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On moving hypersurfaces and the discontinuous ODE-system associated with two-phase flows

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Abstract

We consider the initial value problem $\dot{x}(t) = v(t, x(t))$ for $t \in (a, b)$, $x(t_0) = x_0$ which determines the pathlines of a two-phase flow, i.e. $v = v(t, x)$ is a given velocity field of the type $v(t, x) = \begin{cases} v^+(t, x) \text{ if } x \in \Omega^+(t) \\ v^-(t, x) \text{ if } x \in \Omega^-(t) \end{cases}$ with $\Omega^\pm(t)$ denoting the bulk phases of the two-phase fluid system under consideration. The bulk phases are separated by a moving and deforming interface $\Sigma(t)$ at which v can have jump discontinuities. Since flows with phase change are included, the pathlines are allowed to cross or touch the interface. Imposing a kind of transversality condition at $\Sigma(t)$, which is intimately related to the mass balance in such systems, we show existence and uniqueness of absolutely continuous solutions of the above ODE in case the one-sided velocity fields $v^\pm$ are continuous in (t, x) and locally Lipschitz continuous in x on their respective domain of definition. A main step in proving this result, also interesting in itself, is to freeze the interface movement by means of a particular coordinate transform which requires a tailor-made extension of the intrinsic velocity field underlying a $\mathcal{C}^{1,2}$-family of moving hypersurfaces.

Keywords: discontinuous ODE, moving hypersurfaces, two-phase flows,
Mathematics Subject Classification numbers: 34A36, 53C44, 76T10, 34A60, 53A17.

(Some figures may appear in colour only in the online journal)

1. Introduction

We consider a specific class of discontinuous ODE-systems which appear naturally in the study of two-phase flows, where the underlying situation is as follows: a domain $\Omega \subset \mathbb{R}^n$ is occupied by two immiscible fluid phases, denoted as $\Omega^{\pm}$. These so-called bulk phases are separated by a sharp interface Σ and this interface is allowed to move and deform as time evolves. Hence

$$\Omega \;=\; \Omega^{+}(t) \cup \Omega^{-}(t) \cup \Sigma(t) \quad \text{for all } t \in J$$

with a family of moving hypersurfaces $\{\Sigma(t)\}_{t\in J}$ and $J = (a, b) \subset \mathbb{R}$. While $n = 3$ in concrete physical applications, we consider the general case $n \geqslant 2$.

We will always assume the $\Sigma(t)$ to be $\mathcal{C}^2$-hypersurfaces in $\mathbb{R}^n$. Moreover, the graph of the set-valued map $t \longmapsto \Sigma(t)$, i.e.

$$\mathrm{gr}(\Sigma) := \{(t, x) : x \in \Sigma(t), t \in J\} \subset J \times \mathbb{R}^n$$

is an oriented $\mathcal{C}^1$-hypersurface in $\mathbb{R}^{n+1}$ such that the field $n_\Sigma = n_\Sigma(t, x)$ of (unit) normals to $\Sigma(t) \subset \mathbb{R}^n$ is continuously differentiable on $\mathrm{gr}(\Sigma)$. These assumptions assure that the $\Sigma(t)$ are interrelated by advection with an associated flow field as is explained in detail below. We call this a $\mathcal{C}^{1,2}$-family of moving hypersurfaces; see [21, 23, 25] as well as section 3 below.

In this setting, continuous velocity fields $v^{\pm} = v^{\pm}(t, x)$ are given in the respective bulk phases $\Omega^{\pm}(t)$, where we assume them to have continuous extensions onto

$$G^{\pm} := \mathrm{gr}(\overline{\Omega^{\pm}}) = \{(t, x) : x \in \overline{\Omega^{\pm}(t)}, t \in J\}.$$

Moreover, we assume the $v^{\pm}(t, \cdot)$ to be locally Lipschitz continuous in $\overline{\Omega^{\pm}(t)}$ for all $t \in J$. We then let

$$v(t, x) = \begin{cases} v^{+}(t, x) & \text{if } x \in \Omega^{+}(t),\ t \in J \\ v^{-}(t, x) & \text{if } x \in \Omega^{-}(t),\ t \in J \end{cases} \tag{1}$$

be the corresponding two-phase velocity field. Note that (1) does not define v on $\mathrm{gr}(\Sigma)$. For $t \in J$ and $x \in \Sigma(t)$, both one-sided limits $v^{\pm}(t, x)$ exist by assumption, but they will not coincide in the case of a two-phase flow with phase change (like evaporation from or condensation to a droplet) for which the interface is not a material interface. Indeed, a phase change process is—by definition—a process in which a material point can move from one phase to the other, thus crossing the interface. Since the mass density differs between the adjacent phase, while the mass flux is continuous due to mass conservation, the velocity then has a jump discontinuity at the interface. Hence, whatever definition we give for v on $\mathrm{gr}(\Sigma)$, the resulting function will, in general, be discontinuous at $\mathrm{gr}(\Sigma)$. Consequently, the problem of determining the pathlines of general two-phase flows requires to find physically meaningful assumptions on the discontinuous field $v : J \times \Omega \to \mathbb{R}^n$ such that the initial value problems

$$\dot{x}(t) = v(t, x(t)) \quad \text{for } t \in J, \quad x(t_0) = x_0 \tag{2}$$

are uniquely solvable. Of course, this also requires an appropriate notion for solutions of (2).

Let us note in passing that the question of uniquely defined pathlines for two-phase flows is intimately related to the concept of co-moving control volumes in the two-phase setting. The latter is a very helpful tool already for the mathematical modelling of such systems.

There are a few fundamentally different approaches to the study of problem (2) for discontinuous right-hand sides. The most classical approach employs ODE-techniques, typically trying to obtain a solution from pieces which solve standard ODEs. A second method of investigation replaces the discontinuous single-valued right-hand side v by an appropriate set-valued regularization and studies a differential inclusions instead of an equation. Finally, there is a PDE-based approach due to DiPerna and Lions for (2) with right-hand sides of low regularity. Instead of studying the advection of a fluid particle, governed by the initial value problems (2), the transport of a passive scalar ϕ by the flow field v is considered. The advection of ϕ is governed by the transport equation

$$\partial_t \phi + v \cdot \nabla \phi = 0, \quad t \in J, \ x \in \Omega. \tag{3}$$

Then $\phi(\cdot, x(\cdot; t_0, x_0)) \equiv \phi(t_0, x_0)$, where $x(\cdot, t_0, x_0)$ is the solution of (2), hence the method of characteristics can be applied if (2) is uniquely solvable backwards in time. In their seminal paper [15], DiPerna and Lions initiated the investigation of how the intimate relation between the ODE (2) and the scalar transport equation (3) can be employed to obtain a flow map associated with (2) for velocity fields of low regularity; see [1] for a rather recent overview. But the latter approach does not aim at providing solvability of (2) for every initial value; rather, results on the induced flow in the sense of a set-to-set map (for almost all elements) are obtained.

We are going to employ the approach via differential inclusions in order to obtain existence of solutions and we will briefly explain the concept of set-valued regularization below. The hard part then is to show their uniqueness. Before going into the details, let us provide a brief overview about known uniqueness criteria which apply to certain discontinuous ODE-systems, where we also add information why these approached do not apply to the ODE-system associated with two-phase flows.

Given an open interval $J = (a, b)$ in $\mathbb{R}$, an open set $\Omega \subset \mathbb{R}^n$ and a function $f : J \times \Omega \to \mathbb{R}^n$, we consider the initial value problem (IVP for short)

$$\dot{x}(t) = f(t, x(t)) \quad \text{for } t \in J, \quad x(t_0) = x_0 \tag{4}$$

for $t_0 \in J$ and $x_0 \in \Omega$. By the classical result of Peano [24], problem (4) has a local C^1-solution if f is continuous. If f is discontinuous in t, solutions will typically not be C^1, but absolutely continuous (a.c. for short) such that

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, \mathrm{d}s \quad \text{for all } t \in J. \tag{5}$$

We call such a function $x(\cdot)$ an a.c. solution and, again by a classical result named after C. Carathéodory, existence of local solutions still holds true if f is Lebesgue measurable in t and continuous in x with local integrable bounds, say $|f(t, x)| \leqslant k(t)$ on $J \times \Omega$ with some $k \in L^1(J)$; see for instance [22] for a proof. The solution is also called a Carathéodory solution of (4).

The situation is more involved if f is discontinuous in x, as it happens if f denotes the velocity field in a two-phase flow, i.e. in the case considered in the present paper. More generally, discontinuous ODEs appear in several situations, and possible applications which lead to such cases can be found in [8, 13, 14, 17, 20] and the references given there. If f is discontinuous, possible solutions will not be C^1. Instead, we again look for an absolutely continuous (a.c. for short) function $x(\cdot)$ such that (5) is satisfied. Equivalently, $x(\cdot)$ is a.c. with $x(t_0) = x_0$ and satisfies $\dot{x}(t) = f(t, x(t))$ almost everywhere (a.e.) on J. We call such a function $x(\cdot)$ an a.c. solution of the IVP (4).

There are simple one-dimensional examples which show that (4) may have no solution, a single solution or infinitely many ones. Let $\mathrm{sgn}(r) := r/|r|$ for $r \neq 0$, $\mathrm{sgn}(0) := 0$ denote the standard sign function and consider the discontinuous IVP

$$\dot{x}(t) = \operatorname{sgn}(x(t)) \quad \text{a.e. on } J = \mathbb{R},\ x(0) = 0.$$

Evidently, $x(t) \equiv 0$ is a solution, but also $x(t) = (t-\tau)^+ := \max\{0, t-\tau\}$ is a solution for any $\tau > 0$ as are the functions $-(t-\tau)^+$ for $\tau > 0$. This resembles the classroom example of $\dot{x} = \sqrt{|x|}$, $x(0) = 0$ for non-uniqueness. Now define the modified sign-function $\operatorname{sgn}_\alpha(r) := \operatorname{sgn}(r)$ for $r \neq 0$ but $\operatorname{sgn}_\alpha(0) := \alpha$ with $\alpha \in \mathbb{R}$ and consider the IVP

$$\dot{x}(t) = -\operatorname{sgn}_\alpha(x(t)) \quad \text{a.e. on } J = \mathbb{R},\ x(0) = 0. \tag{6}$$

Assume that $x(\cdot)$ is an a.c. solution of (6). Then

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)| = \dot{x}(t)\ \operatorname{sgn}(x(t)) \quad \text{a.e. on } \{t \in J : x(t) \neq 0\}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)| = 0 \quad \text{a.e. on } N := \{t \in J : x(t) = 0\}.$$

For the latter note that almost all $t \in N$ are simultaneously points of Lebesgue density of N and such that $|x(\cdot)|$ is differentiable at t, hence $\frac{\mathrm{d}}{\mathrm{d}t}|x(t)| = 0$ in these points. Exploiting (6) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)| \leqslant 0 \quad \text{a.e. on } J,$$

hence $x(t) = 0$ for all $t \in [0, \infty)$ since $x(0) = 0$. Consequently, forward uniqueness holds in case $\alpha = 0$, while any choice of $\alpha \neq 0$ leads to a discontinuous ODE without a.c. solution (forward in time).

In order to still build a fruitful theory for discontinuous ODEs, a fundamental step is to relax the concept of a solution in such a way that at least existence of solutions can be guaranteed in appropriate cases. One way to proceed in this direction is to define the set-valued regularization $F : J \times \Omega \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ of $f : J \times \Omega \to \mathbb{R}^n$ according to

$$F(t,x) := \bigcap_{\delta>0} \overline{\operatorname{conv}}\, f\,(t, B_\delta(x) \cap \Omega) \qquad \text{for } t \in J,\ x \in \Omega \tag{7}$$

and to consider the differential inclusion

$$\dot{x} \in F(t, x(t)) \quad \text{for } t \in J, \quad x(t_0) = x_0 \tag{8}$$

instead of (4). In (7), $\overline{\operatorname{conv}}\, A$ denotes the closed convex hull of the set A. In analogy to the single-valued ODE, we call $x(\cdot)$ an a.c. solution of (8) if $x(\cdot)$ is an absolutely continuous function such that $x(t_0) = x_0$ and the inclusion in (8) holds a.e. on J.

It is well known (see [14]) that, given any locally bounded, measurable function f, the map F has the following properties: $F(\cdot, x)$ has a measurable selection for every $x \in \Omega$, $F(t, \cdot)$ is upper semicontinuous (i.e. if A is closed and $F(t, x_n) \cap A \neq \emptyset$ for some x_n with $x_n \to x_0$, then $F(t, x_0) \cap A \neq \emptyset$) for every $t \in J$ and F is locally bounded with closed bounded convex values. Due to theorem 5.2 in [14], this is sufficient for the local-in-time existence of a.c. solutions of the differential inclusion (8) for every $t_0 \in J$ and $x_0 \in \Omega$. If F stems from a discontinuous function f via (7), an a.c. solution of (8) is also called a Krasovskii solution of the discontinuous ODE (4).

A variant of the above concept was introduced by Filippov, considering the more restrictive regularization

$$F(t,x) := \bigcap_{\delta>0} \bigcap_{\lambda_n(N)=0} \overline{\text{conv}}\, f\left(t, B_\delta(x) \cap (\Omega \backslash N)\right) \quad \text{for } t \in J,\ x \in \Omega, \tag{9}$$

where λ_n denotes the n-dimensional Lebesgue measure. In this case, an a.c. solution of (8) is called a Filippov solution of (4) and theorem 8 in chapter 2, section 7 of [17] assures that a (local) a.c. solution of (8) exists for measurable, integrably bounded f.

The difference between the two variants can be explained with the sign function: while the set-valued regularization $\text{Sgn}(\cdot)$ of $\text{sgn}_\alpha(\cdot)$ according to (9) has $\text{Sgn}(0) = [-1, 1]$, independently of the value of $\alpha \in \mathbb{R}$, one always has $\alpha \in \text{Sgn}(0)$ if the latter is defined via (7).

Applied to the two-phase velocity field $f := v$ under consideration, definition (7) and (9) yield the same result, namely

$$F(t,x) = \begin{cases} \{v^+(t,x)\} & \text{if } x \in \Omega^+(t), \\ \text{conv}\{v^+(t,x), v^-(t,x)\} & \text{if } x \in \Sigma(t), \\ \{v^-(t,x)\} & \text{if } x \in \Omega^-(t). \end{cases} \tag{10}$$

This approach yields a non-empty set of solutions to the differential inclusion (8). In general, these are not solutions to the original (single-valued) ODE. Hence a natural follow-up question is, under which conditions at least one of these solutions of (8) is also a solution of (4). This is related to the question, whether the set-valued map F from (7) or (9) coincides with $\{f\}$ in relevant cases.

