Surfaces in Homogeneous Manifolds Generated by Schwarz Reflection

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Zusammenfassung

Minimalflächen sind Flächen, die lokal minimalen Flächeninhalt besitzen. Derartige Formen lassen sich beispielsweise durch Seifenhäute realisieren. Taucht man einen geschlossenen Draht in eine Seifenlauge, so bildet sich eine Seifenhaut, die physikalisch versucht, ihre potentielle Energie zu minimieren.

Die klassische Konstruktion der Schwarz-D-Fläche im euklidischen Raum $\mathbb{E}^3$ lässt sich wie folgt durchführen: Man nehme einen speziellen Kantenzug Γ auf einem dreidimensionalen Würfel und betrachtet die dazugehörige Lösung des Plateau-Problems. Eine solche Plateau-Lösung ist eine verallgemeinerte Minimalfläche $\mathcal{P}$ mit Rand Γ. Das schwarzsche Spiegelungsprinzip liefert, dass durch eine Schwarz-Spiegelung (Drehung an einer Kante von Γ um $180°$) von $\mathcal{P}$ erneut eine verallgemeinerte Minimalfläche entsteht. Durch sukzessives Schwarz-Spiegeln an den Randkanten von Γ, erhält man eine vollständige, eingebettete und dreifach periodische Minimalfläche $\mathcal{S}$ in $\mathbb{E}^3$, die Schwarz-D-Fläche.

In dieser Arbeit verallgemeinern wir dieses Vorgehen auf verschiedene Weise. Zunächst betrachten wir Flächen nicht nur im dreidimensional euklidischen Raum $\mathbb{E}^3$, sondern wir erweitern die Konstruktion auf den euklidischen Raum $\mathbb{E}^k$, die Sphäre $\mathbb{S}^k$, den hyperbolischen Raum $\mathbb{H}^k$ und Produkte von diesen drei Räumen. Wir bezeichnen einen solchen Produktraum mit $\mathbb{X}^n$. Anstelle eines Würfels betrachten wir ein total geodätisches Coxeter-Polytop $P \subset \mathbb{X}^n$. Ein solches Polytop ist konvex, kompakt und hat die Eigenschaft, dass sich seine Facetten in einem Dieder-Winkel von der Form π/k, $k \geq 2$ schneiden. Wir definieren Γ als einen geschlossenen Kantenzug auf P. Anstelle eines Minimalflächenstückes betrachten wir ein allgemeines Flächenstück $\mathcal{P}$, welches eingebettet ist, im Inneren von P liegt und Γ als Randkurve hat. Es ist bekannt, dass die Plateau-Lösungen in $\mathbb{E}^n$ und in $\mathbb{X}^3$ diese Eigenschaften besitzen. In den anderen Fällen ist im Allgemeinen nicht bekannt, ob die Plateau-Lösung eingebettet ist. Durch sukzessives Schwarz-Spiegeln an den Kanten von Γ, erhalten wir eine vollständige Minimalfläche $\mathcal{S}$. Schließlich erhalten wir mit Theorem 4.2.10, dem Hauptresultat der

Arbeit, eine hinreichende und notwendige Bedingung, ob $\mathcal{S}$ eingebettet ist oder Selbstschnitte besitzt.

Im zweiten Kapitel motivieren wir die Betrachtung von Coxeter-Polytopen P. Weiterhin erzeugen die Spiegelungen an den Facetten von P auf natürliche Weise eine Gruppe W, genannt Coxeter-Gruppe. Theorem 2.9.6 besagt, dass jedes total geodätische Coxeter-Polytop den Raum $\mathbb{X}^n$ pflastert. Weiterhin kann man eine solche Pflasterung von $\mathbb{X}^n$ durch P mit dem Coxeter-Komplex $\mathcal{U}(W,P) = W \times P/\sim$ identifizieren, wobei die Äquivalenzrelation $\sim$ das Verkleben der Kopien von P beschreibt.

Im dritten Kapitel betrachten wir sphärischen Coxeter-Gruppen das eindeutig bestimmte längste Element w_0 bezüglich ein Längenfunktion. Mit Hilfe der kanonischen Repräsentation $\rho\colon W \to GL(n)$ untersuchen wir, wann w_0 als $-\operatorname{id}$ wirkt. Ist dies der Fall, so sagen wir, dass w_0 die (-1)-Bedingung erfüllt.

Im vierten Kapitel zeigen wir, dass die Schwarz-Spiegelung an einer Kante von Γ durch die Facetten-Spiegelungen von P ausgedrückt werden kann, sofern die Kante eine sogenannte (-1)-Bedingung erfüllt. Die (-1)-Bedingung besagt im Wesentlichen, dass die Schwarz-Spiegelung als $(-\operatorname{id})$-Element in einem geeigneten Unterraum wirkt bzw. dass sie punktweise Pflastersteine auf Pflastersteine abbildet. Schließlich behandeln wir die Frage, ob die konstruierte Fläche $\mathcal{S}$ eingebettet ist. Dazu stellen wir eine rein gruppentheoretische Bedingung (4.1) auf, die notwendig und hinreichend ist.

Im fünften Kapitel untersuchen wir, ob die konstruierte Fläche $\mathcal{S}$ eingebettet ist. Dazu überprüfen wir (4.1) in $\mathbb{X}^n$ für die Fälle $n = 3$ und $n = 4$. Weiterhin untersuchen wir den n-dimensionalen Raum im Falle $\mathbb{X}^n \in \{\mathbb{S}^n, \mathbb{R}^n, \mathbb{H}^n\}$, wenn P ein n-Simplex ist.

Im sechsten Kapitel verallgemeinern wir die Konstruktion von $\mathcal{S}$. Zum einen schwächen wir die (-1)-Bedingung ab, indem wir fordern, dass die Schwarz-Spiegelung Pflastersteine auf Pflastersteine, aber nicht notwendigerweise punktweise, abbildet. Zum anderen wollen wir die Definition von Γ nicht auf P einschränken, sondern auf die komplette Pflasterung $\mathcal{U}(W,P)$ erweitern. Wir behandeln jeweils zwei ausgewählte Beispiele, um die beiden Verallgemeinerungen vorzuführen.

Contents

1 Introduction

In this thesis, we construct complete embedded surfaces in an n-dimensional Riemannian product space $\mathbb{X}^n$ with factors in $\{\mathbb{S}^k, \mathbb{E}^k, \mathbb{H}^k\}$. The construction is motivated by the classical construction of the Schwarz D surface, Schwarz CLP surface in $\mathbb{E}^3$, and the Lawson surfaces in $\mathbb{S}^3$. We extend the Plateau solution to a polygonal boundary value problem to a complete surface by Schwarz reflection. The main result of this thesis gives a necessary and sufficient condition for the complete surface to be embedded. It can be applied not only to minimal surfaces but to all classes of surfaces with properties which are preserved under Schwarz reflection, e.g, continuous surfaces, discrete surfaces, and discrete minimal surface.

Minimal surfaces are surfaces that locally minimise area. Their study constitutes a central area of research in differential geometry. They have encountered striking applications in other fields: in general relativity in the study of black holes and in cell biology as the elastic properties of membranes. Furthermore, soap films result when a wire frame is dipped into soap solution and, in architecture, tensile structures are closely related to minimal surfaces, e.g., the roofs of the Munich Olympia stadium are inspired by soap films.

Minimal surface theory originates with Lagrange [LSD+73] and Euler who developed the calculus of variations in the 18th century. The existence problem for a minimal surface with a given boundary is named after Joseph Plateau [Pla73], a Belgian physicist who experimented with soap films. Schwarz derived the first successful solution to the Plateau problem for a skew quadrilateral [Sch90]. Over the years, many mathematicians have contributed to the theory of minimal surfaces. In the 19th century, with investigations by Riemann, Weierstrass, and Enneper in complex analysis, methods and results only for special cases of Plateau's problem were known. It was until 1930 that a general solution to Plateau's problem was found independently by Douglas [Dou31] and Radó [Rad30]. Douglas was awarded with one of the two first Fields medals for his solution. In the last century, further mathematicians contributed to this theory, for instance Morrey, Osserman, Gulliver, Meeks and Yau. Furthermore, they extended the view from Euclidean space $\mathbb{E}^3$ to minimal surfaces in Riemannian manifolds and to higher codimension. Morrey showed that the Plateau problem is solvable for what he called homogeneously regular Riemannian manifolds [Mor48]. Osserman [Oss69] and Gulliver [Gul73] stated conditions which exclude true and false branch points. Furthermore, Osserman [Oss02] showed for certain boundary curves that the Plateau solution is embedded in $\mathbb{R}^n$. Meeks and Yau [MY82] proved the embeddedness in three-dimensional manifolds under convexity conditions.

Of particular interest are manifolds with constant curvature, such as the sphere $\mathbb{S}^3$ and hyperbolic space $\mathbb{H}^3$. Lawson [Law70] constructed compact minimal surfaces of arbitrary genus in $\mathbb{S}^3$. Polthier [Pol91] constructed minimal surfaces in $\mathbb{H}^3$ similar to the Schwarz P surface. Rosenberg [Ros02] constructed several minimal surfaces in $\mathbb{S}^2 \times \mathbb{E}$ and $\mathbb{H}^2 \times \mathbb{E}$. The extension of Plateau's problem to higher dimensions, i.e., for k-dimensional surfaces in n-dimensional space, turns out to be much more difficult. Results were obtained in terms of geometric measure theory. Prominent works are due to Almgren [ABS66], and Federer [Fed14].

For a detailed history of the Plateau solution and minimal surfaces see Nitsche's book [Nit75] and Hildebrand [JDK$^+$10].

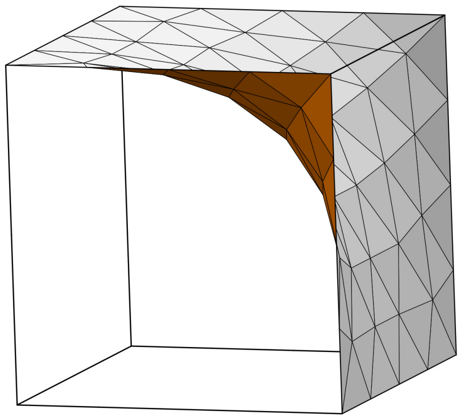


Figure 1.1: The Plateau solution where the boundary is a closed curve on the edge set of a cube.

In the following, we discuss the construction of the Schwarz CLP (crossed layers of parallels) surface to motivate the construction we use in this thesis. Essentially, the Schwarz CLP surface is a surface we get by gluing together

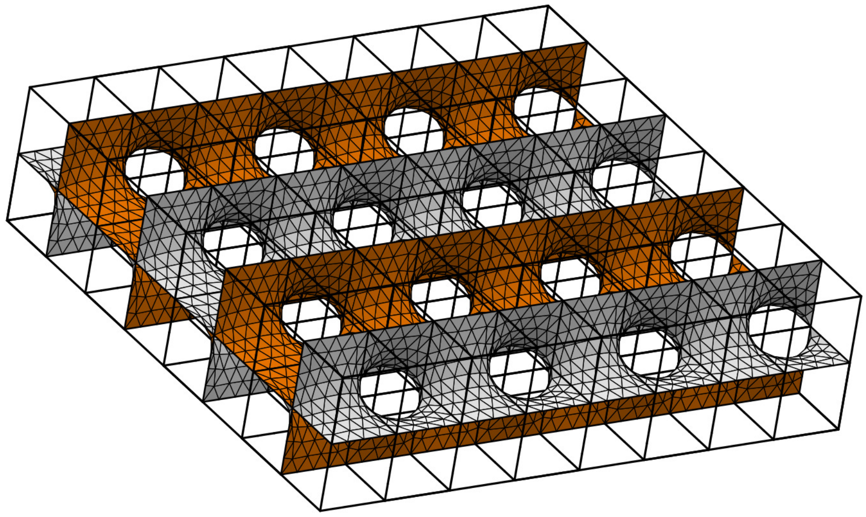


Figure 1.2: The Schwarz CLP surface in $\mathbb{E}^3$. Note that $\mathbb{E}^3$ is naturally tiled by cubes and that a cube is either empty or contains a patch as in Figure 1.1

copies of the Plateau solution, see Figure 1.2. Note that $\mathbb{E}^3$ is tiled by cubes and a cube is either empty or contains a surface patch $\mathcal{P}$ of the CLP surface, see Figure 1.1. We see that the boundary of the surface patch is a Jordan curve Γ consisting of six edges of the cube, i.e., a closed continuous cycle on the edge set of the cube. It can be shown that the surface patch corresponds to the Plateau solution subject to Γ.

Consider four cubes meeting at an edge e. We see that two of the cubes each contain a surface patch and that the other two are empty, see Figure 1.3. One patch can be mapped onto the other by a 180°-rotation about e. The Schwarz reflection principle shows that this rotation about a straight line, called *Schwarz reflection*, extends the surface smoothly. By successively Schwarz reflecting about every edge of Γ, we can construct the CLP surface.

Hence one can construct the CLP surface as follows: take a suitable Jordan curve Γ consisting of edges of a cube, solve the Plateau problem subject to Γ, extend the Plateau solution to a complete surface using Schwarz reflection. In this context, the CLP surface is a surface generated by Schwarz reflection. By considering a different curve, one can similarly construct the Schwarz D surface.

Since Schwarz reflection extends the surface smoothly, we can take any Jordan curve consisting of straight lines and extend the Plateau solution to

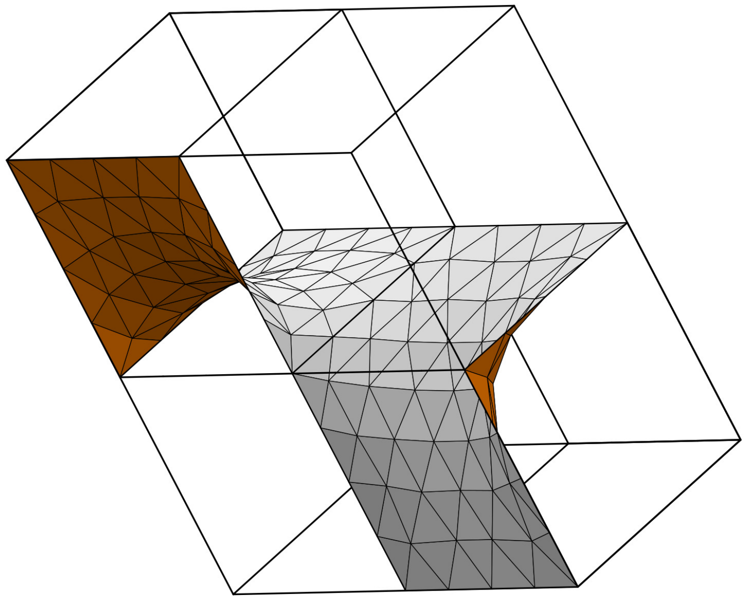


Figure 1.3: The Plateau solution in Figure 1.1 can be extended by Schwarz reflection. The CLP surface in Figure 1.2 can be constructed by successively Schwarz reflecting about all edges of the boundary of the Plateau solution.

a complete minimal surface. This yields a large family of complete minimal surfaces. Of special interest are embedded minimal surfaces, i.e., minimal surfaces without self-intersections, since the ones with self-intersection are abundant, so finding them does not pose a problem.

The goal of this thesis is to construct embedded minimal surfaces in the n-dimensional product space $\mathbb{X}^n$. From both a geometric and algebraic point of view, surfaces generated by Schwarz reflection are not well understood. On the other hand, tessellations generated by facet reflection are very well-known. The main idea is to use the structure of the tessellation of the space $\mathbb{X}^n$ to study surfaces generated by Schwarz reflection. In the example of the CLP surface, Schwarz reflection is compatible with the tessellation of $\mathbb{E}^3$ by cubes, i.e., we see that in Figure 1.3, Schwarz reflection about an edge e can be expressed as the composition of the reflection across the two facets containing e. This is wrong in general: consider a prism in $\mathbb{E}^3$ where the base is an equilateral triangle, see Figure 1.4. Note that in this case Schwarz reflection about an edge e and the composition of the reflections along the facets containing e yield different results.

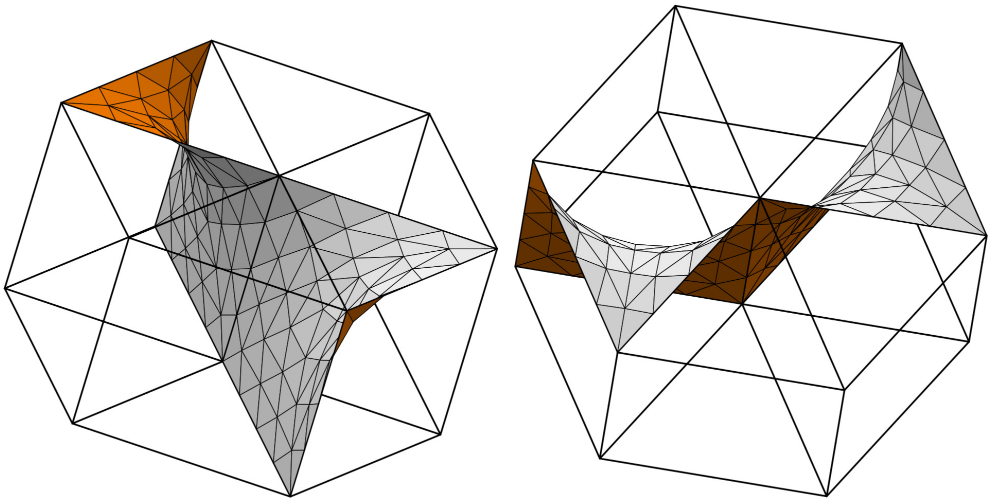


Figure 1.4: On the left hand side, we see that Schwarz reflection about an edge e extends the Plateau solution smoothly. On the right hand side, the composition of the two facets containing e gives a different result than Schwarz reflection. In this case e does not satisfy the (-1)-condition.

Thus we need a condition on an edge e, so that Schwarz reflection about e and the composition of the reflection across the facets containing e coincide. Using the terminology of the (-1)-*condition*. If all edges of Γ satisfy the (-1)-condition, we can identify the group generated by Schwarz reflection as a subgroup of the group generated by the reflections across the facets. The problem in finding embedded minimal surfaces reduces to finding suitable polytopes $P \subset \mathbb{X}^n$. Then, on the edge set of P, we define a curve Γ such that every edge satisfies the (-1)-condition. In this case, the main result of this thesis, Theorem 4.2.10, gives a necessary and sufficient condition such that the minimal surface generated by Schwarz reflection is embedded.

Since the result is purely group-theoretical, we can apply it not only to minimal surfaces, but also to other classes of surfaces which stay invariant under Schwarz reflection, e.g., discrete surface which can be used to approximate surfaces numerically, even if no closed form expressions for this surface are known. Furthermore, the result can be used to formulate existence theorems for surfaces generated by Schwarz reflection. Hence, instead of computing the Plateau solution with respect to Γ, we will inscribe a surface patch $\mathcal{P}$ and assume that $\mathcal{P}$ is embedded and lies in the interior of P. By Dehn's lemma [MY82] and a Lemma of Radó [Oss02], the Plateau solution is embedded in dimension $n = 3$ and in the case $\mathbb{E}^n$ for arbitrary $n \in \mathbb{N}$. Using the maximum principle, one can conclude that the Plateau solution lies in the interior of $P \subset \mathbb{E}^n$ However, note that no general result for $\mathbb{X}^n$ is known such that a Plateau solutions is embedded and lies in the interior of P.

With the ideas above, we give a purely group-theoretical approach for the construction of complete embedded minimal surfaces in $\mathbb{X}^3$ in terms of the Schwarz reflection principle. Our construction extends to n-dimensional homogeneous manifolds, primarily to product spaces of $\mathbb{E}^k$, $\mathbb{S}^k$, and $\mathbb{H}^k$. The problem lies in finding properties for the polytope $P \subset \mathbb{X}^n$ and the Jordan curve Γ such that the surface $\mathcal{S}$ constructed by Schwarz reflection of $\mathcal{P}$ is embedded. Since this approach is not only applicable to minimal surfaces, we will call Schwarz reflection along an edge of P *edge reflection*. We establish properties for P and the group generated by the reflection along the facets of P which results in a tessellation of $\mathbb{X}^n$ tiled by P, in the second chapter. Using the well-understood structure of the tessellation, we establish properties of the Jordan curve which is compatible with the structure of the tessellation. This gives us the means to study the more complicated edge reflections. At the end of the fourth chapter, Theorem 4.2.10, the main result of the thesis, states a purely group-theoretical condition depending on the reflection group generated by the facets of P and edge reflections of Γ to decide whether $\mathcal{S}$ is embedded or has self-intersections.

In the second chapter, we study polytopes in $\mathbb{X}^n \in \{\mathbb{E}^n, \mathbb{S}^n, \mathbb{H}^n\}$ resulting in the definition of special polytopes called *Coxeter polytopes*. Essentially, a Coxeter polytope is a simple polytope where the dihedral angles are of the form π/k, $k \geq 2$. The main result is that a Coxeter polytope tessellates the space $\mathbb{X}^n$ by reflection along its $(n-1)$-dimensional faces, i.e., its facets. For the entire chapter, we closely follow standard theory [DC08], [Bou08], and [Hum90] with minor changes.

We start by introducing the spaces $\mathbb{E}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$ as subsets of Euclidean space $\mathbb{R}^{n+1}$ and study reflections in $\mathbb{X}^n$ as linear automorphisms which fix a hyperplane of a half-space in $\mathbb{R}^{n+1}$. Next, we define a polytope $P^n \subset \mathbb{X}^n \subset \mathbb{R}^{n+1}$ as the intersection of half-spaces in $\mathbb{X}^n$. This can be realised as the intersection of a polyhedral cone (an intersection of linear half-spaces in $\mathbb{R}^{n+1}$) and $\mathbb{X}^n$ if $\mathbb{X}^n = \mathbb{S}^n, \mathbb{H}^n$. This approach allows us to deal with the two spaces, $\mathbb{S}^n$, and $\mathbb{H}^n$ simultaneously. Using the inward-pointing unit normals of the facets of P^n, we define the dihedral angle θ_{ij} between every pair of facets of P^n. On the one hand, if the angles are non-obtuse, then P^n is a simple polytope (which is one of the conditions of a Coxeter polytope), i.e., exactly n facets meet at each vertex of P^n. On the other hand, given numbers θ_{ij}, we can define an n-simplex in $\mathbb{X}^n$. This information can also be stored in the Gram Matrix (c_{ij}) yielding that P^n is, up to isometry, determined by its Gram matrix, i.e., the dihedral angles, in the case $\mathbb{S}^n$ and $\mathbb{H}^n$, and up to homothety in $\mathbb{E}^n$.

Let s_i and s_j be the two reflections across the facets F_i and F_j. If the dihedral angle θ_{ij} enclosed by F_i and F_j is of the form π/k_{ij}, $k_{ij} \geq 2$, we see that successively using s_i and s_j results in the relation $(s_i \circ s_j)^{k_{ij}} = \mathrm{id}$, as seen in Figure 1.5.

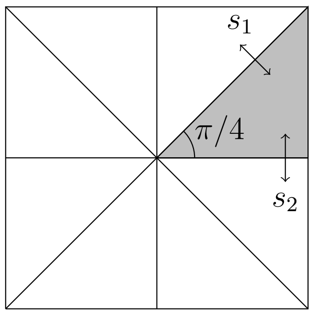


Figure 1.5: The composition $s_1 \circ s_2$ of the two reflection s_1 and s_2 rotates the triangle counter-clockwise by $2 \cdot \pi/4$ thereby fixing the intersection of the facets F_1 and F_2. Thus $(s_1 \circ s_2)^4 = \mathrm{id}$.

Instead of storing the dihedral angles in the Gram matrix (c_{ij}), we can store the order of the reflections in a matrix M, called Coxeter matrix. H.S.M. Coxeter [Cox34] introduced a special diagram called Coxeter diagram which is a graph with numerically labelled edges representing the order of the reflections. The Coxeter graph captures different properties of P^n such as the symmetries of P^n and whether P^n is an n-simplex or a product of simplices. Furthermore, we can associate with the Coxeter matrix M (hence with the Coxeter diagram) a presentation for a group $W = \langle S \mid R \rangle$ where the set S generates W subject to the relations R. Essentially, the generators in S can be associated with the reflections s_i and the relations in R to the order k_{ij} of a pair of reflections. The group W together with its generators S is called a Coxeter system (W, S) and constitutes the main object used in the thesis. Our approach makes use of the fact that simplicial Coxeter groups are completely classified, a result we prove in Section 2.5 and 2.6. Consequently, we can systematically study Coxeter polytopes in $\mathbb{X}^n$.

Using Theorem 2.9.6, we can identify the tessellation of $\mathbb{X}^n$ with the Coxeter complex $\mathcal{U}(W, P^n) = W \times P^n / \sim$, where $\sim$ represents the gluing of the copies of P^n, i.e., the tiles of the tessellation of $\mathbb{X}^n$ or chambers of $\mathcal{U}$. Hence the tessellation of $\mathbb{X}^n$ depends only on the polytope and its corresponding Coxeter system. Furthermore, there is a one-to-one correspondence between elements in W and chambers in $\mathcal{U}$.

Following again standard theory [DC08], [Hum90], and [Bou08] in the third chapter, we define a faithful representation $\rho\colon W \to GL(V)$ for the Coxeter group W called the *canonical representation*. It is constructed by using a bilinear form defined by the Gram matrix of the polytope P^n and allows us to express group elements of W in terms of matrices in $GL(V)$. Furthermore, it shows that every Coxeter group is linear, allowing us to apply properties of linear groups, such as Selberg's Lemma.

Using the canonical representation, we study a distinguished element $w_0 \in W$, called the longest element of a Coxeter group, in which case longest refers to a length function on W. Of particular interest are longest elements w_T of so-called special subgroups W_T of the Coxeter group W. If w_T is mapped onto $-\operatorname{id}_V$, we say that W_T (or the edge e_i which is fixed by W_T) satisfies the (-1)-condition.

In the fourth chapter, we will identify edge reflection j along an edge $e \subset P^n$ with the longest element w_T of a special subgroup T of W provided it satisfies the (-1)-condition. Then edge reflection is compatible with the tessellation of $\mathbb{X}^n$ tiled by P^n.

We will distinguish between three cases which we motivate in the following.

First, consider an isosceles triangle Δ with angles $\pi/2$, $\pi/4$, and $\pi/4$. As in Figure 1.5, we subdivide the square into 8 copies of Δ by using the reflection s_1 and s_2 along two sides of Δ. Let j be the point reflection about the center of the square. It is easy to see that j and $(s_1 \circ s_2)^2$ coincide, see Figure 1.6. Furthermore, $(s_1 \circ s_2)^2$ maps the triangle Δ onto the triangle on the opposite side in the structure of the tessellation of the square. For now, this should motivate the name choice of the longest element.

Second, consider an equilateral triangle Δ with angles $\pi/3$, $\pi/3$, and $\pi/3$. Again, pick two sides of Δ and inspect the reflections s_1 and s_2 along these sides. After successively reflecting s_1 and s_2, we get a hexagon divided by 6 copies of Δ, see Figure 1.7. Notice that $s_1 \circ s_2 \circ s_1$ maps Δ onto the opposite side in the hexagon. Let j be the point reflection about the center of the

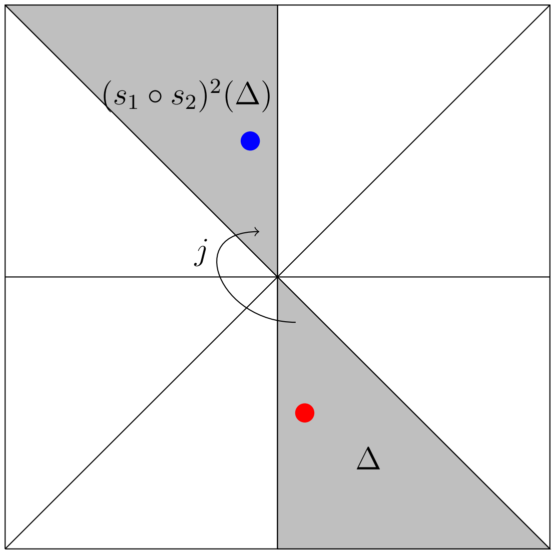


Figure 1.6: The point reflection j about the center of the square coincides point-wise with the composition $(s_1 \circ s_2)^2$ of the reflection along the sides of Δ. Note that j and $(s_1 \circ s_2)^2$ both map the red dot onto the blue dot.

hexagon. Although j and $s_1 \circ s_2 \circ s_1$ map Δ onto the same triangle as a set, they do not coincide point-wise; hence are different.

Third, consider a triangle Δ with angles $\pi/2$, $\pi/3$, and $\pi/6$. Pick the two sides enclosing the angle $\pi/3$ and denote the reflection along these sides with s_1 and s_2. By successively reflecting along s_1 and s_2, we get an equilateral triangle. Let j be the point reflection along the midpoint of the equilateral triangle. We see in Figure 1.8 is not compatible with the structure constructed by the reflection s_1 and s_2.

We study Jordan curves Γ, which we call *edge cycle*, such that the edge reflection along each edge of Γ is compatible with the tessellation, i.e., has a similar behaviour as in the first case above. Thus we decide for every edge e_i of a polytope P^n whether e_i satisfies the (-1)-condition. For this we define a graph $G = (V, E)$, called the *edge reflection graph*, where the vertex set V consists of all vertices of P^n and the edge set E consists of all edges of P^n which satisfy the (-1)-condition. An edge cycle Γ defined on the edge reflection graph naturally respects the structure of the tessellation, i.e., is compatible with the tessellation. Hence we can express every edge reflection as an element in W. Thus the group $\mathcal{J}$ generated by edge reflections along

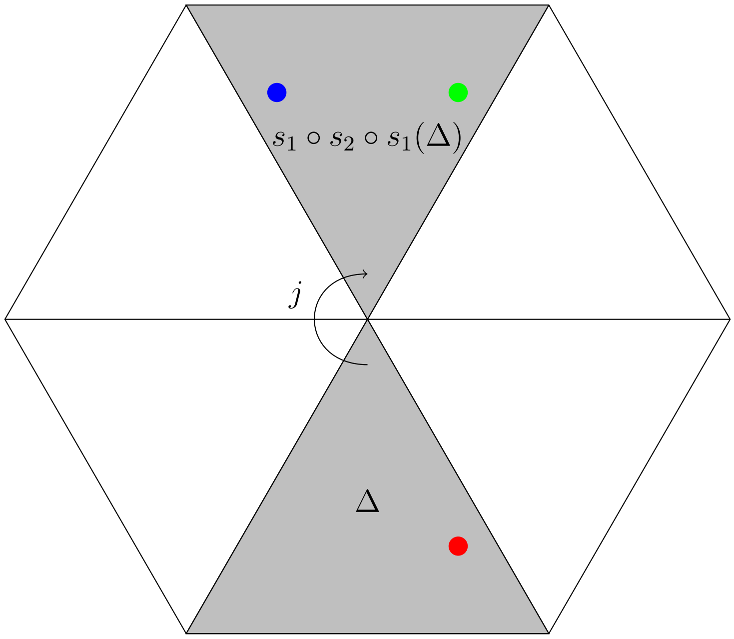


Figure 1.7: The point reflection j about the center of the hexagon and $s_1 \circ s_2 \circ s_1$ map Δ onto the same triangle, but they do not coincide point-wise. Note that j maps the red dot onto the blue dot, while $s_1 \circ s_2 \circ s_1$ maps the red dot onto the green dot. Also compare this case with Figure 1.4.

every edge of Γ is a subgroup of W.

Assume that $\mathcal{P}$ is an embedded surface which lies in the interior of a Coxeter polytope P^n and has boundary Γ, e.g., the Plateau solution. Using edge reflection, we extend the surface patch $\mathcal{P}$ to a complete surface $\mathcal{S}$, e.g., a complete minimal surface. We finish the third chapter with a discussion about the embeddedness of $\mathcal{S}$. The main result of this thesis gives a necessary and sufficient condition which relies only on the groups W, $\mathcal{J}$, and subgroups of those. Theorem 4.2.10 shows that it is sufficient for the embeddedness of $\mathcal{S}$ to check all vertices v_i of Γ.

In the fifth chapter, we will give a short discussion how the construction can be applied to product spaces of $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$ by considering totally geodesic Coxeter polytopes. We discuss how to determine the genus of the surface $\mathcal{S}$; it is defined with respect to the smallest fundamental domain of $\mathcal{S}$. While the determination of this minimal fundamental domain is straightforward in $\mathbb{E}^n$, the construction in $\mathbb{H}^n$ is more involved. The main

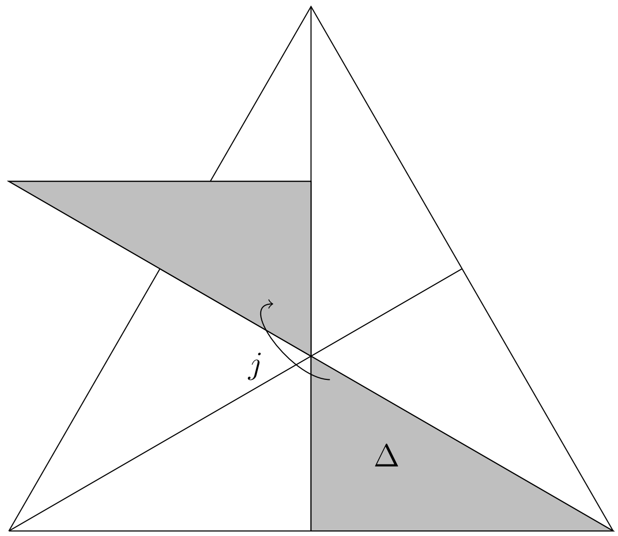


Figure 1.8: The point reflection j about the midpoint of is not mapping Δ onto a copy of Δ in the tessellation generated by s_1 and s_2.

part of the fourth chapter is devoted to the application of Theorem 4.2.10 to 5 three-dimensional and 10 four-dimensional cases to check if a given edge cycle Γ on P^n yields a complete embedded surface $\mathcal{S}$. For most of the specific cases we use GAP (Groups, Algorithm, and Programming), a computer algebra system for computational discrete algebra, to confirm the condition stated in Theorem 4.2.10.

In the last chapter, we will give two generalisations of our the construction. The first generalisation includes the second case discussed above (see Figure 1.7) where edge reflection and the longest element map Δ onto the same set but are not point-wise identical. Then the two mappings coincide after applying a symmetry of P^n. In the case depicted in Figure 1.7, this corresponds to the symmetry of Δ which maps the blue dot onto the green dot. This generalisation can be realised by defining a new action on $\mathcal{U}(W, P)$. We analyse two examples for this generalisation.

For the second generalisation, we allow Γ to consist of edges of a union of simplices or their products, i.e., we define an edge cycle Γ on the Coxeter complex $\mathcal{U}(W, P^n)$. Again, we discuss two examples for this generalisation.

As a summary, we start with a totally geodesic Coxeter simplex P of dimension n in a product space $\mathbb{X}^n$ with factors in $\{\mathbb{S}^k, \mathbb{E}^k, \mathbb{H}^k\}$. The reflection along the facets of P^n naturally gives us a Coxeter system (W, S).

Using P^n and (W, S), we define a tessellation of $\mathbb{X}^n$. After that we define an edge cycle Γ on the edge set of P^n which is compatible with $\mathcal{U}(W, P)$, the tessellation of $\mathbb{X}^n$, i.e., every edge satisfies the (-1)-condition. Consequently, the group $\mathcal{J}$ generated by edge reflection of Γ is a subgroup of W. We inscribe an embedded surface $\mathcal{P}$ which lies in the interior of P^n and has boundary Γ and use $\mathcal{J}$ to construct a complete surface $\mathcal{S}$ in $\mathbb{X}^n$. Theorem 4.2.10 shows whether the surface $\mathcal{S}$ is embedded.

The results and open problems of the thesis can be summarised as follows.

In Section 5.4, we discuss the 5 three-dimensional cases $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{E}$, $\mathbb{E}^3$, $\mathbb{H}^2 \times \mathbb{E}$, and $\mathbb{H}^3$. Assuming that $\mathcal{S}$ is embedded, we list all edge cycles which possibly can be used to construct $\mathcal{S}$. Conversely, except in the case $\mathbb{H}^2 \times \mathbb{E}$, we classify all edge cycles Γ such that the constructed surface $\mathcal{S}$ is embedded.

In Section 5.5, we consider the 10 four-dimensional cases for $\mathbb{X}^n$. Again, assuming that $\mathcal{S}$ is embedded, we give all curves which possibly can be used to construct $\mathcal{S}$. Conversely, we prove for all curves Γ, except $\mathbb{X}^n$ contains a factor $\mathbb{H}^2$ or $\mathbb{H}^3$, that they yield an embedded surface $\mathcal{S}$.

In Section 5.6, we give a brief discussion of the cases $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$. Assuming that $P^n \subset \mathbb{X}^n$ is an n-simplex, we show that there is no curve Γ which yields an embedded surface $\mathcal{S}$ in the case $n \geq 4$. Since every Coxeter polytope on $\mathbb{S}^n$ is an n-simplex, we can omit the assumption in the case $\mathbb{S}^n$. However, for the case $\mathbb{E}^n$ Kürsten showed in [Kür14] that in every dimension a complete embedded surface can be constructed provided that P^n is a cube. In the case $\mathbb{H}^n$ a theorem of Vinberg [Vin85] shows that no compact Coxeter polytope can exists for $n \geq 29$. Using edge reflection, it remains open to show whether a complete embedded surface can be constructed for $\mathbb{S}^n, n \geq 4$, and $\mathbb{H}^n, n \leq 28$.

In Section 6.2., we extend the idea of defining a curve Γ on the tessellation of $\mathbb{X}^n$, i.e., on unions of chambers in $\mathcal{U}(W, P)$. Furthermore, we extend the notion of the (-1)-condition to the second case discussed above where edge reflection and facet reflection coincide set-wise but not point-wise. This gives a wide zoo of curves Γ to study which is not part of the thesis. As examples, we discuss the construction of the Lawson surfaces $\eta_{n,k}$ and for a bowtie-like curve in $\mathbb{H}^2 \times \mathbb{E}$.

2 Tessellation of Homogeneous Manifolds

In this chapter, we first introduce reflections on the three geometries of constant curvature, namely the n-sphere $\mathbb{S}^n$, Euclidean n-space $\mathbb{E}^n$, and hyperbolic n-space $\mathbb{H}^n$. We will use the common notation $\mathbb{X}^n$ for any of these spaces. By scaling the metric appropriately, we may assume that the sectional curvature is 1, 0, or -1 respectively. This section follows with some minor changes Section 6.1 and 6.2 in [DC08].

Second following Section 6.3 in [DC08], we define polytopes $P^n \subset \mathbb{X}^n$ as an intersection of a polyhedral cone in $\mathbb{R}^{n+1}$ and $\mathbb{X}^n$. Furthermore, we show some properties of polytopes with non-obtuse angles, e.g., a polytope with non-obtuse dihedral angles is always simple.

In the third and fourth section, we show that n-simplices are determined by their dihedral angles, which can be decoded in a Gram matrix and Coxeter matrix. The Coxeter matrix can be used to define a natural group structure that admits a formal description of reflections, namely Coxeter groups. These sections are based on Section 6.7, Section 6.8 and Chapter 3 in [DC08].

After that, we will classify the simplicial Coxeter groups. This will allow us to discuss suitable polytopes for the construction of the Coxeter complex $\mathcal{U}(W, P^n)$ in the last two sections. For the classification, we follow Appendix C in [DC08], which is based on Chapter 2 in [Hum90].

Finally, in the last two sections, following Section 6.4 and 6.5 in [DC08] we will construct the Coxeter complex $\mathcal{U}(W, P^n)$ and discuss some of its properties. In the main theorem of this chapter, we will proof that $\mathcal{U}(W, P^n)$ is a manifold homeomorphic to $\mathbb{X}^n$ and tiled by isometric copies of P^n. Furthermore, it shows that the reflection group along the codimension-one facets of P^n is a Coxeter group.

2.1 Reflections in $\mathbb{X}^n$

A *linear reflection* on a finite-dimensional real vector space V is a linear automorphism $r\colon V \to V$ such that $r^2 = \mathrm{id}_V$ and such that the fixed subspace of r is a hyperplane H. Let w be a (-1)-eigenvector and α a linear form on V with kernel H. Then r can be defined by the following formula:

$$r(v) = v - 2\frac{\alpha(v)}{\alpha(w)}w. \tag{2.1}$$

Note that the second term is just twice the projection of v onto w.

Conversely, suppose α is a linear form on V whose kernel is a hyperplane H and $w \in V$ is a vector with $\alpha(w) \neq 0$. Then the linear transformation r defined by (2.1) fixes the hyperplane H, i.e., $\alpha(v) = 0$ for all $v \in H$. Furthermore, with

$$\alpha(r(v)) = \alpha(v) - 2\frac{\alpha(v)}{\alpha(w)}\alpha(w) = -\alpha(v)$$

we get that

$$r(r(v)) = r(v) - 2\frac{\alpha(r(v))}{\alpha(w)}w = v - 2\frac{\alpha(v)}{\alpha(w)}w + 2\frac{\alpha(v)}{\alpha(w)}w = v.$$

Hence $r^2 = \mathrm{id}_V$ and r is a linear reflection.

Since r is linear and $r^2 = \mathrm{id}_V$, we have $\langle rv, rw\rangle = \langle r^2 v, w\rangle = \langle v, w\rangle$ for all $v, w \in V$. Thus r has order 2 in the group $O(V)$ of orthogonal transformations of V.

In the next step, we want to discuss how two distinct linear reflections interact with each other. This will be key for the algebraic generalisation of abstract reflection groups we will define and discuss in more detail in Section 4 ongoing.

Definition 2.1.1. A *dihedral group* is a group generated by two elements of order 2.

Example 2.1.2. *(Finite dihedral groups).* Consider two distinct lines L and L' through the origin in $\mathbb{R}^2$ and let r_L respectively $r_{L'}$ be the reflections across L respectively L'. Furthermore, let θ be the angle between L and

L', see Figure 2.1. Then, $r_L \circ r_{L'}$ describes a rotation through 2θ. Thus, if $\theta = \pi/m$ where $m \geq 2$, then $r_L \circ r_{L'}$ is a rotation through $2\pi/m$. Hence $r_L \circ r_{L'}$ has order m, i.e., $(r_L \circ r_{L'})^m = \mathrm{id}_V$ and the group generated by r_L and $r_{L'}$ is a dihedral group of order m.

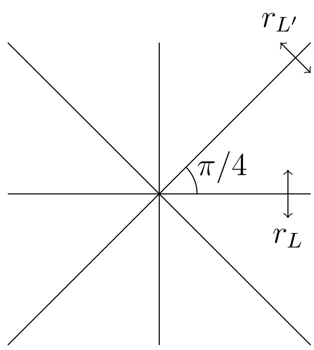


Figure 2.1: The angle between L and L' is $\pi/4$. Hence the reflections r_L and $r_{L'}$ generate a finite dihedral group of order 8. Thus $(r_L \circ r_{L'})^4 = \mathrm{id}_V$.

Example 2.1.3. *(The infinite dihedral group).* Let $V = \mathbb{R}$ and let $r(t) = -t$ and $r'(t) = 2 - t$. Thus r and r' denote the reflections along the points 0 and 1, respectively. Then $r \circ r'$ is a translation by 2 and hence the group generated by r and r' has infinite order, see Figure 2.2.

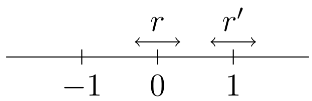


Figure 2.2: The reflections r and r' generate the infinite dihedral group. Since $r \circ r'$ is a translation there is no $m \in \mathbb{N}$ such that $(r \circ r')^m = \mathrm{id}_V$.

In each dimension $n \geq 2$, there are three simply connected, complete Riemannian manifolds of constant sectional curvature: the n-sphere $\mathbb{S}^n$, Euclidean n-space $\mathbb{E}^n$, and the hyperbolic n-space $\mathbb{H}^n$. As mentioned above, we will use the notation $\mathbb{X}^n$ for any of these three spaces. One of the main features of these spaces, important in the study of reflection groups, is that each admits many totally geodesic codimension-one subspaces which we will call *hyperplanes*. Each hyperplane separates the ambient space into two geodesically convex *open half-spaces*. When we refer to a *half-space* we will always mean the closure of an open half-space, i.e., its union with the hyperplane.

We shall briefly review the three standard models below. One focus of this review is that all three spaces are defined in the ambient space $\mathbb{R}^{n+1}$. This allows us to realise reflections in $\mathbb{S}^n$ and $\mathbb{H}^n$ as linear transformation of $\mathbb{R}^{n+1}$. Hence we can deal with the different geometries in a uniform way.

Euclidean n-space $\mathbb{E}^n$

The standard model for n-dimensional Euclidean geometry is the n-dimensional real vector space $\mathbb{R}^n$. A vector in $\mathbb{R}^n$ is an ordered n-tuple $x = (x_1, \dots, x_n)$ of real numbers. The standard inner product on $\mathbb{R}^n$ is a symmetric, positive definite, non-degenerate, bilinear form defined by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

As a linear space, $\mathbb{R}^n$ agrees with its tangent space at any point. Considered as a Riemannian manifold, $\mathbb{R}^n$ has vanishing sectional curvature.

Suppose H is a hyperplane in $\mathbb{R}^n$ through $x_0 \in H$ with unit vector $u \in \mathbb{R}^n$ orthogonal to H. Then the orthogonal reflection r_H across H is given by the formula

$$r_H(x) = x - 2\langle u, x - x_0 \rangle u,$$

where $x - x_0$ is the vector in $\mathbb{R}^n$, which translates the point x_0 to x.

The group of affine automorphisms of $\mathbb{R}^n$ is given as the semi-direct product $\mathbb{R}^n \rtimes GL(n)$, where the action of $GL(n)$ on $\mathbb{R}^n$ is via the standard representation and $\mathbb{R}^n$ is the group of translations of $\mathbb{R}^n$. The subgroup $\mathbb{R}^n \rtimes O(n)$ is the group of isometries of $\mathbb{R}^n$. We can identify $\mathbb{R}^n$ with the affine hyperplane in $\mathbb{R}^{n+1}$ defined by the equation $x_{n+1} = 1$ and will refer to this as $\mathbb{E}^n$.

The n-sphere $\mathbb{S}^n$

We define $\mathbb{S}^n$ as a quadratic surface in $\mathbb{R}^{n+1}$ by $\{x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = 1\}$. The tangent space $T_x\mathbb{S}^n$ is naturally identified with $x^\perp$, the subspace of $\mathbb{R}^{n+1}$ orthogonal to x. The inner product on $\mathbb{R}^{n+1}$ induces an inner product on $T_x\mathbb{S}^n$ and hence, a Riemannian metric on $\mathbb{S}^n$. One can show that $\mathbb{S}^n$ equipped with this metric has constant sectional curvature equal to 1.

A hyperplane (and a half-space) in $\mathbb{S}^n$ is its intersection with a linear hyperplane (and a linear half-space) in $\mathbb{R}^{n+1}$, i.e., a hyperplane is a great subsphere and a half-space is a hemisphere. Suppose H is a hyperplane in $\mathbb{S}^n$ and u is a unit vector in $\mathbb{R}^{n+1}$ orthogonal to H. The spherical reflection r_H of $\mathbb{S}^n$ across H is given by

$$r_H(x) = x - 2\langle u, x\rangle u.$$

We have that r_H is an isometry of $\mathbb{S}^n$ with fixed set H and H is a totally geodesic submanifold of codimension-one in $\mathbb{S}^n$.

The group $O(n+1)$ of linear automorphisms of $\mathbb{R}^n$ is the full group of isometries of $\mathbb{S}^n$.

The hyperbolic n-space $\mathbb{H}^n$

A symmetric bilinear form on an $(n+k)$-dimensional real vector space is said to be of *signature* (n, k), if it has n positive eigenvalues and k negative eigenvalues. A positive semi-definite form on a $(n+1)$-dimensional vector space with precisely one eigenvalue 0 is said to be of corank 1. Let $\mathbb{R}^n_1$ denote the $(n+1)$-dimensional real vector space equipped with the symmetric bilinear form $\langle \cdot, \cdot \rangle_1$ with signature $(n, 1)$ defined by

$$\langle x, y\rangle_1 \coloneqq \sum_{i=1}^{n} x_i y_i - x_{n+1}y_{n+1}.$$

This space is also called the *Minkowski space*. A k-dimensional subspace is called *spacelike*, *timelike*, or *lightlike* if the restriction of the bilinear form to V is, respectively, positive definite, with signature $(k, 1)$, or of corank 1. The quadratic surface defined by $\langle x, y\rangle_1 = -1$ is a two-sheeted hyperboloid. The sheet defined by $x_{n+1} > 0$ is the hyperboloid model of the *hyperbolic n-space* $\mathbb{H}^n$.

Similar to the n-sphere, the tangent space $T_x\mathbb{H}^n$ is naturally defined as $x^\perp$, the subspace of $\mathbb{R}^n_1$ orthogonal to x. Since x is timelike (by $x_{n+1} > 0$) one can show that $x^\perp$ is spacelike. Thus $\mathbb{H}^n$ naturally is a Riemannian manifold. It has constant sectional curvature equal to -1.

A hyperplane in $\mathbb{H}^n$ is the intersection of $\mathbb{H}^n$ with a timelike linear hyperplane of $\mathbb{R}^n_1$. It can be shown that it is a submanifold isometric to $\mathbb{H}^{n-1}$. Similarly, a half-space of $\mathbb{H}^n$ is the intersection of $\mathbb{H}^n$ with a linear half-space of $\mathbb{R}^n_1$ bounded by a timelike hyperplane. Suppose H is a hyperplane in $\mathbb{H}^n$ and u is a unit spacelike vector orthogonal to H. The hyperbolic reflection r_H of $\mathbb{H}^n$ across H is the restriction to $\mathbb{H}^n$ of the orthogonal reflection of $\mathbb{R}^n_1$ defined by

$$r_H(x) = x - 2\langle u, x\rangle_1 u.$$

We have that r_H is an isometry of $\mathbb{H}^n$ with fixed set H and H is a totally geodesic submanifold of codimension one in $\mathbb{H}^n$.

The group $O(n,1)$ of isometries of the bilinear form on $\mathbb{R}^n_1$ has four connected components. There is an index two subgroup $O^+(n,1)$, which preserves the sheets of the hyperboloid. It is the full group of isometries of $\mathbb{H}^n$.

There are several other models of $\mathbb{H}^n$. One of the most useful is the *Poincaré ball model*, where points of $\mathbb{H}^n$ correspond to the points of the open ball and geodesics are circular arcs perpendicular to the bounding sphere. While the hyperboloid model is useful to draw the analogy to $\mathbb{S}^n$ and to classify isometries, the ball model is useful to visualise the geometry, since the metric is conformal to the one on $\mathbb{R}^n$.

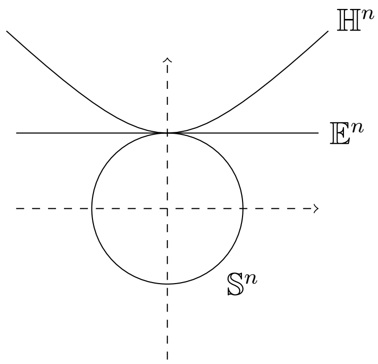


Figure 2.3: The three sets $\mathbb{E}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$ as subsets of $\mathbb{R}^{n+1}$.

As mentioned before, we will use the notation $\mathbb{X}^n$ for any of the spaces $\mathbb{E}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$. Furthermore, we will use the notation $\mathrm{Isom}(\mathbb{X}^n)$ for the isometry group of $\mathbb{X}^n$. Also, we will use the following well-known facts (the proofs can be found in [Rat06]):

Theorem 2.1.4. *(i) The isometry group* $\mathrm{Isom}(\mathbb{X}^n)$ *acts transitively on* $\mathbb{X}^n$*, i.e., for each* $x, y \in \mathbb{X}^n$ *there exists* $g \in \mathrm{Isom}(\mathbb{X}^n)$ *such that* $gx = y$*.*

(ii) Every element of $\mathrm{Isom}(\mathbb{X}^n)$ *can be written as a product of reflections.*

(iii) For each $x \in \mathbb{X}^n$*, the isotropy subgroup* $G_x = \{g \in \mathrm{Isom}(\mathbb{X}^n) \mid gx = x\}$ *of* $\mathrm{Isom}(\mathbb{X})$ *at* x *acts on* $T_x\mathbb{X}^n$ *and gives an identification of the isotropy subgroup with the group of linear isometries of* $T_x\mathbb{X}^n$*, i.e., with* $O(n)$*.*

2.2 Polytopes in $\mathbb{X}^n$

The classical construction of the Schwarz CLP surface is as follows: take a cube in $\mathbb{R}^3$ and define a suitable Jordan curve Γ along the edges of the cube. The Plateau solution yields a minimal surface patch with boundary Γ. After that, use the Schwarz Reflection Principle to construct a complete embedded minimal surface by reflection along the edges defined by Γ.

Our goal is to generalise this construction in three ways. First, we want to consider not only cubes but different polytopes. Second, we extend the construction to dimension n, and third, we extend the construction to product spaces of $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$.

The construction of the Schwarz CLP surface naturally yields a tessellation of $\mathbb{E}^3$ by cubes. This tessellation is constructed by reflection of the cube along its codimension-one faces. Thus it is natural to consider reflection groups of polytopes generated by reflection along their codimension-one faces. The main goal of this chapter is to establish conditions to the polytope such that the polytopes tessellate $\mathbb{X}^n$ and to study the resulting reflection groups.

In this section, we define polytopes and discuss their properties under natural assumptions in order to obtain a tessellation. We show that polytopes with non-obtuse dihedral angles are simple in $\mathbb{X}^n$. In particular, in the spherical case we get that such a polytope is an n-simplex. In the Euclidean case we get that it is either an n-simplex or a product of lower dimensional simplices.

Essentially, we define a polytope in $\mathbb{X}^n$ as an intersection of half-spaces $\mathbb{R}^{n+1}$ with the space $\mathbb{X}^n$. In $\mathbb{S}^n$ and $\mathbb{H}^n$, this can be realised as the intersection of a (linear) polyhedral cone in $\mathbb{R}^n$ with the space $\mathbb{S}^n$ and $\mathbb{H}^n$ This envision motivates the following definition:

Definition 2.2.1. A *polyhedral cone* $C \subset \mathbb{R}^{n+1}$ is the intersection of a finite number of linear half-spaces in $\mathbb{R}^{n+1}$. The *dimension* of the polyhedral cone is the dimension of the affine space it spans, i.e., the dimension of the smallest affine space containing C. We say a polyhedral cone is *essential*, if it contains no line, i.e., the intersection with $\mathbb{S}^n$ contains no antipodal points. An essential polyhedral cone is called *simplicial* if it is $(n+1)$-dimensional and is the intersection of $n+1$ linear half-spaces.

Suppose C is a polyhedral cone and H a supporting hyperplane. Then $C \cap H$ is also a polyhedral cone. We call $C \cap H$ a *face* of C. We call 1-dimensional faces of C *extremal rays*.

Example 2.2.2. Consider the positive octant of $\mathbb{R}^3$. This octant can be described as the intersection of the three half-spaces defined by the equations $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. It has obviously dimension 3 and contains no line. Hence it is a essential simplicial cone in $\mathbb{R}^3$ of dimension 3. Its extremal rays are the coordinate axes.

Definition 2.2.3. A *polytope in* $\mathbb{E}^n$ is a compact intersection of a finite number of half-spaces in $\mathbb{E}^n \subset \mathbb{R}^{n+1}$. A *polytope in* $\mathbb{S}^n$ is the intersection of $\mathbb{S}^n$ with an essential polyhedral cone $C \subset \mathbb{R}^{n+1}$. A *polytope in* $\mathbb{H}^n$ is the intersection of $\mathbb{H}^n$ with a polyhedral cone $C \subset \mathbb{R}^n_1$ such that $C\backslash\{0\}$ is contained in the interior of the positive light cone. We say P is a *polytope in* $\mathbb{X}^n$ if P corresponding properties used above. The *dimension* of P is $\dim(C) - 1$. If C is simplicial, we say that P is an *n-simplex*.

Example 2.2.4. Since the octant of Example 2.2.2 is essential and simplicial, intersecting it with $\mathbb{S}^2$ yields an n-simplex in $\mathbb{S}^2$. More precisely we get a right-angled triangle on $\mathbb{S}^2$.

A 0-dimensional polytope is a *point*; if it is 1-dimensional, it is an *interval*; if it is 2-dimensional, it is a *polygon*. Suppose P is a polytope and H a supporting hyperplane. Then $P \cap H$ is also a convex polytope. We call $P \cap H$ a *face* of P. A 0-dimensional face of P is a *vertex*; a 1-dimensional face is an *edge*; an $(n-1)$-dimensional face is a *facet*. We will denote the dimension of a polytope with a superscript, i.e., a polytope of dimension n is denoted by P^n.

Definition 2.2.5. Suppose H_1 and H_2 are two hyperplanes in $\mathbb{X}^n$ bounding half-spaces E_1 and E_2 with $E_1 \cap E_2 \neq \emptyset$. Let $u_1, u_2 \in \mathbb{R}^{n+1}$ be, respectively, their inward-pointing unit normals at a point $x \in H_1 \cap H_2$. We define $\theta := \pi - \arccos\langle u_1, u_2 \rangle \in [0, \pi]$ as the *dihedral angle* between H_1 and H_2. We say the dihedral angle is *non-obtuse*, if $\theta < \pi/2$. We call θ a *right-angle*, if $\theta = \pi/2$.

Note that in the Euclidean case, the inward-pointing unit normals u_i can also be seen as vectors in $\mathbb{E}^n \cong \mathbb{R}^n$.

Definition 2.2.6. Suppose $\{E_1, \ldots, E_n\}$ is a family of half-spaces in $\mathbb{X}^n$ with non-empty intersection and that $H_1, \ldots, H_n$ are their bounding hyperplanes. A polytope is called *non-obtuse* (respectively *right-angled*), if for every two distinct indices i and j either $H_i \cap H_j = \emptyset$ or the dihedral angle between H_i and H_j is non-obtuse (respectively right-angled).

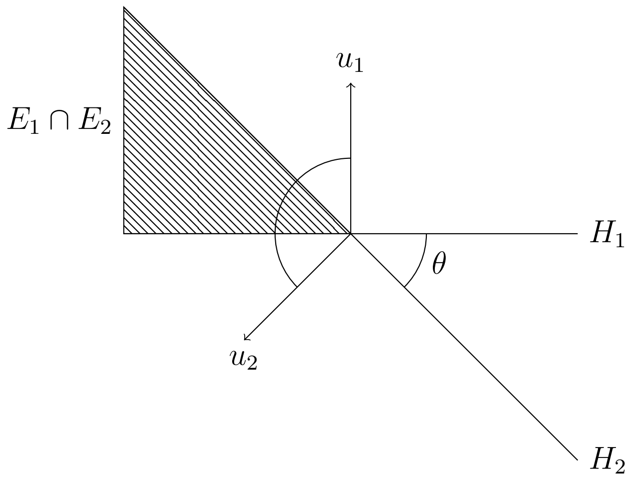


Figure 2.4: The angle between the unit vectors u_1 and u_2 is given by $\arccos\langle u_1, u_2 \rangle$. The dihedral angle between the two hyperplanes H_1 and H_2 orthogonal to u_1, u_2 is given by $\theta = \pi - \arccos\langle u_1, u_2 \rangle$.

Note that in the spherical case the first condition cannot occur since two distinct hyperplanes (i.e., great subspheres) always intersect. In the Euclidean case two hyperplanes do not intersect if and only if they are parallel. Furthermore, the normals $u_i \in \mathbb{R}^{n+1}$ determine the hyperplanes H_i and the condition inward-pointing determines which of the two half-spaces we take. Hence, a polytope P^n, as an intersection of half-spaces, is completely determined by the inward-pointing normals $u_i \in \mathbb{R}^{n+1}$.

Let $P^n \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be an n-simplex and let $\{F_i \mid 1 \leq i \leq n+1\}$ be its facets. Let $u_i \in \mathbb{R}^{n+1}$ be the inward-pointing unit vector normal to F_i. The condition that u_i is *inward-pointing* means that P^n and u_i lie on the same side of the hyperplane defined by the equation $\langle u_i, x\rangle = 0$. Thus P^n is the set of points $x \in \mathbb{S}^n$ satisfying

$$\langle u_i, x\rangle \geq 0 \qquad \text{for} \qquad 1 \leq i \leq n+1.$$

For our next discussion, we will need following linear algebra lemma:

Lemma 2.2.7 ([Bou08], p. 82)**.** *Suppose B is a positive semi-definite symmetric bilinear form on $\mathbb{R}^n$. Let $u_1, \ldots, u_n \in \mathbb{R}^n$ such that $B(u_i, u_j) \leq 0$ for all $i \neq j$.*

(i) Suppose $v = \sum_{i=1}^n c_i u_i$, $c_i \in \mathbb{R}$, is a linear combination of the u_i such that $B(v, v) = 0$. Then

$$B\left(\sum_{i=1}^n |c_i| u_i, \sum_{j=1}^n |c_j| u_j\right) = 0.$$

(ii) If B is non-degenerate and if there is a linear form f such that $f(u_i) > 0$ for all i, then the vectors $u_1, \ldots, u_n$ are linearly independent.

Proof. The first statement is obvious: Since $B(u_i, u_j) \leq 0$ for all $i \neq j$, we have

$$\begin{aligned} B\left(\sum_{i=1}^n |c_i| u_i, \sum_{j=1}^n |c_j| u_j\right) &= \sum_{i=1}^n \sum_{j=1}^n |c_i||c_j| B(u_i, u_j) \\ &\leq \sum_{i=1}^n \sum_{j=1}^n c_i c_j B(u_i, u_j) \\ &= B\left(\sum_{i=1}^n c_i u_i, \sum_{j=1}^n c_j u_j\right). \end{aligned}$$

For the second statement, suppose that B is non-degenerate and $\sum_{i=1}^n c_i u_i = 0$. Using (i) and the non-degeneracy of B, we get $\sum_{i=1}^n |c_i| u_i = 0$. Hence for any linear form f we have $\sum_{i=1}^n |c_i| f(u_i) = 0$. Assuming $f(u_i) > 0$, it follows that all $c_i = 0$, which proves the statement. □

As an immediate consequence we get:

Theorem 2.2.8 ([DC08], Lemma 6.3.3.). *Suppose $P^n \subset \mathbb{S}^n$ is a polytope such that all dihedral angles are non-obtuse. Then P^n is an n-simplex.*

Proof. Let $F_1, \ldots, F_k$ be the codimension-one faces of P^n. Let $u_i \in \mathbb{R}^{n+1}$ be the inward-pointing unit vector normal to F_i. Let x be a point in the interior of P^n and f be the linear form on $\mathbb{R}^{n+1}$ defined by $f(v) = \langle v, x \rangle$. Since x lies of the positive side of the half-space determined by u_i, we have $f(u_i) = \langle u_i, x \rangle > 0$ for all $i = 1, \ldots, k$. Hence by Lemma 2.2.7 (note that we are using Lemma 2.2.7 in the case $n+1$ not n), the vectors $u_1, \ldots, u_k$ are linearly independent. Consequently, $k = n+1$ and the u_i form a basis of $\mathbb{R}^{n+1}$. Thus P^n is an n-simplex. $\square$

Corollary 2.2.9 ([DC08], Corollary 6.3.4.). *Suppose $C \subset \mathbb{R}^{n+1}$ is an essential polyhedral cone with non-obtuse dihedral angles. Then C is a simplicial cone.*

In the case $\mathbb{X}^n \neq \mathbb{S}^n$ it is not necessarily true that a polytope P with non-obtuse angles is an n-simplex. However, it is a simple polytope.

