Classification of BTn-groups over perfectoid rings

Henkel, Timo

(2020)

DOI (TUprints): https://doi.org/10.25534/tuprints-00014223

Publikationstyp: Dissertation

Fachbereich: 04 Fachbereich Mathematik

Quelle des Originals: https://tuprints.ulb.tu-darmstadt.de/14223

Classification of $\mathrm{BT_n}$-groups over perfectoid rings

Zur Erlangung des Grades eines Doktors der Naturwissenschaften (Dr. rer. nat.)
genehmigte Dissertation von Timo Henkel, M. Sc. aus Büren
Tag der Einreichung: 10. September 2020, Tag der Prüfung: 21. Oktober 2020

1. Gutachten: Prof. Dr. Torsten Wedhorn
2. Gutachten: Prof. Dr. Eike Lau
Darmstadt – D 17

TECHNISCHE UNIVERSITÄT DARMSTADT

Fachbereich Mathematik
Algebra

Classification of $\mathrm{BT_n}$-groups over perfectoid rings

genehmigte Dissertation von Timo Henkel, M. Sc.

1. Gutachten: Prof. Dr. Torsten Wedhorn
2. Gutachten: Prof. Dr. Eike Lau

Tag der Einreichung: 10. September 2020
Tag der Prüfung: 21. Oktober 2020

Darmstadt – D 17

Bitte zitieren Sie dieses Dokument als:
URN: urn:nbn:de:tuda-tuprints-142233
URL: http://tuprints.ulb.tu-darmstadt.de/14223

Dieses Dokument wird bereitgestellt von tuprints,
E-Publishing-Service der TU Darmstadt
http://tuprints.ulb.tu-darmstadt.de
tuprints@ulb.tu-darmstadt.de

Zusammenfassung

In dieser Arbeit untersuchen wir p-divisible Gruppen über integralen perfektoiden Ringen, indem wir uns auf die entsprechenden Untergruppen der p^n-Torsionselemente, sogenannte $\mathrm{BT_n}$-Gruppen, konzentrieren. Wir benutzen Resultate von Lau und Anschütz-Le Bras und zeigen, dass solche Gruppen durch semilineare Objekte über dem Tilt des Grundrings beschrieben werden können. Diese Objekte nennen wir $\mathrm{BK_n}$-Moduln. Wir folgern aus dieser Klassifikation, dass in unserem Setting jede $\mathrm{BT_n}$-Gruppe zu einer p-divisiblen Gruppe geliftet werden kann. Für den Fall lokaler perfektoider Ringe finden wir eine explizite Beschreibung dieser Daten in Form von Bahnen bzgl. einer gewissen Gruppenoperation. Durch den Zusammenhang zwischen $\mathrm{BT_1}$-Gruppen und F-Zips erhalten wir hieraus die Klassifikation von F-Zips über perfekten Körpern der Charakteristik p als Spezialfall zurück. Wir beschäftigen uns außerdem mit der Frage nach der Globalisierung dieser Resultate. Wir zeigen, dass sich $\mathrm{BK_n}$-Moduln bezüglich einer bestimmten Topologie verkleben lassen, welche außerdem so fein ist, sodass wir eine Darstellung des klassifizierenden Stacks von $\mathrm{BK_n}$-Moduln als Quotientenstack erhalten. Des Weiteren beschäftigen wir uns mit Eigenschaften unserer Konstruktionen bezüglich der feineren p-vollständigen arc-Topologie. Diese Topologie besitzt eine Basis, die aus Produkten von perfektoiden Bewertungsringen vom Rang ≤ 1 besteht. Es werden schließlich Globalisierungsresultate bezüglich dieser Topologie gezeigt. Insbesondere lassen sich $\mathrm{BK_n}$-Moduln über einem perfekten Ring verkleben und der resultierende Stack besitzt wiederum eine Darstellung als Quotientenstack. Ein analoges Resultat über allgemeinen perfektoiden Ringen wird unter der Annahme einer Vermutung gezeigt.

Contents

Introduction

For an abelian variety A and a prime number p, the associated p-divisible group (also called Barsotti-Tate group or BT-group), defined as the union $\bigcup_{i\in\mathbb{N}} A[p^i]$ of its p-power torsion subgroups, contains much information of A, but is a lot easier to handle. Moreover, most invariants may be checked on $A[p^n]$, for some $n \in \mathbb{N}$, which is a commutative finite locally free group scheme and killed by p^n. Those p-power torsion subgroups are instances of *truncated Barsotti-Tate groups* which are the main subject of interest in this paper.
For k being a perfect field of characteristic p, classical Dieudonné theory provides a semi-linear algebra description of truncated Barsotti-Tate groups of level n ($\mathrm{BT_n}$-groups). This can for example be used to show that every such group in fact arises as the p^n-torsion subgroup of some Barsotti-Tate group. Lau extended these results to perfect rings (i.e. rings of characteristic p such that the ring homomorphism $x \mapsto \sigma(x) := x^p$ is bijective) in [**Lau13**], reproving unpublished results of Gabber by a different method. The overall goal of this work is to study $\mathrm{BT_n}$-groups over perfectoid rings. Those rings arise as natural generalizations of perfect rings to the mixed characteristic world. Recently, in [**ALB19**], Anschütz and Le Bras generalized the existing instances of Dieudonné theory to so called prismatic Dieudonné theory. They obtained results on Dieudonné theory for perfectoid rings, which were partially proven before by Lau in [**Lau18**].
In Chapter 1 we recall the results described above and give a brief overview on perfectoid rings and their relevant properties.
Chapter 2 classifies $\mathrm{BT_n}$-groups over perfectoid rings in terms of semi-linear algebra (so called $\mathrm{BK_n}$-modules) and investigates the consequences for lifting properties of $\mathrm{BT_n}$-groups to BT-groups in this setting.
In Chapter 3, for a given perfectoid ring R, we construct a fully faithful functor from a certain orbit space to $\mathrm{BK_n^{(h,d)}}(R)$, the subcategory of $\mathrm{BK_n}(R)$ whose objects have constant dimension d and height h. For R being a product of local perfectoid rings this functor is shown to be an equivalence. Moreover, the results of this chapter are globalized with respect to a certain Grothendieck topology on the category of perfectoid rings. As a formal consequence, we depict $\mathrm{BK_n^{(h,d)}}$ (and hence also $\mathrm{BT_n^{(h,d)}}$) as a quotient stack.
In Chapter 4 we consider globalization properties of our previous observations with respect to a finer topology than the one investigated in Chapter 3. We recall the *p-complete arc*

topology on the category of perfectoid rings, which was defined by Česnavičius and Scholze in [**CS19**]. This is a variant of the arc topology of Bhatt and Mathew (cf. [**BM18**]) and has a basis consisting of products of rank ≤ 1 perfectoid valuation rings. Assuming a conjecture on gluing vector bundles over completions of perfect rings (Conjecture A), we show that $\mathrm{BK_n}$-modules can also be glued with respect to the p-complete arc topology. By our previous findings, as at the end of Chapter 3, this formally implies that $\mathrm{BK_n^{(h,d)}}$ is given by a quotient stack for this finer topology. When working over a perfect ring, the conjecture and its implications are theorems.

We now provide a more detailed overview:
Chapter 1 starts with prerequisites on perfectoid rings. These are rings R which allow a good notion of *tilting*, i.e. certain information on R can be recovered from the inverse limit perfection $R^\flat := \lim_{x\mapsto x^p} R/p$ which is called the *tilt of* R. In fact, there is a natural map $W(R^\flat) \to R$ (here $W(R^\flat)$ denotes the ring of p-typical Witt vectors over $R^\flat$) whose kernel is generated by a *distinguished element*, say $\xi = (\xi_0, \xi_1, \ldots) \in W(R^\flat)$. Česnavičius and Scholze observed in [**CS19**] that for a fixed perfectoid ring $R = W(R^\flat)/\xi$ the category of perfectoid R-algebras is equivalent to the category of ξ_0-adically complete perfect $R^\flat$-algebras (cf. Proposition 1.1.7). This is the key ingredient used to extend global results on perfect rings to perfectoid rings. For a perfectoid ring R we denote the category of perfectoid R-algebras by Perfd_R. In the second half of Chapter 1 we recall the categorical equivalence between commutative finite locally free p-groups and *torsion Breuil-Kisin-Fargues modules* over perfectoid rings (cf. Theorem 1.2.6). The latter consist of semi-linear algebra data. This result is due to Lau and Anschütz-Le Bras and serves as the central starting point for our further investigations.

In Chapter 2 we analyse $\mathrm{BT_n}$-groups in terms of semi-linear algebra data. The relevant notion is the following: A BK_n-module over a perfectoid ring R (with fixed presentation $R = W(R^\flat)/\xi$) is a triple (M, F, V), where M is a finite projective module over $W_n(R^\flat) = W(R^\flat)/p^n$, and $F\colon M^\sigma \to M$ and $V\colon M \to M^\sigma$ are $W_n(R^\flat)$-linear maps such that $F \circ V = \xi$ and $V \circ F = \xi$. In the case $n = 1$, one additionally demands that $\mathrm{coker}(F)$ is finite projective as an S/ξ-module and that the induced sequence $M/\xi \xrightarrow{\overline{V}} M^\sigma/\xi \xrightarrow{\overline{F}} M/\xi$ is exact. The category of $\mathrm{BK_n}$-modules over R, denoted by $\mathrm{BK_n}(R)$, also admits a fibrewise characterization: A torsion Breuil-Kisin-Fargues module over R is a $\mathrm{BK_n}$-module if and only if it is killed by p^n and all fibres are truncated Dieudonné modules of level n (cf. Lemma 2.2.8). Combining this fact with the fibrewise description of $\mathrm{BT_n}$-groups, we show the following result:

Theorem 1. *(Proposition 2.2.11) Let R be a perfectoid ring. For all $n \in \mathbb{N}$, there is an* *anti-equivalence of categories*

$$\mathrm{BT_n}(R) \cong \mathrm{BK_n}(R).$$

This equivalence is compatible with truncations, base change and preserves height and dimension of the respective objects. At the end of the second chapter we show that $\mathrm{BK_n}$-modules admit *normal representations* (cf. Lemma 2.3.3), which endow such modules with a manageable structure. This has immediate consequences for the truncation functor $\mathrm{BT}(R) \to \mathrm{BT}_n(R)$:

Theorem 2. *(Corollary 2.3.5) Let R be a perfectoid ring and $n \in \mathbb{N}$. The truncation functor*

$$\mathrm{BT}(R) \to \mathrm{BT_n}(R)$$

is essentially surjective, i.e. every $\mathrm{BT_n}$-group over R arises as the p^n-torsion subgroup of some Barsotti-Tate group over R.

We begin Chapter 3 with a short comparison between BK_1-modules and certain F-Zips. The fact, that the classifying stack of the latter admits a presentation as a quotient stack, serves as a motivation for our further studies. We aim to find an explicit description of $\mathrm{BK_n}$-modules over perfectoid rings. For that purpose, we restrict ourselves to $\mathrm{BK_n}$-modules of fixed height h and dimension d for natural numbers $0 \leq d \leq h$. Let $R = W(R^\flat)/\xi$ be a perfectoid ring and let $\mu = \mathrm{diag}(\xi, ..., \xi, 1, ..., 1)$ in $\mathbb{M}_h(W_n(R^\flat))$ (where the first d entries are ξ) and $\mu' = \mathrm{diag}(1, ..., 1, \xi, ..., \xi)$ (where the first d entries are 1). Then $\underline{M}_{\mathrm{id}} = (W_n(R^\flat)^h, x \mapsto \mu\sigma(x), x \mapsto \sigma^{-1}(\mu' x))$ is an example of a BK_n-module of height h and dimension d. Given some element $l \in \mathrm{GL}_h(W_n(R^\flat))$, we can modify $\underline{M}_{\mathrm{id}}$ to a further $\mathrm{BK_n}$-module

$$\underline{M}_l = (W_n(R^\flat)^h, x \mapsto \mu l \sigma(x), x \mapsto \sigma^{-1}(l^{-1}\mu' x)).$$

Hence we obtain a map of sets $\varphi\colon G(R) := \mathrm{GL}_h(W_n(R^\flat)) \to \mathrm{BK_n^{(h,d)}}(R)/{\cong}$, $l \mapsto \underline{M}_l$, where the right hand side denotes the set of isomorphism classes of $\mathrm{BK_n}$-modules of height h and dimension d.

It turns out that the equivalence relation on $G(R)$ defined by this map is given by a group action of a subgroup $E(R) \subseteq G(R) \times G(R)$ via $(A, B) \cdot l = Al\sigma(B^{-1})$ ($E(R)$ depends on n, h and d; since we assume these numbers to be fixed, we neglect them in the notation). This observation upgrades φ to a fully faithful functor $(E\backslash G)(R) := E(R)\backslash G(R) \to \mathrm{BK_n^{(h,d)}}(R)$ of groupoids. Now, using the existence of normal representations, we get the following result:

Theorem 3. *(Proposition 3.2.11) The morphism of prestacks over* Perfd_R

$$(E \setminus G) \to \mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})}, l \mapsto \underline{M}_l$$

is fully faithful and an equivalence on products of local perfectoid R*-algebras. Moreover, locally for the* p*-Zariski topology it is essentially surjective.*

Covers for the p-Zariski topology on Perfd_R in the theorem above are given by maps $A \to A'$ such that $A/p \to A'/p$ is a Zariski cover. We have an analogous notion of p-étale covers and show that $R \mapsto \mathrm{BK}_{\mathrm{n}}(R)$ and $R \mapsto \mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})}(R)$ are stacks for this topology (cf. Proposition 3.3.5). Then, using our previous results, we formally obtain the following depiction as a quotient stack:

Theorem 4. *(Theorem 3.3.7) Let* $n \in \mathbb{N} \cup \{\infty\}$ *and* $0 \leq d \leq h$ *be natural numbers. We have isomorphisms of* p*-étale stacks*

$$[E \setminus G] \cong \mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})} \cong \mathrm{BT}_{\mathrm{n}}^{(\mathrm{h,d})},$$

where $[E \setminus G]$ *denotes the* p*-étale stackification of the prestack* $(E \setminus G)$*.*

In Chapter 4 we aim towards gluing questions concerning a finer topology than the one discussed in Chapter 3. We recall the p-complete arc-topology of Česnavičius and Scholze on Perfd_R, which has a basis consisting of products of perfectoid valuation rings of rank ≤ 1. To show descent for BK_{n}-modules with respect to this topology, we reduce to the following conjecture:

Conjecture A: Let A be a perfectoid ring with presentation $A = W(S)/\xi$ and let $\widehat{(\ )}$ denote ξ_0-adic completion. Then the functor $\mathrm{Perf}_S \to \mathrm{Cat}_1, S' \mapsto \mathrm{LF}(W_n(\widehat{S'}))$ is an arc sheaf.

For a perfect ring A (whence $\xi_0 = 0$ and we do not have to deal with completions), Conjecture A is shown to be a theorem:

Theorem 5. *(Theorem 4.3.8) (Conjecture A for perfect rings) The functor*

$$\mathrm{Perf}_{\mathbb{F}_p} \to \mathrm{Cat}_1, \quad S' \mapsto \mathrm{LF}(W_n(S'))$$

is an arc sheaf.

As indicated above, we obtain the following corollary.

Theorem 6. *(Theorem 4.3.15) Let* R *be a perfectoid ring. Assume that Conjecture A holds or that* R *is perfect. Let* $n \in \mathbb{N} \cup \{\infty\}$*. The functor*

$$\mathrm{BK}_{\mathrm{n}} \colon \mathrm{Perfd}_R \to \mathrm{Groupoids}, \ R' \mapsto \mathrm{BK}_{\mathrm{n}}(R')$$

is a stack for the p-complete arc topology. Moreover, for fixed $0 \leq d \leq h$ the functor

$$\mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})} \colon \mathrm{Perfd}_R \to \mathrm{Groupoids}, \; R' \mapsto \mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})}(R')$$

is a clopen substack of BK_{n}.

Finally, the following theorem is a formal consequence of Theorem 3, Theorem 6 and the fact, that the p-complete arc is finer than the p-étale topology.

Theorem 7. *Let R be a perfectoid ring. Assume that Conjecture A holds or that R is perfect. Let $n \in \mathbb{N} \cup \{\infty\}$ and $0 \leq d \leq h$ be natural numbers. The morphism of prestacks in Theorem 3 induces an isomorphism of p-complete arc stacks*

$$[E \setminus G] \cong \mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})} \cong \mathrm{BT}_{\mathrm{n}}^{(\mathrm{h,d})},$$

where $[E \setminus G]$ denotes the p-complete arc stackification of the prestack $(E \setminus G)$.

Acknowledgements

I deeply thank Torsten Wedhorn for suggesting several interesting topics which led to this dissertation. I am also grateful for his patience in numerous discussions and his support on an academic and private level. I also want to thank Eike Lau for many helpful remarks and suggestions and Kęstutis Česnavičius for answering questions on different topics. Moreover, I thank all members of the Algebra group of TU Darmstadt for their contribution to make the last four years a great time. In particular, I want to thank Jens Hesse, Can Yaylali, Timo Richarz and Paul Kiefer for numerous discussions and the football group.

Notations and assumptions:

- We use the convention $\mathbb{N} = \mathbb{Z}^{\geq 1}$.
- Once and for all, we fix some prime number p.
- For a ring R with $pR = 0$, we denote by $W(R)$ the ring of p-typical Witt vectors of R. For $n \in \mathbb{N}$, the ring of p-typical truncated Witt vectors of length n is denoted by $W_n(R)$. We also use $W_\infty(R)$ for $W(R)$ whenever it is convenient.
- For a ring R with $pR = 0$, we denote the Frobenius morphism $x \mapsto x^p$ by σ. By abuse of notation, we also denote the induced Frobenius morphism on $W_n(S)$ by σ for all $n \in \mathbb{N} \cup \{\infty\}$.

- By abuse of notation, we often denote the image of an element $\xi \in W(S)$ under the map $W(S) \to W_n(S)$ by ξ as well for all $n \in \mathbb{N}$.
- For a ring R, we denote the category of R-algebras by Ring_R and the 1-category of finite projective R-modules by $\mathrm{LF}(R)$.

1 Prerequisites

We recall some general facts about perfectoid rings and briefly state the anti-equivalences obtained in [**Lau18**] (§9 and §10), [**ALB19**] (§ 4 and §5.1) and [**SW19**] (Theorem 17.5.2) which characterize Barsotti-Tate groups and p-groups over perfectoid rings in terms of semi-linear algebra.

1.1 Generalities on perfectoid rings

Our definition of perfectoid rings as quotients of perfect prisms follows [**Lau18**] §8. A nice summary of the relevant notions can also be found in [**CS19**] 2.1.1.

Definition 1.1.1.
1. A ring R with $pR = 0$ is called *perfect* if the Frobenius morphism $\sigma\colon R \to R,\ x \mapsto x^p$ is bijective.
2. For a perfect ring S, an element $\xi = (\xi_0, \xi_1, \dots) \in W(S)$ is called *distinguished* if $\xi_1 \in S^\times$ is a unit and S is ξ_0-adically complete.
3. A ring R is called *perfectoid* if there is an isomorphism $R \cong W(S)/\xi$, where S is a perfect ring and $\xi \in S$ is a distinguished element.
4. For a ring A (resp. an $\mathbb{F}_p$-algebra A) we denote the full subcategory of Ring_A whose objects are perfectoid (resp. perfect) rings by Perfd_A (resp. Perf_A).

