\kappa}[\Gamma(L)_\lambda : K_\lambda \rtimes \Gamma(K_\lambda)] \\
&\times \int_0^\infty \exp(-2\pi v(m/N_\lambda)^2/r)r^{-\frac{l}{2}}\frac{\mathrm{d}r}{r}\int_{\Gamma(\sigma_\lambda K_\lambda)\backslash\mathbb{R}^{l-1}} |\det(\varphi'(1, y_2, \dots, y_l))|\mathrm{d}y_2, \dots, y_l \\
&= a_{F,\lambda}(0)C(\lambda)(2\pi v)^{-\frac{l}{2}} m^{-l} N_\lambda^{l-\kappa}\Gamma(l/2).
\end{aligned}
$$

Hence the constant Fourier coefficient is given by

$$
\begin{aligned}
&\frac{\Gamma(l/2)}{2(2\pi)^{l/2}} \sum_{\lambda\in\Gamma(L)\backslash \operatorname{Iso}_0(L')} a_{F,\lambda}(0)C(\lambda)N_\lambda^{l-\kappa}\sum_{m=1}^\infty m^{\kappa-l}\mathfrak{e}_{m\lambda} \\
&= \frac{\Gamma(l/2)}{2(2\pi)^{l/2}} \sum_{\gamma\in\operatorname{Iso}(L'/L)} \sum_{\substack{\delta\in\operatorname{Iso}(L'/L)\\ \gamma = k_\delta\delta}} N_\delta^{l-\kappa}\zeta_+^{k_\delta}(l-\kappa)\sum_{\lambda\in\pi_L^{-1}(\delta)} a_{F,\lambda}(0)C(\lambda)\mathfrak{e}_\gamma.
\end{aligned}
$$

In particular, this is a constant and independent of v. Hence $\Phi^*(\tau, F)$ is bounded and since it is harmonic of weight 0 it is an invariant vector. □

Remark 6.5.4. Observe that $\Phi^*(\tau, F)$ only depends on the values in the 0-dimensional cusps. In particular, it vanishes on functions that are zero in every 0-dimensional cusp.

Remark 6.5.5. If the weight κ is not the singular weight and F is a cusp form, then $\Phi^*(\tau, F)$ still exists and as in the previous proof one can show that it is harmonic. Moreover, the constant Fourier coefficient can be calculated as above and vanishes. Hence, $\Phi^*(\tau, F)$ decays exponentially and is thus a harmonic and square-integrable of weight k. But this implies that F must be a holomorphic cusp form. This reproduces the result of [Oda78].

Corollary 6.5.6. *For $\delta \in \mathrm{Iso}(L'/L)$ let $a_{F,\delta}(0) := \sum_{\lambda \in \pi^{-1}(\delta)} a_{F,\lambda}(0)C(\lambda)$. Then $\Phi^*(\tau, F)$ vanishes if and only if $a_{F,\delta}(0)$ vanishes for all $\delta \in \mathrm{Iso}(L'/L)$. In particular, if π_L is injective, then the theta lift $\Phi^*(\tau, F)$ vanishes if and only if F vanishes in every cusp.*

Proof. If $\Phi^*(\tau, F)$ vanishes, then the coefficient of $\mathfrak{e}_\gamma$, given by

$$\sum_{\substack{\delta \in \mathrm{Iso}(L'/L) \\ \gamma = k_\delta \delta}} N_\delta^{l-\kappa} \zeta_+^{k_\delta}(l-\kappa) a_{F,\delta}(0),$$

vanishes for all $\gamma \in \mathrm{Iso}(L'/L)$. In particular, if $\gamma \in \mathrm{Iso}(L'/L)$ has maximal order such that $a_{F,\gamma}(0) \neq 0$, then for $k' \in (\mathbb{Z}/N_\gamma\mathbb{Z})^\times$ the coefficient of $\mathfrak{e}_{k'\gamma}$

$$N_\gamma^{l-\kappa} \sum_{k \in (\mathbb{Z}/N_\gamma\mathbb{Z})^\times} \zeta_+^{k^*}(l-\kappa) a_{F,kk'\gamma}(0) = N_\gamma^{l-\kappa} \sum_{k \in (\mathbb{Z}/N_\gamma\mathbb{Z})^\times} \zeta_+^{k'k^*}(l-\kappa) a_{F,k\gamma}(0)$$