Definition 2.2.10. An n-dimensional polytope P^n is called *simple* if exactly n facets (or equivalently n edges) meet at each vertex.

For example, a cube and an n-simplex are simple while an octahedron is not.

As an immediate consequence of Theorem 2.2.8, we get following theorem:

Theorem 2.2.11 ([DC08], Proposition 6.3.9). *Suppose P^n is a convex polytope with non-obtuse dihedral angles in $\mathbb{X}^n$. Then P^n is simple.*

Proof. Let v be a vertex of P^n and let S be a small sphere with midpoint v such that S does not contain any other vertices of P^n. Consider the polytope $P^n \cap S$ and note that the facets of P^n meeting in v have the same dihedral angles as the facets of $P^n \cap S$. By applying Theorem 2.2.8 to the polytope $P^n \cap S$, we conclude that $P^n \cap S$ is an $(n-1)$-simplex. Hence, there are exactly n facets meeting at v. Applying this argument for every vertex of P^n yields that P^n is simple. $\square$

In the Euclidean case, all dihedral angles of a cube $\square \subset \mathbb{E}^n$ are equal to $\pi/2$. However, the cube is a product of simplices, i.e., $\square = [-1, 1]^n$. Thus we need a condition to distinguish between n-simplices and product of simplices.

Definition 2.2.12. A convex subset $X \subset \mathbb{E}^n$ is *reducible* if it is isometric to a product $X' \times X''$, where $X' \subset \mathbb{E}^m$ and $X'' \subset \mathbb{E}^{n-m}$ and neither X' nor X'' are points. The subset X is called *irreducible* if it is not reducible.

Suppose a convex subset $X \subset \mathbb{E}^n$ has a finite number of supporting hyperplanes H_i, $i \in I$. Then X is reducible if there is a non-trivial partition of the index set $I = I' \cap I''$ such that H_i and H_j intersect orthogonally whenever $i \in I'$ and $j \in I''$.

Before we start with the proof, we need a technical lemma from matrix theory.

Definition 2.2.13. An $n \times n$-matrix $A = (a_{ij})$ is called *decomposable* if there is a non-trivial partition of the index set $I \cup J = \{1, \dots, n\}$ such that $a_{ij} = a_{ji} = 0$, whenever $i \in I$ and $j \in J$. Is there no such partition, then A is called *indecomposable*.

Lemma 2.2.14 ([Bou08], p. 83)**.** *Suppose that $A = (a_{ij})$ is an indecomposable, symmetric, positive semi-definite $n \times n$-matrix and that $a_{ij} \leq 0$ for all $i \neq j$.*

(i) If A is degenerate then its corank is 1 and its kernel is spanned by a vector with all coordinates greater than 0.

(ii) In general the smallest eigenvalue of A has multiplicity 1 and this eigenvalue has an eigenvector with all coordinates greater than 0.

Proof. (i) Associated with A there is a quadratic form q on $\mathbb{R}^n$ defined by $q(x) = x^T Ax$. Let N be the null space of q (corresponding to the kernel of A) and suppose $\sum_{i=1}^{n} c_i e_i$ is a non-zero vector in N. By Lemma 2.2.7, we have that $\sum_{i=1}^{n} |c_i| e_i$ also lies in N. Thus, we get that

$$\sum_{j=1}^{n} a_{ij} |c_j| = 0. \tag{2.2}$$

Let I denote the set

$$I \coloneqq \{j \in \{1, \dots, n\} \mid c_j \neq 0\}.$$

Suppose I is a proper subset of $\{1, \dots, n\}$ and fix an index $i \notin I$. We have two cases: for $j \in I$ we have $a_{ij}|c_j| \leq 0$ since $i \neq j$ and for $j \notin I$ we have $a_{ij}|c_j| = 0$ since $|c_j| = 0$. We conclude that a_{ij} vanishes in (2.2).

Thus A is decomposable whenever $I \neq \{1, \dots, n\}$. Since we assume A to be indecomposable, we get $I = \{1, \dots, n\}$. Hence all coordinates of each non-zero vector in N are non-zero. Moreover, if v is a non-zero vector in N, we can assume all its coordinates are positive since we can replace $v = \sum_{i=1}^{n} c_i e_i$ by $\sum_{i=1}^{n} |c_i| e_i$.

Assume $\dim(N) \geq 2$ and take two vectors in N. We can easily construct a linear combination of these two vectors with one component being zero, contradicting the above. Hence $\dim(N) = 1$.

(ii) Apply (i) to $A - \lambda E$, where $\lambda \geq 0$ is the smallest eigenvalue of A. □

Lemma 2.2.15 ([DC08], Lemma 6.3.10)**.** *Suppose $P^n \subset \mathbb{E}^n$ is an irreducible polytope with non-obtuse dihedral angles. Then P^n is an n-simplex.*

Proof. Let $u_1, \dots, u_k$ be the inward-pointing unit normal vectors to the facets of P^n and let $U = (\langle u_i, u_j \rangle)$ be the matrix of inner products of the u_i. Note that this $k \times k$-matrix is positive semi-definite and symmetric. Since P^n is irreducible, U is indecomposable.

By Lemma 2.2.14, there are two cases. Either U is positive definite, in which case $k = n$ and $\{u_1, \dots, u_k\}$ is a basis for $\mathbb{R}^n$. Thus P^n is a simplicial cone. This is impossible since P^n is compact. Or $k = n + 1$ and U has corank 1. After identifying $\mathbb{E}^n$ with $\mathbb{R}^n$, this means that P^n is defined by the inequalities $\langle u_i, x \rangle \geq c_i$ for some $c_i \in \mathbb{R}$, $i = 1, \dots, n + 1$. Hence P^n is an n-simplex. □

Thus we have proven following theorem:

Theorem 2.2.16 ([Cox34])**.** *Suppose $P^n \subset \mathbb{E}^n$ is a polytope and the dihedral angle along any codimension-two face of P^n is non-obtuse. Then P^n is isometric to either an n-simplex or a product of simplices.*

In the hyperbolic case, all we know so far is that a polytope with non-obtuse dihedral angles is simple. As we will be interested in polytopes which tessellate the space by reflection along its facets, a natural assumption is that the dihedral angles are submultiple of π. But, a theorem of Vinberg

(Theorem 2.9.14) states that such polytopes cannot exist in higher dimension. We will postpone the discussion on hyperbolic polytopes until we establish the notion of Coxeter groups.

2.3 The Gram Matrix

In this section, we want to discuss whether an n-simplex in $\mathbb{X}^n$ is determined by its dihedral angles. Later this will help us to see that an n-simplex is determined by a prescribed Coxeter matrix of a Coxeter system. We will discuss the spherical and hyperbolic case first since the Euclidean case is more subtle.

Spherical simplices

In the proof of Lemma 2.2.15, we have used the matrix $U \coloneqq (\langle u_i, u_j\rangle)$ where u_i is a inward-pointing unit normal vector to the facet F_i of P^n. Assume that $P^n \subset \mathbb{S}^n$ is an n-simplex so that $\{u_1, \ldots, u_{n+1}\}$ is a basis of $\mathbb{R}^{n+1}$, by Lemma 2.2.7. Hence U is symmetric, positive definite, and all diagonal entries are 1. The matrix U is also called the *Gram matrix* of P^n. Naturally, the dihedral angle between F_i and F_j is encoded in the Gram matrix of P^n by

$$\theta_{ij} = \pi - \arccos(\langle u_i, u_j\rangle) = \arccos(-\langle u_i, u_j\rangle).$$

By convention, put $\theta_{ii} = \pi$.

Example 2.3.1. Let $\Delta \subset \mathbb{S}^2$ be a right-angled triangle defined by the equations $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. Then the inward-pointing normals $\{(1,0,0), (0,1,0), (0,0,1)\}$ obviously form a basis of $\mathbb{R}^3$. The Gram matrix is given by

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1. \end{pmatrix}$$

Then the dihedral angles are given by $\theta_{ii} = \pi - \arccos(U_{ij}) = \pi$ and $\theta_{ij} = \pi - \arccos(U_{ij}) = \pi/2$, $i \neq j$.

The next lemma shows that the Gram matrix also determines the whole n-simplex up to isometry. Thus the dihedral angles are all we need to know to determine a spherical n-simplex.

Lemma 2.3.2 ([DC08], Lemma 6.8.1)**.** *Let $P^n \subset \mathbb{S}^n$ be an n-simplex. Then P^n is determined, up to isometry, by its Gram matrix respectively by its dihedral angles.*

Proof. Suppose $P^n, \widetilde{P}^n \subset \mathbb{S}^n$ are two n-simplices with the same Gram matrix. Then we have

$$\langle u_i, u_j \rangle = \langle \tilde{u}_i, \tilde{u}_j \rangle. \tag{2.3}$$

Since both $\{u_1, \ldots, u_{n+1}\}$ and $\{\tilde{u}_1, \ldots, \tilde{u}_{n+1}\}$ are bases for $\mathbb{R}^{n+1}$, there is a unique linear automorphism f of $\mathbb{R}^{n+1}$ such that $f(u_i) = \tilde{u}_i$, $1 \leq i \leq n+1$. Thus we get with (2.3) that

$$\langle f(u_i), f(u_j) \rangle = \langle \tilde{u}_i, \tilde{u}_j \rangle = \langle u_i, u_j \rangle.$$

Hence f is an isometry. □

Suppose we prescribe the dihedral angles, i.e., for each unordered pair $i, j \in \{1, \ldots, n+1\}$ we are given numbers $\theta_{ji} = \theta_{ij} \in (0, \pi)$. Is there a spherical n-simplex P^n such that the dihedral angle along $F_i \cap F_j$ is θ_{ij}? If such a simplex exists, its Gram matrix would be the matrix (c_{ij}) defined by

$$c_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\cos(\theta_{ij}) & \text{if } i \neq j. \end{cases} \tag{2.4}$$

Proposition 2.3.3 ([DC08], Proposition 6.8.2)**.** *Suppose that we are given numbers $\theta_{ij} \in (0, \pi)$ for $1 \leq i, j \leq n+1$ and $i \neq j$. Then there is an n-simplex $P^n \subset \mathbb{S}^n$ with dihedral angles (θ_{ij}) if and only if the matrix defined by (2.4) is positive definite.*

Proof. The if part was already discussed above. Thus assume $A = (c_{ij})$ is positive definite. There is a non-singular matrix U such that $U^T U = A$, e.g., the square root of A. Let $u_1, \ldots, u_{n+1}$ be the column vectors of U. Since the diagonal entries of A are all 1, we have that the $\langle u_i, u_i \rangle = 1$ for all i. Thus the u_i are unit vectors. Since the u_i form a basis of $\mathbb{R}^{n+1}$, the polyhedral cone C defined by the inequalities $\langle u_i, x \rangle \geq 0$ is essential and simplicial. Hence the intersection of C with $\mathbb{S}^n$ defines a spherical n-simplex P^n with Gram matrix A. □

Hyperbolic simplices

The hyperbolic case is similar to the spherical case. Let $P^n \subset \mathbb{H}^n$ be an n-simplex and $u_1, \ldots, u_{n+1}$ its unit inward-pointing normals. Note that the u_i are spacelike vectors. Again, the Gram-matrix (c_{ij}) is defined by (2.4). Let J be the $(n+1)\times(n+1)$-matrix with diagonal entries $(1, \ldots, 1, -1)$, and let U be the matrix with column vectors $u_1, \ldots, u_{n+1}$. Then $(c_{ij}) = U^T JU$ is a symmetric, non-degenerate matrix with signature $(n, 1)$, and all diagonal entries are 1.

The proof for the next lemma is virtually the same as in the spherical case.

Lemma 2.3.4 ([DC08], Lemma 6.8.3)**.** *Let $P^n \subset \mathbb{H}^n$ be an n-simplex. Then P^n is determined, up to isometry, by its Gram matrix respectively by its dihedral angles.*

As in the spherical case, suppose we prescribe the dihedral angles, i.e., for each unordered pair $i, j \in \{1, \ldots, n+1\}$ we are given numbers $\theta_{ji} = \theta_{ij} \in (0, \pi)$. Is there a hyperbolic simplex P^n such that the dihedral angle along $F_i \cap F_j$ is θ_{ij}? If such a simplex exists, its Gram matrix would be the matrix (c_{ij}) defined by (2.4).

Proposition 2.3.5 ([DC08], Proposition 6.8.4)**.** *Suppose that we are given numbers $\theta_{ij} \in (0, \pi)$ for $1 \leq i, j \leq n+1$ and $i \neq j$. Then there is an n-simplex $P^n \subset \mathbb{H}^n$ with dihedral angles (θ_{ij}) if and only if the matrix defined by (2.4) has signature $(n, 1)$ and each principal submatrix is positive definite.*

Proof. Note that non-degenerate bilinear forms are classified by their signature. Thus, given any symmetric matrix A with signature $(n, 1)$ there is a non-singular matrix U such that $U^T JU = A$. Apply this to the matrix (c_{ij}) to get U and let $u_1, \ldots, u_{n+1}$ be its column vectors. Since $\langle c_i, c_i\rangle = c_{ii} = 1$, each u_{ii} is spacelike. It follows, that the simplicial cone $C \subset \mathbb{R}^n_1$ defined by $\langle u_i, x\rangle \geq 0$ intersects either the positive or negative light cone. W.l.o.g. it is the positive light cone (otherwise replace u_i by $-u_i$).

Since the principal submatrix $A_{(k)}$ is positive definite, the hyperplane spanned by $\{u_k\}_{i\neq k}$ is spacelike. Let

$$L_k := \bigcap_{i\neq k} u_i^\perp$$

be its orthogonal complement. Since $\{u_k\}_{i\neq k}$ is spacelike, L_k is timelike. Thus the intersection of L_k with C is an extremal ray of C. Since each extremal ray of C lies inside the positive light cone, the entire simplicial cone lies inside the positive light cone. Hence the intersection $\mathbb{H}^n \cap C$ defines a hyperbolic n-simplex. □

Euclidean simplices

The Euclidean case is more subtle than the spherical and hyperbolic case. Suppose $P^n \subset \mathbb{E}^n$ is an n-simplex, $F_1, \dots, F_{n+1}$ are its facets, and $u_i \in \mathbb{R}^n$ the inward-pointing unit normals to F_i. Furthermore, suppose v_i is the vertex of P^n opposite to F_i and H_i is the affine hyperplane spanned by F_i.

Lemma 2.3.6 ([DC08], Lemma 6.8.5)**.** *The vectors $u_1, \dots, u_{n+1}$ determine P^n up to translation and homothety.*

Proof. After translation of P^n, we can identify the vertex v_{n+1} of P^n with the origin in $\mathbb{R}^n$. Then P^n is defined by the inequalities

$$\begin{aligned} \langle u_i, x\rangle &\geq 0 \qquad \text{for } i = 1, \dots, n \\ \langle u_{n+1}, x\rangle &\geq -d, \end{aligned} \tag{2.5}$$

where d is the distance from the hyperplane H_{n+1} to the origin. After scaling by $1/d$, we can put P^n in a form where $d = 1$. □

Since the u_i span $\mathbb{R}^n$ and since there are $n + 1$ such vectors, there is a non-trivial linear relation of the form

$$c_1 u_1 + \cdots + c_{n+1} u_{n+1} = 0 \tag{2.6}$$

and this relation is unique up to scaling.

Lemma 2.3.7 ([DC08], Lemma 6.8.6)**.** *The coefficients c_i in the relation (2.6) are all non-zero and all have the same sign.*

Proof. Suppose P^n is defined by the inequalities in (2.5). For $1 \leq i, j, \leq n$ and $i \neq j$, $\langle u_i, v_i \rangle = 0$ and $\langle u_{n+1}, v_i \rangle = -d$. Take the inner product of both sides of the relation (2.6) with v_i to obtain

$$c_i \langle u_i, v_i \rangle - c_{n+1} d = 0. \tag{2.7}$$

Since $\langle u_i, v_i \rangle > 0$, the coefficients c_i and c_{n+1} are non-zero and both have the same sign (which we could take to be all positive). □

Lemma 2.3.8 ([DC08], Lemma 6.8.7)**.** *Suppose $\{u_1, \dots, u_{n+1}\}$ is a set of $n+1$ unit vectors spanning $\mathbb{R}^n$. Then $\{u_1, \dots, u_{n+1}\}$ is the set of inward-pointing unit normal vectors to a Euclidean n-simplex if and only if the coefficients c_i can all be taken to be positive.*

Proof. Suppose the $\{u_1, \dots, u_{n+1}\}$ satisfies the conclusion of Lemma 2.3.7. Fix a positive number d , let C be the cone defined by the first n inequalities in (2.5), and let P^n be the subset of C defined by the last inequality in (2.5). Since the coefficients c_i are all non-zero, the vectors $u_1, \dots, u_n$ are linearly independent. Hence C is a simplicial cone. We need to show that P^n is an n-simplex.

Let R_i be the extremal ray of C opposite the face defined by $\langle u_i, x \rangle = 0$ and L_i be the line spanned by R_i. Let v_i be the unique point on L_i satisfying $\langle u_{n+1}, v_i \rangle > 0$, for all $1 \leq i \leq n+1$. By (2.7), we get

$$\langle u_i, v_i \rangle = \frac{c_{n+1} d}{c_i}.$$

So if c_i and c_{n+1} have the same sign, then $\langle u_i, v_i \rangle$ is positive. □

The Gram matrix of P^n is defined, just as before, by (2.4). It is the matrix $(\langle u_i, u_j \rangle)_{1 \leq i,j \leq n+1}$. In other words, it is the matrix $A = U^T U$ where U is the $n \times (n+1)$-matrix with the u_i as column vectors. The matrix A has rank n, since the u_i span $\mathbb{R}^n$. Hence A is positive semi-definite of corank 1.

A final observation is that the one-dimensional null space of A is spanned by the column vector $v = (c_1, \dots, c_{n+1})^T$. The reason is that the equation (2.6) can be written as $Uv = 0$. Thus $v^T A v = 0$.

We can now give the necessary and sufficient condition for the existence of a Euclidean n-simplex with prescribed dihedral angles.

Proposition 2.3.9 ([DC08], Proposition 6.8.8)**.** *As before, suppose we are given numbers $\theta_{ij} \in (0, \pi)$ for $1 \leq i, j \leq n+1$ and $i \neq j$. Let (c_{ij}) be the matrix defined by (2.4). Then there is a Euclidean n-simplex P^n with dihedral angles along its codimension-two faces $F_i \cap F_j$ as prescribed by the θ_{ij} if and only if the matrix c_{ij} is positive definite of corank 1 and the null space of (c_{ij}) is spanned by a vector v with positive coordinates $c_1, \ldots, c_n$.*

Proof. Let A be the positive semi-definite matrix (c_{ij}) and let U be its square root. Then A and U have the same null space, namely the line spanned by v . Let $u_1, \ldots, u_{n+1}$ be the column vectors of U. Note that U is the linear transformation taking e_i to u_i.

Since the kernel of U is the line spanned by v , the image of U is the hyperplane H orthogonal to v. Hence, the u_i are unit vectors in the Euclidean space H satisfying the linear equation (2.6) with positive coefficients c_i.

Therefore by Lemma 2.3.8, there is an n-simplex in H with unit inward-pointing normal vectors $u_1, \ldots, u_{n+1}$. □

2.4 Coxeter System

Coxeter groups are abstract groups that admit a formal description in terms of reflections. As we saw in the previous section, an n-simplex in $\mathbb{X}^n$ is determined by its dihedral angles. In view of Example 2.1.2 the order of two distinct reflections is also determined by their dihedral angles. Thus, if we prescribe the order of two distinct reflections in a matrix we are able to characterise the corresponding polytopes via a group representation. This is done by the Coxeter matrix:

Definition 2.4.1. A *Coxeter matrix* $M = (m_{st})$ on a finite set S is a symmetric $(|S| \times |S|)$-matrix with entries in $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ such that $m_{st} = 1$ if $s = t$ and $m_{st} \geq 2$ for $s \neq t$.

One can associate to M a presentation for a group $\widetilde{W}$ as follows. For each $s \in S$, introduce a symbol $\tilde{s}$. Let $\mathcal{I} := \{(s,t) \in S \times S \mid m_{st} \neq \infty\}$. The set of generators for $\widetilde{W}$ is $\tilde{S} = \{\tilde{s}\}_{s \in S}$ and $R := \{(\tilde{s}\tilde{t})^{m_{st}} = e\}_{(s,t) \in \mathcal{I}}$ is the set of relations. As a short hand notation we write $\widetilde{W} = \langle \tilde{S} \mid R \rangle$.

Example 2.4.2. Consider the Coxeter matrix

$$M := \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

The associated presentation to M is $\widetilde{W} := \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = e \rangle$. In Example 2.4.4, we will see that $\widetilde{W}$ is isomorphic to S_3, the symmetric group on a set of 3 elements.

The condition $m_{ss} = 1$ gives that elements of S are *involutions*, i.e., elements of order two and the condition $m_{st} \geq 2$ implies that we can swap a word $stst\cdots$ containing m_{st} letters with a word $tsts\cdots$ containing m_{st} letters. For example, if $m_{st} = 4$: we have $(st)^4 = e$. By multiplying with the element $tsts$, we get the equation $stst = tsts$. Thus we can swap $stst$ for $tsts$.

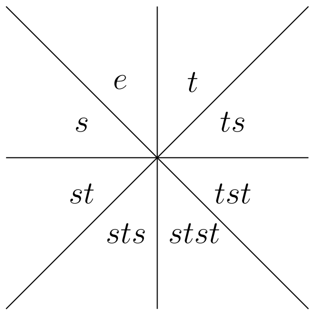


Figure 2.5: Finite dihedral group of order 8 acting on a cone in Euclidean space. Note that the words $stst$ and $tsts$ coincide.

Definition 2.4.3. Let M be a Coxeter matrix and $\widetilde{W}$ the group defined by the presentation associated with M. We say that (W, S) is a *Coxeter system*, if W is isomorphic to the group $\widetilde{W}$, i.e., if the epimorphism $\widetilde{W} \to W$, defined by $\tilde{s} \to s$, is an isomorphism. In this case we say that W is a *Coxeter group* and S is a *fundamental set of generators* or simply *generators*. Furthermore, we call $|S|$ the *rank* of (W, S).

Example 2.4.4. Let S_3 be the symmetric group on a set of 3 elements and s_1 be the transposition (12) and s_2 is the transposition (23). Then, S_3 is generated by s_1 and s_2 and is isomorphic to the group defined by the presentation $\langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = e \rangle$. Now, let s_3 be the transposition (13). Then $s_1 s_3$ and $s_2 s_3$ both have order 3 and s_3 is obviously an involution. Furthermore, S_3 is generated by $\{s_1, s_2, s_3\}$. However, the group defined by the presentation $\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^3 = (s_1s_3)^3 = (s_2s_3)^3 = e\rangle$ is not S_3. We will show later that the group defined by this presentation has infinite order. Thus it cannot be isomorphic to S_3. Hence $(S_3, \{s_1, s_2, s_3\})$ is not a Coxeter system.

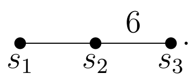
Due to Coxeter, there is a well-known method of encoding the information of a Coxeter matrix into a graph with edges labeled by integers greater than 3 or the symbol ∞.

Definition 2.4.5. Suppose $M = (m_{ij})$ is Coxeter matrix on a set I. We associate to M a graph Γ, called *Coxeter graph*, as follows. The vertex set of Γ is I. A pair of distinct vertices i and j is connected by an edge if and only if $m_{ij} \geq 3$. The edge $\{i, j\}$ is labeled by m_{ij} if $m_{ij} \geq 4$. The graph Γ together with a labeling of its vertices is called the *Coxeter diagram* associated with M. We call the vertices of Γ the *nodes* of the diagram.

Example 2.4.6. Consider the Coxeter matrix

$$M := \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 6 \\ 2 & 6 & 1 \end{pmatrix}.$$

The Coxeter diagram associated with M is given by



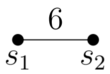
The main advantage of using the Coxeter diagram is that it indicates when W decomposes as a direct product. That is, if S can be partitioned into two non-empty disjoint subsets S' and S'' such that each element in S' commutes with each element of S'', then there exist groups W_1 and W_2 such that $W = W_1 \times W_2$. We will prove this fact in Section 2.7.

Definition 2.4.7. A Coxeter system (W, S) is called *irreducible* or *reducible* according to its Coxeter graph being connected or disconnected.

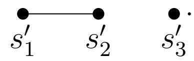
Example 2.4.8. Consider the two Coxeter matrices

$$M := \begin{pmatrix} 1 & 6 \\ 6 & 1 \end{pmatrix} \qquad \text{and} \qquad M' := \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

The corresponding Coxeter graphs are



and



The Coxeter system associated with M is irreducible and the Coxeter system associated with M' is reducible. Furthermore, both groups are isomorphic to the dihedral group of order 12. To see this, one can show that φ defines an isomorphism by the equations $\varphi(s_1) := s_1's_2's_1's_3'$ and $\varphi(s_2) := s_2'$. This shows that a Coxeter group does not uniquely determine the Coxeter matrix, graph, and system.

We denote the *length* of a word with respect to S using the function $\ell_S \colon W \to \mathbb{N}$ as the minimum number k such that for a word $w \in W$ we can write $w = s_{i_1} \cdots s_{i_k}$ for $s_{i_n} \in S$. If the set of generators is clear, we omit the index S from ℓ. If $w = s_1 \cdots s_k$ and $\ell(w) = k$ we say that the word w is *reduced*. A reduced expression for w is not necessarily unique.

Example 2.4.9. Consider the symmetric group S_3 with the representation $\langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = e \rangle$. The word $w = s_1 s_2 s_1 s_2 s_1$ can be shortened due to $(s_1 s_2)^3 = e$. We get $w = s_2$ which cannot be shortened further. Thus s_2 is a reduced expression for w while $s_1 s_2 s_1 s_2 s_1$ is not reduced. The words $s_1 s_2 s_1$ and $s_2 s_1 s_2$ are equal, both cannot be shortened, and both have length 3. Hence they are both reduced.

Proposition 2.4.10 ([Hum90], p. 108)**.** *Some elementary properties of the length function are:*

(i) $\ell(e) = 0$.

(ii) $\ell(w) = 1$ *if and only if* $w \in S$.

(iii) $\ell(w) = \ell(w^{-1})$.

(iv) $\ell(ww') \leq \ell(w) + \ell(w')$.

(v) $\ell(ww') \geq \ell(w) - \ell(w')$.

(vi) $\ell(w) - 1 \leq \ell(sw) \leq \ell(w) + 1$.

Proof. The statements (i) and (ii) are obvious. Let $w = s_1 \cdots s_k$. Then $w^{-1} = s_k \cdots s_1$. Hence, $\ell(w^{-1}) \leq \ell(w)$. Apply the same argument for w^{-1} to get (iii). Let $w' = s'_1 \cdots s'_n$. Then the product has $ww' = s_1 \cdots s_k s'_1 \cdots s'_n$ has length at most $k + n$. Thus we get the statement (iv). Applying (iv) and (iii), we get $\ell(w) = \ell(ww'w'^{-1}) \leq \ell(ww') + \ell(w^{-1}) = \ell(ww') + \ell(w')$. Subtracting $\ell(w')$ yields the statement (v). Statement (vi) is an immediate consequence of (iv) and (v) applied to $\ell(sw)$. □

We introduce three equivalent conditions which are key properties in the study of Coxeter groups:

- *Deletion Condition*: Let $w \in W$ and $s_1, \ldots, s_k \in S$. If $w = s_1 \cdots s_k$ is a word in W with $\ell(w) < k$, then there are indices $i < j$ such that $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$, where $\hat{s}_i$ means we delete this letter.
- *Exchange Condition*: Let $w \in W$ and $s, s_1, \ldots, s_k \in S$. If $w = s_1 \cdots s_k$ is a reduced word, either $\ell(sw) = k + 1$ or else there is an index i such that $w = ss_1 \cdots \hat{s}_i \cdots s_k$.
- *Folding Condition*: Let $w \in W$ and $s, s' \in S$. Suppose that we have $\ell(sw) = \ell(ws') = \ell(w) + 1$. Then either $\ell(sws') = \ell(w) + 2$ or $sws' = w$.

Regarding the Exchange Condition, we distinguish three cases: either $\ell(sw) = \ell(w) + 1$, or $\ell(sw) = \ell(w) - 1$, or $\ell(sw) = \ell(w)$. The meaning of the Exchange Condition is that the third case does not occur and that in the second case we can modify an arbitrary reduced expression of w to get one beginning with s by exchanging one of its letters for an s in front. The meaning of the Deletion Condition is that if we can reduce a word, then we can reduce it by omitting an even number of letters.

Theorem 2.4.11 ([DC08], Theorem 3.3.4)**.** *Let W be a group and S a set of involutions, which generate W. Then (W, S) satisfies the Exchange Condition if and only if (W, S) is a Coxeter group.*

Proof. There are several proofs for the above theorem. Davis uses Cayley graphs in [DC08] which he discusses in Chapter 2. For a purely combinatorial proof, see [BB05]. Humphreys uses in [Hum90] the geometric representation which we will define in the third chapter of this thesis. □

Theorem 2.4.12 ([DC08], Theorem 2.3.16)**.** *Let (W, S) be a Coxeter system. The Deletion Condition, Exchange Condition and Folding Condition are equivalent.*

Proof. Assume the Deletion Condition is true. Let $w = s_1 \cdots s_k$ be a reduced expression and let $s \in S$ such that $\ell(sw) \leq k$. Since $sw = ss_1 \cdots s_k$ is not reduced, the Deletion Condition says we can find a shorter word for sw by deleting two letters. Since $w = s_1 \cdots s_n$ is reduced, both letters cannot belong to the expression in w. Hence s is one of the letters. Thus $sw = \hat{s}s_1 \cdots \hat{s}_i \cdots s_k$ or $w = ss_1 \cdots \hat{s}_i \cdots s_k$.

Assume the Exchange Condition is true and let $w = s_1 \cdots s_k$ be a reduced expression. Furthermore, let $s, s' \in S$ such that $\ell(sw) = \ell(ws') = k + 1$ and $\ell(sws') = k + 2$. Applying the Exchange Condition to the word $s_1 \cdots s_k s'$ and the element s, we see that a letter can be exchanged for an s in front. The exchanged letter cannot be part of $w = s_1 \cdots s_k$, since $\ell(sw) = k + 1$. Hence it must be the final s'. Thus $ss_1 \cdots s_k = s_1 \cdots s_k s'$, i.e., we have $sw = ws'$.

Finally, assume the Folding Condition is true. Suppose $w = s_1 \cdots s_k$ is not reduced. Necessarily $k \geq 2$. We must show that we can delete two letters from w while leaving its length unchanged. A quick induction on k shows that. Assume that the words $s_1 \cdots s_{k-1}$ and $s_2 \cdots s_k$ are reduced (otherwise we are done). Let $w = s_2 \cdots s_{k-1}$. Apply the Folding Condition with $s = s_1$ and $s' = s_k$ wo w. Hence $s_1 w s_k = w$ and w can be shortened by deleting the first and last letters. □

2.5 Finite Coxeter Groups

In the last sections, we saw that Coxeter groups are a natural generalisation and algebraic description for a set of reflections. Hence, if we want to study reflections, it is natural to discuss Coxeter groups. Before we prove the well-known classification result for simplicial Coxeter groups, we introduce some specific examples of Coxeter groups, namely finite irreducible Coxeter groups. They arise for example as symmetry groups of platonic solids and will be used to describe edge reflections and spherical tessellations. The main idea of this section is to get accustomed with all types, since we will heavily refer to them in this thesis ongoing.

Consistent with the classification in [Hum90] the examples are called *types*. For the finite irreducible case, there are three one-parameter families of increasing rank (A_n, B_n, D_n), one one-parameter family of rank 2 ($I_2(m)$), and six exceptional types (E_6, E_7, E_8, F_4, H_3, H_4) of Coxeter groups. In the following, we will discuss the one-parameter families. As we will see in the next chapter, the exceptional types are less important in the setting of edge reflection. Hence we will just list them for the sake of completeness. For the examples, we follow Section 6.7 in [DC08]

Type $I_2(m), m \geq 3$, $m < \infty$

Consider $\mathbb{R}^2$ and an m-sided polygon centered at the origin. If m is even, bisect this polygon with lines joining two opposite vertices and the midpoints of opposite sides. If m is odd, bisect the polygon with lines joining a vertex to the midpoint of the opposite side.

We now subdivided the polygon into $2m$ triangles. Take one of the triangles. The two sides of the triangle containing the origin are intersecting in an angle $\theta = 2\pi/m$. Let H_1, H_2 be the induced hyperplanes of these lines and let s_1, s_2 be the corresponding reflections along the hyperplanes. Note that the composition of the reflections rotates the polygon by an angle of π/m. Hence $(s_1 s_2)^m$ is equal to the identity.

If we take $S := \{s_1, s_2\}$ as a set of generators and $R := \{s_1^2, s_2^2, (s_1 s_2)^m\}$ as the set of relations, we get with $W := \langle S \mid R \rangle$ the *dihedral group* of order $2m$. Obviously, this is a Coxeter group. We say it is of type $I_2(m)$. Its Coxeter matrix is

$$\begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix},$$

and the corresponding Coxeter diagram is given by

$$\underset{s_1}{\bullet}\overset{m}{\text{———}}\underset{s_2}{\bullet}.$$

Type A_n, $n \geq 3$

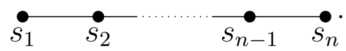
Consider $\mathbb{R}^{n+1}$ and let S_{n+1} be the group of all permutations of $\{1, \dots, n+1\}$, the *symmetric group* on $n+1$ letters. Let S_{n+1} act on $\mathbb{R}^{n+1}$ by permutation of the coordinates. Then the transposition (ij) acts as an orthogonal reflection across the linear hyperplane H_{ij} defined by the equation $x_i = x_j$. Let s_i be the reflection across $H_{i,i+1}$ and set $S \coloneqq \{s_1, \dots, s_n\}$.

Consider the product of the transpositions (12) and (23). It is given by the 3-cycle (123). Hence $s_1 s_2$ has order 3. Similarly, $s_i s_{i+1}$ has order 3 as well. If $|i - j| > 1$ then s_i and s_j correspond to the commuting transpositions $(i, i+1)$ and $(j, j+1)$. Thus $s_i s_j$ has order 2.

It follows that the entries of the Coxeter matrix of (S_n, S) are given by

$$m_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if } |i - j| > 1, \\ 3 & \text{else.} \end{cases}$$

Hence the Coxeter diagram is given by



We say this Coxeter group is of type A_n.

Type B_n, $n \geq 3$

Consider S_n acting on $\mathbb{R}^n$. Another reflection can be defined by sending a coordinate to its negative and fixing all other coordinates. This is a orthogonal reflection across the coordinate hyperplane $x_i = 0$.

Let G_n be the group generated by all sign changes and permutation of the coordinates. It is not hard to see that G_n is the semi-direct product $(\mathbb{Z}\backslash 2\mathbb{Z})^n \rtimes S_n$, where S_n acts on $(\mathbb{Z}\backslash 2\mathbb{Z})^n$ by permuting the factors. Hence G_n consists of $2^n n!$ elements.

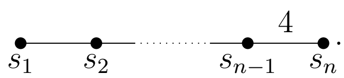
For $1 \leq i \leq n-1$, let s_n denote the orthogonal reflection across the hyperplane $x_i = x_{i+1}$ and let s_n be the reflection across $x_n = 0$. Put $S \coloneqq \{s_1, \dots, s_n\}$. Note that $\{s_1, \dots, s_{n-1}\}$ can be described as the Coxeter group of type A_{n-1}. Thus we just need to determine the order of $s_i s_n$.

The inward-pointing unit normal vector u_i to the wall $x_i = x_{i+1}$ is given by $u_i = \frac{1}{\sqrt{2}}(e_i - e_{i+1})$. The inward-pointing unit normal to $x_n = 0$ is e_n. Hence $\langle e_n, u_i\rangle = 0$ for $i \leq n-2$, while $\langle e_n, u_{n-1}\rangle = -\cos(\pi/4)$. It follows that s_n commutes with s_i for $i \leq n-2$ and that the order of $s_n s_{n-1}$ is 4.

Thus the entries of the Coxeter matrix of the system (G_n, S) are given by

$$m_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if } |i-j| > 1, \\ 3 & |i-j| = 1 \text{ and } i, j \neq n, \\ 4 & \text{if } \{i, j\} = \{n-1, n\}. \end{cases}$$

and the Coxeter diagram is given by



We say this Coxeter group is of type B_n.

Type D_n, $n \geq 4$

There is another reflection group acting on $\mathbb{R}^n$: a subgroup of index 2 in the group of type B_n. Let $H_n \subset G_n$ be the subgroup of all permutations of coordinates and even sign changes. Thus $H_n = (\mathbb{Z}\backslash 2\mathbb{Z})^{n-1} \rtimes S_n$.

This group is generated by the reflections

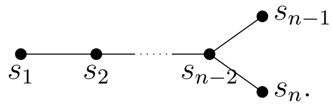
$$e_i - e_j \mapsto -(e_i - e_j) \qquad \text{and} \qquad e_i + e_j \mapsto -(e_i + e_j).$$

Let $S \coloneqq \{s_1, \ldots s_n\}$ where the first $n-1$ reflections are those for A_{n-1}, that is s_i is the reflection across $x_i = x_{i+1}$ and s_n is the reflection across $x_{n-1} = -x_n$. Again, it remains to determine the order of $s_i s_n$.

The inward-pointing normal corresponding to s_{n-2}, s_{n-1}, and s_n are, respectively, $u_{n-2} = \frac{1}{\sqrt{2}}(e_{n-2} - e_{n-1})$, $u_{n-1} = \frac{1}{\sqrt{2}}(e_{n-1} - e_n)$, and $u_n = \frac{1}{\sqrt{2}}(e_{n-1} + e_n)$. A simple calculation gives $\langle u_{n-1}, u_n\rangle = 0$ and $\langle u_{n-2}, u_n\rangle = -1/2$. It is also clear that $\langle u_i, u_n\rangle = 0$ for $i \leq n-2$. Hence the entries m_{ni} are given by

$$m_{ni} = \begin{cases} 2 & \text{if } i \neq n-2, \\ 3 & \text{if } i = n-3. \end{cases}$$

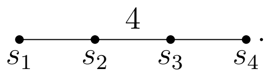
We say that the Coxeter system (H_n, S) is of type D_n. Its Coxeter diagram is given by



Note that following Coxeter types coincide: $A_2 = I_2(3)$, $B_2 = I_2(4)$, and $A_3 = D_3$. Hence the restriction on n to be greater than 3 or 4 in the definition of A_n, B_n, and D_n.

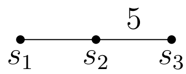
Type F_4

The Coxeter group of type F_4 can be realised as the symmetry group of a regular solid in $\mathbb{R}^4$ having 24 faces which are octahedra. The solid is also called the 24-cell. The group has order 1152 and the Coxeter diagram is given by

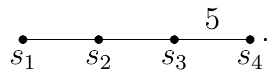


Type H_3 and H_4

The Coxeter groups of type H_3 and H_4 can be realised, respectively, as the symmetry groups of the icosahedron in dimension 3 and the symmetry group of the 120-cell. The 120-cell is another regular solid in $\mathbb{R}^4$ consisting of 120 dodecahedra as its three dimensional faces. The Coxeter diagrams of the Coxeter groups of type H_3 and H_4 are given by

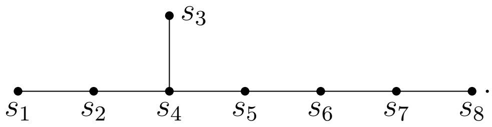


and

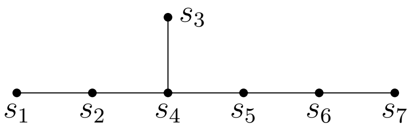


Type E_8, E_7 and E_6

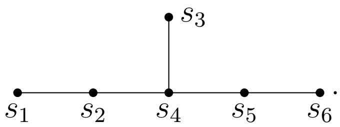
For completeness we will just list the Coxeter graphs of these types. The Coxeter diagram of the Coxeter group of type E_8 is given by



The subgroup of E_8, by omitting the last generator, is the group of type E_7. By omitting the last generator of E_7, we get the Coxeter group of type E_6. The Coxeter diagrams of E_7 and E_6 are given, respectively, by



and



We list all finite Coxeter graphs in Figure 2.6.

2.6 Classification of Simplicial Coxeter Groups

We show that the Coxeter groups we discussed in the last section comprise all finite (or *spherical*) Coxeter groups. For the classification result we follow [DC08] and [Hum90]. Furthermore, we will see how to construct all *Euclidean* and *hyperbolic simplicial* Coxeter groups (i.e., a Coxeter group where the set of generators are the reflections along the facets of a n-simplex). The main work reduces to calculating determinants of matrices, but it gives no hint on how to find all these groups. For a more detailed discussion how to identify these groups see [Bou08].

Let A be a symmetric $n \times n$-matrix. The *minor of order* k of A is the determinant of a submatrix obtained by removing the last k rows and columns from A. We denote it by $A_{(k)}$.

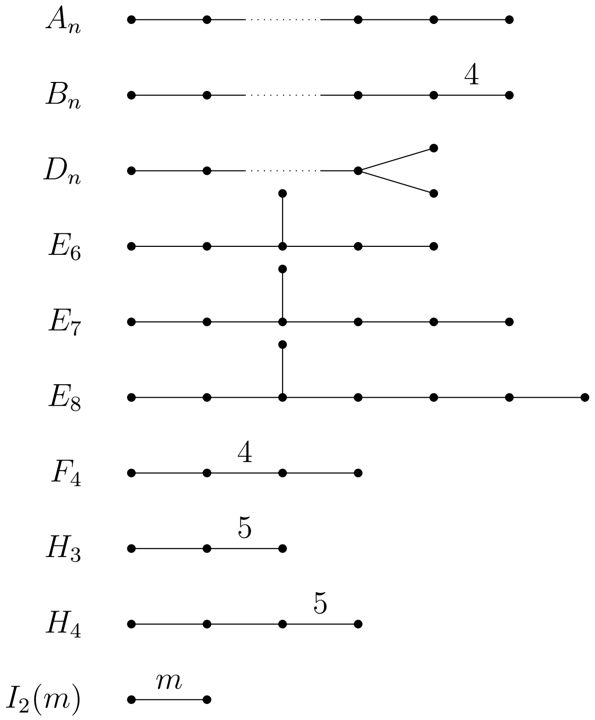


Figure 2.6: The irreducible spherical Coxeter graphs. Note that $m < \infty$ in the spherical case. Furthermore, the index of the type corresponds to the rank of the Coxeter system.

Example 2.6.1. Consider the matrix

$$A \coloneqq \begin{pmatrix} 4 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

We have $A_{(0)} = \det(A) = -5$, $A_{(1)} = 3$, and $A_{(2)} = 4$.

Recall Sylvester's criterion for positive definite matrices:

Theorem 2.6.2. *A matrix A is positive definite if and only if all minors of A are positive.*

Hence, if A is positive definite then the minor $A_{(n-1)}$ is positive. Conversely, if the minor $A_{(n-1)}$ is positive there are only three cases: either it is positive definite, positive semidefinite of corank 1, or non-degenerate with signature $(n, 1)$. Equivalently, $\det(A)$ is positive, vanishes, or is negative, respectively.

Definition 2.6.3. To a Coxeter matrix M on the set $S = \{s_1, \ldots, s_n\}$ of rank n we associate a symmetric $n \times n$-matrix $A(M)$ by setting

$$a_{ij} \coloneqq -\cos(\pi/m_{ij}).$$

We call $A(M)$ the *cosine matrix* of M or of the corresponding Coxeter system. If $m_{ij} = \infty$, we set $a_{ij} = -1$.

Example 2.6.4. Consider the Coxeter matrix

$$M \coloneqq \begin{pmatrix} 1 & 2 & \infty \\ 2 & 1 & 3 \\ \infty & 3 & 1 \end{pmatrix}.$$

The associated cosine matrix is given by

$$A(M) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ -1 & -1/2 & 1 \end{pmatrix}. \tag{2.8}$$

Later, we will see that the three cases for the determinant of the cosine matrix of a Coxeter matrix correspond to spherical, Euclidean, or hyperbolic tessellations. This motivates the following definition:

Definition 2.6.5. We say a Coxeter matrix M is *spherical*, *Euclidean*, or *hyperbolic*, if the cosine matrix of M is, respectively, positive definite, semi-positive definite of corank 1, or non-degenerate with signature $(n, 1)$. In that case, we will also call the Coxeter system, graph, diagram, or group either spherical, Euclidean, or hyperbolic.

First, we look into the Coxeter groups of rank 2. Consider a Coxeter group of type $I_2(m)$. The determinant of the cosine matrix A of $I_2(m)$ is given by

$$\det(A) = \det \begin{pmatrix} 1 & -\cos(\pi/m) \\ -\cos(\pi/m) & 1 \end{pmatrix} = \sin^2(\pi/m) > 0.$$

Hence the Coxeter group of type $I_2(m)$ is spherical. Consider now the Coxeter group associated with the Coxeter graph

$$\bullet\overset{\infty}{\textbf{—}}\bullet.$$

The cosine matrix is given by

$$\det \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 0.$$

Hence this Coxeter group is not spherical. It is easy to see that $(1, -1)$ is an eigenvector to the eigenvalue 2 and $(1, 1)$ is an eigenvector to the eigenvalue 0. Hence the matrix is positive semidefinite of corank 1, i.e., this Coxeter group is Euclidean.

Next, the remaining Coxeter diagrams listed in Figure 2.6 with rank greater than 3 have 5 as the largest label. Hence the relevant values of the cosine function are

$$\cos(\pi/m) = \begin{cases} 1/2 & \text{if } m = 3, \\ \sqrt{2}/2 & \text{if } m = 4, \\ (1+\sqrt{5})/4 & \text{if } m = 5. \end{cases}$$

Since the denominator 2 occurs often, it is convenient to calculate the determinant of $2A$. Furthermore, a glance at Figure 2.6 shows that it is always possible to label vertices of the Coxeter graphs in such a way that the last vertex (numbered n) is connected to only one other vertex (numbered $(n-1)$), and the edge between the n-th vertex and the $(n-1)$-st is labeled with $m = 3$. Let A_i be the minor of order i. Expanding $2A$ along its last row yields that

$$\det(2A) = 2A_{(n-1)} - A_{(n-2)}. \tag{2.9}$$

As an example, consider the Coxeter group of type H_3. The Coxeter diagram of H_3 can be relabelled as follows:

$$\underset{s_1}{\bullet}\overset{5}{\text{———}}\underset{s_2}{\bullet}\text{———}\underset{s_3}{\bullet}\,.$$

The minors $A_{(2)}$ and $A_{(1)}$ are the same as the determinants of the Coxeter graph of types $I_2(5)$ and A_1, respectively. Hence the determinant of the cosine matrix $2A$ is given by

$$\det(2A) = 8\sin^2(\pi/5) - 2 = 3 - \sqrt{5} > 0.$$

Inductively, we can calculate the determinants of the other types, which are given in Table 2.1. So we have proven following lemma:

A_n	B_n	D_n	E_6	E_7	E_8	F_4	H_3	H_4	$I_2(m)$
$n+1$	2	4	3	2	1	1	$3-\sqrt{5}$	$(7-3\sqrt{5})/2$	$4\sin^2(\pi/m)$

Table 2.1: Determinant of $2A$.

Lemma 2.6.6 ([DC08], Lemma C.2.1)**.** *Each Coxeter graph listed in Figure 2.6 is spherical.*

A crucial fact of Lemma 2.6.6 and Theorem 2.6.2 is the following: if we omit a vertex of the graph together with the incident edges, then the resulting Coxeter graph still is spherical. We say that Γ' is a subgraph of Γ, if Γ' is constructed that way. Consider two irreducible Coxeter graphs Γ and Γ'. Motivated by the following lemma, we say that Γ *dominates* Γ' if the underlying graph of Γ' is a subgraph of Γ and the label on each edge of Γ' is smaller than or equal the corresponding edge of Γ. If, in addition, $\Gamma \neq \Gamma'$, we say Γ *strictly dominates* Γ'.

Lemma 2.6.7 ([Hum90], p. 36)**.** *Let Γ be a irreducible Coxeter graph which is positive semidefinite. If Γ strictly dominates Γ', then Γ' is spherical.*

Proof. Let $A = (a_{ij})$ and $A' = (a'_{ij})$ be the $n \times n$ cosine matrix for Γ and Γ' respectively. Since Γ' is a subgraph of Γ, after reordering the vertices of Γ we can assume that A' corresponds to the $k \times k$-matrix in the upper left corner of A for some $k \leq n$. Since Γ dominates Γ', we have that $a_{ij} \leq a'_{ij} \leq 0$ for all $1 \leq i, j \leq k$, $i \neq j$.

Suppose there is a non-zero vector $x \in \mathbb{R}^k$ with $x^T A' x \leq 0$ (i.e., suppose A' is not positive definite). Then we have

$$0 \leq \sum_{i,j=1}^{k} a_{ij}|x_i||x_j| \leq \sum_{i,j=1}^{k} a'_{ij}|x_i||x_j| \leq \sum_{i,j=1}^{k} a_{ij}x_i x_j \leq 0.$$

In particular, all inequalities are equalities. Let $y \coloneqq (|x_1|, \ldots, |x_k|) \in \mathbb{R}^k$ and $z \coloneqq (|x_1|, \ldots, |x_k|, 0 \ldots, 0) \in \mathbb{R}^n$. Hence $0 = z^T A z = y^T A' y = x^T A x$. Thus z lies in the kernel of A. By Lemma 2.2.14, the coordinates of z are all non-zero. Hence $k = n$ and the coordinates of y are non-zero. Since Γ strictly dominates Γ', we have that $a_{ij} < a'_{ij}$ for at least one pair $\{i, j\}$. But this means $z^T A z < y^T A' y$, a contradiction. Thus A' is positive definite and Γ' is spherical. $\square$

Before we start with the classification, we want to list all simplicial Coxeter groups in $\mathbb{E}^n$ and $\mathbb{H}^n$. Consider an n-simplex P^n in $\mathbb{X}^n$. Since it is simple, n facets meet at each vertex of P^n. Hence a reflection group generated by those facets has a vertex as a fixed point. Thus this group must be finite, i.e., spherical. Hence the subgraph of a simplicial Coxeter group is spherical.

Consequently, we can construct all simplicial Coxeter groups by adding a vertex to a spherical Coxeter graph such that if we delete any other vertex, the subgraph will be spherical. By calculating the determinant of the cosine matrix we can assign them to the list of either spherical, Euclidean, or hyperbolic simplicial Coxeter groups. We will justify this method later. For now, we will just list all Euclidean and hyperbolic simplicial Coxeter diagrams in Figure 2.7 and Figure 2.8, respectively. Since we will use some of the Euclidean types in the next proofs, we recommend to study them to get accustomed to these types.

Lemma 2.6.8 ([DC08], Lemma C.2.2)**.** *Each of the diagrams listed in Figure 2.7 is positive semi-definite of corank* 1*, and each of the diagrams listed in Figure 2.8 is non-degenerate with signature* $(n, 1)$*.*

Proof. In the case of $\tilde{A}_1$, we have already done the calculation in (2.8). In all other cases, simply use a suitable labeling, (2.9), and Table 2.1 to confirm. As a random example, for $\tilde{F}_4$, there is a numbering of the vertices of the Coxeter graph such that the relevant subdiagrams are B_4 and A_3. Thus $\det(A) = 2 \cdot 2 - 4 = 0$. Hence the Coxeter system of type $\tilde{F}_4$ is Euclidean. □

Theorem 2.6.9 ([Hum90], p. 37)**.** *The irreducible spherical Coxeter diagrams are listed in Figure 2.6, and the irreducible Euclidean simplicial Coxeter diagrams are listed in Figure 2.7.*

Proof. Suppose Γ is a connected, spherical, or Euclidean Coxeter diagram. Let n be its rank and k the maximum label of the edges. We will use repeatedly Lemma 2.6.7 to show that Γ is listed either in Figure 2.6 or Figure 2.7.

W.l.o.g. let $n \geq 3$, since all Coxeter diagrams of rank smaller than 3 are of type A_1, $I_2(k)$ or $\tilde{A}_1$, which we already checked. In particular, since Γ does not dominate $\tilde{A}_1$, we have $k \neq \infty$. Furthermore, Γ contains no cycles, i.e., Γ is a tree, since Γ does not dominate $\tilde{A}_n$.

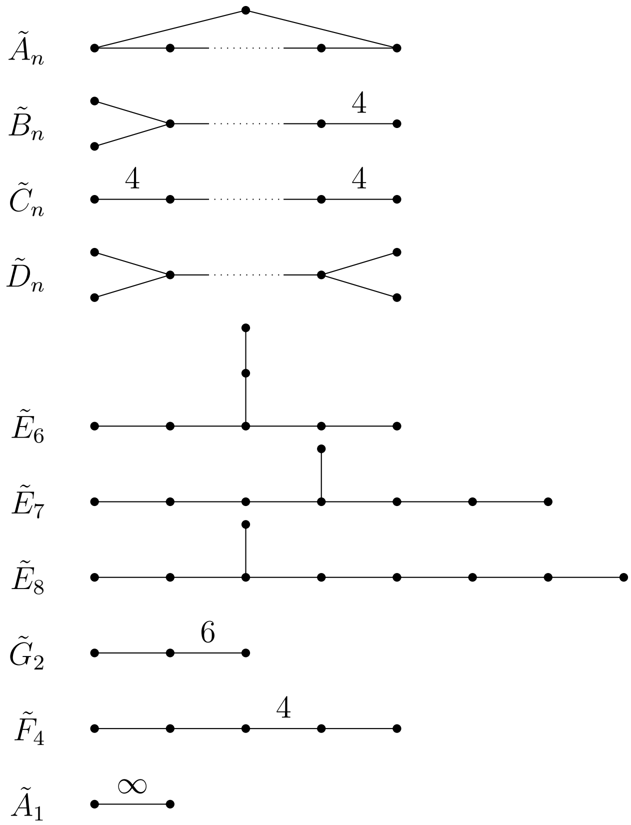


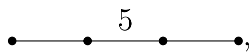
Figure 2.7: The irreducible Euclidean Coxeter graphs. The Euclidean types indicate which spherical type was used to construct it, i.e., the Euclidean Coxeter graph of type $\tilde{A}_n$ is constructed by adding a node to the spherical Coxeter graph of type A_n. Hence the index of the labeling corresponds to the rank plus 1 of the Coxeter system. The only exception is the graph of type $\tilde{G}_2$. It is constructed by adding a vertex and an edge labeled with 3 to the spherical type $I_2(6)$.

Assume that $k = 3$. If $n = 3$, or 4, then $\Gamma = A_3$, A_4, or D_4. Furthermore, assuming Γ has no branch vertex, we get $\Gamma = A_n$. Thus Γ has a branch vertex and $n > 4$. Seeing that Γ does not dominate $\tilde{D}_n$ for $n > 4$, the branch vertex is unique. Since Γ does not dominate $\tilde{D}_4$, the branch vertex is of valence 3. Suppose Γ has three branches with $a \leq b \leq c$ edges along each of the three branches. We have that $a = 1$, $b \leq 2$, and $c \leq 4$, otherwise Γ would dominate $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$, respectively. Then, in all remaining cases Γ equals to either D_n, E_6, E_7 or E_8.

Therefore, assume $k \geq 4$. Since Γ does not dominate $\tilde{C}_n$, only one edge of Γ has a label greater than 3. Furthermore, Γ has no branch vertex since Γ does not dominate $\tilde{B}_n$.

Suppose $k = 4$. If one of the extreme edges of Γ is labeled 4, then $\Gamma = B_n$. Thus the edge with label 4 lies in the interior. Since Γ does not dominate $\tilde{F}_4$, we have $n = 4$. The only remaining case is F_4; hence $\Gamma = F_4$.

Suppose $k = 5$. Since Γ does not dominate the hyperbolic Coxeter graph given by



the edge labeled 5 must be an extreme edge. But, then $\Gamma = H_3$ or H_4. Hence $k \geq 6$. But this is impossible, since Γ does not dominate $\tilde{G}_2$. Thus Γ is listed in Figure 2.6 or Figure 2.7.

□

Theorem 2.6.10 ([DC08], Theorem C.1.4). *The irreducible hyperbolic simplicial Coxeter diagrams are listed in Figure 2.8.*

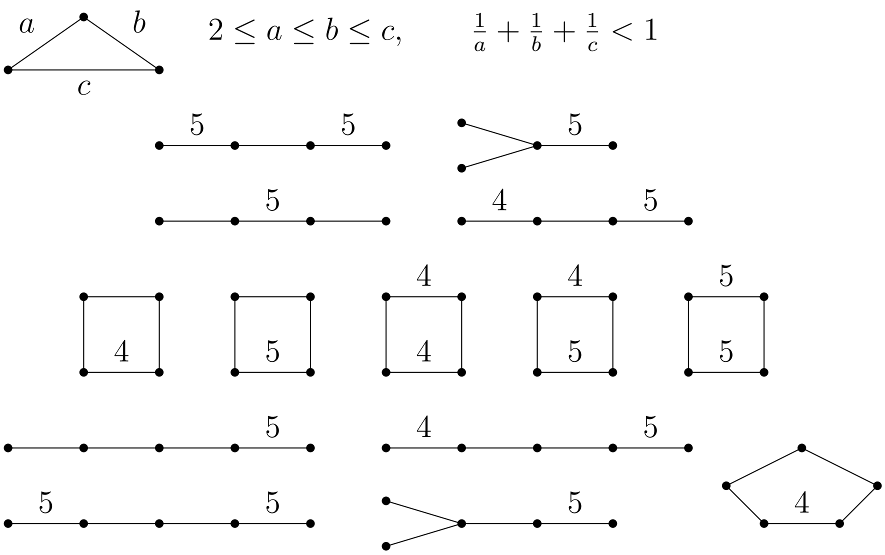


Figure 2.8: The irreducible hyperbolic simplicial Coxeter groups.

Proof. In view of the discussion above, Γ is the diagram of a hyperbolic simplicial Coxeter group if and only if every proper subdiagram is spherical and it does not appear in Figure 2.6 or Figure 2.7. It is simple to check that the diagrams in Figure 2.8 satisfy both conditions. Furthermore, there are no further possibilities. Note that in any connected spherical diagram with rank $n \geq 3$, the largest label is 5. Thus, since every spherical diagram is a tree, Γ cannot contain any proper cycle. Hence Γ is a cycle or a tree. So, we just need to consider adding a node to a graph in Figure 2.6 such that every proper subgraph is spherical. That is how we constructed Figure 2.7 and Figure 2.8. $\square$

2.7 Special Subgroups

Before we start with the construction of the Coxeter complex, i.e., the tessellation of $\mathbb{X}^n$, in the next section, we need a technical result. Special subgroups of Coxeter groups are subgroups, which are generated by a subset of S. Our aim is to prove that special subgroups are again Coxeter groups. To prove this statement, we address a solution by Tits regarding the word problem of Coxeter groups.

We start with a technical lemma regarding dihedral groups.

Lemma 2.7.1 ([DC08], Lemma 3.1.8)**.** *Let $(D, \{s, t\})$ be a dihedral group of order $2n$ where $s \neq t$. Then, every reduced word $w \in D$ with $\ell(w) < n$ has a unique expression.*

Proof. Let w be reduced with $\ell(w) = k$. Since w is reduced, it cannot contain any subword s^2 or t^2. Thus, it must be an alternating word using only the letters s and t. Hence we need to show that

$$\underbrace{stst\cdots}_{\text{length } k} \neq \underbrace{tsts\cdots}_{\text{length } k}.$$

Assume that these expression are equal. By multiplying the equation alternating with s and t we get $(st)^{2k} = e$. Since $k < n$ we have that $m \coloneqq |2k - n| < n$. Thus, we get with $(st)^n = e$ that $(st)^m = e$. This contradicts the minimality of n and proves the assertion. $\square$

The *word problem* for a finitely generated group is the problem of deciding algorithmically whether two words represent the same element, or equivalently a given word is the identity element. In the following, we give Tits' solution to the word problem for Coxeter groups.

Definition 2.7.2. Let (W, S) be a Coxeter system. An *elementary M-operation* on a word in W is one of the following types:

(I) Deleting a subword of the form s_i^2, where $s_i \in S$.

(II) Using a *braid move* on $s_i, s_j \in S$, i.e., replacing an alternating subword of the form $s_i s_j \cdots$ of length m_{ij} by the alternating word $s_j s_i \cdots$, of the same length m_{ij}.

A word is called *M-reduced*, if it cannot be shortened by a sequence of elementary M-operations.

Recall, that a word w of Coxeter system (W, S) is reduced if $w = s_1 \cdots s_k$, $s_i \in S$ and $\ell(w) = k$.

Theorem 2.7.3 ([Tit69])**.** *Suppose (W, S) is a Coxeter System.*

(i) Two reduced words w and w' represent the same element of W if and only if one can be transformed into the other by a sequence of elementary M-operations of type (II).

(ii) A word w is reduced if and only if it is M-reduced.

Proof. (i) Suppose $w, w' \in W$ are reduced with $w = s_1 \cdots s_k$ and $w' = t_1 \cdots t_k$ and $w = w'$. The proof is by induction on $k = \ell(w)$.

If $k = 1$, then the two words are obviously equal and we are done.

Suppose $k > 1$. We divide the proof into two cases. First, assume w and w' start with the same letter, i.e., $s_1 = t_1$. Then, the subwords $s_1 w$ and $s_1 w'$ have length $k - 1$ and can be transformed, by induction, into each other by a sequence of type (II) operations.

Hence, assume $s_1 \neq t_1$ and put $m \coloneqq m(s_1, t_1)$. Since $\ell(t_1 w) < \ell(w)$, we can apply the Exchange Condition to get a reduced expression for w starting with $t_1 s_1$ (the exchanged letter cannot be s_1 by the assumption $t_1 \neq s_1$). Since $\ell(s_1 w) < \ell(w)$, we can again apply the Exchange Condition to get a reduced expression for w starting with $s_1 t_1 s_1$. By Lemma 2.7.1, this can be done $q \leq m$ times. Furthermore, since q is bounded by $\ell(w) = k$, we have that $m \leq \infty$.

Thus, we get a reduced expression for w that starts either with t_1 or s_1 (if m is either odd or even) and ends with s_1 after m steps. If it starts with t_1, we can apply the first case, since this expression for w and w' start with the same letter. Otherwise, if it starts with s_1, we can use a sequence of type (II) operations to get another expression for w starting with t_1 and again apply the first case. This concludes the induction.

(ii) One direction is obvious. If w is reduced, it is M-reduced. Thus suppose $w = s_1 \cdots s_k$ is M-reduced. We will use an induction on $k = \ell(w)$ to show that it is a reduced expression. For $k = 1$ this is clear.

Suppose $k > 1$. By induction, we may assume the word $w' = s_2 \cdots s_k$ is reduced. Suppose w is not reduced. Since $\ell(s_1 w') = \ell(w) \leq k - 1$, the Exchange Condition implies that w' has another reduced expression starting with s_1. By the first statement w' can be transformed by a sequence of M-operations of type (II) into this expression. Hence w can be transformed by a sequence of M-operations to a word starting with $s_1 s_1$. But, this contradicts the assumption that it is M-reduced. Thus w is reduced. □

A consequence of the solution to the word problem for Coxeter groups is that for any element $w \in W$, the set of letters that can occur in a reduced expression is independent of the choice of reduced expression.