Remark 1.1.2. The *tilt* $R^\flat$ of a ring R is the inverse limit perfection of R/pR,

$$R^\flat := \varprojlim_{a \mapsto a^p} R/pR.$$

If R is ϖ-adically complete for a $\varpi \in R$ such that ϖ divides p (this implies that R is p-adically complete by [**SP**] Tag 090T) then we have multiplicative isomorphisms

$$\varprojlim_{a \mapsto a^p} R \xrightarrow{\sim} \varprojlim_{a \mapsto a^p} R/pR \xrightarrow{\sim} \varprojlim_{a \mapsto a^p} R/\varpi R$$

and a multiplicative projection onto the last coordinate $R^\flat \to R$, which is denoted by $a \mapsto a^\sharp$.

Remark 1.1.3. A ring R is perfectoid if and only if the following conditions hold

(a) there is a $\varpi \in R$ such that ϖ^p divides p and R is ϖ-adically complete,

(b) the Frobenius map $R/pR \to R/pR, a \mapsto a^p$ is surjective,

(c) the kernel of the surjection $\theta\colon W(R^\flat) \to R$ (uniquely determined by $\theta([a]) = a^\sharp$) is principal.

Remark 1.1.4.

1. If R is a perfectoid ring, the map θ induces an isomorphism between $W(R^\flat)/\xi$ and R, for a generator ξ of the kernel of θ. Therefore, whenever we have a presentation of a perfectoid ring $R = W(S)/\xi$, we may identify S with $R^\flat$.

2. Let R be a perfectoid ring with presentation $R = W(S)/\xi$. We can find a $\varpi \in R$ (e.g. the image of $[\xi_0^{1/p}]$ under $W(S) \to R$) such that $\varpi^p R = pR$ and $(\varpi^\flat)^p R = \xi_0 R$, where $\varpi^\flat$ denotes an element of $S = R^\flat$ such that $(\varpi^\flat)^\sharp R = \varpi R$ (e.g. $\varpi^\flat = \xi_0^{1/p}$).

3. For a perfectoid ring R with presentation $R = W(S)/\xi$, we have $W_n(S)/\xi = R/p^n$ for all $n \in \mathbb{N}$, and the elements p and $\xi \in W(S)$ are non-zero-divisors. The latter implies that R is p-torsion free if and only if $W_n(S)$ is ξ-torsion free for all (equivalently one) $n \in \mathbb{N}$. Indeed, the $W(S)$-module
$$\{(x, y) \in W(S)^2 \mid \xi x = p^n y\}/\{(p^n z, \xi z) \mid z \in W(S)\}$$
is isomorphic to both $\{x \in W_n(S) \mid \xi x = 0\}$ and $\{y \in R \mid p^n y = 0\}$ via $(x, y) \mapsto x$ and $(x, y) \mapsto y$, respectively.

4. As noted before, every perfectoid ring R is p-adically complete. In particular, given a presentation $R = W(S)/\xi$, we have
$$\mathrm{Max}(R) = \mathrm{Max}(R/p) = \mathrm{Max}(S/\xi_0) = \mathrm{Max}(S) = \mathrm{Max}(W_n(S))$$
for all $n \in \mathbb{N} \cup \{\infty\}$.

Remark 1.1.5. The notion of a perfectoid ring defined above is sometimes called *integral perfectoid ring* to distinguish from the notion of a *perfectoid Tate Huber ring*, which plays the central role in the theory of perfectoid spaces. Given a perfectoid Tate Huber pair (R, R^+), the ring of integral elements R^+ is a perfectoid ring in the sense defined above. However, integral perfectoid rings are more general than rings of integral elements in the adic perfectoid world. For example, if (R, R^+) is a perfectoid Huber pair over $(\mathbb{Q}_p, \mathbb{Z}_p)$ (which implies that p is a non-zero-divisor in R^+), then R^+ is integrally closed in $R = R^+[\frac{1}{p}]$. In contrast, a p-torsion-free integral perfectoid ring A is only *p-integrally closed* in $A[\frac{1}{p}]$ in general (cf. [**CS**] 2.1.7.).

Example 1.1.6. 1. Any perfect ring S is perfectoid since it is isomorphic to $W(S)/p$. In particular, every perfect field of characteristic p is a perfectoid ring.

2. Let C be an algebraically closed complete non-archimedean extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_C$. Choosing a compatible system of p-power roots of p gives an element $p^\flat$ of the perfect ring $\mathcal{O}_C^\flat$ such that $\mathcal{O}_C^\flat$ is $p^\flat$-adically complete. We obtain an isomorphism $W(\mathcal{O}_C^\flat)/([p^\flat]-p) \cong \mathcal{O}_C$ which depicts $\mathcal{O}_C$ as a perfectoid ring.

The following proposition due to Česnavičius and Scholze is an algebraic version of the tilting equivalence for perfectoid spaces of [**Sch12**] and the main reason why global results on perfect rings can be extended to perfectoid rings.

Proposition 1.1.7. *Let R be a perfectoid ring with presentation $R = W(S)/\xi$.*

1. *The pair of mutually quasi-inverse functors*

$$R' \mapsto R'^\flat, \text{ and } S' \mapsto W(S')/\xi$$

gives an equivalence between the category of perfectoid R-algebras and the category of perfect $R^\flat$-algebras that are ξ_0-adically complete.

2. *$R'^\flat$ is $\varpi^\flat$-adically complete for a $\varpi^\flat \in R'^\flat$ with $\varpi^\flat \mid \xi_0$ if and only if R' is ϖ-adically complete for $\varpi := (\varpi^\flat)^\sharp$.*

3. *R' is a valuation ring (resp. of dimension ≤ 1 and with algebraically closed fraction field) if and only if so is $R'^\flat$. In this case the value groups are isomorphic via*

$$\mathrm{Frac}(R'^\flat)^\times/(R'^\flat)^\times \xrightarrow{\sim} \mathrm{Frac}(R')^\times/(R')^\times, \ x \mapsto x^\sharp.$$

Proof. [**CS19**] Proposition 2.1.9. □

We also have pushouts and arbitrary products in the category of perfectoid rings and their formation is compatible with tilting in the following sense:

Proposition 1.1.8. 1. *Let R be a perfectoid ring that is ϖ-adically complete for a $\varpi \in R$ with $\varpi^p \mid p$ and let $(R_i)_{i\in I}$ be a family of ϖ-adically complete perfectoid R-algebras. Then the ϖ-adically completed tensor product $\widehat{\bigotimes}_{i\in I} R_i$ over R is perfectoid and its tilt is the $\varpi^\flat$-adic completed tensor product $\widehat{\bigotimes}_{i\in I} R_i^\flat$ over $R^\flat$ (where $\varpi^\flat$ is an element of $R^\flat$ such that $((\varpi^\flat)^\sharp) = (\varpi)$).*

2. *A product $\prod_{i\in I} B_i$ of $\mathbb{Z}_p$-algebras is perfectoid if and only if so is each B_i, and then $\left(\prod_{i\in I} B_i\right)^\flat \cong \left(\prod_{i\in I} B_i^\flat\right)$.*

Proof. [**CS19**] Proposition 2.1.10. ☐

Remark 1.1.9. Let R be a perfectoid ring with presentation $R = W(S)/\xi$. Then for $\varpi = (\xi_0^{1/p})^\sharp$ and $\varpi^\flat = \xi_0^{1/p}$ we have $(\varpi^p) = (p)$ and $((\varpi^\flat)^p) = (\xi_0)$. Since ϖ-adic and p-adic (resp. $\varpi^\flat$-adic and ξ_0-adic) completions coincide, in this case we can ignore the role of ϖ and $\varpi^\flat$ and work with p and ξ_0 instead:

1. For perfectoid R-algebras $R_1 \leftarrow R_3 \rightarrow R_2$ the p-adic completion $R_1 \widehat{\otimes}_{R_3} R_2$ of $R_1 \otimes_{R_3} R_2$ is perfectoid and it is the pushout in Perfd_R.
2. Its tilt $(R_1 \widehat{\otimes}_{R_3} R_2)^\flat$ is the ξ_0-adic completion of $R_1^\flat \otimes_{R_3^\flat} R_2^\flat$ and this is the pushout in the category of ξ_0-adically complete perfect $R^\flat$-algebras.
3. In particular, for a diagram of perfect rings $B \leftarrow A \rightarrow C$, the ring $B \otimes_A C$ is perfect and the inclusion $\mathrm{Perf} \subseteq \mathrm{Ring}_{\mathbb{F}_p}$ preserves pushouts.

The very last assertion also is a special case of the following remark on colimit perfections which will be important in the further course of this paper.

Remark 1.1.10. We follow [**BIM19**] 3.1. For a ring S with $pS = 0$ we set

$$S_{\mathrm{perf}} = \mathrm{colim}(S \xrightarrow{\sigma} S \xrightarrow{\sigma} \dots).$$

Then S_{perf} is a perfect ring of characteristic p and the natural map $S \rightarrow S_{\mathrm{perf}}$ is universal for maps from S to perfect rings. I.e.

$$\mathrm{Ring}_{\mathbb{F}_p} \rightarrow \mathrm{Perf}_{\mathbb{F}_p}, S \mapsto S_{\mathrm{perf}}$$

is left adjoint to the inclusion $i\colon \mathrm{Perf}_{\mathbb{F}_p} \subseteq \mathrm{Ring}_{\mathbb{F}_p}$. In other words

$$\mathrm{Hom}_{\mathrm{Ring}_{\mathbb{F}_p}}(S, B) = \mathrm{Hom}_{\mathrm{Perf}_{\mathbb{F}_p}}(S_{\mathrm{perf}}, B)$$

for all $\mathbb{F}_p$-algebras S and all perfect rings B. In particular, $S \mapsto S_{\mathrm{perf}}$ commutes with all colimits. Moreover, being itself a filtered colimit, $S \mapsto S_{\mathrm{perf}}$ commutes with finite limits, for example with taking fibre products.

1.2 Short review of Dieudonné theory over perfectoid rings

This section recalls results on classification of p-divisible groups and p-groups in terms of σ-linear algebra. We briefly present the results obtained in [**Lau18**] §9 and §10, [**ALB19**]

§4 and §5, and [**SW19**] Theorem 17.5.2.

Fix a perfectoid ring R with presentation $R = W(S)/\xi$. We denote the category of p-divisible groups over R by $\mathrm{BT}(R)$.

Definition 1.2.1. 1. A *(minuscule) Breuil-Kisin-Fargues module* for R is a triple (M, F, V), where M is a finite projective $W(S)$-module, and $F\colon M^\sigma \to M$ and $V\colon M \to M^\sigma$ are $W(S)$-linear maps such that $V \circ F = \xi$ and $F \circ V = \xi$.

2. We denote the category of Breuil-Kisin-Fargues modules for R by $\mathrm{BK}(R)$ (the morphisms are $W(S)$-linear maps which are compatible with the F's; compatibility with V's is then automatic as explained below).

Note that since $\xi \in W(S)$ is a non-zero-divisor, the maps F and V are injective and hence V is determined by F (and vice versa) via the formula $V(x) = F^{-1}(\xi x)$. We only add the datum to draw the connection to other semi-linear algebra data defined below (cf. Definition 1.2.4).

Theorem 1.2.2. *There is an exact, compatible with base change, anti-equivalence*

$$\begin{aligned} \mathrm{BT}(R) &\cong \mathrm{BK}(R), \\ G &\mapsto \mathbb{M}(G), \end{aligned}$$

admitting an exact quasi-inverse which is compatible with base change as well.

Proof. For $p \neq 2$ this was done by Lau in [**Lau18**] Theorem 9.8. For the general case cf. [**SW19**] Theorem 17.5.2. resp. [**ALB19**] Corollary 4.3.8. □

Definition 1.2.3. Let $\mathrm{BT}_{\mathrm{tor}}(R)$ denote the category of commutative, finite, locally free R-group schemes which are killed by a power of p.

The category $\mathrm{BT}_{\mathrm{tor}}(R)$ is denoted by $\mathrm{pGR}(\operatorname{Spec} R)$ in [**Lau18**]. We justify our notation by Theorem 1.2.6 and Remark 1.2.7 below.

The following definition is taken from [**Lau18**] Definition 10.5. Such objects are introduced as *torsion prismatic Dieudonné modules* in [**ALB19**] Definition 5.1.1.

Definition 1.2.4. 1. A *torsion Breuil-Kisin-Fargues module for R* consists of a triple (M, F, V), where M is a finitely presented $W(S)$-module of projective dimension ≤ 1 which is killed by a power of p, and $M^\sigma \xrightarrow{F} M, M \xrightarrow{V} M^\sigma$ are $W(S)$-linear maps such that $F \circ V = \xi$ and $V \circ F = \xi$.

2. We denote the category of torsion Breuil-Kisin-Fargues modules for R by $\mathrm{BK}_{\mathrm{tor}}(R)$ (the morphisms are $W(S)$-linear maps which are compatible with the F's and the V's).

Remark 1.2.5. If R is p-torsion free (equivalently $W_n(S)$ is ξ-torsion free for some $n \in \mathbb{N}$) for any object $M = (M, F, V)$ of $\mathrm{BK}_{\mathrm{tor}}(R)$, the module M is ξ-torsion free and hence F and V are injective. So as in the case of Breuil-Kisin-Fargues modules, F determines $V\colon M \to M^\sigma$ via the formula $V(x) = F^{-1}(\xi x)$ (and vice versa). Indeed, assume $p^n M = 0$. Then, since M is of projective dimension ≤ 1, it fits into a short exact sequence

$$0 \to P_1 \to P_0 \to M \to 0,$$

where P_1 and P_0 are finite projective $W(S)$-modules s.t. $p^n P_0 \subseteq P_1$. We obtain an injection

$$p^n P_0/p^n P_1 \to P_1/p^n P_1$$

of $W_n(S)$-modules, where the latter is projective and hence $M = p^n P_0/p^n P_1$ is torsion free. But $\xi \in W_n(S)$ is a non-zero-divisor by assumption.

By abuse of notation, we use the same symbol $\mathbb{M}$ for the anti-equivalences in Theorem 1.2.2 and Theorem 1.2.6 below.

Theorem 1.2.6. *There is an exact, compatible with base change, anti-equivalence*

$$\mathrm{BT}_{\mathrm{tor}}(R) \cong \mathrm{BK}_{\mathrm{tor}}(R),$$
$$G \mapsto \mathbb{M}(G),$$

admitting an exact quasi-inverse which is compatible with base change as well.

Proof. For $p \neq 2$ this is due to Lau, cf. [**Lau18**] Theorem 10.12. The general case was proved by Anschütz and Le-Bras in [**ALB19**] Theorem 5.1.4. □

Remark 1.2.7. 1. Zariski-locally on $\mathrm{Spec}(R)$ every $G \in \mathrm{BT}_{\mathrm{tor}}(R)$ can be written as the kernel of an isogeny $H_1 \to H_2$ of p-divisible groups over R (cf. [**BBM82**] Theorem 3.1.1). The functor of the theorem above can Zariski-locally on R be defined as $\mathbb{M}(G) := \mathrm{coker}(\mathbb{M}(H_1) \to \mathbb{M}(H_2))$ using the anti-equivalence of Theorem 1.2.2 (cf. proof of [**Lau18**] Theorem 10.12 and proof of [**ALB19**] Theorem 5.1.4).

2. By [**ALB19**] Proposition 4.3.2 and the explicit description of $\mathbb{M}$ given in Theorem 5.1.4, the anti-equivalence $\mathbb{M}$ extends crystalline Dieudonné theory of [**BBM82**]. Also cf. [**CS**] §4.2.1. for a related discussion.

2 $\mathrm{BT_n}$-groups and $\mathrm{BK_n}$-modules

Let R be a perfectoid ring. In this chapter we investigate the anti-equivalence

$$\mathrm{BT_{tor}}(R) \cong \mathrm{BK_{tor}}(R)$$

(of Theorem 1.2.6) with respect to $\mathrm{BT_n}$-*groups*, which are objects of $\mathrm{BT_{tor}}(R)$ satisfying certain additional properties. Those were investigated by Grothendieck in [**Gro71**] and naturally arise as groups of p^n-torsion points of p-divisible groups over R (and we show at the end of this chapter, that in our setting all $\mathrm{BT_n}$-groups are of this form). Starting with the central case $n = 1$ we describe the corresponding subcategories of $\mathrm{BK_{tor}}(R)$ and establish equivalences of categories.

2.1 $\mathrm{BT_1}$-groups and Dieudonné spaces

Fix a perfectoid ring R with presentation $R = W(S)/\xi$. In the language of [**BS19**] this means that we work over an oriented perfect prism.

2.1.1 Truncated Barsotti-Tate groups of level 1

Definition 2.1.1. Let $\mathrm{BT_1}(R)$ denote the full subcategory of $\mathrm{BT_{tor}}(R)$ consisting of those objects G such that G is annihilated by p and for $G_\circ := G \otimes_R R/p$ the sequence

$$G_\circ \xrightarrow{F} G_\circ^{(p)} \xrightarrow{V} G_\circ$$

is exact (F resp. V denote the Frobenius resp. Verschiebung of $G_\circ$). Objects of $\mathrm{BT_1}(R)$ are called *truncated Barsotti-Tate groups of level* 1 *over* R or $\mathrm{BT_1}$-*groups over* R.

Definition 2.1.2. Let $G = \mathrm{Spec}(A)$ be a $\mathrm{BT_1}$-group over R.

1. The locally constant function $\mathrm{height}\colon \mathrm{Spec}(R/p) \to \mathbb{N}_0$, which is uniquely determined by $s \mapsto \log_p(\dim_{\kappa(s)}(A \otimes_R \kappa(s)))$, for $s \in \mathrm{Max}(R/p)$ with residue field $\kappa(s)$, is said to be the *height of* G. If height is constant and equal to $h \in \mathbb{N}_0$, we say that G has *height* h.

2. The locally constant function $\dim\colon \operatorname{Spec}(R/p) \to \mathbb{N}_0$, which is uniquely determined by $s \mapsto \dim_{\kappa(s)}(\operatorname{Lie}(G \otimes_R \kappa(s)))$, for $s \in \operatorname{Max}(R/p)$ with residue field $\kappa(s)$, is said to be the *dimension of* G. If $\dim$ is constant and equal to $d \in \mathbb{N}_0$, we say that G has *dimension* d.

3. We denote the full subcategory of $\mathrm{BT}_1(R)$ whose objects are BT_1-groups of constant height h and constant dimension d by $\mathrm{BT}_1^{(\mathrm{h,d})}(R)$.

Example 2.1.3. Let G be a p-divisible group over R and let $G[p]$ denote its p-torsion subgroup. Then $G[p]$ is a BT_1-group over R. At the end of this chapter we will see that every object of $\mathrm{BT}_1(R)$ is of this form.

