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Global Attraction to Solitary Waves

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Contents

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0.2 Outline

The long time asymptotics for nonlinear wave equations have been the subject of intensive research, starting with the pioneering papers by Segal [Seg63a, Seg63b], Strauss [Str68], and Morawetz and Strauss [MS72], where the nonlinear scattering and local attraction to zero were considered. Global attraction (for large initial data) to zero may not hold if there are *quasistationary solitary wave solutions* of the form

$$\psi(x,t) = \phi(x)e^{-i\omega t}, \qquad \text{with} \quad \omega \in \mathbb{R}, \quad \lim_{|x|\to\infty} \phi(x) = 0. \tag{0.1}$$

We will call such solutions *solitary waves*. Other appropriate names are *nonlinear eigenfunctions* and *quantum stationary states* (the solution (0.1) is not exactly stationary, but certain observable quantities, such as the charge and current densities, are time-independent indeed).

Existence of such solitary waves was addressed by Strauss in [Str77], and then the orbital stability of solitary waves in a general case has been considered in [GSS87]. The asymptotic stability of solitary waves has been obtained by Soffer and Weinstein [SW90, SW92], Buslaev and Perelman [BP93, BP95], and then by others.

The existing results suggest that the set of orbitally stable solitary waves typically forms a *local attractor*, that is, attracts any finite energy solutions that were initially close to it. Moreover, a natural hypothesis is that the set of all solitary waves forms a *global attractor* of all finite energy solutions. This question is addressed in this paper. We develop required techniques and prove global attraction to solitary waves in several models.

More precisely, for several $\mathbf{U}(1)$-invariant Hamiltonian systems based on the Klein-Gordon equation, we prove that under certain generic assumptions the global attractor of all finite energy solutions is finite-dimensional and coincides with the set of all solitary waves. We prove the convergence to the global attractor in the metric which is just slightly weaker than the convergence in the local energy seminorms.

0.3 Plan of the monograph

We sketch the development of the subject of long-time solitary wave asymptotics for $\mathbf{U}(1)$-invariant Hamiltonian systems and its relation to the Quantum Theory in Chapter 1. The definitions and results on global attraction to solitary waves from the recent papers [KK07a, KK07b, KK08] are presented in Chapter 2. We also give there a very brief sketch of the proof.

In Chapter 3, we formulate the definitions of the attractor and the trajectory attractor in terms of omega-limit points and omega-limit trajectories. The proofs of the attraction to solitary waves in the models we study are given in Chapters 4, 5, and 6. The examples of multifrequency solitary waves are given in Chapter 7.

The existence of solitary waves is addressed in Appendix A. The global well-posedness in the energy space is proved in Appendix B. In Appendix C we briefly derive the local energy decay for the linear Klein-Gordon equation. The relevant results on quasimeasures are given in Appendix D. Finally, in Appendix E, we give a proof of the Titchmarsh Convolution Theoreom.

Chapter 1

History of solitary asymptotics for dispersive systems

1.1 Quantum theory

Bohr's stationary orbits as solitary waves

Let us focus on the behavior of the electron in the Hydrogen atom. According to Bohr's postulates [Boh13], an unperturbed electron runs forever along certain *stationary orbit*, which we denote $|E\rangle$ and call *quantum stationary state*. Once in such a state, the electron has a fixed value of energy E, with the energy not being lost via emitted radiation. Under a perturbation, the electron can jump from one quantum stationary state to another,

$$|E_-\rangle \longmapsto |E_+\rangle, \tag{1.1}$$

emitting or absorbing a quantum of light with the energy equal to the difference of the energies E_+ and E_-. The old quantum theory was based on the quantization condition

$$\oint \mathbf{p} \cdot d\mathbf{q} = 2\pi\hbar n, \qquad n \in \mathbb{N}. \tag{1.2}$$

This condition leads to the values

$$E_n = -\frac{m\,\mathrm{e}^4}{2\hbar^2 n^2}, \qquad n \in \mathbb{N}, \tag{1.3}$$

for the energy levels in Hydrogen, in a good agreement with the experiment. In the above formula, $m > 0$ is the mass of the electron, $\mathrm{e} < 0$ is its charge, $\hbar$ is Planck's constant, and we assume that the units are chosen so that the speed of light is equal to 1.

Apparently, the quantization condition (1.2) did not explain the perpetual circular motion of the electron. According to the classical Electrodynamics, such a motion would be accompanied by the loss of energy via radiation.

In terms of the wavelength $\lambda = \frac{2\pi\hbar}{|\mathbf{p}|}$ of de Broglie's *phase waves* [Bro24], the condition (1.2) states that the length of the classical orbit of the electron is the integer multiple of λ. Following de Broglie's ideas, Schrödinger identified Bohr's *stationary orbits*, or quantum stationary states $|E\rangle$, with the wave functions that have the form

$$\psi(\mathbf{x}, t) = \phi_\omega(\mathbf{x}) e^{-i\omega t}, \qquad \omega = E/\hbar, \tag{1.4}$$

where $\hbar$ is Planck's constant. Physically, the charge and current densities

$$\rho(\mathbf{x}, t) = \mathrm{e}\bar\psi\psi, \qquad \mathbf{j}(\mathbf{x}, t) = \frac{\mathrm{e}}{2i}(\bar\psi \cdot \nabla\psi - \nabla\bar\psi \cdot \psi) \tag{1.5}$$

which correspond to the (quasi)stationary states of the form $\psi(\mathbf{x},t) = \phi_\omega(\mathbf{x})e^{-i\omega t}$ do not depend on time, and therefore the generated electromagnetic field is also stationary and does not carry the energy away from the system, allowing the electron cloud to flow forever around the nucleus.

Bohr's transitions as global attraction to solitary waves

Bohr's second postulate states that the electrons can jump from one quantum stationary state (Bohr's *stationary orbit*) to another. This postulate suggests the dynamical interpretation of Bohr's transitions as long-time attraction

$$\Psi(t) \longrightarrow |E_\pm\rangle, \qquad t \to \pm\infty \tag{1.6}$$

for any trajectory $\Psi(t)$ of the corresponding dynamical system, where the limiting states $|E_\pm\rangle$ depend on the trajectory. Then the *quantum stationary states*, denote them $\mathfrak{S}$, should be viewed as points of the *global attractor*, which we denote $\mathfrak{A}$.

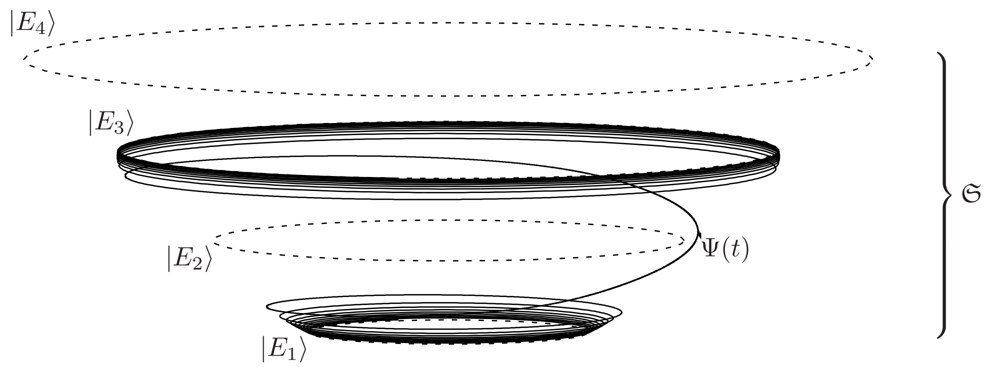

Figure 1.1: $\mathfrak{S}$ is the set of quantum stationary states $|E_n\rangle = \phi_n(x)e^{-i\frac{E_n}{\hbar}t}$, represented by dashed circles. Under a perturbation, the electron wave function $\Psi(t)$ leaves the initial state $|E_3\rangle$ and approaches the final state $|E_1\rangle$ as $t \to +\infty$. The outgoing photon of the energy $h\nu = E_3 - E_1$ is not pictured.

The attraction (1.6) takes the form of the long-time asymptotics

$$\psi(x,t) \sim \phi_{\omega_\pm}(x)e^{-i\omega_\pm t}, \qquad t \to \pm\infty, \tag{1.7}$$

which holds for each finite energy solution. See Figure 1.1. However, because of the superposition principle, the asymptotics of type (1.7) are generally impossible for the linear autonomous equation, be it the Schrödinger equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi - \frac{\mathrm{e}^2}{|\mathbf{x}|}\psi \tag{1.8}$$

or relativistic Schrödinger or Dirac equation in the Coulomb field. An adequate description of this process requires to consider the equation for the electron wave function (Schrödinger or Dirac equation) coupled to the Maxwell system which governs the time evolution of the four-potential $A(x,t) = (\varphi(x,t), \mathbf{A}(x,t))$:

$$\begin{cases} (i\hbar\partial_t - \mathrm{e}\varphi)^2\psi = (c\frac{\hbar}{i}\nabla - \mathrm{e}\mathbf{A})^2\psi + m^2c^4\psi, \\ \Box\varphi = 4\pi\mathrm{e}(\bar\psi\psi - \delta(\mathbf{x})), \qquad \Box\mathbf{A} = 4\pi\mathrm{e}\frac{\bar\psi\cdot\nabla\psi - \nabla\bar\psi\cdot\psi}{2i}. \end{cases} \tag{1.9}$$

Consideration of such a system seems inevitable, because, again by Bohr's postulates, the transitions (1.1) are followed by electromagnetic radiation responsible for the atomic spectra which we observe in the experiment. Moreover, the Lamb shift (a relatively small difference between $2S_{1/2}$ and $2P_{1/2}$ energy levels) can not be explained in terms of the linear Dirac equation in the external Coulomb field. Its theoretical explanation within the Quantum Electrodynamics is based on taking into account the higher order interaction of the electron wave function with the electromagnetic field.

The coupled Maxwell-Schrödinger system was initially introduced in [Sch26]. It is a $\mathbf{U}(1)$-invariant nonlinear Hamiltonian system. Its global well-posedness was considered in [GNS95]. One might expect the following generalization of asymptotics (1.7) for solutions to the coupled Maxwell-Schrödinger (or Maxwell-Dirac) equations:

$$(\psi(x,t), A(x,t)) \sim \left(\phi_{\omega_\pm}(x)e^{-i\omega_\pm t}, A_{\omega_\pm}(x)\right), \qquad t \to \pm\infty. \tag{1.10}$$

The asymptotics (1.10) would mean that the set of all solitary waves

$$\{\left(\phi_\omega e^{-i\omega t}, A_\omega\right) : \omega \in \mathbb{R}\}$$

forms a global attractor for the coupled system. The asymptotics of this form are not available yet in the context of coupled systems. Let us mention that the existence of the solitary waves for the coupled Maxwell-Dirac equations was established in [EGS96].

1.2 Solitary waves as global attractors for dispersive systems

Convergence to a global attractor is well known for dissipative systems, like Navier-Stokes equations (see [BV92, Hen81, Tem97]). For such systems, the global attractor is formed by the *static, stationary states*, and the corresponding asymptotics (1.7) only hold for $t \to +\infty$.

We would like to know whether dispersive Hamiltonian systems could, in the same spirit, possess finite dimensional global attractors, and whether such attractors are formed by the solitary waves. Although there is no dissipation per se, we expect that the attraction is caused by certain friction via the dispersion mechanism (local energy decay). Because of the difficulties posed by the system of interacting Maxwell and Dirac (or Schrödinger) fields (and, in particular, absence of the a priori estimates for such systems), we will work with simpler models which share certain key properties of the coupled Maxwell-Dirac or Maxwell-Schrödinger systems. Let us try to single out these key features:

(1) The system is $\mathbf{U}(1)$*-invariant.*

This invariance leads to the existence of solitary wave solutions $\phi_\omega(x)e^{-i\omega t}$.

(2) The linear part of the system has a dispersive character.

This property provides certain dissipative features in a Hamiltonian system, due to local energy decay via the dispersion mechanism.

(3) The system is nonlinear.

The nonlinearity is needed for the convergence to a single state of the form $\phi_\omega(x)e^{-i\omega t}$. Bohr type transitions to pure eigenstates of the energy operator are impossible in a linear system because of the superposition principle.

We suggest that these are the very features responsible for the global attraction, such as (1.7) or (1.10), to "quantum stationary states".

Remark 1.1. The global attraction (1.7) or (1.10) for $\mathbf{U}(1)$-invariant equations suggests the corresponding extension to general $\mathbf{G}$-invariant equations ($\mathbf{G}$ being the Lie group):

$$\psi(x,t) \sim \psi_\pm(x,t) = e^{\mathbf{\Omega}_\pm t}\phi_\pm(x), \qquad t \to \pm\infty, \tag{1.11}$$

where $\mathbf{\Omega}_\pm$ belong to the corresponding Lie algebra and $e^{\mathbf{\Omega}_\pm t}$ are the one-parameter subgroups. Respectively, the global attractor would consist of the solitary waves (1.11). In particular, for the unitary group

$\mathbf{G} = \mathbf{SU}(3)$, the asymptotics (1.11) relate the "quantum stationary states" to the structure of the corresponding Lie algebra $\mathbf{su}(3)$. On a seemingly related note, let us mention that according to Gell-Mann – Ne'eman theory [GMN64] there is a correspondence between the Lie algebras and the classification of the elementary particles which are the "quantum stationary states". The correspondence has been confirmed experimentally by the discovery of the Omega-Minus Hyperon.

