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A note on the finitization of Abelian and Tauberian theorems

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We present finitary formulations of two well known results concerning infinite series, namely Abel's theorem, which establishes that if a series converges to some limit then its Abel sum converges to the same limit, and Tauber's theorem, which presents a simple condition under which the converse holds. Our approach is inspired by proof theory, and in particular Gödel's functional interpretation, which we use to establish quantitative versions of both of these results.

1 Introduction

In an essay of 2007, Tao discussed the so-called *correspondence principle* between 'soft' and 'hard' analysis, whereby many *infinitary* notions from analysis can be given an equivalent *finitary* formulation ([17]; later published as part of [18]). An important instance of this phenomenon is provided by the simple concept of Cauchy convergence of a sequence $\{c_n\}$:

$$\forall \varepsilon > 0\ \exists N\ \forall m, n \geq N\ (|c_m - c_n| \leq \varepsilon).$$

This corresponds to the finitary notion of $\{c_n\}$ being *metastable*, which is given by the following formula:

$$\forall \varepsilon > 0\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists N\ \forall m, n \in [N; N + g(N)]\ (|c_m - c_n| \leq \varepsilon), \tag{1}$$

where $[N; N+k] := \{N, N+1, \dots, N+k-1, N+k\}$. Roughly speaking, a sequence $\{c_n\}$ is metastable if for any given error $\varepsilon > 0$ it contains a finite regions of stability of any 'size', where size is represented by the function $g : \mathbb{N} \to \mathbb{N}$.

The equivalence of Cauchy convergence and metastability is established via purely logical reasoning, and indeed, as was quickly observed, the correspondence principle as presented in [17] has deep connections with proof theory. More specifically, the finitary variant of an infinitary statement is typically closely related to its *classical functional interpretation* [1], which provides a general method for obtaining quantitative versions of mathematical theorems.

Finitary formulations of infinitary properties play a central role in the *proof mining* program developed by Kohlenbach from the early 1990s [7]. Here, it is often the case that a given mathematical theorem has, in general, no computable realizer (for Cauchy convergence this is demonstrated by the existence of so-called *Specker sequences* [16], which will be discussed further in § 3). On the other hand, the corresponding finitary formulation can typically not only be realized, but a realizer can be directly extracted from a proof that the original property holds. The extraction of a computable bound $\Omega(\varepsilon, g)$ on N in (1)—a so-called *rate of metastability*—is a standard result in this area (e.g., [6, 8, 9, 14]), and techniques from proof theory are often used to give finitizations of more complex statements, including, e.g., the *Bolzano-Weierstrass theorem* [15] and *Ramsey's theorem* [11, 13].

In this article, we apply the aforementioned ideas to study the relationship between two distinct forms of convergence from a finitary perspective, namely (I) the convergence of an infinite series of reals, and (II) the limit as $x \to 1^-$ of the power series it generates, i.e.,

(I) $s_n := \sum_{i=0}^{n} a_i$ as $n \to \infty$, and

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(II) $F(x) := \sum_{i=0}^{\infty} a_i x^i$ as $x \to 1^-$.

Several classic results apply here. *Abel's theorem* covers one direction, and states that if $\lim_{n\to\infty} s_n = s$ then we also have $\lim_{x\to 1^-} F(x) = s$. The converse also holds, subject to the additional condition that $a_n = o(1/n)$, a result due to A. Tauber which is has since become the first and simplest instance of a whole class of results known as *Tauberian theorems*. In this article, we provide new quantitative versions of these two theorems, which take the shape of a route between various forms of metastability.

Though both Abel's and Tauber's theorems are elementary to state and prove, establishing in each case a natural finitary formulation from which the original theorem can be rederived is non-trivial, as is generally the case when it comes to correctly finitizing infinitary statements (an illuminating discussion of the subtleties which arise from the similarly elementary *infinite pigeonhole principle* is given in [5]). We begin by establishing Cauchy variants of Abel and Tauber's theorems which do not explicitly mention limit points. We show that Specker phenomena propagate through both theorems, and as such, in order to give quantitative versions we are forced to consider metastable variants of the associated limiting processes. We then state and prove our finitary theorems, and demonstrate how the original theorems can be reobtained in a uniform way.

Formally speaking, our quantitative theorems are obtained by analysing standard proofs of Abel's and Tauber's theorems using Gödel's functional interpretation, although the underlying logical aspects of this transformation on proofs are not necessary to understand the resulting finitizations. Indeed, the reader will see that the transformed proofs retain the combinatorial core of the originals, but replace infinitary arguments with more precise quantitative reasoning. The usual infinitary theorems can then be derived from ours using purely logical steps.

There are two main motivating factors behind this short article. The first is the fact that Abelian and Tauberian theorems give rise to simple and yet revealing examples of the correspondence principle and related concepts such as metastability, which can be presented in such a way that we are not required to explicitly introduce any proof theoretic concepts (indeed, even the notion of a higher order functional is only needed in § 5 to rederive the original results). As such, it is hoped that our analysis will be of interest to a general mathematical audience. A brief note on the underlying proof theory and the role played by Gödel's functional interpretation is provided in § 6, but this is not required in order to follow the main part of the paper.

More importantly though, we consider the relatively simple results here as paving the way for a more advanced study of theorems of Abelian or Tauberian type, of which those studied here are the simplest. In particular, Tauber's theorem was significantly generalised by Hardy and Littlewood and then by Wiener (cf. [19, 20] and, e.g., [10] for a modern survey covering these and later developments). We conjecture that a wealth of interesting case studies for applied proof theory can be found in this area, and hope that in this article to have taken a first step in this direction.

2 Cauchy variants of convergence properties

We start off with some preliminary mathematical results, with the aim of setting up suitable Cauchy formulations of both Abel's and Tauber's theorem, which will then be analysed over the remainder of the paper. For the sake of completeness, we begin by stating these theorems as they are usually formulated.

