Received: 20 April 2020

DOI: 10.1002/mma.7084

RESEARCH ARTICLE

WILEY

On the transport limit of singularly perturbed convection–diffusion problems on networks

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Communicated by: S. Nicaise

Funding information
German Research Foundation (DFG), Grant/Award Number: TRR146 project C03 and TRR154 project C04; Center for Computational Engineering at TU Darmstadt

We consider singularly perturbed convection–diffusion equations on one-dimensional networks (metric graphs) as well as the transport problems arising in the vanishing diffusion limit. Suitable coupling conditions at inner vertices are derived that guarantee conservation of mass and dissipation of a mathematical energy which allows us to prove stability and well-posedness. For single intervals and appropriately specified initial conditions, it is well-known that the solutions of the convection–diffusion problem converge to that of the transport problem with order $O(\sqrt{\epsilon})$ in the $L^\infty(L^2)$-norm with diffusion $\epsilon \to 0$. In this paper, we prove a corresponding result for problems on one-dimensional networks. The main difficulty in the analysis is that the number and type of coupling conditions changes in the singular limit which gives rise to additional boundary layers at the interior vertices of the network. Since the values of the solution at these network junctions are not known a priori, the asymptotic analysis requires a delicate choice of boundary layer functions that allows to handle these interior layers.

KEYWORDS

asymptotic analysis, diffusion and convection (76R05), partial differential equations on networks, singular perturbations in the context of PDEs (35B25)

MSC CLASSIFICATION

35B25; 35K20; 35R02; 76M45

1 | INTRODUCTION

The transport and diffusion of a chemical substance in the stationary flow of an incompressible fluid through a pipe can be described by

$$a\partial_t u_\epsilon(x,t) + b\partial_x u_\epsilon(x,t) = \epsilon\partial_{xx} u_\epsilon(x,t), \tag{1}$$

which is assumed to hold for $x \in (0,\ell)$ and $t > 0$. Here, u is the concentration of the substance, a and ℓ are the cross-section and length of the pipe, b is the constant flow rate, and $\epsilon > 0$ is the diffusion coefficient. The system is complemented by boundary conditions

$$u_\epsilon(0,t) = \hat{u}_\epsilon^0(t) \quad \text{and} \quad u_\epsilon(\ell,t) = \hat{u}_\epsilon^\ell(t), \tag{2}$$

and by specifying $u_\epsilon(x,0)$ at initial time $t = 0$. A typical application we have in mind is some contaminant transported by water flowing through a pipe network. Then, u denotes the concentration of the contaminant and b the flow rate of the background flow.[1] Another application would be the mixing of gas in a gas transport network[2]; then, b is the volume flow rate of the steady gas flow, and u is the volume fraction of one of the components in the gas mixture.

In the vanishing diffusion limit $\epsilon \to 0$, the flow of the substance in the fluid is characterized by the transport equation

$$a\partial_t u(x,t) + b\partial_x u(x,t) = 0. \tag{3}$$

Assuming $b > 0$, this system is to be complemented by an inflow boundary condition

$$u(0,t) = \hat{u}^0(t) \quad \text{at} \;\; x = 0, \tag{4}$$

while the condition at $x = \ell$ becomes obsolete. For small $\epsilon > 0$, the second boundary condition in (2) therefore gives rise to a boundary layer at the outflow boundary $x = \ell$. In general, the solutions of (1)–(2) may also exhibit initial layers, whose presence can however be avoided by appropriate specification of initial values.

The asymptotic limit of convection–diffusion problems as $\epsilon \to 0$ has been studied intensively in the literature, both from an analytical and a numerical point of view; for details, one may refer, for example, to previous studies.[3-7] Problems with other types of boundary conditions have been considered, for example, in Chacón Rebollo et al.[8] For appropriate initial and boundary data, the solutions of (1)–(2) and (3)–(4) can be shown to satisfy the asymptotic estimate

$$\|u_\epsilon(\cdot,t) - u(\cdot,t)\|_{L^2(0,\ell)} \le C\sqrt{\epsilon}, \tag{5}$$

with a constant C independent of ϵ and t. By considering the corresponding stationary problem, the rate $\sqrt{\epsilon}$ can also be seen to be optimal.

In this paper, we consider convection–diffusion problems in one-dimensional pipe networks. In that case, Equations (1) and (3) are assumed to hold for every single pipe while the boundary conditions (2) and (4) have to be augmented by appropriate coupling conditions at pipe junctions. These can be chosen in order to guarantee conservation of mass across network junctions as well as dissipation of a mathematical energy, which is utilized to ensure the well-posedness of the problems. We refer to previous studies[9-12] for background material on the analysis of partial-differential equations on networks.

The main result of our paper, stated in Theorem 10, is to show that an estimate analogous to (5) also holds for singularly perturbed convection–diffusion problems on networks. One of the main difficulties in the asymptotic analysis here is that the number and type of coupling conditions changes in the singular limit $\epsilon \to 0$. This gives rise to additional internal layers at pipe junctions that need to be handled appropriately. Since the nodal values $\hat{u}_\epsilon$, $\hat{u}$ in Equations (2) and (4) are part of the solution and not prescribed a priori, like the boundary values on a single pipe, a somewhat delicate choice of boundary layer functions at network junctions is required.

The remainder of the manuscript is organized as follows: in Section 2, we introduce our basic notation and then study the convection–diffusion and the transport problem on networks. The choice of suitable coupling conditions ensures conservation of mass at network junctions and dissipation of a mathematical energy, which in turn allows us to establish well-posedness of the problems by semigroup theory. In Section 3, we state and prove our main result, namely, a quantitative estimate similar to (5) for the convergence of solutions to the convection–diffusion problem with vanishing diffusion $\epsilon \to 0$ towards that of the corresponding transport problem. The presentation closes with a short summary.

2 | NOTATION AND PRELIMINARIES

After introducing our basic notation, we formally state the convection–diffusion and the limiting transport problem and study their well-posedness.

2.1 | Basic notation

Following Egger and Kugler,[13] the network is represented by a finite, directed, and connected graph with vertices $\mathcal{V} = \{v_1, \ldots, v_n\}$ and edges $\mathcal{E} = \{e_1, \ldots, e_m\} \subset \mathcal{V} \times \mathcal{V}$. We do not assume a specific topology, and, in particular, circles are admissible. For every edge $e = (v_i, v_j)$, we define two numbers

$$n^e(v_i) = -1 \;\; \text{and} \;\; n^e(v_j) = 1 \tag{6}$$

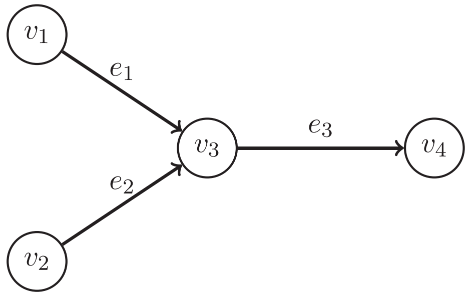

FIGURE 1 A network with three edges $e_1 = (v_1, v_3)$, $e_2 = (v_2, v_3)$, and $e_3 = (v_3, v_4)$; inner vertex $\mathcal{V}_0 = \{v_3\}$; and boundary vertices $\mathcal{V}_\partial = \{v_1, v_2, v_4\}$. The set $\mathcal{E}(v_3) = \{e_1, e_2, e_3\}$ denotes the edges adjacent to the junction v_3. Let the arrows depict the flow direction. Then, we split the set of boundary vertices by $\mathcal{V}_\partial^{in} = \{v_1, v_2\}$ and $\mathcal{V}_\partial^{out} = \{v_4\}$ into inflow and outflow vertices. Similarly, we can split the set $\mathcal{E}(v_3)$ by $\mathcal{E}^{in}(v_3) = \{e_1, e_2\}$ and $\mathcal{E}^{out}(v_3) = \{e_3\}$ into edges that go into or out of the vertex v_3

to indicate the start and end point of the edge, and we set $n^e(v) = 0$ if $v \in \mathcal{V} \backslash \{v_i, v_j\}$. For any $v \in \mathcal{V}$, we define the set of incident edges $\mathcal{E}(v) := \{e \in \mathcal{E} : n^e(v) \neq 0\}$ and distinguish between inner vertices $\mathcal{V}_0 := \{v \in \mathcal{V} : |\mathcal{E}(v)| \geq 2\}$ and boundary vertices $\mathcal{V}_\partial := \mathcal{V} \backslash \mathcal{V}_0$; see Figure 1 for an illustration.