Remark 2.1.4. 1. Let G be a BT_1-group over R with $\mathbb{M}(G) = (M, F, V)$ being the associated $\mathrm{BK}_{\mathrm{tor}}$-module under the anti-equivalence of Theorem 1.2.6. Since $\mathbb{M}$ commutes with base change and extends the Dieudonné theory of [**BBM82**], given a point $s \in \operatorname{Max}(S)$ with residue field κ (which automatically is perfect and of characteristic p), the base change $M_s = (M_s := M \otimes_S \kappa, F_s := F \otimes \mathrm{id}_\kappa, V := V \otimes \mathrm{id}_\kappa)$ of M to κ is a *Dieudonné space* over κ, i.e. M_s is a finite dimensional κ-vector space such that $V_s \circ F_s = 0$, $F_s \circ V_s = 0$ and the sequence

$$M_s \xrightarrow{V_s} M_s^\sigma \xrightarrow{F_s} M_s$$

is exact (note that this implies exactness of $M_s^\sigma \xrightarrow{F_s} M_s \xrightarrow{V_s} M_s^\sigma$ as well).

2. The *height* of a Dieudonné space (M, F, V) over a perfect field k of characteristic p is defined to be the k-dimension of M. The k-dimension of $\operatorname{coker}(F)$ is called *dimension of* M.

3. For a perfect field k of characteristic p we denote the full subcategory of $\mathrm{BK}_{\mathrm{tor}}(k)$ whose objects are Dieudonné spaces over k by $\mathrm{BK}_1(k)$. We obtain a functor $\mathrm{BT}_1(k) \to \mathrm{BK}_1(k)$ which is induced by $\mathbb{M}$. This is a height and dimension preserving anti-equivalence, since the Dieudonné theory of [**BBM82**] §3 provides this result and $\mathbb{M}$ extends this theory.

The following lemma gives a fibrewise characterization of BT_1-groups. This will lead to a description of the essential image of $\mathrm{BT}_1(R)$ under the anti-equivalence of Theorem 1.2.6.

Lemma 2.1.5. *Let $G \in \mathrm{BT}_{\mathrm{tor}}(R)$ such that G is killed by p. Then $G \in \mathrm{BT}_1(R)$ if and only if $G \otimes_R \kappa(s) \in \mathrm{BT}_1(\kappa(s))$ for all $s \in \operatorname{Max}(R/p)$.*

Proof. Noting that $\mathrm{Max}(R) = \mathrm{Max}(R/p)$, the claim of the lemma is a corollary of the following general assertion on finite locally free group schemes. $\square$

Lemma 2.1.6. *Let X be an affine scheme. A sequence of finite locally free group schemes*

$$G' \xrightarrow{\varphi} G \xrightarrow{\psi} G''$$

over X is exact if and only if $\psi \circ \varphi = 0$ and for every closed point $x \in X$ the induced sequence $G'_x \to G_x \to G''_x$ is exact.

Proof. [**dJo93**] Proposition 1.1. $\square$

2.1.2 Modules associated with BT_1-groups

The fibrewise characterization of BT_1-groups over R hints that its essential image under the equivalence $\mathrm{BT}_{\mathrm{tor}}(R) \cong \mathrm{BK}_{\mathrm{tor}}(R)$ consists of those $M \in \mathrm{BK}_{\mathrm{tor}}(R)$ which satisfy $pM = 0$ and whose fibres are Dieudonné spaces for all closed points of $\mathrm{Spec}(S)$. The goal of this section is to show that this essential image is also given via the following global definition.

Definition 2.1.7. Let $\mathrm{BK}_1(R)$ be the full subcategory of $\mathrm{BK}_{\mathrm{tor}}(R)$ whose objects are given by triples $M = (M, F, V)$, such that

(a) M is killed by p (and hence is a finite projective S-module by Lemma 2.1.8 below),

(b) $\mathrm{coker}(F)$ is finite projective as an S/ξ_0-module,

(c) the induced sequence

$$M/\xi_0 \xrightarrow{\overline{V}} M^{\sigma}/\xi_0 \xrightarrow{\overline{F}} M/\xi_0$$

is exact.

Objects of $\mathrm{BK}_1(R)$ will also be called BK_1-modules over R. The lemma below ensures that the underlying S-module of a BK_1-module is finite projective as indicated in (a):

Lemma 2.1.8. *Let M be a finite $W(S)$-module which is killed by p. Then M is of projective dimension 1 as a $W(S)$-module if and only if it is finite projective as an S-module.*

Proof. [**BS17**] Lemma 7.8. $\square$

Remark 2.1.9. 1. If $R = k$ is a perfect field of characteristic p, we have $k = W(k)/p$. Hence $S = k$ and $\xi_0 = 0$ in this case and we recover the definition of Dieudonné spaces over k.

2. More generally, if $R = S$ is a perfect ring, a BK_1-module over R is the same as a *truncated Dieudonné module of level* 1 *over* R as defined in [**Lau13**] Definition 6.1.

3. If R is p-torsion free, an object of $\mathrm{BK}_{\mathrm{tor}}$, which is killed by p, automatically satisfies property (c) of the definition above.

We also have a notion of height and dimension for BK_1-modules:

Definition 2.1.10. Let $M = (M, F, V) \in \mathrm{BK}_1(R)$.

1. The locally constant function $\mathrm{height}\colon \mathrm{Spec}(S/\xi_0) \to \mathbb{N}_0$ uniquely determined by $s \mapsto \dim_{\kappa(s)}(M \otimes_S \kappa(s))$, for $s \in \mathrm{Max}(S/\xi_0)$ with residue field $\kappa(s)$, is said to be the *height of* M. If height is constant and equal to $h \in \mathbb{N}_0$, we say that M has *height* h.

2. The locally constant function $\dim\colon \mathrm{Spec}(S/\xi_0) \to \mathbb{N}_0$ uniquely determined via $s \mapsto \dim_{\kappa(s)}(\mathrm{coker}(F) \otimes_{S/\xi_0} \kappa(s))$, for $s \in \mathrm{Max}(S/\xi_0)$ with residue field $\kappa(s)$, is said to be the *dimension of* M. If $\dim$ is constant and equal to $d \in \mathbb{N}_0$, we say that M has *dimension* d.

3. We denote the full subcategory of $\mathrm{BK}_1(R)$ whose objects are BK_1-modules of constant height h and constant dimension d by $\mathrm{BK}_1^{(\mathrm{h,d})}(R)$.

Example 2.1.11. 1. Let $\mathcal{O}_C = W(\mathcal{O}_C^\flat)/([p^\flat] - p)$ be the ring of integers of an algebraically closed complete non-archimedean extension of $\mathbb{Q}_p$ with tilt $\mathcal{O}_C^\flat$ (cf. Example 1.1.6 2.). In this case we have $\xi_0 = p^\flat$ and consequently, a $\mathrm{BK}_{\mathrm{tor}}$-module over $\mathcal{O}_C$ which is killed by p is uniquely determined by a pair (M, φ), where M is a finite free $\mathcal{O}_C^\flat$-module, and $\varphi\colon M^\sigma \to M$ is a linear map whose cokernel is killed by $p^\flat$ (cf. Remark 1.2.5).

2. For every $a \in \mathcal{O}_C^\flat$ such that $|p^\flat| \leq |a| \leq 1$ the cokernel of the linear map

$$(\mathcal{O}_C^\flat)^\sigma \to \mathcal{O}_C^\flat, x \mapsto a\sigma(x)$$

is killed by $p^\flat$. However, its cokernel $\mathcal{O}_C^\flat/a$ is finite projective as a $\mathcal{O}_C^\flat/p^\flat$-module if and only if it is isomorphic as an $\mathcal{O}_C^\flat/p^\flat$-module to $(\mathcal{O}_C^\flat/p^\flat)^n$ for some $n \in \mathbb{N}_0$ because $\mathcal{O}_C^\flat/p^\flat$ is local. But since multiplication with a is not equal to 0 on $(\mathcal{O}_C^\flat/p^\flat)^n$ if $|a| > |p^\flat|$, this is satisfied if and only if $|a| \in \{|p^\flat|, 1\}$. Hence, up to isomorphism, we only have two different BK_1-modules over $\mathcal{O}_C$ of height 1. They can be separated by their dimension: The case in which $|a| = 1$ is the dimension-0-case and belongs to the BT_1-group $\underline{\mathbb{Z}/p\mathbb{Z}}_{\mathcal{O}_C}$. If $|a| = |p^\flat|$, we are in the dimension-1-case and the module is associated with $\mu_{p,\mathcal{O}_C}$. Both modules can be lifted to BK-modules, which in turn correspond to $\underline{\mathbb{Q}_p/\mathbb{Z}_p}_{\mathcal{O}_C}$ resp. $\mu_{p^\infty,\mathcal{O}_C}$.

For the proposition below, which is the analogue of Lemma 2.1.6 providing a fibrewise characterization of $\mathrm{BT}_1(R)$, and also later in this chapter, we need the following corollaries of Nakayama's lemma.

Lemma 2.1.12. *Let A be a ring and let $g\colon M \to N$ be a morphism of A-modules with M finitely generated and N finite projective. The following assertions are equivalent:*

(i) g *is injective and* $\mathrm{coker}(g)$ *is a finite projective A-module.*

(ii) $g \otimes \mathrm{id}_{\kappa(x)}\colon M \otimes_A \kappa(x) \to N \otimes_A \kappa(x)$ *is injective for all closed points* $x \in \mathrm{Spec}(A)$.

Moreover, g is an isomorphism in both of the following cases:

1. $g \otimes_A \mathrm{id}_{\kappa(x)}\colon M \otimes_A \kappa(x) \to N \otimes_A \kappa(x)$ *is an isomorphism for all closed points x of* $\mathrm{Spec}(A)$.

2. $g \otimes_A \mathrm{id}_{A/I}\colon M/IM \to N/IN$ *is an isomorphism for some ideal $I \subseteq A$ contained in the Jacobson radical of A.*

Proof. 2. follows from 1. which in turn can be obtained by the implication "$(ii) \Rightarrow (i)$" plus a further application of Nakayama's lemma to the finite projective A-module $\mathrm{coker}(g)$. For the equivalence of (i) and (ii) see [**GW10**] Proposition 8.10. The proof presented there shows that if (ii) holds, g is injective and all stalks at maximal ideals of the finitely presented A-module $\mathrm{coker}(g)$ are free. This implies (i) by [**SP**] Tag 00NV. □

Proposition 2.1.13. *Let A be a ring and*

$$M^\bullet = (M' \xrightarrow{F} M \xrightarrow{V} M' \xrightarrow{F} M)$$

be a sequence of finite projective A-modules of equal rank such that $F \circ V = 0$ and $V \circ F = 0$. The following assertions are equivalent:

(i) $M^\bullet \otimes_A \kappa(x)$ *is exact for all* $x \in \mathrm{Max}(A)$,

(ii) $M^\bullet$ *is exact and* $\mathrm{coker}(F)$ *is a finite projective A-module.*

(iii) $M \xrightarrow{V} M' \xrightarrow{F} M$ *is exact and* $\operatorname{coker}(F)$ *is a finite projective* A*-module,*

In this case $\operatorname{im}(V)$ *and* $\ker(F)$ *are direct summands of* M'*,* $\operatorname{im}(F)$ *and* $\ker(V)$ *are direct summands of* M *and* $\operatorname{coker}(V)$ *is a finite projective* A*-module as well.*

Proof. Obviously, (ii) implies (iii). If we assume (iii), so $M \xrightarrow{V} M' \xrightarrow{F} M$ is exact and $\operatorname{coker}(F)$ is finite projective, then $\operatorname{im}(F), \ker(F), \operatorname{coker}(V), \operatorname{im}(V)$ and $\ker(V)$ satisfy the additional assertion of the lemma. Therefore (i) follows because M and M' are of the same rank. So let us assume (i) and consider the exact sequence of A-modules

$$0 \to \operatorname{im}(F) \to M \to \operatorname{coker}(F) \to 0.$$

To show that $\operatorname{coker}(F)$ is a projective A-module, by the lemma above we need establish that for every $x \in \operatorname{Max}(A)$ the map $\operatorname{im}(F) \otimes_A \kappa(x) \to M \otimes_A \kappa(x)$ is injective. By assumption, we have an exact sequence of $\kappa(x)$-vector spaces

$$M \otimes_A \kappa(x) \xrightarrow{V_x} M' \otimes_A \kappa(x) \xrightarrow{F_x} M \otimes_A \kappa(x).$$

Here F_x factors as

$$M' \otimes_A \kappa(x) \xrightarrow{\pi} \operatorname{im}(F) \otimes_A \kappa(x) \to M \otimes_A \kappa(x).$$

Now let $s \in \operatorname{im}(F) \otimes_A \kappa(x)$ which is mapped to 0. By surjectivity of π we can choose $y \in M' \otimes_A \kappa(x)$ with $\pi(y) = s$. Then $F_x(y) = 0$, hence we find $z \in M \otimes_A \kappa(x)$ with $V_x(z) = y$. Therefore $s = \pi(V_x(z)) = 0$ as desired.
Note that we can do the same calculations for V and F interchanged to obtain projectivity of $\operatorname{coker}(V)$. Consequently, $\operatorname{im}(V), \ker(V), \operatorname{im}(F)$ and $\ker(F)$ satisfy the additional assertion of the lemma. The exactness of $M^\bullet$ then follows by applying Nakayama's lemma (cf. Lemma 2.1.12 1.) to $\operatorname{im}(V) \to \ker(F)$ and $\operatorname{im}(F) \to \ker(V)$. □

Remark 2.1.14. Let $R = W(S)/\xi \to R' = W(S')/\xi'$ be a morphism of perfectoid rings. Then the image of ξ under the induced morphism $W(S) \to W(S')$ is a distinguished element of $W(S')$ and we can identify $R' = W(S')/\xi$ (cf. Proposition 1.1.7). By the previous lemma we obtain a base change functor $\mathrm{BK}_1(R) \to \mathrm{BK}_1(R')$ and a functor $\mathrm{BK}_1^{(\mathrm{h,d})}(R) \to \mathrm{BK}_1^{(\mathrm{h,d})}(R')$ for natural numbers $0 \le d \le h$.

Proposition 2.1.15. *Let* $M = (M, F, V) \in \mathrm{BK}_{\mathrm{tor}}(R)$ *such that* $pM = 0$*. Then we have* $M \in \mathrm{BK}_1(R)$ *if and only if* $M \otimes_S \kappa(s) \in \mathrm{BK}_1(\kappa(s))$ *for all* $s \in \operatorname{Max}(S/\xi_0)$*. Moreover, in this case the sequence*

$$M^\sigma/\xi_0 \xrightarrow{\overline{F}} M/\xi_0 \xrightarrow{\overline{V}} M^\sigma/\xi_0$$

is exact as well. In other words, the category $\mathrm{BK}_1(R)$ *is equivalent to the category of triples* (M, F, V)*, where* M *is a finite projective* S*-module and* $F\colon M^\sigma \to M$ *and* $V\colon M \to M^\sigma$ *are* S*-linear maps with* $F \circ V = \xi_0$ *and* $V \circ F = \xi_0$ *and for every* $s \in \mathrm{Max}(S)$ *the sequence*

$$M \otimes_S \kappa(s) \xrightarrow{V_s} M^\sigma \otimes_S \kappa(s) \xrightarrow{F_s} M \otimes_S \kappa(s)$$

is exact.

Proof. This follows from Proposition 2.1.13. □

Proposition 2.1.16. *The anti-equivalence* $\mathbb{M}$ *of Theorem 1.2.6 induces a height and dimension preserving anti-equivalence*

$$\mathrm{BT}_1(R) \cong \mathrm{BK}_1(R),$$

which is compatible with base change.

Proof. Let G be a $\mathrm{BT}_{\mathrm{tor}}$-group over R and $M = \mathbb{M}(G)$ be the associated $\mathrm{BK}_{\mathrm{tor}}$-module over R. Since $\mathbb{M}$ is exact and has an exact quasi-inverse, G is annihilated by p if and only if M is annihilated by p. Since $\mathbb{M}$ and a quasi-inverse are compatible with base change and induce height and dimension preserving anti-equivalences $\mathrm{BT}_1(k) \cong \mathrm{BK}_1(k)$ for every perfect field k, the proposition follows from Lemma 2.1.5 and Lemma 2.1.15. □

2.2 Generalization to $\mathrm{BT_n}$-groups and $\mathrm{BK_n}$-modules

We still fix a perfectoid ring R and a presentation $R = W(S)/\xi$. Similar to the previous section, we determine the essential image of $\mathrm{BT_n}$-groups under the anti-equivalence $\mathbb{M}$ of Theorem 1.2.6.

2.2.1 Modules associated with $\mathrm{BT_n}$-groups

We start with the relevant definition and general remarks on $\mathrm{BT_n}$-groups. Note that our definition of $\mathrm{BT_n}$-groups is not exactly the one given in [**Gro71**] §3, Définition 3.2, but equivalent to it by [**Gro71**] §3, Proposition 2.2.

Definition 2.2.1. For $n \geq 2$ we denote by $\mathrm{BT_n}(R)$ the full subcategory of $\mathrm{BT}_{\mathrm{tor}}(R)$ whose objects are those groups G which are annihilated by p^n and such that the sequence

$$G \xrightarrow{\cdot p^{n-1}} G \xrightarrow{\cdot p} G$$

is exact. Objects of $\mathrm{BT_n}(R)$ are called *truncated Barsotti-Tate groups of level* n *over* R *or* $\mathrm{BT_n}$*-groups over* R.

As in the case for $n = 1$, this definition is compatible with base change and we have a fibrewise description of the category $\mathrm{BT_n}(R)$:

Lemma 2.2.2. *Let $G \in \mathrm{BT_{tor}}(R)$ and $n \in \mathbb{N}$ such that G is annihilated by p^n. Then G is an object of $\mathrm{BT_n}(R)$ if and only if $G \otimes_R \kappa(s)$ is an object of $\mathrm{BT}_n(\kappa(s))$ for all $s \in \mathrm{Max}(R/p)$.*

Proof. Note that $\mathrm{Max}(R) = \mathrm{Max}(R/p)$ and use Lemma 2.1.6. □

Remark 2.2.3. Let $m \geq n$ be two natural numbers and $G \in \mathrm{BT}_m(R)$. The p^n-torsion subgroup $G[p^n]$ of G is an object of $\mathrm{BT}_n(R)$. In particular, the p-torsion subgroup of a $\mathrm{BT_n}$-group is a truncated Barsotti-Tate group of level 1.

Definition 2.2.4. Let $n \in \mathbb{N}$ and $G \in \mathrm{BT}_n(R)$. The *height* (resp. *dimension*) of G is defined to be the height (resp. the dimension) of its p-torsion subgroup $G[p] \in \mathrm{BT}_1(R)$. Moreover, we denote the full subcategory of $\mathrm{BT_n}(R)$ whose objects are $\mathrm{BT_n}$-groups of constant height h and constant dimension d by $\mathrm{BT_n^{(h,d)}}(R)$.

Now we define the full subcategory of $\mathrm{BK_{tor}}(R)$ which will be shown to correspond to $\mathrm{BT_n}(R)$ under the anti-equivalence of Theorem 1.2.6.

Definition 2.2.5. For $n \geq 2$ we denote by $\mathrm{BK_n}(R)$ the full subcategory of $\mathrm{BK_{tor}}(R)$ consisting of objects $M = (M, F, V)$ such that M is killed by p^n and is finite projective as a $W_n(S)$-module.