Besides Maxwell-Dirac system, naturally, there are various nonlinear systems under consideration in the Quantum Physics. One of the simpler nonlinear models is the nonlinear Klein-Gordon equation which takes its origin from the articles by Schiff [Sch51a, Sch51b], in his research on the classical nonlinear meson theory of nuclear forces. The mathematical analysis of this equation has been started by Jörgens and Segal [Jör61, Seg63a], who studied its global well-posedness in the energy space. Since then, this equation (alongside with the nonlinear Schrödinger equation) has been the main playground for developing tools to handle more general nonlinear Hamiltonian systems. The nonlinear Klein-Gordon equation is a natural candidate for having solitary asymptotics (1.7).

Now let us describe the existing results on attractors in the context of dispersive Hamiltonian systems.

Local and global attraction to zero

The asymptotics of type (1.7) were discovered first with $\psi_\pm = 0$ in the scattering theory. Namely, Segal, Morawetz, and Strauss studied the (nonlinear) scattering for solutions of nonlinear Klein-Gordon equation in $\mathbb{R}^3$ [Seg66, Str68, MS72]. We may interpret these results as *local* (referring to small initial data) attraction to zero:

$$\psi(x,t) \sim \psi_\pm = 0, \qquad t \to \pm\infty. \tag{1.12}$$

The asymptotics (1.12) hold on an arbitrary compact set and mean well-known local (in space) energy decay. These results were further extended in [GS79, Kla82, GV85, Hör91]. Apparently, there could be no *global* attraction to zero (*global* referring to arbitrary initial data) if there are solitary wave solutions $\phi_\omega(x)e^{-i\omega t}$.

Existence of solitary waves

The existence of solitary wave solutions of the form

$$\psi_\omega(x,t) = \phi_\omega(x)e^{-i\omega t}, \qquad \omega \in \mathbb{R}, \quad \phi_\omega \in H^1(\mathbb{R}^n), \tag{1.13}$$

to the nonlinear Klein-Gordon equation (and nonlinear Schrödinger equation) in $\mathbb{R}^n$, in a rather generic situation, was established in [Str77] (a more general result was obtained in [BL83a, BL83b]). Typically, such solutions exist for ω from an interval or a collection of intervals of the real line. We denote the set of all solitary waves by $\mathfrak{S}$.

The factor-space $\mathfrak{S}/\mathbf{U}(1)$ in a generic situation is isomorphic to a finite union of intervals. Let us mention that there are numerous results on the existence of solitary wave solutions to nonlinear Hamiltonian systems with $\mathbf{U}(1)$ symmetry. See e.g. [BL84, CV86, ES95].

While all localized stationary solutions to the nonlinear wave equations in spatial dimensions $n \geq 3$ turn out to be unstable [Der64] (the result known as "Derrick's Theorem"), *quasistationary* solitary waves can be orbitally stable. Stability of solitary waves takes its origin from [VK73] and has been extensively studied by Strauss and his school in [Sha83, SS85, Sha85, GSS87].

Local attraction to solitary waves

First results on the asymptotics of type (1.7) with $\omega_\pm \neq 0$ were obtained for the nonlinear $\mathbf{U}(1)$-invariant Schrödinger equation in the context of asymptotic stability. This establishes asymptotics of type (1.7) but only for solutions close to the solitary waves, proving the existence of a *local attractor*. This was first done by Soffer and Weinstein and by Buslaev and Perelman in [SW90, BP93, SW92, BP95], and then developed in [PW97, SW99, Cuc01a, Cuc01b, BS03, Cuc03] and other papers.

Global attraction to solitary waves

The *global attraction* of type (1.7) with $\psi_\pm \neq 0$ and $\omega_\pm = 0$ was established in [Kom91, Kom95, KV96, KSK97, Kom99, KS00] for a number of nonlinear wave problems. There the attractor is the set of all *static* stationary states. Let us mention that this set could be infinite and contain continuous components.

In [Kom03] and [KK07a], the attraction to the set of solitary waves (see Figure 1.2) is proved for the Klein-Gordon field coupled to a nonlinear oscillator. In [KK07b], this result has been generalized for the Klein-Gordon field coupled to several oscillators. In [KK08], this result is extended to higher-dimensional setting for a model with the nonlinear self-interaction of the mean field type. In this monograph, we unify the approach to these models and present their analogues in higher dimensions.

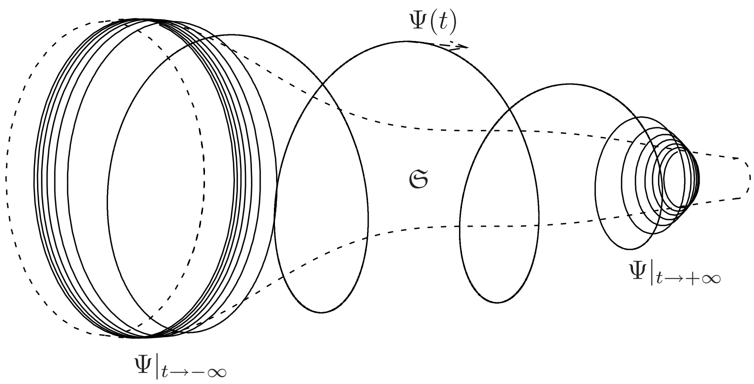

Figure 1.2: For $t \to \pm\infty$, a finite energy solution $\Psi(t)$ approaches the global attractor $\mathfrak{A}$ which coincides with the set of all solitary waves $\mathfrak{S}$.

We are aware of but one recent advance [Tao07] in the field of nontrivial (nonzero) global attractors for Hamiltonian PDEs. In that paper, existence of the global attractor for the nonlinear Schrödinger equation in dimensions $n \geq 5$ was considered. The dispersive (outgoing) wave was explicitly specified using the rapid local energy decay in higher dimensions. The global attractor was proved to be compact, but it was neither identified with the set of solitary waves nor was proved to be finite-dimensional [Tao07, Remark 1.18].

Chapter 2

Description of models and results

2.1 Klein-Gordon with one oscillator

Model

We start with the simplest model, which is the Klein-Gordon equation with the nonlinearity located at a point:

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \delta(x)F(\psi(0,t)), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{2.1}$$

Above, $m > 0$ and F is a nonlinear function describing a nonlinear oscillator at the point $x = 0$. The dots stand for the derivatives in t, and the primes for the derivatives in x. All derivatives and the equation are understood in the sense of distributions. We assume that equation (2.1) is $\mathbf{U}(1)$-invariant, where $\mathbf{U}(1)$ stands for the unitary group $e^{i\theta}$, $\theta \in \mathbb{R} \mod 2\pi$. That is, we assume that

$$F(e^{i\theta}\psi) = e^{i\theta}F(\psi), \qquad \theta \in \mathbb{R}, \quad \psi \in \mathbb{C}. \tag{2.2}$$

This symmetry leads to the charge conservation and to the existence of the solitary wave solutions, which are finite energy solutions of the following form:

$$\psi_\omega(x,t) = \phi_\omega(x)e^{-i\omega t}, \quad \omega \in \mathbb{R}, \quad \phi_\omega \in H^1(\mathbb{R}). \tag{2.3}$$

Above, $H^1(\mathbb{R})$ is the Sobolev space.

If we identify a complex number $\psi = u + iv \in \mathbb{C}$ with the two-dimensional vector $(u,v) \in \mathbb{R}^2$, then, physically, equation (2.1) describes small crosswise oscillations of the infinite string in three-dimensional space (x,u,v) stretched along the x-axis. The string is subject to the action of an "elastic force" $-m^2\psi(x,t)$ and coupled to a nonlinear oscillator of the force $F(\psi)$ attached at the point $x = 0$.

Remark 2.1. In the context of this model, the assumption (2.2) means that the potential $U(\psi)$ is rotation-invariant with respect to the x-axis.

Solitary waves

Definition 2.2. *(1)* The solitary wave solutions (or, briefly, *solitary waves*) of (2.1) are finite energy solutions to (2.1) of the form

$$\psi(x,t) = \phi_\omega(x)e^{-i\omega t}, \quad \text{where} \;\; \omega \in \mathbb{R}, \;\; \phi_\omega \in H^1(\mathbb{R}). \tag{2.4}$$

(2) The set of all solitary wave solutions is denoted by $\mathfrak{S}$:

$$\mathfrak{S} = \{\psi \in C(\mathbb{R}, H^1(\mathbb{R})): \;\; \psi(x,t) = \phi_\omega(x)e^{-i\omega t}, \;\; \omega \in \mathbb{R}, \;\; \phi_\omega \in H^1(\mathbb{R})\}. \tag{2.5}$$

(3) The solitary manifold is the set of corresponding initial data:

$$\mathcal{S} = \left\{(\phi_\omega, -i\omega\phi_\omega)\colon \phi_\omega(x)e^{-i\omega t} \in \mathfrak{S}\right\}. \tag{2.6}$$

Remark 2.3. Since we only consider $\mathbf{U}(1)$-invariant equations, the set $\mathcal{S}$ is invariant under multiplication by $e^{i\theta}$, $\theta \in \mathbb{R}$.

The solitary waves for equation (2.1) are constructed in Appendix A.1. According to Remark A.3, there are numerous nonlinearities leading to the existence of solitary waves.

Hamiltonian structure

We set $\Psi(t) = (\psi(x,t), \pi(x,t)) \in \mathbb{C}^2$ and rewrite equation (2.1) in the vector form:

$$\dot{\Psi}(t) = \begin{bmatrix} 0 & 1 \\ \Delta - m^2 & 0 \end{bmatrix} \Psi(t) + \delta(x) \begin{bmatrix} 0 \\ F(\psi(0,t)) \end{bmatrix}, \tag{2.7}$$

where $x \in \mathbb{R}$ and $t \in \mathbb{R}$. We assume that the nonlinearity F admits a real-valued $\mathbf{U}(1)$-invariant potential, $U(\psi) = w(|\psi|^2)$, for some $w \in C^2(\mathbb{R})$:

$$F(\psi) = -\nabla U(\psi) = -2w'(|\psi|^2)\psi,$$

where the gradient is taken with respect to $(\operatorname{Re}\psi, \operatorname{Im}\psi)$:

$$\nabla U(\psi) = \partial_u U + i\partial_v U, \qquad \psi = u + iv, \quad u,\, v \in \mathbb{R}.$$

Then equation (2.7) can formally be written as a Hamiltonian system,

$$\dot{\Psi}(t) = \mathcal{J}\mathcal{H}'(\Psi), \qquad \mathcal{J} = \begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{bmatrix}, \tag{2.8}$$

where $\mathcal{H}'$ is the variational derivative of the Hamilton functional

$$\mathcal{H}(\Psi) = \frac{1}{2}\int\limits_{\mathbb{R}} \left(|\pi|^2 + |\psi'|^2 + m^2|\psi|^2\right) dx + U(\psi(0)), \qquad \Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix}, \tag{2.9}$$

taken with respect to $(\operatorname{Re}\psi, \operatorname{Im}\psi)$ and $(\operatorname{Re}\pi, \operatorname{Im}\pi)$.

Since (2.7) is $\mathbf{U}(1)$-invariant, the Nöther theorem formally implies that the *charge functional*

$$\mathcal{Q}(\psi, \pi) = \frac{i}{2}\int_{\mathbb{R}} \left(\overline{\psi}\pi - \overline{\pi}\psi\right)\, dx \tag{2.10}$$

is (formally) conserved for solutions $\Psi(t) = \begin{bmatrix} \psi(x,t) \\ \pi(x,t) \end{bmatrix}$ to (2.7).

The phase space

Denote by $\|\cdot\|_{L^2}$ the norm in $L^2(\mathbb{R}^n)$. Let $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, be the Sobolev space with the norm

$$\|\psi\|_{H^s} = \|(m^2 - \Delta)^{s/2}\psi\|_{L^2}. \tag{2.11}$$

For $s \in \mathbb{R}$ and $R > 0$, denote by $H^s_0(\mathbb{B}^n_R)$ the space of distributions from $H^s(\mathbb{R}^n)$ supported in $\mathbb{B}^n_R$ (the ball of radius R in $\mathbb{R}^n$). We denote by $\|\cdot\|_{H^s,R}$ the norm in the space $H^s(\mathbb{B}^n_R)$ which is defined as the dual to $H^{-s}_0(\mathbb{B}_R)$.

Definition 2.4. Let $n \geq 1$.