Theorem 2.1 (Abel's theorem) *Let $\{a_n\}$ be a sequence of reals and suppose that $\sum_{i=0}^{\infty} a_i = s$. Then $\lim_{x\to 1^-} \sum_{i=0}^{\infty} a_i x^i = s$.*

Note that Abel's theorem is often stated with the addition assumption that the radius of convergence r of the power series $\sum_{i=0}^{\infty} a_i x^i$ is exactly 1, as this is the only interesting case: Convergence of $\sum_{i=0}^{\infty} a_i$ implies that $r \geq 1$, and if $r > 1$ then left sided continuity of the power series at $x = 1$ follows trivially from continuity of power series within their radius of convergence.

Theorem 2.2 (Tauber's theorem) *Let $\{a_n\}$ be a sequence of reals with $a_n = o(1/n)$ and suppose that $\lim_{x\to 1^-} \sum_{i=0}^{\infty} a_i x^i = s$. Then $\sum_{i=0}^{\infty} a_i = s$.*

Our preference for Cauchy variants of Abel's and Tauber's theorems lies in the fact that we do not have to directly deal with limits, making the quantifier complexity of the underlying notions of convergence significantly simpler. In particular, we want to formulate the statement that $\lim_{n\to\infty} s_n = \lim_{x\to^-} F(x)$ without mentioning either of the limits explicitly.

Lemma 2.3 *Let* $F : [0,1) \to \mathbb{R}$ *be a function and* $\{s_n\}$ *a sequence of reals. Then each of the following implies* $\lim_{n\to\infty} s_n = \lim_{x\to 1^-} F(x)$*:*

(i) $\{s_n\}$ *converges and* $\lim_{m,n\to\infty} |F(x_m) - s_n| = 0$ *for all* $\{x_m\}$ *in [0,1) with* $\lim_{m\to\infty} x_m = 1$.

(ii) $\lim_{x\to 1^-} F(x)$ *exists and* $\lim_{m,n\to\infty} |F(y_m) - s_n| = 0$ *for some* $\{y_m\}$ *in [0,1) with* $\lim_{m\to\infty} y_n = 1$.

P r o o f . From (i), suppose that $\lim_{n\to\infty} s_n = s$ and $\lim_{m\to\infty} x_m = 1$. Then for any $\varepsilon > 0$ there is a sufficiently large N such that $|F(x_m) - s_n| \leq \frac{\varepsilon}{2}$ for all $m, n \geq N$, and since in addition $|s_n - s| \leq \frac{\varepsilon}{2}$ for all n sufficiently large, it follows that $|F(x_m) - s| \leq \varepsilon$ for $m \geq N$. Since $\{x_m\}$ was arbitrary, we have shown that $\lim_{m\to\infty} F(x_m) = s$ whenever $\lim_{m\to\infty} x_m = 1$ and thus $\lim_{x\to 1^-} F(x) = s$. Similarly, from (ii), if $\lim_{x\to 1^-} F(x) = s$ then in particular $F(y_m) \to s$ and thus for any $\varepsilon > 0$ there is some N with $|F(y_m) - s| \leq \frac{\varepsilon}{2}$ for all $m \geq N$ and some N' with $|F(y_m) - s_n| \leq \frac{\varepsilon}{2}$ for all $m, n \geq N'$. Therefore $|s_n - s| \leq \varepsilon$ for all $n \geq N'$, and so we have shown that $\lim_{n\to\infty} s_n = s$. □

We are now able to give Cauchy variants to both main theorems, which we furnish with direct proofs. These are adapted from standard proofs of Abel's and Tauber's theorem, but are written in a semi-quantitative way which makes certain dependencies clear, with the intention being that that the reader can compare these with our fully quantitative variants in § 4. For the remainder of the paper, we define $\{s_n\}$, $F : [0,1) \to \mathbb{R}$ and $F_\ell : [0,1) \to \mathbb{R}$ for $\ell \in \mathbb{N}$ by

$$s_n := \sum_{i=0}^{n} a_i, \quad F(x) := \sum_{i=0}^{\infty} a_i x^i \quad \text{and} \quad F_\ell(x) := \sum_{i=0}^{\ell} a_i x^i$$

where $\{a_n\}$ will be some given sequence of real numbers, which in practise will always be bounded so that $F(x)$ is well-defined on [0,1).

Theorem 2.4 (Abel's theorem, Cauchy variant) *Let* $\{a_n\}$ *be a sequence of reals such that* $\{s_n\}$ *is Cauchy. Then* $\lim_{m,n\to\infty} |F(x_m) - s_n| = 0$ *for any sequence* $\{x_m\} \in [0,1)$ *with* $\lim_{m\to\infty} x_m = 1$.

P r o o f . We first observe that for any x and ℓ we have

$$F_\ell(x) = s_\ell x^\ell + (1-x) \sum_{i=0}^{\ell-1} s_i x^i. \tag{2}$$

Since $\{s_n\}$ is Cauchy, we must have $\lim_{n\to\infty} a_n = 0$ and so in particular $\{a_n\}$ is bounded, which means that for any $x \in [0,1)$ the power series $F(x)$ converges. But then for any $x \in [0,1)$, since $\lim_{n\to\infty} s_\ell x^\ell = 0$ it follows from (2) that

$$F(x) = (1-x) \sum_{i=0}^{\infty} s_i x^i.$$

Now, using that $(1-x)\sum_{i=0}^{\infty} x^i = 1$ for $x \in [0,1)$ we have

$$|F(x_m) - s_n| = \left| (1-x_m) \sum_{i=0}^{\infty} s_i x_m^i - (1-x_m) \sum_{i=0}^{\infty} s_n x_m^i \right| \leq (1-x_m) \sum_{i=0}^{\infty} |s_i - s_n| x_m^i.$$

Fixing some $\varepsilon > 0$, there exists some N_1 such that $|s_i - s_n| \leq \varepsilon$ for all $i, n \geq N_1$, and thus for $n \geq N_1$:

$$|F(x_m) - s_n| = (1-x_m) \sum_{i=0}^{N_1-1} |s_i - s_n| x_m^i + (1-x_m) \sum_{i=N_1}^{\infty} \varepsilon x_m^i \leq (1-x_m) \sum_{i=0}^{N_1-1} |s_i - s_n| + \varepsilon.$$

Since N_1 is independent of x_m and $\lim_{m\to\infty}(1-x_m) = 0$, there exists some N_2 (dependent on N_1 and a bound on $\{|s_n|\}$) such that the first term on the right hand side is at most ε for any $m \geq N_2$. Thus there exists an N such that $|F(x_m) - s_n| \leq 2\varepsilon$ for all $m, n \geq N$, and so $\lim_{m,n\to\infty} |F(x_m) - s_n| = 0$. □

Theorem 2.5 (Tauber's theorem, Cauchy variant) *Let* $\{a_n\}$ *be a sequence of reals with* $a_n = o(1/n)$ *and suppose that* $\{F(v_m)\}$ *is Cauchy, where* $v_m := 1 - \frac{1}{m}$. *Then* $\lim_{m,n\to\infty} |F(v_m) - s_n| = 0$.

