Isogeometric collocation for nonlinear dynamic analysis of Cosserat rods with frictional contact

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Abstract We present a novel isogeometric collocation method for nonlinear dynamic analysis of three-dimensional, slender, elastic rods. The approach is based on the geometrically exact Cosserat model for rod dynamics. We formulate the governing nonlinear partial differential equations as a first-order problem in time and develop an isogeometric semi-discretization of position, orientation, velocity and angular velocity of the rod centerline as NURBS curves. Collocation then leads to a nonlinear system of first-order ordinary differential equations, which can be solved using standard time integration methods. Furthermore, our model includes viscoelastic damping and a frictional contact formulation. The computational method is validated and its practical applicability shown using several numerical applications of nonlinear rod dynamics.

Keywords Isogeometric analysis · Collocation method · Cosserat rod model · Nonlinear dynamics · Frictional contact

1 Introduction

Modeling and simulation of thin deformable bodies has wide-spread applications in engineering, sciences and animation, such as vibrations of bridges, cables, drill strings and rigs, and machines [1–3], deformation of woven and knitted textiles [4], additively manufactured

structures [5], hair and fiber modeling [6, 7], and bio-dynamic structures such as the double helix of DNA molecules and arterial pathways [8]. In many of these problems, complex dynamic behavior and rod-to-rod contact interactions are essential aspects for the accurate modeling of physical effects and thus accurate, robust and efficient computational discretization methods are required for their numerical solution.

For the static and dynamic modeling of 3-dimensional (3D), slender beam structures subject to large deformations and rotations, the Cosserat rod model [9–11] has been employed successfully in many of the aforementioned problems. It covers nonlinear, geometrically exact deformation behavior, general loading conditions by external forces and moments, and can be extended to include viscous damping [12–14]. It leads to a nonlinear partial differential equation (PDE) in space and time, which usually has to be solved by numerical methods due to its complexity, wherefore various kinds of discretization schemes have been proposed. Most commonly, first a spatial semi-discretization is carried out, either using finite element (FEM) [15, 16] or finite difference methods (FDM) [6, 13, 17], and then time integration is performed using standard methods for ordinary differential equations (ODEs). In [18], *p*-FEM was used in combination with harmonic balance to solve the periodic vibration problem. Dynamics of rod structures, i.e. meshes or nets of interconnected Cosserat rods, were also investigated using these methods [19, 20].

In this work, we apply an isogeometric collocation method for the spatial semi-discretization. Isogeometric analysis (IGA) was first introduced by Hughes et al. in 2005 [21] and has since attracted increasing interest in the computational engineering community. This novel concept aims at bridging the gap between the two largely disjunct domains of computer-aided design

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(CAD) and computational analysis by using spline-based function representations within the numerical discretization scheme. Non-uniform rational B-Splines (NURBS), which are most commonly used for geometry description in CAD, are directly employed for geometry and basis function representation in IGA. Applications of isogeometric methods in the field of computational mechanics of slender structures are particularly advantageous, since they allow a straight-forward CAD integration and mechanical models often require higher-order differentiability of the shape functions. IGA has been successfully applied in novel finite element formulations for beams and rods [3, 22–24], as well as plates and shells [25–27]. Furthermore, it has been shown that isogeometric discretizations have superior approximation properties especially for dynamic and vibration problems, since they avoid the presence of low-frequency optical branches in the frequency spectrum [28–30].

Isogeometric collocation methods were more recently introduced as an alternative to conventional Galerkin or finite element methods [31, 32]. Collocation methods are based on the strong form of equilibrium equations, thus requiring higher continuity of the shape functions – which is enabled by higher-order splines in isogeometric analysis. Isogeometric collocation schemes have been shown to be computationally more efficient than Galerkin methods [33] and have been developed for a wide range of applications, including elasto-dynamics [34], contact problems [35, 36], and various beam and rod formulations [37–40]. Isogeometric collocation of the static Cosserat rod model was introduced in [41], combining the accuracy of isogeometric methods using higher-order spline parameterizations with the efficiency of the collocation approach.

An important aspect in applications such as textile, fiber or hair modeling is the consideration of rod-to-rod and self-contact in the mechanical rod model and a number of contact algorithms for 3D beam and rod finite element models have been proposed for static analysis [4, 42–45]. Most works on dynamic contact from the animation and computer graphics community relate to more approximate contact formulations and are based on finite difference discretizations [7, 17]. In this paper, we extend the frictionless rod-to-rod and self-contact formulation for isogeometric collocation of Cosserat introduced first in [46] to the dynamic regime including Coulomb friction.

The further outline of this paper is as follows: First, we introduce the basic model of Cosserat rod dynamics in Section 2. We review the kinematics and constitutive equations that lead to the governing partial differential equations and include viscous damping into the formulation. Based on the formulation of the prob-

lem as a first-order system in time, we introduce the spline ansatz for the spatial semi-discretization of unknown fields, i.e. centerline positions, rotations, velocities and angular velocities, and apply the isogeometric collocation approach in Section 3. Then we briefly describe time discretization of the resulting first-order ODE system using the conventional implicit Euler and Crank–Nicolson schemes in Section 4. In Section 5, we outline the enhancement of the dynamic rod model with a contact formulation and algorithm, which includes a penalty method for normal contact and friction based on Coulomb’s law. The numerical validation and application of presented methods to several model problems is then discussed in Section 6. Finally, we present a summary of the methods introduced and concluding remarks in Section 7.

2 Dynamic Cosserat rod model

In this Section we briefly introduce the Cosserat rod model, which we use for the mechanical description of the dynamic motion and elastic deformation of slender, elastic, 3-dimensional rods [10, 11].

2.1 Kinematics and constitutive equations

The Cosserat rod theory can be seen as a nonlinear, geometrically exact extension of the spatial Timoshenko beam model, and is thus also based on the assumption that the cross-sections remain undeformed, but not necessarily normal to the tangent of the centerline curve, which accounts also for shear deformation.

In the Cosserat model, a rod is represented as a framed curve (see Fig. 1) and thus its configuration at any time $t \in [t_0, t_1]$ is completely described by its centerline curve, i.e. the line of its mass centroids,

$$\mathbf{r} : [0, L] \times [t_0, t_1] \rightarrow \mathbb{R}^3, \quad (1)$$

and a frame, i.e. a local orthonormal basis field:

$$\mathbf{R} : [0, L] \times [t_0, t_1] \rightarrow SO(3). \quad (2)$$

The centerline curve is arc-length parameterized in the initial configuration given by $\mathbf{r}_0(s) := \mathbf{r}(s, t_0)$, which means that:

$$\|\mathbf{r}'_0(s)\| := \left\| \frac{\partial \mathbf{r}_0}{\partial s} \right\| = 1 \quad \forall s \in [0, L], \quad (3)$$

and thus $L = \int_0^L \|\mathbf{r}'_0(s)\| ds$ is the initial length of the curve.