Corollary 2.7.4 ([DC08], Proposition 4.1.1)**.** *Let $w \in W$ be reduced. Then there is a subset $S(w) = \{s_1, \ldots, s_k\} \subset S$ such that for all reduced expressions $w = s_{i_1} \cdots s_{i_k}$ we have $s_{i_n} \in S(w)$.*

Proof. The elementary M-operations of type (II) do not change the set of letters in a reduced expression. Hence the assertion follows from Theorem 2.7.3. □

With this, we can address special subgroups of Coxeter groups and prove that they are indeed Coxeter groups. Furthermore, we get as a simple corollary that S is a minimal generating set for W. Before we prove this, we state another technical lemma.

Definition 2.7.5. Let (W, S) be a Coxeter system and $T \subset S$. We call the subgroup W_T generated by the elements of T a *special* subgroup of W. Let $w \in W$. We call a w-conjugate of W_T (i.e., $w^{-1}W_T w$) a *parabolic* subgroup of W.

Lemma 2.7.6 ([DC08], Corollory 4.1.2.)**.** *Let (W, S) be a Coxeter system. For each $T \subset S$, W_T consists of those elements $w \in W$ such that $S(w) \subseteq T$.*

Proof. Assume that $w \in W$ is reduced. Since w is reduced w^{-1} is also reduced. Thus

$$S(w^{-1}) = S(w). \tag{2.10}$$

Furthermore, if w and w' are reduced then ww' is not necessarily reduced. But we can get a reduced expression by applying the Deletion Condition, yielding

$$S(ww') \subseteq S(w) \cap S(w'). \tag{2.11}$$

Let $W' := \{w \in W \mid S(w) \subseteq T\}$. Clearly $W' \subseteq W_T$. Furthermore by (2.10) and (2.11), we get that W' is a subgroup of W_T. Since $T \subset W'$ and W_T is the subgroup generated by T, we get that $W_T \subseteq W'$. Hence $W_T = W'$. □

With this we can prove the statement that a special subgroup of a Coxeter group is a Coxeter group itself.

Theorem 2.7.7 ([DC08], Corollary 4.1.3-4.1.5, Theorem 4.1.6)**.** *Let (W, S) be a Coxeter system and $T \subset S$. Furthermore, let W_T be the special subgroup generated by the elements of T.*

(i) It holds that $W_T \cap S = T$. Hence S is a minimal generating set for W.

(ii) The length function ℓ_T agrees with ℓ_S.

(iii) (W_T, T) is a Coxeter system.

Proof. (i) This follows directly by Lemma 2.7.6.

(ii) Suppose $w \in W_T$ is reduced and $w = s_1 \cdots s_k$. By Corollary 2.7.4, each s_i lies in T. Hence $\ell_T(w) = k = \ell_S(w)$.

(iii) Note that W_T is a group generated by T, which is a set of involutions. Thus, to apply Theorem 2.4.11, we need to proof that the Exchange Condition holds for W_T.

By the second statement the length function ℓ_T and ℓ_S agree for all $w \in W_T \subset W$. Let $t \in T$ and $w \in W_T$ such that $\ell_T(tw) \leq \ell_T(w)$. Furthermore, let $w = t_1 \cdots t_k$, $t_i \in T$ be a reduced expression for w. Since (W, S) is a Coxeter system, it satisfies the Exchange Condition. Hence a letter of w can be exchanged for a t in front. Thus (W_T, T) satisfies the Exchange Condition. □

Due to Theorem 2.7.7 and the fact that $m_{ij} = 2$ whenever s_i and s_j commute, we get:

Proposition 2.7.8 ([DC08], Proposition 4.1.7)**.** *Suppose S can be partitioned into two non-empty disjoint subsets S' and S'' such that $m_{ij} = 2$ for all $s_i \in S'$ and $s_j \in S''$. Then $W = W_{S'} \times W_{S''}$.*

Before we start with the construction of the Coxeter complex, we prove another technical lemma, which should motivate the construction in the next section.

Lemma 2.7.9 ([DC08], Theorem 4.1.6)**.** *(i) Let $T, T' \subset S$ and $w, w' \in W$. Then $wW_T \subseteq w'W_{T'}$ if and only if $w^{-1}w' \in W_{T'}$ and $T \subseteq T'$.*

(ii) Let $(T_i)_{i \in I}$ be a family of subsets of S. If

$$T = \bigcap_{i \in I} T_i, \qquad \textit{then} \qquad W_T = \bigcap_{i \in I} W_{T_i}.$$

Proof. (i) The if only part is obvious. Thus, assume $wW_T \subseteq w'W_{T'}$ or equivalently $W_T = w^{-1}w'W_{T'}$. Since $e \in W_T$ there is an element $w_{T'} \in W_{T'}$ such that $e = w^{-1}w'w_{T'}$. Hence, $w^{-1}w' = w_{T'}^{-1} \in W_{T'}$. Thus, $W_T \subseteq W_{T'}$. But, $T = W_T \cap S \subseteq W_{T'} \cap S = T'$. Hence, the assertion follows.

(ii) This one is an immediate consequence of (i). □

2.8 Construction of the Space $\mathcal{U}(W, P^n)$

Consider a tessellation of $\mathbb{R}^2$ by equilateral triangles. We would like to motivate a construction of this tessellation. Let Δ be an equilateral triangle and let v_i and e_i be, respectively, the vertices and edges of Δ. Let s_i be the reflection along e_i. The set $\Delta \cup s_i(\Delta)$ consists of two copies of Δ identified along e_i. Note s_i is an involution, i.e., $s_i^2 = \mathrm{id}_{\mathbb{R}^2}$.

Now consider a vertex v_i of Δ such that $v_i \in e_i \cap e_j$. In the tessellation, 6 copies of Δ meet at v_i. This corresponds to the relation $(s_i s_j)^3 = \mathrm{id}_{\mathbb{R}^2}$.

Let $W = \langle S \mid R \rangle$ where $S := \{s_1, s_2, s_3\}$ and $R := \{s_i^2 = (s_i s_j)^3 = e\}_{1 \leq i \neq j \leq 3}$. By successively reflecting along s_1, s_2, and s_3 and 'glueing' the copies of the initial triangle with respect to the relations above, we get a tessellation consisting of $|W| = \infty$ many copies of Δ. Hence the tessellation of $\mathbb{R}^2$ by equilateral triangles can be seen as a product space $W \times \Delta$ modulo an equivalence relation, which identifies the edges and vertices glued together.

For the general case, let $P^n \subset \mathbb{X}^n$ be a polytope, and $(H_i)_{s_i \in S}$ be the hyperplanes induced by the facets of P^n such that the $s_i \in S$ are the reflection along H_i. In Theorem 2.9.6, we will prove that the group W generated by S yields a Coxeter system (W, S). Furthermore, for each $x \in P^n$, set $S(x) := \{s_i \in S \mid x \in H_i\}$ and let $W_{S(x)}$ be the special subgroup generated by $S(x)$. Note that if x lies in the interior of P^n, then $S(x) = \emptyset$ and $W_{S(x)} = \langle e \rangle$. If $x \in \partial P^n$, then $S(x)$ corresponds to the set of reflections s_i such that x lies in H_i.

Define an equivalence relation $\sim$ on $W \times P^n$ by $(w_1, x) \sim (w_2, y)$ if and only if $x = y$ and $w_1^{-1} w_2 \in W_{S(x)}$ where $W_{S(x)}$ is some special subgroup of W depending on x.

Equip W with the discrete topology and $W \times P^n$ with the product topology and define the quotient space

$$\mathcal{U}(W, P^n) := (W \times P^n)/\sim . \tag{2.12}$$

Definition 2.8.1. We call $\mathcal{U} := \mathcal{U}(W, P^n)$ as in (2.12) the *Coxeter complex* of W and P^n. If the Coxeter complex is homeomorphic to $\mathbb{X}^n$, we call it the *tessellation* of $\mathbb{X}^n$ by P^n using W.

Remark 2.8.2. (*i*) The construction of the Coxeter complex can be generalised to an arbitrary space, a family of subspaces, and an arbitrary group and is not limited to Coxeter systems, see [DC08].

(*ii*) Assume that (W, S) is a Coxeter system of rank $n = |S| - 1$ and P^n is an n-simplex. Then the tessellation naturally is a simplicial complex, see [DC08].

We will write $[w, x]$ for the equivalence class of (w, x) and wP^n for the image of $\{w\} \times P^n$. This is well-defined, since the map $\iota\colon P^n \to \mathcal{U}$ defined by $x \mapsto [e, x]$ clearly is an embedding. Furthermore, the Coxeter group W acts naturally on $\mathcal{U}(W, P^n)$ by $w[w_1, x] = [ww_1, x]$. Assume that $(w_1, x) \sim (w_2, y)$ and $w \in W$. By definition of $\sim$, we have

$$x = y \qquad \text{and} \qquad w_1^{-1}w_2 = w_1^{-1}w^{-1}ww_2 = (ww_1)^{-1}(ww_2) \in W_{S(x)}$$

Thus $(ww_1, x) \sim (ww_2, y)$. Hence the W-action preserves the equivalence relation $\sim$.

We identify P^n with its image under ι and call it the *fundamental chamber*. For any $w \in W$ the image wP^n in $\mathcal{U}$ is called a *chamber* in $\mathcal{U}$. Naturally, the set of chambers is identified with W.

Definition 2.8.3. Suppose a group G acts on a space X and let $C \subset X$ be a closed subset. We say C is a *strict fundamental domain* for G on X if each G-orbit intersects C in exactly one point and if for each point x in the interior of C we have that $Gx \cap C = \{x\}$ or equivalently $X/G \cong C$.

Example 2.8.4. Consider $\mathbb{Z}_2$ acting on $\mathbb{R}^2$ in two different ways. First, let C be the half-space defined by $x \geq 0$ and H be its bounding line $\{x = 0\}$. Assume that $\mathbb{Z}_2$ is a mirror reflection across H. Then $\mathbb{Z}_2 x \cap C = \{x\}$ for all $x \in C$. Hence C is a strict fundamental domain for $\mathbb{Z}_2$ on $\mathbb{R}^2$. Second, let C be as before but let $\mathbb{Z}^2$ be acting as a rotation by π about the origin. Then for every x in the interior of C, we have $\mathbb{Z}_2 x \cap C = \{x\}$. But for any $x \in H \backslash \{0\}$ we have $\mathbb{Z}_2 x \cap C = \{x, -x\}$. Hence C is not a strict fundamental domain for $\mathbb{Z}_2$ on $\mathbb{R}^2$..

Theorem 2.8.5 ([DC08], p. 64-65)**.** *Let $\mathcal{U}(W, P^n)$ be a Coxeter complex. Then, P^n is a strict fundamental domain of W on $\mathcal{U}$.*

Proof. We need to show that $\mathcal{U}/W$ is homeomorphic to P^n. Let $p\colon \mathcal{U} \to P^n$ be the projection map onto the second factor and $\iota\colon P^n \to \mathcal{U}$ the inclusion map. Since $p \circ \iota = \mathrm{id}_{P^n}$, the projection map is a retraction. Furthermore, the orbit relation on $W \times P^n$ is coarser than the equivalence relation $\sim$. Thus p induces a continuous bijection $\overline{p}\colon \mathcal{U}/W \to P^n$. Since p is obviously an open mapping, $\overline{p}$ is a homeomorphism. □

As prerequisites for the main theorem of this chapter, we now establish two properties of the Coxeter complex $\mathcal{U}(W, P^n)$.

Lemma 2.8.6 ([DC08], Lemma 5.1.4)**.** *The Coxeter complex $\mathcal{U}(W, P^n)$ is connected as a topological space.*

Proof. Since $\mathcal{U}(W, P^n) = W \times P^n/\sim$ has the quotient topology, a subset $A \subset \mathcal{U}$ is open (resp. closed) if and only if $A \cap wP^n$ is open (resp. closed) for all chambers wP^n.

Suppose $\emptyset \neq A$ is both open and closed. Since P^n is connected, for any $w \in W$, $A \cap wP^n$ is either wP^n or empty. Thus A is a non-empty union of chambers $A = \bigcup_{v \in V} vP^n$ where V is a non-empty subset of W.

Let $v \in V$ and $s \in S$. If $x \in P^n$ then any open neighbourhood of $[v, x] \in vP^n$ must contain $[vs, x] \in vsP^n$. But S generates W, hence $V = W$ and $A = \mathcal{U}$. □

Furthermore, the Coxeter complex $\mathcal{U}(W, P^n)$ satisfies the following universal property.

Lemma 2.8.7 ([Vin71])**.** *Let (W, S) be a Coxeter system where W is acting on $\mathbb{X}^n$, and $P^n \subset \mathbb{X}^n$ be a polytope with facets F_i. For each $s_i \in S$, let $\mathrm{Fix}_i(\mathbb{X}^n)$ denote the fixed point set of s_i on $\mathbb{X}^n$. Let $f\colon P^n \to \mathbb{X}^n$ be a continuous map such that $f(F_i) \subset \mathrm{Fix}_i(\mathbb{X}^n)$ for all i. Then there is a unique extension of f to a W-equivariant map $\tilde{f}\colon \mathcal{U}(W, P^n) \to \mathbb{X}^n$ given by the formula $\tilde{f}([w, x]) = wf(x)$.*

Proof. Let $g\colon W \times P^n \to \mathbb{X}^n$ be defined by $g((w, x)) = wf(x)$. Then g is continuous and commutes with the action of W. Since $f(F_i) \subset \mathrm{Fix}_i(\mathbb{X}^n)$, the mapping g identifies equivalent points of the equivalence relation $\sim$ we used to construct $\mathcal{U}(W, P^n)$. Hence there exists a continuous mapping

$$\begin{tikzcd}
W \times P^n \arrow[r, "g"] \arrow[d, "\pi"'] & \mathbb{X}^n \\
\mathcal{U}(W, P^n) \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

Figure 2.9: The Coxeter complex satisfies a universal property.

$\tilde{f}\colon \mathcal{U} \to \mathbb{X}^n$ making the diagram in Figure 2.9 commutative. Obviously $\tilde{f}([w, x]) = wf(x)$.

□

2.9 Tessellation of $\mathbb{X}^n$

In this final section of this chapter, we want to discuss when we can identify the Coxeter complex with a tessellation of $\mathbb{X}^n$. First, we construct an atlas, which induces a Riemannian metric on the Coxeter complex. With this we will prove the main theorem.

Definition 2.9.1. An $\mathbb{X}^n$-*structure* on a manifold M^n is an atlas of charts $\{\psi_i\colon U_i \to \mathbb{X}^n\}_{i\in I}$ where

- $\{U_i\}_{i\in I}$ is an open cover of M^n,
- each ψ_i is a homeomorphism onto its image and
- each overlap map $\psi_j\psi_i^{-1}\colon \psi_j(U_i \cap U_j) \to \psi_i(U_i \cap U_j)$ is the restriction of an isometry in $\mathrm{Isom}(\mathbb{X}^n)$.

Note that the Riemannian metric on $\mathbb{X}^n$ induces one on M^n so that each chart ψ_i is an isometry onto its image. Essentially, an $\mathbb{X}^n$-structure on M^n is a Riemannian metric of constant sectional curvature $+1$, 0, or -1.

Example 2.9.2. Let $G \subset \mathrm{Isom}(\mathbb{X}^n)$ such that G acts freely and properly on $\mathbb{X}^n$. Then $\mathbb{X}^n/G$ is a manifold and $\pi\colon \mathbb{X}^n \to \mathbb{X}^n/G$ is a local isometry. For each $x \in \mathbb{X}^n$ choose $r(x) > 0$ so that π maps $B_{r(x)}(x)$ isometrically onto the ball $U_x \coloneqq B_{r(x)}(\pi(x))$. Let $\psi_x\colon U_x \to \mathbb{X}^n$ be the inverse of π

restricted to $B_{r(x)}(x)$. Then $\{U_x\}_{x\in\mathbb{X}^n}$ is an open cover of $\mathbb{X}^n$ and ψ_x maps U_x homeomorphically onto $B_{r(x)}(x)$. Let $x, y \in \mathbb{X}^n$ be arbitrary points such that U_x and U_y overlap and consider the function

$$\psi_y\psi_x^{-1}\colon \psi_x(U_x \cap U_y) \to \psi_y(U_x \cap U_y).$$

Let $w \in \psi_x(U_x \cap U_y)$ and set $z = \psi_y\psi_x^{-1}(w)$. Then $\pi(w) = \pi(z)$. Hence there is a $g \in G$ such that $gw = z$. As g is continuous at w, there is an $\varepsilon > 0$ such that $\psi_y(U_x \cap U_y)$ contains $gB_\varepsilon(w)$. By shrinking ε, we may assume that $\psi_x(U_x \cap U_y)$ contains $B_\varepsilon(w)$. As $\pi g = \pi$, the map $\psi_y^{-1}g$ agrees with ψ_x^{-1} on $B_\varepsilon(w)$. Thus $\psi_y\psi_x^{-1}$ agrees with g on $B_\varepsilon(w)$. Hence $\{\psi_x\colon U_x \to \mathbb{X}^n\}_{x\in\mathbb{X}^n}$ is an $\mathbb{X}^n$-structure on $\mathbb{X}^n/G$.

There are three facts about $\mathbb{X}^n$-structures we will use later. For a proof see [TL97] or [Rat06].

- An $\mathbb{X}^n$-structure on M^n induces one on its universal cover $\widetilde{M}^n$.
- There is a *developing map* $D\colon \widetilde{M}^n \to \mathbb{X}^n$, i.e., a local homeomorphism.
- The manifold M^n with a $\mathbb{X}^n$-structure is complete, if the developing map D is a homeomorphism, i.e., D is a *covering map*.

Suppose $P^n \subset \mathbb{X}^n$ is a convex polytope with codimension-one faces $(F_i)_{i\in I}$. For each $i \in I$, let r_i denote the isometric reflection of $\mathbb{X}^n$ across the hyperplane supported by F_i. Let $\overline{W}$ be the subgroup of $\mathrm{Isom}(\mathbb{X}^n)$ generated by $\{r_i\}_{i\in I}$. We want to assert that P^n is a fundamental domain for the $\overline{W}$-action on $\mathbb{X}^n$ and that $\overline{W}$ is a discrete subgroup of $\mathrm{Isom}(\mathbb{X}^n)$.

One obvious condition is that if the hyperplanes supported by F_i and F_j intersect, then the subgroup $\overline{W}_{ij}$ generated by r_i and r_j is finite and the sector bounded by these hyperplanes which contains P^n must be a fundamental domain for the $\overline{W}_{ij}$-action. As in Example 2.1.2, the condition that the sector is a fundamental domain implies that the dihedral angle between the hyperplanes is a submultiple of π. This forces all the dihedral angles of P^n to be non-obtuse, hence the polytope is simple, by Theorem 2.2.11. The next Definition combines these two conditions:

Definition 2.9.3. Let $P^n \subset \mathbb{X}^n$ be a simple polytope and F_i its facets. We call P^n a *Coxeter polytope* if

- whenever $F_i \cap F_j \neq \emptyset$, the dihedral angle is of the form π/m_{ij} for some natural numbers $m_{ij} \geq 2$ or

- two facets are parallel, i.e., $F_i \cap F_j = \emptyset$.

The matrix (m_{ij}) is a Coxeter matrix for the group $\overline{W}$ generated by $\{r_i\}_{i \in I}$. Let (W, S) be the corresponding Coxeter system, with generating set $S = \{s_i\}_{i \in I}$. In view of Example 2.1.2, the order of $r_i \circ r_j$ is m_{ij} so the function $s_i \to r_i$ extends to a homomorphism $\varphi \colon W \to \overline{W}$. There is a natural mirror structure on P^n: the mirror corresponding to i is F_i. Thus the construction of the Coxeter complex in the last section gives the space $\mathcal{U}(W, P^n)$. As in Lemma 2.8.7, the inclusion $\iota \colon P^n \to \mathbb{X}^n$ gives a φ-equivariant map $\tilde{\iota} \colon \mathcal{U}(W, P^n) \to \mathbb{X}^n$ defined by $[w, x] \to \varphi(w)x$. We want to prove that $\tilde{\iota}$ is a homeomorphism.

Example 2.9.4. (One-dimensional Euclidean space) Let I be a closed interval in $\mathbb{E}^1$ and let r_1 and r_2 be the reflections across its endpoints. Set $m_{12} = \infty$. The group $\overline{W}$ generated by r_1 and r_2 is the infinite dihedral group of Example 2.1.3 The Coxeter group W is also the infinite dihedral group and $\varphi \colon W \to \overline{W}$ is an isomorphism. Thus the map $\tilde{\iota} \colon \mathcal{U}(W, I) \to \mathbb{E}^1$ is a homeomorphism.

Example 2.9.5. (One-dimensional sphere) Suppose I is a circular arc of length π/m and that r_1 and r_2 are the reflections of $\mathbb{S}^1$ across the endpoints of I. The group $\overline{W}$ generated by r_1 and r_2 is the finite dihedral group of order $2m$ of Example 2.1.2. It is easy to see that $\varphi \colon W \to \overline{W}$ is an isomorphism and $\tilde{\iota} \colon \mathcal{U}(W, I) \to \mathbb{S}^1$ a homeomorphism.

We are now ready to prove the main theorem of this chapter.

Theorem 2.9.6 ([DC08], Theorem 6.4.3)**.** *Suppose P^n is a Coxeter polytope in $\mathbb{X}^n$, $n \geq 2$, with facets $\{F_k\}$, and dihedral angles of the form π/m_{ij} with $m_{ij} \geq 2$ whenever $F_i \cap F_j \neq \emptyset$. If $F_i \cap F_j = \emptyset$, put $m_{ij} = \infty$. Let $\overline{W}$ be the group generated by the reflections across the facets F_k. Furthermore, let (W, S) be the Coxeter system defined by the Coxeter matrix (m_{ij}). Then the natural map $\tilde{\iota} \colon \mathcal{U}(W, P^n) \to \mathbb{X}^n$ is a homeomorphism.*

In particular

(i) $\varphi \colon W \to \overline{W}$ is an isomorphism,

(ii) $\overline{W}$ acts properly on $\mathbb{X}^n$, and

(iii) P^n is a strict fundamental domain for the $\overline{W}$-action on $\mathbb{X}^n$.

Proof. We use induction on the dimension n.

Let (s_n) denote the statement of the theorem in case $\mathbb{X}^n = \mathbb{S}^n$ and P^n is a spherical n-simplex σ^n. Let (c_n) denote the statement of the theorem with P^n replaced by an open simplicial cone C_r^n of radius r in $\mathbb{X}^n$ and with $\mathbb{X}^n$ replaced by $B_r(x)$ the open ball of radius r in $\mathbb{X}^n$ about the vertex of C^n. Finally, let (t_n) denote the statement of the theorem. The structure of our induction argument is as follows:

$$(s_n) \implies (c_{n+1}) \implies (t_{n+1}) \implies (s_{n+1}).$$

By Example 2.9.5, (s_1) holds. We now will prove the three implications.

$(s_n) \implies (c_{n+1})$. Suppose $C_r^{n+1} \subset \mathbb{X}^{n+1}$ is a simplicial cone of radius r with non-obtuse dihedral angles of the form π/m_{ij}. Let σ^n be its intersection with $\mathbb{S}^n$. The Coxeter group associated with C_r^{n+1} is the same as the one associated with σ^n. Hence (s_n) implies that $\varphi\colon W \to \overline{W}$ is an isomorphism and $\mathcal{U}(W, \sigma^n)$ is homeomorphic to $\mathbb{S}^n$. Since $\mathbb{S}^n$ is compact, so is $\mathcal{U}(W, \sigma^n)$. Thus W is finite. Since we know that

- C_r^{n+1} is the cone over σ^n,
- $\mathcal{U}(W, C_r^{n+1})$ is the cone over $\mathcal{U}(W, \sigma^n)$, and
- an open ball in $\mathbb{X}^{n+1}$ is the cone over $\mathbb{S}^n$,

it follows that $\tilde{\imath}$ takes $\mathcal{U}(W, C_r^{n+1})$ homeomorphically onto the open ball.

$(c_{n+1}) \implies (t_{n+1})$. Let W, $\overline{W}$, and P^{n+1} be as in the statement of the theorem. We first show that (c_{n+1}) implies that $\mathcal{U}(W, P^{n+1})$ has an $\mathbb{X}^{n+1}$-structure such that $\tilde{\imath}\colon \mathcal{U}(W, P^{n+1}) \to \mathbb{X}^{n+1}$ is a local isometry.

Given $x \in P^{n+1}$, let $S(x)$ denote the set of reflections s_i across the codimension-one faces F_i which contain x. By Theorem 2.7.7, we have that $(W_{S(x)}, S(x))$ is a Coxeter system. Let r_x denote the distance to the nearest face of P^{n+1} which does not contain x and let C_x (resp. B_x) be an open conical neighbourhood (resp. ball) of radius r_x about x in P^{n+1} (resp. $\mathbb{X}^{n+1}$). An open neighbourhood of $[e, x]$ in $\mathcal{U}(W, P^{n+1})$ has the form of $\mathcal{U}(W_{S(x)}, S(x))$. By (c_{n+1}), $\tilde{\imath}$ maps this neighbourhood of $[e, x]$ homeomorphically onto B_x. By Lemma 2.8.7, it maps the w-translate of such a neighbourhood homeomorphically onto $\varphi(w)(B_x)$. This defines an atlas on $\mathcal{U}(W, P^{n+1})$. The atlas gives $\mathcal{U}(W, P^{n+1})$ the structure of a smooth manifold and an $\mathbb{X}^{n+1}$-structure.

The $\mathbb{X}^{n+1}$-structure induces a Riemannian metric of constant sectional curvature on $M^{n+1} := \mathcal{U}(W, P^{n+1})$. Since W acts isometrically on M^{n+1} and since the quotient space $M^{n+1}/W = P^{n+1}$ is compact, M^{n+1} is metrically complete which is equivalent to the condition that the $\mathbb{X}^{n+1}$-structure is complete (see [TL97] or [Rat06]). In other words, the developing map $D\colon \tilde{M}^{n+1} \to \mathbb{X}^{n+1}$ is a covering projection. By Lemma 2.8.6, $\mathcal{U}(W, P^{n+1})$ is connected. Thus, since the developing map is locally given by $\tilde{\iota}\colon \mathcal{U}(W, P^{n+1}) \to \mathbb{X}^{n+1}$ and since $\tilde{\iota}$ is globally defined, $\tilde{\iota}$ must be covered by D, i.e., $\tilde{\iota}$ is also a covering projection. Since $\mathbb{X}^{n+1}$ is simply connected, $\tilde{M}^{n+1} = M^{n+1} = \mathcal{U}(W, P^{n+1})$, $D = \tilde{\iota}$, and $\tilde{\iota}$ is a homeomorphism.

$(t_{n+1}) \implies (s_{n+1})$. The statement (s_{n+1}) is a special case of (t_{n+1}). $\square$

In the remaining part of this chapter, we will state some consequences of Theorem 2.9.6. First of all, part (*iii*) implies that $\mathbb{X}^n$ is tiled by isometric copies of P^n. Thus we can identify $\mathcal{U}(W, P^n)$ as a tessellation of $\mathbb{X}^n$. Furthermore, we have proven that the space $\mathcal{U}(W, P^n)$ is a manifold. Part (*i*) of Theorem 2.9.6 implies that $\overline{W}$ is a Coxeter group and part (*ii*) implies that it is a discrete subgroup of $\mathrm{Isom}(\mathbb{X}^n)$.

Definition 2.9.7. A *geometric reflection group* is the action of a group $\overline{W}$ on $\mathbb{X}^n$, which, as in Theorem 2.9.6, is generated by the reflections across the faces of a simple convex polytope with dihedral angles submultiples of π. The reflection group is called *spherical*, *Euclidean*, or *hyperbolic* as $\mathbb{X}^n$ is, respectively, $\mathbb{S}^n$, $\mathbb{E}^n$, or $\mathbb{H}^n$.

If we combine Theorem 2.9.6 with the three Propositions 2.3.3, 2.3.9, and 2.3.5, we get the following Corollary:

Corollary 2.9.8 ([DC08], Theorem 6.8.12)**.** *Let $M = (m_{ij})$ be a Coxeter matrix over I, W the associated Coxeter group, and $C = (c_{ij})$ the associated cosine matrix. Suppose that no m_{ij} is ∞. Then*

(i) W can be represented as a spherical reflection group generated by reflections across the faces of a spherical simplex with Gram matrix C if and only if C is positive definite.

(ii) W can be represented as a hyperbolic reflection group generated by the faces of a hyperbolic simplex with Gram matrix C if and only if C is non-degenerate with signature $(n, 1)$ and each principle submatrix is positive definite.

(iii) *Suppose, moreover, that M is irreducible. Then W can be represented as a Euclidean reflection group generated by the reflections across the faces of a Euclidean simplex with Gram matrix C if and only if C is positive semidefinite of corank* 1.

Combining Corollary 2.9.8 and the Classification Theorems 2.6.9 and 2.6.10 we get:

Corollary 2.9.9 ([Lan50]). *Any simplicial Coxeter group can be represented as a geometric reflection group with fundamental chamber an n-simplex in either $\mathbb{S}^n$, $\mathbb{E}^n$, or $\mathbb{H}^n$.*

Let $P^2 \subset \mathbb{X}^2$ be an m-gon with interior angles $\alpha_1, \ldots, \alpha_m$ and $\kappa = 1, 0, -1$ be the sectional curvature of $\mathbb{X}^2$. Then the Gauss-Bonnet theorem asserts that

$$\kappa \operatorname{area}(P^2) + \sum_i (\pi - \alpha_i) = 2\pi. \tag{2.13}$$

Hence $\sum_i \alpha_i$ is greater, equal or lesser than $(m-2)\pi$, as $\mathbb{X}^2$ is, respectively, $\mathbb{S}^2$, $\mathbb{E}^2$, or $\mathbb{H}^2$.

Example 2.9.10 (Spherical Triangle groups). Suppose P^2 is a spherical polytope with angles α_i. Since each $\alpha_i \leq \pi/2$, the condition $\sum_i \alpha_i > (m-2)\pi$ forces $m < 4$, i.e., P^2 must be a triangle. We also get this immediately with Lemma 2.2.7. What are the possibilities for α_i?

The inequality $\pi/m_1 + \pi/m_2 + \pi/m_3 > \pi$ can be rewritten as

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} > 1. \tag{2.14}$$

Assume that $m_1 \leq m_2 \leq m_3$, it is easy to see that the only triples (m_1, m_2, m_3) of integer greater that 2 satisfying (2.14) are $(2,3,3)$, $(2,3,4)$, $(2,3,5)$, and $(2,2,m)$ for $m \geq 2$.

By Theorem 2.9.6, each such triple corresponds to a spherical reflection group. The Coxeter groups corresponding to the first three triples are the symmetry groups of the Platonic solids; respectively, the symmetry group of the tetrahedron of type A_3, cube of type B_3, and dodecahedron of type H_3. The Coxeter system corresponding to $(2,2,m)$ is reducible; it is the group of type $A_1 \times I_2(m)$.

Example 2.9.11 (Two-dimensional Euclidean groups)**.** The condition $\sum_i \alpha_i = (m-2)\pi$ forces $m \leq 4$. If $m = 4$ the only possibility is that $m_1 = m_2 = m_3 = m_4 = 2$. In this case P^2 is a rectangle and we get the standard rectangular tilling of $\mathbb{E}^2$. The corresponding Coxeter group is of type $\tilde{A}_1 \times \tilde{A}_1$.

If $m = 3$, the relevant condition is

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = 1 \tag{2.15}$$

The only three possibilities of triples (m_1, m_2, m_3) with $2 \leq m_1 \leq m_2 \leq m_3$ satisfying equation (2.15) are $(2, 3, 6)$, $(2, 4, 4)$, and $(3, 3, 3)$. The corresponding Coxeter groups are, respectively, of type $\tilde{G}_2$, $\tilde{B}_2$, and $\tilde{A}_2$.

Example 2.9.12 (Hyperbolic polygon groups)**.** Given any assignment of angles of the form π/m_i to the vertices of a combinatorial m-gon such that

$$\sum_{i=1}^{m} \frac{1}{m_i} < m - 2,$$

we can find a convex realisation of it in $\mathbb{H}^2$. By Theorem 2.9.6, this yields a corresponding reflection group on $\mathbb{H}^2$. Since a hyperbolic triangle is determined, up to isometry, by its angles, we get a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^2)$, well defined up to conjugation. The corresponding Coxeter graph is listed in Figure 2.8. If $m > 3$, there is a continuous family moduli of hyperbolic polygons with the same angles and hence, a moduli space of representations of the Coxeter group.

A conclusion that can be drawn from these examples is that any assignment of angles of the form π/m_i to the vertices of an m-gon can be realised by a convex polygon in some $\mathbb{X}^2$; in fact, except for finitely many cases, $\mathbb{X}^2 = \mathbb{H}^2$.

This observation is no longer valid in higher-dimensional hyperbolic space as two theorems by Vinberg indicate. We will merely state the results here. The proofs can be found in [DC08] and [Vin85].

Theorem 2.9.13 ([DC08], Corollary 6.11.7.)**.** *Suppose $P^n \subset \mathbb{H}^n$ is a convex (compact) polytope with all dihedral angles equal $\pi/2$. Then $n \leq 4$. In other words, right-angled hyperbolic reflection groups do not exist in dimension $n > 4$.*

Theorem 2.9.14 ([Vin85]). *Suppose $P^n \subset \mathbb{H}^n$ is a convex (compact) polytope with all dihedral angles submultiples of π. Then $n \leq 29$. In other words, hyperbolic reflection groups do not exist in dimension $n > 29$.*

3 Longest Element of a Coxeter group

In this chapter, we want to analyse the longest element w_0 of spherical Coxeter group. We show for which types w_0 satisfies the (-1)-condition. This condition is used in the next chapter to identify edge reflection about an edge of P^n with the longest element of a special subgroup of W.

First, we construct a faithful representation of the Coxeter system (W, S) in $\mathbb{R}^{|S|}$, called the *canonical representation.* This representation allows us to derive properties of our Coxeter system (W, S) and of the longest element of W. We follow Section 6.12 in [DC08], Section 5.3, and 5.4 in [Hum90].

Second, we define the longest element of a Coxeter group. It exists if and only if the Coxeter group is finite. In most cases, the longest element is acting as the $-\operatorname{id}_{\mathbb{R}^{|S|}}$ element in $GL(\mathbb{R}^{|S|})$, using the canonical representation. We show that this is the case if and only if the longest element of W is in the center of W. This section is mainly based on Section 5.6 in [Hum90] and [Ric82].

3.1 Geometric Representation

In this section, we show that the canonical representation $\rho\colon W \to GL(\mathbb{R}^{|S|})$ is a *faithful* representation, i.e., the group homomorphism ρ is injective. The main idea of representation theory is to study abstract algebra in terms of matrices. The canonical representation ρ will be a special reflection invoking the cosine matrix as a bilinear form. Before we define the canonical representation, we need a technical lemma from representation theory.

Definition 3.1.1. Let G be a group, V a real vector space, and $\rho\colon G \to GL(V)$ a linear representation of G. A linear subspace $U \subset V$ is called *G-stable* if $\rho(g)u \in U$ for all $u \in U$ and $g \in G$. We call the restriction of

ρ to a G-stable linear space $U \subset V$ a *subrepresentation.* A representation is said to be *irreducible*, if ρ has exactly two subrepresentations, i.e., the trivial subspaces $\{0\}$ and V are the only G-stable subspaces. If ρ has a proper non-trivial subrepresentation, it is called *reducible.*

Example 3.1.2. A matrix representation of the dihedral group D_8 of order 8 with representation $\langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^4 = e\rangle$ is given by $\rho\colon D_8 \to GL(\mathbb{R}^3)$ where

$$s_1 \mapsto \begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) & 0 \\ \sin(\pi/4) & \cos(\pi/4) & 0 \\ 0 & 0 & 1 \end{pmatrix} =: R, \qquad s_2 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} =: S.$$

The matrix R is a 45°-rotation with respect to the z-axis and S reflects across the plane defined by $x = y$. Let $\mathbb{R}_{xy}$, $\mathbb{R}_z$ be, respectively, the restriction onto the xy-plane, z-component. Obviously $\mathbb{R}_{xy}$ and $\mathbb{R}_z$ are D_8-stable. Thus $\rho_{|\mathbb{R}_{xy}}$ and $\rho_{|\mathbb{R}_z}$ are non-trivial subrepresentations; hence ρ is reducible. However, $\rho_{|\mathbb{R}_{xy}}$ is irreducible because there is no 1-dimensional D_8-stable subspace since a rotation by $\pi/4$ sends no line in $\mathbb{R}_{xy}$ to itself. Furthermore, $\rho_{|\mathbb{R}_z}$ is irreducible since it is acting on a 1-dimensional subspace. Thus ρ can be written as a product of two irreducible subrepresentations.

Lemma 3.1.3 ([Bou08], p. 70)**.** *Let ρ be an irreducible linear representation of a group G on a finite-dimensional vector space V and assume that there is an element $g \in G$ such that $\rho(g)$ is a reflection. Then every endomorphism $u\colon V \to V$ which commutes with $\rho(G)$ is a homothety, i.e., $u = \lambda \cdot \mathrm{id}$, for some constant λ.*

Proof. Let u be an endomorphism of V commuting with $\rho(G)$. Since $\rho(g)$ is a reflection $\mathrm{id}_V - \rho(g)$ is of rank 1. Let D be the one-dimensional image of $\mathrm{id}_V - \rho(g)$ and let $x = (\mathrm{id}_V - \rho(g))(y) \in D$. As u commutes with $\rho(G)$, we have

$$u(x) = u(\mathrm{id}_V - \rho(g))(y) = \mathrm{id}_V - \rho(g)(u(y)) \in D.$$

Hence $u(D) \subset D$. Since D is of dimension one and $u(D) \subset D$, there exists $\lambda \in \mathbb{R}$ such that $u - \lambda \cdot \mathrm{id}_V$ vanishes on D. The kernel K of $u - \lambda \cdot \mathrm{id}_V$ is a subspace of V, invariant under $\rho(G)$, and non-zero as it contains D. By irreducibility of ρ, we have $K = V$. Thus $u = \lambda \cdot \mathrm{id}_V$ and u is a homothety. $\square$

Recall that a bilinear form $B\colon V \times V \to \mathbb{R}$ is *non-degenerate* if $B(v, V) = 0$ implies $v = 0$, i.e., $B\colon V \to V^*$ is an isomorphism.

Definition 3.1.4. Let G be a group, V a real vector space, and $\rho\colon G \to GL(V)$ a linear representation of G. We say a bilinear form $B\colon V \times V$ is *G-invariant* if $B(\rho(g)(v), \rho(g)(w)) = B(v, w)$ for all $v, w \in V$, $g \in G$.

As a consequence of Lemma 3.1.3, two G-invariant bilinear forms on V are proportional.

Lemma 3.1.5 ([Bou08], p. 70)**.** *Let ρ be an irreducible linear representation of a group G on a finite-dimensional vector space V and assume that there is an element $g \in G$ such that $\rho(g)$ is a reflection. Let B be a non-zero G-invariant bilinear form on V. Then*

(i) B is non-degenerate,

(ii) either symmetric or skew-symmetric, and

(iii) every other G-invariant bilinear form B' on V is proportional to B.

Proof. (i) Suppose B is a non-zero G-invariant bilinear form. We define

$$K \coloneqq \{x \in V \mid B(x, y) = 0 \text{ for all } y \in V\}.$$

Since $\rho(g) \in GL(V)$, $\rho(G)$ is bijective with inverse $\rho(g)^{-1}$. Hence for each $y \in V$ there is $y' \in V$ such that $\rho(g)^{-1}(y') = y$. Since B is G-invariant, we have

$$\begin{aligned} 0 = B(x, y) &= B(\rho(g)(x), \rho(g)(y)) \\ &= B(\rho(g)(x), (\rho(g)\rho(g)^{-1})(y')) \\ &= B(\rho(g)(x), y'). \end{aligned}$$

Thus $\rho(g)(x) \in K$. By irreducibility of ρ, we have $K = V$ or $K = \{0\}$. Since B is non-zero, we get $K = \{0\}$. Similarly,

$$K' \coloneqq \{y \in V \mid B(x, y) = 0 \text{ for all } x \in V\} = \{0\}.$$

Thus B is non-degenerate.

(iii) Let B' be another non-zero G-invariant form. Since V is finite dimensional, the non-degenerate form B induces an isomorphism from V to its dual space V^*. It follows that any bilinear form B' can be written as $B'(x, y) = B(u(x), y)$ for some linear endomorphism u. Since B and B' are both G-invariant, u commutes with $\rho(G)$. By Lemma 3.1.3, we get that $u = \lambda \cdot \mathrm{id}$. Hence

$$B'(x, y) = B(\lambda x, y) = \lambda B(x, y).$$

(ii) Apply (iii) to the case where B' is defined by $B'(x, y) = B(y, x)$. This yields

$$B(y, x) = \lambda B(x, y) = \lambda^2 B(y, x).$$

Thus $\lambda^2 = 1$. Hence $\lambda = \pm 1$ and B is either symmetric or skew-symmetric. $\square$

Let (W, S) be a Coxeter system and let $V := \mathbb{R}^{|S|}$ be a real vector space of dimension $|S|$ with basis $\{e_i \mid s_i \in S\}$. We want to define a linear representation $\rho\colon W \to GL(V)$ generated by linear reflections.

Recall Definition (2.4) of the cosine matrix (c_{ij}) associated with the Coxeter matrix M and let B_M be the symmetric bilinear form on V associated with the cosine matrix, i.e., $B_M(e_i, e_j) = c_{ij}$.

For each $s_i \in S$, let H_i be the hyperplane in V defined by

$$H_i := \{x \in V \mid B_M(e_i, x) = 0\}$$

and let $\rho_i := \rho(s_i)\colon V \to V$ be the linear reflection (in the sense of Section 2.1) defined by

$$\rho_i(x) = x - 2B_M(e_i, x)e_i. \tag{3.1}$$

We define the *canonical representation*, by extending the map $S \to GL(V)$ defined by $s_i \mapsto \rho_i$ to a homomorphism $\rho\colon W \to GL(V)$, i.e., for $w = s_{i_1} \cdots s_{i_k} \in W$ we have

$$\rho(w) = \rho(s_{i_1} \cdots s_{i_k}) := \rho_{i_1} \cdots \rho_{i_k}. \tag{3.2}$$

In particular $\rho(e) = \mathrm{id}_V$.

Especially, we have for all $k \leq |S|$ that

$$\begin{aligned} B_M(\rho_k(e_i), \rho_k(e_j)) &= B_M(e_i - 2B_M(e_k, e_i)e_k, e_j - 2B_M(e_k, e_j)e_k) \\ &= c_{ij} - 2c_{ki}c_{kj} - 2c_{kj}c_{ki} + 4c_{ki}c_{kj}c_{kk} \\ &= c_{ij} - 4c_{ki}c_{kj} + 4c_{ki}c_{kj} \\ &= c_{ij} = B_M(e_i, e_j). \end{aligned}$$

Hence by linearity, B_M is $\rho(W)$-invariant.

Example 3.1.6. Consider the Coxeter group W of type $I_2(3)$, i.e.,

$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = e \rangle = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}.$$

Thus $B_M(e_i, e_i) = 1$ and $B_M(e_1, e_2) = \cos(\pi/3) = -1/2$. By using the canonical representation ρ, we get

$$\begin{array}{ccc} \rho(e)(e_1) = e_1 & \rho_1(e_1) = -e_1, & \rho_2(e_1) = e_1 + e_2, \\ (\rho_1\rho_2)(e_1) = e_2, & (\rho_2\rho_1)(e_1) = -e_1 - e_2, & (\rho_1\rho_2\rho_1)(e_1) = -e_2. \end{array}$$

Analogously, we can calculate the action on e_2.

Note that ρ_i is not orthogonal but a linear reflection. In order to show that the map $S \to GL(V)$ defined by $s_i \to \rho_i$ is well defined, i.e., it extends indeed to a homomorphism $\rho\colon W \to GL(V)$, we need to show that relations in R are sent to the identity element of $GL(V)$. Clearly, the relations $s_i^2 = e$ goes to $\rho_i^2 = \mathrm{id}_V$, since ρ_i is a reflection. It remains to show that the relations $(s_i s_j)^{m_{ij}}$ are sent to $(\rho_i \rho_j)^{m_{ij}}$.

Lemma 3.1.7 ([DC08], Lemma 6.12.3)**.** *The order of $\rho_i\rho_j$ is m_{ij}.*

Proof. Let $W_{ij} = \langle \rho_i, \rho_j \rangle$ be the dihedral group generated by ρ_i and ρ_j. The subspace V_{ij} of V spanned by e_i and e_j is W-stable. The restriction of the bilinear form B_M to V_{ij} is given by the matrix

$$\begin{pmatrix} 1 & -\cos(\pi/m_{ij}) \\ -\cos(\pi/m_{ij}) & 1 \end{pmatrix}$$

which is positive definite for $m_{ij} < \infty$. For $m_{ij} = \infty$, we get

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

which is positive semidefinite with kernel the line spanned by the vector $e_i + e_j$. We consider the two cases separately.

Suppose $m_{ij} = \infty$. Thus $B(e_i, e_j) = -1$. If $u \coloneqq e_i + e_j$, then we have $B(u, e_i) = 0 = B(u, e_j)$, so ρ_i and ρ_j fix u. Hence we get

$$(\rho_i\rho_j)(e_i) = \rho_i(e_i - 2B_M(e_j, e_i)e_j) = \rho_i(u + e_j) = \rho(u) + \rho_i(e_j) = 2u + e_i.$$

By iteration, we get for all $n \in \mathbb{Z}$ that

$$(\rho_i\rho_j)^n(e_i) = 2un + e_i.$$

This shows that the order of $\rho_i\rho_j$ is infinite on V_{ij} and therefore also on V.

Suppose now that $m_{ij} < \infty$. Since B_M is positive definite on E_{ij}, we can identify V_{ij} with $\mathbb{R}^2$. Let L_i (resp. L_j) be the line in E_{ij} orthogonal to e_i (resp. e_j). Then the restriction of ρ_i (resp. ρ_j) to E_{ij} is an orthogonal reflection across L_i (resp. L_j). As in Example 2.1.2, we can recognise $\rho_i\rho_j$ as a rotation through the angle $2\pi/m_{ij}$. Hence $\rho_i\rho_j$ has order m_{ij} on V_{ij}. Since B_M is non-degenerate we have that V decomposes as the direct sum of V_{ij} and its orthogonal complement. Both ρ_i and ρ_j fix the orthogonal complement point-wise. Thus $\rho_i\rho_j$ has order m_{ij} on V. □

As a consequence we get immediately:

Theorem 3.1.8 ([DC08], Corollary 6.12.4)**.** *The map $S \to GL(V)$ defined by $s_i \to \rho_i$ extends to a homomorphism $\rho\colon W \to GL(V)$.*

By using the canonical representation, we can show that there is a Coxeter system for any given Coxeter matrix.

Corollary 3.1.9 ([DC08], Corollary 6.12.6)**.** *Suppose M is a Coxeter matrix and W is the group with generating set S defined by the presentation associated with M. Then*

(i) each $s_i \in S$ has order 2,

(ii) the s_i are distinct, and

(iii) s_is_j has order m_{ij}.

Hence (W, S) is a Coxeter system.

Proof. The representation ρ takes s_i to distinct reflections ρ_i on V. This proves (i) and (ii). By Lemma 3.1.7, the order of $\rho_i\rho_j$ is m_{ij}. Hence the order of $s_i s_j$ is also m_{ij}, proving (iii). So M is the associated matrix to the system (W, S), making (W, S) a Coxeter system. □

We will use this representation to state a necessary and sufficient condition for a Coxeter group to be finite. First we show two technical lemmata regarding bilinear forms.

Lemma 3.1.10 ([Bou08], p. 102)**.** *Suppose (W, S) is an irreducible Coxeter system. Let ρ be the canonical representation on $V := \mathbb{R}^{|S|}$ and let K be the kernel of the bilinear form B_M, i.e.,*

$$K = \{x \in V \mid B_M(x, y) = 0 \textit{ for all } y \in V\}.$$

Then W acts trivially on K and every proper W-stable subspace of V is contained in K.

Proof. By (3.1), we get that $\rho_i|_K = \mathrm{id}_K$. Hence W fixes K. Let U be a W-stable subspace of V and suppose that some basis vector e_i lies in U. Since (W, S) is irreducible there is another index $j \in I$ such that $m_{ij} \neq 2$. Hence $B_M(e_i, e_j) \neq 0$. Since U is W-stable $\rho_j(e_i) \in U$. Using (3.1), we see that $\rho_j(e_i) = e_i + ce_j$ for a suitable constant c. Since U is a subspace containing e_i and $\rho_j(e_i)$, we have $e_j \in U$. As (W, S) is irreducible, the Coxeter graph is connected. Thus we can proceed step-by-step and find another index to get all basis vectors, i.e., $U = V$. Hence, if U is a proper subspace, it cannot contain any e_i.

Now consider V as a $\langle\rho_i\rangle$-representation, where $\langle\rho_i\rangle$ denotes the cyclic group of order 2 generated by ρ_i. It decomposes as a direct sum of the ± 1 eigenspaces of ρ_i, i.e., as the direct sum of the line generated by e_i and the hyperplane H_i orthogonal to e_i (with respect to B_M). Since $e_i \notin U$, we have $U \subset H_i$ for all $i \in I$. Thus $U \subset K = \bigcap_i H_i$. □

With this we can distinguish two cases of irreducible Coxeter systems.

Corollary 3.1.11 ([DC08], Corollary 6.12.8.)**.** *Suppose (W, S) is an irreducible Coxeter system.*

(i) If (W, S) is spherical or hyperbolic, then B_M is non-degenerate and the canonical representation on V irreducible, i.e., there is no non-trivial proper W-stable subspace of V.

(ii) If (W, S) is Euclidean, then B_M is degenerate and the canonical representation is not semi-simple, i.e., V has a non-trivial W-stable subspace which is not a direct summand.

Proof. The first statement follows immediately from Lemma 3.1.10, since if B_M is non-degenerate, then $K = \{0\}$. As for (ii), if B_M non-zero and degenerate, then $K \neq V$ and $K \neq \{0\}$. By Lemma 3.1.10, K has no complement which is W-stable. □

Lemma 3.1.12 ([DC08], Lemma 6.6.1, Corollary 6.6.2)**.** *Let V be a real finite dimensional vector space, G a finite group, and $\rho\colon G \to GL(V)$ a representation of G. Then*

(i) there exists a positive definite G-invariant bilinear form on V and

(ii) any representation ρ on V is semi-simple.

Proof. (i) The space of positive definite bilinear forms is convex, i.e., any convex linear combination of positive definite forms is again positive definite. Thus we can define a positive definite bilinear form by averaging over G, i.e., define

$$\widetilde{B}(x, y) \coloneqq \sum_{g \in G} B(\rho(g)(x), \rho(g)(y)).$$

(ii) Suppose $U \subset V$ is a G-stable subspace. Since $\widetilde{B}$ is non-degenerate by Lemma 3.1.5, we have that V is the direct sum of U and its orthogonal complement $U^{\perp}$ relative to $\tilde{B}$. As $\widetilde{B}$ is G-invariant and U is G-stable, we have that $U^{\perp}$ is also G-stable. Thus V is semi-simple. □

We follow up with the proof of criteria for a Coxeter group to be finite.

Theorem 3.1.13 ([DC08], Theorem 6.12.9)**.** *Suppose $M = (m_{ij})$ is a Coxeter matrix on a set I, that (c_{ij}) is its associated cosine matrix defined by (2.4), and that (W, S) is its associated Coxeter system. Then the following statements are equivalent:*

(i) W is a reflection group on $\mathbb{S}^n$, $n = |S| - 1$, such that the elements of S are represented as the reflections across the codimension-one faces of a spherical n-simplex σ^n.

(ii) (c_{ij}) is positive definite.

(iii) W is finite.

Proof. By Theorem 2.9.6, (i) is equivalent to the existence of a spherical n-simplex with dihedral angles of the form π/m_{ij} and by Proposition 2.3.3, this is equivalent to the positive definiteness of (c_{ij}). Hence (i) is equivalent to (ii). Obviously (i) implies (iii). It remains to show that (iii) implies (ii).

Suppose W is finite. The cosine matrix of a reducible Coxeter system is positive definite, if all connected components of its Coxeter graph are positive definite. Hence we can assume W being irreducible. Consider the canonical representation of W on V. Since W is finite we get with Lemma 3.1.12 that V is semi-simple. Thus by Corollary 3.1.11, the bilinear form B_M is non-degenerate and the representation is irreducible. Hence V admits a W-invariant inner product B'. By Lemma 3.1.5, B_M is proportional to B'. Thus B_M is either positive definite or negative definite. Since $B_M(e_i, e_i) = c_{ii} = 1$, the bilinear form B_M is positive definite. Hence (c_{ij}) is positive definite. □

In the following, we want to show that the canonical representation is faithful. For this we want a characterisation for $\ell(ws_i)$ to be greater or smaller than $\ell(w)$, in terms of the action of W on V.

We introduce the *root system* Φ of a Coxeter system (W, S), consisting of the set of unit vectors in V permuted by W using the canonical representation ρ, i.e.,

$$\Phi := \{\rho(w)(e_i) \mid w \in W, s_i \in S\}.$$

These are unit vectors, because B_M is W-invariant on V. Note that $\Phi = -\Phi$, since $\rho_i(e_i) = -e_i$.

We can write $\alpha \in \Phi$ uniquely in the form

$$\alpha = \sum_{s_i \in S} c_i e_i$$

in terms of the basis e_i, with coefficients $c_i \in \mathbb{R}$. We call α *positive* (resp. negative) and write $\alpha > 0$ (resp. $\alpha < 0$) if all $c_i \geq 0$ (resp. $c_i \leq 0$). For example each e_i is positive and $\rho_i(e_i) = -e_i$ is negative.

Example 3.1.14. As in Example 3.1.6 consider the Coxeter group W of type $I_2(3)$ and the action of the canonical representation ρ. Note that all unit vectors e_i permuted by ρ are unit vectors regarding B_M, e.g., $B_M(e_1 + e_2, e_1 + e_2) = 1$. Hence the root system Φ of W is given by

$$\Phi = \{e_1, e_2, -e_1, -e_2, e_1 + e_2, -e_1 - e_2\}.$$

The roots e_1, e_2, and $e_1 + e_2$ are positive and the remaining three are negative.

An immediate consequence of the next theorem is that Φ always decomposes into the set of positive and negative roots, i.e., for a root $\alpha = \sum_{s_i \in S} c_i e_i$ either all c_i are positive or negative.

Theorem 3.1.15 ([Hum90], p. 111-113)**.** *Let (W, S) be a Coxeter system and $w \in W$ and $s_i \in S$. If $\ell(ws_i) > \ell(w)$, then $\rho(w)(e_i) > 0$. If $\ell(ws_i) < \ell(w)$, then $\rho(w)(e_i) < 0$.*

Proof. First, we show that the second statement follows from the first, applied to ws_i in place of w: if $\ell(ws_i) < \ell(w)$, then $\ell((ws_i)s_i) > \ell(ws_i)$, forcing $\rho(ws_i)(e_i) > 0$, i.e., $\rho(w)(-e_i) > 0$, or $\rho(w)(e_i) < 0$.

Second, to prove the first statement, we proceed by induction on $\ell(w) = k$. In case $\ell(w) = 0$, we have $w = e$, and the statement is obviously true. So, assuming $\ell(w) > 0$, we can find an $s_j \in S$ such that $\ell(ws_j) = \ell(w) - 1$ (choose s_j as the last factor of a reduced expression of w). Since $\ell(ws_i) > \ell(w)$ by assumption, we see that $s_i \neq s_j$. Set $T := \{s_i, s_j\}$ such that W_T is a dihedral group. Now we make a crucial choice within the coset wW_T. Consider the set

$$A := \{v \in W \mid v^{-1}w \in W_T \text{ and } \ell(v) + \ell(v^{-1}w) = \ell(w)\}.$$

Obviously we have $w \in A$. Choose $v \in A$ for which $\ell(v)$ is as small as possible, and write $v_T := v^{-1}w \in W_T$. Thus $w = vv_T$ with $\ell(w) = \ell(v) + \ell(v_T)$. The strategy now is to analyse how v and v_T act on roots.

Observe that $ws_j \in A$: indeed, $(s_j w^{-1})w = s_j$ lies in W_T, while $\ell(ws_j) + \ell(s_j) = (\ell(w) - 1) + 1 = \ell(w)$. The choice of v therefore forces $\ell(v) \leq$

$\ell(ws_j) = \ell(w) - 1$. This allow us to apply the induction hypothesis to the pair v, s_i.

Beforehand, we compare the lengths of v and vs_i. Suppose it were true that $\ell(vs_i) < \ell(v)$, i.e., $\ell(vs_i) = \ell(v) - 1$. Then we could calculate as follows:

$$\begin{aligned}
\ell(w) &\leq \ell(vs_i) + \ell((s_i v^{-1})w) \\
&\leq \ell(vs_i) + \ell(s_i v^{-1} w) \\
&= (\ell(v) - 1) + \ell(s_i v^{-1} w) \\
&\leq \ell(v) - 1 + \ell(v^{-1} w) + 1 \\
&= \ell(v) + \ell(v^{-1} w) \\
&= \ell(w).
\end{aligned}$$

So equality holds throughout, forcing $\ell(w) = \ell(vs_i) + \ell((s_i v^{-1})w)$ and therefore $vs_i \in A$, contrary to $\ell(vs_i) < \ell(v)$. This contradiction shows that we must instead have $\ell(vs_i) > \ell(v)$. By induction, we obtain: $\rho(v)(e_i) > 0$. An entirely similar argument shows that $\ell(vs_j) > \ell(v)$ whence $\rho(v)(e_j) > 0$.

Since $w = vv_T$, we will be done if we can show that v_T maps e_i to a non-negative linear combination of e_i and e_j.

We claim that $\ell(v_I s_i) \geq \ell(v_I)$. Otherwise we would have:

$$\begin{aligned}
\ell(ws_i) = \ell(vv^{-1}ws_i) &\leq \ell(v) + \ell(v^{-1}ws_i) = \ell(v) + \ell(v_T s_i) \\
&\leq \ell(v) + \ell(v_T s_i) < \ell(v)\ell(v_T) = \ell(w),
\end{aligned}$$

contrary to $\ell(ws_i) > \ell(w)$. In turn, it follows that any reduced expression for v_T in W_T must end with s_j.

Consider the two possible cases:

First, if $m_{ij} = \infty$, an easy calculation shows that $\rho(v_T)(e_i) = ae_i + be_j$, with $a, b \geq 0$ and $|a - b| = 1$. Indeed, $B(e_i, e_j) = -1$, such that $\rho_j(e_i) = e_i + 2e_j$, $(\rho_i \rho_j)(e_i) = 2e_j + 3e_i$, $(\rho_j \rho_i \rho_j)(e_i) = 3e_i + 4e_j$ and so on.

Second, if $m_{ij} < \infty$, notice that $\ell(v_T) < m_{ij}$. Indeed, m_{ij} is clearly the maximum possible value of ℓ on W_T, and an element of length m_{ij} in W_T has a reduced expression ending with s_i. So v_T can be written as a product of fewer than $m_{ij}/2$ terms $s_i s_j$, possibly preceded by one factor s_j. Direct calculation shows that $\rho(v_T)(e_i)$ is a non-negative linear combination of e_i and e_j. Recall that we are now working in the Euclidean plane, with unit vectors e_i and e_j at angle of $\pi - \pi/m_{ij}$, and $s_i s_j$ rotates e_i through an angle of $2\pi/m_{ij}$ towards e_j. Hence, the rotation involved in v_T move e_i through at most an angle of $\pi - 2\pi/m_{ij}$, still within the positive cone defined by e_i and e_j. If v_T further involves a reflection corresponding to s_j, the resulting vector still lies within this positive cone, because the angle between e_i and the reflecting line is $(\pi/2) - (\pi/m_{ij})$. $\square$

Corollary 3.1.16. *The root system of a Coxeter system (W, S) is the disjoint union of positive roots Φ^+ and negative roots Φ^-.*

Proof. Given $w \in W$ and $s_i \in S$ a root is the image of $\rho(w)(e_i)$. Since either $\ell(ws) > \ell(w)$ or $\ell(ws) < \ell(w)$. We get, by Theorem 3.1.15, $\rho(w)(e_i) \in \Phi^+$ or $\rho(w)(e_i) \in \Phi^-$, respectively. $\square$

As another consequence of Theorem 3.1.15, we get the faithfulness of the canonical representation ρ of a Coxeter System (W, S).

Corollary 3.1.17 ([Hum90], p. 113)**.** *The canonical representation $\rho\colon W \to GL(V)$ of a Coxeter system (W, S) is faithful.*

Proof. Let $w \in \ker \rho$. If $w \neq e$, there exists $s \in S$ for which $\ell(ws) < \ell(w)$. By Theorem 3.1.15, we get that $\rho(w)(e_i) < 0$. But $\rho(w)(e_i) = e_i > 0$, which is a contradiction. Thus ρ is faithful. $\square$

One can show that $\rho(W)$ is a discrete subgroup of $GL(V)$ (see p.130 in [Hum90]). Selberg's Lemma (see Corollary 4 in [Rat06]) asserts that every finitely generated subgroup of $GL(V)$ is virtually torsion-free, i.e., contains a subgroup of finite index which has no non-identity periodic element.

Corollary 3.1.18 ([DC08], Corollary 6.12.12)**.** *Every finitely generated Coxeter group is virtually torsion-free, i.e., contains a torsion-free subgroup of finite index.*

3.2 Longest Element of a Coxeter Group

Our main goal in this section is to define the longest element w_0 of a Coxeter group with respect to the length function. We show that w_0 is uniquely determined and maps all positive roots to their negative. Using the canonical representation, we want to characterise when w_0 is acting as the $-\operatorname{id}_V$ element in $GL(V)$. We show that this is the case if and only if the longest element commutes with all other elements in W, i.e., w_0 lies in the center of W.

We start by observing how a word $w \in W$ acts on the roots of a Coxeter system. The length function gives how w interacts with the positive and negative roots.

Proposition 3.2.1 ([Hum90], p. 114-115)**.** *Let (W, S) be a Coxeter system.*

(i) If $s_i \in S$, then s_i sends e_i to its negative, and acts by permutation on all positive roots (without sign change).

(ii) For any $w \in W$, the length $\ell(w)$ equals the number of positive roots sent by w to negative roots.

Proof. Note that (i) is a special case of (ii); we establish (i) first in order to prove (ii).

(i) Suppose $\alpha > 0$, but $\alpha \neq e_i$. Since all roots are unit vectors, α cannot be a multiple of e_i. We can therefore write

$$\alpha = \sum_{s_j \in S} c_j e_j,$$

where all coefficients are non-negative and some $c_j > 0$, $j \neq i$. Applying s_i to α only modifies the sum by adding some constant multiple of e_i, so the coefficient of α_j remains strictly positive. It follows that $\rho_i(\alpha)$ cannot be a negative root, so it lies in Φ^+ and is obviously distinct from e_i. Thus $\rho_i(\Phi^+ \backslash \{e_i\}) \subset \Phi^+ \backslash \{e_i\}$. Apply s_i to both sides to get the reverse inclusion.

(ii) If $w \in W$, define $n(w)$ to be the number of positive roots sent by w to negative roots, so

$$n(w) := |\Phi(w)|, \text{ where } \Phi(w) := \Phi^+ \cap w^{-1}\Phi^-.$$

Notice that (i) implies that $n(s_i) = 1$ for $s_i \in S$.