Every edge $e \in \mathcal{E}$ has a positive length ℓ^e, and we identify e with the interval $(0, \ell^e)$. The Lebesgue measure on $(0, \ell^e)$ then induces a metric on e, and we denote by $L^2(e) = L^2(0, \ell^e)$ the space of square integrable functions on the edge e. We further use

$$L^2(\mathcal{E}) = L^2(e_1) \times \ldots \times L^2(e_m) = \{u : u^e \in L^2(e) \text{ for all } e \in \mathcal{E}\}$$

to denote the space of square integrable functions on the network. Here and below, $u^e = u|_e$ is the restriction of a function u defined on the whole network to a single edge e. The natural norm and scalar product of the space $L^2(\mathcal{E})$ are given by

$$\|u\|^2_{L^2(\mathcal{E})} = \sum\nolimits_{e \in \mathcal{E}} \|u^e\|^2_{L^2(e)} \quad \text{and} \quad (u, w)_{L^2(\mathcal{E})} = \sum\nolimits_{e \in \mathcal{E}} (u^e, w^e)_{L^2(e)}.$$

We will further make use of the broken Sobolev spaces

$$H^s_{pw}(\mathcal{E}) = \{u \in L^2(\mathcal{E}) : u^e \in H^s(e) \text{ for all } e \in \mathcal{E}\},$$

which are again equipped with the canonical norms and scalar products, defined by

$$\|u\|^2_{H^s_{pw}(\mathcal{E})} = \sum\nolimits_{e \in \mathcal{E}} \|u^e\|^2_{H^s(e)} \quad \text{and} \quad (u, w)_{H^s_{pw}(\mathcal{E})} = \sum\nolimits_{e \in \mathcal{E}} (u^e, w^e)_{H^s(e)}.$$

Here, $H^s(e) \simeq H^s(0, \ell^e)$, $s \geq 0$, are the usual Sobolev spaces on the interval $(0, \ell^e)$ and $\|\cdot\|_{H^s(e)}$ are the canonical norms; see Evans[14] for details. Note that for $s > 1/2$, the functions $u \in H^s_{pw}(\mathcal{E})$ are continuous along edges $e \in \mathcal{E}$, while they may be discontinuous across junctions $v \in \mathcal{V}_0$. The subspace of functions that are also continuous across junctions is denoted by $H^1(\mathcal{E})$. Elements of $H^1(\mathcal{E})$ have a unique value $u(v)$ for every vertex $v \in \mathcal{V}$, and we write $\ell_2(\mathcal{V})$ for the set of possible vertex values.

2.2 | Convection–diffusion problem

We now formally introduce the convection–diffusion problem on networks to be studied, as well as our basic assumptions on the model parameters. A similar problem has been considered in Oppenheimer;[1] also see Mugnolo[12] for further examples.

The transport of the substance along every edge $e \in \mathcal{E}$ shall be described by

$$a^e \partial_t u^e_\epsilon(x, t) + b^e \partial_x u^e_\epsilon(x, t) - \epsilon^e \partial_{xx} u^e_\epsilon(x, t) = 0, \quad x \in e, \, e \in \mathcal{E}, \tag{7}$$

where a^e, b^e, and ϵ^e are appropriate constants; see Assumption 1 below. We further assume the concentration u to be continuous across vertices, that is,

$$u^e_\epsilon(v, t) = \hat{u}^v_\epsilon(t), \quad v \in \mathcal{V}, \, e \in \mathcal{E}(v), \tag{8}$$

for some auxiliary functions $\hat{u}^v_\epsilon(t)$, $v \in \mathcal{V}$ to be determined by the following additional coupling conditions: At pipe junctions $v \in \mathcal{V}_0$, we require that

$$\sum\nolimits_{e \in \mathcal{E}(v)} \left(b^e u^e_\epsilon(v, t) - \epsilon^e \partial_x u^e_\epsilon(v, t)\right) n^e(v) = 0, \qquad v \in \mathcal{V}_0, \tag{9}$$

which expresses the conservation of mass at pipe junctions, and at boundary vertices $v \in \mathcal{V}_\partial$, we explicitly prescribe the concentration by

$$\hat{u}_\epsilon^v(t) = g^v(t), \quad v \in \mathcal{V}_\partial, \tag{10}$$

with g^v denoting the specified boundary values. The above equations are considered for $t > 0$ and complemented by initial conditions

$$u_\epsilon^e(x,0) = u_0^e(x), \quad x \in e, e \in \mathcal{E}. \tag{11}$$

For the analysis of the convection–diffusion problem (7)–(11) which is developed in the rest of the paper, we make the following assumptions on the model parameters.

Assumption 1. On every edge $e \in \mathcal{E}$, the functions a, b, and ϵ are constant and uniformly positive, and at pipe junctions $v \in \mathcal{V}_0$, the flow rate satisfies the conservation condition

$$\sum\nolimits_{e \in \mathcal{E}(v)} b^e n^e(v) = 0, \qquad v \in \mathcal{V}_0, \tag{12}$$

which corresponds to incompressibility of the background flow. We further assume that the diffusion coefficient is bounded by $0 < \epsilon \leq 1$.

Remark 2. The assumption that a and ϵ are piecewise constant could be relaxed with minor changes in the arguments. Since the flow direction changes when changing the orientation of the edge e, the sign of b can always be adopted as desired by appropriate orientation of the edges. The basic assumption on b, therefore, is that it does not vanish. Otherwise, the transport problem (3) degenerates to an ordinary differential equation.

The following theorem establishes well-posedness of the problem under consideration.

Theorem 3. *Let Assumption 1 hold and $T > 0$. Then, for any $u_0 \in H^1(\mathcal{E}) \cap H^2_{pw}(\mathcal{E})$ and $g \in C^2([0,T];\ell_2(\mathcal{V}_\partial))$ satisfying (8)–(10) for some $\hat{u}_0 \in \ell_2(\mathcal{V})$, the system (7)–(11) has a unique classical solution*

$$u_\epsilon \in C^1([0,T];L^2(\mathcal{E})) \cap C^0([0,T];H^1(\mathcal{E}) \cap H^2_{pw}(\mathcal{E}))$$

with $\hat{u}_\epsilon^v(t) = u_\epsilon(v,t)$ defined by (8). Moreover, any solution of (7)–(11) satisfies

$$\frac{d}{dt}\int_{\mathcal{E}} a u_\epsilon \, dx = \sum_{v \in \mathcal{V}_\partial} \left(-b^e g^v + \epsilon^e \partial_x u_\epsilon(v)\right) n^e(v),$$

that is, mass is conserved up to flow over the boundary, as well as the energy identity

$$\frac{1}{2}\frac{d}{dt}\|a^{1/2}u_\epsilon\|^2_{L^2(\mathcal{E})} = -\|\epsilon^{1/2}\partial_x u_\epsilon\|^2_{L^2(\mathcal{E})} + \sum_{v \in \mathcal{V}_\partial} \left(-\frac{1}{2}b^e g^v + \epsilon^e \partial_x u_\epsilon(v)\right) g^v n^e(v).$$

Proof. For later reference, we sketch the main arguments, which allow to apply the Lumer–Phillips theorem of semigroup theory; related results can also be found in previous studies.[12,15,16]

Step 1 *(Homogenization of boundary values).* Let $w(t) \in H^1(\mathcal{E}) \cap H^2_{pw}(\mathcal{E})$ be the unique function that is affine linear on every edge and satisfies $w(v,t) = g^v(t)$ for all $v \in \mathcal{V}_\partial$ as well as $w(v,t) = 0$ for $v \in \mathcal{V}_0$. Then, any solution of the problem can be split into $u_\epsilon = w - z$ with $z(v,t) = 0$ for all $v \in \mathcal{V}_\partial$, $t > 0$, and using the linearity of the problem, one can see that z satisfies

$$a^e \partial_t z^e + b^e \partial_x z^e - \epsilon^e \partial_{xx} z^e = f^e, \quad x \in e, e \in \mathcal{E}, t > 0, \tag{13}$$

with right-hand side $f^e = a^e \partial_t w^e + b^e \partial_x w^e$, as well as the coupling conditions

$$z^e(v,t) = \hat{z}^v(t), \quad v \in \mathcal{V}, e \in \mathcal{E}(v), t > 0. \tag{14}$$

The auxiliary functions $\hat{z}^v(t) = \hat{u}^v(t)$ are here defined by the conservation condition

$$\sum\nolimits_{e\in\mathcal{E}(v)} \left(b^e z^e(v,t) - \epsilon^e \partial_x z^e(v,t)\right) n^e(v) = 0, \quad v \in \mathcal{V}_0, \tag{15}$$

at pipe junctions $v \in \mathcal{V}_0$, and by homogeneous boundary conditions

$$\hat{z}^v(t) = 0, \quad v \in \mathcal{V}_\partial, \tag{16}$$

for the remaining vertices $v \in \mathcal{V}_\partial$. In addition, there holds

$$z^e(x,0) = z_0^e(x), \quad x \in e,\, e \in \mathcal{E}, \tag{17}$$

with $z_0^e(x) = w^e(x,0) - u_0^e(x)$. Let us note that by construction and the regularity assumption on u_0, we have $z_0 \in H^1(\mathcal{E}) \cap H^2_{pw}(\mathcal{E})$ and $z_0(v) = 0$ for all $v \in \mathcal{V}_\partial$.