Fig. 1: Configuration of the Cosserat rod at time t as a framed curve with centerline $\mathbf{r}(s, t)$ and orthonormal frames $\mathbf{R}(s, t) = (\mathbf{g}_1(s, t), \mathbf{g}_2(s, t), \mathbf{g}_3(s, t))$ defining the cross-section orientations (see also [41])

The local frames describe the evolution of the orientation of the cross-section and can be associated with 3D rotation matrices:

$$\mathbf{R}(s, t) = (\mathbf{g}_1(s, t), \mathbf{g}_2(s, t), \mathbf{g}_3(s, t)) \in \mathbb{R}^{3 \times 3} : \quad (4)$$

$$\mathbf{R}^\top \mathbf{R} = \mathbf{I}, \det \mathbf{R} = 1.$$

As in [41], we use unit quaternions, i.e. normalized quadruples of real numbers $\mathbf{q} = (q_1, q_2, q_3, q_4)^\top \in \mathbb{R}^4 : \|\mathbf{q}\| = 1$, for the parameterization of frames resp. rotation matrices:

$$\mathbf{q} : [0, L] \times [t_0, t_1] \rightarrow SO(3) \rightsquigarrow \mathbf{R}(s, t) = \mathbf{R}(\mathbf{q}(s, t)), \quad (5)$$

where

$$\mathbf{R}(\mathbf{q}) = \begin{pmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1 q_2 - q_3 q_4) & 2(q_1 q_3 + q_2 q_4) \\ 2(q_1 q_2 + q_3 q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2 q_3 - q_1 q_4) \\ 2(q_1 q_3 - q_2 q_4) & 2(q_2 q_3 + q_1 q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{pmatrix}. \quad (6)$$

Based on the current (deformed) centerline $\mathbf{r}(s, t)$ and rotation matrix $\mathbf{R}(s, t)$ associated with the frame, as well as the initial (undeformed) configuration $\mathbf{r}_0(s)$ and $\mathbf{R}_0(s) := \mathbf{R}(s, t_0)$, the kinematics of the Cosserat rod can be derived. Dropping the dependency on the arc-length parameter s and time t in the notation, the translational strains are given as:

$$\boldsymbol{\varepsilon} = \mathbf{R}^\top \mathbf{r}' - \mathbf{R}_0^\top \mathbf{r}'_0. \quad (7)$$

Using the curvature vector of the rod:

$$\mathbf{k} = \begin{pmatrix} \mathbf{g}_2'^\top \mathbf{g}_3 \\ \mathbf{g}_3'^\top \mathbf{g}_1 \\ \mathbf{g}_1'^\top \mathbf{g}_2 \end{pmatrix} = 2\mathbf{q}^* \mathbf{q}' \Leftrightarrow [\mathbf{k}]_\times = \mathbf{R}'^\top \mathbf{R}, \quad (8)$$

where $*$ represents the conjugate of a quaternion and $[\cdot]_\times$ the skew-symmetric cross-product matrix, the rotational strains are defined as:

$$\boldsymbol{\kappa} = \mathbf{k} - \mathbf{k}_0. \quad (9)$$

Given these two nonlinear strain vectors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$, and a linear elastic constitutive law, the translational and rotational stress resultants can be computed as:

$$\begin{aligned} \boldsymbol{\sigma} &= \mathbf{A} \boldsymbol{\varepsilon} + \mathbf{B} \boldsymbol{\kappa}, \\ \boldsymbol{\chi} &= \mathbf{B}^\top \boldsymbol{\varepsilon} + \mathbf{C} \boldsymbol{\kappa}. \end{aligned} \quad (10)$$

The constitutive matrices $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{3 \times 3}$ depend on the material and geometry of the cross-sections. Here, we focus on rods with constant, homogeneous, symmetric cross-sections with $\mathbf{A} = \text{diag}\{k_1 GA, k_2 GA, EA\}$, $\mathbf{B} \equiv 0$ and $\mathbf{C} = \text{diag}\{EI_1, EI_2, GJ_T\}$, where E is the Young's and $G = E/(2 + 2\nu)$ the shear modulus of the material, ν the Poisson's ratio, k_1 and k_2 the shear correction factors (here $k_1 = k_2 = \frac{5}{6}$), A the cross-section area, I_1 and I_2 the second moments of area, and J_T the torsion constant. The general derivation of constitutive coefficients for arbitrarily shaped composite cross-sections can be found in [47] and an application of two-layer composite cross-sections with eigentrains to 4D printing simulation is presented in [48].

The stress vectors defined in (10) are now rotated from the local into the global coordinate frame, or in other words, they are transformed from the spatial back to the material configuration, which defines the force and moment resultants:

$$\begin{aligned} \mathbf{n} &= \mathbf{R} \boldsymbol{\sigma}, \\ \mathbf{m} &= \mathbf{R} \boldsymbol{\chi}. \end{aligned} \quad (11)$$

The dynamic rod model furthermore requires the velocity $\mathbf{v}$ of the centerline:

$$\mathbf{v}(s, t) := \dot{\mathbf{r}}(s, t) = \frac{\partial \mathbf{r}}{\partial t}, \quad (12)$$

as well as the angular velocity $\mathbf{w}$ of the frame:

$$\mathbf{w}(s, t) = \begin{pmatrix} \dot{\mathbf{g}}_2^\top \mathbf{g}_3 \\ \dot{\mathbf{g}}_3^\top \mathbf{g}_1 \\ \dot{\mathbf{g}}_1^\top \mathbf{g}_2 \end{pmatrix} = 2\mathbf{q}^* \dot{\mathbf{q}} \Leftrightarrow [\mathbf{w}]_\times = \dot{\mathbf{R}}^\top \mathbf{R}. \quad (13)$$

Using these two quantities, the distributed inertial forces and moments can be computed:

$$\begin{aligned} \mathbf{n}_\rho &= -\rho A \dot{\mathbf{v}}, \\ \mathbf{m}_\rho &= -\rho \mathbf{R}(\mathbf{G} \dot{\mathbf{w}} + \mathbf{w} \times (\mathbf{G} \mathbf{w})). \end{aligned} \quad (14)$$

Here, ρ is the mass density and $\mathbf{G} = \text{diag}\{I_1, I_2, I_P\}$ is the inertia matrix, I_P being the polar moment of area.