We show that $n(w)$ behaves like the length function. The condition $\rho(w)(e_i) > 0$ implies $n(ws_i) = n(w) + 1$, whereas $\rho(w)(e_i) < 0$ implies $n(ws_i) = n(w) - 1$. Indeed, if $\rho(w)(e_i) > 0$, (i) implies that $\Phi(ws_i)$ is the disjoint union of $\rho_i(\Phi(w))$ and $\{e_i\}$. Similarly, if $\rho(w)(e_i) < 0$, we get $\Phi(ws_i) = \rho_i(\Phi(w)\backslash\{e_i\})$ with $e_i \in \Phi(w)$.

Now we proceed by induction on $\ell(w) = k$ to prove that $n(w) = \ell(w)$ for all $w \in W$. This is clear if $\ell(w) = 0$ (and also by (i) if $\ell(w) = 1$). By Theorem 3.1.15, we have $\ell(ws_i) = \ell(w) + 1$ (resp. $\ell(w) - 1$) just when $\rho(w)(e_i) > 0$ (resp. < 0). Combining this with the preceding paragraph and the induction hypothesis completes the proof. □

In the following theorem, we show that the longest element can only exist if the Coxeter group is finite. Indeed, the existence of a longest element is a necessary and sufficient condition for a Coxeter group to be finite. Furthermore, the longest element is an involution and uniquely determined, although in general it has multiple reduced expressions:

Theorem 3.2.2 ([GP00], Proposition 1.5.1)**.** *Let (W, S) be a Coxeter system. The following conditions are equivalent:*

(i) The group W is finite.

(ii) The root system Φ is finite.

(iii) The set $\{\ell(w) \mid w \in W\}$ is finite.

(iv) There exists an element $w_0 \in W$ such that $\ell(s_i w_0) < \ell(w_0)$ for all $s_i \in S$.

Moreover, if these conditions hold, w_0 is uniquely determined, an involution (i.e., $w_0^2 = e$), and $\ell(w) < \ell(w_0)$ for all $w \neq w_0$.

Proof. (i) and (ii) are obviously equivalent and imply (iii).

Assume (iii) and let $w_0' \in W$ be such that $\ell(w) \leq \ell(w_0')$ for all $w \in W$. Then $\ell(w_0') \geq \ell(s_i w_0') = \ell(w_0') - 1$. Hence $\ell(s_i w_0') < \ell(w_0')$ for all $s_i \in S$. Thus w_0' also satisfies condition (iv).

Now assume that $w_0 \in W$ is as in (iv). Then we have $\rho(w_0)(e_i) < 0$ for all $s_i \in S$. Thus every positive root is made negative by w_0. By Proposition 3.2.1, we have $|\Phi| = 2|\Phi^+| = 2\ell(w_0) < \infty$. Thus W is finite. Hence (iv) implies (i).

Next, we show the uniqueness of w_0. Assume there is another element $w \in W$ such that $\ell(s_i w) < \ell(w)$ for all $s_i \in S$. Thus $\rho(w)(e_i) < 0$ for all $s_i \in S$. But w_0 sends all positive roots to their negative roots; hence $\rho(w_0 w)(e_i) > 0$ for all $s_i \in S$. Thus $\ell(s_i w w_0) > \ell(w w_0)$ for all $s_i \in S$. By the Exchange Condition, this can only happen if $ww_0 = 1$ or equivalently $w = w_0^{-1}$. Using the same argument for $w = w_0$, we get $w_0 = w_0^{-1}$. Thus w_0 is unique and an involution.

We have already seen that an element satisfying (iii) satisfies (iv). Hence the uniqueness in (iv) implies that $\ell(w) < \ell(w_0)$ for all $w \neq w_0$. $\square$

Definition 3.2.3. We call w_0 in Proposition 3.2.2 the *longest element* of the Coxeter system (W, S) or the *longest element* of the Coxeter group W.

Example 3.2.4. Let $D_{2n} = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^n = e \rangle$ be the dihedral group of order $2n$. As in Lemma 2.7.1, one can easily construct the element of longest length by alternate multiplying the two generators s_1 and s_2 until we reach the length n. Let w_0 be this word. Consider now the word $s_i w_0$. One can quickly check that either by the relation $s_i^2 = e$ or by the relation $(s_1 s_2)^n = e$, that the word $s_i w_0$ can be shortened. Theorem 3.2.2 yields that w_0 is indeed the longest word of D_{2n}.

Remark 3.2.5. For most Coxeter groups, there is a quick way to construct the element of longest length which we will not discuss in detail (see [Hum90]). A *Coxeter element* $w_c \in W$ of a spherical Coxeter system (W, S) is the product of all $s_i \in S$ in any given order, e.g., if $S = \{s_1, \ldots, s_n\}$ then $w \coloneqq s_1 \cdots s_n$ is a Coxeter element. Although a Coxeter element depends on the permutation of the generators, one can show that they are all conjugate. In particular they have the same order h, called *Coxeter number*. The Coxeter element and Coxeter number are strongly connected to the element of longest length w_0:

- We have the formula $|\Phi| = 2\ell(w_0) = |S|h$.
- If h is even, then $w_0 = w_c^{h/2}$.

A list of the longest elements of all spherical Coxeter systems listed by type is given in Table 3.1.

Coxeter type	Longest element in terms of generators	Length
A_n	$(w_{A_{n-1}})n\cdots 1$	$\binom{n+1}{2}$
B_n	$(1\cdots n)^n$	n^2
D_{2n}	$(1\cdots n)^{n-1}$	$4n^2-2n$
D_{2n+1}	$12w_3\cdots w_n$ where $w_k = k(k-1)\cdots 34\cdots k$	$4n^2+2n$
E_6	612132143215321432534653214325346532	36
E_7	$(1\cdots 7)^9$	63
E_8	$(1\cdots 8)^{15}$	120
F_4	$(1234)^6$	24
H_3	$(123)^5$	15
H_4	$(1234)^{15}$	60
$I_2(2m)$	$(12)^m$	$2m$
$I_2(2m+1)$	$(12)^m1$	$2m+1$

Table 3.1: A representation of the longest element and its length for each irreducible spherical Coxeter group. The number i of the representation denotes the reflection s_i. See [Hum90] and [Fra01].

Since every positive root is made negative by longest element w_0, it is natural to ask whether the longest element is mapped to $-\operatorname{id}_V \in GL(V)$ via the canonical representation. As we have seen in Example 3.1.14, this is not necessarily the case. Since $-\operatorname{id}_V$ lies in $C(GL(V))$ the center of $GL(V)$, one can study whether w_0 lies in the center of W, i.e., w_0 commutes with all $w \in W$. In fact, we show that $\rho(w_0) = -\operatorname{id}_V$ if and only if $w_0 \in C(W)$. We use the terminology of the (-1)-condition and it will be crucial in defining edge reflections. Richardson introduced the (-1)-condition in [Ric82] to classify conjugacy classes of involutions in Coxeter groups.

Definition 3.2.6. Let (W, S) be a Coxeter system. We say (W, S) satisfies the (-1)-*condition*, if there is a $w \in W$ such that $\rho(w)(x) = -x$ for all $x \in V$. If the element w exists, we also say that w satisfies the (-1)-*condition*. A subset T of S satisfies the (-1)-*condition* if the Coxeter group (W_T, T) satisfies the (-1)-condition.

Example 3.2.7. On the one hand, as seen in Example 3.1.6, a Coxeter system of type $I_2(3)$ does not satisfy the (-1)-condition since $(\rho_1\rho_2\rho_1)(e_1) = -e_2$. On the other hand, every right-angled Coxeter system, i.e., $m_{ij} = 2$ for all $i \neq j$, satisfies the (-1)-condition. This can be seen as follows: since $m_{ij} = 2$ for $i \neq j$ we have that $B_M(e_i, e_j) = 0$. By (3.1), we get that x_i, the i-th component of $x \in V$ regarding the basis $\{e_i\}$, is mapped onto its negative, i.e.,

$$\rho_i(x) = x - 2B_M(x, e_i)e_i = x - 2x_i.$$

If $S = \{s_1, \ldots, s_n\}$, then the longest element of a right-angled Coxeter group is given by $s_1 \cdots s_n$. Thus $(\rho_1 \cdots \rho_n)(x) = -x$.

The next lemma shows that an element satisfying the (-1)-condition is indeed the longest element of a Coxeter group.

Lemma 3.2.8 ([Ric82])**.** *Let (W, S) be a Coxeter system.*

(i) If an element w satisfies the (-1)-condition, then w is the longest element. In particular W is finite.

(ii) Let (W_1, S_1), ..., (W_n, S_n) be the irreducible components of (W, S). Then (W, S) satisfies the (-1)-condition if and only if each (W_i, S_i) satisfies the (-1)-condition.

Proof. (i) Since $\rho(w)(x) = -x$ for all $x \in V$, we have that $\rho(w)(e_i) < 0$ for all $s_i \in S$. Thus $\ell(s_i w) < \ell(w)$ for all $s_i \in S$. By Theorem 3.2.2, w is the longest element of W and W is finite.

(ii) This is trivial. □

Assuming that (W, S) is irreducible, we prove that the center is either trivial or consists only of the element of longest length. We start by showing that all elements in center are involutions.

Lemma 3.2.9 ([Hos05])**.** *Let (W, S) be a Coxeter system and $C(W)$ the center of W. Then for each $w \in C(W)$ there exists a spherical subset T of S such that w is the longest element of W_T. In particular, $w^2 = e$ for all $w \in C(W)$.*

Proof. Let $w \in C(W)$ and let $w = s_1 \cdots s_k$ be a reduced expression. Since $w \in C(W)$, we have $ws_1 = s_1 w = s_2 \cdots s_k$. Thus $\ell(ws_1) < \ell(w)$ and $w = s_2 \cdots s_k s_1$ is a reduced expression by the Exchange Condition. By iterating the above argument, we obtain $\ell(ws_i) < \ell(w)$ for all $i = 1, \ldots, k$. Let $T = \{s_1, \ldots, s_k\}$. By Theorem 3.2.2, W_T is finite and w is the longest element of W_T. □

Let (W, S) be a Coxeter system with irreducible components (W_i, S_i). Recall that generators in different components commute. Since a component is either spherical, Euclidean or hyperbolic, we can split $W = W_{\text{Sph}} \times W_{\text{Eucl}} \times W_{\text{Hyp}}$. In particular, we can split the center of W into $C(W) = C(W_{\text{Sph}}) \times C(W_{\text{Eucl}}) \times C(W_{\text{Hyp}})$.

Theorem 3.2.10 ([Hos05]). *Let (W, S) be a Coxeter system and $C(W)$ the center of W. Then $C(W)$ is finite and $C(W) = C(W_{Sph})$.*

Proof. Let T be a spherical subset and w_T be the longest element in W_T. By Lemma 3.2.9, we have $C(W) \subset \{w_T \mid T \text{ is spherical subset of } S\}$, which is finite since S is finite.

Assume that $w \in C(W)$ and $w = s_1 \cdots s_k$ is reduced. Let $T \coloneqq \{s_1, \dots, s_k\}$. Then w is the longest element in W_T. Let $s \in S \backslash T$. Since $w \in C(W)$, we have $ws = sw$ or equivalently $sws = w$. Note that $s \notin T$, thus $ss_i = s_i s$ for all $i = 1, \dots, k$. This means $st = ts$ for all $t \in T$ and $s \in S \backslash T$. Thus W splits as a direct product $W = W_T \times W_{S \backslash T}$. Since W_T is finite, we get that $T \subset W_{\text{Sph}}$. Hence $w \in W_T \subset W_{\text{Sph}}$ for all $w \in C(W)$. Thus $C(W) \subset W_{\text{Sph}}$. Therefore $C(W) = C(W_{\text{Sph}})$. □

As an immediate consequence, we get following corollary and theorem.

Corollary 3.2.11. *The center of an infinite irreducible Coxeter group is trivial.*

Theorem 3.2.12 ([Ric82]). *Let (W, S) be an irreducible Coxeter system, w_0 be the longest element in (W, S), and $C(W)$ the center of W. The Coxeter system (W, S) satisfies the (-1)-condition if and only if $C(W) = \langle w_0 \rangle$. This is the case if and only if the Coxeter graph of (W, S) is one of the following types:*

$$A_1, B_n, D_{2n}, E_7, E_8, F_4, H_3, H_4, \text{ or } I_2(2m). \tag{3.3}$$

The exceptional cases are

$$A_n (n \geq 2), D_{2n+1}, E_6, \text{ and } I_2(2m+1). \tag{3.4}$$

Proof. For each of these types one must determine whether $-\operatorname{id}_V \in \rho(W) \subset GL(V)$. This information can be found in [Fra01] or can be checked directly. □

4 Edge Reflection Group

In this chapter, we want to analyse *edge reflections*, i.e., Schwarz reflections generated by a given Jordan curve Γ defined on the edge set of a Coxeter polytope P^n. By Theorem 2.9.6, the Coxeter group generated by the reflections across the facets of P^n gives us a tessellation of $\mathbb{X}^n$. We show that edge reflection along an edge contained in Γ corresponds to the longest element of the largest special subgroup fixing the edge of P^n if the special subgroup satisfies the (-1)-condition.

First, we justify our approach by identifying edge reflection along an edge of Γ as the longest element of a special subgroup of (W, S), if the edge satisfies the (-1)-condition. At the end of the section, we will define *edge reflection graphs* to visualise when the group generated by edge reflections is a subgroup of W. This will be used in the next chapter to construct $\mathcal{S}$ in $\mathbb{X}^n$.

Second, we discuss the embeddedness problem for our constructed surface $\mathcal{S}$. Theorem 4.2.10 gives a necessary and sufficient condition for $\mathcal{S}$ to be embedded, in terms of a comparison of the Coxeter group (W, S) and comparing with the group $\mathcal{J}$ generated by edge reflections of Γ.

4.1 Edge Reflection Graphs

In this section, we will characterise edge reflection along an edge e of a Coxeter polytope P^n as the longest word of the largest special subgroup W_T of W fixing e, if W_T satisfies the (-1)-condition. We will discuss the geometric meaning of the (-1)-condition and how edge reflection behaves in the Coxeter complex $\mathcal{U}(W, P^n)$. Furthermore, we construct graphs to grasp whether the special subgroups of (W, S) satisfies the (-1)-condition.

A construction of the Schwarz D surface is as follows: we take a cube in $\mathbb{R}^3$ and define a suitable Jordan curve Γ along the edges of the cube.

The Plateau solution $\mathcal{P}$ yields a minimal surface patch with boundary Γ. After that, we use the Schwarz Reflection Principle to construct a complete embedded minimal surface $\mathcal{S}$ by reflection along the edges defined by Γ.

Our goal is to generalise this construction as follows: we take a Coxeter polytope $P^n \subset \mathbb{X}^n$. This gives us a Coxeter system (W, S) and a tessellation $\mathcal{U}(W, P^n)$. Then, we define a suitable edge cycle Γ along the edges of P^n. If the reflection along an edge of Γ satisfies the (-1)-condition, the edge reflection is compatible with $\mathcal{U}$, i.e., chambers of $\mathcal{U}$ are mapped onto chambers. Using edge reflection, we construct a complete surface $\mathcal{S}$ by reflection along the edges of Γ. Eventually, we study whether $\mathcal{S}$ is embedded or has self-intersections.

In the following two sections, we explain and justify this generalisation.

Starting with an irreducible Coxeter system (W, S), we use the canonical representation to construct a faithful representation of W in $GL(\mathbb{R}^{|S|})$. By Corollary 2.9.9, we can construct an n-simplex in $\mathbb{X}^n$. Furthermore, we get with Theorem 2.9.6 a tessellation $\mathcal{U}(W, P^n)$. Next, we define the boundary of our surface patch $\mathcal{P}$ as an edge cycle on P^n.

Definition 4.1.1. An *edge cycle* $\Gamma\colon \mathbb{S}^1 \to P^n \subset \mathbb{X}^n$ is a Jordan curve on P^n , i.e., a simple closed continuous curve consisting only of edges of P^n. We say an edge cycle is *full-dimensional* if it is not contained in a facet of P^n.

Example 4.1.2. Consider a cube $\square := [0, 1]^3 \subset \mathbb{R}^3$ and the path Γ given by following vertices: $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(1, 1, 1)$, $(0, 1, 1)$, $(0, 1, 0)$ (see Figure 4.1). The curve Γ is clearly homeomorphic to $\mathbb{S}^1$, hence a Jordan curve, and Γ is not contained in any of the six facets of $\square$, thus Γ is a full-dimensional edge cycle.

Let $P^n \subset \mathbb{X}^n \subset \mathbb{R}^{n+1}$ be a Coxeter polytope and e and edge of P^n. Intuitively, Schwarz reflection fixes e point-wise in $\mathbb{X}^n$. We like to extend this view to $\mathbb{R}^{n+1}$. Let $U \subset \mathbb{R}^{n+1}$ be the intersection of all supporting hyperplanes for P^n containing e. Note that U is 2-dimensional. We define the Schwarz reflection j such that j fixes U point-wise, i.e., $j(p) = p$ for all $p \in U$, and maps $U^\perp$ onto its negative, i.e., $j(p) = -p$ for all $p \in U^\perp$. Note that these two properties uniquely determine an isometry. To emphasise the dependency on e we write $U(e)$ and $U^\perp(e)$ for the two vector spaces.

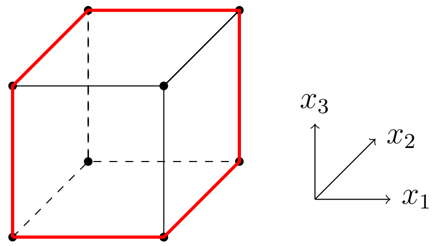


Figure 4.1: The cube and the full-dimensional edge cycle (red) given in Example 4.1.2. This edge cycle is used to construct the Schwarz D surface.

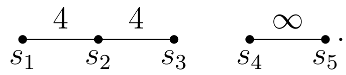
Definition 4.1.3. Assume that e is an edge of a polytope P^n. We call j the *edge reflection about* e if $j(p) = p$ for all $p \in U(e)$ and $j(q) = -q$ for all $q \in U^{\perp}(e)$.

Assume we have a Coxeter polytope $P^n \subset \mathbb{X}^n$ and an edge cycle $\Gamma \subset P^n$ with edges e_i. Let j_i be the edge reflection about e_i. Then edge reflecting about e_i maps P^n onto a copy $j_i P^n$. Our goal is to characterise whether the copy $j_i P^n$ is a chamber in $\mathcal{U}$. If $j_i P^n$ is a chamber in $\mathcal{U}$, the obvious choice with respect to Definition 4.1.3 of an element $w \in W$ coinciding with the edge reflection along e_i would be the longest element of a special subgroup fixing e_i and satisfying the (-1)-condition. Essentially, the (-1)-condition tells us whether edge reflections respect the tessellation. We want to motivate this with the following three cases.

Let $P^3 \subset \mathbb{R}^3$ be a Coxeter prism with triangle $\Delta \subset \mathbb{R}^2$ as base. Depending on the angles of Δ, we want to study the action of the edge reflections about the edges of P^3. Let (W, S) be the associated Coxeter system to Δ, $\mathcal{U}(W, \Delta)$ its Coxeter complex, $(\widetilde{W}, \widetilde{S})$ the associated Coxeter system to P^3, and $\mathcal{U}(\widetilde{W}, P^3)$ its Coxeter complex. By projecting along the z-axis, we can visualise the tessellation $\mathcal{U}(\widetilde{W}, P^3)$ as the tessellation $\mathcal{U}(W, \Delta)$ and edge reflection about an edge $e \subset P^3$ parallel to the z-axis as a point reflection in $\mathcal{U}(W, \Delta)$.

Case 1 - Edge reflection can be expressed as a word in W.

Let Δ be an isosceles triangle with angles $\pi/2$, $\pi/4$, and $\pi/4$. The Coxeter graph of $(\widetilde{W}, \widetilde{S})$ is of type $\tilde{B}_2 \times \tilde{A}_1$ and is given by



In the following, we want to compare edge reflection along an edge to an element of W.

Let H_i be the hyperplane associated with s_i and fix the edge $e_{12} := H_1 \cap H_2$. Consider all prisms meeting in e_{12}. This corresponds to the parabolic subgroup W_{12} generated by s_1 and s_2, see Figure 4.2.

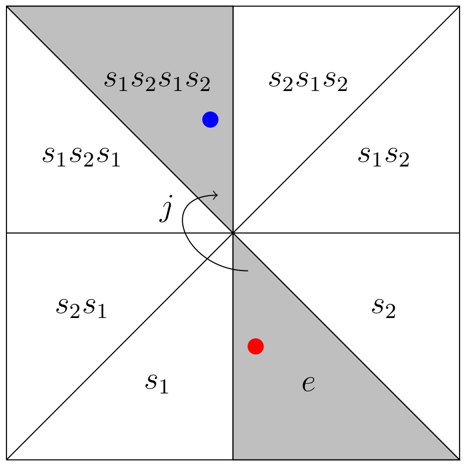
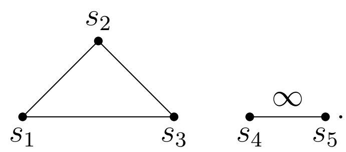


Figure 4.2: Edge reflection j and the word $s_1s_2s_1s_2$ are identical maps. In particular, j and $s_1s_2s_1s_2$ map both the red point onto the blue point.

It is a dihedral group of type $I_2(4)$ and $W_{12}P^3$ consists of 8 prisms meeting in e_{12}. The edge reflection j is the rotation by π about e_{12} and it maps the prism P^3 onto the chamber $s_1s_2s_1s_2P^3$. Note that $s_1s_2s_1s_2$ is the longest element in W_{12} and W_{12} satisfies the (-1)-condition as it is of type $I_2(4)$. Consider the vertices $v_i \in P^3$. In Figure 4.2, we see that $j(v_i) = s_1s_2s_1s_2(v_i)$ for all i. Thus the action of these two mappings on n collinear points is the same. Hence j can be expressed as a composition of reflections s_i, i.e., $j = s_1s_2s_1s_2$.

Case 2 **- Edge reflection can be expressed in** W **up to relabeling.**

Let Δ be an equilateral triangle with angles $\pi/3$, $\pi/3$, and $\pi/3$. The Coxeter graph of $(\widetilde{W}, \widetilde{S})$ is of type $\tilde{A}_2 \times \tilde{A}_1$ and is given by



Let H_i be the hyperplane associated with s_i and fix the edge $e_{12} \coloneqq H_1 \cap H_2$. Consider all prisms meeting in e_{12}. This corresponds to the parabolic subgroup W_{12} generated by s_1 and s_2. It is a dihedral group of type $I_2(3)$ and $W_{12}P$ consists of 6 prisms meeting in e_{12}, see Figure 4.3.

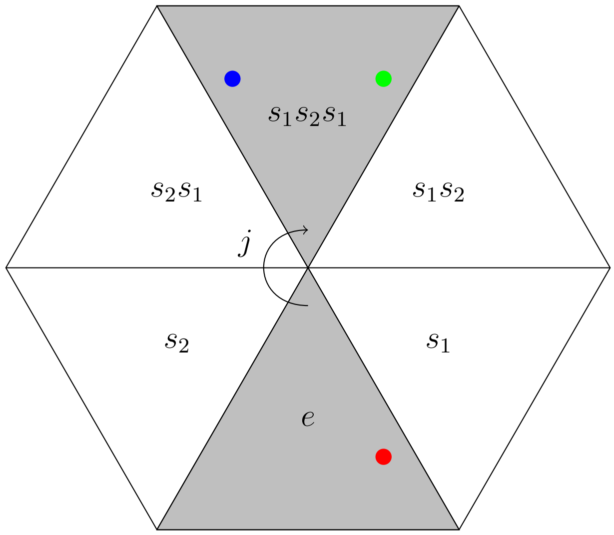
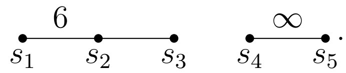


Figure 4.3: The edge reflection j and the word $s_1s_2s_1$ map P^3 onto the same chamber. However, the word $s_1s_2s_1$ interchanges the facets. Thus j maps the red point onto the blue point while $s_1s_2s_1$ maps it onto the green point .

The edge reflection j is the rotation by π along e_{12} and it is mapping the prism P onto the prism $s_1s_2s_1P^3$. Note that $s_1s_2s_1$ is the longest element in W_{12} but W_{12} does not satisfy the (-1)-condition as it is of type $I_2(3)$. Consider the vertices $v_i \in P^3$. We see that $j(v_i) = s_1s_2s_1(v_i)$ for all $v_i \in e_{12}$, but $j(v_i) \neq s_1s_2s_1(v_i)$ for all $v_i \notin e_{12}$. Hence $j \neq s_1s_2s_1$. Furthermore, note that j fixes H_1 and H_2, while the composition $s_1s_2s_1$ interchanges H_1 and H_2. Thus the longest element of W_{12} expresses edge reflection, i.e., $jP^3 = s_1s_2s_1P^3$ holds as an identity of sets, but not point-wise. In Chapter 6, we will see that the relabeling of H_1 and H_2 corresponds to an element in $W \rtimes \mathrm{Aut}(W)$, the semi-direct product of W and its automorphism group.

Case 3 - Edge reflection is not compatible with $\mathcal{U}$.

Let Δ be a triangle with angles $\pi/2$, $\pi/3$, and $\pi/6$. The Coxeter graph of $(\widetilde{W}, \widetilde{S})$ is of type $\tilde{G}_2 \times \tilde{A}_1$ and is given by



Let H_i be the hyperplane associated with s_i and fix the edge $e_{23} := H_2 \cap H_3$. Consider all prisms meeting in e_{23}. This corresponds to the parabolic subgroup W_{12} generated by s_2 and s_3, see Figure 4.4.

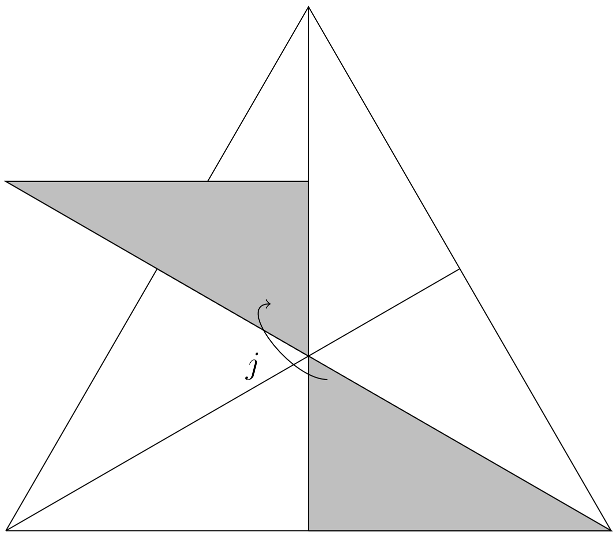


Figure 4.4: Edge reflection does not map P^3 onto a another chamber of $\mathcal{U}(\widetilde{W}, P^3)$

It is the dihedral group of type $I_2(3)$ and $W_{12}P^3$ consists of 6 prisms meeting in e_{23}. Note that $jP^3 \not\subset s_2s_3s_2P^3$.

Motivated by these three cases, we state following theorem and definition.

Theorem 4.1.4. *Let $P^n \subset \mathbb{X}^n$, $n \geq 2$ be a Coxeter polytope, (W, S) its associated Coxeter system, and $\mathcal{U}(W, P^n)$ the Coxeter complex. Let $e_i \subset P^n$ be an edge and let W_T be the largest special subgroup that fixes e. Then the* *edge reflection j about e agrees with the longest element w_T of W_T if and only if W_T satisfies the (-1)-condition.*

Proof. Assume that $j = w_T \in W_T$ and let $p \in U^{\perp}(e)$. Since $jU(e) = U(e)$ and $j(p) = -p$ for all $p \in U^{\perp}(e)$, we have found an element in W_T which is $-\operatorname{id}_{U^{\perp}(e)}$. Hence W_T satisfies the (-1)-condition.

Assume W_T satisfies the (-1)-condition. Note that $n-1$ facets are meeting in e. Since W_T fixes e point-wise, $w_T(p) = p$ for all $p \in U(e)$. As w_T is a composition of reflections acting on $\mathbb{R}^{n+1}$, we get $w_T(p) = p$ for all $p \in U^{\perp}(e)$. Now, W_T satisfies the (-1)-condition, i.e., we have $w_T(p) = -p$ for all $p \in U^{\perp}(e)$. Thus $w_T = j$. $\square$

In general, Theorem 3.2.12 lists whether W_T satisfies the (-1)-condition or not. In dimension $n = 3$ an edge is the intersection of two facets. Let π/m be their dihedral angle; hence W_T is of type $I_2(m)$. If m is even, W_T satisfies the (-1)-condition; if m is odd, W_T does not satisfies the (-1)-condition.

Definition 4.1.5. We say an edge cycle Γ is *compatible* with the tessellation $\mathcal{U}(W, P^n)$, if for every edge $e_i \subset P^n$ the edge reflection j_i along e_i can be expressed as an element of W, i.e., for every edge e_i the largest special subgroup that fixes e_i satisfies the (-1)-condition. In this case, we also say that the edge e_i *satisfies the (-1)-condition.* Let Γ be compatible with $\mathcal{U}(W, P^n)$, we call the subgroup $\mathcal{J} \leq W$ generated by the edge reflections j_i the *edge reflection group* associated with Γ.

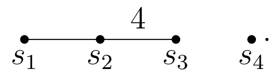
Let $P^n \subset \mathbb{X}^n$ be a Coxeter polytope with vertices v_i, edges e_i, and Coxeter system (W, S). We want to analyse which edges e_i of P^n fall into case 1, i.e., along which edges we can reflect such that the edge cycle is compatible with the tessellation. Our goal is to construct a graph $G = (V, E)$ which we can use to easily spot an edge cycle which satisfies Theorem 4.1.4.

Definition 4.1.6. Let $P^n \subset \mathbb{X}^n$ be a Coxeter polytope with vertices v_i, edges e_i, and Coxeter system (W, S). We define a graph G as follows: the vertex set V of G consists of all vertices v_i of P^n. Two vertices $v_i, v_j \in G$ are connected if the subgroup W_T, which fixes the edge incident to v_i and v_j, satisfies the (-1)-condition, i.e., e_i satisfies the (-1)-condition. We denote with E the edge set of G and call the graph $G = (V, E)$ the *edge reflection graph* associated with P^n.

Essentially, the edge reflection graph $G = (E, V)$ is the polytope itself seen as a graph where edges which do not satisfy the (-1)-condition are omitted.

Example 4.1.7. Let $P^3 \subset \mathbb{S}^3$ be a 3-simplex such that the Coxeter system (W, S) is of type $B_3 \times A_1$, see Figure 4.5

Then the Coxeter diagram is given by



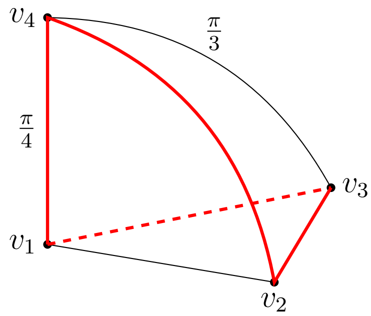


Figure 4.5: A Coxeter polytope $P^3 \subset \mathbb{S}^3$ with Coxeter system of type $B_3 \times A_1$. The dihedral angles are labelled midway of an edge. Dihedral angles of $\pi/2$ are omitted.

Let W_{ij} be the special subgroup generated by all generators of S except s_i and s_j, i.e., W_{ij} is the largest subgroup fixing e_{ij}, the edge incident to v_i and v_j. For example, W_{14} is the group generated by s_2 and s_3; hence of type $I_2(4)$. Inspecting (3.3) in Theorem 3.2.12, we see that $I_2(4)$ satisfies the (-1)-condition. Thus the edge v_1 and v_4 are connected in the edge reflection graph. In Table 4.1, we listed all types of the subgroups W_{ij} and whether they satisfy the (-1)-condition or not.

Thus the edge reflection graph contains all edges except e_{34}. It is given in Figure 4.6. Figure 4.7 shows a compact notation of the edge reflection graph in Figure 4.6. We will use the compact notation ongoing in this thesis.

Let $P^n \subset \mathbb{X}^n$ be a Coxeter polytope and Γ an edge cycle on P^n consisting of the edges e_{ij} where e_{ij} is incident to v_i and v_j. If v_i and v_j are connected in the edge reflection graph, then edge reflection about e_{ij} is compatible with the tessellation $\mathcal{U}(W, P^n)$. Hence by construction we have:

Coxeter group	Type	(-1)-condition
W_{12}	$A_1 \times A_1$	yes
W_{13}	$A_1 \times A_1$	yes
W_{14}	$I_2(4)$	yes
W_{23}	$A_1 \times A_1$	yes
W_{24}	$A_1 \times A_1$	yes
W_{34}	$I_2(3)$	no

Table 4.1: The subgroups and their types of the Coxeter group of type $B_3 \times A_1$ needed to construct the edge reflection graph. Whether the subgroup satisfies the (-1)-condition, can be seen in the list provided in Theorem 3.2.12. In dimension $n = 3$, we have that Coxeter groups of type $I_2(2m)$ satisfy the (-1)-condition and Coxeter groups of type $I_2(2m+1)$ violate the (-1)-condition. Note that a group of type $A_1 \times A_1$ is the same as a group of type $I_2(2)$.

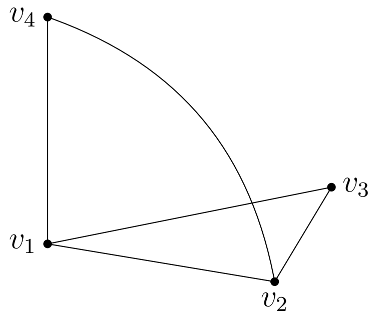


Figure 4.6: The edge reflection graph of the 3-simplex $P^3 \subset \mathbb{S}^3$ of type $B_3 \times A_1$. Compare the graph with P^3 in Figure 4.5. The edge incident to v_3 and v_4 is omitted since it does not satisfy the (-1)-condition.

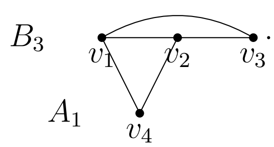


Figure 4.7: A compact notation of the edge reflection graph given in Figure 4.6.

Theorem 4.1.8. *Let $P^n \subset \mathbb{X}^n$ be a Coxeter polytope and Γ an edge cycle on P^n. Then Γ is compatible with $\mathcal{U}(W, P^n)$ if and only if it corresponds to a cycle in the edge reflection graph.*

As in Example 4.1.7, we can construct the edge reflection graph for all Coxeter graphs of a given type. Since the Coxeter graphs are classified by its irreducible components, we want to extend the edge reflection graph by a graphical representation. The goal of this is to construct the whole edge reflection graph only given the irreducible components.

Assume that P^n is an n-simplex in $\mathbb{S}^n$. We will extend the discussion to Euclidean and hyperbolic polytopes shortly. Note that every hyperplane induced by a facet of P^n fixes every vertex but one. Let v_i be the vertices of P^n and renumber the vertices in a way such that the reflection s_i does **not** fix v_i. Let e_{ij} be the edge which is incident to v_i and v_j. Since edge reflection about e_{ij} fixes e_{ij} point-wise, it cannot contain the generators s_i and s_j.

Assume that the Coxeter system (W, S) of P^n is reducible. Thus its Coxeter graph has at least two components, and we can write W as a direct product, i.e., $W = W^1 \times W^2 \times \cdots \times W^n$. Denote with W_{ij} the group generated by all generators of S except s_i and s_j and W_k the group generated by all generators except s_k. There are two cases: either s_i and s_j lie in different irreducible components or they lie in the same component. By Lemma 3.2.8 (*ii*), the group satisfies the (-1)-condition if and only if all components satisfy the (-1)-condition. If s_i and s_j lie in the same component (w.l.o.g in W^1), then W^1_{ij} and all components W^n, $n \neq 1$ not containing s_i and s_j have to satisfy the (-1)-condition. If s_i and s_j lie in different components (w.l.o.g $s_i \in W^1$ and $s_j \in W^2$), then W_i, W_j, and all other components W^n, $n \neq 1, 2$ have to satisfy the (-1)-condition.

With this in mind, we use following graphical representation: draw a circle around the vertex v_i if the group generated by all generators s_j fixing v_i satisfies the (-1)-condition. Furthermore, draw a circle around the type if W satisfies the (-1)-condition. If W is reducible, then the meaning of the circle around a node is that it is connected to another circled node provided all remaining components are circled.

Example 4.1.9. Let $P^3 \subset \mathbb{S}^3$ be a 3-simplex such that the Coxeter system (W, S) is of type $B_3 \times A_1$, compare Example 4.1.7. Let W_{ij} be the special subgroup generated by all generators of S except s_i and s_j and W_i be the special subgroup generated by all generators of S except s_i. Table 4.8 shows the extension of Table 4.1.

Coxeter group	Type	(-1)-condition
W_{12}	$A_1 \times A_1$	yes
W_{13}	$A_1 \times A_1$	yes
W_{14}	$I_2(4)$	yes
W_{23}	$A_1 \times A_1$	yes
W_{24}	$A_1 \times A_1$	yes
W_{34}	$I_2(3)$	no
W_1	$B_2 \times A_1$	yes
W_2	$A_1 \times I_2(4)$	yes
W_3	$A_2 \times A_1$	no
W_4	B_3	yes
W	$B_3 \times A_1$	yes

Figure 4.8: The subgroups and their types of the Coxeter group of type $B_3 \times A_1$ needed to construct the graphical representation of the edge reflection graph. Whether the subgroup satisfies the (-1)-condition, can be seen in the list provided in Theorem 3.2.12.

The graphical extension of the edge reflection graph in Figure 4.7 is given in Figure 4.9.

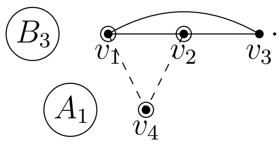


Figure 4.9: The graphical extension of Figure 4.9. The edges e_{14} and e_{24} are shown dashed since they connect different components of W.

If P^n is an Euclidean or hyperbolic n-simplex, the corresponding Coxeter system (W, S) is irreducible. By construction, if the special subgroup W_i not containing s_i is spherical. If W_i satisfies the (-1)-condition, we circle the node v_i in the edge reflection graph. The type is never circled, since only spherical Coxeter groups can satisfy the (-1)-condition. Now assume that P^n is not an n-simplex, i.e., (W, S) is reducible. Since edge reflection about an edge e_i fixes the edge, we need to omit a generator s_j in every irreducible component of (W, S). This results in different choices which can be seen in the edge reflection graph as multiple copies of an irreducible component as seen in the next example.

Example 4.1.10. Consider $\mathbb{E}^3$ tiled by prisms P^3 with an isosceles triangle with angles $\pi/2$, $\pi/4$, and $\pi/4$ as a base. The Coxeter system of P^3 is of type $\tilde{B}^2 \times \tilde{A}_1$, has rank 5, and the Coxeter graph is given in Figure 4.10

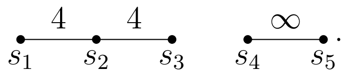


Figure 4.10: The Coxeter diagram for a prism with an isosceles triangle with angles $\pi/2$, $\pi/4$, and $\pi/4$ as a base.

Note that every spherical special subgroup, i.e., every edge, satisfies the (-1)-condition. Thus the edge reflection graph has 6 vertices and 9 edges, since P^3 has 6 vertices and 9 edges.

To represent edge reflection along an edge, we have to omit a generator s_i in every irreducible component, e.g., s_1 and s_4. Thus W_{14}, W_{15}, W_{24}, W_{25}, W_{34}, and W_{35} are the largest subgroups fixing the vertices v_i. Note that W_{12} has infinite order; hence it cannot have an vertex as a fixed point set. The edges are now fixed by the subgroups where we omit 3 generators of S, e.g., e_{12} is fixed by W_{124}, the subgroup generated by all s_i where $i \neq 1, 2, 4$. Hence we can give a graphical representation of the edge reflection graph as shown in Figure 4.11.

The construction becomes more involved as the number of irreducible components increases.

As in the Example 4.1.7, 4.1.9, and 4.1.10, one can construct edge reflection graphs for all irreducible Coxeter graphs listed in Figure 2.6, Figure 2.7, and Figure 2.8. The edge reflection graphs are listed in Figure 4.12, Figure 4.13, and Figure 4.14.

4.2 Embedding Problem

In this section, we will establish the main result of this thesis. Let $\mathcal{P}$ be an embedded surface which lies in the interior of a Coxeter polytope P^n.

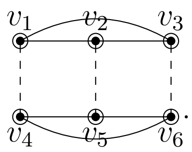


Figure 4.11: The graphical extension of the edge reflection graph consists of two copies (due to the $\tilde{A}_1$ factor) of the edges associated with $\tilde{B}_2$ with the addition that opposite edges, e.g., v_1 and v_4 are also connected (showed as dashed lines). Note that P^3 is a product of a triangle and an interval. This can be seen in the edge reflection graph: in horizontal direction we see a triangle, e.g., e_{12}, e_{13} and e_{23} connecting v_1,v_2, and v_3; in vertical direction we see an interval, e.g., e_{14} connecting v_1 and v_4.

Furthermore, assume that the boundary of $\mathcal{P}$ is Γ. Using edge reflection, we can extend $\mathcal{P}$ to a complete surface $\mathcal{S}$ without boundary on $\mathbb{X}^n$. Theorem 4.2.10 gives a necessary and sufficient condition for $\mathcal{S}$ to be embedded in $\mathbb{X}^n$, provided Γ is compatible with the tessellation $\mathcal{U}(W, P^n)$ of $\mathbb{X}^n$.

Definition 4.2.1. Let $X \subset \mathbb{R}^2$ be compact. A surface $f\colon X \to \mathbb{X}^n$ is an *immersion* if $\operatorname{rank} D_p f = 2$. We say f has *self-intersections* if f is not injective. We call a point $p \in X$ a *self-intersection* if there are two distinct points $p_1, p_2 \in X$ such that $f(p_1) = f(p_2) = x$. The surface f is an *embedding* if f is an injective immersion and homeomorphic onto its image.

For the remainder of this section, let P^n be a Coxeter polytope where $n \geq 3$, (W, S) the corresponding Coxeter system, and $\mathcal{U}(W, P^n)$ the Coxeter complex. Furthermore, let Γ be a full-dimensional edge cycle on P^n compatible with $\mathcal{U}$ and $\mathcal{J}$ the corresponding edge reflection group. Note that in this case $\mathcal{J}$ is a subgroup of W (see Definition 4.1.5).

We want to study whether a given embedded surface $\mathcal{P}\colon X \to P^n$ which lies in the interior of P^n with boundary Γ can be extended to a surface $\mathcal{S} \coloneqq \bigcup_{j \in \mathcal{J}} j\mathcal{P}$ by successive edge reflection. Thus $\mathcal{S}\colon \widetilde{X} \to \mathbb{X}^n$ is a mapping from a topological space $\widetilde{X}$ consisting of $|\mathcal{J}|$-many (possibly infinite) copies of X where boundary points are identified.

The next lemma identifies the space $\widetilde{X}$ with a Coxeter complex $\mathcal{U}(\mathcal{J}, \mathcal{Q}^2)$ where $\mathcal{Q}^2$ is a polygon in $\mathbb{X}^2$. Hence we can identify the extended surface $\mathcal{S}$ as a mapping from $\mathcal{U}(\mathcal{J}, \mathcal{Q}^2)$ to $\mathcal{U}(W, P^n)$.

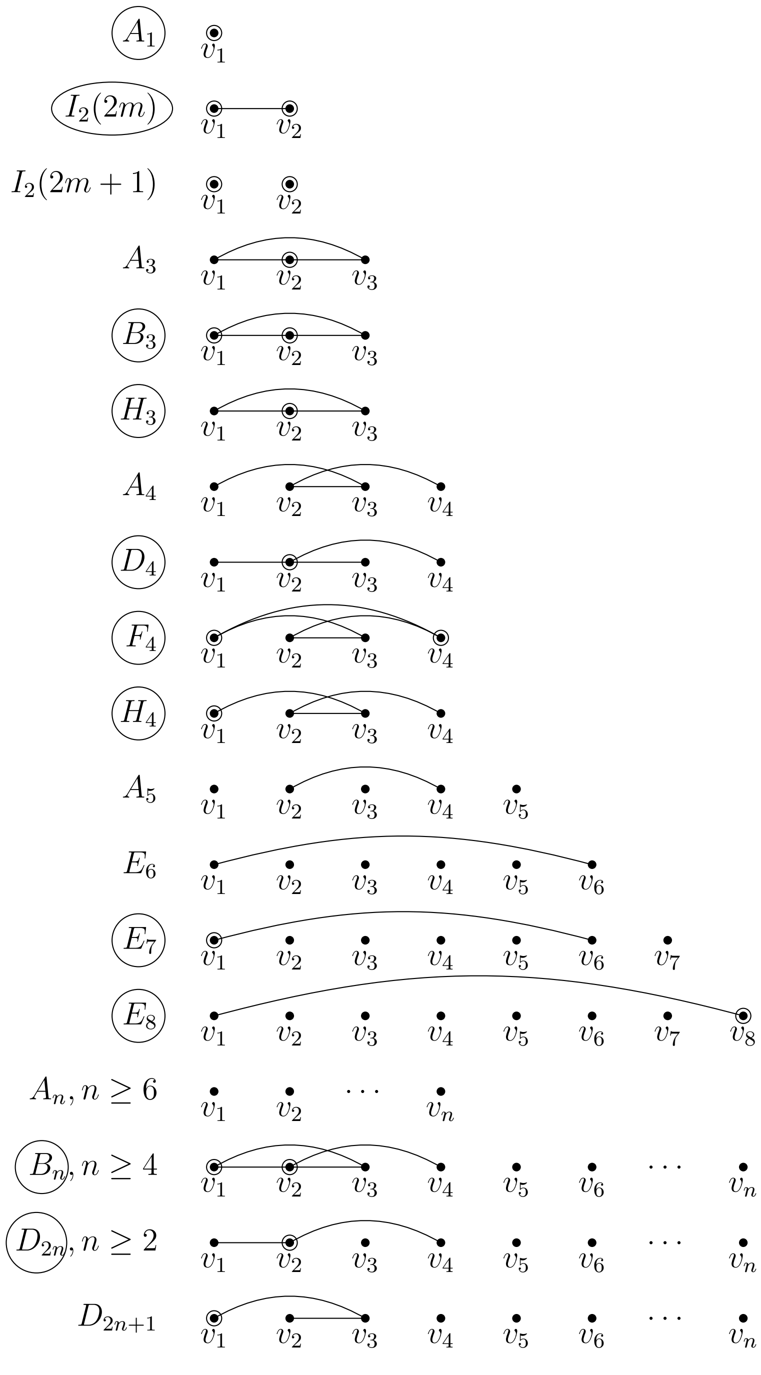


Figure 4.12: The edge reflection graph of all irreducible spherical Coxeter groups. A circle around the type implies that the group satisfies the (-1)-condition. A circle around a node and a circle around an edge implies that the special subgroup generated by omitting these generators satisfies the (-1)-condition.

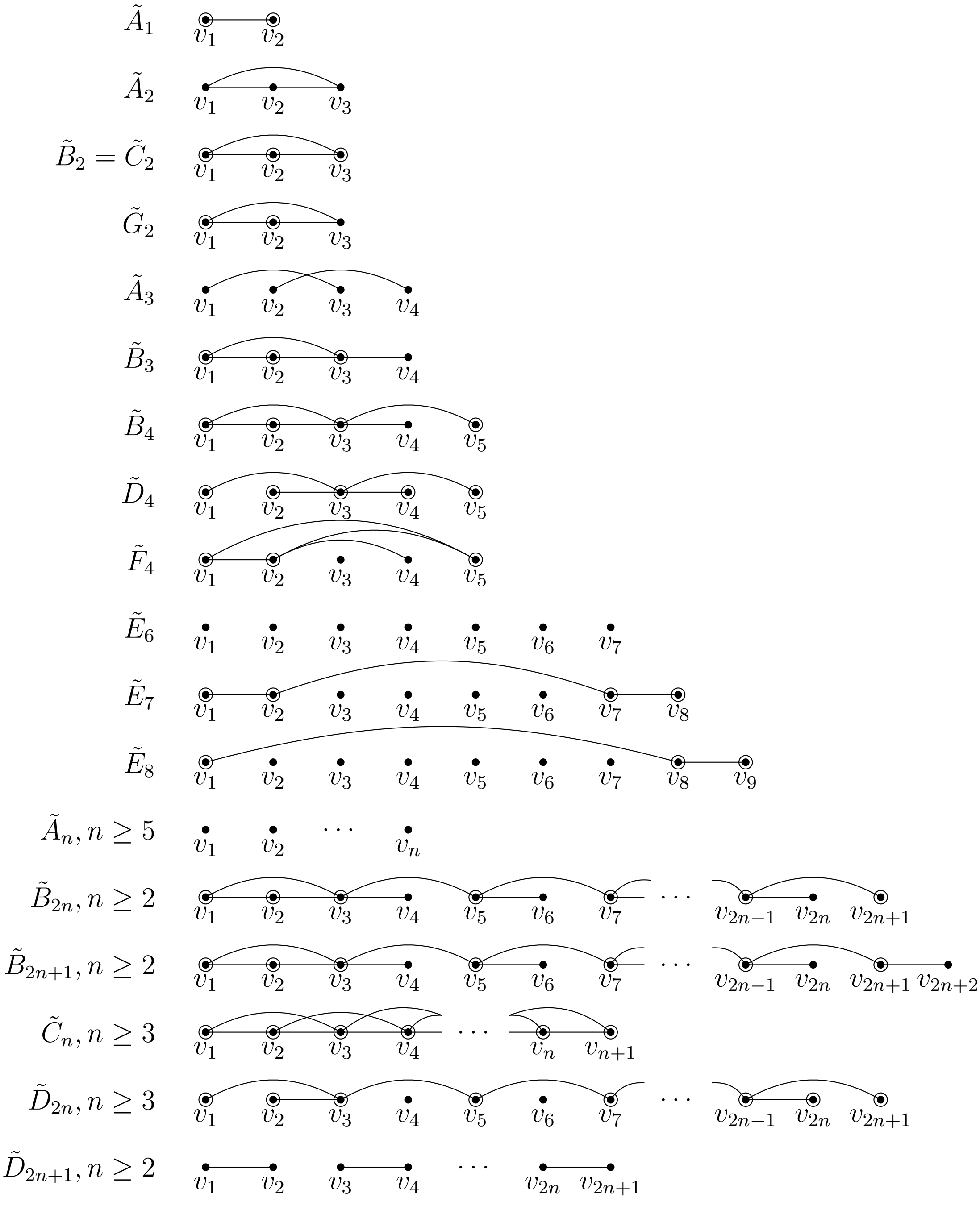


Figure 4.13: The edge reflection graph of all irreducible Euclidean Coxeter groups.

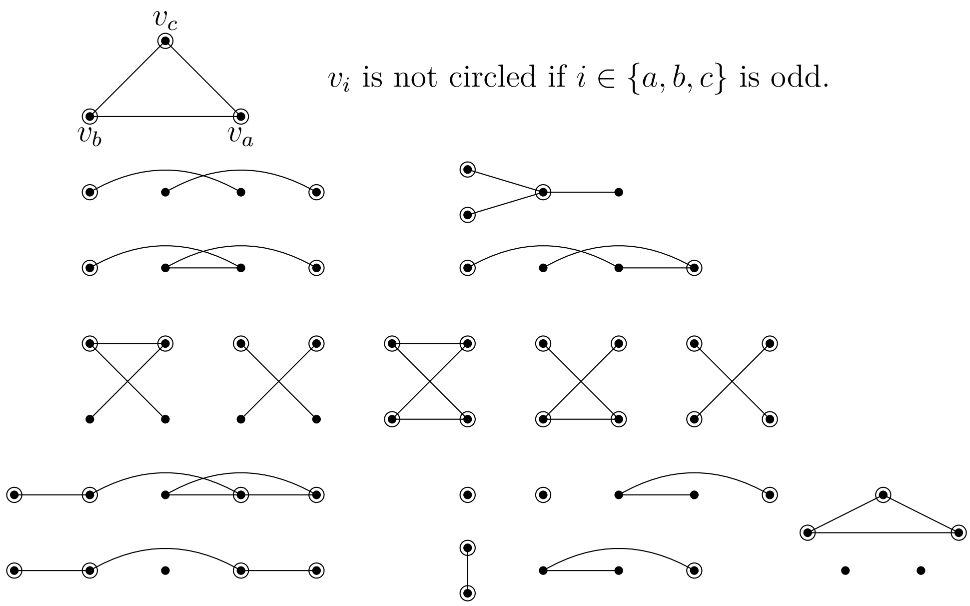


Figure 4.14: The edge reflection graph of all irreducible simplicial hyperbolic Coxeter groups. Compare with Figure 2.8 on page 49.

Lemma 4.2.2. *Let $\mathcal{P}\colon X \to P^n$ be an embedded surface which lies in the interior of P^n with boundary Γ. Then there is a polytope $\mathcal{Q}^2 \subset \mathbb{X}^2$ and a Coxeter system $(\mathcal{J}, T)$ depending only on Γ such that the surface $\mathcal{S}$ obtained by the action of $\mathcal{J}$ on $\mathcal{P}$ is defined on $\mathcal{U}(\mathcal{J}, \mathcal{Q}^2)$. Furthermore, $\mathcal{S}$ maps chambers of $\mathcal{U}(\mathcal{J}, \mathcal{Q}^2)$ onto chambers of $\mathcal{U}(W, P^n)$.*

Proof. The edge cycle Γ determines the edge reflection group $\mathcal{J}$ with representation

$$\mathcal{J} = \langle j_1, \dots, j_n \mid (j_i j_j)^{m_{ij}} = e \rangle$$

where $m_{ij} \in \mathbb{N} \cup \{\infty\}$. Since the j_i are involutions and all j_i are distinct, the matrix (m_{ij}) is a Coxeter matrix. Let $T \coloneqq \{j_1, \dots, j_n\}$, then by Corollary 3.1.9, we get that $(\mathcal{J}, T)$ is a Coxeter system. We can now construct an n-gon $\mathcal{Q}^2$ in $\mathbb{X}^2$ using the angles $\pi/m_{i(i+1)}$ where $m_{n(n+1)} = m_{n1}$. Naturally Γ is homeomorphic to $\partial\mathcal{Q}^2$ and the vertices of $\partial\mathcal{Q}^2$ are identified with the vertices of Γ. With $\mathcal{J}$ and $\mathcal{Q}^2$ we define the Coxeter complex $\mathcal{U}(\mathcal{J}, \mathcal{Q}^2)$ as in (2.12). By construction, $\mathcal{S}$ maps every chamber of $\mathcal{U}(\mathcal{J}, \mathcal{Q}^2)$ onto a chamber of $\mathcal{U}(W, P^n)$. Since Γ is homeomorphic to $\partial\mathcal{Q}^2$ and vertices are identified, the equivalence relation is respected. $\square$

Definition 4.2.3. We call the space $\mathcal{U}(\mathcal{J}, \mathcal{Q}^2)$ constructed in Lemma 4.2.2 the *base space* of Γ.

We want to discuss whether $\mathcal{S}$ is embedded in $\mathbb{X}^n$. Observe that it is enough to show that $\mathcal{S}$ has no self-intersections:

Lemma 4.2.4. *The surface $\mathcal{S}$ is embedded if and only if $\mathcal{S}$ has no self-intersection.*

Proof. If $\mathcal{S}$ is embedded, it is injective; hence has no self-intersections.

Assume $\mathcal{S}$ has no self-intersection, thus $\mathcal{S}$ is injective. Let Y be a union of a finite number of chambers in $\mathcal{U}(\mathcal{J}, \mathcal{Q}^2)$, i.e., $Y \coloneqq \bigcup_{i=1}^{k} \mathcal{Q}^2$. Note that Y is compact. As an injective immersion $\mathcal{S}|_Y$ is an embedding.

Let $K \subset \mathcal{S}(\mathcal{U}(\mathcal{J}, \mathcal{Q}^2))$ be compact and choose Y such that $K \subset \mathcal{S}(Y)$. As $\mathcal{S} \colon Y \to \mathcal{S}(Y)$ is an embedding, $\mathcal{S}^{-1}(K)$ is compact. Hence we get that $\mathcal{S} \colon \mathcal{U}(\mathcal{J}, \mathcal{Q}^2) \to \mathcal{S}(\mathcal{U}(\mathcal{J}, \mathcal{Q}^2))$ is proper. As an proper injective immersion, $\mathcal{S}$ is an embedding. $\square$

Since P^n is a fundamental domain of $\mathcal{U}(W, P^n)$, it is enough to study how $\mathcal{S}$ behaves on P^n. Furthermore, a self-intersection cannot occur in the interior of P^n.

Lemma 4.2.5. *The surface $\mathcal{S}$ has no self-intersections in the interior of any chamber wP^n in $\mathcal{U}(W, P^n)$.*

Proof. Since $\mathcal{P}$ is embedded it has no self-intersection in P^n. By Lemma 4.2.2, $\mathcal{S}$ is maps chambers of $\mathcal{U}(\mathcal{J}, \mathcal{Q}^2)$ onto chambers of $\mathcal{U}(W, P^n)$. Hence $\mathcal{S}$ has no self-intersections in the interior of any chamber in $\mathcal{U}(W, P^n)$. $\square$

By the structure of edge reflections, it suffices to study the vertices of Γ.

Lemma 4.2.6. *The surface $\mathcal{S}$ is embedded if and only if no vertex $v_i \subset \Gamma$ is a self-intersection.*

Proof. Since P^n is a strict fundamental chamber of $\mathcal{U}(W, P^n)$, we only need to consider self-intersection in P^n. As $\mathcal{P}$ intersects ∂P^n only in Γ, a self-intersection of $\mathcal{S}$ can only occur in an edge of P^n or the interior of wP^n. By Lemma 4.2.5, the later is not possible. Suppose $e \subset \Gamma$ is an edge and $\mathcal{S}$ has a self-intersection y in the relative interior of e. Then every copy $j\mathcal{P}$ with $y \in j\mathcal{P}$ contains the entire edge e. Hence every point in e is a self-intersection, in particular the vertices v_i. □

The following lemma states a condition for the surface not to be embedded. We will often use it in the next chapter where we explicitly study if a given edge cycle Γ yields an complete embedded surface $\mathcal{S}$.

Lemma 4.2.7. *Let $e \subset \Gamma$ and $s \in S$ be the reflection in the facet $F \subset P^n$ such that F contains e. If $s \in \mathcal{J}$, then $\mathcal{S}$ is not embedded. Consequently, if the index $[W : \mathcal{J}] = 1$, then $\mathcal{S}$ is not embedded.*

Proof. Let j be the edge reflection along e. Note that

$$n - 1 = \ell(j) \neq \ell(s) = 1.$$

Since $n \geq 3$ the length of j and s are different; hence $j \neq s$. Consequently P^n, sP^n, and jP^n are three chambers containing e. Let p be a point in the relative interior of e. Then there are three points p_1, p_2, $p_3 \in \mathcal{U}(\mathcal{J}, \mathcal{Q}^2)$ lying in different chambers such that $\mathcal{S}(p_1) = \mathcal{S}(p_2) = \mathcal{S}(p_3) = p$. But p lies in the relative interior of a facet in a chamber $j\mathcal{Q}^2$, hence in the base space $\mathcal{U}(\mathcal{J}, \mathcal{Q}^2)$ only two of these points can coincide. Thus there is a point p_i different from the other two making p a self-intersection. By Lemma 4.2.4, we have that $\mathcal{S}$ is not embedded.

If $W = \mathcal{J}$ then $s_i \in \mathcal{J}$ for all i. Thus there is at least a facet containing an edge of Γ. As above, $\mathcal{S}$ is not embedded. □

Lemma 4.2.8. *Let $v \in \Gamma$ be a vertex, $\mathcal{J}$ be the edge reflection group, and $\mathcal{J}(v)$ the special (dihedral) subgroup of $\mathcal{J}$ only using the generators associated with the two edges incident to v. Assume that there is an element $w \in W$ fixing v and $w \in \mathcal{J}$, but $w \notin \mathcal{J}(v)$. Then $\mathcal{S}$ is not embedded.*

Proof. In the base space $\mathcal{U}(\mathcal{J}, \mathcal{Q})$ exactly $|\mathcal{J}(v)|$ chambers are meeting in $\mathcal{P}^{-1}(v)$. Since $w \in \mathcal{J}$ and $w \notin \mathcal{J}(v)$, there is a different chamber in $\mathcal{U}(W, P^n)$. Thus there are at least $(|\mathcal{J}(v)| + 1)$-many points $p_i \in \mathcal{U}(\mathcal{J}, \mathcal{Q})$

such that $\mathcal{S}(p_i) = v$. But only $|\mathcal{J}(v)|$-many points can coincide, making v a self-intersection. □

To use the previous lemma rigorously, we need to prove that the extension of $\mathcal{P}$ by $\mathcal{J}(v)$ yields an embedded minimal surface. This is obvious by construction:

Lemma 4.2.9. *Let v be a vertex in Γ and let $\mathcal{J}(v)$ be the special subgroup generated by the two edge reflections of the edges of Γ containing v. Then the surface patch $\mathcal{S}_{\mathcal{J}(v)} \coloneqq \bigcup_{j \in \mathcal{J}(v)} j\mathcal{P}$ is embedded.*

Proof. Assume that $\mathcal{S}_{\mathcal{J}(v)}$ is not embedded. Hence v is a self-intersection. Thus there is an edge containing v which has self-intersection in its interior. By construction, there is only one $j \in \mathcal{J}$ that fixes this point which contradicts that v is a self-intersection. □

Thus we can state the main result of this chapter and thesis:

Theorem 4.2.10. *Let $P^n \subset \mathbb{X}^n$ be a Coxeter polytope, (W, S) its associated Coxeter system and $\mathcal{U}(W, P^n)$ the Coxeter complex. Let $\Gamma \subset P^n$ be an edge cycle compatible with $\mathcal{U}(W, P^n)$ and $\mathcal{J} \leq W$ the corresponding edge reflection group. Furthermore, let $W(v_i)$ be the largest special subgroup that fixes the vertex v_i and let $\mathcal{J}(v_i)$ be the dihedral group generated by the edges of Γ containing v_i. Let $\mathcal{P}$ be an embedded surface which lies in the interior of P^n with boundary Γ.*

The complete surface $\mathcal{S}$ constructed by extending $\mathcal{P}$ via edge reflection is embedded if and only if

$$\mathcal{J} \cap W(v_i) = \mathcal{J}(v_i) \tag{4.1}$$

for all vertices $v_i \in \Gamma$.

We finish this section with a summary: let P^n be a polytope in $\mathbb{X}^n$. Using the inward-pointing unit normals of the facets of P^n, we can define the dihedral angle θ_{ij} between every pair F_i, F_j of facets of P^n (see Definition 2.2.5). On the one hand, if the angles are non-obtuse, then P^n is simple (Theorem 2.2.11). On the other hand, given numbers θ_{ij}, we can define an n-simplex in $\mathbb{X}^n$ using Proposition 2.3.3, 2.3.9, and 2.3.5. This information can also be stored in the Gram Matrix (c_{ij}) (see (2.4)) yielding that a P^n

is, up to isometry, determined by its Gram matrix in the case $\mathbb{S}^n$ and $\mathbb{H}^n$, and up to translation and homothety in $\mathbb{E}^n$. To get a tessellation of $\mathbb{X}^n$ by P^n, a natural assumption is that the angles are submultiples of π which leads us to the definition of a Coxeter polytope. This naturally leads us to Coxeter systems (W, S) where the set of generators S corresponds to the reflections across the facets of P^n. By Theorem 2.9.6, we have that the space $\mathcal{U}(W, P^n)$ gives us a tessellation of $\mathbb{X}^n$ by P^n.

