Involutions of Kac-Moody Groups

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Dipl.-Math. Max Horn

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1. Referent:	PD dr. Ralf Gramlich
2. Referent:	Prof. Dr. Bernhard Mühlherr
3. Referent:	Prof. Dr. Karl-Hermann Neeb
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Deutsche Zusammenfassung

Historisch sind Involutionen zweifellos von großem Interesse, beispielsweise im Rahmen der Klassifikation der endlichen einfachen Gruppen (in welcher Zentralisatoren von Involutionen eine große Rolle spielen) oder zur Definition von symmetrischen Riemannschen Räumen bzw. von symmetrischen k-Varietäten. Ziel der vorliegenden Arbeit ist das Studium involutorischer Automorphismen reduktiver algebraischer Gruppen und zerfallender Kac-Moody-Gruppen (in diesem Fall soll die Involution die beiden Konjugiertenklassen von Boreluntergruppen vertauschen) in Charakteristik ungleich 2, sowie deren Zentralisatoren.

Die genannten Gruppen haben gemein, dass sie zu einem *Zwillingsgebäude* assoziiert sind. Sei G nun eine solche Gruppe. Ein involutorischer Automorphismus θ von G induziert einen fast-isometrischen Automorphismus des assoziierten Gebäudes $\mathcal{C}$. Dies ermöglicht es, die reichhaltige Strukturtheorie von Gebäuden anzuwenden.

Ein wichtiges Hilfsmittel hierbei ist das so genannte *Flipflop-System* $\mathcal{C}^\theta$, bestehend aus allen Kammern der positiven Hälfte des Gebäudes, welche durch die induzierte Abbildung θ maximal weit abgebildet werden (im Sinne der Kodistanz auf dem Zwillingsgebäude $\mathcal{C}$). Als Teilkammernsystem des Gebäudes $\mathcal{C}_+$ kann man $\mathcal{C}^\theta$ auch als simplizialen Komplex auffassen. Der Zentralisator G_θ von θ in G wirkt auf diesem Komplex.

Sei G eine Gruppe mit Zwillings-BN-Paar (B_+, B_-, N) und Zwillingsgebäude $\mathcal{C} = (\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ und θ eine (fast-)isometrische Involution von $\mathcal{C}$. Die ursprüngliche Motivation für die vorliegende Arbeit beinhaltet die Beantwortung der folgenden Fragen, welches uns im Wesentlichen gelungen ist:

- Wann kann man θ zu einer Involution (oder wenigstens einem beliebigen Automorphismus) der Gruppe liften?
- Wann ist $\mathcal{C}^\theta$ als Kammernsystem zusammenhängend?
- Wann ist $\mathcal{C}^\theta$ ein reiner Simplizialkomplex? Äquivalent, wann ist $\mathcal{C}^\theta$ Kammernsystem einer Inzidenzgeometrie?
- Wenn $\theta \in \mathrm{Aut}(G)$ ist: Wann wirkt der Zentralisator G_θ transitiv auf $\mathcal{C}^\theta$? Allgemeiner, was können wir über die Bahnstruktur aussagen?
- Wann ist G_θ endlich erzeugt?
- Wenn $\mathcal{C}^\theta$ und $\mathcal{C}_+$ übereinstimmen und G_θ transitiv wirkt, erhalten wir eine verallgemeinerte *Iwasawa-Zerlegung* $G = G_\theta B_+$. Wann ist dies möglich?

Abschließend sei erwähnt, dass sich unsere Resultate auf weitere Gruppen mit einem Wurzelgruppendatum im Sinne von [Tit92] (wie z. B. endliche Gruppen vom Lie-Typ) erweitern lassen. In diesem Fall muss die Klasse der betrachteten involutorischen Automorphismen leicht eingeschränkt werden mit der Forderung, dass eine einzelne gewählte Boreluntergruppe B wieder auf eine Boreluntergruppe abgebildet wird (im Falle von Kac-Moody-Gruppen auf eine mit entgegengesetztem Vorzeichen). Wir sprechen dann von einem *Quasiflip* und bezeichnen damit sowohl die Abbildung auf der Gruppe wie auch die auf dem Gebäude.

INTRODUCTION

In this thesis we study involutory automorphisms of reductive algebraic and split Kac-Moody groups over arbitrary fields, or more generally, of groups with a root group system, as defined by Tits [Tit92] (this includes also finite groups of Lie type, for example).

The unifying aspect of all these groups is that to each of them a *twin building* is associated. It turns out that any involutory automorphism θ of a group G as listed above induces an almost isometric automorphism of the associated building $\mathcal{C}$ in a unique way. We call these involutory automorphisms (both of the group and the building) *quasi flips*.

This correspondence is the key insight driving the present work. We can now exploit the rich theory of buildings in general and of twin buildings in particular to derive properties of the building automorphism – and accordingly, via the correspondence we hinted at above, also of the original involutory automorphism θ. We will sketch some of the results in what follows.

Some history

But first, some "historical" background: In hindsight, the study of flips (a special case of our flips, where the building morphism is type preserving) was initiated in the revision of the Phan theorems due to Kok-Wee Phan (see [Pha77a] and [Pha77b]). These play a central role in the classification of finite simple groups.[1] During this effort of reproving and extending Phan's theorems, dubbed also "Phan program", a series of publications was started to reprove and extend the classification theorems by Phan. The original proofs were rather non-conceptual and involved heavy calculations in unitary groups and with generators and relations, which often were even only alluded to be omitted. In the revised program, a geometric approach was used instead, where the groups in question were described as centralizers of involutions – involutions which we today would call flips.

For an overview of the general Phan program, we refer to [BGHS03] and also more recently [Gra]. The case A_n was dealt with in [BS04], the case B_n in [BGHS07] and [GHN07], the case C_n in [GHS03], [Gra04], [GHN06] and [Hor05], the case D_n in [GHNS05].[2]

[1] Phan's results entered the classification via Aschbacher's paper [Asc77].

[2] The $A_3 = D_3$ case also lead to [Hor08], where a specific exception to the Phan theorems is studied

Initially, during the above-mentioned program, somewhat "ad-hoc" choices of suitable involutory automorphisms were made. But it soon became apparent that a deeper systematic reason was hidden below the surface. This connection turned out to be building theory. All involutions that had been used could be understood in terms of the buildings of the involved groups. With this insight, the group recognition and presentation results described above all follow very roughly an argument along the following lines: Given a "target group" G (for which we want to prove a recognition/presentation result), find a group H endowed with a spherical BN-pair and an involutory automorphism θ of H such that G is isomorphic to the centralizer of θ in H, and such that θ also induces an involutory automorphism on the (spherical) building of H. Define a subset $\mathcal{C}^\theta$ of the building (the *flip-flop system*) consisting of all chambers mapped maximally far away by θ. If one can show that $\mathcal{C}^\theta$ is connected and simply connected (a building is a simplicial complex and $\mathcal{C}^\theta$ can be interpreted as a subcomplex), and if moreover G acts transitively on $\mathcal{C}^\theta$, then by Tits' Lemma (see e.g. [Pas85, Lemma 5], [Tit86, Corollary 1]) the group G is finitely presented.

This insight finally made it possible to carry out the Phan program in its full generality as described above. Now, there was a conceptual argument why simple connectedness and transitivity would suffice to derive the desired results on groups. There would be much more to say about this history, but that is far beyond the scope of this introduction, so we stop here now.

Goals

Summarized and simplified, the starting point of the theory of flips was the study of finite groups of Lie type by analyzing (centralizers of) involutory automorphisms via their interaction with the spherical buildings associated to the groups.

The starting point of this thesis was the desire to study arbitrary "flips" θ of some reductive algebraic group G with BN-pair (B_+, B_-, N) of type (W, S) with the vague hope of later extending this to Kac-Moody groups. Originally a *proper* $BN-$*flip* was understood to be an involution which interchanges the Borel groups B_+ and B_- and centralizes the Weyl group W. These would then induce a *proper building flip* of the associated twin building, meaning a permutation of the twin building interchanging the two twin halves isometrically (preserving distances and codistances). Associated to this is the *flip-flop system* $\mathcal{C}^\theta$ consisting of all chambers which are mapped to an opposite chamber by the flip.

Questions that we asked included: When can a building flip be lifted back to a BN-flip (the other direction being straightforward)? What can one say about the flip-flop system in terms of connectedness and transitivity properties of the centralizer G_θ of θ in G? Moreover: When is $\mathcal{C}^\theta$ the chamber system of an incidence geometry? Very early on, there was also the idea of generalizing Iwasawa decompositions in the vein of [HW93].

in detail.

In fact [HW93] turned out to be a major source of inspiration and motivation. This paper deals with the study of involutory algebraic morphisms of the group of $\mathbb{F}$-rational points of connected reductive algebraic groups defined over a field $\mathbb{F}$ – indeed, such an automorphism is a primary example for a (quasi-)flip! Moreover, in [KW92] some results similar to those in [HW93] but applying to Kac-Moody groups over algebraically closed fields in characteristic 0 were given. Our hope was to use building theory to unify and extend these results to (almost) arbitrary algebraic and Kac-Moody groups.

In the end, we managed to achieve most of the goals sketched above and even a lot beyond that: For example, instead of just algebraic groups, we were able to also cover Kac-Moody groups, finite groups of Lie type and other groups. Based on [HW93] we extended our notion of flips to quasi-flips (where the assumption that W is centralized can be dropped) and managed to prove most of the things we originally had hoped to show for type preserving proper flips of algebraic groups for arbitrary quasi-flips of groups with a twin BN-pair.

For all this, [DM07] had a crucial influence. In that beautiful paper, connectedness resp. simple connectedness of certain subsets of buildings is reduced to a study of rank 2 resp. rank 3 residues via an elegant filtration and local-to-global arguments. The results apply in particular to the flip-flop systems from above associated to a large class of interesting involutions (e.g. semi-linear involutions of split algebraic or Kac-Moody groups interchanging a Borel group with an opposite one). Hence part of the present thesis deals with studying connectedness in the rank 2 case. In Sections 3.3 and 4.6 we show that the relevant sets are indeed connected in "most" cases if only single or double bonds exist in the Dynkin diagram of the group.

Unfortunately, it turned out that not all involutions we are interested in allow for a "nice" filtration. Thus, we had to refine the strategy used in [DM07] and replace the simple rank 1 property used there to establish the required filtration by a more complicated rank 2 property, and proving a similar local-to-global result as in loc. cit. (see Chapter 4). Again in Sections 3.3 and 4.6 we show that this property is satisfied in "most" cases if only single or double bonds exist in the Dynkin diagram of the group.

There are several aspects that have not yet been fully settled; for example, we show how to reduce the question about connectedness of the flip-flop system to a rank 2 problem, but have not yet been able to handle all rank 2 cases. Still in several important cases we did, and the remaining are subject of ongoing research.

Structure of this thesis

Chapter 1

In this chapter, we introduce many of the concepts used throughout the present thesis. It by no means attempts to be comprehensive; rather it is meant to settle some notational questions, introduce the fundamentals, and finally provide the interested reader with hints on where to look for further details. Our main reference throughout the entire work is [AB08]. This recent book presents a detailed treatment of the theory of buildings, twin buildings, and groups acting on them. As such, we heartily recommend it to everybody, in particular to readers of the present work.

Chapter 2

Here, we formally introduce *quasi-flips* of twin buildings and groups with a twin BN-pair. The close correspondence between the two concepts is made precise. Various intermediate results are collected and proven there, which are heavily used later on. One of the most important ones certainly is the following (and its group theoretic counterpart):

Theorem 1 (cf. Theorem 2.5.8)**.** *Let θ be a quasi-flip of a twin building "not defined in characteristic 2". Then any chamber c is contained in a θ-stable twin apartment.*

This theorem was inspired by corresponding work done by Aloysius G. Helminck in [HW93], where a similar result is proved for reductive algebraic groups over fields in characteristic different from 2. However, the methods we employ are building theoretic, and thus e.g. also apply to Kac-Moody groups and finite groups of Lie type. In addition, it can be considered as a special case of a more general theorem proved by Bernhard Mühlherr in his PhD thesis [Müh94]; compared to that theorem, however, the present theorem imposes weaker conditions on G and θ.

We then proceed by studying in more detail when the requirements for the precise version of the preceding theorem are satisfied. The chapter concludes with a parameterization of an interesting double coset decomposition. We only give the version for algebraic and Kac-Moody groups from Chapter 6:

Theorem 2 (cf. Corollaries 6.1.4 and 6.2.2 of Proposition 2.7.2)**.** *Suppose G is a connected isotropic reductive algebraic group, or a split Kac-Moody defined over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$, and P a minimal parabolic $\mathbb{F}$-subgroup. Let θ be an abstract involutory automorphism of G (in the case of Kac-Moody groups, interchanging the two conjugacy classes of Borel groups). Let $\{A_i \mid i \in I\}$ be representatives of the $G_\theta(\mathbb{F})$-conjugacy classes of θ-stable maximal $\mathbb{F}$-split tori in G. Then*

$$G_\theta(\mathbb{F})\backslash G(\mathbb{F})/P(\mathbb{F}) \cong \bigcup_{i\in I} W_{G_\theta(\mathbb{F})}(A_i)\backslash W_{G(\mathbb{F})}(A_i).$$

This generalizes a similar statement for algebraic involutions given in [HW93] (which in turn was a generalization of earlier results on special cases by Matsuki [Mat79], Rossmann [Ros79] and Springer [Spr84]). See also [KW92] for a version for Kac-Moody groups over algebraically closed fields in characteristic 0.

Chapter 3

In later chapters, we frequently perform local-to-global and global-to-local arguments. Accordingly, understanding quasi-flips of Moufang buildings of rank 1 and 2 (Moufang sets and Moufang polygons) is of some importance.

In this chapter, we first present some joint work with Tom De Medts and Ralf Gramlich [DMGH09] where we study transitivity properties of quasi-flips of certain rank 1 buildings, namely projective lines over a skew field. This is used in Chapter 5 to study transitivity properties of quasi-flips of locally split groups and buildings in higher rank.

Furthermore, we study quasi-flips of classical generalized quadrangles. Connectedness of the so-called *flip-flop system* is studied and characterized for these buildings. This is then used in Chapter 4.

Chapter 4

We introduce the *flip-flop system* $\mathcal{C}^\theta$ of a quasi-flip θ of a twin building $\mathcal{C}$. This consists of all chambers of the positive half $\mathcal{C}_+$ which are mapped maximally far away. To be precise,

$$\mathcal{C}^\theta := \{c \in \mathcal{C}_+ \mid l^\theta(c) = \min_{d\in\mathcal{C}_+} l^\theta(d)\}.$$

If $\mathcal{C}$ comes from a group G, and θ comes from a quasi-flip of G, then the centralizer G_θ of θ in G naturally acts on $\mathcal{C}^\theta$. This is for example used in Chapter 5.

In Chapter 4 we study the structure of $\mathcal{C}^\theta$. The key questions we investigate are when $\mathcal{C}^\theta$ is connected as a chamber system, and whether it is *residually connected.* We also study homogeneity properties of $\mathcal{C}^\theta$. Our main theorem is the following, obtained by a local-to-global argument and the a careful analysis of quasi-flips of rank 2 buildings.

Theorem 3 (Theorem 4.1.10, joint work with Gramlich and Mühlherr)**.** *Let $\mathbb{F}$ be a field with $\operatorname{char}\mathbb{F} \neq 2$ and let G be an isotropic connected reductive algebraic or a split Kac-Moody group defined over $\mathbb{F}$ and of type (W, S). Let θ be a quasi-flip of G. Assume the diagram is simply laced; or assume that (W, S) is 2-spherical, G is $\mathbb{F}$-locally split, $|\mathbb{F}| > 4$, and no G_2 residues occur.*

Then the flip-flop system $\mathcal{C}^\theta$ is connected and equals the union of all minimal Phan residues, which in turn all have identical spherical type K. The chamber system of K-residues of $\mathcal{C}^\theta$ is connected and residually connected.

To motivate why we are interested in connectedness of $\mathcal{C}^\theta$, let us just mention that it is one of the key points in the proof of Theorem 6.2.5, which, roughly said, states that G_θ is “usually” finitely generated if G is a locally finite Kac-Moody group with 2-spherical diagram.

Chapter 5

Once more, θ is a quasi-flip of a group G, and G_θ the centralizer of θ in G. In this chapter we turn to studying the action of G_θ on the flip-flop system $\mathcal{C}^\theta$ as introduced in Chapter 4.

The main results of this chapter are all based on the idea of generalizing the Iwasawa decomposition of non-compact connected semi-simple real Lie groups to arbitrary groups with a root group system. We make the following definition:

Definition (Definition 5.4.1)**.** A group G with a twin BN-pair (B_+, B_-, N) admits an **Iwasawa decomposition** if there exists an involution $\theta \in \mathrm{Aut}(G)$ which maps B_+ to B_- and satisfies $G = G_\theta B_+$, where $G_\theta := \mathrm{Fix}_G(\theta)$.

Using the local transitivity results of Chapter 5, we arrive at the following, which is one of the motivations for our interest in generalized Iwasawa decompositions:

Theorem 4 (Theorem 5.4.2, joint work with Gramlich and De Medts)**.** *Consider a group G endowed with a system of root groups $\{U_\alpha\}_{\alpha\in\Phi}$ where the root groups generate G (e.g. a Kac-Moody group or a split semi-simple algebraic group), and with an involution θ such that $G = G_\theta B$ is an Iwasawa decomposition of G. Furthermore, let Π be a system of fundamental roots of Φ and for $\{\alpha, \beta\} \subseteq \Pi$ let $X_{\alpha,\beta} := \langle U_\alpha, U_{-\alpha}, U_\beta, U_{-\beta}\rangle$.*

Then θ induces an involution on each $X_{\alpha,\beta}$ and G_θ is the universal enveloping group of the amalgam $((X_{\alpha,\beta})_\theta)_{\{\alpha,\beta\}\subseteq\Pi}$ of fixed point subgroups of the groups $X_{\alpha,\beta}$.

We also characterize when a group actually admits an Iwasawa decomposition in our sense. We give the version for algebraic and Kac-Moody groups from Chapter 6:

Theorem 5 (Corollaries 6.1.6 and 6.2.4 of Theorem 5.4.7; joint work with Gramlich and De Medts)**.** *Let $\mathbb{F}$ be a field and let G be a split connected reductive algebraic or split Kac-Moody group defined over $\mathbb{F}$. The group of $\mathbb{F}$-rational points $G(\mathbb{F})$ admits an Iwasawa decomposition $G(\mathbb{F}) = G_\theta(\mathbb{F})B(\mathbb{F})$ if and only if $\mathbb{F}$ admits an automorphism σ of order 1 or 2 such that*

(1) -1 is not a norm, and

(2) (i) either a sum of norms is a norm, or

(ii) a sum of norms is ε times a norm, where $\varepsilon \in \{+1, -1\}$, (and this case can only occur if all rank 1 subgroups of G are isomorphic to $\mathrm{PSL}_2(\mathbb{F})$),

with respect to the norm map $N_\sigma : \mathbb{F} \to \mathrm{Fix}_{\mathbb{F}}(\sigma) : x \mapsto xx^\sigma$.

Chapter 6

Here, we specialize some of the key results of the preceding chapters to the case of isotropic reductive algebraic and split Kac-Moody groups, with the hope that it is more accessible to readers familiar with either algebraic or Kac-Moody groups, but with less of a background in building theory. As such, it is intended to be readable on its own, without explicitly requiring the knowledge of previous chapters to understand the results presented there.

Above we already described some of the results presented in this chapter but one more should be mentioned:

Theorem 6 (Theorem 6.2.5)**.** *Suppose G is a split Kac-Moody group of type (W, S) over a finite field $\mathbb{F}_q$, $q \geq 5$ and odd, with 2-spherical diagram (and no G_2 residues). Let θ be a quasi-flip of G, i.e., an involutory automorphism of G which interchanges the two conjugacy classes of Borel groups. Then the centralizer G_θ of θ in G is finitely generated.*

The restriction that no G_2 residue may turn up can probably be dropped. This is subject of research in progress by Hendrik Van Maldeghem and the author [HVM].

Appendices

In Appendix A, we present some results obtained with the help of a computer, as well as the program code that was used. These results complement and complete the analysis of quasi-flips of Moufang polygons as performed in Chapters 3 and 4.

In Appendix B we present a list of (in my eyes) interesting open problems that turned up while working on this thesis. These may serve as inspiration and starting point for future research.

In Appendix C, we sketch how to generalize [BS04] from finite fields to arbitrary fields using the methods developed in Section 3.3.1.

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First and foremost I would like to express my deep gratitude towards my primary advisor Ralf Gramlich, who lead me through this project. I am not sure whether I would have lasted through all this without his guidance and constant support. Ralf taught me far more than just mathematics, and always set a great example for all his students.

Furthermore, I am indebted to my second advisor, Bernhard Mühlherr, whose influence was especially essential in Chapter 4. During two stays in Bruxelles and many fruitful discussions, he helped me further my understanding of the "building" aspect of the problems tackled in this thesis.

I also would like to thank Hendrik Van Maldeghem, who taught me a lot about Moufang polygons and worked with me on the local analysis of the flip-flop systems. An extended version of the results found in Section 3.3 is subject of a forthcoming publication.

Thanks also go to to the following people: Tom De Medts, who worked with us on the Iwasawa results and in particular on the Moufang set aspect of that; I learned a lot about Moufang sets from that. My colleagues and friends Andreas Mars and Stefan Witzel gave moral support and were always open for interesting mathematical discussions, but also for much lighter conversation. Aloysius "Loek" Helminck encouraged me to go on with this project, and provided further insights on the results in [HW93]. Our secretary Gerlinde Gehring was always there for me and helped me focus on my work by doing an outstanding job taking care of all the administrative issues that cropped up during my years in Darmstadt. My parents

encouraged me to go on with studying this (to them) arcane, incomprehensible and plain weird thing I am doing, whatever it is, exactly.

Last but far from least, I wish to thank Silke Möser for too many things to list here. Without her, I probably would have gone crazy before this all was over.

CHAPTER
ONE

PRELIMINARIES

In this chapter, we give a brief introduction to some of the key concepts used throughout the present thesis as well as bibliographical references for further reading. Note that we do not strive to be complete in our literature overview. Instead our main reference throughout this chapter (and considerable parts of the rest of the present thesis) is [AB08], and we try to always include a reference pointing there. Consequently, it should be possible to read large parts of this thesis with loc. cit. as exclusive reference. However, we still try to include many original and alternate references.

Almost everything in this chapter is standard, with possibly the exception of parts of Section 1.3. The reader who is already familiar with the concepts introduced below is welcome to skip parts or all of this chapter.

1.1. Coxeter systems

For a general introduction to Coxeter groups and Coxeter systems, we refer to [Bou68] (and its English translation [Bou02]), [Bro89] (and its successor [AB08]), [Hum90], and for a somewhat different approach, [BB05].

Let G be a group. The order of an element $g \in G$ is denoted by $o(g)$.

Definition 1.1.1. A **Coxeter system** is a pair (W, S) consisting of a group W (called **Coxeter group**) and a set $S \subset W$ such that $\langle S \rangle = W$, $s^2 = 1_W \neq s$ for all $s \in S$ and such that the set S and the relators $((st)^{o(st)})_{s,t \in S}$ constitute a presentation of W.

Example 1.1.2. Let n be a natural number. Let S be the set of all transpositions $(i, i+1)$ for $1 \leq i \leq n$. Then $W := \langle S \rangle$ is isomorphic to the symmetric group of all permutations of the set $\{1, \ldots, n+1\}$, and (W, S) is a Coxeter system.

Remark 1.1.3. A Coxeter group W in general does not uniquely determine the Coxeter system (W, S).

Definition 1.1.4. Let (W, S) be a Coxeter system. The matrix $M(S) := (o(st))_{s,t\in S}$ is called the **type** of (W, S). For an element $w \in W$ we put

$$l_S(W) := \min\{k \in \mathbb{N} \mid w = s_1 s_2 \cdots s_k \text{ where } s_i \in S \text{ for } 1 \leq i \leq k\}.$$

The number $l_S(w)$ is called the **length** of w. If S is clear from context, one commonly writes $l(w)$ instead of $l_S(w)$. A word $w = s_1 \cdot s_2 \cdots s_n$, with $s_i \in S$, is called **reduced** if $l(w) = n$.

Definition 1.1.5. For a subset J of S we put $W_J := \langle J \rangle$. This group is commonly referred to as the **parabolic subgroup** of type J.

Proposition 1.1.6 (Theorem 5.5 in [Hum90])**.** *The pair (W_J, J) is again a Coxeter system, obtained from the original one by restricting S to J. For all $w \in W_J$ one has $l_J(w) = l_S(w)$.*

Definition 1.1.7. If W it is finite, we call (W, S), W and S **spherical**. Let n be an integer. If for all subsets J of S of size at most n the Coxeter system (W_J, J) is spherical, we call (W, S), W and S ***n*-spherical**.

Proposition 1.1.8 (Section 5.5 in [Hum90])**.** *A spherical Coxeter system (W, S) admits a unique* ***longest element****, i.e., an element $w_S \in W$ such that $l(w_S) > l(w)$ for all $w \in W \setminus \{w_S\}$. In general, if J is a spherical subset of S, then we denote the longest element of W_J by w_J.*

In Coxeter systems, the **Exchange condition** holds:

Theorem 1.1.9 (Theorem 5.8 in [Hum90])**.** *Let $w = s_1 \cdots s_r$ ($s_i \in S$), not necessarily a reduced expression. Suppose $s \in S$ satisfies $l(ws) < l(w)$. Then there is an index i for which $ws = s_1 \cdots \widehat{s_i} \cdots s_r$ (omitting s_i). If the expression for w is reduced, then i is unique.*

Definition 1.1.10. Let (W, S) be a Coxeter system. An **automorphism of (W, S)** is a group automorphism of W which normalizes S.

1.2. Roots and root systems

For more on root systems, we refer to [AB08, Appendix B], [Bou68] (and its English translation [Bou02]), [CR08, Part I.1], [Wei03, Chapter 3], to name a few.

Let (W, S) be a Coxeter system. In accordance with [AB08, Section 5.5.4], we define the following:

Definition 1.2.1. For each $s \in S$, the set $\alpha_s = \{w \in W \mid l(sw) > l(w)\}$ is a **simple root** of (W, S). A **root** is a set of the form $w.\alpha_s$, where $w \in W$ and α_s is a simple root.

Let $\Pi := \{\alpha_s \mid s \in S\}$ be the set of simple roots of (W, S), let Φ be the set of all roots of (W, S).

Definition 1.2.2. A root $\alpha \in \Phi$ is called **positive** if $\alpha = w.\alpha_s$ and $l(sw) = l(w)+1$; it is called **negative** if $\alpha = w.\alpha_s$ and $l(sw) = l(w) - 1$.

One can show that every root is either positive or negative, and that if $\alpha = w.\alpha_s$ is a positive root, then $-\alpha := W \setminus \alpha = ws.\alpha_s$ is a negative root.

For $\varepsilon \in \{+,-\}$, let Φ_ε denote the set of positive, resp. negative roots of Φ with respect to Π. For a root $\alpha \in \Phi$, denote by s_α the reflection of W which permutes α and $-\alpha$. For each $w \in W$, define $\Phi_w := \{\alpha \in \Phi_+ \mid w.\alpha \in \Phi_-\}$.

Definition 1.2.3. A pair $\{\alpha, \beta\}$ of roots is called **prenilpotent** if $\alpha \cap \beta$ and $(-\alpha) \cap (-\beta)$ are both nonempty.

In that case denote by $[\alpha, \beta]$ the set of all roots γ of Φ such that $\alpha \cap \beta \subseteq \gamma$ and $(-\alpha) \cap (-\beta) \subseteq -\gamma$, and set $]\alpha, \beta[:= [\alpha, \beta] \setminus \{\alpha, \beta\}$.

1.3. Involutions and twisted involutions of Coxeter groups

In the main body of the present work, we frequently need properties of involutions (elements of order 2) of Coxeter groups. In fact, we need to deal with a somewhat wider class of elements, so-called twisted involutions. The following is based on [Spr84, Section 3] (see also [HW93, Section 7]).

Definition 1.3.1. Let (W, S) be a Coxeter system and θ an automorphism of (W, S) of order at most 2. A ***θ*-twisted involution** in W is an element $w \in W$ with $\theta(w) = w^{-1}$. We denote the set of these elements by $\mathrm{Inv}^\theta(W)$.

Thus $\mathrm{Inv}^{\mathrm{Id}}(W)$ is the set of all involutions of W in the ordinary sense.

Lemma 1.3.2. *Let $w \in \mathrm{Inv}^\theta(W)$ be a θ-twisted involution, let $s \in S$ be arbitrary. Then $l(sw) = l(w\theta(s))$. Moreover if $l(sw\theta(s)) = l(w)$ then $sw = w\theta(s)$.*

Proof. Since θ is an automorphism of (W, S), we have $l(w) = l(\theta(w))$ for all $w \in W$. The first equality follows readily:

$$l(sw) = l((sw)^{-1}) = l(w^{-1}s^{-1}) = l(\theta(w)s) = l(w\theta(s)).$$

The second statement is a consequence of [Spr84, Lemma 3.2]. For the convenience of the reader, here is the proof, adapted from the one given in loc. cit.:

Assume $sw < w$ and $l(sw\theta(s)) = l(w)$. Then we may write $w = s_1 \cdots s_h$ with $s_i \in S$, $s_1 = s$, and $l(w) = h$. Then also $w = \theta(w)^{-1} = \theta(s_h) \cdots \theta(s_1)$. Since $sw < w$, we have by the Exchange condition that $sw = \theta(s_h) \cdots \widehat{\theta(s_i)} \cdots \theta(s_1)$ for some i with $1 \leq i \leq h$. If $i > 1$ then $l(sw\theta(s)) < l(w)$ contradicting our hypothesis. Hence $i = 1$ and $sw\theta(s) = w$.

The proof for $sw > w$ is similar. Assume again $w = s_1 \cdots s_h$ with $s_i \in S$, and $l(w) = h$. Then $sw = ss_1 \cdots s_h$. By hypothesis, we have $l(sw\theta(s)) = l(w) < l(sw)$. Therefore the Exchange condition implies that $sw\theta(s) = s_1 \cdots s_h = w$. □

The next statement is [Spr84, Proposition 3.3(a)], which there is stated only for finite reflection groups but (as remarked in loc. cit.) generalizes to Coxeter groups. We give a purely combinatorial proof.

Proposition 1.3.3. *Let $w \in \mathrm{Inv}^\theta(W)$ be a θ-twisted involution. Then there exists a spherical θ-stable subset I of S and $s_1, \ldots, s_h \in S$ such that*

$$w = s_1 \cdots s_h \cdot w_I \cdot \theta(s_h) \cdots \theta(s_1),$$

where $l(w) = l(w_I) + 2h$.

Proof. We prove the claim by induction on $l(w)$ based on the trivial case $w = 1_W$. Let $l(w) > 0$ and assume that the claim holds for all θ-twisted involutions w' with $l(w') < l(w)$. If there exists $s \in S$ with $l(sw\theta(s)) = l(w) - 2$, then by induction there is nothing to show. By Lemma 1.3.2, it remains to deal with the case that for all $s \in S$ with $l(sw) < l(w)$ the identify $sw\theta(s) = w$ holds. By [AB08, Proposition 2.17 and Corollary 2.18] the set $I := \{s \in S \mid l(sw) < l(w)\}$ is spherical and each reduced I-word can occurs as an initial subword of a reduced decomposition of w; in particular, $l(w_I w) = l(w) - (w_I)$. Hence if there exists $s \in S$ such that $l(w_I ws) < l(w_I w)$ then $l(ws) < l(w)$. In this case Lemma 1.3.2 implies $l(\theta(s)w) < l(w)$, thus $\theta(s) \in I$. But then

$$l(\theta(s)ws) = l(\theta(s)w_I w_I ws) \leq l(\theta(s)w_I) + l(w_I ws) = (l(w_I) - 1) + (l(w_I w) - 1) = l(w) - 2,$$

contrary to our hypothesis that $sw\theta(s) = w$ holds. Accordingly for all $s \in S$ we have $l(w_I ws) > l(w_I w)$. Therefore $w_I w = 1_W$ and $w = w_I$. Finally, the observation $\theta(w) = w^{-1} = w$ implies $\theta(I) = I$. □

Remark 1.3.4. In [Ric82], Richardson gives a complete characterization of involutions of Coxeter groups, based on work done in [Deo82] and [How80]. See also [Hum90, Section 8.2] for a brief summary. However, we shall not make use of this in the present work.

1.4. Chamber systems

Chamber systems where introduced by Tits in [Tit81]. See also [AB08, Section 5.2], [BC], [Ron89], [Wei03].

Definition 1.4.1. Let I be a set. A **chamber system** over I is a pair $(C, (\sim_i)_{i \in I})$, where C is a nonempty set whose elements are called **chambers** and where for each $i \in I$, $\sim_i$ is an equivalence relation on the set of chambers such that if $c \sim_i d$ and $c \sim_j d$ then either $i = j$ or $c = d$.

Definition 1.4.2. The **rank** of a chamber system of type I is the cardinality of I.

All chamber systems (and buildings) considered in the present work are assumed to be of finite rank.

Definition 1.4.3. Given $i \in I$ and $c, d \in C$, then c is called ***i*-adjacent** to d if $c \sim_i d$. The chambers c, d are called **adjacent** if they are i-adjacent for some $i \in I$.

For the rest of this section let $\mathcal{C} = (C, (\sim_i)_{i \in I})$ be a chamber system over I.

Definition 1.4.4. A **gallery** in $\mathcal{C}$ is a finite sequence $(c_0, c_1, \ldots, c_k)$ such that $c_\mu \in C$ for all $0 \leq \mu \leq k$ and such that $c_{\mu-1}$ is adjacent to c_μ for all $1 \leq \mu \leq k$. The number k is called the **length** of the gallery. Given a gallery $G = (c_0, c_1, \ldots, c_k)$, we put $\alpha(G) = c_0$ and $\omega(G) = c_k$. If G is a gallery and if $c, d \in \mathcal{C}$ such that $c = \alpha(G), d = \omega(G)$, then we say that G is a **gallery from *c* to *d*** or ***G* joins *c* and *d***.

Definition 1.4.5. The chamber system $\mathcal{C}$ is said to be **connected** if for any two chambers there exists a gallery joining them.

Definition 1.4.6. A gallery G is called **closed** if $\alpha(G) = \omega(G)$. A gallery $G = (c_0, c_1, \ldots, c_k)$ is called **simple** if $c_{\mu-1} \neq c_\mu$ for all $1 \leq \mu \leq k$.

Given a gallery $G = (c_0, c_1, \ldots, c_k)$, G^{-1} denotes the gallery $(c_k, c_{k-1}, \ldots, c_0)$. Furthermore if $H = (c'_0, c'_1, \ldots, c'_l)$ is a gallery such that $\omega(G) = \alpha(H)$, then GH denotes the gallery $(c_0, c_1, \ldots, c_k = c'_0, c'_1, \ldots, c'_l)$.

Definition 1.4.7. Let J be a subset of I. A ***J*-gallery** is a gallery $G = (c_0, c_1, \ldots, c_k)$ such that for each $1 \leq \mu \leq k$ there exists an index $j \in J$ with $c_{\mu-1} \sim_j c_\mu$.

Definition 1.4.8. Given two chambers c, d, we say that c is ***J*-equivalent** to d, if there exists a J-gallery joining c and d; we write $c \sim_J d$ in this case.

Note that since $\sim_i$ is an equivalence relation, c and d are i-adjacent if and only if they are $\{i\}$-equivalent.

Definition 1.4.9. Given a chamber c and a subset J of I, the set $R_J(c) := \{d \in C \mid c \sim_J d\}$ is called the ***J*-residue** of c. If $J = \{i\}$, then $R_J(c)$ is called the ***i*-panel** of c (or the i-panel containing c); a **panel** is an i-panel for some $i \in I$.

Note that $(R_J(c), (\sim_j)_{j \in J})$ is a connected chamber system over J.

Definition 1.4.10. A chamber system $\mathcal{C}$ over I is called **residually connected** if the following holds: For every subset J of I, and for every family of residues $(R_{I \setminus \{j\}})_{j \in J}$ with the property that any two of these residues intersect nontrivially, we have that $\cap_{j \in J} R_{I \setminus \{j\}}$ is an $(I \setminus J)$-residue.

Lemma 1.4.11 (Lemma 3.6.10 in [BC]). *Let $\mathcal{C}$ be a connected chamber system over I. Then $\mathcal{C}$ is residually connected if and only if the following holds: If J, K, L are subsets of I and R_J, R_K, R_L are J-, K-, L-residues which have pairwise nonempty intersection, then $R_J \cap R_K \cap R_L$ is a $(J \cap K \cap L)$-residue.*

Example 1.4.12. Let $\mathbb{F}$ be a field, and V an $(n+1)$-dimensional vector space over $\mathbb{F}$. Denote by $\mathbb{P}(V)$ the **projective space** over V. It consists of all proper nontrivial vector subspaces of V. Let $\mathcal{C}$ be the set of all maximal flags in $\mathbb{P}(V)$, i.e., strictly ascending sequences $V_1 < V_2 < \ldots < V_n$ of elements of $\mathbb{P}(V)$. Then necessarily $\dim(V_i) = i$ for all $i \in \{1, \ldots, n\}$. We call such a maximal flag a *chamber*. Indeed, we get the structure of a chamber system over $I = \{1, \ldots, n\}$ by defining two chambers $(V_1, \ldots, V_n)$ and $(U_1, \ldots, U_n)$ to be i-adjacent if and only if $V_j = U_j$ for all j different from i.

1.5. Buildings

In the present work, we are only interested in (twin) buildings coming from a group with a (twin) BN-pair, i.e., (twin) buildings admitting a strongly transitive group action. Our main reference for (twin) buildings is [AB08]. For detailed treatments of the theory of buildings, we also refer to [Bro89], [Ron89], [Tit74], [Wei03]. For more on twin buildings, see also [Tit92], [Rém02], [Ron02], [Müh02].

Definition 1.5.1. Let (W, S) be a Coxeter system. A **building** of type (W, S) is a pair $(\mathcal{C}, \delta)$ where $\mathcal{C}$ is a nonempty set and $\delta : \mathcal{C} \times \mathcal{C} \to W$ is a **distance function** satisfying the following axioms, where $x, y \in \mathcal{C}$ and $w = \delta(x, y)$:

(Bu1) $w = 1$ if and only if $x = y$;

(Bu2) if $z \in \mathcal{C}$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z) \in \{w, ws\}$, and if furthermore $l(ws) = l(w) + 1$ then $\delta(x, z) = ws$;

(Bu3) if $s \in S$, there exists $z \in \mathcal{C}$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

For a building $(\mathcal{C}, \delta)$ of type (W, S) and $s \in S$, we define a relation $\sim_s$, where $c, d \in \mathcal{C}$ are s-equivalent, i.e., $c \sim_s d$, if and only if $\delta(c, d) \in \{1_W, s\}$. From the axioms above it follows that this is in fact an equivalence relation, and $(\mathcal{C}, (\sim_s)_{s \in S})$ is a chamber system (see [AB08, Section 5.1.1]). One can actually completely reconstruct the building and its distance function from this chamber system. Hence, in the following, we will not distinguish between the building and its chamber system. In particular, we will speak of galleries, residues and panels of a building.

Definition 1.5.2. The **rank** of a building of type (W, S) is $|S|$.

A building is **thick** (resp. **thin**) if for any $s \in S$ and any chamber $c \in C$ there are at least three (resp. exactly two) chambers s-adjacent to c.

Example 1.5.3. Let (W, S) be a Coxeter system. Define $\delta_S : W \times W \to W : (x, y) \mapsto x^{-1}y$. Then δ_S is a distance function and (W, δ_S) is a thin building of type (W, S). It is not hard to see that any thin building of type (W, S) is isometric to this one.

In the present text, all buildings are assumed to be of finite rank and thick.

For any two chambers x and y we define their numerical distance $l(x,y)$ as $l(\delta(x,y))$.

Definition 1.5.4. Suppose $(\mathcal{C},\delta)$ is a building of type (W,S). Then an **apartment** of $\mathcal{C}$ is a subset Σ of $\mathcal{C}$, such that $(\Sigma,\delta|_\Sigma)$ is isometric to (W,δ_S) (cf. Example 1.5.3).

Definition 1.5.5. A building is called **spherical** if its Coxeter system (W,S) is spherical (i.e., finite). In a spherical building, two chambers c,d are called **opposite** if $\delta(c,d)=w_S$, the longest element of (W,S).

Definition 1.5.6 (Cf. Definition 5.35 from [AB08])**.** Let R be a residue of $\mathcal{C}$.

(1) Given $d\in\mathcal{C}$, the unique chamber $c\in R$ at minimal distance from d is called the **projection of $\boldsymbol{d}$ onto $\boldsymbol{R}$** and is denoted by $\operatorname{proj}_R d$.

(2) If S is another residue, we set $\operatorname{proj}_R S := \{\operatorname{proj}_R d \mid d\in S\}$ and call it the **projection of $\boldsymbol{S}$ onto $\boldsymbol{R}$**. Thus $\operatorname{proj}_R S$ is a subset of R.

Note that $\operatorname{proj}_R S$ is actually a residue on its own (cf. Lemma 5.36 in loc. cit.).

Definition 1.5.7. A nonempty subset $M\subset\mathcal{C}$ is called **connected** if for any two chambers $c,d\in M$, there is a gallery between c and d which is completely contained in M. Moreover, M is called **convex** if for any two chambers $c,d\in M$, every minimal gallery joining c and d in $\mathcal{C}$ is contained in M.

For example, $\mathcal{C}$ is connected and convex; and so is every residue. Also, the intersection of a family of convex sets is convex.

Example 1.5.8. The chamber system of a projective space $\mathbb{P}(V)$ as defined in Example 1.4.12 actually is a building, with Coxeter group Sym_{n+1} if V is $(n+1)$-dimensional. See e.g. [AB08, Section 4.3] for details.

1.6. Twin Buildings

Twin buildings generalize spherical buildings in the sense that there is still the notion of two chambers being opposite, only that now *two* buildings are involved, and chambers in one of the two buildings may be opposite to certain chambers in the other building, and vice versa. This is made precise by the following axioms and their consequences.

Definition 1.6.1. A **twin building** *of type* (W,S) is a triple $(\mathcal{C}_+,\mathcal{C}_-,\delta^*)$ consisting of two buildings $(\mathcal{C}_+,\delta_+)$ and $(\mathcal{C}_-,\delta_-)$ of type (W,S) together with a **codistance** function

$$\delta^* : (\mathcal{C}_+\times\mathcal{C}_-)\cup(\mathcal{C}_-\times\mathcal{C}_+)\to W$$

satisfying the following axioms, where $\varepsilon\in\{+,-\}$, $x\in\mathcal{C}_\varepsilon$, $y\in\mathcal{C}_{-\varepsilon}$ and $w=\delta^*(x,y)$:

(Tw1) $\delta^*(y,x)=w^{-1}$;

(Tw2) if $z \in \mathcal{C}_{-\varepsilon}$ is such that $\delta_{-\varepsilon}(y,z) = s \in S$ and $l(ws) = l(w) - 1$, then $\delta^*(x,z) = ws$;

(Tw3) if $s \in S$, there exists $z \in \mathcal{C}_{-\varepsilon}$ such that $\delta_{-\varepsilon}(y,z) = s$ and $\delta^*(x,z) = ws$.

We remind the reader that in this thesis, all buildings are thick and of finite rank. For the rest of this section let $(\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ be a twin building of type (W,S), and $\varepsilon \in \{+,-\}$. For $x \in \mathcal{C}_\varepsilon$ and $y \in \mathcal{C}_{-\varepsilon}$ we put $l^*(x,y) = l(\delta^*(x,y))$.

In view of (Tw1), the other two axioms have the following "left" analogues:

(Tw2') if $z \in \mathcal{C}_\varepsilon$ is such that $\delta_\varepsilon(x,z) = s \in S$ and $l(sw) = l(w) - 1$, then $\delta^*(z,y) = sw$;

(Tw3') if $s \in S$, there exists $z \in \mathcal{C}_\varepsilon$ such that $\delta_\varepsilon(x,z) = s$ and $\delta^*(z,y) = sw$.

As explained in the previous section, the buildings $\mathcal{C}_\varepsilon$ may be viewed as chamber systems over S.

Definition 1.6.2. A **residue** / **panel** / **gallery** in $\mathcal{C}$ is a residue / panel / gallery in either $\mathcal{C}_+$ or $\mathcal{C}_-$.

Definition 1.6.3. We say that two chambers $c \in \mathcal{C}_\varepsilon$ and $d \in \mathcal{C}_{-\varepsilon}$ ($\varepsilon \in \{+,-\}$) are **opposite**, and write c opp d, if $\delta^*(c,d) = 1_W$. Two residues R in $\mathcal{C}_+$ and S in $\mathcal{C}_-$ are called **opposite** if they have the same type and contain opposite chambers.

Definition 1.6.4. A **twin apartment** of a twin building $\mathcal{C}$ is a pair $\Sigma = (\Sigma_+, \Sigma_-)$ such that Σ_+ is an apartment of $\mathcal{C}_+$, Σ_- is an apartment of $\mathcal{C}_-$, and every chamber in $\Sigma_+ \cup \Sigma_-$ is opposite precisely one other chamber in $\Sigma_+ \cup \Sigma_-$.

There is a generalization of the notion of projections from buildings to twin buildings, at least for spherical residues:

Lemma 1.6.5 (E.g. Lemma 5.149 from [AB08])**.** *If R is a residue in $\mathcal{C}_\varepsilon$ of spherical type, and d is a chamber in $\mathcal{C}_{-\varepsilon}$, then there is a unique chamber $c' \in R$ such that $\delta^*(c',d)$ is of maximal length in $\delta^*(R,d)$. This chamber satisfies*

$$\delta^*(c,d) = \delta_\varepsilon(c,c')\delta^*(c',d)$$

for all $c \in R$. We call c' the ***projection*** *of d onto R and denote it by* $\operatorname{proj}_R(d)$.

Using this extended notion of projections, we can also generalize the concept of convexity.

Definition 1.6.6 (Cf. Definition 5.158 from [AB08])**.** A pair (M_+, M_-) of nonempty subsets $M_+ \subseteq \mathcal{C}_+$ and $M_- \subseteq \mathcal{C}_-$ is called **convex** if $\operatorname{proj}_P c \in M_+ \cup M_-$ for any $c \in M_+ \cup M_-$ and any panel $P \subseteq \mathcal{C}_+ \cup \mathcal{C}_-$ that meets $M_+ \cup M_-$.

Remark 1.6.7. An equivalent way of defining convexity is the following: A pair (M_+, M_-) of nonempty subsets of $M_+ \subseteq \mathcal{C}_+$ and $M_- \subseteq \mathcal{C}_-$ is convex if and only if it is closed under projections.

Example 1.6.8. Any spherical building $\mathcal{C}_+$ of type (W,S) admits an (up to isomorphism unique) twinning with a copy $\mathcal{C}_-$ of itself (see [Tit92, Proposition 1] or e.g. [AB08, Example 5.136] for details): For any chamber $c_+ \in \mathcal{C}_+$, denote the copy (the "twin") of c_+ in $\mathcal{C}_-$ by c_-. The distance on $\mathcal{C}_-$ then is defined as $\delta_-(c_-, d_-) := w_0\delta_+(c_+, d_+)w_0$, and the codistance between the two buildings via $\delta^*(c_+, d_-) = \delta_+(c_+, d_+)w_0$ and $\delta^*(d_-, c_+) = w_0\delta_+(d_+, c_+)$, where w_0 is the longest element of W.

In this construction, the two definitions of being opposite, which we once introduced for buildings and once for twin buildings, coincide here in the following sense: If $c_+, d_+ \in \mathcal{C}_+$ are opposite (i.e., $\delta_+(c_+, d_+) = w_0$) if and only if c_+ and d_- are opposite (i.e. $\delta^*(c_+, d_-) = \delta_+(c_+, d_+)w_0 = 1_W$.

Likewise, projections inside $\mathcal{C}_+$ correspond naturally to projections between $\mathcal{C}_+$ and $\mathcal{C}_-$.

Definition 1.6.9. Two residues R and Q (assumed to be spherical if they are in different halves of the building) are called **parallel** if $\mathrm{proj}_R(Q) = R$ and $\mathrm{proj}_Q(R) = Q$.

1.7. BN-pairs

Our main reference for this section is [AB08, Section 6.2], where all claims made below are proved. Another excellent reference is [Tit74]. Finally, [Bou68, Chapter IV] (and its English translation [Bou02]) seem to contain the original definition.

Definition 1.7.1. We call a pair of subgroups B and N of a group G a ***BN*-pair** if B and N generate G, the intersection $T := B \cap N$ is normal in N, and the quotient group $W := N/T$ admits a set S of generators such that the following conditions hold:

(BN1) $wBs \subseteq BwsB \cup BwB$ for all $w \in W$, $s \in S$;

(BN2) $sBs^{-1} \not\subseteq B$ for all $s \in S$.

The group W is called the **Weyl group** associated to the BN-pair. The quadruple (G, B, N, S) is also called a **Tits system**.

We collect some well-known facts about a group G admitting a BN-pair:

- (W,S) is a Coxeter system.
- $G = \bigsqcup_{w\in W} BwB$, the **Bruhat decomposition** of G.
- Any conjugate of B is called a **Borel subgroup**.
- For each subset $J \subset S$ the set $P_J := \bigsqcup_{w\in W_J} BwB$ is a subgroup of G, called **standard parabolic subgroup** of type J. Any conjugate of P_J is called a **parabolic subgroup**.

- A Tits system (G, B, N, S) leads to a building whose set of chambers equals G/B and whose distance function $\delta : G/B \times G/B \to W$ is given by $\delta(gB, hB) = w$ if and only if $Bh^{-1}gB = BwB$.

Example 1.7.2. The standard example to name here is the group $G = \mathrm{SL}_n(\mathbb{F})$ over any field $\mathbb{F}$, with $n \geq 2$. Let B be the group of upper triangular matrices in G, let N be the group of monomial matrices in G. One readily verifies that $G = \langle B, N\rangle$. Now $T = B \cap N$ is the group of diagonal matrices in G, and $W = N/T$ is clearly isomorphic to the group generated by all $n \times n$ permutation matrices, whence isomorphic to Sym_n, the group of all permutations of the set $\{1, \ldots, n\}$. In particular, it is a Coxeter group.

Moreover, G acts naturally on the vector space $V = \mathbb{F}^n$, but also on the projective space $\mathbb{P}(V)$ (see Examples 1.4.12 and 1.5.8). Indeed, B is the stabilizer in G of the maximal flag $\langle e_1\rangle < \langle e_1, e_2\rangle < \ldots < \langle e_1, \ldots, e_n\rangle$, and G acts transitively on the set of all chambers. Thus, one obtains a bijection between the chambers of $\mathbb{P}(V)$ and the coset space G/B, in accordance with the facts we assembled above.

Example 1.7.3. Let G be a connected reductive algebraic group over an algebraically closed field. Take any Borel group B, and any maximal torus T contained in B. Let $N := N_G(T)$. Then (B, N) form a BN-pair in G.

More generally, if G is a connected reductive (possibly non-split) $\mathbb{F}$-group for any field $\mathbb{F}$, then $G(\mathbb{F})$, the group of $\mathbb{F}$-rational points of G, possesses a BN-pair consisting of a minimal parabolic $\mathbb{F}$-subgroup B and the normalizer N of a maximal $\mathbb{F}$-split torus contained in B.

Remark 1.7.4. The preceding example indicates that our choice of calling the conjugates of the subgroup B “Borel subgroup” is somewhat unfortunate, as it is inconsistent with the theory of algebraic groups. In [AB08], the authors use the term *Tits subgroup* instead, which avoids this confusion. However, for the present thesis, we stick with the term Borel subgroup as it seems to be more common in the literature.

1.8. Twin BN-pairs

References include [Tit92, Section 3.2] and [AB08, Section 6.3.3].