2.2 Damping

An important aspect in dynamic simulations is damping, which is a physical dissipation phenomenon occurring in real-life dynamic and vibration behaviour, but also a numerical phenomenon, which may be required to stabilize numerical oscillations. Here, we introduce viscoelastic damping into the model according to [13].

To implement damping, we first need to calculate the translational and rotational strain rates:

$$\begin{aligned}\dot{\boldsymbol{\varepsilon}} &= \frac{\partial \boldsymbol{\varepsilon}}{\partial t} = \dot{\mathbf{R}}^\top \mathbf{r}' + \mathbf{R}^\top \mathbf{v}', \\ \dot{\boldsymbol{\kappa}} &= \frac{\partial \boldsymbol{\kappa}}{\partial t} = \left[\dot{\mathbf{R}}^\top \mathbf{R}' + \mathbf{R}^\top \dot{\mathbf{R}}' \right]_\times.\end{aligned}\quad (15)$$

Then, the translational and rotational stress resultants from (10) can be updated to include dissipation:

$$\begin{aligned}\boldsymbol{\sigma} &= \mathbf{A} \boldsymbol{\varepsilon} + 2 \mathbf{A}^d \dot{\boldsymbol{\varepsilon}}, \\ \boldsymbol{\chi} &= \mathbf{C} \boldsymbol{\kappa} + 2 \mathbf{C}^d \dot{\boldsymbol{\kappa}},\end{aligned}\quad (16)$$

where the shearing/extensional damping matrix is given by $\mathbf{A}^d = \text{diag}\{A_1^d, A_2^d, A_3^d\}$ and the bending/torsional damping matrix by $\mathbf{C}^d = \text{diag}\{C_1^d, C_2^d, C_3^d\}$. A more detailed derivation of those parameters from bulk and shear viscosities of a viscoelastic Kelvin-Voigt model can be found in [14].

Using (16), the internal forces and moments in (11) and all following equations and derivations can be updated to include viscous damping effects.

2.3 Dynamic equilibrium equations

Based on the kinematic quantities derived above, the strong form of the dynamic equations of motion of a Cosserat rod is formulated in terms of equilibria of linear and angular momentum in the initial (material) configuration $\forall (s, t) \in \Omega_T = [0, L] \times [t_0, t_1]$:

$$\begin{aligned}\mathbf{n}_\rho + \mathbf{n}' + \bar{\mathbf{n}} &= \mathbf{0}, \\ \mathbf{m}_\rho + \mathbf{m}' + \mathbf{r}' \times \mathbf{n} + \bar{\mathbf{m}} &= \mathbf{0}.\end{aligned}\quad (17)$$

Here, $\bar{\mathbf{n}}$ and $\bar{\mathbf{m}}$ are external distributed (line) forces and moments acting on the rod.

The partial differential equation problem is completed by boundary conditions of Dirichlet or Neumann type for the two ends of the rod $s = 0$ and $s = L$ for all $t \in [t_0, t_1]$, as well as initial conditions for $s \in [0, L]$ and $t = t_0$. Dirichlet or displacement/rotation boundary conditions are of the form:

$$\mathbf{r}(s, t) = \hat{\mathbf{r}} \quad \wedge \quad \mathbf{q}(s, t) = \hat{\mathbf{q}}, \quad s = 0, L, \quad (18)$$

and Neumann or force/moment boundary conditions read:

$$\mathbf{n}(s, t) = \hat{\mathbf{n}} \quad \wedge \quad \mathbf{m}(s, t) = \hat{\mathbf{m}}, \quad s = 0, L, \quad (19)$$

where $\hat{\mathbf{r}}$, $\hat{\mathbf{q}}$, $\hat{\mathbf{n}}$, and $\hat{\mathbf{m}}$ refer to the prescribed endpoint displacement, rotation, force, and moment. The initial conditions at $t = t_0$ for all $s \in [0, L]$ are:

$$\begin{aligned}\mathbf{r}(s, t_0) &= \mathbf{r}_0(s), \quad \mathbf{q}(s, t_0) = \mathbf{q}_0(s), \\ \mathbf{v}(s, t_0) &= \mathbf{v}_0(s), \quad \mathbf{w}(s, t_0) = \mathbf{w}_0(s).\end{aligned}\quad (20)$$

Furthermore, the algebraic quaternion normalization condition $\|\mathbf{q}(s, t)\| = 1 \Leftrightarrow \mathbf{q}(s, t)^T \mathbf{q}(s, t) = 1$ needs to hold for all $s \in [0, L]$, $t \in [t_0, t_1]$.

3 Isogeometric collocation of dynamic rod

In this Section, we introduce the isogeometric collocation method for the spatial discretization of the unknowns and equation of motion of Cosserat rods. The approach is based on the spatial parameterization of the unknown fields $\mathbf{r}(s, t)$, $\mathbf{q}(s, t)$, $\mathbf{v}(s, t)$, and $\mathbf{w}(s, t)$ using spline (NURBS) functions and the collocation of the equilibrium equations (17).

3.1 NURBS parameterization

The basis of any isogeometric method is the parameterization of geometry and unknowns using B-Splines and Non-Uniform Rational B-Splines (NURBS), which are widely used in computer-aided design [21]. Definitions and properties of B-Splines and NURBS can be found in detail in [49]. Here we briefly introduce the main terminology associated with splines.

B-Splines are piecewise polynomial functions and Non-Uniform Rational B-Spline (NURBS) are piecewise rational function of degree p and order $p + 1$. With $N_i(\xi) : \Omega_0 \rightarrow [0, 1]$, $i = 1, \dots, n$ we denote the spline (B-Spline or NURBS) basis functions, which are defined on the parameter domain $\Omega_0 = [\xi_1, \xi_m] \subset \mathbb{R}$ using a knot vector $\Xi = \{\xi_1, \dots, \xi_m\}$ with $m = n + p + 1$, i.e. a non-decreasing sequence of knots $\xi_i \in \mathbb{R}$ ($i = 1, \dots, m$), $\xi_i \leq \xi_{i+1}$ ($i = 1, \dots, m - 1$). For two distinct knots $\xi_i \neq \xi_{i+1}$ the half-open interval $[\xi_i, \xi_{i+1})$ is called the i -th knot span or element and the total number of nonzero knot spans or elements in Ξ is denoted by ℓ . Typically, only open knot vectors are used in IGA, which means that the first and last knot have multiplicity $p + 1$ and inner knots a multiplicity $1 \leq k \leq p$.