Employing the concept of directional continuity, which goes back to [12, 27], Bressan obtained in [9] existence of solutions for the original initial value problem (4) if $|f(t,x)| \leqslant c$ on $J \times \Omega$ and f is continuous along the cone

$$K_\alpha := \{(t,x) \in \mathbb{R}^{n+1} : |x| \leqslant \alpha\, t\} \quad \text{for some } \alpha > c. \tag{11}$$

Let us indicate how directional continuity can help, while we refer to [9, 14] for more details. First of all, note that the behaviour locally in time is the core point, since continuation of solutions can be done as in the single-valued continuous case. Under the weak assumption of local boundedness of f we also obtain that F from (7) or (9) is locally bounded. Hence, for the local in time existence and uniqueness, it is a harmless restriction to assume that $|f(t,x)| \leqslant c$ on $J \times \Omega$, hence the same for F. Then any solution $x(\cdot)$ of (8) is Lipschitz continuous of constant c. Now let $t \in J$ and assume that there is a sequence $h_n \searrow 0$ such that $x'(t+h_n) \to x'(t)$ and, of course, such that these derivatives exist. Then, for $\alpha > c$, there exits $\delta_n \searrow 0$ such that

$$B_n := B_{\delta_n}((t+h_n, x(t+h_n))) \subset (t, x(t)) + K_\alpha \quad \text{for all } n \in \mathbb{N}.$$

But then, given $\epsilon > 0$, directional continuity of f along K_α implies

$$|f(s,y) - f(t,x(t))| \leqslant \epsilon \quad \text{for } (s,y) \in B_n \text{ if } n \text{ is sufficiently large.}$$

This yields $F(t,x(t)) = \{f(t,x(t))\}$ at such a point $t \in J$. Using the Lusin property of the measurable function $\dot{x}$ and, again, points of Lebesgue density, it is not hard to show that almost all $t \in J$ are of the type above; see [14]. Hence any a.c. solution of (8) with (7) or (9) is already an a.c. solution of (4) if f is (locally) bounded and directionally continuous along K_α with large enough $\alpha > 0$. From the above set inclusion we see that even a joint regularization in (t, x) like

$$F(t,x) := \bigcap_{\delta>0} \overline{\text{conv}}\, f\,(B_\delta((t,x)) \cap [J \times \Omega]) \quad \text{for } t \in J,\ x \in \Omega$$

would work in this situation.

Now, while this approach via directional continuity is very helpful for instance to prove the existence and study qualitative properties of a.c. solutions for differential inclusions with lower semicontinuous right-hand side (see [9, 11]), it does not apply to general two-phase flow velocity fields, since the cone K_α is not related to the discontinuity surface Σ. Note that the simple one-dimensional example of $f(t,x) = a + (b-a)\chi_{[x_0,\infty)}(x)$ with $a, b \in \mathbb{R}$, in which f switches from a to b at $x = x_0$, is only continuous along K_α (for any $\alpha > 0$) if $a = b$, i.e. here directional continuity already implies continuity.

Nevertheless, but with a different and more specific proof, it will turn out that solutions of (9) are solutions of (4) in the situation associated with two-phase flows considered in the present paper. This will be shown in section 6 as part of the proof of the main result.

If the existence of a.c. solutions of (4) or (8) is guaranteed, the next important step then is to find applicable uniqueness criteria, especially in cases where the physics of the problem asks for unique solutions to the initial value problems such as the two-phase flow problem considered here. For ODEs with discontinuous right-hand sides, this typically is the most difficult step. Before giving a brief overview of available results, explaining also why they do not cover the ODE-system associated to two-phase flows, recall that local Lipschitz continuity of f in x is of course sufficient for local existence of a *unique* solution to (4) due to the Picard–Lindelöf theorem. In the latter classical result, f is also assumed to be jointly continuous, which can be relaxed to mere measurability in t. This gives forward and backward uniqueness, but evidently does not apply to f being discontinuous in x.

From a physical point of view, at least forward uniqueness is mandatory. Now if two solutions $x(\cdot), y(\cdot)$ of the same IVP (4) are given, one hence needs to show that $\|x(t) - y(t)\|$ stays zero, being zero at $t = t_0$. The natural idea is to look for differential inequalities, say for $\frac{1}{2}\|x(t) - y(t)\|^2$ which is as regular as the solutions are. For a.c. solutions of (4), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|x(t) - y(t)\|^2 = \langle f(t, x(t)) - f(t, y(t)), x(t) - y(t)\rangle \quad \text{a.e. on } J,$$

where the brackets $\langle \cdot, \cdot \rangle$ denote the standard inner product and $\|\cdot\|$ is the associated (Euclidean) norm. Therefore, a reasonable condition to obtain forward uniqueness is the so-called one-sided Lipschitz continuity, i.e.

$$\langle f(t,x) - f(t,y), x - y\rangle \leqslant k(t)\|x - y\|^2 \quad \text{for all } t \in J,\ x, y \in \Omega \tag{12}$$

with $k \in L^1(J)$. A function f which satisfies (12) is also said to be of dissipative type. Evidently, forward uniqueness then follows by means of Gronwall's lemma, since (12) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|x(t) - y(t)\|^2 \leqslant k(t)\|x(t) - y(t)\|^2 \quad \text{for a.e. } t \in J. \tag{13}$$

Now note that (12) allows for discontinuous f, but it imposes strong restrictions on possible jumps of f. For instance, in case $n = 1$ and if $f \in C^1(\mathbb{R}\backslash\{0\})$ has one-sided limits $a^\pm$ at $x = 0$, then $a^+ \leqslant a^-$ is necessary. Observe that requesting also backward uniqueness in this simple example imposes the condition $a^+ = a^-$, hence continuity of f.

Let us now specialize to the setting which corresponds to two-phase flows and check to what extend one-sided Lipschitz continuity is applicable. The setup is characterized by a right-hand side, which we now denote as v for velocity, being locally Lipschitz (or better) except at a (in general moving and deforming) hypersurface Σ, where v has one-sided limits but with a jump discontinuity across Σ. Lemma 3 in Chapter 2, section 10 of [17] employs one-sided Lipschitz continuity for (2) with right-hand sides as in (1), but with constant (non-moving) $\Sigma \in \mathcal{C}^1$ and $\Omega^\pm$. This yields forward uniqueness of solutions to (2) under the following assumptions: $v^\pm : \overline{\Omega^\pm} \to \mathbb{R}^n$ continuous with continuous (up to the boundary) partial derivatives and such that

$$v_\parallel^+ = v_\parallel^- \quad \text{and} \quad \left\langle v^+ - v^-, n^- \right\rangle \leqslant 0 \quad \text{on } \Sigma, \tag{14}$$

where $v_\parallel^\pm$ denotes the tangential components of $v^\pm$ and n^- is the outer normal to Ω^-. This result does not apply to the situation found for two-phase flows for the following reason: as explained above, in a two-phase flow with phase change, the one-sided limits of the normal velocity at the interface have a jump discontinuity and the difference $(v^+ - v^-) \cdot n_\Sigma$ has no fixed sign; e.g., for a droplet surrounded by its vapour there will, in general, simultaneously be regions of local evaporation as well as regions of local condensation which means that $(v^+ - v^-) \cdot n_\Sigma$ changes its sign along the interface. Let us also note that we are actually looking for conditions guaranteeing forward and backward uniqueness and $(14)_2$ then becomes $\langle v^+ - v^-, n^- \rangle = 0$, hence continuity of $v \cdot n_\Sigma$ at Σ would then be required, ruling out any phase change phenomena.

Another related result in [17] is theorem 2 in chapter 2, section 10 which goes back to [16]. The setting is again with fixed hypersurface Σ and with $v^\pm \in C^1(G^\pm)$. This time, Σ is assumed to be of class $\mathcal{C}^2$, hence $v_{|\Sigma}^+ - v_{|\Sigma}^- \in \mathcal{C}^1(\Sigma)$. By means of a nonlinear coordinate transformation, the condition $(14)_1$ can then be avoided, while condition $(14)_2$ is replaced by

$$\left\langle v^-, n^- \right\rangle > 0 \quad \text{or} \quad \left\langle v^+, n^+ \right\rangle > 0 \quad \text{on } \Sigma, \tag{15}$$

where $n^\pm$ are the outer normals to $\Omega^\pm$. Under this assumption, theorem 2 in chapter 2, section 10 of [17] estabishes forward uniqueness.

Evidently, condition (15) means that at least one of the one-sided limits of v at Σ points into the opposite sub-domain, thus avoiding the case in which each of the one-sided limits points into the sub-domain where it is taken. Note that the latter case would possibly allow for two different solutions emanating from the same point on Σ. While being an interesting result, it does not apply to the two-phase flow case we are aiming at: the normal velocity (relative to the interface) can vanish at Σ, but only simultaneously for both sides due to mass conservation at the interface, cf condition (18) below. Thus condition (15) would rule out the (physically possible) case of having zero normal velocity (relative to the interface).

For the sake of completeness, let us mention that a different criterion for forward uniqueness was established in [10], building on the concept of directional continuity. Theorem 1 in [10] guarantees the existence of a unique forward solution to (4) if f has locally bounded K_α-variation, i.e. locally bounded variations which are computed only in increasing (w.r.t. the cone K_α from (11)) direction; see [10] for details and see paragraph A1 in [14]. As explained above in the context of existence results for discontinuous ODEs, the concept of directional continuity does not apply to the ODE-system associated with two-phase flows since the cone K_α has no relation to the hypersurface of discontinuity.

The main result of the present paper is the wellposedness (with forward and backward uniqueness) of the ODE associated with the velocity field of a two-phase flow under physically meaningful assumptions, where the latter are motivated by the mass conservation jump conditions. The core idea is to establish an energy-type estimate like (13), but with $\|\cdot\|^2$ replaced

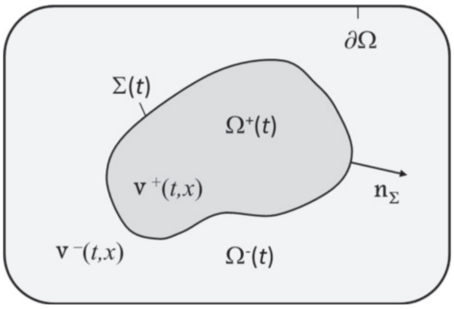

Figure 1. Sketch of a two-phase flow domain with notations.

by a different functional related to the jump conditions in two-phase flows. This requires to first transform the system in a particular manner which not only freezes the moving interface but, in doing so, leaves the jump conditions separated into normal and tangential parts. For this purpose, we prove that the intrinsic normal velocity of a $\mathcal{C}^{1,2}$-family of moving hypersurfaces admits a certain extension which generates a specific flow map which then yields the desired coordinate transform. In order to motivate our assumptions and to state as well as to prove our main result, some background on the physical model as well as some auxiliary results on moving hypersurfaces are hence required.

2. Sharp interface two-phase flow model

Consider the continuum mechanical sharp-interface model for two-phase flows with phase change in a domain $\Omega \subset \mathbb{R}^n$ with bulk phases $\Omega^{\pm}(t)$, separated by a C^2-surface $\Sigma(t)$ such that $\Omega^{+}(t) \cup \Omega^{-}(t) \cup \Sigma(t)$ is a disjoint decomposition of Ω. We assume that $\Sigma(t)$ is an embedded surface in $\mathbb{R}^n$ without boundary; to avoid technical problems with moving contact lines (see [19] concerning mathematical difficulties with moving contact line modelling), we actually restrict to closed surfaces. This typical setup is illustrated in figure 1.

The balances of mass and momentum read

$$\partial_t \rho + \operatorname{div}(\rho v) = 0 \quad \text{in } \Omega \backslash \Sigma, \tag{16}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v - \mathcal{S}) = \rho b \quad \text{in } \Omega \backslash \Sigma, \tag{17}$$

where ρ is the mass density, v the velocity, $\mathcal{S}$ the stress tensor and b denotes body forces. At Σ, the transmission conditions

$$[\![\rho(v - v^{\Sigma})]\!] \cdot n_{\Sigma} = 0 \quad \text{on } \Sigma, \tag{18}$$

$$[\![\rho v \otimes (v - v^{\Sigma}) - \mathcal{S}]\!] \cdot n_{\Sigma} = \operatorname{div}_{\Sigma} \mathcal{S}^{\Sigma} \quad \text{on } \Sigma \tag{19}$$

are valid, where v^{Σ} is the interface velocity, n_{Σ} the interface normal field, $\mathcal{S}^{\Sigma}$ denotes the interface stress tensor and $\operatorname{div}_{\Sigma}$ is the surface divergence. For the sake of completeness recall that the surface divergence of a surface field like $\mathcal{S}^{\Sigma}$ is given as $\operatorname{div}_{\Sigma}\mathcal{S}^{\Sigma} = \operatorname{trace}\nabla_{\Sigma}\mathcal{S}^{\Sigma}$. Here ∇_{Σ} is the surface gradient which is most directly defined via $\nabla_{\Sigma}\mathcal{S}^{\Sigma} := P_{\Sigma}\nabla\mathcal{S}^{\Sigma}_{\text{ext}}$ with $P_{\Sigma} := I - n_{\Sigma} \otimes n_{\Sigma}$ the projector onto the (local) tangent hyperplane and $\mathcal{S}^{\Sigma}_{\text{ext}}$ a differentiable extension of

$\mathcal{S}^\Sigma$ to some neighbourhood of Σ, for example the one which is constant along normal segments. The jump bracket $[\![\cdot]\!]$ is defined as

$$[\![\psi]\!](t,x) := \lim_{h\to 0+} \left(\psi(t, x + hn_\Sigma(t,x)) - \psi(t, x - hn_\Sigma(t,x))\right) \tag{20}$$

for $t \in J$, $x \in \Sigma(t)$. Observe that only the so-called speed of normal displacement $V_\Sigma := v^\Sigma \cdot n_\Sigma$ of $\Sigma(\cdot)$ enters in (18), (19); cf (27) below for a purely kinematic definition of V_Σ.

The system (16)–(19) requires several constitutive relations to arrive at a closed model, i.e. a system of PDEs for the unknown variables ρ, v; see [28] for more details. Here, we are only interested in the flow generated by the two-phase velocity field. For this purpose we need to add an information on the tangential part, where we impose the standard no-slip condition, i.e.

$$[\![P_\Sigma v]\!] = 0 \quad \text{on } \Sigma. \tag{21}$$

We also use $v_\|$ as a shorthand notation for $P_\Sigma v$. Note also that we use 'on Σ' to mean 'for all $(t,x) \in \mathrm{gr}(\Sigma)$'; recall that

$$\mathrm{gr}(\Sigma) = \{(t,x) : x \in \Sigma(t), t \in J\} = \bigcup_{t\in J} \big(\{t\} \times \Sigma(t)\big) \tag{22}$$

denotes the graph of the (set-valued) map $\Sigma : J \subset \mathbb{R} \to 2^{\mathbb{R}^n} \backslash \{\emptyset\}$.