Objects of $\mathrm{BK_n}(R)$ are also called $\mathrm{BK_n}$-modules over R. If $R = S$ is perfect, those are the same as *truncated Dieudonné modules of rank n over R* as defined in [**Lau13**] Definition 6.1. and we have the following classification result:

Proposition 2.2.6. *Assume that R is perfect. The anti-equivalence $\mathbb{M}\colon \mathrm{BT_{tor}}(R) \to \mathrm{BK_{tor}}(R)$ of Theorem 1.2.6 induces an anti-equivalence*

$$\mathrm{BT_n}(R) \cong \mathrm{BK_n}(R)$$

for all $n \geq 1$.

Proof. Since $\mathbb{M}$ extends the Dieudonné theory of [**BBM82**], this is contained in [**Lau13**] Theorem 6.4. □

Now we fix some $n \in \mathbb{N}$.

Remark 2.2.7. For a morphism $R = W(S)/\xi \to R' = W(S')/\xi$ of perfectoid rings we obtain a base change functor $\mathrm{BK_n}(R) \to \mathrm{BK_n}(R')$.

As in the $n = 1$ case, we also have the following fibrewise description of $\mathrm{BK_n}$-modules:

Lemma 2.2.8. *Let $M = (M, F, V) \in \mathrm{BK_{tor}}(R)$ such that $p^n M = 0$. Then $M \in \mathrm{BK_n}(R)$ if and only if $M \otimes_{W_n(S)} W_n(\kappa(s)) \in \mathrm{BK_n}(\kappa(s))$ for all $s \in \mathrm{Max}(R/p) = \mathrm{Max}(S/\xi_0)$.*

Proof. One direction is clear since we have base change functors. Conversely, assume that all fibres of $M = (M, F, V)$ are $\mathrm{BK_n}$-modules. We have to show that M is a finite projective $W_n(S)$-module. The assumption $M \otimes_{W_n(S)} W_n(\kappa(s)) \in \mathrm{BK_n}(\kappa(s))$ implies that $M/p \otimes_S \kappa(s) = (M \otimes_{W_n(S)} W_n(\kappa(s)))/p$ is an object of $\mathrm{BK_1}(\kappa(s))$ for all $s \in \mathrm{Max}(R/p)$ by [**Lau13**] Lemma 6.2. Consequently, $M/p \in \mathrm{BK_1}(R)$ by Lemma 2.1.15 and M/p is a finite projective S-module. By the local flatness criterion we are reduced to show that

$$\mathrm{Tor}_1^{W_n(S)}(S, M) = 0.$$

This is the case if and only if the sequence of $W_n(S)$-modules

$$M \xrightarrow{\cdot p^{n-1}} M \xrightarrow{\cdot p} M$$

is exact. (Apply $M \otimes_{W_n(S)} -$ to the exact sequence $0 \to pW_n(S) \to W_n(S) \to S \to 0$ and use the identification $M/p^{n-1} = M \otimes_{W_n(S)} W_n(S)/p^{n-1}W_n(S) \xrightarrow{\mathrm{id}\otimes \cdot p} M \otimes_{W_n(S)} pW_n(S)$.) But since the anti-equivalence $\mathbb{M}$ (of Theorem 1.2.6) and a quasi-inverse are exact and commute with base change, this is equivalent to the exactness of

$$M \otimes_{W_n(S)} W_n(\kappa(s)) \xrightarrow{\cdot p^{n-1}} M \otimes_{W_n(S)} W_n(\kappa(s)) \xrightarrow{\cdot p} M \otimes_{W_n(S)} W_n(\kappa(s))$$

for all $s \in \mathrm{Max}(W_n(S))$ by Lemma 2.1.6. This holds true because $M \otimes_{W_n(S)} W_n(\kappa(s))$ is a finite projective $W_n(\kappa(s))$-module by assumption. □

Lemma 2.2.9. *Let $m \geq n$ be natural numbers. We have a well defined truncation functor*

$$\mathrm{BK}_m(R) \to \mathrm{BK_n}(R), \quad M \mapsto M \otimes_{W_m(S)} W_n(S).$$

Proof. The assertion is clear if $n \geq 2$. Hence we may assume $n = 1$ and $m = 2$. By Lemma 2.2.8 we are reduced to the case where R is a perfect field of characteristic p and in particular a perfect ring. Then the claim is proved in [**Lau13**] Lemma 6.2. □

Definition 2.2.10. Let M be an object of $\mathrm{BK}_n(R)$. The *height* (resp. *dimension*) of M is defined to be the height (resp. the dimension) of the associated $\mathrm{BK_1}$-module M/p. Moreover, we denote the full subcategory of $\mathrm{BK_n}(R)$ whose objects are $\mathrm{BK_n}$-modules of constant height h and constant dimension d by $\mathrm{BK_n^{(h,d)}}(R)$.

Note that for a morphism of perfectoid rings $R \to R'$ the compatibility of truncation with base change assures that we obtain a base change functor $\mathrm{BK}_{\mathrm{n}}^{(\mathrm{h},\mathrm{d})}(R) \to \mathrm{BK}_{\mathrm{n}}^{(\mathrm{h},\mathrm{d})}(R')$.

The fibrewise description of BK_{n}-modules finally leads to

Proposition 2.2.11. *The anti-equivalence $\mathbb{M}\colon \mathrm{BT}_{\mathrm{tor}}(R) \cong \mathrm{BK}_{\mathrm{tor}}(R)$ of Theorem 1.2.6 induces a height and dimension preserving anti-equivalence $\mathrm{BT}_{\mathrm{n}}(R) \cong \mathrm{BK}_{\mathrm{n}}(R)$, which is compatible with base change and truncation for all $n \geq 1$.*

Proof. This follows formally from Lemma 2.2.2 and Lemma 2.2.8 using the exactness and base change properties of the anti-equivalence $\mathbb{M}$ and the classification in the perfect case as stated in Proposition 2.2.6. □

2.3 Normal representations and lifting properties

We still fix a perfectoid ring R and a presentation $R = W(S)/\xi$. Moreover, we fix some $n \in \mathbb{N} \cup \{\infty\}$ and use the notation $W_\infty(S) := W(S)$, $\mathrm{BK}_\infty(R) := \mathrm{BK}(R)$ and $\mathrm{BT}_\infty(R) := \mathrm{BT}(R)$. The classification of $\mathrm{BT}_{\mathrm{n}}(R)$ via BK_{n}-modules at hand, we now show that any BT_{n}-group can be lifted to a BT-group.

The central observation here is the fact that any BK_{n}-module admits a *normal representation*, which gives us some control over the encoded data. Besides the lifting properties, which are proved in this section, the existence of normal representations will also be crucial when describing the the groupoid of BK_{n}-modules in Chapter 3.

Definition 2.3.1. A *normal representation* for a BK_{n}-module $M = (M, F, V)$ over R is a triple (L, P, Φ) where $L \subseteq M$ and $P \subseteq M$ are direct summands with $M = L \oplus P$ and $\Phi\colon M \to M^\sigma$ is a $W_n(S)$-linear bijection, such that, with respect to the decompositions $M = L \oplus P$ and $M^\sigma = \Phi(L) \oplus \Phi(P)$, we have $V = \alpha \oplus \xi\beta$ and $F = \xi\alpha^{-1} \oplus \beta^{-1}$, where $\alpha = \Phi|_L\colon L \cong \Phi(L)$ and $\beta = \Phi|_P\colon P \cong \Phi(P)$.

Remark 2.3.2.

1. Every triple (L, P, Φ), consisting of finite projective $W_n(S)$-modules L and P and a $W_n(S)$-linear isomorphism $\Phi\colon L \oplus P \cong (L \oplus P)^\sigma$, determines a unique BK_{n}-module M over R, which we call the BK_{n}*-module associated with* (L, P, Φ). In this case (L, P, Φ) is a normal representation for M.
2. For a BK_{n}-module $M = (M, F, V)$ with normal representation (L, P, Φ), the rank of M is given by the sum of the ranks of L and P and the dimension of M is given by the rank of L.

Lemma 2.3.3. *Every* $\mathrm{BK_n}$*-module* (M, F, V) *over* R *admits a normal representation.*

Proof. Let $M = (M, F, V) \in \mathrm{BK_n}(R)$ and $I := \ker(W_n(S) \to S/\xi_0) = (\xi, p)$ which is contained in the Jacobson radical of $W_n(S)$. By **[SP]** Tag 0DYD, the pair $(W_n(S), I)$ even is Henselian since the pairs $(W_n(S), (p))$ and $(S, (\xi_0))$ have this property. Consider $\overline{M} = (\overline{M}, \overline{F}, \overline{V})$, the base change of M to S/ξ_0. Choose submodules $\overline{L} \subseteq \overline{M}$ and $\overline{Q} \subseteq \overline{M}^\sigma$ such that $\overline{M} = \overline{L} \oplus \mathrm{im}(\overline{F})$ and $\overline{M}^\sigma = \mathrm{im}(\overline{V}) \oplus \overline{Q}$. We have isomorphisms of S/ξ_0-modules $\overline{\alpha}\colon \overline{L} \to \mathrm{im}(\overline{V})$ and $\overline{\beta}\colon \mathrm{im}(\overline{F}) \to \overline{Q}$ which are induced by $\overline{V}$ and $\overline{F}^{-1}$ respectively. We claim that we can find a direct summand $Q \subseteq M^\sigma$ such that $Q/IQ = \overline{Q}$. Indeed, let Q' be any finite projective $W_n(S)$-module such that $Q'/IQ' = \overline{Q}$, which exists since the pair $(W_n(S), I)$ is Henselian. We can lift the maps $\overline{Q} \to \overline{M}^\sigma \to \overline{Q}$ to maps $Q' \to M^\sigma \to Q'$ whose composition is an isomorphism by Nakayama. Then $Q \subseteq M^\sigma$, the image of the injection $Q' \to M^\sigma$, does the trick. The same argument shows that we can find a direct summand $L \subseteq M$ such that $L/IL = \overline{L}$. Now consider the commutative diagram

$$\begin{tikzcd}
Q \arrow[r, two heads, "F"] \arrow[d, "/I"] & F(Q) \arrow[r, hook] \arrow[d] & M \arrow[d, "/I"] \\
\overline{Q} \arrow[r, "\overline{\beta}^{-1}"] & \mathrm{im}(\overline{F}) \arrow[r, hook] & \overline{M}.
\end{tikzcd}$$

Since $\overline{\beta}^{-1}$ is bijective, we see that $F(Q)/(IF(Q)) = \mathrm{im}(\overline{F})$.
Now consider the map $F(Q) \oplus L \to M$. This is an isomorphism modulo I and hence itself an isomorphism by Lemma 2.1.12. In particular, $F(Q)$ is a finite projective $W_n(S)$-module. Moreover, $F|_Q\colon Q \to F(Q)$ is an isomorphism since it is an isomorphism modulo I. We set $\beta := (F|_Q)^{-1}\colon F(Q) \to Q$. The same arguments show that $Q \oplus V(L) = M^\sigma$ and that $\alpha := V|_L\colon L \to V(L)$ is bijective. Now define $\Phi\colon M = L \oplus F(Q) \to V(L) \oplus Q = M^\sigma$ to be $\alpha \oplus \beta$. By construction, the triple $(L, F(Q), \Phi)$ then is a normal representation for M. □

Proposition 2.3.4. *For* $m \geq n \in \mathbb{N} \cup \{\infty\}$ *the truncation functor* $\mathrm{BK_m}(R) \to \mathrm{BK}_n(R)$ *is essentially surjective.*

Proof. Let M be a BK_n-module over R with normal representation (L, P, Φ). We choose finite projective $W_m(S)$-modules $\tilde{L}$ and $\tilde{P}$ such that $\tilde{L}/p^n = L$ and $\tilde{P}/p^n = P$ and set $\tilde{M} := \tilde{L} \oplus \tilde{P}$. Moreover, we can lift the map $\tilde{M} \xrightarrow{/p^n} M \xrightarrow{\Phi} M^\sigma$ along the surjection $\tilde{M}^\sigma \to M^\sigma$ to obtain an isomorphism $\tilde{\Phi}\colon \tilde{M} \to \tilde{M}^\sigma$. Now let $\tilde{M} = (\tilde{M}, \tilde{F}, \tilde{V})$ be the $\mathrm{BK_m}$-module over R associated with $(\tilde{L}, \tilde{P}, \tilde{\Phi})$. Then $\tilde{M}/p^n \cong M$ as desired. □

Setting $m = \infty$ and using Proposition 2.2.11 and Theorem 1.2.2, we obtain

Corollary 2.3.5. *Every* $\mathrm{BT_n}$*-group* G *over* R *is isomorphic to* $H[p^n]$ *for some* BT*-group* H *over* R.

3 $\mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})}$ as a quotient stack

The goal of this chapter is to extend results on the classification of F-Zips (cf. Proposition 3.1.3 below) in the direction of perfectoid rings. We construct a fully faithful map from a prestack given by orbit spaces to $\mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})}$ and show that this induces an isomorphism of stacks after stackification for a certain topology.

3.1 BK_1-modules and F-*Zips*

We briefly recall a very small amount of the theory of F-*Zips* developed by Moonen and Wedhorn in [**MW04**] to draw connections to BK_1-modules. Moreover, Pink, Wedhorn and Ziegler depicted the classifying stack of such as a certain quotient stack in [**PWZ15**]. Their observations serve as our motivation to investigate the prestack of BK_1-modules (and more generally the prestack of BK_{n}-modules) over Perfd_R for some perfectoid ring R in the following sections of this work.

Definition 3.1.1. 1. Let R be a ring with $pR = 0$ and let $0 \leq d \leq h$ be integers. An F-*Zip over* R *of type* (h, d) consists of a tuple $(M, C, D, (\varphi_i)_{i=0,1})$, where M is a finite projective R-module of rank h, C (resp. D) is a direct summand of M of rank d (resp. of rank $n - d$) and $\varphi_0\colon M^\sigma/C^\sigma \cong D$ and $\varphi_1\colon C^\sigma \cong M/D$ are R-linear isomorphisms.

2. An *isomorphism of* F-*Zips* $(M, C, D, (\varphi_i)_{i=0,1}) \to (M', C', D', (\varphi'_i)_{i=0,1})$ of type (h, d) is an R-linear isomorphism $f\colon M \to M'$ such that $f(C) = C', f(D) = D'$ and the induced diagrams commute.

3. We denote the associated groupoid of F-Zips of type (h, d) over R by $\text{F-Zip}^{(\mathrm{h,d})}(R)$.

Remark 3.1.2. Let S be a perfect ring. Then there is an equivalence of groupoids

$$\text{F-Zip}^{(\mathrm{h,d})}(S) \cong \mathrm{BK}_1^{(\mathrm{h,d})}(S)$$

given by

$$(M, C, D, (\varphi_i)_{i=0,1}) \mapsto (M, F, V),$$

where F resp. V are induced by φ_0 resp. φ_1^{-1}.

A special case of [**PWZ15**] Proposition 3.11 and Proposition 7.2 can be stated as follows:

Theorem 3.1.3. *Let k be a field of characteristic p. The prestack over Ring_k given by $R \mapsto \text{F-Zip}^{(\mathrm{h,d})}(R)$ is a smooth algebraic stack (for the fpqc topology) of dimension 0 over k. It is isomorphic to the quotient stack*

$$[\tilde{E}^{(h,d)} \setminus \mathrm{GL}_{h,k}],$$

where $\tilde{E} := \tilde{E}^{(h,d)} \subseteq (\mathrm{GL}_{h,k})^2$ for perfect rings S is given by

$$\tilde{E}(S) = \left\{ \left(\begin{pmatrix} J & F \\ 0 & H \end{pmatrix}, \begin{pmatrix} \sigma(J) & 0 \\ \sigma(C) & \sigma(H) \end{pmatrix} \right) \in \mathrm{GL}_h(S)^2 \;\middle|\; \begin{matrix} J \in \mathbb{M}_d(S), H \in \mathbb{M}_{(n-d)}(S) \\ F \in \mathbb{M}_{(d,h-d)}(S), C \in \mathbb{M}_{(h-d,d)}(S) \end{matrix} \right\}.$$

(Since we only describe points of E in the situation of perfect rings, we are free to write C instead of $\sigma(C)$. In fact, for general k-algebras, C (instead of $\sigma(C)$) is the right definition. We nevertheless include the σ here to bring out the relation to our later results.) Here $\tilde{E}$ acts on $\mathrm{GL}_{h,k}$ from the left hand side via $(A, B) \cdot g = AgB^{-1}$.

Moreover, for any perfect field k' of characteristic p over k the groupoid of F-Zips of type (h, d) is isomorphic to the groupoid

$$\tilde{E}(k') \setminus \mathrm{GL}_h(k').$$

3.2 Classification of $\mathrm{BK_n}$-modules of type (h, d)

We fix a perfectoid ring R, a presentation $R = W(S)/\xi$, some $n \in \mathbb{N} \cup \{\infty\}$ and integers $0 \leq d \leq h$. Moreover, we introduce some notation to analyse $\mathrm{BK_n}$-modules of constant height h and constant dimension d.

Let

$$\mu := \mathrm{diag}(\xi, ..., \xi, 1, ..., 1) \in \mathbb{M}_h(W_n(S)),$$

where the first d entries are ξ and the last $(h-d)$ entries are 1. Also set

$$\mu' = \mathrm{diag}(1, ..., 1, \xi, ..., \xi) \in \mathbb{M}_h(W_n(S)),$$

where the first d entries are 1 and the last $(h-d)$ entries are ξ. Moreover, we define $W_n(S)$-linear maps

$$F_{\text{std}}\colon (W_n(S)^h)^\sigma \to W_n(S)^h, x \mapsto \mu\sigma(x)$$

and

$$V_{\text{std}}\colon W_n(S)^h \to (W_n(S)^h)^\sigma, x \mapsto \sigma^{-1}(\mu' x).$$

We claim that this gives a $\mathrm{BK_n}$-module $M_{\text{std}}^{(h,d)} = (W_n(S)^h, F_{\text{std}}, V_{\text{std}})$, which is our standard example for a $\mathrm{BK_n}$-module of height h and dimension d. Indeed, for $n \neq 1$ this is clear and for $n = 1$ the claim follows from the fibre-wise characterization of $\mathrm{BK_1}$-modules, cf. Proposition 2.1.13.

3.2.1 The category of n-quintuples

To classify all $\mathrm{BK_n}$-modules of height h and dimension d, we want to measure its *difference* from $M_{\text{std}}^{(h,d)}$. This *difference* can conveniently be described by considering the category of *n-quintuples of type (h,d)*, defined below, where all variation can locally be pushed into a $W_n(S)$-linear isomorphism. This concept is related to the notion of normal representations of Chapter 2.3. Passing to this category of n-quintuples is just a way to reorganize the data given by a $\mathrm{BK_n}$-module into a form which is more handy for our purposes.

Definition 3.2.1.

1. An *n-quintuple over R* consists of the datum (N, M, F, V, ψ), where N and M are finite projective $W_n(S)$-modules, $F\colon N \to M$ and $V\colon M \to N$ are $W_n(S)$-linear maps such that $F \circ V$ and $V \circ F$ are multiplication with ξ and $\psi\colon M^\sigma \cong N$ is a $W_n(S)$-linear isomorphism.