The initial geometry description of a Cosserat rod, as well as its initial conditions, are now parameterized

Fig. 2: Isogeometric rod parameterization; cubic B-Spline basis functions as shown in (a) with $p = 3, m = 11, n = 7, \Xi = \{0, 0, 0, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1, 1, 1\}$ are used for the isogeometric parameterization of the Cosserat rod in (b) with centerline and rotation quaternions as B-Spline curves (see [41])

using spline curves for the initial centerline position $\mathbf{r}_0$, quaternion $\mathbf{q}_0$, velocity $\mathbf{v}_0$ and angular velocity $\mathbf{w}_0$:

$$\begin{aligned} \mathbf{r}_0 : \Omega_0 &\rightarrow \mathbb{R}^3, & \mathbf{r}_0(\xi) &= \sum_{i=1}^n N_i(\xi) \mathbf{r}_{0,i}, \\ \mathbf{q}_0 : \Omega_0 &\rightarrow \mathbb{R}^4, & \mathbf{q}_0(\xi) &= \sum_{i=1}^n N_i(\xi) \mathbf{q}_{0,i}, \\ \mathbf{v}_0 : \Omega_0 &\rightarrow \mathbb{R}^3, & \mathbf{v}_0(\xi) &= \sum_{i=1}^n N_i(\xi) \mathbf{v}_{0,i}, \\ \mathbf{w}_0 : \Omega_0 &\rightarrow \mathbb{R}^3, & \mathbf{w}_0(\xi) &= \sum_{i=1}^n N_i(\xi) \mathbf{w}_{0,i}, \end{aligned} \quad (21)$$

using control points $\mathbf{r}_{0,i} \in \mathbb{R}^3$, $\mathbf{q}_{0,i} \in \mathbb{R}^4$ such that $\|\mathbf{q}_{0,i}\| = 1$, $\mathbf{v}_{0,i} \in \mathbb{R}^3$, and $\mathbf{w}_{0,i} \in \mathbb{R}^3$ for $i = 1, \dots, n$.

For illustration, the parameterization of a rod using cubic B-Spline basis functions ($p = 3$) with $n = 7$ control points and $\ell = 4$ elements in the knot vector $\Xi = \{0, 0, 0, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1, 1, 1\}$ is shown in Fig. 2. Figure 2a shows the basis functions and Fig. 2b the rod itself in terms of its centerline curve and cross-section frames.

Since the centerline is now parameterized as a spline curve $\mathbf{r}_0(\xi) : \Omega_0 \rightarrow \mathbb{R}^3$ with an arbitrary domain of parameterization $\Omega_0 \subset \mathbb{R}$, it is in general not arc-length parameterized. Thus the derivatives of any vector field $\xi \rightarrow \mathbf{y}(\xi) : [0, 1] \rightarrow \mathbb{R}^d$ required for evaluation of the Cosserat rod model need to be converted to arc-length parameterization using:

$$\begin{aligned} \mathbf{y}' &= \frac{d\mathbf{y}}{ds} = \frac{d\mathbf{y}}{d\xi} \frac{d\xi}{ds} = \mathbf{y}^\nabla \left(\frac{ds}{d\xi} \right)^{-1} \\ &= \mathbf{y}^\nabla \frac{1}{\|\mathbf{r}_0'(\xi)\|} = \frac{1}{J} \mathbf{y}^\nabla, \end{aligned} \quad (22)$$

with $\mathbf{y}^\nabla := \frac{d\mathbf{y}}{d\xi}$ and $J(\xi) := \|\mathbf{r}_0'(\xi)\|$. For instance, it is $\mathbf{r}'(\xi) = \frac{1}{J(\xi)} \mathbf{r}^\nabla(\xi)$.

3.2 Isogeometric collocation method

As in isoparametric finite elements, the unknown centerline position $\mathbf{r}(s, t)$, rotation quaternion $\mathbf{q}(s, t)$, velocity $\mathbf{q}(s, t)$, and angular velocity $\mathbf{w}(s, t)$ in the current/deformed configuration are now also spatially discretized as spline curves $\mathbf{r}_h$, $\mathbf{q}_h$, $\mathbf{v}_h$, and $\mathbf{w}_h$:

$$\begin{aligned} \mathbf{r}_h : \Omega_T &\rightarrow \mathbb{R}^3, & \mathbf{r}_h(s, t) &= \sum_{i=1}^n N_i(s) \mathbf{r}_i(t), \\ \mathbf{q}_h : \Omega_T &\rightarrow \mathbb{R}^4, & \mathbf{q}_h(s, t) &= \sum_{i=1}^n N_i(s) \mathbf{q}_i(t), \\ \mathbf{v}_h : \Omega_T &\rightarrow \mathbb{R}^3, & \mathbf{v}_h(s, t) &= \sum_{i=1}^n N_i(s) \mathbf{v}_i(t), \\ \mathbf{w}_h : \Omega_T &\rightarrow \mathbb{R}^3, & \mathbf{w}_h(s, t) &= \sum_{i=1}^n N_i(s) \mathbf{w}_i(t). \end{aligned} \quad (23)$$

The basis functions N_i here refer to either the same or $p/h/k$ -refined versions of the ones in (21) and the control points $\mathbf{r}_i \in \mathbb{R}^3$, $\mathbf{q}_i \in \mathbb{R}^4$, $\mathbf{v}_i \in \mathbb{R}^3$, and $\mathbf{w}_i \in \mathbb{R}^3$ are arranged into the vector of unknown control points $\mathbf{x}(t) = (\mathbf{r}_i^T(t), \mathbf{q}_i^T(t), \mathbf{v}_i^T(t), \mathbf{w}_i^T(t))_{i=1, \dots, n}^T \in \mathbb{R}^{13n}$.

The main idea behind an isogeometric collocation method, as introduced in [31], is to evaluate the strains, stresses and force/moment resultants based on the discretized unknowns (23) in order to ultimately collocate the equilibrium equations at a given set of discrete collocation points $\{\tau_i\}_{i=1, \dots, n} \in [0, L]$.