vanishes. Thus, for all Dirichlet characters $\chi : (\mathbb{Z}/N_\gamma\mathbb{Z})^\times \to \mathbb{C}^\times$, the sum

$$\begin{aligned} &\sum_{k' \in (\mathbb{Z}/N_\gamma\mathbb{Z})^\times} \chi(k') \sum_{k \in (\mathbb{Z}/N_\gamma\mathbb{Z})^\times} \zeta_+^{k'k^*}(l-\kappa) a_{F,k\gamma}(0) \\ &= \sum_{k' \in (\mathbb{Z}/N_\gamma\mathbb{Z})^\times} \chi(k')\zeta_+^{k'}(l-\kappa) \sum_{k \in (\mathbb{Z}/N_\gamma\mathbb{Z})^\times} \chi(k) a_{F,k\gamma}(0) \\ &= L(l-\kappa, \chi) \sum_{k \in (\mathbb{Z}/N_\gamma\mathbb{Z})^\times} \chi(k) a_{F,k\gamma}(0) \end{aligned}$$

vanishes. Now $L(l-\kappa, \chi) \neq 0$ and hence

$$\sum_{k \in (\mathbb{Z}/N_\gamma\mathbb{Z})^\times} \chi(k) a_{F,k\gamma}(0) = 0$$

for all Dirichlet characters χ. But then we must have $a_{F,k\gamma}(0) = 0$ for all k, since Dirichlet characters form an orthogonal basis of $(\mathbb{Z}/N_\gamma\mathbb{Z})^\times$. □

Corollary 6.5.7. *The theta lift Φ surjects onto the space of holomorphic modular forms of singular weight which are linear combinations of Eisenstein series on the boundary and whose value in a cusp only depends on its image in L'/L, i.e. we have $\mathrm{M}_\kappa^\Phi(\Gamma(L)) = \mathrm{M}_\kappa^\pi(\Gamma(L))$.*

Proof. Let $F : \mathbb{H}_l \to \mathbb{C}$ be a holomorphic modular form which is a linear combination of Eisenstein series on the boundary whose value in a cusp only depends on its image in L'/L. Since Φ and Φ^* are adjoint to each other by Lemma 6.5.1, we can write $F = \Phi(Z, \mathfrak{v}) + G$ for an invariant vector $\mathfrak{v}$ and a holomorphic modular form $G : \mathbb{H}_l \to \mathbb{C}$ of singular weight with $\Phi^*(\tau, G) = 0$. By Corollary 6.5.6 we have $a_{G,\delta}(0) = 0$ for all $\delta \in \mathrm{Iso}(L'/L)$. Moreover, the value of $G = F - \Phi(Z, \mathfrak{v})$ in a cusp only depends on its image in L'/L. Hence we have

$$0 = a_{G,\delta}(0) = \sum_{\lambda \in \pi_L^{-1}(\delta)} a_{G,\lambda}(0) C(\lambda) = a_{G,\tilde{\lambda}}(0) \sum_{\lambda \in \pi_L^{-1}(\delta)} C(\lambda)$$

for some $\tilde{\lambda} \in \pi_L^{-1}(\delta)$. But then $a_{G,\tilde{\lambda}}(0) = 0$ and hence G vanishes in every cusp and thus it is a cusp form on the boundary. But since F and $\Phi(Z, \mathfrak{v})$ are linear combinations of Eisenstein series on the boundary, $G = F - \Phi(Z, \mathfrak{v})$ must vanish on the boundary and hence G is a cusp form. Since we are in singular weight, G must vanish and thus $F = \Phi(Z, \mathfrak{v})$. □

Corollary 6.5.8. *If π_L is injective, then the theta lift Φ surjects onto the space of holomorphic modular forms of singular weight which are linear combinations of Eisenstein series on the boundary, i.e. we have $\mathrm{M}_\kappa^{\Phi}(\Gamma(L)) = M_\kappa^{\partial\,\mathrm{Eis}}(\Gamma(L))$.*

Proof. Since π_L is injective we have $M_\kappa^{\pi}(\Gamma(L)) = M_\kappa^{\partial\,\mathrm{Eis}}(\Gamma(L))$. □