2. A *morphism of n-quintuples* $(N, M, F, V, \psi) \to (N', M', F', V', \psi')$ over R consists of two $W_n(S)$-linear maps $f\colon N \to N'$ and $g\colon M \to M'$ such that the diagrams

$$\begin{tikzcd} M \arrow[r, "V"] \arrow[d, "g"] & N \arrow[r, "F"] \arrow[d, "f"] & M \arrow[d, "g"] \\ M' \arrow[r, "V'"] & N' \arrow[r, "F'"] & M' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M^\sigma \arrow[r, "\psi"] \arrow[d, "g^\sigma"] & N \arrow[d, "f"] \\ (M')^\sigma \arrow[r, "\psi'"] & N' \end{tikzcd}$$

commute.

3. We obtain the category of n-quintuples over R.

Remark 3.2.2. 1. For all $m \geq n$ there is a natural truncation functor from the category of m-quintuples over R to the category of n-quintuples over R

$$(N, M, F, V, \psi) \mapsto (N/p^n, M/p^n, \overline{F}, \overline{V}, \overline{\psi}),$$

where $\overline{F}, \overline{V}$ and $\overline{\psi}$ are the maps induced by reduction modulo p^n.

2. For a morphism of perfectoid rings $R \to R'$ we obtain a base change functor from the category of n-quintuples over R to the category of n-quintuples over R'.

Lemma 3.2.3. *By associating $(M, F, V) \mapsto (M^\sigma, M, F, V, \mathrm{id}_{M^\sigma})$ and $g \mapsto (g^\sigma, g)$, we obtain a fully faithful functor from the category of BK_n-modules over R to the category of n-quintuples over R, which is compatible with base change and truncations. Moreover, for $n \geq 2$ this is an equivalence.*

Proof. For $n \in \mathbb{N}$ (resp. $n = \infty$), the mapping

$$G\colon (N, M, F, V, \psi) \mapsto (M, F', V'), (f, g) \mapsto g$$

where $F'\colon M^\sigma \xrightarrow{\psi} N \xrightarrow{F} M$ and $V'\colon M \xrightarrow{V} N \xrightarrow{\psi^{-1}} M^\sigma$ gives a fully faithful functor from the category of n-quintuples to a subcategory of $\mathrm{BK}_{\mathrm{tor}}(R)$ whose objects have underlying modules killed by p^n and are finite projective as $W_n(S)$-modules (resp. to $\mathrm{BK}(R)$). If $n \geq 2$, this gives a quasi-inverse for the functor of the lemma. Since the inclusion of $\mathrm{BK}_1(R)$ into $\mathrm{BK}_{\mathrm{tor}}(R)$ is fully faithful, the functor from the lemma is fully faithful for $n = 1$ as well. The assertions about base change and truncation can be checked easily. □

Definition 3.2.4. 1. A 1-quintuple (N, M, F, V, ψ) which is in the essential image of the functor described in Lemma 3.2.3 is said to have *type* (h, d) if an associated BK_1-module has type (h, d) (i.e. the BK_1-module has constant height h and constant dimension d).

2. An n-quintuple is said to have *type* (h, d) if its reduction modulo p is a 1-quintuple of type (h, d).

This definition also forces the functor of Lemma 3.2.3 to preserve types.

We give an elementary remark on σ-linear algebra.
Let π denote the canonical $W_n(S)$-linear isomorphism $W_n(S)^h \to (W_n(S)^h)^\sigma, x \mapsto \sigma^{-1}(x)$

and let $f\colon W_n(S)^h \to W_n(S)^h$ be a $W_n(S)$-linear map. The associated $W_n(S)$-linear map $\tilde{f}\colon (W_n(S)^h)^\sigma \to (W_n(S)^h)^\sigma$ is defined by the commutative square

$$\begin{tikzcd} W_n(S)^h \arrow[r, "f"] \arrow[d, "\pi"] & W_n(S)^h \arrow[d, "\pi"] \\ (W_n(S)^h)^\sigma \arrow[r, "\tilde{f}"] & (W_n(S)^h)^\sigma. \end{tikzcd}$$

Beware that $\tilde{f}$ is not equal to f^σ. If $f\colon W_n(S)^h \to W_n(S)^h$ is given by some matrix $A \in \mathbb{M}_h(W_n(S))$, then the diagram

$$\begin{tikzcd} W_n(S)^h \arrow[r, "x\mapsto\sigma(A)x"] \arrow[d, "\pi"] & W_n(S)^h \arrow[d, "\pi"] \\ (W_n(S)^h)^\sigma \arrow[r, "f^\sigma"] & (W_n(S)^h)^\sigma \end{tikzcd}$$

commutes.

The following notions of standard objects play a crucial role on our way to classify n-quintuples and $\mathrm{BK_n}$-modules.

Definition 3.2.5. Let $l \in \mathrm{GL}_h(W_n(S))$.

1. We denote the n-quintuple

$$((W_n(S)^h)^\sigma, W_n(S)^h, F_{\text{std}}, V_{\text{std}}, \tilde{l})$$

   by $\underline{I}_l$. This n-quintuple is called the *n-quintuple (of type (h,d)) associated with l.*

2. Let $F_l := F_{\text{std}} \circ \tilde{l}\colon (W_n(S)^h)^\sigma \to W_n(S)^h$ and $V_l := \tilde{l}^{-1} \circ V_{\text{std}}\colon W_n(S)^h \to (W_n(S)^h)^\sigma$. Then we obtain a $\mathrm{BK_n}$-module[1] of type (h,d)

$$\underline{M}_l := (W_n(S)^h, F_l, V_l)$$

   whose associated n-quintuple is isomorphic to $\underline{I}_l$ in view of Lemma 3.2.3.

[1]This construction indeed gives a $\mathrm{BK_n}$-module. This is clear for $n \neq 1$ and follows by the fibrewise characterization of $\mathrm{BK_1}$-modules (cf. Proposition 2.1.13) and the equations $\ker(V_l) = \ker(V_{\text{std}})$ and $\mathrm{im}(F_l) = \mathrm{im}(F_{\text{std}})$ for the case of R being a perfect field of characteristic p.

3.2.2 Classification via orbits of group operations

Let $\sim$ be the equivalence relation on $\mathrm{GL}_h(W_n(S))$ such that $l \sim l'$ if and only if the n-quintuples $\underline{I}_l$ and $\underline{I}_{l'}$ (or equivalently the BK_n-modules $\underline{M}_l$ and $\underline{M}_{l'}$) are isomorphic. Then the mapping $l \mapsto \underline{M}_l$ induces an injective map of sets

$$\sim \backslash \mathrm{GL}_h(W_n(S)) \to \mathrm{BK}_\mathrm{n}^{(\mathrm{h,d})}(R)/\cong,$$

where the right hand side denotes isomorphism classes of BK_n-modules of type (h,d). The following lemma shows that $\sim$ is induced by a group action on $\mathrm{GL}_h(W_n(S))$ and upgrades the given injection to a fully faithful functor of groupoids.

Lemma 3.2.6. *Let $l, l' \in \mathrm{GL}_h(W_n(S))$. There is a canonical bijection between the set of isomorphisms of n-quintuples $\underline{I}_l \cong \underline{I}_{l'}$ and the set of $(A,B) \in \mathrm{GL}_h(W_n(S)) \times \mathrm{GL}_h(W_n(S))$ with $\mu A = B\mu$, $A\mu' = \mu' B$ and $l' = Al\sigma(B^{-1})$.*

Proof. Let $(\tilde{f}, g) \colon \underline{I}_l \to \underline{I}_{l'}$ be an isomorphism of n-quintuples. We denote the canonical identification $W_n(S)^h \cong (W_n(S)^h)^\sigma, x \mapsto \sigma^{-1}(x)$ by π. Let $g \colon W_n(S)^h \to W_n(S)^h$ be given by $B \in \mathrm{GL}_h(W_n(S))$ and let $A \in \mathrm{GL}_h(W_n(S))$ be the matrix associated with $f \colon W_n(S)^h \to W_n(S)^h$, where f and $\tilde{f}$ relate via the commutative diagram

$$\begin{tikzcd} W_n(S)^h \arrow[r, "f", "x \mapsto Ax"'] \arrow[d, "\pi"] & W_n(S)^h \arrow[d, "\pi"] \\ (W_n(S)^h)^\sigma \arrow[r, "\tilde{f}"] & (W_n(S)^h)^\sigma. \end{tikzcd}$$

The map

$$W_n(S)^h \xrightarrow{\pi} (W_n(S)^h)^\sigma \xrightarrow{F_{\mathrm{std}}} W_n(S)^h$$

is given by $x \mapsto \mu x$ and

$$W_n(S)^h \xrightarrow{V_{\mathrm{std}}} (W_n(S)^h)^\sigma \xrightarrow{\pi^{-1}} W_n(S)^h$$

is given by $x \mapsto \mu' x$. Since $(\tilde{f}, g) \colon \underline{I}_l \cong \underline{I}_{l'}$ is an isomorphism of n-quintuples, the following diagrams commute:

$$\begin{tikzcd} (W_n(S)^h)^\sigma \arrow[r, "F_{\mathrm{std}}"] \arrow[d, "\cong"', "\tilde{f}"] & W_n(S)^h \arrow[d, "\cong"', "g"] \\ (W_n(S)^h)^\sigma \arrow[r, "F_{\mathrm{std}}"] & W_n(S)^h \end{tikzcd}$$

$$
\begin{tikzcd}
W_n(S)^h \arrow[r, "V_{\text{std}}"] \arrow[d, "g", "\cong"'] & (W_n(S)^h)^\sigma \arrow[d, "\tilde{f}", "\cong"'] \\
W_n(S)^h \arrow[r, "V_{\text{std}}"] & (W_n(S)^h)^\sigma
\end{tikzcd}
$$

$$
\begin{tikzcd}
(W_n(S)^h)^\sigma \arrow[r, "\tilde{l}", "\cong"'] \arrow[d, "g^\sigma", "\cong"'] & (W_n(S)^h)^\sigma \arrow[d, "\tilde{f}", "\cong"'] \\
(W_n(S)^h)^\sigma \arrow[r, "\tilde{l'}", "\cong"'] & (W_n(S)^h)^\sigma .
\end{tikzcd}
$$

Now we consider the diagrams when replacing $(W_n(S)^h)^\sigma$ by $W_n(S)^h$ via π. Then the first diagram says that $\mu A = B\mu$, the second diagram means $\mu' B = A\mu'$ and from the last diagram we get $Al = l'\sigma(B)$ or equivalently $l' = Al\sigma(B^{-1})$.
Conversely, assume we have a pair (A, B) as described above. Let $f\colon W_n(S)^h \to W_n(S)^h$ denote the $W_n(S)$-linear isomorphism given by A and $\tilde{f}\colon (W_n(S)^h)^\sigma \to (W_n(S)^h)^\sigma$ be the induced map via the trivialization π. Moreover, let $g\colon W_n(S)^h \to W_n(S)^h$ be the $W_n(S)$-linear isomorphism given by B. Then by the same diagrams as above $(\tilde{f}, g)$ is an isomorphism $\underline{I}_l \cong \underline{I}_{l'}$. $\square$

Remark 3.2.7. 1. The set

$$E(S) := E_n^{(h,d)}(S) := \{(A, B) \in \mathrm{GL}_h(W_n(S))^2 \mid \mu A = B\mu \text{ and } A\mu' = \mu' B\}$$

is a subgroup of $\mathrm{GL}_h(W_n(S)) \times \mathrm{GL}_h(W_n(S))$. Since we will always work with h, d and n fixed, we neglect those indices in the notation of E and hope that this does not lead to any confusion.

2. Elements of $E(S)$ are given by pairs $\left(\begin{pmatrix} J & F \\ \xi C & H \end{pmatrix}, \begin{pmatrix} J & \xi F \\ C & H \end{pmatrix} \right) \in \mathrm{GL}_h(W_n(S))^2$, where

$$J \in \mathbb{M}_d(W_n(S)), H \in \mathbb{M}_{n-d}(W_n(S)), F \in \mathbb{M}_{d,h-d}(W_n(S)), C \in \mathbb{M}_{h-d,d}(W_n(S)).$$

3. One has an action of $E(S)$ on $\mathrm{GL}_h(W_n(S))$ from the left via $(A, B) \cdot l = Al\sigma(B^{-1})$.

4. With this notation we can formulate Lemma 3.2.6 above as follows:
For $l, l' \in \mathrm{GL}_h(W_n(S))$ the transporter $\mathrm{Transp}_E(l, l')$ is in natural bijection to the set of isomorphisms of n-quintuples $\underline{I}_l \cong \underline{I}_{l'}$.

5. As a conclusion we see that the mapping $l \mapsto \underline{M}_l$ induces a fully faithful functor of groupoids

$$E(S) \setminus \mathrm{GL}_h(W_n(S)) \to \mathrm{BK}_\mathrm{n}^{(\mathrm{h,d})}(R).$$

Remark 3.2.8. 1. If $W_n(S)$ is ξ-torsionfree (which is the case if R is p-torsion-free or $n = \infty$), any element (A, B) of $E(S)$ is determined by either of its components. This reflects the fact that in this case for a given BK_n-module V is determined by F and vice versa.

2. If $pR = 0$ and $n = 1$, we recover almost verbatim the classification of F-Zips of type (h, d) as described in Theorem 3.1.3. The difference is that in our result, the σ-twist is included in the group action, whereas in the case of F-Zips, this twist is built into the acting group E.

We say that a map of perfectoid R-algebras $A \to A'$ is a *p-Zariski cover*, if the map $A/p \to A'/p$ is a Zariski cover. The corresponding Grothendieck topology on Perfd_R is called *p-Zariski topology*. Locally for this topology, the described functors are essentially surjective.

Lemma 3.2.9. *Let $\underline{M} = (M, F, V)$ be a BK_n-module over R of type (h, d). Then there exists a p-Zariski cover $R \to R' = W(S')/\xi$ and an $l \in \mathrm{GL}_h(W_n(S'))$ such that $\underline{M} \otimes_R R'$ is isomorphic to $\underline{M}_l \in \mathrm{BK}_\mathrm{n}^{(\mathrm{h,d})}(R')$.*

Proof. By Lemma 2.3.3 we can choose a normal representation (P, Q, γ) for $\underline{M}$, which means that $M = (P \oplus Q)$ is a direct sum of finite projective $W_n(S)$-modules of dimension d, respectively $(n - d)$, the map $\gamma\colon M \to M^\sigma$ is a $W_n(S)$-linear isomorphism and we have $F = \xi\alpha^{-1} \oplus \beta^{-1}$ and $V = \alpha \oplus \xi\beta$ for $\alpha = \gamma|_P$ and $\beta = \gamma|_Q$. Now, choose a Zariski cover $R/p \to B'$ such that the base changes of P and Q via $W_n(S) \to R/p \to B'$ are free. By **[Lau18]** Lemma 8.11 and its proof, we have $B' = R'/p = S'/\xi_0$ for the perfectoid R-algebra $R' = W(S')/\xi$, where $S' = (B')^\flat$. Since $(p, [\xi_0])$ is contained in the Jacobson radical of $W_n(S')$, we see that the base changes of P and Q to $W_n(S')$ are free as well (cf. Lemma 2.1.12, 2.). Hence we may assume P and Q to be free in the first place.
We choose a basis $\mathcal{B}$ for P and a basis $\mathcal{C}$ for Q and let $g\colon M \to (W_n(S))^h$ be the unique $W_n(S)$-linear isomorphism mapping the ordered elements of $\mathcal{B}$ to $e_1, ..., e_d$ and the ordered elements of $\mathcal{C}$ to $e_{d+1}, ..., e_h$, where $(e_1, ..., e_h)$ denotes the standard basis of $(W_n(S))^h$.

Moreover, let $f\colon M^\sigma \to ((W_n(S))^h)^\sigma$ be the unique $W_n(S)$-linear map which maps the ordered elements of $\gamma(\mathcal{B}) \cup \gamma(\mathcal{C})$ to the standard basis of $((W_n(S))^h)^\sigma$ which is given by $(e_1 \otimes 1, ..., e_h \otimes 1)$. Then we obtain a commutative diagram

$$\begin{tikzcd} M \arrow[r, "V"] \arrow[d, "g"] & M^\sigma \arrow[r, "F"] \arrow[d, "f"] & M \arrow[d, "g"] \\ (W_n(S))^h \arrow[r, "V_{\text{std}}"] & ((W_n(S))^h)^\sigma \arrow[r, "F_{\text{std}}"] & (W_n(S))^h, \end{tikzcd}$$

and there exists a $\delta \in \mathrm{GL}_h(W_n(S))$ such that

$$\begin{tikzcd} M^\sigma \arrow[r, "\mathrm{id}_{M^\sigma}"] \arrow[d, "g^\sigma"] & M^\sigma \arrow[d, "f"] \\ ((W_n(S))^h)^\sigma \arrow[r, "\tilde{\delta}"] & ((W_n(S))^h)^\sigma \end{tikzcd}$$

commutes. Hence the pair (f, g) gives an isomorphism of n-quintuples

$$(M^\sigma, M, F, V, \mathrm{id}_{M^\sigma}) \cong (((W_n(S))^h)^\sigma, (W_n(S))^h, F_{\text{std}}, V_{\text{std}}, \tilde{\delta}) = \underline{I}_\delta$$

as desired. $\square$

If $R' = W(S')/\xi$ is a local perfectoid R-algebra, the proof of the lemma above also shows that the fully faithful functor

$$E(S) \setminus \mathrm{GL}_h(W_n(S')) \to \mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})}(R')$$

is essentially surjective (and hence an equivalence). By the following remark, this assertion also holds if we pass to arbitrary products of local perfectoid R-algebras.

Remark 3.2.10. 1. Let $A = \prod_{i\in I} A_i$ be a product of local rings A_i. Then the categories $\mathrm{LF}^d(A)$ and $\prod_{i\in I} \mathrm{LF}^d(A_i)$ of finite projective modules of rank d are equivalent.[2]

[2] Consider the functors $M \mapsto M \otimes_A A_i$ and $(M_i)_{i\in I} \mapsto \prod_{i\in I} M_i$ between the respective categories. Since each A_i is local, every finite projective A_i module of rank d is finite free of rank d. Consequently, the second functor is well defined. Moreover, a free A-module M of rank d is naturally isomorphic to $\prod_{i\in I}(M \otimes_A A_i)$ and for free A_i-modules M_i of rank d we have a natural isomorphism $(\prod_{i\in I} M_i) \otimes_A A_j \cong M_j$ for all $j \in I$.

2. Now let $A = \prod_{i \in I} A_i$ be a product of perfectoid local rings. Then, by Lemma 1.1.8, A is perfectoid and $A^\flat = \prod_{i \in I}(A_i^\flat)$. It follows from 1. that the groupoid $\mathrm{BK}_\mathrm{n}^{(\mathrm{h,d})}(A)$ is equivalent to the groupoid $\prod_{i \in I} \mathrm{BK}_\mathrm{n}^{(\mathrm{h,d})}(A_i)$. This is clear for $n \geq 2$ but also holds for $n = 1$ since all relevant properties can be tested on components.

3. The functor $\mathrm{Perfd}_R \to \mathrm{Groupoids}, R' \mapsto (E \setminus G)(R')$ commutes with arbitrary products. Hence for $A = \prod_{i \in I} A_i$ as in 2., the groupoids $\prod_{i \in I}((E \setminus G)(A_i))$ and $(E \setminus G)(A)$ are equivalent as well.