Here, we apply this idea to the dynamic equations of motion of the rod model, i.e. the equilibria of linear and angular momentum given in (17). In order to obtain a semi-discretized, first-order differential equation in time, we also collocate the defining relations of velocities (12) and angular velocities (13), which are also independently discretized according to (23). This results in the following semi-discretized set of equations for each internal collocation point τ_i , $i = 2, \dots, n-1$ (dropping the dependency on t in the notation):

$$\begin{aligned}\dot{\mathbf{r}}_h(\tau_i) &= \mathbf{v}_h(\tau_i), \\ \dot{\mathbf{q}}_h(\tau_i) &= \frac{1}{2}\mathbf{q}_h(\tau_i)\mathbf{w}_h(\tau_i), \\ \dot{\mathbf{v}}_h(\tau_i) &= \frac{1}{\rho_A}(\mathbf{n}'(\tau_i) + \bar{\mathbf{n}}(\tau_i)), \\ \dot{\mathbf{w}}_h(\tau_i) &= \frac{1}{\rho}\mathbf{G}^{-1}\mathbf{R}^T(\mathbf{m}'(\tau_i) + \mathbf{r}'_h(\tau_i) \times \mathbf{n}(\tau_i) + \bar{\mathbf{m}}(\tau_i)) \\ &\quad - \mathbf{G}^{-1}(\mathbf{w}_h(\tau_i) \times (\mathbf{G}\mathbf{w}_h(\tau_i))).\end{aligned}\tag{24}$$

At the boundary collocation points $\tau_1 = 0$ and $\tau_n = L$, either Neumann boundary conditions for given forces ($\mathbf{n}(\tau_i, t) = \hat{\mathbf{n}}(\tau_i, t)$) and moments ($\mathbf{m}(\tau_i, t) = \hat{\mathbf{m}}(\tau_i, t)$) have to be collocated instead of the equilibria of linear and angular momentum, or Dirichlet boundary conditions for given $\hat{\mathbf{r}}_h(\tau_i, t)$, $\hat{\mathbf{q}}_h(\tau_i, t)$, $\hat{\mathbf{v}}_h(\tau_i, t)$, $\hat{\mathbf{w}}_h(\tau_i, t)$ be prescribed, see [41] for details.

As choice of collocation points, we use the Greville abscissae of the spline knot vector Ξ , which guarantee the stability of the collocation method [31]. Recently, so-called “super-convergence points” have been proposed as an alternative choice, which can provide optimal, Galerkin-like convergence rates and accuracies within the collocation approach [50, 51].

After collocation, we can write the spatially semi-discretized problem as a first order ODE system in a compact form:

$$\dot{\mathbf{x}}_h(t) = \mathbf{f}_h(t, \mathbf{x}(t)) \quad \forall t \in [t_0, t_1].\tag{25}$$

Here $\dot{\mathbf{x}}_h \in \mathbb{R}^{13n}$ refers to the left-hand sides and $\mathbf{f}_h \in \mathbb{R}^{13n}$ to the right-hand sides of (24), together with the boundary conditions combined into one vector for all τ_i , $i = 1, \dots, n$. To obtain the standard formulation as an ODE in terms of the control point vector $\mathbf{x}$, we can write $\dot{\mathbf{x}}_h = \mathbf{M}\dot{\mathbf{x}}$, where $\mathbf{M} \in \mathbb{R}^{13n \times 13n}$ is the collocation matrix containing the basis function evaluations $N_j(\tau_i)$ (including necessary modifications for the boundary conditions).

Remark 1 As mentioned earlier, the quaternion normalization condition $\|\mathbf{q}_h(s, t)\| = 1$ must hold since the quaternions are parameterized as 4-dimensional, real-valued NURBS curves in (23). However, unlike in the static case addressed in [41], it doesn’t have to be enforced separately since the quaternion normalization

becomes an invariant in the dynamic case [52]: The angular velocities can be defined as in (13) through the quaternion multiplication $\mathbf{w}_h = 2\mathbf{q}_h^*\dot{\mathbf{q}}_h$, where $\mathbf{w}_h$ should be seen as a quaternion whose imaginary component is 0. This means that $0 = 2\mathbf{q}_h^T\dot{\mathbf{q}}_h = \frac{\partial}{\partial t}(\mathbf{q}_h^T\mathbf{q}_h) = \frac{\partial}{\partial t}(\|\mathbf{q}_h\|^2)$. Thus, $\|\mathbf{q}_h\| = 1$ is fulfilled through the independent discretization of $\mathbf{w}_h$ and the differential equation $\dot{\mathbf{q}}_h = \frac{1}{2}\mathbf{q}_h\mathbf{w}_h \Leftrightarrow \mathbf{w}_h = 2\mathbf{q}_h^*\dot{\mathbf{q}}_h$ for all $t > 0$ if $\|\mathbf{q}_h(t_0)\| = \|\mathbf{q}_0\| = 1$.

4 Time discretization of dynamic rod model

After spatial semi-discretization of the dynamic equilibrium equations of the Cosserat rod model using an isogeometric collocation method, we arrive at a first order ODE system in the unknown NURBS control points $\mathbf{x}(t)$, given in compact form in (25):

$$\dot{\mathbf{x}}_h(t) := \mathbf{M}\dot{\mathbf{x}}(t) = \mathbf{f}_h(\mathbf{x}(t), t) \quad \forall t \in [t_0, t_1].$$

For the time discretization of this ODE system, we now use standard numerical time integration schemes.

We restrict ourselves to implicit integration methods, since it has been shown that (25) is a stiff problem due to the high-frequency shearing and extension modes present in the rod formulation, even when applied to soft materials such as rubber [13, 53]. Furthermore, as discussed in [35], the enforcement of Neumann boundary conditions is not trivial in isogeometric collocation when explicit methods are used.

Here, we use the two most basic implicit integration methods, namely, the backward Euler and the Crank–Nicolson schemes. In both cases we use a fixed time step size Δt for an incremental solution step from time t_k to $t_{k+1} = t_k + \Delta t$. For the first-order backward Euler (BE) method, equation (25) then becomes:

$$\mathbf{M}\mathbf{x}^{k+1} - \mathbf{M}\mathbf{x}^k - \Delta t \mathbf{f}_h^{k+1} = \mathbf{0},\tag{26}$$

abbreviating $\mathbf{x}^k := \mathbf{x}(t_k)$ and $\mathbf{f}_h^k := \mathbf{f}_h(t_k, \mathbf{x}^k)$ etc. The second-order Crank–Nicolson (CN) scheme leads to:

$$\mathbf{M}\mathbf{x}^{k+1} - \mathbf{M}\mathbf{x}^k - \frac{\Delta t}{2}(\mathbf{f}_h^k + \mathbf{f}_h^{k+1}) = \mathbf{0},\tag{27}$$

requiring function evaluations at both the current and the next time step.