Definition 1.8.1. Let (G, B_+, N, S) and (G, B_-, N, S) be two Tits systems such that $B_+ \cap N = B_- \cap N$, i.e., with equal Weyl groups. Then (B_+, B_-, N) is called a **twin BN-pair** with Weyl group W if the following conditions are satisfied:

(TBN1) $B_\varepsilon w B_{-\varepsilon} s B_{-\varepsilon} = B_\varepsilon w s B_{-\varepsilon}$ for $\varepsilon \in \{+, -\}$ and all $w \in W$, $s \in S$ such that $l(ws) < l(w)$;

(TBN2) $B_+ s \cap B_- = \emptyset$ for all $s \in S$.

In this case, we also say that the tuple (G, B_+, B_-, N, S) is a **twin Tits system**. A twin BN-pair is called **saturated** if $B_+ \cap B_- = T$.

Example 1.8.2 (E.g. Section 6.5 in [AB08])**.** Continuing the $\mathrm{SL}_n(\mathbb{F})$-example from above, we get a twin BN-pair in SL_n by taking as B_+ resp. B_- the upper resp. lower triangular matrices, and for N the monomial matrices. Again, $T = B_+ \cap N = B_- \cap N$ consists of the diagonal matrices. Since $T = B_+ \cap B_-$, this is in fact a saturated twin BN-pair.

Besides the Bruhat decompositions with respect to both B_+ and B_- (a consequence of the fact that (B_+, N) and (B_-, N) are BN-pairs), a group G with a twin BN-pair admits the so-called **Birkhoff decomposition**

$$G = \bigsqcup_{w \in W} B_\varepsilon w B_{-\varepsilon}, \text{ where } \varepsilon \in \{+, -\}.$$

Definition 1.8.3. For $\varepsilon \in \{+, -\}$, any conjugate of B_ε is called a **Borel subgroup** of sign ε. For each subset $J \subset S$ the set $P := \bigsqcup_{w \in W_J} B_\varepsilon w B_\varepsilon$ is a subgroup of G, called **standard parabolic subgroup** of type J and sign ε. Any conjugate of P_J is called **parabolic subgroup** of sign ε.

Remark 1.8.4. A group G with a twin BN-pair hence yields two buildings G/B_+ and G/B_- with distance functions δ_+ and δ_-. Furthermore, using the Birkhoff decomposition we can define the codistance function $\delta^* : (G/B_- \times G/B_+) \cup (G/B_+ \times G/B_-) \to W$ via $\delta^*(gB_-, hB_+) = w$ if and only if $B_+ h^{-1} g B_- = B_+ w B_-$ and $\delta^*(hB_+, gB_-) := (\delta^*(gB_-, hB_+))^{-1}$. The tuple $((G/B_+, \delta_+), (G/B_-, \delta_-), \delta^*)$ then is a twin building, the **twin building associated to $\boldsymbol{G}$**.

Example 1.8.5 (E.g. Sections 6.9 and 6.12 in [AB08])**.** Let $n \geq 2$. Above we have seen that the group $\mathrm{SL}_n(\mathbb{F})$ over any field $\mathbb{F}$ admits a natural twin BN-pair. However, in the above example, B_+ and B_- are conjugate, and each half of the building is spherical. This means that we do not really get anything new from the twinning.

Now consider $\mathrm{SL}_n(\mathbb{F}[t, t^{-1}])$, where $\mathbb{F}$ is an arbitrary field and $\mathbb{F}[t, t^{-1}]$ is the ring of Laurent polynomials over $\mathbb{F}$. Again, we can endow this with the BN-pair consisting of the groups of upper and lower triangular matrices, as well as N equal to the groups of monomial matrices. But there is a second, fundamentally different twin BN-pair: Let B_+ be the set of matrices in $\mathrm{SL}_n(\mathbb{F}[t])$ which are upper triangular modulo t, and likewise let B_- be the set of matrices in $\mathrm{SL}_n(\mathbb{F}[t^{-1}])$ which are upper triangular modulo t^{-1}. Finally, N as before is the set of monomial matrices.

One can now verify that (B_+, B_-, N) indeed constitutes a twin BN-pair. Moreover, using basic matrix calculations, one can readily verify that B_+ and B_- are not conjugate inside G. Indeed, the Weyl group of this twin BN-pair is of type $\widetilde{A}_{n-1}$, in particular infinite. So the two associated buildings are not spherical, and we get a "true" twin building.

1.9. Root group systems

The following definition is based on [AB08, Definition 7.82 and Section 8.6.1], which in turn is derived from [Tit92, Section 3.3]. See also [CR08] for another accessible introduction.

Definition 1.9.1. Let G be a group endowed with a family $\{U_\alpha\}_{\alpha\in\Phi}$ of subgroups, indexed by a root system Φ of type (W,S). Let T be another subgroup of G. Then the triple $(G,\{U_\alpha\}_{\alpha\in\Phi},T)$ is called an **RGD-system of type (W,S)** if it satisfies the conditions below, where $U_\pm := \langle U_\alpha \mid \alpha \in \Phi_\pm\rangle$:

(RGD0) For each $\alpha \in \Phi$, we have $U_\alpha \neq \{1\}$.

(RGD1) For every prenilpotent pair $\{\alpha,\beta\} \subset \Phi$ of distinct roots, we have $[U_\alpha, U_\beta] \subset \langle U_\gamma \mid \gamma \in]\alpha,\beta[\rangle$.

(RGD2) For each $s \in S$ and each $u \in U_{\alpha_s} \setminus \{1\}$, there exist elements u', u'' of $U_{-\alpha_s}$ such that the product $\mu(u) := u'uu''$ conjugates U_β onto $U_{s(\beta)}$ for each $\beta \in \Phi$.

(RGD3) For each $s \in S$ we have $U_{-\alpha_s} \not\subseteq U_+$.

(RGD4) $G = T.\langle U_\alpha \mid \alpha \in \Phi\rangle$.

(RGD5) T normalizes U_α for each $\alpha \in \Phi$, i.e.,

$$T \leq \bigcap_{\alpha\in\Phi} N_G(U_\alpha).$$

Then the U_α are called **root subgroups** and the pair $(\{U_\alpha\}_{\alpha\in\Phi},T)$ is referred to as a **root group datum**.

We state the following without proof, but refer the reader to [Tit92, Proposition 4] or [AB08, Theorem 8.80] for details.

Proposition 1.9.2. *Let $(G,\{U_\alpha\}_{\alpha\in\Phi},T)$ be an RGD-system of type (W,S). Define*

$$\begin{aligned} N &:= T.\langle \mu(u) \mid u \in U_\alpha\setminus\{1\}, \alpha \in \Pi\rangle, \\ B_+ &:= T.U_+, \\ B_- &:= T.U_-. \end{aligned}$$

Then (G,B_+,B_-,N,S) is a saturated twin BN-pair of G with Weyl group $N/T \cong W$. We call it the twin BN-pair ***associated*** *to the root group datum.* □

Hence, to every RGD-system, a (Moufang) twin building is associated in a natural way.

Definition 1.9.3. An RGD-system $(G,\{U_\alpha\}_{\alpha\in\Phi},T)$ is called **faithful** if G operates faithfully on the associated building. It is called **centered** if G is generated by its root groups, and **reduced** if it is both centered and faithful.

Lemma 1.9.4 (E.g. Lemma 8.55 and Section 8.8 in [AB08])**.** *$N_G(U_+) = B_+ = N_G(B_+)$ and $N_G(U_-) = B_- = N_G(B_-)$.*

As a consequence of this Lemma, the following is well-defined:

Definition 1.9.5. Let $B = gB_\varepsilon g^{-1}$ be an arbitrary Borel subgroup, where $\varepsilon \in \{+,-\}$. Then the **unipotent radical** $U(B)$ of B is the corresponding conjugate $B = gU_\pm g^{-1}$ of U_+ or U_-.

Remark 1.9.6. In general, the group $U(B)$ will be neither nilpotent nor a radical, so the name unipotent radical is somewhat misleading. Nevertheless, we chose this name in lack of a better one, and since it is also used like that elsewhere in the literature, e.g. [CM06]. Note also that one can define unipotent radicals geometrically and for arbitrary parabolic subgroups of spherical type, but we do not need this here.

Definition 1.9.7 (See Section 3.3 in [Tit92])**.** For any RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$, denote by G° the quotient of the subgroup $\langle U_\alpha \mid \alpha \in \Phi\rangle$ by its center, and by U_α° the canonical image of U_α in G°. Unless there exists a root orthogonal to all other roots, the canonical homomorphisms $U_\alpha \to U_\alpha^\circ$ are isomorphisms. Then $(G^\circ, \{U_\alpha^\circ\}_{\alpha\in\Phi})$ is a reduced RGD-system with the same associated twin building as $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$. For this reason $(G^\circ, \{U_\alpha^\circ\}_{\alpha\in\Phi})$ is called the **reduction** of $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$.

Definition 1.9.8. We set $X_\alpha := \langle U_\alpha, U_{-\alpha}\rangle$ and $X_{\alpha,\beta} := \langle X_\alpha, X_\beta\rangle$. A root group datum is called **locally split** if the group T is abelian and if for each $\alpha \in \Phi$ there is a field $\mathbb{F}_\alpha$ such that X_α is isomorphic to $\mathrm{SL}_2(\mathbb{F}_\alpha)$ or $\mathrm{PSL}_2(\mathbb{F}_\alpha)$ and $\{U_\alpha, U_{-\alpha}\}$ is isomorphic to its natural root group datum. A locally split root group datum is called **$\mathbb{F}$-locally split** if $\mathbb{F}_\alpha = \mathbb{F}$ for all $\alpha \in \Phi$.

Connected reductive algebraic groups and (split) Kac-Moody groups are examples of groups with a root group datum, cf. Chapter 6.

1.10. Moufang sets and pointed Moufang sets

In this section we give a brief introduction to Moufang sets. The text in this section is an adaption of [DMGH09, Section 5]. For a more complete introduction to Moufang sets, see e.g. [DMS].

In order to be consistent with the standard notation used in the theory of Moufang sets we will always denote the action of a permutation on a set on the right, i.e., we will write $a\varphi$ rather than $\varphi(a)$.

Definition 1.10.1. A **Moufang set** is a set X of size at least two together with a collection of groups $(U_x)_{x\in X}$, such that each U_x is a subgroup of $\mathrm{Sym}(X)$ fixing x and acting regularly (i.e., sharply transitively) on $X \setminus \{x\}$, and such that each U_x permutes the set $\{U_y \mid y \in X\}$ by conjugation. The group $G := \langle U_x \mid x \in X\rangle$ is called the **little projective group** of the Moufang set; the groups U_x are called **root groups**.

Our approach to Moufang sets is taken from [DMW06]. Let $\mathbb{M} = (X, (U_x)_{x \in X})$ be an arbitrary Moufang set, and assume that two of the elements of X are called 0 and ∞. Let $U := X \setminus \{\infty\}$. Each $\alpha \in U_\infty$ is uniquely determined by the image of 0 under α. If $0\alpha = a$, we write $\alpha =: \alpha_a$. Hence $U_\infty = \{\alpha_a \mid a \in U\}$. We make U into a (not necessarily abelian) group with composition $+$ and identity 0, by setting

$$a + b := a\alpha_b\,. \tag{1.1}$$

Clearly, $U \cong U_\infty$. Now let τ be an element of G interchanging 0 and ∞. (Such an element always exists, since G is doubly transitive on X.) By the definition of a Moufang set, we have

$$U_0 = U_\infty^\tau \text{ and } U_a = U_0^{\alpha_a}$$

for all $a \in U$. In particular, the Moufang set $\mathbb{M}$ is completely determined by the group U and the permutation τ; we will denote it by $\mathbb{M} = \mathbb{M}(U, \tau)$.

Remark 1.10.2. In view of equation (1.1), it makes sense to use the convention that $a + \infty = \infty + a = \infty$ for all $a \in U$.

Definition 1.10.3. For each $a \in U$, we define $\gamma_a := \alpha_a^\tau$, i.e., $x\gamma_a = (x\tau^{-1} + a)\tau$ for all $x \in X$. Consequently, $U_0 = \{\gamma_a \mid a \in U\}$.

Definition 1.10.4. For each $a \in U^* = U \backslash \{0\}$, we define a **Hua map** to be

$$h_a := \tau\alpha_a\tau^{-1}\alpha_{-(a\tau^{-1})}\tau\alpha_{-(-(a\tau^{-1}))\tau} \in \mathrm{Sym}(X);$$

if we use the convention of Remark 1.10.2, then we can write this explicitly as $h_a : X \to X : x \mapsto \big((x\tau + a)\tau^{-1} - a\tau^{-1}\big)\tau - \big(-(a\tau^{-1})\big)\tau$. We define the **Hua subgroup** of $\mathbb{M}$ as $H := \langle h_a \mid a \in U^* \rangle$.

Remark 1.10.5. Observe that each h_a fixes the elements 0 and ∞. By [DMW06, Theorem 3.1], the group H equals $G_{0,\infty} := \mathrm{Stab}_G(0, \infty)$, and by [DMW06, Theorem 3.2], the restriction of each Hua map to U is additive, i.e., $H \leq \mathrm{Aut}(U)$.

Definition 1.10.6. For each $a \in U^*$, we define a $\boldsymbol{\mu}$**-map** $\mu_a := \tau^{-1}h_a$.

Note that μ_a is the unique element in the set $U_0^*\alpha_a U_0^*$ interchanging 0 and ∞. In particular, $\mu_a^{-1} = \mu_{-a}$.

Definition 1.10.7. Let $(X, (U_x)_{x \in X})$ and $(Y, (V_y)_{y \in Y})$ be two Moufang sets. A bijection β from X to Y is called an **isomorphism** of Moufang sets, if the induced map $\chi_\beta : \mathrm{Sym}(X) \to \mathrm{Sym}(Y) : g \mapsto \beta^{-1}g\beta$ maps each root group U_x isomorphically onto the corresponding root group $V_{x\beta}$. An **automorphism** of $\mathbb{M} = (X, (U_x)_{x \in X})$ is an isomorphism from $\mathbb{M}$ to itself. The group of all automorphisms of $\mathbb{M}$ will be denoted by $\mathrm{Aut}(\mathbb{M})$.

Now we introduce pointed Moufang sets, which will be Moufang sets with a fixed identity element. We will then, in analogy with the theory of Jordan algebras, introduce the notions of an isotope of a pointed Moufang set, and we will define Jordan isomorphisms between Moufang sets.

Definition 1.10.8. A **pointed Moufang set** is a pair $(\mathbb{M}, e)$, where $\mathbb{M} = \mathbb{M}(U, \rho)$ is a Moufang set and e is an arbitrary element of U^*. The $\boldsymbol{\tau}$**-map** of this pointed Moufang set is $\tau := \mu_{-e} = \mu_e^{-1}$, and the **Hua maps** are the maps $h_a := \tau\mu_a = \mu_{-e}\mu_a$ for all $a \in U^*$. We also define the **opposite Hua maps** $g_a := \tau^{-1}\mu_a = \mu_e\mu_a$ for all $a \in U^*$. Clearly, $\mathbb{M} = \mathbb{M}(U, \tau) = \mathbb{M}(U, \tau^{-1})$.

Definition 1.10.9. Let $(\mathbb{M}, e)$ and $(\mathbb{M}', e')$ be two pointed Moufang sets, with $\mathbb{M} = \mathbb{M}(U, \rho)$ and $\mathbb{M}' = \mathbb{M}(U', \rho')$. A **pointed isomorphism** from $(\mathbb{M}, e)$ to $(\mathbb{M}', e')$ is an isomorphism from U to U' mapping e to e' and extending to a Moufang set isomorphism from $\mathbb{M}$ to $\mathbb{M}'$ (by mapping ∞ to ∞'). A pointed isomorphism from $(\mathbb{M}, e)$ to itself is called a *pointed automorphism* of $(\mathbb{M}, e)$, and the group of all pointed automorphisms is denoted by $\mathrm{Aut}(\mathbb{M}, e)$.

Definition 1.10.10. Let $(\mathbb{M}, e)$ be a pointed Moufang set, and let $a \in U^*$ be arbitrary. Then $(\mathbb{M}, a)$ is called the $\boldsymbol{a}$**-isotope** of $(\mathbb{M}, e)$, or simply an **isotope** if one does not want to specify the element a. The τ-map and the Hua maps of $(\mathbb{M}, a)$ will be denoted by $\tau^{(a)}$ and $h_b^{(a)}$, respectively. Observe that

$$\tau^{(a)} = \mu_{-a} \quad \text{and} \quad h_b^{(a)} = \mu_{-a}\mu_b = h_a^{-1}h_b$$

for all $a, b \in U^*$.

Remark 1.10.11. Our notion of an a-isotope is, in a certain sense, the inverse of the usual notion of an a-isotope in (quadratic) Jordan algebras, where our a-isotope would be called the a^{-1}-isotope (where a^{-1} denotes the inverse in the Jordan algebra) and where $h_b^{(a)} := h_a h_b$. It is, in the general context of Moufang sets, not natural to try to be compatible with this convention, because h_a^{-1} is in general not of the form h_b for some $b \in U^*$. In fact, we have $h_a^{-1} = g_{a\tau}$ for all $a \in U^*$; see [DMW06, Lemma 3.8(i)].

Definition 1.10.12. Let $(\mathbb{M}, e)$ and $(\mathbb{M}', f)$ be two pointed Moufang sets with $\mathbb{M} = \mathbb{M}(U, \rho)$ and $\mathbb{M}' = \mathbb{M}(U', \rho')$, and with Hua maps h_a and h'_a, respectively. An isomorphism φ from U to U' is called a **Jordan isomorphism** if $(bh_a)\varphi = (b\varphi)h'_{a\varphi}$ for all $a, b \in U^*$. A Jordan isomorphism from $(\mathbb{M}, e)$ to $(\mathbb{M}, a)$ is called an **isotopy** from $(\mathbb{M}, e)$ to its a-isotope. Explicitly, a map $\varphi \in \mathrm{Aut}(U)$ is an isotopy if and only if

$$h_a\varphi = \varphi h_{a\varphi}^{(e\varphi)}$$

for all $a \in U^*$. The group of all isotopies from $(\mathbb{M}, e)$ to an isotope is called the **structure group** of $(\mathbb{M}, e)$, and is denoted by $\mathrm{Str}(\mathbb{M}, e)$.

Note that it is not clear whether $\mathrm{Str}(\mathbb{M}, e) \leq \mathrm{Aut}(\mathbb{M})$. Also observe that $G \cap \mathrm{Str}(\mathbb{M}, e) = H$; we call H the **inner structure group** of $(\mathbb{M}, e)$.

CHAPTER

TWO

FLIPS

In Section 2.1, we introduce the concept of *flips* (and their slightly more general siblings, the *quasi-flips*), first in the context of twin buildings, then in the context of groups with twin BN-pairs. Flips are essentially involutory automorphisms which interchange the two halves of a twin building, resp. the conjugacy classes of Borel groups of plus and of minus sign.

We demonstrate the close correspondence between these two kinds of flips (on buildings and on groups) in Section 2.2, where we prove that a twin BN-quasi-flip of a group G with twin BN-pair induces a unique twin building quasi-flip on the twin building associated to G. The converse is shown to hold under certain conditions as well.

A brief detour in Section 2.4 is used to introduce *strong flips*, which are an important special class of flips, for which a lot of the theory developed throughout this thesis simplifies considerably, permitting more uniform and maybe also more elegant approaches. Also, flips that were studied in the past (e.g. as part of the Phan program), have usually been strong.

In Section 2.3 we then briefly present the notion of *steep descent*. This is a basic yet important tool in the further study of flips throughout the rest of this chapter as well as in later parts of the present thesis.

In fact, in the following Section 2.5, we apply steep descent to prove that under some mild conditions, any chamber of a twin building with a quasi-flip θ is contained in a θ-stable apartment. In Section 2.6 we study in some more detail when the aforementioned mild conditions are satisfied.

This all then culminates in Section 2.7, where the main result of this chapter is presented: A double coset decomposition of a group G endowed with a RGD-system and a quasi-flip θ, generalizing previous results on algebraic groups (in characteristic different from 2) and Kac-Moody groups (in characteristic 0, for algebraically closed fields).

2.1. Building flips and BN-flips

Throughout this section, $\mathcal{C} = (\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ denotes a Moufang twin building of type (W, S) (see Section 1.6). Moreover, G is a group acting strongly transitively on $\mathcal{C}$, hence is endowed with a twin BN-pair (B_+, B_-, N) (see Section 1.8).

Building quasi-flips

We now present the definition of a building flip, a concept which has been introduced in [BGHS03], albeit in a different form. Here we give a more general definition compared to what appeared previously in the literature, subsuming all kinds of building flips known to us.

Definition 2.1.1. A **building quasi-flip** of $\mathcal{C}$ is a permutation θ of $\mathcal{C}_+ \cup \mathcal{C}_-$ with the following properties:

(1) $\theta^2 = \mathrm{id}$;

(2) $\theta(\mathcal{C}_+) = \mathcal{C}_-$;

(3) θ preserves adjacency and opposition, i.e., for $\varepsilon \in \{+, -\}$ and for all $x, y \in \mathcal{C}_\varepsilon$, $z \in \mathcal{C}_{-\varepsilon}$ we have $x \sim y$ if and only if $\theta(x) \sim \theta(y)$; and x opp z if and only if $\theta(x)$ opp $\theta(z)$.

If, additionally,

(3*) θ flips the distances and preserves the codistance, i.e., for $\varepsilon \in \{+, -\}$ and for all $x, y \in \mathcal{C}_\varepsilon$, $z \in \mathcal{C}_{-\varepsilon}$ we have $\delta_\varepsilon(x, y) = \delta_{-\varepsilon}(\theta(x), \theta(y))$; and $\delta^*(x, z) = \delta^*(\theta(x), \theta(z))$,

we call θ a **building flip**.

Remark 2.1.2. In Example 1.6.8, we saw that any spherical building $\mathcal{C}_+$ of type (W, S) admits an (up to isomorphism unique) twinning with a copy $\mathcal{C}_-$ of itself.

Let ϕ be an arbitrary involutory automorphism of $\mathcal{C}$. Then we obtain a quasi-flip θ as follows: For $c_+ \in \mathcal{C}_+$, suppose $d_+ = \phi(c_+)$. Then define $\theta(c_+) := d_-$ and $\theta(c_-) := d_+$. This is a well-defined map of order 2, interchanging the halves of the spherical twin building, preserving adjacency by virtue of its definition. Due to the way δ^* was defined, this also implies that θ preserves opposition.

More generally, if ϕ is an *almost isometry* in the sense of [AB08, Section 5.5.1] (that is, it is an isometry up to a permutation of S), we can derive a building quasi-flip from it as sketched above.

Conversely, any quasi-flip θ of a spherical twin building induces an almost isometry ϕ on the positive half of the twin building by setting $\phi(c_+) := \theta(c_-)$. This will be a consequence of Lemma 2.1.4 below.

In view of this remark, we may occasionally talk about quasi-flips of spherical buildings, which by the above are simply almost isometries of order 2.

Historically, the following (spherical) example is the prototype of all flips, see [BS04].

Example 2.1.3. Let V be an $(n+1)$-dimensional vector space over a field $\mathbb{F}$ of characteristic different from 2. Denote by $\mathcal{C}$ the spherical building of type A_n associated to V, which is the chamber complex of the projective space $\mathbb{P}(V)$ of proper nontrivial subspaces of V. (See e.g. [AB08, Section 4.3] for details.)

On this space, consider a non-degenerate unitary or orthogonal form. From this we obtain an involutory map ϕ on the chamber system as follows: Recall that a chamber is a maximal flag of subspaces $c = (V_1 < V_2 < \ldots < V_n)$, where $\dim V_i = i$. We send each V_i to its orthogonal complement $V_i^\perp$ of dimension $n+1-i$. Thus c is mapped to the chamber $(V_n^\perp < \ldots < V_1^\perp)$. The result is a so-called **polarity**.

To see that this is actually a (quasi-)flip in our sense, first recall Remark 2.1.2, which says that any almost isometry of a spherical building induces a quasi-flip of the corresponding twin building. Now, we claim that property (3*) holds, which at first might seem counter-intuitive, as we map i-dimensional subspaces to $(n-i+1)$-dimensional subspaces, which is not type preserving in the spherical setting, but rather induces a diagram automorphism. But (3*) was defined in the twin building context, so taking Remark 2.1.2 into account, we have to check whether $\delta_+(c_+, d_+) = \delta_-(\phi(c_-), \phi(d_-)) = w_0 \delta_+(\phi(c_+), \phi(d_+)) w_0$. It turns out that conjugating by w_0 precisely cancels the type changing effect of the diagram automorphism.

It is clear that property (3*) implies property (3). The converse is not true, but the following holds:

Lemma 2.1.4. *Let θ be a building quasi-flip. Then θ induces an automorphism $\tilde{\theta}$ of the Coxeter system (W, S) of order at most 2, such that for $\varepsilon \in \{+, -\}$ and for all $x, y \in \mathcal{C}_\varepsilon$, $z \in \mathcal{C}_{-\varepsilon}$ we have $\tilde{\theta}(\delta_\varepsilon(x, y)) = \delta_{-\varepsilon}(\theta(x), \theta(y))$; and $\tilde{\theta}(\delta^*(x, z)) = \delta^*(\theta(x), \theta(z))$. In particular, $\tilde{\theta}$ permutes S.*

Proof. Fix a chamber c. By its definition, θ maps panels to panels. Hence for every $s \in S$ there exists $t \in S$, such that the s-panel of c is mapped to the t-panel of $\theta(c)$ and vice versa. We obtain a permutation σ of the set S. We have to prove that this permutation has order at most 2 and extends uniquely to an automorphism $\tilde{\theta}$ of (W, S) satisfying all claimed properties.

We start by arguing that twin apartments are mapped to twin apartments: Let Σ be an arbitrary twin apartment containing c. Let d be the unique chamber in Σ opposite c. From the definition of θ it is clear that it preserves numerical distances, and thus convex sets of either half of the twin building. To conclude that it maps twin apartments to twin apartments, it remains to show that it also preserves the numerical codistance.

For arbitrary $s \in S$, let c' be the projection of d to $P_s(c)$ and d' the projection of c to $P_s(d)$. Then c', d' are contained in Σ and opposite to each other. Since θ

preserves adjacency, there exist $t, r \in S$ such that $\theta(c) \sim_t \theta(c')$ and $\theta(d) \sim_r \theta(d')$. Additionally, as c' is opposite d' but not opposite d, we find that $\theta(c')$ is opposite $\theta(d')$, but not opposite $\theta(d)$. Hence by (Tw2) we must have $\delta^*(\theta(c'), \theta(d)) = t$. But $\delta(\theta(d'), \theta(d)) = r$ and $\delta^*(\theta(c'), \theta(d')) = 1$, thus again by (Tw2), $r = t$, or equivalently, our permutation σ has order at most 2. In particular, $\theta(d')$ and $\theta(c')$ are contained in the twin apartment spanned by $\theta(c)$ and $\theta(d)$. Since c and s were arbitrary, we conclude that $\theta(\Sigma)$ is again a twin apartment.

Now Σ and $\theta(\Sigma)$ are both isomorphic to the Coxeter complex of (W, S), and hence to each other. There is a unique type-preserving isomorphism ι between them mapping c to $\theta(c)$. Accordingly, $\theta \circ \iota$ is an automorphism of Σ fixing c, which induces a well-defined automorphism $\tilde{\theta}_\Sigma$ of (W, S) which corresponds to the permutation σ when restricted to S.

But any chamber d is contained in a twin apartment also containing c. In particular, any panel of the building meets a twin apartment containing c. Since $\tilde{\theta}$ is fully determined by σ, we conclude that every s-panel of the building is mapped to a $\sigma(s)$-panel. The claim follows. □

In the sequel, we also denote, by slight abuse of notation, the induced automorphism $\tilde{\theta}$ of W by θ.

So all in all, the difference between a building flip and a building quasi-flip is that the former is type-preserving, while the latter might additionally involve a diagram automorphism. Or, flips and quasi-flips are related like isometries and almost isometries.

Remark 2.1.5. There are quasi-flips which are not flips. For example, start with an A_n building as in Example 2.1.3, but this time let ϕ be an involutory automorphism of V, say, a reflection. This is an isometry of the spherical building, but on the twin-building, the induced quasi-flip is no longer type preserving.

Alternatively, take a polarity ψ of a generalized quadrangle. As in Example 2.1.3, for this to induce a flip on the twin building, it would have to satisfy

$$\delta_+(c_+, d_+) = \delta_-(\psi(c_-), \psi(d_-)) = w_0 \delta_+(\psi(c_+), \psi(d_+)) w_0.$$

But for a quadrangle, the Weyl group is of type $B_2 = C_2$, and the longest element w_0 is central, hence ψ is a flip if and only if

$$\delta_+(c_+, d_+) = \delta_+(\psi(c_+), \psi(d_+)),$$

which would be true if ψ was an involutory automorphism of the quadrangle, but since it is a polarity, the property does not hold.

Later on, we will be interested in "how far" a quasi-flip moves a chamber. This is captured by the following definition:

Definition 2.1.6. For a chamber c, we call $w = \delta^*(c, \theta(c)) \in W$ the **$\boldsymbol{\theta}$-codistance** of c and write $\delta^\theta(c) := w$. We also set $l^\theta(c) := l(\delta^\theta(c)) = l(w)$, the **numerical $\boldsymbol{\theta}$-codistance**.

The farthest a chamber can possibly be mapped is to an opposite chamber. Flips which admit such chambers have special properties, so we give them a name.

Definition 2.1.7. We call a building (quasi-)flip **proper** if there exists a chamber c with θ-codistance 1_W, that is, $\delta^*(c, \theta(c)) = 1_W$. A proper building flip is also called **Phan involution**.

Generalizing the idea of chambers mapped to opposite ones, we arrive at the following:

Definition 2.1.8. We call a residue R of $\mathcal{C}$ a **Phan residue** if R is opposite $\theta(R)$ (meaning that for every chamber in R there exists a chamber opposite to it in $\theta(R)$, and vice versa). A **minimal Phan Residue** is a Phan residue which is minimal by inclusion, i.e., which does not contain any other Phan residue. Finally, we call a chamber c a **Phan chamber** if c and $\theta(c)$ are opposite.

With the above terminology, a Phan chamber is simply a chamber with θ-codistance 1_W, and a (quasi)-flip is proper if and only if it admits a Phan chamber.

Examples 2.1.9. Assume again the setting described in Example 2.1.3: Let V be an $(n+1)$-dimensional vector space over a field $\mathbb{F}$ of characteristic different from 2. Associated to this is the projective space $\mathbb{P}(V)$ of proper nontrivial subspaces of V, a spherical building.

(1) Assume that f is a non-degenerate unitary, symplectic or orthogonal form of V. As in Example 2.1.3, this induces a (quasi-)flip of $\mathbb{P}(V)$. Now, this flip is proper if and only if there exists a chamber $(V_1 < V_2 < \ldots < V_n)$ which is opposite its image $(V_n^\perp < \ldots < V_1^\perp)$. This holds if and only if $V_i \cap V_{n+1-i}^\perp = \{0\}$ (equivalently $V_i \oplus V_{n+1-i}^\perp = V$) for all $i \in \{1, \ldots, n\}$. This is true if any only if the form is anisotropic (i.e., no vector is orthogonal to itself). Symplectic forms are never anisotropic (there, every vector is orthogonal itself), which leaves the unitary or orthogonal forms.

(2) Consider $V = \mathbb{R}^{n+1}$, endowed with the standard scalar product with respect to some orthogonal basis $e_1, \ldots, e_{n+1}$ of V. Then this form induces a flip, which we claim is a proper flip.

For let $c = (\langle e_1 \rangle < \langle e_1, e_2 \rangle < \ldots < \langle e_1, e_2 \ldots, e_n \rangle)$ be a chamber of $\mathbb{P}(V)$. Then our flip sends $\langle e_1 \rangle \mapsto \langle e_2, \ldots, e_{n+1} \rangle$, $\langle e_1, e_2 \rangle \mapsto \langle e_3, \ldots, e_{n+1} \rangle$, and so on. For this reason, our starting chamber is interchanged with $d = (\langle e_{n+1} \rangle < \ldots < \langle e_3, \ldots, e_{n+1} \rangle < \langle e_2, \ldots, e_{n+1} \rangle)$. One readily verifies that c and d are opposite.

(3) Consider again $V = \mathbb{R}^{2n}$ but this time with a symplectic from $(\cdot,\cdot)$ and corresponding basis $e_1,\dots,e_n,f_1,\dots f_n$ with $(e_i,f_j)=\delta_{ij}$, $(f_i,e_j)=-\delta_{ij}$ and $(e_i,e_j)=(f_i,f_j)=0$ for all $i,j\in\{1,\dots,n\}$. Then any subspace U of odd dimension (e.g. $\langle e_1\rangle$) has a nontrivial radical (i.e., $U\cap U^\perp\neq\{0\}$). So if $V_1<\dots<V_{2n-1}$ is a maximal flag, then $V_1\cap V_1^\perp=V_1$, hence $V_{2n-1}\cap V_1^\perp=V_1\neq\{0\}$. Accordingly, no chamber is mapped to an opposite one, and the flip is improper. In fact, the minimal codistance (maximal distance) one can achieve between a chamber and its image is $s_1\cdot s_3\cdots s_{2n-1}$, assuming the diagram A_{2n} is labeled from 1 to $2n$.

For a detailed analysis of symplectic flips, we refer to [BH08].

(4) Continuing this, there are also proper building *quasi*-flips (not type-preserving): Take the linear map sending each e_i to e_{n-i+1}. We have seen that this induces a quasi-flip which is not a flip, and it swaps the opposite chambers c and d.

Example 2.1.10 (See [PT84])**.** Consider now the permutation $\theta=(15)(24)(36)\in S_6$. Clearly, θ is a nontrivial involution and preserves distances. Since a generalized quadrangle is a building in which the longest element of the Weyl group is central, it also automatically preserves the codistance. Hence θ is a building flip.

However, one easily verifies that for all points p we have that $p\perp\theta(p)$. Thus θ does not map any chamber to an opposite chamber, i.e., it is not a Phan involution. Still, the opposite lines $l_1=(12)(34)(56)$ and $l_2=(13)(26)(45)$ are interchanged.

Hence this is an example of an improper flip.

Example 2.1.11. Assume we are given a quasi-flip θ of a twin building $\mathcal{C}$, and a second twin building $\mathcal{C}'$ of spherical type (W,S). Then $\theta\times\mathrm{id}$ is a quasi-flip of the building $\mathcal{C}\times\mathcal{C}'$, with minimal θ-codistance equal to the minimal θ-codistance of θ times the longest element in W. This shows that for twin buildings which are not irreducible there is no bound on the size of the minimal θ-codistance.

$\boldsymbol{BN}$-quasi-flips

So far our setup was a purely geometric one. We described (quasi-)flips as being, up to a type change, isometries of the involved twin buildings. However, our main motivation to study flips is their application to groups, i.e., to automorphism groups of twin buildings.

Therefore, we now introduce the concept of BN-quasi-flips, as a class of automorphisms of groups with a twin BN-pair. In Section 2.2 we will demonstrate the close correspondence between BN-(quasi-)flips and building (quasi-)flips, justifying the similar choice of names.

Definition 2.1.12. Let G be a group with a twin BN-pair (B_+,B_-,N). An automorphism θ of G is called a ***BN*-quasi-flip** if

(1) $\theta^2=\mathrm{id}$ and

(2) there exists $g \in G$ such that $\theta(B_+) = gB_-g^{-1}$.

Remark 2.1.13. Let $\mathbb{F}$ be a field. Let G be a connected reductive algebraic $\mathbb{F}$-group. If θ is an involutory (abstract) automorphism of $G(\mathbb{F})$ (the group of $\mathbb{F}$-rational points of G), the condition that $\theta(B_+)$ be conjugate to B_- is in fact always satisfied, cf. Fact 6.1.3.

For split Kac-Moody groups, the condition reduces to a dichotomy: Either $\theta(B_+)$ is conjugate to B_- or it is conjugate to B_+, cf. Fact 6.2.1 in Chapter 6.

Example 2.1.14. Let $\mathbb{F}$ be a field and $G = \mathrm{SL}_n(\mathbb{F})$ the special linear group over this field, considered as a matrix group, and endowed with the twin BN-pair B_+ and B_- of upper and lower triangular matrices. Then the **Chevalley involution**, which sends every element $x \in G$ to its transposed inverse ${}^t x^{-1}$, is a clearly an involution. It also interchanges B_+ and B_-, the latter being conjugate to the former. Hence this example constitutes a BN-quasi-flip as defined above.

We now show that the seemingly weak conditions of Definition 2.1.12 imply for a large class of groups that a BN-quasi-flip not only maps B_+ to a conjugate of B_- and vice versa but even maps them to simultaneous conjugates. As a consequence it also induces an automorphism of the Coxeter system (W, S). This is central in Section 2.2 to prove that every BN-quasi-flip induces a building quasi-flip on the twin building associated to the twin BN-pair. To simplify the exposition, we restrict ourselves to saturated twin BN-pairs, but in view of [AB08, Remark 6.83 and following], this restriction is easily overcome.

Proposition 2.1.15. *Let G be a group with saturated twin BN-pair (B_+, B_-, N) of type (W, S), let $T := B_+ \cap B_-$. Let θ be a quasi-flip of G. If the set of chambers fixed by T of the twin building associated to G equals the twin apartment containing B_+ and B_-, then the following hold:*

(1) There exists $x \in G$ such that $\theta(B_\varepsilon) = xB_{-\varepsilon}x^{-1}$ and $\theta(x)x \in T$, where $\varepsilon \in \{+, -\}$.

(2) θ induces a unique automorphism of the Coxeter system (W, S) of order at most 2 (so it normalizes the set S). Specifically, $W \cong NT/T$ and the automorphism is given by $nT \mapsto x^{-1}\theta(n)xT$.

Proof. The normalizer $N_G(T)$ acts on the set of chambers fixed by T of the twin building associated to G. Since by hypothesis this set equals the twin apartment containing B_+ and B_- and since N equals the full stabilizer of this twin apartment (as (B_+, B_-, N) is saturated, cf. [AB08, Definition 6.84]), the equality $N = N_G(T)$ holds.

(1) Recall that by definition of a quasi-flip, $\theta^2 = \mathrm{id}$ and there exists $g \in G$ such that $\theta(B_+) = gB_-g^{-1}$. Moreover by the Birkhoff decomposition, there exist

$b_+ \in B_+$, $b_- \in B_-$ and $n \in N$ such that $\theta(g)g = b_+nb_-$. Then

$$\begin{aligned}\theta(gTg^{-1}) &= \theta(g(B_+ \cap B_-)g^{-1}) \\ &= \theta(g)\theta(B_+)\theta(g^{-1}) \cap \theta(gB_-g^{-1}) \\ &= \theta(g)gB_-g^{-1}\theta(g)^{-1} \cap B_+ \\ &= (b_+nb_-)B_-(b_+nb_-)^{-1} \cap B_+.\end{aligned}$$

Hence for $x := b_+^{-1}\theta(g)$ we have

$$x\theta(T)x^{-1} = b_+^{-1}\theta(gTg^{-1})b_+ = nB_-n^{-1} \cap B_+ \geq T,$$

where the last containment holds because of $n \in N$. Therefore

$$T \leq x\theta(T)x^{-1} \leq x\theta(x\theta(T)x^{-1})x^{-1} = x\theta(x)T(x\theta(x))^{-1}.$$

Accordingly, as T fixes a unique twin apartment, $T = x\theta(x)T(x\theta(x))^{-1}$, i.e., $x\theta(x) \in N_G(T)$. As an immediate consequence $\theta(T) = x\theta(T)x^{-1}$. We note $B_+ = x\theta(g)^{-1}b_+B_+b_+^{-1}\theta(g)x^{-1} = x\theta(B_-)x^{-1}$ thus,

$$B_+ \cap B_- = T = x\theta(T)x^{-1} = x\theta(B_+)x^{-1} \cap x\theta(B_-)x^{-1} = x\theta(x)B_-(x\theta(x))^{-1} \cap B_+.$$

Since B_- is the unique chamber opposite B_+ in the twin apartment fixed by T, this means $x\theta(x) \in N_G(B_-) = B_-$ and, in particular, $\theta(x)B_- = x^{-1}B_-$. Therefore $x\theta(x) \in B_- \cap N_G(T) = T$ and $\theta(B_+) = \theta(x)B_-\theta(x)^{-1} = x^{-1}B_-x$.

(2) Let $X := \{x \in G \mid \theta(B_+) = xB_-x^{-1} \text{ and } \theta(B_-) = xB_+x^{-1}\}$. By (1), this set is nonempty. For $x \in X$, define $\theta_x : g \mapsto x^{-1}\theta(g)x$. Clearly θ_x preserves $T = B_+ \cap B_-$, hence N, thus it induces an automorphism on $W = N/T$ by sending nT to $\theta_x(nT)$.

This automorphism on W does not depend on the choice of x: For if $x' \in X$, then $\theta_x(g) = xx'^{-1}\theta_{x'}(g)x'x^{-1}$. But $xx'^{-1} \in N_G(B_+) \cap N_G(B_-) = B_+ \cap B_- = T$. Thus for all $nT \in N/T$ we have $\theta_x(nT) = \theta_{x'}(nT)$.

It remains to be shown that θ_x normalizes S. For each $s \in S$ the set $P_s := B_+ \cup B_+sB_+$ is a rank 1 parabolic subgroup of positive sign of G. Let n_s be a representative of s in N. Then

$$\theta_x(P_s) = \theta_x(B_+) \cup \theta_x(B_+)\theta_x(n_s)\theta_x(B_+) = B_- \cup B_-\theta_x(n_s)B_-$$

is a parabolic subgroup of negative sign of G: It is a group because it is the image of a subgroup of G under the group automorphism θ_x; it is parabolic because it contains the Borel group B_-. Since it consists of precisely two Bruhat double cosets, it must again be a rank 1 parabolic subgroup. Hence $\theta_x(n_s)$ is a representative of some $s' \in S$. As s and s' are independent of the choice of n_s, the map θ_x permutes S. □

Remark 2.1.16. If the group in Proposition 2.1.15 is endowed with a locally split RGD-system over fields $(\mathbb{K}_\alpha)_{\alpha\in\Phi}$ satisfying $|\mathbb{K}_\alpha| \geq 4$ for each $\alpha \in \Phi$, then by [Cap09, Lemma 4.8] the set of chambers fixed by the torus T equals the twin apartment containing B_+ and B_-.

On the group theoretic level, this condition on T is equivalent to asking that whenever $T^g \leq T$ for some $g \in G$ then we already have $g \in N$. This is for example the case when $N = N_G(T)$ and T is finite.

Analog to building (quasi-)flips, a BN-quasi-flip is a BN-flip if it is type preserving:

Definition 2.1.17. Let G be a group with a twin BN-pair (B_+, B_-, N). A BN-quasi-flip θ of G is a ***BN*-flip** if the induced automorphism $N\widetilde{T}/\widetilde{T} : n\widetilde{T} \mapsto x^{-1}\theta(n)x\widetilde{T}$ is trivial, where $\widetilde{T} := B_+ \cap B_-$ and $x \in G$ such that $\theta(B_\varepsilon) = xB_{-\varepsilon}x^{-1}$ for $\varepsilon \in \{+,-\}$.

Example 2.1.18. The Chevalley involution described in Example 2.1.14 is actually an example of a BN-flip as it centralizes the group W: For we have $W = N/T$, where N are the monomial matrices, and T the diagonal matrices in G. So we compute when θ normalizes the Coxeter group:

$$\theta(nT) = nT \iff {}^t n^{-1}T = nT \iff {}^t nn \in T.$$

But for a monomial matrix n, one readily verifies that ${}^t nn \in T$.

For an example that is not type preserving, i.e., does not centralize the group W, assume $n > 2$ is even and let $J \in G$ denote the matrix with ones on the anti-diagonal and zeros elsewhere. As J has order 2, conjugation by J is an involutory automorphism of G. Being an inner automorphism, it is obvious that B_+ is mapped to a conjugate. Finally, this automorphism does not centralize W. Consider for example the case $n = 4$, then

$$\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}^{-1} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}.$$

Again in analogy to building quasi-flips, we define the notion of a proper quasi-flip.

Definition 2.1.19. We call a BN-quasi-flip **proper** if there exists $h \in G$ such that $\theta(hB_+h^{-1}) = hB_-h^{-1}$.

Geometrically, the above means that θ interchanges the stabilizers of two opposite chambers.

Example 2.1.20. The Chevalley involution from Example 2.1.14 interchanges B_+ and B_-, hence is proper. But as in the case of building quasi-flips, not all BN-quasi-flips are proper. For example, the symplectic building flip from Example 2.1.9 (2.1.9) is improper, and can be lifted to a BN-flip of $\mathrm{SL}_n(\mathbb{F})$ (as is detailed in the next section), which then necessarily is improper.

2.2. Correspondence between building and BN-flips

In this section we investigate the relation between building (quasi-)flips and BN-(quasi-)flips, establishing the close connection between the two concepts which we later frequently exploit in order to apply tools from geometry to solve group theoretic problems and vice versa.

The bulk of this section is joint work with Ralf Gramlich and Bernhard Mühlherr.

We start by showing that every BN-flip induces a building flip in a natural fashion. For this, we use the natural isomorphism $\mathcal{C}_+ \cong G/B_+$, and implicitly also use that $G/B_+ \cong \{gB_+g^{-1} \mid g \in G\}$ (see e.g. [AB08, Section 6.2.4]).

Proposition 2.2.1 (See also [DMGH09, Proposition 3.4])**.** *Let G be a group with a twin BN-pair with associated twin building $\mathcal{C}$. Then any BN-quasi-flip θ of G induces a building quasi-flip $\tilde{\theta}$ of $\mathcal{C}$ by sending gB_ε to $\theta(g)xB_{-\varepsilon}$ for $x \in G$ and $\varepsilon \in \{+,-\}$, as in Proposition 2.1.15. This quasi-flip is unique with the properties that*

(1) for any $g \in G$, and any chamber c, we have $\tilde{\theta}(gc) = \theta(g)\tilde{\theta}(c)$;

(2) $\tilde{\theta}$ maps the chamber stabilized by B_+ to the chamber stabilized by $\theta(B_+)$.

Both θ and $\tilde{\theta}$ induce the same automorphism of (W, S). In particular, θ is a flip (i.e., type preserving) if and only if $\tilde{\theta}$ is. Furthermore, θ is proper if and only if $\tilde{\theta}$ is.

Proof. Recall from Remark 1.8.4 that $\mathcal{C}$ consists of the buildings G/B_ε for $\varepsilon \in \{+,-\}$ with distance functions $\delta_\varepsilon : G/B_\varepsilon \times G/B_\varepsilon \to W$ satisfying $\delta_\varepsilon(gB_\varepsilon, hB_\varepsilon) = w$ if and only if $B_\varepsilon g^{-1}hB_\varepsilon = B_\varepsilon wB_\varepsilon$. These are twinned by the codistance function $\delta_\varepsilon : (G/B_+ \times G/B_-) \cup (G/B_- \times G/B_+) \to W$ satisfying $\delta^*(gB_\varepsilon, hB_{-\varepsilon}) = w$ if and only if $B_\varepsilon g^{-1}hB_{-\varepsilon} = B_\varepsilon wB_{-\varepsilon}$.

By Proposition 2.1.15, there exists $x \in G$ such that $\theta(B_\varepsilon) = xB_{-\varepsilon}x^{-1}$ for $\varepsilon \in \{+,-\}$, and $\theta(x)x \in T$. Define a bijective map $\tilde{\theta}$ between G/B_+ and G/B_- by sending gB_ε to $\theta(g)xB_{-\varepsilon}$. This map is well-defined and has order 2 since

$$\tilde{\theta}(\tilde{\theta}(gB_\varepsilon)) = \tilde{\theta}(\theta(g)xB_{-\varepsilon}) = \theta(\theta(g)x)xB_\varepsilon = g\theta(x)xB_\varepsilon = gB_\varepsilon.$$

Define $\theta_x : g \mapsto x^{-1}\theta(g)x$. Again by Proposition 2.1.15 this induces a well-defined automorphism of the Weyl group $W = N/T$ via $nT \mapsto \theta_x(nT) = \theta_x(n)T$. Now

$$\begin{aligned}
\delta_\varepsilon(gB_\varepsilon, hB_\varepsilon) = w &\iff B_\varepsilon g^{-1}hB_\varepsilon = B_\varepsilon wB_\varepsilon \\
&\iff \theta(B_\varepsilon)\theta(g^{-1}h)\theta(B_\varepsilon) = \theta(B_\varepsilon)\theta(w)\theta(B_\varepsilon) \\
&\iff B_{-\varepsilon}x^{-1}\theta(g^{-1})\theta(h)xB_{-\varepsilon} = B_{-\varepsilon}x^{-1}\theta(w)xB_{-\varepsilon} = B_{-\varepsilon}\theta_x(w)B_{-\varepsilon} \\
&\iff \delta_{-\varepsilon}(\theta(g)xB_{-\varepsilon}, \theta(h)xB_{-\varepsilon}) = \theta_x(w).
\end{aligned}$$

Similarly we find that

$$\delta^*(gB_\varepsilon, hB_{-\varepsilon}) = w \iff \delta^*(\theta(g)xB_{-\varepsilon}, \theta(h)xB_\varepsilon) = \theta_x(w).$$

Therefore, our BN-(quasi-)flip induces a building (quasi-)flip with the claimed properties. Uniqueness follows readily. It is also clear that both induce the same automorphism of (W, S).

Finally, if θ is proper, then there is $b \in G$ such that bB_+b^{-1} is mapped to bB_-b^{-1}. Then the chambers bB_+ and bB_- are opposite, and are interchanged by θ. $\square$

We now turn to the converse question: Given a group G with twin BN-pair and a building (quasi-)flip θ on the associated building, is there a BN-(quasi-)flip inducing θ? If G is generated by its root groups and acts faithfully on the building, the answer is yes, as the following theorem makes precise.

Theorem 2.2.2 (joint work with Gramlich and Mühlherr)**.** *Let $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ be a reduced RGD-system of type (W, S) such that the associated twin building is strictly Moufang (e.g. its diagram contains no isolated nodes). Then any (proper) building quasi-flip of $\mathcal{C}$ induces a (proper) BN-quasi-flip on G. Both induce the same automorphism of (W, S).*

Proof. Suppose θ is a quasi-flip of $\mathcal{C}$. Then it induces an automorphism $\tilde{\theta}$ of $\mathrm{Aut}(\mathcal{C})$ by conjugation: If $g \in \mathrm{Aut}(\mathcal{C})$ is an automorphism of the building, then $\theta \circ g \circ \theta$ is again a building automorphism. We assumed the RGD-system to be reduced, so G acts faithfully on $\mathcal{C}$ and is generated by its root groups. Since we assumed $\mathcal{C}$ to be strictly Moufang, by [AB08, Theorem 8.81 and Proposition 8.82], G is canonically isomorphic to the subgroup $G^\dagger$ of $\mathrm{Aut}(\mathcal{C})$ generated by the root groups of $\mathrm{Aut}(\mathcal{C})$. Since θ normalizes the set of twin apartments resp. the set of twin roots, we deduce that $\tilde{\theta}$ normalizes $G^\dagger$. Hence $\tilde{\theta} \in \mathrm{Aut}(G)$. All properties of a BN-quasi-flip follow readily. It is clear that $\tilde{\theta}$ is type preserving if θ is.

If θ is proper, there exists a chamber c which is mapped by θ to an opposite chamber d. Since the diagonal action of G by left multiplication on $G/B_+ \times G/B_-$ is transitive on the pairs of opposite chambers, we can find $h \in G$ such that $c = hB_+$ and $d = hB_-$. Thus, $\tilde{\theta}$ interchanges hB_+h^{-1} and hB_-h^{-1}. $\square$

A natural question now is what can be said about lifts of quasi-flips to a non-reduced RGD-systems $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$. Here, things are not quite as nice. In particular, for non-centered RGD-systems, many “wild” things can happen, e.g. complicated group extensions may be involved for which the existence of a lifting of a given flip is far from clear.