(1) $\mathscr{X} = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ is the Hilbert space of states $\Psi = (\psi, \pi)$, with the norm

$$\|\Psi\|^2_{\mathscr{X}} = \|\pi\|^2_{L^2} + \|\nabla\psi\|^2_{L^2} + m^2\|\psi\|^2_{L^2} = \|\pi\|^2_{L^2} + \|\psi\|^2_{H^1}.$$

(2) For $\varepsilon \ge 0$, introduce the Banach spaces $\mathscr{X}^{-\varepsilon} = H^{1-\varepsilon}(\mathbb{R}^n) \times H^{-\varepsilon}(\mathbb{R}^n)$ with the norm

$$\|\Psi\|^2_{\mathscr{X}^{-\varepsilon}} = \|(m^2 - \Delta)^{-\varepsilon/2}\Psi\|^2_{\mathscr{X}} = \|\pi\|^2_{H^{-\varepsilon}} + \|\psi\|^2_{H^{1-\varepsilon}}.$$

(3) Define the seminorms

$$\|\Psi\|^2_{\mathscr{X}^{-\varepsilon},R} = \|\pi\|^2_{H^{-\varepsilon},R} + \|\psi\|^2_{H^{1-\varepsilon},R}, \qquad R > 0,$$

and denote by $\mathscr{Y}^{-\varepsilon}$ the Banach space with the norm

$$\|\Psi\|_{\mathscr{Y}^{-\varepsilon}} = \sum_{R=1}^{\infty} 2^{-R}\|\Psi\|_{\mathscr{X}^{-\varepsilon},R} < \infty. \tag{2.12}$$

Lemma 2.5. *For any $\varepsilon > 0$, the embedding $\mathscr{X} \subset \mathscr{Y}^{-\varepsilon}$ is compact.*

Proof. Let $\Psi_j \in \mathscr{X}$, $j \in \mathbb{N}$ be a sequence such that

$$\|\Psi_j\|_{\mathscr{X}} \le C < \infty, \qquad j \in \mathbb{N}. \tag{2.13}$$

It suffices to specify a Cauchy subsequence in Ψ_j considered in the space $\mathscr{Y}^{-\varepsilon}$.

Since $\mathscr{X}$ is a Hilbert space, we can choose a subsequence of Ψ_j which is weakly convergent in $\mathscr{X}$ to some $\Psi_0 \in \mathscr{X}$. Since for any $s > s'$ and $R > 0$ the inclusion $H^s_0(\mathbb{B}^n_R) \subset H^{s'}(\mathbb{R}^n)$ is compact (with $\mathbb{B}^n_R$ being a ball of radius R in $\mathbb{R}^n$), we can choose a smaller subsequence of Ψ_j which converges in the metric $\|\cdot\|_{\mathscr{X}^{-\varepsilon},R}$. By the diagonalization process, we can choose a yet smaller subsequence of Ψ_j, which we denote Ψ_{j_r}, $r \in \mathbb{N}$, which converges in the metric $\|\cdot\|_{\mathscr{X}^{-\varepsilon},R}$, for any $R > 0$.

Let us show that Ψ_{j_r}, $r \in \mathbb{N}$, is a Cauchy sequence in $\mathscr{Y}^{-\varepsilon}$. Pick $\delta > 0$. Choose $R_0 \in \mathbb{N}$ large enough so that $2^{-R_0}C < \delta/4$, where C is from (2.13). Since Ψ_{j_r} is convergent in $\|\cdot\|_{\mathscr{X}^{-\varepsilon},R}$ for any fixed $R > 0$, there is $r_0 \in \mathbb{N}$ such that $\|\Psi_{j_r} - \Psi_{j_{r'}}\|_{\mathscr{X}^{-\varepsilon},R_0} < \delta/2$ for all $r, r' > r_0$. Then, for all $r, r' > r_0$,

$$\begin{aligned}
\|\Psi_{j_r} - \Psi_{j_{r'}}\|_{\mathscr{Y}^{-\varepsilon}} &= \sum_{R=1}^{\infty} 2^{-R}\|\Psi_{j_r} - \Psi_{j_{r'}}\|_{\mathscr{X}^{-\varepsilon},R} \\
&\le \sum_{R=1}^{R_0} 2^{-R}\|\Psi_{j_r} - \Psi_{j_{r'}}\|_{\mathscr{X}^{-\varepsilon},R} + \sum_{R=R_0+1}^{\infty} 2^{-R}\|\Psi_{j_r} - \Psi_{j_{r'}}\|_{\mathscr{X}} \\
&\le \|\Psi_{j_r} - \Psi_{j_{r'}}\|_{\mathscr{X}^{-\varepsilon},R_0} + 2^{-R_0}\cdot 2C < \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\end{aligned}$$

This finishes the proof. □

Equation (2.7) is formally a Hamiltonian system with the phase space $\mathscr{X}$ defined in Definition 2.4 *(1)* (with $n = 1$) and the Hamilton functional $\mathcal{H}$. Both $\mathcal{H}$ and $\mathcal{Q}$ are continuous functionals on $\mathscr{X}$.

Theorem 2.6 (Global attraction for Klein-Gordon equation with one oscillator)**.**
Assume that $F(\psi) = -\nabla U(\psi)$, where

$$U(\psi) = \sum_{l=1}^{p} u_l|\psi|^{2l}, \quad u_l \in \mathbb{R}, \quad u_p > 0, \quad \text{and} \quad p \ge 2. \tag{2.14}$$

For any $(\psi_0, \pi_0) \in \mathscr{X}$*, the solution* $\psi(t)$ *to* (2.1) *with* $(\psi, \dot{\psi})|_{t=0} = (\psi_0, \pi_0)$ *converges to the solitary manifold* $\mathcal{S}$ *in the space* $\mathscr{Y}^{-\varepsilon}$*, for any* $\varepsilon > 0$*:*

$$\lim_{t \to \pm\infty} \operatorname{dist}_{\mathscr{Y}^{-\varepsilon}}((\psi, \dot{\psi})|_t, \mathcal{S}) = 0, \tag{2.15}$$

where $\mathcal{S}$ *is introduced in* (2.6) *and* $\operatorname{dist}_{\mathscr{Y}^{-\varepsilon}}(\Psi, \mathcal{S}) := \inf_{\boldsymbol{s} \in \mathcal{S}} \|\Psi - \boldsymbol{s}\|_{\mathscr{Y}^{-\varepsilon}}$*, with* $\|\cdot\|_{\mathscr{Y}^{-\varepsilon}}$ *introduced in* (2.12)*.*

Remark 2.7. *(1)* The existence of a global solution $\psi(t)$ for any finite energy initial data $(\psi_0, \pi_0) \in H^1 \times L^2$ is proved in Appendix B.

(2) By (2.14), the nonlinearity is of polynomial character and is strictly nonlinear. This condition is crucial in our argument: It will allow us to apply the Titchmarsh convolution theorem.

(3) It suffices to prove Theorem 2.6 for $t \to +\infty$.

(4) For the real initial data, we obtain a real-valued solution $\psi(t)$ to (2.1). Therefore, the convergence (2.15) of $(\psi(t), \dot{\psi}(t))$ to the set of pairs $(\phi_\omega, -i\omega\phi_\omega)$ with $\omega \in \mathbb{R}$ implies that $\psi(t)$ locally converges to zero or a static solution.

(5) As the matter of fact, the convergence (2.15) also holds in the local energy seminorms, and, in particular, in $\mathscr{Y}^{-\varepsilon}$ with $\varepsilon = 0$. The proof based on the technique of quasimeasures is presented in [KK07a].

2.2 Klein-Gordon with several oscillators

Let us consider the Klein-Gordon equation with N nonlinear oscillators located at the points $X_1 < X_2 < \dots < X_N$:

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \sum_{I=1}^{N} \delta(x - X_I) F_I(\psi(X_I, t)), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{2.16}$$

Above, $m > 0$ and F_I are nonlinear functions describing nonlinear oscillators at the points X_I. The dots stand for the derivatives in t, and the primes for the derivatives in x. All derivatives and the equation are understood in the sense of distributions. We assume that equation (2.16) is $\mathbf{U}(1)$-invariant; that is, each $F_I(\psi)$, $1 \le I \le N$, satisfies (2.2):

$$F_I(e^{i\theta}\psi) = e^{i\theta} F_I(\psi), \qquad \theta \in \mathbb{R}, \quad \psi \in \mathbb{C}, \quad 1 \le I \le N.$$

We denote by $\mathcal{X}$ the set of all the locations of oscillators:

$$\mathcal{X} = \{X_1, X_2, \dots, X_N\}. \tag{2.17}$$

We will assume that the oscillator forces F_I admit real-valued $\mathbf{U}(1)$-invariant potentials:

$$F_I(\psi) = -\nabla U_I(\psi), \qquad U_I(\psi) = u_I(|\psi|^2), \qquad u_I \in C^2(\mathbb{R}), \tag{2.18}$$

where u_I are real-valued. The gradient is taken with respect to $(\operatorname{Re}\psi, \operatorname{Im}\psi)$.

Equation (2.16) can formally be written as a Hamiltonian system, with the Hamiltonian

$$\mathcal{H}(\psi, \pi) = \frac{1}{2} \int_{\mathbb{R}} \left(|\pi|^2 + |\psi'|^2 + m^2|\psi|^2\right) dx + \sum_{I=1}^{N} U_I(\psi(X_I)). \tag{2.19}$$

Since (2.16) is $\mathbf{U}(1)$-invariant, the Nöther theorem formally implies that the values of the charge functional $\mathcal{Q}(\psi, \dot{\psi})$ defined in (2.10) are conserved for solutions $\psi(t)$ to (2.16). Both $\mathcal{H}$ and $\mathcal{Q}$ are continuous functionals on the space $\mathscr{X}$ defined in Definition 2.4 *(1)*.

Theorem 2.8 (Global attraction for Klein-Gordon equation with N oscillators)**.**
Assume that for all $1 \le I \le N$, one has $F_I(\psi) = -\nabla U_I(\psi)$, where

$$U_I(\psi) = \sum_{l=1}^{p_I} u_{I,l}|\psi|^{2l}, \quad u_{I,l} \in \mathbb{R}, \quad u_{I,p_I} > 0, \quad \text{and} \quad p_I \ge 2. \tag{2.20}$$

If $N \ge 2$, assume that the intervals $[X_I, X_{I+1}]$, $1 \le I \le N-1$, are so small that $\varDelta := \max\limits_{1\le I\le N-1} |X_{I+1} - X_I|$ satisfies

$$\Big(\frac{\pi^2}{\varDelta^2} + m^2\Big)^{\frac{1}{2}} > m \prod_{l=1}^{N} (2p_l - 1), \tag{2.21}$$

where p_I are exponentials from (2.20). Then for any $(\psi_0, \pi_0) \in \mathscr{X}$ the solution $\psi(t)$ to (2.16) with the initial data $(\psi, \dot\psi)\vert_{t=0} = (\psi_0, \pi_0)$ converges to the solitary manifold $\mathcal{S}$ in the space $\mathscr{Y}^{-\varepsilon}$, for any $\varepsilon > 0$:

$$\lim_{t\to\pm\infty} \operatorname{dist}_{\mathscr{Y}^{-\varepsilon}}((\psi, \dot\psi)\vert_t, \mathcal{S}) = 0, \tag{2.22}$$

where $\operatorname{dist}_{\mathscr{Y}^{-\varepsilon}}(\Psi, \mathcal{S}) := \inf\limits_{\boldsymbol{s}\in\mathcal{S}} \|\Psi - \boldsymbol{s}\|_{\mathscr{Y}^{-\varepsilon}}$.

Remark 2.9. The existence of solitary waves for equation (2.16) is addressed in Appendix A.1.

Remark 2.10. In Section 7.1, we construct counterexamples to the convergence (2.22) in the case when some of F_I are linear (in (2.20), some of p_I are equal to 1) or when (2.21) is not satisfied.

2.3 Klein-Gordon with mean field interaction

To consider the higher dimensional analog of the above results, we substitute the δ-function coupling by the one based on the mean field mechanism. This has to be done because the finite energy solutions to the Klein-Gordon equation in higher dimensions are not necessarily continuous and can not be considered at a particular point.

We consider the complex Klein-Gordon equation with the mean field self-interaction at N points:

$$\ddot\psi(x,t) = \Delta\psi(x,t) - m^2\psi(x,t) + \sum_{I=1}^{N} \rho_I(x) F_I(\langle\rho_I, \psi(\cdot,t)\rangle), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{2.23}$$

where

$$\langle\rho_I, \psi(\cdot,t)\rangle = \int_{\mathbb{R}^n} \bar\rho_I(x)\psi(x,t)\, d^n x.$$

We assume that $\rho_I(x) = \rho(x - X_I)$, where $X_I \in \mathbb{R}^n$ and ρ is a smooth real-valued function from the Schwartz class: $\rho \in \mathscr{S}(\mathbb{R}^n)$, $\rho \not\equiv 0$.

We will assume that the dimension is $n \ge 3$.