The time-discretized systems (26) and (27) are both nonlinear in terms of the unknown $\mathbf{x}^{k+1}$, which is required to evaluate $\mathbf{f}_h^{k+1}$, and thus need to be solved iteratively using Newton’s method.

5 Frictional contact formulation

In the following, we extend the static, frictionless contact formulation for Cosserat rods using isogeometric collocation introduced in [46] to dynamic contact of rods with Coulomb friction.

5.1 Review of basic contact algorithm

For the basic steps required to detect contact points and compute normal contact penalty forces, we follow the approach taken in [46].

First, a coarse-level search for so-called contact candidate pairs is conducted to determine pairs of centerline points, at which contact between two rods or self-contact is likely to take place within the next time step. The basis for searching for these pairs can be the collocation points themselves or points at equidistant intervals. Once such contact candidates are determined, a refined search is conducted to determine the exact closest points on the centerlines of the two rods (rod I and rod J), which will be given in terms of their parameter values $(\hat{\eta}_i^I, \hat{\eta}_j^J)$.

Then the corresponding surface points that might penetrate each other $\mathbf{x}_i^I$ and $\mathbf{x}_j^J$, as well as the normal contact direction $\mathbf{n}_c$ are determined. Finally, the gap function $g = (\mathbf{x}_j^J - \mathbf{x}_i^I)^\top \mathbf{n}_c$, referring to the amount of penetration, is evaluated and the normal contact force can be calculated for each rod of a contact pair:

$$\mathbf{f}_c^I = -R_N \mathbf{n}_c, \quad \mathbf{f}_c^J = R_N \mathbf{n}_c, \quad (28)$$

using:

$$R_N = \begin{cases} 0 & , 0 \leq g, \\ \frac{k_c}{2 p_{reg}} g^2 & , -p_{reg} \leq g < 0, \\ -k_c \left(g + \frac{p_{reg}}{2}\right) & , g < -p_{reg}. \end{cases} \quad (29)$$

where k_c and p_{reg} are penalty parameters used to control the degree of penetration and the magnitude of the reaction force.

The contact penalty forces $\mathbf{f}_c^I$ and $\mathbf{f}_c^J$ then have to be applied to the discretized balance equations of linear momentum (24)₃ of each rod as distributed line loads $\bar{\mathbf{n}}$. Therefore, we follow the approach introduced in [46], where the point forces are distributed over the two nearest collocation points $\tau_k^I < \hat{\eta}_i^I \leq \tau_{k+1}^I$.

5.2 Frictional contact formulation

To incorporate friction into the contact formulation, we follow the approach taken in [42] and calculate the friction forces using Coulomb's law. For a more a much

elaborate discourse of models for dynamic friction phenomena we refer the reader to [54].

The regularized Coulomb's law adopted from [42] accounts for the transition from static to sliding friction and is characterized by the friction coefficient μ and a maximum reversible tangential displacement u_{max} . For each contact pair, first the incremental displacement of each contact point is computed for the motion from time step k to $k+1$:

$$\mathbf{u}_i^{I^{k+1}} = \mathbf{x}_i^{I^{k+1}} - \mathbf{x}_i^{I^k}. \quad (30)$$

Then the incremental relative displacement of the two contact points with respect to each other is calculated:

$$\Delta \mathbf{u}^{k+1} = \mathbf{u}_i^{I^{k+1}} - \mathbf{u}_j^{J^{k+1}}, \quad (31)$$

and its tangential component is computed:

$$\Delta \mathbf{u}_T^{k+1} = (\mathbf{I} - \mathbf{n}_c \otimes \mathbf{n}_c) \Delta \mathbf{u}^{k+1}. \quad (32)$$

In order to determine if the friction case is static or sliding, we first calculate a trial displacement:

$$\mathbf{u}_{T,tr}^{k+1} = \mathbf{u}_T^k + \Delta \mathbf{u}_T^{k+1}, \quad (33)$$

where $\mathbf{u}_T^k$ is the reversible tangential displacement belonging to the current contact pair, but from the previous time step. If the trial displacement is greater than the maximum tangential reversible displacement, the rods are in sliding friction, otherwise in static friction:

$$\mathbf{u}_T^{k+1} = \begin{cases} \mathbf{u}_{T,tr}^{k+1} & , |\mathbf{u}_{T,tr}^{k+1}| \leq u_{max} \\ \frac{u_{max}}{|\mathbf{u}_{T,tr}^{k+1}|} \mathbf{u}_{T,tr}^{k+1} & , |\mathbf{u}_{T,tr}^{k+1}| > u_{max}. \end{cases} \quad (34)$$

The friction force in time step $k+1$ can now be calculated using the reversible tangential displacement, the coefficient of friction μ , as well as the normal contact force R_N :

$$\mathbf{f}_T = \frac{\mu R_N}{u_{max}} \mathbf{u}_T^{k+1}. \quad (35)$$

Finally, the contact penalty forces in (28) are amended with the tangential friction force components:

$$\mathbf{f}_c^I = -R_N \mathbf{n}_c - \mathbf{f}_T, \quad \mathbf{f}_c^J = R_N \mathbf{n}_c + \mathbf{f}_T. \quad (36)$$

Remark 2 In order to calculate the incremental displacements at time step $k+1$, the positions at the previous step k need to be known and stored as well. This is already taken care of, since the rod centerline positions and rotations from the previous load step required to evaluate $\mathbf{x}_i^{I^k}$ have to be available and stored as part of the time integration scheme anyway. However, as also

mentioned in [42], for the relative tangential displacement $\mathbf{u}_T^k$ the issue is that it is defined per contact pair, and the contact pairs have to be re-determined at every time step. Thus, the value of $\mathbf{u}_T^k$ cannot be directly transferred from one time step to the next. Instead, we assign the value of the current $\mathbf{u}_T^k$ into the affected collocation points on each rod at the end of each time step. At the next step, for each contact pair, we take $\mathbf{u}_T^k$ to be the average of the $\mathbf{u}_T^k$ values at the collocation points associated with that pair.

6 Numerical applications

In this Section, we consider a number of numerical examples to validate our computational method for the nonlinear dynamic analysis of slender rods with frictional contact.