3.2.3 Prestacky point of view

In this subsection we reformulate our findings in the language of prestacks. We still fix a perfectoid ring R, a presentation $R = W(S)/\xi$, some $n \in \mathbb{N} \cup \{\infty\}$ and integers $0 \leq d \leq h$. For a morphism $A \to A'$ of perfectoid rings we have base change functors $\mathrm{BK}_\mathrm{n}(A) \to \mathrm{BK}_\mathrm{n}(A')$ and $\mathrm{BK}_\mathrm{n}^{(\mathrm{h,d})}(A) \to \mathrm{BK}_\mathrm{n}^{(\mathrm{h,d})}(A')$ and hence we get prestacks

$$\mathrm{BK}_\mathrm{n} \colon \mathrm{Perfd}_R \to \mathrm{Groupoids}, \quad R' \mapsto \mathrm{BK}_\mathrm{n}(R')$$

and

$$\mathrm{BK}_\mathrm{n}^{(\mathrm{h,d})} \colon \mathrm{Perfd}_R \to \mathrm{Groupoids}, \quad R' \mapsto \mathrm{BK}_\mathrm{n}^{(\mathrm{h,d})}(R').$$

Moreover, the mapping

$$\begin{gathered}(E \setminus G) \colon \mathrm{Perfd}_R \to \mathrm{Groupoids}, \\ R' \mapsto E(W_n(R'^\flat)) \setminus \mathrm{GL}_h(W_n(R'^\flat))\end{gathered}$$

also gives a prestack and the functors

$$E(R'^\flat) \setminus \mathrm{GL}_h(W_n(R'^\flat)) \to \mathrm{BK}_\mathrm{n}^{(\mathrm{h,d})}(R'),$$

defined in the section above, induce a morphism of prestacks

$$(E \setminus G) \to \mathrm{BK}_\mathrm{n}^{(\mathrm{h,d})}.$$

Once again, since we always work with fixed $n \in \mathbb{N} \cup \{\infty\}$ and integers $0 \leq d \leq h$, we expect that there will not arise any confusion by dropping this information in the notation of $(E \setminus G)$.

The discussion up to this point may be summarised as follows:

Proposition 3.2.11. *The morphism of prestacks over* Perfd_R

$$(E \setminus G) \to \mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})}, l \mapsto \underline{M}_l$$

(cf. Definition 3.2.5) is fully faithful on all perfectoid R-algebras and an equivalence on products of local perfectoid R-algebras. Moreover, locally for the p-Zariski topology it is essentially surjective.

Proof. This follows from Remark 3.2.7. 5., Lemma 3.2.9 and Remark 3.2.10. □

3.3 The classifying stack of $\mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})}$-modules

In this section we globalize our previous results. For this purpose, we endow the category Perfd with the étale counterpart of the p-Zariski topology which was used in Lemma 3.2.9. This topology is also considered by Lau in [**Lau18**] §10. A cover is given by a map of perfectoid rings such that reduction modulo p is étale and surjective on spectra. We show that BK_{n} and $\mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})}$ form stacks with respect to this topology and finally depict $\mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})}$ as a quotient stack which is associated to the constructions of the previous chapter. This will also imply that the same results are true for the p-Zariski topology on Perfd_R.

We fix a perfectoid ring R with presentation $R = W(S)/\xi$.

Definition 3.3.1. We say that a map of perfectoid rings $A \to A'$ is *p-étale* if the induced map $A/p \to A'/p$ is étale. If $A/p \to A'/p$ is also faithfully flat, we say that $A \to A'$ is a *p-étale cover*. The Grothendieck topology on Perfd_R whose covers are p-étale covers is called *p-étale topology*.

We observe that p-étale maps and covers are compatible with tilting and taking Witt vectors in the following sense:

Lemma 3.3.2. *Let $A = W(B)/\xi \to A' = W(B')/\xi$ be a p-étale map in* Perfd_R*. Then for all $r, l \in \mathbb{N}$ the induced maps $A/p^r \to A'/p^r$ and $W_l(B/\xi_0^r) \to W_l(B'/\xi_0^r)$ are étale as well. If $A \to A'$ is a p-étale cover, the other maps are surjective on spectra.*

Proof. The maps $A/p^r \to A'/p^r$ are étale by [**Lau18**] Lemma 8.11. The other assertion reduces to the case $l = 1$ by [**LZ04**] Proposition A.8. Then we may assume that $r = p^s$ for some $s \in \mathbb{N}$ and can conclude by considering the commutative square

$$\begin{tikzcd} B/\xi_0 \arrow[r] \arrow[d] & B'/\xi_0 \arrow[d] \\ B/\xi_0^{p^s} \arrow[r] & B'/\xi_0^{p^s}, \end{tikzcd}$$

where the vertical arrows are the isomorphisms induced by $\sigma^s, x \mapsto x^{p^s}$. Now assume that $A/p \to A'/p$ is faithfully flat. Clearly, $A/p^r \to A'/p^r$ is faithfully flat as well. Note that σ is surjective on B/ξ_0^r and hence $W_l(B/\xi_0^r)/pW_l(B/\xi_0^r) = B/\xi_0^r$. The maps $W_l(B/\xi_0^r) \to B/\xi_0^r \leftarrow B/\xi_0$ are homeomorphisms on spectra, which implies the second claim. $\square$

The following lemma shows descent for finite projective modules on Perfd_R with respect to the p-étale topology. The case $n = \infty$ (among further descent results) is proved in [**Lau18**] Lemma 10.9 and the other cases can be proved by similar arguments.

Lemma 3.3.3. *The functor*

$$\mathrm{Perfd}_R \to \mathrm{Cat}_1, \quad A \mapsto \mathrm{LF}(W_n(A^\flat)),$$

where Cat_1 denotes the 2-category of categories, is a stack for the p-étale topology on Perfd_R.

Proof. Let $n \in \mathbb{N}$. First note that $W_n(A^\flat) = \lim_k W_n(A^\flat/\xi_0^k)$, where the transition maps have nilpotent kernels because $[\xi_0^k]W_n(A^\flat) \subseteq W_n(\xi_0^k A^\flat) \subseteq [\xi_0^{k/p^{n-1}}]W_n(A^\flat)$ for all $k \in \mathbb{N}$. Hence, by [**SP**] Tag 0D4B, the natural functor $\mathrm{LF}(W_n(A^\flat)) \to 2\text{-}\lim_k \mathrm{LF}(W_n(A^\flat/\xi_0^k))$ is an equivalence. Therefore, it suffices to show that $A \mapsto \mathrm{LF}(W_n(A^\flat/\xi_0^k))$ is a p-étale stack for all $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and a p-étale cover of perfectoid R-algebras $A = W(B)/\xi \to A' = W(B')/\xi$. The map $W_n(B/\xi_0^k) \to W_n(B'/\xi_0^k)$ is an étale cover by Lemma 3.3.2. Moreover, the natural mappings $W_n(B'/\xi_0^k) \otimes_{W_n(B/\xi_0^k)} W_n(B'/\xi_0^k) \to W_n((B' \otimes_B B')/\xi_0^k)$ and $W_n(B'/\xi_0^k) \otimes_{W_n(B/\xi_0^k)} W_n(B'/\xi_0^k) \otimes_{W_n(B/\xi_0^k)} W_n(B'/\xi_0^k) \to W_n((B' \otimes_B B' \otimes_B B')/\xi_0^k)$ are isomorphisms by [**LZ04**] Corollary A.12. Consequently, the functor $A \mapsto \mathrm{LF}(W_n(A^\flat/\xi_0^k))$ is a p-étale stack on Perfd_R since we have faithfully flat descent for finite projective modules. The case $n = \infty$ now can be deduced by passage to the limit. $\square$

Remark 3.3.4. For a perfectoid R-algebra R' and $n \in \mathbb{N}$ let $\widetilde{\mathrm{BK}_\mathrm{n}}(R')$ denote the full subcategory of $\mathrm{BK}_{\mathrm{tor}}(R')$ consisting of those objects $(M, F, V) \in \mathrm{BK}_{\mathrm{tor}}(R')$ such that M is killed by p^n and finite projective as a $W_n(S)$-module. Moreover, for $n = \infty$ we set $\widetilde{\mathrm{BK}_n}(R') = \mathrm{BK}(R')$. By ignoring non-isomorphisms, we consider $\widetilde{\mathrm{BK}_\mathrm{n}}(R')$ as a groupoid. The lemma above shows that the functor

$$\mathrm{Perfd}_R \to \mathrm{Groupoids}, \ R' \mapsto \widetilde{\mathrm{BK}_\mathrm{n}}(R')$$

is a stack for the p-étale topology because we can descent finite projective modules and morphisms.

For $n \geq 2$ the categories $\widetilde{\mathrm{BK}_\mathrm{n}(R')}$ and $\mathrm{BK}_\mathrm{n}(R')$ are equal. Hence, to show that BK_n is a stack, we only have to care for descent properties in the case $n = 1$ to obtain the first part of the following theorem:

Proposition 3.3.5. *Let $n \in \mathbb{N} \cup \{\infty\}$. The functor*

$$\mathrm{BK_n}\colon \mathrm{Perfd}_R \to \mathrm{Groupoids},\ R' \mapsto \mathrm{BK_n}(R')$$

is a stack for the p-étale topology. Moreover, for fixed $0 \le d \le h$ the functor

$$\mathrm{BK_n^{(h,d)}}\colon \mathrm{Perfd}_R \to \mathrm{Groupoids},\ R' \mapsto \mathrm{BK_n^{(h,d)}}(R')$$

is a clopen substack of $\mathrm{BK_n}$.

Proof. By definition of height and rank for $\mathrm{BK_n}$-modules (via the height and rank of their associated $\mathrm{BK_1}$-module) and the previous remark, we may assume $n = 1$. For the first functor, the relevant properties live over R'/p and can be checked after faithfully flat base change. Moreover, the rank of a finite projective $S' = R'^\flat$-module is determined by its reduction to S'/ξ_0 because $(S', (\xi_0))$ is a Henselian pair. Therefore, also the relevant properties for the second functor can be checked after passing to a p-étale cover. The points $s \in \mathrm{Spec}(R'/p)$ such that $M \otimes_{R'^\flat} \kappa(s) \in \mathrm{BK_1}(\kappa(s))$ is of type (h, d) form a clopen subscheme of $\mathrm{Spec}(R'/p)$. Since $(R', (p))$ is a Henselian pair, the clopen subschemes of $\mathrm{Spec}(R')$ are in natural bijection to the clopen subschemes of $\mathrm{Spec}(R'/p)$ (cf. [**SP**] Tag 09XI) and hence the substack $\mathrm{BK_1^{(h,d)}} \subseteq \mathrm{BK_1}$ is indeed clopen. $\square$

After proving the following general lemma on stackification (which is a version of a lemma suggested by Lau), we will depict $\mathrm{BK_n^{(h,d)}}$ (and hence also $\mathrm{BT_n^{(h,d)}}$) as a quotient stack.

Lemma 3.3.6. *Let $\mathcal{C}$ be a site, let X' be a prestack on $\mathcal{C}$ with stackification X, let Y be a stack on $\mathcal{C}$ and let $f'\colon X' \to Y$ be a fully faithful functor. Then the induced functor $f\colon X \to Y$ is fully faithful. If moreover f' is locally essentially surjective (i.e. for all $A \in \mathcal{C}$ and $a \in Y(A)$ there exists a cover $h\colon A \to \tilde{A}$ such that $\tilde{a} = Y(h)(a)$ is in the essential image of $X'(\tilde{A}) \to Y(\tilde{A})$), then f is an isomorphism of stacks.*

Proof. The Hom-presheaves of X' are sheaves because f' is fully faithful and Y is a stack. This implies that f is fully faithful. Indeed, for some object A of $\mathcal{C}$ and two objects a, b of $X(A)$, the map $\mathrm{Hom}_X(a, b) \to \mathrm{Hom}_Y(a, b)$ is a map of sheaves over A, which is locally an isomorphism because locally a, b come from objects of $X'(A)$ and f' is fully faithful. Hence the map is an isomorphism. For the last assertion, let $a \in Y(A)$ and let $h\colon A \to \tilde{A}$ and $\tilde{a}$ be as in the statement. Since the relevant functors are fully faithful and X is a stack, we can descent an essential preimage of $\tilde{a}$ to obtain an essential preimage of a. $\square$

Theorem 3.3.7. *Let $n \in \mathbb{N} \cup \{\infty\}$ and $0 \leq d \leq h$ be natural numbers. We have isomorphisms of p-étale stacks*

$$[E \setminus G] \cong \mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})} \cong \mathrm{BT}_{\mathrm{n}}^{(\mathrm{h,d})},$$

where $[E \setminus G]$ denotes the p-étale stackification of the prestack $(E \setminus G)$ (cf. Proposition 3.2.11).

Proof. The functor $(E \setminus G) \to \mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})}$ is fully faithful and locally essentially surjective for the p-étale topology (which is finer then the p-Zariski topology) by Proposition 3.2.11. Therefore the claim follows from the lemma above. □

Remark 3.3.8.

1. The depiction of $\mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})}$ as a quotient stack can be obtained for any Grothendieck topology $\mathcal{T}$ on Perfd_R which is finer than or equal to the p-Zariski topology and satisfies gluing for the prestack $\mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})}$. In particular, for the p-Zariski topology.

2. In the case $n = 1$ we get the same results for the *p-faithfully flat topology* on Perfd_R, whose covers are given by maps which are faithfully flat modulo p.

4 The prestack of $\mathrm{BK_n}$-modules and the p-complete arc topology

We fix some perfectoid ring R with presentation $R = W(S)/\xi$, some $n \in \mathbb{N} \cup \{\infty\}$ and natural numbers $0 \leq d \leq h$. In the last chapter we have seen that the prestacks $\mathrm{BK_n}$ and $\mathrm{BK_n^{(h,d)}}$ satisfy descent with respect to the p-étale topology on Perfd_R. The description of $\mathrm{BK_n^{(h,d)}}$ as a quotient stack gives a concrete description of $\mathrm{BK_n}$-modules (and hence also of $\mathrm{BT_n}$-groups) over a perfectoid ring. In this chapter we are interested in descent behaviour of the prestack $\mathrm{BK_n^{(h,d)}}$ with respect to a finer topology which has a very easy basis.

We consider the *p-complete arc topology* defined by Česnavičius and Scholze in [**CS19**] 2.2.1., which is a variant of the *arc topology* introduced by Bhatt and Mathew in [**BM18**]. It is finer than the p-étale topology, admits a basis consisting of products of perfectoid valuation rings and behaves well with respect to tilting. Assuming a conjecture on gluing vector bundles with respect to this topology, we show that $\mathrm{BK_n}$ and $\mathrm{BK_n^{(h,d)}}$ are stacks in this case. As a formal consequence, we can again identify $\mathrm{BK_n^{(h,d)}}$ with the stackification of $(E \setminus G)$. We also approach the mentioned conjecture and prove it in the case of perfect rings.

4.1 General remarks on valuation rings

Following [**BM18**], we recall some general facts on valuation rings. Those rings serve as test objects for the p-complete arc topology on Perfd_R.

Remark 4.1.1. A ring map $f\colon V \to V'$ of valuation rings is called *extension of valuation rings* if the following equivalent conditions are satisfied:

(i) f is faithfully flat.

(ii) f is injective and local.

Remark 4.1.2. Let $g\colon V \to V'$ be an extension of valuation rings and assume $\operatorname{rank}(V) \leq 1$. Let $\mathfrak{m}$ be the maximal ideal of V. The set of prime ideals $\mathfrak{q}' \subseteq V'$ such that $g^{-1}(\mathfrak{q}') = \mathfrak{m}$ has a minimal element $\mathfrak{q}$ and the set of prime ideals $\mathfrak{p}' \subseteq V'$ such that $g^{-1}(\mathfrak{p}') = 0$ has a maximal element $\mathfrak{p}$. Then $V \to (V'/\mathfrak{p})_{\mathfrak{q}}$ is an extension of valuation rings and we have $\operatorname{rank}(V) = \operatorname{rank}((V'/\mathfrak{p})_{\mathfrak{q}})$ (cf. proof of [**BM18**] Proposition 2.1).

Lemma 4.1.3. *Let V be a valuation ring.*

1. *There exists an extension of valuation rings $V \to V'$ such that the fraction field of V' is algebraically closed. Moreover, if the rank of V is finite, we can achieve $\operatorname{rank}(V') = \operatorname{rank}(V)$.*
2. *Assume that V has rank 1 and let $0 \neq t$ be a non-unit in V. Then the t-adic completion $\hat{V}$ of V is a rank 1 valuation ring and $V \to \hat{V}$ is an extension of valuation rings.*

Proof. For part 1 we may choose V' to be the valuation ring of an extension of the valuation of V to $\overline{\operatorname{Frac}(V)}$ (cf. [**BM18**] Lemma 3.27). The assertion that the ranks of V and V' coincide in this case is proven in [**BouCA**] VI §8 Corollary 1 of Proposition 1. For part 2 see [**BM18**] Proposition 6.2. $\square$

The following lemma will ensure that perfectoid valuation rings suffice as test objects for the relevant topologies defined below.

Lemma 4.1.4. *Let V be a p-adically complete valuation ring with algebraically closed fraction field. Then V is perfectoid.*

Proof. If $p = 0$ the assertion is clear since in this case the notions of perfectoid and perfect coincide. So assume that p is a non-zero-divisor in V. Let $\varpi \in V$ with $\varpi^p = p$. Then V is ϖ-adically complete and the map

$$V/\varpi V \to V/\varpi^p V,\ x \mapsto x^p$$

is bijective. Indeed if $x^p \in (\varpi)^p$, then $v(x^p) \leq v(\varpi^p)$ and hence $v(x) \leq v(\varpi)$ which means $x \in (\varpi)$. Moreover, for all $y \in V$ there is some $x \in V$ with $x^p = y$. In particular, the map is surjective. Then the lemma follows from [**BMS18**] Lemma 3.10. $\square$

4.2 The v topology and p-complete arc topology

In this subsection we follow [**BM18**] and [**CS19**] §2.2.1 to define the v topology and the p-complete arc topology and to restate some of their remarks.

4.2.1 Definitions and first properties

Definition 4.2.1. Let $A \to A'$ be a map of rings and let $\varpi \in A$.

1. $A \to A'$ is called *v cover* (cf. [**BM18**] Definition 1.1 (2)) if for any ring map $A \to V$, where V is a valuation ring, there exists an extension of valuation rings $V \to V'$ fitting into a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ V \arrow[r] & V'. \end{tikzcd}$$

2. $A \to A'$ is called *ϖ-complete arc cover* (cf. [**CS19**] 2.2.1) if for any ring map $A \to V$, where V is a ϖ-adically complete valuation ring of rank ≤ 1, there exists an extension of valuation rings $V \to V'$ fitting into a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ V \arrow[r] & V'. \end{tikzcd}$$

3. $A \to A'$ is called *arc cover* if it is a 0-complete arc cover.

If ϖ and ϖ' are two elements of a ring A such that $\varpi' \mid \varpi$, then for an A-algebra A' being ϖ'-adically complete implies being ϖ-adically complete (cf. [**SP**] Tag 090T). In particular, any ϖ-complete arc cover also is a ϖ'-complete arc cover in this case.

Remark 4.2.2. (cf. [**CS19**] 2.2.1) Let $A \to A'$ be an arc cover and $A \to V$ be a ring map from A to a valuation ring V of rank ≤ 1.

