The Navier–Stokes Equations with Elastic Boundary and Boundary Conditions of Friction Type

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Introduction

In the first part of the 19th century a set of partial differential equations has been introduced by Claude-Louis Navier to model the behaviour of a viscous fluid. Before these publications, one focus in mathematical fluid mechanics had been the study of perfect fluids which are modelled by a set of equations introduced by Euler in the 1750s. As depicted in [28], Navier's novum was the addition of a new internal force $\mu(\Delta u + 2\nabla(\operatorname{div}(u)))$ to the Euler equations that takes into account the non-uniformity of the motion of the fluid. His new set of equations proved to be very influential and became known as the Navier–Stokes equations which are used to this day. For an incompressible fluid they have the form

$$\begin{aligned} \partial_t u - \mu\Delta u + (u \cdot \nabla)u + \nabla p = f \quad &\text{in } (0,T) \times \Omega, \\ \operatorname{div} u = 0 \quad &\text{in } (0,T) \times \Omega \end{aligned}$$

where Ω denotes a fixed domain in $\mathbb{R}^n$ that is filled with the incompressible fluid, $\mu > 0$ is the viscosity constant, $u : (0,T) \times \Omega \to \mathbb{R}^n$ denotes the velocity field of the fluid, $p : (0,T) \times \Omega \to \mathbb{R}$ is the pressure of the fluid and $f : (0,T) \times \Omega \to \mathbb{R}^n$ is a given external force. In the case that Ω has a boundary, boundary conditions have to be imposed on u for the sake of uniqueness. One of the most prominent boundary conditions is the homogeneous Dirichlet boundary condition which has the simple form

$$u_{|\partial\Omega} = 0 \quad \text{on } (0,T) \times \partial\Omega$$

and means that the fluid adheres to the boundary. Such a boundary condition is desirable from a purely mathematical point of view since it allows, for instance, the application of well-known inequalities often used in the theory of partial differential equations like Poincaré's and Korn's inequality. It is also heavily used for numerical methods even beyond the field of fluid dynamics like in electromagnetism, where it is frequently used for the analysis of power cables, see [23] for instance. Nevertheless, one challenge in mathematical modeling is the weighing up of physical realism on one hand and mathematical comprehensibility on the other hand. And while the Dirichlet boundary condition has mathematical advantages, it is not completely without paradox from a physical point of view. For example, Hillairet [25] has shown, when imposing a Dirichlet boundary condition, that a body surrounded by a viscous incompressible fluid in a cavity cannot reach the boundary of the cavity, assuming that the cavity has certain geometrical properties. In a later work Hillairet *et al.* [20] have considered a different type of boundary condition, namely a slip boundary condition and have proved that the problem of a lacking collision is circumvented in this case.

In this work we consider two classes of boundary conditions that are more complex but prove to be more physically realistic than the Dirichlet boundary condition. Both boundary conditions have in common that the Cauchy stress tensor

$$\sigma := (2\mu D(u) - pI)\, n \quad \text{with} \quad D(u) := \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$

and where n denotes the exterior unit normal on $\partial\Omega$, plays the role of an internal force which causes the fluid to move on the boundary.

One boundary condition we consider, models friction of the fluid with the boundary. That is to say, that the fluid adheres to the boundary if the tangential component of the stress tensor σ is smaller than a given threshold $\kappa > 0$. However, if σ_τ, the tangential component of σ, becomes large enough the fluid may start to slip in the reverse direction of σ_τ. Mathematically speaking,

$$u_\tau = \begin{cases} 0, & \text{if } |\sigma_\tau| < \kappa \\ -\alpha\sigma_\tau, & \text{if } |\sigma_\tau| = \kappa \text{ with } \alpha \geq 0 \end{cases} \quad \text{and} \quad u_n = 0.$$

The threshold $\kappa > 0$ can be understood as measuring the adherence of the material that $\partial\Omega$ consists of. The larger κ, the easier the fluid adheres to the boundary. Covering the case that $\partial\Omega$ may consist of different materials, we will also consider κ as a function on $\partial\Omega$, not necessarily constant.

The mathematical analysis of this topic goes back to H. Fujita in the nineties of the last century, see [17] and also [31] for further references, where the existence of weak solutions for the stationary and instationary Stokes and Navier–Stokes system as well as the stationary Navier–Stokes case have been proved. A thorough analysis of the stationary Stokes case in bounded smooth domains of $\mathbb{R}^d$, $d \geq 3$, including H^2–results and the Stokes resolvent problem, is due to Saito [31]. To the knowledge of the author, the existence of regular solutions in space, *i.e.*, $u \in L^2(0,T,H^2(\Omega))$, is yet an unsolved problem. However, the existence of strong solutions with respect to time, *i.e.*, $u \in W_2^1(0,T,L^2(\Omega))$, has been shown by Kashiwabara [27] for finite time intervals. Weak instationary solutions in two and three dimensions were also constructed by L. Bălilescu, J. San Martín, and T. Takahashi [6], together with some numerical experiments of Stokes and Navier–Stokes flow in a cylinder with a partly Coulomb boundary condition. In [5] these authors considered the motion of a rigid body in a fluid with Coulomb boundary condition on both the boundary of the container and the rigid body; they proved that the rigid body may touch the boundary of the container in finite time in contrast to the case of Dirichlet boundary conditions for smooth boundaries as mentioned previously.

The second problem we will consider, is a free boundary problem. We consider a cylindric domain Ω_0 that is filled with a fluid at time $t = 0$. We assume that the velocity of the fluid vanishes on the boundary, except on the upper part of it. As time progresses, the upper part of the boundary moves due to the internal force $\sigma \cdot e_n$, as well as due to an additional external force g applied to it. Hence, while the rest of the boundary is rigid, we can view the upper part of the boundary Γ_t as an elastic plate that depends on time. We assume that the vertical displacement of the elastic plate will be tracked by a scalar function h, that describes the height of the upper boundary. Note that we ignore in-plane motions in this model. The first part of the boundary condition on Γ_t reads

$$\partial_t^2 h + \Delta^2 h = \sigma(t) \cdot e_n + g$$

where $\sigma(t) = (2\mu D(u) - pI)n_t$ and n_t is the exterior unit normal on Γ_t. This means that h is the solution to an undamped plate equation with the force on the right-hand side given by $\sigma(t) \cdot e_n + g$. Furthermore, we impose the boundary condition

$$u = V_{\Gamma_t} \text{ on } \Gamma_t$$

where V_{Γ_t} denotes the velocity of Γ_t. Since we ignore in-plane motions, this means that the tangential component of u vanishes and the normal component of u coincides with $\partial_t h$.

This boundary condition has been studied to quite an extent in the literature. Weak solutions in L^2–spaces have been constructed by Chambolle *et al.* [10] when adding a viscous damping term of either the form $\Delta^2 \partial_t h$ or $-\Delta \partial_{tt} h$ in the plate equation. Similar results have been shown by Grandmont [22] in the case of a three-dimensional cavity where one part of the boundary is assumed to be elastic and the other one rigid; weak solutions are shown to exist until a possible epoch where intersections of the two boundaries occurs. Fluid structure interaction problems with a classical nonlinear von Kármán shallow shell of finite thickness that allow for both transversal and lateral displacements are considered by Chueshov and Ryzhkova [13]. Lengeler and Růžička [29] discussed the case of a Koiter shell instead of the flat plate and, hence, replaced the operator Δ^2 by an operator better suited for non-flat boundaries. Similarly, without use of a damping term the existence of weak solutions is guaranteed until a possible point in time where self-intersections of the shell occurs. In the case of a bounded domain and no damping we still have the existence of a contraction semigroup, see Badra and Takahashi [3, Proposition 3.4]. Finally, for (local) strong solutions we refer to Beiraõ da Veiga [7].

This work is structured as follows: In Chapter 1 we introduce basic notations, conventions and functions spaces, as well as some auxiliary results that will be used throughout this work.

In Chapter 2 we consider the free boundary problem in an abstract setting. To be more specific, we consider the boundary problem on the upper half-space that serves as a fixed reference configuration. Here we show the existence of solutions that belong to homogeneous Sobolev spaces. First, we will consider the case of vanishing initial data by using the partial Fourier transform. Secondly, we identify suitable spaces for the initial data. This is accomplished by exploiting the explicit form of the undamped plate equation.

In Chapter 3 we consider a boundary condition of friction type. Since friction is a non-linear phenomenon, this boundary condition is non-linear, too, which will complicate the mathematical analysis and we are mostly interested in retrieving typical results that are well-known in the Dirichlet case. We show existence of weak solutions for the two and three dimensional case, and strong solutions in time for the two dimensional case and also for the three dimensional case if the time interval or the given data are small enough. In the case $d = 2$ or small data, we can also show the existence of a global, strong in time solution. This will be achieved by using the Galerkin method with a proper linearisation of the non-linear boundary condition that depends on the initial value $u(0) = u_0$. Moreover, we show strong-weak uniqueness and consider the limit cases of $\kappa \to \infty$ and $\kappa \to 0$ in a proper sense. The analysis of this problem will be done for the case of manifolds.

In Chapter 4 we consider a combination of the two previous boundary conditions. We will show existence of a weak solution for a free boundary problem. In contrast to the previous chapter, this time $\kappa > 0$ measures the elasticity of the material that the plate consists of. Here we apply the Galerkin method, as well, but due to the fact that the domain on which the solution is defined does not necessarily have a Lipschitz boundary we need to regularize the equation first and use Schauder's fixed-point theorem later.

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Zusammenfassung in deutscher Sprache

In dieser Arbeit betrachten wir die inkompressiblen Navier-Stokes-Gleichungen aus der Strömungsmechanik mit zwei verschiedenen Randbedingungen.

Ein Problem, das wir betrachten, ist ein freies Randwertproblem. Hierbei bewegt sich der obere, elastische Rand eines zylindrischen Gefäßes, welches eine viskose Flüssigkeit beinhaltet, aufgrund einer vom Geschwindigkeitsfeld der Flüssigkeit induzierten, inneren Kraft, sowie aufgrund einer gegebenen äußeren Kraft. In Kapitel 2 werden wir diese Problem in einer festen Referenzkonfiguration, nämlich im oberen Halbraum, betrachten. Mittels partieller Fouriertransformation werden zuerst im Fall verschwindender Anfangsdaten die Existenz von Lösungen zeigen, welche in homogenen Sobolevräumen liegen. Unter Ausnutzung der genauen Form der mathematischen Formulierung der Randbedingung werden wir im anschließenden Schritt geeignete Räume für die Anfangswerte identifizieren.

Die zweite Randbedingung modelliert Reibung einer viskosen Flüssigkeit am Rand. Da Reibung ein nichtlineares Phänomen ist, ist auch die mathematische Formulierung der Randbedingung nichtlinear. Wir sind in Kapitel 3 im Wesentlichen damit beschäftigt klassische Resultate, die zum Beispiel im Fall der Dirichlet-Randbedingung gelten, auch für diese Randbedingungen zu zeigen. Wir zeigen die Existenz von globalen, schwachen Lösungen, die überdies stark in der Zeit sind im zweidimensionalen Fall. Im dreidimensionalen Fall zeigen wir Kurzzeitexistenz von starken Lösungen und Langzeitexistenz, falls die gegebenen Daten klein genug sind. Im dazu verwendeten Galerkinverfahren werden wir hierbei eine geschickte Linearisierung der Randbedingung verwenden, die vom Anfangswert abhängt. Wir zeigen ebenfalls schwach-starke Eindeutigkeit. In der Formulierung der Randbedingung wird das Haftvermögen des Materials, aus dem der Rand besteht, mit dem Parameter $\kappa > 0$ quantifiziert, und wir werden abschließend die Grenzfälle $\kappa \to \infty$ und $\kappa \to 0$ betrachten.

Im Kapitel 4 betrachten wir ein freies Randwertproblem, welches die vorherigen beiden Probleme kombiniert. Hierbei misst $\kappa > 0$ die Elastizität des oberen Randes. Hier werden wir die Existenz einer schwachen Lösung ebenfalls mit dem Galerkinverfahren nachweisen. Da das Gebiet, auf dem das Geschwindigkeitsfeld der Flüssigkeit definiert ist, keinen Lipschitz-Rand haben muss, werden wir das Problem zunächst regularisieren und anschließend ein Fixpunktargument verwenden.

CHAPTER 1

Preliminaries

In this chapter we collect basic notation, conventions and state integral operators and the general function spaces used throughout this work.

1.1. Basic Notation and Conventions

The symbol $\mathbb{Z}$ denotes the set of integers and the subset $\mathbb{N}$ denotes the set of positive integers. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ is the set of all non-negative integers. $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, where $\operatorname{Re} z$ and $\operatorname{Im} z$ are the real- and imaginary part, respectively, of any complex number $z \in \mathbb{C}$. Since we often consider vector-valued functions, we introduce $\mathbb{R}^n$ as the n-ary Cartesian product of $\mathbb{R}$ for all $n \in \mathbb{N}$. The sets $\mathbb{C}^n$, $\mathbb{Z}^n$ and $\mathbb{N}^n$ are defined analogously.

For a complex number $z \in \mathbb{C}$ we define its absolute value by

$$|z| := \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2},$$

and for an element $a \in \mathbb{R}^n$ we define its norm by

$$|a| := \sqrt{\sum_{i=1}^{n} a_i^2}$$

where $a = (a_1, ..., a_n)^T$. For $m \in \{1, ..., n\}$ with $n \in \mathbb{N}$ the expression $e_m \in \mathbb{R}^n$ stands for the m-th basis vector of $\mathbb{R}^n$, *i.e.* for any $i \in \{1, ..., n\}$ the term $(e_m)_i$ equals 1 if $m = i$, and otherwise $(e_m)_i = 0$. The expressions $x \cdot y$ and $\langle x, y \rangle$ for elements $x, y \in \mathbb{R}^n$ denote the standard scalar product of $\mathbb{R}^n$, namely

$$x \cdot y := \langle x, y \rangle := \sum_{i=1}^{n} x_i y_i.$$

We also define an inner product for matrices. For two matrices $A, B \in \mathbb{R}^{n \times m}$ with $n, m \in \mathbb{N}$, we write their Frobenius inner product as

$$A : B := \langle A, B \rangle := \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ij}$$

where a_{ij} denotes the corresponding entry of A for all $i \in \{1, ..., n\}$, $j \in \{1, ..., m\}$. Analogously for b_{ij}. The expression x^β with $x \in \mathbb{R}^n$ and $\beta \in \mathbb{N}_0^n$ is to be understood as $x^\beta := \prod_{i=1}^n x_i^{\beta_i}$.

For all $n \in \mathbb{N}$ with $n \geq 2$ we denote by $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_n > 0\}$ the upper half-space and its boundary $\partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n = 0\} = \mathbb{R}^{n-1} \times \{0\}$. Since $\mathbb{R}^{n-1} \times \{0\}$ is isomorphic to $\mathbb{R}^{n-1}$ we will often identify $\partial\mathbb{R}^n_+$ with $\mathbb{R}^{n-1}$.

For a vector space X with norm $\|\cdot\|$ the set $B_R(a)$ with $a \in X$ and $R > 0$ is the set of all elements $x \in X$ that satisfy $\|x - a\| < R$. Whenever we consider a normed vector space we may write it as tuple $(X, \|\cdot\|_X)$ where $\|\cdot\|_X$ is the corresponding norm that X is equipped with. Since in many cases the norm, we consider the vector space with, will be clear from the context, we omit $\|\cdot\|_X$ and simply write X instead. For a vector space X with dual space X' the term $\langle x', x\rangle_{X',X}$ denotes the dual pairing of an element $x' \in X'$ and an element $x \in X$. The vector spaces considered will mostly be normed vector spaces. When considering a dual space X' of a real or complex, normed vector space $(X, \|\cdot\|_X)$, we tacitly assume that X' is equipped with the induced norm

$$\|x'\|_{X'} := \sup_{x \in X, \|x\|_X \leq 1} |x'(x)|$$

for all $x' \in X'$, if not mentioned differently. With X^n we denote the n-ary Cartesian power of X for $n \in \mathbb{N}$ and, when $x \in X^n$ for $n \in \mathbb{N}$, we will often omit n and simply write $x \in X$ in case there is no confusion.

For a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, we formally define the divergence of f by $\operatorname{div} f = \sum_{i=1}^n \partial_i f_i$ where $\partial_i := \partial_{x_i}$ is the partial derivative in direction x_i for all $i \in \{1, ..., n\}$. We define the matrix-valued function ∇f on Ω such that $(\nabla f)_{ij} = \partial_j f_i$ for all $i, j \in \{1, ..., n\}$ and denote by $\nabla^T f$ its transpose. In contrast, for a scalar-valued function $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ the term $\nabla g = (\partial_1 g, ..., \partial_n g)^T$ denotes a vector-valued function on Ω. Furthermore, the Lapacian of g is given by $\Delta g = \sum_{j=1}^n \partial_j^2 g$ with $\partial_i^2 := \partial_i \partial_i$ for all $i \in \mathbb{N}$. Generally, for all $k \in \mathbb{N}$ the expression ∂_i^k is defined as the application of ∂_i k-times and, even more generally, we set $D^\alpha g := \partial_1^{\alpha_1} ... \partial_n^{\alpha_n} g$ for all $\alpha \in \mathbb{N}_0^n$. Here we set $\partial_i^0 g = g$ for all $i \in \{1, ..., n\}$. The bi-Laplacian of g is defined by $\Delta^2 g = \sum_{i,k=1}^n \partial_k^2 \partial_i^2 g$. In case of a vector-valued function f, the Laplacian of f is vector-valued, too, and is defined via reduction to the scalar case, *i.e.* $(\Delta f)_i := \Delta(f_i)$ for all $i \in \{1, ..., n\}$. If $\tilde{f}$ is another $\mathbb{R}^n$-valued function on Ω, then $(f \cdot \nabla \tilde{f})_j := \sum_{k=1}^n f_k \partial_k \tilde{f}_j$ for all $j \in \{1, ..., n\}$. If, additionally, f or g depends on time, ∂_t stands for the time derivative. We note that these definitions are formal. The functions f and g need to be sufficiently smooth so that the above expressions are well-defined.

In the estimates in this work, there will often appear positive constants like C, c or $\tilde{c}$. These constants are generic constants whose exact value is not important for the purpose of this thesis. For this reason we will use a convention frequently used in the theory of partial differential equations in this work: When estimating a term, such that a constant appears on the right-hand side of the estimate, this constant can replace generic constants on the left-hand side, but, with a misuse of notation, it can still be denoted by one of the replaced constants or by a different generic constant. Using this convention, we can for example write for a positive function a depending on $n \in \mathbb{N}$

$$Ca(n) + c_1 a(n) \leq Ca(n).$$

In a literal sense this estimate is false, but, using this convention, the constant C on the right-hand side replaces the constant $C + c_1$ that we still denote by C. This convention helps to simplify and shorten estimates when the exact value of the constants is of no interest.

Whenever we want to emphasize that a constant C depends on a parameter, say ε, we write $C(\varepsilon)$ or C_ε. If the dependencies of C are clear or not crucial, like in many cases dependencies on the domain or dimension, we often omit the corresponding parameters.

1.2. Function Spaces

Let X, Y be topological vector spaces. The set $\mathcal{L}(X,Y)$ denotes the linear and continuous mappings from X to Y. The set $C(X,Y)$, also written as $C^0(X,Y)$, denotes all continuous functions defined on X and with image in Y. In the particular case $Y = \mathbb{R}$, we abbreviate $C(X,\mathbb{R})$ with $C(X)$. Let now X be an open subset of $\mathbb{R}^n$ and $k \in \mathbb{N}$, then $C^k(X,Y)$ is the subset of $C(X,Y)$ of all k-times continuously Fréchet differentiable functions. In turn, $C^k(\overline{X},Y)$ is the set of all functions in $C^k(X,Y)$ that have a continuous extension to $\overline{X}$ and the same holds true for all derivatives up to order k. With $C^\infty(X,Y)$ we denote the intersection of all spaces $C^k(X,Y)$, $k \in \mathbb{N}$; analogously for $C^\infty(\overline{X},Y)$. With $C_c^\infty(X,Y)$ we denote the subset of $C^\infty(X,Y)$ consisting of functions that have compact support in X and we define $C_c^\infty(\overline{X},Y)$ analogously. If X has the form $X = (a,b)$ with $a \in \mathbb{R}$ and $b \in (a,\infty]$, we also write $C^k(a,b;Y)$ instead of $C^k((a,b),Y)$ for all $k \in \mathbb{N}_0$.

Now we define L^p-spaces. Let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, $p \in [1,\infty]$ and let X be a $\sigma-$finite measure space with measure μ. Then the set $L^p(X,\mathbb{K})$ consists of all measurable, $\mathbb{K}$-valued functions f defined on X such that

$$\|f\|_{L^p(X,\mathbb{K})} := \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} < \infty \quad \text{if } 1 \le p < \infty$$

and

$$\|f\|_{L^\infty(X,\mathbb{K})} := \mu - \operatorname*{ess\,sup}_{x\in X} |f(x)| < \infty \quad \text{if } p = \infty.$$

Note that we identify two elements in $L^p(X,\mathbb{K})$ if they coincide μ-almost everywhere. If $X \subset \mathbb{R}^n$ is open and we consider $\mathbb{R}^n$ with the n-dimensional Lebesgue measure λ, the set $L^p_{\text{loc}}(X,\mathbb{K})$ denotes the measurable, locally integrable $\mathbb{K}$–valued functions on X, *i.e.* all measurable $\mathbb{K}$–valued functions h such that $h \in L^p(K,\mathbb{K})$ for every $K \subset X$ compact. Since the context is usually clear, we will typically omit $\mathbb{K}$ and just write $L^p(X)$. Also, we will often omit the term $d\mu$ in integrals. Furthermore, when $X \subset \mathbb{R}^n$ is open, we tacitly assume that we consider X equipped with the $n-$dimensional Lebesgue measure, if not mentioned differently.

Now we define Banach space–valued L^p–spaces, also known as Bochner spaces. Let Y be a real or complex Banach space, $p \in [1,\infty]$ and let $I \subset \mathbb{R}$ be a real interval in $\mathbb{R}$ equipped with the Lebesgue measure λ. We say that $f : I \to Y$ is a step function if there are Lebesgue measurable sets $B_k \in \mathbb{R}$ and $\alpha_k \in Y$ for all $k \in \{1,...,N\}$ with an $N \in \mathbb{N}$ such that

$$f(t) = \sum_{k=1}^{N} \alpha_k \chi_{B_k}(t) \quad \forall t \in I.$$

The expression χ_A denotes the indicator function of A for any set A such that $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. The space $L^p(I,Y)$ is given by all functions f on I with image in Y such that there exists a sequence of step functions $f_n : I \to Y$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \|f_n(t) - f(t)\|_Y = 0 \quad \text{for almost all } t \in I$$

and the mapping

$$I \ni t \mapsto \|f(t)\|_Y \in \mathbb{R}$$

is an element of $L^p(I)$. We set the norms as

$$\|f\|_{L^p(I,Y)} := \left(\int_I \|f(t)\|_Y^p \; d\lambda\right)^{\frac{1}{p}} \text{ if } 1 \le p < \infty$$

and

$$\|f\|_{L^\infty(I,Y)} := \operatorname*{ess\,sup}_{t\in I} \|f(t)\|_Y.$$

If $I = (a, b)$ with $a \in \mathbb{R}$ and $b \in (a, \infty]$, we also write $L^p(a, b; Y)$ instead of $L^p((a, b), Y)$. We refer to [14, Appendix E] for more information and properties of Bochner spaces.

Now, we consider the inhomogeneous Sobolev spaces $W_p^k(\Omega)$ with $k \in \mathbb{N}$, $p \in [1, \infty]$ and $\Omega \subset \mathbb{R}^n$ open. $W_p^k(\Omega)$ is a subspace of $L^p(\Omega)$ consisting of all $u \in L^p(\Omega)$ such that for all vectors $\alpha \in \mathbb{N}_0^n$ with $|\alpha| := \sum_{i=1}^n \alpha_i \leq k$, u has a weak derivative $D^\alpha u \in L^p(\Omega)$. That is to say, for all these $\alpha \in \mathbb{N}_0^n$ there exists a (unique) $g_\alpha \in L^p(\Omega)$ such that

$$\int_\Omega u \cdot D^\alpha \varphi \; d\lambda = (-1)^{|\alpha|} \int_\Omega g_\alpha \cdot \varphi \; d\lambda \quad \forall \varphi \in C_c^\infty(\Omega).$$

And we set $D^\alpha u := g_\alpha$. The space $W_p^k(\Omega)$ is a Banach space when equipped with the norm

$$\|u\|_{W_p^k(\Omega)} := \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^p(\Omega)}$$

and a Hilbert space in the case $p = 2$, where the inner product is given by

$$\langle u, v\rangle_{W_2^k(\Omega)} := \sum_{|\alpha|\leq k} \langle D^\alpha u, D^\alpha v\rangle_{L^2(\Omega)}.$$

The induced norm of this inner product is equivalent to $\|\cdot\|_{W_2^k(\Omega)}$. From now on, we always write $H^k(\Omega)$ instead of $W_2^k(\Omega)$ and we set $W_p^0(\Omega) := L^p(\Omega)$ for all $k \in \mathbb{N}$, $p \in [1, \infty]$. We refer to [1, Chapter 2-3] for more details on L^p– and Sobolev spaces.

1.3. Convolution and Fourier transform

For more information concerning the integral operators and the underlying function spaces presented in this section we refer to [34, Chapter 25-26].

For $f, g \in L^2(\mathbb{R}^n)$ we define the convolution of f and g labelled with $f * g$ as

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x - y)g(y) \; dy \quad \text{for a.e. } x \in \mathbb{R}^n.$$

The expression 'a.e.' is short for 'almost every'. Similarly, the expression 'a.a.' is short for 'almost all'. With the help of the substitution $y \mapsto x - y$, it is easy to verify that $f * g = g * f$ and via the Cauchy-Schwarz inequality it is easy to see that $f * g \in L^\infty(\mathbb{R}^n)$ with

$$\|f * g\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.$$

Furthermore, if even $f \in C_c^\infty(\mathbb{R}^n)$ holds, then we also have smoothness of $f * g \in C^\infty(\mathbb{R}^n)$. Although the convolution operator can be defined in a more general setting, we will only need it for L^2–functions.

As preparation for the next integral operator, we consider the Schwartz space

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n, \mathbb{C}) \mid \sup_{x\in\mathbb{R}^n} |x^\beta D^\alpha f| < \infty \quad \forall \alpha, \beta \in \mathbb{N}_0^n \right\}$$

and its dual space $\mathcal{S}'(\mathbb{R}^n) := (\mathcal{S}(\mathbb{R}^n))'$, known as the space of tempered distributions. The vector space $\mathcal{S}(\mathbb{R}^n)$ becomes a topological vector space if equipped with the Fréchet metric induced by the family of semi-norms $\sup_{x\in\mathbb{R}^n} |x^\beta D^\alpha \cdot|$ with $\alpha, \beta \in \mathbb{N}_0^n$. It is well-known that the Fourier transform $\mathcal{F}$, formally given by

$$(\mathcal{F}(f))(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx,$$

is a well-defined isomorphism from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ with inverse denoted by $\mathcal{F}^{-1}$. This allows us to define the Fourier transform of a function $g \in \mathcal{S}'(\mathbb{R}^n)$ as

$$\langle \mathcal{F}(g), f\rangle_{S'(\mathbb{R}^n),S(\mathbb{R}^n)} := \langle g, \mathcal{F}(f)\rangle_{\mathcal{S}'(\mathbb{R}^n),\mathcal{S}(\mathbb{R}^n)} \quad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Similarly, we define $\mathcal{F}^{-1}(g)$ which implies that $\mathcal{F}$ is a bijection from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, too. Furthermore, the Fourier transform can also be meaningfully defined on $L^2(\mathbb{R}^n, \mathbb{C})$. In fact, by the Plancherel theorem, it is an up to a constant isometric isomorphism from $L^2(\mathbb{R}^n, \mathbb{C})$ to $L^2(\mathbb{R}^n, \mathbb{C})$ such that the norms $\|\mathcal{F}(\cdot)\|_{L^2(\mathbb{R}^n)}$, $\|\cdot\|_{L^2(\mathbb{R}^n)}$, $\|\mathcal{F}^{-1}(\cdot)\|_{L^2(\mathbb{R}^n)}$ are pairwise equivalent on $L^2(\mathbb{R}^n, \mathbb{C})$.

Sometimes we write $\hat{u}$ instead of $\mathcal{F}(u)$ and elements of $\mathbb{R}^n$ will sometimes be written in the form $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$, and the phase variable ξ as (ξ', ξ_n). Similarly, vector fields $u = (u_1, \dots, u_n)^T$ can be split into $u' = (u_1, \dots, u_{n-1})^T$ and u_n. In particular, we write $0' = (0, \dots, 0) \in \mathbb{R}^{n-1}$. The Fourier transform of the time variable $t \in \mathbb{R}$ is denoted by τ. Since the Fourier transform with respect to the spatial variable will only be used on $\mathbb{R}^{n-1}$, it will, for the sake of simplicity, be denoted by $\mathcal{F}$ rather than $\mathcal{F}'$, its inverse by $\mathcal{F}^{-1}$ rather than $(\mathcal{F}')^{-1}$. However, the phase variable will still be written as ξ'.

1.4. Homogeneous Sobolev Spaces

Given $s \in \mathbb{R}$ we define the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ as

$$\dot{H}^s(\mathbb{R}^n) := \{u \in Z'(\mathbb{R}^n) : \mathcal{F}^{-1}(|\xi|^s \hat{u}) \in L^2(\mathbb{R}^n)\},$$

equipped with the norm

$$\|u\|_{\dot{H}^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(|\xi|^s \hat{u})\|_{L^2(\mathbb{R}^n)}.$$

Here, $Z'(\mathbb{R}^n) := \mathcal{S}'(\mathbb{R}^n)/P(\mathbb{R}^n)$ where $P(\mathbb{R}^n)$ denotes the space of all polynomials on $\mathbb{R}^n$. Equivalently, we can also view $Z'(\mathbb{R}^n)$ as the dual space of

$$\begin{aligned} Z(\mathbb{R}^n) :=& \{f \in \mathcal{S}(\mathbb{R}^n) \mid (D^\alpha \mathcal{F}(f))(0) = 0 \ \ \forall \alpha \in \mathbb{N}_0^n\} \\ =& \{f \in \mathcal{S}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} f \cdot P \, d\lambda = 0 \ \ \forall P \in P(\mathbb{R}^n)\}, \end{aligned}$$

where we consider $Z(\mathbb{R}^n)$ with the topology of $\mathcal{S}(\mathbb{R}^n)$. For the domain $\mathbb{R}^n_+$ we define the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n_+)$ via restriction:

$$\begin{aligned} \dot{H}^s(\mathbb{R}^n_+) &:= \left\{u|_{\mathbb{R}^n_+} : u \in \dot{H}^s(\mathbb{R}^n)\right\}, \\ \|u\|_{\dot{H}^s(\mathbb{R}^n_+)} &:= \inf\left\{\|v\|_{\dot{H}^s(\mathbb{R}^n)} : v|_{\mathbb{R}^n_+} = u\right\}. \end{aligned}$$

We have the following useful properties that we will frequently use.

Proposition 1.4.1. *Let $s \in \mathbb{R}$ and let $\Lambda = (-\Delta)^{1/2}$ denote the operator defined by $\Lambda u = \mathcal{F}^{-1}(|\xi|\mathcal{F}u)$.*

i) $\dot{H}^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

ii) The mapping $\Lambda^\omega \mathcal{F} : \dot{H}^s(\mathbb{R}^n) \to \dot{H}^{s-\omega}(\mathbb{R}^n)$, $u \mapsto \mathcal{F}^{-1}(|\xi|^\omega \mathcal{F}(u))$ is an isomorphism and maps $Z'(\mathbb{R}^n)$ onto itself for all $\omega \in \mathbb{R}$.

iii) $(\dot{H}^s(\mathbb{R}^n), \|\cdot\|_{\dot{H}^s(\mathbb{R}^n)})$ is a Banach space.

iv) If $s > \frac{1}{2}$ then there is a bounded linear mapping $\operatorname{tr} : \dot{H}^s(\mathbb{R}^n_+) \to \dot{H}^{s-\frac{1}{2}}(\partial\mathbb{R}^n_+) \cong \dot{H}^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$, the trace operator, such that $\operatorname{tr}(\phi)(x') = \phi(x', 0)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $0 \notin \operatorname{supp}\mathcal{F}(\phi)$.

For a proof of iv), use that $\mathcal{S}_0 := \{\phi \in \mathcal{S}(\mathbb{R}^n) |\ 0 \notin \operatorname{supp}\mathcal{F}(\phi)\}$ is dense in $\dot{H}^s(\mathbb{R}^n)$ according to [26, Lemma 1.1 i)] and then, mimic the proof of the inhomogeneous case, see [4, Theorem 1.63]. For i), ii) and iii) as well as for more information about homogeneous spaces see [35, Chapter 5].

Concerning Bochner spaces with respect to time over a Banach space X we write $b \in \dot{H}^k(0, T; X)$ meaning that $\partial_t^k b \in L^2(0, T; X)$ for $k \in \mathbb{N}$, $T \in (0, \infty]$. In most cases, X will be a homogeneous Bessel potential space $\dot{H}^s(\Omega)$, where $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$ or $\mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$.

1.5. Sobolev Spaces on Manifolds

In this section, M represents an orientable, smooth, compact, d-dimensional Riemannian manifold with metric g and boundary ∂M for $d \in \{2, 3\}$. Let $\mathfrak{X}(M)$ denote the set of smooth tangential vector fields on M and let $|\cdot|_{g_p}$ denote the norm induced by g_p for $p \in M$. In case there is no confusion, we will abbreviate $|\cdot|_{g_p} = |\cdot|_p$ or even $|\cdot|_{g_p} = |\cdot|$. The same abbreviation will be used for $\langle\cdot,\cdot\rangle_{g_p}$. Let ∇ denote the Levi-Civita connection. For the following, see also [24].

Definition 1.5.1. ***(Sobolev spaces for tangential vector fields)***
Let $q \in [1, \infty)$, $k \in \mathbb{N}$. Then $W^k_q(M)^d$ is defined as the completion of $\mathfrak{X}(M)$ with respect to the norm

$$\|f\|_{W^k_q(M)^d} := \left(\sum_{i=0}^{k} \int_M |\nabla^i f|^q \, dv(g)\right)^{1/q}$$

and in the case $q = \infty$ we set

$$W^k_\infty(M)^d := \{W^k_2(M)^d : \|f\|_{W^k_\infty(M)^d} < \infty\}$$

with

$$\|f\|_{W^k_\infty(M)^d} := \sum_{i=0}^{k} \operatorname{ess\,sup}_{p\in M} |\nabla^i f(p)|_p.$$

Concerning the integrand, in local coordinates we have by using the Einstein summation convention

$$\langle\nabla^l f, \nabla^l h\rangle_g = g_{i_1 i'_1} g^{j_1 j'_1} \cdot \ldots \cdot g^{j_l j'_l} \left(\nabla^l f\right)^{i_1}_{j_1,\ldots,j_l} \left(\nabla^l h\right)^{i'_1}_{j'_1,\ldots,j'_l}$$

where

$$\nabla^l f = \left(\nabla^l f\right)^{i_1}_{j_1,\ldots,j_l} \partial x_{i_1} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_l},$$

$$\nabla^l h = \left(\nabla^l h\right)^{i'_1}_{j'_1,\ldots,j'_l} \partial x_{i'_1} \otimes dx^{j'_1} \otimes \ldots \otimes dx^{j'_l}.$$

If no confusion can occur, we abbreviate $W_q^k(M)^d = W_q^k(M)$ or even $W_q^k(M)^d = W_q^k$. Note that on compact manifolds, metrics are equivalent, see [24, Proposition 2.2]. Thus, the monotonicity of the integral implies the independence of W_p^k from the choice of the metric g, since all these norms of W_q^k are equivalent.

Proposition 1.5.2. *Let $k \in \mathbb{N}$, $q \in (1, \infty)$, and M be a compact, orientable Riemannian manifold with boundary. Then we have: $f \in W_q^k(M)^d$ if and only if $f \in L^q(M)^d$ and there is a sequence $(f_m)_{m\in\mathbb{N}} \subset \mathfrak{X}(M)$ such that*

$$f_m \to f \text{ in } L^q(M)^d \quad \text{as } m \to \infty$$

and

$$\left\|\nabla^j (f_m - f_n)\right\|_{L^q(M)} \to 0 \quad \text{as } m, n \to \infty \; \forall j \in \{1, ..., k\}.$$

Additionally, if one of the above conditions holds true, then $\left|\nabla^j f_m\right|_{m\in\mathbb{N}}$ converges in $L^q(M)$, and we denote the limit by $\left|\nabla^j f\right|$ for all $j \in \{1, ..., k\}$.

See the remark after [24, Definition 2.1]. The following proposition establishes a connection between the Euclidean case and the case of manifolds.

Proposition 1.5.3. ***(Connection with the Euclidean case)*** *Let $(U_i, x_i)_{i\in\{1,...,m\}}$ be an atlas of M such that $x_i(\bar{U}_i)$ is well-defined for $i \in \{1, ..., m\}$ with positively oriented charts, a partition of unity $(\varphi_i)_{i\in\{1,...,m\}}$ subordinated to them and the canonical isomorphism $l_p : T_pM \to \mathbb{R}^d$ at $p \in M$. Then there holds:*

$$\begin{aligned} u \in W_q^k(M)^d &\Leftrightarrow u \in W_q^k(U_i)^d \quad \forall i \in \{1, ..., m\} \\ u \in W_q^k(U_i)^d &\Leftrightarrow l_{x_i^{-1}} \circ u \circ x_i^{-1} \in W_q^k(x_i(U_i))^d \quad \forall i \in \{1, ..., m\}. \end{aligned}$$

The corresponding operators

$$\begin{aligned} &S_i : L^q(M)^d \to L^q(U_i)^d, \;\; u \to u|_{U_i} \\ &\tilde{S} : L^q(U_1)^d \times ... \times L^q(U_m)^d \to L^q(M)^d, \;\; (u_1, ..., u_m) \mapsto \sum_{i=1}^m \varphi_i u_i \\ &T_i : L^q(U_i)^d \to L^q(x_i(U_i))^d, \;\; u \mapsto l_{x_i^{-1}} \circ u \circ x_i^{-1} \\ &T_i^{-1} : L^q(x_i(U_i))^d \to L^q(U_i)^d, \;\; u \mapsto l_{x_i}^{-1} \circ u \circ x_i \end{aligned}$$

are well-defined, linear and continuous. The same holds for the restricted versions

$$\begin{aligned} &S_{i_{|W_q^k(M)}} \in \mathcal{L}\left(W_q^k(M), W_q^k(U_i)\right), \; \tilde{S}_{|W_q^k(U_1)\times...\times W_q^k(U_m)} \in \mathcal{L}\left(W_q^k(U_1) \times ... \times W_q^k(U_m), W_q^k(M)\right) \\ &T_{i_{|W_q^k(U_i)}} \in \mathcal{L}\left(W_q^k(U_i), W_q^k(x_i(U_i))\right), \; T^{-1}_{i_{|W_q^k(x_i(U_i))}} \in \mathcal{L}\left(W_q^k(x_i(U_i)), W_q^k(U_i)\right) \end{aligned}$$

A proof of this statement is given in the appendix, see A.0.1.

Definition 1.5.4. *For $k \in \mathbb{N}$ and $s \in (0, 1)$ we define via real interpolation the fractional Sobolev spaces*

$$W_p^{k+s}(M) := \left(W_p^k(M), W_p^{k+1}(M)\right)_{s,p}.$$

Remark 1.5.5. Using the boundedness of the operators in Proposition 1.5.3, we get by interpolation that Proposition 1.5.3 holds for all real $k \geq 0$. This remark implies that many statements like embeddings for bounded, smooth domains are also valid in the case of compact, smooth manifolds.

Definition 1.5.6. *Let n be the outer normal vector on M. Then for any $a \in T_pM$ with $p \in \partial M$ we define the decomposition*

$$a_n := \langle a, n\rangle\, n \quad \text{and} \quad a_\tau := a - a_n.$$

We mention that there are versions of Korn's and Poincaré's inequality for the case of a manifold M with properties as above, namely

$$\int_M |\nabla u|^2 \le c \int_M |Du|^2 \quad \text{and} \quad \int_M |u|^2 \le c \int_M |\nabla u|^2$$

for every $u \in H^1(M)^d$ such that $u_{|\Gamma_D} = 0$ where $\Gamma_D \subset \partial M$ is open and non-empty. Here we have set $Du := \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$. The proof of Korn's inequality can be found in [12, Theorem 5.5] and for Poincaré's inequality, see [2, Theorem 1.1].

1.6. A Stokes-like Problem

In this section we want to solve a Stokes-like problem. In order to do so, we first consider the problem

$$\begin{aligned} \operatorname{div}(s) &= Z \text{ in } D, \\ s_{|\partial D} &= a \text{ on } \partial D \end{aligned}$$

on a bounded domain D with Lipschitz boundary.

Lemma 1.6.1. *Let $D \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary, and let $a \in W_q^{1-\frac{1}{q}}(\partial D)$, $Z \in L^q(D)$ with $q \in (1,\infty)$ and*

$$\int_D Z = \int_{\partial D} a \cdot n.$$

Then the system

$$\begin{aligned} \operatorname{div}(s) &= Z \text{ in } D, \\ s_{|\partial D} &= a \text{ on } \partial D \end{aligned}$$

has a solution $s \in W_q^1(D)$ such that

$$\|s\|_{W_q^1(D)} \le c\|a\|_{W_q^{1-\frac{1}{q}}(\partial D)} + \|Z\|_{L^q(D)}.$$

Proof. Due to [19, Theorem II.4.3] there exists an extension $A \in W_q^1(D)$ of a such that

$$\begin{aligned} A_{|\partial D} &= a \text{ on } \partial D, \\ \|A\|_{W_q^1(D)} &\le C\|a\|_{W_q^{1-\frac{1}{q}}(\partial D)}. \end{aligned}$$

Because of [19, Lemma III.3.1] and $\int_{\partial D} a \cdot n = \int_D Z$, there exists a $\sigma \in W_{q,0}^1(D)$ such that

$$\begin{aligned} \operatorname{div}\sigma &= -\operatorname{div} A + Z \text{ in } D, \\ \|\sigma\|_{W_q^1(D)} &\le C\left(\|\operatorname{div} A\|_{L^q(D)} + \|Z\|_{L^q(D)}\right). \end{aligned}$$

Now set $s := A + \sigma \in W_q^1(D)$. Then s has the required properties. □

Lemma 1.6.2. *Let $D \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary, and let $a \in H^{\frac{1}{2}}(\partial D)$, $z \in H^1(D)$ with*

$$\int_{\partial D} z \cdot n = \int_{\partial D} a \cdot n,$$

and $\tilde{f} \in (H_0^1(D))'$. Furthermore, let $B \in C^\infty\left(\overline{D}; \mathbb{R}^{3\times 3}\right)$ such that $B(d)$ is invertible for all $d \in \overline{D}$ with $B^{-1} \in C^\infty\left(\overline{D}; \mathbb{R}^{3\times 3}\right)$ and $B^T a = a$ on ∂D. Then the modified Stokes system

$$\begin{aligned} -\Delta \phi^\star + (B \cdot \nabla) p^\star &= \tilde{f} \qquad \text{in } D, \\ \operatorname{div}(B^T \phi^\star) &= \operatorname{div} z \ \text{ in } D, \\ \phi^\star &= a \qquad \text{on } \partial D \end{aligned} \tag{1.1}$$

has a unique, weak solution $\phi^\star \in H^1(D), p^\star \in L^2(D)$ such that $\int_D p^\star = 0$. Furthermore, we have the estimate

$$\|\phi^\star\|_{H^1(D)} + \|p^\star\|_{L^2(D)} \leq C(B)\left(\|\operatorname{div} z\|_{L^2(D)} + \|a\|_{H^{\frac{1}{2}}(\partial D)} + \|\tilde{f}\|_{(H_0^1(D))'}\right).$$

Proof. Because of Lemma 1.6.1 and $\int_{\partial D} a \cdot n = \int_D \operatorname{div} z$, there exists a $s \in H^1(D)$ such that

$$\begin{aligned} \operatorname{div} s &= \operatorname{div} z \text{ in } D, \\ s &= a \qquad \text{on } \partial D, \\ \|s\|_{H^1(D)} &\leq C(\|a\|_{H^{\frac{1}{2}}(\partial D)} + \|\operatorname{div} z\|_{L^2(D)}). \end{aligned} \tag{1.2}$$

We consider the problem

$$\int_D \langle \nabla \tilde{\phi}, \nabla \phi \rangle = -\int_D \langle \nabla\left(B^{-T} s\right), \nabla \phi \rangle + \langle \tilde{f}, \phi \rangle_{(H_0^1(D))', H_0^1(D)}$$

for all $\phi \in H_0^1(D)$ with $\operatorname{div}\left(B^T \phi\right) = 0$. Since the right-hand side defines a linear, bounded operator on $\{\phi \in H_0^1(D) \mid \operatorname{div}(B^T \phi) = 0\}$ we get a unique solution $\tilde{\phi} \in H_0^1(D)$ with $\operatorname{div}\left(B^T \phi\right) = 0$ by Riesz' theorem. Plugging in $\phi = \tilde{\phi}$ and using Hölder's inequality, we deduce the estimate

$$\begin{aligned} \|\tilde{\phi}\|_{H^1(D)} &\leq c\left(\|B^{-T} s\|_{H^1(D)} + \|\tilde{f}\|_{(H_0^1(D))'}\right) \leq C(B)\left(\|s\|_{H^1(D)} + \|\tilde{f}\|_{(H_0^1(D))'}\right) \\ &\leq C(B)\left(\|a\|_{H^{\frac{1}{2}}(\partial D)} + \|\operatorname{div} z\|_{L^2(D)} + \|\tilde{f}\|_{(H_0^1(D))'}\right). \end{aligned}$$

Recalling, that $B^T a = a$ on ∂D by assumption, we set

$$\phi^\star := \tilde{\phi} + B^{-T} s \in H^1(D)$$

and see that $\phi^\star$ satisfies $(1.1)_2$ and $(1.1)_3$, as well as

$$\|\phi^\star\|_{H^1(D)} \leq C(B)\left(\|a\|_{H^{\frac{1}{2}}(\partial D)} + \|\operatorname{div} z\|_{L^2(D)} + \|\tilde{f}\|_{(H_0^1(D))'}\right).$$

Using that B is invertible, and setting $\tilde{\varphi} := B^T \phi$ we obtain that the operator

$$F : H_0^1(D) \to \mathbb{R}, \quad \tilde{\varphi} \mapsto \int_D \left\langle \nabla \phi^\star, \nabla\left(B^{-T} \tilde{\varphi}\right) \right\rangle - \langle \tilde{f}, B^{-T} \tilde{\varphi} \rangle_{(H_0^1(D))', H_0^1(D)}$$

is bounded and vanishes on $\{\tilde{\varphi} \in H_0^1(D) \mid \operatorname{div}(\tilde{\varphi}) = 0\}$. By standard theory, see [19, Theorem III.5.3], there is a unique $p^\star \in L^2(D)$ such that $\int_D p^\star = 0$ and

$$F(\tilde{\varphi}) = \int_D p^\star \operatorname{div} \tilde{\varphi} \quad \forall \tilde{\varphi} \in H_0^1(D). \tag{1.3}$$

Due to [19, Lemma III.3.1] and $\int_D p^\star = 0$, there exists a $\varphi \in H_0^1(D)$ such that

$$\begin{aligned} \operatorname{div} \varphi &= p^\star \qquad\qquad \text{in } D, \\ \|\varphi\|_{H^1(D)} &\leq C \|p^\star\|_{L^2(D)}. \end{aligned}$$

Choosing $\tilde{\varphi} = \varphi$ in (1.3), we conclude

$$\begin{aligned} \|p^\star\|_{L^2(D)}^2 = F(\varphi) &\leq C(B) \left(\|\phi^\star\|_{H^1(D)} + \|\tilde{f}\|_{(H_0^1(D))'} \right) \|\varphi\|_{H^1(D)} \\ &\leq C(B) \left(\|a\|_{H^{\frac{1}{2}}(\partial D)} + \|\operatorname{div} z\|_{L^2(D)} + \|\tilde{f}\|_{(H_0^1(D))'} \right) \|p^\star\|_{L^2(D)} \end{aligned}$$

and thus,

$$\|p^\star\|_{L^2(D)} \leq C(B) \left(\|a\|_{H^{\frac{1}{2}}(\partial D)} + \|\operatorname{div} z\|_{L^2(D)} + \|\tilde{f}\|_{(H_0^1(D))'} \right).$$

Going back from $\tilde{\varphi}$ to ϕ, (1.3) implies that

$$\int_D \langle \nabla \phi^\star, \nabla \phi \rangle - \int_D p^\star \operatorname{div} \left(B^T \phi \right) = \langle \tilde{f}, \phi \rangle_{(H_0^1(D))', H_0^1(D)} \quad \forall \phi \in H_0^1(D),$$

which is the weak formulation of $(1.1)_1$.