3. Moving hypersurfaces and consistent velocity fields

Motivated by the physical background, we employ the following definition of a $\mathcal{C}^{1,2}$-family of moving hypersurfaces which can also be found in [23, 25] and in a similar form in [21]. Let us note that $\mathrm{div}_\Sigma\, \mathcal{S}^\Sigma$ in (19) contains the term $\kappa_\Sigma = \mathrm{div}_\Sigma(-n_\Sigma)$, which is $n-1$ times the mean curvature of Σ. This explains the requirement that all $\Sigma(t)$ are $\mathcal{C}^2$-hypersurfaces in $\mathbb{R}^n$.

Definition 1. Let $J = (a,b) \subset \mathbb{R}$ be an open interval. A family $\{\Sigma(t)\}_{t\in J}$ with $\Sigma(t) \subset \mathbb{R}^n$ is called a $\mathcal{C}^{1,2}$-*family of moving hypersurfaces* if

(a) each $\Sigma(t)$ is an orientable $\mathcal{C}^2$-hypersurface in $\mathbb{R}^n$ with unit normal field denoted as $n_\Sigma(t,\cdot)$;
(b) the graph $\mathcal{M}$ of Σ is a $\mathcal{C}^1$-hypersurface in $\mathbb{R} \times \mathbb{R}^n$;
(c) the unit normal field is continuously differentiable on $\mathcal{M}$, i.e.

$$n_\Sigma \in \mathcal{C}^1(\mathcal{M}).$$

We also need the notion of consistent velocity fields $v^\Sigma : \mathcal{M} \to \mathbb{R}^n$.

Definition 2. Let $J = (a,b) \subset \mathbb{R}$ and $\{\Sigma(t)\}_{t\in J}$ a $\mathcal{C}^{1,2}$-family of moving hypersurfaces in $\mathbb{R}^n$ with graph $\mathcal{M}$. Let $v^\Sigma : \mathcal{M} \to \mathbb{R}^n$ be a continuous velocity field such that the $v^\Sigma(t,\cdot)$ are locally Lipschitz continuous on $\Sigma(t)$ for all $t \in J$. We say that v^Σ and $\mathcal{M}$ are *consistent* (or that v^Σ is consistent to $\mathcal{M}$), if the initial value problems

$$\dot{x}^\Sigma(t) = v^\Sigma\left(t, x^\Sigma(t)\right) \text{ on } J, \quad x^\Sigma(t_0) = x_0 \tag{23}$$

have unique a.c. solutions on J (locally in time, forward and backward) for every $(t_0, x_0) \in \mathcal{M}$.

Note that v^Σ is only given on $\mathcal{M} = \mathrm{gr}(\Sigma)$ in definition 2 above. Hence solvability of (23) on $I \subset J$ implicitly includes the constraint

$$x^\Sigma(t) \in \Sigma(t) \text{ on } I. \tag{24}$$

To characterize consistency, we employ the so-called intermediate cone to $\mathcal{M}$ (see [3]), defined for $(t, x) \in \mathcal{M}$ by

$$T_{\mathcal{M}}(t, x) := \left\{ (\tau, v) : \lim_{h \to 0+} h^{-1} \text{ dist } (x + hv, \Sigma(t + h\tau)) = 0 \right\}. \tag{25}$$

Elements of $T_{\mathcal{M}}(t, x)$ are, in general, *subtangential* to $\mathcal{M}$. At inner points of $\mathcal{M}$ (in the sense of inner points of a surface), the intermediate cone reduces to the set of tangential vectors. Now, as a direct consequence of corollary 5.3 in [14] or theorem 13.2.1 in [26] (see also [5] and the appendix in [7]), the following holds.

Lemma 1. *Let $J = (a, b) \subset \mathbb{R}$ and $\{\Sigma(t)\}_{t \in J}$ be a $\mathcal{C}^{1,2}$-family of moving hypersurfaces in $\mathbb{R}^n$ with graph $\mathcal{M}$. Let $v^\Sigma : \mathcal{M} \to \mathbb{R}^n$ be a continuous velocity field such that the $v^\Sigma(t, \cdot)$ are locally Lipschitz continuous on $\Sigma(t)$ for all $t \in J$. Then v^Σ is consistent to $\mathcal{M}$ iff (if and only if) v^Σ is tangential to $\mathcal{M}$ in the sense that*

$$\left(1, v^\Sigma(t, x)\right) \in T_{\mathcal{M}}(t, x) \quad \text{on } \mathcal{M}. \tag{26}$$

For a $\mathcal{C}^{1,2}$-family $\{\Sigma(t)\}_{t \in J}$ of moving hypersurfaces, V_Σ denotes the *speed of normal displacement* of $\Sigma(\cdot)$ and is defined via the relation

$$\lim_{h \to 0+} \frac{1}{h} \mathrm{dist}(x + hV_\Sigma(t, x) n_\Sigma(t, x), \Sigma(t + h)) = 0 \quad \text{for } t \in J,\ x \in \Sigma(t). \tag{27}$$

More precisely, V_Σ should be named 'speed of normal forward displacement' due to '$h \to 0+$' in (27). But in all cases considered in the present paper, the speed of normal displacement will be the same in forward and in backward direction. Let us note in passing that the definition via (27) is equivalent to the common one which employs curves. Indeed,

$$V_\Sigma(t, x) = \langle \gamma'(t), n_\Sigma(t, \gamma(t)) \rangle$$

for any $\mathcal{C}^1$-curve γ with $\gamma(t) = x$ and $\mathrm{gr}(\gamma) \subset \mathcal{M}$, and the value does not depend on the choice of a particular curve; see chapter 2.5 in [25]. In the literature, V_Σ is often called normal velocity of $\Sigma(\cdot)$, but we prefer to call it the speed of normal displacement since V_Σ is not a velocity field. The definition via (27) clearly shows that V_Σ is a purely kinematic quantity, determined only by the family $\{\Sigma(t)\}_{t \in J}$ of moving interfaces. Its computation is especially simple if $\{\Sigma(t)\}_{t \in J}$ is given by a level set description, i.e.

$$\Sigma(t) = \{x \in \mathbb{R}^n : \phi(t, x) = 0\} \tag{28}$$

with $\phi \in \mathcal{C}^{1,2}(\mathcal{N})$ for some open neighbourhood $\mathcal{N} \subset \mathbb{R} \times \mathbb{R}^n$ of $\mathcal{M}$ such that $\nabla\phi \neq 0$ on $\mathcal{M}$. Then

$$V_\Sigma(t, x) = -\frac{\partial_t \phi(t, x)}{\|\nabla_x \phi(t, x)\|} \quad \text{for } t \in J,\ x \in \Sigma(t). \tag{29}$$

With this notation, the following characterization of consistency holds.

Lemma 2. *Let $J = (a, b) \subset \mathbb{R}$ and $\{\Sigma(t)\}_{t\in J}$ be a $\mathcal{C}^{1,2}$-family of moving hypersurfaces in $\mathbb{R}^n$ with graph $\mathcal{M}$. Let $v^\Sigma : \mathcal{M} \to \mathbb{R}^n$ be a continuous velocity field such that the $v^\Sigma(t, \cdot)$ are locally Lipschitz continuous on $\Sigma(t)$ for all $t \in J$. Then v^Σ is consistent to $\mathcal{M}$ iff*

$$v^\Sigma(t, x) \cdot n_\Sigma(t, x) = V_\Sigma(t, x) \quad \text{on } \mathcal{M}. \tag{30}$$

Proof. We first show that (26) implies (30). Fix $(t_0, x_0) \in \mathcal{M}$ and let $(h_k) \subset \mathbb{R}$ with $0 \neq h_k \to 0+$ be given. Then there are $z_k \in \mathbb{R}^n$ with $z_k \to 0$ such that

$$x_k := x_0 - h_k v^\Sigma_\parallel(t_0, x_0) + h_k z_k \in \Sigma(t_0),$$

since $v^\Sigma_\parallel(t_0, x_0)$ is tangent to $\Sigma(t_0)$ in x_0. By (26) and lemma 1, the solutions of (23) starting in x_k stay in $\mathcal{M}$, i.e.

$$x^\Sigma(t_0 + h_k; t_0, x_k) \in \Sigma(t_0 + h_k) \quad \text{for all } k \geqslant 1.$$

Hence, with $v^\Sigma_n := \langle v^\Sigma, n_\Sigma \rangle n_\Sigma$, we obtain

$$\begin{aligned} \operatorname{dist}(x_0 + h_k v^\Sigma_n(t_0, x_0), \Sigma(t_0 + h_k)) &\leqslant \|x_0 + h_k v^\Sigma_n(t_0, x_0) - x^\Sigma(t_0 + h_k; t_0, x_k)\| \\ &\leqslant \|x_0 + h_k v^\Sigma_n(t_0, x_0) - (x_k + h_k v^\Sigma(t_0, x_k))\| + h_k \delta_k \end{aligned}$$

with some $\delta_k \to 0+$. Therefore,

$$\begin{aligned} &\frac{1}{h_k}\operatorname{dist}(x_0 + h_k v^\Sigma_n(t_0, x_0), \Sigma(t_0 + h_k)) \\ &\quad \leqslant \|v^\Sigma_n(t_0, x_0) + v^\Sigma_\parallel(t_0, x_0) - v^\Sigma(t_0, x_k) - z_k\| + \delta_k \to 0 \quad \text{as } k \to \infty. \end{aligned}$$

This shows that (30) holds at the arbitrarily chosen $(t_0, x_0) \in \mathcal{M}$.

Now we assume (30) to hold. Since $V_\Sigma n_\Sigma$ satisfies (27), the velocity field $v^\Sigma_n := \langle v^\Sigma, n_\Sigma \rangle n_\Sigma$ is consistent to $\mathcal{M}$ due to lemma 1. Hence, with obvious modifications, we can exchange the role of v^Σ and v^Σ_n in the arguments from above to see that

$$\frac{1}{h_k}\operatorname{dist}(x_0 + h_k v^\Sigma(t_0, x_0), \Sigma(t_0 + h_k)) \to 0 \quad \text{as } k \to \infty,$$

hence $(1, v^\Sigma(t_0, x_0)) \in T_\mathcal{M}(t_0, x_0)$. □

The following result is a slight extension of lemma 12 in [19] and provides the existence of a local level set representation of $\mathcal{M} = \operatorname{gr}(\Sigma)$ via a signed distance function.

Lemma 3. *Let $J = (a, b) \subset \mathbb{R}$, $\{\Sigma(t)\}_{t\in J}$ be a $\mathcal{C}^{1,2}$-family of moving hypersurfaces in $\mathbb{R}^n$ and (t_0, x_0) be an inner point of $\mathcal{M} = \operatorname{gr}(\Sigma)$. Then there exists an open neighbourhood $U \subset \mathbb{R}^{n+1}$ of (t_0, x_0) and $\epsilon > 0$ such that the map*

$$X : (\mathcal{M} \cap U) \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}, \quad X(t, x, h) := (t, x + h\, n_\Sigma(t, x))$$

is a diffeomorphism onto its image

$$\mathcal{N}^\epsilon := X((\mathcal{M} \cap U) \times (-\epsilon, \epsilon)) \subset \mathbb{R}^{n+1},$$

i.e. X is invertible there and both X and X^{-1} are $\mathcal{C}^1$. The inverse function has the form

$$X^{-1}(t,x) = (\pi_\Sigma(t,x), d_\Sigma(t,x)) \tag{31}$$

with $\mathcal{C}^1$-functions π_Σ and d_Σ on $\mathcal{N}^\epsilon$. Moreover, $\nabla_x d_\Sigma \in \mathcal{C}^1(\mathcal{N}^\epsilon; \mathbb{R}^n)$ and $\nabla_x d_\Sigma \neq 0$.

Proof. The only point not covered by the proof to lemma 12 in [19] is the additional regularity of $\nabla_x d_\Sigma$, which follows by an argument taken from [25], where it is used for a fixed hypersurface: given a fixed $t \in J$, we have

$$x = \pi_\Sigma(t,x) + d_\Sigma(t,x)\, n_\Sigma(t, \pi_\Sigma(t,x)) \quad \text{on } \Sigma(t),$$

hence

$$d_\Sigma(t,x) = \langle x - \pi_\Sigma(t,x), n_\Sigma(t, \pi_\Sigma(t,x))\rangle$$

by taking inner products with $n_\Sigma(t, \pi_\Sigma(t,x))$. Differentiation as in the time-independent case (see [25]) yields

$$\nabla_x d_\Sigma = n_\Sigma(t, \pi_\Sigma(t,x)), \tag{32}$$

hence the desired regularity of $\nabla_x d_\Sigma$ as well as $\|\nabla_x d_\Sigma\| \equiv 1 \neq 0$ on $\mathcal{N}^\epsilon$. □

The latter result is useful to show that any $\mathcal{C}^{1,2}$-family of moving hypersurfaces has an intrinsic consistent velocity field, allowing for unique solutions.

Corollary 1. *Let $J = (a,b) \subset \mathbb{R}$ and $\{\Sigma(t)\}_{t\in J}$ be a $\mathcal{C}^{1,2}$-family of moving hypersurfaces in $\mathbb{R}^n$ with graph $\mathcal{M}$. Then its speed of normal displacement V_Σ is well-defined with $V_\Sigma \in \mathcal{C}(\mathcal{M})$, $\nabla_\Sigma V_\Sigma \in \mathcal{C}(\mathcal{M}; \mathbb{R}^n)$. Furthermore, the intrinsic velocity field*

$$w^\Sigma(t,x) := V_\Sigma(t,x)\, n_\Sigma(t,x) \quad \textit{for } (t,x) \in \mathcal{M} \tag{33}$$

satisfies $w^\Sigma \in \mathcal{C}(\mathcal{M}; \mathbb{R}^n)$, $\nabla_\Sigma w^\Sigma \in \mathcal{C}(\mathcal{M}; \mathbb{R}^{n\times n})$ and is consistent to $\mathcal{M}$.