Our next step was to analyse edge reflection along a given Jordan curve Γ defined on the edges of P^n. We used the canonical representation ρ defined via (3.1) and (3.2) to define the (-1)-condition (see Definition 3.2.6) for a word in W. Inspecting a special subgroup W_T of W, we could identify with Theorem 4.1.4 the longest element w_T of W_T as the edge reflection about an edge $e \subset \Gamma$ if W_T satisfies the (-1)-condition. Assuming that w_T satisfies the (-1)-condition, edge reflection about e maps chambers onto chambers in the tessellation $\mathcal{U}(W, P^n)$. This led us to Definition 4.1.3 of an edge reflection graphs and Definition 4.1.6 of the edge reflection group $\mathcal{J}$. Using the edge reflection group, we extend an embedded surface $\mathcal{P}$ which lies in the interior of P^n with boundary Γ to a complete surface $\mathcal{S}$. Finally using (4.1) in Theorem 4.2.10, we can decide whether the constructed surface $\mathcal{S}$ is embedded or has self-intersections.

4.3 Minimal Surfaces

In this section, we briefly discuss whether we can use Theorem 4.2.10 to construct complete minimal surfaces. In short, to apply Theorem 4.2.10, we need to show that a Plateau solution $\mathcal{P}$ of the edge cycle Γ exists, lies in the interior of the polytope P^n, and is embedded. It turns out that this is true for $n = 3$ and for $\mathbb{X}^n = \mathbb{R}^n$. In all other cases, so far no general result is known. Showing the four properties mentioned above, we can apply the Schwarz Reflection Principle, i.e., edge reflection to construct a complete minimal surface $\mathcal{S}$. Applying Theorem 4.2.10 shows whether $\mathcal{S}$ is embedded or has self-intersection.

The Plateau problem is one of the classical problems of the Calculus of Variations introduced by Lagrange and Euler in the 18^{th} century. Given a Jordan curve Γ, for us the Plateau solution is a disk-type surface with boundary Γ which locally minimises the area functional. In 1865, Schwarz derived the first successful solution to the Plateau problem for a skew quadrilateral [Sch90]. Many explicit solutions followed, e.g., by Riemann, Weierstrass, and Enneper [Nit75]. It was not until 1930-1931 when Radó [Rad30] and Douglas [Dou31] independently gave a proof for a general contour in $\mathbb{R}^n$. While Radó used conformal mappings, Douglas applied the direct method of the Calculus of Variations to obtain the minimal surface.

In 1948, Morrey showed in [Mor48] that a Plateau problem can be solved also in a *homogeneously regular* manifold. This condition holds in our case since P^n is compact.

In the following, if we mention a Plateau solution, we will always refer to a Plateau solution of Douglas, Radó, or Morrey type. By construction, the Plateau solution has boundary Γ. To apply Theorem 4.2.10, we need to discuss if the Plateau solution is embedded and lies in the interior of P^n.

In general, the Plateau solution can contain isolated singular points which are called *branch points*. As discussed in [Nit75] (p. 330) we can distinguish *true* from *false* branch points. A false branch point can be considered as the result of a poor parametrisation such that after reparametrisation the surface is locally an embedding. However, a true branch point is the origin of a self-intersection. To conclude that the surface is an embedding, we need to exclude both types of branch points.

Osserman [Oss69] ruled out the existence of interior true branch points for minimisers, while Gulliver [Gul73] excluded interior false branch points. The non-existence of boundary branch points is still open in general; however, Gulliver and Lesley [GL73] excluded boundary branch points in case the Jordan curve Γ is regular and real analytic in a real analytic manifold. Unfortunately, in higher dimensions $n \geq 4$ branch points do occur anyway, as an example in $\mathbb{R}^4$ shows (see [Nit75], p. 668).

Assuming that Γ is a contractible Jordan curve on the boundary of a three-dimensional convex manifold, Meeks and Yau [MY82] showed that any solution to Plateau's problem is embedded. Up to date there is no generalisation of this result to higher dimensions $n > 3$. For the result of Meeks and Yau, uniqueness cannot be asserted.

However, using a Lemma of Radó which can be found in [Oss02], one can show that the Plateau solution for boundary curves which are graphs in $\mathbb{R}^n$ is embedded. Using the maximum principle, one can conclude that the Plateau solution lies in the interior of P^n. This implies that in our case of a polytope P^n the Plateau solution lies in its interior. However, for the general case $\mathbb{X}^n$ neither the Lemma of Radó nor the maximum principle can be used to prove that $\mathcal{P}$ is embedded and lies in the interior of P^n.

For a more detailed account of Plateau's problem and minimal surfaces in general, Nitsche's [Nit75] and Hildebrand [JDK+10] is highly recommended.

We conclude that in case $n = 3$ and for $n > 3$ in $\mathbb{R}^n$ the existence of an embedded Plateau solution is guaranteed. In all other cases treated in Chapter 5, our results are derived under the additional assumption that there is a Plateau solution which lies in the interior of P^n and is embedded. Under this condition we can use Theorem 4.2.10 to construct a complete embedded minimal surface $\mathcal{S}$.

5 Complete Surfaces in Homogeneous Manifolds

In this chapter, we analyse Theorem 4.2.10 in detail and discuss whether the constructed surface $\mathcal{S}$ is embedded in $\mathbb{X}^n$. We discuss briefly how we can extend our construction to product spaces and how to compute the genus of $\mathcal{S}$. Finally, we look at the embeddedness of $\mathcal{S}$. We see $\mathbb{X}^n$ as an n-dimensional product space with factors $\mathbb{S}^k$, $\mathbb{E}^k$, and $\mathbb{H}^k$. Note that for an $\mathbb{H}^k$-factor, we have that $k \geq 2$. Furthermore, as the universal cover of $\mathbb{S}^1$ is $\mathbb{E}^1$, we omit the discussion on $\mathbb{S}^1$. Hence, there remain five 3-dimensional cases, namely $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{E}$, $\mathbb{E}^3$, $\mathbb{H}^2 \times \mathbb{E}$, $\mathbb{H}^3$, and ten 4-dimensional cases.

5.1 Product Spaces

We can easily extend our discussion to product spaces. Let P_1^k be a Coxeter polytope in $\mathbb{X}^k$ and P_2^m a Coxeter polytope in $\mathbb{X}^m$. By Theorem 2.9.6, we get Coxeter systems (W_1, S_1), (W_2, S_2), that $\mathcal{U}(W_1, P_1^k)$ is a tessellation of $\mathbb{X}^k$, and that $\mathcal{U}(W_2, P_2^m)$ is a tessellation of $\mathbb{X}^m$. Obviously, the space $\mathcal{U}(W_1, P_1^k) \times \mathcal{U}(W_2, P_2^m)$ is homeomorphic to $\mathbb{X}^k \times \mathbb{X}^m$ and gives us a tessellation. Since the W-action preserves the equivalence relation $\sim$ used to define $\mathcal{U}(W, P^n)$, we have that $\mathcal{U}(W_1 \times W_2, P_1^k \times P_2^m)$ is homeomorphic to $\mathcal{U}(W_1, P_1^k) \times \mathcal{U}(W_2, P_2^m)$. Thus we can extend our discussion onto product manifolds.

Example 5.1.1. As in Example 2.9.5, suppose I_k is a circular arc of length π/k where s_1 and s_2 are the reflections of $\mathbb{S}^1$ across the endpoints of I_k. Let W^1 be the group generated by s_1 and s_2 . Then W^1 is a Coxeter group of type $I_2(k)$ and $\mathcal{U}(W^1, I_k)$ is a tessellation of $\mathbb{S}^1$. Consider another circular arc I_m of length π/m and that t_1 and t_2 are the reflections of $\mathbb{S}^1$ of I_m. Let W^2 be the group generated by t_1 and t_2. Then W^2 is a Coxeter group of type

$I_2(m)$ and $\mathcal{U}(W^2, I_m)$ is a tessellation of $\mathbb{S}^1$. The space $\mathcal{U}(W^1 \times W^2, I_k \times I_m)$ is a tessellation of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ and $I_k \times I_m$ is a rectangle on $\mathbb{S}^1 \times \mathbb{S}^1$.

Remark 5.1.2. (i) Since both circular arcs are constructed using a cone in $\mathbb{R}^2$, the torus $\mathbb{S}^1 \times \mathbb{S}^1$ constructed in Example 5.1.1 is a submanifold of $\mathbb{R}^4$. For a better visualisation one can use that $\mathcal{U}$ is homeomorphic to the flat torus $\mathbb{R}^2/2\pi\mathbb{Z}^2$. Here, the polytope $I_k \times I_m$ is a rectangle with sides of length π/k and π/m.

(ii) In Example 5.1.1, the Coxeter group $W^1 \times W^2$ is of type $I_2(k) \times I_2(m)$. Using the polytope $I_k \times I_m$, we get that $\mathcal{U}(W^1 \times W^2, I_k \times I_m)$ is a tessellation of $\mathbb{S}^1 \times \mathbb{S}^1$. Thus $W^1 \times W^2$ acts on two orthogonal circles. One can also define an 3-simplex P^3 in $\mathbb{S}^3$ with corresponding Coxeter group $I_2(k) \times I_2(m)$. Then the space $\mathcal{U}(W^1 \times W^2, P^3)$ is a tessellation of $\mathbb{S}^3$. Here, $W^1 \times W^2$ can be seen as an action on two orthogonal 2-spheres in $\mathbb{S}^3$. In particular, this shows that two different polytopes can have the same Coxeter group.

Let

$$W := W^1 \times W^2, S := S_1 \cup S_2, P^n := P_1^k \times P_2^m, \text{ and } \mathbb{X}^n := \mathbb{X}^k \times \mathbb{X}^m.$$

Note that the facets of P^n are totally geodesic in $\mathbb{X}^n$. Since $s_1 \in S_1$ and $s_2 \in S_2$ commute, the two corresponding facets are right-angled. By using Theorem 4.2.10, we get a nice result for right-angled 3-dimensional polytopes.

Theorem 5.1.3. *Let P^3 be a right-angled Coxeter polytope in $\mathbb{X}^3$ (or $\mathbb{X}^2 \times \mathbb{E}$). Then every edge cycle Γ leads to an embedded surface $\mathcal{S}$.*

Proof. Let (W, S) be the corresponding Coxeter system of P^3 and Γ an edge cycle in P^3. Furthermore, let v be a vertex of P^3 and $W(v) \leq W$ and $\mathcal{J}(v) \leq \mathcal{J}$ the largest subgroups of W and $\mathcal{J}$ fixing v. Since P^3 is right-angled, 8 chambers wP^3 meet at v in $\mathcal{U}(W, P^3)$. Note that edge reflection along an edge of Γ corresponds to a rotation. Thus $\mathcal{J}$ contains only orientation-preserving mappings. Hence $[W : \mathcal{J}] \geq 2$, i.e., $|\mathcal{J} \cap W(v)| \leq 4$. But, $\mathcal{J}(v)$ is the dihedral group containing at least 4 elements. Since $\mathcal{J}(v)$ is a subgroup of $\mathcal{J} \cap W(v)$, these two groups are equal. By Theorem 4.2.10, edge reflecting the surface $\mathcal{P}$ yields an embedded surface $\mathcal{S}$. □

5.2 Genus of the Surface $\mathcal{S}$

If $\mathcal{S}$ is embedded, we calculate the genus of $\mathcal{S}$ using the Euler characteristic $\chi(\mathcal{S})$ and the Gauss-Bonnet Theorem. By construction, we have a natural subdivision of $\mathcal{S}$. The number of faces is given by the order of the edge reflection group $|\mathcal{J}|$. Since every edge belongs to two surface patches, we have $(|\Gamma||\mathcal{J}|)/2$ edges where $|\Gamma|$ denotes the number of edges in the edge cycle Γ. Let $\mathcal{J}(v_i)$ be the subgroup of $\mathcal{J}$ fixing an vertex $v_i \subset \Gamma$. If we have $|\mathcal{J}|$ patches, we have counted this vertex $|\mathcal{J}(v_i)|$ times for every patch meeting at v_i in $\mathcal{U}(W, P^n)$. Hence we need to divide by the order $|\mathcal{J}(v_i)|$. This needs to be done for every vertex. So the number of vertices of $\mathcal{S}$ is given by $\sum_i |\mathcal{J}|/|\mathcal{J}(v_i)|$ where we sum over the length of the edge cycle Γ. By the Gauss-Bonnet Theorem we obtain:

$$\chi(\mathcal{S}) = |\mathcal{J}| - \frac{|\Gamma||\mathcal{J}|}{2} + \sum_i \frac{|\mathcal{J}|}{|\mathcal{J}(v_i)|} = -\frac{(|\Gamma|-2)|\mathcal{J}|}{2} + \sum_i \frac{|\mathcal{J}|}{|\mathcal{J}(v_i)|}. \quad (5.1)$$

If the surface is orientable, we can calculate the genus g of $\mathcal{S}$ by $\chi(\mathcal{S}) = 2-2g$, i.e., it is given by

$$g = 1 - \frac{\chi(\mathcal{S})}{2} = 1 - \frac{\sum_i \frac{|\mathcal{J}|}{|\mathcal{J}(v_i)|}}{2} + \frac{(|\Gamma|-2)|\mathcal{J}|}{4}. \quad (5.2)$$

Obviously, if $\mathbb{X}^n \neq \mathbb{S}^n$ the surface will not be compact, i.e., $|\mathcal{J}| = \infty$ and $g = \infty$. Hence we need to define the genus differently, e.g., by computing the Euler characteristic in a suitable compact subset of $\mathbb{X}^n$. A possible solution is to calculate the orbifold Euler characteristic of $\mathcal{S}$ in the polytope P^n. In this case, (5.1) reduces to

$$\chi_{\mathrm{orb}}(\mathcal{S}) = 1 - \frac{|\Gamma|}{2} + \sum_i \frac{1}{|\mathcal{J}(v_i)|}.$$

In general, $\chi_{\mathrm{orb}}(\mathcal{S})$ is not a natural number. We want to avoid this and find a group $\mathcal{T}$ such that $\mathcal{S}/\mathcal{T}$ is a manifold. Furthermore, we want $\mathcal{T}$ to be as large as possible. To archive this, $\mathcal{T}$ needs to act freely on $\mathbb{X}^n = \mathbb{R}^n$ or $\mathbb{H}^n$, or equivalently $\mathcal{T}$ needs to be torsion-free (see Theorem 8.2.1 in [Rat06]), i.e., every element has infinite order. By Corollary 3.1.18, every Coxeter group contains a subgroup with finite index which is torsion-free. Our goal is to find the largest torsion-free subgroup with finite index, i.e., the index is as small as possible. In general, finding this group is rather involved, especially in hyperbolic space. Thus we start with the Euclidean case.

In the Euclidean case, any orientation-preserving isometry can be defined as the product of two reflections. It is either a rotation or a translation. Note that a rotation by an angle of π/k, $k \in \mathbb{N}$ closes after $2k$ rotations, i.e., a rotation has torsion. Thus the only orientation-preserving isometries without torsion are translations. We define translation of $\mathcal{U}(W, P^n)$ as follows:

Definition 5.2.1. Let W be Euclidean and let $R \coloneqq \{wsw^{-1} \in W \mid w \in W, s \in S\}$ be the set of *reflections* in $\mathcal{U}$. A *translation* is a torsion-free product of two reflections. We denote the group generated by all translations as $T(\mathcal{U}(W, P^n))$ or simply T, if $\mathcal{U}$ is obvious.

Denote by H a torsion-free subgroup of W with finite index. If H contains orientation-reversing elements, let H^+ be the index 2 subgroup of orientation-preserving elements of H. If H contains only orientation-preserving elements, then $H = H^+$. In both cases, H^+ has finite index in W. Clearly $H^+ \leq T$; hence T has finite index in W. The group $\mathcal{T}$ we are looking for is given by all torsion-free elements of $\mathcal{J}$, i.e., $\mathcal{T} \coloneqq \mathcal{J} \cap T$. To compute the Euler characteristic, we need the number of chambers in $\mathcal{S}/\mathcal{T}$ which is simply given by the index $[\mathcal{J} : \mathcal{T}]$.

In general, translations can reverse the orientation of $\mathcal{S}/\mathcal{T}$. So we need to check whether $\mathcal{S}/\mathcal{T}$ is orientable. If $\mathcal{S}/\mathcal{T}$ is orientable, we can define the Euler characteristic as

$$\chi_{\mathcal{T}}(\mathcal{S}) = [\mathcal{J} : \mathcal{T}] \cdot \chi_{\mathrm{orb}}(\mathcal{S}). \tag{5.3}$$

Otherwise, let $\mathcal{T}^+$ be the index 2 subgroup of orientation-preserving elements. Then the orientable double cover of $\mathcal{S}/\mathcal{T}$ is given by $\mathcal{S}/\mathcal{T}^+$. If $\mathcal{S}/\mathcal{T}$ is non-orientable, the Euler characteristic of $\mathcal{S}/\mathcal{T}^+$ is given by

$$\chi_{\mathcal{T}^+}(\mathcal{S}) = [\mathcal{J} : \mathcal{T}^+] \cdot \chi_{\mathrm{orb}}(\mathcal{S}) = 2[\mathcal{J} : \mathcal{T}] \cdot \chi_{\mathrm{orb}}(\mathcal{S}). \tag{5.4}$$

The genus of $\mathcal{S}/\mathcal{T}$ and $\mathcal{S}/\mathcal{T}^+$, respectively, is given by

$$g_{\mathcal{T}} = 1 - \frac{\chi_{\mathcal{T}}(\mathcal{S})}{2} \qquad \text{and} \qquad g_{\mathcal{T}^+} = 1 - \frac{\chi_{\mathcal{T}^+}(\mathcal{S})}{2}. \tag{5.5}$$

Thus we need to construct $\mathcal{T}$ by finding the largest subgroup of all translations T in $\mathcal{U}$ which are also contained in $\mathcal{J}$ and then calculate the index

$[\mathcal{J} : \mathcal{T}]$. After checking whether $\mathcal{S}/\mathcal{T}$ is orientable or non-orientable, we can compute the Euler characteristic by (5.3) or (5.4), respectively. With the Euler characteristic, we can compute the genus using (5.5).

The hyperbolic case is more subtle. Consider a triangle Δ in $\mathbb{H}^2$ with vertices x, y, and z. Let t_1, t_2, and t_3 be the hyperbolic translations which map x to y, y to z, and z to x, respectively. Note that $t_1t_2t_3$ is the parallel transport along a closed piecewise geodesic curve in $\mathbb{H}^2$. By the Gauss-Bonnet Theorem its holonomy is given by

$$\int_\Delta K \mathrm{d}A = (\alpha + \beta + \gamma) - \pi$$

where K is the Gaussian curvature and α, β, γ are the interior angles of Δ. Thus $t_1t_2t_3$ describes a rotation by the angle $\pi - (\alpha + \beta + \gamma)$. If we define $\mathcal{T}$ as in Definition 5.2.1, the group $\mathcal{T}$ generated by translations is in general not torsion-free, since it can contain rotations. Furthermore, note that the integral vanishes in the Euclidean case and so composition as above yields the identity element.

On the other hand, Corollary 3.1.18 assures that W contains a torsion-free subgroup with finite index. Let T be the maximal torsion-free subgroup in W, i.e., $[W : T]$ is as small as possible. Again, we are interested in all torsion-free elements in $\mathcal{J}$, i.e., we define $\mathcal{T} \coloneqq \mathcal{J} \cap T$. Given the index $[\mathcal{J} : \mathcal{T}]$, if $\mathcal{S}/\mathcal{T}$ is orientable, we can compute the Euler characteristic using (5.3) and if $\mathcal{S}/\mathcal{T}$ is non-orientable, the Euler characteristic is given by (5.4). The genus $g_\mathcal{T}$ is then given by (5.5). By inspecting (5.5), we see that $g_\mathcal{T}$ is an integer if $[\mathcal{J} : \mathcal{T}]$ is a common multiple of all $|\mathcal{J}(v_i)|$. Furthermore, since $\mathcal{J}$ and T are subgroups of W, we have the elementary estimation

$$[\mathcal{J} : \mathcal{T}] = [\mathcal{J} : T \cap \mathcal{J}] \leq [W : T]. \tag{5.6}$$

Motivated by this, denote by $\mathcal{L}(W)$ the lowest common multiple of the orders of all finite subgroups of W and by $\mathcal{M}(W)$ the minimum index of a torsion-free subgroup of W. Assuming that $\mathbb{X}^n = \mathbb{H}^2$ it is shown in [EEK82] that $\mathcal{M}(G)/\mathcal{L}(G) \leq 2$ for any *Fuchsian group* G, i.e., an orientation-preserving group of isometries of $\mathbb{H}^2$. Since every Coxeter group obviously contains an index 2 Fuchsian subgroup, we have

$$\mathcal{M}(G)/\mathcal{L}(G) \leq 4. \tag{5.7}$$

Using (5.6) and (5.7) , we get

$$[\mathcal{J} : \mathcal{T}] \leq 4\mathcal{L}(W). \tag{5.8}$$

This is an upper bound such that $\mathcal{S}/\mathcal{T}$ and $\mathcal{S}/\mathcal{T}^+$ is assured to be an orientable manifold.

Unfortunately, in the case $\mathbb{X}^n = \mathbb{H}^3$ it is shown in [JR98] that in general $\mathcal{M}(W)/\mathcal{L}(W)$ can be arbitrarily large. As the irreducible hyperbolic simplicial Coxeter groups are of rank less than 6 (Theorem 2.9.13) and, as we will see later, all surfaces $\mathcal{S}$ constructed in $\mathbb{H}^4$ have self-intersections, we do not discuss how $\mathcal{M}(W)/\mathcal{L}(W)$ behaves in the case where $n > 3$.

Eventually, assuming that the initial surface patch $\mathcal{P}$ is orientable, is the complete embedded surface $\mathcal{S}$ orientable? Although this fails for general 3-manifolds, e.g., $\mathbb{R}P^2$ is not orientable in $\mathbb{R}P^3$, one can show that it holds for $\mathbb{X}^3$. In the following, we will show that it still holds for even dimensions. Note that edge reflection reverses orientation. Thus $\mathcal{S}$ is non-orientable if and only if there exists an odd number of generators of $\mathcal{J}$ whose composition gives the identity.

Proposition 5.2.2. *Let P^n be a Coxeter polytope with Coxeter system (W, S) and Γ a compatible edge cycle on P^n with edge reflection group $\mathcal{J}$. Assume that the initial surface patch $\mathcal{P}$ is orientable. If n is even, then the complete surface $\mathcal{S}$ is orientable.*

Proof. We use the Deletion Condition to show that the identity element cannot be expressed by a composition of an odd number of generators j_i.

Let W_{T_i} be the largest special subgroup fixing $e_i \subset \Gamma$. Since Γ is compatible, the edge reflection j_i is given by the longest element of W_{T_i}. There are $n-1$ facets meeting at an edge of P^n. Thus the rank of W_{T_i} is odd.

A quick glance at Table 3.1 shows: first, if the rank of an irreducible spherical Coxeter group is even, then the length of the longest element is even. Second, if the rank is odd, then the length of the longest element is odd except in the case D_{2n+1}.

Since j_i is generated by a special Coxeter group satisfying the (-1)-condition, by Theorem 3.2.12, the case D_{2n+1} cannot occur.

If W_{T_i} is irreducible, $\ell_S(j_i)$ is odd. Assume that W_{T_i} is reducible. Note that the length of the longest element is additive in the factor, i.e., we can omit all irreducible factors of even rank. The rank of the remaining subgroup is odd and the rank of each factor is also odd. But, an odd number cannot be subdivided into an even number of odd numbers. Hence the length of j_i is odd.

Let $j \in \mathcal{J}$ be the composition of an odd number of j_i, i.e., $\ell_{\mathcal{J}}(j)$ is odd. Thus we have an expression for j consisting of an odd number of generators s_k which is not necessarily reduced. By the Deletion Condition, a reduced expression can be achieved omitting an even number of generators $s_k \in W$. Hence $\ell_S(j)$ is odd. But $\ell_S(e) = 0$; hence the identity cannot be expressed by a composition of an odd number of generators j_i. Thus $\mathcal{S}$ is orientable. □

The proof obviously fails for odd dimension n. The problem even occurs in dimension 3: let $P^3 \subset \mathbb{S}^3$ be a Coxeter polytope such that its Coxeter system of type F_4. A full-dimensional compatible edge cycle is given by $\Gamma = v_1v_3v_2v_4$ (see Figure 5.1).

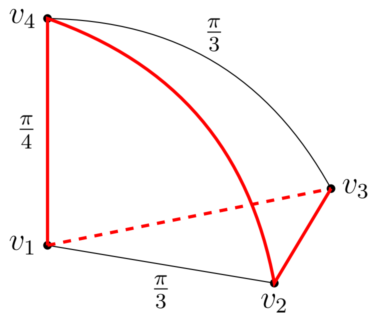


Figure 5.1: A Coxeter polytope $P^3 \subset \mathbb{S}^3$ with Coxeter system of type F_4. The dihedral angles are labelled midway of an edge. Dihedral angles of $\pi/2$ are omitted.

This yields

$$\mathcal{J} = \langle j_1, j_2, j_3, j_4 \rangle = \langle s_2s_4, s_1s_4, s_1s_3, (s_2s_3)^2 \rangle.$$

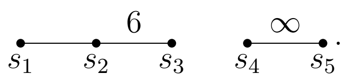
Note that $j_1j_2j_3 = s_2s_3$; hence $(j_1j_2j_3)^2 j_4 = e$. Consequently, the surface $\mathcal{S}$ constructed by reflecting $\mathcal{P}$ using $\mathcal{J}$ is not orientable.

In the subsequent sections, assuming that $\mathcal{S}$ is embedded, we will use (5.1) to calculate the Euler characteristic in the spherical case. In the Euclidean case, we will determine all translations that generate $\mathcal{T}$ and calculate $[\mathcal{J} : \mathcal{T}]$. The Euler characteristic is then given by (5.3). In the case $\mathbb{X}^n = \mathbb{H}^2$, we will use (5.7) to give an upper bound on the fundamental domain. We will omit this discussion for higher dimensional hyperbolic spaces. As the edge reflection may reverse orientation, we need to consider the orientable double cover $\mathcal{S}/\mathcal{T}^+$ and calculate its Euler characteristic. Given the Euler characteristic we calculate the genus of $\mathcal{S}$ using either (5.2) in the spherical case or (5.5) in the Euclidean and hyperbolic case.

5.3 Symmetry of Edge Cycles

In this section, we want to discuss how we distinguish edge cycles. We will identify edge cycles if they are identical after applying a symmetry of the polytope, i.e., two edge cycles Γ_1 and Γ_2 on P^n are *symmetric* if there is a symmetry φ of P^n such that $\varphi(\Gamma_1) = \Gamma_2$.

Example 5.3.1. Let $\Delta \subset \mathbb{E}^2$ be a triangle with angles $\pi/2$, $\pi/3$, $\pi/6$ and consider the prism $P^3 := \Delta \times I$ where $I = [a, b]$ is an interval. Hence the Coxeter group is of type $\tilde{G}_2 \times \tilde{A}_1$ and the Coxeter diagram is given by



Consider the three edge cycles Γ_1, Γ_2, and Γ_3 given in Figure 5.2.

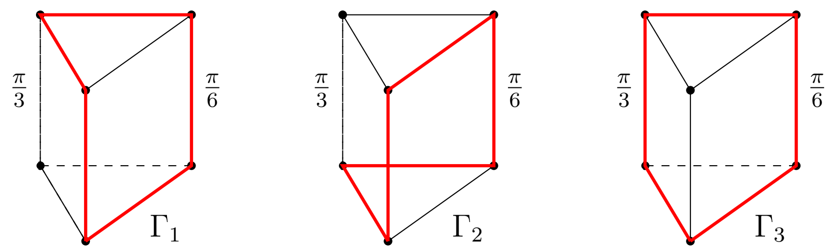


Figure 5.2: The edge cycles on the prism P^3. By the labeling of the Coxeter diagram, we have that s_1, s_2, and s_3 are the reflections along the facets parallel to the vertical edges of P^3 and s_4 and s_5 are the reflections along the respectively lower and upper facet.

The edge cycles Γ_1 and Γ_2 can be mapped onto each other after reflecting through the hyperplane $x_3 = (b - a)/2$ which maps P^3 onto itself. However, Γ_3 is different, since there is no symmetry of the polytope which maps Γ_3 onto Γ_1 or Γ_2.

Note that the reflection across the hyperplane $x_3 = (b-a)/2$ interchanges the lower and upper facet of P^n. Hence it interchanges the reflections s_4 and s_5. Thus this symmetry can be seen as a relabeling of the Coxeter diagram. In Example 5.2, Γ_1 is mapped onto Γ_2 after interchanging the label for the Coxeter diagram as follows:

$$\underset{s_1}{\bullet}\!-\!\!-\!\!-\underset{s_2}{\bullet}\overset{6}{-\!\!-\!\!-}\underset{s_3}{\bullet} \quad \underset{s_4}{\bullet}\overset{\infty}{-\!\!-\!\!-}\underset{s_5}{\bullet} \rightsquigarrow \underset{s_1}{\bullet}\!-\!\!-\!\!-\underset{s_2}{\bullet}\overset{6}{-\!\!-\!\!-}\underset{s_3}{\bullet} \quad \underset{s_5}{\bullet}\overset{\infty}{-\!\!-\!\!-}\underset{s_4}{\bullet}.$$

Furthermore, note that the Coxeter graph stays invariant. This motivates following definition:

Definition 5.3.2. Let (W, S) be a Coxeter system and φ an automorphism of W. We call φ a *diagram automorphism* of (W, S) if $\varphi(S) = S$.

Any diagram automorphism φ, up to relabeling, preserves the associated Coxeter matrix M, i.e., $M_{\varphi(i)\varphi(j)} = M_{ij}$. In other words, the Coxeter matrices M_{ij} and $M_{\varphi(i)\varphi(j)}$ can be associated with the equivalent Coxeter polytope P but with a different labeling of the facets such that all dihedral angles stay invariant. Since every symmetry of P^n permutes the facets of P^n, we can associate a diagram automorphism to a symmetry. In the case $\mathbb{X}^n = \mathbb{S}^n, \mathbb{H}^n$, a polytope is determined up to isometry by its dihedral angles; hence every diagram automorphism corresponds to symmetry.

In the Euclidean case, a polytope is determined by its dihedral angles up to isometry and homothety. This becomes relevant if the Coxeter graph contains two isomorphic irreducible components. Then it is possible to apply a diagram automorphism, but it will not necessarily correspond to a symmetry since the scaling of the facets could be different. For example, consider a rectangle and a square. Both have the same Coxeter graph with two irreducible components. But only in the case of the square, the diagram automorphism interchanging these components results in a symmetry. Thus we get:

Theorem 5.3.3. *Let $P^n \subset \mathbb{X}^n$ be a Coxeter polytope with Coxeter system (W, S). Every symmetry of P^n can be associated with a diagram automorphism of (W, S).*

Conversely, every diagram automorphism of (W, S) corresponds to a symmetry of P^n or, possibly, to a symmetry of its homothetical image $H(P^n)$.

For simplicity, if the Coxeter graph of (W, S) contains two isomorphic components then we will always consider $H(P^n)$ instead of P^n, e.g., we will always consider a cube instead of an rectangle.

Given a Coxeter polytope P^n, our goal is to find all compatible full-dimensional edge cycles on P^n, up to symmetry. This is equivalent to finding all simple circles in the edge reflection graph and checking whether the circle lies in a subgraph representing a facet of P^n. This is done with MATLAB routines we will describe in the appendix in more detail.

Essentially, we are using Johnson's algorithm based on [Joh75] in MATLAB to find all simple edge cycles in the edge reflection graph. Then we delete, using brute-force-methods, all edge cycle which are symmetric and not full-dimensional. After that we translate the edge cycle given in MATLAB to a group in GAP and check (4.1) in GAP for all edges.

In dimension three most calculation can be done by hand. We will do this in detail in the case $\mathbb{S}^3$. For four-dimensional manifolds the calculations become rather tedious. We will use MATLAB and GAP to check the results.

The Todd-Coxeter algorithm is a tool to calculate the index of a subgroup. Provided the index of the subgroup is finite, even for infinite groups the algorithm terminates in a finite number of steps. If the index is too low, we can easily dismiss edge cycles, as the resulting surfaces have self-intersection, e.g., we can use Lemma 4.2.7 in case $[W : \mathcal{J}]$. Thus it is an efficient way to check whether a subgroup coincides with the group itself. GAP mainly uses the Todd-Coxeter algorithm for checking whether a given element lies in the group or not.

5.4 Three-dimensional Manifolds

In this section, we will look at all five three-dimensional cases and discuss whether a given edge cycle yields an embedded surface $\mathcal{S}$.

First, we choose the space. This gives us a list of Coxeter systems. If the 3-manifold is the 3-sphere, the only Coxeter systems to consider are spherical ones. If the space is a product of hyperbolic space and Euclidean space, the Coxeter groups are products of hyperbolic and Euclidean Coxeter groups.

Second, the Coxeter graph gives us an edge reflection graph. With this graph we define compatible edge cycles as closed curves in the edge reflection graph.

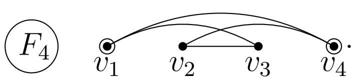
Third, given the edge cycle we can define generators for the edge reflection group $\mathcal{J}$. Finally, we want to apply Theorem 4.2.10 to see whether the edge cycle yields an embedded minimal surface $\mathcal{S}$. Since for an edge v the group $\mathcal{J}(v)$ is a subgroup of $W(v)$ and $\mathcal{J}$, we will either show that there is an element in $\mathcal{J} \cap W(v)$ which is not contained in $\mathcal{J}(v)$, or we will show that the order of these groups coincide. Furthermore, if $\mathcal{S}$ is embedded, we will calculate the genus of $\mathcal{S}$ for a suitable torsion-free subgroup $\mathcal{T}$ of W.

As we go through this section, I strictly recommend having Figure 2.6, Figure 2.7, and Figure 2.8 for the Coxeter graphs and Figure 4.12, Figure 4.13, and Figure 4.14 for the edge reflection graphs at hand, as we will heavily refer to them.

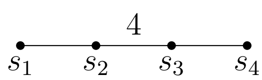
Checking (4.1) of Theorem 4.2.10 and computing the genus of the surface $\mathcal{S}$ can be done efficiently by GAP. In the case $\mathbb{S}^3$, we will do all calculations by hand and at the end use GAP to confirm the results. For all other cases we will just present the results using GAP.

Surfaces in $\mathbb{S}^3$

Let P^3 be a Coxeter polytope in $\mathbb{S}^3$. Then the corresponding Coxeter system (W, S) is spherical with $|S| = 4$. Note that the reflection s_i is defined by the facet not containing the vertex $v_i \in P^3$. If (W, S) is irreducible, Figure 2.6 shows that W is one of the following types: A_4, B_4, D_4, F_4, or H_4. Since Γ is a compatible full-dimensional edge cycle, it is closed and consists of at least 4 edges. Thus we need to find a cycle consisting of at least 4 edges in the corresponding edge reflection graph. Inspecting Figure 4.12, the only edge reflection graph containing a cycle of length 4 is of type F_4, i.e.,



Recall that the Coxeter graph is given by



and consider the edge cycle $\Gamma = v_1v_3v_2v_4$, recall Figure 5.1. Note that, up to symmetry, this is the only compatible full-dimensional edge cycle we can define. Consider the edge connecting v_1 and v_4 in the edge reflection graph. The meaning of the edge is that the special subgroup generated by the remaining generators s_2 and s_3 satisfies the (-1)-condition. This subgroup generated by s_2 and s_3 is of type B_2, hence the longest element is given by $(s_2s_3)^2$ and is a generator of $\mathcal{J}$. The other generators can be constructed similarly and are given by

$$j_1 = s_2s_4, \qquad j_2 = s_1s_4, \qquad j_3 = s_1s_3, \qquad j_4 = (s_2s_3)^2.$$

Since s_2 and s_3 both fix v_1 and v_4, the edge v_1v_4 in the edge reflection graph corresponds to the edge v_1v_4 in Γ. Thus we get the edge cycle $v_1v_3v_2v_4$. In the following, we will show that the surface $\mathcal{S}$ constructed by edge reflecting $\mathcal{P}$ is *not* embedded.

Consider the vertex v_1. Since s_1 does not fix v_1 and j_1, j_4 do not contain s_1, the subgroup $\mathcal{J}(v_1)$ is generated by j_1 and j_4, i.e, $\mathcal{J}(v_1) = \langle j_1, j_4\rangle$. The special subgroup $W(v_1)$ is generated by all s_i which fix v_1, i.e., we have $W(v_1) = \langle s_2, s_3, s_4\rangle$. By Theorem 4.2.10, we need to check if the equation $\mathcal{J} \cap W(v_1) = \mathcal{J}(v_1)$ is satisfied. Consider the element $j_2j_3 = s_3s_4$. It is obviously contained in $\mathcal{J}$ and $W(v_1)$, but not in $\mathcal{J}(v_1)$. The latter can be seen as follows: Since $\mathcal{J}(v_1)$ is a dihedral group, we can compute its order by computing the order of the element $j_1j_4 = s_3s_2s_3s_4$, which is 3. Thus $|\mathcal{J}(v_1)| = 6$ and $\mathcal{J}(v_1)$ is given by

$$\mathcal{J}(v_1) = \{e, s_2s_4, (s_2s_3)^2, s_3s_2s_3s_4, s_4s_3s_2s_3, s_4s_3s_2s_3s_2s_4\}.$$

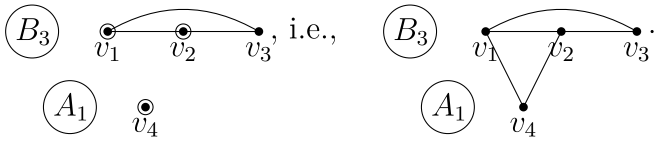
We see that $s_3s_4 \notin \mathcal{J}(v_1)$. By Theorem 4.2.10, the surface $\mathcal{S}$ is not embedded. Furthermore, since $j_1j_2j_4 = s_2s_1(s_2s_3)^2$ has order 3, the identity can be expressed as an composition of edge reflections. Hence $\mathcal{S}$ is non-orientable.

Let (W, S) be reducible. A glance at Figure 4.12 shows that the only Coxeter groups to consider are of type $B_3 \times A_1$, $I_2(m) \times I_2(k)$, $I_2(k) \times A_1 \times A_1$, and $(A_1)^4 = A_1 \times A_1 \times A_1 \times A_1$. Note that for $m = 2$ the types $I_2(m)$ and $A_1 \times A_1$ coincide, i.e, we do not need to consider the types where we can interchange $I_2(m)$ and $A_1 \times A_1$ separately.

We start with the type $B_3 \times A_1$. The edge reflection graph is given by

We see that, up to symmetry, the only edge cycle is given by $\Gamma = v_1v_3v_2v_4$, recall Figure 4.5. The generators of $\mathcal{J}$ are given by

$$j_1 = s_2s_4, \qquad j_2 = s_1s_4, \qquad j_3 = s_1s_3, \qquad j_4 = (s_2s_3)^2.$$



Similarly to the case F_4, we have $j_2 j_3 = s_2 s_3 \in \mathcal{J} \cap W(v_1)$ and $j_2 j_3 \notin \mathcal{J}(v_1)$. Thus $\mathcal{S}$ is not embedded. Since $j_1 j_2 j_4 = s_2 s_1 (s_2 s_3)^2$ has order 3, the surface $\mathcal{S}$ is non-orientable.

Consider a Coxeter system of type $I_2(m) \times I_2(k)$. There are three cases: either m and k are both even, one of them is even and the other one is odd, or both are odd. The edge reflection graphs are given in Figure 5.3

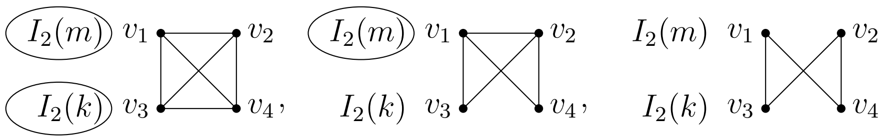


Figure 5.3: The edge reflection graphs of the Coxeter system of type $I_2(m) \times I_2(k)$ where m and k are both even, m even and k odd, and both odd, respectively.

In all cases, the edge cycle $\Gamma_1 = v_1 v_4 v_2 v_3$ is full-dimensional and compatible with $\mathcal{U}$. Furthermore, if m and k are even the edge cycle $\Gamma_2 = v_1 v_2 v_3 v_4$ is also full-dimensional and compatible with $\mathcal{U}$, see Figure 5.4. For Γ_1 the generators for the corresponding edge reflection group $\mathcal{J}_1$ are given by

$$j_1 = s_2 s_3, \qquad j_2 = s_1 s_3, \qquad j_3 = s_1 s_4, \qquad j_4 = s_2 s_4.$$

Consider the vertex v_1. We compare the order of $\mathcal{J}_1(v_1)$ and $\mathcal{J}_1 \cap W(v_1)$. Since $j_1 j_4 = s_3 s_4$ and $(s_3 s_4)^k = e$, the order of $\mathcal{J}_1(v_1)$ is $2k$. The group $W(v_1)$ is of type $A_1 \times I_2(k)$, hence has order $4k$. Note that $\mathcal{J}_1$ consists only of orientation-preserving mappings, i.e., $[W(v_1) \colon \mathcal{J}_1 \cap W(v_1)] \geq 2$. Furthermore, $\mathcal{J}_1(v_1)$ is a subgroup of $\mathcal{J}_1 \cap W(v_1)$ and $|\mathcal{J}_1(v_1)| = 2k$; hence the order of $\mathcal{J}_1 \cap W(v_1)$ is also $2k$. Similarly, one can check the other three vertices. By Theorem 4.2.10, this surface is embedded for all choices m and k. The order of $\mathcal{J}_1$ is $2mk$, since it is an index two subgroup of W which has order $4mk$. Using (5.2), the genus can be computed by

$$g_1 = 1 - \frac{2\frac{2mk}{k} + 2\frac{2mk}{m}}{2} + \frac{(4-2)2mk}{4} = (m-1)(k-1).$$

These surfaces correspond to the surfaces $\xi_{m-1,k-1}$ constructed by Lawson in [Law70].

Let m and k both be even and consider the cycle Γ_2. If $m = k = 2$ then we have a right-angled Coxeter group and Γ_2 yields an embedded surface $\mathcal{S}_2$, by Theorem 5.1.3. Using (5.2), $|\mathcal{J}_2| = 16$, $|\mathcal{J}_2(v_i)| = 4$, and $|\Gamma_2| = 4$, we get that the genus is 1, i.e., we get the Clifford torus. This corresponds to the surface $\tau_{1,1}$ constructed by Lawson [Law70].

Assume that $m > 2$ and $k \geq 2$. The generators for $\mathcal{J}_2$ are given by

$$j_1 = (s_3s_4)^{k/2}, \qquad j_2 = s_1s_3, \qquad j_3 = (s_1s_2)^{m/2}, \qquad j_4 = s_2s_4.$$

The element $j_3j_2 = (s_1s_2)^{m/2}s_1s_3$ fixes v_4 and has order 2: since s_3 commutes with s_1 and s_2 and $(s_1s_2)^{m/2}$ commutes with s_1 and s_2 as it is the longest element, we have $|\mathcal{J}_2(v_4)| = 4$. Note that the only element consisting of two generators of $\mathcal{J}_2$ that has order not equal to two is $j_2j_4 = s_1s_2s_3s_4$. Since j_2j_4 is a Coxeter element, it does not fix any v_i, i.e., every Coxeter element contains every generator s_i. Hence $j_2j_4 \notin W(v_i)$. Furthermore, if $m = k$ then $(j_2j_4)^n \notin W(v_i)$ for all even numbers $n < m = k$. Thus $|\mathcal{J}_2 \cap W(v_2)| = 4$ and the surface $\mathcal{S}_2$ is embedded with genus 1. Assume $k < m$ then $j_1(j_2j_4)^{k/2} = (s_1s_2)^{k/2} \in \mathcal{J}_2 \cap W(v_4)$ which is not contained in

$$\mathcal{J}_2(v_4) = \{e, s_1s_3, (s_1s_2)^{m/2}, (s_1s_2)^{m/2}s_1s_3\}.$$

Hence these surfaces are not embedded.

These surfaces correspond to the surfaces $\tau_{m-1,k-1}$ constructed by Lawson [Law70]. We get that $\tau_{m-1,m-1}$ is embedded for all even $m > 2$. Since the genus of $\tau_{m-1,m-1}$ coincide for all m, they are homeomorphic, i.e., congruent, as stated by Lawson. If $m \neq k$ then $\tau_{m-1,k-1}$ has self-intersections.

For m or k odd, the edge cycle Γ_2 is not compatible with $\mathcal{U}$. Thus we cannot analyse these surfaces with this method. We will review this case later in Chapter 6.

We can summarise the construction in $\mathbb{S}^3$ as follows:

Theorem 5.4.1. *Let P^3 be a Coxeter polytope in $\mathbb{S}^3$ and (W,S) its Coxeter system. Assume that Γ is a compatible full-dimensional edge cycle. Then the surface $\mathcal{S}$ is embedded if and only if (W,S) is of type $I_2(k) \times I_2(m)$ and*

(i) $\Gamma = \Gamma_1 = v_1v_4v_2v_3$ *or*

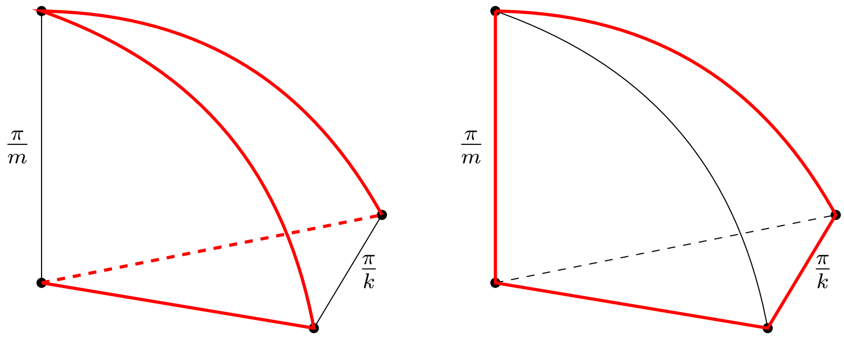


Figure 5.4: The 3-simplex in $\mathbb{S}^3$ corresponding to the Coxeter system of type $I_2(m) \times I_2(k)$. The red edge cycle on the left hand side is Γ_1 and the red edge cycle on the right hand side is Γ_2 of Theorem 5.4.1.

(ii) $m = k$ *even and* $\Gamma = \Gamma_2 = v_1v_2v_3v_4$.

In the case of Γ_1 *the genus of* $\mathcal{S}$ *is* $g_1 = (m-1)(k-1)$*, for* Γ_2 *the genus is* $g_2 = 1$.

Now, let us use GAP to confirm Theorem 5.4.1. First, we need to define the Coxeter group in GAP. We do it as a free group G modulo the relations R. Second, we define for all v_i the subgroups $\mathcal{J}(v_i)$ and $W(v_i)$. Third, we compare the orders of $\mathcal{J} \cap W(v_i)$ and $\mathcal{J}(v_i)$. If the orders coincide for all v_i we compute the genus of the embedded surface $\mathcal{S}$. In the case of F_4 the GAP code can be written as in Listing 5.1.

Listing 5.1: GAP code for group of type F_4. The surface $\mathcal{S}$ is not embedded, since $8 = \mathcal{J}(v_1) \neq W(v_1) = 24$ contradicts (4.1).

```
#Define a free group with 4 generators.
G:=FreeGroup("1","2","3","4");;
#Define a set with relations.
RF4:=[G.1^2,G.2^2,G.3^2,G.4^2,
        (G.1*G.2)^3,(G.2*G.3)^4,(G.3*G.4)^3,
        (G.1*G.3)^2,(G.1*G.4)^2,(G.2*G.4)^2];;
#Define the Coxeter group as the quotient space G/R.
W:=G/RF4;
#Define the subgroups W(v_i).
Wv1:=Subgroup(W, [W.2,W.3,W.4]);;
Wv2:=Subgroup(W, [W.1,W.3,W.4]);;
Wv3:=Subgroup(W, [W.1,W.2,W.4]);;
```

```
Wv4:=Subgroup(W, [W.1,W.2,W.3]);;
#Define the edge reflections j_i and the group 𝒥.
J1:=W.2*W.4;;
J2:=W.1*W.4;;
J3:=W.1*W.3;;
J4:=(W.2*W.3)^2;;
J:=Subgroup(W,[J1,J2,J3,J4]);;
#Determine the order of the subgroups 𝒥(v_i).
Jv1:=Order(Subgroup(J,[J.1,J.4]));
Jv2:=Order(Subgroup(J,[J.2,J.3]));
Jv3:=Order(Subgroup(J,[J.1,J.2]));
Jv4:=Order(Subgroup(J,[J.3,J.4]));
#Determine the order of the intersection 𝒥 ∩ W(v_i).
W1:=Order(Intersection(Wv1,J));
W2:=Order(Intersection(Wv2,J));
W3:=Order(Intersection(Wv3,J));
W4:=Order(Intersection(Wv4,J));
#The result is Jv1 = Jv4 = 8, Jv2 = Jv3 =6, W1 = W4 = 24,
# and W2 = W3 = 6.
#Since 𝒥(v_1) ≠ W(v_1), we have that 𝒮 is not embedded.
```

Consider a Coxeter group of type $I_2(m) \times I_2(k)$. A similar GAP code can be written in this case, see Listing 5.2.

Listing 5.2: GAP code for group of type $I_2(m) \times I_2(k)$. The surface $\mathcal{S}$ is embedded, since $\mathcal{J}(v_i) = W(v_i)$ for all i.

```
#Define a free group, the variables m,k, and the relations.
G:=FreeGroup("1","2");;
m:=10;;
k:=10;;
Rm:=[G.1^2,G.2^2,(G.1*G.2)^m];;
Rk:=[G.1^2,G.2^2,(G.1*G.2)^k];;
#Define the Coxeter group as a direct product.
I2m:=G/Rm;;
I2k:=G/Rk;;
W:=DirectProduct(I2m,I2k);;
#Define the subgroups W(v_i).
Wv1:=Subgroup(W, [W.2,W.3,W.4]);;
Wv2:=Subgroup(W, [W.1,W.3,W.4]);;
Wv3:=Subgroup(W, [W.1,W.2,W.4]);;
```

```
Wv4:=Subgroup(W, [W.1,W.2,W.3]);;
#Define the edge reflections j_i and the group J_2.
j1:=(W.3*W.4)^(k/2);;
j2:=W.1*W.3;;
j3:=(W.1*W.2)^(m/2);;
j4:=W.2*W.4;;
J2:=Subgroup(W,[j1,j2,j3,j4]);;
#Determine the order of the subgroups J(v_i)
Jv1:=Order(Subgroup(J2,[J2.1,J2.4]));
Jv2:=Order(Subgroup(J2,[J2.1,J2.2]));
Jv3:=Order(Subgroup(J2,[J2.3,J2.4]));
Jv4:=Order(Subgroup(J2,[J2.2,J2.3]));
#Determine the order of the intersection J ∩ W(v_i).
W1:=Order(Intersection(Wv1,J2));
W2:=Order(Intersection(Wv2,J2));
W3:=Order(Intersection(Wv3,J2));
W4:=Order(Intersection(Wv4,J2));
#The result is J(v_i) = W(v_i) = 4 for all i.
#Hence the surface S is embedded.
```

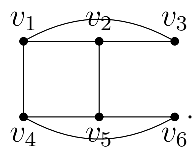
In the appendix, we have included all spherical and Euclidean, irreducible components up to rank 5. For the hyperbolic case, we have included Coxeter groups of type Z_3 and Z_4 (defined below). Furthermore, up to symmetry, all full-dimensional compatible edge cycles for dimension 3 and 4 which lead to an embedded surface $\mathcal{S}$ are listed in the appendix as well.

Surfaces in $\mathbb{S}^2 \times \mathbb{E}$

Let σ^2 be a Coxeter polytope in $\mathbb{S}^2$, i.e., a triangle, and $I = [a, b]$ an interval in $\mathbb{E}$. Then $P^3 := \sigma^2 \times I$ is a Coxeter polytope in $\mathbb{S}^2 \times \mathbb{E}$. The corresponding Coxeter systems are one of the following types: $A_3 \times \tilde{A}_1$, $B_3 \times \tilde{A}_1$, $H_3 \times \tilde{A}_1$, or $I_2(m) \times A_1 \times \tilde{A}_1$. Inspecting the edge reflection graphs of these groups, the groups of type $A_3 \times \tilde{A}_1$ and $H_3 \times \tilde{A}_1$ do not permit a compatible edge cycle.

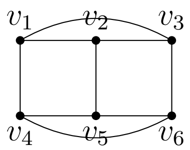
As the operations in the GAP code in Listing 5.1 and Listing 5.2 are mainly based on coset enumeration, we get a problem in defining subgroups with infinite index. Fortunately, the groups $W(v_i)$ and $\mathcal{J}(v_i)$ are both finite. More precisely: since P^n is simple, n facets are meeting at each vertex v_i, hence $W(v_i)$ is a spherical subgroup of rank n and $\mathcal{J}(v_i)$ is a finite dihedral group. Thus for every vertex v_i, we can compute the elements of these two subgroups exactly. All that remains is to check if there is an element in $W(v_i)\backslash\mathcal{J}(v_i)$ contained in $\mathcal{J}$.

Consider a Coxeter group of type $B_3 \times \tilde{A}_1$. The edge reflection graph is given by

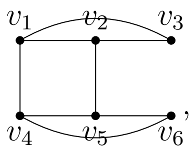


Up to symmetry, there are two full-dimensional compatible edge cycles: $\Gamma_1 = v_1v_2v_5v_6v_4$ and $\Gamma_2 = v_1v_3v_2v_5v_6v_4$. The GAP code in Listing 12 shows that $s_2s_3 \in W(v_4) = \langle s_2, s_3, s_5\rangle$ lies in $\mathcal{J}_i$, $i = 1, 2$, but $s_2s_3 \notin \mathcal{J}_i(v_4) = \{e, (s_2s_3)^2, s_3s_4, s_2s_3s_2s_4\}$. Hence both surfaces $\mathcal{S}_i$ have a self-intersection, i.e., they are not embedded.

Consider a Coxeter group of type $I_2(m) \times A_1 \times \tilde{A}_1$. For m even and odd, the edge reflection graphs are given by



and



respectively. If m is odd then there are two full-dimensional compatible curves $\Gamma_1 = v_1v_4v_6v_5v_2$ and $\Gamma_2 = v_1v_4v_6v_5v_2v_3$. Thus

$$\begin{aligned} \mathcal{J}_1 &= \langle s_2s_3, s_2s_5, s_1s_5, s_1s_3, s_3s_4\rangle \qquad \text{and} \\ \mathcal{J}_2 &= \langle s_2s_3, s_2s_5, s_1s_5, s_1s_3, s_1s_4, s_2s_4\rangle. \end{aligned}$$

We get for $i = 1, 2$ that

$$\begin{aligned} &|\mathcal{J}_i(v_1)| = |\mathcal{J}_i(v_2)| = |\mathcal{J}_i(v_4)| = |\mathcal{J}_i(v_5)| = 4 \qquad \text{and} \\ &|\mathcal{J}_i(v_6)| = |\mathcal{J}_2(v_3)| = 2m. \end{aligned}$$

Furthermore,

$$\begin{aligned} &|W(v_1)| = |W(v_4)| = |W(v_2)| = |W(v_5)| = 8 \qquad \text{and} \\ &|W(v_3)| = |W(v_6)| = 4m. \end{aligned}$$

Note that $[W\colon \mathcal{J}_i] \geq 2$, hence $|\mathcal{J}_i \cap W(v_i)| = |\mathcal{J}_i(v_i)|$ for all i. Thus both $\mathcal{S}_1$ and $\mathcal{S}_2$ are embedded.

Let m be even. With the same argument as above, we get that Γ_1 and Γ_2 yield two embedded surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. But we have two additional full-dimensional edge cycles $\Gamma_3 = v_1v_4v_6v_3v_2$ and $\Gamma_4 = v_1v_4v_5v_6v_3v_2$ resulting in

$$\begin{aligned}\mathcal{J}_3 &= \langle s_2s_3, s_2s_5, (s_1s_2)^{m/2}, s_1s_4, s_3s_4\rangle \qquad \text{and}\\ \mathcal{J}_4 &= \langle s_2s_3, s_3s_5, s_1s_5, (s_1s_2)^{m/2}, s_1s_4, s_3s_4\rangle.\end{aligned}$$

If $m = 2$, we have that Γ_1 is symmetric to Γ_3 and Γ_2 is symmetric to Γ_4; hence assume $m > 2$. Note that

$$\begin{aligned}\mathcal{J}_i(v_6) &= \{e, (s_1s_2)^{m/2}, s_1s_4, s_2(s_1s_2)^{m/2-1}s_4\} \qquad \text{and}\\ W(v_6) &= \langle s_1, s_2, s_4\rangle.\end{aligned}$$

As $j_1j_2j_3 = s_2s_1 \in \mathcal{J}_i$, we get that $|\mathcal{J}_i \cap W(v_6)| > |\mathcal{J}_i(v_6)|$. Thus $\mathcal{S}_3$ and $\mathcal{S}_4$ are not embedded.

Since $\mathcal{J}_i$, $i = 1, 2$, is of infinite order, the genus of $\mathcal{S}_i$ cannot be calculated via (5.2). But we can calculate the genus on a compact domain of $\mathbb{S}^2 \times \mathbb{R}$. The group of translations in the $\mathbb{R}$-factor is obviously generated by the element s_4s_5. Since $s_4s_5 \in \mathcal{J}_i$, we get that $\mathcal{T} = \langle s_4s_5\rangle$. Thus we get that $\mathcal{U}/\mathcal{T} \cong \mathbb{S}^2 \times 2I$. Note that $j_5j_4j_2 = s_4s_5$ in $\mathcal{J}_1$ and $j_5j_3 = s_4s_5$ in $\mathcal{J}_2$. Since we used an odd number of generators j_i in $\mathcal{J}_1$, we see that $\mathcal{S}_1/\mathcal{T}$ is non-orientable. On the other hand, we have used an even number of generators j_i in $\mathcal{J}_2$; hence $\mathcal{S}_2/\mathcal{T}$ is orientable.

We have that $\mathcal{U}/\mathcal{T}$ consists of $8m$ copies of P^3 and $\mathcal{S}_i$ is contained in $4m$ copies, i.e., $[\mathcal{J}\colon \mathcal{T}] = 4m$. Using (5.3) and (5.4), we get for Γ_1 and Γ_2 that

$$\begin{aligned}\chi_{\mathcal{T}^+}(\mathcal{S}_1) &= 2\cdot 4m\left(1 - \frac{5}{2} + \frac{4}{4} + \frac{1}{2m}\right) = -4m + 4 \qquad \text{and}\\ \chi_{\mathcal{T}}(\mathcal{S}_2) &= 4m\left(1 - \frac{6}{2} + \frac{4}{4} + \frac{2}{2m}\right) = -4m + 4\end{aligned}$$

respectively. Hence the genus of $\mathcal{S}_1$ and $\mathcal{S}_2$ is given by

$$\begin{aligned}g_1 &= 1 - \frac{\chi_{\mathcal{T}^+}(\mathcal{S}_1)}{2} = 2m - 1 \qquad \text{and}\\ g_2 &= 1 - \frac{\chi_{\mathcal{T}}(\mathcal{S}_2)}{2} = 2m - 1,\end{aligned}$$

respectively.

Rosenberg used these curves to construct minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ in [Ros02]. Furthermore, Manzano and Plehnert used these curves to construct minimal surface in $\mathbb{S}^2 \times \mathbb{S}^1$ in [MPT16]. Our construction, can also be done in $\mathbb{S}^2 \times \mathbb{S}^1$. Note that in this case it is possible that constructed minimal surface is embedded and non-orientable.

By combining the above, we get:

Theorem 5.4.2. *Let $P^3 = \sigma^2 \times I$ be a Coxeter polytope in $\mathbb{S}^2 \times \mathbb{E}$ where σ is a Coxeter polytope in $\mathbb{S}^2$, i.e., a triangle, and I an interval. Let (W, S) be the Coxeter system corresponding to P^3. Assume that Γ is a compatible full-dimensional edge cycle. Then the surface $\mathcal{S}$ is embedded if and only if W is of type $I_2(m) \times A_1 \times \tilde{A}_1$ and*

(i) $\Gamma = \Gamma_1 = v_1 v_4 v_6 v_5 v_3$ *or*

(ii) $\Gamma = \Gamma_2 = v_1 v_4 v_6 v_5 v_2 v_3$.

On $\mathcal{U}/\mathcal{T}^+ \cong \mathbb{S}^2 \times 4I$ the genus for Γ_1 is $g_1 = 2m - 1$ and on $\mathcal{U}/\mathcal{T} \cong \mathbb{S}^2 \times 2I$ the genus for Γ_2 is $g_2 = 4m - 3$.

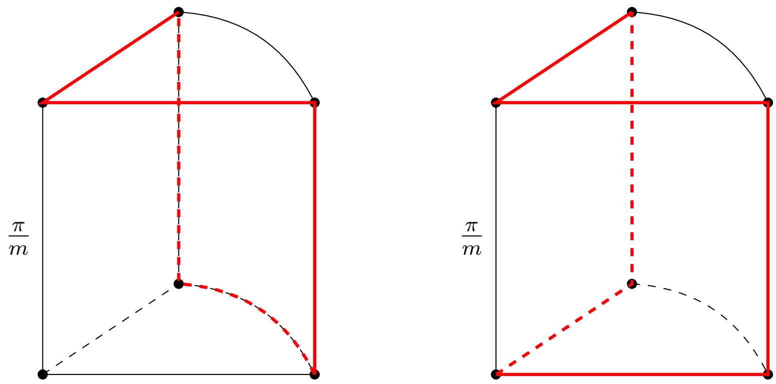


Figure 5.5: The Coxeter polytope in $\mathbb{S}^2 \times \mathbb{E}$ corresponding to the Coxeter system of type $I_2(m) \times A_1 \times \tilde{A}_1$. The red edge cycle on the left hand side is Γ_1 and the red edge cycle on the right hand side is Γ_2 of Theorem 5.4.2.

Surfaces in $\mathbb{E}^3$

In the Euclidean case, a Coxeter polytope P^3 is either an n-simplex or a product of lower dimensional simplices. Thus the corresponding Coxeter system is one of the following types: $\tilde{A}_3$, $\tilde{B}_3$, $\tilde{C}_3$, $\tilde{A}_2 \times \tilde{A}_1$, $\tilde{B}_2 \times \tilde{A}_1$, $\tilde{G}_2 \times \tilde{A}_1$, or $(\tilde{A}_1)^3$ where the first three types correspond to 3-simplices, the next three to prisms, and $(\tilde{A}_1)^3$ to the cube or cuboid.