If the terms on the right-hand side vanish, then by Riesz' theorem $\phi^\star = 0$. This implies $F = 0$ and, since $p^\star \in L^2(D)$ with $\int_D p^\star = 0$ is unique in (1.3), we conclude $p^\star = 0$, too. □

CHAPTER 2

An L^2-Approach to Viscous Flow in the Half Space with a Free Elastic Surface

In this chapter we consider the following linear, coupled system

$$
\begin{aligned}
\partial_t u - \mu\Delta u + \nabla p &= f && \text{in } (0,T)\times\mathbb{R}^n_+,\\
\operatorname{div} u &= 0 && \text{in } (0,T)\times\mathbb{R}^n_+,\\
u|_{x_n=0} &= (0,...,0,\partial_t h + g_1) && \text{on } (0,T)\times\mathbb{R}^{n-1},\\
\partial_t^2 h + \Delta^2 h &= -p_{|x_n=0} + g_2 && \text{on } (0,T)\times\mathbb{R}^{n-1},\\
u|_{t=0} &= u_0 && \text{in } \mathbb{R}^n_+,\\
(h,\partial_t h)|_{t=0} &= (h_0,(u_0)_n) && \text{in } \mathbb{R}^{n-1},
\end{aligned}
\tag{2.1}
$$

where $u : \mathbb{R}^n_+ \times (0,T) \to \mathbb{R}^n$ denotes the velocity of an incompressible, viscous fluid in $\mathbb{R}^n_+$ and p is its pressure coupled with a boundary function h satisfying an undamped Kirchhoff-type plate equation. The constant $\mu > 0$ is the viscosity constant. This problem is a transformed and linearised version of the following more abstract system of equations

$$
\begin{aligned}
\partial_t u - \operatorname{div}(2\mu D(u) - pI) &= f - (u\cdot\nabla)u && \text{in } (0,T)\times\Omega_t,\\
\operatorname{div} u &= 0 && \text{in } (0,T)\times\Omega_t,\\
u &= (0,...,0,\partial_t\tilde{h} + g_1) && \text{on } (0,T)\times\Gamma_t,\\
\partial_t^2\tilde{h} + \Delta^2\tilde{h} &= F + g_2 && \text{on } (0,T)\times\Gamma_t,\\
u &= u_0 && \text{in } \Omega_0,\\
(\tilde{h},\partial_t\tilde{h})|_{t=0} &= (\tilde{h}_0,(u_0)_n) && \text{in } \Gamma_0
\end{aligned}
$$

in a time-dependent domain $\Omega_t = \{x = (x_1,...,x_n) \in \mathbb{R}^n : x_n > \tilde{h}(t,x')\}$, $t \in (0,T)$, with lower free boundary, $\Gamma_t = \{x = (x',x_n) : x_n = \tilde{h}(t,x')\}$. Here, $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ is the deformation tensor, and

$$
F := \langle (2\mu D(u) - pI)n_t, -e_n\rangle, \text{ where } \quad n_t = \frac{(\nabla'\tilde{h},-1)^T}{\sqrt{1+|\nabla'\tilde{h}|^2}},
$$

denotes the force exerted by the incompressible fluid in Ω_t onto the free boundary Γ_t with exterior normal n_t pointing downward. Here ∇' denotes the tangential gradient, *i.e.* $\nabla'\tilde{h} = (\partial_1\tilde{h}, ..., \partial_{n-1}\tilde{h})$. Moreover, at $t = 0$,

$$\Omega_0 := \{x \in \mathbb{R}^n : x_n > \tilde{h}_0(x')\} \text{ and } \Gamma_0 := \{x \in \mathbb{R}^n : x_n = \tilde{h}_0(x')\}.$$

Additionally, a body force g_2 exerts a force from the outside onto the boundary. Hence, the boundary is moving and its motion is tracked by the scalar function $\tilde{h}$ that describes the height of the boundary. On the boundary the fluid velocity u coincides with the velocity of the boundary. That is to say, we assume a no-slip condition for its tangential part, whereas its normal component coincides with $\partial_t h$ modulo an additional force g_1. In the domain Ω_t, the motion of the fluid is described by the Navier-Stokes equations with constant viscosity $\mu > 0$.

The linearised version with fixed boundary is obtained after transforming the equations to a problem in $\mathbb{R}^n_+$ and ignoring all non-linear terms appearing thereby as well as ignoring the convective acceleration term $(u \cdot \nabla)u$. The first part of the term F vanishes due to $\operatorname{div} u = 0$ and $u' = (u_1, \dots, u_{n-1}) = 0$ on $\partial\mathbb{R}^n_+$ so that also $\partial u_n/\partial x_n = 0$ on $\partial\mathbb{R}^n_+$. This way only the pressure term remains, see $(2.1)_4$. For more details on this transformation see [15, Chapter 2], [10] and [22].

Often an additional damping term like $-\partial_t\Delta h$ is introduced in this problem. This term allows the equation to have maximal L^p-regularity as considered by Denk and Saal, see [15], and [9, Chapter 4] in a space-time periodic setting. However, in our case of an unbounded domain even the existence of an L^2-semigroup is doubtful, which is why we will make more basic considerations to solve the equation in the case of non-vanishing initial data. The methods used here allow for a solution in the case $p = 2$ only, since the corresponding multipliers are bounded, but they are not Fourier multipliers for $p \neq 2$. Also, note that although we show the existence of solutions on any finite time interval, we must exclude the case $T = \infty$ since the Fourier transform of a solution of the undamped plate equation, $\partial_t^2 h + \Delta^2 h = 0$, is given by terms involving the cosine and sine functions, which are not square integrable over $(0, \infty)$. Adding a damping term as in [15], solves this issue and allows the use of solution spaces with exponential weights. Additionally, in [15] the damping term also guarantees that the solution belongs to an inhomogeneous Sobolev space, which is not possible in our undamped case; rather we have to work in homogeneous spaces.

The main aim of this chapter is to show the following theorem.

Theorem 2.0.1. *Let $\Gamma := \mathbb{R}^{n-1}$, $T \in (0, \infty)$ and $f \in L^2((0,T) \times \mathbb{R}^n_+)$, $g_2 \in L^2(0,T;\dot{H}^{1/2}(\Gamma))$, $g_1 \in L^2(0,T;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^1(0,T;\dot{H}^{-1/2}(\Gamma))$ with $g_1(0) = 0$, $u_0 \in \dot{H}^2\left(\mathbb{R}^n_+\right)$ with $\operatorname{div} u_0 = 0$, $u_0'|_{\partial\mathbb{R}^n_+} = 0$, and $h_0 \in \dot{H}^{7/2}(\Gamma)$ be given. Then the system (2.1) has a solution*

$$\begin{aligned} &u \in L^2(0,T;\dot{H}^2(\mathbb{R}^n_+)) \cap \dot{H}^1(0,T;L^2(\mathbb{R}^n_+)),\\ &p \in L^2(0,T;\dot{H}^1(\mathbb{R}^n_+)),\\ &h \in \dot{H}^1(0,T;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^2(0,T;\dot{H}^{-1/2}(\Gamma)). \end{aligned}$$

Furthermore, the solution depends continuously on the given data.

This chapter is structured as follows: First, in Section 2.1, we show the existence of solutions in the case of vanishing initial data using partial Fourier transform. Secondly, in Section 2.2, we find admissible spaces for the initial data. Here, we do not use abstract semigroup theory, as usually done, but the specific form of the undamped plate equation.

2.1. The Case of Vanishing Initial Data

First, we assume that we have $f = 0$ in $(2.1)_1$ as well as homogeneous initial data in $(2.1)_5$, *i.e.*, $u(0) = u_0 = 0$. Since $g_1(0) = 0$ due to $(2.1)_{3,5,6}$, we extend the functions $g_1 \in L^2(0,\infty;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^1(0,\infty;\dot{H}^{-1/2}(\Gamma))$ and $g_2 \in L^2(0,\infty;\dot{H}^{1/2}(\Gamma))$ in time by zero to corresponding functions defined on $\mathbb{R}$. Then, we apply the Fourier transform with respect to time and $x' \in \mathbb{R}^{n-1}$, indicated by a hat, to get for $v = \hat{u}$ and $\hat{h}$ the system

$$
\begin{aligned}
(i\tau + \mu|\xi'|^2)v' - \mu\partial_n^2 v' + i\xi'\hat{p} &= 0 && \text{in } \mathbb{R}\times\mathbb{R}^n_+,\\
(i\tau + \mu|\xi'|^2)v_n - \mu\partial_n^2 v_n + \partial_n\hat{p} &= 0 && \text{in } \mathbb{R}\times\mathbb{R}^n_+,\\
i\xi'\cdot v' + \partial_n v_n &= 0 && \text{in } \mathbb{R}\times\mathbb{R}^n_+,\\
(v', v_n)|_{x_n=0} &= (0', i\tau\hat{h} + \hat{g}_1) && \text{on } \mathbb{R}\times\Gamma,\\
(-\tau^2 + |\xi'|^4)\hat{h} &= -\hat{p}|_{x_n=0} + \hat{g}_2 && \text{on } \mathbb{R}\times\Gamma.
\end{aligned}
\tag{2.2}
$$

We note that the following calculations are formal. Applying the divergence onto $(2.1)_1$ and using that $\operatorname{div}(u) = 0$, we deduce $\Delta p = 0$ or, in other words,

$$\partial_n^2\hat{p} - |\xi'|^2\hat{p} = 0,$$

yielding the general solution

$$\hat{p} = \gamma e^{-|\xi'|x_n} \quad \text{for } a.a.\ x_n > 0 \tag{2.3}$$

with a function $\gamma = \gamma(\xi',\tau) \in \mathbb{C}$. Now we can solve $(2.2)_1$ for v'; the generic solution is given by

$$
\begin{aligned}
&v' = \alpha' e^{-Bx_n} - \frac{\xi'\gamma}{\tau}e^{-Ax_n}, \quad \alpha' \in \mathbb{C}^{n-1},\\
&A := |\xi'|, \ \ B := \sqrt{\mu^{-1}i\tau + A^2} \ \ (\operatorname{Re} B > 0).
\end{aligned}
\tag{2.4}
$$

Indeed, this follows from linearity and

$$
\begin{aligned}
(i\tau + \mu|\xi'|^2)e^{-Bx_n} - \mu\partial_n^2 e^{-Bx_n} &= (i\tau + \mu|\xi'|^2)e^{-Bx_n} - \mu B^2 e^{-Bx_n} = 0,\\
((i\tau + \mu|\xi'|^2) - \mu|\xi'|^2)\tfrac{-\xi'\gamma}{\tau}e^{-Ax_n} &= -i\xi'\gamma e^{-Ax_n} = -i\xi'\hat{p}.
\end{aligned}
$$

Concerning α', $(2.2)_4$ implies that

$$\alpha' = \frac{\xi'\gamma}{\tau}.$$

So for now we have

$$v' = \frac{\xi'\gamma}{\tau}(e^{-Bx_n} - e^{-Ax_n}). \tag{2.5}$$

Using this representation, we consider $(2.2)_3$ to determine $\partial_n v_n$:

$$\partial_n v_n = -i\xi'\cdot v' = |\xi'|^2\frac{\gamma}{i\tau}(e^{-Bx_n} - e^{-Ax_n}).$$

This implies

$$\partial_n^2 v_n = A^2 \frac{\gamma}{i\tau}(-Be^{-Bx_n} + Ae^{-Ax_n}).$$

We exploit this representation to solve $(2.2)_2$ and get that

$$v_n = \frac{1}{i\tau + \mu A^2}(-\partial_n \hat{p} + \mu \partial_n^2 v_n) = A\frac{\gamma}{i\tau}e^{-Ax_n} - \frac{1}{B}\frac{\gamma}{i\tau}A^2 e^{-Bx_n}. \tag{2.6}$$

In particular, $v_n|_{x_n=0} = \frac{\gamma A}{i\tau} - \frac{\gamma}{i\tau B}A^2$. Now, we derive two identities for $\hat{h}$ from $(2.2)_4$ and $(2.2)_5$ by using (2.3):

$$\begin{aligned} \hat{h} &= \frac{1}{i\tau}(-\hat{g}_1 + v_n|_{x_n=0}) = \frac{-\hat{g}_1}{i\tau} - \frac{\gamma A}{\tau^2} + \frac{\gamma A^2}{\tau^2 B}, \\ \hat{h} &= \frac{-\hat{p}|_{x_n=0} + \hat{g}_2}{-\tau^2 + A^4} = \frac{-\gamma + \hat{g}_2}{-\tau^2 + A^4}. \end{aligned} \tag{2.7}$$

Combining these identities, we solve for the still unknown term γ to see that

$$\gamma\left(-\frac{A}{\tau^2} + \frac{A^2}{B\tau^2} + \frac{1}{-\tau^2 + A^4}\right) = \frac{\hat{g}_2}{-\tau^2 + A^4} + \frac{\hat{g}_1}{i\tau}.$$

We define

$$\mathcal{N}(A,\tau) := \left(A^4 - \tau^2 - \frac{\tau^2}{A} + \mu i\tau A + \mu i\tau B\right)^{-1} \tag{2.8}$$

and observe the identity

$$\tfrac{-\tau^2 B}{(B-A)A} \cdot (-\tau^2 + A^4)\left(\left\{-\tfrac{A}{\tau^2} + \tfrac{A^2}{B\tau^2}\right\} + \tfrac{1}{-\tau^2+A^4}\right) = (-\tau^2 + A^4) - \tfrac{\tau^2 B}{(B-A)A}.$$

Due to $\frac{-\tau^2 B}{(B-A)A} = \frac{-\tau^2}{A} + \mu i\tau(A+B)$ the above right-hand side equals $(\mathcal{N}(A,\tau))^{-1}$. Now, we conclude that

$$\gamma = \mathcal{N}(A,\tau)\frac{-\tau^2 B}{(B-A)A}\left(\hat{g}_2 + \frac{-\tau^2 + A^4}{i\tau}\hat{g}_1\right) \tag{2.9}$$

$$= \mathcal{N}(A,\tau)\left(\frac{-\tau^2}{A} + \mu i\tau A + \mu i\tau B\right)\left(\hat{g}_2 + \frac{-\tau^2 + A^4}{i\tau}\hat{g}_1\right). \tag{2.10}$$

Summarizing (2.3), (2.5), (2.6), (2.7), (2.9) and (2.10) we obtain a solution $\hat{p}, v, \hat{h}$ in Fourier space by

$$\begin{aligned} \hat{p} &= \gamma e^{-Ax_n} \overset{(2.10)}{=} \left(\frac{-\tau^2}{A} + \mu i\tau A + \mu i\tau B\right)\mathcal{N}(A,\tau)\left(\hat{g}_2 + \frac{-\tau^2 + A^4}{i\tau}\hat{g}_1\right)e^{-Ax_n}, \\ v' &= \frac{i\xi'\gamma}{i\tau}(e^{-Bx_n} - e^{-Ax_n}) \\ &= \frac{i\xi'}{A}\frac{i\tau B}{A}\mathcal{N}(A,\tau)\left(\hat{g}_2 + \frac{-\tau^2 + A^4}{i\tau}\hat{g}_1\right)A\frac{e^{-Bx_n} - e^{-Ax_n}}{B-A}, \\ v_n &= -\frac{i\tau B}{A}\mathcal{N}(A,\tau)\left(\hat{g}_2 + \frac{-\tau^2 + A^4}{i\tau}\hat{g}_1\right)A\frac{e^{-Bx_n} - e^{-Ax_n}}{B-A} \\ &\quad + i\tau\mathcal{N}(A,\tau)\left(\hat{g}_2 + \frac{-\tau^2 + A^4}{i\tau}\hat{g}_1\right)e^{-Bx_n}, \\ \hat{h} &= \frac{-\hat{g}_1}{i\tau} - \frac{\gamma A}{\tau^2} + \frac{\gamma A^2}{\tau^2 B} = \mathcal{N}(A,\tau)\hat{g}_2 + \left((-\tau^2 + A^4)\mathcal{N}(A,\tau) - 1\right)\frac{1}{i\tau}\hat{g}_1. \end{aligned} \tag{2.11}$$

Plugging these identities into (2.2), one can verify that these terms are indeed solutions to (2.2). We now find suitable L^2-multipliers.

Lemma 2.1.1. *The functions*

$$\tau A\mathcal{N}(A,\tau), \quad \frac{\tau^2}{A}\mathcal{N}(A,\tau), \quad \tau^{3/2}\mathcal{N}(A,\tau), \quad (-\tau^2+A^4)\mathcal{N}(A,\tau), \quad \tau B\mathcal{N}(A,\tau)$$

are uniformly bounded with respect to $(A,\tau)\in(0,\infty)\times\mathbb{R}$.

Proof. We note that $B=\operatorname{Re}B+i\frac{\tau}{2\mu\operatorname{Re}B}$ with

$$\operatorname{Re}B=\frac{1}{\sqrt{2}}\sqrt{\sqrt{A^4+\mu^{-2}\tau^2}+A^2}\geq A. \tag{2.12}$$

This is true due to

$$\begin{aligned}\left(\operatorname{Re}B+i\frac{\tau}{2\mu\operatorname{Re}B}\right)^2=&(\operatorname{Re}B)^2-\frac{\tau^2}{4\mu^2(\operatorname{Re}B)^2}+i\tau\mu^{-1}=i\tau\mu^{-1}+\frac{1}{(\operatorname{Re}B)^2}\left((\operatorname{Re}B)^4-\frac{\tau^2}{4\mu^2}\right)\\=&i\tau\mu^{-1}+\frac{1}{(\operatorname{Re}B)^2}\left(\frac{1}{4}\left(2A^4+\mu^{-2}\tau^2+2A^2\sqrt{A^4+\mu^{-2}\tau^2}\right)-\frac{\tau^2}{4\mu^2}\right)\\=&i\tau\mu^{-1}+\frac{1}{(\operatorname{Re}B)^2}\left(\frac{1}{4}\left(2A^4+2A^2\sqrt{A^4+\mu^{-2}\tau^2}\right)\right)\\=&i\tau\mu^{-1}+\frac{A^2(\operatorname{Re}B)^2}{(\operatorname{Re}B)^2}=B^2,\end{aligned}$$

and for a complex number z we always take the root $\sqrt{z}$ that has a non-negative real part. Consequently, $B=\operatorname{Re}B+i\frac{\tau}{2\mu\operatorname{Re}B}$ after taking the square root.

Now, we decompose $\mathcal{N}(A,\tau)^{-1}$ into its real and imaginary part:

$$\begin{aligned}\mathcal{N}(A,\tau)^{-1}=&A^4-\tau^2-\frac{\tau^2}{A}+\mu i\tau B+\mu i\tau A\\=&A^4-\tau^2\left(1+\frac{1}{A}\right)-\mu\tau\operatorname{Im}B+\mu i\tau\operatorname{Re}B+\mu i\tau A\\=&A^4-\tau^2\left(1+\frac{1}{A}+\frac{1}{2\operatorname{Re}B}\right)+\mu i\tau\left(\operatorname{Re}B+A\right).\end{aligned}$$

First, we prove that $\tau A\mathcal{N}(A,\tau)$ is bounded:

$$|\tau A\mathcal{N}(A,\tau)|\leq\frac{|\tau|A}{|\operatorname{Im}(\mathcal{N}(A,\tau)^{-1})|}=\frac{|\tau|A}{|\tau|\mu\left(\operatorname{Re}B+A\right)}\leq\frac{A}{\mu A}=\frac{1}{\mu}.$$

Next, we consider $\frac{\tau^2}{A}\mathcal{N}(A,\tau)$. If $|\tau|\leq A^2$, we use the imaginary part of $\mathcal{N}(A,\tau)^{-1}$ to get as above the estimate

$$\frac{\tau^2}{A}|\mathcal{N}(A,\tau)|\leq\frac{|\tau|A}{|\operatorname{Im}(\mathcal{N}(A,\tau)^{-1})|}\leq\frac{1}{\mu}.$$

Otherwise, if $|\tau| \geq A^2$, we use the real part of $\mathcal{N}(A,\tau)^{-1}$, which leads to the estimate

$$\frac{\tau^2}{A}|\mathcal{N}(A,\tau)| \leq \frac{\tau^2}{A}\frac{1}{|\operatorname{Re}(\mathcal{N}(A,\tau)^{-1})|} = \frac{\tau^2}{A}\frac{1}{\tau^2\left(1+\frac{1}{A}+\frac{1}{2\operatorname{Re}B}\right)-A^4} \leq \frac{\tau^2}{A}\frac{1}{\frac{\tau^2}{A}} = 1.$$

The function $\tau^{3/2}\mathcal{N}(A,\tau)$ is also bounded due to the previous considerations and Young's inequality $|\tau^{3/2}| = \left|\frac{\tau}{\sqrt{A}}\sqrt{\tau A}\right| \leq \frac{\tau^2}{2A} + \frac{|\tau|A}{2}$.

Due to $|B| \leq \mu^{-1/2}\sqrt{|\tau|}+A$ we deduce the boundedness of $\tau B\mathcal{N}(A,\tau)$ from the previous cases. Finally, the boundedness of $(-\tau^2+A^4)\mathcal{N}(A,\tau)$ follows from the identity

$$(-\tau^2+A^4)\mathcal{N}(A,\tau) = 1 - \left(\tfrac{-\tau^2}{A} + \mu i\tau A + \mu i\tau B\right)\mathcal{N}(A,\tau), \tag{2.13}$$

see (2.8) for the definition of $\mathcal{N}(A,\tau)$, and the previous cases. □

To simplify the analysis of the multiplier functions, we reduce the question of their boundedness to a problem in $\mathbb{R}^{n-1}$ rather than in $\mathbb{R}^n_+$ by the following lemma.

Lemma 2.1.2. *As before let $B = \sqrt{\mu^{-1}i\tau + A^2}$ with $A > 0$ and $\tau \in \mathbb{R}$.*

i) Let $f : \mathbb{R}\times\mathbb{R}^{n-1} \to \mathbb{C}$ be measurable, $c > 0$ and $\chi \in \{cA, cB\}$. Then we have the equivalence

$$f(\tau,\xi')e^{-\chi x_n} \in L^2(\mathbb{R}\times\mathbb{R}^n_+) \Longleftrightarrow \frac{1}{\sqrt{\operatorname{Re}\chi}}f(\tau,\xi') \in L^2(\mathbb{R}\times\mathbb{R}^{n-1})$$

with equivalent norms.

ii) There exists a constant $C > 0$ such that

$$A\left|\frac{e^{-Bx_n}-e^{-Ax_n}}{B-A}\right| \leq Ce^{-\frac{1}{2}Ax_n}$$

holds for all $x_n, A > 0$ and $\tau \in \mathbb{R}\backslash\{0\}$.

Proof. i) The claim follows from the calculation

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}^n_+} |f(\tau,\xi')e^{-\chi x_n}|^2 \,\mathrm{d}\xi'\mathrm{d}x_n\mathrm{d}\tau\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\left(\int_0^\infty |e^{-\chi x_n}|^2 dx_n\right)|f(\tau,\xi')|^2\,\mathrm{d}\xi'\mathrm{d}\tau = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\frac{1}{\operatorname{Re}\chi}\,|f(\tau,\xi')|^2\,\mathrm{d}\xi'\mathrm{d}\tau,\end{aligned}$$

where we used Tonelli's theorem in the first identity.

ii) Consider the function

$$\rho : (-\infty,1)\times\mathbb{R} \to \mathbb{R}^2,\ \ (x,y) \mapsto e^x\begin{pmatrix}\cos(y)\\ \sin(y)\end{pmatrix} = \begin{pmatrix}\operatorname{Re}(\exp((x+iy)))\\ \operatorname{Im}(\exp((x+iy)))\end{pmatrix}.$$

It satisfies

$$\|\nabla\rho(x,y)\| = e^x\left\|\begin{pmatrix}\cos(y) & -\sin(y)\\ \sin(y) & \cos(y)\end{pmatrix}\right\| \leq C\ \ \forall (x,y)\in(-\infty,1)\times\mathbb{R}.$$

This means that ρ is globally Lipschitz. Together with $\mathrm{Re}((B-A)x_n) > 0$, see (2.12), this implies

$$\begin{aligned} A\left|\frac{e^{-Bx_n}-e^{-Ax_n}}{B-A}\right| =& \frac{Ae^{-Ax_n}}{|B-A|}|e^{-(B-A)x_n}-1| \le C\frac{Ae^{-Ax_n}}{|B-A|}|B-A|x_n \\ =& CAx_ne^{-Ax_n} \le Ce^{-\frac{1}{2}Ax_n}. \end{aligned}$$

□

Proposition 2.1.3. *Let $g_1 \in L^2(0,\infty;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^1(0,\infty;\dot{H}^{-1/2}(\Gamma))$ with $g_1(0) = 0$ and $g_2 \in L^2(0,\infty;\dot{H}^{1/2}(\Gamma))$ be given. Then the system*

$$\begin{aligned} \partial_t u - \mu\Delta u + \nabla p &= 0 && \text{in } (0,\infty)\times\mathbb{R}^n_+, \\ \operatorname{div} u &= 0 && \text{in } (0,\infty)\times\mathbb{R}^n_+, \\ u|_{x_n=0} &= (0', \partial_t h + g_1) && \text{on } (0,\infty)\times\Gamma, \\ \partial_t^2 h + \Delta^2 h &= -p + g_2 && \text{on } (0,\infty)\times\Gamma \end{aligned}$$

together with initial conditions $u(0) = 0$, $h(0) = \partial_t h(0) = 0$ has a solution

$$\begin{aligned} &u \in \dot{H}^1(0,\infty;L^2(\mathbb{R}^n_+)) \cap L^2(0,\infty;\dot{H}^2(\mathbb{R}^n_+)), \\ &p \in L^2(0,\infty;\dot{H}^1(\mathbb{R}^n_+)), \\ &h \in \dot{H}^1(0,\infty;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^2(0,\infty;\dot{H}^{-1/2}(\Gamma)). \end{aligned}$$

Furthermore, we have the estimate

$$\begin{aligned} &\|u\|_{\dot{H}^1(0,\infty;L^2(\mathbb{R}^n_+))\cap L^2(0,\infty;\dot{H}^2(\mathbb{R}^n_+))} + \|p\|_{L^2(0,\infty;\dot{H}^1(\mathbb{R}^n_+))} \\ &\quad + \|h\|_{\dot{H}^1(0,\infty;\dot{H}^{3/2}(\Gamma))\cap\dot{H}^2(0,\infty;\dot{H}^{-1/2}(\Gamma))} \\ &\le C\left(\|g_1\|_{L^2(0,\infty;\dot{H}^{3/2}(\Gamma))\cap\dot{H}^1(0,\infty;\dot{H}^{-1/2}(\Gamma))} + \|g_2\|_{L^2(0,\infty;\dot{H}^{1/2}(\Gamma))}\right). \end{aligned}$$

The analogous statement holds if the time interval is replaced with $(0,T)$.

Proof. As done at the beginning of Section 2.1 we extend g_1 and g_2 in time to functions defined on $\mathbb{R}$ and then, we derive the solution (2.11). Due to the linearity of the equations, we can distinguish between the cases $g_1 = 0$ and $g_2 = 0$, and treat these cases separately. First, consider the case $g_1 = 0$ in which the solution simplifies to

$$\begin{aligned} \hat{p} &= (1-\mathcal{M}(A,\tau))\hat{g}_2\, e^{-Ax_n}, \\ v' &= \frac{i\xi'}{A}\frac{i\tau B}{A}\mathcal{N}(A,\tau)\hat{g}_2\, A\frac{e^{-Bx_n}-e^{-Ax_n}}{B-A}, \\ v_n &= -\frac{i\tau B}{A}\mathcal{N}(A,\tau)\hat{g}_2\, A\frac{e^{-Bx_n}-e^{-Ax_n}}{B-A} + i\tau\mathcal{N}(A,\tau)\hat{g}_2\, e^{-Bx_n}, \\ \hat{h} &= \mathcal{N}(A,\tau)\hat{g}_2. \end{aligned} \tag{2.14}$$

For the pressure term p we used the identity (2.13), where by Lemma 2.1.1

$$\mathcal{M}(A,\tau) := \mathcal{N}(A,\tau)(-\tau^2 + A^4)$$

is a bounded L^2-multiplier function.

Before we begin with the estimates, we explain how the term $h = \mathcal{F}^{-1}\left(\mathcal{N}(A,\tau)\mathcal{F}(g_2)\right)$, for instance, is to be understood. Assume first that

$$g_2(t,x') = \sum_{j=1}^{N} \varphi_j(t)\alpha_j(x') \in X$$

with

$$X := \left\{ \sum_{j=1}^{M} \tilde{\varphi}_j(t)\tilde{\alpha}_j(x') \mid M \in \mathbb{N},\ \ \tilde{\varphi}_j \in C_c^\infty(\mathbb{R}),\ \ \tilde{\alpha}_j \in \mathcal{S}_0 \quad \forall j \in \{1,...,M\} \right\}$$

and with $\mathcal{S}_0 := \left\{ s_0 \in \mathcal{S}(\mathbb{R}^{n-1}) \mid 0 \notin \operatorname{supp} \mathcal{F}_{\xi'}(s_0) \right\}$. It is easy to verify that $g_2 \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^{n-1})$, and hence $\hat{g}_2 \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^{n-1})$. Furthermore, since $\mathcal{F}_{\xi'}\alpha_j \in \mathcal{S}_0$ vanishes in a neighbourhood of $0 \in \mathbb{R}^{n-1}$ for all $j \in \{1,...,M\}$, the term $\hat{g}_2(\tau,\xi') = \sum_{j=1}^{N} \left(\mathcal{F}_\tau \varphi_j\right)(\tau)(\mathcal{F}_{\xi'}\alpha_j)(\xi')$ vanishes in a neighbourhood of $0 \in \mathbb{R}^n$. With help of Lemma 2.1.1 we know that $\mathcal{N}(|\xi'|,\tau) \in C^\infty(\left(\mathbb{R}^{n-1}\times\mathbb{R}\right)\setminus\{0\},\mathbb{C})$, *i.e.* 0 is the only singularity of $\mathcal{N}$. Nevertheless, since $\hat{g}_2 \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^{n-1})$ vanishes in a neighbourhood of 0, we still deduce $\mathcal{N}\hat{g}_2 \in C^\infty(\mathbb{R}\times\mathbb{R}^{n-1},\mathbb{C})$ and all derivatives in 0 vanish. Due to $\hat{g}_2 \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^{n-1})$ and the fact that $\mathcal{N}$ and all its derivatives have at most polynomial growth for $(|\xi'|,|\tau|) \to \infty$, we have $\mathcal{N}\hat{g}_2 \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^{n-1})$. In total, we have verified that the linear operator $\mathcal{F}^{-1}(\mathcal{N}\mathcal{F}) : X \to Z(\mathbb{R}\times\mathbb{R}^{n-1})$ is well-defined. Assume, for example, that we have shown that

$$\mathcal{F}^{-1}(\mathcal{N}\mathcal{F}) : \left(X, \|\cdot\|_{L^2(\mathbb{R},\dot{H}^{1/2}(\Gamma))}\right) \to \left(Z(\mathbb{R}\times\mathbb{R}^{n-1}), \|\cdot\|_{\dot{H}^1(\mathbb{R};\dot{H}^{3/2}(\Gamma))}\right)$$

is bounded. Then we use that X is dense in $L^2(\mathbb{R},\dot{H}^{1/2}(\Gamma))$, since $\mathcal{S}_0$ is dense in $\dot{H}^s(\Gamma)$ for all $s \in \mathbb{R}$ according to [26, Lemma 1.1 i)], and we can extend $\mathcal{F}^{-1}(\mathcal{N}\mathcal{F})$ continuously to $L^2(\mathbb{R},\dot{H}^{1/2}(\Gamma))$ with image in $\dot{H}^1(\mathbb{R};\dot{H}^{3/2}(\Gamma))$. This is how the term $h = \mathcal{F}^{-1}\left(\mathcal{N}(A,\tau)\mathcal{F}(g_2)\right)$ is to be understood if merely $g_2 \in L^2(\mathbb{R};\dot{H}^{1/2}(\Gamma))$. Similarly for the other terms.

With the previous two lemmata we now show the corresponding estimates.

i) $u' \in L^2(0,\infty;\dot{H}^2(\mathbb{R}^n_+))$: Concerning tangential derivatives $\partial_k\partial_i u_j$, $i,j,k \in \{1,...,n-1\}$, we estimate as follows:

$$\begin{aligned} \|\xi_k\xi_i v_j\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)} &= \left\| \left(\frac{\xi_k}{A}\frac{\xi_i}{A}\frac{i\xi_j}{A}\right) \tau\mathcal{N}AB\,\hat{g}_2\,A\frac{e^{-Bx_n}-e^{-Ax_n}}{B-A} \right\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)} \\ &\le c\left\|\tau B\mathcal{N}A\hat{g}_2 e^{-\frac{1}{2}Ax_n}\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)} \le c\left\|\hat{g}_2\sqrt{A}\right\|_{L^2(\mathbb{R}\times\Gamma)} \\ &\le c\,\|g_2\|_{L^2(\mathbb{R};\dot{H}^{1/2}(\Gamma))}\,. \end{aligned} \tag{2.15}$$

In the first inequality we used Lemma 2.1.2 ii), in the second one we applied Lemma 2.1.2 i) and used the boundedness of $\tau B\mathcal{N}(A,\lambda)$. Note that we can first assume $g_2 \in X$ and then use a density argument as depicted above.
For the normal derivatives $\partial_n^2 u'$ we have

$$\begin{aligned} \partial_n^2 v' &= \frac{i\xi'}{A}\frac{i\tau B}{A}\mathcal{N}\,\hat{g}_2\,A\frac{B^2e^{-Bx_n}-A^2e^{-Ax_n}}{B-A} \\ &= \frac{i\xi'}{A}\frac{i\tau B}{A}\mathcal{N}\hat{g}_2 A\frac{(B^2-A^2)e^{-Bx_n}}{B-A} + \frac{i\xi'}{A}i\tau B\mathcal{N}A\,\hat{g}_2\,A\frac{e^{-Bx_n}-e^{-Ax_n}}{B-A}. \end{aligned}$$

The second summand can be treated as above. Hence, we only have to consider the first one. Using that $|B| \geq \operatorname{Re} B \geq A$, we estimate

$$\begin{aligned}
\left\| \frac{i\xi'}{A} \tau B \mathcal{N} \hat{g}_2 (B+A) e^{-Bx_n} \right\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)} &\leq 2c \left\| \tau B \mathcal{N} \hat{g}_2 B \frac{1}{\sqrt{\operatorname{Re} B}} \right\|_{L^2(\mathbb{R}\times\Gamma)} \\
&\leq c \left\| \frac{\tau B^2 \mathcal{N}}{A} \hat{g}_2 \sqrt{A} \right\|_{L^2(\mathbb{R}\times\Gamma)} = c \left\| \mathcal{N} \left(\frac{-\tau^2 \mu^{-1}}{A} + i\tau A \right) (\hat{g}_2 \sqrt{A}) \right\|_{L^2(\mathbb{R}\times\Gamma)} \\
&\leq c \, \|g_2\|_{L^2(\mathbb{R};\dot{H}^{1/2}(\Gamma))} \, .
\end{aligned} \tag{2.16}$$

For the mixed derivatives $\partial_k \partial_n u_j$, $j,k \in \{1,...,n-1\}$ we have that

$$\begin{aligned}
i\xi_k \partial_n v_j &= \frac{i\xi_k}{A} \frac{i\xi_j}{A} i\tau B \mathcal{N} \hat{g}_2 A \frac{-Be^{-Bx_n} + Ae^{-Ax_n}}{B-A} \\
&= \frac{i\xi_k}{A} \frac{i\xi_j}{A} i\tau B \mathcal{N} \hat{g}_2 A \frac{(A-B)e^{-Bx_n}}{B-A} - \frac{i\xi_k}{A} \frac{i\xi_j}{A} i\tau BA \mathcal{N} \hat{g}_2 A \frac{e^{-Bx_n} - e^{-Ax_n}}{B-A}.
\end{aligned}$$

The first summand can be treated as in (2.16), the second one as in (2.15).

ii) $u_n \in L^2(0,\infty;\dot{H}^2(\mathbb{R}^n_+))$: Since $\operatorname{div} u = 0$ and thus $\partial_k \partial_n u_n = -\sum_{j=1}^{n-1} \partial_k \partial_j u_j \in L^2((0,\infty) \times \mathbb{R}^n_+)$ for all $k \in \{1,...,n\}$ as shown in i), it is only left to consider $\partial_k \partial_i u_n$ for $i,k \neq n$. Moreover, the first summand of v_n can be treated in the same way as v', see (2.14). Therefore, we only have to discuss the second one:

$$\begin{aligned}
\|\xi_k \xi_i \tau \mathcal{N} \hat{g}_2 e^{-Bx_n}\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)} &\leq c \left\| \tau A^2 \mathcal{N} \hat{g}_2 \frac{1}{\sqrt{\operatorname{Re} B}} \right\|_{L^2(\mathbb{R}\times\Gamma)} \\
&\leq c \left\| \tau A \mathcal{N} \hat{g}_2 \sqrt{A} \right\|_{L^2(\mathbb{R}\times\Gamma)} \leq c \, \|g_2\|_{L^2(\mathbb{R};\dot{H}^{1/2}(\Gamma))} \, .
\end{aligned}$$

iii) $p \in L^2(0,\infty;\dot{H}^1(\mathbb{R}^n_+))$: Since $\partial_n \hat{p} = -A\hat{p}$, it suffices to consider the tangential derivatives $\nabla' p$. Since $\mathcal{M}$ is bounded, we get that

$$\|\xi' \hat{p}\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)} = \left\| \xi' (1-\mathcal{M}) \hat{g}_2 e^{-Ax_n} \right\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)} \leq c \|(\hat{g}_2 \sqrt{A})\|_{L^2(\mathbb{R}\times\Gamma)} \leq c \, \|g_2\|_{L^2(\mathbb{R};\dot{H}^{1/2}(\Gamma))} \, .$$

iv) $h \in \dot{H}^1(0,\infty;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^2(0,\infty;\dot{H}^{-1/2}(\Gamma))$: This follows immediately from $\hat{h} = \frac{\mathcal{N}}{\sqrt{A}}(\hat{g}_2 \sqrt{A})$ and the boundedness of $\tau A \mathcal{N} = \frac{\mathcal{N}}{\sqrt{A}}(\tau A^{3/2})$ and $\frac{\tau^2}{A}\mathcal{N} = \frac{\mathcal{N}}{\sqrt{A}} \left(\frac{\tau^2}{\sqrt{A}} \right)$.

v) $u \in \dot{H}^1(0,\infty;L^2(\mathbb{R}^n_+))$: The estimate follows from $\partial_t u = \Delta u - \nabla p \in L^2((0,\infty) \times \mathbb{R}^n_+)$ which is due to *i)*, *ii)* and *iii)*.

If on the other hand $g_2 = 0$, but

$$0 \neq g_1 \in L^2(0,\infty;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^1(0,\infty;\dot{H}^{-1/2}(\Gamma)), \quad g_1(0) = 0,$$

then we make use of the boundedness of $\mathcal{M}(A,\tau) := \mathcal{N}(A,\tau)(-\tau^2 + A^4)$, see Lemma 2.1.1. Consider the set of solutions with $g_2 = 0$, see (2.11):

$$\begin{aligned}
\hat{p} &= \left(\frac{i\tau}{A} + \mu A + \mu B \right) \mathcal{M}(A,\tau) \hat{g}_1 e^{-Ax_n}, \\
v' &= \frac{i\xi'}{A} \frac{B}{A} \mathcal{M}(A,\tau) \hat{g}_1 A \frac{e^{-Bx_n} - e^{-Ax_n}}{B-A}, \\
v_n &= \frac{B}{A} \mathcal{M}(A,\tau) \hat{g}_1 A \frac{e^{-Bx_n} - e^{-Ax_n}}{B-A} + \mathcal{M}(A,\tau) \hat{g}_1 e^{-Bx_n}, \\
\hat{h} &= (\mathcal{M}(A,\tau) - 1) \frac{1}{i\tau} \hat{g}_1.
\end{aligned}$$

Although the following estimates are similar to the previous ones, we present them in some details for the convenience of the reader.

i) $u' \in L^2(0,\infty;\dot H^2(\mathbb{R}^n_+))$: For tangential derivatives $\partial_k\partial_i u_j$, $1 \le i,j,k \le n-1$, we use the inequality

$$\|g_1\|_{\dot H^{1/2}(\mathbb{R};\dot H^{1/2}(\Gamma))} \le \|g_1\|_{L^2(\mathbb{R};\dot H^{3/2}(\Gamma))} + \|g_1\|_{\dot H^1(\mathbb{R};\dot H^{-1/2}(\Gamma))}, \tag{2.17}$$

which is true due to $\sqrt{|\tau A|} = \frac{\sqrt{|\tau|}}{A^{1/4}}A^{3/4} \le \frac{A^{3/2}}{2} + \frac{|\tau|}{2\sqrt{A}}$, and that $|B| \le \left|\frac{\tau}{\mu}\right|^{1/2} + |A|$, to calculate as follows:

$$\begin{aligned}\|\xi_k\xi_i v_j\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)} =& \left\| \left(\frac{\xi_k}{A}\frac{\xi_i}{A}\frac{i\xi_j}{A}\right)\mathcal{M}AB\hat g_1\left(A\frac{e^{-Bx_n}-e^{-Ax_n}}{B-A}\right)\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)}\\ \le& c\left\|AB\hat g_1\frac{1}{\sqrt{A}}\right\|_{L^2(\mathbb{R}\times\Gamma)}\\ \le& c\big(\|g_1\|_{\dot H^{1/2}(\mathbb{R};\dot H^{1/2}(\Gamma))} + \|g_1\|_{L^2(\mathbb{R};\dot H^{3/2}(\Gamma))}\big)\\ \le& c\big(\|g_1\|_{\dot H^1(\mathbb{R};\dot H^{-1/2}(\Gamma))} + \|g_1\|_{L^2(\mathbb{R};\dot H^{3/2}(\Gamma))}\big).\end{aligned} \tag{2.18}$$

For the normal derivatives $\partial_n^2 u'$ we compute:

$$\begin{aligned}\partial_n^2 v' &= \frac{i\xi'}{A}\frac{B}{A}\mathcal{M}\hat g_1 A\frac{B^2e^{-Bx_n}-A^2e^{-Ax_n}}{B-A}\\ &= \frac{i\xi'}{A}\frac{B}{A}\mathcal{M}\hat g_1 A\frac{(B^2-A^2)e^{-Bx_n}}{B-A} + \frac{i\xi'}{A}BA\mathcal{M}\hat g_1 A\frac{e^{-Bx_n}-e^{-Ax_n}}{B-A}.\end{aligned}$$

The second summand can be treated as in (2.18). Hence, we only have to consider the first one. Let $j \in \{1,...,n-1\}$, then

$$\begin{aligned}&\left\|\frac{i\xi_j}{A}B\mathcal{M}\hat g_1(B+A)e^{-Bx_n}\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)} \le 2c\left\|B^2\hat g_1\frac{1}{\sqrt{\operatorname{Re}B}}\right\|_{L^2(\mathbb{R}\times\Gamma)}\\ &= c\left\|\left(\frac{i\tau}{\mu}+A^2\right)\hat g_1\frac{1}{\sqrt{A}}\right\|_{L^2(\mathbb{R}\times\Gamma)} \le c\big(\|g_1\|_{\dot H^1(\mathbb{R};\dot H^{-1/2}(\Gamma))} + \|g_1\|_{L^2(\mathbb{R};\dot H^{3/2}(\Gamma))}\big).\end{aligned} \tag{2.19}$$

For mixed derivatives $\partial_k\partial_n u_j$, $k = 1,...,n-1$, one has

$$\begin{aligned}i\xi_k\partial_n v_j &= \frac{i\xi_k}{A}\frac{i\xi_j}{A}B\mathcal{M}\hat g_1 A\frac{-Be^{-Bx_n}+Ae^{-Ax_n}}{B-A}\\ &= -\frac{i\xi_k}{A}\frac{i\xi_j}{A}B\mathcal{M}\hat g_1 Ae^{-Bx_n} - \frac{i\xi_k}{A}\frac{i\xi_j}{A}BA\mathcal{M}\hat g_1 A\frac{e^{-Bx_n}-e^{-Ax_n}}{B-A}.\end{aligned}$$

The first summand can be treated as in (2.19), the second one as in (2.18).

ii) $u_n \in L^2(0,\infty;\dot H^2(\mathbb{R}^n_+))$: Since $\partial_k\partial_n u_n = -\sum_{j=1}^{n-1}\partial_k\partial_j u_j \in L^2((0,\infty)\times\mathbb{R}^n_+)$ as $\operatorname{div} u = 0$, it is only left to estimate the term $\partial_k\partial_i u_n$ for $i,k \ne n$. The first summand of v_n can be treated in the same way as v_j; thus we only have to consider the second one:

$$\begin{aligned}\|\xi_k\xi_i\mathcal{M}\hat g_1 e^{-Bx_n}\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)} &\le c\left\|A^2\hat g_1\frac{1}{\sqrt{\operatorname{Re}B}}\right\|_{L^2(\mathbb{R}\times\Gamma)}\\ &\le c\left\|A^{3/2}\hat g_1\right\|_{L^2(\mathbb{R}\times\Gamma)} \le c\,\|g_1\|_{L^2(\mathbb{R};\dot H^{3/2}(\Gamma))}.\end{aligned}$$

iii) $p \in L^2(0,\infty;\dot{H}^1(\mathbb{R}^n_+))$: For the tangential derivative $\nabla' p$ we have

$$\begin{aligned}\|\xi'\hat{p}\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)} &\le \left\|\xi'\left(\frac{i\tau}{A}+\mu A+\mu B\right)\hat{g}_1 e^{-Ax_n}\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)}\\ &\le c\left\|\left(\frac{i\tau}{A}+\mu A+\mu B\right)A\hat{g}_1\frac{1}{\sqrt{A}}\right\|_{L^2(\mathbb{R}\times\Gamma)}\\ &\le c\,\|g_1\|_{\dot{H}^1(\mathbb{R};\dot{H}^{-1/2}(\Gamma))}+c\|g_1\|_{L^2(\mathbb{R};\dot{H}^{3/2}(\Gamma)))}.\end{aligned}$$

For the last step we have used (2.17). Since $\partial_n\hat{p} = -A\hat{p}$, we can show $\partial_n p \in L^2(0,\infty;L^2(\mathbb{R}^n_+))$ as above.

iv) $h \in \dot{H}^1(0,\infty;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^2(0,\infty;\dot{H}^{-1/2}(\Gamma))$: This follows immediately from the boundedness of $\mathcal{M}(A,\tau)-1$.

v) $u \in \dot{H}^1(0,\infty;L^2(\mathbb{R}^n_+))$: Here we refer to $\partial_t u = \Delta u - \nabla p \in L^2((0,\infty)\times\mathbb{R}^n_+)$ using the previous cases.

This shows the claim in the case of the time interval $(0,\infty)$.

