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Time-Invariant Control in LQ Optimal Tracking: An Alternative to Output Regulation

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Abstract: We propose a new time-invariant control for linear quadratic tracking problems with references and disturbances generated by linear exo-systems. The control consists of a static feedback and a static pre-filter similar as in output regulation theory (ORT). Instead of forcing the tracking error to converge to zero, a tolerated steady-state error is balanced against the necessary input-energy via a quadratic cost. For the first time in this context, we deduce a time-invariant control from algebraic equations such that necessary optimality conditions are satisfied on infinite horizons. Then, we prove strong optimality for bounded exo-system states. Hence, any other steady-state solution will lead to infinite additional cost. On finite horizons and for arbitrary exo-systems, we prove that our control is an agreeable plan as it approximates the computational expensive, time-varying optimal control of any suitably large horizon. Since our control applies for any initial conditions of the plant and the exo-system, it is well suited for a practical resource-efficient implementation. In this regard, a presented algorithm allows for an easy to carry out control design. Finally, an industrial application indicates the unified treatment of square, under- and over-actuated systems by our approach in contrast to ORT.

Keywords: Linear Quadratic, Optimal Tracking, Linear Systems, Disturbances, Time-Varying References, Under-Actuated, Over-Actuated, Agreeable Plans, Output Regulation

1. INTRODUCTION

Beside stability and robustness, output tracking of given references is a key problem in control of dynamical systems. At present, the growing interest in autonomous vehicles, e.g. Kaminer et al. (1998), and multi-agent systems, e.g. Wieland et al. (2011), offers a wide field of application and emphasizes the rising importance of suitable methods. In this domain, we consider linear disturbed systems like

$$\dot{x} = Ax + Bu + E_d x_{\text{exo}}, \quad (1a)$$
$$y = Cx + D_d x_{\text{exo}} \quad (1b)$$

with states $x \in \mathbb{R}^n$, inputs $u \in \mathbb{R}^m$, outputs $y \in \mathbb{R}^p$ and exogenous states $x_{\text{exo}} \in \mathbb{R}^{n_{\text{exo}}}$ given by the exo-system

$$\dot{x}_{\text{exo}} = S x_{\text{exo}}, \quad (2a)$$
$$w = O x_{\text{exo}} \quad (2b)$$

with references $w \in \mathbb{R}^p$. Typically, asymptotic output tracking $\lim_{t\to\infty} \|y - w\| = 0$ is achieved – output regulation of the tracking error $y - w$, see Trentelman et al. (2001). However, output regulation may be an infeasible problem as the *regulator equations* given by Francis (1977) might be unsolvable. Or, though possible, it may be prohibitively expensive in terms of the required input-energy. Thus, we regard the minimization of the quadratic cost

$$J_T(u(\cdot)) = \tfrac{1}{2} \int_0^T (y - w)^\mathsf{T} Q_y (y - w) + u^\mathsf{T} R u \; \mathrm{d}t \quad (3)$$

with $Q_y \succeq 0$ and $R \succ 0$ instead. This leads to a linear quadratic tracking problem (LQTP). It allows for keeping the tracking error $y - w$ as small as required or possible, i.e. $\lim_{t\to\infty} \|y - w\| \neq 0$, while balancing the necessary input-energy for an efficient operation. On the downside, solving the LQTP for finite T goes along with a higher computational expense. That is, the **time-variant** optimal control $u_T^*(\cdot)$ requires additional memory and processing, see Fig. 1, left. On infinite horizons, $T \to \infty$, the complexity reduces referring to Fig. 1, middle. But this will lead to an unbounded cost for any $u(\cdot)$ in general. Thus, a discussion on optimality is not meaningful, cf. (Anderson and Moore, 2007, Sec. 4.3). To avoid this problem, a different LQTP with a bounded cost is often suggested, e.g. Karimi-Ghartemani et al. (2011). It results in $\lim_{t\to\infty} \|y - w\| = 0$ and, hence, in a rather optimal transition problem. By applying optimality definitions suited for $T \to \infty$, Artstein and Leizarowitz (1985); Leizarowitz (1986) proved *overtaking* optimality for a control structure as Fig. 1, middle, in the absence of disturbances.

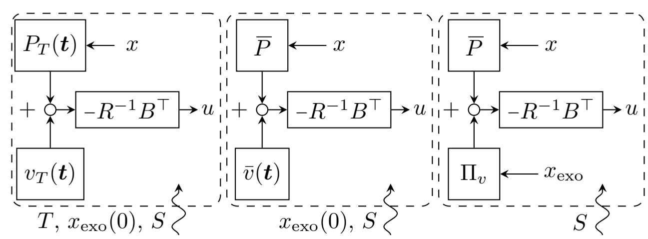

Fig. 1. Control structures and required prior information in LQT: finite T (left) and infinite T (middle), cf. Anderson and Moore (2007); our new approach (right)

From an implementation point of view, a **time-invariant** control is clearly preferable. In output regulation of over-actuated systems, Krener (1992) proposed a paramet-

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ric optimization problem intending to find an "optimal" steady-state solution. However, it is sensitive to the chosen coordinates of (2a) and can be seen to result in suboptimal solutions. In the context of LQTPs, a time-invariant control emerged rather naturally for constant references in Willems and Mareels (2004). With respect to arbitrary exo-systems (2a), the only analogous result known to us is given by the limit solution of the optimal servo problem in Kreindler (1969). To our surprise, this result did not draw much attention and may even be considered unknown by now. A reason may be that necessary optimality conditions are not satisfied and the control performance is not investigated by any means. We achieve converse results in the LQT framework and will prove Kreindler (1969) wrong. Meaning that we will obtain a control in the LQT framework which is not time-variant, does not need to be preprogrammed and exists under weaker conditions.

In view of this discussion, our **main results** with respect to the systems (1a), (2a) and the cost (3) are:

1) A **time-invariant** control $\bar{u}(\cdot)$ consisting of a static feedback and a static pre-filter (see Fig. 1, right)
2) which can be obtained from a simple design algorithm with minimal prior knowledge
3) such that necessary optimality conditions hold on infinite horizons, i.e. $J_\infty(\cdot) = \lim_{T\to\infty} J_T(\cdot)$,
4) $\bar{u}(\cdot)$ is *strongly* optimal for bounded exo-systems
5) and $\bar{u}(\cdot)$ approximates the optimal control $u_T^*(\cdot)$ on finite horizons $[0, T]$ for arbitrary exo-systems.

To begin with, we will give the formulation of LQTPs, assumptions, definitions and preliminary results in Section 2.

In Section 3.1, our **result 1)** is presented. Under an eigenvalue condition, our control follows by the limit $\bar{u}(\cdot) = \lim_{T\to\infty} u_T^*(\cdot)$ of the finite horizon optimal control. In contrast to Kreindler (1969), we achieve **3)** and are able to investigate the optimality of transients and stationary behavior. This involves an explicit derivation of the unique *finitely*-optimal steady-state which we are able to give in contrast to the relevant literature. Besides, we introduce a modification $J_T^\alpha(\cdot)$ of (3) which allows us to specify the rate of convergence α to the finitely-optimal steady-state.

In Section 3.2, we consider **result 4)** by means of an equivalent LQTP with a bounded cost. In the sense of Willems and Mareels (2004), this allows us to give a "rigorous proof of optimality" for bounded time-varying states of (2a) for the first time. As a consequence, any stationary solution of (1a) differing from the finitely-optimal steady-state will lead to a infinitely higher cost (3).

From a practical point of view, the horizon is more likely finite but very large and unknown. In this regard, we show **result 5)** in Section 3.3. We prove that our control satisfies the desirable concept of agreeability. This helps us give a quantitative measure for the approximation quality. If the rate of convergence α is chosen suitably large we will derive indeed that the approximation is as close as desired. Meaning that, with respect to the modified cost $J_T^\alpha(\cdot)$, $\bar{u}_\alpha(\cdot)$ approximates the corresponding optimal control $u_{T,\alpha}^*(\cdot)$ on finite horizons $[0,T]$ as close as desired.

In spite of the technical results, a simple comprehensive design algorithm **2)** is provided in Section 4. It emphasizes that our control is easy to calculate and to implement despite its beneficial properties. These are shown by an illustrative example. An industrial application indicates that our approach applies to under-actuated systems. Besides special cases, a solution for these as Francis (1977) does not exist and explicit approaches like Davison and Davison (2011) are rare. Analogously, our result accounts for over-actuated systems which are of present interest, see the discussion and application under a decoupling constraint in Bernhard and Adamy (2017).