Step 2 *(Generation of a contraction semigroup).* Now, set $\mathcal{X} = L^2(\mathcal{E})$ with norm and scalar product defined by

$$\|u\|_{\mathcal{X}} := \|a^{1/2}u\|_{L^2(\mathcal{E})} \quad \text{and} \quad (u,w)_{\mathcal{X}} := (au,w)_{L^2(\mathcal{E})}.$$

We further introduce the dense subspace

$$\mathcal{D}(\mathcal{A}_\epsilon) := \{z \in H^2_{pw}(\mathcal{E}) : z \text{ satisfies (14)–(16) with some } \hat{z} \in \ell_2(\mathcal{V})\},$$

on which we formally define the linear operator

$$\mathcal{A}_\epsilon : \mathcal{D}(\mathcal{A}_\epsilon) \subset \mathcal{X} \to \mathcal{X}, \qquad \mathcal{A}_\epsilon z|_e := -\frac{1}{a^e}\left(b^e \partial_x z^e - \epsilon^e \partial_{xx} z^e\right). \tag{18}$$

Problem (13)–(17) can then be written as an abstract evolution problem in $\mathcal{X}$, namely,

$$\partial_t z(t) = \mathcal{A}_\epsilon z(t) + f(t), \quad t > 0, \tag{19}$$

$$z(0) = z_0. \tag{20}$$

By construction of f and z_0 and the assumptions on the data, one can immediately see that $f \in C^1([0,T];\mathcal{X})$ and $z_0 \in \mathcal{D}(\mathcal{A}_\epsilon)$. Moreover, the operator $\mathcal{A}_\epsilon$ satisfies

$$\begin{aligned}(\mathcal{A}_\epsilon z, z)_{\mathcal{X}} &= (-b\partial_x z + \epsilon \partial_{xx} z, z)_{L^2(\mathcal{E})} = \sum\nolimits_{e\in\mathcal{E}} (-b^e \partial_x z^e + \epsilon^e \partial_{xx} z^e, z^e)_{L^2(e)} \\ &= \sum\nolimits_{e\in\mathcal{E}} (b^e z^e - \epsilon^e \partial_x z^e, \partial_x z^e)_{L^2(e)} + \sum\nolimits_{v\in\mathcal{V}} \sum\nolimits_{e\in\mathcal{E}(v)} \left(-b^e z^e(v) + \epsilon^e \partial_x z^e(v)\right) z^e(v) n^e(v).\end{aligned}$$

The first term in the last line can be estimated by

$$\begin{aligned}(i) &= \sum\nolimits_{e\in\mathcal{E}} (b^e z^e - \epsilon^e \partial_x z^e, \partial_x z^e)_{L^2(e)} \\ &= \sum\nolimits_{v\in\mathcal{V}} \sum\nolimits_{e\in\mathcal{E}(v)} \frac{1}{2} b^e |z^e(v)|^2 n^e(v) - \sum\nolimits_{e\in\mathcal{E}} \epsilon^e \|\partial_x z^e\|^2_{L^2(e)} = (iii) + (iv).\end{aligned}$$

By rearranging the order of summation and use of the coupling and boundary conditions specified in (14)–(16) as well as the conservation condition (12) for the flow rates, one can see that $(iii) = \frac{1}{2}\sum_{v\in\mathcal{V}} |\hat{z}^v|^2 \sum_{e\in\mathcal{E}(v)} b^e n^e(v) = 0$, and hence,

$$(i) = (iv) = -\|\epsilon^{1/2} \partial_x z\|^2_{L^2(\mathcal{E})}.$$

The second term in the above expression for $(\mathcal{A}_\epsilon z, z)_\mathcal{X}$ can be further evaluated by

$$
\begin{aligned}
(ii) &= \sum\nolimits_{v\in\mathcal{V}} \sum\nolimits_{e\in\mathcal{E}(v)} \left(-b^e z^e(v) + \epsilon^e \partial_x z^e(v)\right) z^e(v) n^e(v) \\
&= \sum\nolimits_{v\in\mathcal{V}} \hat{z}^v \sum\nolimits_{e\in\mathcal{E}(v)} \left(-b^e z^e(v) + \epsilon^e \partial_x z^e(v)\right) n^e(v) = 0,
\end{aligned}
$$

where we again used the coupling and boundary conditions (14)–(16) appearing in the definition of the space $\mathcal{D}(\mathcal{A}_\epsilon)$. In summary, we thus have shown that

$$
(\mathcal{A}_\epsilon z, z)_\mathcal{X} \le -\|\epsilon^{1/2}\partial_x z\|^2_{L^2(\mathcal{E})} \le 0 \quad \text{for all } z \in \mathcal{D}(\mathcal{A}_\epsilon), \tag{21}
$$

from which we deduce that $\mathcal{A}_\epsilon : \mathcal{D}(\mathcal{A}_\epsilon) \subset \mathcal{X} \to \mathcal{X}$ is dissipative, since

$$
\|(\lambda - \mathcal{A}_\epsilon)z\|_\mathcal{X} \|z\|_\mathcal{X} \ge ((\lambda - \mathcal{A}_\epsilon)z, z)_\mathcal{X} = (\lambda z, z)_\mathcal{X} - (\mathcal{A}_\epsilon z, z)_\mathcal{X} \ge \lambda \|z\|^2_\mathcal{X} \tag{22}
$$

for all $\lambda > 0$ and $z \in \mathcal{D}(\mathcal{A}_\epsilon)$. From (22) and the Lax–Milgram lemma, one can further deduce that for any $f \in \mathcal{X} = L^2(\mathcal{E})$, the problem $\lambda z - \mathcal{A}_\epsilon z = f$ has a unique weak solution $z \in H^1_0(\mathcal{E}) = \{w \in H^1(\mathcal{E}) : w|_v = 0 \forall v \in \mathcal{V}_\partial\}$ and $\|z\|_{H^1(\mathcal{E})} \le C(\epsilon)\|f\|_{L^2(\mathcal{E})}$. Moreover, the nodal values $\hat{z} = z|_\mathcal{V}$ are well-defined by the trace theorem for $H^1(\mathcal{E})$. By integrating (7) on each edge $e \in \mathcal{E}$, one can further see that $z \in H^2_{pw}(\mathcal{E})$ as well, and thus, $z \in \mathcal{D}(\mathcal{A}_\epsilon)$; hence, $\lambda - \mathcal{A}_\epsilon : \mathcal{D}(\mathcal{A}_\epsilon) \to \mathcal{X}$ is surjective for any $\lambda > 0$. We can now apply Engel and Nagel,[17, cor. 3.20] which is a variant of the Lumer–Phillips theorem for reflexive Banach spaces, to verify that $\mathcal{A}_\epsilon$ is the generator of a contraction semigroup. This implies the existence of a unique classical solution $z \in C^1([0,T];\mathcal{X}) \cap C([0,T];\mathcal{D}(\mathcal{A}_\epsilon))$ for (19)–(20); see, for example,[17,18] for details.

Step 3 *(Well-posedness).* By combination with the regularity estimate for w constructed in Step 1, one can see that $u = w - z$ is a solution to (7)–(11) with the required regularity. Uniqueness follows by observing that the difference $z = u_1 - u_2$ of any two solutions of (7)–(11) would solve (19)–(20) with $f = 0$ and $z_0 = 0$, which implies $u_1 - u_2 \equiv 0$.

Step 4 *(Conservation of mass and energy identity).* Mass conservation follows by integrating (7) over all pipes, summing up, and using the coupling and boundary conditions (8)–(10) as well as the balance condition (12) for the flow rates, more precisely

$$
\begin{aligned}
\frac{d}{dt}\int_\mathcal{E} a u_\epsilon \, dx &= -\int_\mathcal{E} b\partial_x u_\epsilon \, dx + \int_\mathcal{E} \epsilon \partial_{xx} u_\epsilon \, dx \\
&= \sum\nolimits_{v\in\mathcal{V}} \sum\nolimits_{e\in\mathcal{E}(v)} \left(-b^e u^e_\epsilon(v) + \epsilon^e \partial_x u^e_\epsilon(v)\right) n^e(v) \\
&= \sum\nolimits_{v\in\mathcal{V}_\partial} \left(-b^e g^v + \epsilon^e \partial_x u^e_\epsilon(v)\right) n^e(v).
\end{aligned}
$$

To show the energy identity, we apply similar arguments that have already been used to establish dissipativity of the operator $\mathcal{A}_\epsilon$ above. By first multiplying (7) with u^e_ϵ, integrating over the edges e, applying integration by parts, and using the coupling and boundary conditions (8)–(10), we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|a^{1/2}u_\epsilon\|^2_{L^2(\mathcal{E})} &= (a\partial_t u_\epsilon, u_\epsilon)_{L^2(\mathcal{E})} = -(b\partial_x u_\epsilon, u_\epsilon)_{L^2(\mathcal{E})} + (\epsilon\partial_{xx} u_\epsilon, u_\epsilon)_{L^2(\mathcal{E})} \\
&= (b u_\epsilon, \partial_x u_\epsilon)_{L^2(\mathcal{E})} - (\epsilon \partial_x u_\epsilon, \partial_x u_\epsilon)_{L^2(\mathcal{E})} \\
&\quad + \sum\nolimits_{v\in\mathcal{V}} \sum\nolimits_{e\in\mathcal{E}(v)} \left(-b^e u^e_\epsilon(v) + \epsilon^e \partial_x u^e_\epsilon(v)\right) u^e_\epsilon(v) n^e(v) \\
&= -\|\epsilon^{1/2}\partial_x u_\epsilon\|^2_{L^2(\mathcal{E})} + \sum\nolimits_{v\in\mathcal{V}_\partial} \left(-\frac{1}{2} b^e g^v + \epsilon^e \partial_x u^e_\epsilon(v)\right) g^v n^e(v).
\end{aligned}
$$