P r o o f . Using the basic inequality $1 - x^i \leq i(1 - x)$ for $x \in [0, 1]$, we note that

$$|F_n(v_n) - s_n| \leq \sum_{i=0}^{n} |a_i|(1 - v_n^i) \leq \sum_{i=0}^{n} i|a_i|(1 - v_n) = \frac{1}{n}\sum_{i=0}^{n} i|a_i|. \tag{3}$$

Fixing $\varepsilon > 0$, there exists some N_1 such that $i|a_i| \leq \varepsilon$ for all $i \geq N_1$, and so for $n \geq N_1$ we have

$$|F_n(v_n) - s_n| \leq \frac{1}{n}\sum_{i=0}^{N_1-1} i|a_i| + \frac{1}{n}\sum_{i=N_1}^{n} \varepsilon \leq \frac{1}{n}\sum_{i=0}^{N_1-1} i|a_i| + \varepsilon.$$

It therefore follows that there exists some $N' \geq N_1$ (dependent on a bound on $\{|a_i|\}$ in addition to N_1) such that for all $n \geq N'$ we have $|F_n(v_n) - s_n| \leq 2\varepsilon$. But observing that for $n \geq N'$ we also have

$$|F(v_n) - F_n(v_n)| \leq \sum_{i=n+1}^{\infty} |a_i|v_n^i \leq \varepsilon \sum_{i=n+1}^{\infty} \frac{v_n^i}{i} \leq \frac{\varepsilon}{(n+1)(1 - v_n)} \leq \varepsilon$$

and thus $|F(v_n) - s_n| \leq 3\varepsilon$ for all $n \geq N'$. Finally, since $\{F(v_m)\}$ is Cauchy, there is some N_2 such that $|F(v_m) - F(v_n)| \leq \varepsilon$ for all $m, n \geq N_2$, and so there exists some N such that $|F(v_m) - s_n| \leq 4\varepsilon$ for all $m, n \geq N$, and so $\lim_{m,n\to\infty} |F(x_m) - s_n| = 0$. □

To see that Theorem 2.4 implies Theorem 2.1, suppose that $\lim_{n\to\infty} s_n = s$. Then in particular $\{s_n\}$ is Cauchy, and so $\lim_{m,n\to\infty} |F(x_m) - s_n| = 0$ whenever $\lim_{m\to\infty} x_m = 1$. But then by Lemma 2.3 we have $\lim_{x\to 1^-} F(x) = s$. That Theorem 2.5 implies Theorem 2.2 is similar: Suppose that $a_n = o(1/n)$ and $\lim_{x\to 1^-} F(x) = s$. Since $\lim_{m\to\infty} v_m = 1$ then $\lim_{m\to\infty} F(v_m) = s$ and so in particular $\{F(v_m)\}$ is Cauchy, and thus $\lim_{m,n\to\infty} |F(v_m) - s_n| = 0$. But then by Lemma 2.3 (ii) it follows that $\lim_{n\to\infty} s_n = s$.

3 On Specker sequences

In this short section, we show that Specker phenomena propagate through both Theorems 2.4 & 2.5. For the former this is completely straightforward, but for the latter a little care is needed to construct a suitable sequence satisfying the Tauber condition $a_n = o(1/n)$. Roughly speaking, these results confirm that if we do not have a direct rate of convergence for the input data, then neither do we have a direct rate of convergence for $|F(v_m) - s_n| \to 0$ as $m, n \to \infty$. This justifies our use of the relevant notions of *metastability* instead.

We first recall that a Specker sequence, first introduced in [16], is a *computable*, *monotonically increasing* and *bounded* sequence of rationals $\{q_n\}$ whose limit is not a computable real number. What this means in practice is that the sequence possess neither a computable *rate of convergence* nor a computable *rate of Cauchy convergence*, where by the latter we mean a computable function $\pi : \mathbb{Q}_+ \to \mathbb{N}$ satisfying

$$\forall \varepsilon \in \mathbb{Q}_+, \forall m, n \geq \pi(\varepsilon)(|q_m - q_n| \leq \varepsilon),$$

where here $\mathbb{Q}_+$ denotes the set of all strictly positive rationals.

Proposition 3.1 *There exists some $\{a_n\}$ satisfying the premise of Theorem 2.4, whereby for any $\{x_n\}$ in [0,1) with $\lim_{n\to\infty} x_m = 1$, though $\lim_{m,n\to\infty} |F(x_m) - s_n| = 0$ this has no computable rate of convergence, i.e., there is no computable function $\pi : \mathbb{Q}_+ \to \mathbb{N}$ satisfying*

$$\forall \varepsilon \in \mathbb{Q}_+, \forall m, n \geq \pi(\varepsilon)(|F(x_m) - s_n| \leq \varepsilon).$$

P r o o f . Take any Specker sequence $\{q_n\}$ and define $a_0 := q_0$ and $a_{n+1} := q_{n+1} - q_n$, so that $s_n = q_n$, and so by definition $\{s_n\}$ is Cauchy. Suppose for contradiction there exists some $\{x_m\}$ with $\lim_{m\to\infty} x_m = 1$ such that $|F(x_m) - s_n| \to 0$ as $m, n \to \infty$ with a computable rate of convergence π. Then for any $\varepsilon \in \mathbb{Q}_+$ we have

$$|s_m - s_n| \leq |s_m - F(x_{\pi(\varepsilon/2)})| + |F(x_{\pi(\varepsilon/2)}) - s_n| \leq \varepsilon$$

for all $m, n \geq \pi(\varepsilon/2)$, and thus $\{s_n\}$ has a computable rate of Cauchy convergence, which is false. □