Mathematical notations:
We define constants $\kappa_j = \kappa_j(x(0), x_{\text{exo}}(0)) \in \mathbb{R}^{>0}$ applying to specific initial conditions of systems (1a), (2a) and $M_i \in \mathbb{R}^{>0}$. The zero matrix 0 and identity matrix I have appropriate dimensions. A symmetric matrix is positive (semi-) definite if $X \succ (\succeq) 0$. We denote the spectrum by $\sigma(X)$ and the i-th eigenvalue by $\lambda_i(X)$. The 2-norm of a vector and induced, submultiplicative spectral norm of a matrix are denoted by $\|\cdot\|$. Vector e_i of appropriate dimensions has a one in the i-th row, zeros otherwise.

2. PRELIMINARIES

2.1 Formulation of LQTPs and present assumptions

A main focus of our contribution lies on the two LQTPs:
Linear Quadratic Tracking Problem A. With respect to the cost functional equivalent to (3):

$$J_T(u) = \tfrac{1}{2}\int_0^T \begin{bmatrix} x \\ x_{\text{exo}} \end{bmatrix}^\mathsf{T} \begin{bmatrix} Q & S_w \\ S_w^\mathsf{T} & Q_w \end{bmatrix} \begin{bmatrix} x \\ x_{\text{exo}} \end{bmatrix} + u^\mathsf{T} R u \,\mathrm{d}t \quad (4)$$

with $Q = C^\mathsf{T} Q_y C \succeq 0$, $S_w = C^\mathsf{T} Q_y (D_d - O)$ and $Q_w = (D_d - O)^\mathsf{T} Q_y (D_d - O)$, find the optimal control

1) $u_T^*(\cdot)$ such that $J_T(u_T^*)$
2) $u^*(\cdot)$ such that $J_\infty(u^*) = \lim_{T\to\infty} J_T(u^*)$

is minimal with respect to the disturbed linear dynamics (1a) with initial value $x(0) = x_0$, the exo-dynamics (2a) with $x_{\text{exo}}(0) = x_{\text{exo},0}$ and the references (2b).

We consider LQTP. A 1) and 2) under present standard
Assumption 1. The pair (A, B) is controllable.[1] For some Λ such that $Q = \Lambda^\mathsf{T}\Lambda$, the pair (Λ, A) is observable.[1]

2.2 Preliminary results in LQT on (in)finite horizons

In this section, we recap preliminary results in optimal control theory. We start with a formal definition of optimality followed by necessary conditions. Following Leizarowitz (1986), a control $u(\cdot)$ is *admissible* if it is measurable and integrable on finite intervals. Then, we can give:

Definition 2. (Strong Optimality). If the limit $J_\infty(u^*) = \lim_{T\to\infty} J_T(u^*)$ is finite and $\exists T(u) \geq 0$ for any admissible $u(\cdot)$ such that $\forall T \geq T(u)$: $J_T(u) \geq J_T(u^*)$. Then, the pair (x^*, u^*) is a *strongly optimal* solution of (1a) for a given x_0 in the sense of a minimal cost $J_\infty(u^*)\big|_{x_0}$.

The definition above is given by Carlson (1990) and defines the strongest optimality concept in the domain of

[1] Due to limited space, we cannot present the extensions of our proofs for stabilizability and detectability assumptions here. However, all our results also hold under these weakened assumptions.

infinite horizons, e.g. it implies overtaking optimality. It also directly corresponds to the optimality definition for a finite horizon, i.e. T is fixed in Definition 2. The necessary optimality conditions (NC) for the linear quadratic case and infinite horizons can be adopted from Halkin (1974):

Lemma 3. (Necessary Conditions). Suppose (x^*, u^*) is an optimal solution of LQTP. A 2). Regarding the Hamiltonian $H(x, u, \phi, t) = L(x, u, t) + \phi^\top(t) f(x, u, t)$ with costate $\phi(t) : [0, \infty) \to \mathbb{R}^n$, system dynamics $\dot{x} = f(x, u, t)$ as in (1a) and integrand $L(x, u, t)$ of (4), it necessarily holds

(NC1) $\frac{\partial H}{\partial u}(x^*, u, \phi, t)\big|_{u=u^*} = 0.$
(NC2) $-\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial H}{\partial x}(x, u^*(t), \phi, t)\big|_{x=x^*}$ for some $\phi(0) \in \mathbb{R}^n$.

In order to prove optimality of a control satisfying (NC1-2), we follow the discussion on sufficiency in (Athans and Falb, 1966, Ch. 5). Based on the *calculus of variations*, we introduce $u(\cdot) = u^*(\cdot) + \epsilon\delta u(\cdot)$ and $x(\cdot) = x^*(\cdot) + \epsilon\delta x(\cdot)$ for (x^*, u^*) solving (1a) and some $\epsilon \in \mathbb{R}$. It holds $\dot{\delta x} = A\delta x + B\delta u$, $\delta x(0) = 0$ by superposition. With the i-th variation $\delta^i J_T(u^*, \delta u)$, the Taylor series of $J_T(u)$ in ϵ at $\epsilon = 0$ reads

$$J_T(u^* + \epsilon\delta u) = J_T(u^*) + \epsilon\delta J_T(u^*, \delta u) + \frac{\epsilon^2}{2}\delta^2 J_T(u^*, \delta u). \quad (5)$$

Notice that the remainder vanishes due to the quadratic nature of $J_T(u)$ and linearity of (1a); hence, we have $\delta^i J_T(u^*, \delta u) = 0 \ \forall i > 2$. As $T \to \infty$, the relevant variations are $\delta J_\infty(u^*, \delta u) = \lim_{T\to\infty} -\phi(T)\delta x(T)$ and

$$\delta^2 J_\infty(u^*, \delta u) = \int_0^\infty \delta x^\top Q \delta x + \delta u^\top R \delta u \ \mathrm{d}t. \quad (6)$$

For finite T, it is well known that the first variation satisfies $\delta J_T(u_T^*, \delta u) = 0$ for the optimal control $u_T^*(\cdot)$. In regard of (5), the condition $\delta^2 J_T(u_T^*, \delta u) \geq 0$ for any $\delta u(\cdot) \not\equiv 0$ is sufficient for optimality. On the contrary, it does not necessarily hold $\delta J_\infty(u^*, \delta u) = 0$ for infinite T, since the transversality condition $\phi(T) = 0$ is not necessary anymore. Indeed, this constitutes the key problem to show optimality for LQTP. A 2) since suitable sufficient conditions for our case do not exist. We solve this problem in Section 3.2 by a thorough individual analysis of (5).

In Section 3.3, we will consider the case of a large unknown finite horizon. Then, the concept of *agreeable plans* is useful. Let us denote a control $u_{t_0,t_1}(\cdot)$ defined on $[t_0, t_1]$, $J_{t_0,t_1}(\cdot)$ as its cost, as well as the corresponding optimal control $u^*_{t_0,t_1}(\cdot)$. A definition given in Carlson (1990) reads:

Definition 4. (Agreeability). A control $u(\cdot)$ and the associated solution $x(\cdot)$ of system (1a) are *agreeable* if for any $\theta \in \mathbb{R}^{\geq 0}$, there exists a control $u^*_{\theta,T}(\cdot)$ such that:

$$\lim_{T\to\infty} J_{0,\theta}(u)\big|_{x_0} + J_{\theta,T}(u^*_{\theta,T})\big|_{x(\theta)} - J_{0,T}(u^*_{0,T})\big|_{x_0} = 0. \quad (7)$$

On a suitably large finite horizon, an agreeable plan $u(\cdot)$ applied on $[0, \theta]$ constitutes a negligible increase of cost, if $u(\cdot)$ is finally replaced by the optimal $u^*_{\theta,T}(\cdot)$ on $[\theta, T]$.

Now, we proceed by regarding two optimal control problems which are rather standard. The first forms the basis of our analysis of the limit $\bar{u}(\cdot) = \lim_{T\to\infty} u_T^*(\cdot)$. The solution below extends a standard result of Anderson and Moore (2007) for our case of disturbed LTI-systems.