This yields the desired energy identity and completes the proof of the assertions. □

Remark 4. The energy identity of Theorem 3 yields uniform bounds

$$\frac{1}{2}\|a^{1/2}u_\epsilon\|_{L^\infty(0,T;L^2(\mathcal{E}))} + \|\epsilon^{1/2}\partial_x u_\epsilon\|_{L^2(0,T;L^2(\mathcal{E}))} \le C(u_0, g),$$

which allow to deduce existence and uniqueness of solutions also for less regular boundary and initial data. Similar results could be established alternatively also by Galerkin approximation; see Evans[14, ch. 7] or Dautray and Lions.[19, ch. XVIII]

2.3 | Limiting transport problem

We now turn to the vanishing diffusion limit $\epsilon \to 0$. Since we assumed $b^e > 0$ on every edge $e = (v_1, v_2)$, it is natural to call v_1 the inflow and v_2 the outflow vertex of the edge. For any $v \in \mathcal{V}$, we denote by $\mathcal{E}^{in}(v) = \{e \in \mathcal{E} : e = (\cdot, v)\}$ and $\mathcal{E}^{out}(v) = \{e \in \mathcal{E} : e = (v, \cdot)\}$ the edges that carry flow into or out of the vertex v, and we further split the boundary vertices into the sets $\mathcal{V}_\partial^{in} = \{v \in \mathcal{V}_\partial : |\mathcal{E}^{out}(v)| = 1\}$ and $\mathcal{V}_\partial^{out} = \{v \in \mathcal{V}_\partial : |\mathcal{E}^{in}(v)| = 1\}$; see Figure 1 for an illustration. We then consider the following problem; see Dorn et al.[20] and Egger and Philippi[21] for related results. On every edge $e \in \mathcal{E}$, the transport is described by

$$a^e \partial_t u^e(x,t) + b^e \partial_x u^e(x,t) = 0, \quad x \in e, e \in \mathcal{E}. \tag{23}$$

In contrast to the convection–diffusion problem, we now only need one boundary condition at the inflow boundary of each edge, and accordingly, we set

$$u^e(v,t) = \hat{u}^v(t), \quad v \in \mathcal{V}, e \in \mathcal{E}^{\text{out}}(v), \tag{24}$$

with auxiliary values $\hat{u}^v$ determined by the conservation condition

$$\sum\nolimits_{e\in\mathcal{E}^{in}(v)} b^e u^e(v,t) n^e(v) + \sum\nolimits_{e\in\mathcal{E}^{out}(v)} b^e \hat{u}^v(t) n^e(v) = 0, \quad v \in \mathcal{V}_0, \tag{25}$$

at inner vertices. Note that the vertices in $\mathcal{V}_0$ have at least one inflow and one outflow edge. On the inflow boundary vertices, which only have one outflow edge, we set

$$\hat{u}^v(t) = g^v(t), \quad v \in \mathcal{V}_\partial^{in}. \tag{26}$$

The above equations are assumed to hold for $t > 0$ and complemented by initial conditions

$$u^e(x,0) = u_0^e(x), \quad x \in e, e \in \mathcal{E}. \tag{27}$$

From Equation (25) and the conservation condition (12) for the flow rate b, one can deduce that the nodal values $\hat{u}^v$ at inner vertices $v \in \mathcal{V}_0$ are convex combinations of the concentrations $u^e(v)$, $e \in \mathcal{E}^{in}(v)$ entering the junction v. These *mixtures* serve as inflow values for the pipes $e \in \mathcal{E}^{out}(v)$ with flow leaving the corresponding vertex.

Remark 5. For the asymptotic analysis given in Section 3, it will be convenient to additionally define values $\hat{u}^v$ for the outflow vertices by

$$\hat{u}^v(t) = g^v(t), \quad v \in \mathcal{V}_\partial^{out}, \tag{28}$$

where $g^v, v \in \mathcal{V}_\partial$ are the same boundary data as for the convection–diffusion problem. Note that the values $\hat{u}^v, v \in \mathcal{V}_\partial^{out}$ do not appear in the other equations and therefore are not required for the analysis of the transport problem presented in the sequel.

With similar arguments as in the analysis of the convection–diffusion problem (7)–(11), we can also obtain a well-posedness result for the transport problem (23)–(27).

Theorem 6. *Let Assumption 1 hold and $T > 0$ be given. Then, for any $u_0 \in H^1_{pw}(\mathcal{E})$ and $g \in C^2([0,T]; \ell_2(\mathcal{V}_\partial^{in}))$, satisfying (24)–(26) at $t = 0$ with some $\hat{u}_0 \in \ell_2(\mathcal{V}\setminus\mathcal{V}_\partial^{out})$, the system (23)–(27) has a unique classical solution*

$$u \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1_{pw}(\mathcal{E}))$$

with $\hat{u} \in C^0([0,T]; \ell_2(\mathcal{V} \setminus \mathcal{V}_\partial^{out}))$ defined by (24). Moreover, the solution satisfies

$$\frac{d}{dt}\int_{\mathcal{E}} au\,dx = \sum\nolimits_{v \in \mathcal{V}_\partial^{in}} b^e g^v - \sum\nolimits_{v \in \mathcal{V}_\partial^{out}} b^e u^e(v),$$

that is, mass is conserved up to flow over the boundary, as well as the energy identity

$$\begin{aligned}\frac{d}{dt}\|a^{1/2}u\|_{L^2(\mathcal{E})}^2 = \sum\nolimits_{v \in \mathcal{V}_\partial^{in}} b^e|g^v|^2 - \sum\nolimits_{v \in \mathcal{V}_\partial^{out}} b^e|u^e(v)|^2 \\ - \sum\nolimits_{v \in \mathcal{V}_0}\sum\nolimits_{e \in \mathcal{E}^{in}(v)} b^e\,|u^e(v) - \hat{u}^v|^2.\end{aligned}$$

Proof. One can proceed with similar arguments as in the proof of Theorem 3, and we therefore only sketch the basic steps and the main differences.

Step 1 *(Homogenization of boundary values).* The solution can again be split into two parts $u = w - z$ where $w(t)$, $t > 0$ is a prescribed piecewise linear function in space that satisfies the inflow boundary conditions as well as $w^e(v) = 0$ for all $v \in \mathcal{V} \setminus \mathcal{V}_\partial^{in}$, $e \in \mathcal{E}(v)$, and the function z satisfies the equations with inhomogeneous right-hand side and zero inflow boundary conditions.

Step 2 *(Generation of a contraction semigroup).* We set $\mathcal{X} = L^2(\mathcal{E})$ as before and define the dense subspace

$$\begin{aligned}\mathcal{D}(\mathcal{A}) = \{z \in H_{pw}^1(\mathcal{E}) : z \text{ satisfies (24)–(25) for some } \hat{z} \in \ell_2(\mathcal{V}\backslash\mathcal{V}_\partial^{out}) \\ \text{with } \hat{z}^v = 0 \text{ for } v \in \mathcal{V}_\partial^{in}\},\end{aligned}$$

on which we formally define the linear operator

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}, \quad \mathcal{A}z|_e = -\frac{1}{a^e} b^e \partial_x z^e.$$

The transport problem (23)–(27) can then be written as an abstract evolution problem

$$\partial_t z(t) = \mathcal{A}z(t) + f(t), \quad t > 0, \tag{29}$$

$$z(0) = z_0, \tag{30}$$

with $f(t) = a\partial_t w(t) + b\partial_x w(t)$ and $z_0 = w(0) - u_0$ given. Due to the choice of w and the assumptions on the problem data, one can guarantee that $f \in C^1([0,T]; \mathcal{X})$ and $z_0 \in \mathcal{D}(\mathcal{A})$. Using similar arguments as in the proof of Theorem 3, one can further show that

$$\begin{aligned}(\mathcal{A}z, z)_{\mathcal{X}} &= -\sum\nolimits_{e \in \mathcal{E}} (b^e \partial_x z^e, z^e)_{L^2(e)} \\ &= -\frac{1}{2}\sum\nolimits_{e \in \mathcal{E}} b^e(|z^e(v_o^e)|^2 - |z^e(v_i^e)|^2),\end{aligned}$$

where v_i^e and v_o^e denote the inflow and outflow vertex of the edge $e = (v_i^e, v_o^e)$. By exchanging the order of summation and using the coupling condition (24), we then get

$$(\mathcal{A}z, z)_{\mathcal{X}} = \frac{1}{2}\sum\nolimits_{v \in \mathcal{V}} \left(\sum\nolimits_{e \in \mathcal{E}^{out}(v)} b^e|\hat{z}^v|^2 - \sum\nolimits_{e \in \mathcal{E}^{in}(v)} b^e|z^e(v)|^2\right).$$