In the case of a finite time interval $(0,T)$ we have $g_1 \in L^2(0,T;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^1(0,T;\dot{H}^{-1/2}(\Gamma))$ and $g_2 \in L^2(0,T;\dot{H}^{1/2}(\Gamma))$. Here, we extend g_2 by zero to an element in $L^2(0,\infty;\dot{H}^{1/2}(\Gamma))$ and, since $g_1(0)=0$, we extend g_1 to an element in $L^2(0,\infty;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^1(0,\infty;\dot{H}^{-1/2}(\Gamma))$ by $g_1\circ\varphi$ where $\varphi \in C^0([0,\infty))$ has compact support in $[0,2T]$ and is given by

$$\varphi : (0,\infty)\to\mathbb{R}, \quad t \mapsto \begin{cases} t, & \text{if } t\in(0,T),\\ 2T-t, & \text{if } t\in(T,2T).\end{cases}$$

Note that the conjunction of φ with g_1 preserves the $\dot{H}^1$-regularity with respect to time. Then we apply the above case and restrict the result to the finite time interval. □

2.2. The Case of Non-Vanishing Initial Data

Next, we want to recover non-vanishing initial data starting with $h_0 \in \dot{H}^{7/2}(\Gamma)$. Consider the classical plate equation

$$\begin{aligned}&\partial_t^2\eta_0 + \Delta^2\eta_0 = 0 && \text{in } (0,T)\times\Gamma,\\ &(\eta_0,\partial_t\eta_0)|_{t=0} = (h_0,0) && \text{on } \Gamma.\end{aligned} \tag{2.20}$$

Applying the partial Fourier transform $\mathcal{F}$ with respect to the spatial variable $x' \in \Gamma = \mathbb{R}^{n-1}$, we get a solution

$$\begin{aligned}\eta_0(t,\xi) &= \mathcal{F}^{-1}\big(\cos(A^2t)\mathcal{F}h_0\big) \in L^\infty(0,T;\dot{H}^{7/2}(\Gamma)),\\ \partial_t\eta_0(t,\xi) &= \mathcal{F}^{-1}(-\sin(A^2t)A^2\mathcal{F}h_0) \in L^\infty(0,T;\dot{H}^{3/2}(\Gamma)).\end{aligned} \tag{2.21}$$

Now, the general idea is to replace h from Proposition 2.1.3 with $h+\eta_0$ to satisfy the initial condition and to leave the plate equation unchanged. However, this replacement alters the equation $u_n = \partial_t h$ which is why we need the following proposition:

Proposition 2.2.1. *Let $0 < T < \infty$, $h_0 \in \dot{H}^{7/2}(\Gamma)$, and let $\eta_0 = \mathcal{F}^{-1}(\cos(A^2 t)\mathcal{F}h_0)$ be the solution of the plate equation* (2.21)*. Then the Stokes system*

$$
\begin{aligned}
\partial_t w_0 - \mu\Delta w_0 + \nabla p_0 &= 0 && \text{in } (0,T)\times\mathbb{R}^n_+,\\
\operatorname{div} w_0 &= 0 && \text{in } (0,T)\times\mathbb{R}^n_+,\\
(w_0', (w_0)_n)|_{x_n=0} &= (0', \partial_t\eta_0) && \text{on } (0,T)\times\Gamma,\\
w_0|_{t=0} &= 0 && \text{in } \mathbb{R}^n_+
\end{aligned}
\tag{2.22}
$$

has a solution

$$
w_0 \in L^2(0,T;\dot{H}^2(\mathbb{R}^n_+)) \cap \dot{H}^1(0,T;L^2(\mathbb{R}^n_+)), \quad p_0 \in L^2(0,T;\dot{H}^1(\mathbb{R}^n_+)).
$$

Furthermore, there holds the estimate

$$
\|w_0\|_{L^2(0,T;\dot{H}^2(\mathbb{R}^n_+))\cap\dot{H}^1(0,T;L^2(\mathbb{R}^n_+))} + \|p_0\|_{L^2(0,T;\dot{H}^1(\mathbb{R}^n_+))} \le CT^{1/2}\|h_0\|_{\dot{H}^{7/2}(\Gamma)}. \tag{2.23}
$$

Proof. Choose $\varphi \in C^1_c([0,\infty))$ such that $\varphi = 1$ on $[0,T]$, $\varphi(t) = 0$ for $t > 2T$ and $t|\varphi'(t)| \le c$ on $\mathbb{R}$ with $c > 0$ independent of T. Obviously, we can replace $\partial_t\eta_0(t,\xi)$ with $\varphi(t)\,\partial_t\eta_0(t,\xi)$ in $(2.22)_3$. We want to ensure that

$$
\begin{aligned}
r := \varphi\,\partial_t\eta_0 &\in L^2(0,\infty;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^1(0,\infty;\dot{H}^{-1/2}(\Gamma)),\\
\|r\|_{L^2(0,\infty;\dot{H}^{3/2}(\Gamma))\cap\dot{H}^1(0,\infty;\dot{H}^{-1/2}(\Gamma))} &\le cT^{1/2}\|h_0\|_{\dot{H}^{7/2}(\Gamma)}.
\end{aligned}
\tag{2.24}
$$

Note that $\mathcal{F}$ and $\varphi\partial_t$ commute. Then for the first property we have

$$
|\varphi\,\partial_t\,(\mathcal{F}\eta_0)\;A^{3/2}| = |A^{7/2}\,(\mathcal{F}h_0)\sin(A^2t)\varphi| \le |A^{7/2}\,(\mathcal{F}h_0)\,\varphi| \in L^2((0,\infty)\times\Gamma).
$$

Concerning the space $\dot{H}^1(0,\infty;\dot{H}^{-1/2}(\Gamma))$, we use the estimate $|\sin s| \le |s|$, $s \in \mathbb{R}$, to get that

$$
\begin{aligned}
|A^{-1/2}\partial_t(\varphi\,\partial_t\mathcal{F}\eta_0)| &= |A^{-1/2}\varphi\,\partial_t^2\,(\mathcal{F}\eta_0)\,\varphi + A^{-1/2}\varphi'\,\partial_t\,(\mathcal{F}\eta_0)\,|\\
&= |-A^{-1/2}A^4\,(\mathcal{F}h_0)\cos(A^2t)\varphi - A^{-1/2}A^2\,(\mathcal{F}h_0)\sin(A^2t)\varphi'|\\
&\le |A^{7/2}\,(\mathcal{F}h_0)\,\varphi| + |A^{7/2}\,(\mathcal{F}h_0)\,t\varphi'| \in L^2((0,\infty)\times\Gamma).
\end{aligned}
$$

Now, it is easily seen that r satisfies (2.24).

Since $\partial_t\eta_0(0,x')\varphi(0) = 0$, it is possible to extend $r = \partial_t\eta_0(t,x')\varphi$ by zero to a function in $L^2(\mathbb{R};\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^1(\mathbb{R};\dot{H}^{-1/2}(\Gamma))$ still satisfying an estimate similar to $(2.24)_2$. Then we apply in (2.22), with $(0,T)$ replaced by $\mathbb{R}$, the Fourier transform with respect to time and the first $n-1$ spatial variables to obtain that

$$
\begin{aligned}
(i\tau + \mu|\xi'|^2)\hat{w}_0' - \mu\partial_n^2\hat{w}_0' + i\xi'\hat{p}_0 &= 0 && \text{in } \mathbb{R}\times\mathbb{R}^n_+,\\
(i\tau + \mu|\xi'|^2)(\hat{w}_0)_n - \mu\partial_n^2(\hat{w}_0)_n + \partial_n\hat{p}_0 &= 0 && \text{in } \mathbb{R}\times\mathbb{R}^n_+,\\
i\xi'\cdot\hat{w}_0' + \partial_n(\hat{w}_0)_n &= 0 && \text{in } \mathbb{R}\times\mathbb{R}^n_+,\\
(\hat{w}_0', (\hat{w}_0)_n)|_{x_n=0} &= (0',\hat{r}) && \text{on } \mathbb{R}\times\Gamma.
\end{aligned}
\tag{2.25}
$$

The following calculations resemble those at the beginning of Section 2.1, see (2.4)-(2.6). Since $\operatorname{div} w_0 = 0$, we get from (2.25) that $\Delta p_0 = 0$, *i.e.*

$$\hat{p}_0 = \gamma e^{-Ax_n}$$

for a function $\gamma = \gamma(\xi', \tau)$. Using this identity, $(2.25)_1$ and $(2.25)_4$ we deduce

$$\hat{w}_0' = \frac{\xi'\gamma}{\tau}\left(e^{-Bx_n} - e^{-Ax_n}\right).$$

Now, using $(2.25)_3$, we see that

$$\begin{aligned}
\partial_n(\hat{w}_0)_n &= -i\xi' \cdot \hat{w}_0' = \frac{A^2\gamma}{i\tau}\left(e^{-Bx_n} - e^{-Ax_n}\right), \\
\partial_n^2(\hat{w}_0)_n &= \frac{A^2\gamma}{i\tau}\left(-Be^{-Bx_n} + Ae^{-Ax_n}\right).
\end{aligned}$$

Next, from $(2.25)_2$ we conclude that

$$(\hat{w}_0)_n = \gamma\left(\frac{A}{i\tau}e^{-Ax_n} - \frac{A^2}{Bi\tau}e^{-Bx_n}\right),$$

see (2.6). Finally, $(2.25)_4$ implies $\hat{r} = \gamma\left(\frac{A}{i\tau} - \frac{A^2}{i\tau B}\right) = \gamma\frac{A}{i\tau}\frac{B-A}{B}$. Hence,

$$\gamma = \hat{r}\frac{i\tau}{A}\frac{B}{B-A} = \hat{r}\mu\frac{B(B+A)}{A}.$$

In summary, we obtain a solution

$$\begin{aligned}
\hat{p}_0 &= \frac{\mu B(B+A)}{A}\hat{r}e^{-Ax_n}, \\
\hat{w}_0' &= \frac{i\xi'}{A}\frac{B}{A}\hat{r}\left(A\frac{e^{-Bx_n} - e^{-Ax_n}}{B-A}\right), \\
(\hat{w}_0)_n &= \hat{r}\frac{i\tau}{A}\frac{B}{B-A}\left(\frac{A}{i\tau}e^{-Ax_n} - \frac{A^2}{Bi\tau}e^{-Bx_n}\right) \\
&= -\frac{B}{A}\hat{r}\left(A\frac{e^{-Bx_n} - e^{-Ax_n}}{B-A}\right) + \hat{r}e^{-Bx_n}.
\end{aligned} \tag{2.26}$$

By plugging these terms into (2.25), it is easy to verify that these terms are indeed a solution to (2.25). Now, we can prove *a priori* estimates of this solution in L^2:

i) $w_0' \in L^2(0,\infty;\dot{H}^2(\mathbb{R}^n_+))$: Concerning tangential derivatives $\partial_k\partial_i(w_0)_j$, $i,j,k \in \{1,...,n-1\}$ there holds, since $|B| \le c|\tau|^{1/2} + A$,

$$\begin{aligned}
\|\xi_k\xi_i(\hat{w}_0)_j\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)} &= \left\|\frac{\xi_k}{A}\frac{\xi_i}{A}\frac{\xi_j}{A}BA\hat{r}\left(A\frac{e^{-Bx_n} - e^{-Ax_n}}{B-A}\right)\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)} \\
&\le c\left\|BA\hat{r}\frac{1}{\sqrt{A}}\right\|_{L^2(\mathbb{R}\times\Gamma)} \\
&\le c\|r\|_{\dot{H}^{1/2}(\mathbb{R};\dot{H}^{1/2}(\Gamma))} + c\|r\|_{L^2(\mathbb{R};\dot{H}^{3/2}(\Gamma))}.
\end{aligned}$$

For mixed derivatives $\partial_i\partial_n(w_0)_j$ we see that

$$\begin{aligned}
&\|\xi_i\partial_n(\hat{w}_0)_j\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)}\\
=&\left\|\frac{\xi_i}{A}\frac{\xi_j}{A}B\hat{r}\left(A\frac{-Be^{-Bx_n}+Ae^{-Ax_n}}{B-A}\right)\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)}\\
\leq&\left\|AB\hat{r}\left(-e^{-Bx_n}-A\frac{e^{-Bx_n}-e^{-Ax_n}}{B-A}\right)\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)}\\
\leq&2c\left\|AB\hat{r}\frac{1}{\sqrt{A}}\right\|_{L^2(\mathbb{R}\times\Gamma)}\leq c\|r\|_{\dot{H}^{1/2}(\mathbb{R};\dot{H}^{1/2}(\Gamma))}+c\|r\|_{L^2(\mathbb{R};\dot{H}^{3/2}(\Gamma))}.
\end{aligned}$$

Finally, the normal derivative $\partial_n^2(w_0)_j$ satisfies the estimate

$$\begin{aligned}
\|\partial_n^2(\hat{w}_0)_j\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)}\leq&\left\|\frac{B}{A}\hat{r}\left(A\frac{B^2e^{-Bx_n}-A^2e^{-Ax_n}}{B-A}\right)\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)}\\
\leq&\left\|\frac{B}{A}\hat{r}A\frac{B^2-A^2}{B-A}e^{-Bx_n}\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)}\\
&+\left\|\frac{B}{A}\hat{r}A^2\left(A\frac{e^{-Bx_n}-e^{-Ax_n}}{B-A}\right)\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)}\\
\leq&c\left\|B\hat{r}\frac{B+A}{\sqrt{A}}\right\|_{L^2(\mathbb{R}\times\Gamma)}+c\left\|\frac{B}{A}\hat{r}\frac{A^2}{\sqrt{A}}\right\|_{L^2(\mathbb{R}\times\Gamma)}\\
\leq&3c\left\|B^2\hat{r}\frac{1}{\sqrt{A}}\right\|_{L^2(\mathbb{R}\times\Gamma)}\leq c\|r\|_{\dot{H}^1(\mathbb{R};\dot{H}^{-1/2}(\Gamma))}+c\|r\|_{L^2(\mathbb{R};\dot{H}^{3/2}(\Gamma))},
\end{aligned}$$

since $A\leq|B|$ and $|B|^2\leq c|\tau|+A^2$.

ii) $(w_0)_n\in L^2(0,\infty;\dot{H}^2(\mathbb{R}^n_+))$: Since $\partial_k\partial_n(w_0)_n=-\sum_{j=1}^{n-1}\partial_k\partial_j(w_0)_k$, $k\in\{1,...,n-1\}$, we only have to consider $\partial_k\partial_i(w_0)_n$ for $i,k\in\{1,...,n-1\}$. The first summand in $(2.26)_4$ can be treated as $(\hat{w}_0)_j$ above; so we inspect the second one:

$$\|\xi_i\xi_k\hat{r}e^{-Bx_n}\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)}\leq c\left\|A^2\hat{r}\frac{1}{\sqrt{A}}\right\|_{L^2(\mathbb{R}\times\Gamma)}\leq c\,\|r\|_{L^2(\mathbb{R};\dot{H}^{3/2}(\Gamma))}\,.$$

iii) $p_0\in L^2(0,\infty;\dot{H}^1(\mathbb{R}^n_+))$: Since $\partial_n\hat{p}_0=-A\hat{p}_0$, it suffices to consider the tangential gradient $\nabla'p_0$. Here we get that

$$\begin{aligned}
\|\xi'\hat{p}_0\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)}=&\left\|\mu\frac{\xi'}{A}B(B+A)\hat{r}e^{-Ax_n}\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n_+)}\leq c\left\|B(B+A)\hat{r}\frac{1}{\sqrt{A}}\right\|_{L^2(\mathbb{R}\times\Gamma)}\\
\leq&2c\left\|B^2\hat{r}\frac{1}{\sqrt{A}}\right\|_{L^2(\mathbb{R}\times\Gamma)}\leq c\|r\|_{L^2(\mathbb{R};\dot{H}^{3/2}(\Gamma))}+c\|r\|_{\dot{H}^1(\mathbb{R};\dot{H}^{-1/2}(\Gamma))}.
\end{aligned}$$

iv) $w_0\in\dot{H}^1(0,\infty;L^2(\mathbb{R}^n_+))$: This is immediately implied by $\partial_t w_0=\mu\Delta w_0-\nabla p_0\in L^2((0,\infty)\times\mathbb{R}^n_+)$.

Restricting ourselves to the time interval $(0,T)$, recalling the interpolation inequality (2.17) and also (2.24) we deduce the proposition. □

Now, we reconstruct the initial data $u_0 \in \dot{H}^2(\mathbb{R}^n_+)$ with $\operatorname{div} u_0 = 0$ and $u_0'|_{x_n=0} = 0$, as well as the right-hand side $f \in L^2((0,T)\times\mathbb{R}^n_+)$.

Proposition 2.2.2. *Let $T \in (0,\infty)$, $f \in L^2((0,T)\times\mathbb{R}^n_+)$, and let $u_0 \in \dot{H}^2\left(\mathbb{R}^n_+\right)$ satisfy $\operatorname{div} u_0 = 0$, $u_0'|_{\partial\mathbb{R}^n_+} = 0$. Then the Stokes system*

$$\begin{aligned} \partial_t z - \mu\Delta z + \nabla p_1 &= f && \text{in } (0,T)\times\mathbb{R}^n_+,\\ \operatorname{div} z &= 0 && \text{in } (0,T)\times\mathbb{R}^n_+,\\ (z', z_n)|_{x_n=0} &= (0', \mathcal{F}^{-1}(\cos(A^2 t)(\mathcal{F}u_0)_n)) && \text{on } (0,T)\times\Gamma,\\ z(0) &= u_0 && \text{in } \mathbb{R}^n_+ \end{aligned} \tag{2.27}$$

has a solution

$$z \in L^2(0,T;\dot{H}^2(\mathbb{R}^n_+)) \cap \dot{H}^1(0,T;L^2(\mathbb{R}^n_+)), \quad p_1 \in L^2(0,T;\dot{H}^1(\mathbb{R}^n_+)).$$

Furthermore, we have the estimate

$$\begin{aligned} &\|z\|_{L^2(0,T;\dot{H}^2(\mathbb{R}^n_+))\cap\dot{H}^1(0,T;L^2(\mathbb{R}^n_+))} + \|p_1\|_{L^2(0,T;\dot{H}^1(\mathbb{R}^n_+))}\\ &\qquad \le C\|f\|_{L^2((0,T)\times\mathbb{R}^n_+)} + CT^{1/2}\|u_0\|_{\dot{H}^2(\mathbb{R}^n_+)}. \end{aligned} \tag{2.28}$$

Proof. To get rid of the terms involving u_0, define

$$w_2(t,x',x_n) := \mathcal{F}^{-1}\left(\cos(|\xi'|^2 t)\mathcal{F}u_0(\xi',x_n)\right) \in L^2(0,T;\dot{H}^2(\mathbb{R}^n_+))$$

which satisfies $(2.27)_{2,3,4}$. Moreover, since in $L^2((0,T)\times\mathbb{R}^n_+)$ one has

$$\begin{aligned} \partial_t w_2(t,x',x_n) &= \mathcal{F}^{-1}\left(-\sin(|\xi'|^2 t)|\xi'|^2\mathcal{F}u_0(\xi',x_n)\right),\\ \Delta w_2(t,x',x_n) &= \mathcal{F}^{-1}\left(\cos(|\xi'|^2 t)(-|\xi'|^2\mathcal{F}u_0 + \partial_n^2\mathcal{F}u_0)\right), \end{aligned}$$

the estimate (2.28) holds with $f = 0$, $p_1 = 0$ and z replaced by w_2.

Next, we consider the Stokes system

$$\begin{aligned} \partial_t w_1 - \mu\Delta w_1 + \nabla p_1 &= f - (\partial_t - \mu\Delta)w_2 && \text{in } (0,T)\times\mathbb{R}^n_+,\\ \operatorname{div} w_1 &= 0 && \text{in } (0,T)\times\mathbb{R}^n_+,\\ w_1|_{x_n=0} &= 0 && \text{on } (0,T)\times\Gamma,\\ w_1|_{t=0} &= 0 && \text{in } \mathbb{R}^n_+. \end{aligned}$$

Since the right-hand side lies in $L^2((0,T)\times\mathbb{R}^n_+)$, there exists a strong solution

$$(w_1, p) \in \left(L^2(0,T;H^2(\mathbb{R}^n_+)) \cap H^1(0,T;L^2(\mathbb{R}^n_+))\right) \times L^2(0,T;\dot{H}^1(\mathbb{R}^n_+))$$

with the corresponding maximal regularity estimate (see *e.g.* [16, Corollary 3.6]) with a constant $C > 0$ independent of T, namely

$$\begin{aligned} \|w_1\|_{L^2(0,T;H^2(\mathbb{R}^n_+))} + \|p_1\|_{L^2(0,T;\dot{H}^1(\mathbb{R}^n_+))} &\le C\|f - (\partial_t - \mu\Delta)w_2\|_{L^2((0,T)\times\mathbb{R}^n_+)}\\ &\le C\left(\|f\|_{L^2((0,T)\times\mathbb{R}^n_+)} + T^{1/2}\|u_0\|_{\dot{H}^2(\mathbb{R}^n_+)}\right). \end{aligned}$$

Hence, we deduce that $z := w_1 + w_2 \in L^2(0,T;\dot{H}^2(\mathbb{R}^n_+)) \cap \dot{H}^1(0,T;L^2(\mathbb{R}^n_+))$ and $p_1 \in L^2(0,T;\dot{H}^1(\mathbb{R}^n_+))$ solve (2.27) and obey (2.28). □

As an intermediate step to the main theorem of this chapter, we prove the following corollary.

Corollary 2.2.3. *Let $T \in (0,\infty)$, $f \in L^2((0,T)\times\mathbb{R}^n_+)$, $h_0 \in \dot{H}^{7/2}(\Gamma)$, and $u_0 \in \dot{H}^2(\mathbb{R}^n_+)$ with $\operatorname{div} u_0 = 0$ and $u_0'|_{\partial\mathbb{R}^n_+} = 0$. Then the Stokes system*

$$\begin{aligned} \partial_t z_1 - \mu\Delta z_1 + \nabla q &= f && \text{in } (0,T)\times\mathbb{R}^n_+,\\ \operatorname{div} z_1 &= 0 && \text{in } (0,T)\times\mathbb{R}^n_+,\\ z_1|_{t=0} &= u_0 && \text{in } \mathbb{R}^n_+, \end{aligned} \tag{2.29}$$

together with the boundary condition

$$(z_1', (z_1)_n)|_{x_n=0} = \big(0', \mathcal{F}^{-1}(-\sin(A^2t)A^2\mathcal{F}h_0 + \cos(A^2t)(\mathcal{F}u_0)_n)\big) \tag{2.30}$$

on $(0,T)\times\Gamma$, has a solution

$$z_1 \in L^2(0,T;\dot{H}^2(\mathbb{R}^n_+)) \cap \dot{H}^1(0,T;L^2(\mathbb{R}^n_+)), \quad q \in L^2(0,T;\dot{H}^1(\mathbb{R}^n_+)).$$

Furthermore, we have the estimate

$$\begin{aligned} &\|z_1\|_{L^2(0,T;\dot{H}^2(\mathbb{R}^n_+))\cap\dot{H}^1(0,T;L^2(\mathbb{R}^n_+))} + \|q\|_{L^2(0,T;\dot{H}^1(\mathbb{R}^n_+))}\\ &\le C\|f\|_{L^2((0,T)\times\mathbb{R}^n_+)} + CT^{1/2}\left(\|u_0\|_{\dot{H}^2(\mathbb{R}^n_+)} + \|h_0\|_{\dot{H}^{7/2}(\Gamma)}\right). \end{aligned}$$

Proof. Let (w_0, p_0) be a solution of (2.22) and (z, p_1) be a solution of (2.27). Then, define

$$z_1 := z + w_0 \in L^2(0,T;\dot{H}^2(\mathbb{R}^n_+)) \cap \dot{H}^1(0,T;L^2(\mathbb{R}^n_+))$$

and $q := p_0 + p_1 \in L^2(0,T;\dot{H}^1(\mathbb{R}^n_+))$ to get the desired solution. □

Now, we can prove the main result on existence of solutions.

Theorem 2.2.4. *Let $T \in (0,\infty)$ and $f \in L^2((0,T)\times\mathbb{R}^n_+)$, $g_2 \in L^2(0,T;\dot{H}^{1/2}(\Gamma))$, $g_1 \in L^2(0,T;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^1(0,T;\dot{H}^{-1/2}(\Gamma))$ with $g_1(0) = 0$, $u_0 \in \dot{H}^2\left(\mathbb{R}^n_+\right)$ with $\operatorname{div} u_0 = 0$, $u_0'|_{\partial\mathbb{R}^n_+} = 0$, and $h_0 \in \dot{H}^{7/2}(\Gamma)$ be given. Then the system*

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= f && \text{in } (0,T)\times\mathbb{R}^n_+,\\ \operatorname{div} u &= 0 && \text{in } (0,T)\times\mathbb{R}^n_+,\\ (u', u_n)|_{x_n=0} &= (0', \partial_t h + g_1) && \text{on } (0,T)\times\Gamma,\\ \partial_t^2 h + \Delta^2 h &= -p + g_2 && \text{on } (0,T)\times\Gamma,\\ u(0) &= u_0 && \text{in } \mathbb{R}^n_+,\\ (h(0), \partial_t h(0)) &= (h_0, (u_0)_n) && \text{in } \Gamma \end{aligned} \tag{2.31}$$

has a solution

$$\begin{aligned} &u \in L^2(0,T;\dot{H}^2(\mathbb{R}^n_+)) \cap \dot{H}^1(0,T;L^2(\mathbb{R}^n_+)),\\ &p \in L^2(0,T;\dot{H}^1(\mathbb{R}^n_+)),\\ &h \in \dot{H}^1(0,T;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^2(0,T;\dot{H}^{-1/2}(\Gamma)). \end{aligned}$$

Furthermore, we have the estimate

$$
\begin{aligned}
&\|u\|_{L^2(0,T;\dot{H}^2(\mathbb{R}^n_+))\cap\dot{H}^1(0,T;L^2(\mathbb{R}^n_+))} + \|p\|_{L^2(0,T;\dot{H}^1(\mathbb{R}^n_+))} \\
&\quad + \|h\|_{\dot{H}^1(0,T;\dot{H}^{3/2}(\Gamma))\cap\dot{H}^2(0,T;\dot{H}^{-1/2}(\Gamma))} \\
&\leq C\big(\|f\|_{L^2((0,T)\times\mathbb{R}^n_+)} + \|g_1\|_{L^2(0,T;\dot{H}^{3/2}(\Gamma))\cap\dot{H}^1(0,T;\dot{H}^{-1/2}(\Gamma))} \\
&\quad + \|g_2\|_{L^2(0,T;\dot{H}^{1/2}(\Gamma))}\big) + CT^{1/2}\big(\|u_0\|_{\dot{H}^2(\mathbb{R}^n_+)} + \|h_0\|_{\dot{H}^{7/2}(\Gamma)}\big).
\end{aligned} \tag{2.32}
$$

Note that C in (2.32) is independent of T.

Proof. Let

$$
(z_1, q) \in L^2(0,T;\dot{H}^2(\mathbb{R}^n_+)) \cap \dot{H}^1(0,T;L^2(\mathbb{R}^n_+)) \times L^2(0,T;\dot{H}^1(\mathbb{R}^n_+))
$$

be a solution of (2.29)-(2.30). Due to $q|_{\partial\mathbb{R}^n_+} \in L^2(0,T;\dot{H}^{1/2}(\Gamma))$, Proposition 2.1.3 implies the existence of

$$
\begin{aligned}
&(u^\star, p^\star) \in L^2(0,T;\dot{H}^2(\mathbb{R}^n_+)) \cap \dot{H}^1(0,T;L^2(\mathbb{R}^n_+)) \times L^2(0,T;\dot{H}^1(\mathbb{R}^n_+)), \\
&\quad h^\star \in \dot{H}^1(0,T;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^2(0,T;\dot{H}^{-1/2}(\Gamma))
\end{aligned}
$$

such that

$$
\begin{aligned}
\partial_t u^\star - \mu\Delta u^\star + \nabla p^\star &= 0 && \text{in } (0,T)\times\mathbb{R}^n_+, \\
\operatorname{div} u^\star &= 0 && \text{in } (0,T)\times\mathbb{R}^n_+, \\
u^\star|_{x_n=0} &= (0', \partial_t h^\star + g_1) && \text{on } (0,T)\times\Gamma, \\
\partial_t^2 h^\star + \Delta^2 h^\star &= -p^\star + (g_2 - q) && \text{on } (0,T)\times\Gamma
\end{aligned}
$$

satisfying the initial condition

$$
u^\star|_{t=0} = 0, (h^\star, \partial_t h^\star)|_{t=0} = (0,0)
$$

on $\mathbb{R}^n_+$ and Γ, respectively. To finish the proof define

$$
\begin{aligned}
&u = u^\star + z_1 \in L^2(0,T;\dot{H}^2(\mathbb{R}^n_+)) \cap \dot{H}^1(0,T;L^2(\mathbb{R}^n_+)), \\
&p = p^\star + q \in L^2(0,T;\dot{H}^1(\mathbb{R}^n_+)), \\
&h = h^* + \mathcal{F}^{-1}(\cos(A^2t)\mathcal{F}h_0) + \mathcal{F}^{-1}\big(A^{-2}\sin(A^2t)(\mathcal{F}u_0)_n\big),
\end{aligned}
$$

where $h \in \dot{H}^1(0,T;\dot{H}^{3/2}(\Gamma)) \cap \dot{H}^2(0,T;\dot{H}^{-1/2}(\Gamma))$, to get a solution to (2.31). Note that the term $\mathcal{F}^{-1}(\cos(A^2t)\mathcal{F}h_0)+\mathcal{F}^{-1}\big(A^{-2}\sin(A^2t)(\mathcal{F}u_0)_n\big)$ does not alter the plate equation as described at the beginning of Section 2.2. Also, note that we set $p = p^\star+q$ due to $-p^\star+(g_2-q) = -(p^\star+q)+g_2$. □

Remark 2.2.5. Despite $h \in \dot{H}^2(0,T;\dot{H}^{-1/2}(\Gamma))$ we did not show that h is very regular with respect to space, let alone $h(t) \in \dot{H}^4(\Gamma)$ for almost all $t \in (0,T)$. This is due to the fact that while $\tau^{3/2}\mathcal{N}(A,\tau)$ is bounded, see Lemma 2.1.1, a comparable spatial boundedness like the boundedness of $A^3\mathcal{N}(A,\tau)$ does not hold.

CHAPTER 3

Strong Solutions to the Navier–Stokes Equations on Manifolds with Coulomb Boundary Condition

The Coulomb boundary condition models friction of the fluid with the boundary. Since friction is intrinsically a non-linear phenomenon, so is the mathematical formulation of the boundary condition. This in turn complicates the mathematical analysis for the linear Stokes as well as the nonlinear Navier–Stokes problem, and the results to be expected are less complete than those in the Dirichlet case, for example. The corresponding mathematical model on a bounded domain $\Omega \subset \mathbb{R}^d$ and on a time interval $(0,T)$ with $T > 0$ reads as follows:

$$\begin{aligned} \partial_t u + u \cdot \nabla u - \operatorname{div} \sigma(u,\pi) &= f && \text{in } (0,T)\times\Omega, \\ \operatorname{div} u &= 0 && \text{in } (0,T)\times\Omega, \\ u|_{t=0} &= u_0 && \text{in } \Omega \end{aligned} \tag{3.1}$$

together with the boundary conditions

$$u_\tau = \begin{cases} 0, & \text{if } |(\sigma(u,\pi)n_\tau| < \kappa \\ -\alpha(\sigma(u,\pi)n)_\tau, & \text{if } |(\sigma(u,\pi)n)_\tau| = \kappa \text{ with } \alpha \geq 0 \end{cases} \quad \text{and} \quad u_n = 0 \tag{3.2}$$

on $\partial\Omega \times (0,T)$; here, $\kappa > 0$, $\mu > 0$, and $\sigma(u,\pi)$ denotes the Cauchy stress tensor

$$\sigma(u,\pi) = 2\mu D(u) - \pi I \quad \text{where } D(u) := \tfrac{1}{2}\left(\nabla u + \nabla^T u\right).$$

Moreover, n denotes the exterior normal on $\partial\Omega$ and, for a vector field u on $\overline{\Omega}$, let $u_\tau = u - \langle u, n\rangle n$ be the tangential component of u on $\partial\Omega$.

In this chapter we are mainly concerned to show the exstistence of strong solutions in time. The existence of strong solutions with respect to time, *i.e.*, $u \in W_2^1(0,T;L^2(\Omega))$, has already been shown by Kashiwabara [27] for finite time intervals.

However, we use a method different from the approach in [27] in the sense that our method, additionally, allows to obtain global solutions in the case $d = 2$ and for small data if $d = 3$. Furthermore, weak-strong uniqueness will be proven. Finally, it will be shown that the boundary

condition is in some sense between the Dirichlet and the perfect slip boundary condition - as suggested by its physical motivation; to be more precise, if the key threshold parameter of the Coulomb boundary condition, see $\kappa > 0$ in (3.4) below, converges to 0 or infinity, then a weak solution converges to a weak solution satisfying the perfect slip or Dirichlet condition, respectively. All results will be proven for the Navier–Stokes system of viscous flow on a smooth manifold.

To this aim, we consider the following system of partial differential equations on a smooth, compact, d-dimensional Riemannian manifold (M, g) with boundary ∂M:

$$\begin{cases} \partial_t u + \nabla_u u - \operatorname{div}_M \sigma(u,\pi) = f & \text{in } (0,T)\times \mathring{M}, \\ \operatorname{div} u = 0 & \text{in } (0,T)\times \mathring{M}, \\ u|_{t=0} = u_0 & \text{in } \mathring{M} \end{cases} \tag{3.3}$$

together with the boundary conditions

$$u_n = 0, \;\; \langle (\sigma(u,\pi)n)_\tau, y\rangle_{g_p} \geq \kappa |u_\tau|_{g_p} - \kappa |u_\tau + y|_{g_p} \;\; \forall y \in T_p\partial M \text{ on } \partial M \times (0,T); \tag{3.4}$$

here, $\kappa > 0, \mu > 0$, and $\sigma(u,\pi)$ denotes the Cauchy stress tensor

$$\sigma(u,\pi) = 2\mu D(u) - \pi I \;\text{ where } D(u) := \frac{1}{2}\left(\nabla u + \nabla^T u\right).$$

Moreover, n denotes the exterior normal on ∂M and, for a vector field u on M, let $u_\tau = u - \langle u, n\rangle_{g_p} n$ be the tangential component of u on ∂M at $p \in \partial M$.

There are two further equivalent formulations of the boundary condition (3.4), see the proof below:

a) $|(\sigma(u,\pi)n)_\tau|_{g_p} \leq \kappa$ and

$$u_\tau = \begin{cases} 0, & \text{if } |(\sigma(u,\pi)n_\tau|_{g_p} < \kappa, \\ -\alpha(\sigma(u,\pi)n)_\tau, & \text{if } |(\sigma(u,\pi)n)_\tau|_{g_p} = \kappa \text{ with } \alpha \geq 0 \end{cases}$$

b) $\langle (\sigma(u,\pi)n)_\tau, y\rangle_{g_p} \geq \kappa |u_\tau|_{g_p} - \kappa |u_\tau + y|_{g_p}$ for all $y \in T_p\partial M$

c) $-(\sigma(u,\pi)n)_\tau \in \kappa \partial |u|_{g_p}$

Formulation a) helps to see the physical motivation: The tangential part of the stress-tensor is smaller or equal to a given constant $\kappa > 0$, the threshold value for the onset of slip motion. If it is smaller, then the fluid adheres to the boundary. However, if it equals the given constant, the fluid may start to slip in the direction of the tangential force $-(\sigma(u,\pi)n)_\tau$ on ∂M.

In contrast, formulation b) and c) are better suited for the mathematical analysis. The variational inequality in b) is useful to define weak solutions, whereas c) is used in the context of the theory of maximal monotone operators, see for example [18].

Thereby, $\partial |\cdot|_g$ denotes the subdifferential of $|\cdot|_g$, *i.e.*,

$$\partial |y|_{g_p} = \begin{cases} \left\{ \frac{y}{|y|_{g_p}} \right\}, & \text{if } y \neq 0 \\ \overline{B_1(0)}, & \text{if } y = 0 \end{cases} \qquad \forall p \in \partial M \; \forall y \in T_p\partial M.$$

The proof of this equivalence is given in the following lemma.

Lemma 3.0.1. *The formulations a), b) and c) are equivalent.*

Proof. "a) ⇒ b)": First assume that $|(\sigma(u,\pi)n)_\tau| < \kappa$ which implies $u_\tau = 0$ and due to the Cauchy–Schwarz inequality

$$\langle(\sigma(u,\pi)n)_\tau, y\rangle \geq -\kappa|y| = \kappa|u_\tau| - \kappa|u_\tau + y|.$$

Now assume $|(\sigma(u,\pi)n)_\tau| = \kappa$ which implies $u_\tau = -\alpha(\sigma(u,\pi)n)_\tau$ for an $\alpha \geq 0$. If $\alpha = 0$ we can proceed as before, so assume that $\alpha > 0$. We need to show that

$$\langle(\sigma(u,\pi)n)_\tau, y\rangle_{g_p} \geq \alpha\kappa^2 - \kappa| - \alpha(\sigma(u,\pi)n)_\tau + y|_{g_p} \ \forall y \in T_p\partial M.$$

We replace y with $\alpha((\sigma(u,\pi)n)_\tau + y)$ and divide by α to get that the claim is equivalent to

$$\langle(\sigma(u,\pi)n)_\tau, y\rangle_{g_p} + \kappa^2 \geq \kappa^2 - \kappa|y| \quad \forall y \in T_p\partial M.$$

This is true due to the Cauchy–Schwarz inequality.

"b) ⇒ c)": In the variational inequality choose $y = -(\sigma(u,\pi)n)_\tau$ and multiply by -1 to get that

$$|(\sigma(u,\pi)n)_\tau|^2 \leq \kappa|u_\tau - (\sigma(u,\pi)n)_\tau| - \kappa|u_\tau| \leq \kappa|(\sigma(u,\pi)n)_\tau|$$

which implies $|(\sigma(u,\pi)n)_\tau| \leq \kappa$. Assume that $u_\tau \neq 0$ and choose $y = -u_\tau$ to get

$$\kappa| - u_\tau| \geq \langle(\sigma(u,\pi)n)_\tau, -u_\tau\rangle \geq \kappa| - u_\tau|.$$

Consequently, $\langle(\sigma(u,\pi)n)_\tau, -u_\tau\rangle = \kappa| - u_\tau|$ which means that we have an identity in the Cauchy-Schwarz inequality $\langle(\sigma(u,\pi)n)_\tau, -u_\tau\rangle \leq \kappa| - u_\tau|$ and therefore $(\sigma(u,\pi)n)_\tau = -\beta u_\tau$ for some $\beta \in \mathbb{R}$. Plugging this identity into $\langle(\sigma(u,\pi)n)_\tau, -u_\tau\rangle = \kappa| - u_\tau|$ yields $\beta = \frac{\kappa}{|u_\tau|}$. So, in total, $-(\sigma(u,\pi)n)_\tau = \kappa\frac{u_\tau}{|u_\tau|}$ if $u_\tau \neq 0$.

"c) ⇒ a)": We immediately get $|\sigma(u,\pi)n)_\tau| \leq \kappa$ and $u_\tau = 0$ if $|\sigma(u,\pi)n)_\tau| < \kappa$. If $|(\sigma(u,\pi)n)_\tau| = \kappa$ then in order for the identity $u_\tau = -\alpha(\sigma(u,\pi)n)_\tau$ to hold for an $\alpha \geq 0$, choose $\alpha = 0$ if $u_\tau = 0$ and $\alpha = \frac{|u_\tau|}{\kappa} \geq 0$ if $u_\tau \neq 0$. □

Remark 3.0.2. When generalizing the Navier–Stokes equations from the Euclidean case to the case of manifolds, the question arises how to define the Laplacian to obtain a new one that operates on vector fields. There are many options available that coincide in the Euclidean case, but not necessarily in the case of manifolds. For example one could take the Hodge-Laplacian or the Bochner-Laplacian. In [11] both physical and mathematical reasons are given to show that the choice

$$\begin{aligned}\Delta_{NSt}u :=& 2\,\mathrm{div}_M\, Du = \Delta_{Bochner}u + \mathrm{Ric}(u)\\ &= \left[\Delta_{\mathrm{Hodge}}u^\flat\right]^\sharp + 2\,\mathrm{Ric}(u) \text{ for vector fields } u \text{ with } \mathrm{div}\, u = 0\end{aligned}$$

involving the stress tensor, is the most reasonable one. Here $\flat$ and $\sharp$ denote the musical isomorphisms and Ric denotes the Ricci curvature that vanishes in the Euclidean case. In our case this choice is natural when deriving the weak formulation. Since under this setting, we have for all $u, \varphi \in \mathfrak{X}(M)$ satisfying $u_n = \varphi_n = 0$ on ∂M

$$\begin{aligned}-\int_M \langle\mu\Delta_{NSt}u, \varphi\rangle &= \mu\int_M \langle\nabla u, \nabla\varphi\rangle - \int_{\partial M}\langle(\sigma(u,\pi)n)_\tau, \varphi\rangle\\ &\leq \mu\int_M \langle\nabla u, \nabla\varphi\rangle + \int_{\partial M}\kappa(|u+\varphi| - |u|).\end{aligned}$$

This chapter is structured as follows: In Section 3.1 our main result on strong solutions in time will be proved:

Theorem 3.0.3. *(**Strong solutions in time**)*
Let $u_0 \in H^2(M) \cap H$ *with*

$$H := L^2_\sigma(M) := \big\{ v \in L^2(M)^d : \operatorname{div}_M v = 0 \text{ in } M, v_n = 0 \text{ on } \partial M \big\},$$

$\kappa \in H^{1/2}(\partial M)$ *with* $\kappa > 0$ *a.e. on* ∂M *and* $T \in (0, \infty]$. *Assume that*

$$\Gamma = \Gamma(u_0) := \{ z \in \partial M : u_{0|\partial M}(z) = 0 \},$$

which is defined up to a Lebesgue null set, has a non-empty open subset, and that

$$-2\mu (D\,(u_0)\,n)_\tau = \kappa \frac{u_{0|\partial M}}{|u_{0|\partial M}|} \quad \text{a.e. in } \partial M \backslash \Gamma(u_0).$$

Let $H^1_\Gamma := \{ u \in H^1(M) : u_{|\Gamma(u_0)} = 0 \text{ on } \partial M \}$ *and* $V := H^1_\Gamma \cap H$.

(1) If $d = 2$, $f \in W^1_2(0,T;V')$ *and* $f(0) \in H$, *then there is a unique weak solution* u *of* (3.3) *such that*

$$u \in W^1_\infty(0,T;H) \cap W^1_2(0,T;V), \tag{3.5}$$

and satisfying the energy equality, see (3.8) *later.*

(2) If $d = 3$ *and* $f \in W^1_1(0,T;H)$, *then there exists a solution satisfying* (3.5) *for a short time. Moreover, if the given data is small enough, then there exists a global solution satisfying* (3.5).

After that, weak-strong uniqueness will be shown. In Section 3.2 we consider the limits $\kappa \to 0$ and $\kappa \to \infty$ and derive that weak solutions yield an intermediate case between weak solutions to the Dirichlet case and to the perfect slip case.

3.1. Strong Solutions

The following theorem is the starting point and has been proved for the Euclidean case in [27], [31] and [6, Theorem 2.2]. These results concern existence on finite time intervals. We will give a proof of some claims of it along the way when proving Theorem 3.1.4 on global-in-time strong solutions, see Part I of that proof.

Theorem 3.1.1. *(**Weak solutions**) For dimensions* $d = 2$ *and* $d = 3$ *let* M *be an orientable, smooth, compact Riemannian manifold with metric* g *and boundary* ∂M, *and let* $\kappa \in H^{1/2}(M)$ *be a nonnegative threshold function on* ∂M. *Let* $T \in (0, \infty)$, $u_0 \in H$ *and* $f \in L^2(0,T;\big(H^1_\sigma\big)')$ *with*

$$H^1_\sigma := H \cap H^1(M)$$

be given. Then there exists at least one weak solution of (3.3), i.e., *a vector field*

$$u \in L^\infty(0,T;H) \cap L^2(0,T;H^1_\sigma)$$

such that for all $v \in C_c^1([0;T);H_\sigma^1)$

$$\begin{aligned}(3.6)\qquad &-\int_M \langle u_0, v(0)\rangle - \int_0^T\int_M \big(\langle u, \partial_t v\rangle + \langle \nabla_u v, u\rangle\big) + 2\mu\int_0^T\int_M \langle D(u), D(v)\rangle \\ &+\int_0^T\int_{\partial M} \kappa(|u+v|_g - |u|_g) \geq \int_0^T \langle f, v\rangle_{(H_\sigma^1)',H_\sigma^1}.\end{aligned}$$

Furthermore, $\partial_t u \in L^{4/d}(0,T;\big(H_\sigma^1\big)')$*, and* u *satisfies the energy estimate*

$$\begin{aligned}(3.7)\qquad &\frac{1}{2}\|u(t)\|_{L^2}^2 + 2\mu\int_0^t\int_M |D(u(s))|^2 + \int_0^t\int_{\partial M}\kappa|u(s)|_g \\ &\leq \frac{1}{2}\|u_0\|_{L^2}^2 + \int_0^t \langle f(s), u(s)\rangle_{(H_\sigma^1)',H_\sigma^1} \quad \text{for a.a. } t\in(0,T).\end{aligned}$$

If $d=2$*, this solution is unique and* $u \in C([0;T];H)$*.*

The motivation of the variational inequality (3.6) is based on inequality (3.4). The following lemma considers a smooth approximation of $|\cdot|_p$ on $T_p\partial M$.

Lemma 3.1.2. *Let* $m\in\mathbb{N}$*,* $p\in\partial M$*,*

$$j_{m,p} : T_p\partial M \to \mathbb{R},\ \ x \mapsto \sqrt{|x|_p^2 + \frac{1}{m^2}} - \frac{1}{m},$$

and, for prescribed $z\in T_p\partial M$*, define*

$$k_{m,p} : T_p\partial M \to \mathbb{R},\ \ x \mapsto j_{m,p}(x) - \langle z, x\rangle_p.$$

Then the following properties hold:

(1) $j_{m,p}, k_{m,p}$ *are smooth and convex*

(2) $j_{m,p}(0) = k_{m,p}(0) = 0$

(3) $|j_{m,p}(x) - |x|_p| \leq 1/m \ \ \forall x\in T_p\partial M$

(4) $Gj_{m,p}(x) = \frac{x}{\sqrt{|x|_p^2+m^{-2}}} \ \ \forall x\in T_p\partial M$

(5) $\langle Gj_{m,p}(x), x\rangle_p \geq 0 \ \ \forall x\in T_p\partial M$

(6) $Gk_{m,p}$ *is Lipschitz continuous with a Lipschitz constant independent of* p

Here G *denotes the Gâteaux differential.*

Proof. All points are easy to see. Let us just check the last one. For $x,y\in T_p\partial M$

$$\begin{aligned}|Gk_{m,p}(x) - Gk_{m,p}(y)| &\leq \left|\frac{x}{\sqrt{|x|^2+m^{-2}}} - \frac{x}{\sqrt{|y|^2+m^{-2}}}\right| + \frac{|x-y|}{\sqrt{|y|^2+m^{-2}}} \\ &\leq \frac{|x|}{\sqrt{|x|^2+m^{-2}}}\frac{\left|\sqrt{|y|^2+m^{-2}} - \sqrt{|x|^2+m^{-2}}\right|}{\sqrt{|y|^2+m^{-2}}} + m|x-y| \\ &\leq 2m|x-y|.\end{aligned}$$

Obviously, the Lipschitz constant $2m$ does not depend on p. □

Remark 3.1.3. One problem of the equation is given by the non-smooth absolute value, appearing on the boundary. This problem is tackled by approximating $|\cdot|$ with convex, smooth functions. Since the boundary condition is of the form

$$-(D(u)n)_\tau \in \partial\beta(u)$$

where $\beta = |\cdot|$ is a lower semi-continuous, convex function, it suggests itself to take a look into the theory of maximal monotone operators that deals with boundary conditions of this form, see [8]. Following this line, the suggested approximation for $|\cdot|$ is given by the Yosida approximation, which in our case reads as follows:

$$l_{m,p}(x) := \begin{cases} \frac{m}{2}|x|^2_{g_p} & ,|x|_{g_p} \leq \frac{1}{m} \\ |x|_{g_p} - \frac{1}{2m} & ,|x|_{g_p} > \frac{1}{m} \end{cases} \quad \forall p \in \partial M \; \forall x \in T_p\partial M.$$

However, the problem is that $l_{m,p} \in C^1(T_p\partial M)\backslash C^2(T_p\partial M)$. While this is sufficient to derive the existence of weak solutions as done in [6], it is deficient to derive the existence of strong solutions in time. That is why $j_{m,p} \in C^\infty(T_p\partial M)$ - as in the previous lemma - is used in [27]. We will make an additional modification of j_m and use k_m instead, where the modification will depend on the initial value. In fact, the solution spaces have to be adapted to the initial value.