Proposition 3.2 *There exists $\{a_n\}$ satisfying the premise of Theorem 2.5, whereby though $\lim_{m,n\to\infty} |F(v_m) - s_n| = 0$ this has no computable rate of convergence.*

Proof. Take any Specker sequence $\{q_n\}$ and define $a_0 := q_0$, $a_1 := q_1 - q_0$ and for $n \geq 2$

$$a_n := \frac{q_{m+1} - q_m}{2^{m-1}} \quad \text{for } m := \lceil \log_2(n) \rceil.$$

We first observe that since $2^{m-1} \geq n/2$ we have

$$n|a_n| = \frac{n}{2^{m-1}}(q_{m+1} - q_m) \leq 2(q_{m+1} - q_m) \to 0$$

as $n \to \infty$, and thus $a_n = o(1/n)$. An easy induction establishes that $s_{2^n} = q_{n+1}$ for all $n \geq 1$, where for the induction step we have

$$s_{2^n} = s_{2^{n-1}} + \sum_{i=2^{n-1}+1}^{2^n} a_i = q_n + \sum_{i=2^{n-1}+1}^{2^n} \left(\frac{q_{n+1} - q_n}{2^{n-1}} \right) = q_n + (q_{n+1} - q_n) = q_{n+1}.$$

This implies that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} q_n$, and so in particular by Abel's theorem $\lim_{x\to 1^-} F(x)$ exists and so $\{F(v_n)\}$ is Cauchy. Therefore $\{a_n\}$ satisfies the premise of Theorem 2.5. But now suppose by contradiction we have a computable rate of convergence $\pi : \mathbb{Q}_+ \to \mathbb{N}$ for $\lim_{m,n\to\infty} |F(v_m) - s_n| = 0$. Then as in the proof of Proposition 3.1, for any $\varepsilon \in \mathbb{Q}_+$ we have $|s_m - s_n| \leq \varepsilon$ for all $m, n \geq \pi(\varepsilon/2)$, and therefore $|q_m - q_n| \leq \varepsilon$ for all $m, n \geq \varphi(\varepsilon)$, where $\varphi(\varepsilon) := \lceil \log_2(\pi(\varepsilon/2)) \rceil + 1$. But since φ is computable, this contradicts the assumption that $\{q_n\}$ is a Specker sequence. □

4 The finitary theorems

We now give finitary, quantitative formulations to our Cauchy variants of Abel and Tauber's theorems, presented in each case as a route from a metastable version of the premise to that of the conclusion. These results are finitary in the sense that aside from a global bound on our input data, we only consider finite initial segments of this data, and quantitative in the sense that they provide an explicit method for constructing rates of metastability for the conclusion in terms of rates of metastability from the premises. Moreover, the proofs of both results are also entirely finitistic in nature, appealing to nothing more than simple arithmetic operations.

In the remainder of this paper, we denote by $\omega : \mathbb{Q}_+ \times \mathbb{N}_{>0} \to \mathbb{N}$ some canonical computable function satisfying, for all $\varepsilon \in \mathbb{Q}_+$ and $p \geq 1$:

1. $\omega(\varepsilon, p) \geq p$,
2. if $x \in [0, 1 - \frac{1}{p}]$ then $x^\ell \leq \varepsilon$ whenever $\ell \geq \omega(\varepsilon, p)$.

For instance, using the standard inequality $(1+y)^r \leq e^{yr}$ (for $y \in \mathbb{R}$ and $r > 0$) we could set $\omega(\varepsilon, p) := p \cdot \lceil \log(1/\varepsilon) \rceil$, but for notational simplicity we work directly with ω rather than any specific function. Our first result recalls that a power series has a computable rate of convergence within any compact interval $[0, 1 - \frac{1}{p}] \subset [0, 1)$ given a bound on its coefficients, and thus the function F can be approximated by F_ℓ for ℓ computable in the desired degree of accuracy.

Lemma 4.1 *Let $\{|a_n|\}$ be bounded above by some $L \in \mathbb{N}$. Then for any $\varepsilon \in \mathbb{Q}_+$ and $p \geq 1$ we have $|F(x) - F_\ell(x)| \leq \varepsilon$ whenever $x \in [0, 1 - \frac{1}{p}]$ and $\ell \geq \omega(\frac{\varepsilon}{Lp}, p)$.*

Proof. To see this, we simply observe that

$$|F(x) - F_\ell(x)| = \left| \sum_{i=\ell+1}^{\infty} a_i x^i \right| \leq \sum_{i=\ell+1}^{\infty} |a_i| x^i \leq L\left(\frac{x^{\ell+1}}{1-x} \right) \leq Lpx^{\ell+1} \leq Lpx^\ell \leq \varepsilon$$

where in the last step we use the defining property of ω. □

We now present our finitary theorems, where we recall the notation $[n; k] := \{n, n+1, \ldots, k-1, k\}$ for $n \leq k$, and just $[n; k] := \emptyset$ for $k < n$.

Theorem 4.2 (Finite Abel's theorem) *Let $\{a_n\}$ and $\{x_k\}$ be arbitrary sequences of reals, and $L \in \mathbb{N}$ a bound for $\{|s_n|\}$. Fix some $\varepsilon \in \mathbb{Q}_+$ and $g : \mathbb{N} \to \mathbb{N}$. Suppose that $N_1, N_2 \in \mathbb{N}$ and $p \geq 1$ are such that $|s_i - s_n| \leq \frac{\varepsilon}{4}$ and $\frac{1}{p} \leq 1 - x_m \leq \frac{\varepsilon}{8LN_1}$ for all $i, n \in [N_1; \max\{N + g(N), \ell\}]$ and all $m \in [N_2; N + g(N)]$ where*

1. $N := \max\{N_1, N_2\}$,

2. $\ell := \omega(\frac{\varepsilon}{8Lp}, p)$.