Since $(\tilde{A}_1)^3$ is right-angled, by Theorem 5.1.3, we get that all three full-dimensional compatible edge cycles lead to embedded surfaces. These correspond to the Schwarz D surface, the CLP surface, and the surface related to the Gergonne problem (see Figure 5.6). The edge reflection groups are given by

$$\begin{aligned} \mathcal{J}_D &= \langle s_1s_3, s_1s_6, s_4s_6, s_2s_4, s_2s_5, s_3s_5 \rangle, \\ \mathcal{J}_{CLP} &= \langle s_1s_3, s_1s_6, s_4s_6, s_2s_4, s_4s_5, s_1s_5 \rangle, \qquad \text{and} \\ \mathcal{J}_{GP} &= \langle s_1s_3, s_1s_6, s_4s_6, s_2s_6, s_2s_3, s_2s_5, s_4s_5, s_1s_5 \rangle. \end{aligned}$$

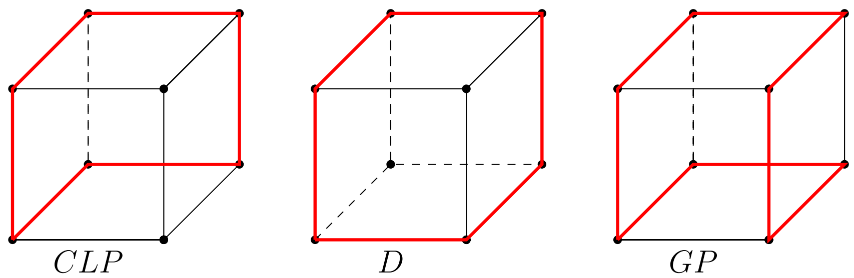


Figure 5.6: The three edge cycles for the group of type $(\tilde{A}_1)^3$.

A glance at the edge reflection graphs yield that the groups of type $\tilde{A}_3$, $\tilde{B}_3$, and $\tilde{A}_2 \times \tilde{A}_1$ do not permit a full-dimensional compatible edge cycle.

We start with the 3-simplex corresponding to the group of type $\tilde{C}_3$. The only edge cycle is $\Gamma = v_1v_3v_4v_2$. Hence

$$\begin{aligned} \mathcal{J} &= \langle s_2s_4, (s_1s_2)^2, s_1s_3, (s_3s_4)^2 \rangle, \\ \mathcal{J}(v_3) &= \{e, s_2s_4, (s_1s_2)^2, s_1s_2s_1s_4\}, \qquad \text{and} \\ W(v_3) &= \langle s_1, s_2, s_4 \rangle. \end{aligned}$$

Furthermore, Listing 13 shows that $s_1s_2 \in \mathcal{J}$. Thus $\mathcal{S}$ is not embedded as $|\mathcal{J} \cap W(v_3)| > |\mathcal{J}(v_3)|$.

Consider a Coxeter group of type $\tilde{B}_2 \times \tilde{A}_1$. There are four possible edge cycles:

$$\begin{aligned}
\Gamma_1 &= v_1v_4v_6v_3v_2, \\
\Gamma_2 &= v_1v_4v_5v_6v_3v_2, \\
\Gamma_3 &= v_1v_4v_5v_2v_3, \qquad \text{and} \\
\Gamma_4 &= v_1v_4v_6v_5v_2v_3.
\end{aligned}$$

Respectively, we get

$$\begin{aligned}
\mathcal{J}_1 &= \langle (s_2s_3)^2, s_2s_5, (s_1s_2)^2, s_1s_4, s_3s_4 \rangle, \\
\mathcal{J}_2 &= \langle (s_2s_3)^2, s_3s_5, s_1s_5, (s_1s_2)^2, s_1s_4, s_3s_4 \rangle, \\
\mathcal{J}_3 &= \langle (s_2s_3)^2, s_3s_5, s_1s_3, s_1s_4, s_2s_4 \rangle, \qquad \text{and} \\
\mathcal{J}_4 &= \langle (s_2s_3)^2, s_2s_5, s_1s_5, s_1s_3, s_1s_4, s_2s_4 \rangle.
\end{aligned}$$

For $\mathcal{J}_3$ we have $j_3j_4j_5 = s_2s_3$ and for $\mathcal{J}_4$ we have $j_4j_5j_6 = s_2s_3$. Thus $s_2s_3 \in \mathcal{J}_3, \mathcal{J}_4$ and obviously $s_2s_3 \in W(v_1) = \langle s_2, s_3, s_4 \rangle$. However $s_2s_3 \notin \mathcal{J}_3(v_1) = \mathcal{J}_4(v_1) = \{e, (s_2s_3)^2, s_2s_4, s_3s_2s_3s_4\}$. So $\mathcal{S}_3$ and $\mathcal{S}_4$ are not embedded.

For $\mathcal{J}_1$ and $\mathcal{J}_2$, Listing 14 shows that there is no element in $W(v_i)\backslash\mathcal{J}(v_i)$ in $\mathcal{J}_1$ and $\mathcal{J}_2$ for all i. Note that $\mathcal{J}_1$ and $\mathcal{J}_2$ cannot contain a word of odd length since all generators have even length.

Eventually, consider the group of type $\tilde{G}_2 \times \tilde{A}_1$. Up to symmetry, there are two full-dimensional compatible edge cycles $\Gamma_1 = v_1v_4v_6v_5v_2$ and $\Gamma_2 = v_1v_4v_6v_5v_2v_3$. We obtain

$$\begin{aligned}
\mathcal{J}_1 &= \langle (s_2s_3)^3, s_2s_5, s_1s_5, s_1s_3, s_3s_4 \rangle \qquad \text{and} \\
\mathcal{J}_2 &= \langle (s_2s_3)^3, s_2s_5, s_1s_5, s_1s_3, s_1s_4, s_2s_4 \rangle
\end{aligned}$$

respectively.

It is easy to see that $j_3j_4j_2 = s_2s_3 \in W(v_1) \cap \mathcal{J}_i$, but $s_2s_3 \notin \mathcal{J}_i(v_1) = \{e, (s_2s_3)^3, s_2s_5, (s_3s_2)^2s_3s_5\}$, see also Listing 15. Hence $\mathcal{S}_1$ and $\mathcal{S}_2$ have self-intersections.

For the genus, we start with the group of type $(\tilde{A}_1)^3$. The translation group T is generated by the elements s_1s_2, s_3s_4, and s_5s_6. It is obviously an index 8 subgroup of W. It is easy to see that in all 3 cases $T \subset \mathcal{J}$ and that at least one generator is given by an odd number of generators j_i, e.g., $j_6j_4j_5 = s_3s_4$ in $\mathcal{J}_D$. Hence $\mathcal{S}/\mathcal{T}$ is non-orientable and we use the orientable double cover $\mathcal{S}/\mathcal{T}^+$. Note that $\mathcal{U}/\mathcal{T}^+ \cong 4I \times 2I \times 2I$ where $I = [a, b]$ are intervals corresponding to the reflections s_1s_2, s_3s_4, and s_5s_6. Thus, $\mathcal{U}/\mathcal{T}^+$ consists of 16 cuboid copies and $\mathcal{S}$ is contained in 8 of them, i.e., $[\mathcal{J} \colon \mathcal{T}^+] = 8$. As $|\Gamma| = 6$ for the Schwarz D surface and the CLP surface, and $|\Gamma| = 8$ for the surface regarding the Geronne problem, we get

$$\begin{aligned} g_D = g_{CLP} &= 1 - \frac{8}{2}\left(1 - \frac{6}{2} + \frac{6}{4}\right) = 3 \qquad \text{and} \\ g_{GP} &= 1 - \frac{8}{2}\left(1 - \frac{8}{2} + \frac{8}{4}\right) = 5. \end{aligned}$$

In case W is of type $\tilde{B}_2 \times \tilde{A}_1$, note that s_1 and $s_2s_3s_2$ (similarly, s_3 and $s_2s_1s_2$) correspond to the reflection along two parallel facets of P^3. Hence we get

$$T = \langle s_1s_2s_3s_2, s_3s_2s_1s_2, s_4s_5 \rangle.$$

A quick GAP check shows that s_4s_5, $s_3s_2s_1s_2s_4s_5$, $s_1s_2s_3s_2s_4s_5 \notin \mathcal{J}_1$, $(s_4s_5)^2 \in \mathcal{J}$, and $T \subset \mathcal{J}_2$. Hence, we have

$$\mathcal{T}_1 = \langle s_1s_2s_3s_2, s_3s_2s_1s_2, (s_4s_5)^2 \rangle \qquad \text{and} \qquad \mathcal{T}_2 = T.$$

Furthermore, we see that $j_5j_6j_1 = s_1s_2s_3s_2$ in $\mathcal{J}_1$ and $\mathcal{J}_2$. Thus in both cases, we use the orientable double cover. We get

$$[\mathcal{J}_1 \colon \mathcal{T}_1^+] = 16 \qquad \text{and} \qquad [\mathcal{J}_2 \colon \mathcal{T}_2^+] = 8.$$

Thus the genus for Γ_1 and Γ_2 is given by

$$\begin{aligned} g_1 &= 1 - \frac{16}{2}\left(1 - \frac{5}{2} + \frac{5}{4}\right) = 3 \qquad \text{and} \\ g_2 &= 1 - \frac{8}{2}\left(1 - \frac{6}{2} + \frac{6}{4}\right) = 3. \end{aligned}$$

Note that the prism corresponding to the Coxeter group of type $\tilde{B}_2 \times \tilde{A}_1$ is just a half-cuboid. The edge cycle Γ_1 is the same edge cycle (after reflection along the edge v_4v_6) as the one used to construct the Schwarz D surface. The edge cycle Γ_2 is the same edge cycle as the one used to construct the CLP surface. Hence we naturally have $g_1 = g_D$ and $g_2 = g_{CLP}$.

By combining the above, we get following theorem.

Theorem 5.4.3. *Let P^3 be a Coxeter polytope in $\mathbb{E}^3$ and (W, S) its Coxeter system. Assume that Γ is a compatible full-dimensional edge cycle. Then the surface $\mathcal{S}$ is embedded if and only if*

(i) *W is of type $\tilde{B}_2 \times \tilde{A}_1$ and $\Gamma = \Gamma_1 = v_1 v_4 v_6 v_3 v_2$ or $\Gamma_2 = v_1 v_4 v_5 v_6 v_3 v_2$, or*

(ii) *W is of type $(\tilde{A}_1)^3$ and $\Gamma = \Gamma_D = v_1 v_2 v_4 v_8 v_7 v_5$, $\Gamma_{CLP} = v_1 v_2 v_4 v_3 v_7 v_5$, or $\Gamma_{GP} = v_1 v_2 v_4 v_8 v_6 v_5 v_7 v_3$.*

The genera are respectively given by

(i) *$g_1 = 3$ on $\mathcal{U}/\mathcal{T}_1^+ \cong 4I \times 2I \times 4I$, $g_2 = 3$ on $\mathcal{U}/\mathcal{T}_2^+ \cong 4I \times 2I \times 2I$,*

(ii) *$g_D = g_{CLP} = 3$, and $g_{GP} = 5$ on $\mathcal{U}/\mathcal{T}^+ \cong 4I \times 2I \times 2I$.*

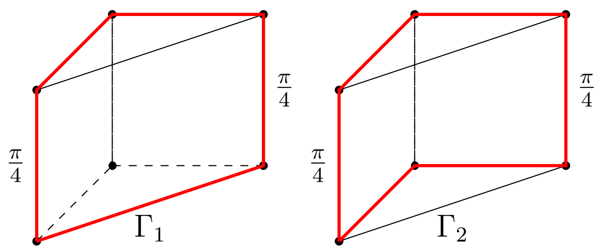
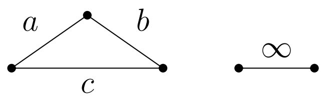


Figure 5.7: The two edge cycles for the group of type $\tilde{B}_2 \times \tilde{A}_1$.

Surfaces in $\mathbb{H}^2 \times \mathbb{E}$

As hyperbolic Coxeter groups are not classified, we will just look at the simplicial hyperbolic Coxeter groups (see Figure 2.8). In the next chapter, we will use unions of simplices to extend our fundamental domain from an n-simplex to a simple polytope.

Let σ^2 be a Coxeter 2-simplex in $\mathbb{H}^2$, i.e., a triangle, and I an interval in $\mathbb{E}$. Then $P^3 := \sigma^2 \times I$ is a Coxeter polytope in $\mathbb{H}^2 \times \mathbb{E}$. There is just a single three parameter family of Coxeter groups with Coxeter graph



where $2 \leq a \leq b \leq c$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$. In the following, we will say that the hyperbolic triangle Coxeter system is of type $Z_3(a, b, c)$. Hence the above Coxeter system is of type $Z_3(a, b, c) \times \tilde{A}_1$.

Since a, b, and c are interchangeable, we may assume that b and c are even, otherwise there are no edge cycles in the edge reflection graph. Similar to the case in $\mathbb{S}^2 \times \mathbb{E}$, we have by symmetry two edge cycles $\Gamma_1 = v_1v_4v_6v_3v_2$ and $\Gamma_2 = v_1v_4v_5v_6v_3v_2$ yielding the edge reflection groups

$$\mathcal{J}_1 = \langle (s_2s_3)^{b/2}, s_2s_5, (s_1s_2)^{c/2}, s_1s_4, s_3s_4 \rangle \qquad \text{and}$$
$$\mathcal{J}_2 = \langle (s_2s_3)^{b/2}, s_3s_5, s_1s_5, (s_1s_2)^{c/2}, s_1s_4, s_3s_4 \rangle.$$

Furthermore, a quick check shows that

$$\begin{aligned}
&|\mathcal{J}_i(v_1)| = |\mathcal{J}_i(v_4)| = |\mathcal{J}_i(v_6)| = |\mathcal{J}_i(v_3)| = 4,\\
&|\mathcal{J}_i(v_2)| = |\mathcal{J}_2(v_5)| = 2a,\\
&|W(v_1)| = |W(v_4)| = 4b,\\
&|W(v_3)| = |W(v_6)| = 4c, \qquad \text{and}\\
&|W(v_2)| = |W(v_5)| = 4a.
\end{aligned}$$

Note that $\mathcal{J}_i$ is generated by elements of even order, hence $\mathcal{J}_i \cap W(v_2)$ and $\mathcal{J}_i \cap W(v_5)$ consist of at most $2a$ elements. Thus the embeddedness of $\mathcal{S}_i$ does not depend on a.

First, consider the group $\mathcal{J}_1$. The edge cycle Γ has two edges in $\mathbb{R}$-direction, i.e., the edge reflection about these edges corresponds to the elements $(s_1s_2)^{c/2}$ and $(s_2s_3)^{b/2}$. We prove that those two edges are self-intersections of $\mathcal{S}_1$ if $b \neq c$. W.l.o.g. let $b < c$. We get

$$j_3(j_4j_2)^{c/2} = (s_1s_2)^c(s_4s_5)^{c/2} = (s_4s_5)^{c/2} \qquad \text{and}$$
$$j_1(j_2j_5)^{b/2} = (s_3s_2)^b(s_4s_5)^{b/2} = (s_4s_5)^{b/2}.$$

Thus $(s_4s_5)^{c/2}$, $(s_4s_5)^{b/2} \in \mathcal{J}_1$. As $b < c$, we get

$$(j_4j_2)^{b/2}(s_5s_4)^{b/2} = (s_1s_2)^{b/2} \in \mathcal{J}_1.$$

Hence $(s_1s_2)^{b/2} \in W(v_4) = \langle s_1, s_2, s_4 \rangle$, but $(s_1s_2)^{b/2}$ is not contained in

$$\mathcal{J}_1(v_4) = \{e, s_2s_5, (s_1s_2)^{c/2}, (s_1s_2)^{c/2-1}s_1s_5\}.$$

Thus $\mathcal{S}_1$ is not embedded. If $b = c$, one needs to check (4.1) of Theorem 4.2.10 to conclude that $\mathcal{S}_1$ is embedded. For fixed number a and b this can be done in GAP (see Listing 16). We have checked this for all choices $a, b \leq 12$, a, b even. For all those choices the surface $\mathcal{S}$ is embedded. We assume that the surface is always embedded provided a, c, b even and $b = c$. But for the general case a less specific argument is needed.

Second, consider the case $\mathcal{J}_2$. Here the problem of embeddedness is more involved. We have

$$\mathcal{J}_2(v_1) = \{e, (s_2s_3)^{b/2}, s_3s_4, (s_2s_3)^{b/2-1}s_2s_4\} \qquad \text{and}$$
$$\mathcal{J}_2(v_6) = \{e, (s_1s_2)^{c/2}, s_1s_4, (s_2s_1)^{c/2-1}s_2s_4\}.$$

If $b = 2$, we see that $s_2s_4 \in \mathcal{J}_2$. The case $c = 2$ is impossible, otherwise we would have a spherical polygon with angles $\pi/2$, $\pi/2$, and π/a. Since $c > 2$ s_2s_4 is not contained in $\mathcal{J}_2(v_6)$. Thus $\mathcal{S}_2$ is not embedded. Assume that $b = c = 4$, a quick GAP check shows that $\mathcal{S}_2$ is embedded.

Remark 5.4.4. Unfortunately, for other choices of b and c, and in general for the hyperbolic products, GAP runs into a coset enumeration problem. A coset enumeration will not finish if the subgroup does not have finite index, and even if it has, it may take many more intermediate cosets than the actual index of the subgroup. To avoid this problem, GAP has a built in 'stop'. If the number of cosets has reached the default value 4096000 GAP will issue an error message.

In general it is easier to show that a given edge cycle yields a surface with self-intersections; hence we will concentrate to exclude these edge cycles.

Eventually, let us continue with the genus. Let $\operatorname{lcm}(a, b, c)$ denote the lowest common multiple of a, b, and c. Using (5.7) for the $\mathbb{H}^2$-product and the generator $(s_4s_5)^{b/2}$ for the $\mathbb{E}$-product, we get a torsion-free subgroup $\mathcal{T}_1$ of W such that $[\mathcal{J}_1 \colon \mathcal{T}_1^+] \leq 8b \operatorname{lcm}(a, b, c)$. Furthermore, as $s_4s_5 \in \mathcal{J}_2$, we have $[\mathcal{J}_2 \colon \mathcal{T}_2] \leq 16 \operatorname{lcm}(a, b, c)$. Note that these are just upper bounds to guarantee that $\mathcal{S}_1/\mathcal{T}_1^+$ and $\mathcal{S}_2/\mathcal{T}_2$ are manifolds. There might be a smaller fundamental domain. By considering orientation, we get that the genera of $\mathcal{S}_1/\mathcal{T}_1^+$ and $\mathcal{S}_2/\mathcal{T}_2$ are respectively given by

$$g_1 = 1 + 2b \cdot \operatorname{lcm}(a, b) \left(1 - \frac{1}{a}\right) \qquad \text{and}$$
$$g_2 = 1 + 8 \cdot \operatorname{lcm}(a, b, c) \left(1 - \frac{1}{a}\right).$$

These curves were used by Rosenberg in [Ros02] and [MRR11].

We obtain the following theorem:

Theorem 5.4.5. *Let $P^3 = \sigma^2 \times I$ be a polytope in $\mathbb{H}^2 \times \mathbb{E}$ where σ^2 is a Coxeter 2-simplex in $\mathbb{H}^2$ and I an interval. Let $Z_3(a,b,c) \times \tilde{A}_1$ be the corresponding Coxeter system of P^3. Assume that Γ is a compatible full-dimensional edge cycle. If $\mathcal{S}$ is embedded then*

(i) $\Gamma = \Gamma_1 = v_1 v_4 v_6 v_3 v_2$ *and* $b = c$ *or*

(ii) $\Gamma = \Gamma_2 = v_1 v_4 v_5 v_6 v_3 v_2$.

If $\Gamma = \Gamma_1$ and $b = c$, then $\mathcal{S}$ is embedded for all choices $a, b \leq 12$. If $\Gamma = \Gamma_2$ and $a = b = c = 4$, then $\mathcal{S}$ is embedded.

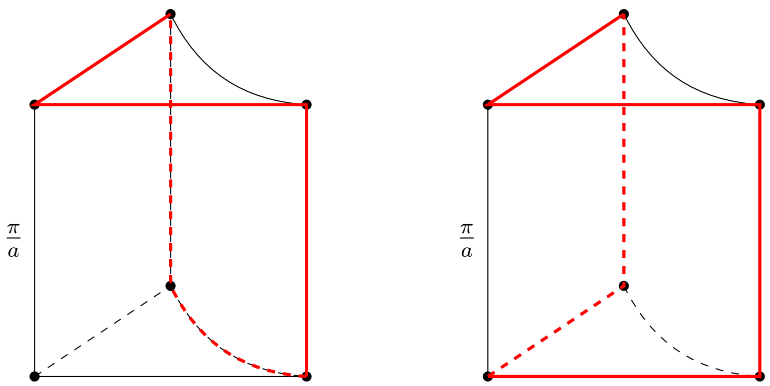


Figure 5.8: The Coxeter polytope in $\mathbb{H}^2 \times \mathbb{E}$ corresponding to the Coxeter system of type $Z_3 \times A_1 \times \tilde{A}_1$. The red edge cycle on the left hand side is Γ_1 and the red edge cycle on the right hand side is Γ_2 of Theorem 5.4.5.

Conjecture 5.4.6. *Let $P^3 = \sigma^2 \times I$ be a polytope in $\mathbb{H}^2 \times \mathbb{E}$ where σ^2 is a Coxeter 2-simplex in $\mathbb{H}^2$ and I an interval. Let $Z_3(a,b,c) \times \tilde{A}_1$ be the corresponding Coxeter system of P^3. Assume that Γ is a compatible full-dimensional edge cycle. If $\Gamma = \Gamma_1 = v_1 v_4 v_6 v_3 v_2$ and $b = c$ then $\mathcal{S}$ is embedded.*

Surfaces in $\mathbb{H}^3$

As in the case $\mathbb{H}^2 \times \mathbb{E}$, the hyperbolic Coxeter groups are not classified. Hence we will only study simplicial hyperbolic Coxeter groups.

Let P^3 be a 3-simplex in $\mathbb{H}^3$. Inspecting the edge reflection graphs in Figure 4.14, the Coxeter graph of the only Coxeter system (W, S) allowing a full-dimensional edge cycle is given in Figure 5.9. In the following, we will say that this Coxeter system is of type Z_4.

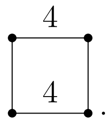


Figure 5.9: The Coxeter graph of a Coxeter group of type Z_4.

The edge cycle $\Gamma = v_1v_2v_3v_4$ yields the edge reflection group

$$\mathcal{J} = \langle (s_3s_4)^2, s_1s_4, (s_1s_2)^2, s_2s_3 \rangle.$$

By symmetry, we just need to check one vertex, e.g., the vertex v_3. We have

$$\begin{aligned} \mathcal{J}(v_3) &= \langle (s_1s_2)^2, s_1s_4 \rangle \qquad \text{and} \\ W(v_3) &= \langle s_1, s_2, s_4 \rangle \end{aligned}$$

where $\mathcal{J}(v_3)$ is a dihedral group of order 8 and $W(v_3)$ is of order 48. Since $\mathcal{J}$ is generated by elements of even order, $\mathcal{J} \cap W(v_3)$ contains at most 24 elements. As $\mathcal{J}(v_3) \leq W(v_3)$, we need to check whether the remaining 16 elements are contained in $\mathcal{J}$. Listing 17 shows that none of these 16 elements are in $\mathcal{J}(v_3)$. Hence $\mathcal{S}$ is embedded.

By Corollary 3.1.18, W contains a torsion-free subgroup with finite index. Let $\mathcal{T}^+$ be the orientation-preserving, torsion-free subgroup of W with finite index. Then we have

$$\chi_{\mathcal{T}^+}(\mathcal{S}) = [\mathcal{J} : \mathcal{T}^+] \left(1 - \frac{4}{2} + \frac{4}{8}\right) = -\frac{[\mathcal{J} : \mathcal{T}^+]}{2}.$$

Then the genus of $\mathcal{S}/\mathcal{T}^+$ is given by

$$g = 1 + \frac{[\mathcal{J} : \mathcal{T}^+]}{4}.$$

Hence we get following theorem:

Theorem 5.4.7. *Let P^3 be a 3-simplex in $\mathbb{H}^3$ and (W, S) its corresponding Coxeter system. Let Γ be a full-dimensional edge cycle on P^3. Then $\mathcal{S}$ is embedded if and only if W is of type Z_4 and $\Gamma = v_1v_2v_3v_4$. Let $\mathcal{T}^+$ be the orientation-preserving, torsion-free subgroup of W with finite index. Then the genus of $\mathcal{S}/\mathcal{T}^+$ is given by $g = 1 + \frac{[\mathcal{J} : \mathcal{T}^+]}{4}$ where $[\mathcal{J} : \mathcal{T}^+]$ needs to be determined.*

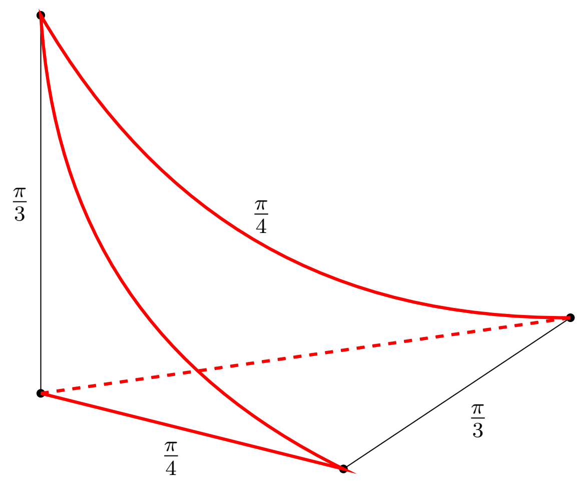


Figure 5.10: The Coxeter polytope in $\mathbb{H}^3$ corresponding to the Coxeter system of type Z_4. The edge cycle in red is the only edge cycle that leads to an embedded surface $\mathcal{S}$.

5.5 Four-dimensional Manifolds

In this section, we will look at all ten four-dimensional cases and discuss whether a given edge cycle Γ yields an embedded surface $\mathcal{S}$. We will proceed as in the previous chapter and see that there are more ways to construct Coxeter groups compared to the three-dimensional case. However, edge reflections in even dimension consist of an odd number of reflections. Hence it is possible that $\mathcal{J}$ is an index 1 subgroup, i.e., $\mathcal{J} = W$. If this is the case, we can apply Lemma 4.2.7 and conclude that $\mathcal{S}$ has self-intersection.

Surfaces in $\mathbb{S}^4$

Let P^4 be a Coxeter polytope in $\mathbb{S}^4$. Note that a full-dimensional edge cycle consists of at least 5 edges. There is no irreducible spherical Coxeter group that permits a compatible full-dimensional edge cycle. We will proceed with the reducible ones. Inspecting the edge reflection graphs, we can define a full-dimensional edge cycle for the following types: $F_4 \times A_1$, $B_3 \times I_2(m)$, and $I_2(m) \times I_2(k) \times A_1$. There is, up to symmetry, one compatible full-dimensional edge cycle that can be defined with $F_4 \times A_1$, one can be defined in $B_3 \times I_2(m)$, and three in $I_2(m) \times I_2(k) \times A_1$.

For $F_4 \times A_1$ we have the edge cycle $\Gamma = v_1v_3v_2v_4v_5$ which gives us the group

$$\mathcal{J} = \langle s_2s_4s_5, s_1s_4s_5, s_1s_3s_5, (s_1s_2s_3)^3, (s_2s_3s_4)^3\rangle.$$

Thus we have

$$\begin{aligned} \mathcal{J}(v_1) &= \{e, (s_2s_3s_4)^2, s_2s_4s_5, (s_3s_4s_2)^2s_3s_4\} \qquad \text{and} \\ W(v_1) &= \langle s_2, s_3, s_4, s_5\rangle. \end{aligned}$$

As $j_3j_2 = s_3s_4 \in \mathcal{J} \cap W(v_1)$ and $j_3j_2 \notin \mathcal{J}(v_1)$, the surface $\mathcal{S}$ is not embedded.

In the case $B_3 \times I_2(m)$, we have one edge cycle $\Gamma = v_1v_3v_2v_4v_5$ for even m and no edge cycle for odd m. We get the groups

$$\begin{aligned} \mathcal{J} &= \langle s_2(s_4s_5)^{m/2}, s_1(s_4s_5)^{m/2}, s_1s_3s_5, (s_1s_2s_3)^3, (s_2s_3)^2s_4\rangle, \\ \mathcal{J}(v_5) &= \{e, (s_1s_2s_3)^3, (s_2s_3)^2s_4, (s_1s_2s_3)^3(s_2s_3)^2s_4\}, \qquad \text{and} \\ W(v_5) &= \langle j_1, j_2, j_3, j_4\rangle. \end{aligned}$$

As $j_2j_1 = s_1s_2 \in \mathcal{J} \cap W(v_5)$ and $j_2j_1 \notin \mathcal{J}(v_5)$, the surface $\mathcal{S}$ is not embedded.

In the case $I_2(m) \times I_2(k) \times A_1$, we get three edge cycles $\Gamma_1 = v_1v_3v_2v_5v_4$, $\Gamma_2 = v_1v_3v_4v_2v_5$, and $\Gamma_3 = v_1v_2v_5v_3v_4$ if both m and k are even, otherwise no edge cycles. This gives us the edge reflection groups

$$\begin{aligned} \mathcal{J}_1 &= \langle s_2s_4s_5, s_1s_4s_5, s_1(s_3s_4)^{k/2}, (s_1s_2)^{m/2}s_3, s_2s_3s_5\rangle, \\ \mathcal{J}_2 &= \langle s_2s_4s_5, (s_1s_2)^{m/2}s_5, s_1s_3s_5, s_1(s_3s_4)^{k/2}, s_2(s_3s_4)^{k/2}\rangle, \qquad \text{and} \\ \mathcal{J}_3 &= \langle (s_3s_4)^{k/2}s_5, s_1(s_3s_4)^{k/2}, (s_1s_2)^{m/2}s_4, (s_1s_2)^{m/2}s_5, s_1s_3s_5\rangle. \end{aligned}$$

A quick check shows $j_1j_2j_5 = s_3 \in \mathcal{J}_1$, $(j_4j_5)^{m/2}j_2 = s_5 \in \mathcal{J}_2$, and $(j_2j_1)^{m/2}j_4 = s_3 \in \mathcal{J}_3$. By Lemma 4.2.7, all surfaces $\mathcal{S}_i$, $i = 1, 2, 3$ have self-intersections.

Theorem 5.5.1. *Let P^4 be a Coxeter polytope in $\mathbb{S}^4$ and (W, S) its Coxeter system. If Γ is a compatible full-dimensional edge cycle on P^4, then the surface $\mathcal{S}$ has self-intersections.*

Surfaces in $\mathbb{S}^3 \times \mathbb{E}$

Let $P^4 := \sigma^3 \times I$ be a prismatic Coxeter polytope in $\mathbb{S}^3 \times \mathbb{E}$ where σ^3 is a Coxeter polytope and I an interval. In general, if a three-dimensional polyhedron has v vertices, e edges, and f faces, then a prismatic polytope has $2v$ vertices, $2e + v$ edges, $2f + e$ faces, $2 + f$ cells. Hence P^4 has 6 vertices, 16 edges, 14 faces, and 6 cells. As a quick check, one can use Euler's formula for n-dimensional simple polytopes (see [DC08], Chapter 16):

$$\chi(P^n) := \sum_{k=0}^{n} (-1)^{k+1} f_k = 1 - (-1)^n$$

where f_k is the number of k-dimensional faces of P^n.

The only Coxeter groups that permits full-dimensional compatible edge cycles on P^4 are of type $F_4 \times \tilde{A}_1$, $B_3 \times A_1 \times \tilde{A}_1$, and $I_2(m) \times I_2(k) \times \tilde{A}_1$. Up to symmetry, there are 2, 7, and 15 (provided $m \neq k$, $m, k \neq 2$, and m, k even) different edge cycles respectively

A quick GAP check shows that most edge reflection groups contain a generator s_i, and hence have self-intersections due to Lemma 4.2.7. The remaining ones is $\Gamma = v_1 v_3 v_2 v_6 v_8 v_5$ in the Coxeter group of type $I_2(m) \times I_2(k) \times \tilde{A}_1$. The groups are defined in GAP code in Listing 18. Similarly to the discussion for the three-dimensional case, one can check (4.1) in Theorem 4.2.10. For simplicity, we will just state the result.

Theorem 5.5.2. *Let $P^4 := \sigma^3 \times I$ be a prismatic Coxeter polytope in $\mathbb{S}^3 \times \mathbb{E}$ where σ^3 is a Coxeter polytope and I an interval and (W, S) its corresponding Coxeter system. Let Γ be a full-dimensional compatible edge cycle on P^4. Then $\mathcal{S}$ is embedded if and only if W is of type $I_2(m) \times I_2(k) \times \tilde{A}_1$, k even, and $\Gamma = v_1 v_3 v_2 v_6 v_8 v_5$. The genus on $\mathcal{U}/\mathcal{T} \cong \mathbb{S}^3 \times 2kI$ is given by $g = 1 + 2k(m-1)$.*

Surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$

Let $P^4 := \sigma_1^2 \times \sigma_2^2$ be a Coxeter duoprism in $\mathbb{S}^2 \times \mathbb{S}^2$ where σ_1^2 and σ_2^2 are two Coxeter polytopes in $\mathbb{S}^2$, i.e., σ_i are triangles. The polytope P^4 has 9 vertices, 18 edges, 15 faces, and 6 cells.

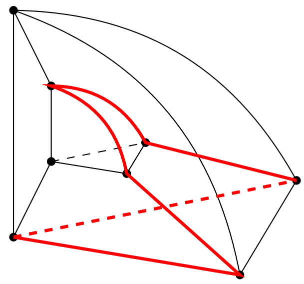


Figure 5.11: A schematic presentation of the Coxeter polytope in $\mathbb{S}^3 \times \mathbb{E}$ corresponding to the Coxeter system of type $I_2(m) \times I_2(k) \times \tilde{A}_1$. The edge cycle in red is the only edge cycle that leads to an embedded surface $\mathcal{S}$.

There are three types of Coxeter groups that permit full-dimensional compatible edge cycles: $B_3 \times B_3$, $B_3 \times I_2(m) \times A_1$, and $I_2(m) \times A_1 \times I_2(k) \times A_1$. Up to symmetry, there are 6, 15, and 40 full-dimensional compatible edge cycles respectively. The edge cycles that yield an embedded surface $\mathcal{S}$ are

$$\begin{aligned} \Gamma_1 &= v_1v_3v_6v_5v_8v_7, \\ \Gamma_2 &= v_1v_2v_3v_9v_6v_5v_4v_7, \qquad \text{and} \\ \Gamma_3 &= v_1v_3v_2v_8v_5v_6v_4v_7 \end{aligned}$$

in the corresponding polytope of the Coxeter group of type $I_2(m) \times A_1 \times I_2(k) \times A_1$. The groups are listed in GAP in Listing 19.

Theorem 5.5.3. *Let $P^4 := \sigma_1^2 \times \sigma_2^2$ be a Coxeter duoprism in $\mathbb{S}^2 \times \mathbb{S}^2$ where σ_1^2 and σ_2^2 are two Coxeter polytopes in $\mathbb{S}^2$, i.e., σ_i is a triangle. Let (W, S) be the corresponding Coxeter system and Γ a full-dimensional compatible edge cycle on P^4. Then $\mathcal{S}$ is embedded if and only if W is of type $I_2(m) \times A_1 \times I_2(k) \times A_1$ and one of the following conditions is satisfied:*

(i) m and k are even, $m = k$, and $\Gamma = \Gamma_1$,

(ii) $m = 2$ and $\Gamma = \Gamma_2$ for all $k \geq 2$, or

(iii) $\Gamma = \Gamma_3$ for all $k, m \geq 2$.

The genus is given by $g_1 = 2m+1$, $g_2 = 6k-3$, and $g_3 = 4(m-1)(k-1)-3$.

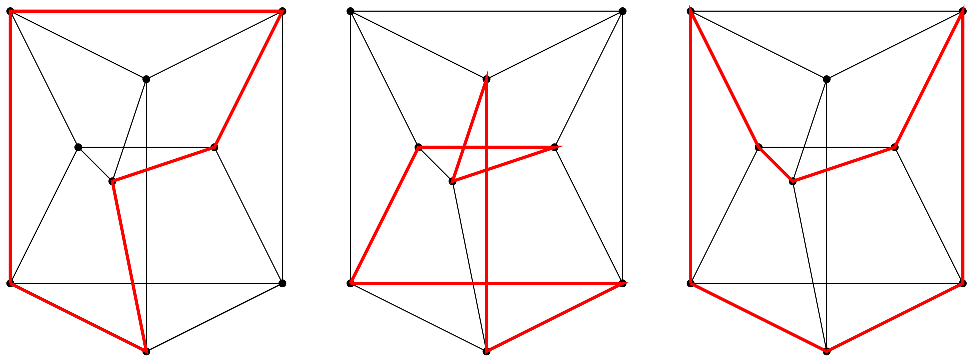


Figure 5.12: The Coxeter polytope in $\mathbb{S}^2 \times \mathbb{S}^2$ corresponding to the Coxeter system of type $I_2(m) \times A_1 \times I_2(k) \times A_1$. The three edge cycles that lead to an embedded surface $\mathcal{S}$ are shown in red.

Surfaces in $\mathbb{S}^2 \times \mathbb{E}^2$

Let $P^4 = \sigma^2 \times P^2$ be a Coxeter duoprism where σ^2 is a Coxeter polytope in $\mathbb{S}^2$ and P^2 is either a Coxeter 2-simplex (i.e. triangle) in $\mathbb{E}^2$ or a rectangle. There are, up to symmetry, 9, 15, 15, 70, 16, and 116 compatible full-dimensional edge cycles for the groups of type $B_3 \times \tilde{G}_2$, $I_2(m) \times A_1 \times \tilde{G}_2$, $\tilde{B}_2 \times B_3$, $I_2(m) \times A_1 \times \tilde{B}_2$, $B_3 \times \tilde{A}_1 \times \tilde{A}_1$, and $I_2(m) \times A_1 \times \tilde{A}_1 \times \tilde{A}_1$ respectively.

The edge cycles that lead to an embedded surface $\mathcal{S}$ are

$$\Gamma_1 = v_1v_2v_5v_8v_9v_7v_4 \qquad \text{and} \qquad \Gamma_2 = v_1v_3v_2v_5v_8v_9v_7v_4$$

for the group of type $I_2(m) \times A_1 \times \tilde{B}_2$ and

$$\begin{aligned} &\Gamma_1 = v_1v_2v_5v_6v_9v_{12}v_{10}, && \Gamma_2 = v_1v_2v_5v_8v_9v_7v_4, \\ &\Gamma_3 = v_1v_3v_2v_5v_8v_9v_7v_4, && \Gamma_4 = v_1v_3v_6v_5v_8v_9v_{12}v_{10}, \qquad \text{and} \\ &\Gamma_5 = v_1v_3v_2v_5v_8v_{11}v_{12}v_{10}v_7v_4 \end{aligned}$$

for the group of type $I_2(m) \times A_1 \times \tilde{A}_1 \times \tilde{A}_1$.

Theorem 5.5.4. *Let $P^4 = \sigma^2 \times P^2$ be a Coxeter duoprism in $\mathbb{S}^2 \times \mathbb{E}^2$ where σ is a Coxeter polytope (i.e. triangle) in $\mathbb{S}^2$ and P^2 is either a Coxeter 2-simplex (i.e. triangle) in $\mathbb{E}^2$ or a rectangle. Let (W, S) be the corresponding Coxeter system and Γ a full-dimensional compatible edge cycle on P^4. Then $\mathcal{S}$ is embedded if and only if one of the following conditions is satisfied:*

(i) *W is of type $I_2(m) \times A_1 \times \tilde{B}_2$ and $\Gamma = \Gamma_1$ or Γ_2,*

(ii) *W is of type $I_2(m) \times A_1 \times \tilde{A}_1 \times \tilde{A}_1$, $m = 2$, and $\Gamma = \Gamma_1$, or*

(iii) *W is of type $I_2(m) \times A_1 \times \tilde{A}_1 \times \tilde{A}_1$ and $\Gamma = \Gamma_2, \Gamma_3, \Gamma_5$, or*

(iv) *W is of type $I_2(m) \times A_1 \times \tilde{A}_1 \times \tilde{A}_1$, m even, and $\Gamma = \Gamma_4$.*

The genera are given by

(i) *$g_1 = 8m - 3$ on $\mathcal{U}/\mathcal{T}_1 \cong \mathbb{S}^2 \times 4I^2$ and $g_2 = 6m - 3$ on $\mathcal{U}/\mathcal{T}_2 \cong \mathbb{S}^2 \times 4I \times 2I$ where I is the length of the short side of the Euclidean triangle.*

(ii) *$g_1 = 13$ on $\mathcal{U}/\mathcal{T}_1 \cong \mathbb{S}^2 \times 4I \times 4I$ where I is the lengths of the side of the Euclidean rectangle,*

(iii) *$g_2 = 8m - 3$ on $\mathcal{U}/\mathcal{T}_2 \cong \mathbb{S}^2 \times 4I \times 4I$, $g_3 = 6m - 3$ on $\mathcal{U}/\mathcal{T}_3 \cong \mathbb{S}^2 \times 4I \times 2I$, $g_5 = 8m - 3$ on $\mathcal{U}/\mathcal{T}_5 \cong \mathbb{S}^2 \times 2I \times 4I$, and*

(iv) *$g_4 = 4m + 1$ on $\mathcal{U}/\mathcal{T}_4 \cong \mathbb{S}^2 \times 2I \times 2mI$.*

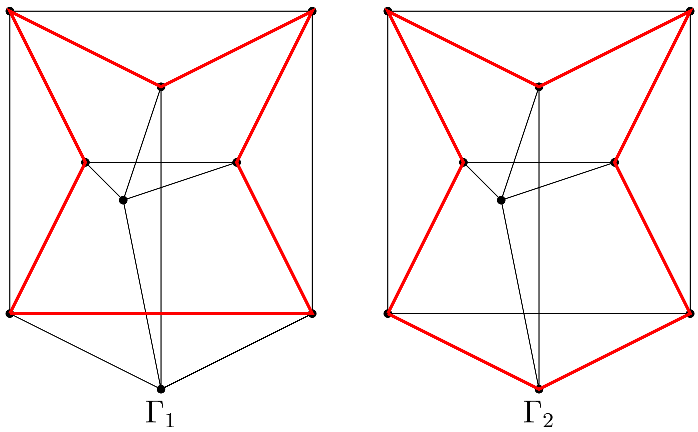


Figure 5.13: The Coxeter polytope in $\mathbb{S}^2 \times \mathbb{E}^2$ corresponding to the Coxeter system of type $I_2(m) \times A_1 \times \tilde{B}_2$. The two edge cycles that lead to an embedded surface $\mathcal{S}$ are shown in red.

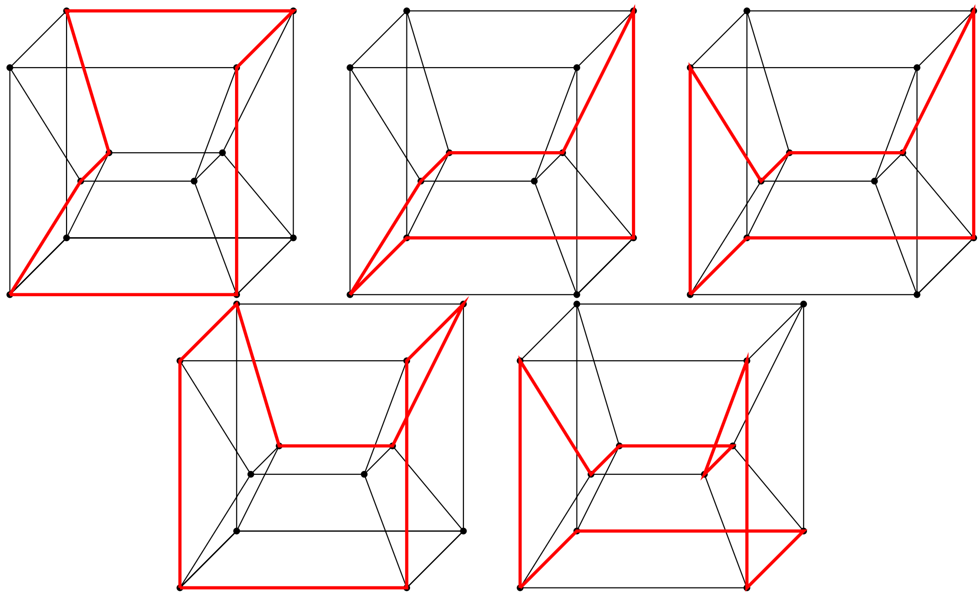


Figure 5.14: The Coxeter polytope in $\mathbb{S}^2 \times \mathbb{E}^2$ corresponding to the Coxeter system of type $I_2(m) \times A_1 \times \tilde{A}_1 \times \tilde{A}_1$. The three edge cycles that lead to an embedded surface $\mathcal{S}$ are shown in red.

Surfaces in $\mathbb{S}^2 \times \mathbb{H}^2$

Let $P^4 = \sigma_1^2 \times \sigma_2^2$ be a Coxeter duoprism in $\mathbb{S}^2 \times \mathbb{H}^2$ where σ_1^2 is a Coxeter polytope in $\mathbb{S}^2$ and σ_2^2 is a Coxeter 2-simplex in $\mathbb{H}^2$. There are, up to symmetry, 6 and 38 full-dimensional compatible edge cycles for the group $B_3 \times Z_3(a,b,c)$ and $I_2(m) \times A_1 \times Z_3(a,b,c)$ respectively.

The edge cycles that possibly lead to an embedded surface $\mathcal{S}$ are

$$\begin{aligned} \Gamma_1 &= v_1v_2v_5v_8v_9v_7v_4 \qquad \text{and} \\ \Gamma_2 &= v_1v_3v_2v_5v_8v_9v_7v_4 \end{aligned}$$

for the group of type $I_2(m) \times A_1 \times Z_3(a,b,c)$.

Theorem 5.5.5. *Let $P^4 = \sigma_1^2 \times \sigma_2^2$ be a Coxeter duoprism in $\mathbb{S}^2 \times \mathbb{H}^2$ where σ_1^2 is a Coxeter polytope in $\mathbb{S}^2$ and σ_2^2 is a Coxeter 2-simplex in $\mathbb{H}^2$, i.e., σ_i^2 is a triangle. Furthermore, let (W,S) be the corresponding Coxeter system and Γ a full-dimensional compatible edge cycle on P^4. Assume that $\mathcal{S}$ is embedded then $\Gamma = \Gamma_1$ or $\Gamma = \Gamma_2$.*

If $\Gamma = \Gamma_1$ or Γ_2, $b = c = 4$ and $a = 4, 8, 12$, then $\mathcal{S}$ is embedded.

If $a = 2, 4, 6$ and $b = c = 4$, then $\mathcal{S}$ has self-intersections. It is possible, that $b = c$ is sufficient for $\mathcal{S}$ to be embedded, but we were not able to use GAP to check (4.1) for $b, c \geq 6$.

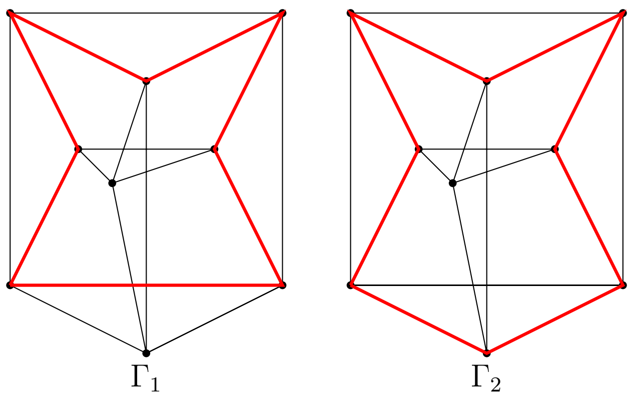


Figure 5.15: The Coxeter polytope in $\mathbb{S}^2 \times \mathbb{H}^2$ corresponding to the Coxeter system of type $I_2(m) \times A_1 \times \tilde{Z}_3$. The two edge cycles that possibly lead to an embedded surface $\mathcal{S}$ are shown in red.

Surfaces in $\mathbb{E}^4$

Let P^4 be Coxeter polytope in $\mathbb{E}^4$. There are, up to symmetry, 1, 14, 40, 15, 6, 116, 16, and 98 compatible full-dimensional edge cycles for the groups of type $\tilde{C}_4$, $\tilde{C}_3 \times \tilde{A}_1$, $\tilde{B}_2 \times \tilde{B}_2$, $\tilde{B}_2 \times \tilde{G}_2$, $\tilde{G}_2 \times \tilde{G}_2$, $\tilde{B}_2 \times \tilde{A}_1 \times \tilde{A}_1$, $\tilde{G}_2 \times \tilde{A}_1 \times \tilde{A}_1$, $(\tilde{A}_1)^4$ respectively.

The edge cycles that lead to an embedded surface $\mathcal{S}$ are:

$$\Gamma_1 = v_1 v_2 v_3 v_4 v_8 v_7 v_6 v_5$$

for the group of type $\tilde{C}_3 \times \tilde{A}_1$,

$$\Gamma_1 = v_1 v_2 v_3 v_6 v_9 v_8 v_7 v_4$$

for the group of type $\tilde{B}_2 \times \tilde{B}_2$,

$$\Gamma_1 = v_1 v_2 v_3 v_6 v_5 v_4 v_7 v_{10}, \qquad \Gamma_2 = v_1 v_2 v_3 v_6 v_9 v_8 v_7 v_4,$$
$$\Gamma_3 = v_1 v_2 v_3 v_6 v_9 v_8 v_7 v_{10}, \qquad \Gamma_4 = v_1 v_2 v_5 v_6 v_9 v_8 v_{11} v_{10},$$
$$\Gamma_5 = v_1 v_2 v_3 v_6 v_9 v_{12} v_{11} v_{10} v_7 v_4, \qquad \Gamma_6 = v_1 v_2 v_3 v_6 v_5 v_4 v_7 v_8 v_9 v_{12} v_{11} v_{10}$$

for the group of type $\tilde{B}_2 \times \tilde{A}_1 \times \tilde{A}_1$, and

$$\Gamma_1 = v_1v_2v_4v_3v_7v_{11}v_9v_5, \qquad \Gamma_2 = v_1v_2v_{14}v_{16}v_{15}v_{11}v_7v_5,$$
$$\Gamma_3 = v_1v_2v_4v_8v_7v_{11}v_{15}v_{13}, \qquad \Gamma_4 = v_1v_2v_4v_3v_7v_{11}v_{12}v_{10}v_9v_5,$$
$$\Gamma_5 = v_1v_2v_{14}v_{13}v_{15}v_{11}v_9v_{10}v_6v_5v_7v_3$$

for the group of type $(\tilde{A}_1)^4$.

Kürsten constructed minimal surface using the curves for a cube in $\mathbb{R}^4$, i.e., when W is of type $(\tilde{A}_1)^4$ in [Kür14]. Similar as in the case $\mathbb{R}^3$, the remaining curves yield homeomorphic surfaces.

Theorem 5.5.6. *Let P^4 be a Coxeter polytope in $\mathbb{E}^4$, (W, S) the corresponding Coxeter system and Γ a full-dimensional compatible edge cycle on P^4. Then $\mathcal{S}$ is embedded if and only if one of the following conditions is satisfied:*

(i) *W is of type $\tilde{C}_3 \times \tilde{A}_1$ and $\Gamma = \Gamma_1$,*

(ii) *W is of type $\tilde{B}_2 \times \tilde{B}_2$ and $\Gamma = \Gamma_1$,*

(iii) *W is of type $\tilde{B}_2 \times \tilde{A}_1 \times \tilde{A}_1$ and $\Gamma = \Gamma_1, \dots, \Gamma_6$,*

(iv) *W is of type $(\tilde{A}_1)^4$ and $\Gamma = \Gamma_1, \dots, \Gamma_5$.*

The genera are given by

(i) *$g_1 = 9$ on $\mathcal{U}/\mathcal{T}_1 \cong (4I \times 2I)^2$ where $I = |v_1v_3|$,*

(ii) *$g_1 = 9$, on $\mathcal{U}/\mathcal{T}_1 \cong 4I_1 \times 2I \times 4I \times 2I$ where $I = |v_1v_2|$,*

(iii) *$g_1 = 9$, $g_2 = 9$, $g_3 = 9$, $g_4 = 9$, $g_5 = 13$, $g_6 = 17$ on $\mathcal{U}/\mathcal{T}_i \cong 4I \times 2I_1 \times 4I_2 \times 2I_2$ where $I_1 = |v_1v_2|$, and $I_2 = |v_1v_3|$,*

(iv) *$g_1 = 9$, $g_2 = 9$, $g_3 = 9$, $g_4 = 13$, and $g_5 = 17$ on $\mathcal{U}/\mathcal{T}_i \cong 4I \times 2I \times 4I \times 2I$ where $I = |v_1v_2|$.*

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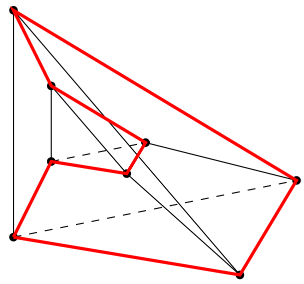


Figure 5.16: The Coxeter polytope in $\mathbb{E}^4$ corresponding to the Coxeter system of type $\tilde{C}_3 \times \tilde{A}_1$. The edge cycle in red is the only edge cycle that leads to an embedded surface $\mathcal{S}$.

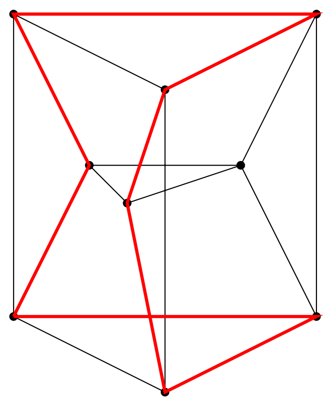


Figure 5.17: The Coxeter polytope in $\mathbb{E}^4$ corresponding to the Coxeter system of type $\tilde{B}_2 \times \tilde{B}_2$. The edge cycle in red is the only edge cycle that leads to an embedded surface $\mathcal{S}$.

Surfaces in $\mathbb{H}^2 \times \mathbb{E}^2$

Let $P^4 = \sigma^2 \times P^2$ be a Coxeter polytope where σ^2 is a Coxeter 2-simplex (triangle) in $\mathbb{H}^2$ and P^2 a Coxeter 2-simplex (triangle) or a rectangle in $\mathbb{E}^2$. There are, up to symmetry, 28, 6, and 48 compatible full-dimensional edge cycles for the groups $Z_3 \times \tilde{B}_2$, $Z_3 \times \tilde{G}_2$, and $Z_3 \times \tilde{A}_1 \times \tilde{A}_1$ respectively.

Theorem 5.5.7. *Let $P^4 = \sigma^2 \times P^2$ be a Coxeter polytope where σ^2 is a Coxeter 2-simplex (triangle) in $\mathbb{H}^2$ and P^2 a Coxeter 2-simplex (triangle) or a rectangle in $\mathbb{E}^2$. Let (W, S) be the corresponding Coxeter system and Γ be a full-dimensional edge cycle on P^4. Assume that $\mathcal{S}$ is embedded then*

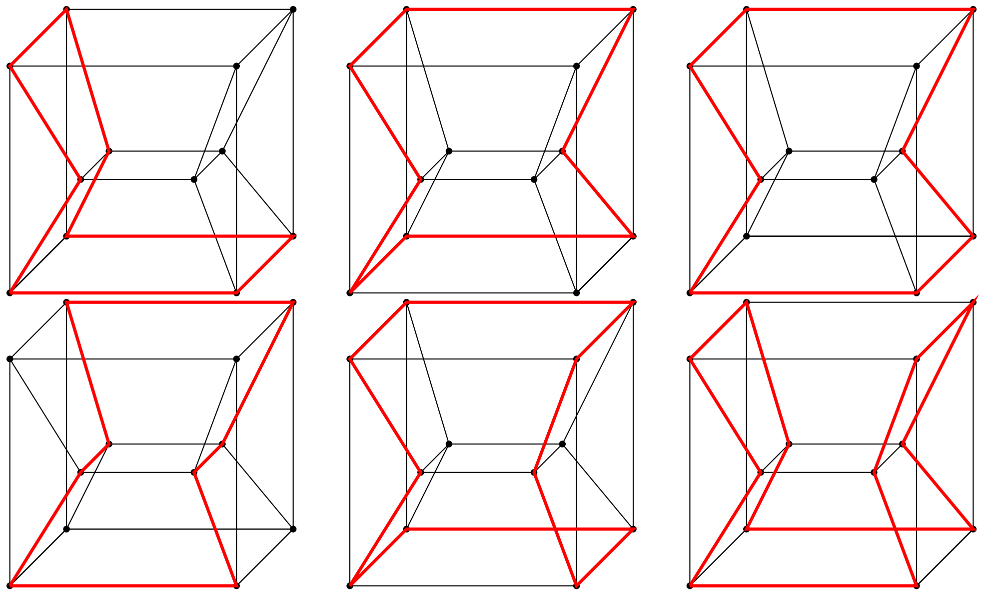


Figure 5.18: The Coxeter polytope in $\mathbb{E}^4$ corresponding to the Coxeter system of type $\tilde{B}_2 \times \tilde{A}_1 \times \tilde{A}_1$. The six edge cycles that lead to an embedded surface $\mathcal{S}$ are shown in red.

(i) W is of type $Z_3 \times \tilde{B}_2$ and

$$\Gamma = \Gamma_1 = v_1v_2v_3v_6v_9v_8v_7 \qquad \textit{or} \qquad \Gamma = \Gamma_2 = v_1v_2v_3v_6v_9v_8v_7v_4.$$

(ii) W is of type $Z_3 \times \tilde{A}_1 \times \tilde{A}_1$ and Γ is one of the following edge cycles:

$$\begin{aligned} \Gamma_1 &= v_1v_2v_3v_6v_5v_4v_7v_{10}, & \Gamma_2 &= v_1v_2v_3v_6v_9v_8v_7v_4 \\ \Gamma_3 &= v_1v_2v_3v_6v_9v_8v_7v_{10}, & \Gamma_4 &= v_1v_4v_6v_9v_8v_{11}v_{12}v_3 \\ \Gamma_5 &= v_1v_2v_3v_6v_9v_{12}v_{11}v_{10}v_7v_4, & \Gamma_6 &= v_1v_2v_3v_6v_5v_4v_7v_8v_9v_{12}v_{11}v_{10}. \end{aligned}$$

Surfaces in $\mathbb{H}^2 \times \mathbb{H}^2$

Let $P^4 = \sigma_1^2 \times \sigma_2^2$ be a Coxeter polytope where σ_1^2 and σ_2^2 are Coxeter 2-simplices (triangles) in $\mathbb{H}^2$. There are up to symmetry 9 compatible full-dimensional edge cycles for the groups $Z_3 \times Z_3$.

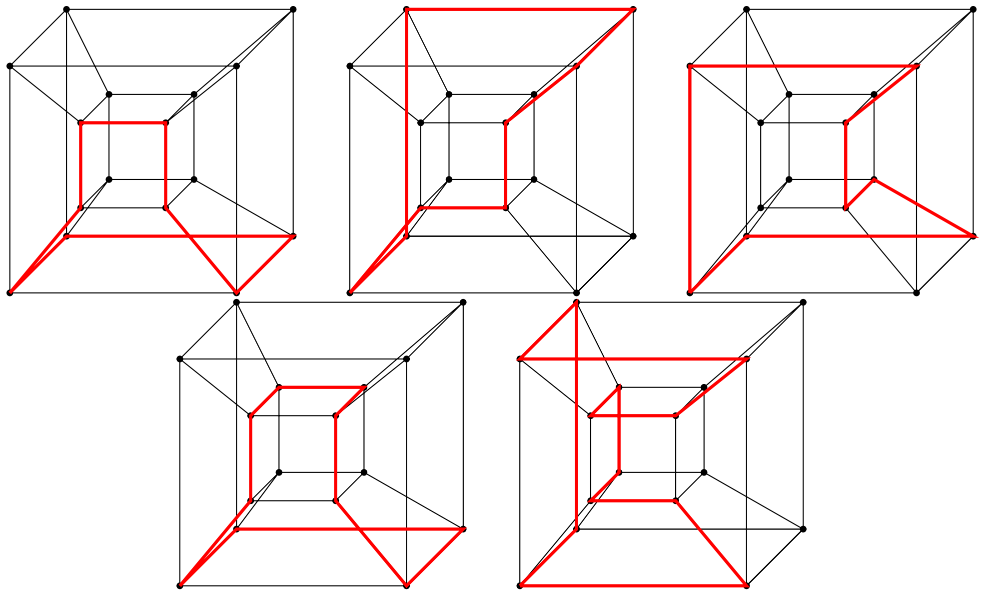


Figure 5.19: The Coxeter polytope in $\mathbb{E}^4$ corresponding to the Coxeter system of type $(\tilde{A}_1)^4$. The five edge cycles that lead to an embedded surface $\mathcal{S}$ are shown in red.

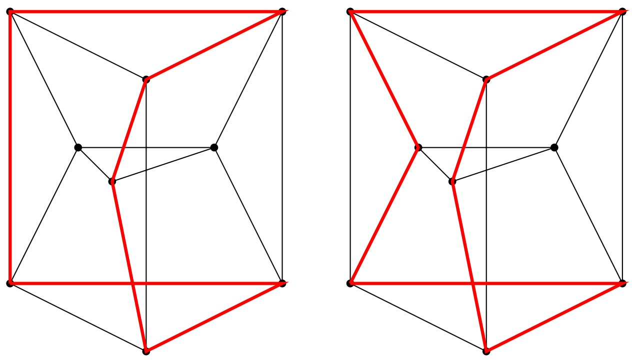


Figure 5.20: The Coxeter polytope in $\mathbb{H}^2 \times \mathbb{E}^2$ corresponding to the Coxeter system of type $Z_3 \times \tilde{B}_2$. The two edge cycles that possibly lead to an embedded surface $\mathcal{S}$ are shown in red.