6.1 Swinging pendulum

As a first validation example, we consider a swinging pendulum rod, that has also been studied in [3, 13, 23]. The rod is initially straight, pinned at one end and free at the other, and its dynamic deformation behavior under gravity loading is investigated.

We choose the parameters and use cases as in [13], assuming an initial rod length of 1 m and differentiating two scenarios, a rubber rod and a steel rod. For the rubber rod, we use a circular cross-section radius $R_r = 0.005$ m, Young's modulus $E_r = 5.0$ MPa, Poisson's ratio $\nu_r = 0.5$, and density $\rho_r = 1100$ kg/m³. For the steel rod, the corresponding parameters are $R_s = 0.001$ m, $E_s = 211$ GPa, $\nu_s = 0.2$, and $\rho_s = 7850$ kg/m³. When damping is used, the parameters as in are: $A_1^d = A_2^d = 0.1$ Ns, $A_3^d = 0.02$ Ns, $C_1^d = C_2^d = 2 \cdot 10^{-4}$ Nm²s, and $C_3^d = 8 \cdot 10^{-6}$ Nm²s. In this way, extension and shearing are more than critically damped, while bending is only lightly damped [13].

First, we investigate the convergence behavior of the dynamic simulation with respect to the isogeometric discretization, see Fig. 3. The error of the centerline displacement at the free end point with respect to a high fidelity simulation is shown for varying $p/h/k$ -refinements of the B-Spline discretization. In all cases, the Crank–Nicolson integration method with a time step size $\Delta t = 0.0001$ s was used and the centerline position was evaluated at $t = 1$ s. The typical convergence rates of isogeometric collocation methods, which were also confirmed for the static discretization of Cosserat rods in [41], can be observed here. Due to the slenderness ratio of the rod of 1:100, shear locking is apparent for low spline degrees, i.e. $p = 2, 3$, and convergence

Fig. 3: Swinging pendulum rod. Convergence of end point displacement at $t = 1$ s with respect to isogeometric discretization

only starts for a fairly high number of elements. For $p = 8$, the initial convergence rate of order 8 cannot be observed for high number of elements, since time-discretization error then dominates over space-discretization error. Overall, this validates the implementation of the isogeometric method for the dynamic rod model.

Furthermore, we want to validate the usage of different time integration schemes. Figure 4a shows snapshots of the centerline of the rod from 0 s to 1 s, evaluated every 0.1 s, comparing the backward Euler (BE) and Crank–Nicolson (CN) methods for a fixed time step size $\Delta t = 0.0001$ s and using an isogeometric discretization with B-Splines using $p = 6$, $\ell = 16$. The effect of the time integration method on the accuracy is clearly visible and we can note that for the CN method our results are in good agreement with the ones obtained in [13], where a finite difference space discretization and higher-order integration methods were used.

Next, we investigate the effects of damping and the beam stiffness on the dynamic displacement behavior. Figure 4b shows snapshots of the centerline of the rod from 0 s to 1 s, comparing the cases of an undamped vs. a damped rubber rod, and Fig. 4c comparing an undamped rubber with an undamped steel rod. In these computations the Crank–Nicolson integration method, with a time-step size of 0.0001s, was used and all fields are discretized with B-Splines using $p = 6$, $\ell = 16$. Furthermore, the x - and y -positions of the damped rod are also plotted in Fig. 5 over a time interval of 10 seconds. The effects of damping on the oscillation amplitudes are clearly visible and the results are in very good agreement with [13].

Fig. 4: Swinging pendulum rod. Snapshots of displaced centerline over time interval from 0 to 1 s.

Altogether, with this first example we have provided a numerical validation of the proposed discretization methods for the Cosserat rod model. Convergence rates of the isogeometric discretization match the theoretical estimates and the visual comparison with reference results shows a good agreement.

6.2 Forced nonlinear vibration

As a second validation example, in which we use nonlinear dynamic analysis for a forced vibration problem,

Fig. 5: Swinging pendulum rod. End point position $\mathbf{r}(1, t)$ of damped rubber rod over time interval from 0 to 10 s

we investigate the planar deformation of a cantilever beam under harmonic loading, see also [16].

The beam is fixed at one end and free at the other, has initial length $L = 1.0$ m, a rectangular cross-section with width $b = 0.06$ m and height $h = 0.04$ m, Young's modulus $E = 71$ GPa, Poisson's ratio $\nu = 0.32$, and mass density $\rho = 2730$ kg/m³. It is excited by an harmonic line load:

$$\bar{\mathbf{n}}(s, t) = 2 \sin \pi s \sin 8\omega_0 t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ kN/m}, \quad (37)$$

where $\omega_0 = 207.0$ 1/s is the lowest eigenfrequency of the beam.

We simulate the forced vibration of the beam using an isogeometric discretization with $p = 4, \ell = 10$ and the Crank–Nicolson time integration scheme with a time step size $\Delta t = 0.0001$ s over 0.06 s, i.e. 2 oscillation periods corresponding to the eigenfrequency ω_0 . The initial velocities and angular velocities are taken as 0. The x -, y - and z -displacements $\mathbf{u}(1, t)$ of the end point of the beam are plotted over time in Fig. 6. We can observe a highly oscillatory, highly nonlinear motion, probably resulting from the nonlinear coupling of vibration modes, which results in external and internal resonances.

Overall, the plots show very good agreement with the ones obtained in [16] and highlight the applicability of the method for nonlinear vibration problems.

6.3 Contacting rods under harmonic loading

Now we move on to the first numerical application involving rod-to-rod contact. We study the behavior of

Fig. 6: Forced nonlinear vibration. End point displacements $\mathbf{u}(1, t)$ over time for forced oscillation

two perpendicular rods, of which one is subject to harmonic loading, causing them to come into contact with each other only temporarily. Since friction is not important in this scenario, we mainly use this example to validate our normal contact formulation for dynamic simulation.