Then we have $|F(x_m) - s_n| \leq \varepsilon$ *for all* $m, n \in [N; N + g(N)]$.

Proof. Fix some $m, n \in [N; N + g(N)]$. We first note that since $m \geq N$ then $m \geq N_2$ and thus $m \in [N_2; N + g(N)]$ from which it follows that $2LN_1(1 - x_m) \leq \frac{\varepsilon}{4}$. Using this together with the fact that $n \in [N_1; \max\{N + g(N), \ell\}]$ and thus $|s_i - s_n| \leq \frac{\varepsilon}{4}$ for any $N_1 \leq i \leq \ell \leq \max\{N + g(N), \ell\}$ we have

$$\begin{aligned}\left|(1 - x_m)\sum_{i=0}^{l-1}(s_i - s_n)x_m^i\right| &\leq (1 - x_m)\sum_{i=0}^{l-1}|s_i - s_n|x_m^i \\ &\leq (1 - x_m)\sum_{i=0}^{N_1-1}|s_i - s_n|x_m^i + (1 - x_m)\sum_{i=N_1}^{l-1}|s_i - s_n|x_m^i \\ &\leq (1 - x_m)\sum_{i=0}^{N_1-1}(|s_i| + |s_n|) + (1 - x_m)\cdot\tfrac{\varepsilon}{4}\sum_{i=N_1}^{l-1}x_m^i \\ &\leq 2LN_1(1 - x_m) + \tfrac{\varepsilon}{4}\cdot(x_m^{N_1} - x_m^{\ell}) \leq \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} = \tfrac{\varepsilon}{2}.\end{aligned} \tag{4}$$

Next, using (2) from the proof of Theorem 2.4 together with (4) we see that

$$\begin{aligned}|F_\ell(x_m) - s_n| &= \left|s_\ell x_m^\ell + (1 - x_m)\sum_{i=0}^{\ell-1}s_i x_m^i - s_n\right| \\ &\leq |s_\ell x_m^\ell| + \left|(1 - x_m)\sum_{i=0}^{\ell-1}(s_i - s_n)x_m^i\right| + \left|(1 - x_m)\sum_{i=0}^{\ell-1}s_n x_m^i - s_n\right| \\ &\leq |s_\ell|x_m^\ell + \tfrac{\varepsilon}{2} + |s_n|x_m^\ell \leq \tfrac{3\varepsilon}{4}\end{aligned} \tag{5}$$

where for the last step we use that $(|s_\ell| + |s_n|)x_m^\ell \leq 2Lx_m^\ell \leq \frac{\varepsilon}{4p} \leq \frac{\varepsilon}{4}$ which holds by definition of ℓ together with the fact that $x_m \in [0, 1 - \frac{1}{p}]$. Finally, observing that $|a_j| = |s_j - s_{j-1}| \leq |s_j| + |s_{j-1}| \leq 2L$ for any $j \in \mathbb{N}$, by Lemma 4.1 and (5) we have $|F(x_m) - s_n| \leq |F(x_m) - F_\ell(x_m)| + |F_\ell(x_m) - s_n| \leq \frac{\varepsilon}{4} + \frac{3\varepsilon}{4} \leq \varepsilon$, which completes the proof. □

Theorem 4.3 (Finite Tauber's theorem) *Let* $\{a_n\}$ *be an arbitrary sequence of reals, and L a bound for* $\{|a_n|\}$. *Define* $v_m := 1 - \frac{1}{m}$, *and fix some* $\varepsilon \in \mathbb{Q}_+$ *and* $g : \mathbb{N} \to \mathbb{N}$. *Suppose that* $N_1, N_2 \in \mathbb{N}$ *are such that* $i|a_i| \leq \frac{\varepsilon}{8}$ *and* $|F(v_m) - F(v_n)| \leq \frac{\varepsilon}{4}$ *for all* $i \in [N_1; \ell]$ *and all* $m, n \in [N_2; N + g(N)]$ *where*

1. $N := \max\{\frac{2LN_1^2}{\varepsilon}, N_2\}$,

2. $\ell := \omega(\frac{\varepsilon}{4Lp}, p)$ *for* $p := N + g(N)$.

Then we have $|F(v_m) - s_n| \leq \varepsilon$ *for all* $m, n \in [N; N + g(N)]$.

Proof. Fix some $m, n \in [N; N + g(N)]$. We first note that since $\frac{2LN_1^2}{\varepsilon} \leq n$ then we have $\frac{LN_1^2}{2n} \leq \frac{\varepsilon}{4}$, and since $N_1 \leq \frac{2LN_1^2}{\varepsilon} \leq n \leq \ell$ then for any $N_1 \leq i \leq n$ we have $i|a_i| \leq \frac{\varepsilon}{8}$. Therefore, using (3) from the proof of Theorem 2.5, we have

$$\begin{aligned}|F_n(v_n) - s_n| &\leq \frac{1}{n}\sum_{i=0}^{n} i|a_i| = \frac{1}{n}\sum_{i=0}^{N_1-1} i|a_i| + \frac{1}{n}\sum_{i=N_1}^{n} i|a_i| \\ &\leq \frac{L}{n}\cdot\frac{1}{2}(N_1 - 1)N_1 + \frac{\varepsilon}{8n}(n - N_1) \leq \frac{LN_1^2}{2n} + \frac{\varepsilon}{8} \leq \frac{3\varepsilon}{8}.\end{aligned} \tag{6}$$

Similarly, we have

$$\begin{aligned} |F_\ell(v_n) - F_n(v_n)| &= \left|\sum_{i=n+1}^{\ell} a_i v_n^i\right| \leq \frac{\varepsilon}{8}\sum_{i=n+1}^{\ell}\frac{v_n^i}{i} \\ &\leq \frac{\varepsilon}{8(n+1)}\sum_{i=n+1}^{\ell} v_n^i \leq \frac{\varepsilon(v_n^{n+1} - v_n^{\ell+1})}{8(n+1)(1-v_n)} \leq \frac{\varepsilon n}{8(n+1)} \leq \frac{\varepsilon}{8}. \end{aligned} \tag{7}$$