Theorem 5.5.8. *Let $P^4 = \sigma_1^2 \times \sigma_2^2$ be a Coxeter polytope where σ_1^2 and σ_2^2 are 2-simplices (triangles) in $\mathbb{H}^2$ and (W,S) the corresponding Coxeter system of type $Z_3 \times Z_3$. Let Γ be a compatible full-dimensional edge cycle*

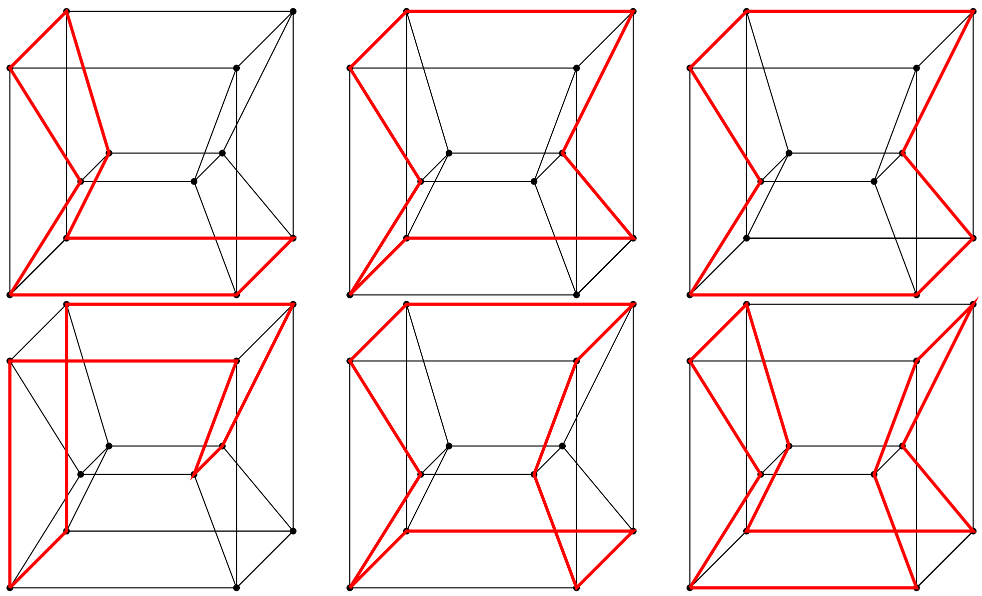


Figure 5.21: The Coxeter polytope in $\mathbb{H}^2 \times \mathbb{E}^2$ corresponding to the Coxeter system of type $Z_3 \times \tilde{A}_1 \times \tilde{A}_1$. The six edge cycles that possibly lead to an embedded surface $\mathcal{S}$ are shown in red.

on P^4. Assume that $\mathcal{S}$ is embedded then Γ is one of the following cycles

$$\Gamma_1 = v_1v_2v_3v_6v_4v_7, \qquad \Gamma_2 = v_1v_2v_5v_6v_9v_7$$
$$\Gamma_3 = v_1v_2v_3v_6v_5v_4v_7 \qquad \Gamma_4 = v_1v_2v_3v_6v_9v_8v_7v_4.$$

Surfaces in $\mathbb{H}^3 \times \mathbb{E}$

Let $P^4 = \sigma^3 \times I$ be a Coxeter polytope where σ^3 is a Coxeter 3-simplex in $\mathbb{H}^3$ and I an interval. Several types permit compatible full-dimensional edge cycles. However, every surface $\mathcal{S}$ has self-intersections.

Theorem 5.5.9. *Let $P^4 = \sigma^3 \times I$ be a Coxeter polytope where σ^3 is a Coxeter 3-simplex in $\mathbb{H}^3$ and I an interval. If Γ is a compatible full-dimensional edge cycle on P^4, then the surface $\mathcal{S}$ has self-intersections.*

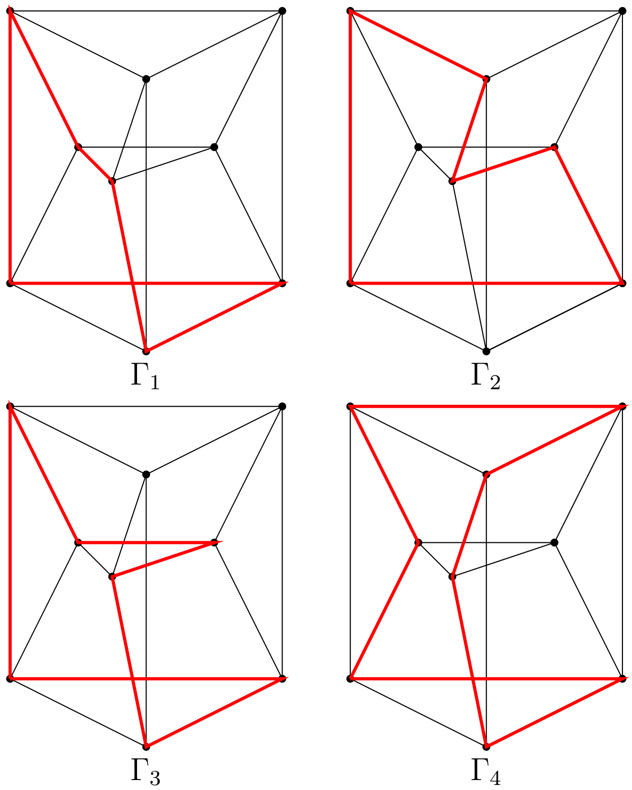


Figure 5.22: The Coxeter polytope in $\mathbb{H}^2 \times \mathbb{H}^2$ corresponding to the Coxeter system of type $Z_3 \times Z_3$. The four edge cycles that possibly lead to an embedded surface $\mathcal{S}$ are shown in red.

Surfaces in $\mathbb{H}^4$

Let σ^4 be a Coxeter 4-simplex in $\mathbb{H}^4$. None of the edge reflection graphs listed in Figure 4.14 permit an full-dimensional edge cycle.

Theorem 5.5.10. *Let P^4 be a Coxeter 4-simplex in $\mathbb{H}^4$. Then no full-dimensional compatible edge cycle can be defined on P^4.*

5.6 The n-dimensional Manifolds

In this section, we examine higher dimensional Coxeter polytopes in $\mathbb{X}^n$. Assume that P^n is an n-simplex in $\mathbb{X}^n$ with Coxeter system (W, S) and Γ

is a full-dimensional edge cycle on P^n. For $n \geq 4$, we will show that the surface $\mathcal{S}$ constructed by edge reflection is not embedded.

We already discussed the hyperbolic case. By Theorem 2.9.13, hyperbolic reflection groups do not exist in dimension $n \geq 5$. Theorem 5.5.10 shows that there are no full-dimensional compatible edge cycles in $\mathbb{H}^4$.

We discuss the Euclidean case first, since it is simpler than the spherical.

Let $P^n \subset \mathbb{E}^n$, $n \geq 4$ be a Coxeter n-simplex and (W, S) its Coxeter system. Thus W is irreducible. Let Γ be a full-dimensional compatible edge cycle. Inspecting the edge reflection graphs in Figure 4.13 shows that Γ can only be defined if W is of type $\tilde{C}_{n-1}$.

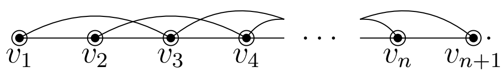
Theorem 5.6.1. *Let $P^n \subset \mathbb{E}^n$, $n \geq 3$ be a Coxeter n-simplex, (W, S) its Coxeter system, and Γ be a full-dimensional compatible edge cycle. Then $\mathcal{S}$ is not embedded.*

Proof. Note that we already discussed the cases $n = 3$ and $n = 4$. For simplicity, we will only show case $n = 5$. The proof for $n \geq 6$ is virtually the same.

As mentioned above, Γ can only be defined if W is of type $\tilde{C}_{n-1}$. Recall that the Coxeter graph is given by

$\tilde{C}_n$ 

and the edge reflection graph is given by

$\tilde{C}_n, n \geq 3$ 

We show case $n = 5$; thus let W be of type $\tilde{C}_4$. The only full-dimensional compatible edge cycle is given by $\Gamma = v_1 v_2 v_4 v_5 v_3$. Hence the generators of the edge reflection group are given by

$$
\begin{aligned}
j_1 &= (s_3 s_4 s_5)^3, & j_2 &= s_1 s_3 s_5, \\
j_3 &= (s_1 s_2 s_3)^3, & j_4 &= (s_1 s_2)^2 s_4, \\
j_5 &= s_2 (s_4 s_5)^2. & &
\end{aligned}
$$

Consider the vertex v_2. Since the words s_1, s_3s_5, and $(s_3s_4s_5)^3$ are involutions and commute, we have

$$\mathcal{J}(v_2) = \langle j_1, j_2 \rangle = \{e, s_1s_3s_5, (s_3s_4s_5)^3, s_1s_3s_5(s_3s_4s_5)^3\}.$$

Now consider the generator $j_3 = (s_1s_2s_3)^3$. The goal is to use successive edge reflection to reduce j_3 to a word w which does not contain the generator s_2; hence $w \in W(v_2) \cap \mathcal{J}$. It remains to check whether $w \in \mathcal{J}(v_2)$. Since s_1 commutes with s_3 we get

$$\begin{aligned} j_4j_3j_2j_4 &= j_4(s_1s_2s_3s_1s_2s_3s_1s_2s_3)(s_1s_3s_5)j_4 \\ &= j_4(s_1s_2s_1s_3s_2s_3s_1s_2s_1)s_5j_4 \\ &= s_4s_2s_3s_2s_3s_2s_5s_4 \\ &= s_4s_3s_5s_4 \end{aligned}$$

where we used $s_2s_3s_2s_3s_2 = s_3$ in the last step. We see that $j_4j_3j_2j_4$ fix the vertex v_2; hence $j_4j_3j_2j_4 \in W(v_2) \cap \mathcal{J}$. But $(j_4j_5j_3)^2 \notin \mathcal{J}(v_2)$. Hence $\mathcal{S}$ is not embedded. $\square$

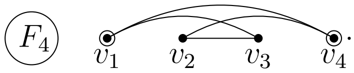
For the case $\mathbb{S}^n$, we can assume that P^n is a Coxeter polytope (instead of an n-simplex), since this implies that P^n is an n-simplex by Lemma 2.2.8.

A glance at the edge reflection graphs in Figure 4.12 shows that the number of edges in an irreducible component is bounded by 4 (for the group of type F_4). Since Γ is full-dimensional, it consists of $n+1$ edges. The goal in the next proof is to show that W connot contain a special subgroup of type F_4 and B_k, $k \geq 3$. Thus for $n \geq 5$, W contains at least 3 components. But in this case, we can find a vertex v_i such that $\mathcal{J}(v_i) \neq W(v_i) \cap \mathcal{J}$. Hence $\mathcal{S}$ is not embedded.

Theorem 5.6.2. *Let P^n be a Coxeter polytope in $\mathbb{S}^n$, $n \geq 4$, with Coxeter system (W, S) and Γ a full-dimensional compatible edge cycle on P^n. Then $\mathcal{S}$ is not embedded.*

Proof. By Lemma 2.2.8, P^n is an n-simplex. Furthermore, the case $n = 4$ is discussed in Theorem 5.5.1. Since Γ is full-dimensional, we can assume that $|\Gamma| = n + 1 \geq 6$. A glance at the edge reflection graphs in Figure 4.12 shows that the number of edges in an irreducible component is bounded by 4; hence W has at least 2 irreducible components.

First, we show that W cannot contain a special subgroup of type F_4. Assume the contrary, and let W^{F_4} be the special subgroup of type F_4 in W. Let s_1, s_2, s_3, and s_4 be the generators of W^{F_4}. Recall that the edge reflection graph of F_4 is given by



Since Γ is full-dimensional, there are exactly 5 edges that contain a vertex of the edge reflection graph of F_4. Namely, we have the path $v_i v_1 v_3 v_2 v_4 v_j$ where v_i and v_j lie in a different component of the edge reflection graph. Let $W = W^{F_4} \times W^2$, W_i^2 be the special subgroup of W^2 not containing s_i, $i \geq 5$, w_0 the longest element in W^2, and w_i the longest element in W_i^2. Then the generators of the edge reflection group $\mathcal{J}$ for the path $v_i v_1 v_3 v_2 v_4 v_j$ are given by

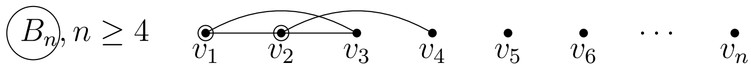
$$j_1 = (s_2 s_3 s_4)^3 w_i, \qquad j_2 = s_2 s_4 w_0, \qquad j_3 = s_1 s_4 w_0,$$
$$j_4 = s_1 s_3 w_0, \qquad j_5 = (s_1 s_2 s_3)^3 w_j.$$

Note that $j_4 j_3 = s_3 s_4 \in W(v_1) \cap \mathcal{J}$. Furthermore, since $(s_2 s_3 s_4)^2$, $s_2 s_4$, w_0, and w_i are involutions and commute, we have

$$\mathcal{J}(v_1) = \langle j_1, j_2 \rangle = \{e, (s_2 s_3 s_4)^3 w_i, s_2 s_4 w_0, (s_2 s_3 s_4)^3 s_2 s_4 w_0 w_i\}.$$

We see that $j_4 j_3 \notin \mathcal{J}(v_1)$; hence $W(v_1) \cap \mathcal{J} \neq \mathcal{J}(v_1)$ and $\mathcal{S}$ has a self-intersection.

Second, we show that W cannot contain a special subgroup of type B_k for all $k \geq 3$. Recall that the edge reflection graph of B_k is given by



Since Γ is full-dimensional, we need only consider the type B_3, otherwise there is a vertex which is not contained in Γ. Similarly to the case for F_4, assume that $W = W^{B_3} \times W^2$. We have the path $v_i v_1 v_3 v_2 v_j$ and the generators are given by

$$j_1 = (s_2 s_3)^2 w_i, \qquad j_2 = s_2 w_0, \qquad j_3 = s_1 w_0, \qquad j_4 = s_1 s_3 w_j.$$

Note that $j_3 j_4 = s_3 w_0 w_j \in W(v_1) \cap \mathcal{J}$. Furthermore, since $s_3 s_2 s_3$, w_0, and w_i are all involutions that commute, we have

$$\mathcal{J}(v_1) = \langle j_1, j_2 \rangle = \{e, (s_2 s_3)^2 w_i, s_2 w_0, s_3 s_2 s_3 w_0 w_i\}.$$

We see that $j_3 j_4 \notin \mathcal{J}(v_1)$; hence $W(v_1) \cap \mathcal{J} \neq \mathcal{J}(v_1)$ and $\mathcal{S}$ has a self-intersection.

Thus W can only contain factors of type $I_2(m)$ and A_1. Since $|\Gamma| \geq 6$, W consists of at least 3 components. Let $W = W^1 \times W^2 \times W^R$ where W^R is the special subgroup that contains all generators that are not in W^1 or W^2. Let v_1 be a vertex which is not fixed by $s_1 \in W^1$. Since there are at least 6 edges there are two edges e_i and e_k which do not contain v_1. Let j_i and j_k be the corresponding edge reflection. Note that j_i and j_k both contain the longest element w_0^1 of W^1 as a subword. Since w_0^1 commutes with all $s_i \in W$, we have that $j_i j_k$ do not contain s_1 as a subword. Hence $j_i j_k$ fix v_1, i.e., $j_i j_k \in W(v_1) \cap \mathcal{J}$. Let j_1 and j_2 be the edge reflection fixing v_1. Thus $\mathcal{J}(v_1) = \langle j_1, j_2 \rangle$. It is easy to see that there is at least one s_m contained as a subword in $j_i j_k$ which is not contained in $j_1 j_2$. Hence $j_i j_k \notin \mathcal{J}(v_1)$ and $\mathcal{S}$ cannot be embedded. □

We get:

Theorem 5.6.3. *Let $P^n \subset \mathbb{X}^n$, $n \geq 4$ be a Coxeter n-simplex and (W, S) its Coxeter system. Let Γ be a full-dimensional compatible edge cycle on P^n. Then $\mathcal{S}$ is not embedded.*

It is interesting to know whether an embedded surface $\mathcal{S}$ can be constructed in $\mathbb{X}^n$, $n \geq 4$ without the assumption that Γ is defined on an n-simplex. Theorem 5.5.6 shows that it is possible to define a cycle Γ on a product of simplices in the case $\mathbb{E}^4$. Furthermore, Kürsten showed in [Kür14] that it is possible to define a full-dimensional edge cycle with embedded surface $\mathcal{S}$ for every dimension n if W is of type $(\tilde{A}_1)$, i.e., on a cube. Theorem 2.9.14 shows that it is not possible in $\mathbb{H}^n$, $n \geq 30$, since Coxeter polytopes can only exist in dimension $n \leq 29$. In case $\mathbb{S}^n$, Theorem 5.6.2 shows that it is not possible for a Coxeter polytope in $\mathbb{S}^n$, $n \geq 4$.

In the next chapter, we will discuss two generalisations for the construction of Γ. However, we will not discuss the n-dimensional cases, using these generalisations.

6 Generalisations

In this chapter, we will generalise the construction of an edge cycle in two ways.

In the first section, we will see that we can define a group action on $\mathcal{U}(W, P^n)$ such that we are able to describe the second case in the discussion of the edge reflection in Chapter 4 (see Figure 4.3). This section is based on Section 9.1 in [DC08].

The second section deals with defining an edge cycle on the edge set of a suitable union of chambers in $\mathcal{U}(W, P^n)$. This will resolve the limitation to an n-simplex in the spherical and hyperbolic space and extend the choices in the Euclidean space.

6.1 Semi-compatible Edge Cycles

Recall the second case in Section 4.1 where edge reflections could be expressed with the Coxeter group W up to a symmetry of the polytope P^n. Furthermore, recall that in Section 5.3 a symmetry of P^n corresponds to a diagram automorphism of W. The main idea is to define a group action on $\mathcal{U}(W, P^n)$ using W and the symmetry group of P^n, i.e, a diagram automorphism of (W, S).

Suppose that D is a group of diagram automorphisms of a Coxeter system (W, S) and put $H \coloneqq W \rtimes D$. The group H acts on $\mathcal{U}(W, P^n)$ as follows: let $h = (w, d) \in H$ and $[u, x] \in \mathcal{U}$, then

$$h \cdot [u, x] = [wd(u), dx]. \tag{6.1}$$

Essentially, the diagram automorphism d interchanges the labeling of the Coxeter graph, i.e., a symmetry interchanges the facets of the polytope, before the element $w \in W$ acts on $\mathcal{U}$. Eventually, the longest element in H corresponds to the edge reflection in this case.

Proposition 6.1.1 ([DC08], Proposition 9.1.7)**.** *The expression for* $h \cdot [u, x]$ *in (6.1) is well-defined and gives an action of* $W \rtimes D$ *on* $\mathcal{U}(W, P^n)$*.*

Proof. Let $S(x) := \{s_i \in S \mid x \in H_i\}$ where H_i is the hyperplane induced by the facet F_i of P^n. The equivalence relation in (2.12), the definition of $\mathcal{U}$, is such that $[uv, x] = [u, x]$ for all $v \in W_{S(x)}$. Hence, to prove that (6.1) is well defined, we must show that by replacing $[u, x]$ with $[uv, x]$ on the left-hand side, the right-hand side remains unchanged.

By the compatibility of the D-action on S and P^n, we have $dS(x) = S(dx)$. Hence

$$h \cdot [uv, x] = [wd(uv), d(x)] = [wd(u)d(v), d(x)] = [wd(u), d(x)] = h \cdot [u, x],$$

where the last equality holds since $h(v) \in W_{S(x)}$.

Suppose $h_1 = (w_1, d_1)$ and $h_2 = (w_2, d_2)$. By definition of multiplication in the semi-direct product, $h_1 h_2 = (w_1 h_1(w_2), h_1 h_2)$. To prove that (6.1) defines an action, we need to show $h_1 \cdot (h_2 \cdot [u, x]) = (h_1 h_2) \cdot [u, x]$. Indeed,

$$\begin{aligned} h_1 \cdot (h_2 \cdot [u, x]) &= h_1 \cdot [w_2 d_2(u), d_2 x] \\ &= [w_1 d_1(w_2 d_2(u)), d_1 d_2 x] \\ &= [w_1 d_1(w_2)(d_1 d_2)(u), d_1 d_2 x] = (h_1 h_2) \cdot [u, x]. \end{aligned}$$

Hence the action defined in (6.1) is well defined. □

Proposition 6.1.2 ([DC08], Proposition 9.1.9)**.** *Given* $x \in P^n$*, the isotropy subgroup* D_x *permutes* $S(x)$*. Hence,* D_x *is a group of diagram automorphisms of* $(W_{S(x)}), S(x))$*. The isotropy subgroup of* H *at the point* $[1, x] \in \mathcal{U}$ *is* $W_{S(x)} \rtimes D_x$*. Furthermore* $\mathcal{U}/H \cong P^n/D$*.*

Proof. The first two sentences are obvious. If (w, d) fixes $[1, x]$, then $(w, d) \cdot [1, x] = [w, dx] = [1, x]$. Thus $d \in D_x$ and $w \in W_{S(x)}$. Furthermore $\mathcal{U}/W \cong P^n$; hence $\mathcal{U}/H \cong P^n/D$. □

The action of H on $\mathcal{U}$ can be interpreted as follows: assuming that $\mathcal{U}(W, P^n)$ is a tessellation of $\mathbb{X}^n$, the elements in D create different images of P^n for each chamber corresponding to the symmetry $d \in D$. Hence, with the action of H, the tessellation $\mathcal{U}$ can be seen as a D-fold cover of each chamber and for each symmetry.

This has some consequences for the construction of $\mathcal{S}$.

Definition 6.1.3. We say an edge cycle Γ is *semi-compatible* with the tessellation $\mathcal{U}(W, P^n)$, if for every edge $e_i \subset P^n$ the edge reflection j_i about e_i can be expressed as an element of $H = W \rtimes D$ and Γ is not compatible, i.e., there is at least one edge e_k where the edge reflection along e_k is an element with $d \neq e$.

The edge reflection about a compatible edge is seen as $W \rtimes \{e\}$ acting on $\mathcal{U}$, which justifies this generalisation. We will again denote with $\mathcal{J}$ the edge reflection group of Γ. Note that $\mathcal{J}$ is now a subgroup of H.

Assume that W is spherical and w_0 is the longest element of W. By Theorem 3.2.12, w_0 is central except in the cases $A_n (n \geq 2)$, D_{2n+1}, E_6, and $I_2(2m+1)$. Recall that w_0 is the element mapping all positive roots unto a negative root and acting as $-\operatorname{id}_V \in GL(V)$ using the canonical representations if and only if w_0 is central. Thus a root α_i is mapped to $-\alpha_i$ if w_0 is central. In the exceptional cases w_0 maps α_i onto $-\alpha_j$ where possibly $i \neq j$. A quick calculation (see [Fra01], p. 7-11) shows that the action of w_0 is exactly the one of a graph automorphism.

Example 6.1.4. Let W be a Coxeter group of type D_5. The only graph automorphism interchanges s_4 and s_5 and fixes all other generators. The longest element w_0 maps α_4 onto $-\alpha_5$, α_5 onto $-\alpha_4$, and the remaining α_i onto $-\alpha_i$.

Using the action of the longest elements on the roots, we can extend the edge reflection graphs (see Figure 4.12-4.14) to semi-compatible edge reflection graphs. If W and every spherical subgroup W_I where $|I| = n-1$ satisfies the (-1)-condition, they coincide. In the remaining cases one needs to check, if there is a graph automorphism $d \in D$ such that $j = (w_0, d)$. The semi-compatible edge graphs are shown in Figure 6.1-6.3.

Furthermore, we can easily extend Lemma 4.2.7 and Theorem 4.2.10. By the action of $H = W \rtimes D$ on $\mathcal{U}(W, P^n)$, it is possible that a chamber is filled with $\mathcal{P}$ multiple times using $\mathcal{J}$. To exclude this, we need to check whether there are two elements $(w_1, d_1), (w_2, d_2) \in \mathcal{J}$ where $w_1, w_2 \in W$ such that $w_1 = w_2$ and $d_1, d_2 \in D$ such that $d_1 \neq d_2$. If this is the case, it remains to check (4.1), to see whether $\mathcal{S}$ is embedded or has self-intersection. If $\mathcal{P}$ is symmetric under the action of D and a chamber is filled multiple times with $\mathcal{P}$ using $\mathcal{J}$, we need to discuss whether $\mathcal{S}$ is a multiple cover.

In the following, we will discuss the cases $\mathbb{S}^3$ and $\mathbb{S}^4$ for semi-compatible edge cycles.

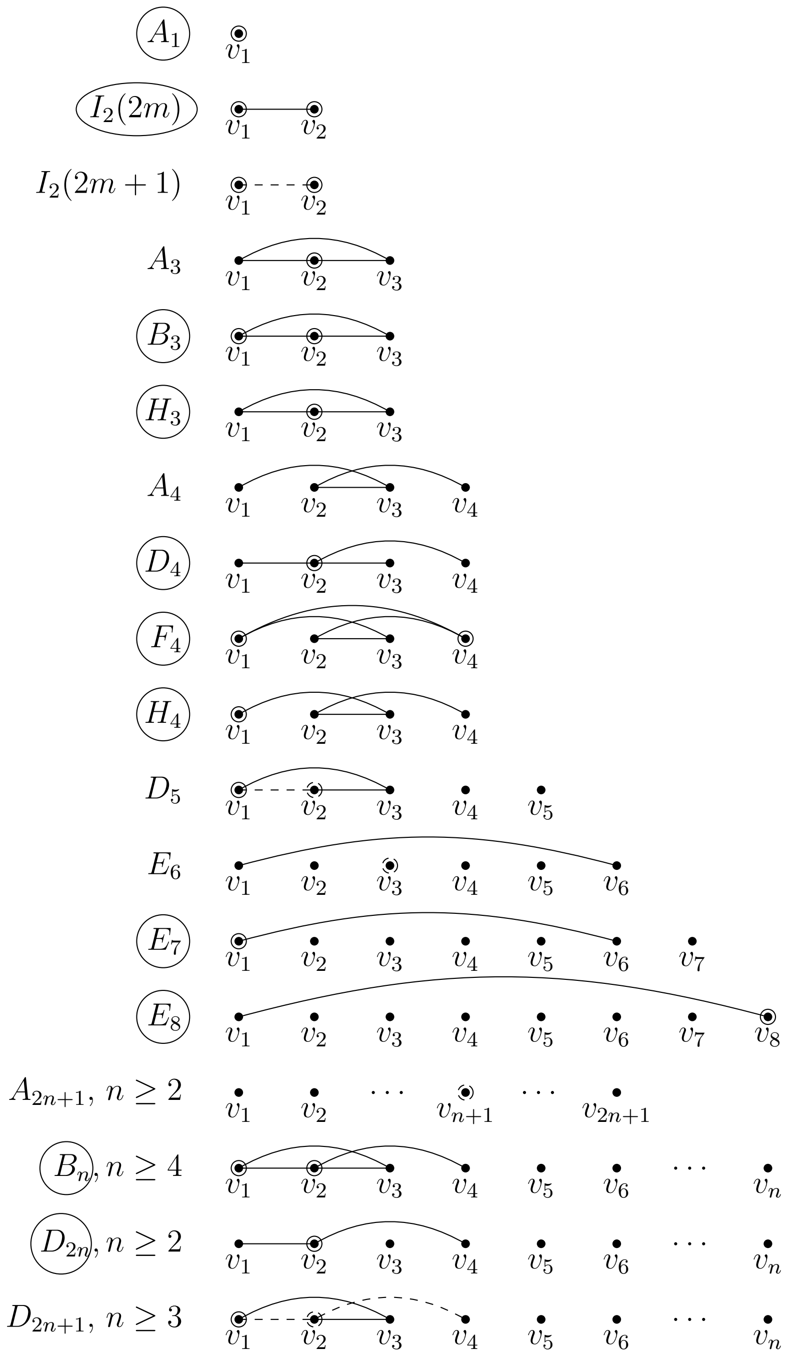


Figure 6.1: The irreducible spherical semi-compatible edge reflection graphs. The extensions of the edge reflection graph in Figure 4.12 are indicated with dashed lines or circles.

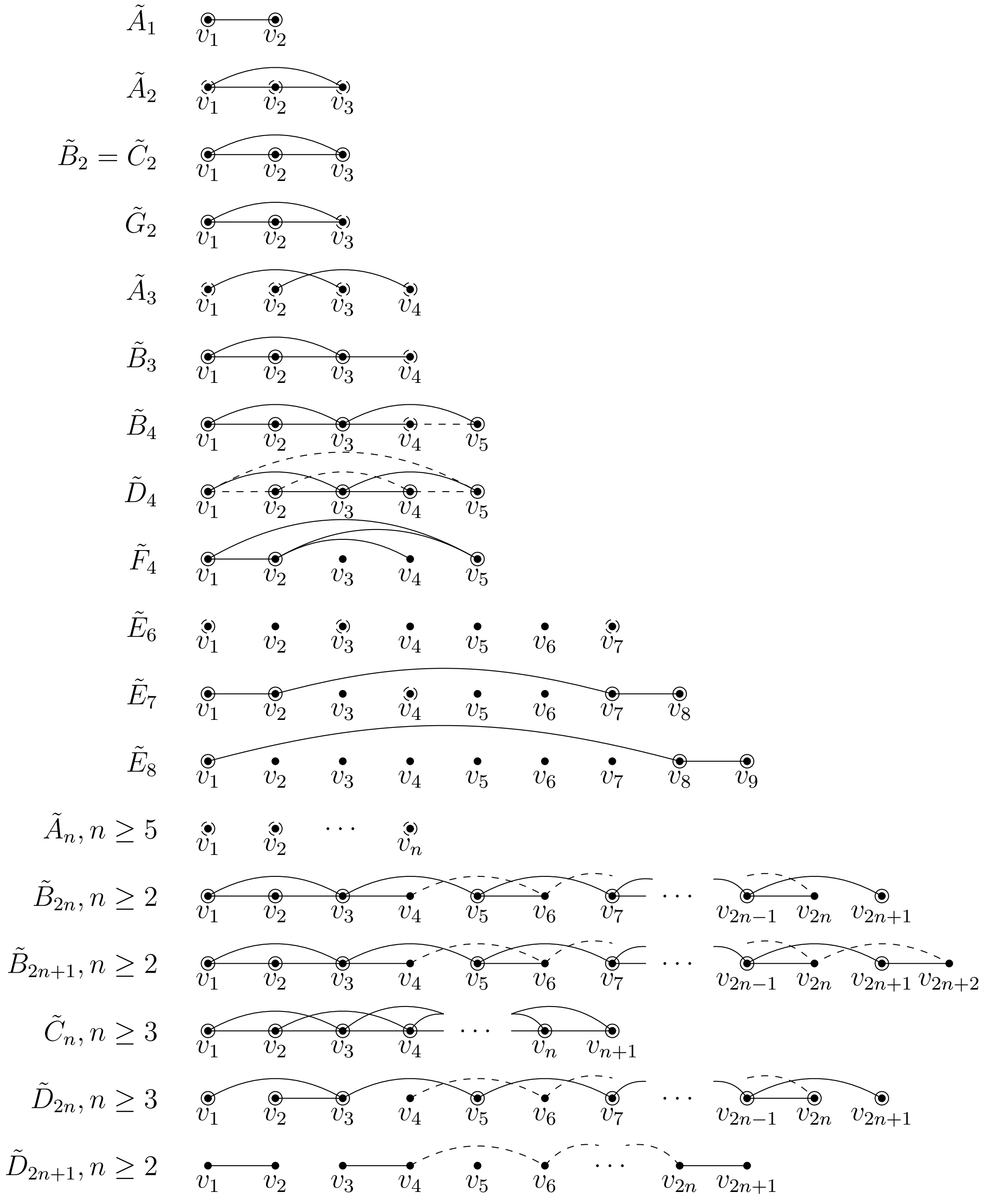


Figure 6.2: The irreducible Euclidean semi-compatible edge reflection graphs.

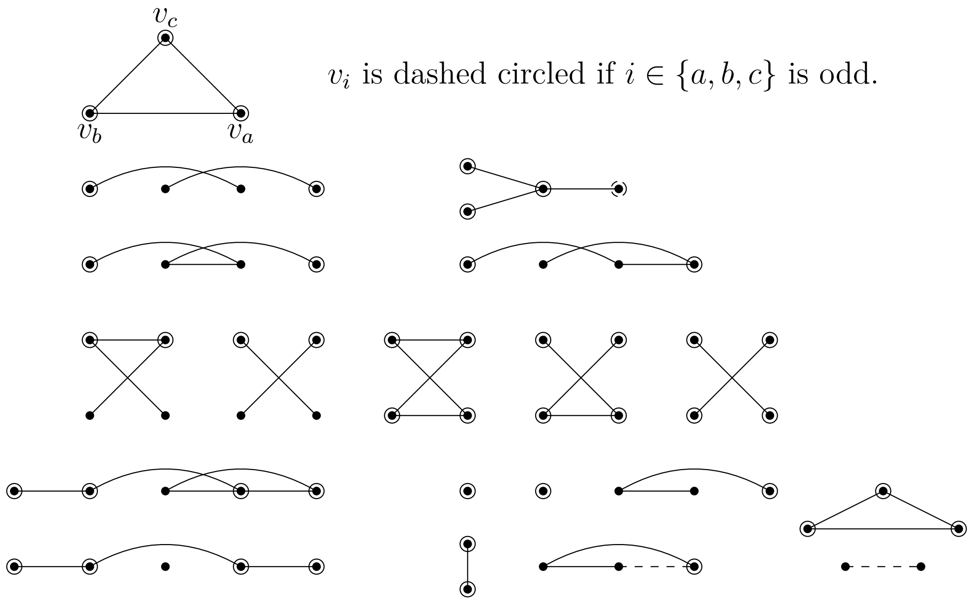
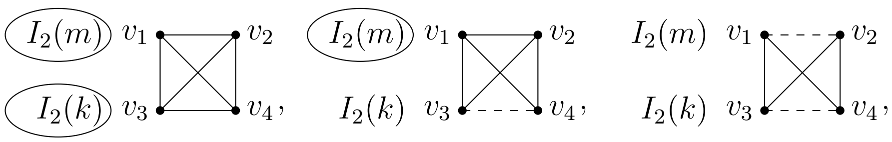


Figure 6.3: The irreducible hyperbolic semi-compatible edge reflection graphs.

Surfaces in $\mathbb{S}^3$

Compare Figure 4.12 and Figure 6.1. The only change we can see for edge reflection graphs up to rank four is the dashed edge for a Coxeter group of type $I_2(2m+1)$. Thus consider a Coxeter system of type $I_2(m) \times I_2(k)$. The semi-compatible edge reflection graphs are given by:



where, respectively, both m and k are even, m is even and k is odd, both m and k are odd. Consider the edge cycle $\Gamma = v_1v_2v_3v_4$. In the first case, we have shown in Theorem 5.4.1 that Γ yields an embedded surface $\mathcal{S}$ if and only if $m = k$.

Assume that m is even and k is odd. Then $H = W \rtimes D$ where W is the Coxeter group generated by the facet reflection s_i and D is generated by the graph automorphism d_{12} interchanging s_1 with s_2. Let $w_{34} := (s_3s_4)^{(k-1)/2}$.

Then Γ yields the edge reflection group $\mathcal{J}$ generated by

$$j_1 = (w_{34}, d_{12}), \quad j_2 = (s_1 s_4, e), \quad j_3 = ((s_1 s_2)^{m/2}, e), \quad j_4 = (s_2 s_3, e).$$

Since $(j_3 j_4)^2 = e$, we have $|\mathcal{J}(v_3)| = 4$. But

$$\begin{aligned}(j_1 j_2)^2 &= j_1 j_2 \cdot (w_{34} s_2 s_4, d_{12}) \\ &= j_1 \cdot (s_1 s_4 w_{34} s_2 s_4, d_{12}) \\ &= (w_{34} s_2 s_4 w_{34} s_1 s_4, e) = (s_2 s_1, e).\end{aligned}$$

Thus $(s_2 s_1, e) \in \mathcal{J}$, but $(s_2 s_1, e) \notin \mathcal{J}(v_3)$ except $m = 2$. If $m = 2$, then $j_2 j_3 j_4 = (s_4 s_3, e)$; hence $(j_2 j_3 j_4)^{(k-1)/2} j_1 = (s_3, d_{12})$. Thus $\mathcal{S}$ is not embedded.

Now, assume m and k are both odd. Then D is generated by d_{12} and d_{34} where d_{ij} interchanges s_i and s_j. Let $w_{12} := (s_1 s_2)^{(k-1)/2}$ and $w_{34} := (s_3 s_4)^{(k-1)/2}$. Thus we have

$$\mathcal{J} = \langle (w_{34}, d_{12}), (s_1 s_4, e), (w_{12}, d_{34}), (s_2 s_3, e) \rangle.$$

As above we have $(s_2 s_1, e) \in \mathcal{J}$; hence $((s_1 s_2)^{(m-1)/2}, e) \in \mathcal{J}$. Furthermore $((s_1 s_2)^{(m-1)/2}, e) \cdot (w_{12}, d_{34}) = (s_1, d_{34})$; hence $\mathcal{S}$ is not embedded.

Theorem 6.1.5. *Let P^3 be a Coxeter 3-simplex in $\mathbb{S}^3$ and (W, S) its Coxeter system. Assume that Γ is a semi-compatible full-dimensional edge cycle. Then the surface $\mathcal{S}$ has self-intersections.*

The semi-direct product construction can also be included in GAP. For the example in $\mathbb{S}^3$, see Listing 6.1.

Listing 6.1: GAP code for the group of type $I_2(m) \times I_2(k)$ and the semi-compatible edge cycle Γ where m and k are both odd.

```
#As before define W.
G:=FreeGroup("1","2");;
m:=3;;
k:=3;;
Rm:=[G.1^2,G.2^2,(G.1*G.2)^m];;
Rk:=[G.1^2,G.2^2,(G.1*G.2)^k];;
I2m:=G/Rm;;
I2k:=G/Rk;;
W:=DirectProduct(I2m,I2k);;
```

```
#Define the graph automorphism d12 and d34.
d12:=GroupHomomorphismByImages(W,W,[W.1,W.2,W.3,W.4],
[W.1,W.2,W.4,W.3]);;
d34:=GroupHomomorphismByImages(W,W,[W.1,W.2,W.3,W.4],
[W.2,W.1,W.3,W.4]);;
#Define D as a subgroup of the automorphism group generated
#by d12 and d34.
D:=Subgroup(AutomorphismGroup(W), [d12,d34]);;
#Define the semi-direct product H.
K:=SemidirectProduct(D,W);;
#This is just a rearrangement of the generators.
Embedding(K,1);;
Embedding(K,2);;
H:=Subgroup(K, [K.3,K.4,K.5,K.6,K.1,K.2]);;
#Define edge reflection group J.
J1:=(H.3*H.4)^((k-1)/2)*H.3*H.6;;
J2:=H.1*H.4;;
J3:=(H.1*H.2)^((m-1)/2)*H.1*H.5;;
J4:=H.2*H.3;;
J:=Subgroup(H,[J1,J2,J3,J4]);;
```

Surfaces in $\mathbb{S}^4$

For $\mathbb{S}^4$ only the edge reflection graphs for the Coxeter group of type $B_3 \times I_2(m)$ and $I_2(m) \times I_2(k) \times A_1$ change.

Let W be a Coxeter group of type $B_3 \times I_2(m)$. The only semi-compatible edge cycle which is not compatible is $\Gamma = v_1v_3v_2v_4v_5$.

Let W be a Coxeter group of type $I_2(m) \times I_2(k) \times A_1$. If m is even and k is odd, we have 4 edge cycles, i.e,

$$\Gamma_1 = v_1v_2v_3v_4v_5 \qquad \Gamma_2 = v_1v_2v_3v_5v_4$$
$$\Gamma_3 = v_1v_3v_2v_4v_5 \qquad \Gamma_4 = v_1v_3v_4v_2v_5.$$

If both m and k are odd then Γ_2 and Γ_4 are symmetric. Using GAP, one can show that all edge cycles lead to a surface with self-intersections.

Theorem 6.1.6. *Let P^4 be a Coxeter polytope in $\mathbb{S}^4$ and (W, S) its Coxeter system. If Γ is a semi-compatible full-dimensional edge cycle on P^4, then the surface $\mathcal{S}$ has self-intersections.*

Surfaces for $|W| = \infty$

So far, any semi-compatible edge cycle which is not compatible yields a surface $\mathcal{S}$ with self-intersections. We do not know whether there is an edge cycle resulting in an embedded surface $\mathcal{S}$. If W contains an Euclidean or hyperbolic factor, i.e., $|W| = \infty$, then the automorphism groups cannot be computed with GAP, using the same approach as in Listing 6.1. Using GAP, the problem with the automorphism group is that it requires the group to be a permutation group or a polycyclic group. For other representation the calculation is very slow or cannot be done.

However, in case $\mathbb{X} = \mathbb{H}^n$, $n \geq 3$, there is no significant change in the edge reflection graphs such that there is a feasible semi-compatible edge cycle.

Theorem 6.1.7. *Let σ^n be a Coxeter n-simplex in $\mathbb{H}^n$, $n \geq 3$ and (W, S) its corresponding Coxeter system. Let Γ be a semi-compatible full-dimensional edge cycle. Then $\mathcal{S}$ has self-intersections.*

6.2 Edge Cycles in $\mathcal{U}(W, P^n)$

So far, we mostly assumed that the Coxeter polytope P^n is an n-simplex. The only exception is if the corresponding Coxeter system has a factor of type $\tilde{A}_1$, e.g., the polytope was a prism or a hypercube. In this section, we will generalise this approach and discuss edge cycles defined on the edge set of $\mathcal{U}(W, P^n)$ and not only on P^n. This allows us to have a much broader approach in defining edge cycles and open up the limitation of P^n being an n-simplex. For example, in $\mathbb{S}^3$ we are able to define the $\eta_{n,k}$-surfaces Lawson constructed in [Law70].

Instead of defining an edge cycle on P^n, we define an edge cycle in $\mathcal{U}(W, P^n)$, i.e., on multiple chambers wP^n. Our goal is to extend this definition by using all chambers of $\mathcal{U}$ and not just the fundamental domain.

Definition 6.2.1. Let $P^n \subset \mathbb{X}^n$ be a Coxeter n-simplex (W, S) its Coxeter system and $\mathcal{U}(W, P^n) = W \times P^n/\sim$ the tessellation of $\mathbb{X}^n$ tiled by P^n. Furthermore, let $G = (V, E)$ be the edge reflection graph of P^n. Set $\mathcal{U}(V) \coloneqq \{wv_i \mid w \in W, v_i \in V\}$, $\mathcal{U}(E) \coloneqq \{we_i \mid w \in W, e_i \in E\}$, and define the graph $\mathcal{U}(G) \coloneqq (\mathcal{U}(V), \mathcal{U}(E))$. We call $\mathcal{U}(G)$ the *edge reflection graph* of $\mathcal{U}(W, P^n)$.

Essentially, the vertex set and edge set of $\mathcal{U}(G)$ consists of all vertices and edges of the tessellation $\mathcal{U}(W, P^n) = W \times P^n/\sim$ of $\mathbb{X}^n$, i.e., all vertices of every chamber where equivalent vertices are identified by $\sim$, and the edge set consists of all edges e_i (again identified by $\sim$) of every chamber which satisfies the (-1)-condition or where the edge reflection j_i can be expressed as an element of $H = W \rtimes D$ in the semi-compatible case.

Example 6.2.2. Let $P^2 \subset \mathbb{S}^2$ such that (W, S) is of type $(A_1)^3$. We have $|V| = 3$ and all edges satisfy the (-1)-condition, thus $|E| = 3$. The tessellation $\mathcal{U}(W, P^n)$ consists of 8 chambers where (after applying a suitable isometry) 4 meet at the north pole and again 4 meet at the south pole. Thus $|\mathcal{U}(V)| = 6$ and $|\mathcal{U}(E)| = 8$.

Let e be an edge of P^n and j be the edge reflection along e. Then we is an edge of the chamber wP^n. Let j' be the edge reflection along we. We can express j' as the conjugation of j using w, i.e., $j' = w^{-1}jw$, by mapping wP^n onto P^n, using j to edge reflect along e, and mapping P^n back onto wP^n.

Definition 6.2.3. We call a cycle Γ in $\mathcal{U}(G)$ *edge cycle* on $\mathcal{U}(W, P^n)$. Furthermore, we call the group $\mathcal{J}$ generated by the edge reflection along the edges of Γ again *edge reflection group* associated with Γ.

Recall the assumptions for the surface $\mathcal{P}$, for an edge cycle defined on P^n. We take an embedded surface $\mathcal{P}$ which lies in the interior of P^n and has boundary Γ. Similarly, we want to define a surface for an edge cycle on $\mathcal{U}(W, P^n)$. Let Γ be an edge cycle in $\mathcal{U}$. In the base space $\mathcal{U}(\mathcal{J}, \mathcal{Q}^2)$, the edge cycle Γ bounds a finite number of chambers, by the Jordan Curve Theorem. Since the surface $\mathcal{S}$ maps chambers onto chambers, we can interpret that Γ is defined on a finite union of chambers U in $\mathcal{U}$. Note that a chamber wP^n is convex, but the union U is not necessarily convex. A feasible assumption is that $\mathcal{P}$ lies in the interior of the $\mathrm{conv}(U)$ the convex hull of U. But conv(U) is not well matched with the structure of $\mathcal{U}$. To use the structure of $\mathcal{U}$, we look at the smallest union of all chambers $\mathcal{C}$ in $\mathcal{U}$ such conv(U) $\subset \mathcal{C}$.

Definition 6.2.4. Let Γ be a curve in $\mathcal{U}$ and U be the union of all chambers $w_i P^n$ bounded by Γ. Furthermore, let $\mathcal{C}$ be the union of chambers such that conv(U) $\subset \mathcal{C}$. We call $\mathcal{C}$ the *chamber hull* of Γ.

If Γ is defined on P^n, the chamber hull is P^n. If U is convex, then $\mathcal{C} = U$.

Example 6.2.5. Let Δ be an equilateral triangle. Then the corresponding Coxeter system (W, S) is of type $\tilde{A}_2$ and $\mathcal{U}(W, \Delta)$ is the tessellation of $\mathbb{E}^2$ by Δ. Define Γ as in Figure 6.4. Then U is given by the union of the four chambers P^n, $s_2\Delta$, $s_1s_2\Delta$, and $s_1s_2s_1\Delta$. Set $T = \{s_1, s_2\}$. Thus the chamber hull $\mathcal{C}$ is given by $W_T\Delta$.

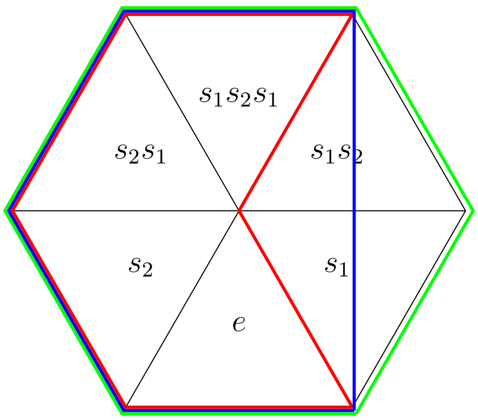


Figure 6.4: The edge cycle Γ is given in red which bounds U. The blue cycle is the boundary conv(U) and the green cycle is the boundary of the chamber hull $\mathcal{C}$.

Assuming that $\mathcal{P}$ lies in the chamber hull $\mathcal{C}$, we can easily adapt Lemma 4.2.7. For Theorem 4.2.10, we need to check if $\mathcal{S}$ has self-intersections in $\mathcal{C}$ and then check (4.1).

Definition 6.2.6. Let Γ be an edge cycle in $\mathcal{U}(G)$. We say that Γ is *full-dimensional* if Γ is not contained in a facet of the chamber hull $\mathcal{C}$.

Surfaces in $\mathbb{S}^3$

The $\eta_{m-1,k-1}$ surfaces constructed by Lawson in [Law70] can be derived from a full-dimensional edge cycle defined on a chamber hull. We will prove that $\eta_{m-1,k-1}$ has self-intersection using the notation of the chamber hull.

Consider a Coxeter polytope $P^3 \subset \mathbb{S}^3$ such that (W, S) is of type $I_2(m) \times I_2(k)$. First, assume that $m = k = 2$. Then $\mathcal{U}(W, P^n)$ is tiled by 16 chambers. Define the full-dimensional edge cycle $\Gamma = v_1 v_4 s_1(v_1) s_1(v_2) v_2 v_3$ which lies in the chamber hull $\mathcal{C} := P^n \cup s_1 P^n$, see Figure 6.5.

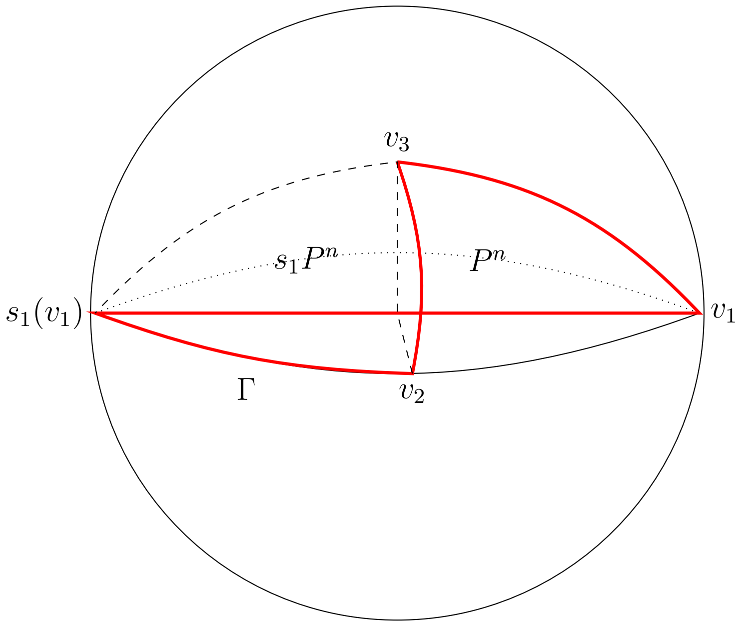


Figure 6.5: The chamber hull $\mathcal{C}$ is the union of the two right-angled chambers $s_1 P^n$ and P^n. The edge cycle Γ (red) is the boundary of the surface $\eta_{1,1}$ constructed by Lawson.

Note that $v_1 v_4 s_1(v_1)$ is one geodesic arc in $\mathbb{S}^3$ and $s_1(v_2) = v_2$; hence Γ can be shortened to $\Gamma = v_1 s_1(v_1) v_2 v_3$. Thus Γ is given by 4 edges. This can also be seen in the edge reflection group. It is given by

$$\begin{aligned}\mathcal{J} &= \langle s_2 s_3, s_1 s_2 s_3 s_1, s_1 s_3 s_4 s_1, s_1 s_4, s_2 s_4 \rangle \\ &= \langle s_2 s_3, s_2 s_3, s_3 s_4, s_1 s_4, s_2 s_4 \rangle \\ &= \langle s_2 s_3, s_3 s_4, s_1 s_4, s_2 s_4 \rangle\end{aligned}$$

where in the second equation we used that all s_i commute. It is easy to see that $s_1s_2, s_1s_3 \in \mathcal{J}$. Consider the vertex v_4 and note that it lies in the relative interior of the edge incident to $v_1s_1(v_1)$. We show that v_4 is a self-intersection. Note that s_1s_2 and s_1s_3 fix v_4. Thus $v_4 \in \mathcal{C}, s_1s_2(\mathcal{C}), s_1s_3(\mathcal{C})$. Hence there are three points p_1, p_2, p_3 in the base space $\mathcal{U}(\mathcal{J}, \mathcal{Q}^2)$ such that $\mathcal{S}(p_1) = \mathcal{S}(p_2) = \mathcal{S}(p_3) = v_4$. Thus v_4 is a self-intersection.

Second, let $T \coloneqq \{s_1, s_2\}$ and define the a chamber hull by $\mathcal{C} = \bigcup_{w \in I} wP^n$ where $I = \{w \in W_T \mid w = e \text{ or every reduced expression of } w \text{ start with } s_1\}$. Essentially, the set I is a quarter sphere; hence convex. Let w_{12} be the longest element in W_T and define the edge cycle $\Gamma \coloneqq v_1w_{12}(v_1)v_2v_3$.

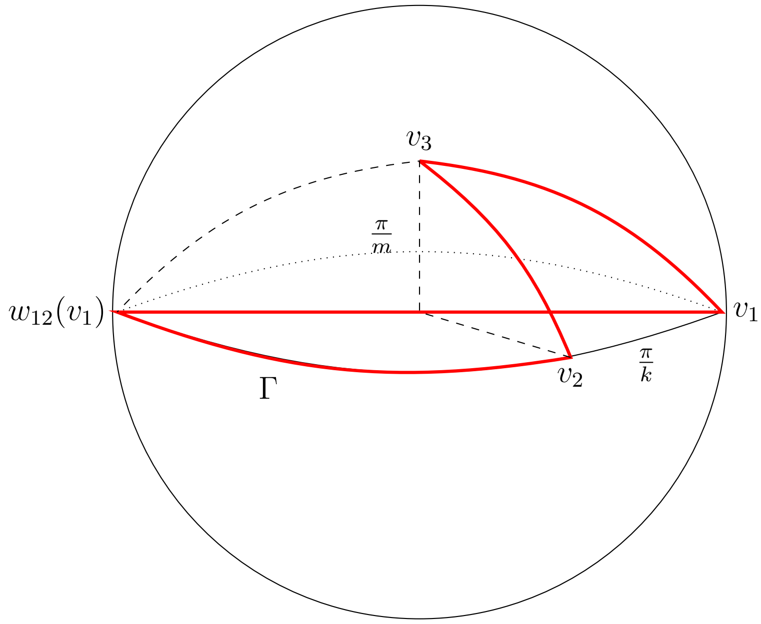


Figure 6.6: The chamber hull $\mathcal{C}$ is the union of all chambers bounded by the edge cycle Γ (red) which is the boundary of the surface $\eta_{m-1,k-1}$ constructed by Lawson.

If k is even, the edge reflection group is given by

$$\mathcal{J} = \langle s_2s_3, (s_3s_4)^{k/2}, s_1s_4, s_2s_4 \rangle.$$

If k is odd, then Γ is no longer compatible but semi-compatible. The edge reflection group is then given by

$$\mathcal{J} = \langle (s_2s_3, e), ((s_3s_4)^{k/2-1}s_3, d_{12}), (s_1s_4, e), (s_2s_4, e) \rangle.$$

As above, we see that s_1s_2, $s_1s_3 \in \mathcal{J}$ respectively (s_1s_2, e), $(s_1s_3, e) \in \mathcal{J}$ making v_4 is a self-intersection.

Surfaces in $\mathbb{H}^2 \times \mathbb{E}$

So far, we have looked at simplices in hyperbolic space. We would like to address a problem mentioned in [Ale16]. Using a Coxeter polytope in $\mathbb{H}^2 \times \mathbb{E}$, we define an edge cycle to construct helicoid-like surfaces. Given a prism in $\mathbb{H}^2 \times \mathbb{E}$ where the base is a 8-gon, Alex defined a bow tie-like curve (see Figure 6.7) on the 8-gon in a tessellation where four 8-gons are incident in every vertex and showed that if a special tessellation can be inscribed, then the Plateau solution extends to a complete minimal surface (see [Ale16], Theorem 2.9.2).

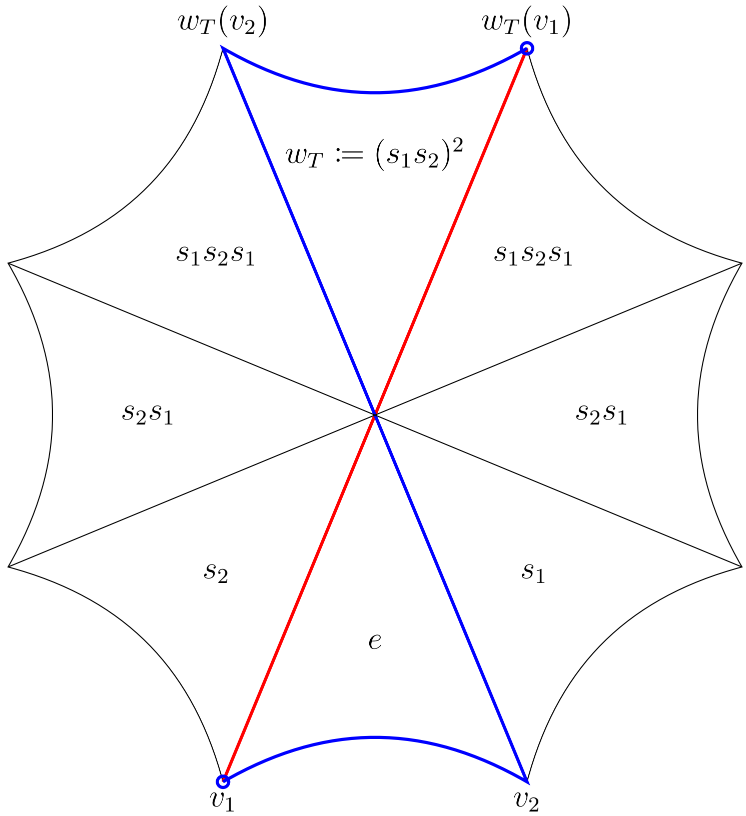
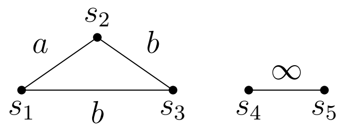


Figure 6.7: The projection of the edge cycle Γ on a hyperbolic 8-gon where edges in height a are blue and the edge in height b is red. The blue circles at the vertices v_1 and $w_T(v_1)$ correspond to vertical lines, e.g., the edge $v_1 v_4$.

We will describe the general problem with a tessellation using $2n$-gon where k of the $2n$-gons are incident in every vertex using the full-compatible edge cycles defined on a convex hull.

Let $P^3 \coloneqq \sigma^2 \times I$ be a Coxeter polytope in $\mathbb{H}^2 \times \mathbb{E}$ such that its corresponding Coxeter system (W, S) has following Coxeter graph



where $2 \leq a \leq b$ and $\frac{1}{a} + \frac{2}{b} < 1.$, i.e., σ^2 is a hyperbolic triangle with angles π/a, π/b, and π/b. Let $\mathcal{U}(W, P^3)$ be the Coxeter complex. Set $T \coloneqq \{s_1, s_2\}$ and let $\mathcal{C} \coloneqq W_T(\sigma^2 \times I)$. Thus $\mathcal{C}$ is the product of a hyperbolic $2a$-gon with the interval I where the $2a$-gon is subdivided barycentrically, i.e., opposite vertices are connected. Hence $\mathcal{U}(W, P^n)$ can be seen as a tessellation by $\mathcal{C}$ where b copies of $\mathcal{C}$ are incident in every copy of $\mathcal{C}$.

We define a bow tie-like edge cycle on P^3 as shown in Figure 6.7. Alex showed that in this construction a and b have to be even. Otherwise the surface has self-intersection (see [Ale16], Proposition 2.9.4). Thus we assume that a and b are even. Let w_T be the longest element of W_T. We define an edge cycle consisting of 6 edges as follows: $\Gamma \coloneqq v_1 v_2 (w_T)(v_2)(w_T)(v_1) w_T(v_4) v_4$. Since a and b are even, Γ is compatible.

The edge reflection group is given by

$$\mathcal{J} = \langle s_3 s_4, s_1 s_4, w_T s_3 s_4 w_T, w_T (s_2 s_3)^{b/2} w_T, s_2 s_5, (s_2 s_3)^{b/2} \rangle.$$

Let $\mathcal{P} \subset \mathcal{C}$ be a surface with boundary Γ. We extend $\mathcal{P}$ using $\mathcal{J}$ to a surface $\mathcal{S}$ and want to check whether $\mathcal{S}$ has self-intersections. First, we will exclude intersections in $\mathcal{C}$, by deriving conditions for a and b. Those conditions will depend whether $\mathcal{P}$ is symmetric under the action of w_T. Note that there are two helicoid-like movements: one is generated by the element $s_1 s_2 s_4 s_5$ and the other by $s_3 s_2 s_4 s_5$. Here $s_1 s_2$ and $s_3 s_2$ rotate the surface by angle of $2\pi/a$ or $2\pi/b$ respectively, and $s_4 s_5$ is an translation by $2|I|$. Consider the subgroup

$$\mathcal{J}' \coloneqq \langle s_1 s_2 s_4 s_5, s_3 s_2 s_4 s_5, (s_2 s_3)^{b/2} \rangle.$$

Since $(s_1 s_2)^a = (s_2 s_3)^b = e$ we have that $(s_4 s_5)^a$, $(s_4 s_5)^b \in \mathcal{J}'$. Furthermore, as $(s_2 s_3)^{b/2} \in \mathcal{J}'$, we get $(s_4 s_5)^{b/2} \in \mathcal{J}'$. If $\mathcal{P}$ is not w_T-symmetric, we get $a = b/2$. Otherwise, we can use the translations $(s_4 s_5)^a$ or $(s_4 s_5)^{b/2}$ to move a rotated version of $\mathcal{P}$ into $\mathcal{C}$ resulting in an self-intersection in the midpoint of the edge $v_2 w_T(v_2)$. Analogously, if $\mathcal{P}$ is w_T-symmetric, we get that $w_T \in \mathcal{J}$; hence $(s_4 s_5)^{a/2} \in \mathcal{J}$ resulting in $a = b$.

If $\mathcal{P}$ is w_T-symmetric, we have $w_T \in \mathcal{J}$. In this case $\mathcal{J}$ coincide with the edge reflection group defined on the polytope $\sigma_2 \times I$ in $\mathbb{H}^2 \times \mathbb{E}$. Theorem 5.4.5 shows that $\mathcal{S}$ is embedded for $a, b \leq 12$. If $\mathcal{P}$ is not w_T-symmetric, we need to check, similarly to Theorem 5.4.5, that (4.1) is satisfied in every vertex of Γ.

Theorem 6.2.7. *Let $P^3 = \sigma_2 \times I$ be a polytope in $\mathbb{H}^2 \times \mathbb{E}$ where σ_2 is a Coxeter 2-simplex in $\mathbb{H}^2$ and I an interval and (W, S) is the Coxeter system of type $Z_3(a, b, b) \times \tilde{A}_1$ corresponding to P^3. Let $T = \{s_1, s_2\}$, w_T the longest element of W_T, and $\mathcal{C} = W_T P^3$ be a convex hull. Furthermore, let $\Gamma \coloneqq v_1 v_2 (w_T)(v_2)(w_T)(v_1) w_T(v_4) v_4$ be a full-dimensional edge cycle on $\mathcal{C}$. Assume that Conjecture 5.4.6 is true. If $\mathcal{P}$ is symmetric under the action of w_T, then $\mathcal{S}$ is embedded if and only if $a = b$. If $\mathcal{P}$ is not w_T-symmetric, then $\mathcal{S}$ is embedded if and only if $a = b/2$.*

Appendix

MATLAB codes

FindElemCircuits.m

We use the Johnson algorithm for finding all simple cycles in an edge reflection graph. Given an adjacency matrix A, **FindElemCircuits.m** finds all simple cycles and stores them in a cell. This is an implementation of Chris Maes.

Listing 2: MATLAB code for finding all simple circles.

```
%Given an adjacency matrix A, FindElemCircuits.m returns
%the number of all simple cycles and a cell containing all
%simple cycles.
%https://gist.github.com/cmaes/1260153
function [numcycles,cycles] = FindElemCircuits(A)
if ~issparse(A)
        A = sparse(A);
end
n = size(A,1);
Blist = cell(n,1);
blocked = false(1,n);
s = 1;
cycles = {};
stack=[];

function unblock(u)
        blocked(u) = false;
        for w=Blist{u}
                if blocked(w)
                        unblock(w)
                end
        end
        Blist{u} = [];
end

function f = circuit(v, s, C)
        f = false;

        stack(end+1) = v;
```

```
		blocked(v) = true;

		for w=find(C(v,:))
			if w == s
				cycles{end+1} = [stack s];
				f = true;
			elseif ~blocked(w)
				if circuit(w, s, C)
					f = true;
				end
			end
		end

		if f
			unblock(v)
		else
			for w = find(C(v,:))
				if ~ismember(v, Blist{w})
					Bnode = Blist{w};
					Blist{w} = [Bnode v];
				end
			end
		end

		stack(end) = [];
end


while s < n

		% Subgraph of G induced by {s, s+1, ..., n}
		F = A;
		F(1:s-1,:) = 0;
		F(:,1:s-1) = 0;

		%components computes the strongly connected
		%components of a graph. This function is implemented
		%in Matlab BGL
		% http://dgleich.github.com/matlab-bgl/
		[ci, sizec] = components(F);
```

```
            if any(sizec >= 2)

                    cycle_components = find(sizec >= 2);
                    least_node =
                    find(ismember(ci, cycle_components),1);
                    comp_nodes =
                    find(ci == ci(least_node));

                    Ak = sparse(n,n);
                    Ak(comp_nodes,comp_nodes) =
                    F(comp_nodes,comp_nodes);

                    s = comp_nodes(1);
                    blocked(comp_nodes) = false;
                    Blist(comp_nodes)
                    = cell(length(comp_nodes),1);
                    circuit(s, s, Ak);
                    s = s + 1;

            else
                    break;
            end
        end

        numcycles = length(cycles);

end
```

EdgeCurve.m

EdgeCurve.m checks if the length of each entry in the cell given by **FindElemCircuits.m** coincides with a given number L_0 and deletes all other curves. Then it converts the cell into a matrix Q containing all edge cycles of length L_0.