Lemma 5. The optimal control of LQTP. A 1) is given by

$$u_T^* = -R^{-1}B^\top (P_T(t)x + v_T(t)) \quad (8)$$

with positive definite $P_T(t) : [0, T] \to \mathbb{R}^{n\times n}$ obtained from the Riccati differential equation (RDE)

$$-\dot{P}_T = P_T A + A^\top P_T - P_T B R^{-1} B^\top P_T + Q \quad (9)$$

for $P_T(T) = 0$, and $v_T(t) : [0, T] \to \mathbb{R}^n$ obtained from

$$-\dot{v}_T = \left(A - BR^{-1}B^\top P_T(t)\right)^\top v_T + (P_T(t)E_d + S_w)\, x_{\text{exo}} \quad (10)$$

for $v_T(T) = 0$. By application of (8), the optimal cost is

$$J_{0,T}(u_T^*)\big|_{x_0} = \tfrac{1}{2}x_0^\top P_T(0)x_0 + x_0^\top v_T(0) + z_T \quad (11)$$

with z_T given by

$$z_T = \int_0^T \tfrac{1}{2}x_{\text{exo}}^\top Q_w x_{\text{exo}} - \tfrac{1}{2}v_T^\top BR^{-1}B^\top v_T + x_{\text{exo}}^\top E_d^\top v_T \ \mathrm{d}\tau.$$

Let us consider the second problem:

Optimal Control Problem B. For $\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$ with initial value $\tilde{x}(t_{k-1}) = \tilde{x}_{k-1}$ and fixed final value $\tilde{x}(t_k) = \tilde{x}_k$, minimize $\widetilde{J}_{k-1,k}(\tilde{u}) = \frac{1}{2}\int_{t_{k-1}}^{t_k} \tilde{x}^\top Q\tilde{x} + \tilde{u}^\top R\tilde{u} \ \mathrm{d}t$.

Its optimal cost will help us reformulate the unbounded cost of an infinite horizon OCP as the sum of finite addends: $\lim_{T\to\infty} \widetilde{J}_T(\tilde{u}) \Leftrightarrow \lim_{K\to\infty} \sum_{k=1}^K \widetilde{J}_{k-1,k}(\tilde{u})$. This is a popular procedure, e.g. Artstein and Leizarowitz (1985), and will be useful in the proof in Section 3.2. Suppose Asmp. 1 holds. The optimal cost of OCP. B is

$$\widetilde{J}_{k-1,k}(\tilde{u}_k^*) = \tfrac{1}{2}\tilde{x}_{k-1}^\top \widetilde{P}_k(0)\tilde{x}_{k-1} + \tfrac{1}{2}\left(\tilde{x}_k - \widetilde{\Phi}_k(\Delta_k, 0)\tilde{x}_{k-1}\right)^\top \cdot \widetilde{W}_{r,k}(\Delta_k, 0)^{-1}\left(\tilde{x}_k - \widetilde{\Phi}_k(\Delta_k, 0)\tilde{x}_{k-1}\right) \quad (12)$$

with $\Delta_k = t_k - t_{k-1}$, $\widetilde{P}_k(\tau) \succ 0$ defined on $\tau \in [0, \Delta_k]$ and given by an RDE (9) for $\widetilde{P}_k(\Delta_k) = 0$. We have the transition matrix $\widetilde{\Phi}_k(\tau, 0)$ of the closed-loop, $\widetilde{A} = A - BR^{-1}B^\top \widetilde{P}_k(\tau)$ and the reachability Gramian $\widetilde{W}_{r,k}(\tau, 0)$ of the pair $(\widetilde{A}, \widetilde{B})$ with $\widetilde{B}\widetilde{B}^\top = BR^{-1}B^\top$. This standard result can be deduced from (Bryson, 1975, p. 160 et seq.).

3. TIME-INVARIANT CONTROL FOR LINEAR QUADRATIC TRACKING

3.1 Infinite horizons: Time-invariant control

In this section, we derive our results 1) and 3). In this regard, we examine the limit $\bar{u}(\cdot) = \lim_{T\to\infty} u_T^*(\cdot)$ of the finite horizon solution $u_T^*(\cdot)$ given by Lemma 5. We show that the necessary conditions in Lemma 3 are satisfied in our case in contrast to Kreindler's (1969). Based on a separation, transient and steady-state optimality is studied. Then, we demonstrate how the rate of convergence α of the closed-loop dynamics can be independently specified. This section extends results of Bernhard and Adamy (2017).

The next theorem provides a time-invariant control $\bar{u}(\cdot)$ in regard of results 1) and 3).

Theorem 6. For any given $t \in \mathbb{R}$, it holds $\lim_{T\to\infty} P_T(t) = \overline{P} \succ 0$ with the algebraic Riccati equation (ARE)

$$\overline{P}A + A^\top \overline{P} - \overline{P}BR^{-1}B^\top \overline{P} + Q = 0 \quad (13)$$

and the closed-loop system matrix $\overline{A} = A - BR^{-1}B^\top \overline{P}$ is asymptotically stable. If

$$\operatorname{Re}\left(\lambda_i(\overline{A})\right) + \operatorname{Re}\left(\lambda_j(S)\right) < 0, \forall i, j \quad (14)$$

then $\bar{v}(t) = \lim_{T\to\infty} v_T(t) = \Pi_v x_{\text{exo}}(t)$ with Π_v given by the Sylvester equation

$$\Pi_v S = -\overline{A}^{\mathsf{T}} \Pi_v - \left(\overline{P} E_d + S_w\right). \quad (15)$$

As $T \to \infty$, the limit of the control law (8) yields

$$\bar{u} = -R^{-1} B^{\mathsf{T}} (\overline{P} x + \Pi_v x_{\text{exo}}) \quad (16)$$

with time-invariant feedback $K = R^{-1} B^{\mathsf{T}} \overline{P}$ and pre-filter $F = -R^{-1} B^{\mathsf{T}} \Pi_v$ leading to $\bar{u} = -Kx + F x_{\text{exo}}$. For the pair $(\bar{x}, \bar{u})$ solving (1a), necessary conditions (NC1) and (NC2) for infinite horizons are satisfied.

Proof. By virtue of Callier and Winkin (1992), for $\alpha < \left|\max_i \mathrm{Re}\left(\lambda_i(\overline{A})\right)\right|$ and a constant M_P as defined in the *mathematical notations* in Section 1, it follows

$$\left\|\widetilde{P}_T(t)\right\| = \left\|P_T(t) - \overline{P}\right\| \leq M_P e^{-2\alpha(T-t)}. \quad (17)$$

With Asmp. 1, $\overline{P}$ always exists and $\overline{A}$ is asymptotically stable since (Q, A) observable, Anderson and Moore (2007).

Regarding the second part, our non-standard proof is more involved. Anderson and Moore (2007) suppose that the limit $\bar{v}(t) = \lim_{T\to\infty} v_T(t)$ is given by the particular integral over $[t, \infty)$ of, in our case,

$$-\dot{\bar{v}} = \left(A - BR^{-1}B^{\mathsf{T}}\overline{P}\right)^{\mathsf{T}} \bar{v} + \left(\overline{P} E_d + S_w\right) x_{\text{exo}} \quad (18)$$

which they justify by $P_T(t) \to \overline{P}$, $T \to \infty$. This is an often adapted result, e.g. Artstein and Leizarowitz (1985). However, with regard to (17), $P_T(\tau) \to \overline{P}$ is clearly not uniformly in $\tau \in [t, T]$, $T \to \infty$ and the justification seems to be inadequate. In the sequel, we shed light on this where our proof is partially presented in Appendix A.

For the solution of (18), we introduce a different, geometric approach. In view of Sylvester equation (15), the existence of a unique Π_v is guaranteed by condition (14) because $\sigma(-\overline{A}) \cap \sigma(S) = \emptyset$, cf. Trentelman et al. (2001). Hence, any solution can be given by superposition, i.e. $\bar{v}(t) = e^{-\overline{A}^{\mathsf{T}} t}\eta + \Pi_v x_{\text{exo}}(t)$ for some $\eta \in \mathbb{R}^n$. By the derivation in Appendix A, it results

$$\left\|v_T(t) - \bar{v}(t)\right\| \leq \left\|e^{-\overline{A}^{\mathsf{T}} t}\eta\right\| + \kappa_v e^{\alpha t} e^{-(\alpha - \alpha_S)T} \quad (19)$$

with $\alpha_S > \max_j \mathrm{Re}(\lambda_{\text{exo},j}(S))$. Since $\left\|e^{-\overline{A}^{\mathsf{T}} t}\right\| \geq M_5 e^{\alpha t}$, we choose $\eta = 0$ obviously. Now, if (14) holds, a suitably small $\alpha_S < \alpha$ can be found such that convergence immediately results for $T \to \infty$ and any given $t \in \mathbb{R}$. Let the pairs $(\bar{x}, \bar{u})$ and (x_T^*, u_T^*) with (8) denote the solutions of (1a) for a given $x_0 \in \mathbb{R}^n$. We have $\lim_{T\to\infty} x_T^*(t) = \bar{x}(t)$ and $\lim_{T\to\infty} u_T^*(t) = \bar{u}(t)$ for any given $t \in \mathbb{R}$, see proof of Lemma 11 for details.