Now, since for any $n \in [N; N + g(N)]$ we have $v_n \in [0, 1 - \frac{1}{p}]$ for $p := N + g(N)$ by definition, it follows by Lemma 4.1 that $|F(v_n) - F_\ell(v_n)| \leq \frac{\varepsilon}{4}$, and thus

$$\begin{aligned} |F(v_m) - s_n| &\leq |F(v_m) - F(v_n)| + |F(v_n) - F_\ell(v_n)| + |F_\ell(v_n) - s_n| \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + |F_\ell(v_n) - F_n(v_n)| + |F_n(v_n) - s_n| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{8} + \frac{3\varepsilon}{8} \leq \varepsilon \end{aligned}$$

where the last line follows from (6) and (7). This completes the proof. □

5 Reobtaining the infinitary variants

We conclude by showing how the theorems of the previous section, though finitary in nature, are strong enough to allow us to reobtain the usual formulations of Abel and Tauber's theorems using purely logical reasoning. We first need a simple but crucial lemma which we use throughout this section.

Lemma 5.1 *Let $P(\varepsilon, X)$ be some predicate on $\varepsilon \in \mathbb{Q}_+$ and finite subsets $X \subset \mathbb{N}$. Then the following two statements are equivalent:*

(a) $\forall \varepsilon \in \mathbb{Q}_+ \, \exists n \in \mathbb{N} \, \forall k \, P(\varepsilon, [n; k])$,

(b) $\forall \varepsilon \in \mathbb{Q}_+ \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists n \in \mathbb{N} \, P(\varepsilon, [n; g(n)])$.

Proof. For $(a) \implies (b)$, if for some $\varepsilon \in \mathbb{Q}_+$ there is some $n \in \mathbb{N}$ satisfying $P(\varepsilon, [n; k])$ then in particular for any $g : \mathbb{N} \to \mathbb{N}$ we have $P(\varepsilon, [n; g(n)])$. To establish $(b) \implies (a)$ we suppose for contradiction that (a) is false, and thus for some $\varepsilon \in \mathbb{Q}_+$ it is the case that for $\forall n \in \mathbb{N} \, \exists k \, \neg P(\varepsilon, [n; k])$. Therefore by the axiom of choice there is some $g : \mathbb{N} \to \mathbb{N}$ satisfying $\forall n \in \mathbb{N} \, \neg P(\varepsilon, [n; g(n)])$, contradicting (b). □

By setting $P(\varepsilon, X) :\iff \forall m, n \in X(|s_m - s_n| \leq \varepsilon)$ the equivalence of Cauchy convergence and metastability in the sense of (1) is a direct corollary of the above lemma—note that the slightly different statement $\forall \varepsilon, g \exists n P(\varepsilon, [n; n + g(n)])$ is just another way of expressing (b). By extending Lemma 5.1 to the various other Cauchy properties involved in our finitary theorems, we are able to prove the original, infinitary variants of those theorems.

Deriving Theorem 2.4 from Theorem 4.2. Suppose that $\{a_n\}$ and $\{x_m\}$ are such that (i) $\{s_n\}$ is Cauchy, (ii) $\lim_{m\to\infty} x_m = 1$. Note that since $\{s_n\}$ convergences then $\{|s_n|\}$ must be bounded above by some L. Now fix some arbitrary $\varepsilon \in \mathbb{Q}_+$ and $g : \mathbb{N} \to \mathbb{N}$. From $\lim_{m\to\infty} x_m = 1$ and Lemma 5.1 we can infer that for any $\delta > 0$ and $h : \mathbb{N} \to \mathbb{N}$ there exists some $n \in \mathbb{N}$ such that

$$\forall m \in [n; h(n)](1 - \delta \leq x_m).$$

Using a weak form of the axiom of choice, let $\Phi(\delta, h)$ be the functional which returns such an n for any given δ and h, and define $f : \mathbb{N} \to \mathbb{N}$ by $f(a) := \max\{M_a + g(M_a), \omega(\frac{\varepsilon}{8Lp_a}, p_a)\}$ where we define

$$\begin{aligned} M_a &:= \max\left\{a, \Phi\left(\frac{\varepsilon}{8La}, h_a\right)\right\} \\ p_a &:= \lceil \max\{1/(1 - x_m) \ : \ m \leq M_a + g(M_a)\}\rceil \\ h_a(b) &:= \max\{a, b\} + g(\max\{a, b\}). \end{aligned}$$

Now, from Cauchyness and hence metastability of $\{s_n\}$ we can infer that there exists some $N_1 \in \mathbb{N}$ such that

$$\forall i, n \in [N_1; f(N_1)](|s_i - s_n| \leq \tfrac{\varepsilon}{4}).$$

Define $N_2 := \Phi(\frac{\varepsilon}{8LN_1}, h_{N_1})$, so that

$$\forall m \in [N_2; h_{N_1}(N_2)]\left(1 - \frac{\varepsilon}{8LN_1} \leq x_m\right).$$

Then setting $N := \max\{N_1, N_2\}$ we see that $M_{N_1} = N$ and therefore $f(N_1) = \max\{N + g(N), \ell\}$ for $\ell = \omega(\frac{\varepsilon}{8Lp}, p)$ with $p = \lceil \max\{1/(1 - x_m) \ : \ m \leq N + g(N)\}\rceil$. We have in addition that $h_{N_1}(N_2) = N + g(N)$. Observing finally that for $m \leq N + g(N)$ we must have $1/(1 - x_m) \leq p$ and thus $\frac{1}{p} \leq 1 - x_m$, we see that N_1, N_2 and p satisfy the premise of Theorem 4.2, and therefore $|F(x_m) - s_n| \leq \varepsilon$ for all $m, n \in [N, N + g(N)]$. But since ε and g were arbitrary, by Lemma 5.1 this means that $\lim_{m,n\to\infty} |F(x_m) - s_n| = 0$.