Proof. Since only local properties are considered, it suffices to consider a fixed $(t_0, x_0) \in \mathcal{M}$ and arbitrarily small neighbourhoods (in $\mathcal{M}$) thereof. Locally, the $\mathcal{C}^{1,2}$-family $\{\Sigma(t)\}_{t\in J}$ of moving hypersurfaces is given as

$$\Sigma(t) \cap B_\epsilon(x_0) = \{x \in B_\epsilon(x_0) : d_\Sigma(t,x) = 0\}$$

with d_Σ from (31) due to lemma 3. Hence, by (29) and (32), the speed of normal displacement is given as

$$V_\Sigma(t,x) = -\,\partial_t d_\Sigma(t,x)$$

in a neighbourhood of (t_0, x_0) in $\mathcal{M}$. Evidently, $\partial_t d_\Sigma \in \mathcal{C}(\mathcal{M})$ by lemma 3, hence $V_\Sigma \in \mathcal{C}(\mathcal{M})$. Since $n_\Sigma \in \mathcal{C}^1(\mathcal{M})$ by assumption on $\{\Sigma(t)\}_{t\in J}$, this also yields $w^\Sigma \in \mathcal{C}(\mathcal{M}; \mathbb{R}^n)$. To see the additional regularity, note that $\nabla_x d_\Sigma$ is $\mathcal{C}^1$ by lemma 3, hence the mixed second order derivatives $\partial_t \partial_{x_k} d_\Sigma$ exist and are continuous. In this case, the order of differentiation can be exchanged due to the theorem of Schwarz[1], thus $\nabla_x \partial_t d_\Sigma$ exists and is continuous on $\mathcal{N}^\epsilon$. Hence

$$\nabla_\Sigma V_\Sigma = -P_\Sigma \nabla_x \partial_t d_\Sigma \in \mathcal{C}(\mathcal{M}; \mathbb{R}^n).$$

Consequently,

$$\nabla_\Sigma w^\Sigma = n_\Sigma \otimes \nabla_\Sigma V_\Sigma + V_\Sigma \nabla_\Sigma n_\Sigma \in \mathcal{C}(\mathcal{M}; \mathbb{R}^{n\times n}).$$

Finally, by definition of V_Σ, the intrinsic velocity field $w^\Sigma = V_\Sigma n_\Sigma$ satisfies

$$(1, w^\Sigma(t,x)) \in T_\mathcal{M}(t,x) \quad \text{on } \mathcal{M}.$$

Hence w^Σ is consistent to $\mathcal{M}$ due to lemma 1; note that the $w^\Sigma(t,\cdot)$ are locally Lipschitz continuous on $\Sigma(t)$ for $t \in J$. □

4. Extension of consistent interface velocities

The proof of wellposedness for the initial value problem (4) in the specific two-phase situation employs a reduction to fixed Σ_0 instead of moving $\Sigma(t)$. This reduction is based on the flow map associated to (4). Recall that if the initial value problems (4) are wellposed, the associated *flow map* (or, simply, *flow*) is the map $\Phi^t_{t_0} : \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$\Phi^t_{t_0}(x_0) := x(t; t_0, x_0), \tag{34}$$

where $x(\cdot; t_0, x_0)$ is the unique solution of (4). Of course, this concept can also be defined locally if (4) only has local (in time) solutions. We call this the flow map associated with the right-hand side f. Below, if the initial time t_0 is fixed, we denote the flow map as Φ^t for better readability. Now, if a $\mathcal{C}^{1,2}$-family of moving hypersurfaces in $\mathbb{R}^n$ is given, there is the intrinsic interface velocity field w^Σ given by (33) and w^Σ is consistent with the regularity as stated in corollary 1. If w denotes a continuous extension of w^Σ from $\mathcal{M} := \mathrm{gr}(\Sigma)$ to some open neighbourhood U of $\mathcal{M}$, being locally Lipschitz continuous in x, say, then the flow map $\Phi^t_{t_0}$ associated with w can be used as a nonlinear coordinate transform which fixes $\Sigma(t)$, since $\Sigma(t) = \Phi^t_{t_0}(\Sigma(t_0))$. But this alone is not sufficient for our purpose, since a curve $\gamma(\cdot)$ which passes through $\Sigma(t_0)$ in normal direction, i.e. $\gamma(s_0) =: x_0 \in \Sigma(t_0)$ and (w.l.o.g.) $\gamma'(s_0) = n_{\Sigma(t_0)}(x_0)$, is mapped into a curve which, while crossing $\Sigma(t)$ in the point $x(t) = \Phi^t_{t_0}(x_0)$, does not pass through $\Sigma(t)$ in normal direction, in general. In other words, the coordinate transform mediated by the flow leaves the interface invariant, but rotates the direction of vector fields, thus mixing tangential and normal parts. To avoid this difficulty, we are going to construct a particular extension of a given consistent interface velocity field which leads to a flow map $\Phi^t_{t_0}$ such that

$$n_{\Sigma(t)}(\Phi^t_{t_0}(y)) = \left[D_y \Phi^t_{t_0}(y)\right] n_{\Sigma(t_0)}(y) \quad \forall\, t_0 \in J,\ y \in \Sigma(t_0),\ t \in J_{t_0,y}, \tag{35}$$

where $J_{t_0,y}$ denotes the interval of existence of the solution to (23) for initial value (t_0, y).

[1] In the following refined version: if $f : B_\epsilon(x_0) \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous with continuous first partial derivatives such that $\partial_1 \partial_2 f(x)$ exists in $B_\epsilon(x_0)$ and is continuous in x_0, then $\partial_2 \partial_1 f(x_0)$ exists and $\partial_1 \partial_2 f(x_0) = \partial_2 \partial_1 f(x_0)$; see section 3.3 in [29] for a proof.

A key step of this extension relies on the following auxiliary result, where $V(r) = \omega_n |r|^n$ and $A(r) = n\omega_n |r|^{n-1}$ with ω_n the volume of $B_1(0) \subset \mathbb{R}^n$. Below, $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

Proposition 1. *Let Σ be a $\mathcal{C}^2$-hypersurface in $\mathbb{R}^n$ without boundary with normal field n. Due to lemma 3, there exists an open neighbourhood $U \subset \mathbb{R}^n$ of Σ such that $\Sigma = \{x \in U : d(x) = 0\}$ with $d \in \mathcal{C}^2(U)$ the signed distance to Σ. Let $\pi \in \mathcal{C}^1(U)$ denote the associated projection*[2]*, i.e. $x = \pi(x) + d(x)n(x)$. Given $f^\Sigma \in \mathcal{C}^1(\Sigma)$ and $g \in \mathcal{C}(U)$, let $\tilde{U} = \{x \in U : B_{|d(x)|}(x) \subset U\}$ which is an open neighbourhood of Σ. Define $f : \tilde{U} \to \mathbb{R}$ via*

$$f(x) = f^\Sigma(\pi(x)) - \frac{d(x)}{V(d(x))} \int_{\|x-y\| \leqslant |d(x)|} g(y)\, \mathrm{d}y \quad \textit{for } x \in \tilde{U}. \tag{36}$$

Then f satisfies

$$\partial_k f(x) = \partial_k (f^\Sigma \circ \pi)(x) + \partial_k d(x) \frac{n-1}{V(d(x))} \int_{\|x-y\| \leqslant |d(x)|} g(y)\, \mathrm{d}y \\ - \partial_k d(x) \frac{n}{A(d(x))} \int_{\|x-y\| = |d(x)|} g(y)\, \mathrm{d}\mathcal{H}^{n-1} + \frac{n}{A(d(x))} \int_{\|x-y\| = |d(x)|} g(y) \frac{x_k - y_k}{d(x)}\, \mathrm{d}\mathcal{H}^{n-1} \tag{37}$$

for all $x \in \tilde{U} \backslash \Sigma$, i.e. all $x \in \tilde{U}$ with $d(x) \neq 0$. Furthermore,

$$\nabla f(x) = \nabla_\Sigma f^\Sigma(x) - g(x)\, n_\Sigma(x) \quad \textit{for } x \in \Sigma. \tag{38}$$

Finally, it holds that $f \in \mathcal{C}^1(\tilde{U})$.

Proof. We consider only the case $x \in \tilde{U}^+ := \{x \in \tilde{U} : d(x) > 0\}$, since this allows for better readability, avoiding the use of $|d(x)|$ instead of $d(x)$; the other case can be treated by the same arguments with obvious modifications. Evidently,

$$f(x) = (f^\Sigma \circ \pi)(x) - \frac{1}{\omega_n d(x)^{n-1}} G(x) \quad \text{for } x \in \tilde{U}^+ \backslash \Sigma \tag{39}$$

with

$$G(x) = \int_{\|x-y\| \leqslant d(x)} g(y)\, \mathrm{d}y.$$

We have

$$\partial_k G(x) = \frac{\mathrm{d}}{\mathrm{d}s} G(x + s\, e_k)_{|s=0} = \left(\frac{\mathrm{d}}{\mathrm{d}s} \int_{\Omega(s)} g(y)\, \mathrm{d}y \right)_{|s=0}$$

and employ the Reynolds' transport theorem to compute $\partial_k G(x)$. For this purpose note that $\Gamma(s) := \partial\Omega(s)$ has the level set representation

$$\Gamma(s) = \{y : \phi(s, y) = 0\} \quad \text{with } \phi(s, y) = \|x + s\, e_k - y\|^2 - d(x + s\, e_k)^2.$$

[2] Actually, π is the metric projection onto Σ, i.e. $\pi(x) \in \Sigma$ with $\|x - \pi(x)\| = d(x)$.

Using (29), a simple calculation shows that $\Gamma(\cdot)$ has normal speed of displacement V_Γ given by

$$V_\Gamma(s,y) = \frac{-\partial_s\phi(s,y)}{\|\nabla_y\phi(s,y)\|} = \frac{d(x+s\,e_k)\frac{d}{ds}d(x+s\,e_k) - x_k + y_k - s}{\|x-y+s\,e_k\|}.$$

Hence

$$\partial_k G(x) = \int_{\Gamma(0)} g(y)\frac{d(x)\partial_k d(x) + y_k - x_k}{\|x-y\|}\,\mathrm{d}\mathcal{H}^{n-1}(y),$$

and therefore

$$\partial_k G(x) = \partial_k d(x)\int_{\|x-y\|=d(x)} g(y)\mathrm{d}\mathcal{H}^{n-1}(y) - \int_{\|x-y\|=d(x)} g(y)\frac{x_k-y_k}{d(x)}\,\mathrm{d}\mathcal{H}^{n-1}(y). \tag{40}$$

Differentiating (39), using (40), yields (37) for all $x\in\tilde{U}^+\backslash\Sigma$.

At $x\in\Sigma$ we have $f(x) = f^\Sigma(\pi(x)) = f^\Sigma(x)$. Hence, for $s>0$,

$$f(x+s\,n) = f(x) - \frac{s}{V(s)}\int_{\|x+s\,n-y\|\leqslant s} g(y)\,\mathrm{d}y$$

with $n := n_\Sigma(x)$. Thus,

$$\left\|\frac{f(x+s\,n)-f(x)}{s} + g(x)\right\| \leqslant \frac{1}{V(s)}\int_{\|x+s\,n-y\|\leqslant s}\|g(x)-g(y)\|\,\mathrm{d}y$$
$$\leqslant \sup\{\|g(x)-g(y)\| : \|x+sn-y\|\leqslant s\} \to 0 \quad \text{as } s\to 0+.$$

It is easy to check (replacing s by $|s|$ at a few places) that the same conclusion holds for $s\to 0-$, hence

$$\frac{\partial f}{\partial n}(x) = -g(x) \quad \text{at } x\in\Sigma. \tag{41}$$

On Σ, we also have $\nabla_\Sigma f(x) = \nabla_\Sigma f^\Sigma(x)$ since $f = f^\Sigma$ there. Together with (41), this yields (38).

To finish the proof, notice first that the $\partial_k f$ are continuous on $\tilde{U}^+\backslash\Sigma$. Indeed, there are two types of averages involved in (37), namely volume averages

$$x\to\frac{1}{V(d(x))}\int_{\|x-y\|\leqslant d(x)} h(y)\,\mathrm{d}y$$

and area averages

$$x\to\frac{1}{A(d(x))}\int_{\|x-y\|=d(x)} h(y)\,\mathrm{d}\mathcal{H}^{n-1}$$

with functions $h\in\mathcal{C}(U)$. The continuity of these maps follows from continuity of h and d by the dominated convergence theorem, if the integrals are rewritten via rescaling as

$$x \to \frac{1}{\omega_n} \int_{\|z\| \leqslant 1} h(x + d(x)z) \ \mathrm{d}z$$

and

$$x \to \frac{n}{\omega_n} \int_{\|z\|=1} h(x + d(x)z) \, \mathrm{d}\mathcal{H}^{n-1}(z).$$

It remains to show that

$$\nabla f(x) \to \nabla_\Sigma f^\Sigma(x_0) - g(x_0) \, n_\Sigma(x_0) \quad \text{for } \tilde{U}^+ \backslash \Sigma \ni x \to x_0 \in \Sigma. \tag{42}$$

For $x \in \tilde{U}^+ \backslash \Sigma$, we have

$$\left\| \frac{1}{V(d(x))} \int_{\|x-y\| \leqslant d(x)} g(y) \ \mathrm{d}y - g(x) \right\| \leqslant \sup_{\|x-y\| \leqslant d(x)} \|g(x) - g(y)\|, \tag{43}$$

$$\left\| \frac{1}{A(d(x))} \int_{\|x-y\|=d(x)} g(y) \, \mathrm{d}\mathcal{H}^{n-1}(y) - g(x) \right\| \leqslant \sup_{\|x-y\|=d(x)} \|g(x) - g(y)\| \tag{44}$$

and

$$\int_{\|x-y\|=d(x)} g(y) \frac{x_k - y_k}{d(x)} \mathrm{d}\mathcal{H}^{n-1}(y) = \int_{\|x-y\|=d(x)} (g(y) - g(x)) \, \frac{x_k - y_k}{d(x)} \mathrm{d}\mathcal{H}^{n-1}(y). \tag{45}$$

For the latter equality, note that

$$\int_{\|x-y\|=d(x)} \frac{x_k - y_k}{d(x)} \, \mathrm{d}\mathcal{H}^{n-1}(y) = \frac{1}{d(x)} \int_{\|z\|=d(x)} z_k \, \mathrm{d}\mathcal{H}^{n-1}(z) = 0.$$

Applying the relations (43)–(45) to (37) immediately yields (42), hence $f \in \mathcal{C}^1(\tilde{U}^+)$. Together with the analogous treatment for $x \in \tilde{U}^-$ and because the limit on Σ is the same for both sides, we obtain $f \in \mathcal{C}^1(\tilde{U})$. □

Let us note in passing that, in vector notation, equation (37) means

$$\begin{aligned} \nabla f(x) = \nabla(f^\Sigma \circ \pi)(x) + n_\Sigma(x) \frac{n-1}{V(d(x))} \int_{\|x-y\| \leqslant |d(x)|} g(y) \ \mathrm{d}y \\ - \ n_\Sigma(x) \frac{n}{A(d(x))} \int_{\|x-y\|=|d(x)|} g(y) \, \mathrm{d}\mathcal{H}^{n-1} \\ - \ \frac{n}{A(d(x))} \int_{\|x-y\|=|d(x)|} g(y) \nu(y) \, \mathrm{d}\mathcal{H}^{n-1} \quad \text{for } x \in \tilde{U} \backslash \Sigma, \end{aligned} \tag{46}$$

where $\nu(\cdot)$ is the outer unit normal to the sphere $\partial B_{d(x)}(x)$.