Corollary 6.5.9. *If L is a maximal lattice of Witt rank 2, then the space $\mathrm{M}_\kappa^{\partial\,\mathrm{Eis}}(\Gamma(L))$ is either 1-dimensional (if L is unimodular) or 0-dimensional (if L is not unimodular). Moreover, if $\kappa = 2, 4, 6, 8, 10, 14$, i.e. $l = 6, 10, 14, 18, 22, 30$, then we have $\mathrm{M}_\kappa(\Gamma(L)) = \mathrm{M}_\kappa^{\partial\,\mathrm{Eis}}(\Gamma(L))$. In the other cases there could be additional holomorphic modular forms that are cusp forms on the boundary.*

Proof. Let L be maximal of signature $(2, l), l \geq 6$. Then L splits two hyperbolic planes over $\mathbb{Z}$ and hence the map π_L is bijective. Thus the space of holomorphic modular forms of singular weight which are linear combinations of Eisenstein series on the boundary has the same dimension as the space of invariant vectors in $\mathbb{C}[L'/L]$. The latter is either 0-dimensional (if L is not unimodular) or 1-dimensional (if L is unimodular). Since we are in singular weight, there are no cusp forms. Since the 1-dimensional boundary components of $\Gamma(L)\backslash\mathbb{H}_l$ are given by $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ and since there are no cusp forms of weight $2, 4, 6, 8, 10, 14$ for $\mathrm{SL}_2(\mathbb{Z})$, we obtain $\mathrm{M}_\kappa(\Gamma(L) =) \mathrm{M}_\kappa^{\partial\,\mathrm{Eis}}(\Gamma(L))$ if $l = 6, 10, 14, 18, 22, 30$. □

Appendix A

Formulas

Lemma A.1. *For $\kappa \in \mathbb{Z}, \kappa \geq 0$ we have*

$$\sum_{j=0}^{\lfloor\frac{\kappa}{2}\rfloor}(-1)^j\binom{\kappa}{2j}\Gamma(1/2+j)\Gamma(1/2+s-j) = 2^{\kappa-2s}\pi\frac{\Gamma(1+2s-\kappa)}{\Gamma(1+s-\kappa)}$$

$$= \sqrt{\pi}\frac{\Gamma(\frac{1}{2}+s-\frac{\kappa}{2})\Gamma(1+s-\frac{\kappa}{2})}{\Gamma(1+s-\kappa)}.$$

Proof. Similar formulas are shown in [O'S18]. We first assume that $s \geq \kappa$ is an integer. Then

$$\sum_{j=0}^{\lfloor\frac{\kappa}{2}\rfloor}(-1)^j\binom{\kappa}{2j}\Gamma(1/2+j)\Gamma(1/2+(s-j))$$

$$= \frac{\pi\kappa!}{4^s}\sum_{j=0}^{\lfloor\frac{\kappa}{2}\rfloor}(-1)^j\frac{1}{(2j)!(\kappa-2j)!}\frac{(2j)!}{j!}\frac{(2s-2j)!}{(s-j)!}$$

$$= \frac{\pi\kappa!(2s-\kappa)!}{4^s s!}\sum_{j=0}^{\lfloor\frac{\kappa}{2}\rfloor}(-1)^j\binom{s}{s-j}\binom{2s-2j}{2s-\kappa}$$

$$= (-1)^{\lfloor\frac{\kappa}{2}\rfloor}\frac{\pi\kappa!(2s-\kappa)!}{4^s s!}\sum_{j=0}^{\lfloor\frac{\kappa}{2}\rfloor}(-1)^j\binom{s}{s-\lfloor\frac{\kappa}{2}\rfloor+j}\binom{2s-2\lfloor\frac{\kappa}{2}\rfloor+2j}{2s-\kappa}$$

$$= (-1)^s\frac{\pi\kappa!(2s-\kappa)!}{4^s s!}\sum_{j=s-\lfloor\frac{\kappa}{2}\rfloor}^{s}(-1)^j\binom{s}{j}\binom{2j}{2s-\kappa}$$

$$= \frac{\pi\kappa!(2s-\kappa)!}{2^{2s-\kappa}s!}\binom{s}{s-\kappa}$$

$$= \frac{\pi(2s-\kappa)!}{2^{2s-\kappa}(s-\kappa)!} = 2^{\kappa-2s}\pi\frac{\Gamma(2s-\kappa+1)}{\Gamma(s-\kappa+1)}$$