But even when restricting to centered but non-faithful RGD-systems, we would have to be able to lift θ to arbitrary central extensions of G. Specifically, Theorem 2.2.2 implies that given an RGD-system and a building quasi-flip on its associated twin building, we can always lift the building quasi-flip to a BN-quasi-flip on the reduction of the RGD-system. But suppose G is a central extension of the group of the reduced RGD-system. Then to get a quasi-flip on G we have to know how to lift the building flip to the center of G, which may not be possible in general. In summary, no general answer to this problem is known to us, and we believe it to be a very difficult problem in general.

However, it is at least possible to lift quasi-flips to universal central extensions of centered RGD-systems:

Corollary 2.2.3. *Given a centered RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$, let $\mathcal{C}$ be the associated twin building. Assume that G is perfect. Then any building quasi-flip of $\mathcal{C}$ induces a BN-quasi-flip on the universal central extension of G.*

Proof. Assume we are given a building quasi-flip θ of $\mathcal{C}$. Since we are in the centered case, by [AB08, Propositions 8.82(2)], the kernel of the action of G on $\mathcal{C}$ coincides with its center. Hence we get a reduced RGD-system for $G/Z(G)$, and by Proposition 2.2.1 we can lift the building quasi-flip θ to a BN-quasi-flip θ' of $G/Z(G)$.

Since G is perfect, also $G/Z(G)$ is perfect. Thus $G/Z(G)$ admits a universal central extension $\widetilde{G}$ (see [Mil72, Chapter 5] for more on universal central extensions). Let $\pi : \widetilde{G} \to G/Z(G)$ be the associated covering map. Then $\theta' \circ \pi$ also is a covering map of $G/Z(G)$, hence by the universality property, there exists an automorphism $\widetilde{\theta}$ of $\widetilde{G}$ such that $\theta' \circ \pi = \pi \circ \widetilde{\theta}$. Since θ' is an involution, we even have

$$\pi = \theta'^2 \circ \pi = \theta' \circ \pi \circ \widetilde{\theta} = \pi \circ \widetilde{\theta}^2.$$

This implies that $\tilde{\theta}$ is an involution as well.

Finally, we obtain a twin-BN-pair of $\widetilde{G}$ by taking the preimages under π of B_+, B_-, T, N and so on. (That this is again a twin-BN-pair is readily verified, as we only replaced everything by central extensions.) Clearly $\tilde{\theta}$ is a quasi-flip with respect to this twin-BN-pair. □

Given the correspondence established in this section, the similar choice of names for building- and BN-quasi-flips is finally justified.

Also, whenever we start with a BN-(quasi-)flips, there is a unique corresponding building (quasi-)flip, so in this case there is no need at all to distinguish between the two notions. This also means that any concept we define for building quasi-flips can be immediately transferred to BN-quasi-flips. Namely, we may say that a BN-quasi-flip *has property X* exactly when the associated building quasi-flip has property X.

We will make frequent and liberal use of this fact in subsequent sections, usually by not distinguishing between building- and BN-quasi-flips explicitly and instead simply using the term “quasi-flips”.

2.3. Steep descent

In preparation for further work later on, we now take a closer look at the possibilities for the θ-codistances that can occur. From this we derive a generic reduction argument that allows us to transfer many questions from the twin-building context to the simpler and more restricted context of spherical buildings.

Lemma 2.3.1. *Let c be an arbitrary chamber, let $w := \delta^\theta(c)$ be its θ-codistance. Then w is a θ-twisted involution, that is, $\theta(w) = w^{-1}$. In particular, if θ is a flip, then $w = \theta(w)$ and w is an involution.*

Proof. By (Tw1) we have $\delta^*(\theta(c), c) = w^{-1}$. Applying Lemma 2.1.4, we obtain

$$w = \delta^*(c, \theta(c)) = \theta(\delta^*(\theta(c), c)) = \theta(w^{-1}). \qquad \square$$

We next give a condition under which a chamber admits a neighboring chamber with lower numerical θ-codistance, i.e., a chamber which is farther away from its image than the original chamber.

Lemma 2.3.2. *Let c be an arbitrary chamber with θ-codistance $w := \delta^\theta(c)$. Assume $s \in S$ satisfies $l(sw\theta(s)) = l(w) - 2$. Then the θ-codistance of all chambers in $P_s(c) \setminus \{c\}$ equals $sw\theta(s)$.*

Proof. Let $d \in P_s(c) \setminus \{c\}$. Since $l(sw) = l(w\theta(s)) = l(w) - 1$, the second twin building axiom (Tw2) implies $\delta^*(c, \theta(d)) = w\theta(s)$. Another application of (Tw2) yields $\delta^*(d, \theta(d)) = sw\theta(s)$. $\square$

The following is an extension of [GM08, Lemma 2] (there, however, no proof is given).

Lemma 2.3.3. *Let $r \in W$ be a θ-twisted involution, and $w \in W$ such that $l(w^{-1}r\theta(w)) = l(r) - 2l(w)$.*

(1) Let $c \in \mathcal{C}$ with $\delta^\theta(c) = r$, and let d be a chamber at distance w from c. Then $\delta^\theta(d) = w^{-1}r\theta(w)$.

(2) Let $d \in \mathcal{C}$ such that $\delta^\theta(d) = w^{-1}r\theta(w)$. Then there exists a unique chamber c with distance w^{-1} from d such that $\delta^\theta(c) = r$.

(3) In either case, the convex hull of d and $\theta(d)$ contains c and $\theta(c)$.

Proof. We fix a minimal decomposition $s_1 \cdots s_n$ of w.

(1) Pick a minimal gallery ($c = c_0 \sim_{s_1} c_1 \sim_{s_2} \cdots \sim_{s_n} c_n = d$) joining c and d. Thus by hypothesis, $l(s_1 r) = l(r) - 1 = l(r\theta(s_1))$. Applying Lemma 2.3.2, we conclude that $\delta^\theta(c_1) = s_1 r\theta(s_1)$. Repeating this argument, we get

$$\delta^\theta(d) = (s_n \cdots s_1) \cdot r \cdot \theta(s_1 \cdots s_n) = w^{-1}r\theta(w).$$

(2) The claim follows by induction on n. For $n = 0$ there is nothing to show. So suppose $n > 0$. By Axiom (Tw3) there is a unique chamber d' which is s_n-adjacent to d such that $\delta^*(d, \theta(d)) = s_n w^{-1}r\theta(w) = (s_{n-1} \cdots s_1) \cdot r \cdot \theta(s_1 \cdots s_n)$. This cannot be a θ-twisted involution (as else, $s_n w^{-1}r\theta(w) = w^{-1}r\theta(ws_n)$, implying that $s_n w^{-1}r\theta(ws_n) = w^{-1}r\theta(w)$, contradicting the hypothesis). Therefore

$$\delta^\theta(d) = s_n w^{-1}r\theta(w)\theta(s_n) = (s_{n-1} \cdots s_1) \cdot r \cdot \theta(s_1 \cdots s_{n-1}).$$

(3) This can be seen as a consequence of (2). However, we give an alternative proof, once more by induction on n. For $n = 0$, nothing has to be shown as $c = d$. Suppose now $n > 0$, and take the same gallery between c and d as in (2). Let P be the s_n-panel around d. We have $\delta^*(d, \theta(d)) = w^{-1}r\theta(w)$ but $\delta^*(c_{n-1}) = s_n w^{-1} r\theta(w)$. Thus, $\mathrm{proj}_P(\theta(d)) = c_{n-1}$. It follows that the convex hull of d and $\theta(d)$ contains c_{n-1} and by symmetry also $\theta(c_{n-1})$, hence also their convex hull. By the induction hypothesis, we are done. □

By Proposition 1.3.3, θ-twisted involutions are conjugate to the longest element of some spherical standard parabolic subgroups of the Weyl group W. We combine this with Lemma 2.3.3 to walk (or, as I like to put it, “descend”) from arbitrary chambers to chambers with spherical θ-codistance.

Lemma 2.3.4. *Let $c \in \mathcal{C}_+$ be an arbitrary chamber with θ-codistance w. Then there exist a spherical subset I of S and an element $w' \in W$ such that the following hold:*

(1) $w = w'w_I\theta(w')^{-1}$ and $l(w) = 2l(w') + l(w_I)$, where w_I is the longest element of W_I.

(2) $w' \leq w$ and $w_I \leq w$ in the Bruhat order.

(3) Every chamber $d \in \mathcal{C}_+$ with $\delta_+(c, d) = w'$ satisfies $\delta^\theta(d) = w_I$.

(4) The convex hull of d and $\theta(d)$ contains c and $\theta(c)$. □

2.4. Strong flips

In this section we present a particularly well-behaved class of quasi-flips, the *strong flips*. Many results which are proven with much labor in this thesis become much simpler when one restricts to this class; conversely, features of these flips may inspire possible generalizations.

The following definition is essentially taken from [DM07, Definition 6.2]; but note that in loc. cit., what we call flip is called involution and what we call strong flip is just called flip.

Definition 2.4.1. Let θ be a building quasi-flip of a twin building $\mathcal{C}$. For any spherical residue R, define the set

$$\mathrm{proj}_R(\theta) := \{c \in R \mid \mathrm{proj}_R(\theta(c)) = c\},$$

where proj_R denotes the projection onto R. If for all panels P of $\mathcal{C}$ we have $\mathrm{proj}_P(\theta) \neq P$, we call θ a **strong quasi-flip**, and say that it has the **Devillers-Mühlherr property**.

We remind the reader that in view of Proposition 2.2.1, we can now also talk about strong BN-quasi-flips.

The importance of this definition is two-fold: Firstly, many arguments can be considerably simplified for strong flips, e.g. stronger descent properties hold, as the following lemma illustrates, or as a look at the beautiful filtration result of [DM07] will reveal. Secondly, many interesting quasi-flips are actually strong, making it worthwhile to study them specifically.

Lemma 2.4.2. *Let θ be a strong quasi-flip, let c be a chamber with θ-codistance w. If $s \in S$ is such that $l(sw) < l(w)$, then there exists a chamber d which is s-adjacent to c and has lower numerical θ-codistance. In particular, strong quasi-flips are proper.*

Proof. If $w = 1_W$ nothing has to be shown. Otherwise, take any $s \in S$ such that $l(sw) < l(w)$ and consider the s-panel P containing c. By Lemma 1.3.2, for every chamber in P, the θ-codistance can only be w, sw (which then equals $w\theta(s)$) or $sw\theta(s)$, all of which are less or equal w in the Bruhat order. Thus $\mathrm{proj}_P(\theta(c)) = c$. But by the Devillers-Mühlherr property, there exists a chamber d in P so that $\mathrm{proj}_P(\theta(d)) \neq d$. Thus the numerical θ-codistance of d is strictly lower than that of c. In particular, we can repeat this process until we reach a Phan chamber. $\square$

Example 2.4.3. The prototypical example of a strong flip is the following: Suppose $\mathbb{F}$ is a field endowed with a nontrivial field involution σ (e.g. the complex numbers with complex conjugation, or a finite field of square order with the corresponding power of the Frobenius automorphism). See also Lemma 6.1.12.

Throughout the present work, we will occasionally mention when results hold for strong flips, or are simpler to prove for them.

2.5. Stable twin apartments

In this section we prove under some mild conditions the existence of θ-stable (twin) apartments around any chamber c. Here as usual θ is a quasi-flip of a Moufang twin building $\mathcal{C}$. From this we derive a nice double coset decomposition of groups with a twin BN-pair admitting a BN-quasi-flip.

The condition we are going to impose will be that all root groups are uniquely 2-divisible. This generalizes the idea of a group being defined over a field of characteristic different from 2, in the sense that all algebraic and Kac-Moody groups defined over such a field satisfy it.

Definition 2.5.1. Let n be an integer greater than 1. A group G is called ***n*-divisible** if for each $g \in G$ there exists $h \in G$ such that $h^n = g$. If h is unique with that property, we call G **uniquely *n*-divisible**.

Note that we do not require G to be abelian, as is usually the case in the literature when defining n-divisibility. Also, often in the literature, n is required to be prime. But clearly G is n-divisible if and only if G is p-divisible for each prime p dividing n.

The following proposition is one of the key ingredients of the main result of this section.

Proposition 2.5.2. *Let $\mathbb{M} = (X, (U_x)_{x\in X})$ be a Moufang set. If the root groups U_x are uniquely 2-divisible, then an involutory automorphism of $\mathbb{M}$ fixing a point necessarily fixes a second point.*

Proof. Suppose ϕ is an involutory automorphism (i.e., a permutation) of $\mathbb{M}$ which fixes a point, say ∞; so $U_\infty^\phi = U_\infty$. As in Section 1.10, we will always denote the action of a permutation on a set on the right, i.e., we will write $a.\phi$ rather than $\phi(a)$.

Let a be any element of $\mathbb{M}$ different from ∞. If $a = a.\phi$, we have found a second fixed point and are done. So assume $a \neq a.\phi$. Since U_∞ acts simply transitively on $X \setminus \{\infty\}$, there exists a unique $g \in U_\infty$ such that $a.g = a.\phi$. Choose $h \in U_\infty$ such that $h^2 = g$. We claim that $a.h$ is a fixed point. Indeed

$$(a.g).g^{-1} = a = (a.g).\phi = a.g\phi g\phi = (a.g).g^\phi.$$

Since U_∞ acts simply transitively, we have $g^\phi = g^{-1}$, and as U_∞ is uniquely 2-divisible this implies $h^\phi = h^{-1}$ as well. Therefore

$$(a.h).\phi = (a.\phi).h^\phi = (a.g).h^\phi = (a.h^2).h^{-1} = a.h.$$ □

Remark 2.5.3. For abelian root groups the statement above can be easily extended to finite automorphism groups Γ: If Γ fixes one point and the root groups are $|\Gamma|$-divisible, then Γ fixes a second point. It is an interesting question whether one can extend this to non-abelian root groups.

An alternative way of stating Proposition 2.5.2 is that an involutory automorphism of a rank 1 building with 2-divisible root groups which fixes a chamber, also fixes an opposite chamber. The following proposition extends this to spherical Moufang buildings of higher rank, from which the existence of θ-stable apartments follows immediately.

Proposition 2.5.4. *Given a spherical RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$, let $\mathcal{C}$ be the associated spherical building. Assume all root groups U_α are uniquely 2-divisible. Let θ be a quasi-flip of G which fixes some Borel subgroup B. Then there exists a Borel subgroup B' opposite B (i.e., B and B' intersect in a torus) which is fixed by θ. Geometrically, let c be the chamber corresponding to B, then there exists a chamber c' fixed by θ and opposite c.*

Proof. In the following, we take the geometric viewpoint, where it is easier to argue. So, θ is a building quasi-flip of $\mathcal{C}$ in the sense of Remark 2.1.2 (resp. an almost isometry, as defined in [AB08, Section 5.5.1]). Then θ induces an automorphism of (W, S) of order at most 2. Denote by $\mathcal{I}$ the set of θ-orbits in S. For each $I \in \mathcal{I}$ we show that the residue $R_I(c)$ contains a chamber c_I fixed by θ and opposite c in that residue, i.e., $\delta(c, c_I) = w_I$, where w_I denotes the longest element of $W_I = \langle I\rangle$. If $|I| = 1$, this is Proposition 2.5.2. So assume $|I| = 2$, say, $I = \{s, t\}$. Then $R_I(c)$ is a Moufang n-gon which is normalized by θ. We construct a gallery $(c_0, \dots, c_{m-1})$ of length $m := \lceil \frac{n+1}{2} \rceil$ with $c_0 := c$: For c_1 choose any chamber different from but

s-adjacent to c_0. If $m = 2$, stop here; else, choose for c_2 any chamber different from but t-adjacent to c_1, and so on, alternating between s- and t-adjacent chambers.

Consider now the θ-stable gallery $(\theta(c_{m-1}), \dots, c, \dots, c_{m-1})$ of length $2m-1$. If n is even, then $2m-1 = n+1$, and c_{m-1} is opposite $\theta(c_{m-1})$, and for that reason they span a θ-stable apartment containing c. (See Figure 2.1a.)

If n is odd, some more effort is needed. Let $P := P_t(c_{m-1})$ be the t-panel containing c_{m-1}. Then by construction, P and $\theta(P)$ are opposite panels in $R_I(c)$. By composing θ and the projection map from $\theta(P)$ to P, we obtain an automorphism $\theta' = \mathrm{proj}_P \circ \theta$ of the Moufang set P of order at most 2. Clearly θ' fixes c_{m-1}. Hence by Proposition 2.5.2 there is a second chamber $c_m \in P$ fixed by θ'. But then $(\theta(c_m), \dots, c, \dots, c_m)$ is a θ-stable gallery of length $2m+1 = n+2$, and c_m is opposite to $\theta(c_{m-1})$, and the two of them span a θ-stable apartment containing c. (See Figure 2.1b.)

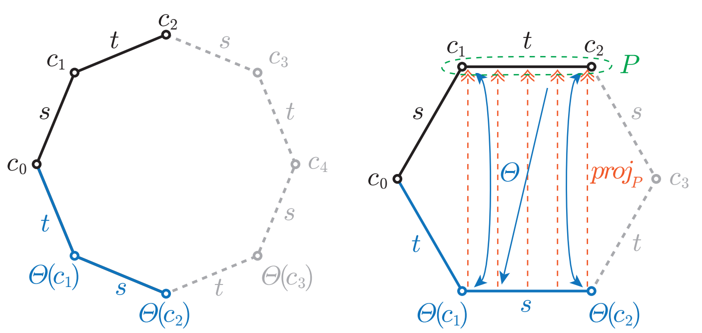
In either case, we obtain a θ-stable apartment containing c, which then necessarily contains a unique chamber opposite c and also fixed by θ.

Now the longest element w_0 of W can be written as a product of the longest words $w_I, I \in \mathcal{I}$ (see [Ste68a, 1.32]). So assume $w_0 = w_{I_1} w_{I_2} \cdots w_{I_k}$. Starting in c, by the above we can find a chamber d_1 fixed by θ at distance w_{I_1} from c. We proceed to find a chamber d_2 fixed by θ and at distance w_{I_2} from d_1, hence distance $w_{I_1} w_{I_2}$ from c. We repeat this until we finally reach a chamber d_k, fixed by θ and at distance w_0 from c, i.e., opposite c. This yields the desired θ-stable apartment since opposite chambers determine a unique apartment. □

Example 2.5.5. To illustrate the 2-divisibility condition, we sketch an example: Let G be a split algebraic group over a field $\mathbb{F}$, and $\mathcal{C}$ the associated spherical Moufang building. Then all root groups are parametrized by the additive group of $\mathbb{F}$. Hence they are uniquely 2-divisible if and only if $\operatorname{char} \mathbb{F} \neq 2$.

In arbitrary spherical Moufang buildings the classification of Moufang polygons shows that here all root groups are in a sense additive groups of (vector spaces over) fields; hence we get a similar condition on the characteristic of some underlying field. We make this precise in Section 2.6.

Remark 2.5.6. The condition on the root groups in Propositions 2.5.2 and 2.5.4 is



(a) Moufang quadrangle (b) Moufang projective plane

Figure 2.1.: Constructing a θ-stable apartment inside Moufang polygons.

essential in the following sense: Take any spherical Moufang building $\mathcal{C}$ associated to some $\mathbb{F}_q$-locally split RGD-system with $q = 2^n$. The root groups then are not 2-divisible, they even admit 2-torsion. Let α be a positive root, and take an arbitrary nontrivial element u in U_α. Then u is an involutory automorphism of $\mathcal{C}$, fixing the chamber c stabilized by B_+. Now u acts on the set $P \setminus \{c\}$, where P is any panel intersecting the root α only in c. But P has odd size $q+1$, and we know that u fixes c. As U_α acts sharply transitively, u cannot fix any other chambers in P. In particular, u cannot fix any apartment in $\mathcal{C}$.

On the other hand, the 2-divisibility condition is not strictly necessary: Take the Fano plane, the projective plane over $\mathbb{F}_2$. This projective plane admits an up to isomorphism unique polarity, which then is a building flip. In fact it is a proper building flip and one readily verifies that each chamber is contained in an apartment stabilized by the polarity. In fact this generalizes to arbitrary polarities of projective planes in characteristic 2, using arguments similar to those used in Section 4.6.3. The key observation here is that in this situation, every line contains at least one absolute point (see [Bae46, Theorem 1] for the finite case, which can be generalized to the general case using Moufang set arguments).

Remark 2.5.7. The statements of Propositions 2.5.2 and 2.5.4 were inspired by [Müh94, Section 3.5] (sadly, this thesis was never published and hence is difficult to obtain and not as well-known as it should be). In loc. cit., fixed points of an arbitrary finite group of automorphisms are considered. In the present work, we focus on the special case of a single involutory automorphism. This enables us to employ different methods for the proofs and get somewhat "better" results, at the loss of a great deal of generality. To get a flavor of the improvement, here are two examples:

First, applying Lemma 3.5.4 from loc. cit. to the root groups of a suitable Moufang set yields a result similar in spirit to Proposition 2.5.2. But by specializing to the case $|\Gamma| = 2$, we are able to reduce the assumptions one has to impose on the root groups; in particular, no nilpotency has to be assumed.

Secondly, Theorem 3.5.5 of loc. cit. is very similar to Proposition 2.5.4. However, the conditions imposed there are less explicit and less practical than ours. For example, the unipotent radicals (cf. Definition 1.9.5) of the Borel subgroups must satisfy certain filtration conditions, which are not known in general. Compared to this, 2-divisibility of the root groups is in many cases known or easy to verify.

We finally conclude for any flip θ and any chamber c the existence of θ-stable apartments containing c, provided the root groups are uniquely 2-divisible.

Theorem 2.5.8. *Let θ be a quasi-flip of an RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$, let $\mathcal{C}$ be the associated twin building. Assume all root groups U_α are uniquely 2-divisible. Then for any Borel subgroup B of G, there exists a θ-stable conjugate of T in B. Geometrically, for any chamber c, there exists a θ-stable twin apartment containing c.*

Proof. By Lemma 2.3.4 there exist a spherical subset I of S and a chamber $d \in \mathcal{C}_+$ such that $\delta^\theta(d) = w_I$ and the convex hull of d and $\theta(d)$ contains c and $\theta(c)$.

Since $\delta^\theta(d) = w_I$, the spherical I-residue $R_I(d)$ is opposite to its image under θ, and thus is a Phan residue. Therefore, if we compose θ with the projection map from $\theta(R_I(d)) = R_I(\theta(d))$ to $R_I(d)$, we obtain an involutory almost isometry θ' of the spherical building $R_I(d)$. Clearly, θ' fixes the chamber d. We can now apply Proposition 2.5.4 to find a second chamber d' in $R_I(d)$ fixed by θ' and opposite d in $R_I(d)$. That is,

$$\delta_+(d, d') = w_I = \delta^*(d, \theta(d)) = \delta^*(d', \theta(d')),$$

therefore $\delta^*(d, \theta(d')) = 1_W$. It follows that the convex hull of d and $\theta(d')$ defines an apartment Σ. One readily verifies that also d' and $\theta(d)$ are contained in Σ, which hence is θ-stable and contains c. $\square$

2.6. 2-divisible root groups

In this section, we investigate when root groups are uniquely 2-divisible. For locally split RGD-systems, this is easy: all root groups are isomorphic to the additive group of the underlying field, and hence are uniquely 2-divisible if and only if the field does not have characteristic 2, if and only if the root groups are 2-torsion free.

It turns out that a similar statement holds for root groups occurring in Moufang polygons. For this we exploit the classification of Moufang polygons given in [TW02].

Proposition 2.6.1. *Let $\mathbb{M} = (X, (U_x)_{x \in X})$ be a Moufang set occurring in a Moufang polygon. If $U = U_\infty$ is 2-torsion free, then it is uniquely 2-divisible.*

Proof. We follow the explicit enumeration of all Moufang polygons presented in [TW02, Chapter 16]. We recommend to simultaneously look at loc. cit. while reading this proof.

It will become apparent that all root groups in Moufang polygons are essentially either the additive group of a field, a vector space over a field, or a sub- or supergroup of one of these. Hence the root groups will be 2-torsion free if and only if the underlying field is not of characteristic 2. We will implicitly use this fact below.

Triangles $\mathcal{T}(A)$. A is an alternative division ring, the root groups are parametrized by its additive group, which is abelian (in fact it is a vector space over some field K). Hence they are uniquely 2-divisible if and only if $\operatorname{char} A = \operatorname{char} \mathbb{K} \neq 2$.

Quadrangles $\mathcal{Q}_\mathcal{I}(K, K_0, \sigma)$ of involutory type. K is a field or skew-field, σ an involution of K and K_0 is an additive subgroup of K containing 1. Two of the root groups are parametrized by the additive group of K, the other two by K_0. If $\operatorname{char} K \neq 2$, then by [TW02, Remark 11.2] we have $K_0 = \operatorname{Fix}_K(\sigma)$. So $x \in K_0$ if and only if $\frac{x}{2} \in K_0$. The claim follows.

Quadrangles $\mathcal{Q}_\mathcal{Q}(K, L_0, q)$ of quadratic form type. K is a field, L_0 is a vector space over K. The root groups are parametrized by the additive group of K resp. by L_0. Again the claim follows readily.

Quadrangles $\mathcal{Q}_{\mathcal{D}}(K, K_0, L_0)$ of indifferent type. K is a field of characteristic 2, and K_0 and L_0 are additive subgroups of K containing 1, which parametrize the root groups. So all root groups admit 2-torsion.

Quadrangles $\mathcal{Q}_{\mathcal{P}}(K, K_0, \sigma, L_0, q)$ of pseudo-quadratic form type. K is a field or a skew-field, L_0 is a right vector space over K. Also, q is an anisotropic pseudo-quadratic form on L_0 (see (11.16) and (11.17) in loc. cit.) Following (11.24) in loc. cit., we define the group

$$T = \{(a, t) \in L_0 \times K \mid q(a) - t \in K_0\}$$

with group operation given by

$$(a, t) + (b, u) := (a + b, t + u + f(b, a)),$$

where f is a skew-hermitian form on L_0 such that

$$q(a + b) \equiv q(a) + q(b) + f(a, b) \pmod{K_0}. \tag{2.1}$$

Then the root groups are parametrized by the additive group of K resp. by T. Note that T has the additive group of K as a subgroup, i.e., $\{0\} \times K \leq T$. So we get 2-torsion in the root groups if $\operatorname{char} K = 2$.

Suppose now that $\operatorname{char} K \neq 2$. We prove that T is uniquely 2-divisible: Given any element $(a, t) \in T$, we easily compute the unique element $(b, u) \in L_0 \times K$ such that $(a, t) = 2(b, u)$:

$$(b, u) = (a/2, t/2 - f(b, b)/2) = (a/2, t/2 - f(a/2, a/2)/2).$$

To see that $(b, u) \in T$, we use that $q(2b) = q(a) \equiv t \pmod{K_0}$ and compute

$$\begin{aligned} q(b) - u &\equiv q(b) - (t/2 - f(b, b)/2) \\ &\equiv \frac{1}{2}(2q(b) + f(b, b) - q(2b)) \overset{(2.1)}{\equiv} 0 \pmod{K_0}. \end{aligned}$$

Quadrangles $\mathcal{Q}_{\mathcal{E}}(K, L_0, q)$ of type E_6, E_7 and E_8. K is a field, L_0 is a vector space over K. X_0 is another vector space over K, and g some function from $X_0 \times X_0$ to K. Let S be the group with underlying set $X_0 \times K$ and group operation given by

$$(a, s) + (b, t) = (a + b, s + t + g(a, b))$$

for all $a, b \in X_0$ and $s, t \in K$. The root groups are then parametrized by S and L_0, which are 2-torsion free if $\operatorname{char} K \neq 2$. In that case, given an arbitrary element $(a, s) \in S$, an easy computation shows that $(b, t) := (a/2, s/2 - g(a/2, a/2)/2)$ is the unique element of S satisfying $2(b, t) = (a, s)$.

Quadrangles $\mathcal{Q}_{\mathcal{F}}(K, L_0, q)$ of type F_4. K is a field of characteristic 2 and L_0 a vector space over K. Furthermore, a certain subfield F of K is defined (see (14.3) in loc. cit.). Then the root groups are parametrized by $X_0 \oplus K$ and $W_0 \oplus F$ (where X_0 and W_0 are certain vector spaces over F, which is however irrelevant for us in this context). Since K and hence F are of characteristic 2, all root groups admit 2-torsion.

Hexagons $\mathcal{H}(J, F, \#)$. By [TW02, Definition 15.16], F is a field and J a vector space over F, and the root groups are parametrized by these. The claim follows.

Octagons $\mathcal{O}(K, \sigma)$. K is a field of characteristic 2, and $K_\sigma^{(2)}$ is a group on the set $K \times K$ which has the additive group $K = K \times \{0\}$ as a subgroup. The root groups are parametrized by the additive group of K resp. by $K_\sigma^{(2)}$, both of which admit 2-torsion. $\square$

It would be nice to have a general argument for the above, which does not rely on the classification of Moufang polygons, and which might be applicable in a broader context. In the finite case, things are quite easy.

Lemma 2.6.2. *Let U be a finite group. Then U is uniquely 2-divisible if and only if U has odd order.*

Proof. If U has even order, then it contains an involution x. Hence $x^2 = 1 = 1^2$ but $x \neq 1$, so U is not uniquely 2-divisible.

If U has odd order, then every element x has odd order, say $n = 2k - 1$. Then $y := x^k$ satisfies $y^2 = x^{2k} = x$. Moreover, any element z which squares to x generates a cyclic group of odd order which contains x and hence y. But in such a group, squaring is a group automorphism, hence $y = z$. $\square$

One might hope to generalize this idea to infinite root groups. A natural idea would be to generalize "odd order" to "2-torsion free", in analogy to Proposition 2.6.1. However, the following example shows that there are infinite Moufang sets for which the root groups are abelian and 2-torsion free, yet not 2-divisible.

Example 2.6.3. Consider any imperfect field $\mathbb{F}$ of characteristic 2 (e.g. the field $\mathbb{F}_2((t))$ of Laurent series in t over the finite field $\mathbb{F}_2$). Then the sharply 2-transitive Moufang set $AG(1, \mathbb{F})$ has root groups isomorphic to $\mathbb{F}^*$, which is abelian and 2-torsion free but not 2-divisible: Squaring is just the Frobenius map of this field, which is not surjective in an imperfect field (in our example, there is no square root of t).

Morally, what we learn from this section is that fields of characteristic 2 cause nothing but trouble when dealing with flips.

2.7. Double coset decomposition

In this section we present a double coset decomposition result, generalizing resp. adapting [HW93]. There, $\mathbb{F}$-involutions of algebraic groups are considered, where $\mathbb{F}$ is a field of characteristic different from 2. We extend this to quasi-flips of groups with a twin BN-pair with uniquely 2-divisible root groups. The results in loc. cit. in turn refine Springer [Spr84], which deals with algebraically closed fields, and also Rossmann [Ros79] and Matsuki [Mat79] for $\mathbb{F} = \mathbb{R}$. See also [KW92] for a result on Kac-Moody groups over algebraically closed fields of characteristic 0.

Our approach is based primarily on building theoretic arguments, unlike previous proofs. This allows us to treat the subject in a unified way, and works for arbitrary quasi-flips which satisfy the conclusion of Theorem 2.5.8. This extends previous work in various ways. For example, in the context of algebraic groups we obtain the results from [HW93], but also cover semi-linear automorphisms (think of this as a $\mathbb{F}$-linear automorphism composed with a field automorphism of $\mathbb{F}$, where $\operatorname{char}\mathbb{F} \neq 2$). Similarly, we generalize [KW92, Proposition 5.14], which deals with Kac-Moody groups over algebraically closed fields of characteristic 0, to arbitrary fields of characteristic different from 2.

We begin by adapting some tools from [HW93].

Lemma 2.7.1 (Adaption of [HW93, Lemma 2.4, Part 2])**.** *Let θ be a quasi-flip of an RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ of type (W, S). Two θ-stable twin apartments in the associated twin building containing a common chamber c are conjugate by an element of G_θ fixing c.*

Proof. Let Σ and Σ' be two θ-stable twin apartments with nonempty intersection $\Sigma \cap \Sigma'$ containing the chamber c. Then also $\theta(c) \in \Sigma \cap \Sigma'$. The unipotent radical U (cf. Definition 1.9.5) of the Borel subgroup B stabilizing c acts sharply transitively on the twin apartments containing c. Hence there exists a unique $u \in U$ mapping Σ to Σ', i.e., $\Sigma' = u\Sigma$. Being a building automorphism fixing c, u stabilizes the set $\Sigma \cap \Sigma'$ chamber-wise. In particular, u fixes $\theta(c)$, hence $u \in U(\theta(c)) = \theta(U)$, equivalently, $\theta(u) \in U$. Since Σ, Σ' are θ-stable,

$$u\Sigma = \Sigma' = \theta(\Sigma') = \theta(u\Sigma) = \theta(u)\Sigma.$$

Since $u, \theta(u) \in U$, and since u was unique, we conclude that $u = \theta(u) \in G_\theta$. □

Proposition 2.7.2 (Adaption of [HW93, Proposition 6.10])**.** *Let θ be a quasi-flip of an RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ of type (W, S) such that every chamber is contained in a θ-stable twin apartment. Let $\{\Sigma_i \mid i \in I\}$ be representatives of the G_θ-conjugacy classes of θ-stable twin apartments in $\mathcal{C}$ (resp. θ-stable maximal tori in G). If B is a Borel group of the RGD-system, then*

$$G_\theta\backslash G/B \cong \bigcup_{i\in I} W_{G_\theta}(\Sigma_i)\backslash W_G(\Sigma_i).$$

Proof. Assume we are given two chambers c, c' of the building G/B which are G_θ-conjugate, say $c' = gc$ for $g \in G_\theta$. By our hypotheses, there is a θ-stable twin apartment Σ containing c, hence $g\Sigma$ is a θ-stable twin apartment containing c'. Accordingly, any two G_θ-conjugate chambers are contained in G_θ-conjugate θ-stable twin apartments. Note that G_θ-conjugacy classes of θ-stable twin apartments are in one-to-one correspondence with the G_θ-conjugacy classes of θ-stable maximal tori in G.

Furthermore, by Lemma 2.7.1, if two θ-stable apartments intersect, then they are already G_θ-conjugate. Hence every chamber lies in a unique G_θ-orbit of θ-stable apartments, represented by some Σ_i. Therefore, the orbits of G_θ on the building G/B, i.e., $G_\theta\backslash G/B$, can be parametrized via the Σ_i and the G_θ-orbits on the chambers in each Σ_i.

The chambers of each Σ_i are in turn parametrized by $W_G(\Sigma_i) = \operatorname{Stab}(\Sigma_i)/\operatorname{Fix}(\Sigma_i)$. Taking the G_θ-action into account, we can parametrize the G_θ-conjugacy classes of chambers in Σ_i by $W_{G_\theta}(\Sigma_i)\backslash W_G(\Sigma_i)$ which yields the claimed decomposition. $\square$

We immediately obtain the following corollary (see also Corollaries 6.1.4 and 6.2.2 for applications to algebraic and Kac-Moody groups):

Corollary 2.7.3 (of Theorem 2.5.8 and Proposition 2.7.2)**.** *Let θ be a quasi-flip of an RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ of type (W, S) where all root groups are uniquely 2-divisible. Then with the notation from Proposition 2.7.2, we have*

$$G_\theta\backslash G/B \cong \bigcup_{i\in I} W_{G_\theta}(\Sigma_i)\backslash W_G(\Sigma_i).$$

Corollary 2.7.4 (of Corollary 2.7.3 and Proposition 2.6.1)**.** *Let θ be a quasi-flip of a 2-spherical RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ of type (W, S), with no isolated nodes in the diagram, and with 2-torsion free root groups. Then with the notation from Proposition 2.7.2, we have*

$$G_\theta\backslash G/B \cong \bigcup_{i\in I} W_{G_\theta}(\Sigma_i)\backslash W_G(\Sigma_i).$$

An alternative parameterization of this double coset decomposition is given in [HW93], refining a result by Springer [Spr84]. In [HW93, Remark 6.11] the authors sketch how to derive this from the parameterization we gave above. We adapt and closely follow that remark in the following. A special case of this occurs again on page 87, Equation 5.2.

Proposition 2.7.5 (Adaption of [HW93, Proposition 6.8])**.** *Let θ be a quasi-flip of an RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ of type (W, S) such that every chamber is contained in a θ-stable twin apartment. Let B be a Borel group stabilizing a chamber c, let Σ be a θ-stable twin apartment containing c. Then*

$$G_\theta\backslash G/B \cong W_{G/G_\theta}(\Sigma) = \{G_\theta g Z_G(\Sigma) \mid g^{-1}\theta(g) \in N_G(\Sigma)\}.$$

Proof. We claim that every (G_θ, B) double coset $G_\theta hB$ has a representative g satisfying $g^{-1}\theta(g) \in N_G(\Sigma)$: Let Σ' be a θ-stable twin apartment containing $h.c$. By strong transitivity, there exists $g \in G$ such that $\Sigma' = g.\Sigma$ and $h.c = g.c$, hence $gB = hB$. Then $\theta(g.\Sigma) = \theta(g).\Sigma = g.\Sigma$, i.e., $g^{-1}\theta(g) \in N_G(\Sigma)$.

Now g is unique up to right translation by $Z_G(\Sigma)$ and left translation by G_θ. So if we put $W_{G/G_\theta}(\Sigma) = \{G_\theta g Z_G(\Sigma) \mid g^{-1}\theta(g) \in N_G(\Sigma)\}$, then $G_\theta \backslash G/B \cong W_{G/G_\theta}(\Sigma)$ in such a way that $G_\theta gB \leftrightarrow G_\theta g Z_G(\Sigma)$ if $g^{-1}\theta(g) \in N_G(\Sigma)$. $\square$

CHAPTER

THREE

FLIPS IN RANK 1 AND 2

In this chapter, we present some results about flips of Moufang buildings of rank 1 and 2. These are of some interest on their own, but in general will enable us to prove things about higher-rank flips by reducing to results on lower-rank flips.

In rank 1, the correct viewpoint is to study involutory automorphisms of Moufang sets, resp. of rank 1 groups. For this, we first focus our attention on the most basic case, namely involutory automorphisms of $\mathrm{SL}_2(\mathbb{F})$ and $\mathrm{PSL}_2(\mathbb{F})$ where $\mathbb{F}$ is an arbitrary field. This case suffices to deal with locally split groups, such as split algebraic or Kac-Moody groups. This is detailed in Section 3.1.

For flips of arbitrary Moufang sets, the situation is not as good. Still, we give some results in Section 3.2. The aim is to show how one might be able (albeit with difficulties) to extend the theory to groups beyond $\mathbb{F}$-locally split ones. As a first step we present some results for $\mathrm{SL}_2(\mathbb{D})$ where $\mathbb{D}$ is a division ring.

In Section 3.3, we study involutory automorphisms of classical generalized quadrangles, a special class of Moufang buildings of rank 2 (Moufang polygons). There, we primarily investigate when the chamber system of chambers mapped far away by a flip is connected. Knowing this will be crucial in Chapter 4. This is the result of joint work by Hendrik Van Maldeghem and the author, see [HVM]. We hope to eventually be able to treat arbitrary Moufang quadrangles and hexagons, but this is work in progress.

3.1. Flips of $\mathrm{SL}_2(\mathbb{F})$ and $\mathrm{PSL}_2(\mathbb{F})$

In Section 3.1.1 we classify all involutory automorphisms of $\mathrm{SL}_2(\mathbb{F})$ and $\mathrm{PSL}_2(\mathbb{F})$. We exploit that all automorphisms of $\mathrm{PSL}_2(\mathbb{F})$ are induced by automorphisms of $\mathrm{SL}_2(\mathbb{F})$, which follows from the fact that SL_2 is perfect if $|\mathbb{F}| \geq 4$ and is easily verified over the fields of two and three elements. Alternatively one can use the classification of endomorphisms of Steinberg groups or apply the results in [RWW87]. Hence it suffices to study flips of $\mathrm{SL}_2(\mathbb{F})$. In Section 3.1.2 we compute the fixed point groups of these flips. This work will be used in Chapter 5 where we strive to determine global transitivity properties of flips from their local (rank 1) behavior.

3.1.1. Classifying flips of $\mathrm{SL}_2(\mathbb{F})$

In order to be able to understand involutory automorphisms of $G := \mathrm{SL}_2(\mathbb{F})$, consider $\mathrm{SL}_2(\mathbb{F})$ as a matrix group acting on its natural module. Let T denote the subgroup of diagonal matrices, which is a maximal torus of G. Let U_+ resp. U_- denote the subgroups of strictly upper resp. lower triangular unipotent matrices, which are root subgroups with respect to the root system of type A_1 associated to T. The standard Borel subgroups of G then are the groups $B_+ := T.U_+$ and $B_- := T.U_-$ (the subgroups of all upper resp. lower matrices). Finally, setting $N := N_G(T)$ we obtain a (twin)-BN-pair (B_+, B_-, N) as defined in Section 1.8.

Recall that two automorphisms ϕ, ψ of G are **conjugate** if they are conjugate within $\mathrm{Aut}(G)$, that is, there exists $\omega \in \mathrm{Aut}(G)$ such that $\phi = \omega \circ \psi \circ \omega^{-1}$. For $A \in G$, denote by Int_A the inner automorphism $x \mapsto AxA^{-1}$. By slight abuse of notation, we also use this notation if $A \in \mathrm{GL}_2(\mathbb{F})$.

Lemma 3.1.1. *Every involutory automorphism of $G = \mathrm{SL}_2(\mathbb{F})$ is conjugate to an involutory automorphism which interchanges B_+ and B_-.*

Proof. The automorphisms of $\mathrm{SL}_2(\mathbb{F})$ are determined in [SW28] (a gap in the proof given there is closed in [Hua48]): Any automorphism θ of G is obtained by composing a field automorphism σ with Int_g, where $g \in \mathrm{GL}_2(\mathbb{F})$. One readily observes that if θ has order 2, σ has order at most 2. Furthermore, from $\theta^2(x) = \mathrm{Int}_{gg^\sigma}(x) = x$ follows that $gg^\sigma = \lambda I$ for some $\lambda \in \mathbb{F}^*$ (i.e., gg^σ is an element of the center of $\mathrm{GL}_2(\mathbb{F})$). Finally, since σ maps B_+ to itself, θ maps any conjugate of B_+ again to a conjugate.

Assume that for all $h \in G$, θ maps the group hB_+h^{-1} to itself, equivalently, maps hB_-h^{-1} to itself. We claim that θ then is the identity: For $h \in G$ and $\varepsilon \in \{+, -\}$, we have $\theta(hB_\varepsilon h^{-1}) = hB_\varepsilon h^{-1}$ which implies $h^{-1}gh^\sigma \in N_G(B_+) \cap N_G(B_-) = B_+ \cap B_- = T$. Setting $h = I$, we conclude $g \in T$, say $g = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ with $t \in \mathbb{F}^*$. Setting, $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, from $h^{-1}gh^\sigma = \begin{pmatrix} t & t-t^{-1} \\ 0 & t^{-1} \end{pmatrix} \in T$, we deduce $t = t^{-1}$, hence $g = tI$. Thus, for all $h \in G$ we have $h^{-1}h^\sigma \in T$. Setting $h_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ with $x \in \mathbb{F}$ arbitrary, $h^{-1}h^\sigma = \begin{pmatrix} 1 & x^\sigma - x \\ 0 & 1 \end{pmatrix} \in T$ implies $\sigma = \mathrm{id}_\mathbb{F}$, thus θ is indeed the identity map.

Hence if θ is an involution, there exists $a \in G$ such that aB_+a^{-1} is distinct from its image under θ. Then $\theta' := \mathrm{Int}_a \circ \theta \circ (\mathrm{Int}_a)^{-1}$ is an involution conjugate to θ which maps B_+ to a conjugate yB_+y^{-1} different from B_+. There exists a unique $u \in U_+$ such that $yB_+y^{-1} = u^{-1}B_-u$. Then $\theta'' := \mathrm{Int}_u \circ \theta' \circ (\mathrm{Int}_u)^{-1}$ is an involution conjugate to θ which interchanges B_+ and B_-. □

Lemma 3.1.2. *Any involutory automorphism θ of G which interchanges B_+ and B_- is of the form*

$$\theta : X \mapsto \begin{pmatrix} 0 & 1 \\ \delta & 0 \end{pmatrix} X^\sigma \begin{pmatrix} 0 & \delta^{-1} \\ 1 & 0 \end{pmatrix},$$

where σ is a field automorphism of order at most 2, and $\delta \in \mathrm{Fix}_{\mathbb{F}^*}(\sigma)$.

Proof. We know that $\theta = \mathrm{Int}_g \circ \sigma$ for some $\sigma \in \mathrm{Aut}(\mathbb{F})$ of order at most 2, and some $g \in \mathrm{GL}_2(\mathbb{F})$. Since θ interchanges B_+ and B_-, it follows that θ stabilizes $T = B_+ \cap B_-$ and interchanges U_+ and U_-. As σ stabilizes U_+ and U_-, we conclude that Int_g must interchange U_+ and U_-. One readily computes that then $g = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ for some $b, c \in \mathbb{F}^*$. Moreover, $\theta^2 = \mathrm{Id}$ implies $gg^\sigma = \lambda I$ for some $\lambda \in \mathbb{F}^*$. But $gg^\sigma = \left(\begin{smallmatrix} bc^\sigma & 0 \\ 0 & b^\sigma c \end{smallmatrix}\right)$. Hence $\lambda = bc^\sigma = b^\sigma c \in \mathrm{Fix}_{\mathbb{F}^*}(\sigma)$. Setting $\delta := \frac{c}{b} = \frac{\lambda}{bb^\sigma} \in \mathrm{Fix}_{\mathbb{F}^*}(\sigma)$, we observe that g and $\left(\begin{smallmatrix} 0 & 1 \\ \delta & 0 \end{smallmatrix}\right)$ induce the same automorphism. □

The preceding lemma motivates the following definition:

Definition 3.1.3. For $\sigma \in \mathrm{Aut}(\mathbb{F})$ of order at most 2 and $\delta \in \mathrm{Fix}_{\mathbb{F}^*}(\sigma)$ we define the **standard involution**

$$\theta_{\delta,\sigma} : \mathrm{SL}_2(\mathbb{F}) \to \mathrm{SL}_2(\mathbb{F}) : X \mapsto \theta_{\delta,\sigma}(X) = x_\delta X^\sigma x_\delta^{-1},$$

where $x_\delta := \left(\begin{smallmatrix} 0 & 1 \\ \delta & 0 \end{smallmatrix}\right)$. Equivalently, $\theta_{\delta,\sigma} = \mathrm{Int}_{x_\delta} \circ \sigma$.

By slight abuse of notation, we will use the same symbol $\theta_{\delta,\sigma}$ to denote the induced flip on $\mathrm{PSL}_2(\mathbb{F})$. Altogether, we have proved the following in this section:

Proposition 3.1.4. *For every involutory automorphism of* $\mathrm{SL}_2(\mathbb{F})$ *or* $\mathrm{PSL}_2(\mathbb{F})$ *there exist* $\sigma \in \mathrm{Aut}(\mathbb{F})$ *of order at most* 2 *and* $\delta \in \mathrm{Fix}_{\mathbb{F}^*}(\sigma)$ *such that* θ *is conjugate to* $\theta_{\delta,\sigma}$*.*
□

We now describe when two standard involutions are conjugate.

Proposition 3.1.5. *Two standard involutions* $\theta_{\delta,\sigma}$ *and* $\theta_{\varepsilon,\tau}$ *are conjugate if and only if there exists* $\rho \in \mathrm{Aut}(\mathbb{F})$ *such that* $\sigma = \rho\tau\rho^{-1}$ *and* $\delta/\varepsilon^\rho \in N_\sigma(\mathbb{F}^*)$*, where* $N_\sigma(x) := xx^\sigma$*.*

Proof. If $\theta_{\delta,\sigma}$ and $\theta_{\varepsilon,\tau}$ are conjugate, there exists an automorphism $\phi = \mathrm{Int}_g \circ \rho$, with $\rho \in \mathrm{Aut}(\mathbb{F})$ and $g \in \mathrm{GL}_2(\mathbb{F})$, such that our two standard involutions are conjugate by ϕ. That is,

$$\begin{aligned} & \theta_{\delta,\sigma} \circ \phi = \phi \circ \theta_{\varepsilon,\tau} \\ \Longleftrightarrow & (\mathrm{Int}_{x_\delta} \circ \sigma) \circ (\mathrm{Int}_g \circ \rho) = (\mathrm{Int}_g \circ \rho) \circ (\mathrm{Int}_{x_\varepsilon} \circ \tau) \\ \Longleftrightarrow & \mathrm{Int}_{x_\delta} \circ \mathrm{Int}_{g^\sigma} \circ \sigma \circ \rho = \mathrm{Int}_g \circ \mathrm{Int}_{x_\varepsilon^\rho} \circ \rho \circ \tau \\ \Longleftrightarrow & \mathrm{Int}_{x_\delta} \circ \mathrm{Int}_{g^\sigma} \circ \sigma = \mathrm{Int}_g \circ \mathrm{Int}_{x_\varepsilon^\rho} \circ (\rho\tau\rho^{-1}), \end{aligned}$$

where x_δ, x_ε as in Definition 3.1.3. Hence we must have $\sigma = \rho\tau\rho^{-1}$ (to see this, note that $\mathrm{Aut}(G)$ is the semi-direct product of the normal subgroup of "inner" automorphisms induced by $\mathrm{GL}_2(\mathbb{F})$, and the field automorphisms). Accordingly, $\mathrm{Int}_{x_\delta} \circ \mathrm{Int}_{g^\sigma} = \mathrm{Int}_g \circ \mathrm{Int}_{x_\varepsilon^\rho}$, which implies $x_\delta g^\sigma = \lambda g x_\varepsilon^\rho$ for some $\lambda \in \mathbb{F}^*$. Setting $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, this is equivalent to

$$\begin{pmatrix} c^\sigma & d^\sigma \\ \delta a^\sigma & \delta b^\sigma \end{pmatrix} = \lambda \begin{pmatrix} \varepsilon^\rho b & a \\ \varepsilon^\rho d & c \end{pmatrix},$$

from which we deduce by comparing coefficients that $\frac{\delta}{\varepsilon^\rho} = N_\sigma(\lambda)$.

Conversely, suppose there exist $\rho \in \operatorname{Aut}(\mathbb{F})$ and $\lambda \in \mathbb{F}^*$ such that $\sigma = \rho\tau\rho^{-1}$ and $\delta/\varepsilon^\rho = N_\sigma(\lambda)$. Set $g := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ and $\phi = \operatorname{Int}_g \circ \rho$. Then $\theta_{\delta,\sigma} = \phi \circ \theta_{\varepsilon,\tau} \circ \phi^{-1}$. $\square$

Corollary 3.1.6. *Let* $\operatorname{Inv}(\mathbb{F})$ *denote a set of representatives of the conjugacy classes of automorphisms of* $\mathbb{F}$ *of order at most* 2. *Then the conjugacy classes of involutory automorphisms of* G *correspond one-to-one to the disjoint union*

$$\bigsqcup_{\sigma \in \operatorname{Inv}(\mathbb{F})} \operatorname{Fix}_{\mathbb{F}^*}(\sigma)/N_\sigma(\mathbb{F}^*), \qquad \textit{where } N_\sigma(x) := xx^\sigma.$$

3.1.2. Centralizers of flips

We now turn our attention to the centralizers of a given flip θ, which will be of interest in Chapter 5. We compute

$$C_{\mathrm{SL}_2(\mathbb{F})}(\theta_{\delta,\sigma}) = \left\{ \begin{pmatrix} u & v \\ w & x \end{pmatrix} \middle| \begin{pmatrix} 0 & 1 \\ \delta & 0 \end{pmatrix} \begin{pmatrix} u^\sigma & v^\sigma \\ w^\sigma & x^\sigma \end{pmatrix} \begin{pmatrix} 0 & \delta^{-1} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u & v \\ w & x \end{pmatrix}, ux - vw = 1 \right\}.$$

It is now easy to verify that

$$K_{\delta,\sigma} := C_{\mathrm{SL}_2(\mathbb{F})}(\theta_{\delta,\sigma}) = \left\{ \begin{pmatrix} u & v \\ \delta v^\sigma & u^\sigma \end{pmatrix} \mid uu^\sigma - \delta vv^\sigma = 1 \right\},$$

which is precisely the group preserving the σ-sesquilinear form

$$f_{\delta,\sigma}(x,y) := x^T \begin{pmatrix} -\delta & 0 \\ 0 & 1 \end{pmatrix} y^\sigma, \quad x, y \in \mathbb{F}^2,$$

on the vector space $\mathbb{F}^2$ and its associated σ-quadratic form $q_{\delta,\sigma}$ given by $q_{\delta,\sigma}(x) := f_{\delta,\sigma}(x,x)$. This alternative characterization will turn out to be quite useful.