Inspection of the above proof in the time-dependent case shows that the following result is an immediate corollary to proposition 1.

Corollary 2. *Let* $J = (a, b) \subset \mathbb{R}$ *and* $\{\Sigma(t)\}_{t\in J}$ *be a* $\mathcal{C}^{1,2}$*-family of moving hypersurfaces without boundary in* $\mathbb{R}^n$ *with graph* $\mathcal{M}$*. By lemma 3, there exists an open neighbourhood* $\mathcal{N} \subset \mathbb{R}^{n+1}$ *of* $\mathcal{M}$ *such that* $\{\Sigma(t)\}_{t\in J}$ *has a level set representation with signed distance function* d_Σ *such that* $d_\Sigma \in \mathcal{C}^1(\mathcal{N})$ *and* $\nabla_x d_\Sigma \in \mathcal{C}^1(\mathcal{N}; \mathbb{R}^n)$*. Let* $\pi_\Sigma \in \mathcal{C}^1(\mathcal{N})$ *denote the associated family of projections onto* $\Sigma(\cdot)$ *characterized by*

$$x = \pi_\Sigma(t, x) + d_\Sigma(t, x)\, n_\Sigma(t, x) \quad \text{for all } (t, x) \in \mathcal{N}.$$

Given $f^\Sigma \in \mathcal{C}(\mathcal{M})$ *with* $\nabla_\Sigma f^\Sigma \in \mathcal{C}(\mathcal{M})$ *and* $g \in \mathcal{C}(\mathcal{N})$*, let* $\mathcal{U}$ *with* $\mathcal{M} \subset \mathcal{U} \subset \mathcal{N}$ *be open and so small that* $(t, x) \in \mathcal{U}$ *implies* $\{t\} \times B_{|d(t,x)|}(x) \subset \mathcal{N}$*. Define* $f : \mathcal{U} \to \mathbb{R}$ *by means of*

$$f(t, x) = f^\Sigma(t, \pi(t, x)) - \frac{d(t, x)}{V(d(t, x))} \int_{\|x-y\|\leqslant|d(t,x)|} g(t, y)\, \mathrm{d}y \quad \text{on } \mathcal{U} \tag{47}$$

with $d := d_\Sigma$ *and* $\pi := \pi_\Sigma$*. Then* $f \in \mathcal{C}(\mathcal{U})$ *and* $f(t, \cdot) \in \mathcal{C}^1(\mathcal{U}^t)$*, where* $\mathcal{U}^t := \{x \in \mathbb{R}^n : (t, x) \in \mathcal{U}\}$ *is an open neighbourhood of* $\Sigma(t)$*. Moreover, the spatial derivatives* $\partial_{x_k} f$ *are given by (37) on* $\mathcal{M}$*, and by (38) on* $\mathcal{U}\backslash\mathcal{M}$ *with obvious modifications in form of the additional variable* t*.*

We are now able to prove the following key extension result.

Lemma 4. *Let* $J = (a, b) \subset \mathbb{R}$ *and* $\{\Sigma(t)\}_{t\in J}$ *be a* $\mathcal{C}^{1,2}$*-family of moving hypersurfaces without boundary in* $\mathbb{R}^n$ *with* $\mathcal{M} = \mathrm{gr}(\Sigma)$*. Let* $v^\Sigma \in \mathcal{C}(\mathcal{M}; \mathbb{R}^n)$ *be consistent to* $\mathcal{M}$ *with* $\nabla_\Sigma v^\Sigma \in \mathcal{C}(\mathcal{M}; \mathbb{R}^{n\times n})$*. Then there exists a neighbourhood* $\mathcal{U}$ *of* $\mathcal{M}$ *and an extension* $\hat{v}^\Sigma : \mathcal{U} \to \mathbb{R}^n$ *of* v^Σ *being jointly continuous and locally Lipschitz continuous in* x *such that, with* $\Phi^t_{t_0}$ *the (local) flow map associated to* $\hat{v}^\Sigma$*, the evolution of the normal field satisfies (35). In particular, the intrinsic surface velocity* $v^\Sigma = V_\Sigma n_\Sigma$ *admits such an extension.*

Proof. Since the statement is about local properties of the desired extension, we may consider a small neighbourhood $U_\epsilon = (\eta - \epsilon, \eta + \epsilon) \times B_\epsilon(\xi)$ of a point $(\eta, \xi) \in \mathcal{M}$ in which the moving hypersurfaces are given by means of the signed distance function from lemma 3. We then extend the given function v^Σ from $\mathcal{M} \cap U_\epsilon$ to a function $\hat{v}^\Sigma$ on U_ϵ by means of

$$\hat{v}^\Sigma(t, x) := v^\Sigma(t, \pi(t, x)) - \frac{d(t, x)}{V(d(t, x))} F(t, x) \tag{48}$$

with

$$F(t, x) = \int_{B_{|d(t,x)|}(x)} \sum_{k=1}^{n-1} \left\langle \frac{\partial v^\Sigma}{\partial \tau_k}(t, \pi(t, y)), n_\Sigma(t, \pi(t, y)) \right\rangle \tau_k(t, \pi(t, y))\, \mathrm{d}y, \tag{49}$$

where $d := d_\Sigma$, $\pi := \pi_\Sigma$ is the projection from lemma 3 and

$$\{\tau_k(t, x) : k = 1, \ldots, n-1\} \quad \text{for } (t, x) \in \mathcal{M} \cap U_\epsilon \tag{50}$$

is an orthonormal basis of the tangent space to $\Sigma(t)$ at the point x, depending continuously differentiable on $(t, x) \in \mathcal{M} \cap U_\epsilon$. Note that we obtain such an orthonormal basis with the desired regularity by applying the Gram–Schmidt orthonormalization procedure to the system

$$\{\tau_k^0 - \langle \tau_k^0, n_\Sigma(t, x)\rangle\, n_\Sigma(t, x) \, : \, k = 1, \ldots, n-1\} \tag{51}$$

with $\{\tau_k^0 : k = 1, \dots, n-1\}$ being a basis of the tangent space to $\Sigma(\eta)$ at the point ξ. By choosing $\epsilon > 0$ sufficiently small, this is a system of linearly independent vectors on $\mathcal{M} \cap U_\epsilon$ and the elements depend continuously differentiable on (t, x) since n_Σ has this regularity.

Now observe that the components of $\hat{v}^\Sigma(t, x)$ in (48) are precisely of the type as given in (47) and the integrand in (49) is continuous due to our assumptions on $\{\Sigma(t)\}_{t\in J}$ and v^Σ. Therefore, by corollary 2, $\hat{v}^\Sigma$ is continuous in $\mathcal{M} \cap U_\epsilon$ and the $\hat{v}^\Sigma(t, \cdot)$ are continuously differentiable in a neighbourhood of $\Sigma(t)$. In particular, $\hat{v}^\Sigma$ is jointly continuous and locally Lipschitz continuous in x and, hence, the initial value problems (4) are uniquely solvable for right-hand side $\hat{v}^\Sigma$, at least locally in time. Consequently, the associated flow map $\Phi^t_{t_0}$ is welldefined. Moreover, $\Phi^t_{t_0}$ is invertible with inverse $\Phi^{t_0}_t$, hence a diffeomorphism due to the regularity of $\hat{v}^\Sigma$. Thus, $D_y\Phi^t_{t_0}(y)$ is invertible.

Furthermore, by corollary 2, we also know that $\hat{v}^\Sigma$ satisfies

$$\frac{\partial \hat{v}^\Sigma}{\partial n_\Sigma}(t, x) = -\sum_{k=1}^{n-1} \left\langle \frac{\partial v^\Sigma}{\partial \tau_k}(t, x), n_\Sigma(t, x) \right\rangle \tau_k(t, x) \quad \text{for } (t, x) \in \mathcal{M} \cap U_\epsilon. \tag{52}$$

In order to prove (35), we consider the equivalent relation

$$\left[D_y\Phi^t_{t_0}(y)\right]^{-1} n_{\Sigma(t)}(\Phi^t_{t_0}(y)) = n_{\Sigma(t_0)}(y) \quad \forall\, t_0 \in J, y \in \Sigma(t_0), t \in J_{t_0,y}. \tag{53}$$

Evidently, equation (53) holds for $t = t_0$. Therefore, it holds for all $t \in J_{t_0,y}$, if we show that the t-derivative of the left-hand side vanishes. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[D_y\Phi^t_{t_0}(y)\right]^{-1} n_{\Sigma(t)}(\Phi^t_{t_0}(y)) = &-\left[D_y\Phi^t_{t_0}(y)\right]^{-1} \partial_t D_y\Phi^t_{t_0}(y) \left[D_y\Phi^t_{t_0}(y)\right]^{-1} n_{\Sigma(t)}(\Phi^t_{t_0}(y))\\ &+ \left[D_y\Phi^t_{t_0}(y)\right]^{-1} \frac{\mathrm{d}}{\mathrm{d}t} n_{\Sigma(t)}(\Phi^t_{t_0}(y)).\end{aligned}$$

We now employ Schwarz' theorem to get

$$\partial_t D_y\Phi^t_{t_0}(y) = D_y\partial_t\Phi^t_{t_0}(y) = D_y\hat{v}^\Sigma(t, \Phi^t_{t_0}(y)) = \nabla_x\hat{v}^\Sigma(t, \Phi^t_{t_0}(y))\, D_y\Phi^t_{t_0}(y)$$

which yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[D_y\Phi^t_{t_0}(y)\right]^{-1} n_{\Sigma(t)}(\Phi^t_{t_0}(y))\\ &\quad = \left[D_y\Phi^t_{t_0}(y)\right]^{-1} \left(\frac{\mathrm{d}}{\mathrm{d}t} n_{\Sigma(t)}(\Phi^t_{t_0}(y)) - \nabla_x\hat{v}^\Sigma(t, \Phi^t_{t_0}(y))\, n_{\Sigma(t)}(\Phi^t_{t_0}(y)) \right).\end{aligned}$$

Due to theorem 4 in [19] (extended from hypersurfaces in $\mathbb{R}^3$ to $\mathbb{R}^n$; see [18] for a formal proof which employs a level set representation of $\Sigma(\cdot)$), the Lagrangian derivative of the normal field satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} n_{\Sigma(t)}(\Phi^t_{t_0}(y)) = -\sum_{k=1}^{n-1} \left\langle \frac{\partial v^\Sigma}{\partial \tau_k}(t, \Phi^t_{t_0}(y)), n_{\Sigma(t)}(\Phi^t_{t_0}(y)) \right\rangle \tau_k(t, \Phi^t_{t_0}(y)). \tag{54}$$

This relation, together with the normal derivative of $\hat{v}^\Sigma$ according to (52) shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[D_y\Phi^t_{t_0}(y)\right]^{-1} n_{\Sigma(t)}(\Phi^t_{t_0}(y)) = 0$$

along the solution of (23), hence (35) holds. □

5. The ODE-system associated with a two-phase flow

Let us briefly recall the setup. Given the domain $\Omega \subset \mathbb{R}^n$ of a two-phase flow and a time interval $J = (a, b)$, we consider a $\mathcal{C}^{1,2}$-family of moving closed hypersurfaces $\{\Sigma(t)\}_{t \in J}$ which decomposes Ω into disjoint sets according to $\Omega = \Omega^+(t) \cup \Omega^-(t) \cup \Sigma(t)$. We let $G^\pm = \mathrm{gr}(\overline{\Omega^\pm})$ and $v^\pm : G^\pm \to \mathbb{R}^n$ be continuous vector fields which are locally Lipschitz continuous in x, separately on G^+, respectively G^-. We also assume at most linear growth of v in x, i.e.

$$|v^\pm(t,x)| \leqslant c\,(1 + |x|) \quad \text{for all}\, t \in J,\ \ x \in \Omega^\pm(t) \tag{55}$$

with some $c > 0$. We denote by v without superscript the set-valued map defined by

$$v(t,x) = \begin{cases} \{v^+(t,x)\} & \text{if}\, x \in \Omega^+(t), \\ \{v^+(t,x), v^-(t,x)\} & \text{if}\, x \in \Sigma(t), \\ \{v^-(t,x)\} & \text{if}\, x \in \Omega^-(t). \end{cases} \tag{56}$$

Note that v is not single-valued on $\mathcal{M} = \mathrm{gr}(\Sigma)$, but attains two possibly distinct values there, i.e. v is set-valued on $\mathcal{M}$. We then study the discontinuous differential equation

$$\dot{x}(t) = v\,(t, x(t)) \quad \text{on } J, \quad x(t_0) = x_0 \tag{57}$$

for $t_0 \in J$, $x_0 \in \Omega$. Let us note that the growth condition (55) implies *a priori* bounds for possible solutions which guarantee that any local solution can be continued up to the boundary of $J \times \Omega$, assuming that the moving hypersurfaces are given on all of J. In particular, in case $\Omega = \mathbb{R}^n$ the solution exists globally. Let us also mention that uniqueness of solutions, which is the core point to be proven, is a *local property* of the ODE-system.

Note that we are slightly abusing notation in (57), since it should actually read

$$\dot{x}(t) \in v\,(t, x(t)) \quad \text{on } J, \quad x(t_0) = x_0.$$

But this is not relevant if, along the solution, the set $v(t, x(t))$ is a singleton except for t from a set of Lebesgue measure zero, as it will turn out to be the case here. We hence stick to (57) and employ the following solution concept.

Definition 3. We call an absolutely continuous function $x : J \to \mathbb{R}^n$ an a.c. solution of (57), if $x(t_0) = x_0$, $N := \{t \in J : v\,(t, x(t)) \text{ is set} - \text{valued}\}$ is a set of Lebesgue measure zero and $\dot{x}(t) = v\,(t, x(t))$ a.e. on $J \backslash N$.