Theorem 3.1.4. ***(Strong solutions in time)*** *Let $u_0 \in H^2(M) \cap H$, $\kappa \in H^{1/2}(\partial M)$ with $\kappa > 0$ a.e. on ∂M and $T \in (0, \infty]$. Assume that*

$$\Gamma := \Gamma(u_0) := \{z \in \partial M : u_{0|\partial M}(z) = 0\},$$

which is defined up to a Lebesgue null set, has a non-empty open subset, and that

$$-2\mu(D(u_0)n)_\tau = \kappa \frac{u_{0|\partial M}}{|u_{0|\partial M}|} \quad a.e. \text{ in } \partial M\backslash\Gamma(u_0).$$

Let

$$H^1_\Gamma := \{u \in H^1(M) : u_{|\Gamma(u_0)} = 0 \text{ on } \partial M\}$$

and

$$V := H \cap H^1_\Gamma.$$

(1) If $d = 2$, $f \in W^1_2(0,T;V')$ and $f(0) \in H$, then there exists a unique weak solution u of (3.3) in the sense of (3.6) such that

$$u \in W^1_\infty(0,T;H) \cap W^1_2(0,T;V),$$

and satisfying the energy equality

$$\frac{1}{2}\|u(t)\|^2_H + 2\mu\int_0^t\int_M |Du|^2 + \int_0^t\int_{\partial M\backslash\Gamma} \kappa|u_{|\partial M}| = \frac{1}{2}\|u_0\|^2_H + \int_0^t\int_M \langle f, u\rangle \tag{3.8}$$

for all $t \in (0,T)$.

(2) Let $d = 3$ and $f \in W_1^1(0,T;H)$.

(i) There is a $T' \in (0,T)$ such that there exists a $u \in W_\infty^1(0,T';H) \cap W_2^1(0,T';V)$ such that (3.6) and (3.8) hold true with T' instead of T.

(ii) There is a constant $\varepsilon_* > 0$ such that if $f \in L^\infty(0,T;V')$ and

$$\|u_0\|_{H^2} + \|f\|_{L^\infty(0,T;V')} + \|f(0)\|_H + \|\partial_t f\|_{L^1(0,T;H)} \leq \varepsilon_*,$$

then (i) holds true on $[0,T)$.

Proof. Let $(v_j)_{j\in\mathbb{N}} \subset H^2 \cap V$ be such that $\operatorname{span}\{v_j : j \in \mathbb{N}\}$ is dense both in V and $V \cap H^2$. Furthermore, let $V_m := \operatorname{span}\{v_1, ..., v_m\}$ for all $m \in \mathbb{N}$. Choose a sequence $(u_{m,0})_{m\in\mathbb{N}}$ such that $u_{m,0} \in V_m$ for all $m \in \mathbb{N}$, $u_{m,0} \to u_0$ in $V \cap H^2$ as $m \to \infty$ and $\|u_{m,0}\|_{H^2} \leq 2\|u_0\|_{H^2}$. Given the functions $j_{m,p}(x)$ as in Lemma 3.1.2 we choose a smooth cutoff function $l \in C_c^\infty([0,T))$ with $l(0) = 1$ and define

$$k_{m,p}(x,t) := j_{m,p}(x) - l(t)\left\langle Gj_{m,p}(u_{m,0_{|\partial M}}) + \tfrac{2\mu}{\kappa}(D(u_{m,0})_{|\partial M} n)_\tau \,,\, x\right\rangle. \tag{3.9}$$

For the existence of weak solutions the simpler setting $k_{m,p}(x,t) = j_{m,p}(x)$ suffices. However, in order to control the term $\|\partial_t u_m(0)\|_H$ later and to get strong solutions in time, see (3.19) below, the second term is needed. The introduction of l will ensure the existence of a strong solution if $T = \infty$, assuming $d = 2$ or small given data.

Again we abbreviate $j_{m,p} = j_m$ and $k_{m,p} = k_m$. Note that

$$Gk_m(u_{m,0}|_{\partial M}, 0) = -\tfrac{2\mu}{\kappa}(D\,(u_{m,0})_{|\partial M}\, n)_\tau. \tag{3.10}$$

Since $u_{m,0} \to u_0$ in $H^2(M)$, we also have $u_{m,0|\partial M} \to u_{0|\partial M}$ in $L^2(\partial M)$. Hence, there is a subsequence $(u_{m_j,0})_{j\in\mathbb{N}}$ of $(u_{m,0})_{m\in\mathbb{N}}$, which we denote by $(u_{m,0})_{m\in\mathbb{N}}$ again, such that $u_{m,0|\partial M} \to u_{0|\partial M}$ almost everywhere in ∂M.

So, we get that

$$Gj_m(u_{m,0}) = \frac{u_{m,0|\partial M}}{\sqrt{|u_{m,0|\partial M}|^2 + m^{-2}}} \longrightarrow \frac{u_0}{|u_0|} \text{ a.e. in } \partial M \backslash \Gamma(u_0)$$

as $m \to \infty$ for $u_{m,0} \to u_0(z) \to u_0(z)$ for a.a. $z \in M\backslash\Gamma$. Since $|Gj_m(u_{m,0})| \leq 1$ on ∂M, we obtain from the Coulomb boundary condition (3.2) and by the dominated convergence theorem that

$$\kappa Gj_m(u_{m,0}) \longrightarrow \kappa\frac{u_0}{|u_0|} = -2\mu(D(u_0)n)_\tau \text{ in } L^2(\partial M\backslash\Gamma).$$

Due to the convergence $u_{m,0} \to u_0$ in $H^2(M)$ as $m \to \infty$ this implies that

$$\delta(m) := \int_{\partial M\backslash\Gamma} \left|\kappa Gj_m(u_{m,0}) + 2\mu\,(D(u_{m,0})n)_\tau\right|^2 \longrightarrow 0 \text{ as } m \to \infty. \tag{3.11}$$

Part I: A priori estimates for a weak solution. We consider the three mappings $A, B, C_{m,t} : V \to V'$ with

$$\begin{aligned}\langle Au, v\rangle_{V',V} &:= 2\mu \int_M \langle Du, Dv\rangle,\\ \langle Bu, v\rangle_{V',V} &:= -\int_M \langle \nabla_u v, u\rangle,\\ \langle C_{m,t}(u), v\rangle_{V',V} &:= \int_{\partial M} \kappa \langle Gk_m(u,t), v\rangle\end{aligned}$$

for all $m \in \mathbb{N}, t \in [0,T], u, v \in V$. Since $f \in C([0,T]; V')$ or $f \in C([0,T']; V')$ for all $T' > 0$ if $T = \infty$, we get that the ordinary differential equation

$$\begin{aligned}\langle \partial_t u_m(t), v\rangle_{V',V} + \langle Au_m(t) + Bu_m(t) + C_{m,t}(u_m(t)), v\rangle_{V',V} &= \langle f(t), v\rangle_{V',V} \quad \forall v \in V_m\\ u_m(0) &= u_{m,0}\end{aligned} \tag{3.12}$$

has a unique local solution $u_m \in C^1([0,T^\star); V_m)$ for a $T^\star \le T$. Here, we used that $Gk_m(\cdot,t)$ is Lipschitz continuous according to Lemma 3.1.2 and smooth with respect to time. The first line of (3.12) is equivalent to

$$\begin{aligned}\int_M \langle \partial_t u_m, v\rangle - \int_M \langle \nabla_{u_m} v, u_m\rangle + 2\mu \int_M \langle Du_m, Dv\rangle\\ + \int_{\partial M} \kappa \langle Gk_m(u_m, \cdot), v\rangle = \langle f, v\rangle_{V',V} \quad \forall v \in V_m.\end{aligned} \tag{3.13}$$

For $v = u_m$ we get, using $\int_M \langle \nabla_{u_m} u_m, u_m\rangle = 0$ and shifting the boundary integral to the right-hand side, that

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2_{L^2(M)} + 2\mu \int_M |D(u_m)|^2 = -\int_{\partial M} \kappa \langle Gk_m(u_m,\cdot), u_m\rangle + \langle f, u_m\rangle_{V',V}. \tag{3.14}$$

Integration over $(0,t)$ for $t \le T^*$ and (3.9) yield

$$\begin{aligned}&\frac{1}{2}\|u_m(t)\|^2_{L^2} + 2\mu\|Du_m\|^2_{L^2(0,t;H)}\\ &\quad= \frac{1}{2}\|u_{m,0}\|^2_{L^2} - \int_0^t\int_{\partial M} \kappa\langle Gk_m(u_m,\cdot), u_m\rangle + \int_0^t \langle f_m, u_m\rangle_{V',V}\\ &\quad= \frac{1}{2}\|u_{m,0}\|^2_{L^2} - \int_0^t\int_{\partial M} \kappa \underbrace{\langle Gj_m(u_m), u_m\rangle}_{\ge 0}\\ &\qquad+ \int_0^t\int_{\partial M} l\left\langle \kappa Gj_m(u_{m,0}) + 2\mu\left(D\left(u_{m,0}\right)n\right)_\tau, u_m\right\rangle + \int_0^t \langle f, u_m\rangle_{V',V}.\end{aligned}$$

Since in the boundary integral u_m vanishes on $\Gamma(u_0)$, we estimate the last two integrals on the

right-hand side as follows (here $L_* = \|l\|^2_{L^2(0,\infty)}$) by using Young's inequality:

$$\begin{aligned}
&\int_0^t \int_{\partial M} l \left\langle \kappa G j_m(u_{m,0}) + 2\mu \left(D\left(u_{m,0}\right) n\right)_\tau, u_m \right\rangle + \int_0^t \langle f, u_m \rangle_{V',V} \\
&\quad \leq \int_0^t \int_{\partial M \setminus \Gamma(u_0)} \left(\frac{l^2}{2\varepsilon} |\kappa G j_m(u_{m,0}) + 2\mu \left(D\left(u_{m,0}\right) n\right)_\tau|^2 + \frac{\varepsilon}{2} |u_m|^2 \right) \\
&\qquad + \frac{1}{2\varepsilon} \|f\|^2_{L^2(0,T;V')} + \frac{\varepsilon}{2} \|u_m\|^2_{L^2(0,t;V)} \\
&\quad \leq \frac{L_*}{2\varepsilon} \int_{\partial M \setminus \Gamma(u_0)} \left(|\kappa G j_m(u_{m,0}) + 2\mu \left(D\left(u_{m,0}\right) n\right)_\tau|^2 \right) \\
&\qquad + \frac{\varepsilon}{2} \int_0^t \int_{\partial M \setminus \Gamma(u_0)} |u_m|^2 + \frac{1}{2\varepsilon} \|f\|^2_{L^2(0,T;V')} + \frac{\varepsilon}{2} \|u_m\|^2_{L^2(0,t;V)} \\
&\quad \overset{(3.11)}{\leq} \frac{L_*}{2\varepsilon}\, \delta(m) + \frac{\varepsilon}{2} \int_0^t \int_{\partial M} |u_m|^2 + \frac{1}{2\varepsilon} \|f\|^2_{L^2(0,T;V')} + \frac{\varepsilon}{2} \|u_m\|^2_{L^2(0,t;V)} \\
&\quad \overset{\text{Korn}}{\leq} \frac{L_*}{2\varepsilon}\, \delta(m) + \frac{1}{2\varepsilon} \|f\|^2_{L^2(0,T;V')} + c\varepsilon 2\mu \|Du_m\|^2_{L^2(0,t;L^2)} \quad \forall t \in (0, T^*).
\end{aligned}$$

Choose $\varepsilon > 0$ such that $c\varepsilon < 1/2$ to obtain after taking the supremum over t and the absorption of the last term on the right-hand side that

$$\sup_{0 \leq \tau \leq t} \|u_m(\tau)\|^2_{L^2} + 2\mu \|Du_m\|^2_{L^2(0,t;L^2)} \leq C \left(\|u_{m,0}\|^2_{L^2} + \delta(m) + \|f\|^2_{L^2(0,T;V')} \right). \tag{3.15}$$

The right-hand side is bounded by a constant independent of time and m; therefore, we may choose $T^* = T$. Furthermore, $(u_m)_{m \in \mathbb{N}}$ is bounded in

$$C([0,T); H) \text{ and } L^\infty(0,T;H) \cap L^2(0,T;V). \tag{3.16}$$

Part II: A priori estimates for a strong solution. Considering (3.13) once again we apply $\frac{d}{dt}$ to get the variational identity

$$\begin{aligned}
&\int_M \langle \partial_t^2 u_m, v \rangle - \int_M \left(\langle \nabla_{\partial_t u_m} v, u_m \rangle + \langle \nabla_{u_m} v, \partial_t u_m \rangle - 2\mu \langle D(\partial_t u_m), Dv \rangle \right) \\
&\quad + \int_{\partial M} \kappa \left\langle \partial_t G k_m(u_m, \cdot), v \right\rangle = \langle \partial_t f, v \rangle_{L^2} \quad \forall v \in V_m.
\end{aligned}$$

Now, choose $v = \partial_t u_m(t)$ and obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\partial_t u_m(t)\|^2_{L^2} - \int_M \langle \nabla_{\partial_t u_m} \partial_t u_m, u_m \rangle + 2\mu \|\partial_t Du_m\|^2_{L^2} + \int_{\partial M} \kappa \langle \partial_t G k_m(u_m, \cdot), \partial_t u_m \rangle \\
= \langle \partial_t f, \partial_t u_m \rangle_{L^2}
\end{aligned}$$

Concerning the last term on the left-hand side, note that due to the convexity of j_m

$$\begin{aligned}
&\kappa \langle \partial_t G k_m(u_m, \cdot), \partial_t u_m \rangle \\
&\quad = \kappa \langle G^2 j_m(u_m) \partial_t u_m, \partial_t u_m \rangle - l'(t) \left\langle \kappa G j_{m,p}(u_{m,0}) + 2\mu ((Du_{m,0}) n)_\tau, \partial_t u_m \right\rangle \\
&\quad \geq -l'(t) \left\langle \kappa G j_{m,p}(u_{m,0}) + 2\mu ((Du_{m,0}) n)_\tau, \partial_t u_m \right\rangle.
\end{aligned}$$

So, we get that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_t u_m(t)\|_{L^2}^2 + 2\mu\|\partial_t Du_m\|_{L^2}^2 &\leq \int_M \big(\,|\langle\nabla_{\partial_t u_m} u_m, \partial_t u_m\rangle| + |\langle\partial_t f, \partial_t u_m\rangle_{L^2}|\big)\\ &\quad + |l'(t)|\int_{\partial M} |\,\langle \kappa G j_{m,p}(u_{m,0}) + 2\mu((Du_{m,0})n)_\tau, \partial_t u_m\rangle\,|.\end{aligned} \tag{3.17}$$

Integration over time yields

$$\begin{aligned}&\frac{1}{2}\|\partial_t u_m(t)\|_{L^2}^2 + 2\mu\int_0^t \|\partial_t Du_m\|_{L^2}^2\\ &\quad\leq \int_0^t\int_M \big(\,|\langle\nabla_{\partial_t u_m} u_m, \partial_t u_m\rangle| + |\langle\partial_t f, \partial_t u_m\rangle_{L^2}|\big) + \|\partial_t u_m(0)\|_{L^2}^2\\ &\qquad + \int_0^t |l'|\int_{\partial M} \big|\,\langle \kappa G j_{m,p}(u_{m,0}) + 2\mu((Du_{m,0})n)_\tau, \partial_t u_m\rangle\,\big|.\end{aligned} \tag{3.18}$$

We look at the last two summands. Starting with the second one, we have by Young's, Korn's, and Poincaré's inequality that

$$\int_0^t |l'|\int_{\partial M} \big|\langle \kappa G_{j,m}(u_{m,0}) + 2\mu(Du_{m,0}n)_\tau, \partial_t u_m\rangle\big| \leq C(\varepsilon)\|l'\|_{L^2(0,\infty)}^2\delta(m) + \varepsilon\int_0^t \|\partial_t Du_m\|_H^2.$$

Note that the last term can be absorbed by the left-hand side of (3.18) if $\varepsilon > 0$ is small enough. In order to estimate $\|\partial_t u_m(0)\|_H^2$, we use (3.13) with $t = 0$ and $v = \partial_t u_m(0)$ to get

$$\begin{aligned}\|\partial_t u_m(0)\|_H^2 =& -2\mu\int_M \langle Du_{m,0}, D\partial_t u_m(0)\rangle + \int_M \big\langle\nabla_{u_{m,0}}\partial_t u_m(0), u_{m,0}\big\rangle\\ &- \int_{\partial M}\kappa\langle Gk_m(u_{m,0},0), \partial_t u_m(0)\rangle + \int_M \langle f(0), \partial_t u_m(0)\rangle.\end{aligned} \tag{3.19}$$

We have $Gk_m(u_{m,0},0) = -\dfrac{2\mu}{\kappa}(Du_{m,0}n)_\tau$, see (3.10), and by integration by parts

$$\int_M \langle Du_{m,0}, D\partial_t u_m(0)\rangle = -\int_M \langle\Delta u_{m,0}, \partial_t u_m(0)\rangle + \int_{\partial M}\langle(Du_{m,0}n)_\tau, \partial_t u_m(0)\rangle.$$

Plugging these two identities into (3.19) all the boundary terms are cancelled and it follows that

$$\begin{aligned}\|\partial_t u_m(0)\|_{L^2}^2 =& 2\mu\int_M \langle\Delta u_{m,0}, \partial_t u_m(0)\rangle + \int_M \big\langle\nabla_{u_{m,0}}\partial_t u_m(0), u_{m,0}\big\rangle + \int_M \langle f(0), \partial_t u_m(0)\rangle\\ \leq& c(\|u_{m,0}\|_{H^2} + \underbrace{\|u_{m,0}\|_{L^\infty}}_{\leq\|u_{m,0}\|_{H^2}}\|u_{m,0}\|_{H^1} + \|f(0)\|_{L^2})\|\partial_t u_m(0)\|_{L^2}.\end{aligned}$$

Consequently,

$$\|\partial_t u_m(0)\|_{L^2} \leq c\left(\|u_0\|_{H^2} + \|u_0\|_{H^2}^2 + \|f(0)\|_{L^2}\right). \tag{3.20}$$

In total, (3.18) becomes

$$\frac{1}{2}\|\partial_t u_m(t)\|_{L^2}^2 + \mu\int_0^t \|\partial_t Du_m\|_{L^2}^2 \le \int_0^t\int_M (|\langle\nabla_{\partial_t u_m} u_m, \partial_t u_m\rangle| + |\langle\partial_t f, \partial_t u_m\rangle_{L^2}|) + C(\|u_0\|_{H^2} + \|u_0\|_{H^2}^2 + \|f(0)\|_{L^2} + \delta(m)). \tag{3.21}$$

Proof of (1)*:* Since there are no boundary terms in (3.21) we can, from this point on, proceed for the proof of (1) by copying the proof from the Dirichlet case, see [33, Chapter III, Theorem 3.5]. However, for the sake of completion we still give a detailed proof. Since $d = 2$ we use

$$\|\partial_t u_m\|_{L^4}^2 \le C\|\partial_t u_m\|_{H^{\frac{1}{2}}}^2 \le C\|\partial_t u_m\|_H\|\partial_t u_m\|_V$$

and $\partial_t f \in L^2(0,T;V')$ to estimate (3.21) further:

$$\begin{aligned}
&\frac{1}{2}\|\partial_t u_m(t)\|_{L^2}^2 + \mu\int_0^t \|\partial_t Du_m\|_{L^2}^2 \\
\le& C\int_0^t \|\partial_t u_m\|_H\|\partial_t u_m\|_V\|u_m\|_V + C + C\|\partial_t f\|_{L^2(0,T;V')}^2 + \frac{\varepsilon}{2}\|\partial_t u_m\|_{L^2(0,T;V')}^2 \\
\le& C\int_0^t \|\partial_t u_m\|_H^2\|u_m\|_V^2 + \varepsilon\|\partial_t u_m\|_{L^2(0,T;V)}^2 + C + C\|\partial_t f\|_{L^2(0,T;V')}^2
\end{aligned} \tag{3.22}$$

where we used Young's inequality. By Korn's and Poincaré's inequality and by choosing $\varepsilon > 0$ small enough, we get

$$\frac{1}{2}\|\partial_t u_m(t)\|_{L^2}^2 + \frac{\mu}{2}\int_0^t \|\partial_t u_m\|_V^2 \le C\int_0^t \|\partial_t u_m\|_H^2\|u_m\|_V^2 + C + C\|\partial_t f\|_{L^2(0,T;V')}^2. \tag{3.23}$$

Gronwall's lemma implies

$$\sup_{0\le t<T}\|\partial_t u_m(t)\|_H^2 + \frac{\mu}{2}\int_0^t \|\partial_t u_m\|_V^2 \le (C + C\|\partial_t f\|_{L^2(0,T;V')}^2)\exp\left(\int_0^T \|u_m\|_V^2\right).$$

Since $(u_m)_{m\in\mathbb{N}}$ is bounded in $L^2(0,T;V)$ according to (3.16), we conclude that $(u_m)_{m\in\mathbb{N}}$ is bounded in $W_2^1(0,T;V)\cap W_\infty^1(0,T;H)$. Therefore, there is a subsequence of $(u_m)_{m\in\mathbb{N}}$ that we denote by $(u_m)_{m\in\mathbb{N}}$ again such that

$$u_m \rightharpoonup u \text{ in } W_2^1(0,T;V) \text{ as } m\to\infty.$$

Since

$$\left\{z \in W_2^1(0,T;V) \,|\, \sup_{0\le t\le T}\|\partial_t z\|_H \le R\right\}$$

is closed, convex and thus weakly closed by Mazur's lemma for all $R > 0$, we also deduce that $\partial_t u \in L^\infty(0,T;H)$.

Proof of (2)(*i*)*:* Now let $d = 3$. Since $(u_m)_{m\in\mathbb{N}}$ is bounded in $L^2(0,T;V)$ uniformly with respect to time, we have that

$$\begin{aligned}
\|Du_m(t)\|_{L^2} &\le C\|Du_{m,0}\|_{L^2} + C\left(\|\partial_t Du_m\|_{L^2(0,t;L^2)} + \|Du_m\|_{L^2(0,t;L^2)}\right) \\
&\le C + C\|\partial_t Du_m\|_{L^2(0,t;L^2)},
\end{aligned} \tag{3.24}$$

with $C > 0$ independent of time. Next, (3.21) implies with $\|u_m\|_{L^4} \leq c\|u_m\|_{L^2}^{1/4}\|u_m\|_{H^1}^{3/4}$ that

$$\begin{aligned}
a_m(t) :=& \sup_{0\leq\tau\leq t} \|\partial_t u_m(\tau)\|_{L^2}^2 + 2\mu \int_0^t \|\partial_t Du_m(\tau)\|_{L^2}^2 \\
\leq& C\int_0^t \|\partial_t u_m\|_{L^2}^{1/2}\|\partial_t u_m\|_V^{3/2}\|u_m\|_V + \frac{1}{\varepsilon}\left(\int_0^t \|\partial_t f\|_{L^2}\right)^2 + \varepsilon \sup_{0\leq\tau\leq t}\|\partial_t u_m(\tau)\|_{L^2}^2 + C \\
\leq& \frac{1}{\varepsilon}\int_0^t \|\partial_t u_m\|_{L^2}^2\|Du_m\|_{L^2}^4 + \frac{1}{\varepsilon}\int_0^t \|\partial_t u_m\|_{L^2}^2\|u_m\|_{L^2}^4 + \varepsilon\int_0^t \|\partial_t Du_m\|_{L^2}^2 \\
&+ \varepsilon\int_0^t \|\partial_t u_m\|_{L^2}^2 + C + \varepsilon a_m(t) \\
\overset{(3.24)}{\leq}& \frac{C}{\varepsilon}\int_0^t a_m(\cdot)(1 + a_m(\cdot)^2) + c\varepsilon a_m(t) + C \\
\leq& \frac{C}{\varepsilon} t(1 + a_m(t)^3) + c\varepsilon a_m(t) + C.
\end{aligned}$$

Choosing $\varepsilon > 0$ small enough, we conclude

$$a_m(t) \leq ct(1 + a_m(t)^3) + \tilde{C} \tag{3.25}$$

for all $t \in [0, T)$ and $\tilde{C}$ depends on ε, $\|u_0\|_{H^2}$ and $\|f\|_{W_1^1(0,T;L^2)}$. We choose $T' < T$ small enough such that

$$cT'(1 + 8\tilde{C}^3) < \tilde{C}.$$

Assume for the purpose of contradiction that $a_m(T') \geq 2\tilde{C}$ for an $m \in \mathbb{N}$ and consider $T_0 := \sup\{0 \leq t \leq T' \,|\, a_m(t) < 2\tilde{C}\}$. Since $a_m : [0, T'] \to \mathbb{R}$ is continuous, we have that $a_m(T_0) = 2\tilde{C}$ and $a_m(t)$ converges to $a_m(0) = \|\partial_t u_m(0)\|_{L^2}^2 \leq \tilde{C} < 2\tilde{C}$ as t goes to 0 and therefore $T_0 > 0$. By (3.25) we get the contradiction

$$2\tilde{C} = a_m(T_0) \leq cT_0(1 + a_m(T_0)^3) + \tilde{C} \leq cT'(1 + 8\tilde{C}^3) + \tilde{C} < 2\tilde{C}.$$

Therefore, we conclude that $a_m(t) \leq 2\tilde{C}$ for all $m \in \mathbb{N}$. This means that $(\partial_t u_m)_{m\in\mathbb{N}}$ is bounded in $L^\infty(0, T'; L^2(M)) \cap L^2(0, T'; V)$.

Proof of (2)(*ii*)*:* Since we have (3.17) and (3.20) the proof is very similar to the Dirichlet case again, see [33, Chapter III, Theorem 3.7]. However, boundary terms appear this time, but they can be made small. Actually, from (3.17) we deduce that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_t u_m(t)\|_H^2 + \mu\|\partial_t Du_m\|_{L^2}^2 \leq& C\|\partial_t u_m(t)\|_V^2\|u_m(t)\|_V + \|\partial_t u_m(t)\|_{L^2}\|\partial_t f(t)\|_{L^2} \\
&+ C|l'(t)|^2\delta(m).
\end{aligned}$$

Using the inequalities of Poincaré and Korn, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_t u_m(t)\|_{L^2}^2 + (\mu - C_1\|u_m(t)\|_V)\|\partial_t u_m(t)\|_V^2 \\
&\leq \|\partial_t u_m(t)\|_{L^2}\|\partial_t f(t)\|_{L^2} + C|l'(t)|^2\delta(m).
\end{aligned} \tag{3.26}$$

If $\varepsilon_* > 0$ is small enough, then $\mu - C_1\|u_{m,0}\|_V > \mu - 2C_1\|u_0\|_{H^2} > \mu - 2C_1\varepsilon_* > 0$. So, by continuity,

$$\exists T_m \in (0,T]\ \forall t \in [0,T_m) : \mu - C_1\|u_m(t)\|_V \geq 0$$

and $\mu - C_1\|u_m(T_m)\|_V = 0$ or $T = T_m$. By (3.26) we have

$$\frac{d}{dt}\|\partial_t u_m(t)\|_H^2 \leq 2\|\partial_t f(t)\|_H\|\partial_t u_m(t)\|_H + C|l'(t)|^2\delta(m)\ \forall t \in [0,T_m)$$

and thus

$$\partial_t(1+\|\partial_t u_m(t)\|_H^2) \leq \|\partial_t f(t)\|_H(1+\|\partial_t u_m(t)\|_H^2) + C|l'(t)|^2\delta(m)\ \forall t \in [0,T_m).$$

This in turn implies by Gronwall's lemma that for all $t \in [0,T_m)$

$$\begin{aligned} 1+\|\partial_t u_m(t)\|_H^2 \leq& (1+\|\partial_t u_m(0)\|_H^2)\exp\left(\int_0^t \|\partial_t f\|_H\ ds\right) \\ &+C\delta(m)\int_0^t |l'(s)|^2\exp\left(\int_s^t \|\partial_t f(\tau)\|_H\ d\tau\right)\ ds \\ \overset{(3.20)}{\leq}& (1+d^2+C\delta(m)\|l'\|_{L^2(0,\infty)}^2)\exp\left(\int_0^t \|\partial_t f\|_H\ ds\right). \end{aligned} \tag{3.27}$$

with $d := c(\|u_0\|_{H^2} + \|u_0\|_{H^2}^2 + \|f(0)\|_H)$. From (3.14) and Lemma 3.1.2 (4) with $\delta(m)$ as in (3.11) we see, by Young's inequality, that for $t \in [0,T_m)$

$$\begin{aligned} &\mu\|Du_m(t)\|_H^2 \\ \leq& \|f(t)\|_{V'}\|u_m(t)\|_V + l(t)\int_{\partial M}\kappa\left|\left\langle Gj_m(u_{m,0}) + \tfrac{2\mu}{\kappa}(D(u_{m,0})n)_\tau, u_m\right\rangle\right| - (u_m(t),\partial_t u_m(t))_H \\ \leq& \frac{1}{2\varepsilon}\|f(t)\|_{V'}^2 + \frac{\varepsilon}{2}\|u_m(t)\|_V^2 + \frac{1}{2\varepsilon}\delta(m) + \frac{\varepsilon}{2}\|u_m(t)\|_V^2 + \|u_m(t)\|_H\|\partial_t u_m(t)\|_H \\ \overset{(3.15)}{\leq}& \frac{1}{2\varepsilon}\|f(t)\|_{V'}^2 + \varepsilon\|u_m(t)\|_V^2 + \frac{1}{2\varepsilon}\delta(m) + C\left(\|u_0\|_H + \sqrt{\delta(m)} + \|f\|_{L^2(0,T;V')}\right)\|\partial_t u_m(t)\|_H \\ \overset{(3.27)}{\leq}& \frac{1}{2\varepsilon}\|f\|_{L^\infty(0,T;V')}^2 + \varepsilon\|u_m(t)\|_V^2 + \frac{1}{2\varepsilon}\delta(m) \\ &+ C\left(\|u_0\|_H + \sqrt{\delta(m)} + \|f\|_{L^2(0,T;V')}\right)(1+d^2+C\delta(m))\exp\left(\int_0^T \|\partial_t f\|_H\ ds\right). \end{aligned}$$

Since $\Gamma(u_0)$ has positive surface measure, we can use the inequalities of Poincaré and of Korn. Choosing $\varepsilon > 0$ small enough and considering only large $m \in \mathbb{N}$ such that $\delta(m)$ becomes small enough, we get for $\varepsilon_* > 0$ small enough:

$$\mu - C_1\|u_m(t)\|_V \geq c > 0\ \forall t \in [0,T_m).$$

Hence, we can choose $T = T_m$. And so by (3.26), we obtain that

$$\frac{1}{2}\frac{d}{dt}\|\partial_t u_m(t)\|_H^2 + c\|\partial_t u_m(t)\|_V^2 \leq \|\partial_t u_m(t)\|_H\|\partial_t f(t)\|_H + C|l'(t)|^2\delta(m).$$

This implies that $\partial_t u_m$ is bounded in $L^2(0,T;V) \cap L^\infty(0,T;H)$ uniformly with respect to m.

Part III: Passing to the limit. In total, we have seen that $(u_m)_{m\in\mathbb{N}}$ is bounded in $W_2^1(0,T;V) \cap W_\infty^1(0,T;H)$. Consequently, there is a subsequence of $(u_m)_{m\in\mathbb{N}}$, called $(u_m)_{m\in\mathbb{N}}$ again, such that

$$\begin{aligned} u_m &\overset{\star}{\rightharpoonup} u \quad \text{in } W_\infty^1(0,T;H), \\ u_m &\rightharpoonup u \quad \text{in } W_2^1(0,T;V) \end{aligned} \tag{3.28}$$

for a $u \in W_2^1(0,T;V) \cap W_\infty^1(0,T;H)$. Define for $\alpha \in (0,1)$

$$V^\alpha := \big(L^2(M)^d, H^1(M)^d\big)_{\alpha,2} \cap L^2_\sigma(M).$$

According to [30, Theorem 16.1], the embedding $V \hookrightarrow V^\alpha$ is compact. Hence, the convergence result $(3.28)_2$ implies by the compactness lemma of Aubin-Lions, see [32, Proposition 1.3, p. 106], that

$$u_m \to u \text{ in } L^2(0,T';V^\alpha)$$

and, if $\alpha > \frac{1}{2}$,

$$u_{m|\partial M} \to u_{|\partial M} \text{ in } L^2(0,T';L^2(\partial M)). \tag{3.29}$$

Now, integrate (3.13) over $(0,T)$ to get for all $v \in C_c^1([0;T);V_m)$ that

$$\begin{aligned} \int_0^T\!\!\int_M \langle \partial_t u_m, v\rangle - \int_0^T\!\!\int_M \big(\langle \nabla_{u_m} v, u_m\rangle - 2\mu\langle Du_m, Dv\rangle\big) \\ + \int_0^T\!\!\int_{\partial M} \kappa \,\langle Gk_m(u_m,\cdot), v\rangle = \int_0^T \langle f, v\rangle_{V',V}. \end{aligned}$$

Since k_m is convex, we have

$$\langle Gk_m(u_m,\cdot), v\rangle = \langle Gk_m(u_m,\cdot), v + u_m - u_m\rangle \le k_m(v+u_m,\cdot) - k_m(u_m,\cdot).$$

Summarizing the last two results, we get that

$$\begin{aligned} \int_0^T\!\!\int_M \langle \partial_t u_m, v\rangle - \int_0^T\!\!\int_M \big(\,\langle \nabla_{u_m} v, u_m\rangle - 2\mu\langle Du_m, Dv\rangle\big) \\ + \int_0^T\!\!\int_{\partial M} \kappa\big(k_m(v+u_m,\cdot) - k_m(u_m,\cdot)\big) \ge \int_0^T \langle f, v\rangle_{V',V}. \end{aligned}$$

Due to (3.28) the linear terms pose no problem when considering the limit $m \to \infty$. Because of Lemma 3.1.2 (3) and (3.11), we have

$$\int_0^T\!\!\int_{\partial M} k_m(v,\cdot) \to \int_0^T\!\!\int_{\partial M} |v_{|\partial M}| \quad \text{for } m \to \infty$$

for all $v \in C_c^1([0;T);V_{m_0})$ for each fixed $m_0 \in \mathbb{N}$. Combining this with (3.29), we can pass to the limit in the terms on the boundary as well. Finally, since

$$u_m \to u \text{ in } L^2(0,T';V^\alpha) \hookrightarrow L^2(0,T';L^4(M)) \text{ as } m \to \infty$$

for all $\alpha \in \left(\frac{3}{4}, 1\right)$ and for all $0 \leq T' \leq T$ finite, the convergence of the nonlinear term $\langle \nabla_{u_m} v, u_m \rangle$ to $\langle \nabla_u v, u \rangle$ follows from Hölder's inequality:

$$\begin{aligned}\int_0^T \int_M |\langle \nabla_{u_m} v, u_m \rangle - \langle \nabla_u v, u \rangle| \leq \int_0^T \int_M |\langle \nabla_{u_m - u} v, u_m \rangle| + |\langle \nabla_u v, u_m - u \rangle| \\ \leq \|u_m - u\|_{L^2(0,T';L^4(M))} \|\nabla v\|_{L^\infty(0,T';L^2(M))} (\|u_m\|_{L^2(0,T';L^4(M))} + \|u\|_{L^2(0,T';L^4(M))}) \to 0\end{aligned}$$

as $m \to \infty$, where we used that v has compact support with respect to time. By these arguments we arrive at the variational inequality

$$\begin{aligned}\int_0^T \int_M \langle \partial_t u, v \rangle - \int_0^T \int_M \big(\langle \nabla_u v, u \rangle - 2\mu \langle Du, Dv \rangle \big) \\ + \int_0^T \int_{\partial M} \kappa \big(|v + u| - |u| \big) \geq \int_0^T \langle f, v \rangle_{V',V}\end{aligned}$$

for all $v \in C_c^1([0;T); V_{m_0})$. Now, let $v \in C_c^1([0;T); V)$ and P_{m_0} denote the orthogonal projection from V onto V_{m_0}. Replacing v in the above inequality by $P_{m_0} v$ and using that

$$P_{m_0} v \to v \text{ in } C^1([0,T'); V) \quad \text{as } m_0 \to \infty,$$

we obtain the inequality for all $v \in C_c^1([0;T); V)$. Due to

$$\int_0^T \int_M \langle \partial_t u, v \rangle = - \int_0^T \int_M \langle u, \partial_t v \rangle - \int_M \langle u_0, v(0) \rangle$$

for all $v \in C_c^1([0,T); V)$, this shows the existence of a weak solution u with $u \in W_2^1(0,T;V) \cap W_\infty^1(0,T;H)$.

Part IV: Energy equality. Concerning the energy equality (3.8) let $u \in W_2^1(0,T;H) \cap L^\infty(0,T;V)$ be a strong solution. Since $\partial_t u \in L^2(0,T;H)$, we can write the inequality solved by u in the form

$$\begin{aligned}\int_0^T \int_M \langle \partial_t u, v \rangle - \int_0^T \int_M \langle \nabla_u v, u \rangle + 2\mu \int_0^t \int_M \langle Du, Dv \rangle \\ + \int_0^T \int_{\partial M} \kappa (|u + v| - |u|) \geq \int_0^T \int_M \langle f, v \rangle \ \ \forall v \in C_c^1([0;T); V).\end{aligned} \tag{3.30}$$

Via density this inequality holds also for $v \in L^2(0,T;V)$. Now let $t \in (0,T)$, and choose

$$v(s) = \begin{cases} u(s), & \text{if } s \in [0,t] \\ 0, & \text{if } s \in (t,T) \end{cases} \in L^2(0,T;V)$$

and use

$$\int_0^T \int_{\partial M} \kappa \left(|u + v| - |u| \right) = \int_0^t \int_{\partial M} \kappa |u|$$

to get from (3.30) that

$$\int_0^t \int_M \langle \partial_t u, u \rangle + 2\mu \int_0^t \int_M |Du|^2 + \int_0^t \int_{\partial M} \kappa |u| \geq \int_0^t \int_M \langle f, u \rangle.$$

Next, test with $-v$, use

$$\int_0^T \int_{\partial M} \kappa \left(|u - v| - |u| \right) = - \int_0^t \int_{\partial M} \kappa |u|$$

and multiply (3.30) by -1 to see that

$$\int_0^t \int_M \langle \partial_t u, u \rangle + 2\mu \int_0^t \int_M |Du|^2 + \int_0^t \int_{\partial M} \kappa |u| \leq \int_0^t \int_M \langle f, u \rangle.$$

These two inequalities imply that

$$\int_0^t \int_M \langle \partial_t u, u \rangle + 2\mu \int_0^t \int_M |Du|^2 + \int_0^t \int_{\partial M} \kappa |u| = \int_0^t \int_M \langle f, u \rangle.$$

Since $\int_0^t \int_M \langle \partial_t u, u \rangle = \frac{1}{2} \|u(t)\|_{L^2}^2 - \frac{1}{2} \|u_0\|_{L^2}^2$, the energy equality (3.8) is proved. The uniqueness of strong solutions follows from Theorem 3.1.5. □

Uniqueness follows from the next proposition about weak-strong uniqueness.

Theorem 3.1.5. *(**Weak-strong uniqueness**) Let v_1 and v_2 be weak solutions of (3.3), (3.4) with data $f \in L^2(0,T;V')$ and $u_0 \in H$ such that $v_1 \in W_2^1(0,T;H) \cap L^\infty(0,T;V)$ with $T \in (0,\infty)$ and v_2 satisfies the energy inequality (3.7) and $\partial_t v_2 \in L^{4/d}(0,T;V')$. Then $v_1 = v_2$.*

Proof. If $d = 2$, then even weak solutions satisfying the energy inequality are unique, see [6, Theorem 2.2]. So we focus on the case $d = 3$, but the case $d = 2$ is also included in the following.

Let $w := v_1 - v_2$. Then we have because of the energy estimate

$$\begin{aligned}
&\frac{1}{2} \|w(t)\|_H^2 + 2\mu \int_0^t \|Dw(\tau)\|_H^2 \\
&= \frac{1}{2} \|v_1(t)\|_H^2 + 2\mu \int_0^t \|Dv_1(\tau)\|_H^2 - (v_1(t), v_2(t))_H - 4\mu \int_0^t (Dv_1(\tau), Dv_2(\tau))_H \\
&\quad + \frac{1}{2} \|v_2(t)\|_H^2 + 2\mu \int_0^t \|Dv_2(\tau)\|_H^2 \\
&\leq \|u_0\|_H^2 + \int_0^t \langle f(\tau), v_1(\tau) \rangle_{V',V} + \int_0^t \langle f(\tau), v_2(\tau) \rangle_{V',V} - \int_0^t \int_{\partial M} \kappa (|v_1| + |v_2|) \\
&\quad - (v_1(t), v_2(t))_H - 4\mu \int_0^t (Dv_1(\tau), Dv_2(\tau))_H.
\end{aligned}$$

We want to show that

$$\frac{1}{2} \|w(t)\|_H^2 + 2\mu \int_0^t \|Dw(\tau)\|_H^2 \leq \int_0^t \int_M \left| \left\langle \nabla_{w(\tau)} v_1(\tau), w(\tau) \right\rangle \right|. \tag{3.31}$$

So it suffices to prove that

$$\begin{aligned}
&(v_1(t), v_2(t))_H + 4\mu \int_0^t (Dv_1(\tau), Dv_2(\tau))_H \\
\geq &\|u_0\|_H^2 + \int_0^t \langle f(\tau), v_1(\tau) \rangle_{V',V} + \int_0^t \langle f(\tau), v_2(\tau) \rangle_{V',V} - \int_0^t \int_M \left\langle \nabla_{w(\tau)} v_1(\tau), w(\tau) \right\rangle \\
&- \int_0^t \int_{\partial M} \kappa \left(|v_1| + |v_2| \right).
\end{aligned}$$

Since $v_1 \in L^\infty(0,T;V) \cap W^1_2(0,T;H) \hookrightarrow L^4(0,T;V) \cap W^1_2(0,T;H)$, there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset C^\infty([0,T];V)$ with $u_k \to v_1$ in $L^4(0,T;V) \cap W^1_2(0,T;H)$ for $k \to \infty$. Then $g_k(t) := \langle v_2(t), u_k(t)\rangle_{V',V} \in W^1_{4/3}(0,T)$, as $v_2 \in W^1_{4/3}(0,T;V')$ and

$$g_k'(t) = \langle v_2(t), \partial_t u_k(t)\rangle_{V',V} + \langle \partial_t v_2(t), u_k(t)\rangle_{V',V}.$$

Since v_2 is a weak solution, we get from (3.6) with u_k as test function that

$$\begin{aligned}
&(v_2(t), u_k(t))_H = \langle v_2(t), u_k(t)\rangle_{V',V} \\
= \;& \langle u_0, u_k(0)\rangle_{V',V} + \int_0^t \langle \partial_t u_k(\tau), v_2(\tau)\rangle_{V',V} + \int_0^t \langle \partial_t v_2(\tau), u_k(\tau)\rangle_{V',V} \\
\overset{v_2 \text{ sol.}}{\geq} \;& \langle u_0, u_k(0)\rangle_{V',V} + \int_0^t \langle \partial_t u_k(\tau), v_2(\tau)\rangle_{V',V} + \int_0^t \int_M \langle \nabla_{v_2} u_k, v_2\rangle \\
& - 2\mu \int_0^t \int_M \langle Dv_2, Du_k\rangle + \int_0^t \int_{\partial M} \kappa\left(|v_2| - |u_k + v_2|\right) + \int_0^t \langle f(\tau), u_k(\tau)\rangle_{V',V}.
\end{aligned}$$

Passing to the limit, we get that

$$\begin{aligned}
(v_2(t),v_1(t))_H \geq \;& \|u_0\|_H^2 + \int_0^t \langle \partial_t v_1(\tau), v_2(\tau)\rangle_{V',V} + \int_0^t \int_M \langle \nabla_{v_2} v_1, v_2\rangle \\
& - 2\mu \int_0^t \int_M \langle Dv_2, Dv_1\rangle + \int_0^t \int_{\partial M} \kappa\left(|v_2| - |v_1 + v_2|\right) + \int_0^t \langle f(\tau), v_1(\tau)\rangle_{V',V}.
\end{aligned}$$

On the other hand, since $\partial_t v_1 \in L^2(0,T;V')$, we conclude from (3.6) with v_2 as test function that

$$\begin{aligned}
\int_0^t \langle \partial_t v_1(\tau), v_2(\tau)\rangle_{V',V} \geq \;& \int_0^t \langle f(\tau), v_2(\tau)\rangle_{V',V} + \int_0^t \int_M \langle \nabla_{v_1} v_2, v_1\rangle \\
& - 2\mu \int_0^t \int_M \langle Dv_1, Dv_2\rangle + \int_0^t \int_{\partial M} \kappa\left(|v_1| - |v_1 + v_2|\right).
\end{aligned}$$

Together with the previous estimate, this yields

$$\begin{aligned}
(v_2(t), v_1(t))_H \geq \;& \|u_0\|_H^2 + \int_0^t \langle f(\tau), v_1(\tau)\rangle_{V',V} + \int_0^t \langle f(\tau), v_2(\tau)\rangle_{V',V} \\
& + \int_0^t \int_M \langle \nabla_{v_2} v_1, v_2\rangle + \int_0^t \int_M \langle \nabla_{v_1} v_2, v_1\rangle \\
& + \int_0^t \int_{\partial M} \kappa(\underbrace{|v_1| + |v_2| - 2|v_1 + v_2|}_{\geq -|v_1| - |v_2|}) - 4\mu \int_0^t \int_M \langle Dv_1, Dv_2\rangle.
\end{aligned}$$

Finally, since

$$\int_0^T \int_M \left\langle \nabla_{w(\tau)} v_1(\tau), w(\tau)\right\rangle = \int_0^T \int_M \langle \nabla_{v_1} v_2, v_1\rangle + \int_0^T \int_M \langle \nabla_{v_2} v_1, v_2\rangle,$$

we have proven (3.31).

By the Gagliardo-Nirenberg inequality $\|w\|_{L^3} \leq c\|w\|_{L^2}^{1/2}\|w\|_{H^1}^{1/2}$ we see

$$\begin{aligned}
&\frac{1}{2}\|w(t)\|_H^2 + 2\mu\int_0^t \|Dw\|_H^2 \\
\leq& c\int_0^t \|v_1\|_{L^6}\|w\|_{L^3}\|\nabla w\|_{L^2} \leq c\int_0^t \|v_1\|_{L^6}\|w\|_{L^2}^{1/2}\|w\|_{H^1}^{3/2} \\
\leq& \frac{c}{\varepsilon}\int_0^t \|v_1\|_{L^6}^4\|w\|_{L^2}^2 + c\varepsilon\int_0^t \|\nabla w\|_{L^2}^2 + c\varepsilon\int_0^t \|w\|_{L^2}^2 \\
\leq& \frac{c}{\varepsilon}\int_0^t \left(\|v_1\|_{L^6}^4 \sup_{s\in(0,\tau)}\|w(s)\|_{L^2}^2\right) d\tau + c\varepsilon\int_0^t \|\nabla w\|_{L^2}^2 + c\varepsilon T \sup_{s\in(0,t)}\|w(s)\|_{L^2}^2.
\end{aligned}$$

Since $v_1 \in L^\infty(0,T;V) \hookrightarrow L^4(0,T;L^6(M)^3)$ and $w \in L^\infty(0,T;H)$, the right-hand side is finite. Using Korn's inequality and choosing $\varepsilon > 0$ small enough, we get by absorption that

$$\sup_{s\in(0,t)}\|w(s)\|_H^2 \leq c\int_0^t \|v_1(\tau)\|_{L^6}^4 \sup_{s\in(0,\tau)}\|w(s)\|_{L^2}^2\, d\tau.$$

By Gronwall's lemma, we conclude $w = 0$ for $w(0) = 0$. □

Remark 3.1.6. Let us make a concluding remark concerning the regularity of the equation. Consider the simple case of $M = \mathbb{R}^2_+$, homogeneous initial data and the right-hand side is given by $f(t,x,y) = (y,0)$ for all $(t,x,y) \in (0,\infty)\times\mathbb{R}^2_+$. It is easy to check that in the Dirichlet case a smooth solution is given by the function $u(t,x,y) = (ty,0)$ for all $(t,x,y) \in (0,\infty)\times\mathbb{R}^2_+$.