For $\mathrm{PSL}_2(\mathbb{F})$, the situation is slightly different. Denote by Z the center of $\mathrm{SL}_2(\mathbb{F})$, so $\mathrm{PSL}_2(\mathbb{F}) = \mathrm{SL}_2(\mathbb{F})/Z$, accordingly the centralizer of θ in $\mathrm{PSL}_2(\mathbb{F})$ is $C_{\mathrm{PSL}_2(\mathbb{F})}(\theta) = \{gZ \in \mathrm{PSL}_2(\mathbb{F}) \mid (gZ)^\theta = gZ\}$. We are mainly interested in the action of this centralizer on $\mathbb{P}_1(\mathbb{F})$. Since the action of $\mathrm{PSL}_2(\mathbb{F})$ is induced by that of $\mathrm{SL}_2(\mathbb{F})$, this boils down to studying the preimage of the centralizer in $\mathrm{SL}_2(\mathbb{F})$, which suggests the following definition:

Definition 3.1.7. Let θ be an automorphism of $\mathrm{SL}_2(\mathbb{F})$. We define the **projective centralizer** of θ in $\mathrm{SL}_2(\mathbb{F})$ as the group $PC_{\mathrm{SL}_2(\mathbb{F})}(\theta) := \{g \in \mathrm{SL}_2(\mathbb{F}) \mid g^\theta \in gZ\}$, which is the preimage of $C_{\mathrm{PSL}_2}(\theta)$ in $\mathrm{SL}_2(\mathbb{F})$ under the canonical projection $\pi : \mathrm{SL}_2 \to \mathrm{PSL}_2$.

We compute

$$PK_{\delta,\sigma} := PC_{\mathrm{SL}_2(\mathbb{F})}(\theta_{\delta,\sigma}) = \left\{ \begin{pmatrix} \varepsilon u & \varepsilon v \\ \delta v^\sigma & u^\sigma \end{pmatrix} \mid uu^\sigma - \delta vv^\sigma = \varepsilon,\ \varepsilon \in \{+1, -1\} \right\}.$$

While $K_{\delta,\sigma}$ preserves the σ-sesquilinear form $f_{\delta,\sigma}(x,y)$ and its associated σ-quadratic form $q_{\delta,\sigma}(x)$, the group $PK_{\delta,\sigma}$ preserves these forms only up to sign.

3.2. Flips of Moufang sets

We now turn our attention to the general rank 1 case, i.e., the study of flips of Moufang sets. The results presented here are due to Tom De Medts, cf. [DMGH09, Section 5]. We closely follow the notation introduced in Section 1.10.

The goal of this section is to characterize the involutions $\theta \in \mathrm{Aut}(G)$ interchanging U_∞ and U_0. Such an involution θ maps each α_a to some $\gamma_{a\varphi}$ and each γ_b to some $\alpha_{b\psi}$. Since $\theta \in \mathrm{Aut}(G)$, we have $\varphi, \psi \in \mathrm{Aut}(U)$. Moreover, $\theta^2 = \mathrm{id}$ implies $\psi = \varphi^{-1}$. In particular, θ is competely determined by φ. More precisely, for each $\varphi \in \mathrm{Aut}(U)$, we define

$$\theta_\varphi : U_\infty \cup U_0 \to U_0 \cup U_\infty : \begin{cases} \alpha_a \mapsto \gamma_{a\varphi} \\ \gamma_a \mapsto \alpha_{a\varphi^{-1}}. \end{cases}$$

The question is when θ_φ extends to an automorphism of G. Observe that if θ_φ extends, then this extension is unique and is involutory, since θ is involutory on $U_\infty \cup U_0$ and $G = \langle U_\infty \cup U_0 \rangle$.

Proposition 3.2.1. *Let $\varphi \in \mathrm{Aut}(U)$. Then θ_φ extends to an (involutory) automorphism of G if and only if $(\varphi\tau)^2 = \mathrm{id}$. Moreover, if this is the case, then $\varphi \in \mathrm{Aut}(\mathbb{M})$.*

Proof. Let $\theta := \theta_\varphi$ and $\beta := \varphi\tau$. Assume first that θ extends to an automorphism χ of G. Then

$$\chi(U_a) = \chi(U_0^{\alpha_a}) = \chi(U_0)^{\chi(\alpha_a)} = U_\infty^{\gamma_{a\varphi}} = U_{a\varphi\tau} = U_{a\beta} \tag{3.1}$$

for all $a \in U$. Since θ^2 is the identity on $U_\infty \cup U_0$ and since $G = \langle U_\infty, U_0 \rangle$, this implies that $\chi^2 = 1$ and hence $\beta^2 = 1$.

Conversely, assume that $\beta^2 = 1$, and let χ_β be as in Definition 1.10.7. Then for all $a \in U$,

$$\begin{aligned} \chi_\beta(\alpha_a) &= \alpha_a^{\varphi\tau} = \alpha_{a\varphi}^{\tau} = \gamma_{a\varphi}\,, \\ \chi_\beta(\gamma_a) &= \gamma_a^{\varphi\tau} = \gamma_a^{\tau^{-1}\varphi^{-1}} = \alpha_a^{\varphi^{-1}} = \alpha_{a\varphi^{-1}}\,; \end{aligned}$$

hence χ_β and θ coincide on $U_\infty \cup U_0$. Note that χ_β is an (inner) automorphism of $\mathrm{Sym}(X)$, and hence the same calculation as in equation (3.1) (with χ_β in place of χ) shows that $\beta \in \mathrm{Aut}(\mathbb{M})$. Hence the restriction of χ_β to G is an automorphism of G; this is the (unique) extension of θ to an element of $\mathrm{Aut}(G)$.

Finally, since we have just shown that $\beta \in \mathrm{Aut}(\mathbb{M})$ and since obviously $\tau \in \mathrm{Aut}(\mathbb{M})$, we conclude that $\varphi \in \mathrm{Aut}(\mathbb{M})$ as well. $\square$

Definition 3.2.2. An automorphism $\varphi \in \mathrm{Aut}(U)$ with the property that $(\varphi\tau)^2 = 1$ will be called a **flip automorphism** of $\mathbb{M}$.

The following theorem gives important information about such flip automorphisms. In accordance with the usual conventions employed in theory of Moufang sets and with our notation from Section 1.10, we will always denote the action of a permutation on a set on the right, i.e., we will write $a\varphi$ rather than $\varphi(a)$.

Theorem 3.2.3. *Let $\mathbb{M}$ be a Moufang set, and let φ be a flip automorphism of $\mathbb{M}$. Then*

$$g_{a\varphi} = \varphi \cdot h_a \cdot \varphi$$

for all $a \in U^*$. Moreover, if e is an identity element of $\mathbb{M}$, i.e., $\tau = \mu_{-e}$, then $\varphi \in \operatorname{Str}(\mathbb{M}, e) \cap \operatorname{Aut}(\mathbb{M})$.

Proof. For each $a \in U^*$, the map g_a is the Hua map of a with τ replaced by τ^{-1}, and hence $g_{a\varphi} = \tau^{-1}\alpha_{a\varphi}\tau\alpha_{-a\varphi\tau}\tau^{-1}\alpha_{-(-(-a\varphi\tau))\tau^{-1}}$ for all $a \in U^*$. Using the facts that $\alpha_a^\varphi = \alpha_{a\varphi}$, $\varphi\tau = \tau^{-1}\varphi^{-1}$ and $(-a)\varphi = -a\varphi$ several times, we get $\varphi^{-1}g_{a\varphi} = \tau\alpha_a\tau^{-1}\alpha_{-a\tau^{-1}}\tau\alpha_{-(-(-a\tau^{-1}))\tau}\varphi = h_a\varphi$. In particular, if e is an identity element of $\mathbb{M}$, then $h_e = 1$ and hence $\varphi^{-1}g_{e\varphi} = \varphi$. It follows that $\varphi g_{e\varphi}^{-1} g_{a\varphi} = h_a\varphi$ for all $a \in U^*$. However, $g_{e\varphi}^{-1}g_{a\varphi} = (\mu_e\mu_{e\varphi})^{-1}(\mu_e\mu_{a\varphi}) = (\mu_{-e}\mu_{e\varphi})^{-1}(\mu_{-e}\mu_{a\varphi}) = h_{e\varphi}^{-1}h_{a\varphi} = h_{a\varphi}^{(e\varphi)}$ and hence $h_a\varphi = \varphi h_{a\varphi}^{(e\varphi)}$ for all $a \in U^*$, proving that $\varphi \in \operatorname{Str}(\mathbb{M}, e)$. The fact that $\varphi \in \operatorname{Aut}(\mathbb{M})$ was shown in Proposition 3.2.1 above. □

We will now illustrate the strength of Theorem 3.2.3 by explicitly determining all flips of $\mathrm{PSL}_2(\mathbb{D})$, where $\mathbb{D}$ is a field or a skew field.

Proposition 3.2.4. *Let $\mathbb{D}$ be an arbitrary field or skew field, and let $\mathbb{M} = \mathbb{M}(\mathbb{D})$ be the corresponding Moufang set, i.e., the Moufang set $\mathbb{M} = \mathbb{M}(U, \tau)$ where $U := (\mathbb{D}, +)$ and $\tau : \mathbb{D}^* \to \mathbb{D}^* : x \mapsto -x^{-1}$.*

(i) *Let φ be a flip automorphism of $\mathbb{M}$. Then there exists an automorphism or anti-automorphism σ of $\mathbb{D}$ and an element $\varepsilon \in \operatorname{Fix}_{\mathbb{D}}(\sigma)$ such that $x\varphi = \varepsilon\sigma(x)$ for all $x \in \mathbb{D}$. If σ is an automorphism, then $\sigma^2(x) = \varepsilon^{-1}x\varepsilon$ for all $x \in \mathbb{D}$; if σ is an anti-automorphism, then $\sigma^2 = 1$.*

(ii) *Conversely, suppose that either σ is an anti-automorphism of order 2 and $\varepsilon \in \operatorname{Fix}_{\mathbb{D}}(\sigma)$ is arbitrary, or σ is an automorphism such that $\sigma^2(x) = \varepsilon^{-1}x\varepsilon$ for some $\varepsilon \in \operatorname{Fix}_{\mathbb{D}}(\sigma)$. Then the map $\varphi : \mathbb{D} \to \mathbb{D} : x \mapsto \varepsilon\sigma(x)$ is a flip automorphism of $\mathbb{M}$.*

Proof. (i) Observe that $1 \in \mathbb{D}^*$ is an identity element of $\mathbb{M}$; also note that $\tau^2 = \mathrm{id}$. For all $a, b \in U^*$, we have $bh_a = aba$. The condition $(\varphi\tau)^2 = 1$ translates to

$$(a^{-1})\varphi = (a\varphi^{-1})^{-1} \tag{3.2}$$

for all $a \in \mathbb{D}^*$. Let $\varepsilon := 1\varphi$; then $bh_a^{(1\varphi)} = bh_{1\varphi}^{-1}h_a = a\varepsilon^{-1}b\varepsilon^{-1}a$ for all $a, b \in U^*$. By Theorem 3.2.3, $\varphi \in \operatorname{Str}(\mathbb{M}, e)$, which means that $bh_a\varphi = b\varphi h_{a\varphi}^{(1\varphi)}$ for all $a, b \in U^*$, or explicitly, $(aba)\varphi = a\varphi \cdot \varepsilon^{-1} \cdot b\varphi \cdot \varepsilon^{-1} \cdot a\varphi$ for all $a, b \in \mathbb{D}^*$. Now let $\sigma(a) := \varepsilon^{-1} \cdot a\varphi$ for all $a \in \mathbb{D}$. Then $\sigma \in \operatorname{Aut}(U)$, and the previous equation can be rewritten as $\sigma(aba) = \sigma(a)\sigma(b)\sigma(a)$ for all $a, b \in \mathbb{D}$, i.e., σ is a Jordan automorphism of $\mathbb{D}$. It is a well known result by Jacobson and Rickart [JR50] (see also [Jac68, page 2]), which simply amounts to calculating that $\big(\sigma(ab) - \sigma(a)\sigma(b)\big) \cdot \big(\sigma(ab) - \sigma(b)\sigma(a)\big) = 0$, that σ is either an automorphism or an anti-automorphism of $\mathbb{D}$. Now by equation (3.2), we have

$(\varepsilon^{-1})\varphi = (\varepsilon\varphi^{-1})^{-1} = 1^{-1} = 1$, and hence $\sigma(\varepsilon^{-1}) = \varepsilon^{-1}$; since σ is an automorphism or anti-automorphism, it follows that $\sigma(\varepsilon) = \varepsilon$. Finally, again by equation (3.2), we obtain $\sigma(\varepsilon\sigma(a)) = \sigma(a\varphi) = \sigma\Big(((a^{-1})\varphi^{-1})^{-1}\Big) = \sigma((a^{-1})\varphi^{-1})^{-1} = (\varepsilon^{-1}a^{-1})^{-1} = a\varepsilon$ for all $a \in \mathbb{D}^*$. If σ is an automorphism, then this can be rewritten as $\varepsilon\sigma^2(a) = a\varepsilon$, or $\sigma^2(a) = \varepsilon^{-1}a\varepsilon$; if σ is an anti-automorphism, we get $\sigma^2(a)\varepsilon = a\varepsilon$, i.e., $\sigma^2 = 1$.

(ii) It suffices to check that equation (3.2) holds. This amounts to checking that $\varepsilon\sigma(a^{-1}) = (\sigma^{-1}(\varepsilon^{-1}a))^{-1}$ for all $a \in \mathbb{D}$. It is straightforward to check that this is valid in both cases. □

By [RWW87] the flips of $\mathrm{SL}_2(\mathbb{D})$ are just the lifts of the flips of $\mathrm{PSL}_2(\mathbb{D})$.

3.3. Classical quadrangles

In this section we study involutory automorphisms of classical quadrangles. First, we give some auxiliary results on Moufang sets in Section 3.3.1.

In Section 3.3.2 we give an algebraic description of classical quadrangles which we use throughout the rest of this section. Our main reference for classical quadrangles is [VM98, Section 2.3].

For any involutory automorphism θ of a generalized quadrangle, we define the **flip-flop system** R^θ as the set of all chambers c for which $\delta(c, \theta(c))$ is maximal among all chambers (here δ is the Weyl metric on the quadrangle). This is a special case of a more general definition in Chapter 4. We will show that when our quadrangle is defined over a field of characteristic different from 2, the flip-flop system R^θ is connected regardless of the choice of θ.

Note: We only deal with classical quadrangles defined over commutative fields in characteristic different from 2. With some effort it should be possible to refine the arguments to work over skew fields, and (at least the connectedness result) in characteristic 2. However, the arguments become a lot more involved. Since we are primarily interested in the split case, we decided not to try to achieve full generality here.

3.3.1. Some auxiliary results on Moufang sets

We refer the reader to Section 1.10 for the basics about Moufang sets. Here, we only present some non-standard extensions to the theory, which while basic and relatively simple, mostly seem to not be in the literature.

For convenience, we will write $\mathbb{M}(X, U)$ as a shorthand for $(X, (U_x)_{x\in X})$.

Definition 3.3.1. A **(proper) Moufang subset** of a Moufang set $\mathbb{M}(X, U)$ is a Moufang set $\mathbb{M}(Y, V)$ such that Y is a (proper) subset of X and for all $y \in Y$ we have $V_y \leq U_y$. We also write $\mathbb{M}(Y, V) \leq \mathbb{M}(X, U)$ (resp. $\mathbb{M}(Y, V) < \mathbb{M}(X, U)$ if Y is a proper Moufang subset).

We extend this notion slightly:

Definition 3.3.2. A **generalized Moufang subset** of a Moufang set $\mathbb{M}(X,U)$ is subset Y of X such that if $|Y| \geq 2$, then there exists a Moufang subset $\mathbb{M}(Y,V)$ of $\mathbb{M}(X,U)$ on V. We also write $Y \leq \mathbb{M}(X,U)$ (and $Y < \mathbb{M}(X,U)$ if Y is a proper subset of X).

Remark 3.3.3. A related concept is that of a root subgroup, see [Seg08, Section 3].

A well-known fact is that (generalized) Moufang subsets occur naturally as fixed point sets of automorphisms of Moufang sets:

Lemma 3.3.4. *Let $\mathbb{M}(X,U)$ be a Moufang set, let σ be an automorphism of $\mathbb{M}(X,U)$. Denote by Y the set of fixed points of σ. Then Y is a generalized Moufang subset of $\mathbb{M}(X,U)$.*

Proof. If $|Y| < 2$ there is nothing to show. So suppose $|Y| \geq 2$. For every $y \in Y$, choose an element y' in Y different from y and set $V_y := \{g \in U_y \mid y'.g \in Y\}$. We need to verify that for each $y \in Y$ the set V_y is well-defined (i.e., independent of the choice of y'), is a subgroup of U_y and acts sharply transitively on $Y \setminus \{y\}$.

So first observe that for each $y \in Y$, we have $y'.V_y = Y \setminus \{y\}$, due to the way we defined V_y and since U_y acts sharply transitively on $X \setminus \{y\}$. Then for all $g \in V_y$ we have

$$y'.g^\sigma = y'.(\sigma^{-1} g \sigma) = y'.g\sigma = y'.g$$

and thus by regularity, $g = g^\sigma$.

To see that V_y maps Y to Y (and hence forms a group), assume the existence of some $g \in V_y$ and some $y'' \in Y$ such that $y''.g =: z \notin Y$. But this yields a contradiction:

$$z = y''.g = y''.(\sigma^{-1} g \sigma) = y''.g\sigma = z.\sigma \neq z.$$

Hence V_y is the unique subgroup of U_y acting sharply transitively on $Y \setminus \{y\}$ for each $y \in Y$. It follows that V_y is independent of the choice of y', and permutes the set $\{V_x \mid x \in Y\}$ by conjugation. □

The intersection of two generalized Moufang subsets of a given Moufang set is again a generalized Moufang subset.

Lemma 3.3.5. *Let $\mathbb{M}(X,U)$ be a Moufang set, and let Y and Z be two generalized Moufang subsets. Then $Y \cap Z$ is a generalized Moufang subset.*

Proof. Set $A := Y \cap Z$. If $|A| < 2$, there is nothing to show. So assume A contains at least two distinct elements 0 and ∞. Then Y and Z are the base sets of two Moufang subsets $\mathbb{M}(Y,V)$ and $\mathbb{M}(Z,W)$ of $\mathbb{M}(X,U)$. Set $B_\infty := V_\infty \cap W_\infty$. Since X is a Moufang set, there exists for each $a \in A$ a unique element $g \in U_\infty$ which maps 0 to a. Therefore g must also be contained in V_∞ and W_∞ and hence in B_∞. Thus B_∞ acts regularly on $A \setminus \{\infty\}$. Also, g permutes the B_x by conjugation, for we have

$$B_x^g = (V_x \cap W_x)^g = V_x^g \cap W_x^g = V_{xg} \cap W_{xg} = B_{xg}.$$

Since 0 and ∞ were arbitrary, this completes the proof. □

Next we observe that Moufang subsets can be at most about "half as big" as the Moufang set they are contained in. In particular, the order of the root subgroups V_x must divide the order of the original root groups U_x, and hence have at least index 2. The following lemma makes this precise.

Lemma 3.3.6. *Let $\mathbb{M}(X, U)$ be a Moufang set and Y a proper generalized Moufang subset. Then the following holds:*

(1) $|Y| \leq |X \setminus Y| + 1$.

(2) If X is infinite then $X \setminus Y$ cannot be finite.

(3) If X is finite, denote by p the smallest prime dividing $|X| - 1$. Then

$$p(|Y| - 1) \leq |X| - 1 \leq \frac{p}{p-1}|X \setminus Y|.$$

(4) If X is finite then $2|Y| \leq |X| + 1$ and $|X| \leq 2|X \setminus Y| + 1$.

Proof. (1) The claim is trivial if $|Y| \leq 1$. So assume w.l.o.g. that Y contains two distinct elements 0 and ∞. Since $Y \subsetneq X$, we know in fact that $V_0 < U_0$. Thus $|U_0 : V_0| \geq 2$, which implies that $|V_0| \leq |U_0 \setminus V_0|$. But $U_0 \setminus V_0$ is in natural bijection with $X \setminus Y$ (identify each element g in the first set with $0g$). Likewise, $Y \setminus \{\infty\}$ is in natural bijection with V_0.

(2) If X was infinite and $X \setminus Y$ finite, then Y would be infinite, contradicting (1).

(3) If X is finite, then $|X| = |U_0| + 1$ and $|Y| = |V_0| + 1$, thus p is the smallest prime dividing $|U_0|$, and so $|U_0 : V_0| \geq p$. For this reason $p(|Y| - 1) \leq |X| - 1$. By adding $(p - 1)|X|$ to both sides, and dividing by $p - 1$, we get the second inequality.

(4) Follows from (3) by using that $2 \leq p$. □

As a consequence, a Moufang set cannot be the union of two of its proper generalized Moufang subsets, unless it is very small. This is still true if we allow adding one extra point to the union.

Lemma 3.3.7. *Let $\mathbb{M}(X, U)$ be Moufang set, and let Y and Z be two proper generalized Moufang subsets. Then if $X = Y \cup Z \cup \{a\}$ for some $a \in X$, we have $|X| \leq 5$.*

Proof. Suppose $X = Y \cup Z \cup \{a\}$ and $|X| > 5$.

By Lemma 3.3.6, if X is infinite then both Y and Z and their complements must be infinite. If X is finite, then $5 < |X| \leq 2|X \setminus Y| + 1$ implies that $|X \setminus Y| > 2$. In either case, both $X \setminus Y \subseteq Z \setminus Y \cup \{a\}$ and $X \setminus Z \subseteq Y \setminus Z \cup \{a\}$ each contain at least three elements. Hence $Z \setminus Y$, $Y \setminus Z$, Y and Z all contain at least two elements each. In particular, Y and Z form Moufang subsets $\mathbb{M}(Y, V)$ and $\mathbb{M}(Z, W)$.

Fix two distinct elements 0 and ∞ in $Y \setminus Z$ and choose any $g \in U_\infty$ which maps 0 into $Z \setminus Y$. Clearly, $g \notin V_\infty$ and in fact g maps $Y \setminus \{\infty\}$ into a subset of $Z \setminus Y \cup \{a\}$ (because U_∞ acts regularly on $X \setminus \{\infty\}$, and $V_\infty < U_\infty$ already acts regularly on $Y \setminus \{\infty\}$). Now Yg is again a Moufang set with root groups V_y^g. Hence by Lemma 3.3.5, $Z' := Z \cap Yg$ is a generalized Moufang subset. But then $Y' := Z'g^{-1}$ is also a generalized Moufang subset.

If $a \in Y \cup Z$, then $Y' = Y \setminus \{\infty\}$ and by Lemma 3.3.6, we obtain $|Y| \le 2|Y \setminus Y'| + 1 \le 3$ and by symmetry $|Z| \le 3$, thus $|X| \le 6$. However if $|X| = 6 = 5 + 1$, the only Moufang subsets are of size 2 and 1, so a Moufang set of size 6 cannot be covered by two Moufang subsets and a single point.

If $a \notin Y \cup Z$, then we may have $Y' = Y \setminus \{ag^{-1}, \infty\}$. Again by Lemma 3.3.6, we obtain $|Y| \le 2|Y \setminus Y'| + 1 \le 5$. By symmetry also $|Z| \le 5$ hence $|X| \le 11$.

The remaining possibilities can be excluded via the classification of finite Moufang sets (see [HKS72] and [Shu72]), or via direct computations (using a simple computer program). Briefly sketched, the arguments are as follows: For a Moufang set of size $n + 1$, any subset has to have size $m + 1$ with m dividing n. For $6 = 5 + 1$, $8 = 7 + 1$ and $10 = 9 + 1$, it hence is clear that two subsets plus one point cannot cover everything. Moreover, the Moufang sets of size 7 and 11 are sharply transitive, and have no nontrivial Moufang subsets. The case where $|X| = 9 = 8 + 1$ is the hardest to exclude, as it could potentially have a subset of size $5 = 4 + 1$, but none of the three Moufang sets of size 9 has a Moufang subset of size 5.[1] $\square$

3.3.2. Common setting

We follow precisely the setting in [VM98, Section 2.3.1], and will omit some details given there. The reader may hence wish to consult loc. cit. parallel to reading this section.

Let $\mathbb{K}$ be a skew field, $\operatorname{char} \mathbb{K} \neq 2$, and σ an anti-automorphism of order at most 2 (thus if $\sigma = \mathrm{id}$, then $\mathbb{K}$ is commutative). Let V be a – not necessarily finite-dimensional – right vector space over $\mathbb{K}$ and let $g : V \times V \to \mathbb{K}$ be a $(\sigma, 1)$-linear form. We define $f : V \times V \to \mathbb{K}$ as follows:

$$f(x, y) = g(x, y) + g(y, x).$$

Then f is a (σ-)Hermitian form. Denote $\mathbb{K}_\sigma := \{t^\sigma - t \mid t \in \mathbb{K}\}$. We define $q : V \to K/K_\sigma$ as

$$q(x) = g(x, x) + \mathbb{K}_\sigma,$$

for all $x \in V$. Then q is a σ-quadratic form.

Assume now that q is non-degenerate and has Witt index 2. We obtain a classical generalized quadrangle Γ by taking the totally isotropic 1-spaces as points, and the

[1] Alternatively, one can use that for $|X| = 9$ and $|Y| = |Z| = 5$, $a \notin Y \cup Z$ implies $|Y \cap Z| \ge 2$. But then one can choose distinct points 0, ∞ in $Y \cap Z$, and gets that the root groups V_∞ and W_∞ cover U_∞ except for one element. But then $|U_\infty| \le 4$.

totally isotropic 2-spaces as lines. One can show that two points $\langle v\rangle$ and $\langle w\rangle$ are incident if and only if $f(v,w)=0$.

Proposition 3.3.8. *Let q be a non-degenerate σ-quadratic form as above, of Witt index 2, with corresponding generalized quadrangle Γ. Let θ be a collineation of Γ of order 2. Let p_1,p_2,p_3,p_4 be four points of Γ spanning a θ-stable thin subquadrangle, where p_1 is opposite p_4, and p_2 is opposite p_3. Then the following hold:*

(1) There exists a vector subspace V_0 of V, a direct sum decomposition

$$V = e_1\mathbb{K} \bigoplus e_2\mathbb{K} \bigoplus e_3\mathbb{K} \bigoplus e_4\mathbb{K} \bigoplus V_0$$

and a non-degenerate anisotropic σ-quadratic form $q_0 : V_0 \to \mathbb{K}/\mathbb{K}_\sigma$, such that for all $v = e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + v_0$ with $x_i \in \mathbb{K}$, $i \in \{1,2,3,4\}$ and $v_0 \in V_0$,

$$q(v) = x_1^\sigma x_4 + x_2^\sigma x_3 + q_0(v_0).$$

Moreover, $p_i = \langle e_i\rangle$ for $i \in \{1,2,3,4\}$, and $f(e_i,e_j)$ equals 0 unless $\{i,j\} = \{1,4\}$ or $\{i,j\} = \{2,3\}$, in which case it equals 1.

(2) Denote $\widetilde{V} := \langle e_1,e_2,e_3,e_4\rangle$. There exist $A \in \mathrm{GL}(\widetilde{\mathrm{V}})$, $D \in \mathrm{GL}(\mathrm{V}_0)$, $\gamma \in \mathrm{Aut}(\mathbb{K})$ of order at most 2 and $\lambda \in C(\mathbb{K})^*$, such that for all $\widetilde{v} \in \widetilde{V}$, $v_0 \in V_0$,

$$\theta(\widetilde{v} + v_0) = A\widetilde{v}^\gamma + Dv_0^\gamma,$$

and $AA^\gamma = \lambda I$, $DD^\gamma = \lambda I$.

Proof. (1) This is [VM98, Proposition 2.3.4]. In particular,

$$V_0 := \{v \in V \mid f(v,e_i) = 0, \text{ for } i = 1,2,3,4\}.$$

By choosing appropriate scalar multiples of the e_i, we can ensure the condition on $f(e_i,e_j)$.

(2) By Proposition 4.6.5 in loc. cit., θ is induced by a projective semilinear transformation of the underlying vector space V. Hence there is $T \in \mathrm{GL}(\mathrm{V})$ and $\gamma \in \mathrm{Aut}(\mathbb{K})$ such that $\theta(\langle v\rangle) = T\langle v\rangle^\gamma$ for all isotropic vectors v. Since θ has order 2, γ has at most order 2, and there exists $\lambda \in C(\mathbb{K})^*$ such that $TT^\gamma = \lambda I$. By slight abuse of notation, we also use θ to denote the semilinear map $v \mapsto Tv^\gamma$ on V.

Now $\widetilde{V}$ is θ-stable by hypothesis, and orthogonal to V_0 by definition of the latter. Since θ is a collineation, V_0 must also be θ-stable. Therefore we can block-decompose T as stated. □

3.3.3. Direct descent

The following results will be useful in Chapter 4 to prove the so-called "direct descent" property. We give them here because they also yield connectedness of the flip-flop system if no Phan chambers exists, and in general their proofs perfectly fit in with the rest of this section.

Proposition 3.3.9. *Consider a classical quadrangle over a skew field $\mathbb{K}$, in which a line L exists such that L is opposite to $\theta(L)$, and such that all points p on L are collinear to $\theta(p)$. Then $\mathbb{K}$ is commutative. If* $\operatorname{char}\mathbb{K} \neq 2$ *or* $\dim(V_0) = 0$, *then every point is collinear to its image.*

Proof. Take two arbitrary points p_1, p_2 on L. Then $p_1, p_2, \theta(p_1), \theta(p_2)$ form a θ-stable thin quadrangle. Then by Proposition 3.3.8, there are isotropic vectors e_1, e_2, e_3, e_4 such that $p_1 = \langle e_1 \rangle$, $\theta(p_1) = \langle e_2 \rangle$, $\theta(p_2) = \langle e_4 \rangle$, $p_2 = \langle e_3 \rangle$, and

$$\theta(\widetilde{v} + v_0) = A\widetilde{v}^\gamma + Dv_0^\gamma,$$

where $A \in \mathrm{GL}(\widetilde{\mathrm{V}})$, $D \in \mathrm{GL}(\mathrm{V}_0)$, $\gamma \in \mathrm{Aut}(\mathbb{K})$ of order at most 2 and $\lambda \in C(\mathbb{K})^*$. In particular, since θ swaps $\langle e_1 \rangle$ with $\langle e_2 \rangle$, and $\langle e_3 \rangle$ with $\langle e_4 \rangle$, we have

$$A = \begin{pmatrix} & b & & \\ a & & & \\ & & & d \\ & & c & \end{pmatrix}, \qquad a, b, c, d \in \mathbb{K}^*.$$

Since θ is only determined up to a scalar factor, we may choose $a = 1$; from $AA^\gamma = \lambda I$, we deduce $\lambda = b = b^\gamma = dc^\gamma = cd^\gamma$, hence $\lambda^\gamma = \lambda$.

By hypothesis, every point $p \in L = \langle e_1, e_3 \rangle$ is collinear to its image. So for all $\mu, \nu \in \mathbb{K}$:

$$\begin{aligned} 0 &= f(e_1\nu + e_3\mu,\ \theta(e_1\nu + e_3\mu)) = f(e_1\nu + e_3\mu,\ e_2\nu^\gamma + e_4c\mu^\gamma) \\ &= f(e_1\nu, e_2\nu^\gamma) + f(e_3\mu, e_2\nu^\gamma) + f(e_1\nu, e_4c\mu^\gamma) + f(e_3\mu, e_4c\mu^\gamma) \\ &= \mu^\sigma\nu^\gamma + \nu^\sigma c\mu^\gamma. \end{aligned}$$

Setting $\mu = \nu = 1$ we find $c = -1$, hence $d = -\lambda$. For $\nu = 1$ and μ arbitrary, we obtain $\gamma = \sigma$. Since γ is an automorphism, but σ an anti-automorphism, we conclude that $\mathbb{K}$ is commutative. All in all, we get

$$\theta(e_1v_1 + e_2v_2 + e_3v_3 + e_4v_4 + v_0) = e_1\lambda v_2^\sigma + e_2v_1^\sigma - e_3\lambda v_4^\sigma - e_4v_3^\sigma + Dv_0^\sigma. \tag{3.3}$$

Showing that every point of the quadrangle is collinear to its image is equivalent to showing that for every isotropic vector v with $\theta(\langle v \rangle) \neq \langle v \rangle$, we have $f(\theta(v), v) = 0$.

Let $v = e_1v_1 + e_2v_2 + e_3v_3 + e_4v_4 + v_0 \in V$ arbitrary. We compute

$$\begin{aligned} f(\theta(v), v) &= f(e_1\lambda v_2^\sigma + e_2v_1^\sigma - e_3\lambda v_4^\sigma - e_4v_3^\sigma + Dv_0^\sigma,\ e_1v_1 + e_2v_2 + e_3v_3 + e_4v_4 + v_0) \\ &= f(e_1\lambda v_2^\sigma,\ e_4v_4) + f(e_2v_1^\sigma,\ e_3v_3) + f(-e_3\lambda v_4^\sigma,\ e_2v_2) + f(-e_4v_3^\sigma,\ e_1v_1) \\ &\qquad + f(Dv_0^\sigma, v_0) \\ &= v_2\lambda v_4 - v_3v_1 + v_1v_3 - v_4\lambda v_2 + f(Dv_0^\sigma, v_0) \\ &= f(Dv_0^\sigma, v_0). \end{aligned}$$

To prove that $f(Dv_0^\sigma, v_0)$ is zero for all $v_0 \in V_0$, choose $x \in \mathbb{K}$ such that $x \equiv q_0(v_0) \pmod{\mathbb{K}_\sigma}$. Set $v := e_1x - e_4 + v_0$; this is an isotropic vector. Similarly, choose $y \in \mathbb{K}$ such that $y \equiv q_0(Dv_0^\sigma) \pmod{\mathbb{K}_\sigma}$ and set $w := e_1 f(v_0, Dv_0^\sigma) + e_2 y - e_3 + Dv_0^\sigma$. This is also an isotropic vector, and $\langle v\rangle$ is collinear to $\langle w\rangle$, since

$$f(v, w) = f(-e_4, e_1 f(v_0, Dv_0^\sigma)) + f(v_0, Dv_0^\sigma) = 0.$$

Since θ maps lines to lines, it follows that

$$\begin{aligned} 0 = f(\theta(v), \theta(w)) &= f(e_2x^\sigma + e_3\lambda + Dv_0^\sigma,\ e_1\lambda y^\sigma + e_2 f(v_0, Dv_0^\sigma)^\sigma + e_4 + \theta(Dv_0^\sigma)) \\ &= \lambda f(v_0, Dv_0^\sigma)^\sigma + f(Dv_0^\sigma, \lambda v_0) \\ &= 2\lambda f(Dv_0^\sigma, v_0). \end{aligned}$$

Hence if $\operatorname{char}\mathbb{K} \neq 2$, we indeed have $f(Dv_0^\sigma, v_0) = 0$ and therefore as claimed, $f(\theta(v), v) = 0$ for all $v \in V$. $\square$

Remark 3.3.10. If $\operatorname{char}\mathbb{K} = 2$ one can construct examples which otherwise satisfy the assumptions of Proposition 3.3.9 but where points exist that are mapped to opposite ones. For example choose $\mathbb{K} = \mathbb{F}_4$, $\dim(V_0) = 1$ and $\gamma = \sigma$ equal to the Frobenius automorphism. Consider the automorphism in equation (3.3) with $D = \lambda = 1$. Then the point $\langle e_1 + e_4\alpha + v_0\rangle$ is not collinear to its image $\langle e_2 + e_3\alpha + v_0\rangle$, where $\alpha \in \mathbb{F}_4 \setminus \mathbb{F}_2$ and $v_0 \in V_0 \setminus \{0\}$.

Proposition 3.3.11. *Consider a classical quadrangle over a skew field $\mathbb{K}$ in which a point p exists such that p is opposite to its image $\theta(p)$, and such that all lines through p contain a fixed point. If $\dim(V_0) > 0$, or if $\operatorname{char}\mathbb{K} \neq 2$ and $\mathbb{K}$ is commutative, then every line contains a fixed point.*

Proof. Take two arbitrary lines M, N through p. Denote by q_M resp. q_N the (by our hypothesis that p is opposite $\theta(p)$ unique) fixed points on M resp. N. Then $p, q_M, \theta(p), q_N$ form a θ-stable thin quadrangle. By Proposition 3.3.8, there are isotropic vectors e_1, e_2, e_3, e_4 such that $p = \langle e_1\rangle$, $q_M = \langle e_2\rangle$, $\theta(p) = \langle e_4\rangle$, $q_N = \langle e_3\rangle$, and we have

$$\theta(\widetilde{v} + v_0) = A\widetilde{v}^\gamma + Dv_0^\gamma,$$

where $A \in \mathrm{GL}(\widetilde{\mathrm{V}})$, $D \in \mathrm{GL}(\mathrm{V}_0)$, $\gamma \in \mathrm{Aut}(\mathbb{K})$ of order at most 2 and $\lambda \in C(\mathbb{K})^*$. In particular, since θ fixes $q_M = \langle e_2\rangle$ and $q_N = \langle e_3\rangle$ but interchanges $p = \langle e_1\rangle$ and $\theta(p) = \langle e_4\rangle$,

$$A = \begin{pmatrix} & & & d \\ & b & & \\ & & c & \\ a & & & \end{pmatrix}, \qquad a, b, c, d \in \mathbb{K}^*.$$

Since θ is only determined up to a scalar factor, choose $a = 1$; from $AA^\gamma = \lambda I$, we deduce $\lambda = d^\gamma = bb^\gamma = cc^\gamma = d$, hence $\lambda^\gamma = \lambda$.

By hypothesis, all lines through $\langle e_1\rangle$ contain a fixed point. In other words, for any isotropic vector v collinear to but different from e_1, there exists a scalar α_v such that $\langle e_1\alpha_v + v\rangle$ is a fixed point:

$$\langle e_1\alpha_v + v\rangle = \theta(\langle e_1\alpha_v + v\rangle) = \langle e_4\alpha_v^\gamma + \theta(v)\rangle.$$

Consequently, there exists $\beta_v \in \mathbb{K}^*$ such that

$$(e_1\alpha_v + v)\beta_v = e_4\alpha_v^\gamma + \theta(v) \iff \theta(v) - v\beta_v = e_1\alpha_v\beta_v - e_4\alpha_v^\gamma \in \langle e_1, e_4\rangle. \tag{3.4}$$

We now make a case distinction based on V_0.

$V_0 \neq 0$: We start by showing that D is a multiple of the identity matrix. Let $v_0 \in V_0$ be arbitrary, let $x \in \mathbb{K}$ be so that $q(v_0) \equiv x \pmod{K_\sigma}$. Then $v := e_2x - e_3 + v_0$ is a nontrivial isotropic vector, collinear to but different from e_1. In particular, the value β_v as above is defined. We compute:

$$\begin{aligned}\theta(v) - v\beta_v &= \theta(e_2x - e_3 + v_0) - (e_2x - e_3 + v_0)\beta_v \\ &= (e_2bx^\gamma - e_3c + Dv_0^\gamma) - e_2x\beta_v + e_3\beta_v - v_0\beta_v \\ &= e_2(bx^\gamma - x\beta_v) + e_3(\beta_v - c) + (Dv_0^\gamma - v_0\beta_v).\end{aligned}$$

But by Equation 3.4 we have $\theta(v) - v\beta_v \in \langle e_1, e_4\rangle$. Consequently, by comparing coefficients, $\beta_v = c$, and since v_0 was arbitrary, $D = cI$ and $\gamma : x \mapsto c^{-1}xc$. From $(bx^\gamma - x\beta_v) = 0$ then follows that $b = c$. Finally, $\lambda = cc^\gamma = c^2$.

Summarizing the above, for all $v \in V$,

$$\theta(v) = e_1v_4c^2 + e_2v_2c + e_3v_3c + e_4v_1 + v_0c.$$

Hence we have $\theta(v') = v'c$ for all v' in the hyperplane $V' := \langle e_1c + e_4, e_2, e_3, V_0\rangle$. But every line intersects this hyperplane, thus contains a fixed point.

$V_0 = 0$: In this case we must have $\sigma \neq \mathrm{id}$. From now on we will assume $\mathbb{K}$ is commutative and $\operatorname{char}\mathbb{K} \neq 2$. Since θ preserves the form q, we have $\lambda = c^\sigma b = b^\sigma c = \lambda^\sigma$. Pick $\alpha \in \mathbb{K}^*$ such that $\alpha^\sigma = -\alpha$. Then $v_\varepsilon := \langle e_2 + e_3\alpha\rangle$ is an isotropic vector collinear to but different from e_1, and $\theta(v_\alpha) = \langle e_2b + e_3c\alpha^\gamma\rangle$. By hypothesis the line $\langle e_1, v_\alpha\rangle$ contains a fixed point. From Equation 3.4 we deduce that this fixed point must be $\langle v\rangle$, implying $\alpha b = c\alpha^\gamma$. We distinguish two subcases:

$\gamma = \mathrm{id}$: From $\alpha b = c\alpha^\gamma$ follows $b = c$, thus as in the case $V_0 \neq 0$ one verifies that $\theta(v') = v'c$ for all v' in the hyperplane $V' := \langle e_1c + e_4, e_2, e_3\rangle$, and the claim follows.

$\gamma \neq \mathrm{id}$: For all $\varepsilon \in \mathrm{Fix}_\mathbb{K}(\sigma)$ we have $(\alpha\varepsilon)^\sigma = -\alpha\varepsilon$, and so we must in fact have $\alpha\varepsilon b = c(\alpha\varepsilon)^\gamma$, i.e., $\mathrm{Fix}_\mathbb{K}(\sigma) \subseteq \mathrm{Fix}_\mathbb{K}(\gamma)$. Since $\gamma \neq \mathrm{id}$ one readily concludes that $\gamma = \sigma$. Now $\alpha b = c\alpha^\gamma$ reduces to $b = -c$. But $cc^\gamma = \lambda = c^\sigma b = -c^\gamma c$, a contradiction as $\operatorname{char}\mathbb{K} \neq 2$. □

Remark 3.3.12. If $\operatorname{char}\mathbb{K} = 2$ and $V_0 = 0$ then there are quadrangles that satisfy the conditions of Proposition 3.3.11 yet still admit lines without a fixed point.

Moreover, it seems quite likely that the assumption that $\mathbb{K}$ is commutative in the case $V_0 = 0$ can be dropped. However, we didn't try very hard to work this out as we have to make this assumption in other places anyway.

3.3.4. R^θ is almost always connected

Recall that a Phan chamber is a chamber which θ maps to an opposite one. We call a point resp. a line *bad* if it is not fixed but incident to a fixed line resp. point. We call a point resp. a line *good* if it is neither fixed nor bad, equivalently, if it is not incident to any fixed element.

Lemma 3.3.13. *Let R be a Moufang quadrangle. Assume that the point and line orders are greater than 4, and that a Phan chamber $(\tilde{p}, L)$ exists. Then R^θ is connected if $(\tilde{p}, L)$ is connected within R^θ to every Phan chamber (r, M) satisfying the following properties:*

(1) M is opposite L;

(2) $p := \operatorname{proj}_L(r)$ is good, hence $p' := \operatorname{proj}_L \theta(p)$ is good;

(3) $r' := \operatorname{proj}_M(p')$ is bad.

Proof. Let (s, K) be an arbitrary Phan chamber. We prove that it is connected to $(\tilde{p}, L)$ by a gallery of Phan chambers.

If L and K are equal or meet in a good point, we are done. If not, then s, being a good point on K, is not contained in L. Since the line order is greater than 4, by Lemma 3.3.6, s is contained in at least two good lines. Since we are in a quadrangle, it follows that there is a good line K' (possibly $K' = K$) through s not meeting L. The chambers (s, K) and (s, K') are adjacent, so it suffices to connect the latter to $(\tilde{p}, L)$ to establish our claim.

Since the point order is greater than 4, by Lemma 3.3.7, there is a good point r on K' which projects to a good point p on L. If the projection line is good, we are done. Otherwise, it now suffices to connect the Phan chambers (p, L) and (r, K') via a gallery in R^θ.

Denote by p' the projection of $\theta(p)$ to L. Since p is good, $p \neq p'$, and so r and p' are opposite. Consider the pencils of r and of p'. Again by Lemma 3.3.7, there is a good line M through r meeting a good line through p'. Denote by r' the projection

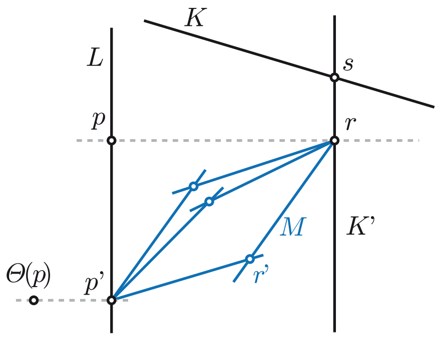


Figure 3.1.: The construction from Lemma 3.3.13.

of p' to M. If r' is a good point, we have constructed a connection and are done. If it is bad, we can invoke our hypothesis and are done as well. □

Lemma 3.3.14. *Let R be a classical quadrangle over a field $\mathbb{K}$, let p,p' be two collinear good points such that $p, p', \theta(p), \theta(p')$ form a thin subquadrangle. Then the following hold:*

(1) There are isotropic vectors e_1, e_2, e_3, e_4 in V such that $p = \langle e_1 \rangle$, $p' = \langle e_2 \rangle$, $\theta(p) = \langle e_4 \rangle$, $\theta(p') = \langle e_3 \rangle$. Denote $V_0 := \langle e_1, e_2, e_3, e_4 \rangle^\perp$. There exist $b, c, \lambda \in \mathbb{K}^*$, $D \in \mathrm{GL}(V_0)$, and $\gamma \in \mathrm{Aut}(\mathbb{K})$ of order at most 2, such that for all $v_1, v_2, v_3, v_4 \in \mathbb{K}$, $v_0 \in V_0$,

$$\theta(e_1 v_1 + e_2 v_2 + e_3 v_3 + e_4 v_4 + v_0) = e_1 \lambda v_4^\gamma + e_2 c v_3^\gamma + e_3 b v_2^\gamma + e_4 v_1^\gamma + D v_0^\gamma,$$

and $\lambda = \lambda^\gamma = cb^\gamma = bc^\gamma = c^\sigma b = b^\sigma c$, $DD^\gamma = \lambda I$.

(2) $\lambda = \lambda^\sigma$.

(3) The field $\mathrm{Fix}_{\mathbb{K}}(\sigma) \cap \mathrm{Fix}_{\mathbb{K}}(\gamma)$ has index at most 4 in $\mathbb{K}$.

Proof. (1) The first part is a consequence of Proposition 3.3.8(2). In fact, let A be as in 3.3.8(2), then it immediately follows that

$$A = \begin{pmatrix} & & & d \\ & & c & \\ & b & & \\ a & & & \end{pmatrix}, \qquad \text{where } a, b, c, d \in \mathbb{K}^*.$$

Since θ is only determined up to a scalar factor, we may choose $a = 1$; from $AA^\gamma = \lambda I$, we deduce $\lambda = d = cb^\gamma = bc^\gamma = d^\gamma$, hence $\lambda = \lambda^\gamma$.

(2) To see that $\lambda = \lambda^\sigma$, consider the isotropic vectors $e_1 + e_2$, $e_3 - e_4$, $e_1 + e_4$ and $e_2 - e_4$:

$$0 = f(e_1 + e_2, e_3 - e_4) = f(\theta(e_1 + e_2), \theta(e_3 - e_4)) = f(e_3 b + e_4, -e_1 \lambda + e_2 c) = b^\sigma c - \lambda$$
$$0 = f(e_1 + e_3, e_2 - e_4) = f(\theta(e_1 + e_3), \theta(e_2 - e_4)) = f(e_2 c + e_4, -e_1 \lambda + e_3 b) = c^\sigma b - \lambda,$$

proving that $\lambda = c^\sigma b = (b^\sigma c)^\sigma = \lambda^\sigma$.

(3) We now prove the claim that $\mathrm{Fix}_{\mathbb{K}}(\sigma) \cap \mathrm{Fix}_{\mathbb{K}}(\gamma)$ has index at most 4 in $\mathbb{K}$. If $\sigma = \mathrm{id}$, this is trivial. Else, pick $x \in \mathbb{K}_\sigma$ and set $v_x := e_1 + e_4 x$. Then $q(v_x) = x \equiv 0 \pmod{\mathbb{K})_\sigma}$, so v_x is an isotropic vector. Thus also $\theta(v_x) = e_1 \lambda x^\gamma + e_4$ is isotropic, whence

$$0 = f(\theta(v_x), \theta(v_x)) = f(e_1 \lambda x^\gamma + e_4, e_1 \lambda x^\gamma + e_4) = (\lambda x^\gamma)^\sigma + \lambda x^\gamma.$$

As $\lambda = \lambda^\sigma$, this is equivalent to $x^{\gamma\sigma} = -x^\gamma = x^{\sigma\gamma}$, for all $x \in \mathbb{K}_\sigma$.

Denote $\mathbb{F} := \mathrm{Fix}(\sigma)$. If $\mathrm{char}\, \mathbb{K} = 2$, then $\mathbb{K}_\sigma = \mathbb{F}$ (it is easy to see that $\mathbb{K}_\sigma \subseteq \mathbb{F}$; but also that $\mathbb{F} \cdot \mathbb{K}_\sigma = \mathbb{K}_\sigma$). Accordingly γ induces an involutory automorphism of $\mathbb{F}$, and so $\mathrm{Fix}_{\mathbb{K}}(\sigma) \cap \mathrm{Fix}_{\mathbb{K}}(\gamma)$ has at most index 2 in $\mathbb{F}$, ergo at most index 4 in $\mathbb{K}$.

If $\operatorname{char} \mathbb{K} \neq 2$, pick $\alpha \in \mathbb{K}^*$ such that $\alpha^\sigma = -\alpha$. We claim that $\mathbb{K}_\sigma = \alpha\mathbb{F}$: If $x \in \mathbb{K}_\sigma$, then $x = t^\sigma - t$ for some $t \in \mathbb{K}$, hence $x^\sigma = -x$. On the other hand, if $x \in \alpha\mathbb{F}$, then $x = t^\sigma - t$ for $t = -\frac{x}{2}$.

We already know that σ and γ commute on $\mathbb{K}_\sigma = \alpha\mathbb{F}$. If $y \in \mathbb{F}$, then $y = (\alpha^{-1})(\alpha y)$. Since $\alpha^{-1}, \alpha y \in \alpha\mathbb{F}$, we have $y^{\gamma\sigma} = y^{\sigma\gamma}$ as well. Hence σ and γ commute on $\mathbb{F}$ as well, and the claim follows. □

The preceding two lemmas finally enable us to prove connectedness of R^θ under the assumption that a Phan chamber exists.

Proposition 3.3.15. *Let R be a classical quadrangle over a field $\mathbb{K}$, let θ be an involutory automorphism of R. If θ admits a Phan chamber, and $|\mathbb{K}| > 9$, $|\mathbb{K}| \neq 16$, then R^θ is connected.*

Proof. By Lemma 3.3.13, it suffices to fix a Phan chamber $(\widetilde{p}, L)$, and then prove that it is connected to every Phan chamber (r, M) satisfying

(1) M is opposite L;

(2) $p := \mathrm{proj}_L(r)$ is good, hence $p' := \mathrm{proj}_L \theta(p)$ is good;

(3) $r' := \mathrm{proj}_M(p')$ is bad.