We assume that (2.23) is $\mathbf{U}(1)$-invariant:

$$F_I(e^{i\theta} z) = e^{i\theta} F_I(z), \qquad z \in \mathbb{C}, \quad \theta \in \mathbb{R}, \quad 1 \le I \le N.$$

We also assume that F_I admit real-valued $\mathbf{U}(1)$-invariant potentials:

$$F_I(z) = -\nabla U_I(z), \qquad U_I(z) = u_I(|z|^2), \qquad u_I \in C^2(\mathbb{R}),$$

where u_I are real-valued.

Solitary waves

The set $\mathfrak{S}$ of solitary wave solutions to (2.23) is defined similarly to Definition 2.2 as the set of all finite energy solutions of the form $\psi(x,t)=\phi_\omega(x)e^{-i\omega t}$ with $\phi_\omega\in H^1(\mathbb{R}^n)$:

$$\mathfrak{S}=\{\psi\in C^1(\mathbb{R},H^1(\mathbb{R}^n))\colon\ \psi(x,t)=\phi_\omega(x)e^{-i\omega t},\ \omega\in\mathbb{R},\ \phi_\omega\in H^1(\mathbb{R}^n)\}. \tag{2.24}$$

The solitary manifold is the set of the corresponding initial data:

$$\mathcal{S}=\{(\phi_\omega,-i\omega\phi_\omega)\colon\ \phi_\omega e^{-i\omega t}\in\mathfrak{S}\}. \tag{2.25}$$

Equation (2.23) can formally be written as a Hamiltonian system with the phase space $\mathscr{X}$ defined in Definition 2.4 *(1)* and the Hamiltonian

$$\mathcal{H}(\psi,\pi)=\frac{1}{2}\int_{\mathbb{R}^n}\left(|\pi|^2+|\nabla\psi|^2+m^2|\psi|^2\right)d^nx+\sum_{I=1}^{N}U_I(\langle\rho_I,\psi\rangle). \tag{2.26}$$

Due to $\mathbf{U}(1)$-invariance of (2.23), the functional $\mathcal{Q}(\psi,\dot\psi)$ (defined in (2.10)) is conserved (formally) for the solutions of (2.23). Both $\mathcal{H}$ and $\mathcal{Q}$ are continuous functionals on $\mathscr{X}$.

Let

$$Z_\rho=\{\omega\in\mathbb{R}\backslash[-m,m]\colon\ \hat\rho(\xi)=0 \text{ for all } \xi\in\mathbb{R}^n \text{ such that } m^2+\xi^2=\omega^2\}. \tag{2.27}$$

Above, $\xi^2=|\xi|^2$.

Define

$$\varSigma_I(x,\omega)=\mathcal{F}_{\xi\to x}\Big[\frac{\hat\rho_I(\xi)}{\xi^2+m^2-\omega^2}\Big]=\mathcal{F}_{\xi\to x}\Big[\frac{e^{-i\xi\cdot X_I}\hat\rho(\xi)}{\xi^2+m^2-\omega^2}\Big],\qquad \omega\in\mathbb{C}^+\cup(-m,m), \tag{2.28}$$

where $\mathbb{C}^+=\{\omega\in\mathbb{C}\colon\ \operatorname{Im}\omega>0\}$. Note that $\varSigma_I(\cdot,\omega)$ is an analytic function of $\omega\in\mathbb{C}^+$ with the values in $\mathscr{S}(\mathbb{R}^n)$. Since $|\varSigma_I(x,\omega)|\le\text{const}|\operatorname{Im}\omega|^{-1}$ for $\omega\in\mathbb{C}^+$, we can extend for any $x\in\mathbb{R}^n$ the function $\varSigma_I(x,\omega)$ to the entire real line $\omega\in\mathbb{R}$ as a boundary trace:

$$\varSigma_I(x,\omega)=\lim_{\epsilon\to0+}\varSigma_I(x,\omega+i\epsilon),\qquad \omega\in\mathbb{R}, \tag{2.29}$$

where the limit holds in the sense of tempered distributions.

Definition 2.11. For $1\le I\le N$, $1\le J\le N$, define

$$\sigma_{IJ}(\omega)=\langle\rho_I,\varSigma_J(\cdot,\omega)\rangle=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{e^{i(X_I-X_J)\cdot\xi}|\hat\rho(\xi)|^2}{\xi^2+m^2-(\omega+i0)^2}\,d^n\xi. \tag{2.30}$$

Let

$$Z_\sigma^N=\Big\{\omega\colon\ \det_{1\le I,J\le N}\sigma_{IJ}(\omega)=0\Big\}.$$

More generally, for $N'\le N$, define

$$Z_\sigma^{N'}=\Big\{\omega\colon\ \exists\,\mathcal{I},\mathcal{J}\subset\{1,\dots,N\},\ |\mathcal{I}|=|\mathcal{J}|=N',\ \det_{I\in\mathcal{I},J\in\mathcal{J}}\sigma_{IJ}(\omega)=0\Big\}.$$

Denote

$$Z_\sigma^*=\bigcup_{1\le N'\le N}Z_\sigma^{N'}. \tag{2.31}$$

Assumption 2.12. Z_σ^* is a discrete set of points, and $Z_\sigma^*\cap([-m,m]\cup Z_\rho)=\emptyset$.

Above, Z_ρ is defined in (2.27) and Z^*_σ is defined in (2.31).

Remark 2.13. Assume that ξ_0 and X_I, $1 \le I \le N$, are such that the matrix

$$S_{IJ} = e^{i\xi_0 \cdot (X_I - X_J)}$$

is non-degenerate and all its $N' \times N'$ minors, $1 \le N' \le N$, are also non-zero. Then, if ρ is such that $\hat\rho \ne 0$ and $\hat\rho(\xi)$ is concentrated in a sufficiently small neighborhood of $\xi = \xi_0$, one has

$$Z^*_\sigma \cap [-m, m] = \emptyset.$$

Therefore, $Z^*_\sigma \cap ([-m, m] \cup Z_\rho) = \emptyset$ ($Z_\rho = \emptyset$ since $\hat\rho \ne 0$).

Remark 2.14. The local well-posedness of (2.23) in the energy space, Theorem B.16, is similar to Theorem B.1, but easier to prove. The local well-posedness of (2.23) is immediate since the nonlinearity in the right-hand side in (2.23) belongs to $H^1(\mathbb{R}^n)$. The global well-posedness follows from the a priori bound on $\|(\psi, \dot\psi)\|_{\mathscr{X}}$ which is a consequence of the energy conservation and the bound $\inf_{z \in \mathbb{C}} U(z) > -\infty$.

Theorem 2.15 (Global attraction for Klein-Gordon with mean field interaction)**.**
Assume that for all $1 \le I \le N$, one has $F_I(z) = -\nabla U_I(z)$, where

$$U_I(z) = \sum_{l=1}^{p} u_{I,l} |z|^{2l}, \quad u_{I,l} \in \mathbb{R}, \quad u_{I,p_I} > 0, \quad \text{and} \quad p_I \ge 2. \tag{2.32}$$

Assume that the coupling function $\rho(x)$ and the points X_I, $1 \le I \le N$, are such that Assumption 2.12 is satisfied.

Then for any $(\psi_0, \pi_0) \in \mathscr{X}$ the solution $\psi(t)$ to equation (2.23) with the initial data $(\psi, \dot\psi)|_{t=0} = (\psi_0, \pi_0)$ converges to the solitary manifold $\mathcal{S}$ in the space $\mathscr{Y}^{-\varepsilon}$, for any $\varepsilon > 0$:

$$\lim_{t \to \pm\infty} \operatorname{dist}_{\mathscr{Y}^{-\varepsilon}}((\psi, \dot\psi)|_t, \mathcal{S}) = 0, \tag{2.33}$$

where $\operatorname{dist}_{\mathscr{Y}^{-\varepsilon}}(\Psi, \mathcal{S}) := \inf_{\boldsymbol{s} \in \mathcal{S}} \|\Psi - \boldsymbol{s}\|_{\mathscr{Y}^{-\varepsilon}}$*, with* $\|\cdot\|_{\mathscr{Y}^{-\varepsilon}}$ *introduced in* (2.12).

Remark 2.16. We assume that $n \ge 3$. In this case, $\Sigma_I(x, \omega)$ defined in (2.29) is smooth near $\omega = \pm m$, and hence is a multiplicator in $\mathscr{S}'(\mathbb{R})$.

Remark 2.17. We do not know whether the $\mathscr{Y}^{-\varepsilon}$-convergence with $\varepsilon > 0$ stated in this theorem could be improved to the $\mathscr{Y}^0$-convergence.

Chapter 3

Attractors

The consideration in this section applies to models (2.16) and (2.23). We assume that the equations are written in the Hamiltonian form such as (2.8):

$$\dot{\Psi}(t) = \mathcal{J}\mathcal{H}'(\Psi). \tag{3.1}$$

We will follow the notations of the general theory of attractors [CV02].

Assumption 3.1. We assume that (3.1) satisfies the following properties:

(1) For any $\Psi_0 \in \mathscr{X}$, there is a unique solution $\Psi \in C(\mathbb{R}, \mathscr{X})$ to (3.1) with $\Psi|_{t=0} = \Psi_0$.

(2) For each $\Psi_0 \in \mathscr{X}$, there is a constant $C_{\Psi_0} < \infty$ such that $\sup\limits_{t\in\mathbb{R}} \|\Psi|_t\|_{\mathscr{X}} \le C_{\Psi_0}$.

(3) There is $\varepsilon_0 > 0$ such that for any $T > 0$, any $\varepsilon \in (0, \varepsilon_0)$, and any $\Psi_j \in C(\mathbb{R}, \mathscr{X})$, $j \in \mathbb{N}$, which are solutions to (3.1) satisfying

$$\sup_{j\in\mathbb{N}} \|\Psi_j|_{t=0}\|_{\mathscr{X}} < \infty \quad \text{and} \quad \Psi_j|_{t=0} \xrightarrow[j\to\infty]{\mathscr{Y}^{-\varepsilon}} X_0 \in \mathscr{X},$$

there is the convergence

$$\Psi_j \xrightarrow[j\to\infty]{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} X,$$

where $X \in C(\mathbb{R}, \mathscr{X})$ is the solution to (3.1) with $X|_{t=0} = X_0$.

(4) For any $\varepsilon > 0$, the embedding $\mathscr{X} \subset \mathscr{Y}^{-\varepsilon}$ is compact.

The space $C_b([-T, T], \mathscr{Y}^{-\varepsilon})$ appearing above is equipped with the the sup-norm

$$\|\Psi\|_{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} := \sup_{t\in[-T,T]} \|\Psi|_t\|_{\mathscr{Y}^{-\varepsilon}}.$$

Remark 3.2. The solutions to (2.16) and (2.23) satisfy the conditions *(1)*, *(2)*, and *(3)* of Assumption 3.1 due to Theorem B.1 and Theorem B.16 (see Appendix B). In the case of (2.16), one can take $\varepsilon_0 = 1/2$; for (2.23), one can take any $\varepsilon_0 > 0$.

The condition *(4)* is satisfied for $\mathscr{X}$, $\mathscr{Y}^{-\varepsilon}$ defined in Definition 2.4 due to Lemma 2.5.

Let S_τ be the time shift operator acting on $C(\mathbb{R}, \mathscr{S}')$:

$$S_\tau \Psi(t) = \Psi(\tau + t). \tag{3.2}$$

3.1 Omega-limit points and omega-limit trajectories

Definition 3.3 (Omega-limit point of a trajectory)**.** Let $\Psi \in C(\mathbb{R}, \mathscr{X})$ be a solution to (3.1). We will say that $X_0 \in \mathscr{X}$ is the *omega-limit point* of Ψ, $X_0 \in \omega(\Psi)$, if there is a sequence $t_j \to +\infty$ such that

$$\Psi|_{t_j} \xrightarrow[j\to\infty]{\mathscr{S}'} X_0.$$

This definition could be reformulated as follows:

$$\omega(\Psi) = \bigcap_{t\ge 0} \left[\bigcup_{s\ge t} \Psi(s) \right],$$

where $[\ \cdot\ ]$ denotes the closure of the set in the topology of $\mathscr{S}'$.

Lemma 3.4. *Let $\Psi \in C_b(\mathbb{R}, \mathscr{X})$. For any sequence $\{t_j \colon j \in \mathbb{N}\}$ and any $\varepsilon > 0$,*

$$\Psi|_{t_j} \xrightarrow[j\to\infty]{\mathscr{S}'} X_0 \quad \textit{if and only if} \quad \Psi|_{t_j} \xrightarrow[j\to\infty]{\mathscr{Y}^{-\varepsilon}} X_0.$$

In either case, $X_0 \in \mathscr{X}$.

Due to the a priori bounds in $\mathscr{X}$ for solutions to (3.1) (Assumption 3.1 *(2)*), this lemma shows that the $\mathscr{S}'$-convergence in Definition 3.3 could be substituted by the $\mathscr{Y}^{-\varepsilon}$-convergence.