This shows the identity for integers $s \geq \kappa$ (in fact this also shows the identity for integers $\kappa > s \geq \lfloor \kappa/2 \rfloor$ since then both sides vanish). Now observe using $\Gamma(z+1) = z\Gamma(z)$ and the duplication formula that

$$\begin{aligned} 2^{\kappa-2s}\pi \frac{\Gamma(1+2s-\kappa)}{\Gamma(1+s-\kappa)} &= \sqrt{\pi}\frac{\Gamma(\frac{1}{2}+s-\frac{\kappa}{2})\Gamma(1+s-\frac{\kappa}{2})}{\Gamma(1+s-\kappa)} \\ &= \Gamma(1/2+s-\kappa)p(s) \end{aligned}$$

for some polynomial p. Similarly, the left hand side is given by

$$\sum_{j=0}^{\lfloor \frac{\kappa}{2} \rfloor} (-1)^j \binom{\kappa}{2j} \Gamma(1/2+j)\Gamma(1/2+s-j) = \Gamma(1/2+s-\kappa)q(s)$$

for some polynomial q. Now we have seen that $p(s) = q(s)$ for infinitely many integers and thus $p(s) = q(s)$ for all $s \in \mathbb{C}$. □

Lemma A.2 ([Bor98, Corollary 14.2]). *If C and $\kappa - h$ are integers such that $0 < C < \kappa - h$, then*

$$\sum_{j+m=C} \frac{(-1)^j(\kappa-h+m-j-1)!}{j!(\kappa-h-2j)!m!} = 0$$

Lemma A.3. *We have*

$$\begin{aligned} &\sum_{h=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j}{(4\pi|y|)^j j!} \binom{\kappa}{h} \frac{(\kappa-h)!}{(\kappa-h-2j)!} \left(\frac{y}{|y|}\right)^{\kappa-h} K_{-\frac{1}{2}+\kappa-h-j}(2\pi|y|) \\ &= 2^{\kappa-1} y^{-\frac{1}{2}} e^{-2\pi y}. \end{aligned}$$

if $y > 0$ and the left hand side vanishes for $y < 0$.

Proof. This is done in [Bor98, Theorem 14.3]. We use the formula

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{0 \leq m} (2z)^{-m} \frac{(n+m)!}{m!(n-m)!}$$

to obtain that the sum over j is given by (ignoring the terms indepenend of j)

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{(-1)^j e^{-2\pi|y|}}{(4\pi|y|)^j j!(\kappa-h-2j)!\sqrt{4|y|}} \sum_{0 \leq m} (4\pi|y|)^{-m} \frac{(\kappa-h+m-j-1)!}{(\kappa-h-j-m-1)!m!} \\ &= \frac{e^{-2\pi|y|}}{\sqrt{4|y|}} \sum_{C=0}^{\infty} \frac{(4\pi|y|)^{-C}}{(\kappa-h-C-1)!} \sum_{j+m=C} \frac{(-1)^j(\kappa-h+m-j-1)!}{j!(\kappa-h-2j)!m!} \end{aligned}$$

and observe that the terms with $m \neq 0, j \neq 0$ cancel out according to the previous lemma. Hence we obtain

$$\sum_{h=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j}{(4\pi|y|)^j j!} \binom{\kappa}{h} \frac{(\kappa-h)!}{(\kappa-h-2j)!} \left(\frac{y}{|y|}\right)^{\kappa-h} K_{-\frac{1}{2}+\kappa-h-j}(2\pi|y|)$$

$$= \frac{1}{2}\sum_{h=0}^{\infty}\binom{\kappa}{h}\left(\frac{y}{|y|}\right)^{\kappa-h}|y|^{-\frac{1}{2}}e^{-2\pi|y|}.$$

Now the sum over h vanishes for $y < 0$ and is equal to 2^κ for $y > 0$, which shows the result. □