In our case of Coulomb boundary condition the condition $2\mu|(D(u)n)_\tau| \leq \kappa$ translates to $t \leq \frac{\kappa}{\mu}$. Hence, on the time interval $(0,\frac{\kappa}{\mu})$ and only on this interval the solution to the Dirichlet case is also a solution to the Coulomb case. Hence, the question arises what happens at $t = \frac{\kappa}{\mu}$. Assume that there is an extension v to a larger time interval $(0,T_1)$ that is smooth with respect to space. Then, due to $2\mu|(D(v)n)_\tau| \leq \kappa$ on $(0,T_1)$ and $2\mu(D(v)n)_\tau = \mu t$ on $(0,\frac{\kappa}{\mu})$, $(D(v)n)_\tau$ cannot be classically differentiable at $t = \frac{\kappa}{\mu}$ with respect to time. Thus there can not be an extension that is both smooth with respect to both space and time even though the right-hand side is.

3.2. Weak Solutions

The two propositions of this section aim to demonstrate that the Coulomb boundary condition lies in between the Dirichlet and the perfect slip case.

Proposition 3.2.1. ***(Connection with the Dirichlet boundary condition)*** *Let $0 < T \leq \infty$, $u_0 \in L^2_\sigma(M)$, $f \in L^1(0,T;V') \cap L^2(0,T;V')$ and $(\kappa_n)_{n\in\mathbb{N}} \subset H^{1/2}(\partial M)$ be a sequence converging to infinity in the sense*

$$\operatorname*{ess\,sup}_{x\in\partial M} \frac{1}{\kappa_n(x)} \to 0 \quad \text{as } n \to \infty.$$

Consider the sequence $(u_{\kappa_n})_{n\in\mathbb{N}}$ of weak solutions of (3.3), (3.4) satisfying the energy inequality (3.7) with κ_n instead of κ and $u_{\kappa_n}(0) = u_0$. Then there exists a weak solution $u \in L^\infty(0,T;L^2_\sigma(M)) \cap L^2(0,T;V)$ of the Navier–Stokes system with Dirichlet boundary condition,

$$\begin{aligned}
\partial_t u + \nabla_u u - \operatorname{div} D(u) + \nabla_M p = f \quad &\text{in } (0,T)\times\mathring{M} \\
\operatorname{div}_M u = 0 \quad &\text{in } (0,T)\times\mathring{M} \\
u_{|\partial M} = 0 \quad &\text{on } (0,T)\times\partial M \\
u(0) = u_0 \ &\text{in } M,
\end{aligned}$$

such that $u_{\kappa_n} \overset{*}{\rightharpoonup} u$ *(up to a subsequence) in* $L^2(0,T;V) \cap L^\infty(0,T;H)$ *for* $n \to \infty$*. Moreover,* u *satisfies the global energy estimate*

$$\|u\|^2_{L^\infty(0,T;L^2)} + \mu\|Du\|^2_{L^2(0,T;L^2)} \leq 2\|u_0\|^2_{L^2} + C(\|f\|^2_{L^1(0,T;V')} + \|f\|^2_{L^2(0,T;V')}) \tag{3.32}$$

with a constant $C = C(\mu) > 0$*.*

Proof. For $u_\kappa = u_{\kappa_n}$ we get with $t \to T$ from the energy estimate (3.7)

$$\begin{aligned} &\frac{1}{2}\|u_\kappa\|^2_{L^\infty(0,T;L^2)} + 2\mu\|Du_\kappa\|^2_{L^2(0,T;L^2)} + \int_0^T \int_{\partial M} \kappa|u_\kappa(s)|_g \\ &\leq \frac{1}{2}\|u_0\|^2_{L^2} + \int_0^T \|f\|_{V'}\|u_\kappa\|_V. \end{aligned} \tag{3.33}$$

By Korn's inequality, there exists a constant $c = c(M) > 0$

$$\begin{aligned} \int_0^T \|f\|_{V'}\|u_\kappa\|_V &\leq c\int_0^T \|f\|_{V'}\big(\|u_\kappa\|_{L^2} + \|Du_\kappa\|_{L^2}\big) \\ &\leq \frac{1}{4}\|u_\kappa\|^2_{L^\infty(0,T;L^2)} + \mu\|Du_\kappa\|^2_{L^2(0,T;L^2)} + c\|f\|^2_{L^1(0,T;V')} + \frac{c}{\mu}\|f\|^2_{L^2(0,T;V')}. \end{aligned}$$

The terms involving u_κ and Du_κ can be absorbed by terms on the left-hand side of the energy estimate (3.33), and we get the global energy estimate

$$\begin{aligned} &\|u_\kappa\|^2_{L^\infty(0,T;L^2)} + \mu\|Du_\kappa\|^2_{L^2(0,T;L^2)} + \int_0^T \int_{\partial M} \kappa|u_\kappa(s)|_g \\ &\qquad \leq 2\|u_0\|^2_{L^2} + C(\|f\|^2_{L^1(0,T;V')} + \|f\|^2_{L^2(0,T;V')}). \end{aligned} \tag{3.34}$$

Note that $C = C(\mu)$ does not depend on κ. Hence, $(u_\kappa)_{\kappa\in\mathbb{N}} \subset L^\infty(0,T;H) \cap L^2(0,T;V)$ is bounded and, thus, (up to a subsequence) there exists a $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ such that

$$u_\kappa \overset{\star}{\rightharpoonup} u \quad \text{for } \kappa \to \infty \quad \text{in } L^\infty(0,T;L^2_\sigma(M)) \cap L^2(0,T;V).$$

The estimate (3.34) also implies that

$$C \geq \int_0^T \int_{\partial M} \kappa|u_\kappa(s)|_g \geq \operatorname*{ess\,inf}_{x\in\partial M} \kappa \cdot \int_0^T \int_{\partial M} |u_\kappa(s)|_g$$

for a $C > 0$ independent of κ. Since $\operatorname{ess\,inf}_{x\in\partial M} \kappa_n(x) \to \infty$ as $n \to \infty$, it must hold that $u_\kappa \to 0$ in $L^1(0,T;L^1(\partial M))$. Due to $(u_\kappa)_{|\partial M} \rightharpoonup u_{|\partial M}$ in $L^2(0,T;L^2(\partial M)) \hookrightarrow L^1(0,T;L^1(\partial M))$, it follows that $u_{|\partial M} = 0$.

Now, it suffices to prove that u is a weak solution of the Navier–Stokes system. Since u_κ is a weak solution (satisfying (3.6)), we have

$$\begin{aligned} &-\int_M \langle u_0(x), v(0,x)\rangle - \int_0^T \int_M \big(\langle u_\kappa, \partial_t v\rangle + \langle \nabla_{u_\kappa} v, u_\kappa\rangle\big) \\ &+ 2\mu\int_0^T \int_M \langle Du_\kappa, Dv\rangle + \int_0^T \int_{\partial M} \kappa(|u_\kappa + v| - |u_\kappa|) \geq \int_0^T \langle f, v\rangle_{V',V} \end{aligned}$$

for all $v \in C_c^1([0,T);V)$. Multiplying this inequality with -1 and replacing v by $-v$, we receive that

$$
\begin{aligned}
&-\int_M \langle u_0(x), v(0,x)\rangle - \int_0^T \int_M \big(\langle u_\kappa, \partial_t v\rangle + \langle \nabla_{u_\kappa} v, u_\kappa\rangle\big) \\
&+ 2\mu \int_0^T \int_M \langle Du_\kappa, Dv\rangle - \int_0^T \int_{\partial M} \kappa(|u_\kappa - v| - |u_\kappa|) \leq \int_0^T \langle f, v\rangle_{V',V}.
\end{aligned}
$$

for all $v \in C_c^1([0,T);V)$. Note that in the boundary integral above integration with respect to time is on a bounded interval in $[0,T)$, since the test function v has compact support in time. If we take $v \in C_c^1([0,T);V_0)$ with $V_0 := \{v \in V : v_{|\partial M} = 0\}$ as set of test functions in both inequalities, then the boundary terms vanish and, thus,

$$
\begin{aligned}
&-\int_M \langle u_0(x), v(0,x)\rangle - \int_0^T \int_M \big(\langle u_\kappa, \partial_t v\rangle + \langle \nabla_{u_\kappa} v, u_\kappa\rangle\big) + 2\mu \int_0^T \int_M \langle Du_\kappa, Dv\rangle \\
&\quad = \int_0^T \langle f, v\rangle_{V',V} \quad \forall v \in C_c^1([0,T);V_0).
\end{aligned}
$$

Passing to the limit, we obtain

$$
\begin{aligned}
&-\int_M \langle u_0(x), v(0,x)\rangle - \int_0^T \int_M \big(\langle u, \partial v_t\rangle + \langle \nabla_u v, u\ \rangle\big) + 2\mu \int_0^T \int_M \langle Du, Dv\rangle \\
&\quad = \int_0^T \langle f, v\rangle_{V',V} \quad \forall v \in C_c^1([0,T);V_0).
\end{aligned} \tag{3.35}
$$

We have already proven that $u|_{\partial M} = 0$. So, u is a weak solution of the Navier–Stokes equations with homogeneous Dirichlet boundary conditions. Moreover, by the weak convergence properties of u_k, (3.7) yields the energy estimate for u with the term involving κ being omitted, and also (3.32). □

In accordance with the previous proposition numerical examples in [6] show that, with increasing κ, the tangential stress on the boundary is increasing and the tangential velocity is decreasing. We consider the case $\kappa \to 0$ in the next proposition.

Proposition 3.2.2. ***(Connection with the perfect slip boundary condition)*** *Let $0 < T \leq \infty$, $u_0 \in L^2_\sigma(M)$, and $f \in L^1(0,T;V') \cap L^2(0,T;V')$. Consider a sequence of arbitrarily chosen weak solutions $(u_{\kappa_n})_{n\in\mathbb{N}}$ of (3.3), (3.4) satisfying the energy inequality (3.7) such that (with $\kappa_n = \kappa$)*

$$
\kappa \to 0 \quad \text{in } L^q(\partial M), 1 < q < \infty, \text{ for } d = 2, \qquad \kappa \to 0 \quad \text{in } L^{\frac{4}{3}}(\partial M) \text{ for } d = 3.
$$

Then there exists a vector field $u \in L^\infty(0,T;L^2_\sigma(M)) \cap L^2(0,T;V)$ such that $u_\kappa \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;L^2_\sigma(M)) \cap L^2(0,T;V)$ for $\kappa \to 0$ (up to a subsequence) where u is a weak solution of the Navier–Stokes system satisfying a pure Navier slip boundary condition:

$$
\begin{aligned}
\partial_t u + \nabla_u u - \operatorname{div} D(u) + \nabla_M p &= f && \text{in } (0,T)\times \mathring{M} \\
\operatorname{div}_M u &= 0 && \text{in } (0,T)\times \mathring{M} \\
u_{n_{|\partial M}} &= 0 && \text{on } (0,T)\times \partial M \\
((Du)n)_\tau &= 0 && \text{on } (0,T)\times \partial M \\
u(0) &= u_0 && \text{in } M.
\end{aligned}
$$

Proof. By analogy to the proof of Proposition 3.2.1, we have that (u_κ) is bounded in $L^\infty(0,T;L^2_\sigma(M)) \cap L^2(0,T;V)$ and satisfies the global energy estimate (3.34). Hence, there exists $u \in L^\infty(0,T;L^2_\sigma(M)) \cap L^2(0,T;V)$ such that

$$u_\kappa \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^2_\sigma(M)) \cap L^2(0,T;V) \text{ for } \kappa \to 0.$$

In particular, $u_{\kappa|\partial M}$ is bounded in $L^2(0,T;H^{\frac{1}{2}}(\partial M))$. Thus, using that $H^{1/2}(\partial M) \hookrightarrow L^4(\partial M)$ if $d = 3$ and $H^{1/2}(\partial M) \hookrightarrow L^{q'}(\partial M)$ if $d = 2$ (where q' is the conjugate exponent to q), we see that

$$\|\kappa u_\kappa|_{\partial M}\|_{L^1(0,T';L^1(\partial M))} \to 0 \ \text{ for } \kappa \to 0$$

and

$$\|\kappa(u_\kappa|_{\partial M} + v)\|_{L^1(0,T';L^1(\partial M))} \to 0 \ \text{ for } \kappa \to 0 \ \ \forall v \in C^1_c([0,T);V).$$

Considering the equation solved by u_κ and passing to the limit, we obtain that u satisfies the weak variational formulation of the Navier–Stokes system locally in M, cf. (3.35). Thus, u is a weak solution of the Navier–Stokes system satisfying the perfect slip boundary condition. □

CHAPTER 4

The Hybrid Model

In this chapter we consider a combination of the problems investigated in the two previous chapters. Let ω denote a bounded Lipschitz domain in $\mathbb{R}^2$. We consider a domain filled with a viscous fluid:

$$\Omega_{h_0} = \{(x, y, z) \in \mathbb{R}^3 |\ \ (x, y) \in \omega, 0 < z < 1 + h_0(x, y)\}$$

where h_0 is a given vertical displacement of the elastic, upper part of the boundary from the reference state $\omega \times \{1\}$. We denote the rigid part of the boundary of Ω_{h_0} by $\Gamma_0 := (\omega \times \{0\}) \cup (\partial\omega \times [0, 1])$, which consists of the lower part of the boundary, as well as of the lateral part. Only the upper part of $\partial\Omega_{h_0}$ is subject to change in time which is described by a function h that is to be determined. So while Ω_{h_0} denotes the domain occupied by the fluid at the initial time $t = 0$,

$$\Omega_{h(t)} = \{(x, y, z) \in \mathbb{R}^3 |\ \ (x, y) \in \omega, 0 < z < 1 + h(t, x, y)\}$$

denotes the domain occupied by the fluid at time t. The equations that describe the interplay between the fluid flow and the elastic boundary, expressed by (u, p) and h, respectively, are given by

$$\partial_t u - \operatorname{div}(2\mu D(u) - pI) = f - (u \cdot \nabla)u \qquad \text{in } (0, T) \times \Omega_h, \tag{4.1}$$

$$\operatorname{div} u = 0 \qquad \text{in } (0, T) \times \Omega_h, \tag{4.2}$$

$$u(t, \cdot) = 0 \qquad \text{on } \Gamma_0, \tag{4.3}$$

$$u(t, x, y, 1 + h(t, x, y)) = (0, 0, \partial_t h(t, x, y))^T \qquad \text{on } (0, T) \times \omega, \tag{4.4}$$

$$h = \frac{\partial h}{\partial n_\omega} = 0 \qquad \text{on } \partial\omega, \tag{4.5}$$

$$u_{|t=0} = u_0 \qquad \text{in } \Omega_{h_0}, \tag{4.6}$$

$$(h, \partial_t h)_{|t=0} = (h_0, h_1) \qquad \text{in } \omega. \tag{4.7}$$

with $Du := \frac{1}{2}(\nabla u + \nabla u^T)$ and n_ω denoting the outer unit normal on $\partial\omega$. On $\omega \times (0, T)$ we have for given $\lambda, \kappa > 0$ that

$$\partial_t h = \begin{cases} 0, & \text{if } |\partial_t^2 h + \Delta^2 h + \lambda\Delta^2\partial_t h - (T_f)_3| < \kappa \\ -\alpha(\partial_t^2 h + \Delta^2 h + \lambda\Delta^2\partial_t h - (T_f)_3), & \text{if } |\partial_t^2 h + \Delta^2 h + \lambda\Delta^2\partial_t h - (T_f)_3| = \kappa \end{cases} \tag{4.8}$$

for $\alpha = \alpha(t,x,y) \geq 0$ which is equivalent to

$$\left(\partial_t^2 h + \Delta^2 h + \lambda\Delta^2\partial_t h - (T_f)_3\right)\eta \geq \kappa|\partial_t h| - \kappa|\partial_t h + \eta| \quad \forall \eta \in \mathbb{R}. \tag{4.9}$$

For a proof of the equivalence we refer to the proof of Lemma 3.0.1. Here $\mu > 0$ denotes the viscosity of the fluid. (4.1) − (4.2) are the incompressible Navier-Stokes equations with a given external force f. (4.3) describes that the fluid does not move on the rigid part Γ_0. Since the fluid adheres to the plate we postulate (4.4). Here T_f is the surface force exerted by the fluid on the structure given by

$$\int_\omega T_f(t,x,y)\cdot v(t,x,y,1+h(t,x,y))\,dx\,dy = \int_{\partial\Omega_{h(t)}\setminus\Gamma_0} (-2\mu D(u)\cdot n_t + pn_t)\cdot v\,d\sigma \tag{4.10}$$

for all $v \in C_c^\infty(\overline{\Omega}_{h(t)})^3$, where $n_t = \dfrac{(-\nabla' h, 1)^T}{\sqrt{1+|\nabla' h|^2}}$ is the outer unit normal at $\partial\Omega_{h(t)}\setminus\Gamma_0$.

The boundary condition (4.8) describes that if the force $\partial_t^2 h + \Delta^2 h + \lambda\Delta^2\partial_t h - (T_f)_3$, which for $\lambda = 0$ was equal to the exterior force g_2 in Chapter 2, is smaller than the threshold κ, then h and therefore the domain does not change in time. However, if the force becomes large enough, the upper boundary may start to change in time and it does so in the opposite direction of the force. So κ can be viewed as a constant measuring the elasticity of the plate which is the upper boundary. If the material that the plate consists of is very elastic, κ is very small. If κ is very large, the plate is harder to move. Note that in contrast to Chapter 2 we have introduced a damping term $\lambda\Delta^2\partial_t h$ that will help us to get $h \in C([0,T]\times\overline{\omega})$. For simplicity we assume in the following that $\lambda = 1$.

In this chapter we show the existence of weak solutions. First, we will formally derive a priori estimates and identify suitable solution spaces in Section 4.1. In Section 4.2, we rigorously define the solution spaces and gather results about trace operators associated with the domain $\Omega_{h(t)}$. In Section 4.3, we define weak solutions before we show their existence in Section 4.4.

We want to emphasize that we follow the work of Chambolle, Desjardins, Esteban and Grandmont, see [10], except that we have an additional nonlinear boundary term. This term does not require any significant changes from the mentioned paper but in order for this chapter to be self-contained and comprehensible we still give a detailed proof of the existence of a weak solutions that is in some points more detailed than in [10].

4.1. Energy Estimate

Formally multiplying (4.1) with u and integrating over Ω_h leads to

$$\begin{aligned}&\int_{\Omega_{h(t)}} \partial_t u\cdot u + \int_{\Omega_h(t)} (u\cdot\nabla)u\cdot u + 2\mu\int_{\Omega_{h(t)}} |D(u)|^2 + \int_{\partial\Omega_{h(t)}\setminus\Gamma_0} (-2\mu D(u)\cdot n_t + pn_t)\cdot u\\ =&\int_{\Omega_{h(t)}} f\cdot u.\end{aligned} \tag{4.11}$$

Due to Reynolds transport theorem we have that

$$\frac{d}{dt}\int_{\Omega_{h(t)}} \frac{1}{2}|u|^2 = \int_{\Omega_{h(t)}} \frac{d}{dt}\left(\frac{1}{2}|u|^2\right) + \int_{\partial\Omega_{h(t)}\setminus\Gamma_0} \frac{1}{2}|u|^2 v\cdot n_t$$

with v denoting the velocity of the area element which equals u. Hence, using $\operatorname{div} u = 0$ and the divergence theorem, we deduce that

$$\int_{\Omega_h(t)} (u \cdot \nabla) u \cdot u = \frac{1}{2} \int_{\partial\Omega_{h(t)}} |u|^2 u \cdot n_t = \frac{d}{dt} \int_{\Omega_{h(t)}} \frac{1}{2}|u|^2 - \int_{\Omega_{h(t)}} \frac{d}{dt} \left(\frac{1}{2}|u|^2 \right).$$

Plugging this into (4.11), we get that

$$\frac{1}{2}\frac{d}{dt} \int_{\Omega_{h(t)}} |u|^2 + 2\mu \int_{\Omega_{h(t)}} |D(u)|^2 + \int_{\partial\Omega_{h(t)} \setminus \Gamma_0} (-2\mu D(u) \cdot n_t + p n_t) \cdot u = \int_{\Omega_{h(t)}} f \cdot u. \tag{4.12}$$

Now due to the definition of T_f given by (4.10) we calculate for the boundary term

$$\begin{aligned} &\int_{\partial\Omega_{h(t)} \setminus \Gamma_0} (-2\mu D(u) \cdot n_t + p n_t) \cdot u = \int_\omega T_f \cdot u(t, x, y, 1 + h(t, x, y)) \\ &\overset{(4.4)}{=} \int_\omega (T_f)_3 \partial_t h = \int_\omega -\big(\partial_t^2 h + \Delta^2 h + \Delta^2 \partial_t h - (T_f)_3\big)\partial_t h \\ &\quad + \int_\omega \big(\partial_t^2 h + \Delta^2 h + \Delta^2 \partial_t h\big)\partial_t h. \end{aligned} \tag{4.13}$$

We rewrite the last integral like this

$$\int_\omega \left(\frac{1}{2}\frac{d}{dt}|\partial_t h|^2 + \frac{1}{2}\frac{d}{dt}|\Delta h|^2 + |\Delta \partial_t h|^2 \right).$$

Applying (4.9) to the second last integral yields

$$\int_\omega -\big(\partial_t^2 h + \Delta^2 h + \Delta^2 \partial_t h - (T_f)_3\big)\partial_t h \leq \int_\omega \big(\kappa|2\partial_t h| - \kappa|\partial_t h|\big) = \int_\omega \kappa|\partial_t h|.$$

We also obtain the reverse inequality if we consider $y = -\partial_t h$ in (4.9). Hence we conclude

$$\int_\omega -\big(\partial_t^2 h + \Delta^2 h + \Delta^2 \partial_t h - (T_f)_3\big)\partial_t h = \int_\omega \kappa|\partial_t h|.$$

Plugging these identities into (4.12), we get the equality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \int_{\Omega_{h(t)}} |u|^2 + 2\mu \int_{\Omega_{h(t)}} |D(u)|^2 + \int_\omega \left(\frac{1}{2}\frac{d}{dt}|\partial_t h|^2 + \frac{1}{2}\frac{d}{dt}|\Delta h|^2 + |\Delta \partial_t h|^2 + \kappa|\partial_t h| \right) \\ &= \int_{\Omega_{h(t)}} f \cdot u. \end{aligned}.$$

Integrating over time yields for all $t > 0$

$$\begin{aligned} &\frac{1}{2} \int_{\Omega_{h(t)}} |u(t)|^2 + 2\mu \int_0^t \int_{\Omega_{h(s)}} |D(u)|^2 \, ds + \frac{1}{2} \int_\omega \big(|\partial_t h(t)|^2 + |\Delta h(t)|^2\big) \\ &\quad + \int_0^t \int_\omega \big(|\Delta \partial_t h|^2 + \kappa|\partial_t h|\big) \\ &= \frac{1}{2} \int_{\Omega_{h_0}} |u_0|^2 + \frac{1}{2} \int_\omega \big(|h_1|^2 + |\Delta h_0|^2\big) + \int_0^t \int_{\Omega_{h(s)}} f \cdot u. \end{aligned}$$

Using Young's and Hölder's inequality, we deduce for all $t > 0$

$$\begin{aligned} v(t) :=& \frac{1}{2}\left(\|u(t)\|^2_{L^2(\Omega_{h(t)})} + \|\partial_t h(t)\|^2_{L^2(\omega)}\right) + \frac{1}{2}\|\Delta h(t)\|^2_{L^2(\omega)} \\ &+ 2\mu\int_0^t \|D(u)\|^2_{L^2(\Omega_{h(s)})}\,ds + \|\Delta\partial_t h\|^2_{L^2((0,t)\times\omega)} + \kappa\|\partial_t h\|_{L^1((0,t)\times\omega)} \\ \leq& \frac{1}{2}\int_0^t \|u(s)\|^2_{L^2(\Omega_{h(s)})} + \frac{1}{2}\left(\|u_0\|^2_{L^2(\Omega_{h_0})} + \|h_1\|^2_{L^2(\omega)} + \|\Delta h_0\|^2_{L^2(\omega)}\right) \\ &+ \frac{1}{2}\int_0^t \|f(s)\|^2_{L^2(\Omega_{h(s)})}\,ds. \end{aligned} \tag{4.14}$$

If we define $a_0 := \frac{1}{2}\big(\|u_0\|^2_{L^2(\Omega_{h_0})} + \|h_1\|^2_{L^2(\omega)} + \|\Delta h_0\|^2_{L^2(\omega)}\big)$, then above inequality implies

$$v(t) \leq a_0 + \frac{1}{2}\int_0^t \|f(s)\|^2_{L^2(\Omega_{h(s)})}\,ds + \int_0^t v(s)\,ds.$$

Applying Grönwall's lemma, we get

$$\begin{aligned} v(t) &\leq a_0 + \frac{1}{2}\int_0^t \|f(s)\|^2_{L^2(\Omega_{h(s)})}\,ds + \int_0^t \left(a_0 + \frac{1}{2}\int_0^s \|f(\tau)\|^2_{L^2(\Omega_{h(\tau)})}\,d\tau\right) e^{t-s}\,ds \\ &= e^t a_0 + \frac{1}{2}\int_0^t \|f(s)\|^2_{L^2(\Omega_{h(s)})}\,ds + \frac{1}{2}\int_0^t \left(\int_0^s \|f(\tau)\|^2_{L^2(\Omega_{h(\tau)})}\,d\tau\right) e^{t-s}\,ds \\ &= e^t a_0 + \frac{1}{2}\int_0^t \|f(s)\|^2_{L^2(\Omega_{h(s)})} e^{t-s} ds. \end{aligned}$$

In the first equality we used that $\int_0^t e^{t-s}\,ds = e^t - 1$ and in the second one we used integration by parts. Plugging in the definitions yields

$$\begin{aligned} &\frac{1}{2}\left(\|u(t)\|^2_{L^2(\Omega_{h(t)})} + \|\partial_t h(t)\|^2_{L^2(\omega)}\right) + \frac{1}{2}\|\Delta h(t)\|^2_{L^2(\omega)} \\ &+ 2\mu\int_0^t \|D(u)\|^2_{L^2(\Omega_{h(s)})}\,ds + \|\Delta\partial_t h\|^2_{L^2((0,t)\times\omega)} + \kappa\|\partial_t h\|_{L^1((0,t)\times\omega)} \\ \leq& \frac{e^t}{2}\left(\|u_0\|^2_{L^2(\Omega_{h_0})} + \|h_1\|^2_{L^2(\omega)} + \|\Delta h_0\|^2_{L^2(\omega)}\right) + \frac{1}{2}\int_0^t \|f(s)\|^2_{L^2(\Omega_{h(s)})} e^{t-s} \end{aligned} \tag{4.15}$$

for all $t > 0$. Hence, it is natural to assume for the given data that $f \in L^2(0,T;L^2(\mathbb{R}^3))$, $u_0 \in L^2(\Omega_{h_0}), h_0 \in H^2_0(\omega)$ and $h_1 \in L^2(\omega)$. Due to the fact that we do not know how large $\Omega_{h(t)}$ is, we defined f on $\mathbb{R}^3$, which is not restrictive given that the expansion by zero preserves L^2-integrability. Via (4.15) these properties, in turn, imply that

$$u \in L^\infty(0,T;L^2(\Omega_{h(t)})),\;\; D(u) \in L^2(0,T;L^2(\Omega_{h(t)})),\;\; h \in W^{1,\infty}(0,T;L^2(\omega)) \cap H^1(0,T;H^2_0(\omega)).$$

For a rigours definition of these spaces, see the next section. Due to $u(t,\cdot) = 0$ on Γ_0, it is tempting to use Korn's inequality in order to get $u \in L^2(0,T;H^1(\Omega_{h(t)}))$. However, here, we encounter the main problem in solving the equations (4.1)-(4.8) in a weak sense. While $h \in H^1(0,T;H^2_0(\omega)) \subset C([0,T]\times\overline{\omega})$ is continuous, h does not need to be Lipschitz continuous. Hence, $\Omega_{h(t)}$ is not necessarily a domain with Lipschitz boundary. Nevertheless, due to the continuity of h, $\Omega_{h(t)}$ is

open and uniformly bounded. Therefore, we can find an $M \in \mathbb{N}$ such that $B_M := \omega \times (0, M) \supset \Omega_{h(t)}$ for all $t \in [0, T]$. Now, we define

$$\overline{u} = \begin{cases} u & \text{in } \Omega_{h(t)}, \\ (0, 0, \partial_t h)^T & \text{in } B_M \backslash \Omega_{h(t)}. \end{cases}$$

Due to the interface condition (4.4) and the previous properties, we see that $D(\overline{u}) \in L^2((0, T) \times B_M)$. Ergo, we can apply Korn's inequality in B_M to get $\overline{u} \in L^2(0, T; H^1(B_M))$, and by restriction we still conclude that $u \in L^2(0, T; H^1(\Omega_{h(t)}))$.

4.2. Solution Spaces and Trace Operators

Due to the fact that $\Omega_{h(t)} \subset B_M$ is not necessarily Lipschitz we need to carefully define the solution spaces. Let $T > 0$ and $\delta \in C([0, T] \times \overline{\omega})$ such that for $M \geq \beta > 0$ we have $M \geq 1 + \delta(t, x, y) \geq \beta > 0$ for all $(t, x, y) \in [0, T] \times \overline{\omega}$ and such that $\delta = 0$ on $\partial\omega$. Then, the set

$$\Omega_{\delta(t)} := \{(x, y, z) \in \mathbb{R}^3 |\ (x, y) \in \omega,\ 0 < z < 1 + \delta(t, x, y)\ \forall (x, y) \in \omega, z \in \mathbb{R}\}$$

is open and contained in $B_M = \omega \times (0, M)$. Furthermore, we define the open set

$$\hat{\Omega}_\delta := \bigcup_{t \in (0,T)} \{t\} \times \Omega_{\delta(t)}.$$

Recalling $\Gamma_0 = (\omega \times \{0\}) \cup (\partial\omega \times [0, 1])$, we define the following spaces

$$\begin{aligned}
H^1_{0,\Gamma_0}(\Omega_{\delta(t)}) &:= \{v \in H^1(\Omega_{\delta(t)}) |\ u_{|\Gamma_0} = 0\}, \\
L^2(0, T; H^1(\Omega_{\delta(t)})) &:= \{v \in L^2(\hat{\Omega}_\delta) |\ \nabla v \in L^2(\hat{\Omega}_\delta)\}, \\
L^2(0, T; H^1_0(\Omega_{\delta(t)})) &:= \overline{C^\infty_c(\hat{\Omega}_\delta)}^{L^2(0,T;H^1(\Omega_{\delta(t)}))}, \\
\mathcal{V}_\delta &:= \{v \in C^1(\overline{\hat{\Omega}_\delta}) |\ \operatorname{div} v = 0,\ v = 0 \text{ on } (0, T) \times \Gamma_0\}, \\
V_\delta &:= \overline{\mathcal{V}_\delta}^{L^2(0,T;H^1(\Omega_{\delta(t)}))}, \\
L^\infty(0, T; L^2(\Omega_{\delta(t)})) &:= \{v \in L^2(\hat{\Omega}_\delta) |\ \operatorname*{ess\,sup}_{t \in (0,T)} \|v\|_{L^2(\Omega_{\delta(t)})} < \infty\}, \\
V &:= \{v \in L^2(0, T; H^1(B_M)) | \operatorname{div} v = 0,\ v = 0 \text{ on } (0, T) \times (\Gamma_0 \cup (\partial\omega \times (1, M)))\}.
\end{aligned}$$

Even though $\Omega_{\delta(t)}$ might not necessarily be Lipschitz, one can still show that

$$V_\delta = \{v \in L^2(0, T; H^1(\Omega_{\delta(t)}))|\ \operatorname{div} v = 0,\ v = 0 \text{ on } (0, T) \times \Gamma_0\},$$

see [1, Theorem 3.22, p. 68], by using that $\Omega_{\delta(t)}$ is locally the graph of a function. The following lemmata treat the definition and properties of the trace on $\partial\Omega_{\delta(t)} \backslash \Gamma_0$. For these lemmata always assume that δ is as at the beginning of the section.

Lemma 4.2.1. *For fixed $t \in [0, T]$, the linear mapping*

$$\gamma_{\delta(t)} : C^1(\overline{B_M}) \to C^0(\overline{\omega}),\ v \mapsto v(x, y, 1 + \delta(t, x, y))$$

can be extended to a linear continuous mapping from $H^1(B_M)$ to $L^2(\omega)$.

Similarly, the same mapping but defined on $C^1(\overline{\Omega}_{\delta(t)})$ can be extended to $H^1(\Omega_{\delta(t)})$.

Additionally, we have $\gamma_{\delta(t)}(v) \in L^2(0, T; L^2(\omega))$ for all $v \in L^2(0, T; H^1(\Omega_{\delta(t)}))$.

The proof of this lemma, as well as the proofs of the following lemmata can be found in [10]. Using the trace operator $\gamma_{\delta(t)}$, we can now rewrite $H_0^1(\Omega_{\delta(t)})$ if we additionally assume that $\delta \in C^0([0,T]; H^1(\omega))$.

Lemma 4.2.2. *Additionally, let $\delta \in C^0([0,T]; C^0(\overline{\omega}) \cap H^1(\omega))$, then*

$$H_0^1(\Omega_{\delta(t)}) = \{v \in H^1_{0,\Gamma_0}(\Omega_{\delta(t)}) |\ \gamma_{\delta(t)}(v) = 0\}.$$

In particular, $v \in L^2(0,T; H_0^1(\Omega_{\delta(t)}))$ if and only if $v \in L^2(0,T; H^1_{0,\Gamma_0}(\Omega_{\delta(t)}))$ and $\gamma_{\delta(t)}(v(t)) = 0$ for almost all $t \in (0,T)$.

In the next lemma, we consider an extension operator we have already used at the end of the previous section.

Lemma 4.2.3. *Additionally, let $\delta \in C^0([0,T]; C^0(\overline{\omega}) \cap H^1(\omega))$ and let $b \in L^2(0,T; H_0^1(\omega))$ and $v \in V_\delta$ such that $\gamma_{\delta(t)}(v(t)) = (0,0,b)^T$ for a.e. $t \in (0,T)$. The function defined by*

$$\overline{v} = \begin{cases} v & \text{in } \hat{\Omega}_\delta, \\ (0,0,b)^T & \text{in } ((0,T) \times B_M) \backslash \hat{\Omega}_\delta \end{cases}$$

belongs to V and

$$\|\overline{v}\|_V \leq C(\|v\|_{V_\delta} + \|b\|_{L^2(0,T;H^1(\omega))})$$

where $C > 0$ depends only on M.

We have the existence of a lifting operator:

Lemma 4.2.4. *For every $\phi \in H_0^1(\omega)$ there exists a $w \in H^1_{0,\Gamma_0}(\Omega_{\delta(t)})$ such that*

$$\gamma_{\delta(t)}(w) = \phi \quad \text{and} \quad \|w\|_{H^1(\Omega_{\delta(t)})} \leq C_\beta \|\phi\|_{H^1(\omega)}.$$

The next lemma gives a meaning to the normal trace on the boundary:

Lemma 4.2.5. *For every $t \in (0,T)$ there exists a linear continuous operator*

$$\gamma^n_{\delta(t)} : \{v \in L^2(\Omega_{\delta(t)}) |\ \operatorname{div} v \in L^2(\Omega_{\delta(t)})\} \to (H_0^1(\omega))'$$

with

$$\gamma^n_{\delta(t)}(v) = v(t,x,y,1+\delta(t,x,y)) \cdot n_t \quad \forall (x,y) \in \omega,\ \forall v \in C^\infty(\overline{\Omega}_{\delta(t)}).$$

The following lemma justifies Korn's inequality.

Lemma 4.2.6. *For all*

$$v, u \in \{w \in V_\delta |\ \exists b \in L^2(0,T; H_0^1(\omega)) \text{ such that } \gamma_{\delta(t)}(w(t)) = (0,0,b)^T \text{ for a.e. } t \in (0,T)\}$$

we have

$$2\int_{\Omega_{\delta(t)}} D(u) : D(v) = \int_{\Omega_{\delta(t)}} \nabla u : \nabla v, \quad \text{for a.e. } t \in (0,T);$$

thus,

$$\sqrt{2}\|D(u)\|_{L^2(\Omega_{\delta(t)})} = \|\nabla u\|_{L^2(\Omega_{\delta(t)})} \quad \text{for a.e. } t \in (0,T).$$

We also have Poincaré's inequality due to the boundedness of $\omega \times B_M$.

Lemma 4.2.7. *Let $v \in H^1_{0,\Gamma_0}(\Omega_{\delta(t)})$. Then*

$$\|v\|_{L^2(\Omega_{\delta(t)})} \leq M \|\nabla v\|_{L^2(\Omega_{\delta(t)})}.$$

4.3. Weak Formulation

Having done the fundamental work of defining the solution spaces, we are now in the position to define weak solutions. Let $u_0 \in L^2(\Omega_{h_0})$, $h_1 \in L^2(\omega)$ and $h_0 \in H_0^2(\omega) \subset C^0(\overline{\omega})$ with

$$\begin{aligned} \min_{\overline{\omega}}(1+h_0) &> 0, \\ \operatorname{div} u_0 &= 0 \text{ in } \Omega_{h_0}, \\ u_0 \cdot n &= 0 \text{ on } \Gamma_0, \\ \gamma^n_{h_0}(u_0) &= (0,0,h_1)^T \cdot n_0 \text{ on } \omega, \\ \int_\omega h_1(x,y) &= 0. \end{aligned} \tag{4.16}$$

The first assumption on h_0 is natural since we assume that there is no intersection of the free boundary with the rigid boundary at the start. In fact the reason why the existence of a global solution for all times will not be shown is due to such a possible self-intersection of the boundary. The other conditions are compatibility conditions. The last one is due to $(4.16)_2$ and $(4.16)_4$.

Definition 4.3.1. *We call (u,h) a weak solution of (4.1) − (4.8) if*

i) $u \in V_h \cap L^\infty(0,T;L^2(\Omega_{h(t)}))$,

ii) $h \in W^{1,\infty}(0,T;L^2(\omega)) \cap H^1(0,T;H_0^2(\omega))$,

iii) $\gamma_{h(t)}(u(t)) = (0,0,\partial_t h)^T$ *on* ω, *for a.e.* $t \in (0,T)$,

iv) for all $(\phi,b) \in \mathcal{V}_h \times C^1([0,T];H_0^2(\omega))$ *such that* $\phi(t,x,y,1+h(t,x,y)) = (0,0,b(t,x,y))^T$ *for all* $(t,x,y) \in [0,T]\times\omega$, *the inequality*

$$\begin{aligned} &\int_{\Omega_{h(t)}} u(t)\cdot\phi(t) - \int_0^t\int_{\Omega_{h(s)}} u\cdot\partial_t\phi + 2\mu\int_0^t\int_{\Omega_{h(s)}} D(u):D(\phi) \\ &+ \int_0^t\int_{\Omega_{h(s)}} (u\cdot\nabla)u\cdot\phi - \int_0^t\int_\omega (\partial_t h)^2 b + \int_\omega \partial_t h(t)b(t) \\ &- \int_0^t\int_\omega \partial_t h\partial_t b + \int_0^t\int_\omega \Delta\partial_t h\Delta b + \int_0^t\int_\omega \kappa\big(|\partial_t h| - |\partial_t h - b|\big) + \int_0^t\int_\omega \Delta h\cdot\Delta b \\ \leq &\int_0^t\int_{\Omega_{h(t)}} f\cdot\phi + \int_{\Omega_{h_0}} u_0\cdot\phi(0) + \int_\omega h_1 b(0) \end{aligned} \tag{4.17}$$

holds for almost all $t \in (0,T)$.

The term $-\int_0^t\int_\omega(\partial_t h)^2 b$ in (4.17) is explained by a formal application of Reynolds transport theorem:

$$\int_{\Omega_{h(t)}} u(t)\phi(t) - \int_0^t\int_{\Omega_{h(s)}} u\partial_t\phi - \int_{\Omega_{h_0}} u_0\phi(0) = \int_0^t\int_{\Omega_{h(s)}} \partial_t u\phi + \int_0^t\int_\omega (\partial_t h)^2 b.$$

Using the identity

$$\frac{1}{2}\int_{\Omega_{h(t)}} (u\cdot\nabla)u\cdot\phi = -\frac{1}{2}\int_{\Omega_{h(t)}} (u\cdot\nabla)\phi\cdot u + \frac{1}{2}\int_\omega (\partial_t h)^2 b$$

we can also write

(4.18)

$$\begin{aligned}
&\int_{\Omega_{h(t)}} u(t)\cdot\phi(t) - \int_0^t\int_{\Omega_{h(s)}} u\cdot\partial_t\phi + 2\mu\int_0^t\int_{\Omega_{h(s)}} D(u):D(\phi)\\
&+\frac{1}{2}\int_0^t\int_{\Omega_{h(s)}}(u\cdot\nabla)u\cdot\phi - \frac{1}{2}\int_0^t\int_{\Omega_{h(s)}}(u\cdot\nabla)\phi\cdot u - \frac{1}{2}\int_0^t\int_\omega(\partial_t h)^2 b\\
&+\int_\omega \partial_t h(t)b(t) - \int_0^t\int_\omega \partial_t h\partial_t b + \int_\omega\int_0^t \Delta\partial_t h\Delta b + \int_0^t\int_\omega \kappa\big(|\partial_t h| - |\partial_t h - b|\big) + \int_0^t\int_\omega \Delta h\cdot\Delta b\\
&\leq \int_0^t\int_{\Omega_{h(t)}} f\cdot\phi + \int_{\Omega_{h_0}} u_0\cdot\phi(0) + \int_\omega h_1 b(0).
\end{aligned}$$

Later, it will become clear why we use (4.18) rather than (4.17).

4.4. Existence of Weak Solutions

This section is dedicated to the proof of our main theorem. Because the proof is very long, we will give a brief overview of it. Since the main problem, as mentioned at the end of Section 4.1, is that $\Omega_{h(t)}$ is not Lipschitz, we will regularize the problem in Step 1. Here, we will regularize the initial data. Since h is part of the solution, we obviously cannot regularize it before we show its existence. So instead we replace instances of h in the weak formulation by a function δ, which we then go on to regularize in the first step. In this step, we will also derive convergence statements that will help us in the last step to receive solutions to the original weak formulation.

In the Step 2, we state the regularized and almost linearised formulation that is given on $\Omega_{\delta_\varepsilon^\star}$, where $\delta_\varepsilon^\star$ is a regularization of δ derived in the first step. This domain is no longer dependent on the solution, yet it still depends on time, which is why we transform the system to a system defined on the reference configuration $\omega\times(0,1)$.

The aim in the Step 3 is to show the existence of a solution to the regularized, almost linearised and transformed sytem. Here we employ the Galerkin method. First we will construct suitable bases by using Stokes-like systems that we use in solving the corresponding ordinary differential equations. After solving these equations by showing that the corresponding mass matrix is positive definite, we derive a priori estimates, similar to Section 4.1 as preparation to pass to the limit.

However, in order to go to the limit, we still need to derive estimates of the terms with time-derivatives of the highest order, which is done in Step 4. This step is very technical and we will use properties of the Galerkin basis.

Finally, in Step 5 we will use the result of Step 4 in conjunction with the Aubin-Lions lemma to go to the limit and obtain a solution to the linearised and regularized system. However, this system still depends on δ that we used to replace instances of h.

Then, in Step 6, we revoke this replacement by using Schauder's fixed point theorem on the solution operator from the previous step. This allows us to replace $\delta_\varepsilon^\star$ with $h_\varepsilon^\star$, with $h_\varepsilon^\star$ being a regularization of h_ε. But we sill have a linearised and regularized system at hand that is defined on $\Omega_{h_\varepsilon^\star}$.

In Step 7, we use the convergences, mentioned in the first step, as well as additional strong L^2-convergences of u_ε and $\partial_t h_\varepsilon$, that we get at the start of this step, in order to take the limit $\varepsilon\to 0$ and conclude the proof. Whenever we go to the limit in a weak formulation, we have to be careful when the test functions depend on the parameter, in which regard we want to take the limit. This problem will occur in the last two steps and will be tackled by first considering a large enough class of test functions that do not depend on this parameter and by using a density argument afterwards.

Theorem 4.4.1. *Let $f \in L^2_{loc}((0,\infty); L^2(\mathbb{R}^3))$, $u_0 \in L^2(\Omega_{h_0})$, $h_1 \in L^2(\omega)$ and $h_0 \in H^2_0(\omega)$ with*

$$\begin{aligned} \min_{\overline{\omega}}(1+h_0) &> 0, \\ \operatorname{div} u_0 &= 0 \text{ in } \Omega_{h_0}, \\ u_0 \cdot n &= 0 \text{ on } \Gamma_0, \\ \gamma^n_{h_0}(u_0) &= (0,0,h_1)^T \cdot n_0 \text{ on } \omega, \\ \int_\omega h_1(x,y) &= 0. \end{aligned}$$

Then, there exists a $T^* \in (0,\infty]$ and a weak solution (u,h) of $(4.1)-(4.8)$ on $[0,T]$ for all $T < T^*$. This solution satisfies the estimate

$$\begin{aligned} &\|u\|_{L^\infty(0,T;L^2(\Omega_{h(t)}))} + \|\nabla u\|_{L^2(0,T;L^2(\Omega_{h(t)}))} + \|\partial_t h\|_{L^\infty(0,T;L^2(\omega))} + \|\Delta h\|_{H^1(0,T;L^2(\omega))} \\ \leq & C(T, \|u_0\|_{L^2(\Omega_{h_0})}, \|f\|_{L^2((0,T)\times\mathbb{R}^3)}, \|h_0\|_{H^2_0(\omega)}, \|h_1\|_{L^2(\omega)}) \end{aligned}$$

with a constant $C > 0$.

Proof. Step 1: The regularization procedure

First, we regularize the problem starting with the initial data. We find a sequence $(h_0^\varepsilon)_{\varepsilon>0} \subset C^\infty_c(\omega)$ such that $\int_\omega h_0^\varepsilon = \int_\omega h_0$ for all $\varepsilon > 0$ and $h_0^\varepsilon \to h_0$ in $H^2_0(\omega)$ for $\varepsilon \to 0$. Due to $\min_{\overline{\omega}}(1+h_0) > 0$ there is a $\beta > 0$ such that $\min_{\overline{w}}(1+h_0) \geq 2\beta > 0$. Due to the continuous embedding $H^2_0(\omega) \hookrightarrow C(\overline{\omega})$, we also have that

$$\min_{\overline{\omega}}(1+h_0^\varepsilon) \geq \frac{3\beta}{2} \tag{4.19}$$

given that $\varepsilon > 0$ is small enough.

Now, we define $\overline{u}_0$ by

$$\overline{u}_0 = \begin{cases} u_0 & \text{in } \Omega_{h_0}, \\ (0,0,h_1)^T & \text{in } B_{M+1}\backslash\Omega_{h_0}. \end{cases}$$

Because of $\operatorname{div}(u_0) = 0$ in Ω_{h_0}, $\operatorname{div}(0,0,h_1) = 0$ in $B_{M+1}\backslash\Omega_{h_0}$ and $\gamma^n_{h_0}(u_0) = (0,0,h_1)^T \cdot n_0$, we also have that $\operatorname{div}(\overline{u}_0) = 0$ in B_{M+1}. For $0 < \delta < 1$, we define the scaling

$$u_0^{[1+\delta]}(x,y,z) = (((1+\delta)(\overline{u}_0)_1, (1+\delta)(\overline{u}_0)_2, (\overline{u}_0)_3)(x,y,(1+\delta)z))^T.$$

It is easy to check that $u_0^{[1+\delta]}$ is also divergence free. Due to $h_0 \in C(\overline{\omega})$, the set

$$Z := \left\{(x,y,z) \subset \mathbb{R}^3 |\ (x,y) \in \omega, (1-\delta/2)(1+h_0(x,y)) < z < (1+\delta)(1+h_0(x,y))\right\}$$

is open and a neighbourhood of $\partial\Omega_{h_0}\backslash\Gamma_0$. In particular, we have that $u_0^{[1+\delta]} = (0,0,h_1)^T$ on Z. Due to $h_0^\varepsilon \to h_0$ in $C(\overline{\omega})$, we get for $\varepsilon > 0$ small enough the inclusion

$$Z_\varepsilon := \left\{(x,y,z) \subset \mathbb{R}^3 |\ (x,y) \in \omega, z \in (1-\delta/4)(1+h_0(x,y)) < z < (1+\delta/2)(1+h_0(x,y))\right\} \subset Z.$$

Hence, provided that $\varepsilon > 0$ is small enough, we also have $u_0^{[1+\delta]} = (0,0,h_1)^T$ on Z_ε, which is a neighbourhood of $\partial\Omega_{h_0^\varepsilon}\backslash\Gamma_0$.