Proof. Since $\mathscr{Y}^{-\varepsilon} \subset \mathscr{S}'$, it suffices to prove that the convergence $\Psi|_{t_j} \xrightarrow{\mathscr{S}'} X_0$ as $j \to \infty$ implies the convergence $\Psi|_{t_j} \xrightarrow{\mathscr{Y}^{-\varepsilon}} X_0$, $j \to \infty$, for any $\varepsilon > 0$.

Since the set $\Psi|_{t_j}$ is bounded in the Hilbert space $\mathscr{X}$, it contains a weakly convergent (in $\mathscr{X}$) subsequence. On the other hand, any such subsequence would have to converge (weakly) to X_0. It follows that $X_0 \in \mathscr{X}$ and $\Psi|_{t_j}$ weakly converges to X_0.

Assume that the sequence $\Psi|_{t_j}$ does not converge to X_0 in the metric $\|\cdot\|_{\mathscr{Y}^{-\varepsilon}}$. Then there is $\delta > 0$ and a subsequence $\Psi|_{t_{j_r}}$, $r \in \mathbb{N}$, such that $\|\Psi|_{t_{j_r}} - X_0\|_{\mathscr{Y}^{-\varepsilon}} > \delta$ for all $r \in \mathbb{N}$. On the other hand, due to the compactness of the inclusion $\mathscr{X} \subset \mathscr{Y}^{-\varepsilon}$ (Assumption 3.1 *(4)*), a sequence $\Psi|_{t_{j_r}}$ would have to contain a subsequence $\Psi|_{t_{j_{r_s}}}$, $s \in \mathbb{N}$, convergent in $\mathscr{Y}^{-\varepsilon}$, whose limit would have to coincide with $X_0 \in \mathscr{X} \subset \mathscr{Y}^{-\varepsilon}$.
□

Definition 3.5 (Omega-limit set)**.** The ω-limit set of a set $\mathcal{B} \subset \mathscr{X}$ is defined by

$$\omega(\mathcal{B}) = \{X_0 \colon \exists \Psi \in C(\mathbb{R}, \mathscr{X}),\ \dot{\Psi} = \mathcal{J}\mathcal{H}'(\Psi),\ \Psi|_{t=0} \in \mathcal{B},\ X_0 \in \omega(\Psi)\} = \bigcup_{\Psi\colon \Psi|_{t=0} \in \mathcal{B}} \omega(\Psi).$$

This definition could be restated as

$$\omega(\mathcal{B}) = \bigcap_{t\ge 0} \left[\bigcup_{s\ge t} W(s)\mathcal{B} \right],$$

where $W(t)$ is the dynamical group of equation (3.1) and $[\ \cdot\ ]$ denotes the closure of the set in the topology of $\mathscr{S}'$.

Definition 3.6 (Omega-limit trajectory)**.** Let $\Psi \in C(\mathbb{R}, \mathscr{X})$ be a solution to (3.1). We call $X \in C(\mathbb{R}, \mathscr{X})$ *the omega-limit trajectory* of Ψ if there is a sequence $t_j \to +\infty$ such that

$$S_{t_j}\Psi \xrightarrow[j\to\infty]{\mathscr{S}'} X.$$

In this definition, the convergence is in the topology of the space of tempered distributions over space-time. The following lemma shows that this convergence could be substituted by the $C_b([-T,T],\mathscr{Y}^{-\varepsilon})$-convergence.

Lemma 3.7. *Let $\Psi \in C(\mathbb{R},\mathscr{X})$ be a solution to either* (2.16) *or* (2.23) *written in the form* (3.1)*. For any sequence $\{t_j\colon j\in\mathbb{N}\}$ and any $\varepsilon>0$,*

$$S_{t_j}\Psi \xrightarrow[j\to\infty]{\mathscr{S}'} X \quad \textit{if and only if} \quad S_{t_j}\Psi \xrightarrow[j\to\infty]{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} X, \quad \forall T>0.$$

In either case, $X\in C(\mathbb{R},\mathscr{X})$.

Proof. It suffices to prove that the convergence $S_{t_j}\Psi \xrightarrow{\mathscr{S}'} X$ implies the convergence $S_{t_j}\Psi \xrightarrow{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} X$, for all $T>0$; the converse statement is trivial.

Thus, we assume that

$$S_{t_j}\Psi \xrightarrow{\mathscr{S}'} X, \qquad j\to\infty. \tag{3.3}$$

Assume that, contrary to the statement of the lemma, there is $T>0$, $\varepsilon>0$, and a subsequence t_{j_r}, $r\in\mathbb{N}$, such that $S_{t_{j_r}}\Psi$ does not converge to X in the topology of $C_b([-T,T],\mathscr{Y}^{-\varepsilon})$. It means that there is $\delta>0$ such that

$$\sup_{|t|\le T}\|(S_{t_r}\Psi - X)|_t\|_{\mathscr{Y}^{-\varepsilon}} > \delta, \qquad r\in\mathbb{N}. \tag{3.4}$$

By Assumption 3.1 *(2)*, $\sup_{t\in\mathbb{R}}\|\Psi|_t\|_{\mathscr{X}}<\infty$. Therefore, we can choose a subsequence of $\Psi|_{t_{j_r}}$ weakly convergent to some $Y_0\in\mathscr{X}$. By Assumption 3.1 *(4)*, we can choose a smaller subsequence, denoted $\Psi|_{t_{j_{r_m}}}$, $m\in\mathbb{N}$, which converges to Y_0 in the norm $\|\cdot\|_{\mathscr{Y}^{-\varepsilon}}$. Let $Y\in C(\mathbb{R},\mathscr{X})$ be a solution to equation (3.1) with the initial data $Y|_{t=0}=Y_0$, which exists due to Assumption 3.1 *(1)*. Due to the continuous dependence on the initial data (Assumption 3.1 *(3)*),

$$S_{t_{j_{r_m}}}\Psi \xrightarrow[m\to\infty]{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} Y. \tag{3.5}$$

The convergence (3.3) implies that $X=Y$ for $|t|\le T$, and we see that (3.5) contradicts (3.4). This contradiction finishes the proof. □

Now let us prove the existence of omega-limit trajectories.

Proposition 3.8 (Existence of omega-limit trajectories)**.** *Let $\Psi\in C(\mathbb{R},\mathscr{X})$ be a solution to equation* (3.1) *with the initial data $\Psi|_{t=0}=\Psi_0\in\mathscr{X}$.*

(1) Let $\varepsilon\in(0,\varepsilon_0)$. For any sequence $t_j\to+\infty$ there exists a subsequence t_{j_r}, $r\in\mathbb{N}$, such that, for any $T>0$,

$$S_{t_{j_r}}\Psi \xrightarrow[r\to\infty]{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} X,$$

for some $X\in C(\mathbb{R},\mathscr{X})$.

(2) $X(t)$ satisfies (3.1)*:*

$$\dot X = \mathcal{J}\mathcal{H}'(X),$$

which is understood in the sense of distributions.

(3) There is the bound

$$\sup_{t\in\mathbb{R}}\|X|_t\|_{\mathscr{X}}<\infty.$$

Remark 3.9. According to Definition 3.6, the function X appearing as a limit in Part *(1)* is called *omega-limit trajectory.*

Proof. First, let us note that for any $\Psi_0 \in \mathscr{X}$, Assumption 3.1 *(1)* provides a solution $\Psi \in C(\mathbb{R}, \mathscr{X})$ to equation (3.1) with the initial data $\Psi|_{t=0} = \Psi_0$.

Let $t_j > 0$, $j \in \mathbb{N}$ be a sequence such that $t_j \to \infty$. Fix $\varepsilon \in (0, \varepsilon_0)$, with ε_0 from Assumption 3.1 *(3)*. Since $\Psi|_{t_j}$ are bounded in $\mathscr{X}$ (Assumption 3.1 *(2)*) and the embedding $\mathscr{X} \subset \mathscr{Y}^{-\varepsilon}$ is compact by Assumption 3.1 *(4)*, we can pick a subsequence t_{j_r}, $r \in \mathbb{N}$, of $\{t_j \colon j \in \mathbb{N}\}$ such that

$$\Psi|_{t_{j_r}} \xrightarrow[r\to\infty]{\mathscr{Y}^{-\varepsilon}} X_0, \tag{3.6}$$

for some $X_0 \in \mathscr{X}$. By Assumption 3.1 *(1)*, there is a solution $X \in C(\mathbb{R}, \mathscr{X})$ to equation (3.1) with the initial data $X|_{t=0} = X_0 \in \mathscr{X}$.

Let S_τ be the time shift operators on $C(\mathbb{R}, \mathscr{S}')$, introduced in (3.2). By (3.6) and the continuous dependence of solutions on initial data (Assumption 3.1 *(3)*), for any $T > 0$, there is the convergence

$$S_{t_{j_r}} \Psi \xrightarrow[r\to\infty]{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} X. \tag{3.7}$$

This finishes the proof of Part *(1)*.

The limit (3.7), combined with equation (3.1), proves Part *(2)*.

Part *(3)* of the Proposition follows from $X|_{t=0} = X_0 \in \mathscr{X}$ and Assumption 3.1 *(2)*. □

3.2 Global attractor and trajectory attractor

Definition 3.10 (Global attractor)**.** The attractor $\mathcal{A} \subset \mathscr{X}$ is the set of the initial data of all omega-limit trajectories:

$$\mathcal{A} = \omega(\mathscr{X}) = \bigcup_{\Psi(0)\in\mathscr{X}} \omega(\Psi).$$

Definition 3.11 (Trajectory attractor)**.** The trajectory attractor (or *path attractor*) $\mathfrak{A}$ of equation (3.1) is the set of all omega-limit trajectories of all finite energy solutions:

$$\mathfrak{A} = \{X \in C(\mathbb{R}, \mathscr{X}) \colon \exists \Psi \in C(\mathbb{R}, \mathscr{X}),\ \dot{\Psi} = \mathcal{J}\mathcal{H}'(\Psi),\ \exists t_j \to \infty,\quad S_{t_j}\Psi \xrightarrow[j\to\infty]{\mathscr{S}'} X\}.$$

Lemma 3.12. *There is the following relation between $\mathcal{A}$ and $\mathfrak{A}$:*

$$\mathcal{A} = \{\Psi|_{t=0} \colon \Psi \in \mathfrak{A}\},$$
$$\mathfrak{A} = \{\Psi \in C(\mathbb{R}, \mathscr{X}) \colon \dot{\Psi} = \mathcal{J}\mathcal{H}'(\Psi),\ \Psi|_{t=0} \in \mathcal{A}\}.$$

Proof. Assume that $X_0 \in \mathcal{A}$. By Definition 3.10, this means that there is $\Psi \in C(\mathbb{R}, \mathscr{X})$ which is a solution to (3.1) and a sequence $t_j \to +\infty$ such that

$$\Psi|_{t_j} \xrightarrow[j\to\infty]{\mathscr{S}'} X_0.$$

By Lemma 3.4,

$$\Psi|_{t_j} \xrightarrow[j\to\infty]{\mathscr{Y}^{-\varepsilon}} X_0 \in \mathscr{X}.$$

Let $X \in C(\mathbb{R}, \mathscr{X})$ be a solution to (3.1) with the initial data $X|_{t=0} = X_0$. Then, due to the continuous dependence on the initial data (Assumption 3.1 *(3)*), for any $T > 0$ and $\varepsilon > 0$,

$$S_{t_j}\Psi \xrightarrow[j\to\infty]{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} X.$$

By Definition 3.11, $X \in \mathfrak{A}$.

Now, conversely, assume that $X \in \mathfrak{A}$. Then there is $\Psi \in C(\mathbb{R}, \mathscr{X})$ and a sequence $t_j \to +\infty$ such that $S_{t_j}\Psi \xrightarrow{\mathscr{S}'} X$, and, by Lemma 3.7, for any $T > 0$ and $\varepsilon > 0$,

$$S_{t_j}\Psi \xrightarrow[j\to\infty]{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} X.$$

In particular,

$$\big(S_{t_j}\Psi\big)|_{t=0} = \Psi|_{t_j} \xrightarrow[j\to\infty]{\mathscr{Y}^{-\varepsilon}} X|_{t=0}.$$

By Definition 3.10, $X|_{t=0} \in \mathcal{A}$. □

Lemma 3.13. *Let $\Psi \in C(\mathbb{R}, \mathscr{X})$ be a solution to $\dot{\Psi} = \mathcal{J}\mathcal{H}'(\Psi)$. Then, for any $\varepsilon > 0$,*

$$\Psi(t) \xrightarrow[t\to\infty]{\mathscr{Y}^{-\varepsilon}} \mathcal{A}.$$

Proof. Assume that, on the contrary, there are $\delta > 0$, a solution $\Psi \in C(\mathbb{R}, \mathscr{X})$ to (3.1), and a sequence $t_j \to \infty$ such that

$$\operatorname{dist}_{\mathscr{Y}^{-\varepsilon}}(\Psi|_{t_j}, \mathcal{A}) \geq \delta. \tag{3.8}$$

Since $\Psi|_{t_j}$ are bounded in $\mathscr{X}$, there is a subsequence $\Psi|_{t_{j_r}}$, $r \in \mathbb{N}$, which converges in the topology of $\mathscr{Y}^{-\varepsilon}$ to some X_0. By Definition 3.10, $X_0 \in \mathcal{A}$, contradicting (3.8). □

Lemma 3.14. *If the set $\mathfrak{A}$ of all omega-limit trajectories coincides with the set $\mathfrak{S}$ of all solitary waves, then, for any finite energy solution $\Psi(t) \in C(\mathbb{R}, \mathscr{X})$ and any $\varepsilon > 0$, one has*

$$\Psi|_t \xrightarrow[t\to\infty]{\mathscr{Y}^{-\varepsilon}} \mathcal{S}.$$

Proof. Since all omega-limit trajectories are solitary waves, Lemma 3.12 implies that

$$\mathcal{A} = \{X(0)\colon X \in \mathfrak{A}\} = \mathcal{S}.$$

Now the proof follows from Lemma 3.13. □

Chapter 4

Klein-Gordon with one oscillator

In this chapter, we give the proof of the global attraction to solitary waves for equation (2.1),

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \delta(x)F(\psi(0,t)), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{4.1}$$

which describes the Klein-Gordon string interacting with a nonlinear oscillator located at the origin (Theorem 2.6).