Using the fact that $\hat{z}^v$ for $v \in \mathcal{V}_0$ is a convex combination of the values $z^e(v)$, $e \in \mathcal{E}^{in}(v)$, we can estimate the first term in this identity by Jensen's inequality, which yields

$$\sum\nolimits_{e \in \mathcal{E}^{out}(v)} b^e|\hat{z}^v|^2 \le \sum\nolimits_{e \in \mathcal{E}^{in}(v)} b^e|z^e(v)|^2$$

for all $v \in \mathcal{V}_0$. As a consequence, we obtain the inequality

$$(\mathcal{A}z, z)_{\mathcal{X}} \leq \frac{1}{2}\sum\nolimits_{v\in\mathcal{V}_\partial^{in}} b^e|z^e(v)|^2 - \frac{1}{2}\sum\nolimits_{v\in\mathcal{V}_\partial^{out}} b^e|z^e(v)|^2 \leq 0, \tag{31}$$

where we used that z vanishes at the inflow vertices $v \in \mathcal{V}_\partial^{in}$ in the last inequality. From this estimate and the same argument as in (22), we deduce that $\mathcal{A}$ is dissipative. We now show that $\lambda - \mathcal{A}$ is surjective for $\lambda > 0$. For any $f \in \mathcal{X}$ and given nodal values $\hat{z}^v$, $v \in \mathcal{V}_0 \cup \mathcal{V}_\partial^{in}$, we can solve $\lambda z^e + \frac{1}{a^e} b^e \partial_x z^e = f^e$ analytically by integration on every edge $e = (v_i^e, v_o^e) \simeq (0, \ell^e)$. This yields

$$z^e(x) = \hat{z}^{v_i^e} e^{-\frac{a^e}{b^e}\lambda x} + \int_0^x \frac{a^e}{b^e} f(s) e^{-\frac{a^e}{b^e}\lambda(x-s)}\, ds =: \hat{z}^{v_i^e} e^{-\frac{a^e}{b^e}\lambda x} + F^e(x), \tag{32}$$

where we used the coupling condition (24) to specify the constant of integration. Using $\hat{z}^v = 0$ for $v \in \mathcal{V}_\partial^{in}$ and inserting these local solutions into the flux balance condition (25) leads to a linear system of equations for $\hat{z}^v$, $v \in \mathcal{V}_0$, given by

$$\sum\nolimits_{e\in\mathcal{E}^{out}(v)} b^e \hat{z}^v - \sum\nolimits_{e\in\mathcal{E}^{in}(v)} b^e \hat{z}^{v_i^e} e^{-\frac{a^e}{b^e}\lambda \ell^e} = \sum\nolimits_{e\in\mathcal{E}^{in}(v)} F^e(\ell^e), \quad v \in \mathcal{V}_0.$$

Due to condition (12), the system matrix for this linear system can be seen to be strictly diagonally dominant, and hence, the nodal values $\hat{z}^v$, $v \in \mathcal{V}_0$ are uniquely determined. By construction, the function z defined by (32) lies in $\mathcal{D}(\mathcal{A})$ and $\lambda z - \mathcal{A}z = f$, which shows that $\lambda - A$ is surjective. By Engel and Nagel,[17, cor. 3.20] we thus know that $\mathcal{A}$ is the generator of a contraction semigroup which guarantees the existence of a unique classical solution $z \in C^1([0,T]; X) \cap C([0,T]; \mathcal{D}(\mathcal{A}))$ of (23)–(27).

Steps 3 and 4 *(Existence, uniqueness, and further properties).* The existence of a unique solution $u = w - z$ for problem (23)–(27) is now established with the same arguments as in the proof of Theorem 3. Mass conservation again directly follows by integrating (23) over all pipes, summing up, and using the coupling and inflow boundary conditions (24)–(26) as well as conservation condition (12), more precisely

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{E}} au\, dx &= \int_{\mathcal{E}} b\partial_x u\, dx \\ &= \sum\nolimits_{v\in\mathcal{V}}\sum\nolimits_{e\in\mathcal{E}(v)} b^e u^e(v) n^e(v) = \sum\nolimits_{v\in\mathcal{V}_\partial^{in}} b^e g^v - \sum\nolimits_{v\in\mathcal{V}_\partial^{out}} b^e u^e(v).\end{aligned}$$

The energy identity can be derived by multiplying (23) with u^e, integrating over all edges, summing up, and again using the coupling and inflow boundary conditions (24)–(26) as well as (12). This yields

$$\begin{aligned}\frac{d}{dt}\|a^{1/2}u\|_{L^2(\mathcal{E})}^2 &= 2(a\partial_t u, u)_{L^2(\mathcal{E})} = -2(b\partial_x u, u)_{L^2(\mathcal{E})} = -\sum\nolimits_{v\in\mathcal{V}}\sum\nolimits_{e\in\mathcal{E}(v)} b^e|u^e(v)|^2 \\ &= \sum\nolimits_{v\in\mathcal{V}}\left(\sum\nolimits_{e\in\mathcal{E}^{out}(v)} b^e|\hat{u}^v|^2 - \sum\nolimits_{e\in\mathcal{E}^{in}(v)} b^e|u^e(v)|^2\right).\end{aligned}$$

By the coupling condition (25) and the conservation condition (12) for the flow field, one can see that at inner vertices $v \in \mathcal{V}_0$, there holds

$$\begin{aligned}\sum\nolimits_{e\in\mathcal{E}^{out}(v)} b^e|\hat{u}^v|^2 - \sum\nolimits_{e\in\mathcal{E}^{in}(v)} b^e|u^e(v)|^2 &= \sum\nolimits_{e\in\mathcal{E}^{in}(v)} b^e\left(|\hat{u}^v|^2 - |u^e(v)|^2\right) \\ &= -\sum\nolimits_{e\in\mathcal{E}^{in}(v)} b^e|u^e(v) - \hat{u}^v|^2,\end{aligned}$$

where we used the fact that

$$\sum\nolimits_{e\in\mathcal{E}^{in}(v)} b^e|\hat{u}^v|^2 = \sum\nolimits_{e\in\mathcal{E}^{in}(v)} b^e u^e(v)\hat{u}^v.$$

Together with the inflow boundary conditions (26), we then obtain the energy identity. □

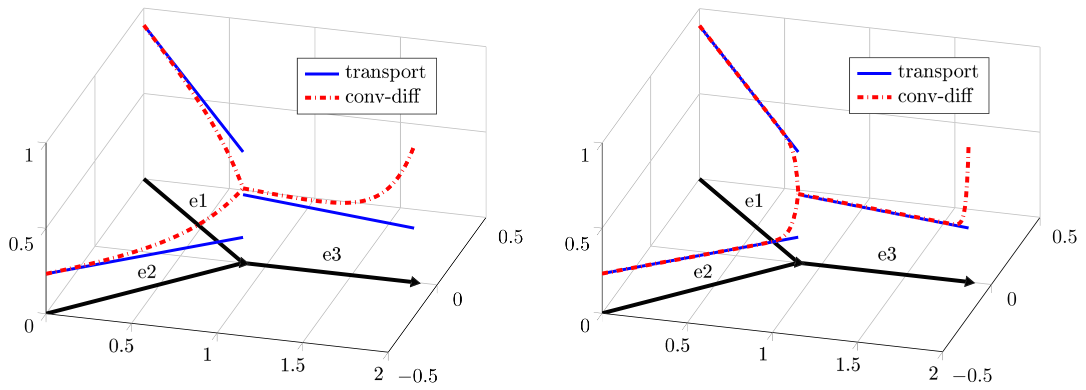

FIGURE 2 Snapshots of typical solutions u and u_ϵ of the transport problem (blue) and the convection–diffusion problem (red, dashed) for different values of ϵ (left, large; right, small). The different continuity conditions and the occurrence of boundary layers for small ϵ are clearly visible [Colour figure can be viewed at wileyonlinelibrary.com]

2.4 | Comparison of the coupling conditions

Before we proceed, let us briefly comment on the coupling conditions. For the convection–diffusion problem with $\epsilon > 0$, the number of coupling conditions at a junction $v \in \mathcal{V}_0$ is $|\mathcal{E}(v)| + 1$, which suffices to guarantee continuity of the solution and conservation of mass at the junction. For the transport problem, on the other hand, the number of coupling conditions is $|\mathcal{E}^{out}(v)| + 1$ which only suffices to guarantee conservation of mass at the junction and to prescribe the concentrations at the outflow edges. The concentration $u^e(v)$, $e \in \mathcal{E}^{in}(v)$ on edges with flows into the junctions will however usually deviate from the mixing value $\hat{u}^v$. In this case, the mixing at pipe junctions generates dissipation, which amounts to the inequality resulting from the application of Jensen's inequality in Step 2 of the proof of the previous lemma. In Figure 2, we display typical solutions u_ϵ and u for the convection–diffusion and the limiting transport problem. One can clearly see the generation of boundary layers in u_ϵ as $\epsilon \to 0$ and the fact that transport solutions u are usually discontinuous across junctions.