We want to show that

$$\chi_{\Omega_{h_0^\delta}} u_0^{[1+\delta]} \to \chi_{\Omega_{h_0}} u_0 \text{ in } L^2(B_M) \text{ as } \delta \to 0.$$

We use that in Hilbert spaces strong convergence is equivalent to weak convergence and convergence of the norm. For the convergence of the norm we have

$$\begin{aligned}
&\int_{B_M} \left|\chi_{\Omega_{h_0^\delta}} u_0^{[1+\delta]}\right|^2 \\
=&\int_{\Omega_{h_0^\delta}} \left|((1+\delta)(\overline{u}_0)_1, (1+\delta)(\overline{u}_0)_2, (\overline{u}_0)_3)(x, y, (1+\delta)z)^T\right|^2 \\
=&(1+\delta)\int_{B_M} \left|\chi_{\Omega_{\frac{h_0^\delta}{1+\delta}}} \left((\overline{u}_0)_1, (\overline{u}_0)_2, \frac{1}{1+\delta}(\overline{u}_0)_3\right)^T\right|^2 \to \int_{B_M} \left|\chi_{\Omega_{h_0}} \overline{u}_0\right|^2 = \int_{B_M} \left|\chi_{\Omega_{h_0}} u_0\right|^2
\end{aligned}$$

as $\delta \to 0$. Here we used the dominated convergence theorem and the uniform convergence of $(h_0^\delta)_{\delta>0}$ to h_0. The weak convergence follows in a similar way: For $\varphi \in C_c^\infty(B_M)$ we have

$$\begin{aligned}
&\int_{B_M} \langle \chi_{\Omega_{h_0^\delta}} u_0^{[1+\delta]}, \varphi\rangle = \int_{B_M} \left\langle \chi_{\Omega_{\frac{h_0^\delta}{1+\delta}}} ((\overline{u}_0)_1, (\overline{u}_0)_2, \frac{1}{1+\delta}(\overline{u}_0)_3)^T, \varphi\left(x, y, \frac{z}{1+\delta}\right)\right\rangle \\
&\to \int_{B_M} \left\langle \chi_{\Omega_{h_0}} u_0, \varphi\right\rangle \text{ as } \delta \to 0.
\end{aligned}$$

Actually, it suffices to consider elements of the dense subset $C_c^\infty(B_M) \subset L^2(B_M)$; this is justified by the Banach-Steinhaus theorem and the fact that we already have convergence and thus boundedness of the norm. Now, let $(h_1^\varepsilon)_{\varepsilon>0} \subset C^\infty(\omega)$ be such that $h_1^\varepsilon \to h_1$ in $L^2(\omega)$ and $\int_\omega h_1^\varepsilon = 0$ for all $\varepsilon > 0$. Moreover, we can regularize $u_0^{[1+\delta]}$ in $L^2(B_{M+1})$ by functions $(u_0^\varepsilon)_{\varepsilon>0} \subset C^\infty(B_{M+1})$ with $\operatorname{div} u_0^\varepsilon = 0$, $u_0^\varepsilon = 0$ on Γ_0 and $u_0^\varepsilon(x, y, 1 + h_0^\varepsilon(x, y)) = (0, 0, h_1^\varepsilon)^T$ on ω. The latter can be assumed due to $u_0^{[1+\delta]} = (0, 0, h_1)^T$ on Z_ε. In summary, we get that

- $\operatorname{div} u_0^\varepsilon = 0$,
- $u_0^\varepsilon(x, y, 1 + h_0^\varepsilon(x, y)) = (0, 0, h_1^\varepsilon)^T$ for all $(x, y) \in \omega$,
- $\int_\omega h_1^\varepsilon = 0$,
- $u_0^\varepsilon = 0$ on Γ_0,
- $\int_\omega h_0^\varepsilon = \int_\omega h_0$

and the convergences

$$\begin{aligned}
\chi_{\Omega_{h_0^\varepsilon}} u_0^\varepsilon \to \chi_{\Omega_{h_0}} u_0 &\text{ in } L^2(B_M),\\
h_1^\varepsilon \to h_1 \qquad &\text{in } L^2(\omega),\\
h_0^\varepsilon \to h_0 \qquad &\text{in } H_0^2(\omega) \text{ as } \varepsilon \searrow 0.
\end{aligned}$$

Since the unknown domain $\Omega_{h(t)}$ depends on the solution h, we first replace it with a regularized domain $\Omega_{\delta_\varepsilon^*(t)}$ and use a fixed point argument later. To be more precise, let $\delta \in H^1(0, T; C^0(\overline{\omega}) \cap H_0^1(\omega))$ such that $\delta_{|t=0} = h_0^\varepsilon$ and $M \geq 1 + \delta(t, x, y) \geq \beta > 0$ for all $(t, x, y) \in [0, T] \times \overline{\omega}$ with β as

before, *i.e.* $\min_{\overline{\omega}}(1+h_0) \geq 2\beta > 0$, and $M \in \mathbb{N}$ will be chosen later.

We take a space-time regularization $\delta_\varepsilon^\star := N_\varepsilon(\delta) \in C_c^\infty([0,T]\times\omega)$ of δ such that

(4.20)
$$\begin{aligned}
&N_\varepsilon(h_\varepsilon) \to \delta && \text{in } C^0([0,T]\times\overline{\omega}) \text{ as } \varepsilon\to 0 && \text{if } h_\varepsilon \to \delta && \text{in } C^0([0,T]\times\overline{\omega}) \text{ as } \varepsilon\to 0\\
&\partial_t N_\varepsilon(h_\varepsilon) \to \partial_t\delta && \text{in } L^2(0,T;L^2(\omega)) \text{ as } \varepsilon\to 0 && \text{if } \partial_t h_\varepsilon \to \partial_t\delta && \text{in } L^2(0,T;L^2(\omega)) \text{ as } \varepsilon\to 0\\
&N_\varepsilon(\delta)_{|t=0} = h_0^\varepsilon
\end{aligned}$$

for any sequence $(h_\varepsilon)_{\varepsilon>0} \subset H^1(0,T;C^0(\overline{\omega})\cap H_0^1(\omega))$ with $h_{\varepsilon|t=0} = h_0^\varepsilon$ for all $\varepsilon > 0$.

In order to address potential confusion due to the difference between $N_\varepsilon(h_\varepsilon)$ and $N_\varepsilon(\delta)$, we give a small outlook: We will replace instances of h with $N_\varepsilon(\delta)$ in the weak formulation for a fixed δ with properties as described above. Later, we will find a fixed point h_ε for all $\varepsilon > 0$ fixed and small enough. Given that we have found a fixed point, we will then replace the term $N_\varepsilon(\delta)$ with $N_\varepsilon(h_\varepsilon)$ and the above convergences will be helpful when considering the limit as ε goes to 0.

The operator N_ε can be constructed as follows: Consider for each $\varepsilon > 0$ a regularization

$$\begin{aligned}
&S_\varepsilon : H^1(0,T;C^0(\overline{\omega})\cap H_0^1(\omega)) \longrightarrow C^\infty([0,T];C_0^\infty(\omega)) \text{ such that}\\
&\|S_\varepsilon(z) - z\|_{H^1(0,T;C^0(\overline{\omega})\cap H_0^1(\omega))} \leq \varepsilon \quad \forall z \in H^1(0,T;C^0(\overline{\omega})\cap H_0^1(\omega))
\end{aligned}$$

for all $\varepsilon > 0$. Due to the fact that $C^\infty([0,T];C_0^\infty(\omega))$ is dense in $H^1(0,T;C^0(\overline{\omega})\cap H_0^1(\omega))$ such a regularizing function exists, and we can also assume that for all $z \in H^1(0,T;C^0(\overline{\omega})\cap H_0^1(\omega))$ with $z_{|t=0} = 0$, also $(S_\varepsilon(z))_{|t=0} = 0$ holds. The latter implies that if we set

$$N_\varepsilon(\delta) := S_\varepsilon(\delta - \delta_{|t=0}) + h_0^\varepsilon \quad \forall \delta \in H^1(0,T;C^0(\overline{\omega})\cap H_0^1(\omega)), \tag{4.21}$$

we have $N_\varepsilon(\delta)_{|t=0} = h_0^\varepsilon$ for all $\delta \in H^1(0,T;C^0(\overline{\omega})\cap H_0^1(\omega))$. To check the first convergence statement we estimate

$$\begin{aligned}
&\|N_\varepsilon(h_\varepsilon) - \delta\|_{C^0([0,T]\times\overline{\omega})} = \|S_\varepsilon(h_\varepsilon - h_{\varepsilon|t=0}) + h_0^\varepsilon - \delta\|_{C^0([0,T]\times\overline{\omega})}\\
&\leq \|S_\varepsilon(h_\varepsilon - h_{\varepsilon|t=0}) - (h_\varepsilon - h_{\varepsilon|t=0})\|_{C^0([0,T]\times\overline{\omega})} + \|(h_\varepsilon - h_{\varepsilon|t=0}) + h_0^\varepsilon - \delta\|_{C^0([0,T]\times\overline{\omega})}\\
&\leq \varepsilon + \|h_\varepsilon - \delta\|_{C^0([0,T]\times\overline{\omega})} \to 0
\end{aligned}$$

if $h_\varepsilon \to \delta$ in $C^0([0,T]\times\overline{\omega})$ as $\varepsilon \to 0$ and $h_{\varepsilon|t=0} = h_0^\varepsilon$ for all $\varepsilon > 0$. For the second convergence we have

$$\begin{aligned}
&\|\partial_t N_\varepsilon(h_\varepsilon) - \partial_t\delta\|_{L^2(0,T;L^2(\omega))} = \|\partial_t S_\varepsilon(h_\varepsilon - (h_\varepsilon)_{|t=0}) - \partial_t\delta\|_{L^2(0,T;L^2(\omega))}\\
&\leq \|\partial_t S_\varepsilon(h_\varepsilon - (h_\varepsilon)_{|t=0}) - \partial_t(h_\varepsilon - (h_\varepsilon)_{|t=0})\|_{L^2(0,T;L^2(\omega))} + \|\partial_t h_\varepsilon - \partial_t\delta\|_{L^2(0,T;L^2(\omega))}\\
&\leq \|S_\varepsilon(h_\varepsilon - (h_\varepsilon)_{|t=0}) - (h_\varepsilon - (h_\varepsilon)_{|t=0})\|_{H^1(0,T;L^2(\omega))} + \|\partial_t h_\varepsilon - \partial_t\delta\|_{L^2(0,T;L^2(\omega))} \to 0
\end{aligned}$$

if $(\partial_t h_\varepsilon)_{\varepsilon>0}$ converges to $\partial_t\delta$ in $L^2(0,T;L^2(\omega))$ as $\varepsilon \to 0$.

Due to $\|N_\varepsilon(\delta) - \delta\|_{C^0([0,T]\times\overline{\omega})} = \|S_\varepsilon(\delta - \delta_{|t=0}) - (\delta - h_0^\varepsilon)\|_{C^0([0,T]\times\overline{\omega})} \leq \varepsilon$ and $M \geq 1 + \delta \geq \beta$ on $[0,T]\times\overline{\omega}$ we can assume that

$$2M \geq 1 + \delta_\varepsilon^\star \geq \frac{\beta}{2} \text{ on } [0,T]\times\overline{\omega} \tag{4.22}$$

by considering $\varepsilon > 0$ small enough.

Furthermore, in order to regularize the non-linearity we also introduce the space-time regularization $v_\varepsilon^\star := R_\varepsilon(v) \in C^\infty((0,T) \times B_{2M})$ for $v \in L^2(0,T; L^2(B_{2M}))$ such that

$$R_\varepsilon(v_\varepsilon) \to v \text{ in } L^2(0,T; L^2(B_{2M})) \text{ as } \varepsilon \to 0 \text{ if } v_\varepsilon \to v \text{ in } L^2(0,T; L^2(B_{2M})) \text{ as } \varepsilon \to 0. \tag{4.23}$$

For example, we can set $R_\varepsilon(v) = \varphi_\varepsilon * E_0(v)$ with $(\varphi_\varepsilon)_{\varepsilon>0} \subset C^\infty(\mathbb{R}^4)$ being an approximate identity with compact support in $(0,T) \times B_{2M}$, and E_0 denotes the extension by zero on $\mathbb{R}^4$.

Step 2: The approximate, almost linearised problem

We consider the following, almost linearised, approximate problem: Find $(u_\varepsilon, h_\varepsilon)$ such that

i) $u_\varepsilon \in V_{\delta_\varepsilon^\star} \cap L^\infty(0,T; L^2(\Omega_{\delta_\varepsilon^\star(t)}))$,

ii) $h_\varepsilon \in W^{1,\infty}(0,T; L^2(\omega)) \cap H^1(0,T; H_0^2(\omega))$,

iii) $u_\varepsilon(t,x,y,1+\delta_\varepsilon^\star(t,x,y)) = (0,0,\partial_t h_\varepsilon(t,x,y))^T$ on $[0,T] \times \omega$,

iv) $\partial_t u_\varepsilon \in L^2(0,T; L^2(\Omega_{\delta_\varepsilon^\star(t)}))$,

v) $\partial_t^2 h_\varepsilon \in L^2(0,T; L^2(\omega))$,

vi) $u_\varepsilon(0) = u_0^\varepsilon$, $h_\varepsilon(0) = h_0^\varepsilon$ and $\partial_t h_\varepsilon(0) = h_1^\varepsilon$,

vii) for all $\phi_\varepsilon \in V_{\delta_\varepsilon^\star}$, $b \in L^2(0,T; H_0^2(\omega))$ such that $\phi_\varepsilon(t,x,y,1+\delta_\varepsilon^\star(t,x,y)) = (0,0,b(t,x,y))^T$ on $[0,T] \times \omega$ we have

$$\begin{aligned} &\int_0^t \int_{\Omega_{\delta_\varepsilon^\star(s)}} \partial_t u_\varepsilon \cdot \phi_\varepsilon + \mu \int_0^t \int_{\Omega_{\delta_\varepsilon^\star(s)}} \nabla u_\varepsilon : \nabla \phi_\varepsilon + \frac{1}{2} \int_0^t \int_{\Omega_{\delta_\varepsilon^\star(s)}} (v_\varepsilon^\star \cdot \nabla) u_\varepsilon \cdot \phi_\varepsilon \\ &- \frac{1}{2} \int_0^t \int_{\Omega_{\delta_\varepsilon^\star(s)}} (v_\varepsilon^\star \cdot \nabla) \phi_\varepsilon \cdot u_\varepsilon + \frac{1}{2} \int_0^t \int_\omega \partial_t h_\varepsilon (\partial_t \delta_\varepsilon^\star) b + \int_0^t \int_\omega \partial_t^2 h_\varepsilon b + \int_0^t \int_\omega \Delta \partial_t h_\varepsilon \Delta b \\ &+ \int_0^t \int_\omega \Delta h_\varepsilon \Delta b + \int_0^t \int_\omega \langle \kappa G j_\varepsilon(\partial_t h_\varepsilon), b \rangle = \int_0^t \int_{\Omega_{\delta_\varepsilon^\star(s)}} f \cdot \phi_\varepsilon. \end{aligned} \tag{4.24}$$

Here, we linearised the term $\int_0^t \int_\omega \kappa(|\partial_t h| - |\partial_t h - b|)$ with respect to b via $\int_0^t \int_\omega \kappa \langle G j_\varepsilon(\partial_t h_\varepsilon), b \rangle$ with $G j_\varepsilon$ being the derivative of $j_\varepsilon : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sqrt{x^2 + \varepsilon^2}$, *i.e.*

$$G j_\varepsilon(\partial_t h_\varepsilon) = \frac{\partial_t h_\varepsilon}{\sqrt{|\partial_t h_\varepsilon|^2 + \varepsilon^2}},$$

similar to Chapter 3. Also we wrote $\int_{\Omega_{\delta_\varepsilon^\star(s)}} \nabla u_\varepsilon : \nabla \phi_\varepsilon$ instead of $2 \int_{\Omega_{\delta_\varepsilon^\star(s)}} Du_\varepsilon : D\phi_\varepsilon$ which is possible due to Lemma 4.2.6. We used the regularizations mentioned before. That is to say, $\delta_\varepsilon^\star = N_\varepsilon(\delta)$ with δ as described shortly before (4.20) and $v_\varepsilon^\star = R_\varepsilon(v)$ for an arbitrary $v \in L^2(0,T; L^2(B_{2M}))$. Furthermore, comparing (4.24) with (4.18), we reversed the integration by parts with respect to time. For h_ε, we used

$$-\int_0^t \int_\omega \partial_t h_\varepsilon \partial_t b + \int_\omega \partial_t h_\varepsilon(t) b(t) - \int_\omega h_1^\varepsilon b(0) = \int_0^t \int_\omega (\partial_t^2 h_\varepsilon) b \tag{4.25}$$

which explains the new term $\int_0^t \int_\omega (\partial_t^2 h_\varepsilon) b$ in (4.24). For u_ε, we used Reynolds transport theorem to get

$$\begin{aligned}\int_{\Omega_{\delta_\varepsilon^\star(t)}} u_\varepsilon(t)\phi(t) - \int_0^t \int_{\Omega_{\delta_\varepsilon^\star(s)}} u_\varepsilon \partial_t \phi_\varepsilon - \int_{\Omega_{h_0^\varepsilon}} u_0^\varepsilon \phi_\varepsilon(0) = \int_0^t \int_{\Omega_{\delta_\varepsilon^\star(s)}} \partial_t u_\varepsilon \phi_\varepsilon \\ + 2 \cdot \frac{1}{2} \int_0^t \int_\omega \partial_t h_\varepsilon (\partial_t \delta_\varepsilon^\star) b.\end{aligned} \tag{4.26}$$

This in combination with the replacement of the term $-\frac{1}{2}\int_0^t \int_\omega (\partial_t h)^2 b$ in (4.18) by the more regular term $-\frac{1}{2}\int_0^t \int_\omega \partial_t h_\varepsilon (\partial_t \delta_\varepsilon^\star) b$ explains the term $\frac{1}{2}\int_0^t \int_\omega \partial_t h_\varepsilon (\partial_t \delta_\varepsilon^\star) b$ in (4.24).

Note that we are allowed to take the solution $(u_\varepsilon, \partial_t h_\varepsilon)$ as a test function. This yields

$$\begin{aligned}&\frac{1}{2}\|u_\varepsilon(t)\|^2_{\Omega_{\delta_\varepsilon^\star(t)}} + \mu \int_0^t \|\nabla u_\varepsilon\|^2_{L^2(\Omega_{\delta_\varepsilon^\star(s)})} + \frac{1}{2}\|\partial_t h_\varepsilon(t)\|^2_{L^2(\omega)} + \int_0^t \|\Delta \partial_t h_\varepsilon\|^2_{L^2(\omega)} + \frac{1}{2}\|\Delta h_\varepsilon(t)\|^2_{L^2(\omega)} \\ &+ \int_0^t \int_\omega \kappa \frac{|\partial_t h_\varepsilon|^2}{\sqrt{|\partial_t h_\varepsilon|^2 + \varepsilon^2}} = \int_0^t \int_{\Omega_{\delta_\varepsilon^\star(s)}} f \cdot u_\varepsilon + \frac{1}{2}\|u_0^\varepsilon\|^2_{L^2(\Omega_{h_0^\varepsilon})} + \frac{1}{2}\|\Delta h_0\|^2_{L^2(\omega)} + \frac{1}{2}\|h_1\|^2_{L^2(\omega)}.\end{aligned}$$

Here, we used that

$$\int_{\Omega_{\delta_\varepsilon^\star(s)}} \partial_t u_\varepsilon \cdot u_\varepsilon + \frac{1}{2}\int_\omega (\partial_t h_\varepsilon)^2 \partial_t \delta_\varepsilon^\star = \frac{1}{2}\frac{d}{dt}\int_{\Omega_{\delta_\varepsilon^\star(s)}} |u_\varepsilon|^2,$$

which is true since the upper boundary of $\Omega_{\delta_\varepsilon^\star(t)}$ moves at the velocity $(0,0,\partial_t \delta_\varepsilon^\star)^T$ and $u_\varepsilon(t,x,y,1+\delta_\varepsilon^\star(t,x,y)) = (0,0,\partial_t \delta_\varepsilon(t,x,y))^T$ on ω, see iii) on page 62. Similar to the considerations in Section 4.1, we get

$$\begin{aligned}\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_{\delta_\varepsilon^\star(t)}))} + \|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\Omega_{\delta_\varepsilon^\star(t)}))} + \|\partial_t h_\varepsilon\|_{L^\infty(0,T;H^1(\omega))} \\ + \|\Delta h_\varepsilon\|_{H^1(0,T;L^2(\omega))} \leq C\end{aligned} \tag{4.27}$$

with $C > 0$ depending only on the data and T but not on $\varepsilon > 0$, M or β. Due to ω being bounded, we also deduce by Poincaré's inequality, see Lemma 4.2.7, that u_ε is bounded in $L^2(0,T;H^1(\Omega_{\delta_\varepsilon^\star(t)}))$ by a constant independent of $\varepsilon > 0$.

Now, we transform the domain $\Omega_{\delta_\varepsilon^\star(t)}$ to our reference configuration $\omega \times (0,1)$ via

$$\chi_\varepsilon(t) : \omega \times (0,1) \to \Omega_{\delta_\varepsilon^\star(t)}, \quad (x,y,z) \mapsto (x,y,z(1+\delta_\varepsilon^\star(t,x,y)))^T.$$

Note that $\chi_\varepsilon(t)$ is a smooth diffeomorphism with $\det \nabla \chi_\varepsilon(t) = 1 + \delta_\varepsilon^\star(t)$ for all $t \in (0,T)$. For the sake of abbreviation we set $\underline{\rho_\varepsilon}(t) := \rho \circ \chi_\varepsilon(t)$ for a function ρ in the following. Then, the transformed system reads

$$\begin{aligned}&\int_0^t \int_{\omega\times(0,1)} \partial_t \underline{u_\varepsilon} \cdot \underline{\phi_\varepsilon}(1+\delta_\varepsilon^\star) - \int_0^t \int_{\omega\times(0,1)} \partial_3 \underline{u_\varepsilon} \cdot \underline{\phi} z \partial_t \delta_\varepsilon^\star + \mu \int_0^t \int_{\omega\times(0,1)} \left(\nabla \underline{u_\varepsilon}\, A_\varepsilon\right) : \nabla \underline{\phi_\varepsilon} \\ &+ \frac{1}{2}\int_0^t \int_{\omega\times(0,1)} (\underline{v_\varepsilon^\star} \cdot (B_\varepsilon \nabla)) \underline{u_\varepsilon} \cdot \underline{\phi_\varepsilon} - \frac{1}{2}\int_0^t \int_{\omega\times(0,1)} (\underline{v_\varepsilon^\star} \cdot (B_\varepsilon \nabla)) \underline{\phi_\varepsilon} \cdot \underline{u_\varepsilon} \\ &+ \frac{1}{2}\int_0^t \int_\omega \partial_t h_\varepsilon (\partial_t \delta_\varepsilon^\star) b + \int_0^t \int_\omega \partial_t^2 h_\varepsilon b + \int_0^t \int_\omega \Delta \partial_t h_\varepsilon \Delta b + \int_0^t \int_\omega \Delta h_\varepsilon \Delta b \\ &+ \int_0^t \int_\omega \kappa \langle G j_\varepsilon(\partial_t h_\varepsilon), b\rangle = \int_0^t \int_{\omega\times(0,1)} \underline{f} \cdot \underline{\phi_\varepsilon}(1+\delta_\varepsilon^\star)\end{aligned} \tag{4.28}$$

for all $\underline{\phi_\varepsilon} \in L^2(0,T;H^1_{0,\Gamma_0}(\omega\times(0,1)))$, $b \in L^2(0,T;H^2_0(\omega))$ such that

$$\underline{\phi_\varepsilon}(t,x,y,1) = (0,0,b(t,x,y))^T \text{ on } \omega, \;\; \operatorname{div}(B_\varepsilon^T \underline{\phi_\varepsilon}) = 0 \text{ in } \omega\times(0,1).$$

Here we have set $B_\varepsilon := (\nabla\chi_\varepsilon)^{-T}(1+\delta^\star_\varepsilon)$, $A_\varepsilon := \frac{1}{1+\delta^\star_\varepsilon} B_\varepsilon^T B_\varepsilon$ and $(\underline{v^\star_\varepsilon}\cdot(B_\varepsilon\nabla))\rho := (\nabla\rho)\, B^T \underline{v^\star_\varepsilon}$. The interface condition transfers to

$$\underline{u_\varepsilon}(t,x,y,1) = (0,0,\partial_t h_\varepsilon(t,x,y))^T \;\; \forall (x,y)\in\omega.$$

Step 3: The Galerkin procedure for the approximate, almost linearised problem
Now we construct a Galerkin basis $(\psi^0_i)_{i\in\mathbb{N}}$ of the space $\{v\in H^1_0(\omega\times(0,1))|\ \operatorname{div} v = 0 \text{ in } \omega\times(0,1)\}$ by the eigenfunctions of the Stokes problem, see [33, Chapter 1, Section 2.6],

$$\begin{aligned} -\Delta\psi^0_i + \nabla p^0_i &= \lambda_i \psi^0_i && \text{in } \omega\times(0,1),\\ \operatorname{div}\psi^0_i &= 0 && \text{in } \omega\times(0,1),\\ \psi^0_i &= 0 && \text{on } \partial(\omega\times(0,1)). \end{aligned}$$

Setting $\phi^{0,\varepsilon}_i = \phi^{0,\varepsilon}_i(t) := (B^T_\varepsilon(t))^{-1}\psi^0_i$, we get for any $t\in[0,T]$ a basis of the space

$$\{v\in H^1_0(\omega\times(0,1))|\ \operatorname{div}(B^T_\varepsilon v) = 0 \text{ in } \omega\times(0,1)\}.$$

Furthermore, we take a basis $(\xi_j)_{j\in\mathbb{N}}$ of $\{b\in H^2_0(\omega)|\ \int_\omega b = 0\}$, and we construct $(\phi^{1,\varepsilon}_j)_{j\in\mathbb{N}}$ such that $\operatorname{div}(B^T_\varepsilon\phi^{1,\varepsilon}_j) = 0$ and $\phi^{1,\varepsilon}_j(t,x,y,1) = (0,0,\xi_j(x,y))^T$ for all $(x,y)\in\omega$. This can be achieved by solving the modified Stokes problem

$$\begin{aligned} -\Delta\phi^{1,\varepsilon}_j + (B_\varepsilon\cdot\nabla)p^{1,\varepsilon}_j &= 0 && \text{in } \omega\times(0,1),\\ \operatorname{div}(B^T_\varepsilon\phi^{1,\varepsilon}_j) &= 0 && \text{in } \omega\times(0,1),\\ \phi^{1,\varepsilon}_j &= \begin{cases} 0 & \text{on } \Gamma_0,\\ (0,0,\xi_j)^T & \text{on } \partial(\omega\times(0,1))\backslash\Gamma_0 \end{cases} \end{aligned} \tag{4.29}$$

in a weak sense. This follows from Lemma 1.6.2, since $B^T_\varepsilon(0,0,\xi_j)^T = (0,0,\xi_j)^T$ on ω and

$$\int_{\partial(\omega\times(0,1))\backslash\Gamma_0} (0,0,\xi_j)^T\cdot n_{\partial(\omega\times(0,1))\backslash\Gamma_0} = \int_{\partial(\omega\times(0,1))\backslash\Gamma_0} (0,0,\xi_j)^T\cdot e_3 = \int_\omega \xi_j = 0$$

for all $j\in\mathbb{N}$. Furthermore, we also get the estimate

$$\|\phi^{1,\varepsilon}_j\|_{H^1(\omega\times(0,1))} + \|p^{1,\varepsilon}_j\|_{L^2(\omega\times(0,1))} \le C_\varepsilon \|\xi_j\|_{H^{\frac12}(\omega)}. \tag{4.30}$$

Note that both $\phi^{1,\varepsilon}_i$ and $\phi^{0,\varepsilon}_i$ are smooth with respect to time since B_ε is smooth with respect to time.

Now, we consider the equations

$$\begin{aligned} &\int_{\omega\times(0,1)} \partial_t \underline{u_\varepsilon}^{m,n}\cdot\phi^{0,\varepsilon}_i(1+\delta^\star_\varepsilon) - \int_{\omega\times(0,1)} \partial_3\underline{u_\varepsilon}^{m,n}\cdot\phi^{0,\varepsilon}_i z\partial_t\delta^\star_\varepsilon\\ &+\mu\int_{\omega\times(0,1)} \nabla\underline{u_\varepsilon}^{m,n}\, A_\varepsilon : \nabla\phi^{0,\varepsilon}_i + \frac12\int_{\omega\times(0,1)} (\underline{v^\star_\varepsilon}\cdot(B_\varepsilon\nabla))\underline{u_\varepsilon}^{m,n}\cdot\phi^{0,\varepsilon}_i\\ &-\frac12\int_{\omega\times(0,1)} (\underline{v^\star_\varepsilon}\cdot(B_\varepsilon\nabla))\phi^{0,\varepsilon}_i\cdot\underline{u_\varepsilon}^{m,n} = \int_{\omega\times(0,1)} \underline{f}\cdot\phi^{0,\varepsilon}_i(1+\delta^\star_\varepsilon) \;\; \forall i\in\{1,...,m\} \end{aligned} \tag{4.31}$$

and

$$
\begin{aligned}
&\int_{\omega\times(0,1)} \partial_t \underline{u_\varepsilon}^{m,n} \cdot \phi_j^{1,\varepsilon}(1+\delta_\varepsilon^\star) - \int_{\omega\times(0,1)} \partial_3 \underline{u_\varepsilon}^{m,n} \cdot \phi_j^{1,\varepsilon} z \partial_t \delta_\varepsilon^\star \\
&+ \mu \int_{\omega\times(0,1)} \nabla \underline{u_\varepsilon}^{m,n}\, A_\varepsilon : \nabla \phi_j^{1,\varepsilon} + \frac{1}{2}\int_{\omega\times(0,1)} (\underline{v_\varepsilon^\star}\cdot(B_\varepsilon\nabla))\underline{u_\varepsilon}^{m,n}\cdot\phi_j^{1,\varepsilon} \\
&- \frac{1}{2}\int_{\omega\times(0,1)} (\underline{v_\varepsilon^\star}\cdot(B_\varepsilon\nabla))\phi_j^{1,\varepsilon}\cdot\underline{u_\varepsilon}^{m,n} + \frac{1}{2}\int_\omega \partial_t h_\varepsilon^n(\partial_t\delta_\varepsilon^\star)\xi_j + \int_\omega(\partial_t^2 h_\varepsilon^n)\xi_j + \int_\omega(\Delta\partial_t h_\varepsilon^n)\Delta\xi_j \\
&+ \int_\omega \Delta h_\varepsilon^n \Delta\xi_j + \int_\omega \kappa\langle G j_\varepsilon(\partial_t h_\varepsilon^n),\xi_j\rangle = \int_{\omega\times(0,1)} \underline{f}\cdot\phi_j^{1,\varepsilon}(1+\delta_\varepsilon^\star) \quad \forall j\in\{1,...,n\}
\end{aligned} \tag{4.32}
$$

combined with the initial conditions

$$
h_\varepsilon^n(0) = h_0^\varepsilon,\ \ \underline{u_\varepsilon}^{m,n}(0) = \underline{u_{\varepsilon,0}}^{m,n},\ \ \partial_t h_\varepsilon^n(0) = h_{\varepsilon,1}^n,
$$

here $\underline{u_{\varepsilon,0}}^{m,n}$ and $h_{\varepsilon,1}^n$ denote the orthogonal projections of $\underline{u_0}^\varepsilon$ and h_1^ε onto the finite dimensional spaces $\operatorname{span}(\phi_i^{0,\varepsilon},\phi_j^{1,\varepsilon})_{i\in\{1,...,m\},j\in\{1,...,n\}}$ and $\operatorname{span}(\xi_j)_{j\in\{1,...,n\}}$, respectively. Due to the smoothness of the involved functions with respect to time we get the existence of a solution $(\underline{u_\varepsilon}^{m,n}, h_\varepsilon^n)$ of the form

$$
h_\varepsilon^n(t) = \sum_{j=1}^n \beta_j(t)\xi_j + h_0^\varepsilon \quad\text{and}\quad \underline{u_\varepsilon}^{m,n}(t) = \sum_{i=1}^m \alpha_i(t)\phi_i^{0,\varepsilon} + \sum_{j=1}^n \dot\beta_j(t)\phi_j^{1,\varepsilon} \tag{4.33}
$$

on $[0,T]$. In order to prove the unique existence of the solution we first plug (4.33) into (4.31) and (4.32) to have a system that we want to solve for $\tilde\alpha := (\alpha_1,...,\alpha_m)$ and $\tilde\beta := (\beta_1,...,\beta_n)$. This, in turn, can be done by reducing it to a system of first order. For this purpose let $s_1 := \tilde\beta$ and $s_2 := \dot{\tilde\beta}$. The first order system (without initial conditions) reads

$$
\begin{pmatrix} I_{n\times n} & 0_{n\times(n+m)} \\ 0_{(n+m)\times n} & M_\varepsilon(t)+N \end{pmatrix}
\begin{pmatrix} \dot s_1 \\ \dot{\tilde\alpha} \\ \dot s_2 \end{pmatrix}
= \begin{pmatrix} s_2 \\ F_{\varepsilon,t}(s_1,\tilde\alpha,s_2) \end{pmatrix}.
$$

Here, $F_{\varepsilon,t}$ denotes all lower-order terms that do not involve $\dot{\tilde\alpha}$ or $\ddot{\tilde\beta} = \dot s_2$. $F_{\varepsilon,t}$ is smooth with respect to time, because B_ε is smooth with respect to time. $M_\varepsilon(t)+N \in \mathbb{R}^{(n+m)\times(n+m)}$ is given by

$$
\begin{aligned}
&M_\varepsilon(t)+N = \\
&\begin{pmatrix} \left(\int \phi_l^{0,\varepsilon}\cdot\phi_k^{0,\varepsilon}(1+\delta_\varepsilon^\star)\right)_{l,k\in\{1,...,m\}} & \left(\int \phi_l^{0,\varepsilon}\cdot\phi_k^{1,\varepsilon}(1+\delta_\varepsilon^\star)\right)_{l\in\{1,...,m\},k\in\{1,...,n\}} \\ \left(\int \phi_k^{0,\varepsilon}\cdot\phi_l^{1,\varepsilon}(1+\delta_\varepsilon^\star)\right)_{l\in\{1,...,n\},k\in\{1,...,m\}} & \left(\int \phi_l^{1,\varepsilon}\cdot\phi_k^{1,\varepsilon}(1+\delta_\varepsilon^\star)\right)_{l,k\in\{1,...,n\}} \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & \left(\int_\omega \xi_k\xi_l\right)_{l,k\in\{1,...,n\}} \end{pmatrix},
\end{aligned}
$$

where the integrals in $M_\varepsilon(t)$ are taken over $\omega \times (0,1)$. $M_\varepsilon(t)$ is smooth with respect to time, too, and by using Hölder's inequality one can check that $M_\varepsilon(t)$ is positive semidefinite. Moreover, since $(\phi_l^{0,\varepsilon})_{l\in\mathbb{N}}$ and $(\xi_l)_{l\in\mathbb{N}}$ are bases the matrices $\left(\int_{\omega\times(0,1)} \phi_l^{0,\varepsilon}\cdot\phi_k^{0,\varepsilon}(1+\delta_\varepsilon^\star)\right)_{l,k\in\{1,\dots,m\}}$ and $\left(\int_\omega \xi_k\xi_l\right)_{l,k\in\{1,\dots,n\}}$ are positive definite. In order to see that $M_\varepsilon(t)+N$ is positive definite, we note that the semidefiniteness is clear and for an $x\in\mathbb{R}^{n+m}$ with $x_1 := (x_1,\dots,x_m), x_2 := (x_{m+1},\dots,x_{m+n})$ satisfying

$$0 = \langle (M_\varepsilon(t)+N)\,x, x\rangle$$

we use

$$0 = \langle M_\varepsilon(t)x, x\rangle + \langle Nx, x\rangle \geq \langle Nx, x\rangle \geq 0$$

to conclude $\langle Nx, x\rangle = 0$. Since $\left(\int_\omega \xi_k\xi_l\right)_{l,k\in\{1,\dots,n\}}$ is positive definite, we see that $x_2 = 0$ and thus

$$0 = \langle M_\varepsilon(t)x, x\rangle = \left\langle \left(\int_{\omega\times(0,1)} \phi_l^{0,\varepsilon}\cdot\phi_k^{0,\varepsilon}(1+\delta_\varepsilon^\star)\right)_{l,k\in\{1,\dots,m\}} x_1, x_1\right\rangle,$$

which implies $x_1 = 0$, and therefore $x = 0$. In particular, $M_\varepsilon(t)+N$ is invertible and therefore we deduce the existence of a unique solution on a time interval $[0,T_{m,n}]$ of the form as in (4.33) that satisfies the mentioned initial conditions.

Now, we multiply equation (4.31) with α_i and sum the resulting equations over $i\in\{1,\dots,m\}$. Then, we multiply (4.32) with $\dot\beta_j$ and add those equations over $j\in\{1,\dots,n\}$. Adding those two terms yields

(4.34)
$$\begin{aligned}
&\int_{\omega\times(0,1)} \partial_t \underline{u_\varepsilon}^{m,n}\cdot \underline{u_\varepsilon}^{m,n}(1+\delta_\varepsilon^\star) - \int_{\omega\times(0,1)} \partial_3 \underline{u_\varepsilon}^{m,n}\cdot \underline{u_\varepsilon}^{m,n} z\partial_t\delta_\varepsilon^\star\\
&+\mu\int_{\omega\times(0,1)} \nabla \underline{u_\varepsilon}^{m,n}\, A_\varepsilon : \nabla \underline{u_\varepsilon}^{m,n} + \frac{1}{2}\int_\omega \partial_t h_\varepsilon^n(\partial_t\delta_\varepsilon^\star)\partial_t h_\varepsilon^n + \int_\omega (\partial_t^2 h_\varepsilon^n)\partial_t h_\varepsilon^n\\
&+\int_\omega (\Delta\partial_t h_\varepsilon^n)\Delta\partial_t h_\varepsilon^n + \int_\omega \Delta h_\varepsilon^n\Delta\partial_t h_\varepsilon^n + \int_\omega \kappa\langle Gj_\varepsilon(\partial_t h_\varepsilon^n), \partial_t h_\varepsilon^n\rangle = \int_{\omega\times(0,1)} \underline{f}\cdot \underline{u_\varepsilon}^{m,n}(1+\delta_\varepsilon^\star).
\end{aligned}$$

Due to

$$\partial_t(1+\delta_\varepsilon^\star) = \operatorname{div}(0,0,z\partial_t\delta_\varepsilon^\star)^T$$

and with $n_{\omega\times(0,1)}$ denoting the exterior normal vector of $\omega\times(0,1)$, we have that

$$\begin{aligned}
&\frac{1}{2}\int_{\omega\times(0,1)} |\underline{u_\varepsilon}^{m,n}|^2\partial_t(1+\delta_\varepsilon^\star) = \frac{1}{2}\int_{\omega\times(0,1)} |\underline{u_\varepsilon}^{m,n}|^2 \operatorname{div}(0,0,z\partial_t\delta_\varepsilon^\star)^T\\
=&\frac{1}{2}\int_{\partial(\omega\times(0,1))\setminus\Gamma_0} |\underline{u_\varepsilon}^{m,n}|^2(0,0,z\partial_t\delta_\varepsilon^\star)^T\cdot n_{\omega\times(0,1)} - \frac{1}{2}\int_{\omega\times(0,1)} \nabla|\underline{u_\varepsilon}^{m,n}|^2\cdot(0,0,z\partial_t\delta_\varepsilon^\star)^T\\
=&\frac{1}{2}\int_{\partial(\omega\times(0,1))\setminus\Gamma_0} |\underline{u_\varepsilon}^{m,n}|^2(0,0,z\partial_t\delta_\varepsilon^\star)^T\cdot n_{\omega\times(0,1)} - \int_{\omega\times(0,1)} \partial_3 \underline{u_\varepsilon}^{m,n}\cdot \underline{u_\varepsilon}^{m,n} z\partial_t\delta_\varepsilon^\star\\
=&\frac{1}{2}\int_\omega (\partial_t h_\varepsilon^n)^2\partial_t\delta_\varepsilon^\star - \int_{\omega\times(0,1)} \partial_3 \underline{u_\varepsilon}^{m,n}\cdot \underline{u_\varepsilon}^{m,n} z\partial_t\delta_\varepsilon^\star.
\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\omega\times(0,1)} |\underline{u_\varepsilon}^{m,n}|^2(1+\delta_\varepsilon^\star) \\
&= \int_{\omega\times(0,1)} \partial_t \underline{u_\varepsilon}^{m,n}\cdot \underline{u_\varepsilon}^{m,n}(1+\delta_\varepsilon^\star) + \frac{1}{2}\int_{\omega\times(0,1)} |\underline{u_\varepsilon}^{m,n}|^2\partial_t(1+\delta_\varepsilon^\star) \\
&= \int_{\omega\times(0,1)} \partial_t \underline{u_\varepsilon}^{m,n}\cdot \underline{u_\varepsilon}^{m,n}(1+\delta_\varepsilon^\star) + \frac{1}{2}\int_{\omega} (\partial_t h_\varepsilon^n)^2\partial_t\delta_\varepsilon^\star - \int_{\omega\times(0,1)} \partial_3 \underline{u_\varepsilon}^{m,n}\cdot \underline{u_\varepsilon}^{m,n} z\partial_t\delta_\varepsilon^\star.
\end{aligned}$$

Plugging this into (4.34), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\omega\times(0,1)} |\underline{u_\varepsilon}^{m,n}|^2(1+\delta_\varepsilon^\star) + \mu\int_{\omega\times(0,1)} \nabla \underline{u_\varepsilon}^{m,n}\, A_\varepsilon : \nabla \underline{u_\varepsilon}^{m,n} + \int_\omega (\partial_t^2 h_\varepsilon^n)\partial_t h_\varepsilon^n + \int_\omega (\Delta\partial_t h_\varepsilon^n)^2 \\
&+ \int_\omega \Delta h_\varepsilon^n \Delta\partial_t h_\varepsilon^n + \int_\omega \kappa\langle G j_\varepsilon(\partial_t h_\varepsilon^n), \partial_t h_\varepsilon^n\rangle = \int_{\omega\times(0,1)} \underline{f}\cdot \underline{u_\varepsilon}^{m,n}(1+\delta_\varepsilon^\star).
\end{aligned}$$

Similarly as done in Section 4.1 along with $\langle G j_\varepsilon(\partial_t h_\varepsilon^n), \partial_t h_\varepsilon^n\rangle \geq 0$ and using that A_ε is elliptic with a constant independent of m or n, we can integrate in time and apply Grönwall's inequality to conclude that

$$\begin{aligned}
&\|\underline{u_\varepsilon}^{m,n}\|_{L^\infty(0,T;L^2(\omega\times(0,1)))} + \|\nabla \underline{u_\varepsilon}^{m,n}\|_{L^2(0,T;L^2(\omega\times(0,1)))} + \|\partial_t h_\varepsilon^n\|_{L^\infty(0,T;L^2(\omega))} \\
&+ \|\Delta h_\varepsilon^n\|_{H^1(0,T;L^2(\omega))} \leq C(T, \|u_0\|_{L^2(\Omega_{h_0})}, \|f\|_{L^2((0,T)\times\mathbb{R}^3)}, \|h_0\|_{H_0^2(\omega)}, \|h_1\|_{L^2(\omega)}, \varepsilon, \beta, M).
\end{aligned} \tag{4.35}$$

Since the constant $C>0$ is independent of $T_{m,n}$, the solution indeed exists on $[0,T]$, which justifies replacing $T_{m,n}$ by T on the left-hand side in (4.35). Nevertheless, C still depends on ε, β and M. However, if we return to the problem in the deformed configuration we obtain - similarly as done in Section 4.1 - that

$$\begin{aligned}
&\|u_\varepsilon^{m,n}\|_{L^\infty(0,T;L^2(\Omega_{\delta_\varepsilon^\star(t)}))} + \|\nabla u_\varepsilon^{m,n}\|_{L^2(0,T;L^2(\Omega_{\delta_\varepsilon^\star(t)}))} + \|\partial_t h_\varepsilon^n\|_{L^\infty(0,T;L^2(\omega))} \\
&+ \|\Delta h_\varepsilon^n\|_{H^1(0,T;L^2(\omega))} \leq C(T, \|u_0\|_{L^2(\Omega_{h_0})}, \|f\|_{L^2((0,T)\times\mathbb{R}^3)}, \|h_0\|_{H_0^2(\omega)}, \|h_1\|_{L^2(\omega)})
\end{aligned} \tag{4.36}$$

with $C>0$ independent of ε, β and M.

Step 4: Uniform estimates of time derivatives in m, n

In order to obtain a solution for $m, n \to \infty$, we still need additional estimates on the terms with the largest time derivatives. To be more specific, we need the estimate

$$\|\partial_t \underline{u_\varepsilon}^{m,n}\|_{L^2(0,T;L^2(\omega\times(0,1)))} + \|\partial_{tt} h_\varepsilon^n\|_{L^2(0,T;L^2(\omega))} \leq \tilde{C}(\varepsilon)$$

with $\tilde{C}(\varepsilon)>0$ independent of m, n but possibly dependent on the given data, β, M and ε.