Since p and $p' = \mathrm{proj}_L \theta(p)$ are good and collinear, Lemma 3.3.14 yields a description of θ and f using a convenient basis of the underlying vector space V.

We construct the desired connection by showing that there is a good point on M which projects to a good point on L via a good projection line. For this, we need to characterize three subsets of the point row of M: The bad points; the points which project to a bad point on L; and those for which the projection line to L is bad.

The projections of $p = \langle e_1 \rangle$ resp. $p' = \langle e_2 \rangle$ to M are $r = \langle m \rangle$ resp. $r' = \langle m' \rangle$. Furthermore, p is not collinear to r', and p' is not collinear to r. For $\mu \in \mathbb{K}$ let $x_\mu := m\mu + m'$. Then $\langle x_\mu \rangle \in M$, and all points of M except for r are obtained in this way. The projection of $\langle x_\mu \rangle$ to L is $\langle y_\mu \rangle$, where $y_\mu := e_1\mu^\sigma - e_2$.

We now study in detail the three subsets of M mentioned earlier (actually, for (2) and (3) we left out one element, but that is irrelevant for our purposes):

(1) The set $\mathfrak{A} := \{\mu \in \mathbb{K} \mid \langle x_\mu \rangle \text{ is a bad point}\}$, i.e., the set of all μ for which x_μ is collinear to $\theta(x_\mu)$, corresponds to the solutions of the following equation:

$$\begin{aligned} 0 &= f(x_\mu, \theta(x_\mu)) = f(m\mu + m', \theta(m\mu + m')) \\ &= f(m\mu, \theta(m\mu)) + f(m\mu, \theta(m')) + f(m', \theta(m\mu)) + f(m', \theta(m')) \\ &= \mu^\sigma f(m, \theta(m))\mu^\gamma + \mu^\sigma f(m, \theta(m')) + f(m', \theta(m))\mu^\gamma + f(m', \theta(m')). \end{aligned}$$

Since $r' = \langle m' \rangle = \langle x_0 \rangle$ is bad, $f(m', \theta(m')) = 0$. So

$$\mathfrak{A} = \{\mu \in \mathbb{K} \mid \mu^{\sigma\gamma}\mu \cdot f(m, \theta(m)) + \mu^{\sigma\gamma} \cdot f(m, \theta(m')) + \mu \cdot f(m', \theta(m)) = 0\}. \quad (3.5)$$

Note that $\mathfrak{A}$ is a proper subset of $\mathbb{K}$, as $r = \langle m \rangle$ is a good point, implying $f(m, \theta(m)) \neq 0$, hence the defining equation of $\mathfrak{A}$ does not vanish everywhere.

(2) The set $\mathfrak{B} := \{\mu \in \mathbb{K} \mid \langle y_\mu \rangle$ is a bad point$\}$, i.e., the set of all μ for which y_μ is collinear to $\theta(y_\mu)$, corresponds to the solutions of the following equation:

$$\begin{aligned} 0 &= f(y_\mu, \theta(y_\mu)) \\ &= f(e_1\mu^\sigma - e_2,\ -e_3 b + e_4 \mu^{\sigma\gamma}) \\ &= \mu\mu^{\sigma\gamma} + b \\ \implies \mathfrak{B} &= \{\mu \in \mathbb{K} \mid \mu\mu^{\sigma\gamma} + b = 0\}. \end{aligned} \quad (3.6)$$

Note that $p' = \langle e_2 \rangle = \langle y_0 \rangle$ is a good point, and indeed, clearly $0 \notin \mathfrak{B}$.

(3) The set $\mathfrak{C} := \{\mu \in \mathbb{K} \mid \langle x_\mu, y_\mu \rangle$ is not a good line$\}$, can be characterized as follows: The line $\langle x_\mu, y_\mu \rangle$ is not good if it contains a fixed point; equivalently, a point which is collinear to both $\theta(x_\mu)$ and $\theta(y_\mu)$. Thus the line $\langle x_\mu, y_\mu \rangle$ is not good if and only if there exists $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{0\}$ such that

$$\begin{aligned} &f(x_\mu\alpha + y_\mu\beta,\ \theta(x_\mu)) = 0 = f(x_\mu\alpha + y_\mu\beta,\ \theta(y_\mu)) \\ \iff &Z_\mu \left(\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right) = \left(\begin{smallmatrix}0 \\ 0\end{smallmatrix}\right), \text{ where } Z_\mu := \begin{pmatrix} f(x_\mu, \theta(x_\mu)) & f(y_\mu, \theta(x_\mu)) \\ f(x_\mu, \theta(y_\mu)) & f(y_\mu, \theta(y_\mu)) \end{pmatrix}. \end{aligned}$$

But this is equivalent to $\det(Z_\mu) = 0$. Hence

$$\mathfrak{C} := \{\mu \in \mathbb{K} \mid f(x_\mu, \theta(x_\mu)) \cdot f(y_\mu, \theta(y_\mu)) = f(y_\mu, \theta(x_\mu)) \cdot f(x_\mu, \theta(y_\mu))\}. \quad (3.7)$$

We now argue that $\det(Z_0) \neq 0$ and hence $0 \notin \mathfrak{C}$: We know that $r' = \langle m' \rangle = \langle x_0 \rangle$ is bad, hence $f(x_0, \theta(x_0)) = 0$. But $f(y_0, \theta(y_0)) \neq 0$ since $p' = \langle e_2 \rangle = \langle y_0 \rangle$ is good. Moreover $f(y_0, \theta(x_0)) \neq 0$, for else $\langle y_0 \rangle$ would be collinear to and different from both $\langle x_0 \rangle$ and $\langle \theta(x_0) \rangle$, which are distinct but collinear points, consequently $\langle y_0 \rangle, \langle x_0 \rangle, \langle \theta(x_0) \rangle$ would form a triangle, which is impossible.

Let $\mathbb{F} := \mathrm{Fix}_\mathbb{K}(\sigma) \cap \mathrm{Fix}_\mathbb{K}(\gamma)$. By Lemma 3.3.14, $\mathbb{F}$ is a subfield of $\mathbb{K}$ of index at most 4. Over $\mathbb{F}$, the defining equations for the sets $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ become nonzero polynomial equations in μ of degree 2, 2 and 4, respectively (as the terms $f(x_\mu, \theta(x_\mu))$, $f(x_\mu, \theta(y_\mu))$ etc. become quadratic polynomials over $\mathbb{F}$). In particular $|(\mathfrak{A} \cup \mathfrak{B} \cup \mathfrak{C}) \cap \mathbb{F}| \leq 2+2+4 = 8$. Thus if $|\mathbb{F}| > 8$, there exists $\mu \in \mathbb{F}$ such that none of the equations hold. But then x_μ, y_μ and $\langle x_\mu, y_\mu \rangle$ all are good. For this reason, (r, M) and (p, L) are connected if $|\mathbb{K}| > 8^4 = 4096$, in particular over all infinite fields.

If $\mathbb{K}$ is a finite field, it is well-known that $\mathrm{Aut}(\mathbb{K})$ is cyclic and generated by the Frobenius automorphism $x \mapsto x^p$, where $p = \mathrm{char}\,\mathbb{K}$. Hence $\mathbb{K}$ admits a (necessarily unique) involutory automorphisms if and only if the order of $\mathbb{K}$ is a square.

If $|\mathbb{K}|$ is not a square, $\mathbb{F} = \mathbb{K}$ and we get a connection if $|\mathbb{K}| > 8$. If $|\mathbb{K}|$ is a square, say q^2, then $x \mapsto x^q$ is the unique automorphism of order 2. Hence Equations 3.5 and 3.6 are polynomial equations in μ of degree at most $q + 1$, and Equation 3.7 is a polynomial equation of degree at most $2q + 2$, implying $|\mathfrak{A} \cup \mathfrak{B} \cup \mathfrak{C}| \leq 4q + 4$. Therefore we can construct a connection if $|\mathbb{K}| = q^2 > 4q + 4$, equivalently if $q > 4$ and thus $|\mathbb{K}| > 16$. $\square$

Finally, we prove that R^θ is always connected if no Phan chamber exists.

Proposition 3.3.16. *Let R be a classical quadrangle over a field $\mathbb{K}$, $\operatorname{char} \mathbb{K} \neq 2$, let θ be an involutory automorphism of R. If θ does not admit a Phan chamber, then R^θ is connected.*

Proof. Since no Phan chamber exists, we can use Proposition 3.3.11 to conclude the following: Either a point p exists which is opposite to $\theta(p)$. Then all lines through p must contain a fixed point (else we would get a Phan chamber). Or else every point p is collinear to its image. Then p is in fact contained in a line fixed by θ: If p is not fixed, then $\langle p, \theta(p) \rangle$ is the (unique) fixed line through p. If p is fixed, take an arbitrary point p' collinear to but different from $p = \theta(p)$, then p' is also collinear to $\theta(p')$; since no triangles exist, we conclude that θ then fixes all lines through p.

So up to duality we may assume that all points are contained in a fixed line. If θ fixes all points, it is the identity and we are done. Else, pick a non-fixed point p. Then any line L through p different from $\langle p, \theta(p) \rangle$ is mapped by θ to an opposite one. We conclude that R^θ consists of all chambers of this kind: A non-fixed point p and a line L which is mapped to an opposite line $\theta(L)$.

Given two such chambers (p_1, L_1) an (p_2, L_2), we construct a connection within R^θ as follows: If L_1 and L_2 contain a common point, then it is necessarily a non-fixed point and we have the desired connection. If they are opposite, two cases are possible: First, there is a non-fixed projection line from a point on L_1 to a point on L_2, yielding a connection (the intersection points must be non-fixed, and the projection line, being non-fixed, cannot contain a fixed point, as there can be no triangles). Or secondly, all projection lines are fixed. In that case, pick a line L_2' through p_2 which is different from L_2 and $\langle p_2, \theta(p_2) \rangle$ (it exists because we are in a thick quadrangle). Then there must be non-fixed projection lines between L_1 and L_2', whence we have reduced to the first case and are done. $\square$

CHAPTER

FOUR

STRUCTURE OF FLIP-FLOP SYSTEMS

Throughout this chapter, $\mathcal{C} = (\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ is a twin building of type (W, S), and θ is a quasi-flip of $\mathcal{C}$.

4.1. Flip-flop systems

In this chapter we study certain chamber subsystems of $\mathcal{C}$ which are associated to the building quasi-flip θ.

On the one hand, we study minimal Phan residues (recall that a Phan residue is a residue which is mapped by θ to an opposite one). In some sense, the study of minimal Phan residues is local. For example, a priori, we cannot relate two different minimal Phan residues. Moreover, one might consist of a single chamber while another could be a much larger residue, possibly even non-spherical.

Despite this, we will show that under suitable assumptions, all minimal Phan residues have identical type. From the point of view of groups, this is equivalent to all minimal θ-split parabolic subgroups having equal type.[1] This is known to be true for algebraic involutions of algebraic groups, see e.g. [HW93].

On the other hand, we study the so-called flip-flop system consisting of all chambers which are mapped as far away as possible, globally. To make this precise, consider the following definitions:

Definition 4.1.1. Let θ be a quasi-flip of a twin building $\mathcal{C}$, let R be any residue of $\mathcal{C}_+$ (in particular, R might equal $\mathcal{C}_+$). The **minimal numerical $\boldsymbol{\theta}$-codistance** of R is the value $\min_{c\in R}\ l^\theta(c) = \min_{c\in R}\ l(c, \theta(c))$.

With the above, the set of chambers which are mapped "as far away as possible" by a quasi-flip can be described as follows.

[1] Recall that a parabolic subgroup P is θ-split if $P \cap \theta(P)$ is a maximal Levi factor in both P and $\theta(P)$.

Definition 4.1.2. Let θ be a quasi-flip of a twin building $\mathcal{C}$, let R be a residue of $\mathcal{C}_+$. The **induced flip-flop system** R^θ on R associated to θ is the (sub)chamber system

$$R^\theta := \{c \in R \mid l^\theta(c) = \min_{d \in R} \ l^\theta(d)\}$$

with the equivalence relations inherited from $\mathcal{C}_+$. In particular, for $R = \mathcal{C}_+$ the **flip-flop system** associated to θ is the (sub)chamber system $\mathcal{C}^\theta := \mathcal{C}_+^\theta$.

For this globally defined chamber subsystem, we can now for example ask whether it is connected. For a wide class of twin buildings, we give a positive answer to this question. Moreover, we show that in many cases, the flip-flop system coincides with the union of all minimal Phan residues. This then yields the homogeneity result on minimal Phan residues we already mentioned above.

Remark 4.1.3. If θ is a strong quasi-flip, then by Lemma 2.4.2, the minimal numerical θ-codistance is 1_W, as there exist Phan chambers. Moreover, minimal Phan residues always are Phan chambers. So in this case, homogeneity indeed holds.

Example 4.1.4. The following example originally comes from [BS04].

Let V be an $(n+1)$-dimensional vector space ($n \geq 2$) over a finite field $\mathbb{F}$ with a field automorphism σ of order 2. Consider the building $\mathcal{C}$ of type A_n associated to V, i.e., the projective space $\mathbb{P}(V)$. Here, the chambers are maximal flags consisting of nontrivial proper subspaces of V. Fix a basis of V. The group $\mathrm{SL}(V) = \mathrm{SL}_{n+1}(\mathbb{F})$ acts strongly transitively on the building $\mathcal{C}$, where B_+ and B_-, the subgroups of upper resp. lower triangular matrices, stabilize opposite chambers.

The σ-twisted Chevalley involution $\theta \colon x \mapsto {}^t(x^\sigma)^{-1}$ is a proper BN-quasi-flip (as it interchanges B_+ and B_-). The induced proper building quasi-flip sends vector subspaces to their orthogonal complement with respect to the standard σ-sesquilinear form f on V ("standard" regarding our fixed basis).

The flip-flop system then consists of all chambers (i.e., maximal flags) which are mapped to opposite flags. These maximal flags are precisely those where all involved vector subspaces are non-degenerate with respect to f. It is not hard to see that any flag consisting of non-degenerate subspaces can be extended to a chamber (a maximal flag consisting of n proper non-trivial subspaces). In fact, by [BS04, Corollary 2.4], the flip-flop system is residually connected if $|\mathbb{F}| > 4$.

Once we understand homogeneity and connectedness, we turn to the question whether the chamber system $\mathcal{C}^\theta$ is residually connected (cf. Section 1.4). We establish this for the special case that the flip is K-homogeneous (meaning all minimal Phan residues have type K) and $|K| \leq 2$. In general, we prove that the so-called K-residue chamber system $\mathcal{C}_K^\theta$ (see Definition 4.5.1) is residually connected. From this, residual connectedness of $\mathcal{C}^\theta$ would follow if one could solve a problem in Coxeter systems (see Section 4.5 for details).

The importance of this is that when $\mathcal{C}^\theta$ is residually connected, one can construct a so-called *synthetic* or *incidence geometry* from it (in the sense of Buekenhout and Tits, see e.g. [BC]), the **flip-flop geometry** $\mathcal{G}^\theta := \mathcal{G}(\mathcal{C}^\theta)$ associated to θ. In all examples known to us, $\mathcal{G}^\theta$ is in fact a geometry.

The following property is the corner stone of our approach to proving all the results hinted at above:

Definition 4.1.5. Let R be a residue. We say that **direct descent into $\boldsymbol{R^\theta}$** is possible if for any chamber c in R there exists a gallery in R from c to a chamber in R^θ with the property that l^θ (as defined in Section 2.1) is strictly decreasing along the gallery.

Remark 4.1.6. If θ is a strong quasi-flip, then by Lemma 2.4.2 direct descent is possible for all residues. In fact, quasi-flips which allow direct descent for all residues may be thought of as generalizing strong quasi-flips.

Indeed in [DM07], several of the results we present here have been elegantly proven for strong flips using filtrations of buildings. In fact the local-to-global connectivity and homogeneity results in this chapter can be considered as generalizations of corresponding results for strong flips in loc. cit.; but in addition, we perform a rank 2 analysis which could independently be combined (for strong flips) with the Devillers-Mühlherr filtration.

The following holds (note that we do not require the twin building to be Moufang):

Theorem 4.1.7 (joint work with Gramlich and Mühlherr)**.** *Let θ be a quasi-flip of a twin building $\mathcal{C}$ such that for all rank 2 residues R, direct descent into R^θ is possible and R^θ is connected.*

Then the flip-flop system $\mathcal{C}^\theta$ is connected and equals the union of all minimal Phan residues. The minimal Phan residues all have identical spherical type K, or equivalently, δ^θ takes on the constant value w_K on all chambers of $\mathcal{C}^\theta$, where w_K is the longest element of the spherical Coxeter system (W_K, K). Moreover, the chamber system $\mathcal{C}^\theta_K$ of K-residues of $\mathcal{C}^\theta$ is connected and residually connected.

Proof. Combine Proposition 4.4.4 and 4.5.3. □

In Section 4.6 we investigate closer for which rank 2 Moufang buildings the conditions of Theorem 4.1.7 are satisfied. This culminates in the following theorem, which we prove in Section 4.7:

Theorem 4.1.8 (joint work with Gramlich and Mühlherr)**.** *Let θ be a quasi-flip of an RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ of type (W, S), where all root groups U_α are uniquely 2-divisible. Assume the diagram is simply laced; or assume that the RGD-system is 2-spherical, $\mathbb{F}$-locally split, $|\mathbb{F}| > 4$, and no G_2 residues occur.*

Then for all rank 2 residues R, direct descent into R^θ is possible and R^θ is connected.

Theorem 4.1.8 should in fact extend to most 2-spherical buildings (with the exception of some small rank 2 cases). This is subject of ongoing research by Hendrik Van Maldeghem and the author [HVM]. We conjecture the following:

Conjecture 4.1.9. *Let θ be a quasi-flip of an RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ of type (W, S), where all root groups U_α are uniquely 2-divisible. Assume the diagram is 2-spherical, and all rank 2 residues are not included in a finite list of exceptions.*

Then for all rank 2 residues R, direct descent into R^θ is possible and R^θ is connected.

Combining Theorems 4.1.7 and 4.1.8 we conclude.

Theorem 4.1.10. *Let θ be a quasi-flip of an RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ of type (W, S), where all root groups U_α are uniquely 2-divisible. Assume the diagram is simply laced; or assume that the RGD-system is 2-spherical, $\mathbb{F}$-locally split, $|\mathbb{F}| > 4$, and no G_2 residues occur.*

Then the flip-flop system $\mathcal{C}^\theta$ is connected and equals the union of all minimal Phan residues, which in turn all have identical spherical type K. The chamber system of K-residues of $\mathcal{C}^\theta$ is connected and residually connected.

Remark 4.1.11. This partially answers the question posed in [BGHS03] regarding whether the flip-flop system is geometric in general; residual connectedness implies this.

4.2. Outline of the proof

In Section 4.3 we prove the following facts, without any assumptions on the twin building or the quasi-flip: Any minimal Phan residue R is spherical, and if R is of type J, the θ-codistance must be constant and equal to the longest element of W_J.

In Section 4.4 we assume that for any rank 2 residue R, the induced flip-flop system R^θ is connected, and that direct descent into R^θ is possible. Under these assumptions, $\mathcal{C}^\theta$ is homogeneous and inherits connectedness from $\mathcal{C}_+$, as defined below:

Definition 4.2.1. A quasi-flip θ is called **homogeneous** or ***K*-homogeneous** if all minimal Phan residues have identical type K.

Definition 4.2.2. Let $\mathcal{C}$, $\mathcal{C}'$ be chamber systems such that $\mathcal{C}' \subseteq \mathcal{C}$ and the equivalence relations on $\mathcal{C}'$ are obtained by restricting those on $\mathcal{C}$. We say $\mathcal{C}'$ **inherits connectedness** from $\mathcal{C}$ if any two chambers c, d in $\mathcal{C}$ are connected by a J-gallery in $\mathcal{C}'$ if and only if they are connected by a J-gallery in $\mathcal{C}$.

Assume we are given two distinct minimal Phan residues R and R' of types I and I', resp., and pick chambers c and c' from each. Choose any minimal gallery connecting the two and denote its type by J. We deform this J-gallery via a series of local transformations (inside rank 2 residues) to a new J-gallery γ on which the numerical θ-codistance is constant. This then implies by Lemma 4.4.1 that the θ-codistance is constant. Hence R and R' must have been of equal type $I = I'$. So $\mathcal{C}^\theta$ is I-homogeneous and inherits connectedness from $\mathcal{C}_+$ as claimed.

In Section 4.5 we prove the following: If $\mathcal{C}^\theta$ is K-homogeneous and inherits connectedness from $\mathcal{C}_+$, then the chamber system $\mathcal{C}^\theta_K$ of K-residues of $\mathcal{C}^\theta$ is residually connected, which will complete the proof of Theorem 4.1.7.

In Section 4.6 we turn to studying what happens in rank 2 residues, with the goal of determining explicit criteria on the rank 2 residues, which imply direct descent into and connectedness of the induced flip-flop systems. Here, we assume the Moufang property on the building, which implies that the rank 2 residues we need to study are in fact Moufang polygons. For example, for A_2 residues not in characteristic 2, the desired properties hold.

Finally, in Section 4.7 the proof of Theorem 4.1.8 and the other main results of this chapter are presented.

4.3. Minimal Phan residues

Recall that a Phan residue is a residue which θ maps to an opposite residue. In this section we characterize Phan residues which are minimal with respect to inclusion.

Lemma 4.3.1 (Lemma 5.140(1) in [AB08]). *For $\varepsilon \in \{+,-\}$ let $x \in \mathcal{C}_\varepsilon$ and $y, z \in \mathcal{C}_{-\varepsilon}$. Then*

$$\delta^*(x,z) \leq \delta^*(x,y) \cdot \delta_{-\varepsilon}(y,z)$$

in the Bruhat order.

In Lemma 2.3.1 we saw that if w is the θ-codistance of a chamber, then w is a θ-twisted involution, so $\theta(w) = w^{-1}$. We will make frequent use of this fact in what follows below.

Lemma 4.3.2. *Let R be a Phan residue of type I. Then the θ-codistance on R has image in W_I.*

Proof. Take any chamber $c \in R$ and let $w = \delta^\theta(c)$ denote its θ-codistance. Since R is a Phan residue, there exists $d \in R$ opposite $\theta(c)$. Applying Lemma 4.3.1 with $x = \theta(c)$, $y = d$, $z = c$ yields

$$w^{-1} = \theta(w) = \delta^*(\theta(c), c) \leq \delta^*(\theta(c), d) \cdot \delta_{-\varepsilon}(d, c) = 1_W \cdot \delta_{-\varepsilon}(c,d)^{-1}.$$

Since $c, d \in R$, we get $w \in W_I$ and the claim follows. $\square$

Lemma 4.3.3. *Let R_1 and R_2 be Phan residues of type I_1 and I_2 with nonempty intersection. Then $R_1 \cap R_2$ is also a Phan residue.*

Proof. Pick any chamber $c \in R_1 \cap R_2$. It suffices to find a chamber $d \in R_1 \cap R_2$ such that $\delta^*(c, \theta(d)) = 1_W$. By Lemma 4.3.2 and since $c \in R_1 \cap R_2$, we have $w := \delta_\theta(c) \in W_{I_1} \cap W_{I_2} = W_{I_1 \cap I_2}$. Walking from c along any gallery of type w, we arrive at a chamber d with $\delta_+(c, d) = w$. Applying Lemma 4.3.1 with $x = c$, $y = \theta(c)$ and $z = \theta(d)$, we deduce

$$\delta^*(c, \theta(d)) \leq \delta^*(c, \theta(c)) \cdot \delta_-(\theta(c), \theta(d)) = w \cdot \theta(\delta_+(c, d)) = w \cdot \theta(w) = 1_W,$$

hence $\delta^*(c, \theta(d)) = 1_W$. Accordingly $R_1 \cap R_2$ is a Phan residue. □

Lemma 4.3.4. *Minimal (by inclusion) Phan residues are spherical. In particular, spherical Phan residues exist.*

Proof. Let R be a Phan residue of type J. Take any chamber c in R, and denote its θ-codistance by w. By Lemma 2.3.4, there exist a spherical subset I of S, an element $w' \in W$ with $w' \leq w$ in the Bruhat order, and a chamber $c' \in \mathcal{C}_+$ such that $\delta_+(c, c') = w'$ and $\delta^\theta(c') = w_I$. By Lemma 4.3.2, $w \in W_J$. As $w' < w$ in the Bruhat order, it is contained in W_J. Hence $c' \in R$.

The I-residue $R_I(c')$ around c' is spherical. Moreover, it is a Phan-residue: Pick a chamber $d \in R_I(c')$ such that $\delta_\varepsilon(c', d) = w_I$, hence $\delta_{-\varepsilon}(\theta(c'), \theta(d)) = \theta(w_I) = w_I^{-1} = w_I$ (the latter equality holds because w_I is a θ-codistance, hence a θ-twisted involution, but also a regular involution). Applying Lemma 4.3.1 with $x = c'$, $y = \theta(c')$, $z = \theta(d)$, yields $\delta^*(c', \theta(d)) = 1_W$. Hence every Phan residue contains spherical Phan residues, and the claim follows. □

Lemma 4.3.5. *Let R be a minimal Phan residue of type I. Then I is spherical and the θ-codistance on R is constant and equal to w_I, the longest element of W_I.*

Proof. That I is spherical follows from Lemma 4.3.4. Assume that there exists a chamber $c \in R$ such that the θ-codistance w of c is different from w_I.

By Lemma 2.3.4, there exist a spherical subset J of S, an element $w' \in W$, and a chamber $c' \in \mathcal{C}_+$ such that $\delta_+(c, c') = w'$ and $\delta^\theta(c') = w_J$. Moreover, $w' \leq w$ and $w_J \leq w$ in the Bruhat order. Yet by Lemma 4.3.2, $w \in W_I$, and by assumption $w \neq w_I$, thus $w_J \leq w < w_I$ and so $J \subsetneq I$. Then the J-residue $R_J(c')$ around c' would be a Phan-residue contained in R, but strictly smaller than it, contradicting the minimality of R. □

As a first immediate application, we highlight how K-homogeneity influences the structure of the flip-flop system.

Lemma 4.3.6. *If θ is a K-homogeneous quasi-flip, then for all chambers c, $w_K \leq \delta^\theta(c)$ in the Bruhat order, and the flip-flop system $\mathcal{C}^\theta$ equals the union of all minimal Phan residues.*

Proof. Let c be an arbitrary chamber, denote its θ-codistance by w.

By Lemma 2.3.4, there exist a spherical subset J of S, an element $w' \in W$, and a chamber $c' \in \mathcal{C}_+$ such that $\delta_+(c, c') = w'$ and $\delta^\theta(c') = w_J$. Moreover, w' and w_J are less or equal w in the Bruhat order.

Now assume $w_K \not\leq w_J$, therefore $K \not\subseteq J$. It follows that the residue $R_J(c')$ is a spherical Phan residue which contains no K-residue. But by K-homogeneity, any Phan residue contains a Phan residue of type K, which is a contradiction. Hence $w_K \leq w_J \leq w = \delta^\theta(c)$.

In particular, if $c \in \mathcal{C}^\theta$, then $w_K \leq \delta^\theta(c)$; but since the numerical θ-codistance of c is globally minimal, we must in fact have $\delta^\theta(c) = w_K$. Then $R_K(c)$ is a minimal Phan residue of type K. Moreover, by Lemma 4.3.5 we have $R_K(c) \subset \mathcal{C}^\theta$, so $\mathcal{C}^\theta$ is the union of all minimal Phan residues. □

4.4. Homogeneity and inherited connectedness

We now establish that the flip-flop system is homogeneous and inherits connectedness if for all rank 2 residues R, direct descent into R^θ is possible and R^θ is connected. First, we prove a little lemma which shows that on the connected components of the flip-flop system the θ-codistance is constant.

Lemma 4.4.1. *If two adjacent chambers have equal numerical θ-codistance, then they have equal θ-codistance.*

Proof. Consider two s-adjacent chambers c and c' with equal numerical θ-codistance and set $v := \delta^*(c)$, $v' := \delta^*(c')$. We have $l(v) = l(v')$ by assumption. But then $v = v'$ (and we are done), or $v' = sv\theta(s)$. But by Lemma 1.3.2, $l(sv\theta(s)) = l(v)$ implies $sv = v\theta(s)$ and hence $v' = v$ after all. □

The actual heart of our proof is the following useful lemma.

Lemma 4.4.2. *Let θ be a quasi-flip of a twin building $\mathcal{C}$ such that for all rank 2 residues R, direct descent is possible and R^θ is connected.*

Let (c_0, c_1, c_2) be a gallery such that the numerical θ-codistance of c_1 is at least as big as that of c_2 and exceeds that of c_0. Then there exists a gallery γ from c_0 to c_2 such that the numerical θ-codistance of all chambers in $\gamma \setminus \{c_2\}$ is lower than that of c_1 (see Figure 4.1).

Figure 4.1.: A peak and a short plateau with bypasses in the θ-codistance of a gallery. Higher numerical θ-codistance is reflected by chambers being depicted farther upwards.

Proof. Clearly there is a rank 2 residue R which completely contains (c_0, c_1, c_2).

Since by hypothesis direct descent into R^θ is possible in R, there is a gallery γ_0 from c_0 to a chamber c_0' in R^θ such that l^θ is strictly decreasing along this gallery. In particular, the numerical θ-codistance of any chamber in this gallery is at most equal to that of c_0 and hence strictly less than that of c_1. Likewise we find a gallery γ_2 from c_2 to a chamber c_2' in R^θ on which l^θ is strictly decreasing. Hence for all chambers in this gallery different from c_2, the numerical θ-codistance is strictly less than that of c_2 and hence c_1.

Finally, R^θ is connected by hypothesis, therefore there exists a gallery γ_1 in R^θ connecting c_0' and c_2'. We conclude that $\gamma = \gamma_0\gamma_1\gamma_2^{-1}$ is a gallery with the desired properties. □

The key idea is now to repeatedly invoke the preceding lemma.

Proposition 4.4.3. *Let θ be a quasi-flip of a twin building $\mathcal{C}$ such that for all rank 2 residues R direct descent is possible and R^θ is connected. Then for any residue Q of $\mathcal{C}_+$, direct descent is possible and Q^θ inherits connectedness from Q. In particular, Q^θ is connected.*

Proof. Choose any $c \in Q$ and $d \in Q^\theta$. Pick a minimal gallery $\gamma = (c_0, c_1, \ldots, c_n)$ between c and d, so $c_0 = c$ and $c_n = d$. By minimality of γ and convexity of Q we have $\gamma \subseteq Q$.

Our goal is to transform γ via a series of local transformations into a gallery γ' such that the numerical θ-codistance of c is at least as big as that of any other chamber in γ', and such that if γ is a J-gallery, then so is γ'.

Denote by m the maximal numerical θ-codistance among all chambers in γ, and denote by X_γ the set of all chambers in γ with numerical θ-codistance m. Assume $c = c_0 \notin X_\gamma$. Let c_i be the chamber from X_γ which is closest to c_0 along γ, and consider the subgallery (c_{i-1}, c_i, c_{i+1}). By our choice, $l^\theta(c_{i-1}) < l^\theta(c_i)$, and we can apply Lemma 4.4.2 to obtain a gallery $\widehat{\gamma}$ from c_{i-1} to c_{i+1} bypassing c_i and containing no chambers in X_γ except for possibly c_{i+1}. Now substitute the subgallery (c_{i-1}, c_i, c_{i+1}) in γ by $\widehat{\gamma}$ to produce a new gallery with one element less at numerical θ-codistance m. We repeat this process $|X_\gamma|$ times, until we arrive at a gallery γ' between c_0 and c_n with maximal numerical θ-codistance strictly less than m.

Take now γ' as our new gallery γ. Repeating the above finitely many times (bounded by the initial value of m), we arrive at a gallery $\gamma \subset Q$ where the set X_γ contains $c_0 = c$.

Hence we may assume from now on that $c \in X_\gamma$, so all chambers in γ have numerical θ-codistance at most $l^\theta(c)$. If in addition $c \in Q^\theta$, then this implies that $\gamma \subset Q^\theta$ and Q^θ inherits connectedness from Q (as our construction above transforms J-galleries into J-galleries).

To prove that direct descent is possible, we proceed by induction on $n := l^\theta(c) - l^\theta(d)$. If $n = 0$, then $c \in Q^\theta$ and direct descent from c into Q^θ is trivially possible.

Assume now that $n > 0$, and that direct descent into Q^θ is possible for all chambers with numerical θ-codistance less than $l^\theta(c)$. Since $l^\theta(c) > l^\theta(d)$, we have $d = c_n \notin X_\gamma$. Denote by i the lowest index such that $c_i \in X_\gamma$ but $c_{i+1} \notin X_\gamma$.

If $i > 0$, we apply Lemma 4.4.2, this time coming from the right, to (c_{i+1}, c_i, c_{i-1}) (clearly c_{i+1} has lower numerical θ-codistance than c_0). This yields a gallery $\widehat{\gamma}$ from c_{i+1} to c_{i-1} which bypasses c_i and all chambers in $\widehat{\gamma}$ except for c_{i-1} have lower numerical θ-codistance than c_0. Define a new gallery γ' by substituting $\widehat{\gamma}$ for (c_{i+1}, c_i, c_{i-1}) in γ. In γ', clearly c_{i-1} now has the same property as c_i had in γ.

By repeating this process i times, we arrive at a gallery where $c = c_0$ has bigger numerical θ-codistance than c_1. By our induction hypothesis, direct descent from c_1 into Q^θ is possible, thus we can also directly descend from c_0 into Q^θ. □

The following is an immediate consequence.

Proposition 4.4.4. *Let θ be a quasi-flip of a twin building $\mathcal{C}$ such that for all rank 2 residues R direct descent is possible and R^θ is connected. Then θ is homogeneous and $\mathcal{C}^\theta$ inherits connectedness from $\mathcal{C}_+$.*

Proof. Applying Proposition 4.4.3 to $Q = \mathcal{C}_+$, we find that direct descent into $\mathcal{C}^\theta$ is possible and that $\mathcal{C}^\theta$ inherits connectedness from $\mathcal{C}_+$. Hence $\mathcal{C}^\theta$ is connected, and so by Lemma 4.4.1 the θ-codistance on $\mathcal{C}^\theta$ is constant and equals some element $w \in W$. By Lemma 2.3.4, there exists $K \subseteq S$ such that K is spherical and $w = w_K$.

Let R be an arbitrary minimal Phan residue of type I. By Lemma 4.3.5, I is spherical and the θ-codistance on R is constant and equal to w_I. But from any chamber c in R we can directly descend into $\mathcal{C}^\theta$. Yet the only way one could shorten the element w_I is with some $s \in I$, hence we actually stay inside R, where the θ-codistance is constant. Altogether this proves that $R \subset \mathcal{C}^\theta$, $I = K$ and θ is K-homogeneous. □

4.5. Residual connectedness

I am not aware of a good reference for the following definition, but the objects described in it are certainly not new. See for example [BC, Theorem 14.6.3] for a similar definition.

Definition 4.5.1. Let $\mathcal{C}$ be a chamber system over I, let K be a subset of I. Then we define the ***K*-residue chamber system** $\mathcal{C}_K$ over $I' := I \setminus K$ as follows: The chambers are the K-residues $R_K(c)$ in $\mathcal{C}$, and for $i \in I'$, two chambers $R_K(c), R_K(d)$ of $\mathcal{C}_K$ are i-adjacent if and only if both are contained in the same $(K \cup \{i\})$-residue.

Remark 4.5.2. This corresponds to going from the building G/B to the coset geometry G/P, where P is a parabolic subgroup of type K containing B. In our case, P is in fact a minimal θ-split parabolic subgroup. Geometrically, we move from a complete flag variety (say, all proper nontrivial subspaces of a vector space) to a partial flag variety (where we omit some types of subspaces, as determined by K).

Proposition 4.5.3. *Let θ be a K-homogeneous quasi-flip of a Moufang twin building $\mathcal{C}$ of type (W, S) such that $\mathcal{C}^\theta$ inherits connectedness from $\mathcal{C}_+$. Then the K-residue chamber system $\mathcal{C}^\theta_K$ is residually connected.*

Proof. Let $I' = S \setminus K$ denote the type set of $\mathcal{C}^\theta_K$. To establish that $\mathcal{C}^\theta_K$ is residually connected, consider an arbitrary finite family of residues $(R_j)_{j\in J}$ in $\mathcal{C}^\theta_K$, where $J \subseteq I'$ and each R_j is of type $I' \setminus \{j\}$, and such that all R_j have pairwise nonempty intersections. We have to prove that the intersection of all R_j is nonempty.

For each j in J, define the completion $\overline{R}_j$ of R_j in $\mathcal{C}_+$ as the unique $(S\setminus\{j\})$-residue of $\mathcal{C}_+$ which contains R_j (viewed as a subset of $\mathcal{C}_+$). These completions are Phan residues (i.e., $\overline{R}_j$ is opposite $\theta(\overline{R}_j)$), for they contain K-residues of chambers in $\mathcal{C}^\theta$.

Since $\mathcal{C}_+$ is residually connected, $\overline{R}_J := \cap_{j\in J}\overline{R}_j$ is a nonempty $(S \setminus J)$-residue of $\mathcal{C}_+$. By Lemma 4.3.3, it is again a Phan residue, hence contains a minimal Phan residue of type K. Using Lemma 4.3.6, we conclude that it intersects $\mathcal{C}^\theta$ nontrivially.

Set $R_J := \overline{R}_J \cap \mathcal{C}^\theta$. Let $c \in R_J$ and $j \in J$ be arbitrary. Pick a chamber d in $R_j \subset \overline{R}_j \subset \mathcal{C}^\theta$. By virtue of their definition, c and d are connected by an $(I \setminus \{j\})$-gallery in $\mathcal{C}_+$. As $\mathcal{C}^\theta$ inherits connectedness from $\mathcal{C}_+$, they are also connected by an $(I\setminus\{j\})$-gallery in $\mathcal{C}^\theta$, and we see that $c \in R_j$. Since c and j were arbitrary, it follows that $R_J \subset \cap_{j\in J}R_j$. But R_J is nonempty, hence the claim follows. □

If $K = \emptyset$, then Proposition 4.5.3 states that $\mathcal{C}^\theta$ itself is residually connected. If $K \neq \emptyset$, it is in general unknown whether this is the case. However, for the special case that $|K| \leq 2$ and that we have direct descent in all residues, this is true by a simple argument in Coxeter groups, as the following shows:

Proposition 4.5.4. *Let θ be a quasi-flip of a twin building $\mathcal{C}$ of type (W, S) such that for all rank 2 residues R, direct descent is possible and R^θ is connected. If θ is K-homogeneous with $|K| \leq 2$, then $\mathcal{C}^\theta$ is residually connected.*

Proof. For $i \in \{1, 2, 3\}$, let R_i be a residue of type J_i in $\mathcal{C}^\theta$, such that for each $j \in \{1, 2, 3\}$ we have $R_i \cap R_j \neq \emptyset$. To show residual connectedness of $\mathcal{C}^\theta$, by Lemma 1.4.11 it suffices to show that $R_{123} := R_1 \cap R_2 \cap R_3$ is nonempty and connected.

Denote by $\overline{R}_i$ the convex hull of R_i in $\mathcal{C}_+$. By residual connectedness of the building, we have $\overline{R}_{123} := \overline{R}_1 \cap \overline{R}_2 \cap \overline{R}_3 \neq \emptyset$, so we can pick a chamber c in $\overline{R}_{123}$. Applying Proposition 4.4.3 to $R_1 \cap R_2$ we can directly descend from c into $\mathcal{C}^\theta$ via a $(J_1 \cap J_2)$-gallery. By symmetry we can do likewise via a $(J_2\cap J_3)$-gallery or a $(J_1\cap J_3)$-gallery. More precisely, and denoting the θ-codistance of c by w, this means that there are words $w_{ij} \in W_{J_i\cap J_j}$ such that for each there is a directly descending gallery of type w_{ij} from c with θ-codistance w to a chamber with θ-codistance w_K. Hence there are subwords $w'_{ij} \leq w_{ij}$ such that $w = w_{ij}w_K\theta(w'_{ij})$ and $l(w) = l(w_{ij})+l(w_K)+l(\theta(w'_{ij}))$.[2]

Set $X := J_1 \cap J_2 \cap J_3$ and $Y := X \cup K$. By what we just observed, $w_{ij} \in W_Y$ for all i, j. Therefore, $w_{ij} \in W_{J_i\cap J_j\cap Y}$. Since $|K| \leq 2$ by hypothesis, we can find $i, j \in \{1, 2, 3\}$ such that $K \cap J_i \cap J_j = \emptyset$, hence in fact $w_{ij} \in W_X$. We conclude that

[2] In fact these subwords are uniquely determined by the requirement that at each step there must be a θ-twisted involution occurring as θ-codistance, but this is not of importance here.

we can directly descend from c into $\mathcal{C}^\theta$ via a gallery of type w_{ij}, staying within $\overline{R}_{123}$. It follows that $\overline{R}_{123} \cap \mathcal{C}^\theta \neq \emptyset$, and accordingly $R_{123} \neq \emptyset$. Finally Proposition 4.4.3 implies that $R_{123} = \overline{R}^\theta_{123}$ is connected. □

In Example 2.1.9 (2.1.9) we saw that K can be arbitrary large, even for irreducible buildings. It would be interesting to extend the above result to arbitrary quasi-flips, or find counterexamples. But at least it covers the important case of proper quasi-flips (for which $K = \emptyset$).

4.6. Rank 2 residues

In this section we study when rank 2 residues satisfy the prerequisites of Theorem 4.1.7. That is, when direct descent into R^θ is possible and when R^θ is connected.

We will focus solely on the case of 2-spherical Moufang twin buildings. The main reason is that for tree residues, there is only one unique minimal gallery between any two given chambers. Hence our idea of bypassing problematic chambers inside rank 2 residues alone cannot work there in general. However, 2-spherical twin buildings are already sufficiently interesting objects (including all spherical and all affine buildings without direct factors of type $\tilde{A}_1$).

Recall that for a residue R of the positive half of the twin building, the induced flip-flop system R^θ consists of all chambers in R with minimal numerical θ-codistance, that is,

$$R^\theta := \{c \in R \mid l^\theta(c) = \min_{d \in R} l^\theta(d)\}.$$

Furthermore recall that *direct descent into* R^θ is possible if for any chamber in R there exists a gallery in R to a chamber in R^θ with the property that l^θ is strictly decreasing along that gallery.

In the rest of this chapter, R will be an arbitrary rank 2 residue of the positive half of the twin building. By 2-sphericity it is actually a Moufang polygon.

It turns out to be important to study the projection residue $Q := \operatorname{proj}_R(\theta(R))$ and the induced map $\theta' := \operatorname{proj}_Q \circ \theta$ on Q. The motivation for that is the following alternative characterization of R^θ (which holds for arbitrary spherical residues R):

Proposition 4.6.1 (Proposition 3.5 in [DGM])**.** *Let R be a spherical residue in $\mathcal{C}_+$, set $Q := \operatorname{proj}_R(\theta(R))$. Then R^θ consists of all chambers c such that both $l_+(c, d)$ and $l_+(d, \theta'(d))$ are simultaneously maximal, where $d := \operatorname{proj}_Q(c)$.*

Proof. Let I be the type set of R, let $c \in R$ be arbitrary. Denote by d the projection of c to Q. By definition, $\theta'(d)$ is the projection of $\theta(d)$ to Q. Moreover,

$$\delta_+(c, \theta'(d)) = \delta_+(c, d) \cdot \delta_+(d, \theta'(d)).$$

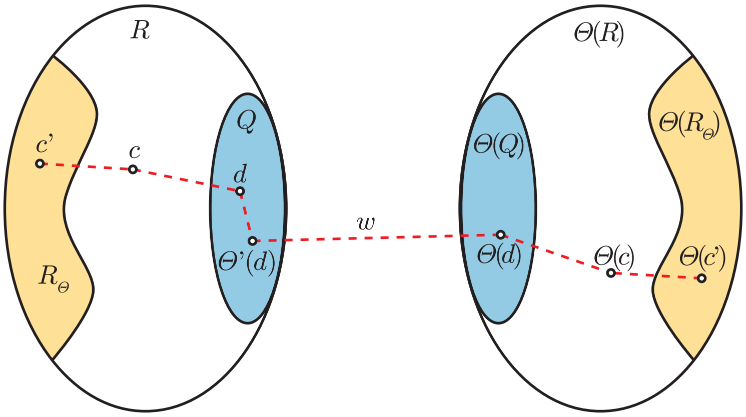


Figure 4.2.: A spherical residue R with R^θ and Q depicted for $Q \neq R$.

Let w denote the codistance of R and $\theta(R)$ (equivalently, that of Q and $\theta(Q)$, or that of $\theta(d)$ and $\theta'(d)$). By repeatedly applying Lemma 1.6.5, we get (see also Figure 4.2):

$$\begin{aligned}\delta^*(c,\theta(c)) &= \delta_+(c,d)\cdot\delta^*(d,\theta(c))\\ &= \delta_+(c,d)\cdot\delta^*(d,\theta(d))\cdot\delta_-(\theta(d),\theta(c))\\ &= \delta_+(c,d)\cdot\delta_+(d,\theta'(d))\cdot\underbrace{\delta^*(\theta'(d),\theta(d))}_{w}\cdot\delta_-(\theta(d),\theta(c)).\end{aligned}$$

Now θ is a quasi-flip and therefore preserves numerical distances. Moreover, w is the maximal element of the double coset $W_I w W_I$ (cf. [AB08, Lemma 5.148]). We conclude that $l^\theta(c) = l(w) - 2l_+(c,d) - l_+(d,\theta'(d))$. Thus c is in R^θ if and only if $f(c) := 2l_+(c,d) + l_+(d,\theta'(d))$ is maximal.

Denote the maximal numerical distance of any chamber in R to Q by n. Now we can always find a chamber c' which has the same projection d to Q as c has, but has maximal distance from Q. Hence if c is in R^θ, then $l_+(c,d)$ must equal n. That is, if $c \in R^\theta$ then $f(c) = n + l_+(d,\theta'(d))$.

Denote the maximal numerical distance of any chamber x in Q from its image $\theta'(x)$ by m. Denote by Q_θ the set of chambers in Q which θ' moves maximally. Hence the maximal value $l_+(d,\theta'(d))$ can attain is m, hence $f(c)$ is at most $m+n$.

We conclude that $c \in R^\theta$ if and only if $f(c) = m+n$ if and only if c is opposite Q and $\mathrm{proj}_Q(c)$ is in Q_θ. The claim follows. □

As we are dealing exclusively with rank 2 residues R, we define and distinguish three cases, based on the rank of $Q = \mathrm{proj}_R(\theta(R))$:

(1) R is **θ-orthogonal** if the rank of Q is 0, i.e., Q consists of a single chamber.

(2) R is **θ-acute** if the rank of Q is 1, i.e., Q is a panel of R.

(3) R is **θ-parallel** if R is parallel to $\theta(R)$ and thus $Q = R$.

In the next three sections, we study each case in detail. Specifically, we analyze under which conditions R^θ is connected, and when direct descent into R^θ is possible.

4.6.1. R is a θ-orthogonal rank 2 residue

If the rank of $Q = \mathrm{proj}_R(\theta(R))$ equals 0, then Q consist of a single chamber c. Then by Proposition 4.6.1, R^θ equals the set c^{op} of chambers opposite some fixed chamber c. This set and the following result are well-known:

Proposition 4.6.2 (Proposition 7 in [Abr96]; see also [AVM99])**.** *The geometry opposite a chamber in a Moufang polygon is connected if and only if the polygon is not associated to any of the groups $C_2(\mathbb{F}_2) \cong \mathrm{Sp}_4(\mathbb{F}_2)$, $G_2(\mathbb{F}_2)$, $G_2(\mathbb{F}_3)$ or ${}^2F_4(\mathbb{F}_2)$.*

Furthermore, we readily observe the following:

Proposition 4.6.3. *If R is θ-orthogonal, then direct descent into R^θ is possible.*

Proof. For any chamber d in R, if d is not in $R^\theta = c^{op}$, hence not opposite c, then we can find a chamber d' adjacent to d which is farther away from c than d is. By Proposition 4.6.1, d' has numerical θ-codistance strictly smaller than that of d. By repeating this finitely many times, we directly descend into R^θ. $\square$

4.6.2. R is a θ-acute rank 2 residue

If R is θ-acute, then the rank of $Q = \mathrm{proj}_R(\theta(R))$ is 1, i.e., Q is a panel. Since we are in a Moufang polygon, Q is endowed with the structure of a Moufang set. Recall that θ induces an automorphism of the Moufang set Q, namely $\theta' := \mathrm{proj}_Q \circ \theta$.

Define $T := Q \setminus Q_\theta$, the complement of the induced flip-flop system in Q. Equivalently, T is the complement of the set of elements moved maximally by θ'. Hence if $\theta' = \mathrm{id}|_Q$, then T is the empty set; otherwise it is the set of chambers in Q fixed by θ'. In either case, by Lemma 3.3.4, T is a proper generalized Moufang subset of Q.

Proposition 4.6.4. *If R is θ-acute, then direct descent into R^θ is possible.*

Proof. Proposition 4.6.1 implies that the chambers in R^θ are those which are opposite the residue Q and which are projected onto a chamber in $Q_\theta = Q \setminus T$.

Direct descent into R^θ is possible by a similar argument as in Proposition 4.6.3: For any chamber c in R, if c is not opposite T, then we find an adjacent chamber c' which is farther away from Q, and hence has lower numerical θ-codistance. If c is already opposite Q, then it is contained in a panel P opposite Q. Then P must contain a chamber c'' which is also opposite Q and projects to a chamber in Q_θ, and which hence is in R^θ. $\square$

We now turn to the question whether R^θ is connected as a chamber system. For digons this is trivially true: The incidence graph of the point-line geometry of a digon is a complete bipartite graph. The flip-flop system is a subset of the points and lines (containing elements of both types, as it consists of chambers). Therefore the induced incidence graph for the flip-flop system is once again a complete bipartite graph, thus connected. So we next consider triangles.

Proposition 4.6.5. *If R is a θ-acute Moufang projective plane, then R^θ is connected.*

Proof. By invoking duality, we may assume that Q corresponds to a line K, and hence T corresponds to a proper subset of the point row of K. By Proposition 4.6.1, the elements of R^θ are all chambers (p, L) where p is not on K, and L meets K in a point q outside of T.

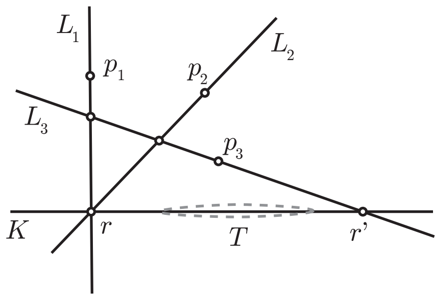


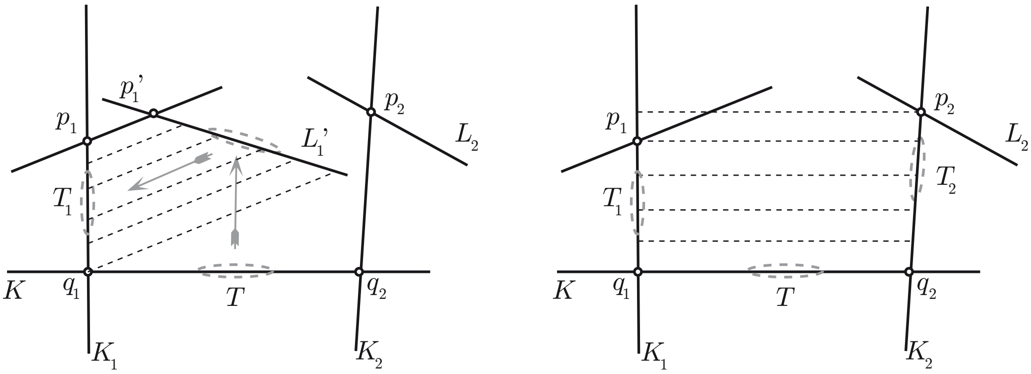
Figure 4.3.: Connecting good chambers inside a θ-acute Moufang projective plane.