Deriving Theorem 2.5 from Theorem 2.2. Suppose that $\{a_n\}$ is a sequence of reals such that (i) $a_n = o(1/n)$ and (ii) $\{F(v_m)\}$ is Cauchy. From (i) we must have that $\{|a_n|\}$ is bounded above by some L. Now fix some arbitrary $\varepsilon \in \mathbb{Q}_+$ and $g : \mathbb{N} \to \mathbb{N}$. From Cauchyness and hence metastability of $\{F(v_n)\}$ we can infer that for any $h : \mathbb{N} \to \mathbb{N}$ there exists some $k \in \mathbb{N}$ such that

$$\forall m, n \in [k; h(k)](|F(v_m) - F(v_n)| \leq \tfrac{\varepsilon}{4}).$$

Let $\Psi(h)$ be the functional which returns such a k for any given h, and define $f : \mathbb{N} \to \mathbb{N}$ by $f(a) := \omega(\frac{\varepsilon}{4Lp_a}, p_a)$ where we define

$$p_a := M_a + g(M_a)$$

$$M_a := \max\left\{\frac{2La^2}{\varepsilon}, \Psi(h_a)\right\}$$

$$h_a(b) := \max\left\{\frac{2La^2}{\varepsilon}, b\right\} + g\left(\max\left\{\frac{2La^2}{\varepsilon}, b\right\}\right).$$

From $\lim_{n\to\infty} n|a_n| = 0$ and Lemma 5.1 we infer that there exists some $N_1 \in \mathbb{N}$ such that

$$\forall i \in [N_1; f(N_1)](i|a_i| \leq \tfrac{\varepsilon}{8}).$$

Define $N_2 := \Psi(h_{N_1})$ so that

$$\forall m, n \in [N_2, h_{N_1}(N_2)](|F(v_m) - F(v_n)| \leq \tfrac{\varepsilon}{4}).$$

Then setting $N := M_{N_1} = \max\{\frac{2LN_1^2}{\varepsilon}, N_2\}$ we see that $f(N_1) = \omega(\frac{\varepsilon}{4Lp}, p)$ for $p = N + g(N)$ and $h_{N_1}(N_2) = N + g(N)$, and therefore N_1 and N_2 satisfy the premise of Theorem 4.3. Therefore $|F(v_m) - s_n| \leq \varepsilon$ for all $m, n \in [N; N + g(N)]$, and since ε and g were arbitrary, this means by Lemma 5.1 that $\lim_{m,n\to\infty} |F(v_m) - s_n| = 0$.

6 General proof theoretic remarks

In this final section we give some deeper insights into proof theoretic aspects of our paper. We have deliberately suppressed this aspect of the work so far, in order to make our main results as accessible and self-contained as possible. However, we now outline briefly how the preceding results can be connected to both Gödel's functional interpretation and the 'proof mining' program.

6.1 Our finitary results and Gödel's functional interpretation

The main quantitative results in this paper were obtained by carrying out an analysis of the original proofs of both Abel's and Tauber's theorems using the classical Gödel functional interpretation (i.e., the combination of the usual functional interpretation with a negative translation), which as already mentioned in the introduction constitutes a formal technique for obtaining finitary version of infinitary theorems. In both cases, the resulting realizing terms were simple enough that the core *combinatorial part* of the analysis could be presented in a traditional mathematical style, in particular without reference to higher-order functionals. This gave rise to § 4 and Theorems 4.2 & 4.3.

The full analysis, in which higher-order rates of metastability for the conclusions of each theorem are constructed in terms of corresponding rates for the premises, follows by appealing to the results of § 5. In particular, the routes from Theorem 4.2 to 2.4 and from Theorem 4.3 to 2.5 use, in both cases, a simple form of *finite bar recursion* of length two, an explanation for which follows below. Though in this paper we do not work in any formal systems, the proofs of § 5 essentially use just classical predicate logic together with the quantifier-free axiom of choice, and in this sense, our finitary theorems imply the infinitary versions over a very weak base theory, and using just elementary logical reasoning.

We now make these ideas more precise, though we do assume some familiarity with the classical functional interpretation (background can be found in [7]). We first point out that in Lemma 5.1, (b) corresponds to the classical functional interpretation (before the final Skolemization) of (a). Let us now focus on the proof of Tauber's theorem as stated in Theorem 2.5. A slight modification of the proof of this theorem allows us to establish that for any $\varepsilon > 0$ we have

$$\exists N_1\ \forall k\ P_1(\tfrac{\varepsilon}{8}, [N_1; k]) \wedge \exists N_2\ \forall k\ P_2(\tfrac{\varepsilon}{4}, [N_2; k]) \implies \exists N \forall k\ Q(\varepsilon, [N; k]) \tag{8}$$

where here the first conjunct of the premise represents the statement $a_n = o(1/n)$, i.e.,

$$P_1(\delta, [n; k]) :\Longleftrightarrow \forall i \in [n; k](i|a_i| \leq \delta),$$

and in an analogous way the second conjunct and the conclusion represent the statements that $\{F(v_n)\}$ is Cauchy and that $\lim_{m,n\to\infty} |F(v_m) - s_n| = 0$, respectively. Q represent the Π_3-statements that $\{F(v_n)\}$ is Cauchy resp. $\lim_{m,n\to\infty} |F(v_m) - s_n| = 0$. Now, the classical functional interpretation of the negative translation of (8), i.e.,

$$\exists N_1\ \forall k\ P_1(\tfrac{\varepsilon}{8}, [N_1; k]) \wedge \exists N_2\ \forall k\ P_2(\tfrac{\varepsilon}{4}, [N_2; k]) \implies \neg\neg\exists N \forall k\ Q(\varepsilon, [N; k]) \tag{9}$$

asks for terms $r_i(N_1, N_2, g)$ for $i = 1, 2$ and $s(N_1, N_2, g)$ satisfying

$$\begin{aligned} &P_1(\tfrac{\varepsilon}{8}, [N_1; r_1(N_1, N_2, g)] \wedge P_2(\tfrac{\varepsilon}{4}, [N_2; r_2(N_1, N_2, g)])) \\ &\quad \implies Q(\varepsilon, [s(N_1, N_2, g); s(N_1, N_2, g) + g(s(N_1, N_2, g))]). \end{aligned} \tag{10}$$

Our finitary variant of Tauber's theorem (Theorem 4.3) corresponds to the extraction of such terms from the proof of (8), where these terms also depend on the global parameters ε together with a bound L for $\{|a_n|\}$.