We present the argument from [Kom03] and [KK07a], slightly shortened since we prove the convergence to the attractor in the $\mathscr{Y}^{-\varepsilon}$-norm with $\varepsilon > 0$ (as opposed to the convergence in the local energy norm $\mathscr{Y}^0$ proved in [Kom03] and [KK07a]). This argument illustrates the main common points of the arguments for other models considered in this monograph.

Pick the initial data

$$(\psi_0, \pi_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}). \tag{4.2}$$

According to Theorem B.1 *(1)* there exists a global solution to (4.1), which we denote $\psi(x,t)$, with the initial data

$$(\psi, \dot{\psi})|_{t=0} = (\psi_0, \pi_0). \tag{4.3}$$

By Theorem B.1 *(4)*,

$$(\psi, \dot{\psi}) \in C_b(\mathbb{R}, \mathscr{X}). \tag{4.4}$$

4.1 Compactness and omega-limit trajectories

We fix $\varepsilon \in (0, 1/2)$. According to Proposition 3.8 applied to the model (4.1) (see Remark 3.2), for any sequence $t_j \to +\infty$ there exists a subsequence t_{j_r}, $r \in \mathbb{N}$, such that, for any $T > 0$,

$$S_{t_{j_r}}(\psi, \dot{\psi}) \xrightarrow[r\to\infty]{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} (\beta, \dot{\beta}), \tag{4.5}$$

for some $\beta \in C(\mathbb{R}, H^1(\mathbb{R}))$ with $\dot{\beta} \in C(\mathbb{R}, L^2(\mathbb{R}))$. Recall that the space $\mathscr{Y}^{-\varepsilon}$ is introduced in Definition 2.4 *(3)* and S_τ is the time shift operator (3.2) defined on $C(\mathbb{R}, \mathscr{S}')$.

The function $\beta(x,t)$ satisfies the equation

$$\ddot{\beta}(x,t) = \beta''(x,t) - m^2\beta(x,t) + \delta(x)F(\beta(x,t)), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{4.6}$$

which is understood in the sense of distributions. There is the bound

$$\sup_{t\in\mathbb{R}} \|(\beta, \dot{\beta})|_t\|_{\mathscr{X}} < \infty. \tag{4.7}$$

By Lemma 3.14, to conclude the proof of Theorem 2.6, it suffices to check that every omega-limit trajectory $\beta(x,t)$ belongs to the set of solitary waves.

Let χ be the solution to the linear Klein-Gordon equation with the initial data (4.2):

$$\ddot{\chi}(x,t)=\chi''(x,t)-m^2\chi(x,t),\qquad (\chi,\dot{\chi})\big|_{t=0}=(\psi_0(x),\pi_0(x)). \tag{4.8}$$

Lemma 4.1 (Local energy decay of the dispersive component)**.** *There is a local energy decay for χ:*

$$\lim_{t\to\infty}\|(\chi,\dot{\chi})|_t\|_{\mathscr{Y}^{-\varepsilon}}=0,\qquad \forall\varepsilon\ge 0. \tag{4.9}$$

See Corollary C.2 in Appendix C.

Remark 4.2. Lemma 4.1 means that the dispersive component χ does not give any contribution to the omega-limit trajectories (see Definition 3.6).

4.2 Absolute continuity for large frequencies

Define

$$\varphi(x,t)=\begin{cases}0, & t<0;\\ \psi(x,t)-\chi(x,t), & t\ge 0.\end{cases} \tag{4.10}$$

Then $\varphi(x,t)$ solves the following Cauchy problem:

$$\ddot{\varphi}(x,t)=\varphi''(x,t)-m^2\varphi(x,t)+\delta(x)f(t),\qquad (\varphi,\dot{\varphi})\big|_{t\le 0}=(0,0), \tag{4.11}$$

where

$$f(t):=\Theta(t)F(\psi(0,t)),\qquad t\in\mathbb{R}, \tag{4.12}$$

where $\Theta(t)$ is the Heaviside step function. Recall that $(\psi,\dot{\psi})\in C_b(\mathbb{R},\mathscr{X})$ by (4.4). On the other hand, since $\chi(x,t)$ is a finite energy solution to the free Klein-Gordon equation, we also have $(\chi,\dot{\chi})\in C_b(\mathbb{R},\mathscr{X})$. It follows that $\varphi(x,t)=\Theta(t)(\psi(x,t)-\chi(x,t))$ is finite in the energy norm:

$$(\varphi,\dot{\varphi})\in C_b(\mathbb{R},\mathscr{X}),\qquad t\in\mathbb{R}. \tag{4.13}$$

Let $k(\omega)$ be the analytic function with the domain $D:=\mathbb{C}\backslash((-\infty,-m]\cup[m,+\infty))$ such that

$$k(\omega)=\sqrt{\omega^2-m^2},\qquad \operatorname{Im}k(\omega)>0,\qquad \omega\in D. \tag{4.14}$$

Let us also denote its limit values at the real axis by

$$k_\pm(\omega):=k(\omega\pm i0),\qquad \omega\in\mathbb{R}. \tag{4.15}$$

As illustrated on Figure 4.1 (where all square roots take positive values), we have:

$$\begin{array}{lll} k_-(\omega)=k_+(\omega) & \text{for} & -m\le\omega\le m,\\ k_-(\omega)=-k_+(\omega) & \text{for} & \omega\in\mathbb{R}\backslash(-m,m),\\ \omega\,k_+(\omega)>0 & \text{for} & \omega\in\mathbb{R}\backslash[-m,m]. \end{array} \tag{4.16}$$

Let us consider the Fourier transform

$$\tilde{\varphi}(x,\omega)=\mathcal{F}_{t\to\omega}[\varphi(x,t)]=\int_0^\infty e^{i\omega t}\varphi(x,t)\,dt,\qquad (x,\omega)\in\mathbb{R}^2. \tag{4.17}$$

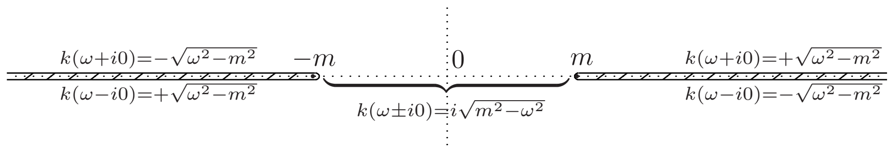

Figure 4.1: Domain D and the boundary values $k_\pm(\omega) := k(\omega \pm i0)$, $\omega \in \mathbb{R}$.

This is a continuous function of $x \in \mathbb{R}$ with values in tempered distributions of $\omega \in \mathbb{R}$, which satisfies the following equation (Cf. (4.11)):

$$-\omega^2\tilde{\varphi}(x,\omega) = \partial_x^2\tilde{\varphi}(x,\omega) - m^2\tilde{\varphi}(x,\omega) + \delta(x)\tilde{f}(\omega), \qquad (x,\omega) \in \mathbb{R}^2, \tag{4.18}$$

where

$$\tilde{f}(\omega) = \mathcal{F}_{t\to\omega}[f(t)](\omega) = \int_0^\infty e^{i\omega t} f(t)\,dt, \qquad \omega \in \mathbb{R}. \tag{4.19}$$

Proposition 4.3 (Spectral representation)**.** *There is the following relation:*

$$\tilde{\varphi}(x,\omega) = -\frac{e^{ik_+(\omega)|x|}}{2ik_+(\omega)}\tilde{f}(\omega), \qquad x \in \mathbb{R}, \quad \omega \in \mathbb{R}\backslash\{\pm m\}. \tag{4.20}$$

Proof. According to (4.10), $\varphi|_{t\le 0} \equiv 0$, hence the formula (4.17) could be extended to

$$\omega \in \mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\},$$

defining complex Fourier transform of $\varphi(x,t)$:

$$\tilde{\varphi}(x,\omega) = \int_0^\infty e^{i\omega t}\varphi(x,t)\,dt, \qquad x \in \mathbb{R}, \quad \operatorname{Im}\omega \ge 0. \tag{4.21}$$

Similarly, since $f|_{t<0} = 0$, the formula (4.19) could be extended to $\omega \in \mathbb{C}^+$:

$$\tilde{f}(\omega) = \int_0^\infty e^{i\omega t} f(t)\,dt, \qquad \operatorname{Im}\omega \ge 0. \tag{4.22}$$

Due to (4.13), $\tilde{\varphi}(\cdot,\omega)$ is an H^1-valued analytic function of $\omega \in \mathbb{C}^+$, and, by (4.11), it satisfies

$$-\omega^2\tilde{\varphi}(x,\omega) = \partial_x^2\tilde{\varphi}(x,\omega) - m^2\tilde{\varphi}(x,\omega) + \delta(x)\tilde{f}(\omega), \quad \operatorname{Im}\omega \ge 0. \tag{4.23}$$

For $\omega \in \mathbb{C}^+$, the solution $\tilde{\varphi}(x,\omega)$, could be written as a linear combination of the fundamental solutions

$$G_\pm(x,\omega) = \frac{e^{\pm ik(\omega)|x|}}{\pm 2ik(\omega)}, \qquad x \in \mathbb{R}, \quad \omega \in D,$$

with $k(\omega)$ defined in (4.14) and D plotted on Figure 4.1. These fundamental solutions satisfy

$$G_\pm''(x,\omega) + (\omega^2 - m^2)G_\pm(x,\omega) = \delta(x), \qquad x \in \mathbb{R}, \quad \omega \in D.$$

We use the standard "limiting absorption principle" for the selection of the appropriate fundamental solution. We proved that, for $\omega \in \mathbb{C}^+$, $\tilde{\varphi}(\cdot,\omega) \in H^1$; on the other hand, for $\omega \in \mathbb{C}^+$, only the function $G_+(\cdot,\omega)$ is in H^1 due to the definition (4.14), while $G_-(\cdot,\omega)$ is not. This proves that

$$\tilde{\varphi}(x,\omega) = -G_+(x,\omega)\tilde{f}(\omega) = -\frac{e^{ik(\omega)|x|}}{2ik(\omega)}\tilde{f}(\omega), \qquad \omega \in \mathbb{C}^+. \tag{4.24}$$

Lemma 4.4. *(1)*

$$\tilde{f}(\omega)=\lim_{\epsilon\to 0+}\tilde{f}(\omega+i\epsilon),\qquad \omega\in\mathbb{R}, \tag{4.25}$$

with the convergence in $\mathscr{S}'(\mathbb{R})$.

(2)

$$\tilde{\varphi}(x,\omega)=\lim_{\epsilon\to 0+}\tilde{\varphi}(x,\omega+i\epsilon),\qquad \omega\in\mathbb{R}, \tag{4.26}$$

with the convergence in $\mathscr{S}'(\mathbb{R},H^1(\mathbb{R}))$.

Proof. By (4.4), the function $\psi(\cdot,t)$ is bounded in $H^1(\mathbb{R})$, uniformly in $t\in\mathbb{R}$. By the Sobolev embedding, $\psi(0,\cdot)\in C_b(\mathbb{R})$, hence the function $f(t)=\Theta(t)F(\psi(0,t))$ is bounded:

$$f(\cdot)\in C_b(\mathbb{R}). \tag{4.27}$$

Since $f|_{t<0}\equiv 0$, one has:

$$f(t)=\lim_{\epsilon\to 0+}f(t)e^{-\epsilon t},$$

where the convergence is in $C_b(\mathbb{R})$. The convergence (4.25) takes place by the continuity of the Fourier transform in the space of tempered distributions.