We are interested in physically relevant conditions on v and Σ such that (57) has unique a.c. solutions, locally in time, for every initial value. Motivated by (18), we impose the transmission condition

$$\rho^+(v^+ - v^\Sigma) \cdot n_\Sigma = \rho^-(v^- - v^\Sigma) \cdot n_\Sigma \quad \text{on } \mathcal{M} \tag{58}$$

with locally Lipschitz functions $\rho^\pm : \mathcal{M} \to (0, \infty)$. The main point here is that $\rho^\pm(t,x) > 0$ on $\mathcal{M}$. Note also that we can assume w.l.o.g. that $\rho^\pm$ are locally Lipschitz functions $\rho^\pm : G^\pm \to (0, \infty)$, since we can extend $\rho^\pm$ from $\mathcal{M}$ to $G^\pm$ via $\rho^\pm(t,x) := \rho^\pm(t, \pi_\Sigma(t,x))$ such that the extension is still locally Lipschitz continuous due to lemma 3. In addition, we assume (21) to hold, i.e. the tangential parts of $v^{\pm}$ satisfy

$$v_{\parallel}^{+} = v_{\parallel}^{-} \quad \text{on } \mathcal{M}. \tag{59}$$

Since v^{Σ} enters our assumptions only via $V_{\Sigma} = v^{\Sigma} \cdot n_{\Sigma}$, we may assume

$$v_{\parallel}^{\Sigma} = v_{\parallel}^{\pm}. \tag{60}$$

Let us explain how the physically motivated assumptions (58) and (59) can be exploited to obtain uniqueness of solutions to (57). Of course, the only difficulty occurs if a solution reaches the interface. Now observe that $\rho^{\pm}$ is strictly positive by assumption, hence (58) implies the transversality-type condition

$$\mathrm{sgn}_0\left((v^{+} - v^{\Sigma}) \cdot n_{\Sigma}\right) = \mathrm{sgn}_0\left((v^{-} - v^{\Sigma}) \cdot n_{\Sigma}\right) \quad \text{on } \mathcal{M}. \tag{61}$$

This indicates that no problem will occur if a solution hits $\Sigma(t)$ with non-zero normal velocity relative to the interface: by (61), the solution cannot stay inside Σ, but continues into the opposite phase; cf. the explanations around (15). Thus the relevant case is when $x_0 \in \Sigma(t_0)$ and $v^{\pm}(t_0, x_0) \cdot n_{\Sigma} = v^{\Sigma}(t_0, x_0) \cdot n_{\Sigma}$. In this case, a solution could follow Σ or tangentially enter one of the bulk phases.

To treat this case, we shall freeze the moving interface by a suitable nonlinear time-dependent coordinate transform and, by another nonlinear transform, reduce to the case of a planar fixed interface.

By the tailor-made transformations developed in section 4, this can be done in such a manner that Σ is transformed into $\mathbb{R}^{n-1} \times \{0\}$ and the jump conditions (58) and (59) become

$$\rho^{+} v_{n}^{+} = \rho^{-} v_{n}^{-} \quad \text{and} \quad v_{\parallel}^{\pm} = 0 \quad \text{for } t \in J,\ x_n = 0. \tag{62}$$

Observe that under these assumptions, the underlying differential equation is not reduced to a one-dimensional problem. This is illustrated in figure 2 which refers to the two-dimensional and autonomous case $v^{\pm} = v^{\pm}(x)$. A solution $x(\cdot)$ can cross the interface at isolated times $t_k \in J$ and the main question then is whether there can be infinitely many such points t_k with a finite accumulation point, i.e. such that $t_k \nearrow t_{\infty} \in J$. If this is true than $x(t_k) \to x_{\infty} \in \Sigma(t_{\infty})$ and, necessarily, $0 \in F(x_{\infty})$. Hence $x(\cdot)$ can be continued as a solution with $x(t) \equiv x_{\infty}$ for $t > t_{\infty}$, but there may be another continuation as a solution, say $\bar{x}(\cdot)$, which leaves x_{∞} at a later time. Evidently, this cannot happen in such a way that $\bar{x}(t) \notin \Sigma(t)$ on some interval $(t_{\infty}, t_{\infty} + \delta)$ with $\delta > 0$ since this would contradict the unique solvability inside G^{+} or G^{-} which holds due to the separate local Lipschitz continuity of $v^{\pm}$. Consequently, such a continuation $\bar{x}(\cdot)$ would cross Σ at infinitely many times $\bar{t}_k$ such that $t_{\infty} < \cdots < \bar{t}_{k+1} < \bar{t}_k < \cdots < \bar{t}_1$. Then, qualitatively, the trajectory after t_{∞} looks like the one in figure 1, but with time reversed. Consequently, since we aim at both forward and backward uniqueness, this can only be avoided if already the existence of infinitely many isolated points t_k with $t_k \nearrow t_{\infty}$ cannot occur. Moreover, for the same reason it suffices to show that for initial value $x_0 \in \Sigma(t_0)$ such that $0 \in F(x_0)$ there is a unique (forward) solution which is then given by $\bar{x}(t) \equiv x_0$.

It is now instructive to see how this unique solvability can be obtained in the autonomous case, in which (62) can be employed in a rather direct way. So, after transformation to the case $\Sigma(t) \equiv \mathbb{R}^{n-1} \times \{0\}$, we assume that the initial value satisfies $x_{0,n} = 0$ and is such that $0 \in F(x_0)$, hence $\bar{x}(t) \equiv x_0$ is one solution of (57). Now suppose that $x(\cdot)$ is another solution of (57). We are going to rescale time in such a way that the arcs of the trajectory lying in Ω^{+} or

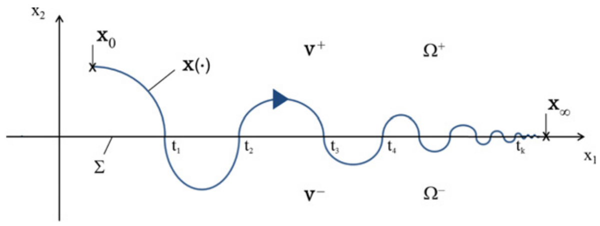

Figure 2. Trajectory of a possible solution coming to rest at x_∞.

Ω^-, respectively, are traversed with differently modified speed in such a way that the velocity becomes continuous at Σ. For this purpose, let τ be an a.c. solution of

$$\tau'(t) \in R(x(\tau(t))) \quad \text{for } t \in J,\ \tau(t_0) = t_0 \tag{63}$$

with the set-valued map $R : \Omega \to 2^{\mathrm{IR}} \backslash \{\emptyset\}$ defined as

$$R(\xi) = \begin{cases} \{\rho^+(\xi)\} & \text{if } \xi_n > 0, \\ \text{conv}\{\rho^+(\xi), \rho^-(\xi)\} & \text{if } \xi_n = 0, \\ \{\rho^-(\xi)\} & \text{if } \xi_n < 0. \end{cases} \tag{64}$$

Concerning the solvability of the differential inclusion (63) notice that $H = R \circ x$ is the multivalued regularization in the sense of (7) or (9) of, say, the discontinuous function $h(\tau) := \rho^+(x(\tau))$ if $x_n(\tau) \geqslant 0$ and $h(\tau) := \rho^-(x(\tau))$ if $x_n(\tau) < 0$. Hence (63) has an a.c. solution according to the results mentioned in the introduction. Moreover, the discontinuous function $h : J \to \mathbb{R}$ satisfies condition (15), hence forward uniqueness holds, i.e. there is a unique a.c. solution $\tau : J \to \mathbb{R}$ of (63). Consider the a.c. function $y(t) := x(\tau(t))$ for $t \in J$. It follows immediately that $y(\cdot)$ satisfies

$$y'(t) \in v(y(t))R(y(t)) \quad \text{a.e. on } J,\ y(t_0) = x_0. \tag{65}$$

Hence

$$y'(t) = \rho(y(t))v(y(t)) \quad \text{a.e. on } \{t \in J : y_n(t) \neq 0\}. \tag{66}$$

On the other hand, using points of Lebesgue density,

$$v_n^{\pm}(y(t)) = 0 \quad \text{a.e. on } N := \{t \in J : y_n(t) = 0\} \tag{67}$$

since $\rho^{\pm} > 0$. Thus $v^{\pm}(y(t)) = 0$ for those $t \in N$ due to (62) and then $v(y(t))R(y(t)) = \{0\}$. Consequently, the a.c. function $y(\cdot)$ satisfies

$$y'(t) = \rho(y(t))v(y(t)) \quad \text{a.e. on } J,\ y(t_0) = x_0. \tag{68}$$

But the function ρv is also continuous at $\{x : x_n = 0\}$ due to (62), and one easily checks that ρv is locally Lipschitz, also across the interface $x_n = 0$. Furthermore, $y(t) \not\equiv x_0$, hence $y(\cdot)$ and $\bar{y}(t) \equiv x_0$ are different solutions of (68), a contradiction. Thus a solution $x(\cdot)$ which leaves x_0 cannot exits.

This approach does not work in the time-dependent case. To indicate the core idea for proving uniqueness in the non-autonomous case, let two solutions of (57) be given, where the relevant case is when the two solutions, say for $t > t_0$ close to t_0, lie in different phases. We call these solutions $x^{\pm}(\cdot)$ and assume that $x^{\pm}(t) \in \Omega^{\pm}(t)$ on $[t_0, t_0 + \delta]$. In order to estimate the growth of $|x^{+}(t) - x^{-}(t)|$, the (local) Lipschitz continuity of $v^{\pm}$ is to be exploited. For estimating $|v^{+}(t, x^{+}(t)) - v^{-}(t, x^{+}(t))|$, the terms $\mp v^{\pm}(t, x^{\Sigma}(t))$ at an intermediate point $x^{\Sigma}(t) \in \Sigma(t)$ need to be added such that separate estimates in $\Omega^{\pm}(t)$ become possible. But adding $\mp v^{\pm}(t, x^{\Sigma}(t))$ is not a zero addition, while $\mp(\rho^{\pm} v^{\pm})(t, x^{\Sigma}(t))$ would be. Therefore, instead of $|x^{+}(t) - x^{-}(t)|$, we consider the functional

$$|\rho^{+}(t, x^{\Sigma}(t))x_n^{+}(t) - \rho^{-}(t, x^{\Sigma}(t))(t)x_n^{-}(t)| + |x_{\|}^{+}(t) - x_{\|}^{-}(t)|$$

with an appropriate function $x^{\Sigma}(t)$ in the uniqueness proof below.

6. Wellposedness of the ODE-system from two-phase flows

We now state and prove the main result of this paper.

Theorem 1. *Let $J = (a, b) \subset \mathbb{R}$ and $\{\Sigma(t)\}_{t \in J}$ be a $\mathcal{C}^{1,2}$-family of moving hypersurfaces in $\mathbb{R}^n$ without boundary which divide an open set $\Omega \subset \mathbb{R}^n$ into $\Omega^{+}(t) \cup \Omega^{-}(t) \cup \Sigma(t)$ for all $t \in J$ with time-dependent bulk phases $\Omega^{\pm}(t)$. Let*

$$v^{\pm} : \mathrm{gr}\left(\overline{\Omega^{\pm}(\cdot)}\right) \to \mathbb{R}^n$$

be continuous in (t, x) and locally Lipschitz continuous in x such that (58) and (59) are valid, where $v^{\Sigma} := V_{\Sigma} n_{\Sigma}$ is the consistent intrinsic interface velocity associated to $\{\Sigma(t)\}_{t \in J}$. Then, for given $t_0 \in J$ and $x_0 \in \Omega$, the initial value problem (57) has a unique a.c. solution, locally in time. This solution is also the unique Filippov solution of (57). If, in addition, the growth condition (55) is satisfied, this solution exists on all of J.

Proof. The proof is given in several steps, building on the idea explained above for the uniqueness part.

Step 1. *Existence of solutions.*

In the specific situation under consideration, one can easily see that F from (7) is given by (10), i.e.

$$F(t, x) = \begin{cases} \{v^{+}(t, x)\} & \text{if } x \in \Omega^{+}(t), \\ \mathrm{conv}\{v^{+}(t, x), v^{-}(t, x)\} & \text{if } x \in \Sigma(t), \\ \{v^{-}(t, x)\} & \text{if } x \in \Omega^{-}(t). \end{cases}$$

This set-valued map is even jointly upper semicontinuous such that classical existence results for differential inclusions apply; see [2, 14]. Therefore, concerning the existence part, it only remains to show that any a.c. solution $x(\cdot)$ of (8) with F from (10) is actually an a.c. solution of (57). For this purpose, we will show that

$$M := \{t \in J : F(t, x(t)) \, \text{is set-valued}\}$$

is a Lebesgue null set. Evidently, $M \subset N := \{t \in J : x(t) \in \Sigma(t)\}$, since for $t \in J \backslash N$ it holds that $F(t, x) = \{v(t, x)\}$, hence $\dot{x}(t) = v\,(t, x(t))$ a.e. on $J \backslash N$. Since $x(\cdot)$ is a.c., the derivative $\dot{x}(t)$

exists a.e. on J, in particular a.e. on N. Given a (local) level set representation of Σ according to (28), we have

$$\phi(t, x(t)) = 0 \quad \text{on } N,$$

hence also

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\phi(t, x(t)) = \partial_t\phi(t, x(t)) + \dot{x}(t) \cdot \nabla_x\phi(t, x(t)) \quad \text{a.e. on } N.$$

Note that such a level set representation exists at least locally due to our regularity assumptions on Σ by lemma 3. Using (29), this implies

$$\dot{x}(t) \cdot n_\Sigma(t, x(t)) = V_\Sigma(t, x(t)) \quad \text{a.e. on } N.$$

On the other hand,

$$P_\Sigma\dot{x}(t) \in P_\Sigma F(t, x(t)) = \{v_\parallel^\pm(t, x(t))\}$$

due to (59). Therefore, employing (60), we obtain

$$\dot{x}(t) = V_\Sigma(t, x(t))\, n_\Sigma(t, x(t)) + v_\parallel^\pm(t, x(t)) = v^\Sigma(t, x(t)) \quad \text{a.e. on } N. \tag{69}$$

Consequently,

$$v^\Sigma(t, x(t)) \in \mathrm{conv}\{v^+(t, x(t)), v^-(t, x(t))\} \quad \text{for all } t \in N_0,$$

where $N_0 \subset N$ has $\lambda_1(N \backslash N_0) = 0$. Taking inner product with n_Σ, this implies (with a slight abuse of notation)

$$0 \in \left(\mathrm{conv}\{(v^+ - v^\Sigma) \cdot n_\Sigma, (v^- - v^\Sigma) \cdot n_\Sigma\}\right)(t, x(t)) \quad \text{for all } t \in N_0.$$

For fixed $t \in N_0$, two cases are hence possible: either

$$\left(v^+ - v^\Sigma\right) \cdot n_\Sigma \leqslant 0 \leqslant \left(v^- - v^\Sigma\right) \cdot n_\Sigma \quad \text{at } (t, x(t)) \tag{70}$$

or the same with v^+, v^- exchanged. Because of the transversality-type condition (61), a strict inequality is not possible in (70), hence

$$v^+ \cdot n_\Sigma = v^\Sigma \cdot n_\Sigma = v^- \cdot n_\Sigma \quad \text{at } (t, x(t)).$$

To sum up, it therefore holds that

$$v^+(t, x(t)) = v^-(t, x(t)) \quad \text{for all } t \in N_0,$$

hence $t \in N_0$ implies $t \notin M$, i.e. $M \subset N \backslash N_0$ and thus M is a null set. Note that, up to here, less regularity of $v^\pm$ would be sufficient, say measurability in t and local Lipschitz continuity in x.