Based on the sweep method $\phi = \overline{P}\bar{x} + \Pi_v x_{\text{exo}}$, e.g. Bryson (1975), it is easy to verify that $\bar{u}$ satisfies (NC1) while (NC2) is guaranteed on the basis of the ARE (13) and the Sylvester equation (15). □

As we see, in our new approach, necessary optimality conditions are satisfied. On the contrary, notice that (NC2) is not met in Kreindler (1969) which prohibits any optimality analysis. The condition (14) was proven by (Kreindler, 1969, p. 468) to be necessary and sufficient for the existence of their proposed control law. In our case, this is true for the convergence $\lim_{T\to\infty} v_T(t) = \bar{v}(t)$. However, our control $\bar{u}(\cdot)$ may exist when those in Kreindler (1969) or (Anderson and Moore, 2007, Sec. 4.3) do **not**. This happens when condition (14) is violated but $\sigma(-\overline{A}) \cap \sigma(S) = \emptyset$ holds, which is sufficient for the existence of $\bar{u}(\cdot)$.

In the sequel, we analyze $(\bar{x}, \bar{u})$ and give a brief first optimality study. Based on the **explicit** steady-state $x_s^*(t) = \Pi_x x_{\text{exo}}(t)$ with

$$\Pi_x S = (A - BK)\,\Pi_x + BF + E_d, \quad (20)$$

we apply a state transformation $\bar{x} = \tilde{x} + \Pi_x x_{\text{exo}}$ to the closed-loop of (1a) with control law (16). This clearly divides transients $\tilde{x}$ and stationary behavior x_s^* which we will analyze individually. Regarding (16), we have

$$\bar{u} = -K\tilde{x} + \Gamma_s x_{\text{exo}} = \tilde{u}^* + u_s^* \quad (21)$$

with $\Gamma_s = -K\Pi_x + F$. Notice that Π_x is uniquely given if $\sigma(A - BK) \cap \sigma(S) = \emptyset$. But, this can be assumed without loss of generality (wlog) since the stable part of (2a) does not contribute to the stationary behavior and can be "deleted" (Trentelman et al., 2001, Sec. 9.1, p. 198).

Regarding the transients $\tilde{x}$, we propose the linear quadratic regulator (LQR) problem:

Optimal Control Problem C. (Optimal Transients). Minimize $\widetilde{J}_\infty(\tilde{u}(\cdot)) = \frac{1}{2}\int_0^\infty \tilde{x}^{\mathsf{T}} Q \tilde{x} + \tilde{u}^{\mathsf{T}} R \tilde{u}\ \mathrm{d}t$ for $\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$ with initial value $\tilde{x}(0) = \tilde{x}_0$.

Clearly, $\tilde{x}$ solves the given differential equation for $\tilde{x}(0) = x_0 - \Pi_x x_{\text{exo},0}$. The optimal control is the well known LQR $\tilde{u}^* = -R^{-1}B^{\mathsf{T}}\overline{P}\tilde{x}$ with $\overline{P}$ given by ARE (13). Hence, $\tilde{u}^*$ in (21) guarantees an optimal transient $x \to \Pi_x x_{\text{exo}}$, $t \to \infty$.

Considering the stationary behavior, we show that $u_s^* = (-K\Pi_x + F)\, x_{\text{exo}}$ leads to a *finitely*-optimal steady-state $x_s^* = \Pi_x x_{\text{exo}}$. Thus, it solves the following OCP:

Optimal Control Problem D. (Finite Optimality). Solve LQTP. A 1) for **any** finite $T > 0$, initial value $x(0) = \Pi_x x_{\text{exo},0}$ and fixed final value $x(T) = \Pi_x x_{\text{exo}}(T)$.

Since (NC1), (NC2) hold and $\delta J_T(u_s^*, \delta u) = 0$ due to the fixed end-point, optimality can be verified by sufficiency with the help of (6) for $Q \succeq 0$, $R \succ 0$. In addition, $x_s^* = \Pi_x x_{\text{exo}}$ is *unique*. This can be shown by a stationary analysis of the Hamiltonian system which is given by (1a) and (NC2). In conclusion, (x_s^*, u_s^*) uniquely satisfies the weakest notion of optimality, i.e. finite optimality, in the domain of infinite horizons, see Carlson and Haurie (1987).

Apparently, the optimal transients $\tilde{x}$ and the finitely-optimal steady-state $\Pi_x x_{\text{exo}}$ are both induced by the same choice of Q_y and R. However, different or even conflicting requirements for both may be present. Furthermore, for unstable exo-systems, it would be desirable to guarantee (14) which was left open by Kreindler (1969). In this context, it is desirable to influence the transients, i.e. to specify the rate of convergence $\alpha \in \mathbb{R}^{>0}$ of the closed-loop dynamics which was given so far by

$$\alpha < \left|\max_i \mathrm{Re}(\lambda_i(A - BR^{-1}B^{\mathsf{T}}\overline{P}))\right|. \quad (22)$$

At the same time, this should not affect the desired finitely-optimal steady-state $\Pi_x x_{\text{exo}}$ specified by the choice of Q_y, R. In this regard, we formulate an **expanded cost**

$$J_T^\alpha(u) = J_T(u) + \int_0^T \alpha(x - \Pi_x x_{\text{exo}})^{\mathsf{T}} P_\alpha (x - \Pi_x x_{\text{exo}})\ \mathrm{d}t$$

for a specified $\alpha > 0$ and P_α obtained from

$$P_\alpha(A + \alpha I) + (A + \alpha I)^{\mathsf{T}} P_\alpha - P_\alpha B R^{-1} B^{\mathsf{T}} P_\alpha + Q = 0. \quad (23)$$

We remark that the expansion of the cost vanishes for the steady-state $x_s^* = \Pi_x x_{\text{exo}}$. This already shows that (x_s^*, u_s^*) is still finitely-optimal in view of OCP. D with $J_T^\alpha(u)$. In this light, the next lemma is formulated.

Lemma 7. Suppose $(Q, A + \alpha I)$ is observable. Furthermore, P_α is obtained from ARE (23) for a specified $\alpha > 0$ and $\overline{P}$, Π_v are given by Theorem 6 as well as Π_x by Sylvester equation (20). Then, the rate of convergence $\alpha < \left|\max_i \text{Re}(\lambda_i(A - BR^{-1}B^\mathsf{T} P_\alpha))\right|$ is guaranteed by

$$\bar{u}_\alpha = -R^{-1}B^\mathsf{T}\left(P_\alpha(x - \Pi_x x_{\text{exo}}) + (\overline{P}\Pi_x + \Pi_v)x_{\text{exo}}\right).$$

For the choice $\alpha \geq \max_j \text{Re}\{\lambda_j(S)\}$, the condition (14) holds. In view of LQTP. A 2) with expanded cost $J_\infty^\alpha(u) = \lim_{T\to\infty} J_T^\alpha(u)$, (NC1-2) are satisfied. Both, $\bar{u}_\alpha(\cdot)$ and $\bar{u}(\cdot)$, lead to the same unique steady-state, i.e. $\Pi_{x,\alpha} = \Pi_x$ and $\bar{u}_\alpha\big|_{\tilde{x}=0} = \bar{u}\big|_{\tilde{x}=0} = u_s^*$ with $\tilde{x} = x - \Pi_x x_{\text{exo}}$. □

In view of the rate of convergence, we extended a result in the LQR framework (Anderson and Moore, 2007, Sec. 3.5). In our framework, the stationary behavior is untouched.

Keep in mind that Lemma 7 is based on a closed LQTP formulation. In Section 3.3, this will be utilized to compare $\bar{u}_\alpha(\cdot)$ to the optimal control $u_{T,\alpha}^*(\cdot)$ of LQTP. A 1) with expanded cost $J_T^\alpha(u)$ on a finite interval $[0, T]$.

3.2 Optimality for bounded references and disturbances

Considering LQTP. A 2), we will prove strong optimality of the time-invariant control $\bar{u}(\cdot)$ given by Theorem 6. We require bounded references and disturbances, i.e.

Assumption 8. All eigenvalues with $\text{Re}\left(\lambda_{\text{exo},j}(S)\right) = 0$ are semi-simple and $\sigma(S) \cap \mathbb{C}^+ = \emptyset$.

We remark that a boundedness assumption such as 8 is standard in optimality analyses on infinite horizons, cf. Leizarowitz (1986), Carlson and Haurie (1987) and others.