Definition A.4. Let $N \in \mathbb{N}$ and $c \in \mathbb{Z}/N\mathbb{Z}$. Define the *modified zeta functions*

$$\zeta_+^c(s) := \sum_{\substack{n\equiv c \bmod N\\ n>1}} \frac{1}{n^s}, \quad \zeta^c(s) := \sum_{n\equiv c \bmod N} \frac{1}{n^s}.$$

They both have a meromorphic continuation to all $s \in \mathbb{C}$. For a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ define

$$L(\chi, s) := \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} = \sum_{c\in\mathbb{Z}/N\mathbb{Z}} \chi(c)\zeta_+^c(s).$$

We have $\zeta_+^c(s) = N^{-s}\zeta(s, c/N)$ for $0 \le c < N$, where $\zeta(s, x)$ is the *Hurwitz zeta function*. It is related to the Bernoulli polynomials $B_\kappa(x)$ by

$$\zeta(-\kappa, x) = -\frac{B_{\kappa+1}(x)}{\kappa+1}$$

for integral $\kappa \ge 0$, which implies (using $B_{\kappa+1}(1-x) = (-1)^\kappa B_{\kappa+1}(x)$, see [OLBC10, 24.4.3]) $\zeta_+^c(-\kappa) + (-1)^\kappa\zeta_+^{-c}(-\kappa) = 0$. Further we have

$$\zeta^c(s) = \zeta_+^c(s) + e^{-\pi i s}\zeta_+^{-c}(s)$$

For integers κ this yields

$$\zeta^c(\kappa) = \zeta_+^c(\kappa) + (-1)^\kappa\zeta_+^{-c}(\kappa)$$

and $\zeta^{-c}(\kappa) = (-1)^\kappa\zeta^c(\kappa)$. Now let $g = \gcd(c, N), c = gc', N = gN'$. Then we have

$$\zeta_+^c(s) = \sum_{\substack{n\equiv c \bmod N\\ n>0}} \frac{1}{n^s} = \sum_{\substack{n\equiv gc' \bmod gN'\\ n>0}} \frac{1}{n^s} = g^{-s}\sum_{\substack{n\equiv c' \bmod N'\\ n>0}} \frac{1}{n^s} = g^{-s}\zeta_+^{c'}(s)$$

and similarly $\zeta^c(s) = g^{-s}\zeta^{c'}(s)$.

Lemma A.5. *For $\beta \in \mathbb{Z}/N\mathbb{Z}$ we have*

$$\frac{\Gamma(1-2s-\kappa)}{\Gamma(1-s-\kappa)}2^{2s}\pi^s \sum_{b,c\in\mathbb{Z}/N\mathbb{Z}} e\left(\frac{bc}{N}\right)(\delta_{\beta,b} + (-1)^\kappa\delta_{-\beta,b})\zeta_+^c(1-2s-\kappa)$$
$$= \frac{N^{2s+\kappa}\Gamma(s+\kappa)}{(-2\pi i)^\kappa\pi^s}(\zeta_+^\beta(2s+\kappa) + (-1)^\kappa\zeta_+^{-\beta}(2s+\kappa)).$$

Proof. One first shows the formula

$$\sum_{c \in \mathbb{Z}/N\mathbb{Z}} e\left(\frac{bc}{N}\right) \zeta_+^c(1-2s-\kappa) = \frac{\Gamma(2s+\kappa)N^{2s+\kappa}}{(-2\pi i)^\kappa (2\pi)^{2s}} e^{\pi i s} \zeta^b(2s+\kappa)$$

which follows from the functional equation of the Hurwitz zeta function [OLBC10, 25.11.16]. Now use the duplication formula of the Gamma function to show

$$\frac{\Gamma(1-2s-\kappa)\Gamma(2s+\kappa)}{\Gamma(1-s-\kappa)} = \Gamma(s+\kappa)\frac{\sin(\pi s)}{\sin(2\pi s)} = \frac{\Gamma(s+\kappa)}{2\cos(\pi s)}.$$

Combined with

$$\zeta^b(2s+\kappa) + (-1)^\kappa \zeta^{-b}(2s+\kappa) = 2\cos(\pi s) e^{-\pi i s}(\zeta_+^b(2s+\kappa) + (-1)^\kappa \zeta_+^{-b}(2s+\kappa))$$

this shows the result. □

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