1. By Remark 4.1.2 we can always find an extension of rank ≤ 1 valuation rings $V \to V'$ and a morphism $A' \to V'$ making the relevant diagram commute. In particular, one could demand in Definition 4.2.1 2. that V' has rank ≤ 1.

2. By Lemma 4.1.3 1. we can always find an extension of rank ≤ 1 valuation rings $V \to V'$ and a morphism $A' \to V'$ making the respective diagram commute, where $\mathrm{Quot}(V')$ is algebraically closed. Consequently, we also do not lose any generality when we assume $\mathrm{Quot}(V)$ to be algebraically closed in the first place.

3. If $\varpi \in A$ and $A \to A'$ is a ϖ-complete arc cover, we can also pass to ϖ-adic completions of V and V' by Lemma 4.1.3 2..

Since ϖ-adic completion preserves algebraic closedness (cf. [**BGR84**], §3.4), we can draw the following conclusion (again cf. [**CS19**] 2.2.1):

A map of rings $A \to A'$ is a ϖ-complete arc cover if and only if for any ring map $A \to V$, where V is a ϖ-adically complete rank ≤ 1 valuation ring with algebraically closed fraction field, there exists an extension of ϖ-adically complete rank ≤ 1 valuation rings $V \to V'$, where V' has an algebraically closed fraction field, fitting into a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ V \arrow[r] & V'. \end{tikzcd}$$

Similarly, a map of rings $A \to A'$ is a v cover if and only if for any ring map $A \to V$, where V is a valuation ring with algebraically closed fraction field, there exists an extension of valuation rings $V \to V'$, where V' has an algebraically closed fraction field, fitting into a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ V \arrow[r] & V'. \end{tikzcd}$$

The following examples are taken from [**CS19**] 2.2.1.

Example 4.2.3. Let $A \to A'$ be a map of perfectoid rings.

1. If $A \to A'$ is faithfully flat, then it is an arc cover.
2. If $A \to A'$ is faithfully flat modulo powers of p, then it is a p-complete arc cover. In particular, every p-étale cover is a p-complete arc cover.

Remark 4.2.4. Let A be a ring and $\varpi \in A$.

1. If $A \to A'$ is an isomorphism, then it is a v cover (resp. a ϖ-complete arc cover).
2. If $A \to A'$ is a v cover (resp. a ϖ-complete arc cover) and $A \to B$ is a map of rings, then $B \to B \otimes_A A'$ (resp. $B \to B \widehat{\otimes}_A A'$ (ϖ-adic completion)) is a v cover (resp. a ϖ-complete arc cover).
3. If $A \to A'$ and $A' \to A''$ are v covers (resp. ϖ-complete arc covers), then $A \to A''$ is a v cover (resp. a ϖ-complete arc cover).

By the remark above, for a ring R (resp. for an $\mathbb{F}_p$-algebra S) we obtain Grothendieck topologies on Ring_R and Perfd_R, (resp. on Perf_S). In particular, we will always endow Perfd_R with the p-complete arc topology.

Example 4.2.5. The arc topology is strictly finer than the v topology: Let V be a valuation ring and $\mathfrak{p} \subseteq V$ be a prime ideal. Then $V \to V_{\mathfrak{p}} \times V/\mathfrak{p}$ can be shown to be an arc cover (cf. [**BM18**] Corollary 2.9). If $0 \neq \mathfrak{p}$ is a non maximal prime, then $V \to V_{\mathfrak{p}} \times V/\mathfrak{p}$ cannot be a v cover (note that in this case V has rank ≥ 2). Indeed, assume that there exists an extension of valuation rings $V \to W$ such that the diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\mathrm{id}_V"'] & V_{\mathfrak{p}} \times V/\mathfrak{p} \arrow[d] \\ V \arrow[r] & W \end{tikzcd}$$

commutes. Since $\mathrm{Spec}(W)$ is connected the map $V_{\mathfrak{p}} \times V/\mathfrak{p} \to W$ has to factor through $V_{\mathfrak{p}}$ or $V/\mathfrak{p}$. But this is impossible since $\mathrm{Spec}(V_{\mathfrak{p}}) \to \mathrm{Spec}(V)$ and $\mathrm{Spec}(V/\mathfrak{p}) \to \mathrm{Spec}(V)$ are not surjective in this case.

The following lemma will be important for reducing questions about ϖ-complete arc covers to arc covers.

Lemma 4.2.6. *Let $A \to A'$ be a ϖ-complete arc cover for some $\varpi \in A$. Then $A \to A' \times A[\frac{1}{\varpi}]$ is an arc cover.*

Proof. If $\varpi = 0$, there is nothing to show. Let $A \to V$ be a map from A to a rank ≤ 1 valuation ring V. If ϖ is a unit in V we may choose $V' = V$ and $A' \times A[\frac{1}{\varpi}] \to A[\frac{1}{\varpi}] \to V'$ to create a commutative square. So we may assume that ϖ is not a unit in V (and in particular the rank of V is equal to 1). Then we are done by Lemma 4.1.3 2. □

The following lemma ensures in combination with Lemma 4.1.4 that the p-complete arc topology on Perfd_R has a basis consisting of products of perfectoid valuation rings of rank ≤ 1. In particular, it has a basis of products of perfectoid local rings for which we obtained an explicit description of $\mathrm{BK}_{\mathrm{n}}^{(\mathrm{h,d})}$ in Chapter 3.

Lemma 4.2.7. *Every ring A with a $\varpi \in A$ has a ϖ-complete arc cover $A \to \prod_{i \in I} V_i$ with the V_i being ϖ-adically complete valuation rings of rank ≤ 1 with algebraically closed fraction fields.*

Proof. [**CS19**] Lemma 2.2.3. □

4.2.2 Behaviour under tilting and colimit perfection

We now discuss how ϖ-complete arc covers and v covers behave under tilting and colimit perfection of rings. It turns out that they are not affected by either of these operations.

Lemma 4.2.8. *A map $A \to A'$ of perfectoid rings is a ϖ-complete arc cover for a $\varpi \in A$ with $\varpi^p \mid p$ such that A and A' are ϖ-adically complete if and only if $A^\flat \to A'^\flat$ is a $\varpi^\flat$-complete arc cover (where as before $\varpi^\flat \in A^\flat$ is some element such that $(\varpi^\flat)^\sharp$ is a unit multiple of ϖ, cf. Remark 1.1.4).*

Proof. **[CS19]** Lemma 2.2.2. ☐

Lemma 4.2.9. *Let $f\colon A \to A'$ be a map of $\mathbb{F}_p$-algebras. Then f is an arc cover (resp. v cover) if and only if $f_{\mathrm{perf}}\colon A_{\mathrm{perf}} \to A'_{\mathrm{perf}}$ is an arc cover (resp. v cover).*

Proof. We only prove the case of arc covers since the other proof is similar. Assume that f is an arc cover and consider a map $A_{\mathrm{perf}} \to V$ where V is a rank ≤ 1 valuation ring with algebraically closed fraction field (in particular, V is perfect). Choose some extension of rank ≤ 1 valuation rings with algebraically closed fraction field $V \to V'$ and a morphism $A' \to V'$ such that

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & A' \arrow[d] \arrow[ddr] & \\
A_{\mathrm{perf}} \arrow[r, "f_{\mathrm{perf}}"] \arrow[d] & A'_{\mathrm{perf}} & \\
V \arrow[rr] & & V'
\end{tikzcd}$$

commutes. But V' is perfect as well and hence we obtain a unique map $A'_{\mathrm{perf}} \to V'$ which makes the diagram

$$\begin{tikzcd}
A_{\mathrm{perf}} \arrow[r, "f_{\mathrm{perf}}"] \arrow[d] & A'_{\mathrm{perf}} \arrow[d] \\
V \arrow[r] & V'
\end{tikzcd}$$

commute since this can be tested by pre-composing with $A \to A_{\mathrm{perf}}$ (universal property of $A \to A_{\mathrm{perf}}$).
On the other hand, assume that f_{perf} is an arc cover. Then the claim follows similarly by the fact that every map $A \to V$ for a perfect valuation ring V factors as $A \to A_{\mathrm{perf}} \to V$. ☐

4.3 p-complete arc descent

Fix some perfectoid ring R with presentation $R = W(S)/\xi$ and some $n \in \mathbb{N} \cup \{\infty\}$. In this chapter we consider the question of p-complete arc descent for the functor

$$\mathrm{Perfd}_R \to \mathrm{Groupoids}, \quad R' \mapsto \mathrm{BK_n}(R').$$

We show this to have an affirmative answer if we knew

$$\mathrm{Perfd}_R \to \mathrm{Cat}_1, \quad R' \mapsto \mathrm{LF}(W_n(R'^\flat))$$

to be a p-complete arc sheaf. The latter in turn holds true if we assume

Conjecture A: Let A be a perfectoid ring with presentation $A = W(S)/\xi$ and let $\widehat{(\ )}$ denote ξ_0-adic completion. Then the functor $\mathrm{Perf}_S \to \mathrm{Cat}_1, S' \mapsto \mathrm{LF}(W_n(\widehat{S'}))$ is an arc sheaf.

If Conjecture A holds for all $n \in \mathbb{N}$, it also holds for $n = \infty$ since 2-limits of sheaves are sheaves and $\mathrm{LF}(\quad)$ commutes with cofiltered limits whose transition maps have nilpotent kernels (cf. [**SP**] Tag 0D4B).

After recalling general facts about descent in our context, we adapt arguments of the proof of [**CS19**] Lemma 4.2.6 to approach Conjecture A and present partial results on its proof. In particular, we give a full proof over perfect rings. Finally, we show the claimed implications of Conjecture A and in particular depict $\mathrm{BK_n^{(h,d)}}$ and $\mathrm{BT_n^{(h,d)}}$ as the quotient stack $[E \setminus G]$ for the p-complete arc topology.

4.3.1 General remarks on p-complete arc descent

In this subsection we follow [**BM18**] §3. Let R be a ring. For a map of rings $A \to A'$ in Ring_R (resp. a map of perfectoid rings $A \to A'$ in Perfd_R) we write $A'^{\bullet/A}$ for the Čech nerve

$$A' \rightrightarrows A' \otimes_A A' \substack{\to\\\to\\\to} \dots \text{ (resp. } A' \rightrightarrows A'\widehat{\otimes}_A A' \substack{\to\\\to\\\to} \dots)$$

(here $\widehat{\otimes}$ denotes the p-adically completed tensor product, which ensures that we obtain pushouts in Perfd_R as discussed in Remark 1.1.9).
Let $\mathcal{C}$ be an ∞-category that has small limits. The only ∞-categories which are relevant for this paper are Cat_1, the $(2,1)$-category of 1-categories, where we only admit isomorphisms of functors as 2-morphisms, and $\mathrm{Groupoids}$, the $(2,1)$-category of 1-groupoids. We just

use the general setting to match notation of [**BM18**].
For a functor $\mathcal{F}\colon \mathrm{Ring}_R \to \mathcal{C}$ (resp. $\mathcal{F}\colon \mathrm{Perfd}_R \to \mathcal{C}$) we write $\mathcal{F}(A'^{\bullet/A})$ for the termwise application of $\mathcal{F}$ to $A'^{\bullet/A}$.
If $pR = 0$, the category Perf_R of perfect R-algebras will also play an important role for our considerations. Since $\mathrm{Perf}_R \subseteq \mathrm{Ring}_R$ preserves pushouts, we proceed in this case with the same definitions as for Ring_R.

Definition 4.3.1. Let R be a ring and $\varpi \in R$ be an element.

1. A functor $\mathcal{F}\colon \mathrm{Ring}_R \to \mathcal{C}$ is said to be a v *sheaf* (respective ϖ*-complete arc sheaf*) if $\mathcal{F}$ carries finite products to finite products and for every v cover (resp. ϖ-complete arc cover) $A \to A'$ in Ring_R the natural map

$$\mathcal{F}(A) \to \lim(\mathcal{F}(A'^{\bullet/A}))$$

   is an equivalence.

2. We make analogous definitions as in 1. with Ring_R substituted by Perfd_R.

3. A functor $\mathcal{F}\colon \mathrm{Ring}_R \to \mathcal{C}$ is called *finitary* if for every tower of R-algebras $(R_i)_{i\in I}$ (indexed over some filtered partially ordered set), we have

$$\mathcal{F}(\varinjlim_i R_i) \cong \varinjlim_i \mathcal{F}(R_i).$$

If $\mathcal{F}$ is a functor which is a v sheaf (resp. is a ϖ-complete arc sheaf), we will also say that $\mathcal{F}$ satisfies v *descent* (resp. ϖ*-complete arc descent*).

Remark 4.3.2. If we assume $\mathcal{C} = \mathrm{Cat}_1$, the limits and filtered colimits of $\mathcal{C}$ considered as an ∞-category coincide with the 2-limits and filtered 2-colimits of Cat_1. Indeed, in both cases the limits and colimits can be calculated as homotopy limits and homotopy colimits, where equivalences of categories serve as weak equivalences. Hence the 2-limits and 2-colimits are characterized by homotopy limits of 1-truncated spaces, whereas limits and colimits in the ∞-categorical setting are described by homotopy limits of (not necessarily truncated) spaces. However, the full subcategory of spaces which is spanned by 1-truncated spaces is stable under limits (cf. [**HTT**] Proposition 5.5.6.5) and filtered colimits (cf. [**HTT**] Example 7.3.4.4), showing the claim. Hence in the case $\mathcal{C} = \mathrm{Cat}_1$, the objects of the $\lim$-category in the definition of sheaves above are given by pairs (M, α), where M is an object of $\mathcal{F}(A')$ and α is an isomorphism of M over $A' \otimes_A A'$ (resp. over $A'\widehat{\otimes}_A A'$) satisfying a cocycle condition over $A' \otimes_A A' \otimes_A A'$ (resp. over $A'\widehat{\otimes}_A A'\widehat{\otimes}_A A'$). In particular, the sheaf condition in this case is the same as the usual condition for stacks.

Remark 4.3.3. 1. Every v sheaf (resp. ϖ-complete arc sheaf) with values in Cat_1 or $\mathrm{Groupoids}$ automatically satisfies v descent (resp. ϖ-complete arc descent) with respect to hypercoverings. Indeed, this holds if and only if $\mathrm{Hom}_{\mathcal{C}}(x, \cdot)$ satisfies hyperdescent for all objects $x \in \mathcal{C}$. But for any $k \in \mathbb{N}$ sheaves of k-truncated spaces are automatically hypercomplete, cf. [**HTT**], Sec. 6.5.3.

2. The $(2, 1)$-category Cat_1 is *compactly generated by cotruncated objects* (cf. [**BM18**] Example 3.5 (3)). Hence the results of the following theorem, which characterize the difference between v sheaves and arc sheaves in terms of the arc covers described in Example 4.2.5, holds for the functors we consider in the upcoming sections.

Theorem 4.3.4. *Let R be a ring, $\mathcal{C}$ an ∞-category which is compactly generated by cotruncated objects and let $\mathcal{F}\colon \mathrm{Ring}_R \to \mathcal{C}$ be a finitary functor satisfying v descent. Then the following are equivalent:*

(i) $\mathcal{F}$ satisfies arc descent.

(ii) $\mathcal{F}$ satisfies aic-v-excision, i.e. for every valuation ring V with algebraically closed fraction field and prime ideal $\mathfrak{p} \subseteq V$, the square in $\mathcal{C}$

$$\begin{array}{ccc} \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V/\mathfrak{p}) \\ \downarrow & & \downarrow \\ \mathcal{F}(V_{\mathfrak{p}}) & \longrightarrow & \mathcal{F}(\kappa(\mathfrak{p})) \end{array}$$

is cartesian.

Proof. [**BM18**] Theorem 4.1. □

For the sake of clarity, we will assume from now on that $\mathcal{C} = \mathrm{Cat}_1$ or $\mathcal{C} = \mathrm{Groupoids}$.

We need the following lemma which provides a criterion to decide when an arc sheaf already is a ϖ-complete arc sheaf.

Lemma 4.3.5. *Let R be a ring and let $\mathcal{F}\colon \mathrm{Ring}_R \to \mathcal{C}$ be a functor which carries finite products to finite products. Fix some $\varpi \in R$ and assume that $\mathcal{F}(S[\frac{1}{\varpi}])$ is the terminal category for all R-algebras S. Then $\mathcal{F}$ satisfies ϖ-complete arc descent if and only if it satisfies arc descent.*

Proof. One direction is clear. So assume that $\mathcal{F}$ satisfies arc descent and let $S \to S'$ be a ϖ-complete arc cover. By Lemma 4.2.6, $S \to S'' := S' \times S[\frac{1}{\varpi}]$ is an arc cover. Since tensor products commute with finite products of rings, we have

$$\mathcal{F}(S'^{\bullet/S}) \cong \mathcal{F}(S''^{\bullet/S})$$

and consequently

$$2\text{-}\lim(\mathcal{F}(S'^{\bullet/S}) \cong 2\text{-}\lim(\mathcal{F}(S''^{\bullet/S})) \cong \mathcal{F}(S)$$

as desired.

□

The following lemma reduces descent questions over Perf_R to questions over Ring_R and vice versa by use of the colimit perfection.

Lemma 4.3.6. *(cf. [**BS17**] Proposition 4.5) Let $\mathcal{F}\colon \mathrm{Perf}_{\mathbb{F}_p} \to \mathcal{C}$ be a functor and let $\mathcal{F}'\colon \mathrm{Ring}_{\mathbb{F}_p} \to \mathcal{C}$ be the functor defined by $\mathcal{F}'(S) = \mathcal{F}(S_{\mathrm{perf}})$. Then $\mathcal{F}$ is a v sheaf (resp. an arc sheaf) if and only if $\mathcal{F}'$ is a v sheaf (resp. an arc sheaf).*

Proof. In the case of v descent, this is verbatim the proof of [**BS17**] Proposition 4.5, where the target category is the infinity category of spaces. For arc descent we only have to use the additional fact that for an arc cover $f\colon S \to S'$ the colimit perfection $f_{\mathrm{perf}}\colon S_{\mathrm{perf}} \to S'_{\mathrm{perf}}$ is an arc cover as well. But this was proved in Lemma 4.2.9. □

4.3.2 Notes on p-complete arc descent for finite projective modules

We would like to show that the functor

$$\mathrm{Perfd}_{\mathbb{Z}_p} \to \mathrm{Cat}_1, \quad A' \mapsto \mathrm{LF}(W_n(A'^{\flat}))$$

satisfies p-complete arc descent for $n \in \mathbb{N} \cup \{\infty\}$ (where we again use the convention $W_\infty(\ ) = W(\ )$). Fixing some perfectoid ring R with presentation $R = W(R^\flat)/\xi$ and passing to the tilt (cf. Lemma 1.1.8 and Lemma 4.2.8), this question reduces to showing that the functor

$$\mathrm{Perf}_{R^\flat} \to \mathrm{Cat}_1, S' \mapsto \mathrm{LF}(W_n(\widehat{S'}))$$

(where $\widehat{(\ )}$ denotes ξ_0-adic completion) satisfies ξ_0-complete arc descent. Then by Lemma 4.3.5 we only have to check the sheaf property for arc covers and hence it is left to establish the validity of

Conjecture A: Let A be a perfectoid ring with presentation $A = W(S)/\xi$ and let $\widehat{(\ )}$ denote ξ_0-adic completion. Then the functor $\mathrm{Perf}_S \to \mathrm{Cat}_1, S' \mapsto \mathrm{LF}(W_n(\widehat{S'}))$ is an arc sheaf.