Listing 3: MATLAB code for selecting all simple circles of fixed length.

```
%Given a length L0 and the numcycles, cylces from
```

```
%FindElemCircuits.m, EdgeCurve.m returns a matrix with
%all cycles of length L0.
function Q = EdgeCurve(L0,numcycles,cycles)
        Q=zeros(numcycles,L0+1);
        for i = 1:numcycles
                D=length(cell2mat(cycles(numcycles-i+1)));
                if D == L0+1
                        Q(numcycles-i+1,:) =
                        cell2mat(cycles(numcycles-i+1));
                else
                        Q(numcycles-i+1,:) = [];
                end
        end
end
```

GenerateSymmetry.m

Given a permutation matrix T representing a subset of symmetries, **GenerateSymmetry.m** constructs all possible combinations, by applying each permutation onto another. Multiple combinations are then deleted and the resulting permutation matrix is named S. The order of the symmetry group can be calculated by hand and one can compare it with the column size of S. This can be used as a safety check to check if the permutations in T indeed generates S.

Listing 4: MATLAB code for generating symmetries.

```
%Given a matrix containing a subset of symmetries T,
%GenerateSymmetry.m finds all symmetries S that are
%generated by T.
function S = GenerateSymmetry(T)
        [~,h] = size(T);
        v = [1:1:h];
        S=v;
        if isequal(T,S)
                S=T;
        else
                while ~isequal(S,T)
                        S=T;
```

```
            [s,h] = size(S);
            for i=1:1:s
                a = T(i,:);
                for j=1:1:s
                    b = T(j,:);
                    c = zeros(1,h);
                    for k = 1:1:h
                        c(a(k)) = b(k);
                    end
                T = unique([T;c], 'rows');
                end
            end
        end
    end
end
```

Symmetry.m and SymmetryAll.m

Given a permutation matrix S with two rows (the identity and the permutation) and a matrix Q containing all edge cycle of a fixed length, **Symmetry.m** applies a permutations in S to a single curve in Q and deletes duplicates, i.e., a symmetric curve. This is done for all curves. The routine **SymmetryAll.m** applies **Symmetry.m** for all symmetries for a Coxeter polytope P^n. All non-symmetric curves are stored in a matrix P. The routine applies all symmetries of P^n to curve and checks whether a duplicate was generated. Since this is done for all curves, these routines are rather slow for higher dimensional problems. In the three-dimensional case the routine **SymmetryAll.m** took seconds. For the tesseract where Q stores about 30000 curves and the symmetry group has order 384, the routine took about a minute.

Listing 5: MATLAB code for deleting symmetric edge reflection curves subject to one symmetry.

```
%Given a matrix Q and a permutation matrix S,
%Symmetry.m lists all rows of Q that are different
%subject to the symmetry S.
function P = Symmetry(Q,S)
len = size(Q,1);
for n = size(Q,1):-1:1
```

```
        Q(n,:) = flip(Q(n,:));
        Q = unique(Q,'rows','stable');
        if size(Q,1) == len
                Q(n,:) = flip(Q(n,:));
        else
                len = size(Q,1);
        end
end
for i = size(Q,1):-1:1
T = Q(i,:);
        for k = length(S):-1:1
                T(T==S(1,k)) = S(1,k) + length(S);
        end
        Q(i,:) = T;
        for k = length(S):-1:1
                T(T==S(1,k)+length(S)) = S(2,k);
        end
        Q(i,:) = T;
        Q = unique(Q,'rows','stable');
        if size(Q,1) == len
                Q = Q(:,1:size(Q,2)-1);
                j=0;
                len2 = size(Q,1);
                while j <size(Q,2)
                        j=j+1;
                        Q(i,:) =circshift(Q(i,:),1);
                        Q = unique(Q,'rows','stable');
                        if size(Q,1) == len
                                Q(i,:) = flip(Q(i,:));
                                Q = unique(Q,'rows','stable');
                                if size(Q,1) == len
                                        Q(i,:) = flip(Q(i,:));
                                else
                                        len = size(Q,1);
                                        j=size(Q,2);
                                end
                        else
                                len = size(Q,1);
                                j=size(Q,2);
                        end
```

```
            end
            Q = [Q,Q(:,1)];
            if len2 == len
                for k = length(S):-1:1
                    T(T==S(1,k)) = S(1,k) +
                    length(S);
                end
                Q(i,:) = T;
                for k = length(S):-1:1
                    T(T==S(1,k)+length(S)) = S(2,k);
                end
                Q(i,:) = T;
            end
            else
                len=size(Q,1);
            end
        end
        P=Q;
end
```

Fulldimensional.m

Given a matrix M, e.g., Q and P obtained by **EdgeCurve.m** and **SymmetryAll.m**, respectively, the routine **Fulldimensional.m** checks if the curve is contained in a facet F of P^n. Since the vertex set of the facet usually contains more edges than the vertex set of P^n which are not contained in F, I find it easier the check whether none of the vertices not contained in F are contained in the curve.

CreateCell.m and GAP.m

CreateCell.m creates a $k \times 2$ cell array out of the adjacency matrix A representing the edge reflection graph where k is the number of directed edges in P^n. The second dimension of the array needs to be filled with edge reflection as it is used in GAP. The routine **GAP.m** then translates all curves given in a matrix P. The result is a string array which can be copied to GAP.

Listing 6: MATLAB code for deleting symmetric edge reflection edge cycles subject to a number of symmetries.

```
%Given a matrix Q and a permutation matrix S,
%SymmetryAll.m lists all rows of Q that are different
%subject to the symmetries S.
%SymmetryAll.m uses the function Symmetry.m.
function P = SymmetryAll(Q,S)
        P=[];
        v=[1:1:size(S,2)];
        while ~isequal(size(P,1),size(Q,1))
                P=Q;
                for i=1:1:size(S,1)
                        Q = Symmetry(Q,[v;S(i,:)]);
                end
        end
        P=Q;
end
```

Listing 7: MATLAB code for deleting all edge cycles that are not full-dimensional.

```
%Given a matrix M and a set of numbers s,
%Fulldimensional.m deletes all rows which do not contain
%all numbers in s.
%If F is a facet of P^n, s consists of all
%numbers not contained in F.
function K = Fulldimensional(M,s)
        h = 0;
        len = size(M,1);
        for i =len:-1:1
                for k=1:length(s)
                        h = h + sum(M(i,:)==s(k));
                end
                if h == 0
                        M(i,:) = [];
                else
                h = 0;
                end
        end
        K=M;
```

```
end
```

Listing 8: MATLAB code creating a list of all edges in P^n.

```
%CreateCell.m creates a cell with all edges of P^n
%given by the adjecency matrix A.
function C = CreateCell(A)
        k=1;
        C = cell(sum(sum(A)),2);
        for i=1:size(A,1)
                for j=1:size(A,2)
                        if A(i,j) == 1
                                C{k,1} = [i,j];
                                k=k+1;
                        end
                end
        end
end
```

Listing 9: MATLAB code for creating the edge reflection group given in GAP from a cycle given in MATLAB.

```
%Given a cell C with all edges and their edge reflection
%given in a GAP code, a matrix P, and a
%starting length L0, GAP.m translates all rows
%in P to edge reflection groups in GAP.
function G = GAP(P,C,L0)
        G = cell(size(P,1),size(P,2)-1);
        for j=1:size(P,2)-1
                for i=1:size(P,1)
                        for k=1:size(C,1)
                                if P(i,j) == C{k,1}(1)
                                && P(i,j+1) == C{k,1}(2)
                                        G{i,j} = C{k,2};
                                end
                        end
                end
        end
        x = string(([1:1:size(P,1)]'+L0*ones(size(P,1),1)));
        G = join(G,",");
```

```
        J = "J";
        Sub = ":=Subgroup(W, [";
        End = "]);;";
        G = append(J,x,Sub,G,End);
        %fprintf('%s\n', G);
end
```

Listing 10: MATLAB code which gives a GAP code to check the index of $\mathcal{J}_i$ in W for all i. It also gives a list of GAP commands which ask if a list of elements is containd in $\mathcal{J}_i$.

```
%CheckEmb.m gets a number c, a set of numbers S, and
% a number L. It returns a string array E containing a GAP
%code with all permuted elements in S of length L in
%the edge reflection group J_c and a string
%array I containing a GAP code which computes the index
%of J_k, k = 1,...,c in W.
function [E,I] = CheckEmb(c,S,L)
        if length(S) == 1
                x = string(nchoosek([1:1:S],L));
                %x = string(permn([1:1:S],L));
        else
        for k=1:L
                x = string(nchoosek(S,k));
                %x = string(permn(S,k));
        end
        end
        M = append("W.",x);
        d = [1:1:c]';
        d = string(d);
        c = string(c);
        if L >1
                        M = join(M,"*");
        end
        E = append(M," in J",c,";");
        I = append("Index(W,J",d,");");
        %fprintf('%s\n', E);
        %fprintf('%s\n', I);
end
```

GAP codes

Listing 11 contains all types of irreducible Coxeter groups as they can be implemented in GAP. Products can be defined using **DirectProduct** function in GAP. The remaining listings contain all checks done in $\mathbb{H}^3$ and all four-dimensional cases.

Listing 11: GAP code for all relevant groups up to rank 5.

```
# Group of type $A_1$.
G:=FreeGroup("1");;
R:=[G.1^2];;
A1:=G/R;;

# Group of type $\tilde{A}_1$.
G:=FreeGroup("1","2");;
RInf:=[G.1^2,G.2^2];;
AInf:=G/RInf;;

# Groups of type $I_2(m)$ and $I_2(k)$.
G:=FreeGroup("1","2");;
m:=4;;
k:=4;;
Rm:=[G.1^2,G.2^2,(G.1*G.2)^m];;
Rk:=[G.1^2,G.2^2,(G.1*G.2)^k];;
I2m:=G/Rm;;
I2k:=G/Rk;;

# Group of type $\tilde{A}_2$, $\tilde{B}_2$, and $\tilde{G}_2$.
G:=FreeGroup("1","2","3");;
RA:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^3,(G.2*G.3)^3,(G.1*G.3)^3];;
RB:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^4,(G.2*G.3)^4,(G.1*G.3)^2];;
RG:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^3,(G.2*G.3)^6,(G.1*G.3)^2];;
A2Inf:=G/RA;;
B2Inf:=G/RB;;
G2Inf:=G/RH;;

# Groups of type $A_3$, $B_3$, and $H_3$.
G:=FreeGroup("1","2","3");;
```

```
RA:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^3,(G.2*G.3)^3,(G.1*G.3)^2];;
RB:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^3,(G.2*G.3)^4,(G.1*G.3)^2];;
RH:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^3,(G.2*G.3)^5,(G.1*G.3)^2];;
A3:=G/RA;;
B3:=G/RB;;
H3:=G/RH;;

# Groups of type Ã3, B̃3, C̃3, and D̃3.
G:=FreeGroup("1","2","3","4");;
RA:=[G.1^2,G.2^2,G.3^2,G.4^2,(G.1*G.2)^3,(G.2*G.3)^3,
(G.3*G.4)^3,(G.1*G.3)^2,(G.1*G.4)^3,(G.2*G.4)^2];;
RB:=[G.1^2,G.2^2,G.3^2,G.4^2,(G.1*G.2)^2,(G.2*G.3)^3,
(G.3*G.4)^4,(G.1*G.3)^3,(G.1*G.4)^2,(G.2*G.4)^2];;
RC:=[G.1^2,G.2^2,G.3^2,G.4^2,(G.1*G.2)^4,(G.2*G.3)^3,
(G.3*G.4)^4,(G.1*G.3)^2,(G.1*G.4)^2,(G.2*G.4)^2];;
A3Inf:=G/RA;;
B3Inf:=G/RB;;
C3Inf:=G/RC;;

# Groups of type A4, B4, D4, F4, and H4.
G:=FreeGroup("1","2","3","4");;
RA:=[G.1^2,G.2^2,G.3^2,G.4^2,(G.1*G.2)^3,(G.2*G.3)^3,
(G.3*G.4)^3,(G.1*G.3)^2,(G.1*G.4)^2,(G.2*G.4)^2];;
RB:=[G.1^2,G.2^2,G.3^2,G.4^2,(G.1*G.2)^3,(G.2*G.3)^3,
(G.3*G.4)^4,(G.1*G.3)^2,(G.1*G.4)^2,(G.2*G.4)^2];;
RF:=[G.1^2,G.2^2,G.3^2,G.4^2,(G.1*G.2)^3,(G.2*G.3)^4,
(G.3*G.4)^3,(G.1*G.3)^2,(G.1*G.4)^2,(G.2*G.4)^2];;
RD:=[G.1^2,G.2^2,G.3^2,G.4^2,(G.1*G.2)^3,(G.2*G.3)^3,
(G.3*G.4)^2,(G.1*G.3)^2,(G.1*G.4)^2,(G.2*G.4)^3];;
RH:=[G.1^2,G.2^2,G.3^2,G.4^2,(G.1*G.2)^3,(G.2*G.3)^3,
(G.3*G.4)^5,(G.1*G.3)^2,(G.1*G.4)^2,(G.2*G.4)^2];;
A4:=G/RA;;
B4:=G/RB;;
D4:=G/RD;;
F4:=G/RF;;
H4:=G/RH;;


# Groups of type Ã4, B̃4, C̃4, D̃4 and F̃4.
```

```
G:=FreeGroup("1","2","3","4","5");;
RA:=[G.1^2,G.2^2,G.3^2,G.4^2,G.5^2,(G.1*G.2)^3,(G.1*G.3)^2,
(G.1*G.4)^2,(G.1*G.5)^3,(G.2*G.3)^3,(G.2*G.4)^2,(G.2*G.5)^2,
(G.3*G.4)^3,(G.3*G.5)^2,(G.4*G.5)^3];;
RB:=[G.1^2,G.2^2,G.3^2,G.4^2,G.5^2,(G.1*G.2)^2,(G.1*G.3)^3,
(G.1*G.4)^2,(G.1*G.5)^2,(G.2*G.3)^3,(G.2*G.4)^2,(G.2*G.5)^2,
(G.3*G.4)^3,(G.3*G.5)^2,(G.4*G.5)^4];;
RC:=[G.1^2,G.2^2,G.3^2,G.4^2,G.5^2,(G.1*G.2)^4,(G.1*G.3)^2,
(G.1*G.4)^2,(G.1*G.5)^2,(G.2*G.3)^3,(G.2*G.4)^2,(G.2*G.5)^2,
(G.3*G.4)^3,(G.3*G.5)^2,(G.4*G.5)^4];;
RD:=[G.1^2,G.2^2,G.3^2,G.4^2,G.5^2,(G.1*G.2)^2,(G.1*G.3)^3,
(G.1*G.4)^2,(G.1*G.5)^2,(G.2*G.3)^3,(G.2*G.4)^2,(G.2*G.5)^2,
(G.3*G.4)^3,(G.3*G.5)^3,(G.4*G.5)^2];;
RF:=[G.1^2,G.2^2,G.3^2,G.4^2,G.5^2,(G.1*G.2)^3,(G.1*G.3)^2,
(G.1*G.4)^2,(G.1*G.5)^2,(G.2*G.3)^4,(G.2*G.4)^2,(G.2*G.5)^2,
(G.3*G.4)^3,(G.3*G.5)^2,(G.4*G.5)^4];;
A4:=G/RA;;
B4:=G/RB;;
C4:=G/RC;;
D4:=G/RD;;
F4:=G/RF;;

# Groups of type A5, B5, and D5.
G:=FreeGroup("1","2","3","4","5");;
RA:=[G.1^2,G.2^2,G.3^2,G.4^2,G.5^2,(G.1*G.2)^3,(G.1*G.3)^2,
(G.1*G.4)^2,(G.1*G.5)^2,(G.2*G.3)^3,(G.2*G.4)^2,(G.2*G.5)^2,
(G.3*G.4)^3,(G.3*G.5)^2,(G.4*G.5)^3];;
RB:=[G.1^2,G.2^2,G.3^2,G.4^2,G.5^2,(G.1*G.2)^3,(G.1*G.3)^2,
(G.1*G.4)^2,(G.1*G.5)^2,(G.2*G.3)^3,(G.2*G.4)^2,(G.2*G.5)^2,
(G.3*G.4)^3,(G.3*G.5)^2,(G.4*G.5)^4];;
RD:=[G.1^2,G.2^2,G.3^2,G.4^2,G.5^2,(G.1*G.2)^2,(G.1*G.3)^3,
(G.1*G.4)^2,(G.1*G.5)^2,(G.2*G.3)^3,(G.2*G.4)^2,(G.2*G.5)^2,
(G.3*G.4)^3,(G.3*G.5)^2,(G.4*G.5)^3];;
A5:=G/RA;;
B5:=G/RB;;
D5:=G/RD;;

# Group of type Z3.
G:=FreeGroup("1","2","3");;
a:=4;;
```

```
b:=6;;
c:=6;;
R:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^c,(G.1*G.3)^a,(G.2*G.3)^b];;
Z3:=G/R;;

# Group of type Z_4.
G:=FreeGroup("1","2","3","4");;
R:=[G.1^2,G.2^2,G.3^2,G.4^2,(G.1*G.2)^4,(G.1*G.3)^3,
(G.1*G.4)^2,(G.2*G.3)^2,(G.2*G.4)^3,(G.3*G.4)^4];;
Z4:=G/R;;
```

Listing 12: GAP code for group of type $B_3 \times \tilde{A}_1$. Both surfaces are not embedded, since we found an element in $\mathcal{J}$ which also lies in $W(v_i)\backslash\mathcal{J}(v_i)$.

```
#Define free groups with 3 and 2 generators.
G1:=FreeGroup("1","2","3");;
G2:=FreeGroup("4","5");;
#Define a set with relations.
RB3:=[G1.1^2,G1.2^2,G1.3^2,
(G1.1*G1.2)^3,(G1.2*G1.3)^4, (G1.1*G1.3)^2];;
RAInf:=[G2.1^2,G2.2^2];;
#Define the direct product of G1/RB3 and G2/AInf.
B3:=G1/RB3;;
AInf:=G2/RAInf;;
W:=DirectProduct(B3,AInf);;
#Define the groups J_i, i = 1,2.
J1:=Subgroup(W, [(W.2*W.3)^2,W.2*W.5,W.1*W.5,
W.1*W.3,W.3*W.4]);;
J2:=Subgroup(W, [(W.2*W.3)^2,W.2*W.5,W.1*W.5,
W.1*W.3,W.1*W.4,W.2*W.4]);;
#Shows that the element s_2s_3 lies in J_i.
W.2*W.3 in J1;
W.2*W.3 in J2;
```

Listing 13: GAP code for group of type $\tilde{C}_3$.

```
G:=FreeGroup("1","2","3","4");;
RC3:=[G.1^2,G.2^2,G.3^2,G.4^2,
(G.1*G.2)^4,(G.2*G.3)^3,(G.3*G.4)^4,
```

```
(G.1*G.3)^2,(G.1*G.4)^2,(G.2*G.4)^2];;
W:=G/RC3;;
J:=Subgroup(W, [W.2*W.4,(W.1*W.2)^2,W.1*W.3,(W.3*W.4)^2]);;
W.1*W.2 in J;
```

Listing 14: GAP code for group of type $\tilde{B}_2 \times \tilde{A}_1$.

```
G:=FreeGroup("1","2");;
RAInf:=[G.1^2,G.2^2];;
AInf:=G/RAInf;;
G:=FreeGroup("1","2","3");;
RB3:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^4,
(G.1*G.3)^2,(G.2*G.3)^4];;
B3:=G/RB3;;
W:=DirectProduct(B3,AInf);;
J1:=Subgroup(W, [(W.2*W.3)^2,W.2*W.5,(W.1*W.2)^2,
W.1*W.4,W.3*W.4]);;
J2:=Subgroup(W, [(W.2*W.3)^2,W.3*W.5,W.1*W.5,
(W.1*W.2)^2,W.1*W.4,W.3*W.4]);;
W.1*W.2 in J1;
W.2*W.1 in J1;
W.1*W.5 in J1;
W.2*W.3 in J1;
W.3*W.2 in J1;
W.2*W.4 in J1;
W.3*W.5 in J1;
W.1*W.2*W.1*W.4 in J1;
W.2*W.1*W.2*W.5 in J1;
W.2*W.3*W.2*W.5 in J1;
W.3*W.2*W.3*W.4 in J1;
W.1*W.2 in J2;
W.2*W.1 in J2;
W.2*W.3 in J2;
W.2*W.4 in J2;
W.2*W.5 in J2;
W.3*W.2 in J2;
W.1*W.2*W.1*W.4 in J2;
W.1*W.2*W.1*W.5 in J2;
W.3*W.2*W.3*W.4 in J2;
W.3*W.2*W.3*W.5 in J2;
```

Listing 15: GAP code for group of type $\tilde{G}_2 \times \tilde{A}_1$.

```
G:=FreeGroup("1","2");;
RAInf:=[G.1^2,G.2^2];;
AInf:=G/RAInf;;
G:=FreeGroup("1","2","3");;
RG2:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^3,
(G.1*G.3)^2,(G.2*G.3)^6];;
G2:=G/RG2;;
W:=DirectProduct(G2,AInf);;
J1:=Subgroup(W, [(W.2*W.3)^3,W.2*W.5,W.1*W.5,W.1*W.3,
W.3*W.4]);;
J2:=Subgroup(W, [(W.2*W.3)^3,W.2*W.5,W.1*W.5,W.1*W.3,W.1*W.4,
W.2*W.4]);;
W.2*W.3 in J1;
W.2*W.3 in J1;
```

Listing 16: GAP code for group of type $\tilde{Z}_3 \times \tilde{A}_1$.

```
G:=FreeGroup("1","2");;
RAInf:=[G.1^2,G.2^2];;
AInf:=G/RAInf;;
G:=FreeGroup("1","2","3");;
a:=4;;
b:=6;;
c:=6;;
RZ3:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^c,
(G.1*G.3)^a,(G.2*G.3)^b];;
Z3:=G/RZ3;;
W:=DirectProduct(Z3,AInf);;
J1:=Subgroup(W, [(W.2*W.3)^(b/2),W.2*W.5,(W.1*W.2)^(c/2),
W.1*W.4,W.3*W.4]);;
J2:=Subgroup(W, [(W.2*W.3)^(b/2),W.3*W.5,W.1*W.5,
(W.1*W.2)^(c/2),W.1*W.4,W.3*W.4]);;
```

Listing 17: GAP code the for the case in $\mathbb{H}^3$.

```
G:=FreeGroup("1","2","3","4");;
R:=[G.1^2,G.2^2,G.3^2,G.4^2,
(G.1*G.2)^4,(G.1*G.3)^3,(G.1*G.4)^2,
(G.2*G.3)^2,(G.2*G.4)^3,(G.3*G.4)^4];;
```

```
W:=G/R;;
J:=Subgroup(W, [(W.3*W.4)^2,W.1*W.4,(W.1*W.2)^2,W.2*W.3]);;
W.1*W.2 in J;
W.2*W.4 in J;
W.2*W.1 in J;
W.4*W.2 in J;
W.2*W.4*W.2*W.1 in J;
W.2*W.1*W.4*W.2 in J;
W.1*W.4*W.2*W.1 in J;
W.1*W.2*W.1*W.4 in J;
W.1*W.2*W.4*W.2 in J;
W.2*W.4*W.2*W.1*W.2*W.4 in J;
W.2*W.1*W.2*W.4*W.2*W.1 in J;
W.2*W.1*W.4*W.2*W.1*W.2 in J;
W.1*W.2*W.1*W.2*W.4*W.2 in J;
W.1*W.2*W.1*W.4*W.2*W.1 in J;
W.1*W.2*W.4*W.2*W.1*W.2 in J;
W.1*W.2*W.1*W.4*W.2*W.1*W.2*W.4 in J;
```

Listing 18: GAP code for group of type $I_2(m) \times I_2(k) \times \tilde{A}_1$.

```
G:=FreeGroup("1","2");;
RInf:=[G.1^2,G.2^2];;
AInf:=G/RInf;;
m:=4;;
k:=6;;
Rm:=[G.1^2,G.2^2,(G.1*G.2)^m];;
Rk:=[G.1^2,G.2^2,(G.1*G.2)^k];;
I2m:=G/Rm;;
I2k:=G/Rk;;
W:=DirectProduct(I2m,I2k,AInf);;
J1:=Subgroup(W, [W.2*W.4*W.5,W.1*W.4*W.5,W.1*(W.3*W.4)^(k/2),
W.1*W.3*W.6,W.2*W.3*W.6,W.2*(W.3*W.4)^(k/2)]);;
```

Listing 19: GAP code for group of type $I_2(m) \times A_1 \times I_2(k) \times A_1$.

```
G:=FreeGroup("1");;
R:=[G.1^2];;
A1:=G/R;;
G:=FreeGroup("1","2");;
```

```
m:=6;;
k:=6;;
Rm:=[G.1^2,G.2^2,(G.1*G.2)^m];;
Rk:=[G.1^2,G.2^2,(G.1*G.2)^k];;
I2m:=G/Rm;;
I2k:=G/Rk;;
W:=DirectProduct(I2m,A1,I2k,A1);;
J1:=Subgroup(W, [W.2*W.5*W.6,(W.1*W.2)^(m/2)*W.6,W.1*W.4*W.6,
W.1*W.3*W.4,W.3*(W.4*W.5)^(k/2),W.2*W.3*W.5]);;
J2:=Subgroup(W, [W.3*W.5*W.6,W.1*W.5*W.6,
(W.1*W.2)^(m/2)*W.5,(W.1*W.2)^(m/2)*W.4,W.1*W.4*W.6,
W.3*W.4*W.6,W.2*W.3*W.4,W.2*W.3*W.5]);;
J3:=Subgroup(W, [W.2*W.5*W.6,W.1*W.5*W.6,W.1*W.3*W.5,
W.1*W.3*W.4,W.1*W.4*W.6,W.2*W.4*W.6,W.2*W.3*W.4,
W.2*W.3*W.5]);;
Index(W,J1);
Index(W,J2);
Index(W,J3);
```

Listing 20: GAP code for group of type $I_2(m) \times A_1 \times \tilde{B}_2$.

```
G:=FreeGroup("1");;
R:=[G.1^2];;
A1:=G/R;;
G:=FreeGroup("1","2");;
m:=2;;
Rm:=[G.1^2,G.2^2,(G.1*G.2)^m];;
I2m:=G/Rm;;
G:=FreeGroup("1","2","3");;
RB:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^4,(G.2*G.3)^4,
(G.1*G.3)^2];;
B2:=G/RB;;
W:=DirectProduct(I2m,A1,B2);;
J1:=Subgroup(W, [W.3*(W.5*W.6)^2,W.1*W.3*W.6,W.1*W.3*W.4,
W.1*(W.4*W.5)^2,W.2*(W.4*W.5)^2,W.2*W.3*W.4,W.2*W.3*W.6]);;
J2:=Subgroup(W, [W.2*(W.5*W.6)^2,W.1*(W.5*W.6)^2,
W.1*W.3*W.6,W.1*W.3*W.4,W.1*(W.4*W.5)^2,W.2*(W.4*W.5)^2,
W.2*W.3*W.4,W.2*W.3*W.6]);;
```

Listing 21: GAP code for group of type $I_2(m) \times A_1 \times \tilde{A}_1 \times \tilde{A}_1$.

```
G:=FreeGroup("1");;
R:=[G.1^2];;
A1:=G/R;;
G:=FreeGroup("1","2");;
RInf:=[G.1^2,G.2^2];;
AInf:=G/RInf;;
G:=FreeGroup("1","2");;
m:=5;;
Rm:=[G.1^2,G.2^2,(G.1*G.2)^m];;
I2m:=G/Rm;;
W:=DirectProduct(I2m,A1,AInf,AInf);;
J1:=Subgroup(W, [W.3*W.4*W.6,W.1*W.3*W.4,W.1*W.4*W.7,
(W.1*W.2)^(m/2)*W.7,(W.1*W.2)^(m/2)*W.5,W.2*W.5*W.6,
W.2*W.3*W.6]);;
J2:=Subgroup(W, [W.3*W.4*W.6,W.1*W.3*W.4,W.1*W.3*W.7,
W.1*W.5*W.7,W.2*W.5*W.7,W.2*W.3*W.7,W.2*W.3*W.4]);;
J3:=Subgroup(W, [W.2*W.4*W.6,W.1*W.4*W.6,W.1*W.3*W.4,
W.1*W.3*W.7,W.1*W.5*W.7,W.2*W.5*W.7,W.2*W.3*W.7,
W.2*W.3*W.4]);;
J4:=Subgroup(W, [W.2*W.4*W.6,(W.1*W.2)^(m/2)*W.4,
W.1*W.4*W.7,W.1*W.3*W.7,W.1*W.5*W.7,
(W.1*W.2)^(m/2)*W.5,W.2*W.5*W.6,W.2*W.3*W.6]);;
J5:=Subgroup(W, [W.2*W.4*W.6,W.1*W.4*W.6,W.1*W.3*W.4,
W.1*W.3*W.7,W.1*W.3*W.5,W.1*W.5*W.6,W.2*W.5*W.6,
W.2*W.3*W.5,W.2*W.3*W.7,W.2*W.3*W.4]);;
 T:=Subgroup(W, [W.4*W.5,W.6*W.7]);;
 J1T:=Intersection(J1,T);;
J2T:=Intersection(J2,T);;
 J3T:=Intersection(J3,T);;
J4T:=Intersection(J4,T);;
J5T:=Intersection(J5,T);;
 Index(J2,J2T);
 Index(J3,J3T);
Index(J4,J4T);
Index(J5,J5T);
```

Listing 22: GAP code for group of type $I_2(m) \times A_1 \times Z_3$.

```
G:=FreeGroup("1");;
```

```
R:=[G.1^2];;
A1:=G/R;;
G:=FreeGroup("1","2");;
m:=2;;
Rm:=[G.1^2,G.2^2,(G.1*G.2)^m];;
I2m:=G/Rm;;
G:=FreeGroup("1","2","3");;
a:=12;;
b:=4;;
c:=4;;
RZ:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^a,(G.1*G.3)^c,(G.2*G.3)^b];;
Z3:=G/RZ;;
W:=DirectProduct(I2m,A1,Z3);;
J1:=Subgroup(W, [W.3*(W.5*W.6)^(b/2),W.1*W.3*W.6,W.1*W.3*W.4,
W.1*(W.4*W.5)^(c/2),W.2*(W.4*W.5)^(c/2),W.2*W.3*W.4,
W.2*W.3*W.6]);;
J2:=Subgroup(W, [W.2*(W.5*W.6)^(b/2),W.1*(W.5*W.6)^(b/2),
W.1*W.3*W.6,W.1*W.3*W.4,W.1*(W.4*W.5)^(c/2),
W.2*(W.4*W.5)^(c/2),W.2*W.3*W.4,W.2*W.3*W.6]);;
```

Listing 23: GAP code for group of type $\tilde{C}_3 \times \tilde{A}_1$.

```
G:=FreeGroup("1","2","3","4");;
RC3:=[G.1^2,G.2^2,G.3^2,G.4^2,
(G.1*G.2)^4,(G.1*G.3)^2,(G.1*G.4)^2,
(G.2*G.3)^3,(G.2*G.4)^2,(G.3*G.4)^4];;
C3:=G/RC3;;
G:=FreeGroup("1","2");;
RInf:=[G.1^2,G.2^2];;
AInf:=G/RInf;;
W:=DirectProduct(C3,AInf);;
J1:=Subgroup(W, [(W.3*W.4)^2*W.5,W.1*W.4*W.5,
(W.1*W.2)^2*W.5,(W.1*W.2*W.3)^3,(W.1*W.2)^2*W.6,
W.1*W.4*W.6,(W.3*W.4)^2*W.6,(W.2*W.3*W.4)^3]);;
```

Listing 24: GAP code for group of type $\tilde{B}_2 \times \tilde{B}_2$.

```
G:=FreeGroup("1","2","3");;
RB2:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^4,(G.2*G.3)^4,(G.1*G.3)^2];;
B2:=G/RB2;;
```

```
W:=DirectProduct(B2,B2);;
J1:=Subgroup(W, [W.3*(W.5*W.6)^2,W.1*(W.5*W.6)^2,
(W.1*W.2)^2*W.6,(W.1*W.2)^2*W.4,W.1*(W.4*W.5)^2,
W.3*(W.4*W.5)^2,(W.2*W.3)^2*W.4,(W.2*W.3)^2*W.6]);;
```

Listing 25: GAP code for group of type $\tilde{B}_2 \times \tilde{A}_1 \times \tilde{A}_1$.

```
G:=FreeGroup("1","2");;
RInf:=[G.1^2,G.2^2];;
AInf:=G/RInf;;
G:=FreeGroup("1","2","3");;
RB2:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^4,(G.2*G.3)^4,(G.1*G.3)^2];;
B2:=G/RB2;;
W:=DirectProduct(B2,AInf,AInf);;
J1:=Subgroup(W, [W.3*W.4*W.6,W.1*W.4*W.6,(W.1*W.2)^2*W.4,
W.1*W.4*W.7,W.3*W.4*W.7,(W.2*W.3)^2*W.7,(W.2*W.3)^2*W.5,
(W.2*W.3)^2*W.6]);;
J2:=Subgroup(W, [W.3*W.4*W.6,W.1*W.4*W.6,(W.1*W.2)^2*W.4,
(W.1*W.2)^2*W.7,W.1*W.5*W.7,W.3*W.5*W.7,(W.2*W.3)^2*W.7,
(W.2*W.3)^2*W.4]);;
J3:=Subgroup(W, [W.3*W.4*W.6,W.1*W.4*W.6,(W.1*W.2)^2*W.4,
(W.1*W.2)^2*W.7,W.1*W.5*W.7,W.3*W.5*W.7,(W.2*W.3)^2*W.5,
(W.2*W.3)^2*W.6]);;
J4:=Subgroup(W, [W.3*W.4*W.6,W.1*W.3*W.4,W.1*W.4*W.7,
(W.1*W.2)^2*W.7,W.1*W.5*W.7,W.1*W.3*W.5,W.3*W.5*W.6,
(W.2*W.3)^2*W.6]);;
J5:=Subgroup(W, [W.3*W.4*W.6,W.1*W.4*W.6,(W.1*W.2)^2*W.4,
(W.1*W.2)^2*W.7,(W.1*W.2)^2*W.5,W.1*W.5*W.6,W.3*W.5*W.6,
(W.2*W.3)^2*W.5,(W.2*W.3)^2*W.7,(W.2*W.3)^2*W.4]);;
J6:=Subgroup(W, [W.3*W.4*W.6,W.1*W.4*W.6,(W.1*W.2)^2*W.4,
W.1*W.4*W.7,W.3*W.4*W.7,(W.2*W.3)^2*W.7,W.3*W.5*W.7,
W.1*W.5*W.7,(W.1*W.2)^2*W.5,W.1*W.5*W.6,W.3*W.5*W.6,
(W.2*W.3)^2*W.6]);;
```

Listing 26: GAP code for group of type $\tilde{A}_1 \times \tilde{A}_1 \times \tilde{A}_1 \times \tilde{A}_1$.

```
G:=FreeGroup("1","2");;
RInf:=[G.1^2,G.2^2];;
AInf:=G/RInf;;
W:=DirectProduct(AInf,AInf,AInf,AInf);;
```

```
J1:=Subgroup(W, [W.3*W.5*W.7,W.2*W.5*W.7,W.4*W.5*W.7,
W.1*W.4*W.5,W.1*W.4*W.8,W.1*W.6*W.8,W.1*W.3*W.8,
W.1*W.3*W.5]);;
J2:=Subgroup(W, [W.3*W.5*W.7,W.2*W.3*W.7,W.2*W.6*W.7,
W.4*W.6*W.7,W.1*W.4*W.6,W.1*W.4*W.8,W.1*W.5*W.8,
W.1*W.3*W.5]);;
J3:=Subgroup(W, [W.3*W.5*W.7,W.2*W.5*W.7,W.2*W.4*W.5,
W.4*W.5*W.8,W.1*W.4*W.8,W.1*W.4*W.6,W.1*W.6*W.7,
W.1*W.3*W.7]);;
J4:=Subgroup(W, [W.3*W.5*W.7,W.2*W.5*W.7,W.4*W.5*W.7,
W.1*W.4*W.5,W.1*W.4*W.8,W.4*W.6*W.8,W.2*W.6*W.8,
W.3*W.6*W.8,W.1*W.3*W.8,W.1*W.3*W.5]);;
J5:=Subgroup(W, [W.3*W.5*W.7,W.2*W.3*W.7,W.3*W.6*W.7,
W.1*W.6*W.7,W.1*W.4*W.6,W.1*W.6*W.8,W.3*W.6*W.8,
W.2*W.3*W.8,W.3*W.5*W.8,W.1*W.5*W.8,W.1*W.4*W.5,
W.1*W.5*W.7]);;
```

Listing 27: GAP code for group of type $Z_3 \times \tilde{B}_2$.

```
G:=FreeGroup("1","2","3");
RB:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^4,(G.2*G.3)^4,(G.1*G.3)^2];
B2:=G/RB;
a:=4;;
b:=4;;
c:=4;;
RZ:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^c,(G.1*G.3)^a,(G.2*G.3)^b];;
Z3:=G/RZ;;
W:=DirectProduct(Z3,B2);;
J1:=Subgroup(W, [W.3*(W.5*W.6)^2,W.1*(W.5*W.6)^2,
(W.1*W.2)^(c/2)*W.6,(W.1*W.2)^(c/2)*W.4,
W.1*(W.4*W.5)^2,W.3*(W.4*W.5)^2,(W.2*W.3)^(b/2)*W.5]);;
J2:=Subgroup(W, [W.3*(W.5*W.6)^2,W.1*(W.5*W.6)^2,
(W.1*W.2)^(c/2)*W.6,(W.1*W.2)^(c/2)*W.4,
W.1*(W.4*W.5)^2,W.3*(W.4*W.5)^2,(W.2*W.3)^(b/2)*W.4,
(W.2*W.3)^(b/2)*W.6]);;
```

Listing 28: GAP code for group of type $Z_3 \times \tilde{A}_1 \times \tilde{A}_1$.

```
G:=FreeGroup("1","2","3");
a:=4;;
```

```
b:=4;;
c:=4;;
RZ:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^c,(G.1*G.3)^a,(G.2*G.3)^b];;
Z3:=G/RZ;;
G:=FreeGroup("1","2");;
RInf:=[G.1^2,G.2^2];;
AInf:=G/RInf;;
W:=DirectProduct(Z3,AInf,AInf);;
J1:=Subgroup(W, [W.3*W.4*W.6,W.1*W.4*W.6,(W.1*W.2)^(c/2)*W.4,
W.1*W.4*W.7,W.3*W.4*W.7,(W.2*W.3)^(b/2)*W.7,
(W.2*W.3)^(b/2)*W.5,(W.2*W.3)^(b/2)*W.6]);;
J2:=Subgroup(W, [W.3*W.4*W.6,W.1*W.4*W.6,
(W.1*W.2)^(c/2)*W.4,(W.1*W.2)^(c/2)*W.7,W.1*W.5*W.7,
W.3*W.5*W.7,(W.2*W.3)^(b/2)*W.7,(W.2*W.3)^(b/2)*W.4]);;
J3:=Subgroup(W, [W.3*W.4*W.6,W.1*W.4*W.6,
(W.1*W.2)^(c/2)*W.4,(W.1*W.2)^(c/2)*W.7,W.1*W.5*W.7,
W.3*W.5*W.7,(W.2*W.3)^(b/2)*W.5,(W.2*W.3)^(b/2)*W.6]);;
J4:=Subgroup(W, [W.2*W.4*W.6,(W.2*W.3)^(b/2)*W.4,
W.2*W.4*W.7,(W.1*W.2)^(c/2)*W.7,W.1*W.5*W.7,
(W.1*W.3)^(a/2)*W.5,W.1*W.5*W.6,(W.1*W.2)^(c/2)*W.6]);;
J5:=Subgroup(W, [W.3*W.4*W.6,W.1*W.4*W.6,
(W.1*W.2)^(c/2)*W.4,(W.1*W.2)^(c/2)*W.7,(W.1*W.2)^2*W.5,
W.1*W.5*W.6,W.3*W.5*W.6,(W.2*W.3)^(b/2)*W.5,
(W.2*W.3)^(b/2)*W.7,(W.2*W.3)^(b/2)*W.4]);;
J6:=Subgroup(W, [W.3*W.4*W.6,W.1*W.4*W.6,
(W.1*W.2)^(c/2)*W.4,W.1*W.4*W.7,W.3*W.4*W.7,
(W.2*W.3)^(b/2)*W.7,W.3*W.5*W.7,W.1*W.5*W.7,
(W.1*W.2)^2*W.5,W.1*W.5*W.6,W.3*W.5*W.6,
(W.2*W.3)^(b/2)*W.6]);;
```

Listing 29: GAP code for group of type $Z_3 \times \tilde{A}_1 \times \tilde{A}_!$.

```
G:=FreeGroup("1","2","3");
a:=4;;
b:=4;;
c:=4;;
d:=4;;
e:=4;;
f:=4;;
RZ:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^c,(G.1*G.3)^a,(G.2*G.3)^b];;
```

```
RZ2:=[G.1^2,G.2^2,G.3^2,(G.1*G.2)^f,(G.1*G.3)^d,(G.2*G.3)^e];;
Z3:=G/RZ;;
Z32:=G/RZ2;;
W:=DirectProduct(Z3,Z32);;
J1:=Subgroup(W, [W.3*(W.5*W.6)^(e/2),W.1*(W.5*W.6)^(e/2),
(W.1*W.2)^(c/2)*W.6,W.2*(W.4*W.6)^(d/2),(W.2*W.3)^(b/2)*W.4,
(W.2*W.3)^(b/2)*W.5]);;
J2:=Subgroup(W, [W.3*(W.5*W.6)^(e/2),(W.1*W.3)^(a/2)*W.6,
W.1*(W.4*W.6)^(d/2),(W.1*W.2)^(c/2)*W.4,W.2*(W.4*W.5)^(f/2),
(W.2*W.3)^(b/2)*W.5]);;
J3:=Subgroup(W, [W.3*(W.5*W.6)^(e/2),W.1*(W.5*W.6)^(e/2),
(W.1*W.2)^(c/2)*W.6,W.1*(W.4*W.6)^(d/2),W.3*(W.4*W.6)^(d/2),
(W.2*W.3)^(b/2)*W.4,(W.2*W.3)^(b/2)*W.5]);;
J4:=Subgroup(W, [W.3*(W.5*W.6)^(e/2),W.1*(W.5*W.6)^(e/2),
(W.1*W.2)^(c/2)*W.6,(W.1*W.2)^(c/2)*W.4,W.1*(W.4*W.5)^(f/2),
W.3*(W.4*W.5)^(f/2),(W.2*W.3)^(b/2)*W.4,
(W.2*W.3)^(b/2)*W.6]);;
```

Glossary

(-1)-condition

Let (W, S) be a Coxeter system. We say (W, S) satisfies the (-1)-*condition*, if there is a $w \in W$ such that $\rho(w)(x) = -x$ for all $x \in V$.

$\mathbb{X}^n$-structure

An $\mathbb{X}^n$-*structure* on a manifold M^n is an atlas of charts $\{\psi_i \colon U_i \to \mathbb{X}^n\}_{i \in I}$ where

- $\{U_i\}_{i \in I}$ is an open cover of M^n,
- each ψ_i is a homeomorphism onto its image and
- each overlap map $\psi_j \psi_i^{-1} \colon \psi_j(U_i \cap U_j) \to \psi_i(U_i \cap U_j)$ is the restriction of an isometry in $\mathrm{Isom}(\mathbb{X}^n)$.

Base space

We call the space $\mathcal{U}(\mathcal{J}, \mathcal{Q}^2)$ constructed in Lemma 4.2.2 the *base space* of Γ.

Bilinear form, G-invariant

Let G be a group, V a real vector space, and $\rho \colon G \to GL(V)$ a linear representation of G. We say a bilinear form $B \colon V \times V$ is G-*invariant* if $B(\rho(g)(v), \rho(g)(w)) = B(v, w)$ for all $v, w \in V$, $g \in G$.

Canonical representation

For each $s_i \in S$, let H_i be the hyperplane in V defined by

$$H_i \coloneqq \{x \in V \mid B_M(e_i, x) = 0\}$$

and let $\rho_i \coloneqq \rho(s_i)\colon V \to V$ be the linear reflection (in the sense of Section 2.1) defined by

$$\rho_i(x) = x - 2B_M(e_i, x)e_i.$$

We define the *canonical representation*, by extending the map $S \to GL(V)$ defined by $s_i \mapsto \rho_i$ to a homomorphism $\rho\colon W \to GL(V)$, i.e., for $w = s_{i_1} \cdots s_{i_k} \in W$ we have

$$\rho(w) = \rho(s_{i_1} \cdots s_{i_k}) \coloneqq \rho_{i_1} \cdots \rho_{i_k}.$$

Chamber hull

Let Γ be a curve in $\mathcal{U}$ and U be the smallest union of all chambers $w_i P^n$ bounded by Γ. Furthermore, let $\mathcal{C}$ be the union of chambers such that $\mathrm{conv}(U) \subset \mathcal{C}$. We call $\mathcal{C}$ the *chamber hull* of Γ.

Convex Subset, irreducible

A convex subset $X \subset \mathbb{E}^n$ is called *irreducible* if it is not reducible.

Convex Subset, reducible

A convex subset $X \subset \mathbb{E}^n$ is *reducible* if it is isometric to a product $X' \times X''$, where $X' \subset \mathbb{E}^m$ and $X'' \subset \mathbb{E}^{n-m}$ and neither X' nor X'' are points.

Cosine matrix

To a Coxeter matrix M on the set $S = \{s_1, \ldots, s_n\}$ of rank n we associate a symmetric $n \times n$-matrix $A(M)$ by setting

$$a_{ij} \coloneqq -\cos(\pi/m_{ij}).$$

We call $A(M)$ the *cosine matrix* of M or of the corresponding Coxeter system. If $m_{ij} = \infty$, we set $a_{ij} = -1$.

Covering map

If a developing map is a a homeomorphism, it is called *covering map*.

Coxeter complex

We call $\mathcal{U} \coloneqq \mathcal{U}(W, P^n) \coloneqq (W \times P^n)/\sim$ as in (2.12) the *Coxeter complex* of W and P^n.

Coxeter complex, tessellation

If the Coxeter complex is homeomorphic to $\mathbb{X}^n$, we call it the *tessellation* of $\mathbb{X}^n$ by P^n using W.

Coxeter Diagram

A *Coxeter diagram* is a Coxeter graph together with a labeling of its vertices.

Coxeter graph

Suppose $M = (m_{ij})$ is Coxeter matrix on a set I. We associate to M a graph Γ, called *Coxeter graph*, as follows. The vertex set of Γ is I. A pair of distinct vertices i and j is connected by an edge if and only if $m_{ij} \geq 3$. The edge $\{i, j\}$ is labeled by m_{ij} if $m_{ij} \geq 4$.

Coxeter group

The group W of a Coxeter system (W, S) is called *Coxeter group*.

Coxeter group, Coxeter element

A *Coxeter element* is the is the product of all $s_i \in S$ in any given order.

Coxeter group, Coxeter number

The *Coxeter number* is the order of a Coxeter element.

Coxeter group, Deletion Condition

Let $w \in W$ and $s_1, \ldots, s_k \in S$. If $w = s_1 \ldots s_k$ is a word in W with $\ell(w) < k$, then there are indices $i < j$ such that $w = s_1 \ldots \hat{s}_i \ldots \hat{s}_j \ldots s_k$, where $\hat{s}_i$ means we delete this letter.

Coxeter group, Exchange Condition

Let $w \in W$ and $s, s_1, \ldots, s_k \in S$. If $w = s_1 \ldots s_k$ is a reduced word, either $\ell(sw) = k + 1$ or else there is an index i such that $w = ss_1 \ldots \hat{s}_i \ldots s_k$.

Coxeter group, Folding Condition

Let $w \in W$ and $s, s' \in S$. Suppose that we have $\ell(sw) = \ell(ws') = \ell(w) + 1$. Then either $\ell(sws') = \ell(w) + 2$ or $sws' = w$.

Coxeter group, generator

The set S of a Coxeter system (W, S) is called the *set of generators.*

Coxeter group, Longest Element

The *longest element* of a Coxeter group W is the uniquely determined element with longest length with respect to the length function ℓ.

Coxeter group, parabolic subgroup

Let $w \in W$. We call a w-conjugate of a special subgroup W_T (i.e., $w^{-1}W_T w$) a *parabolic subgroup* of W.

Coxeter group, reduced word

A word w of a Coxeter group is *reduced*, if If $w = s_1 \cdots s_k$ and $\ell(w) = k$ where ℓ is the length of w.

Coxeter group, special subgroup

Let (W, S) be a Coxeter system and $T \subset S$. We call the subgroup W_T generated by the elements of T a *special subgroup* of W.

Coxeter group, word

A *word* is an element of the Coxeter group.

Coxeter group, word length

The *length* of a word with respect to S is denoted by using the function $\ell_S \colon W \to \mathbb{N}$ as the minimum number k such that for a word $w \in W$ we can write $w = s_{i_1} \cdots s_{i_k}$ for $s_{i_n} \in S$. If the set of generators is clear the index S from ℓ is omitted.

Coxeter Matrix

A *Coxeter matrix* $M = (m_{st})$ on a finite set S is a symmetric $(|S| \times |S|)$-matrix with entries in $\overline{\mathbb{N}} \coloneqq \mathbb{N} \cup \{\infty\}$ such that $m_{st} = 1$ if $s = t$ and $m_{st} \geq 2$ for $s \neq t$.

Coxeter matrix, Euclidean

A Coxeter matrix M is called *Euclidean*, if the cosine of M is semi-positive definite of corank 1.

Coxeter matrix, hyperbolic

A Coxeter matrix M is called *hyperbolic*, if the cosine of M is non-degenerate with signature $(n, 1)$.

Coxeter matrix, spherical

A Coxeter matrix M is called *spherical*, if the cosine of M is positive definite.

Coxeter polytope

Let $P^n \subset \mathbb{X}^n$ be a simple polytope and F_i its facets. We call P^n a *Coxeter polytope* if

- whenever $F_i \cap F_j \neq \emptyset$, the dihedral angle is of the form π/m_{ij} for some natural numbers $m_{ij} \geq 2$ or
- two facets are parallel, i.e., $F_i \cap F_j = \emptyset$.

Coxeter System

Let M be a Coxeter matrix and $\widetilde{W}$ the group defined by the presentation associated with M. We say that (W, S) is a *Coxeter system*, if W is isomorphic to the group $\widetilde{W}$, i.e., if the epimorphism $\widetilde{W} \to W$, defined by $\tilde{s} \to s$, is an isomorphism.

Coxeter system, M-reduced word

A word is called *M-reduced*, if it cannot be shortened by a sequence of elementary M-operations.

Coxeter system, diagram automorphism

Let (W, S) be a Coxeter system and φ an automorphism of W. We call φ a *diagram automorphism* of (W, S) if $\varphi(S) = S$.

Coxeter system, elementary M-operation

An *elementary M-operation* on a word in W is one of the following types:

(I) Deleting a subword of the form s_i^2, where $s_i \in S$.

(II) Using a *braid move* on $s_i, s_j \in S$, i.e., replacing an alternating subword of the form $s_i s_j \cdots$ of length m_{ij} by the alternating word $s_j s_i \cdots$, of the same length m_{ij}.

Coxeter system, irreducible

A Coxeter system is *irreducible* if its Coxeter graph is connected.

Coxeter system, Rank

The *rank* of a Coxeter system (W, S) is the cardinality of set of generators S.

Coxeter system, reducible

A Coxeter system is *reducible* if its Coxeter graph is disconnected.

Developing map

A *developing map* $D\colon \widetilde{M}^n \to \mathbb{X}^n$ is a local homeomorphism.

Dihedral Angle

Suppose H_1 and H_2 are two hyperplanes in $\mathbb{X}^n$ bounding half-spaces E_1 and E_2 with $E_1 \cap E_2 \neq \emptyset$. Let u_1 and u_2 be, respectively, their inward-pointing unit normals at a point $x \in H_1 \cap H_2$. The *dihedral angle* between H_1 and H_2 is given by $\theta := \pi - \arccos\langle u_1, u_2\rangle \in [0, \pi]$.

Dihedral angle, non-obtuse

A dihedral angle is *non-obtuse*, if it is smaller than $\pi/2$.

Dihedral angle, right-angled

A dihedral angle is *right-angled*, if it is exactly $\pi/2$.

Dihedral group

A *dihedral group* is a group generated by two elements of order 2.

Edge cycle

An *edge cycle* $\Gamma\colon \mathbb{S}^1 \to P^n \subset \mathbb{X}^n$ is a Jordan curve on P^n, i.e., a simple closed continuous curve consisting only of edges of P^n.

Edge cycle, compatible

We say an edge cycle Γ is *compatible* with the tessellation $\mathcal{U}(W, P^n)$, if for every edge $e_i \subset P^n$ the edge reflection j_i along e_i can be expressed as an element of W, i.e., for every edge e_i the largest special subgroup that fixes e_i satisfies the (-1)-condition.

Edge cycle, full-dimensional

We say an edge cycle is *full-dimensional* if it is not contained in a facet of P^n.

Edge cycle, semi-compatible

We say an edge cycle Γ is *semi-compatible* with the tessellation $\mathcal{U}(W, P^n)$, if for every edge $e_i \subset P^n$ the edge reflection j_i along e_i can be expressed as an element of $H = W \rtimes D$.

Edge cycle, tessellation

A cycle Γ in the edge reflection graph is called *edge cycle* on $\mathcal{U}$.

Edge cycle, tessellation full-dimensional

Let Γ be an edge cycle in $\mathcal{U}(G)$. We say that Γ is *full-dimensional* if Γ is not contained in a facet of the chamber hull $\mathcal{C}$.

Edge reflection

Assume that e is an edge of a polytope P^n. We call j the *edge reflection about* e, if $j(p) = p$ for all $p \in e$ and $j(q) = -q$ for all $q \in e^{\perp}$.

Edge reflection graph

Let $P^n \subset \mathbb{X}^n$ be a Coxeter polytope with vertices v_i, edges e_i, and Coxeter system (W, S). We define a graph G as follows: the vertex set V of G consists of all vertices v_i of P^n. Two vertices $v_i, v_j \in G$ are connected if the subgroup W_T, which fixes the edge incident to v_i and v_j, satisfies the (-1)-condition, i.e., e_i satisfies the (-1)-condition. We denote with E the edge set of G and call the graph $G = (V, E)$ the *edge reflection graph* associated with P^n.

Edge reflection graph, tessellation

Let $P^n \subset \mathbb{X}^n$ be a Coxeter n-simplex (W, S) its Coxeter system and $\mathcal{U}(W, P^n) = W \times P^n / \sim$ the tessellation of $\mathbb{X}^n$ tiled by P^n. Furthermore, let $G = (V, E)$ be the edge reflection graph of P^n. Set $\mathcal{U}(V) \coloneqq \{wv_i \mid w \in W, v_i \in V\}$, $\mathcal{U}(E) \coloneqq \{we_i \mid w \in W, e_i \in E\}$, and define the graph $\mathcal{U}(G) \coloneqq (\mathcal{U}(V), \mathcal{U}(E))$. We call $\mathcal{U}(G)$ the *edge reflection graph* of $\mathcal{U}(W, P^n)$.

Edge reflection group

The *edge reflection group* $\mathcal{J}$ is the group generated by all edge reflections j_i along the edges $e_i \subset \Gamma$.

Euler characteristic, surface $\mathcal{S}$

The *Euler characteristic* of $\mathcal{S}$ constructed on $\mathbb{S}^n$ is given by the formula

$$\chi(\mathcal{S}) = |\mathcal{J}| - \frac{|\Gamma||\mathcal{J}|}{2} + \sum_i \frac{|\mathcal{J}|}{|\mathcal{J}(v_i)|} = -\frac{(|\Gamma|-2)|\mathcal{J}|}{2} + \sum_i \frac{|\mathcal{J}|}{|\mathcal{J}(v_i)|}.$$

Genus

The *genus* of an closed orientable surface $\mathcal{S}$ is given by $g = 1 - \chi(\mathcal{S})/2$ and of an closed non-orientable surface it is given by $g = 2 - \chi(\mathcal{S})$ where χ is the Euler characteristic of $\mathcal{S}$.

Geometric reflection group

A *geometric reflection group* is the action of a group $\overline{W}$ on $\mathbb{X}^n$, which, as in Theorem 2.9.6, is generated by the reflections across the faces of a simple convex polytope with dihedral angles submultiples of π.

Gram matrix

The *Gram matrix* U of a polytope P^n is defined by $U \coloneqq (\langle u_i, u_j \rangle)$ where u_i is a inward-pointing unit normal vector of the facet F_i of P^n.

Involution

An *involution* is a group element of order two.

Linear reflection

A *linear reflection* on a finite-dimensional real vector space V is a linear automorphism $r\colon V \to V$ such that $r^2 = \mathrm{id}_V$ and such that the fixed subspace of r is a hyperplane.

Linear subspace, G-stable

Let G be a group, V a real vector space, and $\rho\colon G \to GL(V)$ a linear representation of G. A linear subspace $U \subset V$ is called G-*stable* if $\rho(g)u \in U$ for all $u \in U$ and $g \in G$.

Matrix, decomposable

An $n \times n$-matrix $A = (a_{ij})$ is called *decomposable* if there is a non-trivial partition of the index set $I \cup J = \{1, \ldots, n\}$ such that $a_{ij} = a_{ji} = 0$, whenever $i \in I$ and $j \in J$.

Matrix, indecomposable

A matrix is called *indecomposable*, if it is not decomposable.

Polyhedral cone

A *polyhedral cone* $C \subset \mathbb{R}^{n+1}$ is the intersection of a finite number of linear half-spaces in $\mathbb{R}^{n+1}$.

Polyhedral Cone, dimension

The *dimension* of a polyhedral cone C is the dimension of the affine space it spans, i.e., the dimension of the smallest affine space containing C.

Polyhedral cone, essential

A polyhedral cone C is *essential*, if it contains no line, i.e., the intersection with $\mathbb{S}^n$ contains no antipodal points.

Polyhedral cone, extremal ray

An *extremal ray* of a polyhedral cone is a 1-dimensional face.

Polyhedral cone, face

A *face* of a polyhedral cone is the intersection of the cone with a supporting hyperplane.

Polyhedral cone, simplicial

An essential polyhedral cone is called *simplicial*, if it is $(n+1)$-dimensional and is the intersection of $n+1$ linear half-spaces.

Polytope

A *polytope in* $\mathbb{E}^n$ is a compact intersection of a finite number of half-spaces in $\mathbb{E}^n \subset \mathbb{R}^{n+1}$. A *polytope in* $\mathbb{S}^n$ is the intersection of $\mathbb{S}^n$ with an essential polyhedral cone $C \subset \mathbb{R}^{n+1}$. A *polytope in* $\mathbb{H}^n$ is the intersection of $\mathbb{H}^n$ with a polyhedral cone $C \subset \mathbb{R}^n_1$ such that $C \backslash \{0\}$ is contained in the interior of the positive light cone.

Polytope, dimension

The *dimension* of a polytope is $\dim(C) - 1$ where C is the polyhedral cone.

Polytope, edge

An *edge* is a 1-dimensional face of a polytope.

Polytope, Euler characteristic

The *Euler characteristic* is given by Euler's formula for n-dimensional simple polytopes:

$$\chi(P^n) := \sum_{k=0}^{n}(-1)^{k+1} f_k = 1 - (-1)^n$$

where f_k is the number of k-dimensional faces of P^n.

Polytope, face

A *face* of a polytope P is the intersection of P with a supporting hyperplane.

Polytope, facet

A *facet* is an $(n-1)$-dimensional face of a polytope.

Polytope, non-obtuse

A polytope is called non-obtuse, if for every two distinct indices i and j either $H_i \cap H_j = \emptyset$ or the dihedral angle between H_i and H_j is non-obtuse.

Polytope, right-angled

A polytope is called right-angled, if for every two distinct indices i and j either $H_i \cap H_j = \emptyset$ or the dihedral angle between H_i and H_j is right-angled.

Polytope, simple

An n-dimensional polytope P^n is called *simple* if exactly n facets (or equivalently n edges) meet at each vertex.

Polytope, simplex

An n-*simplex* is an n-dimensional polytope where the polyhedral cone is simplicial.

Polytope, vertex

A *vertex* is a 0-dimensional face of a polytope.

Reflection

Let (W, S) a Coxeter system. The set of *reflections* is defined by $R \coloneqq \{wsw^{-1} \in W \mid w \in W, s \in S\}$.

Representation

A *representation* of a group W is a map $\rho\colon W \to GL(n)$.

Representation, faithful

A group representation ρ is *faithful*, if the group homomorphism ρ is injective.

Representation, irreducible

A representation is said to be *irreducible*, if ρ has exactly two subrepresentations, i.e., the trivial subspaces $\{0\}$ and V are the only G-stable subspaces.

Representation, reducible

If a representation ρ has a proper non-trivial subrepresentation, it is called *reducible*.

Representation, semi-simple

A representation is called *semi-simple*, if V has a non-trivial W-stable subspace which is not a direct summand.

Representation, Subrepresentation

We call the restriction of ρ to a G-stable linear space $U \subset V$ a *subrepresentation*.

Root system

The *root system* Φ of a Coxeter system (W, S) consists of the set of unit vectors in V permuted by W using the canonical representation ρ, i.e.,

$$\Phi \coloneqq \{\rho(w)(e_i) \mid w \in W, s_i \in S\}.$$

Root system, positive and negative root

We can write $\alpha \in \Phi$ uniquely in the form

$$\alpha = \sum_{s_i \in S} c_i e_i$$

in terms of the basis e_i, with coefficients $c_i \in \mathbb{R}$. We call α *positive* (resp. negative) and write $\alpha > 0$ (resp. $\alpha < 0$) if all $c_i \geq 0$ (resp. $c_i \leq 0$).

Root system, root

A *root* is an element of the root system.

Strict fundamental domain

Suppose a group G acts on a space X and let $C \subset X$ be a closed subset. We say C is a *strict fundamental domain* for G on X if each G-orbit intersects C in exactly one point and if for each point x in the interior of C we have that $Gx \cap C = \{x\}$ or equivalently $X/G \cong C$.

Surface, embedding

The surface $f\colon X \to \mathbb{X}^n$ is an *embedding* if f is an injective immersion and homeomorphic onto its image.

Surface, immersion

A surface $f\colon X \to \mathbb{X}^n$ is an *immersion* if $\operatorname{rank} D_p f = 2$.

Surface, self-intersection

We say $f\colon X \to \mathbb{X}^n$ has *self-intersections* if f is not injective. We call a point $p \in X$ a *self-intersection* if there are two distinct points $p_1, p_2 \in X$ such that $f(p_1) = f(p_2) = x$.

Translation

A *translation* is a torsion-free product of two reflections. We denote the group generated by all translations as $T(\mathcal{U}(W, P^n))$ or simply T, if $\mathcal{U}$ is clear.

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