Now we need to consider $\tilde{\varphi}(x,\omega)$ for $\omega\in\mathbb{R}$. Since $\varphi\in C_b(\mathbb{R},H^1(\mathbb{R}))$ by (4.13) and $\varphi|_{t\le 0}\equiv 0$, we have

$$\varphi(x,t)=\lim_{\epsilon\to 0+}\varphi(x,t)e^{-\epsilon t}, \tag{4.28}$$

where the convergence holds in e.g. $\mathscr{S}'(\mathbb{R},H^1(\mathbb{R}))$, which is the space of H^1-valued tempered distributions. The Fourier transform $\tilde{\varphi}(x,\omega)=\mathcal{F}_{t\to\omega}[\varphi(x,t)]$ is defined as a tempered H^1-valued distribution of $\omega\in\mathbb{R}$. As it follows from (4.28) and the continuity of the Fourier transform in $\mathscr{S}'(\mathbb{R})$, for each $x\in\mathbb{R}$, the function $\tilde{\varphi}(x,\omega)$ of $\omega\in\mathbb{R}$ can be considered as the boundary value of the analytic function $\tilde{\varphi}(x,\omega)$, $\omega\in\mathbb{C}^+$. This proves (4.26). □

Now we can extend the relation (4.24) to $\omega\in\mathbb{R}$. We use Lemma 4.4 *(1)* and *(2)* to take the limit $\operatorname{Im}\omega\to 0+$ in both sides of the relation (4.24), and keep in mind that $k(\omega)$ is smooth for $\omega\in\mathbb{C}^+\cup\mathbb{R}\backslash\{\pm m\}$ and hence is a multiplicator in the space of distributions. □

Remark 4.5. One can use the fact that for each $x\in\mathbb{R}$, the distribution $\tilde{\varphi}(x,\omega)$ is a quasimeasure (see Remark 4.6), while the factor in (4.24) is a multiplicator in the *space of quasimeasures* for all $\omega\in\mathbb{C}^+\cup\mathbb{R}$. Then the formula (4.20) follows for $\omega\in\mathbb{R}$.

Remark 4.6. A tempered distribution $\mu(\omega)\in\mathscr{S}'(\mathbb{R})$ is called a *quasimeasure* if

$$\check{\mu}(t)=\mathcal{F}^{-1}_{\omega\to t}[\mu(\omega)]\in C_b(\mathbb{R}).$$

For more details on quasimeasures and multiplicators in the space of quasimeasures, see Appendix D.

Proposition 4.7 (Absolute continuity of spectrum)**.** *The distribution* $\tilde{f}(\omega)$ *is absolutely continuous for* $\omega\in\mathbb{R}\backslash[-m,m]$*, and moreover*

$$\int_{\mathbb{R}\backslash[-m,m]}|\tilde{f}(\omega)|^2\frac{d\omega}{\omega k_+(\omega)}<\infty, \tag{4.29}$$

where $\omega k_+(\omega)>0$ *for* $\omega\in\mathbb{R}\backslash[-m,m]$ *by* (4.16).

Proof. We use the Paley-Wiener arguments. Namely, the Parseval identity and (4.13) imply that

$$\int_{\mathbb{R}}\|\tilde{\varphi}(\cdot,\omega+i\epsilon)\|^2_{L^2}\,d\omega=2\pi\int_0^\infty e^{-2\epsilon t}\|\varphi(\cdot,t)\|^2_{L^2}\,dt\le\frac{\mathrm{const}}{\epsilon},\quad \epsilon>0. \tag{4.30}$$

On the other hand, we can calculate the term in the left-hand side of (4.30) exactly. According to (4.24),

$$\tilde{\varphi}(x, \omega + i\epsilon) = -\frac{e^{ik(\omega+i\epsilon)|x|}}{2ik(\omega + i\epsilon)} \tilde{f}(\omega + i\epsilon),$$

hence (4.30) results in

$$\epsilon \int_{\mathbb{R}} \frac{\|e^{ik(\omega+i\epsilon)|x|}\|_{L^2}^2}{|k(\omega + i\epsilon)|^2} |\tilde{f}(\omega + i\epsilon)|^2 \, d\omega \le \text{const}, \qquad \epsilon > 0. \tag{4.31}$$

Here is a crucial observation about the norm of $e^{ik(\omega+i\epsilon)|x|}$.

Lemma 4.8. *(1) For $\omega \in \mathbb{R}\backslash(-m, m)$,*

$$\lim_{\epsilon \to 0+} \epsilon \frac{\|e^{ik(\omega+i\epsilon)|x|}\|_{L^2}^2}{|\omega k(\omega + i\epsilon)|^2} = \frac{1}{\omega k_+(\omega)}. \tag{4.32}$$

(2) For any $\delta > 0$ there exists $\epsilon_\delta > 0$ such that for $\omega \in \mathbb{R}\backslash[-m - \delta, m + \delta]$ and $\epsilon \in (0, \epsilon_\delta)$,

$$\epsilon \frac{\|e^{ik(\omega+i\epsilon)|x|}\|_{L^2}^2}{|\omega k(\omega + i\epsilon)|^2} \ge \frac{1}{2\omega k_+(\omega)}. \tag{4.33}$$

Remark 4.9. The asymptotic behavior of the L^2-norm of $e^{ik(\omega+i\epsilon)}$ stated in the lemma is easy to understand: for $\omega \in \mathbb{R}\backslash[-m, m]$, this norm is finite for $\epsilon > 0$ due to the small positive imaginary part of $k(\omega + i\epsilon)$, but it becomes unboundedly large when $\epsilon \to 0+$. Let us also mention that the expression in the left-hand side of (4.32) is easy to evaluate in the momentum space. Since

$$\mathcal{F}_{x \to \xi}\left[\frac{e^{ik(\omega+i\epsilon)|x|}}{2\omega k(\omega + i\epsilon)}\right] = \frac{1}{\xi^2 + m^2 - (\omega + i\epsilon)^2} = \frac{1}{\xi^2 - k_1^2},$$

where $k_1 = k(\omega + i\epsilon) \in \mathbb{C}^+$, we have:

$$\frac{\|e^{ik(\omega+i\epsilon)|x|}\|_{L^2}^2}{4|\omega k(\omega + i\epsilon)|^2} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{d\xi}{|\xi^2 - k_1^2|^2} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{d\xi}{(\xi + k_1)(\xi - k_1)(\xi + \bar{k}_1)(\xi - \bar{k}_1)}.$$

Closing the contour of integration at $\xi \to +i\infty$ and using the Cauchy Residue Theorem (note that $k_1 \in \mathbb{C}^+$ and $-\bar{k}_1 \in \mathbb{C}^+$), one gets:

$$\frac{\|e^{ik(\omega+i\epsilon)|x|}\|_{L^2}^2}{4|\omega k(\omega + i\epsilon)|^2} = \frac{i}{2(k_1^2 - \bar{k}_1^2)} \left(\frac{1}{k_1} + \frac{1}{\bar{k}_1}\right).$$

The relation (4.32) follows after we note that $k_1^2 - \bar{k}_1^2 = (\omega + i\epsilon)^2 - (\omega - i\epsilon)^2 = 4i\omega\epsilon$.

Substituting (4.33) into (4.31), we get:

$$\int_{|\omega| \ge m+\delta} |\tilde{f}(\omega + i\epsilon)|^2 \frac{d\omega}{\omega k_+(\omega)} \le 2C, \qquad 0 < \epsilon < \epsilon_\delta, \tag{4.34}$$

with the same C as in (4.31). We conclude that for each $\delta > 0$ the set of functions

$$g_{\delta,\epsilon}(\omega) = \frac{\tilde{f}(\omega + i\epsilon)}{|\omega k_+(\omega)|^{1/2}}, \qquad 0 < \epsilon < \epsilon_\delta,$$

defined for $\omega \in \Omega_\delta$, is bounded in $L^2(\mathbb{R}\backslash[-m - \delta, m + \delta])$, and hence is weakly compact. The convergence of the distributions (4.26) implies the following weak convergence in $L^2(\mathbb{R}\backslash[-m - \delta, m + \delta])$:

$$g_{\delta,\epsilon} \rightharpoonup g_\delta, \qquad \epsilon \to 0+,$$

where the limit function $g_\delta(\omega)$ coincides with the distribution $\tilde{f}(\omega)|\omega k_+(\omega)|^{-1/2}$ restricted onto $\mathbb{R}\backslash[-m - \delta, m + \delta]$. It remains to note that, by (4.34), the norms of all functions g_δ, $\delta > 0$, are bounded in $L^2(\mathbb{R}\backslash[-m - \delta, m + \delta])$ by a constant independent on δ, hence (4.29) follows. □

4.3 Spectral analysis of omega-limit trajectories

By Lemma 4.1, as $t \to \infty$, the dispersive component $\chi(\cdot, t)$ converges to zero in $\mathscr{Y}^{-\varepsilon}$, for any $\varepsilon \geq 0$. On the other hand, according to (4.5), the functions $\psi(x, t_{j_r} + t)$ converge to $\beta(x, t)$ as $r \to \infty$, in the topology of $C_b([-T, T], \mathscr{Y}^{-\varepsilon})$, for any $T > 0$ and $0 < \varepsilon < 1/2$. Hence, the functions $\varphi(x, t_{j_r} + t) = \Theta(t_{j_r} + t)(\psi(x, t_{j_r} + t) - \chi(x, t_{j_r} + t))$ also converge to $\beta(x, t)$:

$$\varphi(x, t_{j_r} + t) \xrightarrow[r\to\infty]{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} \beta(x, t), \tag{4.35}$$

for any $T > 0$ and $0 < \varepsilon < 1/2$.

For brevity, we denote

$$\upbeta(t) := \beta(0, t), \tag{4.36}$$

$$g(t) := F(\beta(0, t)). \tag{4.37}$$

By (4.6), the function $\tilde{\beta}(x, \omega)$, which is the Fourier transform of $\beta(x, t)$, satisfies the equation

$$-\omega^2 \tilde{\beta}(x, \omega) = \tilde{\beta}''(x, \omega) - m^2 \tilde{\beta}(x, \omega) + \delta(x)\tilde{g}(\omega), \qquad (x, \omega) \in \mathbb{R}^2, \tag{4.38}$$

valid in the sense of tempered distributions of $(x, \omega) \in \mathbb{R}^2$. Above, $\tilde{g}(\omega)$ is the Fourier transform of $g(t)$. According to (4.7), $\tilde{\beta}(x, \omega)$ is a continuous function of $x \in \mathbb{R}$ with values in tempered distributions of $\omega \in \mathbb{R}$.

Lemma 4.10. *Let $u \in \mathscr{S}'(\mathbb{R})$ and $\{t_j \colon j \in \mathbb{N}\}$ be such that $\lim_{j\to\infty} t_j = \infty$. If*

$$e^{i\omega t_j} u \xrightarrow{\mathscr{S}'} v \in \mathscr{S}'(\mathbb{R}) \tag{4.39}$$

and $u|_{\mathcal{I}} \in L^1_{loc}(\mathcal{I})$ for some open set $\mathcal{I} \subset \mathbb{R}$, then $v|_{\mathcal{I}} = 0$.

Proof. Pick any $\zeta \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} \zeta \subset \mathcal{I}$. Then, due to the convergence (4.39), $\langle \zeta, e^{i\omega t_{j_r}} u \rangle \longrightarrow \langle \zeta, v \rangle$. On the other hand, $\langle \zeta, e^{i\omega t_{j_r}} u \rangle = \mathcal{F}_{\omega\to t}[\zeta(\omega)u(\omega)](t_{j_r}) \to 0$, as the Fourier transform of the L^1-function ζu. It follows that $\langle \zeta, v \rangle = 0$. Since ζ is an arbitrary smooth function with support in $\mathcal{I}$, we are done. □

Lemma 4.11 (Compactness of spectrum)**.**

$$\operatorname{supp} \tilde{\upbeta} \subset [-m, m].$$

Proof. By (4.35), for any $x \in \mathbb{R}$, we have:

$$\varphi(x, t_{j_r} + t) \xrightarrow{\mathscr{S}'} \beta(x, t), \qquad t \in \mathbb{R}. \tag{4.40}$$

Since $\varphi(x, t_j + t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} e^{-i\omega t_j} \tilde{\varphi}(x, \omega)\, d\omega$, the relation (4.40) implies that, for any $x \in \mathbb{R}$,

$$e^{-i\omega t_{j_r}} \tilde{\varphi}(x, \omega) \xrightarrow{\mathscr{S}'} \tilde{\beta}(x, \omega), \qquad r \to \infty. \tag{4.41}$$

By Proposition 4.7, $\tilde{\varphi}(0, \omega)$ is locally L^2 for $\omega \in \mathbb{R}\backslash[-m, m]$. Therefore, the convergence (4.41) and Lemma 4.10 show that $\tilde{\upbeta}(\omega) := \tilde{\beta}(0, \omega)$ vanishes for $\omega \in \mathbb{R}\backslash[-m, m]$. □