This proves the following:

Proposition 4.3.7. *If we assume Conjecture A, then the functor*

$$\mathrm{Perfd}_{\mathbb{Z}_p} \to \mathrm{Cat}_1, A' \mapsto \mathrm{LF}(W_n(A'^{\flat}))$$

satisfies p-complete arc descent.

In this subsection we approach Conjecture A. First, we establish its validity for perfect rings (i.e. where $\xi_0 = 0$ and we do not have to pass to the completion). Here we use the results on perfect schemes obtained by Bhatt and Scholze in [**BS17**] and combine them with Theorem 4.3.4.

Note that every faithfully flat map of rings is an arc cover (cf. [**CS**] 2.2.1. Example (1)). Therefore our results generalize gluing for finite projective modules along faithfully flat maps.

Theorem 4.3.8. *(Conjecture A for perfect rings) The functor*

$$\mathrm{Perf}_{\mathbb{F}_p} \to \mathrm{Cat}_1, \quad S' \mapsto \mathrm{LF}(W_n(S'))$$

is an arc sheaf.

Proof. By Lemma 4.3.6 we have to show that

$$\mathrm{Ring}_{\mathbb{F}_p} \to \mathrm{Cat}_1, \quad S' \mapsto \mathrm{LF}(W_n(S'_{\mathrm{perf}}))$$

satisfies arc descent. Note that this functor is a finitary v sheaf which satisfies aic-v-excision (all of this properties are explained below) and hence it is an arc sheaf by Theorem 4.3.4. □

Lemma 4.3.9. *The functor*

$$\mathrm{Ring}_{\mathbb{F}_p} \to \mathrm{Cat}_1, \quad S' \mapsto \mathrm{LF}(W_n(S'_{\mathrm{perf}}))$$

is finitary.

Proof. This follows from the general lemma on colimits of finite projective modules below and the fact that filtered colimits commute with colimit perfections and filtered colimits of perfect rings commute with $W_n(\quad)$. □

Lemma 4.3.10. *Suppose that $A = \mathrm{colim}_{\lambda\in\Lambda} A_\lambda$ is a directed colimit of rings. Then we have an equivalence of categories*

$$\mathrm{LF}(A) \cong 2\text{-}\,\mathrm{colim}_\lambda \mathrm{LF}(A_\lambda).$$

Proof. Cf. [**SP**] Tag 05N7 and for the flat part [**SP**] Tag 02JO(3). □

The following theorem is a direct consequence of the results obtained in [**BS17**] §4.

Theorem 4.3.11. *The functor $\mathrm{Ring}_{\mathbb{F}_p} \to \mathrm{Cat}_1,\ S' \mapsto \mathrm{LF}(W_n(S'_{\mathrm{perf}}))$ is a v sheaf. Equivalently, (by Lemma 4.3.6) the functor $\mathrm{Perf}_{\mathbb{F}_p} \to \mathrm{Cat}_1,\ S' \mapsto \mathrm{LF}(W_n(S'))$ is a v sheaf.*

Proof. In [**BS17**] Theorem 4.1.(ii) it is proved that assigning to a perfect ring S' the groupoid of finite projective $W_n(S')$-modules is a v sheaf. By [**BS17**] Theorem 4.1.(i) for a perfect ring S and fixed finite projective $W_n(S)$-module M the functor $S' \mapsto M \otimes_{W_n(S)} W_n(S')$ is a v sheaf on Perf_S. As in [**BS17**] Corollary 4.4. this can be applied to Hom-bundles to deduce descent for homomorphisms of finite projective modules as well. □

Lemma 4.3.12. *The functor $\mathrm{Ring}_{\mathbb{F}_p} \to \mathrm{Cat}_1, S \mapsto \mathrm{LF}(W_n(S_{\mathrm{perf}}))$ satisfies aic-v-excision.*

Proof. Let V be a valuation ring with algebraically closed fraction field and let $\mathfrak{p}$ be a prime ideal of V. Since $V = V/\mathfrak{p} \times_{\kappa(\mathfrak{p})} V_\mathfrak{p}$ (cf. [**BM18**] Proposition 2.8), the claim of the lemma follows from [**SP**] Tag 0D2J and the fact that fibre products commute with colimit perfection and fibre products of perfect rings commute with $W_n(\quad)$. □

We provide further explanations on Conjecture A, where we for simplicity restrict to the case $n = 1$. Let A be a perfectoid ring with presentation $A = W(S)/\xi$ and let $\widehat{(\ )}$ denote ξ_0-adic completion.

Remark 4.3.13. If the functors

$$\mathrm{Perf}_S \to \mathrm{Cat}_1, \quad S' \mapsto \mathrm{LF}(S'/\xi_0^k)$$

are arc sheaves for all $k \in \mathbb{N}$, then

$$\mathrm{Perf}_S \to \mathrm{Cat}_1, \quad S' \mapsto \mathrm{LF}(\widehat{S'})$$

also is an arc sheaf. This is true because 2-limits of sheaves are sheaves and we have $\mathrm{LF}(\widehat{S'}) = 2\text{-}\lim_k(\mathrm{LF}(S'/\xi_0^k))$ by [**SP**] Tag 0D4B. However this does not imply that Conjecture A is an immediate consequence of Theorem 4.3.8 (Conjecture A for perfect rings) because the rings S'/ξ_0^k are not perfect.

Remark 4.3.14. Assume we are given an arc cover $S \to S'$. Let $\mathrm{DD}\left(\widehat{S}'/\widehat{S}\right)$ denote the category of finite projective $\widehat{S}'$-modules which are endowed with a descent datum (with respect to $\widehat{S} \to \widehat{S}'$). To verify the sheaf condition of Conjecture A in this case, we have to show that the functor

$$\mathrm{LF}(\widehat{S}) \to \mathrm{DD}\left(\widehat{S}'/\widehat{S}\right)$$

mapping a finite projective $\widehat{S}$-module M to $M \otimes_{\widehat{S}} \widehat{S}'$, endowed with the canonical descent datum, is an equivalence. We know this to be fully faithful because the functor $S \mapsto \widehat{S}$ is an arc sheaf for the arc topology on Perf_S (cf. proof of [**CS19**] Lemma 4.2.6).

4.3.3 p-complete arc descent and $\mathrm{BK_n}$-modules

We fix some perfectoid ring R with presentation $R = W(S)/\xi$ and evaluate the consequences of our findings for $\mathrm{BK_n}$-modules.

Theorem 4.3.15. *Assume that Conjecture A holds or that R is perfect. Let $n \in \mathbb{N} \cup \{\infty\}$. The functor*

$$\mathrm{BK_n}\colon\ \mathrm{Perfd}_R \to \mathrm{Groupoids},\ R' \mapsto \mathrm{BK_n}(R')$$

is a stack for the p-complete arc topology. Moreover, for fixed $0 \le d \le h$ the functor

$$\mathrm{BK_n^{(h,d)}}\colon\ \mathrm{Perfd}_R \to \mathrm{Groupoids},\ R' \mapsto \mathrm{BK_n^{(h,d)}}(R')$$

is a clopen substack of $\mathrm{BK_n}$.

Proof. By definition of height and rank for $\mathrm{BK_n}$-modules (via the height and rank of their associated $\mathrm{BK_1}$-module) and Remark 3.3.4, we may assume $n = 1$. Let $A \to A'$ be a p-complete arc cover of perfectoid R-algebras. Let $(M, F, V) \in \widetilde{\mathrm{BK_1}}(A)$ and assume that the base change of M along $A \to A'$ to an object of $\widetilde{\mathrm{BK_1}}(A')$ is a $\mathrm{BK_1}$-module over A'. We want to show that M is a $\mathrm{BK_1}$-module over A in this case. This can be checked on residue fields of maximal ideals (cf. Proposition 2.1.15). So pick such a residue field κ, which automatically is a perfect field of characteristic p. The element $\xi_0 \in A^\flat$ is mapped to 0 under $A^\flat \to \kappa$ because $A^\flat$ is ξ_0-adically complete. The fact that $A^\flat \to A'^\flat$ is a ξ_0-complete arc cover (cf. Lemma 4.2.8) provides an algebraically closed (in particular perfect) field κ' fitting into a commutative diagram

$$\begin{tikzcd} A^\flat \arrow[r] \arrow[d] & A'^\flat \arrow[d] \\ \kappa \arrow[r] & \kappa'. \end{tikzcd}$$

Hence we may assume $A = \kappa = A^\flat$ and $A' = \kappa' = A'^\flat$ and $A \to A'$ being faithfully flat. So exactness of the relevant sequence can be checked after base change to κ', where it holds by assumption. Moreover, since ranks of finite projective modules are determined by their fibres, we can use the same argument to see that $\mathrm{BK}_1^{(\mathrm{h,d})}$ is a substack of BK_1. Finally, let R' be a perfectoid R-algebra and $M \in \mathrm{BK}_1(R')$. The points $s \in \mathrm{Spec}(R'/p)$ such that $M \otimes_{R'^\flat} \kappa(s) \in \mathrm{BK}_1(\kappa(s))$ is of type (h,d) form a clopen subscheme of $\mathrm{Spec}(R'/p)$. Since $(R', (p))$ is a Henselian pair, the clopen subschemes of $\mathrm{Spec}(R')$ are in natural bijection to the clopen subschemes of $\mathrm{Spec}(R'/p)$ (cf. [**SP**] Tag 09XI) and hence the substack $\mathrm{BK}_1^{(\mathrm{h,d})} \subseteq \mathrm{BK}_1$ is indeed clopen. □

In view of Theorem 2.2.11 and Theorem 1.2.2 we obtain analogous results for truncated Barsotti-Tate groups and Barsotti-Tate groups:

Theorem 4.3.16. *Assume that Conjecture A holds or that R is perfect. Let $n \in \mathbb{N} \cup \{\infty\}$. The functor*

$$\mathrm{BT_n} \colon \mathrm{Perfd}_R \to \mathrm{Groupoids}, \ R' \mapsto \mathrm{BT_n}(R')$$

is a stack for the p-complete arc topology. Moreover, for fixed $0 \le d \le h$ the functor

$$\mathrm{BT_n^{(h,d)}} \colon \mathrm{Perfd}_R \to \mathrm{Groupoids}, \ R' \mapsto \mathrm{BT_n^{(h,d)}}(R')$$

is a clopen substack of $\mathrm{BT_n}$.

Theorem 4.3.17. *Assume that Conjecture A holds or that R is perfect. Let $n \in \mathbb{N} \cup \{\infty\}$ and $0 \le d \le h$ be natural numbers. We have isomorphisms of p-complete arc stacks*

$$[E \setminus G] \cong \mathrm{BK_n^{(h,d)}} \cong \mathrm{BT_n^{(h,d)}},$$

where $[E \setminus G]$ denotes the p-complete arc stackification of the prestack $(E \setminus G)$ (cf. Proposition 3.2.11).

Proof. Since $\mathrm{BK_n^{(h,d)}}$ is a sheaf in this case, the claim follows from Lemma 3.3.6 and Proposition 3.2.11. □

Remark 4.3.18. Since every faithfully flat map of rings is an arc cover, in the case of perfect rings this result extends the classification of F-Zips (cf. Theorem 3.1.3).

4.4 The v topology for perfectoid spaces

There are analytic analogues of the questions considered in this chapter, namely those that arise in the world of perfectoid spaces. We give a brief discussion following [**SW19**].

Let $\mathrm{Perfd}^{\mathrm{a}}$ denote the category of *perfectoid Huber pairs*. As noted in Remark 1.1.5, for a perfectoid Huber pair (R, R^+) the ring of integral elements R^+ is a perfectoid ring in the sense of Definition 1.1.1.

A map $(R, R^+) \to (R', R'^+)$ in $\mathrm{Perfd}^{\mathrm{a}}$ is called a v *cover*, if the associated map on adic spectra $\mathrm{Spa}(R', R'^+) \to \mathrm{Spa}(R, R^+)$ is surjective. This notion is related to our prior definition of v covers for perfectoid rings:
A map of perfectoid rings $A \to A'$ is a v cover in $\mathrm{Perfd}_{\mathbb{Z}_p}$ if and only if the induced map $\mathrm{Spa}(A', A') \to \mathrm{Spa}(A, A)$ is surjective.

Similar to our previous explanation, we can consider the prestack

$$\mathrm{Perfd}^{\mathrm{a}} \to \mathrm{Groupoids}, \quad (R, R^+) \mapsto \mathrm{BK}_1(R^+),$$

and ask if this is a stack for the v topology. Once this was proved, one could go on and try to construct an associated *diamond* in the sense of Scholze (cf. [**SW19**] 8.3). However, it is not clear to the author whether or not the prestack

$$\mathrm{Perfd}^{\mathrm{a}} \to \mathrm{Cat}_1, (R, R^+) \mapsto \mathrm{LF}(R^+)$$

even is a stack for the *analytic topology* on $\mathrm{Perfd}^{\mathrm{a}}$, which is coarser than the v topology. The corresponding non-integral statement is true, i.e. the prestack

$$\mathrm{Perfd}^{\mathrm{a}} \to \mathrm{Cat}_1, (R, R^+) \mapsto \mathrm{LF}(R)$$

is a stack for the analytic topology (cf. [**Ked17**] Theorem 1.4.2) and even for the v topology (cf. [**SW19**] Lemma 17.1.8).
The main problem when trying to adapt the proof to an integral version is the fact that the integral structure sheaf of an affinoid perfectoid space is not *acyclic* but only satisfies *almost vanishing* for its higher cohomology groups (cf. [**Sch12**] Theorem 6.3). This is closely related to the observation that the pushout of the diagram

$$(B, B^+) \leftarrow (A, A^+) \to (C, C^+)$$

in $\mathrm{Perfd}^{\mathrm{a}}$ is given by the perfectoid Huber pair (D, D^+), where D^+ is the completion of the *integral closure* of $B^+ \otimes_{A^+} C^+$ in $B \otimes_A C$ (cf. [**SW19**] Proposition 5.1.5. (2)). In particular, D^+ is in general not equal to the pushout of

$$B^+ \leftarrow A^+ \rightarrow C^+$$

in $\mathrm{Perfd}_{\mathbb{Z}_p}$, which is the p-adic completion of $B^+ \otimes_{A^+} C^+$.

Bibliography

[ALB19] Johannes Anschütz and Arthur-César Le Bras, *Prismatic Dieudonné theory*, preprint (2019). Available at https://arxiv.org/abs/1907.10525v2.

[BBM82] Pierre Berthelot, Lawrence Breen, William Messing, *Théorie de Dieudonné cristalline II*. Lecture Notes in Mathematics, vol. 930, Springer-Verlag, Berlin, 1982 (French).

[BGR84] S. Bosch, U. Günzer and R. Remmert, *Non-archimedean analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 261, Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry.

[BIM19] Bhargav Bhatt, Srikanth B. Iyengar and Linquan Ma, *Regular rings and perfect(oid) algebras*, Comm. Algebra 47 (2019), no. 6, 2367–2383, DOI 10.1080/00927872.2018.1524009.

[BM18] Bhargav Bhatt and Akhil Mathew, *The arc-topology*, preprint (2018). Available at https://arxiv.org/abs/1807.04725v3.

[BMS18] Bhargav Bhatt, Matthew Morrow, Peter Scholze, *Integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. 128 (2018), 219-397, DOI 10.1007/s10240-019-00102-z.

[BouCA] Nicolaus Bourbaki, *Commutative Algebra: Chapters 1-7*, Springer-Verlag, 1989.

[BS17] Bhargav Bhatt and Peter Scholze, *Projectivity of the Witt vector affine Grassmanian*, Invent. Math. 209 (2017), no. 2, 329-423, DOI 10.1007/s00222-016-0710-4.

[BS19] Bhargav Bhatt and Peter Scholze, *Prisms and Prismatic Cohomology*, preprint (2019). Available at https://arxiv.org/abs/1905.08229.

[CS19] Kęstutis Česnavičius and Peter Scholze, *Purity for flat cohomology*, preprint (2019). Available at https://arxiv.org/abs/1912.10932.

[dJo93] A.J. de Jong, *Finite locally free group schemes in characteristic p and Dieudonné modules*, Invent. Math., 114(1): 89-137, 1993.

[EGAI] Alexander Grothendieck, Jean. A. Dieudonné , *Eléments de Géométie Algébrique I*, Grundlehren der mathematischen Wissenschaften, Springer-Verlag, 1971, ISBN 3-540-05113-9.

[GW10] Ulrich Görtz, Torsten Wedhorn, *Algebraic Geometry I*, Advanced Lectures in Mathematics. Schemes with examples and exercises. Vieweg + Teubner, Wiesbaden, 2010. doi: 10.1007/978- 3- 8348-9722-0.

[Gro71] Alexandre Grothendieck, *Groupes de Barsotti-Tate et cristaux de Dieudonné* , Actes duCongrès International des Mathématiciens (Nice, 1970), Tome 1, Gauthier-Villars, Paris, 1971, pp. 431–436.

[HTT] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Ked17] Kiran S. Kedlaya, *Sheaves, Stacks and Shtukas*, preprint (2017). Available at http://swc.math.arizona.edu/aws/2017/2017KedlayaNotes.pdf.

[Lau13] Eike Lau, *Smoothness of the truncated display functor*, J. Amer. Math. Soc. 26 (2013), no. 1, 129-165, DOI 10.1090/S0894-0347-2012-00744-9.

[Lau18] Eike Lau, *Dieudonné theory over semiperfect rings and perfectoid rings*, Compos. Math.154(2018), no. 9, 1974-2004, DOI 10.1112/s0010437x18007352.

[LZ04] Andreas Langer, Thomas Zink, *De Rham–Witt Cohomology for a Proper and Smooth Morphism*, Journal of the Institute of Mathematics of Jussieu, 3(2), 231-314. DOI 10.1017/S1474748004000088

[MW04] Ben Moonen, Torsten Wedhorn, *Discrete invariants of varieties in positive characteristic*, Int. Math. Res. Not., (72):3855–3903, 2004.

[PWZ15] Richard Pink, Torsten Wedhorn and Paul Ziegler, *F-zips with additional structure*, Pacific Journal of Mathematics., vol. 274 (2015),no. 1, 183-236, DOI 10.2140/pjm.2015.274.183.

[Sch12] Peter Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. 116 (2012), 245-313, DOI 10.1007/s10240-012-0042-x.

[SW19] Peter Scholze, Jared Weinstein: *Berkeley lectures on p-adic geometry*, preprint (2019). Available at http://www.math.uni-bonn.de/people/scholze/Berkeley.pdf.

[SP] A.J. de Jong et al., *The Stacks Project*. Available at http://stacks.math.columbia.edu.

Wissenschaftlicher Werdegang

2011	Abitur, Liebfrauengymnasium Büren
2011-2014	B. Sc. in Mathematik (Nebenfach Elektrotechnik), Universität Paderborn
2014	Auslandssemester, Western Michigan University, Kalamazoo, USA
2014-2016	M. Sc. in Mathematik, Universität Paderborn
2016-2020	wissenschaftlicher Mitarbeiter, Fachbereich Mathematik der Technischen Universität Darmstadt
2020	Promotion in Mathematik, Technischen Universität Darmstadt