3 | ASYMPTOTIC ANALYSIS

We will now show that the solutions of the convection–diffusion problem (7)–(11) converge to that of the transport problem (23)–(27) with rate $\mathcal{O}(\sqrt{\epsilon})$. We will closely follow the arguments of the proof for the corresponding result for a single edge, which can be found in Roos et al.[6], pp159–166; see Bobisud[22] for the original reference. Following Roos et al.,[6] we start with establishing some preliminary results that will be required for the proof.

3.1 | Auxiliary results

As a first step, we establish a weak maximum principle for solutions of convection–diffusion problems on networks. Without further mentioning, we will always assume that Assumption 1 holds in the following auxiliary results.

Lemma 7. *Let* $u \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1(\mathcal{E}) \cap H^2_{pw}(\mathcal{E}))$ *satisfy*

$$a^e \partial_t u^e + b^e \partial_x u^e - \epsilon^e \partial_{xx} u^e \geq 0, \quad e \in \mathcal{E}, \tag{33}$$

$$\sum\nolimits_{e \in \mathcal{E}(v)} \epsilon^e \partial_x u^e(v) n^e(v) = 0, \quad v \in \mathcal{V}_0, \tag{34}$$

$$u(v) \geq 0, \quad v \in \mathcal{V}_\partial, \tag{35}$$

for all $0 < t < T$, *as well as the initial conditions*

$$u^e(x,0) \geq 0, \quad x \in e, e \in \mathcal{E}. \tag{36}$$

Then, the function u is nonnegative, that is, $u \geq 0$ *on* $\mathcal{E}$ *for all* $t \in [0, T]$.

Proof. We multiply the differential inequality (33) by the test function $w := \min(0, u) \leq 0$, integrate over all edges $e \in \mathcal{E}$, and use integration-by-parts for the spatial derivative terms, similar as in the proof of Theorems 3 and 6. This leads to

$$0 \geq (a\partial_t u, w)_{L^2(\mathcal{E})} + (b\partial_x u, w)_{L^2(\mathcal{E})} - (\epsilon\partial_{xx} u, w)_{L^2(\mathcal{E})}$$
$$= (a\partial_t u, w)_{L^2(\mathcal{E})} - (bu, \partial_x w)_{L^2(\mathcal{E})} + (\epsilon\partial_x u, \partial_x w)_{L^2(\mathcal{E})},$$

where we used continuity of u and w across junctions, the conservation condition (12) for the flow rates, as well as (34) and the fact that $u \geq 0$ on the boundary, and hence, $w = 0$ at vertices $v \in \mathcal{V}_\partial$. Next, observe that $w(t) \equiv 0$, and thus, also $\partial_x w(t) \equiv 0$, on the set where u is nonnegative, and $w \equiv u$ on the complement $\mathcal{E}_-(t) = \{x : u(x,t) < 0\}$. From this and the previous inequality, we immediately deduce that

$$0 \geq (a\partial_t u, u)_{L^2(\mathcal{E}_-(t))} - (bu, \partial_x u)_{L^2(\mathcal{E}_-(t))} + (\epsilon\partial_x u, \partial_x u)_{L^2(\mathcal{E}_-(t))} \geq (a\partial_t u, u)_{L^2(\mathcal{E}_-(t))},$$

where we used that $bu\partial_x u = \frac{b}{2}\partial_x |u|^2$ and the fact that possible coupling and boundary terms appearing when integrating this expression drop out due to continuity of u across junctions; furthermore, we employed the flow conservation condition (12) for b and the fact that $u = 0$ on the boundary of $\mathcal{E}_-$ due to its definition and (35). Let us note that $\mathcal{E}_-(0) = \emptyset$, since $u(0) \geq 0$. By the fundamental theorem of calculus, we thus obtain

$$\int_{\mathcal{E}_-(t)} a|u(t)|^2\, dx = \int_0^t \frac{d}{dt}\int_{\mathcal{E}_-(s)} a|u(s)|^2\, dx\, ds = \int_0^t 2(a\partial_t u(s), u(s))_{\mathcal{E}_-(s)}\, ds \leq 0,$$

where we used that $u = 0$ on the boundary of $\mathcal{E}_-(s)$ for the second identity. As a consequence, we obtain $\mathcal{E}_-(t) = \emptyset$, and hence, $u(t) \geq 0$ for all $0 \leq t \leq T$. ☐

Using the weak maximum principle, we can show the following uniform bounds.

Lemma 8. *The solution of problem (7)–(11) is uniformly bounded by* $|u_\epsilon(x,t)| + |\partial_t u_\epsilon(x,t)| \leq C_u$ *for all* $x \in \mathcal{E}$, $t \in [0,T]$ *with* C_u *independent of* ϵ.

Proof. The boundedness of u_ϵ follows from the weak maximum principle with the usual arguments; see, for example, Evans.[14], ch. 7 By defining $w^e(x,t) := \max(\|u_0\|_\infty, \max|g^v(t)|) \pm u^e_\epsilon(x,t)$, we immediately see that w satisfies all conditions of Lemma 7 and is thus nonnegative. Consequently, u_ϵ is bounded independently of ϵ. By linearity of the problem, one can further see that $z_\epsilon = \partial_t u_\epsilon$ again solves (7)–(11), but with with boundary data $z(v) = \partial_t g^v$ on $\mathcal{V}_\partial$ and initial data $z^e_\epsilon(0) = \partial_t u^e_\epsilon(0) = -\frac{1}{a^e}\left(b^e\partial_x u^e_0 - \epsilon^e\partial_{xx} u^e_0\right)$. The boundedness of $z_\epsilon = \partial_t u_\epsilon$ then follows from the assumptions on the problem data with the same reasoning as above. ☐

Lemma 9. *Let* u_ϵ *denote the solution of problem (7)–(11). Then,*

$$|\partial_x u^e_\epsilon(v^e_i, t)| \leq K, \qquad t \in (0,T),$$

for all edges $e = (v^e_i, v^e_o)$ *with uniform constant* K *independent of* ϵ.

Proof. For every edge $e = (v^e_i, v^e_o) \simeq (0, \ell^e)$, we define $w^e(x,t) := Kx + \hat{u}^{v^e_i}_\epsilon(t) - u^e_\epsilon(x,t)$, where K is a positive constant to be chosen later. From Lemma 8, we know that u_ϵ and $\partial_t u_\epsilon$ and hence by (8) also $\hat{u}^v_\epsilon$ and $\partial_t \hat{u}^v_\epsilon$ are bounded independently of ϵ by a uniform constant C_u. Then, for any $K \geq \frac{a^e}{b^e} C_u$, we have

$$a^e\partial_t w^e + b^e\partial_x w^e - \epsilon^e\partial_{xx} w^e = a^e\partial_t \hat{u}^{v^e_i}_\epsilon + b^e K \geq 0.$$

If we further assume that $K \geq \max_{x\in\mathcal{E}} |\partial_x u_0(x)|$, then

$$w^e(x,0) = Kx + \hat{u}^{v^e_i}_\epsilon(0) - u^e_\epsilon(x,0) = Kx + u^e_0(0) - u^e_0(x)$$
$$= Kx - \int_0^x \partial_x u^e_0(s) ds \geq Kx - \max_{x\in e}|\partial_x u_0(x)|x \geq 0.$$

Using Lemma 8, we may further assume $K \geq 2C_u/\min_{e\in\mathcal{E}} \ell^e$, and deduce that

$$w^e(v_i^e, t) = 0 \text{ and } w^e(v_o^e, t) = K \cdot \ell^e + \hat{u}_\epsilon^{v_i^e}(t) - u_\epsilon^e(v_o^e, t) \geq 0,$$

since u_ϵ is assumed to be continuous across network junctions. The weak maximum principle then yields $w^e \geq 0$ for all $t \in [0, T]$, and consequently,

$$u_\epsilon^e(x, t) - \hat{u}_\epsilon^{v_i^e}(t) \leq Kx.$$

This implies that $\partial_x u_\epsilon^e(v_i^e, t) = \lim_{x\to 0} \frac{u_\epsilon^e(x,t) - \hat{u}_\epsilon^{v_i^e}(t)}{x} \leq K$ for all $t \in (0, T)$. From the construction, one can see that K can be chosen independent of ϵ. In a similar manner, by defining $w^e(x, t) := Kx + u_\epsilon^e(x, t) - \hat{u}_\epsilon^{v_i^e}(t)$, one can show that $-\partial_x u_\epsilon^e(v_i^e, t) \leq K$ for all $t \in (0, T)$. In summary, we thus have proven that $|\partial_x u_\epsilon^e(v_i^e, t)| \leq K$ for all $t \in (0, T)$ with a constant K that is independent of ϵ. Since the network is finite, K can be chosen independent of $e \in \mathcal{E}$ as well. □

3.2 | Asymptotic estimates

With the auxiliary results derived in the previous section, we are now in the position to prove our main result.