Since $J_\infty(\bar{u})$ will be unbounded in general (Anderson and Moore, 2007, Sec. 4.3), we will introduce a modified bounded cost which leads to an equivalent LQTP. This allows us to meet the conditions of Definition 2. A common idea is to subtract a function $\mu(T)$ with $\lim_{T\to\infty}\mu(T) = \infty$ such that the difference $\widehat{J}_T(u) = J_T(u) - \mu(T)$ is bounded for some $u(\cdot)$, $T \to \infty$. Clearly, this results in a shift of the zero level of the cost and will not affect the optimal $u^*(\cdot)$. Thus, LQTP. A 2) with $\widehat{J}_\infty(u)$ and $J_\infty(u)$ are **equivalent**.

For constant references, Willems and Mareels (2004) suggested subtracting the cost of the constant steady-state. Based on our explicit derivation of the time-varying steady-state $\Pi_x x_{\text{exo}}$ in (20), we are able to apply this idea. The steady-state cost of (x_s^*, u_s^*) is given by

$$\mu = \tfrac{1}{2}\int_0^T x_{\text{exo}}^\mathsf{T}\left(\Pi_x^\mathsf{T} Q \Pi_x + 2\Pi_x^\mathsf{T} S_w + \Gamma_s^\mathsf{T} R \Gamma_s + Q_w\right) x_{\text{exo}}\, \text{d}t.$$

Clearly, this equals $\mu(T) = J_{0,T}(\bar{u})\big|_{\Pi_x x_{\text{exo}}(0)}$. By deriving $\mu(T)$ and $J_{0,T}(\bar{u})$ in a similar manner as (11), we can calculate $\widehat{J}_\infty(\bar{u}) = \lim_{T\to\infty} J_{0,T}(\bar{u}) - \mu(T)$ which yields

$$\widehat{J}_\infty(\bar{u})\big|_{x_0} = \tfrac{1}{2}x_0^\mathsf{T}\overline{P}x_0 + x_0^\mathsf{T}\Pi_v x_{\text{exo},0} - x_{\text{exo},0}^\mathsf{T}\left(\tfrac{1}{2}\Pi_x^\mathsf{T}\overline{P}\Pi_x + \Pi_x^\mathsf{T}\Pi_v\right)x_{\text{exo},0}. \quad (24)$$

Since $\widehat{J}_\infty(\bar{u})$ is bounded, we meet the conditions of Definition 2. This allows us to examine if our solution candidate $\bar{u}(\cdot)$ given in Theorem 6 is strongly optimal with respect to the **equivalent** LQTP. A 2) with cost $\widehat{J}_\infty(u)$. In the next theorem, we prove result 4).

Theorem 9. If Asmp. 8 holds, then $\bar{u}(\cdot)$ given by (16) is a unique strongly optimal control of LQTP. A 2) with cost $\widehat{J}_\infty(u) = \lim_{T\to\infty}(J_T(u) - \bar{\mu}(T))$. The optimal trajectory $\bar{x}$ converges to $\Pi_x x_{\text{exo}}$, $t \to \infty$, where Π_x is given by (20).

Proof. Since $\bar{u}(\cdot)$ satisfies (NC1-2), we concentrate on sufficiency and base the analysis on the calculus of variations as introduced in Section 2.2. Wlog, we can choose $\epsilon = 1$. Based on (5) for $\widehat{J}_T(u)$, we will examine

$$\widehat{J}_T(\bar{u} + \delta u) - \widehat{J}_T(\bar{u}) = \delta J_T(\delta u) + \delta^2 J_T(\delta u). \quad (25)$$

As $T \to \infty$, we will obtain for any $\delta u(\cdot) \not\equiv 0$ that (25) is either bounded below by a positive constant or tends to $+\infty$. Either way, uniqueness and strong optimality of $\bar{u}(\cdot)$ follow since $\widehat{J}_\infty(\bar{u})$ is bounded and a $T(u)$ as in Definition 2 always exists.

Similar to Halkin (1974), it proves useful to define a strictly increasing sequence $t_0, t_1, \ldots$ such that $[0, \infty) = \cup_{k=1,2,\ldots}[t_{k-1}, t_k]$. For $k \to \infty$, we need to examine $\delta J_{t_k}(\bar{u}, \delta u) = -\phi(t_k)^\mathsf{T}\delta x(t_k)$ and $\delta^2 J_{t_k}(\bar{u}, \delta u) \geq 0$ given by (6) with upper integral limit t_k. Notice that $\left\|\phi(t_k)\right\| = \left\|\overline{P}\bar{x}(t_k) + \Pi_v x_{\text{exo}}(t_k)\right\|$ is uniformly bounded in t_k by Asmp. 8. We analyze (25) in a three-part case study:

Suppose $\lim_{k\to\infty}\left\|\delta x(t_k)\right\| = 0$. It follows $\delta J_\infty(\bar{u}, \delta u) = \lim_{k\to\infty}\delta J_{t_k}(\bar{u}, \delta u) = 0$. For any $\delta u(\cdot) \not\equiv 0$, $\exists t_0 \in \mathbb{R}^{>0}$ such that $\left\|\delta x(t_0)\right\| \neq 0$. Let us introduce the optimal control $\delta u^*(\cdot)$ of the related OCP. C on $[t_0, \infty)$. Notice that it always exists under Asmp. 1. Then, we have $\delta^2 J_\infty(\bar{u}, \delta u) \geq \delta^2 J_\infty(\bar{u}, \delta u^*) = \delta x(t_0)^\mathsf{T}\overline{P}\delta x(t_0)$. Since $\overline{P} \succ 0$ due to Asmp. 1, $\exists M_6 \in \mathbb{R}^{>0}$: $\delta^2 J_\infty(\bar{u}, \delta u) > M_6$.

Suppose $\left\|\delta x(t_k)\right\| < M_7\ \forall k$, $\lim_{k\to\infty}\left\|\delta x(t_k)\right\| \neq 0$. We have $\delta J_\infty(\bar{u}, \delta u) > -M_8$. Applying the idea introduced in the context of OCP. B in Section 2.2, we find

$$\delta^2 J_{t_K}(\bar{u}, \delta u) \geq \sum_{k=1}^{K}\widetilde{J}_{k-1,k}(\delta u_k^*) \geq 0,\ \forall K \quad (26)$$

with $\widetilde{J}_{k-1,k}(\delta u_k^*) > 0$ given by (12) and $\tilde{x}_k = \delta x(t_k)$. Since $\widetilde{P}_k(0) \succ 0$, the positive addends do not vanish for $K \to \infty$ and the series tends to $+\infty$. Consequently, $\lim_{K\to\infty}\widehat{J}_{t_K}(\bar{u} + \delta u) - \widehat{J}_{t_K}(\bar{u}) = +\infty$.

Suppose $\lim_{k\to\infty}\left\|\delta x(t_k)\right\| = \infty$. Then $\delta J_{t_k}(\bar{u}, \delta u)$ may tend to $-\infty$ as a linear function of $\delta x(t_k)$. In contrast, the right-hand side of (26) tends to $+\infty$ quadratically as $\left\|\delta x(t_{k-1})\right\|$ and/or $\left\|\delta x(t_k)\right\| \to \infty$. Thus, we accomplish $\lim_{K\to\infty}\widehat{J}_{t_K}(\bar{u} + \delta u) - \widehat{J}_{t_K}(\bar{u}) = +\infty$.

Reminding the discussion at the beginning of the proof, it results that $\bar{u}(\cdot)$ is a unique strongly optimal control and $\lim_{t\to\infty}(\bar{x} - \Pi_x x_{\text{exo}}) = 0$ follows from Section 3.1. □

As a direct consequence of our proof, we can conclude:

Corollary 10. With $\lim_{t\to\infty}\left\|\bar{x} - \Pi_x x_{\text{exo}}\right\| = 0$ based on Theorem 9, it holds for any other (x, u) with $u(\cdot) \not\equiv \bar{u}(\cdot)$:

1) If $\lim_{t\to\infty}\left\|x - \Pi_x x_{\text{exo}}\right\| \neq 0$, then $\lim_{T\to\infty}\left(J_T(u) - J_T(\bar{u})\right) = +\infty$, else
2) $\lim_{T\to\infty}\left(J_T(u) - J_T(\bar{u})\right) > \kappa(x_0, x_{\text{exo},0}) \in \mathbb{R}^{>0}$. □

Hence, any deviation in the steady-state results in an infinite increase of cost while any deviation in the transients yields at least a constant increase. For the control structure Fig. 1, middle, this was also shown by Leizarowitz (1986).