Let (p_1, L_1) and (p_2, L_2) be two such chambers. Assume L_1 and L_2 meet in some point r. If r is a point outside K, we have a connection and are done. So assume $r \in K$. By Lemma 3.3.6, there must be a second point r' in $Q \setminus T$, different from r. Then there exists a chamber (p_3, L_3) with p_3 not on K and L_3 meeting K in r (see Figure 4.3). By what we said previously, this new chamber is connected inside R^θ to our two original chambers. □

Proposition 4.6.6. *Let R be a θ-acute Moufang quadrangle of order (s, t). Then R^θ is connected unless the order (s, t) of R is $(2, 2)$, $(2, 4)$, $(4, 2)$, $(3, 3)$ or $(4, 4)$ (i.e., associated to one of the groups $C_2(\mathbb{F}_q) \cong \mathrm{Sp}_4(\mathbb{F}_q)$ for $q \in \{2, 3, 4\}$ or ${}^2A_3(\mathbb{F}_2) \cong \mathrm{PGU}_4(\mathbb{F}_2)$).*

Proof. By invoking duality, we may assume that Q corresponds to a line K, and hence T corresponds to a (proper) subset of the point row of K. By Proposition 4.6.1, the elements in R^θ are all chambers (p, L) where L is opposite K and p projects to a point on K outside of T. We call points and lines satisfying these properties *good* points and lines. Observe that all but one of the lines in the pencil of a good point are good lines. Moreover, by Lemma 3.3.6, we have $|K \setminus T| \geq 2$, therefore every good line contains at least two good points.

Take two such chambers (p_1, L_1) and (p_2, L_2). Denote the projections of p_1 and p_2 to K by q_1 and q_2, and the projection lines by K_1 and K_2. If q_1 and q_2 are equal, use the fact that L_2 contains a second good point p_2'. Then the chambers (p_2, L_2) and (p_2', L_2) are connected in R^θ, and p_2' projects to a point on K different from q_1. Hence it suffices to deal with the case where q_1 and q_2 differ to establish our claim in full generality.



(a) Projecting from K to L_1' to K_1 to obtain T_1 (b) Projecting between K_1 and K_2

Figure 4.4.: Connecting good chambers inside a θ-acute Moufang quadrangle.

Again by Lemma 3.3.6, there must be a second point p_1' on L_1 which does not project to T. Take any line L_1' through that point not meeting K. Then project T to L_1' and from there to K_1, to obtain a set T_1 of "bad" points on K_1 (see Figure 4.4a). Note that all projection lines between L_1' and K_1, except for the one through q_1, are opposite K. All points on K_1 which are not in T_1 and are not q_1 are good points, and by construction reachable from (p_1, L_1).

By symmetry, we can do the same with the second chamber to obtain a similar subset T_2 of K_2. Finally, we project from K_1 to K_2 (see Figure 4.4b) and apply Lemma 3.3.7 if the point order is at least 5: K_1 cannot be covered by T_1, the projection of T_2, and the single point q_1 (which also is the projection of q_2). This implies that there must be a projection line between K_1 and K_2 but different from K which meets those two lines in good points. This line clearly does not meet K, hence is good, and so we have constructed a suitable connection between our two starting chambers within R^θ.

This leaves a finite number of potential exceptions for quadrangles, namely the Moufang quadrangles satisfying $s + t \leq 8$. By computer calculations (see Appendix A.1), it turns out that the only counterexamples exist in the quadrangles of order $(2, 2)$, $(2, 4)$ (and its dual), $(3, 3)$ and $(4, 4)$ – these are the smallest existing Moufang quadrangles. $\square$

For Moufang hexagons (and possibly octagons), I believe that a similar statement holds, but no general proof is known to me at this time. However, the following counting argument proves connectedness for most finite Moufang hexagons.

Proposition 4.6.7. *Let R be a θ-acute finite Moufang hexagon of order (s,t). Then R^θ is connected if s and t both are at least 19 and are not divisible by 2 or 3.*

Sketch of proof. By invoking duality, we may assume that Q corresponds to a line K, and hence T corresponds to a (proper) subset of the point row of K. By Proposition 4.6.1, the elements in R^θ are all chambers (p, L) where L is opposite K and p projects to a point on K outside of T. We call points and lines satisfying these properties *good* points and lines.

The number of good lines hence equals the number of lines opposite K, which is s^2t^3. Let $x := |T| - 1$. Then each good line contains $s + 1 - |T| = s - x$ good points. Hence R^θ consists of $(s - x)s^2t^3$ good chambers.

On the other hand, starting at a good line L, one can reach at least

$$\begin{aligned} & 1 + (s-x)(t-1) + (s-x)(t-1)(s-x-1)(t-1) \\ & \quad + (t-1)(s-x-1)(t-1)(s-x-1)(t-1) \\ = \quad & t \cdot \Big(1 + (s-x-1)(t-1) + (s-x-1)^2(t-1)^2\Big) \end{aligned}$$

good lines. Therefore, the size of a connected component of R^θ is at least $(s - x)$ times this number. Dividing the number of good chambers by this number, we obtain an upper bound on the number of connected components:

$$\#\text{connected components} \leq \frac{s^2t^2}{1 + (s-x-1)(t-1) + (s-x-1)^2(t-1)^2}. \tag{4.1}$$

If s and t are not divisible by 2 and 3, then by Lemma 3.3.6, we get $x \leq \frac{s}{5}$ (as 5 is the smallest prime number bigger than 2 and 3). Combining this with inequality (4.1) yields that for s and t at least 19, the number of connected components is less than 2. $\square$

Remark 4.6.8. A similar counting approach can be used for finite Moufang quadrangles, but yields worse bounds than Proposition 4.6.6, and does not cover characteristic 2. For this reason a counting argument also fails for Moufang octagons as these only exist in characteristic 2.

4.6.3. R is a θ-parallel rank 2 residue

R is θ-parallel if the rank of $Q = \text{proj}_R(\theta(R))$ is 2, i.e., $Q = R$. Hence θ induces an (anti-)automorphism $\theta' := \text{proj}_R \circ \theta$ of R. As R is a Moufang polygon, θ' is either the identity (in which case $R = R^\theta$ and nothing has to be done), an involutory automorphism of the underlying point-line geometry (mapping points to points and lines to lines), or a polarity (interchanging the types). We will deal with the latter two cases separately.

In the next few pages, we completely treat Moufang projective planes, and give partial results on Moufang quadrangles and hexagons. For digons, all claims are trivial: The incidence graph of the point-line geometry of a digon is a complete bipartite graph. The flip-flop system is a subset of the points and lines (containing elements of both types, as it consists of chambers). Therefore the induced incidence graph for the flip-flop system is once again a complete bipartite graph, thus connected. Direct descent is possible because any point (or line) is incident with all lines (points) of the flip-flop system.

θ' is a polarity

Given a polarity θ', recall that an **absolute element** is a point or line which is mapped by the polarity to an incident line or point, that is, x is absolute if and only if $x \sim \theta'(x)$.

Remark 4.6.9. The only Moufang quadrangles that admit polarities are defined in characteristic 2, i.e., with root groups which are not uniquely 2-divisible; see e.g. Theorem 7.3.2 and Corollary 7.4.3 in [VM98].

For Moufang hexagons, polarities exist only for the mixed hexagons over fields admitting a Tits endomorphism, cf. Theorem 7.3.4 in loc. cit., which implies characteristic 3.

Finally by Theorem 7.3.6 in loc. cit. Moufang octagons do not admit polarities at all.

In view of this remark, the following proposition deals with characteristic 2 exclusively.

Proposition 4.6.10. *Suppose R is a θ-parallel Moufang quadrangle and θ' a polarity. Then R^θ consists of Phan chambers and direct descent is possible. Furthermore, R^θ is connected if the order of the quadrangle is not $(2, 2)$.*

Proof. Any line L contains at most one absolute point: For assume p_1 and p_2 in L are absolute points. Then $\theta'(p_1)$ and $\theta'(p_2)$ would be lines meeting in the point $\theta'(L)$. Since no triangles may exist, we conclude that $\theta'(L) \in L$, and so L is absolute and $p_1 = p_2 = \theta'(L)$.

Dually, every point is contained in at most one absolute line. Thus, we see that every chamber consisting of two absolute elements is adjacent to a chamber with only one absolute element, and any such chamber in turn is adjacent to a chamber with two non-absolute elements. Finally, a chamber consisting of a non-absolute point and a non-absolute line is mapped by θ' to an opposite chamber. This proves that direct descent is possible and that R^θ consists of Phan chambers.

Given two chambers (p_1, L_1) and (p_2, L_2) in R^θ, we want to construct a connection inside R^θ. If $p_1 \in L_2$ or $p_2 \in L_1$, we are done. If that is not the case, we may up to duality assume that L_1 is opposite L_2.

If the order of the quadrangle is $(2, 2)$, then there is a polarity for which R^θ consists of two connected components, each forming a pentagon.[3] This polarity can be easily understood by looking at Figure 4.5: It interchanges each outer "corner" with the

[3]In fact this is true for all 36 polarities, as they are all conjugate, but we do not need this here.

opposite outer edge; each point on the middle of an outer edge is mapped to the line spanned by it and the opposite corner; and hence each inner point is interchanged with the "curved" line partially encircling it. One connected component contains all chambers which consists of a "corner" and an outer edge; the other contains the chambers made from inner points and the "curved" lines between them.

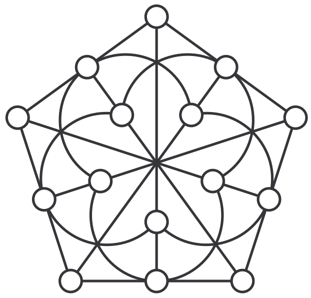


Figure 4.5.: Collinearity graph of the generalized quadrangle of order $(2, 2)$.

Assume now that the order is at least $(3, 3)$. Every line contains at most one absolute point, hence L_1 and L_2 each contain at least three non-absolute points. Hence we can find two good points a_1 and b_1 on L_1 which project to good points a_2 and b_2 on L_2. If one of the projection lines is non-absolute, we are done. So assume that both $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$ are absolute. Take a second non-absolute line L_3 through the third good point c_1 on L_1, one which does not intersect L_2. Then project a_2 and b_2 to L_3. Both projection lines must be good (as they are different from the unique absolute lines through a_2 and b_2). At most one of a_2 and b_2 can project to an absolute point on L_3 (as it contains at most one); hence we obtain a good connection. $\square$

What remains is the case of Moufang projective planes, where polarities occur plentifully.

Proposition 4.6.11. *Suppose R is a θ-parallel Moufang projective plane and θ' a polarity. Then R^θ consists of Phan chambers. Moreover, it is connected if the plane is different from $\mathbb{P}^2(\mathbb{F}_4)$, i.e., not associated to the group $A_2(\mathbb{F}_4) \cong \mathrm{SL}_3(\mathbb{F}_4)$.*

Proof. First we establish the existence of a chamber in R opposite to its image under θ'. Assume there are no absolute points, hence no absolute lines. Then every chamber is mapped to an opposite one. So assume there is an absolute point p. Then exactly one line L through p is absolute, namely $L = \theta'(p)$. So there are lines L', L'' through p which are not absolute. If there is a non-absolute point p' on L', the chamber (p', L') is mapped to an opposite one. If all points on L' are absolute, then in fact all absolute points must be on L' (if there was an absolute point p' outside L', then its absolute line would meet L' in a second absolute point, which is impossible since absolute lines contain exactly one absolute point). Hence we find a non-absolute point p'' on L'' and are again done by choosing the chamber (p'', L'').

Take two chambers (p_1, L_1) and (p_2, L_2) in R^θ. If they are not opposite, then they are connected inside R^θ. E.g. if p_1 is on L_2, then (p_1, L_2) is a chamber in R^θ.

Hence we may assume that (p_1, L_1) and (p_2, L_2) are opposite. Assume furthermore that the order of the plane is at least 5. The set of absolute lines through p_1 forms a proper generalized Moufang subset of the pencil of p_1 (to see this, just apply θ' to the pencil, then project back, and use Lemma 3.3.4); so does the set of absolute points on L_2. Projecting the pencil of p_1 to L_2 and applying Lemma 3.3.7, there must be a non-absolute line through p_1 which meets L_2 in a non-absolute point, and we have the desired connection.

For $\mathbb{P}^2(\mathbb{F}_2)$, we can use the same argument, since by counting we see that the proper generalized Moufang subset must have size one (a nontrivial involution on a set of three elements has one fixed point). For $\mathbb{P}^2(\mathbb{F}_3)$ we give a computer proof in Appendix A.2. $\square$

Remark 4.6.12. For $\mathbb{P}^2(\mathbb{F}_4)$, there is a counterexample. Namely, take a hermitian resp. unitary polarity of $\mathbb{F}_4^3$. Then R^θ consists of four connected components, each containing six chambers (more precisely, each component contains three points and three lines forming a complete bipartite graph, cf. [BS04, Section 2]).

We next consider the direct descent property in Moufang projective planes. Here, [Bae46] originally inspired us to prove Proposition 2.5.2, which readily implies the following (which corresponds to Corollary 2 in loc. cit.):

Lemma 4.6.13. *Let θ' be a polarity of a Moufang projective plane R defined over an alternative division algebra A. If θ' admits a non-absolute line L such that all points on L are absolute, then the characteristic of A is* 2*.*

Proof. Assume there exists a line L with all points on it being absolute. Pick any such point p and then pick a non-absolute line K through it different from L. We induce an automorphism φ on the point set of K by composing θ' with the projection map, i.e., $\varphi(x) := \mathrm{proj}_K(\theta'(x))$ for all $x \in K$. Clearly φ fixes only the point p. But by Proposition 2.5.2 this means that the additive group of A is not uniquely 2-divisible. Hence the characteristic of A is 2. $\square$

We can now deduce the desired result in characteristic different from 2. In characteristic 2, there are counterexamples, see Remark 4.6.15. Accordingly, this is the best we can hope for without imposing further restrictions on the plane or the polarity.

Proposition 4.6.14. *Suppose R is a θ-parallel Moufang projective plane and θ' a polarity. Then direct descent is possible if the root groups are uniquely* 2*-divisible.*

Proof. Let (p, L) be a chamber in R. If $\theta'(p) = L$, i.e., the chamber is fixed by θ', then for any other line L' through p, the chamber (p, L') is not fixed, and hence the θ-codistance is reduced.

Assume next that p is an absolute point, but L is a non-absolute line. By Lemma 4.6.13, there must be a non-absolute point p' on L. Therefore, the chamber (p', L) is mapped to an opposite chamber, and we arrive in R^θ. $\square$

Remark 4.6.15. By [Bae46], in characteristic 2, there are polarities of finite Desarguesian projective planes for which all absolute points are collinear, and in fact, form the point row of a single line L. By Appendix A in loc. cit., this is in particular always the case if the order of the field is not a square. In this scenario, the chamber (p, L) (for any $p \in L$), consisting of an absolute point and a non-absolute line, is neither fixed nor mapped to an opposite chamber (so not contained in R^θ). But all adjacent chambers also contain an absolute point, hence are not contained in R^θ. So direct descent is impossible.

Remark 4.6.16. We have not dealt with Moufang hexagons and octagons. However, we have strong reasons to believe that connectedness and direct descent hold at least for hexagons. This is subject of ongoing research by Hendrik Van Maldeghem and the author [HVM].

θ' is an involutory automorphism

Proposition 4.6.17. *Suppose R is a θ-parallel Moufang projective plane and θ' an involutory automorphism. Then R^θ is connected and direct descent is possible.*

Proof. Since θ' is not the identity, there must be a line L moved by θ' to a different line. Any such line L contains a unique fixed point, namely the point where L and $\theta'(L)$ meet. Likewise, any non-fixed point is on a unique fixed line. Consequently, adjacent to a fixed chamber we always find a chamber consisting of one fixed and one non-fixed element; and adjacent to such a chamber, we always find a chamber where both elements are non-fixed, i.e., a chamber which is mapped to an opposite one. Thus R_θ contains all chambers mapped to an opposite chamber, and direct descent is possible.

To see that R^θ is connected, let (p_1, L_1) and (p_2, L_2) be two chambers in R^θ and consider the set of all lines from p_1 to points on L_2. Exactly one of these is fixed by θ', and exactly one meets the unique fixed point on L_2. Hence any of the remaining lines through p_1 is non-fixed and meets L_2 in a non-fixed point, and we are done. □

For quadrangles, we can currently only deal with the classical quadrangles using the results from Section 3.3. This covers all finite Moufang quadrangles.

Proposition 4.6.18. *Suppose R is a θ-parallel classical quadrangle and θ' an involutory automorphism. Then R^θ is connected unless the order (s, t) of R is $(2, 2)$, $(2, 4)$, $(4, 2)$, $(3, 3)$ or $(4, 4)$ (i.e., associated to one of the groups $C_2(\mathbb{F}_q) \cong \mathrm{Sp}_4(\mathbb{F}_q)$ for $q \in \{2, 3, 4\}$ or ${}^2A_3(\mathbb{F}_2) \cong \mathrm{PGU}_4(\mathbb{F}_2)$).*

Proof. In Section 3.3.4 we proved this when the size of the underlying field $\mathbb{K}$ is bigger than 9 and different from 16, in particular for infinite quadrangles.

This leaves the following orders: (s, s) for $s \in \{2, 3, 4, 5, 7, 8, 9, 16\}$; (s^2, s) for $s \in \{2, 3, 4\}$; and (s^2, s^3) for $s \in \{2, 3, 4\}$. These are dealt with in Appendix A.3 by machine computations. □

We conjecture the following result to hold in general:

Conjecture 4.6.19. *Suppose R is a θ-parallel Moufang quadrangle and θ' an involutory automorphism. Then R^θ is connected unless the order (s,t) of R is $(2,2)$, $(2,4)$, $(4,2)$, $(3,3)$ or $(4,4)$ (i.e., associated to one of the groups $C_2(\mathbb{F}_q) \cong \mathrm{Sp}_4(\mathbb{F}_q)$ for $q \in \{2,3,4\}$ or ${}^2A_3(\mathbb{F}_2) \cong \mathrm{PGU}_4(\mathbb{F}_2)$).*

As with Moufang projective planes, we can prove that direct descent is possible if the characteristic is different from 2. Unlike there, however, we currently are not aware of any actual counterexamples, so this might simply be due to our proof being deficient.

Proposition 4.6.20. *Suppose R is a θ-parallel classical quadrangle and θ' an involutory automorphism. Then direct descent is possible if the root groups are uniquely 2-divisible.*

Proof. For direct descent in quadrangles, two things can go wrong:

(1) There might be a chamber (p,L) fixed by θ such that all adjacent chambers are fixed by θ as well. But then θ is a μ-map, i.e., the product of two root elations, and we are in characteristic 2, as only then μ-maps can be involutory.

To see that θ is a μ-map, choose a second fixed point q on L, and a second fixed line M through p. Then choose a line K through q different from L and a point r on M different from p. Now, (K,L,M) defines a unique root of the quadrangle, and there is a unique root elation ϕ associated to that root which maps r to $\theta(r) \in \theta(M) = M$. Being a root elation, it also fixes the point row of L and the pencils of q and p. Likewise, there is a unique root elation ψ associated to the root $(q,p,\theta(r))$ which sends the line K to $\theta(K) \ni \theta(q) = q$, and which fixes the pencil of p and the point rows of L and M. Accordingly, $\psi \circ \phi$ fixes p and its pencil, L and its point row, and sends r to $\theta(r)$ and K to $\theta(K)$. But (q,K) and (r,M) are opposite, hence define an apartment, on which θ and $\psi \circ \phi$ coincide; but the two maps also coincide on all chambers adjacent to (p,L). For this reason and by rigidity of thick spherical buildings (see e.g. [AB08, Corollary 5.206])), θ equals $\psi \circ \phi$.

(2) Up to duality, there could be a good line L (i.e., L is not incident to $\theta(L)$) such that all points on it are bad (collinear to their image). Hence each point p on L is on a unique θ-fixed line L_p. Then (p,L) is a chamber with numerical θ-codistance 3, and all chambers adjacent to it have numerical θ-codistance at most 3, so we cannot descend further. But in Propositions 3.3.9 and 3.3.11 we proved that if this situation occurs in a classical quadrangle (defined in characteristic different from 2), then *all* points are collinear to their image, hence the maximal numerical θ-codistance is 3. □

Again, we strongly believe that the above holds in the general case:

Conjecture 4.6.21. *Suppose R is a θ-parallel Moufang quadrangle and θ' an involutory automorphism. Then direct descent is possible if the root groups are uniquely 2-divisible.*

The question of whether R^θ is connected for Moufang hexagons, or whether direct descent is possible, is still open. Dealing with these and the remaining (exceptional) quadrangles is subject of ongoing research by Hendrik Van Maldeghem and the author [HVM]. For classical hexagons (i.e., split Cayley hexagons), some promising partial results already have been achieved.

Finally, nothing is known to us regarding Moufang octagons, but since these only occur in characteristic 2, we do not lose too much (as we have to exclude characteristic 2 in many other places anyway). Nevertheless, it would be interesting to at least know whether R^θ is connected for octagons, as we then could apply this to strong quasi-flips, where direct descent is always possible.

4.7. Statement of the main theorems

Combining all we have done in the preceding section, we arrive at the following main theorems:

Theorem 4.7.1 (joint work with Van Maldeghem). *Let R be a Moufang projective plane of order different from 4, or a classical quadrangle of order (s,t), $st > 16$. Let θ be an involutory automorphism or a polarity of R. Then R^θ is connected. If furthermore the root groups are uniquely 2-divisible, then direct descent into R^θ is possible.*

As an immediate consequence, we obtain a generalization of Theorem 4.1.8:

Theorem 4.7.2 (joint work with Gramlich and Mühlherr). *Let θ be a quasi-flip of an RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ of type (W,S), where all root groups U_α are uniquely 2-divisible. Assume all rank 2 residues of the associated twin building $\mathcal{C}$ are projective planes, or classical quadrangles of order (s,t), $st > 16$.*

Then for all rank 2 residues R, direct descent into R^θ is possible and R^θ is connected.

Combining this with Theorem 4.1.7 yields the following version of Theorem 4.1.10:

Theorem 4.7.3 (joint work with Gramlich and Mühlherr). *Let θ be a quasi-flip of an RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ of type (W,S), where all root groups U_α are uniquely 2-divisible. Assume all rank 2 residues of the associated twin building $\mathcal{C}$ are projective planes, or classical quadrangles of order (s,t), $st > 16$.*

Then the flip-flop system $\mathcal{C}^\theta$ is connected and equals the union of all minimal Phan residues which in turn all have identical spherical type K. The chamber system of K-residues of $\mathcal{C}^\theta$ is connected and residually connected.

As already stated previously, I believe that the above can be extended to also cover residues isomorphic to split hexagons (thus extending the results above to almost all split groups in characteristic different from 2).

Finally, we contrast this with the following beautiful result on strong flips, due to Devillers and Mühlherr. By restricting to strong flips, many of the technical complications we had to deal with can be avoided (for example, direct descent is implied by Lemma 2.4.2). This enables them to deduce connectedness of $\mathcal{C}^\theta$ from rank 2 connectedness (as we did, although we also explicitly computed when this local connectedness holds). But moreover, they can conclude simple connectedness from studying rank 3 residues.

Theorem 4.7.4 (Proposition 6.6 in [DM07])**.** *Let $\mathcal{C}$ be a Moufang twin building, let θ be a strong flip of $\mathcal{C}$. Suppose the following conditions are satisfied:*

(1) $\mathcal{C}$ is 3-spherical (if $J \subset S$ is of cardinality at most 3, then J is spherical).

(2) For all rank 2 residues R of $\mathcal{C}_+$, the chamber system R^θ is connected.

(3) For all rank 3 residues R of $\mathcal{C}_+$, the chamber system R^θ is simply connected.

Then $\mathcal{C}^\theta$ is simply connected.

CHAPTER

FIVE

TRANSITIVE ACTIONS ON FLIP-FLOP SYSTEMS

Let G be a group with twin BN-pair, let θ be a quasi-flip of G. In this chapter we study transitivity properties of the action of G_θ, the centralizer of θ in G, on the building and on the flip-flop system $\mathcal{C}^\theta$ as defined in Chapter 4. Our motivation for doing so is that given a sufficiently "nice" transitive action of our group G, we can derive many interesting properties from this. For example, presentations of the centralizer G_θ of θ (Theorem 5.4.2), generalized Iwasawa decompositions (cf. Theorem 5.4.7), lattices in Kac-Moody groups (cf. Theorem 6.2.7 due to Gramlich and Mühlherr) or finite generation of G_θ (cf. Theorem 6.2.5 in Chapter 6).

The work presented in this chapter is partially based on joint work with Tom De Medts and Ralf Gramlich in [DMGH09], specifically the parts on rank 1 transitivity in Section 5.3 and on Iwasawa decompositions in Section 5.4. The results on lattices are due to Bernhard Mühlherr and Ralf Gramlich [GM08].

5.1. Transitivity

In this section, θ is a quasi-flip of a group G with twin BN-pair, and G_θ is the group of elements in G fixed by θ. Recall from Proposition 2.2.1 that θ induces a building quasi-flip on the twin building $\mathcal{C}$ associated to G, which we also denote by θ.

Definition 5.1.1. For $w \in W$, set

$$\mathcal{C}_w^\theta := \{c \in \mathcal{C}_+ \mid \delta^\theta(c) = w\},$$

where $\delta^\theta(c) := \delta^*(c, \theta(c))$ as defined in Section 2.1.

Since the θ-codistance of a chamber is unique, we obtain a partition of the positive half of the building:

$$\mathcal{C}_+ = \bigsqcup_{w \in W} \mathcal{C}_w^\theta. \tag{5.1}$$

Remark 5.1.2. While we defined the $\mathcal{C}_w^\theta$ to be subsets of the positive half of the twin building, we could just as well have defined them as subsets of the negative half. Either way, we obtain essentially identical results, as θ is a bijection between both halves which respects the partition given above (up to a relabeling, as $\delta^\theta(\theta(c)) = \theta(\delta^\theta(c))$), and all other properties we will be interested in later.

For practical purposes, we are only interested in those $\mathcal{C}_w^\theta$ which are nonempty. This motivates the following definition.

Definition 5.1.3. Denote the set of all θ-codistances by

$$\mathrm{Inv}^\theta(\mathcal{C}) := \{w \in W \mid \text{ there exists a chamber } c \in \mathcal{C} \text{ such that } \delta^\theta(c) = w\}.$$

In other words, $w \in \mathrm{Inv}^\theta(\mathcal{C})$ if and only if $\mathcal{C}_w^\theta$ is nonempty. As we have seen in Lemma 2.3.1, all $w \in \mathrm{Inv}^\theta(\mathcal{C})$ are θ-twisted involutions, i.e., $\theta(w) = w^{-1}$ and $\mathrm{Inv}^\theta(\mathcal{C}) \subseteq \mathrm{Inv}^\theta(W)$.

Remark 5.1.4. For a proper quasi-flip (i.e., a quasi-flip admitting a Phan chamber), $\mathcal{C}_{1_W}^\theta$ coincides with the flip-flop system $\mathcal{C}^\theta$ defined in Chapter 4. In general, the flip-flop system is the union of those $\mathcal{C}_w^\theta$ for which $l(w)$ is minimal among all $w \in \mathrm{Inv}^\theta(W)$. The case where there is a unique w such that $\mathcal{C}^\theta = \mathcal{C}_w^\theta$ is of particular interest. Currently, we know no example where this is not the case.

The action of G_θ on the building (obtained by restricting the action of G) preserves the decomposition given in (5.1), and thus induces an action on each $\mathcal{C}_w^\theta$ since for $g \in G_\theta$,

$$\delta^\theta(g.c) = \delta^*(g.c, \theta(g.c)) = \delta^*(g.c, g.\theta(c)) = \delta^*(c, \theta(c)) = \delta^\theta(c).$$

By this and the preceding remark, G_θ also acts on the flip-flop system. As was explained in the introduction of this chapter, we are interested in transitivity properties of this action, motivating the following definition.

Definition 5.1.5. We call θ **flip-flop transitive** if G_θ acts transitively on $\mathcal{C}^\theta$. We call θ **distance transitive** if for each $w \in W$ the group G_θ acts transitively on $\mathcal{C}_w^\theta$. Finally, we call θ **building transitive** if the group G_θ acts transitively on $\mathcal{C}_+$.

Examples 5.1.6. Let $\mathbb{F}$ be a field with a field automorphism σ of order at most 2. Let $G = \mathrm{SL}_2(\mathbb{F})$, let B_+ resp. B_- be the subgroups of upper resp. lower triangular matrices in G, and let θ be the σ-twisted Chevalley involution, i.e., the map $x \mapsto ({}^t x^{-1})^\sigma$. In the notation of Section 3.1, $\theta = \theta_{-1,\sigma}$.

(a) If $\mathbb{F} = \mathbb{C}$ and σ is complex conjugation, then $G_\theta = \mathrm{SU}_2(\mathbb{C})$, and by the well-known Iwasawa decomposition (see [Iwa49], or most books on Lie groups, such as [Hel78] or [Kna02]), $G = G_\theta B_+ = G_\theta B_-$ (cf. Corollary 5.4.6). Hence this flip is building transitive.

(b) Let q be a prime power. If $\mathbb{F}$ is the finite field of order q^2, and σ its unique involutory automorphism $x \mapsto x^q$, then by Proposition 5.3.6, it is flip-flop transitive. But by Corollary 5.4.6, it is not building transitive.

(c) If $(\mathbb{F}, \sigma) = (\mathbb{Q}, \mathrm{id})$, then $G_\theta = \mathrm{SO}_2(\mathbb{Q})$ and θ is far from being building or even flip-flop transitive. This follows from Proposition 5.3.4, as in $\mathbb{Q}$ the sum of two squares is not generally a square. See also [HW93, Examples 4.12 and 6.12].

If $\mathcal{C}^\theta$ is the disjoint union of several sets $\mathcal{C}^\theta_w$, there is no chance G_θ could act transitively on it. On the other hand, if $\mathcal{C}^\theta$ equals one of the $\mathcal{C}^\theta_w$, then distance transitivity trivially implies flip-flop transitivity. We study some cases of the converse question (i.e., finding conditions under which flip-flop transitivity implies distance transitivity) in Section 5.5.

Remark 5.1.7. One may ask when $\mathcal{C}^\theta = \mathcal{C}^\theta_w$. In Chapter 4 we studied conditions under which a quasi-flip θ is K-homogeneous, i.e., when all minimal Phan residues have identical type K. Then by Lemma 4.3.5, K is spherical and δ^θ is constant and equal to w_K on any minimal Phan residue of type K. Consequently, for a K-homogeneous quasi-flip, the flip-flop system $\mathcal{C}^\theta$ equals $\mathcal{C}^\theta_{w_K}$ (and also is a union of K-residues).

Remark 5.1.8. In view of the preceding remark, another kind of transitivity comes to mind: For a K-homogeneous flip, it would also be interesting to know whether G_θ is transitive on the K-residue chamber system $\mathcal{C}^\theta_K$ (cf. Definition 4.5.1). However with the exception of a brief observation in Section 6.1.3 we do not study this question in the present thesis.

Suppose now G_θ acts distance transitively, and let B be a Borel subgroup of G stabilizing some chamber $c \in \mathcal{C}_+$. For each $w \in \mathrm{Inv}^\theta(W)$ choose a representative $g_w \in G$ with the property that $g_w.c \in \mathcal{C}^\theta_w$. Then by distance transitivity we have

$$\mathcal{C}_+ = \bigsqcup_{w \in \mathrm{Inv}^\theta(W)} G_\theta g_w.c.$$

And thus on the group level, using that $\mathcal{C}_+ \cong G/B$,

$$G = \bigsqcup_{w \in \mathrm{Inv}^\theta(W)} G_\theta g_w B. \tag{5.2}$$

Remark 5.1.9. This can be considered as a special case of the Springer parameterization of the double coset decomposition given in Proposition 2.7.5. We sketch this relation under the assumption that every chamber is contained in a θ-stable twin apartment (cf. Section 2.5). Fix a θ-stable twin apartment Σ containing c, and assume that the g_w were chosen so that $g_w.\Sigma$ is again θ-stable (in view of strong transitivity of G and the assumption that every chamber is contained in a θ-stable apartment, this is certainly possible). Then $\theta(g_w.\Sigma) = g_w.\Sigma$, therefore $g_w^{-1}\theta(g_w) \in N_G(\Sigma)$. This should make the correspondence between the two decompositions clear.

The decomposition from Equation (5.2) works without the assumption on the existence of θ-stable twin-apartments. The assumption that G_θ acts distance transitively can also be dropped, but then we may have to take multiple representatives g_w for each $\mathcal{C}_w^\theta$. Finally, observe (again without further restrictions on g_w) that

$$w = \delta^\theta(g_w.c) = \delta^*(g_w.c, \theta(g_w.c)) = \delta^*(c, g_w^{-1}\theta(g_w).\theta(c)).$$

So, in a way, $g_w^{-1}\theta(g_w)$ encodes how much $\delta^\theta(g_w.c)$ differs from $\delta^\theta(c) = \delta^*(c, \theta(c))$. If $g_w^{-1}\theta(g_w) \in N_G(\Sigma)$, then we can make this observation precise, which is exactly how one obtains, geometrically, the statement of Proposition 2.7.5.

Remark 5.1.10. In the case of $\mathbb{F}$-rational points $G(\mathbb{F})$ of an algebraic group, resp. for the associated spherical building, and an $\mathbb{F}$-linear quasi-flip θ (in the sense Section 6.1.3), by [HW93, Proposition 4.11] (see also Section 6.1.3), all minimal θ-split parabolic $\mathbb{F}$-subgroups are G_θ-conjugate over the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$. That is, G_θ acts transitively on the flip-flop system in the building of $G(\overline{\mathbb{F}})$. In particular, the quasi-flip is homogeneous in the sense of Chapter 4. In the smaller building associated to $G(\mathbb{F})$, flip-flop transitivity may no longer hold, however homogeneity is inherited.

Inspired by the proof of [GW, Theorem 7.1] we now determine a bound on the number of G_θ-orbits on a given $\mathcal{C}_w^\theta$, $w \in \mathrm{Inv}^\theta(W)$. Namely, under the assumption that there every chamber is contained in a θ-stable twin apartment, one can obtain good initial bounds by studying what happens in a single θ-stable torus (i.e., a subgroup conjugate to $B_+ \cap B_-$ and stabilized by θ).

For this, we need to introduce some notation.

Definition 5.1.11. To every quasi-flip θ of a group G, we can assign the **twist map** $\tau_\theta : G \to G : g \mapsto \theta(g^{-1})g$ (note that this is in general not a homomorphism). Moreover, the **θ-twisted action** of G on itself is given by $y *_\tau g := \theta(g)^{-1}yg$. A **$\theta$-twisted T-orbit** is the orbit of a group T under the θ-twisted action.

Lemma 5.1.12. *Let θ be a quasi-flip of an RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$, let $\mathcal{C}$ be the associated twin building, and suppose that every chamber is contained in a θ-stable twin apartment. Then for each $w \in \mathrm{Inv}^\theta(W)$, there exists $a \in G$ such that each G_θ-orbit on $\mathcal{C}_w^\theta$ corresponds to a unique twisted aTa^{-1}-orbit on $\tau_\theta(G) \cap aTa^{-1}$. In particular, if T is finite, there are only finitely many G_θ-orbits on every $\mathcal{C}_w^\theta$.*

Proof. Choose any $w \in \mathrm{Inv}^\theta(W)$ and any chamber $c \in \mathcal{C}_w^\theta$. By Theorem 2.5.8, there exists a θ-stable twin apartment Σ containing c. The stabilizer T' of the pair (c, Σ) is conjugate to T so $T' = aTa^{-1}$ for $a \in G$.

Let $c' \in \mathcal{C}_w^\theta$ be arbitrary, and choose a θ-stable twin apartment Σ' containing c'. By strong transitivity of G, there exists $g \in G$ such that $g.c = c'$ and $g.\Sigma = \Sigma'$. Therefore $g.\theta(c) \in \Sigma'$, and

$$w = \delta^*(c', \theta(c')) = \delta^*(c, \theta(c)) = \delta^*(g.c, g.\theta(c)) = \delta^*(c', g.\theta(c)).$$

Since there is a unique chamber in Σ' at any given codistance from c', we conclude that $g.\theta(c) = \theta(c')$. Setting $x := \tau_\theta(g) = \theta(g^{-1})g$, we compute

$$x.c = \theta(g^{-1})g.c = \theta(g^{-1}).c' = \theta(g^{-1}.\theta(c')) = \theta(\theta(c)) = c,$$

and similarly $x.\theta(c) = \theta(c)$ and $x.\Sigma = \Sigma$, thus $x \in T'$.

The group element g chosen above is of course not unique. Indeed, for any $t \in T'$ we have $gt.c = c' = g.c$ and $gt.\Sigma = \Sigma' = g.\Sigma$, and these are all elements of G with this property. We have $\tau(g) \in \tau(G) \cap T'$.

Let $c'' \in \mathcal{C}_w^\theta$ be another chamber in $\mathcal{C}_w^\theta$, contained in a θ-stable twin apartment Σ'' and assume h was chosen for (c'', Σ'') as g was chosen above.

By Lemma 2.7.1), c' and c'' are in the same G_θ-orbit if and only if $(c', \Sigma') = gT'.(c, \Sigma)$ and $(c'', \Sigma'') = hT'.(c, \Sigma)$ are in the same G_θ-orbit. This is the case if and only if $hT'g^{-1} \cap G_\theta \neq \emptyset$, if and only if $1 \in \tau_\theta(hT'g^{-1})$. This is in turn equivalent to $\tau_\theta(g) \in \tau_\theta(hT') = \tau_\theta(h) *_\tau T'$.

Hence c' and c'' are in the same G_θ orbit if and only if the θ-twisted orbits $\tau_\theta(h) *_\tau T'$ and $\tau_\theta(g) *_\tau T'$ are equal. □

This lemma has various useful applications. We will use it in Chapter 6 to prove a criterion on when G_θ is finitely generated, see Theorem 6.2.5. It can also be used to extend the result from [GM08], which states conditions when the centralizer G_θ of a strong flip θ of a locally finite Kac-Moody group is a lattice in the completion of the ambient Kac-Moody group, to arbitrary proper quasi-flips. See Theorem 6.2.7.

We conclude this section with some observations on building transitive quasi-flips.

Since G_θ preserves the partition of the building given in 5.1, building transitivity implies that there exists $w \in W$ such that $\mathcal{C}_+ = \mathcal{C}_w^\theta$ and hence also $\mathcal{C}_w^\theta = \mathcal{C}^\theta$. Accordingly building transitivity implies flip-flop transitivity and distance transitivity.

Another nice consequence of building transitivity is that it implies that the Weyl group is centralized, at least if the quasi-flip is proper, as the following lemma shows.

Lemma 5.1.13. *Suppose θ is a quasi-flip for which the θ-codistance is constant. If θ is proper, then it is a flip, i.e., it centralizes the Weyl group.*

Proof. By hypothesis, we have $\delta^\theta(c) = 1_W$ for all chambers c. Let c, d be two arbitrary s-adjacent chambers for some $s \in S$. Then $\theta(c)$ and $\theta(d)$ are $\theta(s)$-adjacent. Since $\delta^*(c, \theta(c)) = 1_W = \delta^*(d, \theta(d))$, Axiom (Tw2) implies $s = \theta(s)$. □

That the preceding lemma only applies to proper quasi-flips is not actually a severe restriction, as the following lemma illustrates.

Lemma 5.1.14. *Suppose θ is a quasi-flip of a twin building $\mathcal{C}$ of type (W, S). If $\mathcal{C}_+ = \mathcal{C}_w^\theta$ for some $w \in W$ (equivalently, if δ^θ is constant), then there exists a θ-stable spherical subset $K \subseteq S$ such that $w = w_K$. Moreover, θ induces the identity on $S \setminus K$ and all generators in K commute with all generators in $S \setminus K$.*

Proof. By Lemma 4.3.5, we know that there is a θ-stable spherical subset K of S such that w equals the longest word of K. Pick $s \in S$ and any chamber c. By Axiom (Tw3) there is a chamber d which is s-adjacent to c such that $\delta^*(d, \theta(c)) = sw \neq w$. Thus $\delta^\theta(d) \in \{sw, sw\theta(s)\}$. But by hypothesis, $\delta^\theta(d) = w$. We conclude that $sw\theta(s) = w$, or equivalently $\theta(s) = w^{-1}sw$ for all $s \in S$.

Suppose now that $s \notin K$. Since $\theta(s) = wsw = w_K s w_K$, the Exchange condition (resp. the Deletion condition) and the fact that K and hence $S \setminus K$ are θ-stable, implies that $\theta(s) = s$, and so $sw = ws$.

For all $v \in W_K$, we have $v \leq w_K = w$ in the Bruhat order. Thus clearly $svs \leq sws = w$ (see e.g. [Hum90, Proposition 5.9] and its proof). Consequently, $sws \in W_K$, and W_K is normal in W. This implies that K and S belong to two distinct components of the diagram of (W, S). □

Remark 5.1.15. In view of Lemma 5.1.14, irreducibility is necessary in Lemma 5.1.14. Otherwise, we easily produce counterexamples: Take a spherical twin building $\mathcal{C}$ admitting a proper quasi-flip θ for which all chambers are Phan chambers. Then take the product $\mathcal{C} \times \mathcal{C}$ of the twin building with itself, and define a quasi-flip $\theta' := \theta \times (-\operatorname{id})$ on the result (where by $(-\operatorname{id})$ we mean the map which interchanges a chamber c_+ with its "twin" c_- (cf. Example 1.6.8). Clearly θ' cannot be proper, yet δ^θ is constant and equal to $1_W \times w_0$, where w_0 is the longest element of the Weyl group of $\mathcal{C}$.

In fact, looking at the proof of the Lemma 5.1.14, this is all that can happen: If $\mathcal{C}_+ = \mathcal{C}^\theta_{w_K}$, then our building splits into two direct factors: One spherical factor of type K, on which our quasi-flip restricts to a "trivial" quasi-flip (coming from the identity, as in the previous paragraph), and one factor on which we get a proper building quasi-flip, which then is a flip by Lemma 5.1.13.

5.2. A local criterion for transitivity

It is well-known and easy to see that an adjacency-preserving action of a group G on a connected chamber system $\mathcal{C}$ over I is transitive if and only if there exists a chamber $c \in \mathcal{C}$ such that for each $i \in I$ the normalizer $\operatorname{Stab}_G(P_i(c))$ acts transitively on the i-panel $P_i(c)$ of $\mathcal{C}$ containing c. For completeness, we provide a short proof nevertheless.

Proposition 5.2.1. *Let $\mathcal{C}$ be a connected chamber system and let G be a group of automorphisms of $\mathcal{C}$. Then G acts transitively on $\mathcal{C}$ if and only if there exists a chamber $c \in \mathcal{C}$ such that for each panel P containing c the stabilizer $\operatorname{Stab}_G(P)$ acts transitively on P.*

Proof. It suffices to show that for any chamber d there exists $g \in G$ mapping c to d. Since $\mathcal{C}$ is connected, we can find a minimal gallery $c = c_0 \sim_{i_1} c_1 \sim_{i_2} c_2 \cdots c_{n-1} \sim_{i_n} c_n = d$ from c to d. We prove the claim by induction on n. For $n = 0$, we have $c = d$ and nothing has to be shown. Else, assume we know the claim holds for $n - 1$.

Then there exists $g_0 \in G$ mapping c to c_{n-1}. Let $d' := g_0^{-1}d$ be the preimage of d under this map. Then $c = g_0^{-1}c_{n-1} \sim_{i_n} g_0^{-1}d = d'$. By hypothesis, $\mathrm{Stab}_G(P_{i_n}(c))$ acts transitively on $P_{i_n}(c)$, so there exists $g_1 \in \mathrm{Stab}_G(P_{i_n}(c))$ mapping c to d'. Hence $g := g_0 g_1$ maps c to d as $g.c = (g_0 g_1).c = g_0.(g_1.c) = g_0.d' = d$. □

Corollary 5.2.2. *Let G be a group with a twin BN-pair, let $\mathcal{C}$ be the associated twin building. Suppose θ is a quasi-flip of G and G_θ the group of all elements fixed by θ. Then G_θ acts transitively on the positive (resp. negative) half of $\mathcal{C}$ if and only if there exists a chamber c such that for each panel P containing c the stabilizer $\mathrm{Stab}_{G_\theta}(P)$ acts transitively on P.*

Corollary 5.2.3. *Let G be a group with a twin BN-pair, let $\mathcal{C}$ be the associated twin building. Suppose θ is a quasi-flip of G and G_θ the group of all elements fixed by θ. Moreover, assume that the flip-flop system $\mathcal{C}^\theta$ is connected. Then G_θ acts transitively on $\mathcal{C}^\theta$ if and only if there exists a chamber $c \in \mathcal{C}^\theta$ such that for each panel P in $\mathcal{C}^\theta$ containing c the stabilizer $\mathrm{Stab}_{G_\theta}(P)$ acts transitively on P.*

5.3. Transitivity in rank 1

By the preceding section, it is natural to study transitivity properties of rank 1 groups, i.e., of Moufang sets. For this, we build on the work done in Chapter 3.

Specifically, for a nontrivial involutory automorphism θ of a Moufang set (briefly: a flip), two questions are of interest to us:

(1) When does G_θ act transitively on the flip-flop system $\mathcal{C}^\theta$? As we are looking at rank 1 (i.e., a Moufang set), $\mathcal{C}^\theta$ consists of all points not fixed by θ.

(2) When does the flip-flop system equal the whole Moufang set? This is of high relevance when studying Iwasawa decompositions.

In Section 5.3.1, we focus on projective lines over fields, the simplest kind of Moufang sets, associated to the group PSL_2 resp. SL_2. While being far from general, this already suffices to deal with flips of all locally split higher-rank groups, in particular, split algebraic and Kac-Moody groups. We will soon see that flips of projective lines correspond closely to sesquilinear forms, and the flip-flop system then to the anisotropic points of this form. This simple insight is the key to our analysis.

We also present some limited results on flips of projective lines over skew fields, using Moufang set techniques, in Section 5.3.2. This might serve as encouragement for future work in this direction.

The results presented in this section are based on joint work with Tom De Medts and Ralf Gramlich in [DMGH09], but have been extended.

5.3.1. Transitivity in rank 1: SL_2 and PSL_2

In this section, we closely follow the notation from Section 3.1. In particular, $\mathbb{F}$ is a field, G is $\mathrm{SL}_2(\mathbb{F})$ or $\mathrm{PSL}_2(\mathbb{F})$, and θ is an arbitrary nontrivial involutory automorphism (i.e., a flip) of G. Hence θ also induces a flip of the associated Moufang set, the projective line $\mathbb{P}^1(\mathbb{F}) \cong G/B_+$. Again, we denote by $V = \mathbb{F}^2$ the standard module of G.

In this setting, we obtain the following description of the flip-flop system associated to θ: We know that it consists of the set of 1-dimensional vector subspaces of V (corresponding to points of $\mathbb{P}^1(\mathbb{F})$) which are moved by θ. In Section 3.1, we saw that every flip is conjugate to a standard flip $\theta_{\delta,\sigma}$, which in turn is closely related to the sesquilinear form $f_{\delta,\sigma}$ (and its associated σ-quadratic form $q_{\delta,\sigma}$). For $G = \mathrm{SL}_2$, the algebraic version of this connection is that the fixed point group of the flip corresponds to the subgroup of elements preserving the form. Geometrically, the flip θ is the polarity induced by the form $f_{\delta,\sigma}$. Accordingly, the fixed points are those 1-dimensional subspaces which are isotropic with respect to this this form while the elements of the flip-flop system are the anisotropic 1-dimensional subspaces.

This observation together with the rest of Section 3.1 allows us to characterize when $\theta_{\delta,\sigma}$ is flip-flop transitive, i.e., when its fixed point group acts transitively on the anisotropic 1-dimensional subspaces.

Remark 5.3.1. In the following, all results are written with $G = \mathrm{PSL}_2(\mathbb{F})$ in mind. The corresponding results for $G = \mathrm{SL}_2(\mathbb{F})$ can be obtained by replacing $PK_{\delta,\sigma}$ by $K_{\delta,\sigma}$ and restricting ε to $+1$ in both statements and proofs.

Definition 5.3.2. Let $\mathbb{F}$ be a field with an automorphism σ of order at most 2. We define the **norm map** N_σ as follows: $N_\sigma : \mathbb{F} \to \mathrm{Fix}_{\mathbb{F}}(\sigma) : a \mapsto aa^\sigma$.

Thus for $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in V$ we have $q_{\delta,\sigma}(\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)) = N_\sigma(b) - \delta N_\sigma(a)$.

Lemma 5.3.3. *A flip-flop transitive flip of G is is conjugate to $\theta_{-\varepsilon,\sigma}$ with $\varepsilon \in \{+1, -1\}$.*

Proof. By Proposition 3.1.4, it suffices to deal with flips of the form $\theta_{\delta,\sigma}$. The flip-flop system corresponds to the 1-dimensional subspaces which are anisotropic with respect to $q_{\delta,\sigma}$. Clearly $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ are anisotropic.

By the assumed transitivity of $PK_{\delta,\sigma}$, there exists $g \in PK_{\delta,\sigma}$ such that $g\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$ for some $x \in \mathbb{F}^*$. Since $PK_{\delta,\sigma}$ preserves the form $q_{\delta,\sigma}$ up to a sign $\varepsilon \in \{+1, -1\}$, we have

$$1 = q_{\delta,\sigma}(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)) = \varepsilon q_{\delta,\sigma}(g\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)) = \varepsilon q_{\delta,\sigma}(\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)) = -\varepsilon\delta N_\sigma(x),$$

thus $\delta = -\varepsilon N_\sigma(x^{-1})$. Let $X := \left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)$, and set $\mathrm{Int}_X(g) := XgX^{-1}$, the inner automorphism induced by X. Then $\mathrm{Int}_X \circ \theta_{\delta,\sigma} \circ \mathrm{Int}_X^{-1}$ equals $\theta_{-\varepsilon,\sigma}$. □

By the preceding lemma, it suffices to determine when $\theta_{1,\sigma}$ and $\theta_{-1,\sigma}$ are flip-flop transitive.

Proposition 5.3.4. *The flip $\theta_{-1,\sigma}$ (resp. $\theta_{+1,\sigma}$) is flip-flop transitive if and only if the sum (resp. difference) of any two norms is ε times a norm, for $\varepsilon \in \{+1,-1\}$.*

Proof. We present the argument for $\delta = -1$; the case $\delta = +1$ works completely analogously.

Assume $\theta_{-1,\sigma}$ is flip-flop transitive. Take an arbitrary anisotropic vector $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathbb{F}^2$. By transitivity, there exists $g \in PK_{-1,\sigma}$ such that $g\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right)$ for some $x \in \mathbb{F}^*$. Since $PK_{\delta,\sigma}$ preserves the form q up to a sign $\varepsilon \in \{+1,-1\}$, we have

$$N_\sigma(a) + N_\sigma(b) = -\delta N_\sigma(a) + N_\sigma(b) = q_{\delta,\sigma}(\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)) = \varepsilon q_{\delta,\sigma}(g\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)) = \varepsilon q_{\delta,\sigma}(\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right)) = \varepsilon N_\sigma(x),$$

proving that a sum of two norms is ε times a norm.