It remains to show *uniqueness of a.c. solutions,* where we start with forward uniqueness. For this purpose, let $x(\cdot)$ and $\overline{x}(\cdot)$ be two (distinct) a.c. solutions of (57) with common initial value x_0. Local-in-time (forward and backward) uniqueness is clear in case $x_0 \notin \Sigma(t_0)$. So, we may assume $x_0 \in \Sigma(t_0)$ and have to show that $x(t) = \overline{x}(t)$ on $[t_0, t_0 + \delta]$ for some $\delta > 0$.

Step 2. Reduction to fixed Σ and vanishing tangential part of v.

Let $\hat{v}^\Sigma$ be the extension of $v^\Sigma : \mathcal{M} \to \mathbb{R}^n$ provided by lemma 4. Considering only local wellposedness, we may assume that v^Σ and then also $\hat{v}^\Sigma$ are bounded and that $\hat{v}^\Sigma$ is given on all of $J \times \mathbb{R}^n$. Hence $\hat{v}^\Sigma$ generates a global flow $\Phi^t_{t_0} : \mathbb{R}^n \to \mathbb{R}^n$ via $\Phi^t_{t_0}(y_0) := y(t; t_0, y_0)$, where $y(\cdot, t_0, y_0)$ is the unique global solution of

$$\dot{y}(t) = \hat{v}^\Sigma(t, y(t)) \quad \text{on } J,\ y(t_0) = y_0. \tag{71}$$

Note that the flow $\Phi^t_{t_0}$ leaves $\Sigma(\cdot)$ invariant, which means that

$$\Sigma(t) = \Phi^t_{t_0}(\Sigma(t_0)) \quad \text{for all } t, t_0 \in J. \tag{72}$$

This follows by lemma 2, since the vector field v^Σ is consistent to $\mathcal{M}$. Moreover, $\Phi^t_{t_0}$ also leaves $\Omega^+(\cdot)$, respectively $\Omega^-(\cdot)$ invariant since solutions cannot cross $\Sigma(\cdot)$ due to unique solvability. Now $x(\cdot)$ is a a.c. solution of (57) iff the a.c. function $y(\cdot)$, implicitly defined by

$$x(t) = \Phi^t_{t_0}(y(t)), \tag{73}$$

solves the initial value problem

$$\dot{y}(t) = f(t, y(t)) \quad \text{a.e. on } J,\ y(t_0) = x_0 \tag{74}$$

with right-hand side f given by

$$f(t, y) := \left[D_y \Phi^t_{t_0}(y)\right]^{-1} \cdot \left(v(t, \Phi^t_{t_0}(y)) - \hat{v}^\Sigma(t, \Phi^t_{t_0}(y))\right). \tag{75}$$

Note that f is discontinuous at $(\Phi^t_{t_0})^{-1}(\Sigma(t)) = \Sigma(t_0) =: \Sigma_0$, but the $f^\pm$ given by the right-hand side of (75) on $\Omega^\pm(t_0)$ have the same regularity as the $v^\pm$, with continuous extensions onto the closure of $\Omega^\pm(t_0)$. To rewrite the transmission condition (58), let

$$\hat{\rho}^\pm(t, y) := \rho^\pm(t, \Phi^t_{t_0}(y)), \quad n(y) := n_{\Sigma_0}(y)$$

and note that the $\hat{\rho}^\pm$ have the same regularity as the $\rho^\pm$. Then, for $y \in \Sigma_0$,

$$\hat{\rho}^+(t, y) f^+(t, y) \cdot n(y) = \rho^+(t, x) \left[D_y \Phi^t_{t_0}(y)\right]^{-1} (v^+(t, x) - \hat{v}^\Sigma(t, x)) \cdot n(y)$$

with $x = \Phi^t_{t_0}(y) \in \Sigma(t)$. Due to (60), we have

$$v^+(t, x) - \hat{v}^\Sigma(t, x) = \left\langle v^+(t, x) - \hat{v}^\Sigma(t, x), n_\Sigma(t, x) \right\rangle n_\Sigma(t, x),$$

hence (with shorthand notation)

$$\left(\hat{\rho}^+ f^+\right)(t, y) \cdot n(y) = \left(\rho^+ \left\langle v^+ - \hat{v}^\Sigma, n_\Sigma \right\rangle\right)(t, x) \left[D_y \Phi^t_{t_0}(y)\right]^{-1} n_\Sigma(t, x) \cdot n(y).$$

Rewriting $\left(\hat{\rho}^- f^-\right)(t, y) \cdot n(y)$ in an analogous way, we see that (58) becomes

$$\hat{\rho}^+ f^+ \cdot n = \hat{\rho}^- f^- \cdot n \quad \text{on } \Sigma_0, \tag{76}$$

and the transversality condition (61) becomes

$$\mathrm{sgn}_0(f^+ \cdot n) = \mathrm{sgn}_0(f^- \cdot n) \quad \text{on } \Sigma_0. \tag{77}$$

We did not need the specific form of the extension $\hat{v}^\Sigma$ for the normal part, but it is required for treating the tangential parts. In fact, with

$$f^\pm(t,y) = \left[D_y \Phi^t_{t_0}(y)\right]^{-1} \left(v^\pm(t,x) - v^\Sigma(t,x)\right)$$

for $y \in \Sigma_0$ and $x = \Phi^t_{t_0}(y) \in \Sigma(t)$, condition (60) implies

$$\begin{aligned} f^\pm(t,y) &= \left[D_y \Phi^t_{t_0}(y)\right]^{-1} \left\langle v^\pm(t,x) - v^\Sigma(t,x), n_\Sigma(t,x)\right\rangle \, n_\Sigma(t,x) \\ &= \left\langle v^\pm(t,x) - v^\Sigma(t,x), n_\Sigma(t,x)\right\rangle \left[D_y \Phi^t_{t_0}(y)\right]^{-1} n_\Sigma(t,x) \\ &= \left\langle v^\pm(t,x) - v^\Sigma(t,x), n_\Sigma(t,x)\right\rangle \, n_{\Sigma(t_0)}(y) \end{aligned}$$

by (35). Consequently, condition (59) becomes

$$f^+_\| = f^-_\| = 0 \quad \text{on } \Sigma_0. \tag{78}$$

Step 3. *Reduction to* $\Sigma \equiv \mathbb{R}^{n-1} \times \{0\}$.

By a translation and a rotation, we may assume $x_0 = 0$ and $n(0) = e_n$, the nth Cartesian base vector. We are only interested in a local result, hence may assume that Σ_0 is a graph over $\mathbb{R}^{n-1}$ for a height function h, i.e.

$$\Sigma_0 = \{x = (x', x_n) : x_n = h(x')\} \tag{79}$$

with the notation $x' = (x_1, \ldots, x_{n-1})$. Consider the nonlinear transformation

$$x = \begin{bmatrix} x' \\ x_n \end{bmatrix} \to H(x) = \begin{bmatrix} x' - x_n \nabla_{x'} h(x') / \left(1 + \|\nabla_{x'} h(x')\|^2\right)^{1/2} \\ h(x') + x_n / \left(1 + \|\nabla_{x'} h(x')\|^2\right)^{1/2} \end{bmatrix}. \tag{80}$$

For sufficiently small $\varepsilon, r > 0$, H is a diffeomorphism from

$$[\mathbb{R}^{n-1} \times (-\varepsilon, \varepsilon)] \cap B_r(0)$$

onto its image

$$\mathcal{N} := H([\mathbb{R}^{n-1} \times (-\varepsilon, \varepsilon)] \cap B_r(0)),$$

which is a neighbourhood of $0 \in \mathbb{R}^n$. Given any solution $y(\cdot)$ of (74) starting at $x_0 = 0 \in \Sigma_0$, this solution stays inside $\mathcal{N}$ for $t \in (-\delta, \delta)$, where $\delta > 0$ can be chosen independently of the solution due to the local boundedness of f. The coordinate transformation induced by H yields an a.c. function $x(\cdot)$ via

$$y(t) = H(x(t)), \tag{81}$$

which is an a.c. solution of

$$\dot{x}(t) = g\,(t, x(t)) \quad \text{on } J_\delta := (-\delta, \delta),\ x(0) = 0, \tag{82}$$

where $g : J_\delta \times \big([\mathbb{R}^{n-1} \times (-\varepsilon, \varepsilon)] \cap B_r(0)\big) \to \mathbb{R}^n$ is given as

$$g(t,x) = \begin{cases} g^+(t,x) & \text{if } x_n \geqslant 0 \\ g^-(t,x) & \text{if } x_n < 0 \end{cases} \tag{83}$$

with $g^\pm$ given by

$$g^\pm(t,x) = H'(x)^{-1} f^\pm(t, H(x)) \quad \text{for } x \in \mathbb{R}^n_\pm.$$

Note that the specific definition of g as g^+ for $x_n = 0$ in (83) is arbitrary and the concrete choice of the values there plays no role. Note also that $g^+ : J_\delta \times \mathbb{R}^n_+ \to \mathbb{R}^n$ and $g^- : J_\delta \times \mathbb{R}^n_- \to \mathbb{R}^n$ are jointly continuous and locally Lipschitz continuous in x, where $\mathbb{R}^n_\pm$ denote the closed halfspaces $\{x_n \geqslant 0\}$ and $\{x_n \leqslant 0\}$, respectively.

Evidently, $y \in \Sigma_0$ iff $x_n = 0$ and for such $y = (x', h(x'))$ we have

$$n(y) = \frac{1}{(1 + \|\nabla_{x'} h(x')\|^2)^{1/2}} \begin{bmatrix} -\nabla_{x'} h(x') \\ 1 \end{bmatrix} \quad \text{for } y = (x', h(x')). \tag{84}$$

Given $t \in J_\delta$, $x = (x', 0)$ and $y = H(x) \in \Sigma_0$, it holds that

$$\begin{aligned} \hat\rho^\pm(t,y) f^\pm(t,y) \cdot n(y) &= \hat\rho^\pm(t, H(x)) \left\langle H'(x) g^\pm(t,x), \begin{bmatrix} -\nabla_{x'} h(x') \\ 1 \end{bmatrix} \right\rangle (1 + \|\nabla_{x'} h(x')\|^2)^{-1/2} \\ &= \hat\rho^\pm(t, H(x)) \left\langle g^\pm(t,x), H'(x)^\mathsf{T} \begin{bmatrix} -\nabla_{x'} h(x') \\ 1 \end{bmatrix} \right\rangle (1 + \|\nabla_{x'} h(x')\|^2)^{-1/2}. \end{aligned}$$

Now note that

$$H'(x) = \left[\begin{array}{c|c} I_{n-1} & \begin{matrix} n_1(x', h(x')) \\ \vdots \\ n_{n-1}(x', h(x')) \end{matrix} \\ \hline \nabla_{x'} h(x')^\mathsf{T} & n_n(x', h(x')) \end{array}\right] \quad \text{for } x = (x', 0) \tag{85}$$

with $n(x', h(x')) = n(y)$ from (84), hence

$$H'(x)^\mathsf{T} \begin{bmatrix} -\nabla_{x'} h(x') \\ 1 \end{bmatrix} = (1 + \|\nabla_{x'} h(x')\|^2)^{1/2}\, e_n.$$

Consequently,

$$\hat\rho^\pm(t,y) f^\pm(t,y) \cdot n(y) = \tilde\rho^\pm(t,x) \left\langle g^\pm(t,x), e_n \right\rangle = \tilde\rho^\pm(t,x)\, g_n^\pm(t,x)$$

with

$$\tilde\rho^\pm(t,x) := \hat\rho^\pm(t, H(x)) \quad \text{for } t \in J_\delta,\ x \in [\mathbb{R}^{n-1} \times (-\varepsilon, \varepsilon)] \cap B_r(0). \tag{86}$$

This shows that the transmission condition (76) becomes

$$\tilde\rho^+(t,x) g_n^+(t,x) = \tilde\rho^-(t,x) g_n^-(t,x) \quad \text{for } t \in J_\delta,\ x_n = 0 \tag{87}$$

with locally Lipschitz continuous $\tilde\rho^\pm : J_\delta \times \mathbb{R}^n_\pm \to (0, \infty)$.

Concerning the transformed version of (78), observe that, for $x = (x', 0)$,

$$g^{\pm}(t, x) = H'(x)^{-1} f^{\pm}(t, H(x)) = H'(x)^{-1} \left(\lambda^{\pm}(t, x)\, n(H(x))\right)$$

with certain $\lambda^{\pm}(t, x) \in \mathbb{R}$ due to (78). Hence

$$g^{\pm}(t, x) = \lambda^{\pm}(t, x) H'(x)^{-1} n(H(x)).$$

Now note that

$$H'(x', 0) e_n = n(x', h(x'))$$

by (85) with $n(x', h(x')) = n(y)$ from (84). Therefore,

$$g^{\pm}(t, x) = \lambda^{\pm}(t, x) e_n,$$

which implies

$$g_k^{\pm}(t, x) = 0 \quad \text{for } t \in J,\ x_n = 0,\ k = 1, \ldots, n-1. \tag{88}$$

As the result of this step, we may assume that $\Sigma(t) \equiv \mathbb{R}^{n-1} \times \{0\}$ and the new (discontinuous) right-hand side g has the same regularity as v, i.e. the $g^{\pm}$ are continuous on $J \times \mathbb{R}^n_{\pm}$ and the $g^{\pm}(t, \cdot)$ are locally Lipschitz continuous on $\mathbb{R}^n_{\pm}$. Furthermore, g satisfies the conditions (87) and (88).

It remains to show that (82) is uniquely solvable to the right on $[t_0, t_0 + \delta]$ for some $\delta > 0$.