3.3 Finite horizons: Agreeability and approximation

From a practical point of view, especially if the boundedness Asmp. 8 does not hold, the horizon is potentially very large and not exactly known but most likely finite. In this context, we prove $\bar{u}(\cdot)$ to be agreeable. This allows for analyzing its approximative quality of the optimal control $u^*_T(\cdot)$ on a finite horizon $[0,T]$. To begin with, we present another result on convergence:

Lemma 11. Define the solution $\bar{x}(\cdot)$ of system (1a) for $\bar{u}(\cdot)$ given by (16) and analogously $x^*_T(\cdot)$ for $u^*_T(\cdot)$ given by (8). If condition (14) is satisfied, it holds $\lim_{T\to\infty} x^*_T(t) = \bar{x}(t)$ and $\lim_{T\to\infty} u^*_T(t) = \bar{u}(t)$ uniformly in $t \in [0,\tau]$ for any given $\tau \in \mathbb{R}^{>0}$.

Proof. Considering Appendix A, we have

$$\left\|x^*_T(t) - \bar{x}(t)\right\| \le \kappa_x e^{\alpha t} e^{-(\alpha-\alpha_S)T}, \tag{27a}$$

$$\left\|u^*_T(t) - \bar{u}(t)\right\| \le \kappa_u e^{\alpha t} e^{-(\alpha-\alpha_S)T} \tag{27b}$$

with $\alpha_S < \alpha$ as given in proof of Theorem 6. Hence, uniform convergence in t follows on any finite $[0,\tau]$. □

We are now able to prove agreeability as in Definition 4:

Theorem 12. If the condition (14) holds, the control $\bar{u}(\cdot)$ given by (16) is an agreeable plan of LQTP. A 1).

Proof. We need to verify (7) for any $\theta \in \mathbb{R}$. Considering $\lim_{T\to\infty} x^*_T(\theta) = \bar{x}(\theta)$, it can be shown that $\lim_{T\to\infty} J_{\theta,T}(u^*_{\theta,T})\big|_{\bar{x}(\theta)} - J_{\theta,T}(u^*_T)\big|_{x^*_T(\theta)} = 0$ for any θ by means of (11). Then, it is left to verify (7) on $[0,\theta]$. For the integrand $L(x,u,t)$ of (4), we have $\lim_{T\to\infty} L(x^*_T,u^*_T,t) - L(\bar{x},\bar{u},t) = 0$ uniformly in $t \in [0,\theta]$ due to Lemma 11. Hence, limit operation and integration commute which results in $\lim_{T\to\infty} J_{0,\theta}(\bar{u}) - J_{0,\theta}(u^*_T) = 0\ \forall x_0$. □

The proof indicates that $\bar{u}(\cdot)$ leads to a close approximation of $u^*_T(\cdot)$ and a negligible additional cost on some interval $[0,\theta]$ for a suitably large T. However, this raises two questions in regard of result 5). How large may θ be chosen for a **fixed** T and how close will the approximation be on $[0,T]$?

In this context, we regard the expanded cost $J^\alpha_{0,T}(u)$ as introduced in Section 3.1 and the control $\bar{u}_\alpha(\cdot)$ given in Lemma 7. For these, the previous results hold. This allows us to show how the approximation on $[0,T]$ depends on the rate of convergence α of the closed-loop dynamics. In order to be able to specify α freely, we assume that the conditions of Lemma 7 are satisfied. Answering the questions, we firstly show that the gap $[\theta,T]$ can be chosen arbitrarily small for a suitably large α. Secondly, we will derive that the additional cost $\Delta J^\alpha_{0,T} = J^\alpha_{0,T}(\bar{u}_\alpha) - J^\alpha_{0,T}(u^*_{T,\alpha}) \ge 0$ on $[0,T]$ will be arbitrarily small in a similar way.

Regarding the proof of Theorem 12, we point out that $\Delta J^\alpha_{0,\theta} \le \epsilon$ holds for a given $\epsilon > 0$ if $\left\|u^*_{T,\alpha}(t) - \bar{u}_\alpha(t)\right\| \le \epsilon_u$ holds on $[0,\theta]$ for a suitably small $\epsilon_u > 0$. By means of (27b), we have to consider a choice of θ such that

$$T - \theta \ge \tfrac{\alpha_S}{\alpha} T + \tfrac{1}{\alpha} \ln\left(\tfrac{\kappa_u(\alpha)}{\epsilon_u}\right) > 0. \tag{28}$$

This defines an upper bound $\theta \le \Theta(\alpha)$. Notice that we have $\kappa_u(\alpha)$ since $M_P(\alpha)$ in (17) for P_α given by (23). Apparently, $M_P(\alpha)$ does **not** depend exponentially on α which can be shown by eigenvalue bounds of P_α given in Kim and Park (2000). Since the term in the middle of (28) vanishes for $\alpha \to \infty$, we conclude that $\Delta J^\alpha_{0,\theta} \le \epsilon$ holds for any arbitrarily small gap $[\theta,T]$ if α is suitably large.

With regard to $\Delta J^\alpha_{0,T}$, it is only left to examine $\Delta J^\alpha_{\theta,T}$. Since $J^\alpha_{\theta,T}(u)$ depends on α, it seems possible that $\Delta J^\alpha_{\theta,T} \to \infty$, $\alpha \to \infty$. But this is infeasible, as we will show by analysis of $J^\alpha_{\theta,T}(\bar{u}_\alpha)$. Remember that the steady-state pair $(\Pi_x x_{\text{exo}}, u^*_s)$ is unaffected by the choice of α where u^*_s is given in (21), see Lemma 7. With $\tilde{x}_\alpha = \bar{x}_\alpha - \Pi_x x_{\text{exo}}$ and $\tilde{u}_\alpha = \bar{u}_\alpha - u^*_s$, the expanded cost reads on $[\theta,T]$:

$$J^\alpha_{\theta,T}(\bar{u}_\alpha) = J_{\theta,T}(\bar{u}_\alpha) + \tfrac{1}{2}\int_\theta^T 2\alpha \tilde{x}_\alpha^\mathsf{T} P_\alpha \tilde{x}_\alpha \,\mathrm{d}t.$$

We remind that $(\tilde{x}_\alpha, \tilde{u}_\alpha)$ is optimal for OCP. C with $Q + 2\alpha P_\alpha$ and R. Hence, we can find an upper bound for the integral term: $\frac{1}{2}\tilde{x}_\alpha^\mathsf{T}(\theta) P_\alpha \tilde{x}_\alpha(\theta)$. It holds $\left\|P_\alpha\right\| \le M_P(\alpha)$ and $\left\|\tilde{x}_\alpha(\theta)\right\| \to 0$ exponentially in α. Thus, we find that the upper bound converges to zero as $\alpha \to \infty$ and the integral term vanishes. In addition, we have $\lim_{\alpha\to\infty} L(\bar{x}_\alpha, \bar{u}_\alpha, t) = L(\Pi_x x_{\text{exo}}, u^*_s, t)$ uniformly in $t \in [\theta,T]$. As a consequence: $\lim_{\alpha\to\infty} J^\alpha_{\theta,T}(\bar{u}_\alpha)\big|_{\bar{x}_\alpha(\theta)} = J_{\theta,T}(u^*_s)\big|_{\Pi_x x_{\text{exo}}(\theta)}$, which are the bounded steady-state cost on $[\theta,T]$. Due to the boundedness, it follows $\lim_{\alpha\to\infty} J^\alpha_{\Theta(\alpha),T}(\bar{u}_\alpha) = 0$ since $[\Theta(\alpha),T]$ vanishes as $\alpha \to \infty$ based on (28) $\forall \epsilon_u > 0$. Hence, we can conclude $-\epsilon \le \lim_{\alpha\to\infty} \Delta J^\alpha_{\Theta(\alpha),T} \le 0$.

We are now able to clarify the **result 5)**. Since $\epsilon > 0$ can be freely chosen, it holds for any $\epsilon_\Delta > 0$ and some $\alpha(\epsilon_\Delta)$:

$$\Delta J^\alpha_{0,T} = J^\alpha_{0,T}(\bar{u}_\alpha) - J^\alpha_{0,T}(u^*_{T,\alpha}) \le \epsilon_\Delta. \tag{29}$$

By means of Lemma 7, we are able to specify the rate of convergence α of the closed-loop dynamics. Hence, a suitable choice of α allows for an arbitrarily small performance loss $\Delta J^\alpha_{0,T}$ of $\bar{u}_\alpha(\cdot)$ in comparison to the optimal $u^*_{T,\alpha}(\cdot)$. In Section 4, we will see that a **moderate** α can already suffice for a satisfactory small relative loss $\Delta J^\alpha_{0,T} / J^\alpha_{0,T}(u^*_{T,\alpha})$. Hence, if a fast closed-loop is required anyway, one can expect that the relative loss will be small. Then, from a practical point of view, the implementation of the optimal $u^*_{T,\alpha}(\cdot)$ is not reasonable given the high computational expense and the restriction to a single fixed, exactly known in advance horizon.