Conversely, suppose that the sum of two arbitrary norms is known to be ε times a norm. It suffices to show that for any anisotropic vector $v = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathbb{F}^2$ there exists $g \in PK_{1,\sigma}$ mapping v to a nonzero multiple of $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Choose x such that $\varepsilon N_\sigma(x) = N_\sigma(a) + N_\sigma(b) = q_{\delta,\sigma}(v)$ for $\varepsilon \in \{+1,-1\}$, and note that $x \neq 0$ since v is anisotropic. Thus the equation

$$\begin{pmatrix} \frac{b}{x} & -\frac{a}{x} \\ -\varepsilon\left(\frac{a}{x}\right)^\sigma & \varepsilon\left(\frac{b}{x}\right)^\sigma \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

finishes the proof, as the matrix on the left hand side of the equation is in $PK_{-1,\sigma}$. $\square$

Example 5.3.5. Consider $G = \mathrm{PSL}_2(\mathbb{R})$. Then both $\theta_{-1,\mathrm{id}}$ and $\theta_{+1,\mathrm{id}}$ are flip-flop transitive: In the first case, we have to verify that the sum of two squares is again a square, which is certainly true (so this flip is in fact also flip-flop transitive over $\mathrm{SL}_2(\mathbb{R})$). In the second case, we readily see that the difference of two squares is always either a square or minus a square, hence by the the preceding proposition, this flip is also flip-flop transitive (but only over $\mathrm{PSL}_2(\mathbb{F})$, not over $\mathrm{SL}_2(\mathbb{R})$). However, the two flips are certainly not conjugate, because for the first one, all points are anisotropic (and the flip-flop system equals the projective line) while in the second case $\langle\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\rangle$ and $\langle\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)\rangle$ are isotropic points.

In the special case that -1 is a norm, we can refine this requirement a bit. The proof for the following Proposition was partially inspired by [AG06, Lemma 4.2]. Recall from Proposition 3.1.5, if two standard flips $\theta_{\delta,\sigma}$ and $\theta_{\varepsilon,\tau}$ are conjugate, then there exists $\rho \in \mathrm{Aut}(\mathbb{F})$ such that $\sigma = \rho\tau\rho^{-1}$. An immediate consequence is that $N_\sigma(\mathbb{F}) = (N_\tau(\mathbb{F}))^\rho$. Therefore, the property "-1 is a norm" is well-defined for an arbitrary flip of SL_2 or PSL_2.

Proposition 5.3.6. *Consider a flip θ conjugate to some standard involution $\theta_{\delta,\sigma}$, and suppose -1 is a norm with respect to θ. Then θ is conjugate to $\theta_{-1,\sigma}$. Moreover, θ is flip-flop transitive if and only if the norm map N_σ is surjective (onto $\mathrm{Fix}_{\mathbb{F}}(\sigma)$), or $\sigma = \mathrm{id}$ and $\mathrm{char}\,\mathbb{F} = 2$.*

Proof. By Propositions 3.1.4 and 5.3.4, our flip is conjugate to $\theta_{-1,\sigma}$ or $\theta_{+1,\sigma}$. Since -1 is a norm, these two are actually conjugate as well.

Suppose our flip is flip-flop transitive. By Proposition 5.3.4 and since -1 is a norm, sums and differences of norms are again norms. Moreover, products and quotients of (nonzero) norms are norms. We conclude that the norms $N_\sigma(\mathbb{F})$ form a subfield of $\mathrm{Fix}_{\mathbb{F}}(\sigma)$.

Pick $x \in \mathbb{F}$ such that $x^\sigma \neq -x$ (this is always possible unless $\sigma = \mathrm{id}$ and $\operatorname{char}\mathbb{F} = 2$). Since the norms form a subfield, for any $y \in \mathrm{Fix}_{\mathbb{F}}(\sigma)$ we have that

$$N_\sigma(xy+1) - N_\sigma(xy) - N_\sigma(1) = xy + (xy)^\sigma = (x + x^\sigma)y$$

is again a norm. Since $(x + x^\sigma) \neq 0$, we conclude that $\mathrm{Fix}_{\mathbb{F}}(\sigma) = N_\sigma(\mathbb{F})$.

Conversely, if $\sigma = \mathrm{id}$ and $\operatorname{char}\mathbb{F} = 2$, then the norms (i.e., squares) form a subfield of $\mathbb{F}$ (as squaring is a field endomorphism in characteristic 2). In particular, their sums are again norms. On the other hand, if the norm function is surjective, we have $\mathrm{Fix}_{\mathbb{F}}(\sigma) = N_\sigma(\mathbb{F})$, and thus the sum of norms is a norm. In either case, by Proposition 5.3.4, $\theta_{-1,\delta}$ is flip-flop transitive. $\square$

Remark 5.3.7. If $\sigma = \mathrm{id}$ and $\operatorname{char}\mathbb{F} \neq 2$, then the norm map is surjective if and only if $\mathbb{F}$ is quadratically closed.

For finite fields with nontrivial field automorphism of order 2 it is well-known that the associated norm map is always surjective.

We now turn to the second question: When is a flip building transitive, i.e., when does its centralizer act transitively on the whole projective line? This is equivalent to asking when the associated form is anisotropic, which immediately rules out all finite fields $\mathbb{F}_q$ with $q \not\equiv 3 \bmod 4$ (because then, -1 is a square). To be precise, we obtain the following:

Proposition 5.3.8. *A flip is transitive on the whole projective line if and only if it is conjugate to $\theta_{-1,\sigma}$, the sum of two norms is ε times a norm for $\varepsilon \in \{+1, -1\}$, and -1 is not a norm.*

Proof. Let θ be a flip. We have to verify two things: The flip-flop system must equal the whole projective line, and the flip must be flip-flop transitive.

By Lemma 5.3.3 transitivity implies that our flip is conjugate to either $\theta_{-1,\sigma}$ or $\theta_{+1,\sigma}$. For the latter, $\binom{1}{1}$ is isotropic while $\binom{1}{0}$ is not, hence we cannot have a transitive action on the whole projective line preserving the form. So θ must be conjugate to $\theta_{-1,\sigma}$.

If -1 was a norm, say $-1 = N_\sigma(x)$, then the vector $\binom{1}{x}$ would be isotropic, and the flip-flop system would not equal the whole projective line. Conversely, if there is a nonzero isotropic vector $\binom{a}{b}$, then $0 = N_\sigma(a) + N_\sigma(b)$, and w.l.o.g. $b \neq 0$, hence $N_\sigma(\frac{a}{b}) = -1$.

The claim now follows from Proposition 5.3.4. $\square$

5.3.2. Transitivity in rank 1: Moufang flips

This section builds on the work done in Section 3.2 and the basic setup presented in Section 1.10, and deals with the transitivity of a restricted class of flips of the Moufang set associated to PSL_2 over a skew field. It is based on [DMGH09, Section 5]. The results have been obtained independently by the author of the present thesis using matrix computations: Essentially, one can extend the work done in the previous section and in Section 3.1 to skew fields, but the computations become a lot more involved without providing a major new insight. However, we prefer to include this Moufang set based approach as it may be more conceptual than the one based on matrix computations. Additionally, it illustrates how Moufang set theory can help in solving this problem.

Definition 5.3.9. If $\tau^2 = \mathrm{id}$, then $\varphi = 1$ is a flip automorphism. We will call the corresponding automorphism θ_1 of G (as defined in Section 3.2) the **obvious flip**. Observe that θ_1 is just conjugation by τ.

Definition 5.3.10. A flip automorphism $\varphi \in \mathrm{Aut}(U)$ is called **fully transitive** if the group G_{θ_φ} is transitive on X.

Let $\mathbb{M} = \mathbb{M}(U, \tau)$ be a Moufang set with $\tau^2 = \mathrm{id}$. Then the obvious flip θ_1 is fully transitive if and only if $C_G(\tau)$ is transitive on X because $G_{\theta_1} = C_G(\tau)$.

Lemma 5.3.11. *Let $\mathbb{M} = \mathbb{M}(U, \tau)$ be a Moufang set with $\tau^2 = \mathrm{id}$, and assume that the obvious flip is fully transitive. Then τ has no fixed points.*

Proof. Assume that $a\tau = a$ for some $a \in U^*$. Let $g \in C_G(\tau)$ be such that $0g = a$. Then $\infty g = 0\tau g = 0g\tau = a\tau = a = 0g$ and hence $\infty = 0$, a contradiction. □

We now examine the transitivity of the obvious flip for $\mathbb{M}(\mathbb{D})$ where $\mathbb{D}$ is an arbitrary skew field.

Definition 5.3.12. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{D})$, then the Dieudonné determinant $\det(g) \in \mathbb{D}^*/[\mathbb{D}^*, \mathbb{D}^*]$ is defined as

$$\det(g) := \begin{vmatrix} a & b \\ c & d \end{vmatrix} := \begin{cases} ad - aca^{-1}b & \text{if } a \neq 0\,; \\ -cb & \text{if } a = 0\,; \end{cases}$$

see [Die43]. Then $\mathrm{SL}_2(\mathbb{D})$ is precisely the kernel of the Dieudonné determinant, i.e., a matrix $g \in \mathrm{GL}_2(\mathbb{D})$ lies in $\mathrm{SL}_2(\mathbb{D})$ if and only if $\det(g) \in [\mathbb{D}^*, \mathbb{D}^*]$. Also observe that $\det(\lambda g) \equiv \det(g\lambda) \equiv \lambda^2 \det(g) \bmod [\mathbb{D}^*, \mathbb{D}^*]$ for all $\lambda \in \mathbb{D}^*$.

Lemma 5.3.13. *Let $G = \mathrm{SL}_2(\mathbb{D})$ and let $\tau = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in G$. Then*

$$C_G(\tau) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; \begin{matrix} a^2 + aba^{-1}b \in [\mathbb{D}^*, \mathbb{D}^*] & \text{if } a \neq 0 \\ b^2 \in [\mathbb{D}^*, \mathbb{D}^*] & \text{if } a = 0 \end{matrix} \right\};$$

$$PC_G(\tau) = \left\{ \begin{pmatrix} \epsilon a & \epsilon b \\ -b & a \end{pmatrix} \;\middle|\; \begin{matrix} \epsilon \cdot (a^2 + aba^{-1}b) \in [\mathbb{D}^*, \mathbb{D}^*] & \text{if } a \neq 0 \\ \epsilon \cdot b^2 \in [\mathbb{D}^*, \mathbb{D}^*] & \text{if } a = 0 \end{matrix} \quad \text{where } \epsilon = \pm 1 \right\}.$$

Proof. This is a straightforward calculation. $\square$

Proposition 5.3.14. *Let $G = \mathrm{SL}_2(\mathbb{D})$ and let $\tau = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in G$. Let X be the projective line over $\mathbb{D}$, i.e., $X = \{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\mathbb{D} \neq 0 \mid a, b \in \mathbb{D}\}$. Then the following are equivalent:*

(1) $C_G(\tau)$ is transitive on X;

(2) $a^2 + aba^{-1}b \in (\mathbb{D}^*)^2\ [\mathbb{D}^*, \mathbb{D}^*]$ for all $a, b \in \mathbb{D}^*$;

(3) $1 + a^2 \in (\mathbb{D}^*)^2\ [\mathbb{D}^*, \mathbb{D}^*]$ for all $a \in \mathbb{D}^*$.

Proof. Since $a^2 + aba^{-1}b = a^2(1 + a^{-1}ba^{-1}b)$, we have $a^2 + aba^{-1}b \in (\mathbb{D}^*)^2\ [\mathbb{D}^*, \mathbb{D}^*]$ if and only if $1 + a^{-1}ba^{-1}b \in (\mathbb{D}^*)^2\ [\mathbb{D}^*, \mathbb{D}^*]$. Equivalence between (ii) and (iii) follows by replacing $a^{-1}b$ by a in the latter term.

Assume now that (ii) holds. Let $a, b \in \mathbb{D}^*$ be arbitrary; we want to show that there exists some $g \in C_G(\tau)$ mapping $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)$ for some $z \in \mathbb{D}^*$. By (ii), we know that there is some $c \in \mathbb{D}^*$ such that $b^{-2} + b^{-1}a^{-1}ba^{-1} \equiv c^{-2} \bmod [\mathbb{D}^*, \mathbb{D}^*]$. Let $g := \left(\begin{smallmatrix} cb^{-1} & ca^{-1} \\ -ca^{-1} & cb^{-1} \end{smallmatrix}\right)$. Then $\det(g) \equiv c^2(b^{-2} + b^{-1}a^{-1}ba^{-1}) \equiv 1$, i.e., $g \in G$. Moreover, $g\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)$ for $z = c(b^{-1}a + a^{-1}b)$, proving that $C_G(\tau)$ acts transitively on X.

Conversely, assume that $C_G(\tau)$ acts transitively on X. Let $a, b \in \mathbb{D}^*$ be arbitrary; then there exists some $g \in C_G(\tau)$ mapping $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\mathbb{D}$ to $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\mathbb{D}$, i.e., there is some $z \in \mathbb{D}^*$ such that g maps $\left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$. By Lemma 5.3.13, we know that g has the form $g = \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)$ with $x^2 + xyx^{-1}y \in [\mathbb{D}^*, \mathbb{D}^*]$. Then $g\left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} xz \\ -yz \end{smallmatrix}\right)$, and hence $a = xz$ and $b = -yz$. Hence $a^2 + aba^{-1}b = xzxz + xzyx^{-1}yz = xzx^{-1} \cdot (x^2 + xyx^{-1}y) \cdot z$, and since $x^2 + xyx^{-1}y \in [\mathbb{D}^*, \mathbb{D}^*]$, this implies $a^2 + aba^{-1}b \equiv xzx^{-1}z \equiv z^2 \bmod [\mathbb{D}^*, \mathbb{D}^*]$. Since $a, b \in \mathbb{D}^*$ were arbitrary, this proves (ii). $\square$

Proposition 5.3.15. *Let $G = \mathrm{PSL}_2(\mathbb{D})$, let $\tau = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{D})$, and let $\tilde{\tau}$ be the image of τ in G. Let X be the projective line over $\mathbb{D}$, i.e., $X = \{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\mathbb{D} \mid a, b \in \mathbb{D}, \text{not both zero}\}$. Then the following are equivalent:*

(1) $C_G(\tilde{\tau})$ is transitive on X;

(2) $PC_G(\tau)$ is transitive on X;

(3) $a^2 + aba^{-1}b \in \{\pm 1\} \cdot (\mathbb{D}^*)^2\ [\mathbb{D}^*, \mathbb{D}^*]$ for all $a, b \in \mathbb{D}^*$;

(4) $1 + a^2 \in \{\pm 1\} \cdot (\mathbb{D}^*)^2\ [\mathbb{D}^*, \mathbb{D}^*]$ for all $a \in \mathbb{D}^*$.

Proof. The equivalence between (i) and (ii) follows immediately from the definition of the projective centralizer $PC_G(\tau)$. The other equivalences are shown exactly as in the proof of Proposition 5.3.14 above. $\square$

Corollary 5.3.16. (i) *Let $G = \mathrm{SL}_2(\mathbb{D})$, and assume that for all $a \in \mathbb{D}^*$, we have $1 + h_a \in H$. Then $C_G(\tau)$ acts transitively on X.*

(ii) *Let $G = \mathrm{PSL}_2(\mathbb{D})$, and assume that for all $a \in \mathbb{D}^*$ we have $1 + h_a \in \{\pm 1\} \cdot H$. Then $C_G(\tilde{\tau})$ acts transitively on X.*

Proof. We only show (i). The proof of (ii) is completely similar. So let $a \in \mathbb{D}^*$ be arbitrary, and assume that $1 + h_a = h \in H$. Then $1 + 1h_a = 1h$, i.e., $1 + a^2 = 1h$. Write $h = h_{x_1} \cdots h_{x_n}$ with $x_1, \ldots, x_n \in \mathbb{D}^*$. Then $1h = x_n \cdots x_1 \cdot 1 \cdot x_1 \cdots x_n \equiv (x_1 \cdots x_n)^2 \bmod [\mathbb{D}^*, \mathbb{D}^*]$, and hence $1 + a^2 = 1h \in (\mathbb{D}^*)^2\ [\mathbb{D}^*, \mathbb{D}^*]$. So (iii) of Proposition 5.3.14 holds, and therefore the group $C_G(\tau)$ acts transitively on X. □

A natural extension of the study of the obvious flip would be to study its close relatives, the *semi-obvious* flips, which are obtained by composing the obvious flip with an automorphism or anti-automorphism of $\mathbb{D}$.

5.4. Iwasawa decompositions

The work in this section is based on joint work Tom De Medts and Ralf Gramlich in [DMGH09].

The Iwasawa decomposition of a connected semisimple complex Lie group or a connected semisimple split real Lie group is one of the most fundamental observations of classical Lie theory. It implies that the geometry of a connected semisimple complex resp. split real Lie group G is controlled by any maximal compact subgroup K. Examples are Weyl's unitarian trick in the representation theory of Lie groups, or the transitive action of K on the Tits building G/B. In the case of the connected semisimple split real Lie group of type G_2 the latter implies the existence of an interesting epimorphism from the real building of type G_2, the split Cayley hexagon, onto the real building of type A_2, the real projective plane, by means of the epimorphism $\mathrm{SO}_4(\mathbb{R}) \to \mathrm{SO}_3(\mathbb{R})$, cf. [Gra98]. This epimorphism cannot be described using the group of type G_2 because it is quasisimple.

To be able to transfer these ideas to a broader class of groups, we extend the notion of an Iwasawa decomposition in the following way:

Definition 5.4.1. A group G with a twin BN-pair (B_+, B_-, N) admits an **Iwasawa decomposition** if there exists a proper building-transitive quasi-flip θ of G.

In other words, G admits an Iwasawa decomposition if there exists $\theta \in \mathrm{Aut}(G)$ which maps some positive Borel group to an opposite one, and such that moreover $G = G_\theta B_+$ where G_θ is the centralizer of θ in G.

Our interest in Iwasawa decompositions stems from the presentation by generators and relations (in the non-finitely presented case usually formulated as a universal enveloping result of an amalgam) of an arbitrary group acting with a fundamental domain on some simply connected simplicial complex, which is implied by Tits' Lemma [Pas85, Lemma 5], [Tit86, Corollary 1]. The transitive action of a compact real form of a complex Lie group or complex Kac-Moody group on the associated complex building gives particularly nice presentations as studied in [GGH] and [Gra06]; the compact real form is the universal enveloping group of the amalgam consisting of the rank 1 and 2 subgroups with respect to a system of fundamental roots. The following theorem is the main amalgamation result of the present work.

Theorem 5.4.2 (joint work with Gramlich and De Medts, see [DMGH09])**.** *Consider a centered RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ with an involution θ such that $G = G_\theta B$ is an Iwasawa decomposition of G (cf. Definition 5.4.1). Furthermore, let Π be a system of fundamental roots of Φ and for $\{\alpha, \beta\} \subseteq \Pi$ let $X_{\alpha,\beta} := \langle U_\alpha, U_{-\alpha}, U_\beta, U_{-\beta}\rangle$.*

Then θ induces an involution on each $X_{\alpha,\beta}$ and G_θ is the universal enveloping group of the amalgam $((X_{\alpha,\beta})_\theta)_{\{\alpha,\beta\}\subseteq\Pi}$ of fixed point subgroups of the groups $X_{\alpha,\beta}$.

Proof. By Lemma 5.1.13 the involution θ induces an involution of each group $X_{\alpha,\beta}$. By the Iwasawa decomposition the group G_θ acts with a fundamental domain on the simplicial complex Δ associated to G/B, the flag complex of G/B. Choose F to be a fundamental domain of Δ stabilized by the torus T of G, so that the stabilizers of the simplices of F of dimension 0 and one with respect to the natural action of G on Δ are exactly the groups $(X_\alpha)_\theta T$ and $(X_{\alpha\beta})_\theta T$. By the simple connectedness of building geometries of rank at least 3 (cf. [Bro89, Theorem IV.5.2] or [Tit74, Theorem 13.32]) and Tits' Lemma (see e.g. [Pas85, Lemma 5], [Tit86, Corollary 1]) the group G_θ equals the universal enveloping group of the amalgam $((X_{\alpha\beta})_\theta T)_{\alpha,\beta\in\Pi}$. Finally, by [GLS95, Lemma 29.3] the torus T can be reconstructed from the rank 2 tori $T_{\alpha\beta}$, $\alpha, \beta \in \Pi$, and so the group G actually equals the universal enveloping group of the amalgam $((X_{\alpha\beta})_\theta)_{\alpha,\beta\in\Pi}$. □

Iwasawa decompositions have been studied for all kinds of groups (cf. [Bel], [Krö]) and over real closed fields (cf. [Gro72]). In this section we characterize the fields $\mathbb{F}$ for which a group with an $\mathbb{F}$-locally split root group datum admits an Iwasawa decomposition, cf. Definition 5.4.1 and Theorem 5.4.7. We point out that this class of groups contains the class of groups of $\mathbb{F}$-rational points of a connected reductive algebraic group defined over $\mathbb{F}$ (cf. [Spr98]) as well as the class of split Kac-Moody groups over $\mathbb{F}$ (cf. [Rém02], [Tit87]).

For the next definition recall that any Cartan–Chevalley involution of $(\mathrm{P})\mathrm{SL}_2(\mathbb{F})$ is given, resp. induced by the transpose-inverse automorphism with respect to the choice of a basis of the natural $\mathrm{SL}_2(\mathbb{F})$-module $\mathbb{F}^2$.

Definition 5.4.3. Let $\mathbb{F}$ be a field, let σ be an automorphism of $\mathbb{F}$ of order at most 2, let $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ be an $\mathbb{F}$-locally split RGD-system. We call an automorphism θ of G a **$\boldsymbol{\sigma}$-twisted Chevalley involution** of G if it satisfies for all $\alpha \in \Phi$:

(1) $\theta^2 = \mathrm{id}_G$,

(2) $U_\alpha^\theta = U_{-\alpha}$, and

(3) $\theta \circ \sigma$ induces the standard Chevalley involution (resp. its image under the canonical projection) on $X_\alpha := \langle U_\alpha, U_{-\alpha}\rangle \cong (\mathrm{P})\mathrm{SL}_2(\mathbb{F})$.

All split Kac-Moody groups admit σ-twisted Chevalley involutions (in particular, the classical Chevalley involution and its twist under an involutory field automorphism) by combining a sign automorphism with a field automorphism, see [CM05, Section 8.2]. Likewise for all split reductive algebraic groups. Also groups with 2-spherical $\mathbb{F}$-locally split root group datum over a field with at least four elements meet this condition:

Lemma 5.4.4. *Let $\mathbb{F}$ be a field with at least four elements, let σ be an automorphism of $\mathbb{F}$ of order at most 2, let $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ be a centered, 2-spherical and $\mathbb{F}$-locally split RGD-system. Then G admits a σ-twisted Chevalley involution.*

Proof. By [AM97] (and also by the unpublished manuscript [Müh96]) the group G is a universal enveloping group of the amalgam $\bigcup_{\alpha,\beta\in\Pi} X_{\alpha,\beta}$ for a system Π of fundamental roots of Φ, so that any automorphism of $\bigcup_{\alpha,\beta\in\Pi} X_{\alpha,\beta}$ induces an automorphism of G. For each pair $\alpha, \beta \in \Pi$ the Chevalley involution of the split reductive algebraic group $X_{\alpha,\beta}$ composed with σ induces automorphisms θ_α on X_α and θ_β on X_β satisfying the criteria for a σ-twisted Chevalley involution. Therefore there exists an involution of the amalgam $\bigcup_{\alpha,\beta\in\Pi} X_{\alpha,\beta}$ inducing θ_α on X_α for each $\alpha \in \Phi$. Consequently there exists an involution θ on its universal enveloping group G inducing θ_α on each subgroup X_α. This involution θ of G by construction is a σ-twisted Chevalley involution of G. $\square$

What makes σ-twisted Chevalley involutions interesting is that they are flips. In particular they centralize the Weyl group. Hence we can apply our full machinery to them.

Proposition 5.4.5. *Any σ-twisted Chevalley involution θ of a group G is a BN-flip.*

Proof. By definition, θ is an involution. Furthermore, the Borel subgroup B_+ is generated by T and the set of root groups associated to the positive root system $\Phi_+ \subset \Phi$. More precisely, $B = T.\langle U_\alpha \mid \alpha \in \Phi_+\rangle$. Since $T = \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$ by [CR08, Corollary 5.3], the involution θ stabilizes T and maps B_+ to $B_- = T.\langle U_{-\alpha} \mid \alpha \in \Phi_+\rangle$. Finally, θ acts trivially on $W = N/T$ as each root α of the root lattice of W is mapped onto its negative $-\alpha$, which means that the reflection given by α is mapped onto the reflection given by $-\alpha$, which is identical to the reflection given by α. $\square$

The following corollary is a direct consequence of Proposition 5.3.8, once applied to SL_2 (by restricting ε to 1), and once to PSL_2.

Corollary 5.4.6 (joint work with De Medts and Gramlich)**.** *The group $\mathrm{PSL}_2(\mathbb{F})$ resp. $\mathrm{SL}_2(\mathbb{F})$ admits an Iwasawa decomposition if and only if $\mathbb{F}$ admits an automorphism σ of order at most 2 such that*

(1) -1 is not a norm, and

(2) a sum of norms is a norm (in the $\mathrm{SL}_2(\mathbb{F})$ case), resp. a sum of norms is ε times a norm, where $\varepsilon \in \{+1, -1\}$ (in the $\mathrm{PSL}_2(\mathbb{F})$ case),

with respect to the norm map $N_\sigma : \mathbb{F} \to \mathrm{Fix}_{\mathbb{F}}(\sigma) : x \mapsto xx^\sigma$. □

We finally have assembled all tools required to prove our main result in this section.

Theorem 5.4.7 (joint work with Gramlich and De Medts)**.** *Let* $\mathbb{F}$ *be a field and let* $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ *be an* $\mathbb{F}$*-locally split RGD-system. The group* G *admits an Iwasawa decomposition if and only if* $\mathbb{F}$ *admits an automorphism* σ *of order at most* 2 *such that*

(1) -1 *is not a norm, and*

(2) (i) if there exists a rank 1 *subgroup* $\langle U_\alpha, U_{-\alpha}\rangle$ *of* G *isomorphic to* $\mathrm{SL}_2(\mathbb{F})$*, then a sum of norms is a norm, or*

(ii) if each rank 1 *subgroup* $\langle U_\alpha, U_{-\alpha}\rangle$ *of* G *is isomorphic to* $\mathrm{PSL}_2(\mathbb{F})$*, then a sum of norms is* ± 1 *times a norm,*

with respect to the norm map $N_\sigma : \mathbb{F} \to \mathrm{Fix}_{\mathbb{F}}(\sigma) : x \mapsto xx^\sigma$*, and*

(3) G *admits a* σ*-twisted Chevalley involution.*

Proof. Assume the existence of an Iwasawa decomposition of G. By definition there exists an involution θ of G such that $G = G_\theta B_+$. Hence any Borel subgroup of G is mapped onto an opposite one, so that by Lemma 5.1.13 the involution θ centralizes the Weyl group N/T and, for any simple root α, normalizes the group $X_\alpha := \langle U_\alpha, U_{-\alpha}\rangle$, which by $\mathbb{F}$-local splitness is isomorphic to (P)$\mathrm{SL}_2(\mathbb{F})$. In particular the restriction $\theta_{|X_\alpha}$ of θ to X_α is a BN-flip.

We now argue that this restricted BN-flip induces an Iwasawa decomposition of X_α. Let P_α be the panel of the building corresponding to the root α. By Corollary 5.2.3 we know that $(G_{P_\alpha})_\theta = G_{P_\alpha} \cap G_\theta$ acts transitively on P_α, and it remains to show that this is also the case for $(X_\alpha)_\theta = X_\alpha \cap G_\theta$. First observe that $P_{-\alpha} = \theta(P_\alpha)$ and hence $(G_{P_\alpha})_\theta$ also stabilizes the panel $P_{-\alpha}$. For, if $g \in (G_{P_\alpha})_\theta$, then $g.P_{-\alpha} = g.\theta(P_\alpha) = \theta(g.P_\alpha) = \theta(P_\alpha) = P_{-\alpha}$ and so $g \in (G_{P_\alpha})_\theta = G_{P_\alpha} \cap G_{P_{-\alpha}} \cap G_\theta$. If $x \in (G_{P_\alpha})_\theta$ stabilizes the chamber B_+ in P_α, then $x.B_- = x.\theta(B_+) = \theta(x.B_+) = \theta(B_+) = B_-$. We conclude that $x \in B_+ \cap B_- = T$. Moreover, the group $U_\alpha < X_\alpha$ stabilizes B_+ and acts transitively on $P_{-\alpha}$. Thus, in fact $(G_{P_\alpha})_\theta = (X_\alpha T) \cap G_\theta$. Any $t \in T \setminus X_\alpha$ acts trivially on P_α. Hence, since $(G_{P_\alpha})_\theta$ acts transitively on P_α, so does $(X_\alpha)_\theta$. Accordingly X_α admits an Iwasawa decomposition.

Therefore, by Corollary 5.4.6 below, the field $\mathbb{F}$ admits an automorphism σ with the required properties.

For the converse implication, let θ be the σ-twisted Chevalley involution of G. For each $\alpha \in \Phi$ the involution θ induces a BN-flip θ_α on X_α. By Proposition 5.3.8 below, these induced flips are transitive. Hence by Corollary 5.2.3, we have $G = G_\theta B_+$, proving that G admits an Iwasawa decomposition. □

Corollaries 6.1.6 and 6.2.4 specialize this theorem to the case of algebraic and Kac-Moody groups.

Remark 5.4.8. All split rank 2 groups are known. This follows from the classification of Moufang polygons (see [TW02] and also the enumeration in Section 2.6), but also more elementary by pre-classification results (e.g. by results on Chevalley groups, see [Ste68b]). In particular, their rank 1 groups are not isomorphic to $\mathrm{PSL}_2(\mathbb{F})$, except for $\mathrm{PSL}_2(\mathbb{F}) \times \mathrm{PSL}_2(\mathbb{F})$ or $\mathrm{PSL}_2(\mathbb{F}) \times \mathrm{SL}_2(\mathbb{F})$.

Thus, if all rank 1 groups are isomorphic to $\mathrm{PSL}_2(\mathbb{F})$, then we can deduce that the diagram of the group must be right angled, i.e., any two nodes are either not joined by an edge, or by an edge with infinity as label. Such examples can be obtained by taking arbitrary direct products of $\mathrm{PSL}_2(\mathbb{F})$ with itself, or using certain free constructions (see e.g. [CR08, Example 2.8] or for more details, [RR06]).

In view of the above remark, we obtain the following corollary:

Corollary 5.4.9. *Let $\mathbb{F}$ be a field, let $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ be an $\mathbb{F}$-locally split 2-spherical RGD-system without isolated nodes in the diagram. The group G admits an Iwasawa decomposition if and only if $\mathbb{F}$ admits an automorphism σ of order at most 2 such that*

(1) -1 is not a norm (in particular, $\operatorname{char}\mathbb{F} \neq 2$) and,

(2) a sum of norms is a norm,

with respect to the norm map $N_\sigma : \mathbb{F} \to \mathrm{Fix}_{\mathbb{F}}(\sigma) : x \mapsto xx^\sigma$, and

(3) G admits a σ-twisted Chevalley involution.

5.4.1. Fields admitting Iwasawa decompositions

Besides the widely known Iwasawa decompositions over real closed fields (see [Gro72]) and the field of complex numbers there exist lots of fields admitting automorphisms that satisfy the conditions of Corollary 5.4.6. Note that any pythagorean formally real field $\mathbb{F}$ satisfies the conditions of Corollary 5.4.6 with respect to the identity automorphism as does $\mathbb{F}[\sqrt{-1}]$ with respect to the nontrivial Galois automorphism. In the $\mathrm{PSL}_2(\mathbb{F})$ case the finite fields $\mathbb{F}_q$ with $q \equiv 3 \bmod 4$ yield additional examples.

Quite a number of properties of pythagorean and formally real fields are known, see [Lam73], [Lam05], [Raj93].

Remark 5.4.10. (1) A field is formally real pythagorean if and only if its Witt group is torsionfree, see [Lam05, Theorem VIII.4.1].

(2) A field is formally real pythagorean if and only if it is the intersection of a nonempty family of euclidean subfields of its algebraic closure, see [Lam05, Theorem VIII.4.4].

(3) If a field $\mathbb{F}$ is formally real pythagorean, then so is the field $\mathbb{F}((t))$ of formal Laurent series, see [Raj93, Theorem 18.9].

(4) If a field $\mathbb{F}$ is real closed, then the field $\mathbb{F}((t_1))\cdots((t_n))$ is pythagorean and has 2^{n+1} square classes, see [Raj93, Theorem 18.9].

(5) If $\mathbb{F}$ is pythagorean but not formally real, then $\mathbb{F}$ is quadratically closed, see [Raj93, Theorem 16.4]. In particular, the intersection of the real numbers with the field of the numbers which are constructible with straightedge and compass, is pythagorean and formally real.

(6) If $\mathbb{F}$ is a field in which -1 is not a square, then it is pythagorean (and hence formally real) if and only if $\mathbb{F}$ does not admit any cyclic extension of order 4, see [DD65].

Inspired by classical Lie theory and the passage from complex Lie groups to their split real forms, the question arises whether an Iwasawa decomposition $G = G_\theta B$ of a group G with an $\mathbb{F}$-locally split root group datum with respect to an involution θ involving a nontrivial field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ always implies the existence of an Iwasawa decomposition over the field $\mathrm{Fix}_{\mathbb{F}}(\sigma)$ with respect to an involution involving the trivial field automorphism on $\mathrm{Fix}_{\mathbb{F}}(\sigma)$. The following example shows that this is generally not the case.

Example 5.4.11. Let $\mathbb{F}$ be a formally real field which is not pythagorean and admits four square classes. Such fields exist, see for example [Szy75]. This means that exactly two square classes contain absolutely positive elements, so that there exists a unique ordering. Choose a positive non-square element $w \in \mathbb{F}$. Set $\alpha := \sqrt{-w}$ and $\tilde{\mathbb{F}} := \mathbb{F}[\alpha]$. Then

$$N(x_0 + \alpha x_1) + N(y_0 + \alpha y_1) = x_0^2 + wx_1^2 + y_0^2 + wy_1^2,$$

which is a non-negative number, hence either a square or a square multiple of w. Hence there exist z_0 and z_1 in $\mathbb{F}$ such that

$$N(x_0 + \alpha x_1) + N(y_0 + \alpha y_1) = x_0^2 + wx_1^2 + y_0^2 + wy_1^2 = z_0^2 + wz_1^2 = N(z_0 + \alpha z_1)$$

and thus the field $\tilde{\mathbb{F}}$ together with the nontrivial Galois automorphism satisfies the conditions of Corollary 5.4.6 while $\mathbb{F}$ together with the identity does not.

5.5. More on flips of locally split groups

For strong flips of locally split groups, flip-flop transitivity implies distance transitivity.

Lemma 5.5.1 (Gramlich and Mühlherr)**.** *Suppose θ is a strong and flip-flop transitive quasi-flip of a locally split RGD-system $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$, then θ is distance transitive.*

Proof. Let c and d be chambers of $\mathcal{C}_+$ with identical θ-codistance w. If $w = 1_W$ then we are transitive on $\mathcal{C}_w^\theta$ by hypothesis. Assume now by means of induction that transitivity has already been established for each θ-codistance of shorter length. There exists $s \in S$ such that $l(sw) < l(w)$. By Lemma 1.3.2 either $sw = w\theta(s)$ or $sw\theta(s)$ is a θ-twisted involution of shorter length than w; denote it by w'. Accordingly by Lemma 2.4.2 there is a chamber $a \sim_s c$ with $\delta^\theta(a) = w'$ and also a chamber $b \sim_s d$ with $\delta^\theta(b) = w'$. By induction there exists an element $g \in \Gamma$ mapping a to b. Thus both d and $g(c)$ are contained in the same s-panel P_s.

Viewing P_s as the geometry of 1-dimensional subspaces of a 2-dimensional vector space endowed with a nontrivial σ-sesquilinear form f, both d and $g(c)$ correspond to f-singular 1-dimensional subspaces. By Witt's Theorem there exists an element $h \in \Gamma$ mapping $g(c)$ onto d. □

Remark 5.5.2. The hypothesis of the preceding lemma can be somewhat weakened by replacing "strong" with the requirement that *uniform descent* is possible, as given in the following definition. By Lemma 2.4.2 every strong quasi-flip allows uniform descent.

Definition 5.5.3. Let θ be a quasi-flip. For a chamber c, we define the **descent set** $D_\theta(c) := \{s \in S \mid \exists d \in P_s(c) : l^\theta(d) < l^\theta(c)\}$. We say θ allows **uniform descent** if for any two chambers c, d with equal θ-codistance the sets $D_\theta(c)$ and $D_\theta(d)$ coincide.

CHAPTER
SIX

APPLICATIONS TO ALGEBRAIC AND KAC-MOODY GROUPS

Throughout this whole chapter, we will always assume all fields to be of characteristic different from 2 unless stated differently.

6.1. Algebraic groups

In this section we present applications of the results in the preceding chapters of the present work to reductive linear algebraic groups. For an introduction to linear algebraic groups, we refer to [Hum75], [Bor91], [Spr98].

Let G be a connected reductive linear algebraic group defined over an infinite field $\mathbb{F}$. We denote the set of $\mathbb{F}$-rational points of G by $G(\mathbb{F})$. In particular, we identify G with $G(\overline{\mathbb{F}})$, where $\overline{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$.

Assume that G is isotropic over $\mathbb{F}$, i.e., some proper nontrivial parabolic subgroup of G is defined over $\mathbb{F}$. Let T be a maximal $\mathbb{F}$-split $\mathbb{F}$-torus. By Borel and Tits [BT65], there exists a family of root groups $\{U_\alpha\}_{\alpha\in\Phi}$, indexed by the relative root system Φ of $(G(\mathbb{F}), T(\mathbb{F}))$, such that $(G(\mathbb{F}), \{U_\alpha\}_{\alpha\in\Phi}, T(\mathbb{F}))$ is an RGD-system. For details see e.g. [BT72, Section 6], [AB08, Section 7.9].

In particular, G admits a (twin) BN-pair. In general, the group B (which we somewhat misleadingly have also called "Borel group" in previous chapters) will be the group $P(\mathbb{F})$ of $\mathbb{F}$-rational points of a minimal parabolic $\mathbb{F}$-subgroup P of G.

The root groups of $G(\mathbb{F})$ are uniquely 2-divisible if and only if $\operatorname{char}\mathbb{F} \neq 2$. Indeed, if the $\mathbb{F}$-rank of G is at least 2, this follows from Proposition 2.6.1. But in general this is true because the root groups are vector spaces over $\mathbb{F}$ or extensions of a vector space over $\mathbb{F}$ by another such vector space (see e.g. [BT73, Section 8]). Yet another argument can be seen in the following sketched proof.

Lemma 6.1.1. *Let $U_\alpha(\mathbb{F})$ be a root group of a connected reductive group defined over a field $\mathbb{F}$. Then $U_\alpha(\mathbb{F})$ is uniquely 2-divisible if and only if* $\operatorname{char}\mathbb{F} \neq 2$.

Sketch of proof. Note that $U_\alpha(\overline{\mathbb{F}})$ is connected (see e.g. [Hum75, 26.3]). Take an arbitrary element $u \in U_\alpha(\mathbb{F})$, and let $V := \overline{\langle u \rangle} \cap G(\mathbb{F})$ be the Zariski closure of the group generated by u inside $G(\overline{\mathbb{F}})$ intersected with $G(\mathbb{F})$. Since $U_\alpha(\overline{\mathbb{F}})$ is closed, V is a subgroup of it and hence of $U_\alpha(\mathbb{F})$. If $\operatorname{char}\mathbb{F} = 0$, then V is a connected 1-dimensional unipotent group isomorphic to the additive group of $\mathbb{F}$. If $\operatorname{char}\mathbb{F} = p > 2$, then V is a finite cyclic p-group. In either case there exists a unique $v \in V$ such that $v^2 = u$. Uniqueness then follows by the observation that $\overline{\langle u \rangle} = \overline{\langle v \rangle}$. □

6.1.1. Quasi-flips of algebraic groups

In this section we show that we can apply the full machinery developed in this thesis to arbitrary involutory automorphisms of connected reductive groups or of finite groups of Lie type. Readers who are only interested in the consequences for algebraic groups which we can draw from this may wish to skip to the next section.

Let $\mathbb{F}$ be an infinite field and G a simple algebraic group defined over $\mathbb{F}$ and of positive $\mathbb{F}$-rank (i.e., isotropic). It is a deep result by Borel and Tits [BT73] (see also [Ste73] for an English summary of the centrals results) that abstract automorphisms of the group $G(\mathbb{F})$ of $\mathbb{F}$-rational points split into a product of a field automorphism, an $\mathbb{F}$-isogeny and the inverse of a purely inseparable central isogeny (the latter can be omitted if G is simply connected or adjoint).

As a consequence, any abstract automorphism of $G(\mathbb{F})$ maps (minimal) parabolic $\mathbb{F}$-subgroups to such groups of the same type. In fact, this even extends to semisimple groups (and from there to reductive groups):

Proposition 6.1.2 (Proposition 7.2 in [BT73])**.** *Let G be a connected reductive linear algebraic group over an infinite field $\mathbb{F}$. Then any abstract automorphism of $G(\mathbb{F})$ maps parabolic $\mathbb{F}$-subgroups again to parabolic $\mathbb{F}$-subgroups (in particular, minimal parabolic $\mathbb{F}$-subgroups are mapped to parabolic $\mathbb{F}$-subgroups).*

Over finite fields, reductive algebraic groups become split, so that there exist Borel $\mathbb{F}$-subgroups. However, the connected component becomes trivial, so studying $G(\mathbb{F})$ is not the right approach. The correct viewpoint is to study *finite groups of Lie type* (see e.g. [Car72]), which are obtained from algebraic groups via Lang's theorem. Let G now be a finite group of Lie type coming from a reductive algebraic $\mathbb{F}$-group. Let $p := \operatorname{char}\mathbb{F}$. By [Che55] one knows that the Borel subgroups can be abstractly described as the normalizers of p-Sylow subgroups, and hence any abstract automorphism will map Borel groups to Borel groups (see also [Ste60]). All in all, we get:

Fact 6.1.3. *Let $\mathbb{F}$ be an infinite field and G the group of $\mathbb{F}$-rational points of a semisimple algebraic group defined over $\mathbb{F}$ and of positive $\mathbb{F}$-rank (i.e., isotropic). Then any abstract automorphism of G of order 2 is a quasi-flip as defined in Chapter 2.*

The same holds if G is a finite group of Lie type.

6.1.2. Applications to algebraic groups

Here, we generalize [HW93, Proposition 6.10] in the sense that we also cover non-linear automorphisms. Proposition 6.8 in loc. cit. can be generalized analogously.

Corollary 6.1.4 (of Corollary 2.7.3 and Proposition 2.6.1)**.** *Suppose G is a connected isotropic reductive algebraic group defined over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$, and P a minimal parabolic $\mathbb{F}$-subgroup. Let θ be an abstract involutory automorphism of G. Let $\{A_i \mid i \in I\}$ be representatives of the $G_\theta(\mathbb{F})$-conjugacy classes of θ-stable maximal $\mathbb{F}$-split tori in G. Then*

$$G_\theta(\mathbb{F})\backslash G(\mathbb{F})/P(\mathbb{F}) \cong \bigcup_{i\in I} W_{G_\theta(\mathbb{F})}(A_i)\backslash W_{G(\mathbb{F})}(A_i).$$

In Chapter 5 we studied Iwasawa decompositions of groups with a twin BN-pair. The work done there applies as follows to split algebraic groups:

Definition 6.1.5 (Cf. 5.4.1)**.** A reductive algebraic group G defined over $\mathbb{F}$ admits an **Iwasawa decomposition** if there exists an abstract involutory automorphism θ of $G(\mathbb{F})$ and a minimal parabolic $\mathbb{F}$-subgroup P of G such that $\theta(P)$ is opposite P and $G(\mathbb{F}) = G_\theta(\mathbb{F})P(\mathbb{F})$ where $G_\theta(\mathbb{F})$ is the centralizer of θ in $G(\mathbb{F})$.

Corollary 6.1.6 (of Theorem 5.4.7; joint work with Gramlich and De Medts, see [DMGH09])**.** *Let $\mathbb{F}$ be a field and let G be a split connected reductive algebraic group defined over $\mathbb{F}$. The group of $\mathbb{F}$-rational points $G(\mathbb{F})$ admits an Iwasawa decomposition $G(\mathbb{F}) = G_\theta(\mathbb{F})B(\mathbb{F})$ if and only if $\mathbb{F}$ admits an automorphism σ of order at most 2 such that*

(1) -1 is not a norm, and

(2) (i) either a sum of norms is a norm, or

(ii) a sum of norms is ε times a norm, where $\varepsilon \in \{+1, -1\}$, (and this case can only occur if all rank 1 subgroups of G are isomorphic to $\mathrm{PSL}_2(\mathbb{F})$),

with respect to the norm map $N_\sigma : \mathbb{F} \to \mathrm{Fix}_{\mathbb{F}}(\sigma) : x \mapsto xx^\sigma$.

Remark 6.1.7. In Section 5.4.1, we gave examples and some extra details on fields satisfying the criteria given above. Note that in view of Remark 5.4.8, condition (2)(ii) becomes vacuous if there are no isolated nodes in the diagram of $G(\mathbb{F})$.

Let G be a connected reductive $\mathbb{F}$-group, and let θ be an abstract involutory automorphism of $G(\mathbb{F})$. In [HW93] it is shown that all minimal θ-split parabolic $\mathbb{F}$-subgroups have equal type, provided θ is an algebraic morphism (see Propositions 4.8 and 4.11 in loc. cit.).

Part of our work in Chapter 4 can be considered to be a variation of this. More precisely, it implies the following (recall that $\mathbb{F}$ is assumed to be infinite):

Proposition 6.1.8. *Let G be a connected reductive $\mathbb{F}$-group, and let θ be an abstract involutory automorphism of $G(\mathbb{F})$. If the diagram of $G(\mathbb{F})$ contains no triple bonds (i.e., there are no residues isomorphic to G_2), then all minimal θ-split parabolic $\mathbb{F}$-subgroups have equal type.*

Proof. Minimal θ-split parabolic $\mathbb{F}$-subgroups correspond one-to-one to minimal Phan residues of the same type (cf. Section 4.3). By our hypotheses and Theorem 4.1.10, all minimal Phan residues have equal type. □

6.1.3. Linear flips

Definition 6.1.9. Let G be the group of $\mathbb{F}$-rational points of a connected reductive $\mathbb{F}$-group. A abstract involutory automorphism θ of G which is an (algebraic) $\mathbb{F}$-morphism is called **$\mathbb{F}$-linear quasi-flip**.

Involutory $\mathbb{F}$-morphisms are well-understood objects, and their theory heavily influenced us during the creation of the present thesis, in particular [HW93]. As such, we do not say much about these here, but rather just give one brief example.

The following observation is due to Helminck and Wang, as a consequence of Propositions 4.8 and 4.11 of [HW93]. (See also Proposition 3.16 of [Hel97] and [Spr86].) Recall that G is identified with the rational points over $\overline{\mathbb{F}}$, the algebraic closure of $\mathbb{F}$.

Fact 6.1.10. *Let G be a connected reductive linear group, let θ be an $\mathbb{F}$-linear quasi-flip of G. Then all minimal θ-split parabolic $\mathbb{F}$-subgroups of G are conjugate under G_θ.*

Geometrically, this means that G_θ acts transitively on the set of minimal Phan residues of the building G/B (where B is a Borel subgroup of G). In the special case that the minimal θ-split parabolic $\mathbb{F}$-subgroups are Borel subgroups, this means that G_θ acts transitively on the flip-flop system.

6.1.4. Semi-linear flips

Definition 6.1.11. Let G be a reductive linear $\mathbb{F}$-group. A **semi-linear quasi-flip** of G is an involutory $\mathbb{F}$-automorphism of G composed with a field automorphism of $\mathbb{F}$.

Note that the field automorphism necessarily has order at most 2.

Lemma 6.1.12. *Semi-linear quasi-flips of split reductive groups are strong, i.e., possess the Devillers-Mühlherr property.*

Proof. Informally, the argument is that in a split group, projecting from a panel to another panel is "linear", and therefore composing this with a semi-linear map can not be the identity.

Less informally, let G be a split reductive group and θ a semi-linear flip of G. Assume P is a panel in $\mathcal{C}_+$. We have to show that there exists $c \in P$ such that $\mathrm{proj}_P(\theta(c)) \neq c$. If P is not parallel to its image $\theta(P)$ (i.e., the projection from $\theta(P)$ to P is a single chamber), then this is clearly the case. So we may assume that P and $\theta(P)$ are parallel. We will now assume that θ maps all chambers in P to their projection on $\theta(c)$, i.e., that θ does not possess the Devillers-Mühlherr property, and will lead this to a contradiction.

Let c be a chamber in P and denote its θ-codistance by w. By Theorem 2.5.8 we can choose a θ-stable twin apartment Σ containing c and hence $\theta(c)$.[1] Denote the unique chamber in $\Sigma \cap P$ different from c by c' and let $\alpha = (\alpha_+, \alpha_-)$ be the twin root in Σ containing c but not c'. By our assumption that θ does not possess the Devillers-Mühlherr property we have $\delta^\theta(c) = w = \delta^\theta(c')$.

We claim that $\alpha_- \cap \theta(P) = \{\theta(c)\}$. Indeed $\theta(c)$ resp. $\theta(c')$ is the unique chamber in Σ at codistance w from c resp. c'. The reflection s_α corresponding to α interchanges c and c', therefore it must also swap $\theta(c)$ and $\theta(c')$, so $\theta(P)$ is in the boundary of α. By convexity of twin roots we deduce that $\theta(c) \in \alpha$, as it is closer to c than $\theta(c')$ is.

Consequently $\theta(\alpha_+) = \alpha_-$ and thus $\theta(\alpha) = \alpha$. Let U_α be the root group corresponding to α. By rigidity of twin buildings $\theta(U_\alpha) = U_{\theta(\alpha)} = U_\alpha$.

Let $d \in P \setminus \{c\}$. For any $u \in U_\alpha$ we have $u.d \in P \setminus \{c\}$. Denote the projection of d to $\theta(P)$ by d'. By our standing assumption, $d' = \theta(d)$. Since any element of G preserves distances and codistances, in particular u must preserve projections. Therefore $u.d = \mathrm{proj}_P(u.d') = \theta(u.d') = \theta(u).d$. But U_α acts sharply transitively on $P \setminus \{c\}$, accordingly $u = \theta(u)$ and $U_\alpha \subset G_\theta$.

Since G is split, U_α is isomorphic to the additive group of $\mathbb{F}$. Now θ involves a field automorphism σ of $\mathbb{F}$. Denote $U' := (\theta \circ \sigma)(U_\alpha)$. By rigidity of spherical buildings, $U' = U_\beta$ for some root β. But then the non-linear map $\sigma : U_\beta \to U_\alpha$ would have to be the inverse of the linear map $(\theta \circ \sigma) : U_\alpha \to U_\beta$, which is absurd. $\square$

Remark 6.1.13. The proof we just gave applies also applies to split Kac-Moody group and their semi-linear automorphisms. The latter may be defined completely analogously as for algebraic groups, thanks to the solution of the isomorphism problem of split Kac-Moody groups, see [Cap05], [CM05], [CM06].

6.2. Groups of Kac-Moody type

In the following, we give only a very rough overview on Kac-Moody groups. The interested reader may find a gentle introduction in [AB08, Section 8.11], or [CR08, Section 3.3]. The original references for the kind of Kac-Moody groups we consider are of course [Tit87] and [Tit92].

[1] While Theorem 2.5.8 requires characteristic different from 2, one can show that such a twin apartment always exists if the quasi-flip is semi-linear. However, for the sake of simplicity of the exposition, we just appeal to our standing convention and assume the characteristic to be different from 2.

Chevalley's work made it possible to define groups over arbitrary fields analogously to the complex semisimple Lie groups by defining these groups in terms of a group functor or group scheme. Paralleling this ground breaking work, Tits introduced in [Tit87] a similar description of Kac-Moody groups. Roughly speaking, he introduced a group functor

$$\mathcal{G}_B : \mathrm{RING} \to \mathrm{GROUP}$$

from the category of commutative rings with unit into the category of groups depending on a integral finite root basis B. This group functor has the property that for a field $\mathbb{F}$, the group $\mathcal{G}_B(\mathbb{F})$ possesses a natural root group datum, with root groups isomorphic to the additive group of $\mathbb{F}$. This functor is called the **Tits functor**. If $\mathbb{F}$ is a field, the group $\mathcal{G}_B(\mathbb{F})$ is called **split Kac-Moody group** over $\mathbb{F}$.