Formally, in order to derive Theorem 2.5 from (10) (i.e., Theorem 4.3), we combined the conjunction of (metastable variants) of the assumptions of Theorem 2.5 into a metastable version of the premise of (9), i.e., we used terms which witnessed the functional interpretation of the following implication:

$$\begin{aligned} &\neg\neg\exists N_1 \forall k\ P_1(\tfrac{\varepsilon}{8}, [N_1; k]) \wedge \neg\neg\exists N_2 \forall k\ P_2(\tfrac{\varepsilon}{4}, [N_2; k]) \\ &\quad \implies \neg\neg(\exists N_1\ \forall k\ P_1(\tfrac{\varepsilon}{8}, [N_1; k]) \wedge \exists N_2\ \forall k\ P_2(\tfrac{\varepsilon}{4}, [N_2; k])). \end{aligned}$$

This an instance of the so-called *finite double negation shift*, and as shown in [12], this is interpreted using a form of *finite bar recursion* (cf. also [4]). While the specific details of our particular case are given in § 5, our aim here is to emphasise that our rederivation of the infinitary theorems from their finitary counterparts uses a simple version of a much more fundamental scheme, which has recently been connected to game theory (cf. [2, 3]). The above analysis is also valid for Abel's theorem, although here we require a *dependent* version of the double negation shift, which is nevertheless still solved using finite bar recursion.

6.2 Numerical results via proof mining

We conclude with a simple illustration of how our quantitative formulation of Abel's theorem can be used to obtain concrete rates of metastability in the style of traditional proof mining. Since it is clear that the proofs of both Abel's and Tauber's theorems can be formalised in Peano arithmetic, it follows immediately from the main soundness theorem of the classical functional interpretation that converting rates of metastability for the premises to one for the conclusion can be done in Gödel's System T. However, in the concrete example that follows, the actual bounds we obtain are extremely simple, as is typically the case in proof mining.

Let us take the following simple consequence of Abel's theorem, which follows directly from Theorem 2.4, using the fact that whenever $\{a_n\}$ is a sequence of *positive* reals then $\{s_n\}$ is monotonically increasing, and thus is Cauchy whenever it is bounded above:

Proposition 6.1 *Let $\{a_n\}$ be a sequence of positive reals whose partial sums $\{s_n\}$ are bounded above. Then* $\lim_{m,n\to\infty} |F(v_m) - s_n| = 0$.

An analysis of this result using Theorem 4.2 together with ideas from § 5 yields the following.

Proposition 6.2 *Let $\{a_n\}$ be a sequence of positive reals and L a bound for the partial sums $\{s_n\}$. Then for any $\varepsilon \in \mathbb{Q}_+$ and $g : \mathbb{N} \to \mathbb{N}$ we have*

$$\exists N \leq \Gamma_L(\varepsilon, g)\, \forall m, n \in [N, N + g(N)]\, (|F(v_m) - s_n)| \leq \varepsilon)$$

for $\Gamma_L(\varepsilon, g)$ given as follows:

1. $\Gamma_L(\varepsilon, g) := \left\lceil \frac{8Lf^{(\lceil 4L/\varepsilon\rceil)}(0)}{\varepsilon} \right\rceil$,
2. $f(a) := p_a \cdot \left\lceil \log(\frac{8Lp_a}{\varepsilon}) \right\rceil$,
3. $p_a := \tilde{g}(\lceil \frac{8La}{\varepsilon} \rceil)$,

where $f^{(k)}(x)$ denotes the k-times iteration of f applied to x, and $\tilde{g}(x)$ is defined by $\tilde{g}(x) := x + g(x)$.

Proof. Following closely and using notation from the proof of Theorem 2.4 from Theorem 4.2 given in § 5, we first note that setting $x_m := v_m = 1 - \frac{1}{m}$, for any $\delta > 0$ and $h : \mathbb{N} \to \mathbb{N}$ we trivially have

$$\forall m \in [n; h(n)](1 - \delta \leq v_m)$$

for $n := \lceil 1/\delta \rceil$, and thus we can define $\Phi(\delta, h) := \lceil 1/\delta \rceil$. Therefore in this case, $M_a = \max\{a, \lceil 8La/\varepsilon \rceil\} = \lceil 8La/\varepsilon \rceil$ and $p_a = M_a + g(M_a) = \tilde{g}(\lceil 8La/\varepsilon \rceil)$, and thus using our explicit definition of $\omega(\varepsilon, p) = p \cdot \lceil \log(1/\varepsilon) \rceil$ from § 4, we see that $f(a) = \max\{p_a, \omega(\varepsilon/8Lp_a, p_a)\} = p_a \cdot \lceil \log(8Lp_a/\varepsilon) \rceil$. Now, it is a well-known fact from proof mining (cf. [7, Proposition 2.26]) that for any monotone increasing $\{s_n\}$ bounded above by some L, a bound on the corresponding rate of metastability, i.e.,

$$\forall \varepsilon' \in \mathbb{Q}_+\, \forall f' : \mathbb{N} \to \mathbb{N}\, \exists N' \leq \Psi(\varepsilon', f') \forall m \in [N'; f'(N')](|s_m - s_n| \leq \varepsilon')$$

is given by $\Psi(\varepsilon', f') := f'^{\lceil (L/\varepsilon')\rceil}(0)$. Thus in this case, setting $\varepsilon' := \frac{\varepsilon}{4}$ and $f' = f$ we would have $N_1 \leq f^{\lceil (4L/\varepsilon)\rceil}(0)$ and therefore $N_2 := \Phi(\frac{\varepsilon}{8LN_1}, h_{N_1}) = \left\lceil \frac{8LN_1}{\varepsilon} \right\rceil \leq \left\lceil \frac{8Lf^{\lceil (4L/\varepsilon)\rceil}(0)}{\varepsilon} \right\rceil$. Therefore by Theorem 4.2, $N := \max\{N_1, N_2\} = N_2$ satisfies

$$\forall m, n \in [N; N + g(N)]\, (|F(v_m) - s_n)| \leq \varepsilon).$$

Backtracking through the above definitions yields the given bound on N. □

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