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Application of a reduced basis method for an efficient treatment of structural mechanics problems

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For numerous problems in structural mechanics, a repeated solution of partial differential equations (PDEs), varying certain input parameters, is necessary. Solving the PDE for a large number of different input parameter sets using a full-dimensional finite element method, requires repeated solving of large systems of equations and, thus, leads to a high computational effort. The aim of model order reduction techniques is to reduce the computational complexity in such calculations. In order to achieve this, the idea of the reduced basis method [1–3] is to replace the high-dimensional model with a lower dimensional model, which is realized by forming a basis of solutions of the full problem for selected parameter sets. Key to determining suitable parameter sets is an appropriate error estimator.

1 Introduction

The reduced basis method (RBM) can be applied to various problems in structural mechanics. For instance, Huynh [4, 5] makes use of the method to calculate stress intensity factors efficiently, and for an application to fracture problems. In this work, the RBM is employed for the optimization of functionally graded adhesive joint designs with regard to the minimization of stress peaks and gradients within the adhesive. The stresses can be calculated rapidly for a large number of material parameter variations so that the Young's modulus distribution within the adhesive is optimized efficiently.

2 Reduced Basis Method for functionally graded single lap joints

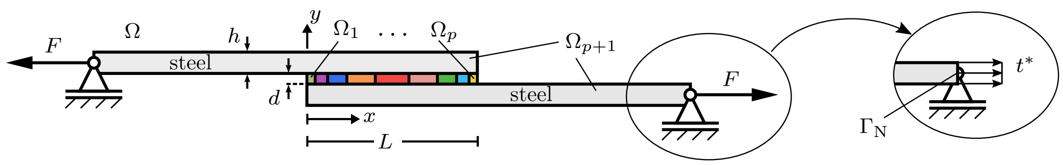

Fig. 1: Single-lap joint with graded adhesive. Domain $\Omega = \bigcup_{q=1}^{p+1} \Omega_q$. Parameters used for the calculation are $F = 500\,\mathrm{N}$, $L = 20\,\mathrm{mm}$, $d = 2\,\mathrm{mm}$, $h = 5\,\mathrm{mm}$, plane strain with unit out-of-plane width, $E_{\text{steel}} = 210\,\mathrm{GPa}$, $E_{\text{adhesive}} \in [2500\,\mathrm{MPa}, 6500\,\mathrm{MPa}]$, $\nu = 0.3$.

At the bi-material notch of adhesive joints, high stress concentrations (singularities) are expected. The state of the art is the gradation of the adhesive in order to reduce stress peaks. Specifically, this requires a repeated solution of a partial differential equation (PDE) for different Young's modulus distributions within the adhesive. Let $\boldsymbol{\mu} = [E_1, E_2, \dots, E_p] \in \mathcal{P}$ be the parameter vector, where E_q is the Young's modulus in domain Ω_q and $\mathcal{P} := [E_{\text{min}}, E_{\text{max}}]^p$ is the set of possible parameter combinations. Then the weak form of the parametrized PDE reads:

$$\text{Find } \boldsymbol{u} \in X \quad \text{with} \quad a(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}) = f(\boldsymbol{v}; \boldsymbol{\mu}) \qquad \forall\, \boldsymbol{v} \in X, \tag{1}$$

$$\text{where} \quad a(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}) = \int_\Omega \left(\underline{\boldsymbol{C}}(\boldsymbol{\mu}) : \boldsymbol{\varepsilon}(\boldsymbol{u})\right) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}\Omega \quad \text{and} \quad f(\boldsymbol{v}) = \int_{\Gamma_{\mathrm{N}}} \boldsymbol{t}^* \cdot \boldsymbol{v}\, \mathrm{d}\Gamma_{\mathrm{N}}. \tag{2}$$

An important requirement for the application of the RBM is the parameter separability of the bilinear and linear form in (2). The linear form $f(\cdot)$ does not depend on $\boldsymbol{\mu}$ and the bilinear form can be written as

$$a(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}) = \sum_{q=1}^{p+1} E_{\mathrm{q}} \int_{\Omega_{\mathrm{q}}} \left(\underline{\boldsymbol{C}}^{\text{red}} : \boldsymbol{\varepsilon}(\boldsymbol{u})\right) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}\Omega_{\mathrm{q}}. \tag{3}$$

Then the high-fidelity discretization reads $\mathbf{A}_H(\boldsymbol{\mu})\mathbf{u}_H(\boldsymbol{\mu}) = \mathbf{f}_H$ with $\mathbf{A}_H(\boldsymbol{\mu}) = \sum_{q=1}^{p+1} E_{\mathrm{q}}\, \mathbf{A}_H^q$, where $\mathbf{A}_H^q$ is parameter-independent and has to be assembled only once. Using that, solution snapshots $\{\mathbf{u}_H(\boldsymbol{\mu}^{(1)}), \dots, \mathbf{u}_H(\boldsymbol{\mu}^{(N)})\}$ are generated, where $\boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(N)}$ are chosen by evaluating an efficient a-posteriori error estimator. The reduced system matrix is obtained from the projection: $\mathbf{A}_N(\boldsymbol{\mu}) = \boldsymbol{\Phi}_N^T \mathbf{A}_H(\boldsymbol{\mu}) \boldsymbol{\Phi}_N$, where $\boldsymbol{\Phi}_N$ contains coefficients of snapshots in terms of high-dimensional basis. Then, the low dimensional RB system reads $\mathbf{A}_N(\boldsymbol{\mu})\mathbf{u}_N(\boldsymbol{\mu}) = \mathbf{f}_N$, where $N \ll H$.