Both rods have a length of $L = 1.0$ m and circular cross-sections with radius $R = 0.01$ m. They are perpendicular to each other, with rod 2 directly beneath

rod 1 at an initial gap of the centerlines of 0.05 m. Both rods are made from aluminium, having Young's modulus $E = 71$ GPa, density $\rho = 2730$ kg/m³, and Poisson's ratio $\nu = 0.32$. The damping parameters are $A_1^d = A_2^d = 0.1$ Ns, $A_3^d = 0.02$ Ns, $C_1^d = C_2^d = 0.1$ Nm²s, and $C_3^d = 0.004$ Nm²s. Both rods are clamped at one end and a harmonic load $\hat{\mathbf{n}}_z = -10 \sin 5t$ N is applied at the free end of the top rod in z -direction. Each rod is discretized with $p = 4$ and $\ell = 16$. We use points at equal intervals on the rod for our coarse contact search, and contact parameters $k_c = 5000$ N/m, $p_{reg} = 10^{-6}$ m and $\mu = 0$. The simulation is run for a total time of 5 seconds with a time step size of 0.001 s.

As rod 1 moves up and down, it triggers an oscillatory motion in the second rod by coming in and out of contact with it. Figure 7 shows four snapshots of the most characteristic states of deformation. From Fig. 8, which shows the end point z -positions of both rods, we can clearly see the perturbation of the second rod as soon as the first comes into contact with it. Once contact is lost and rod 1 continues to move upward, rod 2 keeps oscillating, but due to damping the magnitude of oscillation diminishes and is virtually zero when rod 1 comes down again. In this way, a stable periodic motion is established, as can be seen in Fig. 8, altogether showing that our framework with the contact implementation works well in the dynamic regime.

6.4 Frictional contact problems

Next, we introduce and investigate two model problems for dynamic motion of rods with frictional, sliding contact.

6.4.1 Swinging rod in frictional contact

As a first example with frictional contact, we study the highly nonlinear dynamic motion of a rod under a gravity load and investigate the influence of the friction coefficient μ .

The rod is pinned at one end and allowed to swing downwards under the influence of gravity. It has length $L = 1.0$ m, cross-section radius $R = 0.005$ m, Young's modulus $E = 50$ MPa, and Poisson's ratio $\nu = 0.5$. Unlike in Section 6.1, however, the swinging motion is hindered by the presence of a second, stiffer rod placed perpendicular to and directly beneath the first rod. The stiffer rod is clamped at both ends and has a radius of 0.01 m and Young's modulus 5.0 GPa. Both rods have density $\rho = 1100$ kg/m³. The spatial discretization is done using $p = 4$ and $\ell = 16$ and the contact parameters k_c and p_{reg} are set to 500 N/m and $5 \cdot 10^{-6}$ m

Fig. 7: Contacting rods under harmonic loading. The snapshots are taken at the following points: (a) rod 1 first contacts rod 2; (b) rod 1 and rod 2 are at their bottom most positions; (c) both rods return to their original unperturbed positions; (d) rod 1 attains its highest amplitude in the upward direction while rod 2 remains at its original position

Fig. 8: Contacting rods under harmonic loading. Centerline end point z -positions of two perpendicular rods $\mathbf{r}_z^1(1, t)$ and $\mathbf{r}_z^2(1, t)$ in time interval 0 to 5 s

respectively. The Crank–Nicolson integration scheme is used with a time step size of 0.001 s.

We vary the friction coefficient and observe its impact on the final position of the pinned rod after $t = 1.0$ s, see Figure 9. Due to an increase in μ more energy is dissipated and the ability of the first rod to slide past the second rod is restrained. For $\mu = 0$ and $\mu = 0.1$, the rod still manages to finally slip past the stiff rod, but for higher values of μ the rod gets “stuck” in different

Fig. 9: Swinging rod in frictional contact. All snapshots are taken at $t = 1$ s and highlight the influence of the coefficient of friction μ

positions. At the highest value of μ , the rod is closest to its horizontal position, which is what one would expect.

6.4.2 Three rods in sliding frictional contact

With the second application including frictional contact, we want to validate the effect of the choice of fric-

Fig. 10: Three rods in sliding frictional contact. Displaced rods after $t = 5$ s for $\mu = 0$ (red), $\mu = 0.2$ (green), and $\mu = 0.4$ (blue)

tion coefficient more quantitatively and also show that our formulation can handle multiple contacts correctly.

The use case involves three rods of length $L = 1.0$ m, with the first rod perpendicular to and underneath the other two parallel rods. All rods have cross-section radius $R = 0.01$ m, Young's modulus $E = 1.0$ MPa, Poisson's ratio $\nu = 0.5$, density $\rho = 1100$ kg/m³, and no damping is applied. An isogeometric discretization with $p = 4$ and $\ell = 10$ is used, as well as time integration using the backward Euler method with a time step size of 0.02 s over a total integration time of 5.0 s. Each rod is clamped at one end and free at the other, and an upward-directed point load of 0.005 N is applied to the first rod, pushing it onto the other two. For the contact treatment, parameters k_c and p_{reg} are set to 500 N/m and 0.0001 m respectively.

The friction coefficient μ is now varied for each simulation and Fig. 10 demonstrates this visually by showing three end configurations at $t = 5$ s for increasing values of μ . In Figure 11 the end point position of the first rod is plotted over the friction coefficient μ . In all three principal directions, the magnitude of displacement in the case of frictionless contact is the highest. As μ is increased, the displacement decreases in a (roughly) linear fashion, which is due to the frictional force acting against the forced motion of the first rod.

Altogether, both applications with frictional contact provide a good qualitative and quantitative validation of our methods and implementation and show that also complex scenarios, where the system behavior depends on the friction coefficient in a highly nonlinear way, or with multiple contacts can be handled in a robust and accurate manner.

7 Summary and conclusions

We have presented an isogeometric collocation method for the nonlinear dynamic analysis of spatial rods, including frictional contact. Our approach is based on the dynamic model of mechanical deformation of geometrically exact Cosserat rods, for which we have introduced

Fig. 11: Three rods in sliding frictional contact. End point displacement of first rod at $t = 5$ s for varying coefficient of friction μ

a spatial semi-discretization based on the concept of isogeometric collocation. The balance equations of linear and angular momentum were re-formulated as a first-order system in time, and the additional kinematic variables, centerline velocities and angular velocities, along with the primary unknowns, centerline positions and rotations, were discretized as spline curves. Collocation of the equilibrium equations at Greville abscissae leads to a stiff, nonlinear ODE system, which was then integrated using standard implicit numerical time integration schemes.

The method was validated successfully through several numerical studies and computational applications. Particularly important for many practical applications is the ability to resolve rod-to-rod contact, which was successfully included into our approach by a frictional contact formulation. Overall, the presented computational method shows good potential for efficient and accurate nonlinear dynamic rod modeling of challenging applications, including dynamic contacts with friction and highly nonlinear system behavior.

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