Theorem 10. *Let Assumption 1 hold. Further, let u_ϵ be the solution of problem (7)–(11) and u be the solution of the corresponding limit problem (23)–(28). Then,*

$$\|u_\epsilon - u\|_{L^\infty(0,T;L^2(\mathcal{E}))} \leq C(T)\sqrt{\epsilon}, \tag{37}$$

with a constant C(T) depending on T but not on the parameter $0 < \epsilon \leq 1$.

Proof. The proof follows the arguments given in Roos et al.[6], pp159–166. Since we require particular boundary layer functions for junctions $v \in \mathcal{V}_0$, we present the result in detail.

Step 1. For every $e \in \mathcal{E}$ with $e = (v_i^e, v_o^e) \simeq (0, \ell^e)$, we define a boundary layer function

$$w_\epsilon^e(x, t) = \left(\hat{u}^{v_o^e}(t) - u^e(v_o^e, t)\right) e^{-b^e(\ell^e - x)/\epsilon^e}; \tag{38}$$

see Figure 2 for an illustration of the boundary layers in the solution of problem (7)–(11) that motivates this particular construction. We immediately obtain

$$b^e \partial_x w_\epsilon^e - \epsilon^e \partial_{xx} w_\epsilon^e = 0, \tag{39}$$

and $\|w_\epsilon\|_{L^\infty(0,T;L^2(\mathcal{E}))} \leq C\sqrt{\epsilon}$, where we used that u, and thus, also $\hat{u}^v$ are uniformly bounded according to Theorem 6. Further estimates for w_ϵ and its spatial derivatives can be found in Dobrowolski and Roos.[23] The error between u_ϵ and u can then be split into

$$\begin{aligned}\|u_\epsilon - u\|_{L^\infty(0,T;L^2(\mathcal{E}))} &\leq \|u_\epsilon - u - w_\epsilon\|_{L^\infty(0,T;L^2(\mathcal{E}))} + \|w_\epsilon\|_{L^\infty(0,T;L^2(\mathcal{E}))} \\ &\leq \|u_\epsilon - u - w_\epsilon\|_{L^\infty(0,T;L^2(\mathcal{E}))} + C\sqrt{\epsilon}.\end{aligned}$$

Step 2. For ease of notation, we introduce $\eta_\epsilon := u_\epsilon - u - w_\epsilon$ and investigate the values of η_ϵ at time $t = 0$ and at the vertices $v \in \mathcal{V}$ of the network. For $t = 0$, we have

$$\begin{aligned}\eta_\epsilon^e(x, 0) &= u_\epsilon^e(x, 0) - u^e(x, 0) - \left(\hat{u}^{v_o^e}(0) - u^e(v_o^e, 0)\right) e^{-b^e(\ell^e - x)/\epsilon^e} \\ &= u_0^e(x) - u_0^e(x) - \left(u_0^e(v_o^e) - u_0^e(v_o^e)\right) e^{-b^e(\ell^e - x)/\epsilon^e} = 0,\end{aligned} \tag{40}$$

where we used that u_ϵ and u have the same initial value u_0 which is continuous across junctions $v \in \mathcal{V}_0$ and $g^v(0) = u_0(v)$ for $v \in \mathcal{V}_\partial$ due to the compatibility conditions of initial and boundary values. For inflow boundary vertices $v \in \mathcal{V}_\partial^{in}$ and $e = (v, v_o^e)$, we obtain

$$\eta_\epsilon^e(v, t) = g^v(t) - g^v(t) - \left(\hat{u}^{v_o^e}(t) - u^e(v_o^e, t)\right) e^{-b^e\ell^e/\epsilon^e} \leq C'\epsilon, \tag{41}$$

where C' is a constant independent of ϵ. For outflow boundary vertices $v \in \mathcal{V}_\partial^{out}$ and the corresponding edge $e = (v_i^e, v)$, we obtain

$$\eta_\epsilon^e(v,t) = g^v(t) - u^e(v,t) - \left(g^v(t) - u^e(v,t)\right) = 0. \tag{42}$$

At inner vertices $v \in \mathcal{V}_0$, on the other hand, there holds

$$\eta_\epsilon^e(v,t) = \hat{u}_\epsilon^v(t) - \hat{u}^v(t), \quad e = (v_i^e, v) \in \mathcal{E}^{in}(v), \tag{43}$$

$$\eta_\epsilon^e(v,t) = \hat{u}_\epsilon^v(t) - \hat{u}^v(t) - \left(\hat{u}^{v_o^e}(t) - u^e(v_o^e,t)\right) e^{-b^e\ell^e/\epsilon^e}, \quad e = (v, v_o^e) \in \mathcal{E}^{out}(v). \tag{44}$$

Step 3. Inserting η_ϵ into the convection–diffusion equation (7) and testing with η_ϵ yield

$$\begin{aligned}(a\partial_t\eta_\epsilon, \eta_\epsilon)_{L^2(\mathcal{E})} &= -(b\partial_x\eta_\epsilon, \eta_\epsilon)_{L^2(\mathcal{E})} + (\epsilon\partial_{xx}\eta_\epsilon, \eta_\epsilon)_{L^2(\mathcal{E})} + (\epsilon\partial_{xx}u, \eta_\epsilon)_{L^2(\mathcal{E})} \\ &\quad - (a\partial_t w_\epsilon, \eta_\epsilon)_{L^2(\mathcal{E})} = (i) + (ii) + (iii) + (iv),\end{aligned}$$

where we used the identity (39). The individual terms are now estimated separately.

Step 3(i). The first term can be transformed into

$$(i) = -\sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)}\frac{1}{2}b^e|\eta_\epsilon^e(v)|^2 n^e(v) = \sum_{v\in\mathcal{V}}(*).$$

For internal vertices $v \in \mathcal{V}_0$ using (43)–(44), we obtain

$$\begin{aligned}(*) &= \sum_{e\in\mathcal{E}^{out}(v)}\frac{1}{2}b^e\left(\hat{u}_\epsilon^v - \hat{u}^v - \left(\hat{u}^{v_o^e} - u^e(v_o^e)\right)e^{-b^e\ell^e/\epsilon^e}\right)^2 \\ &\quad - \sum_{e\in\mathcal{E}^{in}(v)}\frac{1}{2}b^e(\hat{u}_\epsilon^v - \hat{u}^v)^2 \\ &= \sum_{e\in\mathcal{E}^{out}(v)}\frac{1}{2}b^e\left(\hat{u}_\epsilon^v - \hat{u}^v\right)^2 - \sum_{e\in\mathcal{E}^{in}(v)}\frac{1}{2}b^e\left(\hat{u}_\epsilon^v - \hat{u}^v\right)^2 \\ &\quad - \sum_{e\in\mathcal{E}^{out}(v)}b^e\left(\hat{u}_\epsilon^v - \hat{u}^v\right)\left(\hat{u}^{v_o^e} - u^e(v_o^e)\right)e^{-b^e\ell^e/\epsilon^e} \\ &\quad + \sum_{e\in\mathcal{E}^{out}(v)}\frac{1}{2}b^e\left(\hat{u}^{v_o^e} - u^e(v_o^e)\right)^2 e^{-2b^e\ell^e/\epsilon^e} \\ &\le C\sum_{e\in\mathcal{E}^{out}(v)}b^e(e^{-b^e\ell^e/\epsilon^e} + e^{-2b^e\ell^e/\epsilon^e}) \le C'\epsilon.\end{aligned}$$

Here, we additionally used the conservation property (12) of the volume flow rates and the uniform boundedness of u_ϵ stated in Lemma 8. For inflow boundary vertices $v \in \mathcal{V}_\partial^{in}$, we know from (41) that $\eta_\epsilon^e(v,t) \le C'\epsilon$ on $e = (v, v_o^e)$, and hence $(*) \le Cb^e\epsilon$, and for outflow boundary vertices $v \in \mathcal{V}_\partial^{out}$, we have $\eta_\epsilon^e(v,t) = 0$ by (42), and thus, $(*) = 0$ there. In summary, we thus obtain $(i) \le C''\epsilon$ with constant C'' independent of ϵ.