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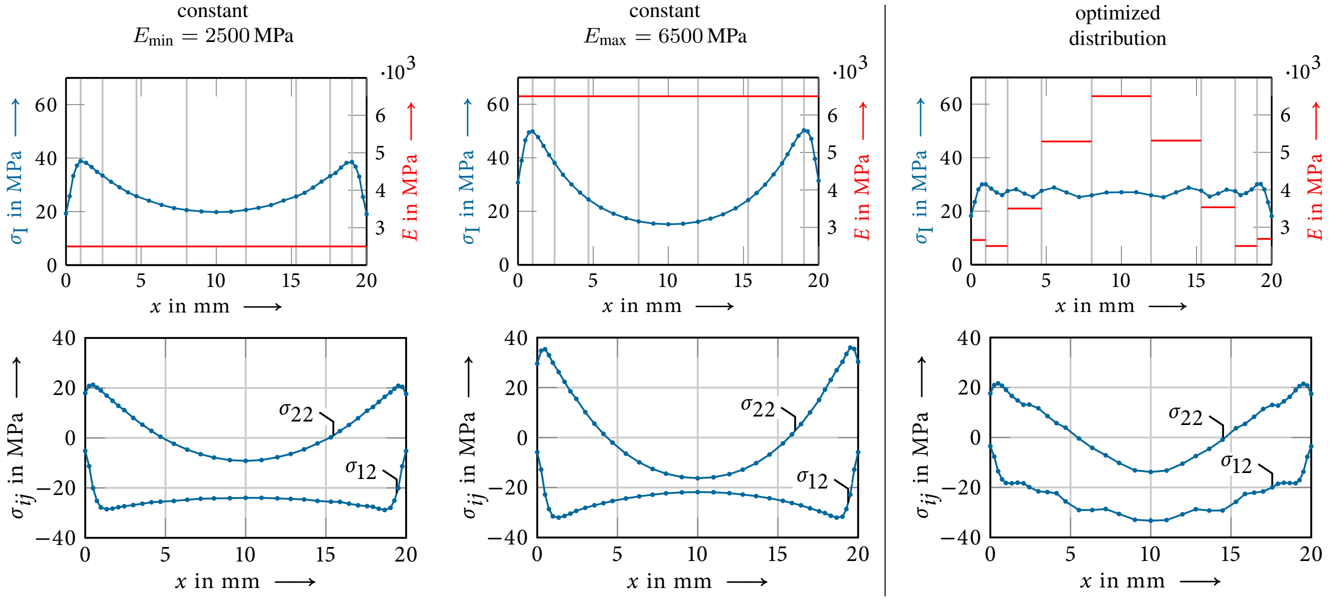

Fig. 2: Stress curves σ_{12}, σ_{22} and maximum principal stress σ_{I} (left axes) for constant and optimized distribution of Young's moduli (right axes) within the adhesive. The stresses are evaluated along the vertical center of the adhesive.

3 Results and discussion

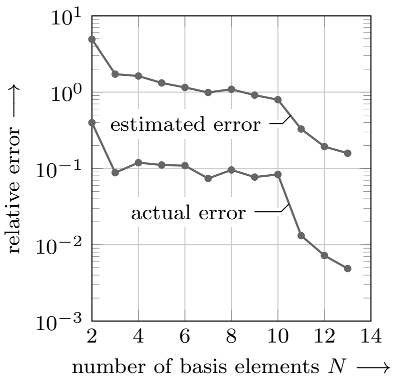

Fig. 3: Estimated and actual error.

Fig. 2 shows the shear and normal stresses σ_{12} and σ_{22} as well as the maximum principal stress $\sigma_{\mathrm{I}} := \frac{\sigma_{22}}{2} + \sqrt{\left(\frac{\sigma_{22}}{2}\right)^2 + (\sigma_{12})^2}$ along the length of the adhesive. The adhesive is divided into $p = 9$ subdomains, which equals the length of the parameter vector $\boldsymbol{\mu} = [E_1, ..., E_9]$. The first and second column show the stress curves for constant Young's moduli of the adhesive, specifically the lower and upper bound of the accepted interval $\mathcal{P}$. The third column shows the optimized distribution of Young's moduli and the associated stress curves. The maximum principal stress σ_{I}, in regard to which the optimization has been performed, is much smoother than in the constant cases. Especially, the peaks near the bi-material notches are reduced considerably.
Within the optimization, the repeated solution of the parametrized PDE has been performed using the RBM. Fig. 3 shows the development of the error estimator and the actual error of the RBM in comparison to a high-dimensional finite element method (FEM) with $H = 955$ degrees of freedom. It can be seen that the error estimator works well as an upper bound for the actual error and the error decreases by increasing the number of basis elements within the reduced basis (RB). It is remarkable that a RB with dimension $N = 13$ yields a sufficiently small error, so that only 13 high dimensional FEMs are necessary for a varying parameter-set $\boldsymbol{\mu}$ of 9 Young's moduli. Once the RB has been created, only linear equation systems of dimension 13×13 have to be solved for each desired parameter configuration.

4 Conclusion and outlook

The Reduced Basis Method is very efficient for the repeated solution of parametrized PDE. The idea is to create a model with reduced dimension instead of solving the full-dimensional model frequently. The method works well for the optimization of the stress distribution in functionally graded adhesive joints. In further works, the transfer to other parametrized structural mechanics problems is conceivable. For example, fracture mechanics problems with crack length Δa as parameter could be treated by transformation from a parameter independent to a dependent domain $\Omega \rightarrow \Omega(\Delta a)$.

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