In order to achieve this estimate, we multiply (4.31) with $\dot{\alpha}_i$ and sum these equations over $i \in \{1, ..., m\}$. In a similar fashion, we multiply (4.32) with $\ddot{\beta}_j$ and sum the resulting equations over $j \in \{1, ..., n\}$. For the sake of abbreviation, we omit the indices ε, n and m in the following

and obtain, after adding the two sums,

$$
\begin{aligned}
&\int_{\omega\times(0,1)}\left|\sum_{i=1}^{n+m}\dot\lambda_i\phi_i\right|^2(1+\delta^\star)-\int_{\omega\times(0,1)}\partial_3\underline{u}\cdot\sum_{i=1}^{n+m}\dot\lambda_i\phi_i z\partial_t\delta^\star\\
&+\int_{\omega\times(0,1)}\sum_{i=1}^{n+m}\dot\lambda_i\phi_i\cdot\sum_{i=1}^{n+m}\lambda_i\partial_t\phi_i(1+\delta^\star)+\mu\int_{\omega\times(0,1)}\nabla\underline{u}A:\nabla\left(\sum_{i=1}^{n+m}\dot\lambda_i\phi_i\right)\\
&+\frac{1}{2}\int_{\omega\times(0,1)}(v^\star\cdot(B\nabla))\underline{u}\cdot\sum_{i=1}^{n+m}\dot\lambda_i\phi_i-\frac{1}{2}\int_{\omega\times(0,1)}(v^\star\cdot(B\nabla))\sum_{i=1}^{n+m}\dot\lambda_i\phi_i\cdot\underline{u}\\
&+\frac{1}{2}\int_\omega\partial_t h\partial_t\delta^\star\partial_t^2h+\int_\omega(\partial_t^2h)^2+\int_\omega\Delta\partial_t h\Delta\partial_t^2h+\int_\omega\Delta h\Delta\partial_t^2h+\int_\omega\langle\kappa Gj(\partial_t h),\partial_t^2h\rangle\\
=&\int_{\omega\times(0,1)}\underline{f}\cdot\sum_{i=1}^{n+m}\dot\lambda_i\phi_i(1+\delta^\star)
\end{aligned}
\tag{4.37}
$$

with

$$
\lambda_i:=\begin{cases}\alpha_i, & i\in\{1,...,m\},\\ \dot\beta_{i-m}, & i\in\{m+1,...,n+m\},\end{cases}
$$

and

$$
\phi_i:=\begin{cases}\phi_i^0, & i\in\{1,...,m\},\\ \phi_{i-m}^1, & i\in\{m+1,...,n+m\}.\end{cases}
$$

Hence, we get that

$$
\begin{aligned}
&\left\|\sum_{i=1}^{n+m}\dot\lambda_i\phi_i\sqrt{1+\delta^\star}\right\|^2_{L^2(\omega\times(0,1))}+\frac{\mu}{2}\frac{d}{dt}\int_{\omega\times(0,1)}\nabla\underline{u}A:\nabla\underline{u}+\|\partial_t^2h\|^2_{L^2(\omega)}+\frac{1}{2}\frac{d}{dt}\int_\omega(\Delta\partial_t h)^2\\
=&-\int_{\omega\times(0,1)}\sum_{i=1}^{n+m}\dot\lambda_i\phi_i\sum_{i=1}^{n+m}\lambda_i\partial_t\phi_i(1+\delta^\star)+\frac{\mu}{2}\int_{\omega\times(0,1)}\nabla\underline{u}\partial_tA:\nabla\underline{u}\\
&+\mu\int_{\omega\times(0,1)}\nabla\underline{u}A:\nabla\sum_{i=1}^{n+m}\lambda_i\partial_t\phi_i-\frac{1}{2}\int_{\omega\times(0,1)}(v^\star\cdot(B\nabla))\underline{u}\cdot\sum_{i=1}^{n+m}\dot\lambda_i\phi_i\\
&+\frac{1}{2}\int_{\omega\times(0,1)}(v^\star\cdot(B\nabla))\sum_{i=1}^{n+m}\dot\lambda_i\phi_i\cdot\underline{u}+\int_{\omega\times(0,1)}\partial_3\underline{u}\cdot\sum_{i=1}^{n+m}\dot\lambda_i\phi_i z\partial_t\delta^\star-\frac{1}{2}\int_\omega\partial_t h\partial_t\delta^\star\partial_t^2h\\
&+\int_\omega(\Delta\partial_t h)^2-\frac{d}{dt}\int_\omega\Delta h\Delta\partial_t h-\int_\omega\kappa\langle Gj(\partial_t h),\partial_t^2h\rangle+\int_{\omega\times(0,1)}\underline{f}\cdot\sum_{i=1}^{n+m}\dot\lambda_i\phi_i(1+\delta^\star).
\end{aligned}
$$

After integration in time, we obtain that

$$\begin{aligned}
&\int_0^t \left\| \sum_{i=1}^{n+m} \dot\lambda_i \phi_i \sqrt{1+\delta^\star} \right\|^2_{L^2(\omega\times(0,1))} + \frac{\mu}{2}\int_{\omega\times(0,1)} \nabla \underline{u}(t) A(t) : \nabla \underline{u}(t) + \int_0^t \|\partial_t^2 h\|^2_{L^2(\omega)} \\
&+ \frac{1}{2}\|\Delta \partial_t h(t)\|^2_{L^2(\omega)} \\
=&\frac{\mu}{2}\int_{\omega\times(0,1)} \nabla \underline{u}(0) A(0) : \nabla \underline{u}(0) + \frac{1}{2}\|\Delta \partial_t h(0)\|^2_{L^2(\omega)} + \int_\omega \Delta h(0) \Delta \partial_t h(0) \\
&- \int_0^t \int_{\omega\times(0,1)} \sum_{i=1}^{n+m} \dot\lambda_i \phi_i \sum_{i=1}^{n+m} \lambda_i \partial_t \phi_i (1+\delta^\star) + \frac{\mu}{2}\int_0^t \int_{\omega\times(0,1)} \nabla \underline{u} \partial_t A : \nabla \underline{u} \\
&+ \mu \int_0^t \int_{\omega\times(0,1)} \nabla \underline{u}\, A : \nabla \sum_{i=1}^{n+m} \lambda_i \partial_t \phi_i - \frac{1}{2}\int_0^t \int_{\omega\times(0,1)} (\underline{v}^\star \cdot (B\nabla)) \underline{u} \cdot \sum_{i=1}^{n+m} \dot\lambda_i \phi_i \\
&+ \frac{1}{2}\int_0^t \int_{\omega\times(0,1)} (\underline{v}^\star \cdot (B\nabla)) \sum_{i=1}^{n+m} \dot\lambda_i \phi_i \cdot \underline{u} + \int_0^t \int_{\omega\times(0,1)} \partial_3 \underline{u} \cdot \sum_{i=1}^{n+m} \dot\lambda_i \phi_i z \partial_t \delta^\star \\
&- \frac{1}{2}\int_0^t \int_\omega \partial_t h \partial_t \delta^\star \partial_t^2 h + \int_0^t \|\Delta \partial_t h\|^2_{L^2(\omega)} - \int_\omega \Delta h(t) \Delta \partial_t h(t) - \int_0^t \int_\omega \langle \kappa G j(\partial_t h), \partial_t^2 h \rangle \\
&+ \int_0^t \int_{\omega\times(0,1)} \underline{f} \cdot \sum_{i=1}^{n+m} \dot\lambda_i \phi_i (1+\delta^\star).
\end{aligned}$$

Now, we have to estimate every term on the right-hand side by a constant independent of m and n, but possibly depending on ε, M , β and on δ and v. The first three pose no problem since they depend on the initial data that are smooth. For the fourth term, Hölder's and Young's inequality imply that

$$\begin{aligned}
&\left| \int_0^t \int_{\omega\times(0,1)} \sum_{i=1}^{n+m} \dot\lambda_i \phi_i \sum_{i=1}^{n+m} \lambda_i \partial_t \phi_i (1+\delta^\star) \right| \\
\leq& \frac{1}{8} \left\| \sum_{i=1}^{n+m} \dot\lambda_i \phi_i \sqrt{1+\delta^\star} \right\|^2_{L^2(0,t;L^2(\omega\times(0,1)))} + C \left\| \sum_{i=1}^{n+m} \lambda_i \partial_t \phi_i \right\|^2_{L^2(0,t;L^2(\omega\times(0,1)))}.
\end{aligned} \tag{4.38}$$

We estimate the second summand of the right-hand side of (4.38). Remembering that $\phi_i^{0,\varepsilon} = (B_\varepsilon)^{-T}\psi_i^0$ we get that

$$\sum_{i=1}^{n+m} \lambda_i \partial_t \phi_i = \sum_{i=1}^{m} \alpha_i \partial_t B^{-T} \psi_i^0 + \sum_{j=1}^{n} \dot\beta_j \partial_t \phi_j^1 = \sum_{i=1}^{m} \alpha_i \left(\partial_t B^{-T}\right) B^T \phi_i^0 + \sum_{j=1}^{n} \dot\beta_j \partial_t \phi_j^1. \tag{4.39}$$

Concerning the second term we recall that ϕ_j^1 solves the Stokes-like problem (4.29), and, hence, $\sum_{j=1}^n \dot\beta_j \partial_t \phi_j^1$ solves

$$\begin{aligned}
-\Delta\big(\sum_{j=1}^{n} \dot\beta_j \partial_t \phi_j^1\big) + (B\nabla)\big(\sum_{j=1}^{n} \dot\beta_j \partial_t p_j^1\big) &= -(\partial_t B\nabla) \sum_{j=1}^{n} \dot\beta_j p_j^1 \qquad \text{in } \omega\times(0,1), \\
\operatorname{div}\big(B^T \sum_{j=1}^{n} \dot\beta_j \partial_t \phi_j^1\big) &= -\operatorname{div}\big(\partial_t B^T \sum_{j=1}^{n} \dot\beta_j \phi_j^1\big) \text{ in } \omega\times(0,1), \\
\sum_{j=1}^{n} \dot\beta_j \partial_t \phi_j^1 &= 0 \qquad\qquad\qquad\qquad \text{on } \partial(\omega\times(0,1))
\end{aligned}$$

in the weak sense. Due to Lemma 1.6.2 and the estimate

$$\begin{aligned}\left\langle -(\partial_t B\nabla)\sum_{j=1}^n \dot\beta_j p_j^1, \phi\right\rangle_{(H_0^1)',H_0^1} &= \int_{\omega\times(0,1)} \sum_{j=1}^n \dot\beta_j p_j^1 \operatorname{div}\left((\partial_t B)\,\phi\right)\\ &\le C_\varepsilon \Big\|\sum_{j=1}^n \dot\beta_j p_j^1\Big\|_{L^2(\omega\times(0,1))}\|\phi\|_{H^1(\omega\times(0,1))}\end{aligned}$$

for all $\phi\in H_0^1(\omega\times(0,1))$, we get that

$$\left\|\sum_{j=1}^n \dot\beta_j \partial_t\phi_j^1\right\|_{H^1(\omega\times(0,1))} \le C\left(\left\|\sum_{j=1}^n \dot\beta_j p_j^1\right\|_{L^2(\omega\times(0,1))} + \left\|\sum_{j=1}^n \dot\beta_j \phi_j^1\right\|_{H^1(\omega\times(0,1))}\right).$$

Now, using the equation solved by (ϕ_j^1, p_j) and their continuous dependence on the given data, see (4.30), we can estimate further:

$$\left\|\sum_{j=1}^n \dot\beta_j p_j^1\right\|_{L^2(\omega\times(0,1))} + \left\|\sum_{j=1}^n \dot\beta_j \phi_j^1\right\|_{H^1(\omega\times(0,1))} \le C\left\|\sum_{j=1}^n \dot\beta_j \xi_j\right\|_{H^{1/2}(\omega)} = C\|\partial_t h\|_{H^{1/2}(\omega)}. \tag{4.40}$$

Combining the last two estimates, we have

$$\left\|\sum_{j=1}^n \dot\beta_j \partial_t\phi_j^1\right\|_{L^2(0,t;H^1(\omega\times(0,1)))} \le C\|\partial_t h\|_{L^2(0,t;H^{1/2}(\omega))} \le C\|\partial_t h\|_{L^2(0,t;H_0^2(\omega))} \overset{(4.35)}{\le} C. \tag{4.41}$$

Treating the first term in (4.39) we have that

$$\begin{aligned}&\left(\partial_t B^{-T}\right) B^T\\ =&\begin{pmatrix} \frac{-\partial_t\delta^\star}{(1+\delta^\star)^2} & 0 & 0\\ 0 & \frac{-\partial_t\delta^\star}{(1+\delta^\star)^2} & 0\\ z\frac{(1+\delta^\star)\partial_t\partial_x\delta^\star - \partial_x\delta^\star\partial_t\delta^\star}{(1+\delta^\star)^2} & z\frac{(1+\delta^\star)\partial_t\partial_y\delta^\star - \partial_y\delta^\star\partial_t\delta^\star}{(1+\delta^\star)^2} & 0\end{pmatrix}\begin{pmatrix} 1+\delta^\star & 0 & 0\\ 0 & 1+\delta^\star & 0\\ -z\partial_x\delta^\star & -z\partial_y\delta^\star & 1\end{pmatrix}\\ =&\frac{1}{1+\delta^\star}\begin{pmatrix} -\partial_t\delta^\star & 0 & 0\\ 0 & -\partial_t\delta^\star & 0\\ z\left((1+\delta^\star)\partial_t\partial_x\delta^\star - \partial_x\delta^\star\partial_t\delta^\star\right) & z\left((1+\delta^\star)\partial_t\partial_y\delta^\star - \partial_y\delta^\star\partial_t\delta^\star\right) & 0\end{pmatrix},\end{aligned}$$

which is bounded on $(0,T)\times(\omega\times(0,1))$ due to the smoothness and boundedness of $\delta^\star$. The bound may depend on ε though. Hence, we get that

$$\begin{aligned}\left\|\sum_{i=1}^m \alpha_i\left(\partial_t B^{-T}\right)B^T\phi_i^0\right\|_{L^2(0,t;H^1(\omega\times(0,1)))} &\le C\left\|\sum_{i=1}^m \alpha_i\phi_i^0\right\|_{L^2(0,t;H^1(\omega\times(0,1)))}\\ &\le C\|\underline{u}\|_{L^2(0,t;H^1(\omega\times(0,1)))} + C\left\|\sum_{j=1}^n \dot\beta_j(t)\phi_j^1\right\|_{L^2(0,t;H^1(\omega\times(0,1)))} \le C,\end{aligned} \tag{4.42}$$

where we used the estimate (4.40) for the term involving $\dot{\beta}$ and (4.35) for the term involving $\underline{u}$.

Therefore, we get from (4.38) that

$$\left|\int_0^t \int_{\omega\times(0,1)} \sum_{i=1}^{n+m} \dot{\lambda}_i\phi_i \sum_{i=1}^{n+m} \lambda_i \partial_t\phi_i(1+\delta^\star)\right| \leq \frac{1}{8}\left\|\sum_{i=1}^{n+m}\dot{\lambda}_i\phi_i\right\|^2_{L^2(0,t;L^2(\omega\times(0,1)))} + C.$$

The term

$$\frac{\mu}{2}\int_0^t \int_{\omega\times(0,1)} \nabla\underline{u}\partial_t A : \nabla\underline{u}$$

is bounded independently of m, n due to $\underline{u}$ being bounded uniformly in $L^2(0,t;H^1(\omega\times(0,1)))$, see (4.35), and due to $\partial_t A$ being bounded in $L^\infty((0,t)\times\omega\times(0,1))$ due to $\delta^\star$ being independent of m, n. Concerning the term

$$\mu\int_0^t \int_{\omega\times(0,1)} \nabla\underline{u}\, A : \nabla\left(\sum_{i=1}^{n+m}\lambda_i\partial_t\phi_i\right)$$

we use that

$$\begin{aligned}
&\left\|\sum_{i=1}^{n+m}\lambda_i\partial_t\phi_i\right\|_{L^2(0,t;H^1(\omega\times(0,1)))}\\
\leq&\left\|\sum_{i=1}^{m}\alpha_i\partial_t\phi_i^0\right\|_{L^2(0,t;H^1(\omega\times(0,1)))} + \left\|\sum_{j=1}^{n}\dot{\beta}_j\partial_t\phi_j^1\right\|_{L^2(0,t;H^1(\omega\times(0,1)))}\\
=&\left\|\sum_{i=1}^{m}\alpha_i\left(\partial_t B^{-T}\right)B^T\phi_i^0\right\|_{L^2(0,t;H^1(\omega\times(0,1)))} + \left\|\sum_{j=1}^{n}\dot{\beta}_j\partial_t\phi_j^1\right\|_{L^2(0,t;H^1(\omega\times(0,1)))}
\end{aligned} \tag{4.43}$$

is bounded due to (4.42) and (4.41). For the next term, that involves $v^\star = R_\varepsilon(v) = \varphi_\varepsilon * E_0(v)$, see (4.23) and the remark after, we use

$$\begin{aligned}
\|\underline{v^\star}\|_{L^\infty((0,t)\times\omega\times(0,1))} &\leq \|\varphi_\varepsilon * E_0(v)\|_{L^\infty((0,t)\times\mathbb{R}^3)} \leq C\|\varphi_\varepsilon\|_{L^2((0,t)\times B_{2M})}\|v\|_{L^2((0,t)\times B_{2M})}\\
&\leq C_\varepsilon\|v\|_{L^2((0,t)\times B_{2M})}
\end{aligned}$$

and the boundedness of $\underline{u}$ in $L^2(0,t;H^1(\omega\times(0,1)))$, to get that

$$\begin{aligned}
&\left|\frac{1}{2}\int_0^t\int_{\omega\times(0,1)}(\underline{v^\star}\cdot(B\nabla))\underline{u}\cdot\sum_{i=1}^{n+m}\dot{\lambda}_i\phi_i\right|\\
\leq& C\|v\|^2_{L^2((0,t)\times B_{2M})} + \frac{1}{8}\left\|\sum_{i=1}^{m+n}\dot{\lambda}_i\phi_i\sqrt{1+\delta^\star}\right\|_{L^2(0,t;L^2(\omega\times(0,1)))}.
\end{aligned}$$

For the second integral involving $v^\star$, apply integration by parts to the next term to get

$$\begin{aligned}
&\frac{1}{2}\int_0^t\int_{\omega\times(0,1)}(\underline{v^\star}\cdot(B\nabla))\sum_{i=1}^{n+m}\dot{\lambda}_i\phi_i\cdot\underline{u}\\
=&-\frac{1}{2}\int_0^t\int_{\omega\times(0,1)}\sum_{j=1}^{3}(\partial_j(B^T\underline{v^\star})_j\underline{u})\cdot\sum_{i=1}^{n+m}\dot{\lambda}_i\phi_i + \frac{1}{2}\int_0^t\int_{\partial(\omega\times(0,1))}\left(\underline{u}\cdot\sum_{i=1}^{n+m}\dot{\lambda}_i\phi_i\right)(\underline{v^\star}\cdot Bn_{\omega\times(0,1)}).
\end{aligned}$$

The first integral on the right-hand side can be estimated as

$$\left|\frac{1}{2}\int_0^t\int_{\omega\times(0,1)}\sum_{j=1}^{3}(\partial_j(B^T\underline{v}^\star)_j\underline{u})\cdot\sum_{i=1}^{n+m}\dot{\lambda}_i\phi_i\right|\leq C+\frac{1}{8}\left\|\sum_{i=1}^{n+m}\dot{\lambda}_i\phi_i\sqrt{1+\delta^\star}\right\|^2,$$

while, for the second one, we can write

$$\begin{aligned}&\left|\frac{1}{2}\int_0^t\int_{\partial(\omega\times(0,1))}\left(\underline{u}\cdot\sum_{i=1}^{n+m}\dot{\lambda}_i\phi_i\right)(\underline{v}^\star\cdot Bn_{\omega\times(0,1)})\right|\\ =&\left|\frac{1}{2}\int_0^t\int_\omega\partial_t h\partial_t^2 h(\underline{v}^\star\cdot Bn_{\omega\times(0,1)})\right|\leq C+\frac{1}{6}\|\partial_t^2 h\|^2_{L^2(0,t;L^2(\omega))},\end{aligned}$$

where we used that $\partial_t h$ is bounded in $L^2(0,t;L^2(\omega))$ according to (4.35). For the next term, we easily have

$$\left|\int_0^t\int_{\omega\times(0,1)}\partial_3\underline{u}\cdot\sum_{i=1}^{n+m}\dot{\lambda}_i\phi_i z\partial_t\delta^\star\right|\leq C+\frac{1}{8}\left\|\sum_{i=1}^{n+m}\dot{\lambda}_i\phi_i\sqrt{1+\delta^\star}\right\|^2_{L^2(0,t;L^2(\omega\times(0,1)))}.$$

Concerning integrals involving h, we start with the estimate

$$\left|\frac{1}{2}\int_0^t\int_\omega\partial_t h\partial_t\delta^\star\partial_t^2 h\right|\leq C+\frac{1}{8}\|\partial_t^2 h\|^2_{L^2(0,t;L^2(\omega))}.$$

The term $\int_0^t\|\Delta\partial_t h\|^2_{L^2(\omega)}$ is bounded in view of (4.35). For the next term, we have due to $H^1(0,T;L^2(\omega))\hookrightarrow L^\infty(0,T;L^2(\omega))$ and (4.35), that

$$\left|\int_\omega\Delta h(t)\Delta\partial_t h(t)\right|\leq\frac{1}{4}\|\Delta\partial_t h(t)\|^2_{L^2(\omega)}+C\|\Delta h(t)\|^2_{L^2(\omega)}\leq\frac{1}{4}\|\Delta\partial_t h(t)\|^2_{L^2(\omega)}+C.$$

Moreover, due to the boundedness of ω,

$$\left|\int_0^t\int_\omega\langle\kappa Gj(\partial_t h),\partial_t^2 h\rangle\right|\leq\int_0^t\int_\omega\kappa\cdot|\partial_t^2 h|\leq C\cdot\kappa\cdot t+\frac{1}{100}\|\partial_t^2 h\|^2_{L^2(0,t;L^2(\omega))}.$$

Finally, the estimate

$$\left|\int_0^t\int_{\omega\times(0,1)}\underline{f}\cdot\sum_{i=1}^{m+n}\dot{\lambda}_i\phi_i 1+\delta^\star\right|\leq C+\frac{1}{8}\left\|\sum_{i=1}^{m+n}\dot{\lambda}_i\phi_i\sqrt{1+\delta^\star}\right\|^2_{L^2(0,t;L^2(\omega))}$$

holds. So, in total, we conclude

$$\left\|\sum_{i=1}^{n+m}\dot{\lambda}_i\phi_i\sqrt{1+\delta^\star}\right\|^2_{L^2(0,t;L^2(\omega\times(0,1)))}+\|\partial_t^2 h\|^2_{L^2(0,t;L^2(\omega))}+\frac{1}{2}\|\Delta\partial_t h(t)\|^2_{L^2(\omega)}\leq C.$$

Step 5: Convergence of $\underline{u_\varepsilon}^{m,n}, h^n$ to a weak solution

We return to the original notation, that is to say we do not skip m, n and ε. Due to $\partial_t \underline{u_\varepsilon}^{m,n} = \sum_{i=1}^{m+n} \dot{\lambda}_i \phi_i + \sum_{i=1}^{m+n} \lambda_i \partial_t \phi_i$, with the second term bounded in $L^2(0,T;L^2(\omega\times(0,1)))$ because of (4.43), and $\sqrt{1+\delta^\star} \geq \sqrt{\frac{\beta}{2}}$ we have shown that

$$\|\partial_t \underline{u_\varepsilon}^{m,n}\|_{L^2(0,T;L^2(\omega\times(0,1)))} + \|\partial_t^2 h_\varepsilon^n\|_{L^2(0,T;L^2(\omega))} \leq \tilde{C}(\varepsilon). \tag{4.44}$$

The bounds from (4.35) and (4.44) imply that there is a subsequence of $((\underline{u_\varepsilon}^{m,n})_{m,n\in\mathbb{N}}, (h_\varepsilon^n)_{n\in\mathbb{N}})$ which we denote by $((\underline{u_\varepsilon}^{m,n})_{m,n\in\mathbb{N}}, (h_\varepsilon^n)_{n\in\mathbb{N}})$ again such that

$$\begin{aligned}
\partial_t \underline{u_\varepsilon}^{m,n} &\rightharpoonup \partial_t \underline{u_\varepsilon} && \text{in } L^2(0,T;L^2(\omega\times(0,1))),\\
\underline{u_\varepsilon}^{m,n} &\rightharpoonup \underline{u_\varepsilon} && \text{in } L^2(0,T;H^1(\omega\times(0,1))),\\
\partial_t^2 h_\varepsilon^n &\rightharpoonup \partial_t^2 h_\varepsilon && \text{in } L^2(0,T;L^2(\omega)),\\
\Delta h_\varepsilon^n &\rightharpoonup \Delta h_\varepsilon && \text{in } H^1(0,T;L^2(\omega)),\\
\partial_t h_\varepsilon^n &\to \partial_t h_\varepsilon && \text{in } L^2(0,T;L^2(\omega)),\\
\partial_t h_\varepsilon^n &\to \partial_t h_\varepsilon && \text{pointwise almost everywhere on } (0,T)\times\omega
\end{aligned} \tag{4.45}$$

as $m,n\to\infty$ for some $(u_\varepsilon, h_\varepsilon) \in L^2(0,T;H^1_{0,\Gamma_0}(\omega\times(0,1))) \times \big(H^2(0,T;L^2(\omega)) \cap H^1(0,T;H^2_0(\omega))\big)$. The last convergence is implied by the second to last (up to a subsequence), which, in turn, is implied by the two preceding ones and by the Aubin-Lions lemma. If we integrate the equations (4.31) and (4.32) with respect to time, we obtain the equations

$$\begin{aligned}
&\int_0^t \int_{\omega\times(0,1)} \partial_t \underline{u_\varepsilon}^{m,n} \cdot \phi_i^{0,\varepsilon}(1+\delta_\varepsilon^\star) - \int_0^t \int_{\omega\times(0,1)} \partial_3 \underline{u_\varepsilon}^{m,n} \cdot \phi_i^{0,\varepsilon} z \partial_t \delta_\varepsilon^\star\\
&+ \mu \int_0^t \int_{\omega\times(0,1)} \nabla \underline{u_\varepsilon}^{m,n}\, A_\varepsilon : \nabla \phi_i^{0,\varepsilon} + \frac{1}{2}\int_0^t \int_{\omega\times(0,1)} (\underline{v_\varepsilon^\star}\cdot(B_\varepsilon\nabla))\underline{u_\varepsilon}^{m,n}\cdot\phi_i^{0,\varepsilon}\\
&- \frac{1}{2}\int_0^t \int_{\omega\times(0,1)} (\underline{v_\varepsilon^\star}\cdot(B_\varepsilon\nabla))\phi_i^{0,\varepsilon}\cdot\underline{u_\varepsilon}^{m,n} = \int_0^t \int_{\omega\times(0,1)} \underline{f}\cdot\phi_i^{0,\varepsilon}(1+\delta_\varepsilon^\star)\ \ \forall i\in\{1,...,m\}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \int_{\omega\times(0,1)} \partial_t \underline{u_\varepsilon}^{m,n} \cdot \phi_j^{1,\varepsilon}(1+\delta_\varepsilon^\star) - \int_0^t \int_{\omega\times(0,1)} \partial_3 \underline{u_\varepsilon}^{m,n} \cdot \phi_j^{1,\varepsilon} z \partial_t \delta_\varepsilon^\star\\
&+ \mu \int_0^t \int_{\omega\times(0,1)} \nabla \underline{u_\varepsilon}^{m,n}\, A_\varepsilon : \nabla \phi_j^{1,\varepsilon} + \frac{1}{2}\int_0^t \int_{\omega\times(0,1)} (\underline{v_\varepsilon^\star}\cdot(B_\varepsilon\nabla))\underline{u_\varepsilon}^{m,n}\cdot\phi_j^{1,\varepsilon}\\
&- \frac{1}{2}\int_0^t \int_{\omega\times(0,1)} (\underline{v_\varepsilon^\star}\cdot(B_\varepsilon\nabla))\phi_j^{1,\varepsilon}\cdot\underline{u_\varepsilon}^{m,n} + \frac{1}{2}\int_0^t \int_\omega \partial_t h_\varepsilon^n(\partial_t\delta_\varepsilon^\star)\xi_j + \int_0^t\int_\omega (\partial_t^2 h_\varepsilon^n)\xi_j\\
&+ \int_0^t \int_\omega (\Delta\partial_t h_\varepsilon^n)\Delta\xi_j + \int_0^t\int_\omega \Delta h_\varepsilon^n \Delta\xi_j + \int_0^t\int_\omega \kappa\langle G j_\varepsilon(\partial_t h_\varepsilon^n), \xi_j\rangle = \int_0^t\int_{\omega\times(0,1)} \underline{f}\cdot\phi_j^{1,\varepsilon}(1+\delta_\varepsilon^\star)
\end{aligned}$$

for all $j\in\{1,...,n\}$. Using the above convergences we can send m and n to infinity. For the convergence

$$\int_0^t\int_\omega \langle \kappa G j_\varepsilon(\partial_t h_\varepsilon^n), \xi_j\rangle \longrightarrow \int_0^t\int_\omega \langle \kappa G j_\varepsilon(\partial_t h_\varepsilon), \xi_j\rangle \ \text{ as } n\to\infty$$

for all $j \in \mathbb{N}$ we use that $|Gj_\varepsilon(\partial_t h_\varepsilon^n)| \leq 1$, the convergence of $(\partial_t h_\varepsilon^n)_{n\in\mathbb{N}}$ to $\partial_t h_\varepsilon$ pointwise almost everywhere on $(0,T)\times\omega$ and the dominated convergence theorem.

For the interface condition we note that by construction, see (4.33), we have

$$\underline{u_\varepsilon}^{m,n}(t,x,y,1) = (0,0,\partial_t h_\varepsilon^n(t,x,y))^T \text{ almost everywhere on } (0,T)\times\omega \ \ \forall m,n\in\mathbb{N}.$$

The first two convergences in (4.45) imply via the Aubin-Lions lemma that, as m and n go to infinity, we have

$$\underline{u_\varepsilon}^{m,n} \to \underline{u_\varepsilon} \text{ in } L^2(0,T;H^{3/4}(\omega\times(0,1)))$$

and thus

$$\underline{u_\varepsilon}^{m,n}_{|\omega\times\{1\}} \to \underline{u_\varepsilon}_{|\omega\times\{1\}} \text{ in } L^2(0,T;L^2(\omega)).$$

We already know that $\partial_t h_\varepsilon^n \to \partial_t h_\varepsilon$ in $L^2(0,T;L^2(\omega))$ as $n\to\infty$. Therefore, by possibly choosing a suitable subsequence, we deduce

$$\underline{u_\varepsilon}(t,x,y,1) = (0,0,\partial_t h_\varepsilon(t,x,y))^T \text{ almost everywhere on } (0,T)\times\omega.$$

For the initial values, we have by design for $m,n\to\infty$

$$\begin{aligned}\underline{u_\varepsilon}^{m,n}(0) = \underline{u_{\varepsilon,0}}^{m,n} &\longrightarrow \underline{u_0}^{\varepsilon} \text{ in } L^2(\omega\times(0,1))\\ \partial_t h_\varepsilon^n(0) = h_{\varepsilon,1}^n &\longrightarrow h_\varepsilon^1 \qquad \text{in } L^2(\omega).\end{aligned}$$

Due to $H^1(0,T;L^2(X)) \hookrightarrow C([0,T];L^2(X))$ with $X\in\{\omega,\omega\times(0,1)\}$, we deduce that $\underline{u_\varepsilon}(0) = \underline{u_0}^{\varepsilon}$ and $h_\varepsilon(0) = h_0^\varepsilon$.

Now, in order to combine the equations (4.31) and (4.32), and to solve the weak formulation (4.28) on $\omega\times(0,1)$ as a whole, we take a relevant test function $(\underline{\phi_\varepsilon}, b) \in L^2(0,T;H^1_{0,\Gamma_0}(\omega\times(0,1)))\times L^2(0,T;H_0^2(\omega))$, see (4.28).

Let $P_N(b) = \sum_{j=1}^N \tilde{b}_j(t)\xi_j$ denote the projection of b onto $\text{span}(\xi_j)_{j\in\{1,\dots,N\}}$. As before, $\phi_j^{1,\varepsilon}$ denotes the solution of the stationary Stokes-like system (4.29) and $\phi_b^{1,\varepsilon}$ denotes the weak solution to the similar system but with $(0,0,b)^T$ instead of $(0,0,\xi_j)^T$. Due to the linearity of this system and due to the continuous dependence of the solution on the data, we get

$$\left\| \sum_{j=1}^N \tilde{b}_j\phi_j^{1,\varepsilon} - \phi_b^{1,\varepsilon} \right\|_{L^2(0,T;H^1(\omega\times(0,1)))} \leq C\|P_N(b)-b\|_{L^2(0,T;H^{1/2}(\omega))} \to 0 \quad \text{as } N\to\infty.$$

This enables us to multiply (4.32) with $\tilde{b}_j(t)$, integrate in time, take the limit m and n to infinity for the solution, as justified above, take the sum over $j\in\{1,\dots,N\}$ and then take the limit N to infinity for the test function. This allows us to basically replace ξ_j with b and $\phi_j^{1,\varepsilon}$ with $\phi_b^{1,\varepsilon}$ in (4.32).

Then, we consider the Dirichlet part $\phi^0 := \underline{\phi_\varepsilon} - \phi_b^{1,\varepsilon}$ which satisfies $\phi^0 = 0$ on $\partial(\omega\times(0,1))$ and $\text{div}(B_\varepsilon^T\phi^0) = 0$. For this reason, we can consider the projection of $P_M(\phi^0) = \sum_{i=1}^M \tilde{\beta}_i(t)\phi_i^{0,\varepsilon}$ onto $\text{span}(\phi_i^{0,\varepsilon})_{i\in\{1,\dots,M\}}$. Analogously to before, we multiply (4.31) with $\tilde{\beta}_i(t)$, integrate over time, take the limit m and n to infinity for the solution, take the sum over $i\in\{1,\dots,M\}$ and then take the limit M to infinity for the test function.

Next using that $\underline{\phi_\varepsilon} = \phi^0 + \phi_b^{1,\varepsilon}$ we can add the two resulting equations to get that $(\underline{u_\varepsilon}, h_\varepsilon)$ solves (4.24). Since the diffeomorphism χ_ε is smooth, we obtain a weak solution for the problem (4.24), formulated on $\Omega_{\delta_\varepsilon^\star}$, by setting $(u_\varepsilon, h_\varepsilon) = (\underline{u}_\varepsilon \circ \chi_\varepsilon^{-1}, h_\varepsilon)$.

Moreover, weak solutions of (4.24) are unique. For the proof, take two solutions $(u_\varepsilon^1, h_\varepsilon^1), (u_\varepsilon^2, h_\varepsilon^2)$ of (4.24) to the same given data and initial values. In the equation that is solved by the difference $(u_\varepsilon^1 - u_\varepsilon^2, h_\varepsilon^1 - h_\varepsilon^2)$ we test with the difference itself. After integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_{\delta_\varepsilon^\star(t)}} |(u_\varepsilon^1 - u_\varepsilon^2)(t)|^2 + \mu \int_0^t \int_{\Omega_{\delta_\varepsilon^\star(s)}} |\nabla(u_\varepsilon^1 - u_\varepsilon^2)|^2 \\
&+ \frac{1}{2}\int_\omega |\partial_t(h_\varepsilon^1 - h_\varepsilon^2)(t)|^2 + \frac{1}{2}\int_\omega |\Delta(h_\varepsilon^1 - h_\varepsilon^2)(t)|^2 + \int_0^t \int_\omega |\Delta(h_\varepsilon^1 - h_\varepsilon^2)|^2 \\
&+ \int_0^t \int_\omega \kappa \left\langle Gj_\varepsilon(\partial_t h_\varepsilon^1) - Gj_\varepsilon(\partial_t h_\varepsilon^2), \partial_t h_\varepsilon^1 - \partial_t h_\varepsilon^2 \right\rangle = 0.
\end{aligned}$$

Since Gj_ε is the derivative of a convex function, it is monotonically increasing. Therefore, every term on the right-hand side is non-negative and we conclude that $u_\varepsilon^1 = u_\varepsilon^2$ and $\partial_t h_\varepsilon^1 = \partial_t h_\varepsilon^2$. The latter implies $h_\varepsilon^1 = h_\varepsilon^2$ due to $h_1(0) = h_2(0)$.

Step 6: The fixed point problem of F_ε

To sum up: For a pair (δ, v) with $\delta \in H^1(0,T; C^0(\overline{\omega}) \cap H_0^1(\omega))$, $M \geq 1 + \delta \geq \beta > 0$ on $[0,T] \times \overline{\omega}$ and $v \in L^2(0,T; L^2(B_{2M}))$ we have shown the existence of a unique solution $(u_\varepsilon, h_\varepsilon)$ to the almost linearised problem (4.24). This solution satisfies the estimates, see a priori estimate (4.27),

$$\begin{aligned}
&\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_{\delta_\varepsilon^\star(t)}))} + \|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\Omega_{\delta_\varepsilon^\star(t)}))} + \|\partial_t h_\varepsilon\|_{L^\infty(0,T;L^2(\omega))} \\
&+ \|\Delta h_\varepsilon\|_{H^1(0,T;L^2(\omega))} \leq C(T, \|u_0\|_{L^2(\Omega_{h_0})}, \|f\|_{L^2((0,T)\times\mathbb{R}^3)}, \|h_0\|_{H_0^2(\omega)}, \|h_1\|_{L^2(\omega)}).
\end{aligned} \tag{4.46}$$

with a constant $C > 0$ independent of $\varepsilon > 0$ and

$$\|\partial_t \underline{u_\varepsilon}\|_{L^2(0,T;L^2(\omega\times(0,1)))} + \|\partial_t^2 h_\varepsilon\|_{L^2(0,T;L^2(\omega))} \leq \tilde{C}(\varepsilon), \tag{4.47}$$

which is due to (4.44), the corresponding weak convergences in (4.45) and the weakly lower semi-continuity of the norm. For $T^* \in (0,T]$ let

$$Y := H^1(0,T^*; C^0(\overline{\omega}) \cap H_0^1(\omega)) \times L^2(0,T^*; L^2(B_{2M})).$$

We consider the solution operator F_ε:

$$\begin{aligned}
F_\varepsilon : B_M^{\beta,\varepsilon} := \left\{(\delta, v) \in Y \middle| \ \|(\delta, v)\|_Y \leq C_M, \beta \leq 1 + \delta(t,x,y) \leq M \text{ on } [0,T^*] \times \overline{\omega}, \delta_{|t=0} = h_0^\varepsilon\right\} &\to Y \\
(\delta, v) &\mapsto (h_\varepsilon, \overline{u}_\varepsilon)
\end{aligned}$$

with $(u_\varepsilon, h_\varepsilon)$ denoting the solution for the data (δ, v) and with

$$\overline{u}_\varepsilon = \begin{cases} u_\varepsilon & \text{in } \Omega_{\delta_\varepsilon^\star(t)}, \\ (0,0,\partial_t h_\varepsilon)^T & \text{in } B_{2M} \backslash \Omega_{\delta_\varepsilon^\star(t)}. \end{cases}$$

It is easy to check that $B_M^{\beta,\varepsilon}$ is non-empty, closed and convex. If we can show that $F_\varepsilon(B_M^{\beta,\varepsilon}) \subset B_M^{\beta,\varepsilon}$, that $F_\varepsilon(B_M^{\beta,\varepsilon})$ is relatively compact and that F_ε is continuous, we get the existence of at least one

fixed point by Schauder's fixed-point theorem, see [21, Corollary 10.2]. Note that $\beta > 0$ was already chosen such that $\min_{\overline{w}}(1 + h_0) \geq 2\beta > 0$, but M and T^* still have to be chosen large and small enough, respectively, such that F_ε satisfies the mentioned properties, if C_M is chosen large enough, as well.

6.1. F_ε is a self-map

First, we want to show that $F_\varepsilon(B_M^{\beta,\varepsilon}) \subset B_M^{\beta,\varepsilon}$ if $M > 0$ is large enough. Due to (4.46) and $H^1(0,T;H^2(\omega)) \hookrightarrow C^0([0,T] \times \overline{\omega})$, we conclude that

$$\sup_{[0,T]\times\overline{\omega}} (1 + h_\varepsilon) \leq M,$$

if M is chosen large enough in accordance with the given data. Furthermore, it is easy to see that

$$\|\overline{u}_\varepsilon\|_{L^2(0,T;L^2(B_{2M}))} \leq \|u_\varepsilon\|_{L^2(0,T;L^2(\Omega_{\delta_\varepsilon^\star(t)}))} + 2M\|\partial_t h_\varepsilon\|_{L^2(0,T;L^2(\omega))} \overset{(4.46)}{\leq} C_1 \cdot M.$$

Due to $h_\varepsilon \in H^1(0,T;H^2(\omega)) \hookrightarrow H^1(0,T;C^0(\overline{\omega}))$, we also get $\|h_\varepsilon\|_{H^1(0,T;C^0(\overline{\omega})\cap H_0^1(\omega))} \leq C_2$ from (4.46). Hence,

$$\|(h_\varepsilon, \overline{u}_\varepsilon)\|_Y \leq C_1 \cdot M + C_2 =: C_M.$$

Since h_ε is bounded in $H^1(0,T;H^2(\omega))$ by a constant independent of ε, $H^1(0,T;H^2(\omega)) \hookrightarrow C([0,T] \times \overline{\omega})$ and $\min_{\overline{\omega}}(1 + h_0^\varepsilon) \geq \frac{3}{2}\beta$, see (4.19), there is a time $T^* > 0$ such that

$$\beta \leq \min_{[0,T^*]\times\overline{\omega}} (1 + h_\varepsilon)$$

for $\varepsilon > 0$ small enough. In total, we have verified that $F_\varepsilon(B_M^{\beta,\varepsilon}) \subset B_M^{\beta,\varepsilon}$.

Step 6.2: $F_\varepsilon(B_M^{\beta,\varepsilon})$ is relatively compact

Now we show that $F_\varepsilon(B_M^{\beta,\varepsilon}) \subset Y$ is relatively compact. Let $(\delta_n, v_n)_{n\in\mathbb{N}} \subset B_M^{\beta,\varepsilon}$ and let $(h_\varepsilon^n, u_\varepsilon^n) = F_\varepsilon(\delta_n, v_n)$ denote the solution to (4.24). Via the estimates (4.46) and (4.47), we get that $(\partial_t h_\varepsilon^n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T^*;H_0^2(\omega))$ and $(\partial_t^2 h_\varepsilon^n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T^*;L^2(\omega))$. Since $H_0^2(\omega)$ is compactly embedded in $H_0^1(\omega)\cap C^0(\overline{\omega})$, we get that $(\partial_t h_\varepsilon^n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T^*;H^1(\omega)\cap C^0(\overline{\omega}))$. Moreover, we have that $\left(\underline{u_\varepsilon}^n\right)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T^*;H^1(\omega\times(0,1)))$ and $\left(\partial_t\underline{u_\varepsilon}^n\right)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T^*;L^2(\omega\times(0,1)))$. Hence, $\underline{u_\varepsilon}^n$ is relatively compact in $L^2(0,T^*;L^2(\omega\times(0,1)))$ and we denote the limit of the corresponding subsequence with $\underline{u_\varepsilon}$.

Since $(\delta_n, v_n) \in B_M^{\beta,\varepsilon}$ we have for the corresponding regularization $\delta_{n,\varepsilon}^\star$ of δ_n, see (4.21) for the definition,

$$\begin{aligned}\|\delta_{n,\varepsilon}^\star\|_{H^1(0,T^*;H^1(\omega))} &= \|N_\varepsilon(\delta_n)\|_{H^1(0,T^*;H^1(\omega))} \\ &\leq \|S_\varepsilon(\delta_n - h_0^\varepsilon) - (\delta_n - h_0^\varepsilon)\|_{H^1(0,T^*;H^1(\omega))} + \|\delta_n\|_{H^1(0,T^*;H^1(\omega))} \leq \varepsilon + C_M\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, due to the compact embedding $H^1(0,T^*;H^1(\omega)) \hookrightarrow L^2((0,T^*)\times\omega)$ we get that there is a subsequence of $(\delta_{n,\varepsilon}^\star)_{n\in\mathbb{N}}$, that we assume to be the sequence itself, that converges pointwise almost everywhere on $(0,T^*)\times\omega$ to an element $\tilde{\delta}_\varepsilon \in H^1(0,T^*;H^1(\omega))$ as n goes to infinity. Due to

$$2M \geq 1 + \delta_{n,\varepsilon}^\star \geq \frac{\beta}{2} \text{ on } [0,T]\times\overline{\omega},$$

see (4.22), we also have the same estimate for the limit $\tilde{\delta}_\varepsilon$. Now let

$$\chi_\varepsilon^n(t) : \omega \times (0,1) \longrightarrow \Omega_{\delta^\star_{n,\varepsilon}(t)},\ (x,y,z) \mapsto (x,y,z(1+\delta^\star_{n,\varepsilon}(t,x,y)))^T \ \ \forall t \in (0,T^*)$$

and $\chi_\varepsilon(t)$ is defined similarly but with $\tilde{\delta}_\varepsilon$ instead of $\delta^\star_{n,\varepsilon}$. We want to show that

$$\chi_{\Omega_{\delta^\star_{n,\varepsilon}(t)}}(\underline{u_\varepsilon} \circ (\chi_\varepsilon^n)^{-1}) \to \chi_{\Omega_{\tilde{\delta}_\varepsilon(t)}}(\underline{u_\varepsilon} \circ \chi_\varepsilon^{-1}) \text{ in } L^2(0,T^*;L^2(B_{2M})) \text{ as } n \to \infty. \tag{4.48}$$

Concerning the convergence of the norms, we have that

$$\begin{aligned}
&\int_{(0,T)\times B_{2M}} |\chi_{\Omega_{\delta^\star_{n,\varepsilon}(t)}}(\underline{u_\varepsilon} \circ (\chi_\varepsilon^n)^{-1})|^2 = \int_{(0,T)\times \Omega_{\delta^\star_{n,\varepsilon}(t)}} |\underline{u_\varepsilon} \circ (\chi_\varepsilon^n)^{-1}|^2 \\
&= \int_{(0,T)\times \Omega_{\tilde{\delta}_\varepsilon(t)}} |\underline{u_\varepsilon} \circ (\chi_\varepsilon)^{-1}|^2 \det(\nabla(\chi_\varepsilon^n \circ \chi_\varepsilon^{-1})) = \int_{(0,T)\times \Omega_{\tilde{\delta}_\varepsilon(t)}} \left|\underline{u_\varepsilon} \circ (\chi_\varepsilon)^{-1}\right|^2 \frac{1+\delta^\star_{n,\varepsilon}}{1+\tilde{\delta}_\varepsilon}.
\end{aligned}$$

Due to the pointwise convergence of $\left(\delta^\star_{n,\varepsilon}\right)_{n\in\mathbb{N}}$ to $\tilde{\delta}_\varepsilon$ a.e. on $(0,T)\times\omega$ as $n \to \infty$ and

$$\frac{1+\delta^\star_{n,\varepsilon}}{1+\tilde{\delta}_\varepsilon} \leq \frac{2M}{\beta/2},$$

we get by the dominated convergence theorem that for $n \to \infty$

$$\begin{aligned}
&\int_{(0,T^*)\times B_{2M}} |\chi_{\Omega_{\delta^\star_{n,\varepsilon}(t)}}(\underline{u_\varepsilon} \circ (\chi_\varepsilon^n)^{-1})|^2 \to \int_{(0,T^*)\times \Omega_{\tilde{\delta}_\varepsilon(t)}} \left|\underline{u_\varepsilon} \circ (\chi_\varepsilon)^{-1}\right|^2 \\
&= \int_{(0,T^*)\times B_{2M}} |\chi_{\Omega_{\tilde{\delta}_\varepsilon(t)}}(\underline{u_\varepsilon} \circ \chi_\varepsilon^{-1})|^2.
\end{aligned}$$

For the weak convergence let $\varphi \in C_c^\infty((0,T)\times B_{2M})$ be arbitrary. Then we calculate

$$\begin{aligned}
&\int_{(0,T^*)\times B_{2M}} \langle \chi_{\Omega_{\delta^\star_{n,\varepsilon}(t)}}(\underline{u_\varepsilon} \circ (\chi_\varepsilon^n)^{-1}), \varphi\rangle \\
&= \int_{(0,T^*)\times \Omega_{\tilde{\delta}_\varepsilon}} \langle \underline{u_\varepsilon} \circ (\chi_\varepsilon)^{-1}, \varphi \circ (\chi_\varepsilon^n \circ \chi_\varepsilon^{-1})\rangle \det(\nabla(\chi_\varepsilon^n \circ \chi_\varepsilon^{-1})).
\end{aligned}$$

Applying the dominated convergence theorem again, yields the desired weak convergence. In total, we obtain the strong convergence (4.48). Hence, (4.48) in combination with

$$\begin{aligned}
&\|\chi_{\Omega_{\delta^\star_{n,\varepsilon}(t)}} u_\varepsilon^n - \chi_{\Omega_{\delta^\star_{n,\varepsilon}(t)}} \underline{u_\varepsilon} \circ (\chi_\varepsilon^n)^{-1}\|_{L^2(0,T^*;L^2(B_{2M}))} = \|u_\varepsilon^n - \underline{u_\varepsilon} \circ (\chi_\varepsilon^n)^{-1}\|_{L^2(0,T^*;L^2(\Omega_{\delta^\star_{n,\varepsilon}(t)}))} \\
&= \left\|(\underline{u_\varepsilon}^n - \underline{u_\varepsilon})\sqrt{1+\delta^\star_{n,\varepsilon}}\right\|_{L^2(0,T^*;L^2(\omega\times(0,1)))} \leq \sqrt{2M}\|\underline{u_\varepsilon}^n - \underline{u_\varepsilon}\|_{L^2(0,T^*;L^2(\omega\times(0,1)))} \to 0 \text{ as } n \to \infty
\end{aligned}$$

implies that $(\chi_{\Omega_{\delta^\star_{n,\varepsilon}(t)}} u_\varepsilon^n)_{n\in\mathbb{N}}$ converges in $L^2(0,T^*;L^2(B_{2M}))$. Therefore, since $\left(\chi_{\Omega_{\delta^\star_{n,\varepsilon}(t)}} u_\varepsilon^n\right)$ is relatively compact in $L^2(0,T^*;L^2(B_{2M}))$ and $(\partial_t h_\varepsilon^n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T^*;H_0^1(\omega)\cap C^0(\omega))$, as previously shown, we also deduce that $(\overline{u}_\varepsilon^n)_{n\in\mathbb{N}}$ is relatively compact in $L^2((0,T^*)\times B_{2M})$. This implies that $F_\varepsilon(B_M^{\beta,\varepsilon}) \subset Y$ is relatively compact.