Being equipped with a root group datum, any Kac-Moody group also possesses a twin BN-pair. However, unlike algebraic groups, the two groups B_+ and B_- are in general not conjugate anymore; rather, they are conjugate if and only if the Weyl group of the RGD-system is finite.

Recently, Caprace and Mühlherr have determined all abstract automorphisms of (infinite) split Kac-Moody groups over almost arbitrary ground fields (only $\mathbb{F}_2$ and $\mathbb{F}_3$ require some extra care), see [Cap05], [CM05], [CM06]. Their result essentially states (in the irreducible case) that any such automorphism splits into the product of an inner automorphism, a sign automorphism (the identity map or the Chevalley involution, which interchanges the conjugacy classes of positive and negative Borel groups), a diagonal automorphism, a graph automorphism (generalizing diagram automorphisms), and a field automorphism. In the general case, the statement is more complicated. Nevertheless, the following holds:

Fact 6.2.1. *Let G be an infinite Kac-Moody group over some field $\mathbb{F}$, $|\mathbb{F}| \geq 4$. Then any abstract automorphism of G maps Borel subgroups to Borel subgroups. In particular, any involutory automorphism of G either preserves the sign of all Borel subgroups, or it interchanges plus and minus type Borel subgroups.*

This follows from [Cap05, Theorem 4.1], which implies that any group automorphism of G induces an automorphism of the root group datum of G.

We can now generalize [KW92, Proposition 5.15] (which in loc. cit. was only stated for Kac-Moody groups over algebraically closed fields in characteristic 0) as follows:

Corollary 6.2.2 (of Corollary 2.7.3, Proposition 2.6.1 and Example 2.5.5)**.** *Suppose G is a split Kac-Moody group defined over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$. Let θ be an involutory automorphism of G interchanging the two conjugacy classes of Borel groups of G. Then with the notation from Proposition 2.7.2, we have*

$$G_\theta\backslash G/B \cong \bigcup_{i\in I} W_{G_\theta}(A_i)\backslash W_G(A_i).$$

In Chapter 5 we studied Iwasawa decompositions of groups with a twin BN-pair. The work done there applies as follows to split Kac-Moody groups:

Definition 6.2.3 (Cf. 5.4.1)**.** A Kac-Moody group G defined $\mathbb{F}$ admits an **Iwasawa decomposition** if there exists an abstract involutory automorphism θ of G and a Borel subgroup B of G such that $\theta(B)$ is opposite B and $G = G_\theta B$ where G_θ is the centralizer of θ in G.

Corollary 6.2.4 (of Theorem 5.4.7; joint work with Gramlich and De Medts, see [DMGH09])**.** *Let $\mathbb{F}$ be a field and let G be a split Kac-Moody group over $\mathbb{F}$. Then G admits an Iwasawa decomposition $G = G_\theta B$ if and only if $\mathbb{F}$ admits an automorphism σ of order at most 2 such that*

(1) -1 is not a norm, and

(2) (i) either a sum of norms is a norm, or

(ii) a sum of norms is ε times a norm, where $\varepsilon \in \{+1, -1\}$, (and this case can only occur if all rank 1 subgroups of G are isomorphic to $\mathrm{PSL}_2(\mathbb{F})$),

with respect to the norm map $N_\sigma : \mathbb{F} \to \mathrm{Fix}_{\mathbb{F}}(\sigma) : x \mapsto xx^\sigma$.

In Section 5.4.1, we gave examples and some extra details on fields satisfying the criteria given above.

6.2.1. Locally finite Kac-Moody groups

In this section, we collect a few results about locally finite Kac-Moody groups.

The following is known for certain special cases. E.g. when θ is semi-linear and the diagram is spherical this follows from Lang's theorem (and G_θ then is in fact a finite group), even for odd q. See also Remark 6.2.8 below.

Theorem 6.2.5. *Suppose G is a split Kac-Moody group of type (W, S) over a finite field $\mathbb{F}_q$, $q \geq 5$ and odd, with 2-spherical diagram (and no G_2 residue). Let θ be a quasi-flip of G, i.e., an involutory automorphism of G which interchanges the two conjugacy classes of Borel groups. Then the centralizer G_θ of θ in G is finitely generated.*

Proof. For any $c \in \mathcal{C}$, the stabilizer $\mathrm{Stab}_{G_\theta}(c)$ also stabilizes $\theta(c)$, hence is contained in $\mathrm{Stab}_G(c) \cap \mathrm{Stab}_G(\theta(c))$. Thus it is a bounded subgroup (i.e., the intersection of two spherical parabolic subgroups of opposite sign) as defined in [CM06]. Since G is locally finite, Corollary 3.8 in loc. cit. implies that $\mathrm{Stab}_{G_\theta}(c)$ is finite. By the same argument, the torus $T = B_+ \cap B_-$ is finite. Since q is odd, Theorem 2.5.8 ensures that all chambers are contained in a θ-stable apartment. Thus by Lemma 5.1.12, G_θ acts with finitely many orbits $O_1, \ldots, O_n$ on $\mathcal{C}^\theta$.

Choose a chamber $c_1 \in O_1$. For each $i \in \{2, \ldots, n\}$ pick a chamber $c_i \in O_i$ such that $l(c_1, c_i)$ is minimal among all chambers in O_i. Set $m := 1 + \max_{i\in\{2,\ldots,n\}} l(c_1, c_i)$, and let X be the set of all chambers at distance at most m from c_1. Clearly X contains all the c_i and all their panels. Since our building is locally finite, this is a finite set. By construction, X intersects all G_θ orbits.

Let $Y := \{g \in G_\theta \mid g.X \cap X \neq \emptyset\}$. Since X is finite and all chamber stabilizers in G_θ are finite, Y is also a finite set. Let $H := \langle Y \rangle$ and consider the set $H.X$. This is readily seen to be connected. Assume there was $c \in \mathcal{C}^\theta \setminus H.X$. Since $\mathcal{C}^\theta$ is connected by Theorem 4.1.10, we can choose a minimal gallery inside $\mathcal{C}^\theta$ from c to some chamber in $H.X$. By following this gallery, we find a chamber c' outside $H.X$ but adjacent to a chamber d inside $H.X$.

But by definition of H and X, there must be some $h \in H$ and some orbit representative c_i such that $d = h.c_i$. But then $d \in h.X$, and by construction also all panels of d are contained in $h.X$, thus in particular $c' \in h.X$. Contradiction, hence $G_\theta.X = H.X$. Since moreover H contains $\mathrm{Stab}_{G_\theta}(c_i)$ for all orbit representatives c_i, we conclude that $G_\theta = H = \langle Y \rangle$ is finitely generated. □

Remark 6.2.6. The exclusion of G_2 residues is a deficiency of Theorem 4.1.10, which hopefully can be removed in the future.

On the other hand, the restriction to characteristic 2 is partially due to Theorem 2.5.8 which we use to construct θ-stable apartments around arbitrary chambers. There is no hope of improving this bound using our methods as long as θ is not further restricted. However if θ is assumed to be semi-linear then the existence of θ-stable apartments is actually guaranteed regardless of the characteristic. Thus in this case it suffices to know that $\mathcal{C}^\theta$ is connected (as a substitute for Theorem 4.1.10) to conclude that G_θ is finitely generated.

Lattices

We will demonstrate that Theorem 6.2.5 is in some sense sharp (with the preceding remark in mind) by sketching that for Kac-Moody groups which are not 2-spherical the group G_θ will in general not be finitely generated. For this we first need another result showing that G_θ is a *lattice* in $\overline{G}_+$. Recall that a **lattice** is a discrete subgroup Γ of a locally compact group G with the property that $\Gamma \backslash G$ is endowed with a finite G-invariant measure. Moreover $\overline{G}_+$ denotes the topological completion of G as defined in [CR09, Section 1.2], which there is shown to be a locally compact group.

The theorem we state now is a slightly modified version of a result by Gramlich and Mühlherr [GM08]. We merely extend the class of morphisms to which it applies. In loc. cit. the result is given only for distance transitive flips with the Devillers-Mühlherr property. We can replace this assumption by Lemma 5.1.12 (combined with Theorem 2.5.8) at the price of having to restrict to characteristic different from 2. Moreover, G_θ will is a discrete subgroup because $G_\theta \cap U_+ = G_\theta \cap U_+ \cap U_-$ is a bounded subgroup of G, hence finite. With this in mind it is straightforward to adjust the proof given in loc. cit. to the version of the theorem we present here.

Theorem 6.2.7 (Gramlich and Mühlherr, 2007)**.** *Suppose G is a split Kac-Moody group of type (W, S) over a finite field $\mathbb{F}_q$, $q \geq 5$ and odd. Let θ be a quasi-flip of G, i.e., an involutory automorphism of G which interchanges the two conjugacy classes of Borel groups. Then the centralizer G_θ of θ in G is a lattice in the group $\overline{G}_+$ if the series $\sum_{w \in W} \frac{1}{q^{l(w)}}$ converges.* ■

Let G be a locally finite Kac-Moody group of type (W, S). Then G is always finitely generated. For G_θ to be finitely generated, we had to assume that G is 2-spherical. In fact Caprace, Gramlich, and Mühlherr have recently observed that G_θ may not be finitely generated if G is not 2-spherical: Let T be a tree residue of the building, then $G.T$ is a simplicial tree by [DJ02, Proposition 2.1]. The key insight is the following: The action of the lattice G_θ on the simplicial tree $G.T$ is minimal but there are infinitely many G_θ-orbits on $G.T$ if $\mathrm{Inv}^\theta(T) = \{\delta^\theta(c) \in W | c \in T\}$ is infinite (which for instance is the case if θ is a semi-linear flip). From [Bas93, Proposition 7.9] (also [BL01, Proposition 5.6]) it follows that the lattice G_θ cannot be finitely generated.

If G is 2-spherical, then it is finitely presented. For G_θ to be finitely presented, in general we need G to be at least 3-spherical.[2] This "gap" between G and G_θ is believed to extend to higher finiteness properties.

Remark 6.2.8. By [DJ02] in the 2-spherical case the full automorphism group of the building associated to locally finite Kac-Moody group G has Kazhdan's property (T) provided the ground field is sufficiently large (e.g. if its order is greater than $1764^n/25$, where n is the dimension of the building). Since G_θ is a lattice in G it also has property (T) by [BdlHV08, Theorem 1.7.1] and in particular is finitely generated by [BdlHV08, Theorem 1.3.1]. Note that the bounds in loc. cit. are known to be not optimal.

[2] Here is a brief argument for this, at least in the affine case: By [GM08] the group G_θ is a lattice. Hence by [Mar91, Chapter IX] it is S-arithmetic in the ambient semisimple Lie group (the completion of the affine Kac-Moody group). Now [BW07] states that an S-arithmetic subgroup of a split semisimple algebraic group over a function field is of type F_2 if and only if the corresponding affine diagram is 3-spherical. So, if the diagram is 2-spherical, but not 3-spherical, then the group is not finitely presented.

APPENDIX

A

COMPUTER RESULTS

In this appendix, we present results obtained with the help of machine computations, as well as the computer code that was used. All computations were performed with the help of GAP [GAP08].

A.1. Connectedness of R^θ: θ-acute quadrangles

In this section we study the geometry opposite a Moufang subset of a panel in several low-order generalized quadrangles, namely those of order $(3, 9)$, $(4, 8)$ and $(4, 16)$. To do this, we first produce a computer representation of the points and lines, by loading the file `quadrangle.gap` into GAP, with the variables (n, q) set to $(4, 3)$, $(5, 2)$ and $(4, 4)$, respectively.

It is easy to compute the geometry opposite a chamber, a point or a line with this code. If we fix a panel, say the point row of a line, then this opposite geometry gets smaller the more points we have to be opposite of. Hence it suffices to take all maximal Moufang subsets of the panel, then show that for each of them the opposite geometry is connected. The code in `quadrangle-acute.gap` does just that, printing out any "counterexamples" it finds. By running it we confirmed that the subsets of the quadrangles described above are always connected.

A.2. Connectedness of R^θ: θ-parallel projective planes

In Proposition 4.6.11, we studied connectedness of the flip-flop system associated to polarities of projective planes. One case was left open, namely the polarities of the projective plane over the field with three elements. While this is a tiny example and certainly could be handled by manual computations, we present some computer code dealing with this problem for this and other projective planes.

In file `triangle.gap`, we compute the projective plane $\mathcal{T}(q)$ over the field $\mathbb{F}_q$, where q is a prime power set by the user. Then, in file `triangle-invs.gap`, all type

Order	Description	Max l^θ	$\lvert P\rvert$	$\lvert L\rvert$	$\lvert C\rvert$	#comps
2	Projective plane	-	7	7	21	
	linear automorphism	3	4	4	8	1
	linear polarity	3	3	3	6	1
3	Projective plane	-	13	13	52	
	linear automorphism	3	8	8	24	1
	linear polarity	3	9	9	24	1
4	Projective plane	-	21	21	105	
	linear automorphism	3	16	16	64	1
	linear polarity	3	15	15	60	1
	semilinear automorphism	3	14	14	56	1
	semilinear polarity	3	12	12	24	4
5	Projective plane	-	31	31	186	
	linear automorphism	3	24	24	120	1
	linear polarity	3	25	25	120	1
7	Projective plane	-	57	57	456	
	linear automorphism	3	48	48	336	1
	linear polarity	3	49	49	336	1
8	Projective plane	-	73	73	657	
	linear automorphism	3	64	64	512	1
	linear polarity	3	63	63	504	1
9	Projective plane	-	91	91	910	
	linear automorphism	3	80	80	720	1
	linear polarity	3	81	81	720	1
	semilinear automorphism	3	78	78	702	1
	semilinear polarity	3	63	63	378	1

Table A.1.: Sizes of the flip-flop systems in various finite projective planes.

preserving automorphisms of the triangle are determined, as well as an orthogonal polarity. This suffices to compute the full extended automorphism group of $\mathcal{T}(q)$ and in there all conjugacy classes of involutions. Finally, for each conjugacy class, a representative is chosen and the flip-flop system is computed. In Table A.1, we present the computed results for several small projective planes. Note that it confirms that the only exception for connectedness of R^θ occurs in the projective plane of order 4 with a semilinear projectivity.

A.3. Connectedness of R^θ: θ-parallel quadrangles

In this section, we complete the proof of Proposition 4.6.18. For this, we study involutory automorphisms of certain low order generalized quadrangles. Specifically, for each conjugacy class of involutions, we pick a representative θ and determine whether R^θ (the set of chambers moved maximally by θ) is connected as a chamber system. The quadrangles we need to consider are of the following orders: (s,s) for $s \in \{2,3,4,5,7,8,9,16\}$; (s^2,s) for $s \in \{2,3,4\}$; and (s^2,s^3) for $s \in \{2,3,4\}$.

The code works as follows: In the files `quadrangle.gap` and `quadrangle2.gap`, there is code which computes internal representations of quadrangles of orthogonal and unitary type. Then in `quadrangle-invs.gap`, representatives for the conjugacy

(s,s)	Description	Max l^θ	$\|P\|$	$\|L\|$	$\|C\|$	#comps
(2,2)	Quadrangle	-	15	15	45	
	linear	3	12	8	24	1
	linear	3	8	12	24	1
	linear	4	8	8	16	2
(3,3)	Quadrangle	-	40	40	160	
	linear	3	30	40	120	1
	linear	3	24	32	96	1
	linear	4	36	24	96	1
	linear	4	18	24	48	3
(4,4)	Quadrangle	-	85	85	425	
	linear	3	64	80	320	1
	linear	3	80	64	320	1
	linear	4	64	64	256	1
	semilinear	4	40	40	80	10
(5,5)	Quadrangle	-	156	156	936	
	linear	3	130	156	780	1
	linear	3	120	144	720	1
	linear	4	150	120	720	1
	linear	4	100	120	480	1
(7,7)	Quadrangle	-	400	400	3200	
	linear	3	350	400	2800	1
	linear	3	336	384	2688	1
	linear	4	392	336	2688	1
	linear	4	294	336	2016	1
(8,8)	Quadrangle	-	585	585	5265	
	linear	3	576	512	4608	1
	linear	3	512	576	4608	1
	linear	4	512	512	4096	1
(9,9)	Quadrangle	-	820	820	8200	
	linear	3	738	820	7380	1
	linear	3	720	800	7200	1
	linear	4	810	720	7200	1
	linear	4	648	720	5760	1
	semilinear	4	540	540	3240	1
(11,11)	Quadrangle	-	1464	1464	17568	
	linear	3	1342	1464	16104	1
	linear	3	1320	1440	15840	1
	linear	4	1452	1320	15840	1
	linear	4	1210	1320	13200	1
(13,13)	Quadrangle	-	2380	2380	33320	
	linear	3	2210	2380	30940	1
	linear	3	2184	2352	30576	1
	linear	4	2366	2184	30576	1
	linear	4	2028	2184	26208	1
(16,16)	Quadrangle	-	4369	4369	74273	
	linear	3	4352	4096	69632	1
	linear	3	4096	4352	69632	1
	linear	4	4096	4096	65536	1
	semilinear	4	3264	3264	39168	1

(s^2,s)	Description	Max l^θ	$\|P\|$	$\|L\|$	$\|C\|$	#comps
(4,2)	Quadrangle	-	45	27	135	
	linear	3	32	24	96	1
	semilinear	3	30	12	60	1
	semilinear	4	24	12	48	1
	linear	4	16	16	32	4
(9,3)	Quadrangle	-	280	112	1120	
	linear	3	252	112	1008	1
	semilinear	3	240	72	720	1
	semilinear	4	270	72	720	1
	semilinear	4	216	72	576	1
	linear	4	144	96	576	1
	linear	4	180	60	360	1
(16,4)	Quadrangle	-	1105	325	5525	
	linear	3	1024	320	5120	1
	semilinear	3	1020	240	4080	1
	semilinear	4	960	240	3840	1
	linear	4	768	256	3072	1
(25,5)	Quadrangle	-	3276	756	19656	
	linear	3	3150	756	18900	1
	semilinear	3	3120	600	15600	1
	semilinear	4	3250	600	15600	1
	semilinear	4	3000	600	14400	1
	linear	4	2400	720	14400	1
	linear	4	2600	520	10400	1

(s^2,s^3)	Description	Max l^θ	$\|P\|$	$\|L\|$	$\|C\|$	#comps
(4,8)	Quadrangle	-	165	297	1485	
	linear	3	128	288	1152	1
	semilinear	4	120	180	720	1
	linear	4	64	256	512	1
(9,27)	Quadrangle	-	2440	6832	68320	
	linear	3	2160	6720	60480	1
	semilinear	4	2160	5832	51840	1
	linear	4	1512	6048	36288	1
(16,64)	Quadrangle	-	17425	66625	1132625	
	linear	3	16384	66560	1064960	1
	semilinear	4	16320	61200	979200	1
	linear	4	12288	65536	786432	1

Table A.2.: Sizes of the flip-flop systems in various finite quadrangles.

classes of type-preserving involutions are chosen. The latter is mostly done by brute force, instead of relying on classification results for these involutions, as to simplify the code and reduce the risk for errors. An exception is made for the quadrangles of order (s^2, s^3) in even characteristic, where we use results from [YP92] and [AS76] in order to be able to compute all involutions for the quadrangle of order $(16, 64)$.

See Table A.2 for a summary of the output of the GAP code. For each of the quadrangles above (and some more), it gives the size (i.e., the number of points, lines and chambers) of the quadrangle. Moreover, for each class of involutory automorphisms of these quadrangles, it states whether it is linear or semilinear, and presents the maximal numerical θ-distance a chamber is moved, the size (i.e., the number of points, lines and chambers) of the corresponding flip-flop system R^θ, and its number of connected components. It is evident that there is only one connected component in R^θ, except for the quadrangles of order $(2,2)$, $(3,3)$, $(4,4)$ and $(4,2)$, as claimed.

A.4. GAP code

triangle.gap

```
# Projective plane of order q
#q := 3;
n := 3;
F := GF(q);
G := GL(n,q);
V := FullRowSpace(F,n);

MAX_DIST := 3;
s := q;
t := q;

Print("Projective plane of order ", q, "... ");

# Compute all points, lines and chambers of the triangle
points := Orbit(G, [[1,0,0]] * One(F), OnSubspacesByCanonicalBasis);;
Print(Size(points), " points, ");
lines := Orbit(G, [[1,0,0], [0,1,0]] * One(F), OnSubspacesByCanonicalBasis);;
Print(Size(lines), " lines, ");

Read("common.gap");

Assert(0, Size(points) = 1+q+q^2);
Assert(0, Size(lines) = 1+q+q^2);
Assert(0, Size(chambers) = (1+q)*(1+q+q^2));
```

quadrangle.gap

```
# Quadrangles of order (q^2,q) and (q^2,q^3)
#q:=2;
#n := 4;
#n := 5;
F := GF(q^2);
G := GU(n,q);
V := FullRowSpace(F,n);

MAX_DIST := 4;
```

```
s := q^2;
t := q^(1 + (n-4)*2);    # q if n=4, and q^3 if n=5

Print("Quadrangle of order (", s, ",", t, ")... ");

# Element x such that x^(q+1) = -1
x := First(F, g -> g^(q+1) = -One(F));

if n = 4 then
    point := [[1,x,0,0]] * One(F);
    line := [[1,x,0,0], [0,0,1,x^-1]] * One(F);
else
    point := [[1,x,0,0,0]] * One(F);
    line := [[1,x,0,0,0], [0,0,0,1,x^-1]] * One(F);
fi;

# Compute all points, lines and chambers of the quadrangle
points := Set(Orbit(G,point,OnSubspacesByCanonicalBasis));;
Print(Size(points), " points, ");
lines := Set(Orbit(G,line,OnSubspacesByCanonicalBasis));;
Print(Size(lines), " lines, ");

Read("common.gap");

Assert(0, Size(points) = (1+s)*(1+s*t));
Assert(0, Size(lines) = (1+t)*(1+s*t));
Assert(0, Size(chambers) = (1+s)*(1+t)*(1+s*t));
```

quadrangle2.gap

```
# Quadrangles of order (q,q)
#q:=2;
n := 5;
F := GF(q);
G := GO(n,q);
V := FullRowSpace(F,n);

MAX_DIST := 4;
s := q;
t := q;

Print("Quadrangle of order (", s, ",", t, ")... ");

# Compute the product of two vectors w.r.t. the form used by GAP
bmat := InvariantBilinearForm(G).matrix;
Prod := function(u, v) return u*bmat*TransposedMat(v); end;;

# Compute the product of two vectors w.r.t. the form used by GAP
qmat := InvariantQuadraticForm(G).matrix;
QForm := function(u) return u*qmat*TransposedMat(u); end;;

# Compute all points, lines and chambers of the quadrangle
point := [[0,1,0,0,0]] * One(F);
points := Set(Filtered(Orbit(SL(n,q),point,OnSubspacesByCanonicalBasis), p -> QForm(p)=[[Zero(F)]]));;
Print(Size(points), " points, ");

point2 := First(points, p->IsZero(p[1][1]) and IsZero(p[1][2]) and Prod(point,p)=[[Zero(F)]]);;
line := [point[1], point2[1]];
lines := Set(Orbit(G,line,OnSubspacesByCanonicalBasis));;
Print(Size(lines), " lines, ");

Read("common.gap");

Assert(0, Size(points) = (1+s)*(1+s*t));
Assert(0, Size(lines) = (1+t)*(1+s*t));
```

```
Assert(0, Size(chambers) = (1+s)*(1+t)*(1+s*t));
```

common.gap

```
# Compute all chambers
chambers := Set(Union(List([1..Size(lines)], l->
            List( NormedRowVectors(VectorSpace(F, lines[l])),
                    v-> [Position(points, [v]), l] )
            )));;
Print(Size(chambers), " chambers\n");

# Determine all point rows and pencils. Used to speed up computations later on.
pencil := List(points, p->[]);;
pointrow := List(lines, l->[]);;
for c in chambers do
    AddSet(pencil[c[1]], c[2]);
    AddSet(pointrow[c[2]], c[1]);
od;
```

quadrangle-acute.gap

```
n:=4;q:=3;
#n:=5;q:=2;
#n:=4;q:=4;
Read("quadrangle.gap");
# Look both at a pencil, and a point row.
panels := [List(pencil[1], l -> [1,l]), List(pointrow[1], p -> [p,1])];
for panel in panels do
    # Size of moufang subset minus one must divide size of this set minus one.
    # Use this to determine the maximal size 'k' a Moufang subset could possibly have.
    k := 1+(Size(panel)-1)/(FactorsInt(Size(panel)-1)[1]);
    subsets := Filtered(Combinations(panel,k), comb -> IsSubset(comb, panel{[1,2]}));;
    tmp := Filtered(chambers, c->IsOpposite(panel[1],c) and IsOpposite(panel[2],c));;
    for T in subsets do
        TOp := Filtered(tmp, c->ForAll(T, cT->IsOpposite(cT,c)));;
        if NrComponentsChamberSet(TOp) <> 1 then Display(T); fi;
    od;
od;
```

triangle-invs.gap

```
Read("poly-utils.gap");    # Some helper coder

# Compute full automorphism group:
#
# 1) Determine a permutation presentation of PGU(n,q) on our points
phi := ActionHomomorphism(G, points, OnSubspacesByCanonicalBasis);
# 2) Compute action of frobenius automorphism
gfrob := Permutation(FrobeniusAutomorphism(F), points, OnTuplesTuples);
# 3) The full automorphism group:
Gperm := Image(phi);
autG := Group(Concatenation(GeneratorsOfGroup(Gperm), [gfrob]));

#
# Compute the extended automorphism group
#

# Function which computes the orthogonal complement of a vector
# w.r.t. the form given by the identity matrix.
PolarOfVector := function(v)
    local mat;
    mat := MutableCopyMat(BaseMat([ [v[2],-v[1],0], [v[3],0,-v[1]], [0,v[3],-v[2]] ]*One(F)));
```

```
    TriangulizeMat(mat);
    return mat;
end;;

# Compute the polars of all the chambers, for *some* polarity. From this,
# we can get *all* polarities, since the product of two polarities is a
# type preserving automorphism, so in autG.
p2l := List(points, p->Position(lines, PolarOfVector(p[1])));;
l2p := List([1..Size(lines)], l->Position(p2l, l));;
tau := PermList(List(chambers, c -> Position(chambers,[l2p[c[2]], p2l[c[1]]])));

# Extended automorphism group
psi:=ActionHomomorphism(autG, chambers, OnChambers);
autGOnChambers := Image(psi);
Assert(0, autGOnChambers^tau=autGOnChambers);
extG := Group(Concatenation(GeneratorsOfGroup(autGOnChambers), [tau]));

# Determine representatives for all involutions
Print("Computing involution representatives... ");
invs := Filtered(List(ConjugacyClasses(extG),Representative), x->Order(x)=2);;
Print(Size(invs), " involution classes\n");

Read("analyze-invs.gap");
```

quadrangle-invs.gap

```
Read("poly-utils.gap");    # Some helper coder

# Compute full automorphism group:
#
# 1) Determine a permutation presentation of PGU(n,q) on our points
phi := ActionHomomorphism(G, points, OnSubspacesByCanonicalBasis);
# 2) Compute action of frobenius automorphism
gfrob := Permutation(FrobeniusAutomorphism(F), points, OnTuplesTuples);
# 3) The full automorphism group:
Gperm := Image(phi);
autG := Group(Concatenation(GeneratorsOfGroup(Gperm), [gfrob]));

# Determine representatives for all involutions
Print("Computing involution representatives... ");

# By Aschbacher & Seitz, 19.8, "Involutions in Chevalley groups over fields of even order",
# we know for n=5 and even q that there are three involution classes:
# Two inner ones (determined in a paper by Park and Yoo), plus a field automorphism.
if n = 5 and IsEvenInt(q) then
    inv1 := IdentityMat(n, F); inv1[1][4] := One(F); inv1[2][5] := One(F);
    inv2 := IdentityMat(n, F); inv2[1][5] := One(F);
    invs := [ Image(phi,inv1), Image(phi,inv2), gfrob^(Order(gfrob)/2) ];
# TODO: Do something similar for n=4; and for odd q
else
    # By default, we use brute force to find all involutions
    invs := Filtered(List(ConjugacyClasses(autG),Representative), x->Order(x)=2);;
fi;
Print(Size(invs), " involution classes\n");

psi:=ActionHomomorphism(autG, chambers, OnChambers);
autGOnChambers := Image(psi);
invs := Image(psi,invs);

Read("analyze-invs.gap");
```

analyze-invs.gap

```
# Analyze the involutions
c_all := [];  c_dist := [];
for i in [1..Size(invs)] do
    Print("Involution ", i, ": ");
    tmp := invs[i];
    if tmp in autGOnChambers then Print("type preserving; ");
    else Print("polarity; "); tmp := tmp*tau; fi;
    tmp := PreImagesRepresentative(psi, tmp);
    if tmp in Gperm then Print("linear; ");
    else Print("semilinear; "); fi;

    # Sort all elements according to their theta-dist
    c_all[i] := List([0..MAX_DIST], x->[]);
    for c in [1..Size(chambers)] do
        tmp := Distance(chambers[c],chambers[c^invs[i]]);
        Add(c_all[i][1+tmp], chambers[c]);
    od;

    Print("maximal theta-dist... ");
    c_dist[i] := First(Reversed([0..MAX_DIST]), j -> not IsEmpty(c_all[i][j+1]));
    Display(c_dist[i]);

    Print("  elements of geometry... ");
    tmp := c_all[i][c_dist[i]+1];
    Print(Size(Set(tmp, c->c[1])), " points, ");
    Print(Size(Set(tmp, c->c[2])), " lines, ");
    Print(Size(tmp), " chambers; ");
    Print(NrComponentsChamberSet(tmp), " component(s)\n");

    Print("        fixed elements... ");
    tmp := c_all[i][1];
    Print(Size(Set(tmp, c->c[1])), " points, ");
    Print(Size(Set(tmp, c->c[2])), " lines, ");
    Print(Size(tmp), " chambers\n");
od;
```

poly-utils.gap

```
OnFlagByCanonicalBasis := function(flag,g)
    return List(flag, x-> OnSubspacesByCanonicalBasis(x,g));
end;;

OnChambers := function(c,g)
    return [c[1]^g, Position(pointrow, OnSets(pointrow[c[2]],g))];
end;;

# Compute connected components.
ComponentsChamberSet := function(chambers)
    local comps, c, tmp;
    # List of components. Each component is a pair of lists. The first contains
    # all points, the second all lines in that component.
    comps := [];
    for c in chambers do
        tmp := Filtered([1..Size(comps)], i -> c[1] in comps[i][1]
                                        or c[2] in comps[i][2]);
        if Size(tmp) = 0 then # Start a new component
            Add(comps, [ [c[1]], [c[2]] ] );
        elif Size(tmp) = 1 then    # Add to existing component
            AddSet(comps[tmp[1]][1], c[1] );
            AddSet(comps[tmp[1]][2], c[2] );
        else # Merge multiple components
            Assert(0, Size(tmp) = 2);    # Can only be two components!
            UniteSet(comps[tmp[1]][1], comps[tmp[2]][1] );
```

```
            UniteSet(comps[tmp[1]][2], comps[tmp[2]][2] );
            # Remove the second component
            Remove(comps, tmp[2]);
        fi;
    od;
    return comps;
end;;

NrComponentsChamberSet := function(chambers)
    return Size(ComponentsChamberSet(chambers));
end;;

# Generic distance function
Distance := function(c1, c2)
    if c1 = c2 then
        return 0;
    elif c1[1] = c2[1] or c1[2] = c2[2] then
        return 1;
    elif c1[1] in pointrow[c2[2]] or c2[1] in pointrow[c1[2]] then
        return 2;
    elif c1[2] in pencil[c2[1]] or c2[2] in pencil[c1[1]] then
        return 2;
    elif ForAny(pointrow[c1[2]], p->p in pointrow[c2[2]]) then
        return 3;
    elif ForAny(pencil[c1[1]], l->l in pencil[c2[1]]) then
        return 3;
    else
        return 4;
    fi;
end;;

IsOpposite := function(c1, c2)
    return Distance(c1,c2) = MAX_DIST;
end;;
```

APPENDIX

B

OPEN PROBLEMS

In the following, we present a list of open problems and questions that arose during the preparation of this thesis, roughly ordered by the corresponding chapter and section.

Flips

(1) Are there examples of groups G with twin BN-pair and a building quasi-flip of the associated building, which cannot be lifted to a BN-quasi-flip of G? (See Theorem 2.2.2 and the discussion afterwards.)

(2) Involutions of a non-spherical twin building which do not interchange its halves are not quasi-flips. The problem is that for these maps, there seems to be no way to make use of the extra information provided by the twinning. Hence one is reduced to the (rather broad) theory of general buildings. But even for an involutory automorphisms of an arbitrary building, one can introduce the notion of a θ-distance. At least for affine buildings, it seems possible to derive results e.g. on θ-stable twin apartments, by using the spherical building at infinity, and subsequently double coset decompositions.

(3) Extend the parameterization of the double coset decomposition $G_\theta\backslash G/B$ from Section 2.7 to $G_\theta\backslash G/P$ where P is an arbitrary (spherical) parabolic subgroup. This should be straightforward. Geometrically, one could argue with (spherical) residues instead of chambers. By going from a Borel subgroup to a larger parabolic subgroup, G_θ-orbits may fuse. One would expect that

$$G_\theta\backslash G/P \cong \bigsqcup_{T\in\mathcal{T}} W_{G_\theta}(T)\backslash W_G(T)/W_P(T)$$

where $\mathcal{T}$ is a set of representatives of the $G_\theta \times P$-conjugacy classes of θ-stable tori.

(4) In the case of algebraic groups over algebraically closed fields, there is a unique "largest" (open and dense) orbit in $G_\theta\backslash G/B$, the "big cell", see e.g. [HW93, Sections 4 and 9]. This has been extended to Kac-Moody groups in characteristic 0 in [KW92].

It would be interesting to study this for arbitrary Kac-Moody groups. The weak Zariski topology used in [KW92] can be extended to arbitrary Kac-Moody groups, but there many of its useful properties are not known. So, can one use this or some other "nice" topology such that there is a unique open and dense orbit? Can one understand orbit closure in this topology, and say meaningful things about the general orbit structure?

Flips in rank 1 and 2

(1) Study flips of a wider class of Moufang sets: E.g. all finite Moufang sets, or even all Moufang sets occurring in 2-spherical buildings. In particular, their transitivity properties are of interest: Both transitivity on the moved chambers (the flip-flop system of the Moufang set) as well as the fixed chambers.

Structure of flip-flop systems

(1) Given a K-homogeneous quasi-flip θ of an irreducible twin-building of type (W, S), what can we say about K in relation to S, other than that it is spherical? For linear flips of algebraic groups this is answered by classifying Satake diagrams. For example, are there quasi-flips such that $s \in S$ exists where $K \cup \{s\}$ is *not* spherical? Does the theory of Satake diagrams extend to (split) Kac-Moody groups?

(2) In Proposition 4.5.4 we prove for K-homogeneous quasi-flips satisfying a rank 2 condition that $\mathcal{C}^\theta$ is residually connected if $|K| \leq 2$. Can this bound on K be improved or even dropped, possibly after adding more hypotheses? Also (counter)examples would be of interest, i.e., quasi-flips for which the rank 2 condition is met, yet $\mathcal{C}^\theta$ is not residually connected.

(3) An answer to the following question about Coxeter systems would give an affirmative answer to the preceding question, and imply that $\mathcal{C}^\theta$ is always residually connected: Suppose we are given a Coxeter system (W, S), an automorphism θ of (W, S) of order at most 2 and a spherical and θ-invariant proper subset K of S. Moreover for all $s \in K$ we have $sw_K = w_K\theta(s)$.

There is a poset structure on the set of all θ-twisted involutions (cf. [Spr84], [RS90]): Starting with a θ-twisted involution w, given a generator $s \in S$, then exactly one of sw and $sw\theta(s)$ is a θ-twisted involution different from w. We write $\underline{s}w$ for this and set $w < \underline{s}w$ if $l(sw) < l(w)$, otherwise $\underline{s}w < w$.

Let X be a subset of S, let K_1, K_2, K_3 be subsets of $K \setminus X$ such that $K_1 \cap K_2 = K_1 \cap K_3 = K_2 \cap K_3 = \emptyset$. Suppose w is a θ-twisted involution above w_K in the poset we just described. If there are directly ascending chains from w_K to w inside each of $X \cup K_i$, can we prove that there is such a chain inside X?

Of course, if any of the K_i is empty (e.g. if $|K| \leq 2$), then this is obviously true. So one needs to deal with the case where all K_i are nonempty.

(4) Study connectedness of R^θ and direct descent properties of those Moufang polygons we did not cover in Section 4.6, in particular Moufang hexagons. This is subject of ongoing research by Hendrik Van Maldeghem and the author [HVM].

Transitive actions on flip-flop systems

(1) Are there quasi-flips for which the flip-flop system $\mathcal{C}^\theta$ is the union of two or more distinct sets $\mathcal{C}^\theta_w$ as defined in Section 5.1? (See also Remark 5.1.4.) Say $\mathcal{C}^\theta = \bigcup_{w \in X} \mathcal{C}^\theta_w$, then we must have $l(w_1) = l(w_2)$ for any $w_1, w_2 \in X$. Furthermore, as a consequence of Lemma 2.3.4 for each $w \in X$ there exists a θ-stable spherical subset K_w of S such that w is the longest element in $\langle K_w \rangle$. The results in Chapter 4 further restrict the possible diagrams of the involved building, if any such example even exists.

(2) Are there flip-flop transitive flips which are not distance transitive?

(3) Study transitivity properties of quasi-flips of rank 2 buildings. This might be easier (and in some cases yield more insights) than the rank 1 (Moufang set) case, and would still allow to give local-to-global transitivity results for quasi-flips of two-spherical buildings.

(4) Find examples of flips that do not allow uniform descent (cf. Definition 5.5.3). More specifically, find examples that admit direct descent into $\mathcal{C}^\theta$ (so if $c \notin \mathcal{C}^\theta$, then $D(c) \neq \emptyset$), but not uniformly. Give criteria as to when a quasi-flip allows (does not allow) uniform descent.

Knowing more about this would help answering the preceding question on transitivity (together with knowledge on the transitivity in rank 1 and 2).

More open problems

(1) Let G be a locally finite Kac-Moody group of non-spherical type, θ a quasi-flip, and B a Borel group. It is notable that B and G_θ have several interesting properties in common: Both are in general lattices in a completion of G (cf. Theorem 6.2.7, see [GM08]). To B we can associate the chamber system c^{op}

of chambers opposite the chamber c stabilized by B, to G_θ the flip-flop system $\mathcal{C}^\theta$; for both we are interested in connectedness and transitivity properties.

For both we get closely related double coset decompositions $B \backslash G/B$ (the Bruhat decomposition) and $G_\theta \backslash G/B$ (cf. Section 2.7) – in each case the orbits can be parameterized by (quotients of) Weyl groups. This similarity is not a coincidence: For the former, the fact that the twin building can be covered by the chambers contained in twin apartments containing c is central, for the latter that the building can be covered by the chambers of all θ-stable twin apartments is relevant (plus the fact that two intersecting θ-stable twin apartments are G_θ-conjugate).

The following question comes to mind: Can we generalize this to the study of double coset spaces $\Gamma \backslash G/B$ where Γ is an arbitrary lattice in G? One idea would be to try to define a suitable replacement for the twin apartments containing c on the one hand and θ-stable twin apartments on the other hand.

(2) For many applications, one is interested in the interplay of two commuting involutory automorphisms θ and σ. For algebraic groups this has been researched e.g. in [Hel88], [HW93], [HS01] [HS04]. It would be worthwhile to extend this to Kac-Moody groups or even arbitrary groups with a root group datum.

APPENDIX

C

PHAN THEORY USING MOUFANG SETS

In many ways [BS04], can be considered the origin of the theory of flips, laying the foundation for what we now call Phan theory. There, a special case of a flip of a (spherical) building is described (albeit in disguise, and the term "flip" is not even used). The geometric setup was already briefly sketched in Example 4.1.4.

In loc. cit., only finite fields are covered, using counting arguments which fail over infinite fields. In this appendix, we briefly sketch how these counting arguments can be replaced by Moufang set arguments as in Section 3.3.1, e.g. Lemma 3.3.5. We prove the following:

Theorem C.1. *Let $\mathcal{C}$ be a Moufang twin building of type A_n, $n \geq 3$, and assume that all panels contain more than* 10 *elements. Let θ be a proper flip for which direct descent into $\mathcal{C}^\theta$ is possible. If $\mathcal{C}^\theta$ is residually connected, then $\mathcal{C}^\theta$ is simply connected.*

Note that by the results in Chapter 4, $\mathcal{C}^\theta$ is residually connected and direct descent is possible if all root groups are uniquely 2-divisible, or if the flip is semi-linear. The latter is the case for the flip used in [BS04].

By the classification of spherical Moufang buildings, it is known that an A_n building for $n \geq 3$ comes from a (left or right) vector space over a skew field $\mathbb{K}$. Based on this knowledge, Tits' Lemma and Theorem C.1 yield a presentation of the group $\mathrm{SU}_{n+1}(\mathbb{K})$ as an amalgam of unitary subgroups $\mathrm{SU}_2(\mathbb{K})$ and $\mathrm{SU}_3(\mathbb{K})$. A classification of Phan amalgams over arbitrary (skew) fields would then imply a Phan-type theorem of type A_n for $\mathrm{SU}_{n+1}(\mathbb{K})$. All in all, we get:

Corollary C.2. *Let $\mathcal{C}$ be a Moufang twin building of type A_n, $n \geq 3$, defined over a skew field $\mathbb{K}$ with $|\mathbb{K}| > 10$. Let θ be a proper flip. If $\operatorname{char} \mathbb{K} \neq 2$ or if θ is semi-linear, then $\mathcal{C}^\theta$ is residually connected and simply connected.*

The remainder of this appendix is dedicated to proving Theorem C.1. Throughout, $\mathcal{C}$ denotes a Moufang building of type A_n endowed with a flip θ. Let (W, S) be the corresponding Coxeter system of type A_n. We will adopt the view of a building as an incidence (pre)geometry for our arguments (see [BC] or [Pas94]). In particular, elements of type 1 and 2 of the building $\mathcal{C}$ will be called **points** and **lines**, respectively (i.e., they correspond to chamber residues of type $S \setminus \{1\}$ and $S \setminus \{2\}$, respectively). We denote the incidence pregeometry coming from $\mathcal{C}^\theta$ by $\mathcal{G}^\theta$.[1] Its points and lines form a subset of those of the building and to avoid confusion we will refer to them as $\mathcal{G}^\theta$-points and $\mathcal{G}^\theta$-lines. If two $\mathcal{G}^\theta$-points are joined by a $\mathcal{G}^\theta$-line, we will call them $\mathcal{G}^\theta$-collinear. Note that we drop the twin building point of view and instead consider $\mathcal{C}$ as a plain spherical building. Consequently since θ is a proper flip, it maps elements of type i to elements of type $n-i+1$.

Flip-flop systems coming from A_n buildings

Recall that the collinearity graph Γ associated with $\mathcal{C}^\theta$ is the graph on $\mathcal{G}^\theta$-points in which two points are adjacent whenever they are incident to a common $\mathcal{G}^\theta$-line.

The remark before [BS04, Lemma 2.2] that every line of the geometry contains q^2-q points of course makes no sense for infinite fields. We instead use the following:

Lemma C.3. *The points (resp. lines)* not *in $\mathcal{G}^\theta$ of the pointrow (resp. pencil) of a $\mathcal{G}^\theta$-line form a proper generalized Moufang subset.*

Proof. We argue for the pointrow of a line, the dual case is similar. Apply θ to the points in the pointrow of a $\mathcal{G}^\theta$-line, then project them back. This induces a permutation of the pointrow compatible with the Moufang structure on it. The points which are fixed are precisely those not in $\mathcal{G}^\theta$. By Lemma 3.3.4 this is a generalized Moufang subset X. Since $\mathcal{G}^\theta$ is a geometry (due to being residually connected), the line L contains a $\mathcal{G}^\theta$-point, whence X is a proper subset. □

Lemma C.4 (Lemma 2.2 in loc. cit.)**.** *If L is a $\mathcal{G}^\theta$-line and p a $\mathcal{G}^\theta$-point, then the $\mathcal{G}^\theta$-points on L which are $\mathcal{G}^\theta$-collinear to p lie in the complement of two proper generalized Moufang subsets. In particular, if L contains more than 5 points then p is $\mathcal{G}^\theta$-collinear to a $\mathcal{G}^\theta$-point on L.*

Proof. Take any $\mathcal{G}^\theta$-line L' containing p, take any $\mathcal{G}^\theta$-point p' on L. By the building axioms, the two flags (p, L') and (p', L) lie in a common apartment (an n-simplex). In particular, p is collinear to every point on L, and every line in the pencil of p intersects L. Applying Lemma C.3 and projecting suitably, it follows that the $\mathcal{G}^\theta$-points of L which are $\mathcal{G}^\theta$-collinear to p lie in the complement of two proper generalized Moufang subsets. Finally Lemma 3.3.7 implies that if there are more than 5 points on L, this complement is non-empty. □

From now we will assume that all panels contain at least 11 elements, which follows if the underlying (skew) field $\mathbb{K}$ satisfies $|\mathbb{K}| \geq 10$.

[1] In [BS04], $\mathcal{G}^\theta$ is called $\mathcal{N}$

Simple connectivity

The following is heavily based on Section 3 of [BS04], where a description of homotopy in incidence geometry and some important Lemmas are given; it is best to read up there first before attempting to understand the following. Note that the proofs of Lemma 3.2, Corollary 3.3 and Lemma 3.4 in loc. cit. apply almost verbatimly to our setup.[2] Therefore, we merely have to adapt Lemmas 3.5, 3.6 and 3.7 to complete the proof of Theorem C.1.

Before we start, we need the following auxiliary result. We omit the proof which is similar to that of Lemma 3.3.7.

Lemma C.5. *Let $\mathbb{M}(X, U)$ be a Moufang set, and let Y_1, Y_2, Y_3 be three proper generalized Moufang subsets. Then $X = Y_1 \cup Y_2 \cup Y_3$ implies $|X| \leq 10$.* ∎

Note that for the projective line over the field $\mathbb{F}_{q^2}$, this is equivalent to asking $q^2 + 1 > 10$, i.e., $q > 3$ (the same bound as in [BS04]).

In the following, we identify lines, planes, hyperplanes etc. with the collection of all points incident with them. This is easily justified e.g. by identification of the elements of type i with proper subspaces of dimension i of $\mathbb{K}^{n+1}$.

We call two points x and y *perpendicular* and write $x \perp y$ if $x = \operatorname{proj}_{\langle x,y\rangle} \theta(y)$, equivalently, if $x \in \theta(y)$. Since θ is an involution, this is clearly a symmetric relation. Note that $x \in \mathcal{G}^\theta$ if and only if $x \notin \theta(x)$. The following observation is an immediate consequence of the definition:

Lemma C.6. *If two $\mathcal{G}^\theta$-points are perpendicular, then they are $\mathcal{G}^\theta$-collinear.* ∎

Lemma C.7 (Lemma 3.5 in loc. cit.)**.** *Every triangle in Γ is decomposable.*

Proof. Let $\gamma = abca$ be a triangle (3-cycle) in Γ. If the plane $U = \langle a, b, c\rangle$ is in $\mathcal{G}^\theta$ (i.e., $\theta(U) \cap U = 0$) then γ is geometric. So suppose $U \notin \mathcal{G}^\theta$. Then $U \cap \theta(U)$ must be a point outside $\mathcal{G}^\theta$ (it cannot be a line, since U contains $\mathcal{G}^\theta$-lines). If $n \geq 4$, we can use the direct descent property to find an element in $\mathcal{G}^\theta$ of type ≥ 4 incident to U, which then necessarily is incident to all of a, b, c, and hence γ is geometric.

This leaves us with $n = 3$. We will first deal with the case where two points on γ (say, a and b) are perpendicular. In this case we say that γ is of *perp type.* Let $W = \theta(\langle a, c\rangle)$, which is a line of $\mathcal{G}^\theta$. Any $\mathcal{G}^\theta$-point $d \in W$ is by construction perpendicular to both a and c, hence $\mathcal{G}^\theta$-collinear to both. By Lemma C.4, the points in the complement of two proper generalized Moufang subsets of W are also $\mathcal{G}^\theta$-collinear with b. If d is a $\mathcal{G}^\theta$-point on W that is $\mathcal{G}^\theta$-collinear with b then we say that d is *good* if the triangle $dbcd$ is geometric, and that it is *bad* otherwise.

[2] In Lemma 3.4, a somewhat hidden induction argument is used which assumes that the residue of a hyperplane is isomorphic to a flip-flop geometry of lower rank; however, that assumption is not necessary, residually connectedness and a small refinement of the proof suffice.

We claim that the bad points form a proper generalized Moufang subset of W. Indeed, if d is bad then the plane π spanned by b, c, d is not mapped to an opposite plane by θ, that is $s := \theta(\pi) \cap \pi$ is non-empty. Since π contains $\mathcal{G}^\theta$-lines, s must be a point outside $\mathcal{G}^\theta$. Now $s \in \theta(\langle b, c\rangle)$ which is a $\mathcal{G}^\theta$-line. But the points outside $\mathcal{G}^\theta$ on a $\mathcal{G}^\theta$-line form a proper generalized Moufang subset Y. Since the bad d bijectively correspond to the s in Y, the claim follows.

Using Lemma C.5 and the hypothesis that all panels contain more than 10 elements, we conclude that a good point d exists. Since a is perpendicular to b by assumption and d by construction, and since $\langle b, d\rangle$ is a $\mathcal{G}^\theta$-line, the plane $\langle a, b, d\rangle$ is an element of $\mathcal{G}^\theta$. Hence *abda* is a geometric triangle. Similarly *adca* is geometric, since d is perpendicular to a and c. Also, *dbcd* is geometric, since d is good. Hence $\gamma = abca$ is decomposable.

Finally, let $\gamma = abca$ be arbitrary. Let $W = \theta(\langle a, c\rangle)$. By Lemma C.4, the points in the complement of two proper generalized Moufang subsets of W are also $\mathcal{G}^\theta$-collinear with b . Let d be one of these points. Then all three triangles *abda*, *dbcd* and *adca* are of perp type, hence decomposable by the above. We conclude that all triangles γ are decomposable. □

Lemma C.8 (Lemma 3.6 in loc. cit.)**.** *Every* 4*-cycle in* Γ *is decomposable.*

Proof. Let $\gamma = abcda$ be a 4-cycle. By Lemma C.4 (resp. its proof), the $\mathcal{G}^\theta$-points on L which are $\mathcal{G}^\theta$-collinear to both c and d simultaneously form the complement of three proper generalized Moufang subsets. By Lemma C.5 and since we assumed all panels to contain more than 10 points, we conclude the existence of a $\mathcal{G}^\theta$-point p which is $\mathcal{G}^\theta$-collinear to both c and d.

Now it follows by Lemma C.7 that γ is decomposable, since it is the product of the shorter cycles *apda*, *bcpb* and *cdpc*. □

Lemma C.9 (Lemma 3.7 in loc. cit.)**.** *Every* 5*-cycle in* Γ *is decomposable.*

Proof. Let $\gamma = abcdea$ be a 5-cycle. By Lemma C.4, d is $\mathcal{G}^\theta$-collinear to some $\mathcal{G}^\theta$-point p on the line $\langle a, b\rangle$.

Now it follows by Lemmas C.7 and C.8 that γ is decomposable, since it is the product of the shorter cycles *bcdpb* and *apdea*. □

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INDEX

Wissenschaftlicher Werdegang

24. April 1980	Geburt in Darmstadt
1990 – 1993	Gymnasium in Darmstadt
1993 – 1999	Gymnasium in Lahnstein
1999	Abitur
2000 – 2005	Studium der Mathematik an der TU Darmstadt
19. Okt. 2005	Mathematikdiplom TU Darmstadt
Nov. 2005 – März 2009	Promotionsstudium an der TU Darmstadt
Nov. 2005 – Okt. 2009	Wissenschaftlicher Mitarbeiter in Lehre und Forschung am Fachbereich Mathematik der TU Darmstadt
April 2009	Promotion