Step 4. *Local forward uniqueness for (82).*

The next argument exploits the physically motivated transmission condition (58), respectively (87). Let $x(\cdot)$ and $\overline{x}(\cdot)$ be two solutions of (82). We then let

$$\phi(t) = |\rho(t) x_n(t) - \overline{\rho}(t) \overline{x}_n(t)| + \|x_{\|}(t) - \overline{x}_{\|}(t)\| \quad \text{on } J, \tag{89}$$

where $\|\cdot\|$ denotes the Euclidean norm, $x_{\|} = (x_1, \ldots, x_{n-1}, 0)$ is the tangential part of x,

$$\rho(t) = \begin{cases} \tilde{\rho}^{+}\left(t, x^{\Sigma}(t)\right) & \text{if } x_n(t) \geqslant 0 \\ \tilde{\rho}^{-}\left(t, x^{\Sigma}(t)\right) & \text{if } x_n(t) < 0 \end{cases} \tag{90}$$

with $\tilde{\rho}^{\pm}$ from (86) and

$$x^{\Sigma}(t) = \frac{1}{2}\left(x_{\|}(t) + \overline{x}_{\|}(t)\right). \tag{91}$$

Let $\overline{\rho}(t)$ be defined analogously, i.e.

$$\overline{\rho}(t) = \begin{cases} \tilde{\rho}^{+}\left(t, x^{\Sigma}(t)\right) & \text{if } \overline{x}_n(t) \geqslant 0 \\ \tilde{\rho}^{-}\left(t, x^{\Sigma}(t)\right) & \text{if } \overline{x}_n(t) < 0. \end{cases} \tag{92}$$

Evidently, $\phi(\cdot)$ is locally Lipschitz continuous, hence ϕ is a.c. and a.e. differentiable on J. Let

$$J_0 = \{t \in J : \rho'(t), \overline{\rho}'(t), x'(t), \overline{x}'(t)\, \text{exist}\}. \tag{93}$$

We are going to show that $\phi' \leqslant K\phi$ a.e. on J for some $K > 0$ and it suffices to show this a.e. on J_0. We distinguish four different cases, where we start by considering $\tau \in J_0$ such that

$x_n(\tau) < 0$, $\overline{x}_n(\tau) < 0$. Then $x_n(t) < 0$, $\overline{x}_n(t) < 0$ in a neighbourhood of τ, hence $\rho(t) = \overline{\rho}(t)$ there. This implies

$$\phi(t) = \rho(t)|x_n(t) - \overline{x}_n(t)| + \|x_\|(t) - \overline{x}_\|(t)\| \quad \text{near } \tau,$$

hence

$$|\phi'(t)| \leqslant |\rho'(t)|\,|x_n(t) - \overline{x}_n(t)| + \rho(t)|g_n(t, x(t)) - g_n(t, \overline{x}(t))| + \|g_\|(t, x(t)) - g_\|(t, \overline{x}(t))\|.$$

Consequently, using the Lipschitz continuity of g and $\|x\| \leqslant |x_n| + \|x_\|\|$,

$$|\phi'(t)| \leqslant \left(\left|\frac{\rho'(t)}{\rho(t)}\right| + L\left(1 + \frac{1}{\rho(t)}\right)\right)\rho(t)|x_n(t) - \overline{x}_n(t)| + L(1 + \rho(t))\|x_\|(t) - \overline{x}_\|(t)\|, \tag{94}$$

and therefore

$$\phi'(\tau) \leqslant K\,\phi(\tau) \tag{95}$$

with

$$K := \max_J \left(\left|\frac{\rho'}{\rho}\right| + L\left(\rho + 1 + \frac{1}{\rho}\right)\right); \tag{96}$$

note that $\rho(\cdot)$ is (locally) bounded from below by some $\alpha > 0$.

Next, we consider $\tau \in A := \{t \in J_0 : x_n(t) \geqslant 0, \overline{x}_n(t) \geqslant 0\}$, where it suffices to consider those points τ which are points of Lebesgue density of A. Given such τ, we have

$$\phi'(\tau) = \lim_{k\to\infty} \frac{\phi(t_k) - \phi(\tau)}{t_k - \tau}$$

for every sequence $t_k \to \tau$ with $t_k \neq \tau$. Since τ is a point of Lebesgue density of A, we find such a sequence (t_k) in A. Then

$$\phi(t_k) = \rho(t_k)|x_n(t_k) - \overline{x}_n(t_k)| + \|x_\|(t_k) - \overline{x}_\|(t_k)\|,$$

since $\rho(t_k) = \overline{\rho}(t_k)$. Hence $\phi'(t)$ can be estimated in the same way as above, i.e. (95) holds also for such τ.

In the remaining two cases, the two solutions are assumed to run in different phases. Since both cases can be treated in exactly the same way, we only consider $\tau \in B := \{t \in J_0 : x_n(t) \geqslant 0, \overline{x}_n(t) < 0\}$. In fact, it suffices to consider points τ of Lebesgue density of B. In the considered case, we have

$$\phi(t) = \rho(t)x_n(t) - \overline{\rho}(t)\overline{x}_n(t) + \|x_\|(t) - \overline{x}_\|(t)\| \quad \text{for } t \in B. \tag{97}$$

Hence, since $\phi'(\tau)$ exists and can be obtained from difference quotients with $t_k \in B$, we obtain

$$\begin{aligned}\phi'(\tau) = {} & \frac{\rho'(\tau)}{\rho(\tau)}\rho(\tau)x_n(\tau) - \frac{\overline{\rho}'(\tau)}{\overline{\rho}(\tau)}\overline{\rho}(\tau)\overline{x}_n(\tau) \\ & + \rho(\tau)g_n^+(\tau, x(\tau)) - \overline{\rho}(\tau)g_n^-(\tau, \overline{x}(\tau)) + \frac{\mathrm{d}}{\mathrm{d}t}\|x_\|(\tau) - \overline{x}_\|(\tau)\|.\end{aligned} \tag{98}$$

By means of (87), we have

$$\rho(\tau)g_n^+(\tau, x(\tau)) - \overline{\rho}(\tau)g_n^-(\tau, \overline{x}(\tau))$$
$$= \rho(\tau)\left(g_n^+(\tau, x(\tau)) - g_n^+\left(\tau, x^\Sigma(\tau)\right)\right) - \overline{\rho}(\tau)\left(g_n^-(\tau, \overline{x}(\tau)) - g_n^-\left(\tau, x^\Sigma(\tau)\right)\right).$$

Using the Lipschitz continuity of g as well as $\|x(t) - x^\Sigma(t)\| \leqslant \|x(t) - \overline{x}(t)\|$ and the corresponding inequality for $\|\overline{x}(t) - x^\Sigma(t)\|$, equation (98) implies

$$|\phi'(\tau)| \leqslant \left(\left|\frac{\rho'(\tau)}{\rho(\tau)}\right| + \left|\frac{\overline{\rho}'(\tau)}{\overline{\rho}(\tau)}\right|\right) |\rho(\tau)x_n(\tau) - \overline{\rho}(\tau)\overline{x}_n(\tau)| + L(1 + 2|\rho|_\infty)\|x(\tau) - \overline{x}(\tau)\|;$$

recall that $-\overline{x}_n(\tau) > 0$. Splitting $x(\tau)$ and $\overline{x}(\tau)$ into their normal and tangential parts, this yields (95) with

$$K := \max_J \left(\left|\frac{\rho'}{\rho}\right| + \left|\frac{\overline{\rho}'}{\overline{\rho}}\right| + L\left(1 + |\rho|_\infty + |\overline{\rho}|_\infty\right)\left(1 + \frac{1}{\rho} + \frac{1}{\overline{\rho}}\right)\right). \tag{99}$$

Recall that both $\rho(\cdot)$, $\overline{\rho}(\cdot)$ are (locally) bounded from below by some $\alpha > 0$.

Consequently, inequality (95) holds a.e. on J with a common $K > 0$, thus $\phi(t) = 0$ on J by Gronwall's lemma, since $\phi(0) = 0$. This means

$$\rho(t)x_n(t) = \overline{\rho}(t)\overline{x}_n(t) \quad \text{and} \quad x_\|(t) = \overline{x}_\|(t) \quad \text{on } J. \tag{100}$$

To finish this step of the proof, consider the energy functional

$$\psi(t) = \frac{1}{2}\|x(t) - \overline{x}(t)\|^2 \quad \text{for } t \in J.$$

Evidently, using $(100)_2$, we get

$$\psi'(t) = (x_n(t) - \overline{x}_n(t))\,(g_n(t, x(t)) - g_n(t, \overline{x}(t))) \quad \text{on } J. \tag{101}$$

By $(100)_1$ and the non-degeneracy of $\rho(\cdot)$ and $\overline{\rho}(\cdot)$, both $x_n(\cdot)$ and $\overline{x}_n(\cdot)$ run either in $\mathbb{R}^n_+$ or in $\mathbb{R}^n_-$. Hence the second argument of g_n in (101) is always in the same halfspace, i.e. g_n is either g_n^+ or g_n^-. By the Lipschitz continuity of $g^\pm$ on $J \times \mathbb{R}^n_\pm$, we obtain $\psi'(t) \leqslant 2L\psi(t)$ on J. Hence $\psi(t) = 0$ on J, i.e. $x(t) = \overline{x}(t)$ on J which ends the proof.

Step 5. *Local backward uniqueness.*

The vector field $\tilde{v} := -v$ satisfies all assumptions of theorem 1 if v^Σ is replaced by $-v^\Sigma$. Hence (57) with $-v$ instead of v and the backward moving $\Sigma(\cdot)$ has unique local forward solvability. Reversing time, this yield unique local backward solvability of the original problem. □

7. Outlook and open problems

It is unclear whether in theorem 1 the condition (58) of mass-conservation type, i.e.

$$\rho^+(v^+ - v^\Sigma)\cdot n_\Sigma = \rho^-(v^- - v^\Sigma)\cdot n_\Sigma \quad \text{on } \mathcal{M}$$

with locally Lipschitz functions $\rho^\pm : \mathcal{M} \to (0, \infty)$, can be replaced by the transversality-type condition (61), i.e.

$$\mathrm{sgn}_0\left((v^+ - v^\Sigma)\cdot n_\Sigma\right) = \mathrm{sgn}_0\left((v^- - v^\Sigma)\cdot n_\Sigma\right) \quad \text{on } \mathcal{M}.$$

Since $\rho^\pm$ is only needed on $\mathcal{M}$, it is tempting to use the locally Lipschitz function $\big((v^+ - v^\Sigma)\cdot n_\Sigma\big)_{|\mathcal{M}}$ as a substitute for ρ^- and vice versa, exploiting the trivial equality

$$(v^+ - v^\Sigma)\cdot n_\Sigma\,(v^- - v^\Sigma)\cdot n_\Sigma = (v^- - v^\Sigma)\cdot n_\Sigma\,(v^+ - v^\Sigma)\cdot n_\Sigma \quad \text{on } \mathcal{M}.$$

But $(v^\pm - v^\Sigma)\cdot n_\Sigma(t,x) = 0$ is possible on $\mathcal{M}$ and precisely these points are relevant: at the remaining points, condition (15) of theorem 2 in chapter 2, section 10 of [17] is satisfied after transformation to fixed, planar interface $\mathbb{R}^{n-1}\times\{0\}$, cf. the introduction and section 5.

Condition (58) contains significantly more information than (61). Note, e.g., that for differentiable velocity fields $v^\pm$, v^Σ, condition (58) implies

$$\rho^+\nabla_\Sigma\left((v^+ - v^\Sigma)\cdot n_\Sigma\right) = \rho^-\nabla_\Sigma\left((v^- - v^\Sigma)\cdot n_\Sigma\right)$$

at those $(t,x)\in\mathcal{M}$ where $(v^+ - v^\Sigma)\cdot n_\Sigma = (v^- - v^\Sigma)\cdot n_\Sigma = 0$.

The **first open problem** therefore is:

- Prove or disprove theorem 1 with (58) replaced by (61), possibly under appropriate additional assumptions.

In the present paper we consider the ODE-system associated with two-phase flows under the no-slip condition (21) at the interface, i.e.

$$[v_\parallel] = 0 \quad \text{on } \Sigma.$$

While this is the standard setting considered in the overwhelming majority of publications, it is not a necessary model assumption. Indeed, condition (21) is just one possible closure, relating the difference between the tangential parts of the one-sided bulk and interface velocities with the respective one-sided bulk stress in such a way that the second law of thermodynamics holds, i.e. such that a slippage of the two bulk fluids against each other generates a *positive* entropy production. If a linear closure is used, the one-sided Navier slip conditions

$$(v^\pm - v^\Sigma)_\parallel + \alpha^\pm(\mathcal{S}^\pm n^\pm)_\parallel = 0 \quad \text{with } \alpha^\pm \geqslant 0$$

result, where $n^\pm$ are the outer unit normals to $\Omega^\pm$; see [4, 6]. While this requires a proper definition of the interfacial velocity v^Σ, a non-trivial task, it implies the jump condition

$$(v^+ - v^-)_\parallel = \alpha^-(\mathcal{S}^- n^-)_\parallel - \alpha^+(\mathcal{S}^+ n^+)_\parallel,$$

showing that a physically sound condition for a possible tangential velocity jump is required in order to enforce the second law of thermodynamics. This should lead to indications for formulating conditions for admissible tangentially discontinuous velocity fields. Let us note in passing that tangential jumps of the right-hand side of discontinuous ODEs are common in applications from control theory, where this can lead to solutions which slide tangentially along the hypersurface of discontinuity with a velocity which is a convex combination of the two adjacent bulk velocity vectors. More on this topic can be found, e.g., in [13, 17]. Notice, however, that the approach in [17] to deal with a non-vanishing so-called discontinuity vector $h(t,x) := v^+(t,x) - v^-(t,x)$ on $\mathcal{M}$ consists in a nonlinear coordinate transformation which changes h to become aligned with the normal direction. But this transformation mixes normal and tangential components of v, thus destroying the mass-conservation type condition (58).

The **second open problem** therefore is:

- Generalize theorem 1, allowing for an appropriate class of two-phase flow velocities v without imposing $v_\parallel^+ = v_\parallel^-$.

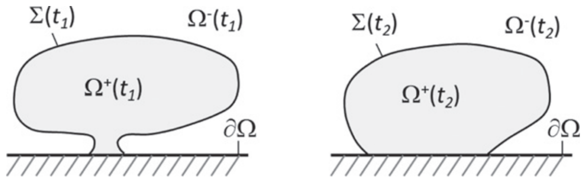

Figure 3. Sketch of a droplet wetting a solid support ($t_1 < t_2$).

In the present paper we consider ODE-systems with moving discontinuity surfaces which are closed hypersurfaces, i.e. without boundary. While this covers a large class of two-phase flows, in wetting applications the fluid interface has contact with parts of the boundary of Ω, typically at a solid wall. Such a situation is illustrated in figure 3, showing a droplet which is sketched at time t_1 shortly after its initial contact with the solid support and at a later instant t_2, already closer to an equilibrium state which would be a spherical cap with material dependent contact angle in cases without external forces such as gravity.

A generalization of theorem 1 to such a case with moving contact lines requires to extend the setup to discontinuous ODEs on closed sets. Indeed, one needs to consider the flow on the closed set $\overline{\Omega}$, allowing for $x_0 \in \partial\Omega$. The approach via set-valued regularizations would still work, since sufficiently general existence results for differential inclusions on closed sets are available; see [2, 3, 14] as well as [5] in cases with time-dependent flow domain. The velocity fields then need to obey certain sub-tangentiality constraints which will be fulfilled if $\partial\Omega$ ('the support') is non-permeable, since then v is tangential to $\partial\Omega$ there. The principle idea to impose and exploit a jump condition of mass-conservation type still seems applicable, but a proof—if it works—will be technically even more involved then the one given above. We leave this as future work, hence close with

The **third open problem**:

- Extend theorem 1 to cover two-phase flows in which the discontinuity surface touches the domain boundary.

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