Step 6.3: F_ε is continuous

Lastly, we prove that F_ε is continuous. So let $B_M^{\beta,\varepsilon} \supseteq (\delta_n, v_n) \to (\delta, v) \in B_M^{\beta,\varepsilon}$ in Y as $n \to \infty$. We want to show that any subsequence of $(F_\varepsilon(\delta_n, v_n))_{n\in\mathbb{N}}$ has a subsequence that converges to $F_\varepsilon(\delta, v)$, which implies that $(F_\varepsilon(\delta_n, v_n))_{n\in\mathbb{N}}$ converges to $F_\varepsilon(\delta, v)$. Since any subsequence of $(\delta_n, v_n)_{n\in\mathbb{N}}$ converges to (δ, v) in Y, too, it is enough to consider only the (sub)sequence $(F_\varepsilon(\delta_n, v_n))_{n\in\mathbb{N}}$.

We have already shown that a subsequence of $F_\varepsilon(\delta_n, v_n) =: (h_\varepsilon^n, \overline{u}_\varepsilon^n)$, which we assume to coincide with the sequence itself again, converges to an element $(\tilde{h}_\varepsilon, \tilde{\overline{u}}_\varepsilon)$ in Y. Moreover, since $(h_\varepsilon^n, \overline{u}_\varepsilon^n)$ is a solution of the almost linearised system (4.24), it is also bounded due to the a priori estimates (4.46) and (4.47). This also implies that (up to a subsequence) $(h_\varepsilon^n, \overline{u}_\varepsilon^n)_{n\in\mathbb{N}}$ converges weakly in $H^1(0,T^*; H_0^2(\omega)) \times L^2(0,T^*; H^1(B_{2M}))$ and $(\partial_t^2 h_\varepsilon^n, \chi_{\Omega_{\delta^\star_{n,\varepsilon}(t)}} \partial_t u_\varepsilon^n)$ converges weakly in $L^2(0,T^*; L^2(\omega)) \times L^2(0,T; L^2(B_{2M}))$.

So, we want to show that $(\tilde{h}_\varepsilon, \tilde{\overline{u}}_\varepsilon) = F_\varepsilon(\delta, v)$. That is to say $(\tilde{h}_\varepsilon, \tilde{\overline{u}}_\varepsilon)$ is a solution to the almost linearised problem (4.24) associated with (δ, v). Since the solution to this problem is unique, this indeed implies $(\tilde{h}_\varepsilon, \tilde{\overline{u}}_\varepsilon) = F_\varepsilon(\delta, v)$. It is easy to see that the mentioned convergences imply

$$\begin{aligned} \tilde{\overline{u}}_\varepsilon &= (0,0,\partial_t \tilde{h}_\varepsilon)^T && \text{in } B_{2M}\backslash\Omega_{\delta_\varepsilon^\star(t)}, \\ \tilde{\overline{u}}_\varepsilon(t,x,y,1+\delta_\varepsilon^\star(t,x,y)) &= (0,0,\partial_t \tilde{h}_\varepsilon(t,x,y))^T && \text{on } [0,T^*]\times\omega, \\ \operatorname{div} \tilde{\overline{u}}_\varepsilon &= 0 && \text{in } B_{2M} \end{aligned}$$

Now, we consider the system solved by $(h_\varepsilon^n, \overline{u}_\varepsilon^n)$: For all $\phi_\varepsilon^n \in V_{\delta_\varepsilon^\star}$, $b \in L^2(0,T^*; H_0^2(\omega))$ such that $\phi_\varepsilon(t,x,y,1+\delta_{n,\varepsilon}^\star(t,x,y)) = (0,0,b(t,x,y))^T$ on $[0,T^*]\times\omega$ we have

$$\begin{aligned} &\int_0^t \int_{\Omega_{\delta^\star_{n,\varepsilon}(s)}} \partial_t u_\varepsilon^n \cdot \phi_\varepsilon^n + \mu \int_0^t \int_{\Omega_{\delta^\star_{n,\varepsilon}(s)}} \nabla u_\varepsilon^n : \nabla \phi_\varepsilon^n + \frac{1}{2}\int_0^t \int_{\Omega_{\delta^\star_{n,\varepsilon}(s)}} (v_{n,\varepsilon}^\star \cdot \nabla) u_\varepsilon^n \cdot \phi_\varepsilon^n \\ &-\frac{1}{2}\int_0^t \int_{\Omega_{\delta^\star_{n,\varepsilon}(s)}} (v_{n,\varepsilon}^\star \cdot \nabla)\phi_\varepsilon^n \cdot u_\varepsilon^n + \frac{1}{2}\int_0^t \int_\omega \partial_t h_\varepsilon^n (\partial_t \delta_{n,\varepsilon}^\star) b + \int_0^t \int_\omega \partial_t^2 h_\varepsilon^n b + \int_0^t \int_\omega \Delta \partial_t h_\varepsilon^n \Delta b \\ &+\int_0^t \int_\omega \Delta h_\varepsilon^n \Delta b + \int_0^t \int_\omega \langle \kappa G j_\varepsilon(\partial_t h_\varepsilon^n), b\rangle = \int_0^t \int_{\Omega_{\delta^\star_{n,\varepsilon}(s)}} f \cdot \phi_\varepsilon^n. \end{aligned} \tag{4.49}$$

We want to consider the limit as n goes to infinity, but the test functions depend on n. In order to overcome this problem, we first consider the test function

$$\phi_\varepsilon^0 \in C_c^\infty\left(\bigcup_{t\in(0,T^*)} \{t\}\times\Omega_{\delta_\varepsilon^\star(t)}\right), \quad \operatorname{div}\phi_\varepsilon^0 = 0.$$

Then, $(\phi_\varepsilon^0, 0)$ is an admissible test function for large n due to the convergence of $(\delta_n)_{n\in\mathbb{N}}$ to δ in $H^1(0,T^*; C^0(\overline{\omega}) \cap H_0^1(\omega))$ as n goes to infinity, which implies uniform convergence and therefore, with the help of (4.20), for n large enough

$$\phi_\varepsilon^0(t,x,y,1+\delta_{n,\varepsilon}^\star(x,y)) = 0 \text{ on } \omega.$$

For an element $b \in L^2(0,T^*; H_0^2(\omega))$ with $\int_\omega b = 0$ there exists a $z \in L^2(0,T^*; H^1(\omega\times(0,\frac{\beta}{2}))$ such that

$$\begin{aligned} \operatorname{div} z &= 0 && \text{in } (0,T^*)\times\omega\times\left(0,\frac{\beta}{2}\right), \\ z_{|\partial(\omega\times(0,\frac{\beta}{2}))} &= \begin{cases} (0,0,b)^T & \text{on } (0,T^*)\times\omega\times\{\frac{\beta}{2}\}, \\ 0 & \text{else}, \end{cases} \end{aligned} \tag{4.50}$$

see Lemma 1.6.1. Then, we consider

$$\phi_\varepsilon^1(b) := \begin{cases} (0,0,b)^T & \text{on } B_{2M}\backslash(\omega \times (0, \frac{\beta}{2})), \\ z & \text{on } \omega \times (0, \frac{\beta}{2}). \end{cases}$$

The pair $(\phi_\varepsilon^1(b), b)$ is an admissible test function, too, since $1 + \delta_{n,\varepsilon}^\star \geq \frac{\beta}{2}$ on $(0, T^*) \times \omega$ for all $n \in \mathbb{N}$.

For these special choices of test functions $(\phi_\varepsilon^0, 0)$ and $(\phi_\varepsilon^1(b), b)$ that do not depend on n, we can send n to infinity in the equation. Here, we use the convergences of $(h_\varepsilon^n, u_\varepsilon^n)$, as we have done in the finite dimensional case. Let us remark that, in order to go to the limit in the most problematic term $\int_0^t \int_{\Omega_{\delta_{n,\varepsilon}^\star(s)}} (v_{n,\varepsilon}^\star \cdot \nabla) u_\varepsilon^n \cdot \phi_\varepsilon$ we use that, see (4.23) and the remark after,

$$\|v_\varepsilon^\star - v_{n,\varepsilon}^\star\|_{L^\infty((0,t)\times B_{2M})} = \|\varphi_\varepsilon * E_0(v - v_n)\|_{L^\infty((0,t)\times B_{2M})} \leq \|\varphi_\varepsilon\|_{L^2(\mathbb{R}^4)} \|v - v_n\|_{L^2(0,T;B_{2M})} \to 0$$

as $n \to \infty$. So, we obtain that $(\tilde{h}_\varepsilon, \tilde{\overline{u}}_\varepsilon)$ is a solution to the limit problem—at least for the chosen set of test functions.

For a general test function $(\phi, b) \in V_{\delta_\varepsilon^\star} \times L^2(0, T^*; H_0^2(\omega))$ with

$$\phi_\varepsilon(t, x, y, 1 + \delta_\varepsilon^\star(t, x, y)) = (0, 0, b(t, x, y))^T$$

on $[0, T^*] \times \omega$, we use that $(\phi_\varepsilon, b) = ((\phi_\varepsilon - \phi_\varepsilon^1(b)) + \phi_\varepsilon^1(b), 0 + b)$ and the first summand $\phi_\varepsilon - \phi_\varepsilon^1(b)$ can be approximated by functions of the same type as ϕ_ε^0. This shows that F_ε is continuous. Therefore, by Schauder's fixed-point theorem F_ε has at least one fixed point $(h_\varepsilon, \overline{u}_\varepsilon)$.

To avoid possible confusion concerning the expression $h_\varepsilon^\star$ in the following we want to make it clear that, now that we have found a fixed point, this term equals $h_\varepsilon^\star = N_\varepsilon(h_\varepsilon)$ with h_ε as one component of the fixed point $(h_\varepsilon, \overline{u}_\varepsilon)$. Hence, $h_\varepsilon^\star$ is not to be understood as $N_\varepsilon(h)$ with h denoting a limit of $(h_\varepsilon)_{\varepsilon>0}$. Similarly, $u_\varepsilon^\star = R_\varepsilon(\overline{u}_\varepsilon)$.

Step 7: The limit for $\varepsilon \to 0$

In order to take the limit $\varepsilon \to 0$, we make the following claim:

For all $\tau > 0$ small enough, we have that

$$\int_0^{T^*} \int_{B_{2M}} \chi_{\Omega_{h_\varepsilon^\star(t)}} |\overline{u}_\varepsilon(t, \cdot) - \overline{u}_\varepsilon(t - \tau, \cdot)|^2 \, dt + \int_0^{T^*} \int_\omega (\partial_t h(t, \cdot) - \partial_t h_\varepsilon(t - \tau, \cdot))^2 \, dt \leq C\sqrt{\tau} \tag{4.51}$$

and

$$\int_0^{T^*} \int_{B_{2M}} |\chi_{\Omega_{h_\varepsilon^\star(t)}} \overline{u}_\varepsilon(t, \cdot) - \chi_{\Omega_{h_\varepsilon^\star(t-\tau)}} \overline{u}_\varepsilon(t - \tau, \cdot)|^2 \, dt \leq C\sqrt{\tau} \tag{4.52}$$

with $C > 0$ independent of ε and where $h_\varepsilon(t) := h_0^\varepsilon$, $\partial_t h_\varepsilon(t) := 0$ and $\overline{u}_\varepsilon(t) := 0$ if $t < 0$.

Since the proof of this claim is almost word for word the same as the one in [10], Lemma 9, we only treat here the new, additional term

$$\int_0^{T^*} \int_\omega \langle G j_\varepsilon(\partial_t h_\varepsilon^n), b \rangle.$$

The key idea in the proof is to test the equation solved by $(u_\varepsilon, h_\varepsilon)$ with the special test function (ϕ_ε, b) depending on the solution:

$$(\phi_\varepsilon, b) = \left(\int_{t-\tau}^{t} (((1+\delta)(\overline{u}_\varepsilon)_1, (1+\delta)(\overline{u}_\varepsilon)_2, (\overline{u}_\varepsilon)_3)(s,x,y,(1+\delta)z))^T \, ds, \int_{t-\tau}^{t} \partial_t h_\varepsilon(s,x,y) \, ds \right)$$

for some $\delta > 0$. After inserting this test function, the resulting terms are estimated by bounds of the form $C\sqrt{\tau}$ with C independent of ε. Here, the only new term is $\int_0^{T^*} \int_\omega \langle G j_\varepsilon(\partial_t h_\varepsilon^n), \int_{t-\tau}^{t} \partial_t h_\varepsilon(s,x,y) \, ds \rangle$, which we can estimate as follows

$$\begin{aligned} \left| \int_0^{T^*} \int_\omega \langle G j_\varepsilon(\partial_t h_\varepsilon^n), \int_{t-\tau}^{t} \partial_t h_\varepsilon(s,x,y) \, ds \rangle \right| &\leq \int_0^{T^*} \int_\omega |G j_\varepsilon(\partial_t h_\varepsilon^n)| \left(\int_0^{T^*} |\partial_t h_\varepsilon(s,x,y)|^2 \, ds \right)^{1/2} \sqrt{\tau} \\ &\leq C T^* \|\partial_t h_\varepsilon\|_{L^2(0,T^*;L^2(\omega))} \sqrt{\tau} \leq C(T^*)\sqrt{\tau}, \end{aligned}$$

where we used that $\|\partial_t h_\varepsilon\|_{L^2(0,T^*;L^2(\omega))}$ is bounded independently of ε, see (4.46). For the estimation of the other terms see [10].

In the next step we use the Fréchet-Kolmogorov theorem. Here (4.51) and (4.52) imply the equicontinuity of $(\partial_t h_\varepsilon)_{\varepsilon>0}$ and $\left(\chi_{\Omega_{h_\varepsilon^\star(t)}} \overline{u}_\varepsilon\right)_{\varepsilon>0}$ in $L^2(0,T^*;L^2(\omega))$ and $L^2(0,T^*;L^2(B_{2M}))$, respectively. Since $(0,T^*)\times\omega$ and $(0,T^*)\times B_{2M}$ are bounded, the equitightness is trivial. Thus we deduce that $(\partial_t h_\varepsilon)_{\varepsilon>0}$ is relatively compact in $L^2(0,T^*;L^2(\omega))$ and $\left(\chi_{\Omega_{h_\varepsilon^\star(t)}} \overline{u}_\varepsilon\right)_{\varepsilon>0}$ is relatively compact in $L^2(0,T^*;L^2(B_{2M}))$. This implies that $(\overline{u}_\varepsilon)_{\varepsilon>0}$ is relatively compact in $L^2(0,T^*;L^2(B_{2M}))$.

From these considerations, as well as from the estimates in (4.46) where the constant does not depend on ε, the following convergences (of a suitable subsequence which we identify with the sequence itself) follow:

$$\begin{aligned} h_\varepsilon &\to h && \text{in } C^0([0,T^*]\times\overline{\omega}), \\ h_\varepsilon &\rightharpoonup h && \text{in } H^1(0,T^*;H_0^2(\omega)), \\ \partial_t h_\varepsilon &\to \partial_t h && \text{in } L^2(0,T^*;L^2(\omega)), \\ \overline{u}_\varepsilon &\to \tilde{u} && \text{in } L^2(0,T^*;L^2(B_{2M})), \\ \chi_{\Omega_{h_\varepsilon^\star(t)}} u_\varepsilon &\to \chi_{\Omega_{h(t)}} \tilde{u} && \text{in } L^2(0,T^*;L^2(B_{2M})), \\ \overline{u}_\varepsilon &\rightharpoonup \tilde{u} && \text{in } L^2(0,T^*;H^1_{0,\Gamma_0}(B_{2M})), \\ h_\varepsilon^\star &\to h && \text{in } C^0([0,T^*]\times\overline{\omega}), \\ \partial_t h_\varepsilon^\star &\to \partial_t h && \text{in } L^2(0,T^*;L^2(\omega)), \\ \overline{u}_\varepsilon^\star &\to \tilde{u} && \text{in } L^2(0,T^*;L^2(B_{2M})) \end{aligned} \tag{4.53}$$

as ε goes to 0 with $(h,\tilde{u})$ denoting the limit. The last three convergences follow from the way we regularized h_ε and u_ε, see (4.20) and (4.23), and from the corresponding convergences without regularization. Furthermore, $(\chi_{\Omega_{h_\varepsilon^\star(t)}} \nabla u_\varepsilon)_{\varepsilon>0}$ converges (up to a subsequence) weakly to a function $q \in (L^2(0,T;L^2(B_{2M})))^{3\times 3}$ due to (4.46) as $\varepsilon \to 0$. Let $\varphi \in (L^2((0,T^*)\times B_{2M}))^{3\times 3}$ such that $\operatorname{supp}\varphi \subset \bigcup_{t\in(0,T^*)} \{t\} \times (B_{2M} \backslash \Omega_{h(t)})$. Then, we have

$$\begin{aligned} &\left| \int_{(0,T^*)\times B_{2M}} \chi_{\Omega_{h_\varepsilon^\star(t)}} \nabla u_\varepsilon : \varphi \right| = \left| \int_{(0,T^*)\times B_{2M}} \chi_{\Omega_{h_\varepsilon^\star(t)}} \nabla u_\varepsilon : \varphi(\chi_{\Omega_{h_\varepsilon^\star(t)}} - \chi_{\Omega_{h(t)}}) \right| \\ &\leq \left\| \chi_{\Omega_{h_\varepsilon^\star(t)}} \nabla u_\varepsilon \right\|_{L^2((0,T^*)\times B_{2M})} \left\| \varphi(\chi_{\Omega_{h_\varepsilon^\star(t)}} - \chi_{\Omega_{h(t)}}) \right\|_{L^2((0,T^*)\times B_{2M})} \\ &\overset{(4.46)}{\leq} C \left\| \varphi(\chi_{\Omega_{h_\varepsilon^\star(t)}} - \chi_{\Omega_{h(t)}}) \right\|_{L^2((0,T^*)\times B_{2M})} \longrightarrow 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

We used the convergence of $(h_\varepsilon^\star)_{\varepsilon>0}$ to h in $C^0([0,T]\times\overline{\omega})$ as $\varepsilon\to 0$. Uniqueness of the limit implies that $q=0$ on $\bigcup_{t\in(0,T^*)}\{t\}\times(B_{2M}\backslash\Omega_{h(t)})$.

Furthermore let $\psi\in L^2(0,T^*;H^1_{0,\Gamma_0}(B_{2M}))$ with $\operatorname{supp}\psi\subset\bigcup_{t\in(0,T^*)}\{t\}\times\Omega_{h(t)}$. In a similar fashion as before, we have

$$\begin{aligned}\left|\int_{(0,T^*)\times B_{2M}}(\nabla\tilde{u}-q):\nabla\psi\right| &\longleftarrow \left|\int_{(0,T^*)\times B_{2M}}(\nabla\overline{u}_\varepsilon-\chi_{\Omega_{h_\varepsilon^\star(t)}}\nabla u_\varepsilon):\nabla\psi\right|\\ &= \left|\int_{(0,T^*)\times B_{2M}}(\nabla\overline{u}_\varepsilon-\chi_{\Omega_{h_\varepsilon^\star(t)}}\nabla\overline{u}_\varepsilon):\nabla\psi\right|\\ &\le \|\nabla\overline{u}_\varepsilon\|_{L^2((0,T^*)\times B_{2M})}\left\||\chi_{\Omega_{h(t)}}-\chi_{\Omega_{h_\varepsilon^\star(t)}}|\,|\nabla\psi|\right\|_{L^2((0,T^*)\times B_{2M})}\\ &\le C\left\||\chi_{\Omega_{h(t)}}-\chi_{\Omega_{h_\varepsilon^\star(t)}}|\,|\nabla\psi|\right\|_{L^2((0,T^*)\times B_{2M})}\to 0\quad\text{as }\varepsilon\to 0.\end{aligned}$$

Therefore, $q=\nabla\tilde{u}$ on $\bigcup_{t\in(0,T^*)}\{t\}\times\Omega_{h(t)}$. So, in total,

$$\chi_{\Omega_{h_\varepsilon^\star(t)}}\nabla u_\varepsilon\rightharpoonup q=\chi_{\Omega_{h(t)}}\nabla\tilde{u}\text{ in }L^2(0,T^*;L^2(B_{2M})).\tag{4.54}$$

Next, we proceed to the limit in the equation

$$u_\varepsilon(t,x,y,1+h_\varepsilon^\star(t,x,y))=(0,0,\partial_t h_\varepsilon(t,x,y))^T\text{ on }[0,T^*]\times\omega.$$

We already know that the right-hand side converges to $(0,0,\partial_t h)^T$ in $L^2(0,T^*;L^2(\omega))$. We consider

$$w_\varepsilon=\begin{cases}(0,0,\partial_t h_\varepsilon)^T & \text{in } B_{2M}\backslash(\omega\times(0,\frac{\beta}{2}))\\ z_{\partial_t h_\varepsilon} & \text{in }\omega\times(0,\frac{\beta}{2})\end{cases}$$

with $z_{\partial_t h_\varepsilon}\in L^2(0,T^*;H^1_{0,\Gamma_0}(\omega\times(0,\frac{\beta}{2})))$ solving (4.50) but with b replaced with $\partial_t h_\varepsilon$. Then, $(\overline{u}_\varepsilon-w_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(0,T^*;H^1(B_{2M}))$ with a constant independent of ε. This and (4.53) imply that a subsequence of $(\overline{u}_\varepsilon-w_\varepsilon)_{\varepsilon>0}$ converges weakly to $w_0:=\tilde{u}-w$ with

$$w:=\begin{cases}(0,0,\partial_t h)^T & \text{in } B_{2M}\backslash(\omega\times(0,\frac{\beta}{2}))\\ z_{\partial_t h} & \text{in }\omega\times(0,\frac{\beta}{2}).\end{cases}$$

in $L^2(0,T^*;H^1(B_{2M}))$. Let $\delta>0$ be arbitrary. Due to the uniform convergence of $(h_\varepsilon^\star)_{\varepsilon>0}$ to h, we get that

$$\exists\tilde{\varepsilon}>0\ \forall\varepsilon\in(0,\tilde{\varepsilon}]\ \forall t\in(0,T^*)\ :\ \Omega_{h(t)+\delta}\supset\Omega_{h_\varepsilon(t)}.$$

This implies that

$$\overline{u}_\varepsilon=(0,0,\partial_t h_\varepsilon)^T\text{ on }B_{2M}\backslash\Omega_{h(t)+\delta}\subset B_{2M}\backslash\Omega_{h_\varepsilon(t)}.$$

for all $\varepsilon\in(0,\tilde{\varepsilon}]$ and $t\in(0,T^*)$. We also have that

$$w_\varepsilon=(0,0,\partial_t h_\varepsilon)^T\text{ on }B_{2M}\backslash\Omega_{h(t)+\delta}\subset B_{2M}\backslash(\omega\times(0,\frac{\beta}{2}))$$

because of $1+h \geq \beta$ on $[0,T^*]\times\omega$. Since $\delta > 0$ was arbitrary, we deduce that $w_0 = 0$ in $\bigcup_{t\in(0,T^*)}\{t\}\times(B_{2M}\backslash\Omega_{h(t)})$. Due to $1+h_\varepsilon^\star \geq \frac{\beta}{2}$ on $[0,T^*]\times\omega$, we also deduce that $w_0 = 0$ on Γ_0. This readily implies that

$$w_0 = 0 \text{ in } \bigcup_{t\in(0,T^*)} \{t\}\times(B_{2M}\backslash\Omega_{h(t)}) \quad \text{and} \quad w_0 \in L^2(0,T^*;H_0^1(\Omega_{h(t)})).$$

Due to Lemma 4.2.2, it follows that $\gamma_{h(t)}(w_0) = 0$. Since $\gamma_{h(t)}(w) = (0,0,\partial_t h)^T$, it must be the case that

$$\gamma_{h(t)}(\tilde{u}) = (0,0,\partial_t h)^T.$$

This shows that the interface condition is satisfied.

Now we take the limit in the equation solved by $(u_\varepsilon, h_\varepsilon)$. Recall that $(u_\varepsilon, h_\varepsilon)$ solves

$$\begin{aligned}
&\int_0^t\int_{\Omega_{h_\varepsilon^\star(s)}} \partial_t u_\varepsilon\cdot\phi_\varepsilon + \mu\int_0^t\int_{\Omega_{h_\varepsilon^\star(s)}} \nabla u_\varepsilon : \nabla\phi_\varepsilon + \frac{1}{2}\int_0^t\int_{\Omega_{h_\varepsilon^\star(s)}} (u_\varepsilon^\star\cdot\nabla)u_\varepsilon\cdot\phi_\varepsilon \\
&-\frac{1}{2}\int_0^t\int_{\Omega_{h_\varepsilon^\star(s)}} (u_\varepsilon^\star\cdot\nabla)\phi_\varepsilon\cdot u_\varepsilon + \frac{1}{2}\int_0^t\int_\omega \partial_t h_\varepsilon(\partial_t h_\varepsilon^\star)b + \int_0^t\int_\omega \partial_t^2 h_\varepsilon b + \int_0^t\int_\omega \Delta\partial_t h_\varepsilon\Delta b \\
&+\int_0^t\int_\omega \Delta h_\varepsilon\Delta b + \int_0^t\int_\omega \langle\kappa G j_\varepsilon(\partial_t h_\varepsilon), b\rangle = \int_0^t\int_{\Omega_{h_\varepsilon^\star(s)}} f\cdot\phi_\varepsilon.
\end{aligned} \tag{4.55}$$

for all $\phi_\varepsilon \in V_{h_\varepsilon^\star}$, $b \in L^2(0,T;H_0^2(\omega))$ such that $\phi_\varepsilon(t,x,y,1+h_\varepsilon^\star(t,x,y)) = (0,0,b(t,x,y))^T$ on $[0,T^*]\times\omega$. Since Gj_ε is the derivative of the convex function $j_\varepsilon : \mathbb{R}\to\mathbb{R},\ x\mapsto\sqrt{x^2+\varepsilon^2}$, we get

$$\langle Gj_\varepsilon(\partial_t h_\varepsilon), -b\rangle = \langle Gj_\varepsilon(\partial_t h_\varepsilon), -b+\partial_t h_\varepsilon - \partial_t h_\varepsilon\rangle \leq \sqrt{(\partial_t h_\varepsilon - b)^2+\varepsilon^2} - \sqrt{\partial_t h_\varepsilon^2+\varepsilon^2}$$

which is equivalent to

$$\sqrt{\partial_t h_\varepsilon^2+\varepsilon^2} - \sqrt{(\partial_t h_\varepsilon - b)^2+\varepsilon^2} \leq \langle Gj_\varepsilon(\partial_t h_\varepsilon), b\rangle\,.$$

If we plug this into (4.55) and use integration by parts with respect to time for the first term and for the term involving $\partial_t^2 h_\varepsilon$, see (4.26) and (4.25) (with δ replaced by h), we end up with the inequality

$$\begin{aligned}
&\int_{\Omega_{h_\varepsilon^\star(t)}} u_\varepsilon(t)\cdot\phi_\varepsilon(t) - \int_0^t\int_{\Omega_{h_\varepsilon^\star(s)}} u_\varepsilon\cdot\partial_t\phi_\varepsilon + \mu\int_0^t\int_{\Omega_{h_\varepsilon^\star(s)}} \nabla u_\varepsilon : \nabla\phi_\varepsilon \\
&+\frac{1}{2}\int_0^t\int_{\Omega_{h_\varepsilon^\star(s)}} (u_\varepsilon^\star\cdot\nabla)u_\varepsilon\cdot\phi_\varepsilon - \frac{1}{2}\int_0^t\int_{\Omega_{h_\varepsilon^\star(s)}} (u_\varepsilon^\star\cdot\nabla)\phi_\varepsilon\cdot u_\varepsilon - \frac{1}{2}\int_0^t\int_\omega \partial_t h_\varepsilon(\partial_t h_\varepsilon^\star)b \\
&+\int_\omega \partial_t h_\varepsilon(t)b(t) - \int_0^t\int_\omega \partial_t h_\varepsilon\partial_t b + \int_0^t\int_\omega \Delta\partial_t h_\varepsilon\Delta b + \int_0^t\int_\omega \Delta h_\varepsilon\Delta b \\
&+\int_0^t\int_\omega \kappa\left(\sqrt{\partial_t h_\varepsilon^2+\varepsilon^2} - \sqrt{(\partial_t h_\varepsilon - b)^2+\varepsilon^2}\right) \\
&\leq \int_0^t\int_{\Omega_{h_\varepsilon^\star(s)}} f\cdot\phi_\varepsilon + \int_\omega h_1^\varepsilon b(0) + \int_{\Omega_{h_0^\varepsilon}} u_0^\varepsilon\phi_\varepsilon(0)
\end{aligned} \tag{4.56}$$

for a.e. $t \in (0,T^*)$ and for all $(\phi_\varepsilon, b) \in \mathcal{V}_{h_\varepsilon^\star} \times C^1([0,T^*]; H_0^2(\omega))$ with $\phi_\varepsilon(t,x,y,1+h_\varepsilon^\star) = (0,0,b)^T$ on $[0,T] \times \omega$.

As in the variational problem (4.49), we have to deal with the problem that the test functions depend on ε. We solve this issue in a similar manner: We consider

$$\phi^0 \in C_c^\infty \left(\bigcup_{t\in(0,T^*)} \{t\} \times \Omega_{h(t)} \right), \quad \operatorname{div} \phi^0 = 0.$$

Then, $(\phi^0, 0)$ is an admissible test function for small ε since $\phi^0 \in \mathcal{V}_{h_\varepsilon^\star}$ for $\varepsilon > 0$ sufficiently small due to the uniform convergence of $h_\varepsilon^\star$ to h as ε goes to 0. For an element $b \in C^1([0,T^*]; H_0^2(\omega))$ with $\int_\omega b = 0$, we use the continuous embedding $H^2(\omega) \hookrightarrow W_4^{\frac{3}{4}}(\omega)$ to deduce that there exists a $z \in L^\infty(0,T^*; W_4^1(\omega \times (0,\frac{\beta}{2})) \subset L^\infty((0,T^*) \times (\omega \times (0,\frac{\beta}{2})))$ such that

$$\begin{aligned} \operatorname{div} z &= 0 && \text{in } (0,T^*) \times \omega \times \left(0, \frac{\beta}{2}\right), \\ z_{|\partial(\omega\times(0,\frac{\beta}{2}))} &= \begin{cases} (0,0,b)^T & \text{on } (0,T^*) \times \omega \times \{\frac{\beta}{2}\}, \\ 0 & \text{else}, \end{cases} \end{aligned}$$

according to Lemma 1.6.1. Then, we consider

$$\phi^1(b) := \begin{cases} (0,0,b)^T & \text{on } B_{2M} \backslash (\omega \times (0,\frac{\beta}{2}))), \\ z & \text{on } \omega \times (0,\frac{\beta}{2}). \end{cases}$$

The pair $(\phi^1(b), b)$ is an admissible test function, too, since $1 + \delta_\varepsilon^\star \geq \frac{\beta}{2}$ on $(0,T^*) \times \omega$ for all $\varepsilon > 0$. For this class of test functions, we can go to the limit in (4.56) due to the convergences (4.53) and (4.54). Here, let us consider the most problematic term

$$\begin{aligned} &\int_0^t \int_{\Omega_{h_\varepsilon^\star(s)}} (u_\varepsilon^\star \cdot \nabla) u_\varepsilon \cdot \phi = \int_0^t \int_{B_{2M}} \chi_{\Omega_{h_\varepsilon^\star(s)}} (u_\varepsilon^\star \cdot \nabla) u_\varepsilon \cdot \phi \\ &= \int_0^t \int_{B_{2M}} \chi_{\Omega_{h_\varepsilon^\star(s)}} (\tilde{u} \cdot \nabla) u_\varepsilon \cdot \phi + \int_0^t \int_{B_{2M}} \chi_{\Omega_{h_\varepsilon^\star(s)}} \left((u_\varepsilon^\star - \tilde{u}) \cdot \nabla\right) u_\varepsilon \cdot \phi. \end{aligned}$$

The first term converges to the correct limit due to (4.54) and the second summand converges to 0 because of the strong convergence $\overline{u}_\varepsilon^\star \to \tilde{u}$ in $L^2(0,T^*; L^2(B_{2M}))$ and the boundedness of $(\chi_{\Omega_{h_\varepsilon^\star(s)}} \nabla u_\varepsilon)_{\varepsilon>0}$ in $L^2(0,T^*; L^2(B_{2M}))$. Note that this argument only works, because $\phi \in L^\infty((0,T^*) \times (\omega \times (0,\frac{\beta}{2})))$; this, in retrospect, explains why the previous divergence problem was solved for an exponent larger than 2. Concerning the term $\int_0^t \int_{\Omega_{h_\varepsilon^\star(s)}} (u_\varepsilon^\star \cdot \nabla)\phi \cdot u_\varepsilon$, we estimate

$$\begin{aligned} &\int_0^t \int_{B_{2M}} \left| \chi_{\Omega_{h_\varepsilon^\star(s)}} (u_\varepsilon^\star \cdot \nabla)\phi \cdot u_\varepsilon - \chi_{\Omega_{h(s)}} (\tilde{u} \cdot \nabla)\phi \cdot \tilde{u} \right| \\ \leq &\int_0^t \int_{B_{2M}} \left| (\chi_{\Omega_{h_\varepsilon^\star(s)}} u_\varepsilon^\star - \chi_{\Omega_{h(s)}} \tilde{u}) \cdot \nabla)\phi \cdot \chi_{\Omega_{h_\varepsilon^\star(s)}} u_\varepsilon \right| + \left| (\chi_{\Omega_{h(s)}} \tilde{u} \cdot \nabla)\phi \cdot (\chi_{\Omega_{h(s)}} \tilde{u} - \chi_{\Omega_{h_\varepsilon^\star(s)}} u_\varepsilon) \right| \\ \leq &\|\chi_{\Omega_{h_\varepsilon^\star(s)}} u_\varepsilon^\star - \chi_{\Omega_{h(s)}} \tilde{u}\|_{L^2((0,t)\times B_{2M})} \|\nabla\phi\|_{L^\infty(0,t;L^4(B_{2M}))} \\ &\cdot \left(\|\overline{u}_\varepsilon\|_{L^2(0,t;L^4(B_{2M}))} + \|\chi_{\Omega_{h(s)}} \tilde{u}\|_{L^2(0,t;L^4(B_{2M}))} \right) \\ &+ \|\chi_{\Omega_{h_\varepsilon^\star(s)}} (u_\varepsilon^\star - u_\varepsilon)\|_{L^2((0,t)\times B_{2M})} \|\nabla\phi\|_{L^\infty(0,t;L^4(B_{2M}))} \|\chi_{\Omega_{h(s)}} u\|_{L^2(0,t;L^4(B_{2M}))}, \end{aligned}$$

which converges to 0 as $\varepsilon \to 0$ due to the boundedness of $(\overline{u}_\varepsilon)_\varepsilon$ in $L^2(0,t;L^4(B_{2M}))$ implied by $(4.53)_6$. For the convergence of the non-linear boundary term, due to

$$\partial_t h_\varepsilon \to \partial_t h \text{ in } L^2(0,T;L^2(\omega)) \hookrightarrow L^1(0,T;L^1(\omega)) \text{ as } \varepsilon \to 0$$

and $\sqrt{x^2+\varepsilon^2} \leq |x| + \varepsilon$ for all $x \in \mathbb{R}$, we also get the convergence of the boundary term

$$\int_0^t \int_\omega \kappa \left(\sqrt{\partial_t h_\varepsilon^2 + \varepsilon^2} - \sqrt{(\partial_t h_\varepsilon - b)^2 + \varepsilon^2} \right) \longrightarrow \int_0^t \int_\omega \kappa(|\partial_t h| - |\partial_t h_\varepsilon - b|) \quad \text{as} \quad \varepsilon \to 0.$$

This shows that $(\tilde{u}, h)$ is a solution of (4.18) for these special test functions.

Now, for a general test function $(\phi, b) \in \mathcal{V}_h \times C^1([0,T^*]; H_0^2(\omega))$ with $\phi(t,x,y,1+h(t,x,y)) = (0,0,b(t,x,y))$ in $[0,T] \times \omega$ we write $\phi = (\phi - \phi^1(b)) + \phi^1(b)$ with $\phi^1(b)$ as before. And since the Dirichlet part $\phi - \phi^1(b)$ can be approximated by functions of the same type as ϕ^0, we can recover the weak formulation (4.18) for general test functions. This shows that $(\tilde{u}, h)$ is a weak solution of (4.1) – (4.8) which concludes the proof. □

Remark 4.4.2. We remark that it if a weak solution is not a global solution, *i.e.* if it cannot be extended past a finite time interval $(0,T')$, then $\lim_{t \to T'} \min_{\overline{\omega}}(1+h) = 0$. That is to say that an intersection of the upper free boundary part with the lower rigid boundary part occurs. A proof of this can be found in [10]. The main ingredient for this proof is the a priori estimate. The a priori estimate in our case, see (4.15), is even slightly better than the one in [10] due to the additional term $\kappa \|\partial_t h\|_{L^1((0,t)\times\omega)}$ on the left-hand side. This is why the proof in our case is identical.

APPENDIX A

Appendix

Here, we give the proof of Proposition 1.5.3.

Proposition A.0.1. *(**Connection with the Euclidean case**) Let $(U_i, x_i)_{i\in\{1,...,m\}}$ be an atlas of M such that $x_i(\bar{U}_i)$ is well-defined for $i \in \{1, ..., m\}$ with positively oriented charts, a partition of unity $(\varphi_i)_{i\in\{1,...,m\}}$ subordinated to them and the canonical isomorphism $l_p : T_pM \to \mathbb{R}^d$ at $p \in M$. Then there holds:*

$$\begin{aligned} u \in W_q^k(M)^d &\Leftrightarrow u \in W_q^k(U_i)^d \quad \forall i \in \{1, ..., m\} \\ u \in W_q^k(U_i)^d &\Leftrightarrow l_{x_i^{-1}} \circ u \circ x_i^{-1} \in W_q^k(x_i(U_i))^d \quad \forall i \in \{1, ..., m\}. \end{aligned}$$

The corresponding operators

$$\begin{aligned} &S_i : L^q(M)^d \to L^q(U_i), \;\; u \to u|_{U_i} \\ &\tilde{S} : L^q(U_1)^d \times ... \times L^q(U_m)^d \to L^q(M)^d, \;\; (u_1, ..., u_m) \mapsto \sum_{i=1}^m \varphi_i u_i \\ &T_i : L^q(U_i)^d \to L^q(x_i(U_i))^d, \;\; u \mapsto l_{x_i^{-1}} \circ u \circ x_i^{-1} \\ &T_i^{-1} : L^q(x_i(U_i))^d \to L^q(U_i)^d, \;\; u \mapsto l_{x_i}^{-1} \circ u \circ x_i \end{aligned}$$

are well-defined, linear and continuous. The same holds for the restricted versions

$$\begin{aligned} &S_{i_{|W_q^k(M)}} \in \mathcal{L}\left(W_q^k(M), W_q^k(U_i)\right), \; \tilde{S}_{|W_q^k(U_1)\times...\times W_q^k(U_m)} \in \mathcal{L}\left(W_q^k(U_1) \times ... \times W_q^k(U_m), W_q^k(M)\right) \\ &T_{i_{|W_q^k(U_i)}} \in \mathcal{L}\left(W_q^k(U_i), W_q^k(x_i(U_i))\right), \; T^{-1}_{i_{|W_q^k(x_i(U_i))}} \in \mathcal{L}\left(W_q^k(x_i(U_i)), W_q^k(U_i)\right) \end{aligned}$$

Proof. In the following we will always make use of Proposition 1.5.2.
First, let $u \in W_q^k(M)^d$ and (U, x) be an arbitrary chart.
The compactness of $\overline{U}$ implies that

$$\exists c, C > 0 \; \forall p \in x\left(\overline{U}\right) : c < \sqrt{\det G\left(x^{-1}(p)\right)} < C \tag{A.1}$$

with $G_{ij}(x) := g_{ij}(x)$ for all $i,j \in \{1,..,d\}, x \in M$. Since the charts are positively oriented

$$\det \nabla \left(x \circ x_l^{-1}(p)\right) > 0 \; \forall p \in x_l\left(\bar{U} \cap \bar{U}_l\right) \; \forall l \in \{1,...,m\}.$$

Once again the compactness of $\bar{U}$ implies that

$$\exists c, C > 0 \; \forall p \in x_l\left(\bar{U} \cap \bar{U}_l\right) : c < \det D\left(x \circ x_l^{-1}(p)\right) < C \; \forall l \in \{1,...,m\}. \tag{A.2}$$

As a first step, we show $u \in L^q(U)$. Indeed,

$$\begin{aligned}
\|u\|_{L^q(U)}^q &= \int_{x(U)} \left|u \circ x^{-1}\right|^q \sqrt{\det G\left(x^{-1}\right)} \overset{(A.1)}{\cong} \int_{x(U)} \left|u \circ x^{-1}\right|^q \\
&= \int_{x(U)} \sum_{l=1}^m \varphi_l \circ x^{-1} \left|u \circ x^{-1}\right|^q \\
&\leq c \sum_{l=1}^m \int_{x_l(U \cap U_l)} \varphi_l \circ x_l^{-1} \left|u \circ x_l^{-1}\right|^q \det \nabla \left(x \circ x_l^{-1}\right) \qquad (A.3) \\
&\overset{(A.2)}{\leq} c \sum_{l=1}^m \int_{x_l(U \cap U_l)} \varphi_l \circ x_l^{-1} \left|u \circ x_l^{-1}\right|^q \leq c \sum_{l=1}^m \int_{x_l(U_l)} \varphi_l \circ x_l^{-1} \left|u \circ x_l^{-1}\right|^q \\
&\overset{(A.1)}{\leq} c \sum_{l=1}^m \int_{x_l(U_l)} \varphi_l \circ x_l^{-1}(x) \left|u \circ x_l^{-1}(x)\right|^q \sqrt{\det G\left(x_l^{-1}(x)\right)} dx = c \|u\|_{L^q(M)}^q .
\end{aligned}$$

Choose $(f_m)_{m \in \mathbb{N}} \subseteq C^\infty(M)$ such that $f_m \to u$ in $L^q(M)$ for $m \to \infty$ and

$$\left\|\nabla^j (f_m - f_n)\right\|_{L^q(M)} \to 0 \text{ for } m,n \to \infty \; \forall j \in \{1,...,k\}.$$

By restriction $(f_m)_{m \in \mathbb{N}} \subseteq C^\infty(U)$ and

$$\|f_m - u\|_{L^q(U)}^q \overset{(A.3)}{\leq} c \|f_m - u\|_{L^q(M)}^q \to 0 \text{ for } m \to \infty$$

as well as

$$\left\| \left| \nabla_U^j (f_m - f_n) \right|_g \right\|_{L^q(U)} \lesssim \left\| \left| \nabla_M^j (f_m - f_n) \right|_g \right\|_{L^q(M)} \to 0 \text{ for } m,n \to \infty$$

where we used $\nabla^j_{M_{|U}} = \nabla^j_U$. Hence $u \in W_q^k(U)$.
Now let $u \in W_q^k(U_j)$ for all $j \in \{1,...,m\}$. Then

$$\|u\|_{L^q(M)}^q \cong \sum_{l=1}^m \int_{x_l(U_l)} \varphi_l \circ x_l^{-1}(x) \left|u \circ x_l^{-1}(x)\right|^q dx \leq C \sum_{l=1}^m \|u\|_{L^q(U_l)}^q.$$

Choose $\left(f_m^l\right)_{m \in \mathbb{N}} \subseteq C^\infty(U_l)$ such that $f_m^l \to u$ in $L^q(U_l)$ for $m \to \infty$ and

$$\left\| \left| \nabla_{U_l}^j \left(f_m^l - f_n^l\right) \right|_g \right\|_{L^q(U_l)} \to 0 \text{ for } m,n \to \infty \; \forall j \in \{1,...,k\}.$$

Set $g_n := \sum_{l=1}^m \varphi_l f_n^l \in C^\infty(M)$ to get that

$$\begin{aligned}\|g_n - u\|_{L^q(M)}^q &= \left\| \sum_{l=1}^m \varphi_l f_n^l - \sum_{l=1}^m \varphi_l u \right\|_{L^q(M)}^q \leq c \sum_{l=1}^m \left\| \varphi_l \left(f_n^l - u \right) \right\|_{L^q(M)}^q \\ &\overset{(A.3)}{\leq} c \sum_{l,j=1}^m \left\| \varphi_l \left(f_n^l - u \right) \right\|_{L^q(U_j)}^q = c \sum_{l,j=1}^m \left\| \varphi_l \left(f_n^l - u \right) \right\|_{L^q(U_j \cap U_l)}^q \\ &\leq c \sum_{l,j=1}^m \left\| \varphi_l \left(f_n^l - u \right) \right\|_{L^q(U_l)}^q \leq cm \sum_{l=1}^m \left\| f_n^l - u \right\|_{L^q(U_l)}^q \to 0 \text{ for } n \to \infty.\end{aligned}$$

Keeping in mind that $\nabla^j_{M_{|U_l}} = \nabla^j_{U_l}$, we obtain for an arbitrary $j \in \{1, ..., k\}$ that

$$\begin{aligned}&\left\| \left| \nabla_M^j (g_m - g_n) \right|_g \right\|_{L^q(M)}^q \\ &\leq c \sum_{l=1}^m \left\| \left| \nabla_M^j (g_m - g_n) \right|_g \right\|_{L^q(U_l)}^q \leq c \sum_{s,l=1}^m \left\| \left| \nabla_M^j \left(\varphi_s \left(f_m^s - f_n^s \right) \right) \right|_g \right\|_{L^q(U_l)}^q \\ &= c \sum_{s,l=1}^m \left\| \left| \nabla_M^j \left(\varphi_s \left(f_m^s - f_n^s \right) \right) \right|_g \right\|_{L^q(U_l \cap U_s)}^q \leq c \sum_{s,l=1}^m \left\| \left| \nabla_M^j \left(\varphi_s \left(f_m^s - f_n^s \right) \right) \right|_g \right\|_{L^q(U_s)}^q \\ &\leq cm \sum_{s=1}^m \sum_{i=0}^j \left\| \left| \nabla_M^i \left(f_m^s - f_n^s \right) \right|_g \right\|_{L^q(U_s)}^q = cm \sum_{s=1}^m \sum_{i=0}^j \left\| \left| \nabla_{U_s}^i \left(f_m^s - f_n^s \right) \right|_g \right\|_{L^q(U_s)}^q \to 0\end{aligned}$$

for $m, n \to \infty$. Hence $u \in W_q^k(M)$.
Locally we have that

$$|v(p)|_{g_p}^2 = \sum_{i,j=1}^n v_i(p) v_j(p) g_{ij}(p) = \langle l_p(v(p)), G(p) l_p(v(p)) \rangle = \left| \sqrt{G(p)} l_p(v(p)) \right|^2$$

is equivalent to $|l_p(v(p))|^2$ such that the constants in this norm equivalence are independent of p, because $\bar{U} \ni p \mapsto \sqrt{G(p)} \in \mathbb{R}^{d \times d}$ and $\bar{U} \ni p \mapsto \sqrt{G^{-1}(p)} \in \mathbb{R}^{d \times d}$ are continuous and $\bar{U}$ is compact. As a consequence, we easily see $u \in L^q(U)$ if and only if $l_{x^{-1}} \circ u \circ x^{-1} \in L^q(x(U))$.
If $k = 1$, then we use additionally

$$|(\nabla_U f) \circ x^{-1}|_{g_{x^{-1}}} = \langle (GF)(x^{-1}), (FG^{-1})(x^{-1}) \rangle \; \forall f \in \mathfrak{X}(U)$$

with $F_{k,l} := \dfrac{\partial f^l}{\partial x^k} + \sum_{s=1}^d f^s \Gamma_{ks}^l$, where Γ denotes the Christoffel symbols and the f^l are the coordinates of f. Using an approximation argument as before and Proposition 1.5.2 we obtain the claim. The assertion for $k \geq 1$ follows inductively. □

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