Step 3(ii). Using integration-by-parts, we can transform the second term into

$$\begin{aligned}(ii) &= -\sum_{e\in\mathcal{E}}(\epsilon^e\partial_x\eta_\epsilon^e, \partial_x\eta_\epsilon^e)_{L^2(e)} + \sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)}\epsilon^e\partial_x\eta_\epsilon^e(v)\eta_\epsilon^e(v)n^e(v) \\ &\le \sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)}\epsilon^e\partial_x\eta_\epsilon^e(v)\eta_\epsilon^e(v)n^e(v) = \sum_{v\in\mathcal{V}}(**).\end{aligned}$$

At inner vertices $v \in \mathcal{V}_0$, we again use (43)–(44) to obtain

$$\begin{aligned}(**) &= \sum_{e\in\mathcal{E}(v)}\epsilon^e\partial_x u_\epsilon^e(v)(\hat{u}_\epsilon^v - \hat{u}^v)n^e(v) - \sum_{e\in\mathcal{E}^{out}(v)}\epsilon^e\partial_x\eta_\epsilon^e(v)w_\epsilon^e(v) \\ &\quad - \sum_{e\in\mathcal{E}(v)}\epsilon^e(\partial_x u^e(v) + \partial_x w_\epsilon^e(v))(\hat{u}_\epsilon^v - \hat{u}^v)n^e(v) = (a) + (b) + (c).\end{aligned}$$

From (8), (9), and (12), we deduce that

$$\sum_{e\in\mathcal{E}(v)} \epsilon^e \partial_x u_\epsilon^e(v,t) n^e(v) = 0, \quad \text{for all } v \in \mathcal{V}_0, \tag{45}$$

and hence, the the term (a) vanishes. Inserting the definition of η_ϵ, we further obtain

$$(b) = -\sum_{e\in\mathcal{E}^{out}(v)} \epsilon^e \left(\partial_x u_\epsilon^e(v) - \partial_x u^e(v) - \partial_x w_\epsilon^e(v)\right) w_\epsilon^e(v).$$

From Lemma 9, we know that $\partial_x u_\epsilon^e(v)$ is bounded uniformly for all $e \in \mathcal{E}^{out}(v)$, and the derivative $\partial_x u^e$ is also bounded independently of ϵ. Furthermore, the spatial derivative $\partial_x w_\epsilon^e(v)$ can be bounded by $(C/\epsilon^e)e^{-b^e\ell^e/\epsilon^e}$ for all $e \in \mathcal{E}^{out}(v)$; see (47). From these bounds, we conclude that $(b) \leq C(\epsilon^e+1)e^{-b^e\ell^e/\epsilon^e} \leq C'\epsilon$ with constant C' independent of ϵ. To estimate the term (c), we observe that $\partial_x u$ and $\hat{u}_\epsilon$ are bounded independently of ϵ; see Theorem 6 and Lemma 8. Consequently,

$$(c_1) = -\sum_{e\in\mathcal{E}(v)} \epsilon^e \partial_x u^e(v)(\hat{u}_\epsilon^v - \hat{u}^v) n^e(v) \leq C\epsilon.$$

For the spatial derivative $\partial_x w_\epsilon^e(v)$, we further obtain

$$\partial_x w_\epsilon^e(v) = \frac{b^e}{\epsilon^e}(\hat{u}^v - u^e(v)), \quad e \in \mathcal{E}^{in}(v), \tag{46}$$

$$\partial_x w_\epsilon^e(v) = \frac{b^e}{\epsilon^e}(\hat{u}^{v_o^e} - u^e(v_o^e))e^{-b^e\ell^e/\epsilon^e}, \quad e \in \mathcal{E}^{out}(v), \tag{47}$$

which allows us to rewrite

$$\begin{aligned}(c_2) &= -\sum_{e\in\mathcal{E}(v)} \epsilon^e \partial_x w_\epsilon^e(v)(\hat{u}_\epsilon^v - \hat{u}^v) n^e(v) \\ &= -\sum_{e\in\mathcal{E}^{in}(v)} b^e(\hat{u}^v - u^e(v))(\hat{u}_\epsilon^v - \hat{u}^v) \\ &\quad + \sum_{e\in\mathcal{E}^{out}(v)} b^e(\hat{u}^{v_o^e} - u^e(v_o^e))e^{-b^e\ell^e/\epsilon^e}(\hat{u}_\epsilon^v - \hat{u}^v).\end{aligned}$$

Now, the first term on the right-hand side vanishes due to the coupling conditions (24)–(25) and the conservation condition (12) for the flow rates. The uniform bounds for u, $\hat{u}^v$ and u_ϵ, $\hat{u}_\epsilon^v$ then allow to bound $(c_2) \leq C'\epsilon$, and hence, $(c) \leq C''\epsilon$ with C'' independent of ϵ. By combination of the estimates for (a), (b), and (c), we obtain $\sum_{v\in\mathcal{V}_0}(**) \leq C\epsilon$. For the remaining boundary vertices $v \in \mathcal{V}_\partial$, we use (41)–(42) to see that

$$\sum_{v\in\mathcal{V}_\partial}(**) = \sum_{v\in\mathcal{V}_\partial^{in}} \epsilon^e \partial_x \eta_\epsilon^e(v)\eta^e(v) n^e(v) \leq C'\epsilon,$$

since $\partial_x \eta_\epsilon^e(v)$, $v \in \mathcal{V}_\partial^{in}$ is bounded independently of ϵ by Theorem 6, Lemma 9, and (47). In summary, we thus obtain $(ii) \leq C\epsilon$ with a constant C independent of ϵ.

Step 3(iii). Integration-by-parts and Young's inequality yield

$$\begin{aligned}(iii) &= -(\epsilon\partial_x u, \partial_x \eta_\epsilon)_{L^2(\mathcal{E})} + \sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)} \epsilon^e \partial_x u^e(v)\eta_\epsilon^e(v) n^e(v) \\ &\leq \frac{1}{2}\|\epsilon^{1/2}\partial_x u\|_{L^2(\mathcal{E})}^2 + \frac{1}{2}\|\epsilon^{1/2}\partial_x \eta_\epsilon\|_{L^2(\mathcal{E})}^2 + \sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)} \epsilon^e \partial_x u^e(v)\eta_\epsilon^e(v) n^e(v).\end{aligned}$$

The first term is bounded by $C\epsilon$, the second term can be absorbed into (ii), and the boundary terms can be estimated by $C\epsilon$, since $\partial_x u$ and η_ϵ are uniformly bounded; see Theorem 6 and (41)–(44). In summary, we thus obtain $(iii) \leq C\epsilon$.

Step 3(iv). Using Young's inequality, we have

$$-(a\partial_t w_\epsilon, \eta_\epsilon)_{L^2(\mathcal{E})} \leq \frac{1}{2}\|a^{1/2}\partial_t w_\epsilon\|_{L^2(\mathcal{E})}^2 + \frac{1}{2}\|a^{1/2}\eta_\epsilon\|_{L^2(\mathcal{E})}^2.$$

By the uniform bounds for $\partial_t u$ and $\partial_t u_\epsilon$, we can estimate the first term by

$$\|a^{1/2}\partial_t w_\epsilon\|^2_{L^2(e)} = \int_0^{\ell^e} a^e\left(\partial_t \hat{u}^{v^e_o}(t) - \partial_t u^e(v^e_o, t)\right)^2 e^{-2b^e(\ell^e - x)/\epsilon^e}\,dx \le C'\epsilon,$$

and since the graph is finite, this estimate translates to the whole network.

Step 4. By combination of the estimates for the terms (i)–(iv), we finally obtain

$$\frac{1}{2}\frac{d}{dt}\|a^{1/2}\eta_\epsilon\|^2_{L^2(\mathcal{E})} = (a\partial_t\eta_\epsilon, \eta_\epsilon)_{L^2(\mathcal{E})} \le C'\epsilon + \frac{1}{2}\|a^{1/2}\eta_\epsilon\|^2_{L^2(\mathcal{E})}.$$

An application of Gronwall's lemma then immediately yields

$$\|\eta_\epsilon(t)\|_{L^2(\mathcal{E})} \le 2a^{-1}_{\min} C' e^t \epsilon \le C(T)\epsilon,$$

with $a_{\min} = \min_{e\in\mathcal{E}} a^e$ and constant $C(T) = 2a^{-1}_{\min} C' e^T$ that is independent of ϵ and t. Together with Step 1, this completes the proof of the theorem. □

3.3 | Summary

The previous theorem shows that the asymptotic analysis of convection–diffusion problems can be extended almost verbatim to networks, if appropriate coupling conditions and corresponding boundary layer functions are defined at the network junctions. By considering stationary problems or networks consisting only of a single pipe, one can see that the rate of the theorem can again not be improved.

Before closing the presentation, let us mention some directions for further research: a natural next step would be to consider numerical approximations for singularly perturbed convection–diffusion problems on networks. Based on the analysis given in this paper, we would expect that most of the results available for a single pipe, see Roos et al.[6] and the references given there, can be extended to networks. We would also expect that the convergence of the semigroup approach of Bardos[24] can be extended to the network setting quite naturally. Another point of interest might be to consider nonlinear problems and the asymptotic convergence in different metrics, which should be possible in the framework of entropy methods; we refer to Jüngel[25] for an introduction to the field.

ACKNOWLEDGEMENTS

The authors are grateful for support by the German Research Foundation (DFG) via grants TRR154 project C04 and TRR146 project C03 and via the Center for Computational Engineering at TU Darmstadt. Open access funding enabled and organized by Projekt DEAL.

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How to cite this article: Egger H, Philippi N. On the transport limit of singularly perturbed convection-diffusion problems on networks. *Math Meth Appl Sci*. 2021;44:5005–5020. https://doi.org/10.1002/mma.7084