4. DESIGN ALGORITHM & SIMULATION RESULTS

For a comprehensive overview, we give a design algorithm:

Algorithm 1. For LQTP.A, the time-invariant control

$$\bar{u} = -R^{-1}B^\mathsf{T}\left(P_\alpha(x - \Pi_x x_{\text{exo}}) + (\overline{P}\Pi_x + \Pi_v)x_{\text{exo}}\right)$$

is strongly optimal (if Asmp. 8 holds) on $[0,\infty)$, is agreeable and approximates the optimal control $u^*_T(\cdot)$ on any $[0,T]$. It is applicable for any initial conditions x_0, $x_{\text{exo},0}$ and obtained from a 4-step algorithm:

1) Specify $Q_y \succeq 0$ and $R \succ 0$, calculate Q, S_w in (4)
2) Solve (13)→ $\overline{P}$, (15)→ Π_v, (20)→ Π_x
3) Analyze tracking performance, e.g. $\Pi_x x_{\text{exo}}(\cdot)$ for relevant $x_{\text{exo},0}$. If unsatisfactory: return to 1)

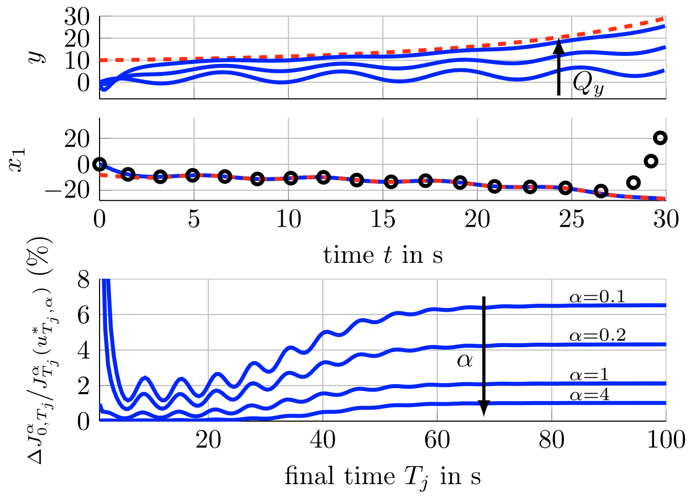

Fig. 2. Output tracking (various Q_y), state trajectories for $\bar{u}$ (—) and u^*_T (o), relative additional cost (various α)

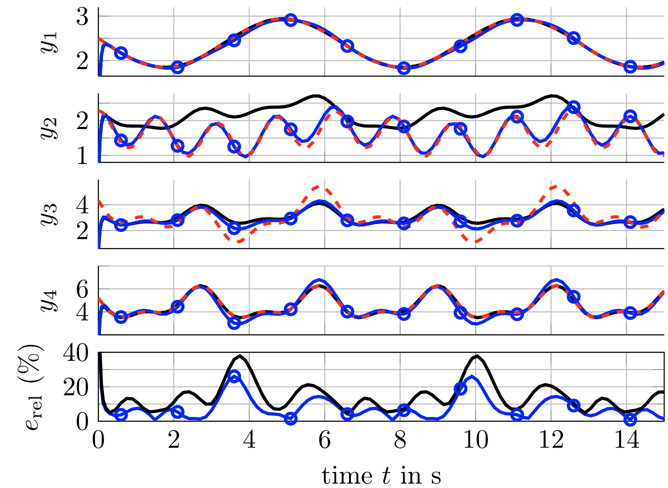

Fig. 3. Furnace system: outputs y_i for $\bar{u}$ (-o-), u_{asym} (—) and desired w_i (- -), relative tracking error

4) Analyze transient performance, e.g. $\max_i \text{Re}(\lambda_i(A - BR^{-1}B^{\mathsf{T}}\overline{P}))$. If satisfactory: $P_\alpha = \overline{P}$, else: either
   (a) Specify desired α: Solve ARE (23) $\rightarrow P_\alpha$ $\Rightarrow$ a suitable α gives a close approximation of the optimal $u^*_{T,\alpha}(\cdot)$ for $J^\alpha_T(\cdot)$ on $[0, T]$, or
   (b) Specify Q_α, R_α: Solve corresponding ARE (13)$\rightarrow$ $P_\alpha \Rightarrow$ optimal transient with respect to OCP. C.

In the sequel, two examples are given. First, we demonstrate the properties of our approach by an illustrative example. Second, our approach is applied to an underactuated industrial system.

Example 13. We consider a second-order LTI SISO-system

$$(C, A, B, E_d) = \left(\begin{bmatrix} -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \right)$$

which is unstable and non-minimum phase. It is desired to track the exponentially growing reference $w(t) = 9 + \mathrm{e}^{0.1t}$ in regard of sinusoidal disturbances $E_d x_{\text{exo}}(t) = [-2\cos(t)\ 3\sin(t)]^{\mathsf{T}}$. For any $x_{\text{exo}}(0)$ and $x(0)$, $\bar{u}(\cdot)$ obtained from Algorithm 1 for step 4a) is agreeable and approximates the optimal solution $u^*_T(\cdot)$ on $[0, T]$ given by Lemma 5. The upper plot in Fig. 2 shows the tracking result for $\bar{u}^{[i]}(\cdot)$, $i = 1, 2, 3$ determined for different costs $J^{[i]}_\infty(\cdot)$, i.e. $Q^{[i]}_y = \{1, 5, 30\}$ and $R^{[i]} = 1$, based on Theorem 6. As expected, the higher the tracking error is weighted by increasing $Q^{[i]}_y$, the closer the output $y^{[i]}(t)$ follows the desired $w(t)$ (- -) and the stronger the disturbance is attenuated. In other words, the ratio Q_y, R allows for balancing tracking error against necessary input-energy. Thus, it particularly enables us to save input-energy if a trajectory is too costly to follow asymptotically. The second plot shows the trajectory $x^*_{1,T}$ induced by the optimal control $u^*_T(\cdot)$ of $J^{[3]}_T(\cdot)$ on $[0, T]$, $T = 30$s and the agreeable $\bar{x}^{[3]}_1$ induced by $\bar{u}^{[3]}$. Besides, the stationary finitely-optimal trajectory $e^{\mathsf{T}}_1 \Pi_x x_{\text{exo}}(t)$ (- -) is displayed which can be explicitly calculated by (20). The former both converge from $x_1(0)$ to the latter, which is indeed the limit $\bar{x}_1 \rightarrow e^{\mathsf{T}}_1 \Pi_x x_{\text{exo}}(t)$, $t \rightarrow \infty$. Due to the transversality condition $\phi(T) = 0$, the optimal $x^*_{1,T}$ pulls away at the very end which finally causes the additional cost $\Delta J_{0,T} = J_T(\bar{u}) - J_T(u^*_T) \geq 0$.

However, if in addition a certain rate of convergence α of the closed-loop dynamics is required then we have to consider the expanded cost $J_{T,\alpha}(\cdot)$ with $Q_y = 30$ for comparison. Hence, for the same simulation example and $\alpha \in \{0.1, 0.2, 1, 4\}$, we determine different $u^*_{T_j,\alpha}(\cdot)$ based on Lemma 5, each applying to only one specific horizon with final time $T_j \in [1s, 100s]$, $j = 1, \ldots, 180$. Then, we compare our control $\bar{u}_\alpha(\cdot)$ to each $u^*_{T_j,\alpha}(\cdot)$, i.e. we calculate the relative additional cost $\Delta J^{\alpha}_{0,T_j}/J^{\alpha}_{T_j}(u^*_{T_j,\alpha})$ on each horizon $[0, T_j]$. The lower plot in Fig. 2 verifies the decrease of the relative additional cost for any T_j if α increases. For a moderate choice $\alpha = 4$, $\bar{u}_\alpha$ already approximates $u^*_{T_j,\alpha}$ by only $\leq 1\%$ additional cost. This emphasizes our quantitative approximation result (29).

Example 14. A 8-th order boiler furnace system is regarded. It consists of $i \in \{1, 2, 3, 4\}$ coupled heating coils, whose temperatures y_i are measured, with a burner u_i each, cf. Davison and Davison (2011) where the dense matrices (C, A, B) are also given. Similarly, we assume an actuator loss: only burner u_1 and u_4, $m = 2$, are available and system (1a) is under-actuated, i.e. $\text{rank}\,(B) < \text{rank}\,(C)$, $p = 4$. While Davison and Davison (2011) are restricted to constant references, we consider time-varying, periodic desired values $w(t) = O x_{\text{exo}}$ with

$$x^{\mathsf{T}}_{\text{exo}}(t) = [5\ \cos(t)\ \sin(t)\ \cos(2t)\ \sin(2t)\ \cos(4t)\ \sin(4t)],$$

$$O = \frac{1}{10} \begin{bmatrix} 4.85 & 1.04 & 5.36 & 0 & 0 & 0.04 & 0 \\ 3.21 & 1.65 & 0 & 0 & 0 & 5.12 & 0 \\ 6.02 & 8.83 & 0 & 4.33 & 9.04 & 0 & 6.79 \\ 9.13 & 0 & 0 & 6.30 & 9.83 & 0 & 5.66 \end{bmatrix};$$

O was randomly sparsely generated. The goal is to minimize the tracking error; hence, we choose $Q_y = 1000 \cdot \text{diag}(3, 1, 1, 1)$ and $R = {}^{I}\!/_{1000}$. To avoid undesirable fast closed-loop dynamics, we carry out Algorithm 1 for step 4b) $Q_\alpha = 100 \cdot I$, $R_\alpha = I$. Then, $\bar{u}$ leads to an optimal transient OCP. C and a stationary strongly optimal trajectory $\Pi_x x_{\text{exo}}$, cf. Theorem 9. Solving the *regulator equations*, we obtain $u_{\text{asym}}(\cdot)$ for asymptotic tracking $\lim_{t\to\infty} y_i - w_i = 0$, $i \in \{1, 4\}$, which is applied for comparison. This gives the best overall performance among all the feasible output pairs for asymptotic tracking. The results are shown in Fig. 3. While the performance for y_1, y_3 and y_4 is comparable, $\bar{u}(\cdot)$ also achieves a close tracking of w_2 where $u_{\text{asym}}(\cdot)$ fails. The relative tracking error $e_{\text{rel}} = {\|y-w\|}/{\|w\|}$ indicates that $\bar{u}(\cdot)$ works significantly better. It leads to an average relative error of only $< 9\%$ per time period. This is nearly half compared to $u_{\text{asym}}(\cdot)$. In conclusion, though only two inputs can be used the tracking performance of four outputs is satisfactory. Of course, there exists a lower bound of e_{rel} depending on the system structure and references w. Hence, at times, a satisfying performance may be unattainable. Nonetheless, by a suitable weighting we can approach this lower bound as close as desired.

Appendix A. PROOF OF THEOREM 6 & LEMMA 11

Due to the space restrictions, it is impossible to show all the details of the derivations. Besides, the given bounds might not be the closest possible but suffice for our analysis. All constants M_i, κ_j are defined in Section 1. In view of **Theorem 6**, the solution $z(t)$ of $\dot{z} = \overline{A}z + BR^{-1}B^{\mathsf{T}}\widetilde{P}_T(t)z$ for any initial $z(t_0)$, $t \in [t_0, T]$ satisfies

$$\left\|z(t)\right\| \leq M_{\overline{A}}\left\|z(t_0)\right\| + \int_{t_0}^{t} \kappa_1\, \mathrm{e}^{-2\alpha(T-\tau)} \left\|z(\tau)\right\| \,\mathrm{d}\tau$$

based on (17) and $\left\| \mathrm{e}^{\overline{A}(t-t_0)} \right\| \leq M_{\overline{A}}$ due to asymptotic stability of $\overline{A}$. By means of the *Gronwall-Bellman inequality*, e.g. (Khalil, 2002, p. 651), the inequality can be solved for $\left\|z(t)\right\|$ and we can obtain $\left\|z(t)\right\| \leq M_1\left\|z(t_0)\right\|$. Since $z(t) = \Phi(t, t_0)z(t_0)$ with arbitrary $z(t_0)$, the transition matrix $\Phi(t, t_0)$ of the closed loop $A - BR^{-1}B^{\mathsf{T}}P_T(t)$ satisfies:

$$\left\|\Phi(t, t_0)\right\| = \max_{z(t_0)\neq 0} \frac{\left\|z(t)\right\|}{\left\|z(t_0)\right\|} \leq M_{\Phi}. \tag{A.1}$$

Introducing $\dot{\bar{z}} = \overline{A}\bar{z}$ for $\bar{z}(t_0) = z(t_0)$, we consider $\tilde{z} = z - \bar{z}$, $\tilde{z}(t_0) = 0$. For $\dot{\tilde{z}} = \overline{A}\tilde{z} + BR^{-1}B^{\mathsf{T}}\widetilde{P}_T(t)z$, it follows $\left\|\tilde{z}(t)\right\| \leq \int_{t_0}^{t} M_2\left\|\widetilde{P}_T(\tau)\right\|\left\|z(\tau)\right\| \,\mathrm{d}\tau$. With (17) and boundedness of $z(t)$, we have $\left\|\tilde{z}(t)\right\| \leq M_{\Delta\Phi}\, \mathrm{e}^{-2\alpha(T-t)} \left\|z(t_0)\right\|$. Since $\tilde{z}(t) = (\Phi(t, t_0) - \mathrm{e}^{\overline{A}(t-t_0)})z(t_0)$, it results

$$\left\|\widetilde{\Phi}(t, t_0)\right\| = \left\|\Phi(t, t_0) - \mathrm{e}^{\overline{A}(t-t_0)} \right\| \leq M_{\Delta\Phi}\, \mathrm{e}^{-2\alpha(T-t)}. \tag{A.2}$$

We have $\left\|x_{\text{exo}}(t)\right\| \leq \kappa_{\text{exo}}\, \mathrm{e}^{\alpha_S t}$ for $\alpha_S > \max_j \operatorname{Re}(\lambda_{\text{exo},j}(S))$. The solution $v_T(t) = \int_t^T \Phi^{\mathsf{T}}(\tau, t)(P_T(\tau)E_d + S_w)x_{\text{exo}}(\tau) \,\mathrm{d}\tau$ of (10) is easily derived. It satisfies $\left\|v_T(t)\right\| \leq \kappa_1\, \mathrm{e}^{\alpha_S T}$ with (A.1) and uniformly bounded $\left\|P_T(t)\right\| \leq M_3$ by virtue of (17). With differential equations (10), (18), we give

$$-\dot{\tilde{v}}_T = \overline{A}^{\mathsf{T}}\tilde{v}_T - \widetilde{P}_T(t)BR^{-1}B^{\mathsf{T}}v_T(t) + \widetilde{P}_T(t)E_d x_{\text{exo}}$$

for $\tilde{v}_T(t) = v_T(t) - \bar{v}(t)$ with final value $\tilde{v}_T(T) = -\mathrm{e}^{-\overline{A}^{\mathsf{T}}T}\eta - \Pi_v x_{\text{exo}}(T)$. With respect to $\left\| \mathrm{e}^{\overline{A}^{\mathsf{T}}(\tau-t)} \right\| \leq M_4\, \mathrm{e}^{-\alpha(\tau-t)}$ and the prior results, an analysis of $\left\|\tilde{v}_T(t)\right\|$ yields (19).

In regard of **Lemma 11**, $x_T^*(t) - \bar{x}(t)$ is given by

$$x_T^*(t) - \bar{x}(t) = \widetilde{\Phi}(t, 0)x_0 + \int_0^t \widetilde{\Phi}(t, \tau)(-BR^{-1}B^{\mathsf{T}}\bar{v} + E_d x_{\text{exo}}) + \Phi(t, \tau)(-BR^{-1}B^{\mathsf{T}})(v_T - \bar{v}) \,\mathrm{d}\tau.$$

By virtue of (19) for $\eta = 0$, (A.1) and (A.2), it holds (27a). With (20), we can write $\left\|\bar{x}\right\| = \left\|\tilde{x} + \Pi_x x_{\text{exo}}\right\| \leq \kappa_2 + \kappa_3\, \mathrm{e}^{\alpha_S t}$. Then $P_T(t)x_T^* - \overline{P}\bar{x} = \widetilde{P}_T(t)\bar{x} + P_T(t)(x_T^* - \bar{x})$ with $\left\|P_T(t)\right\| \leq M_3$, (17), (19) and (27a) lead to (27b).

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