Lemma 4.12 (Spectral representation for β)**.** *The distribution $\tilde{\beta}(x, \omega)$ admits the following representation:*

$$\tilde{\beta}(x, \omega) = -\frac{e^{ik_+(\omega)|x|}}{2ik_+(\omega)} \tilde{g}(\omega), \qquad x \in \mathbb{R}, \quad \omega \in \mathbb{R}\backslash\{\pm m\}. \tag{4.42}$$

Proof. Due to (4.5), we also have

$$f(t_{j_r}+t) := F(\psi(0,t_{j_r}+t)) \overset{\mathscr{S}'}{\longrightarrow} F(\beta(0,t)) =: g(t),$$

hence, due to the continuity of the Fourier transform in $\mathscr{S}'$,

$$e^{i\omega t_{j_r}}\tilde{f}(\omega) \overset{\mathscr{S}'}{\longrightarrow} \tilde{g}(\omega), \qquad \omega\in\mathbb{R}. \tag{4.43}$$

Now the statement of the lemma can be proved by starting with the relation (4.20) proved in Proposition 4.3 and applying the limits (4.41) and (4.43). When taking the limits, we use the fact that $k(\omega)$ is smooth for $\omega\in\mathbb{R}\backslash\{\pm m\}$ and hence the expression $\frac{e^{ik(\omega)|x|}}{2ik(\omega)}$, $\omega\in\mathbb{R}\backslash\{\pm m\}$, is a multiplicator in $\mathscr{S}'$ away from $\omega=\pm m$. □

Lemma 4.13. *The points $\omega=\pm m$ can not be isolated points of the support of $\tilde{g}(\omega)$.*

Proof. Let us assume that, on the contrary, $\omega_0=m$ or $-m$ is an isolated point of the support of $\tilde{g}$. Pick an open neighborhood U of ω_0 such that $U\cap\operatorname{supp}\tilde{g}=\{\omega_0\}$. Pick $\zeta\in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\zeta\tilde{g}\subset U$, $\zeta(\omega_0)=1$. Then

$$\zeta(\omega)\tilde{g}(\omega)=M\delta(\omega-m), \qquad M\in\mathbb{C}\backslash\{0\}, \tag{4.44}$$

where the derivatives of $\delta(\omega-m)$ do not appear since $\check{\zeta}*g(t)$ is bounded. By (4.42), we have, for any $x\in\mathbb{R}$, $U\cap\operatorname{supp}\tilde{\beta}(x,\cdot)\subset\{\omega_0\}$, hence

$$\zeta(\omega)\tilde{\beta}(x,\omega)=\delta(\omega-\omega_0)b(x), \qquad b\in H^1(\mathbb{R}). \tag{4.45}$$

Again, the terms with the derivatives of $\delta(\omega-\omega_0)$ are prohibited since $\langle\alpha,\check{\zeta}*\beta(\cdot,t)\rangle$ are bounded for any $\alpha\in C_0^\infty(\mathbb{R})$. The inclusion $b(x)\in H^1(\mathbb{R})$ is due to $\tilde{\beta}\in\mathscr{S}'(\mathbb{R},H^1(\mathbb{R}))$.

Multiplying (4.38) by $\zeta(\omega)$ and taking into account (4.44), (4.45), and the relation $\omega_0^2=m^2$, we see that the distribution $b(x)$ satisfies the equation

$$0=b''(x)+M\delta(x).$$

$M\neq 0$ would lead to $b\notin H^1(\mathbb{R})$, contradicting the inclusion $\tilde{\beta}\in\mathscr{S}'(\mathbb{R},H^1(\mathbb{R}))$. This contradiction shows that $\omega=\pm m$ can not be isolated points of the support of $\tilde{g}$, finishing the proof. □

Lemma 4.14. $\operatorname{supp}\tilde{g}(\cdot)\subset\operatorname{supp}\tilde{\beta}$.

Proof. By Lemma 4.12,

$$\operatorname{supp}\tilde{g}(\cdot)\subset\operatorname{supp}\tilde{\beta}\cup\{\pm m\}.$$

Now the statement of the lemma follows from Lemma 4.13. □

Lemma 4.15 (Reduction to point spectrum)**.** *Either* $\operatorname{supp}\tilde{\beta}=\{\omega_\star\}$ *for some* $\omega_\star\in[-m,m]$ *or* $\tilde{\beta}=0$.

Proof. By (2.20), the Fourier transform $\tilde{g}(\omega)$ of $g(t):=F(\beta(0,t))$ is given by

$$\tilde{g}=-\sum_{l=1}^{p}2l\,u_l\,\underbrace{(\tilde{\beta}*\tilde{\bar{\beta}})*\ \dots\ *(\tilde{\beta}*\tilde{\bar{\beta}})}_{l-1}*\tilde{\beta}. \tag{4.46}$$

Now we will use the Titchmarsh Convolution Theorem [Tit26] which could be stated as follows:

For any u, $v\in\mathscr{E}'(\mathbb{R})$, $\sup\operatorname{supp}(u*v)=\sup\operatorname{supp}u+\sup\operatorname{supp}v$.

Above, $\mathscr{E}'(\mathbb{R})$ is the space of compactly supported distributions. For more details and a proof, see Appendix E.

Applying the Titchmarsh Convolution Theorem to the convolutions in (4.46), we obtain the following equality:

$$\sup\operatorname{supp}\tilde{g} \geq \sup\operatorname{supp}\tilde{\upbeta} + (p-1)(\sup\operatorname{supp}\tilde{\upbeta} - \inf\operatorname{supp}\tilde{\upbeta}). \tag{4.47}$$

We used the relation

$$\sup\operatorname{supp}\tilde{\bar{\upbeta}} = -\inf\operatorname{supp}\tilde{\upbeta}.$$

We wrote "$\geq$" in (4.47) because of possible cancellations in the summation in the right-hand side of (4.46). Note that the Titchmarsh Theorem is applicable to each summand in the right-hand side of (4.46) since by Lemma 4.11 the function $\tilde{\upbeta}$ is compactly supported ($\operatorname{supp}\tilde{\upbeta} \subset [-m, m]$).

Comparing (4.47) with the statement of Lemma 4.14, we conclude that

$$(p-1)(\sup\operatorname{supp}\tilde{\upbeta} - \inf\operatorname{supp}\tilde{\upbeta}) = 0. \tag{4.48}$$

Since $p \geq 2$ by (2.20) (which means that the oscillator at $x = 0$ is nonlinear), we conclude that $\operatorname{supp}\tilde{\upbeta}$ consists of at most a single point $\omega_\star \subset [-m, m]$. □

Lemma 4.16. *$\beta(x,t)$ is a solitary wave:*

$$\beta(x,t) = \phi(x)e^{-i\omega_\star t},$$

where $\omega_\star \in (-m, m)$ and $\phi \in H^1(\mathbb{R})$ satisfies

$$-\omega_\star^2\phi(x) = \phi''(x) - m^2\phi(x) + \delta(x)F(\phi(0)), \qquad x \in \mathbb{R}. \tag{4.49}$$

Proof. By Lemma 4.15, $\operatorname{supp}\tilde{\upbeta} \subset \{\omega_\star\}$, with $\omega_\star \in [-m, m]$. Therefore,

$$\tilde{\upbeta}(\omega) = a_1\delta(\omega - \omega_\star), \qquad \text{with some } \ a_1 \in \mathbb{C}. \tag{4.50}$$

Note that the derivatives $\delta^{(k)}(\omega - \omega_\star)$, $k \geq 1$ do not enter the expression for $\tilde{\upbeta}(\omega)$ since $\upbeta(t) = \beta(0,t)$ is a bounded continuous function of t due to the bound (4.7). The relation (4.50), together with (4.46), yield that

$$\tilde{g}(\omega) = g_1\delta(\omega - \omega_\star), \qquad \text{with some } \ g_1 \in \mathbb{C}. \tag{4.51}$$

Now Lemma 4.12 implies that the omega-limit trajectory $\beta(x,t)$ is a solitary wave:

$$\beta(x,t) = \phi(x)e^{-i\omega_\star t}.$$

Since $\tilde{\beta}(x,\omega)$ solves (4.38), $\phi(x)$ satisfies (4.49).

Remark 4.17. By Lemma 4.13, $\omega_\star = \pm m$ could only correspond to the zero solution.

□

According to Lemma 3.14, Lemma 4.16 completes the proof of (2.15).

Chapter 5

Klein-Gordon with several oscillators

In this Chapter, we are going to prove Theorem 2.8, which states the global attraction to the set of solitary waves for all finite energy solutions to the equation

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \sum_{I=1}^{N} \delta(x - X_I)F_I(\psi(X_I,t)), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{5.1}$$

which describes the Klein-Gordon field interacting with oscillators F_I, $1 \le I \le N$, located at the points $X_I \in \mathbb{R}$.

Pick

$$(\psi_0, \pi_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}). \tag{5.2}$$

According to Theorem B.1 *(1)* there exists a global solution to (5.1), which we denote $\psi(x,t)$, with the initial data

$$(\psi, \dot{\psi})|_{t=0} = (\psi_0, \pi_0). \tag{5.3}$$

By Theorem B.1 *(4)*,

$$(\psi, \dot{\psi}) \in C_b(\mathbb{R}, \mathscr{X}). \tag{5.4}$$

5.1 Compactness

Fix $\varepsilon \in (0, 1/2)$. Proposition 3.8, applied to the model (5.1), states that for any sequence $t_j \to +\infty$ there exists a subsequence t_{j_r}, $r \in \mathbb{N}$, such that, for all $T > 0$,

$$S_{t_{j_r}}(\psi, \dot{\psi}) \xrightarrow[r\to\infty]{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} (\beta, \dot{\beta}), \tag{5.5}$$

for some $\beta \in C(\mathbb{R}, H^1(\mathbb{R}))$ with $\dot{\beta} \in C(\mathbb{R}, L^2(\mathbb{R}))$. The function $\beta(x,t)$ satisfies the equation (5.1),

$$\ddot{\beta}(x,t) = \beta''(x,t) - m^2\beta(x,t) + \sum_{I=1}^{N} \delta(x - X_I)F_I(\beta(x,t)), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{5.6}$$

which is understood in the sense of distributions, and obeys the bound

$$\sup_{t\in\mathbb{R}} \|(\beta, \dot{\beta})|_t\|_{\mathscr{X}} < \infty. \tag{5.7}$$

By Lemma 3.14, to conclude the proof of Theorem 2.8, it suffices to check that every omega-limit trajectory $\beta(x,t)$ belongs to the set of solitary waves.

Let $\chi(x,t)$ be the solution to the following Cauchy problem:

$$\ddot{\chi}(x,t) = \chi'' - m^2\chi, \qquad (\chi,\dot{\chi})|_{t=0} = (\psi_0(x), \pi_0(x)), \tag{5.8}$$

where $(\psi_0(x), \pi_0(x))$ is the initial data from (5.2).

Let us denote

$$f_I(t) := \Theta(t) F_I(\psi(X_I,t)), \qquad t \in \mathbb{R}. \tag{5.9}$$

Since $(\psi(x,t), \dot{\psi}(x,t)) \in C_b(\mathbb{R}, \mathscr{X})$ by (5.4), one has $\psi(X_I,\cdot) \in C_b(\mathbb{R})$ for $1 \le I \le N$ by the Sobolev embedding, and hence $f_I(t) \in C_b(\mathbb{R})$.

Define

$$\varphi(x,t) = \begin{cases} 0, & t < 0; \\ \psi(x,t) - \chi(x,t), & t \ge 0. \end{cases}$$

Then $\varphi(x,t)$ satisfies

$$\ddot{\varphi}(x,t) = \partial_x^2 \varphi(x,t) - m^2 \varphi(x,t) + \sum_{I=1}^{N} \delta(x - X_I) f_I(t), \qquad t \ge 0, \tag{5.10}$$

with $(\varphi, \dot{\varphi})|_{t\le 0} = (0,0)$. With both $(\psi, \dot{\psi})$ and $(\chi, \dot{\chi}) \in C_b(\mathbb{R}, \mathscr{X})$, one also has the same inclusion for $\varphi(x,t) = \Theta(t)(\psi(x,t) - \chi(x,t))$:

$$(\varphi(x,t), \dot{\varphi}(x,t)) \in C_b(\mathbb{R}, \mathscr{X}), \qquad t \in \mathbb{R}. \tag{5.11}$$

5.2 Spectral representation

Proposition 5.1. *There is the following representation for φ in terms of f_I, $1 \le I \le N$:*

$$\tilde{\varphi}(x,\omega) = -\sum_{I=1}^{N} \frac{e^{ik_+(\omega)|x - X_I|}}{2ik_+(\omega)} \tilde{f}_I(\omega), \qquad \omega \in \mathbb{R}. \tag{5.12}$$

The function $k_+(\omega)$ is defined in (4.16).

Proof. Let us analyze the complex Fourier transforms of $\varphi(x,t)$:

$$\tilde{\varphi}(x,\omega) = \mathcal{F}_{t\to\omega}[\varphi(x,t)] = \int_0^\infty e^{i\omega t} \varphi(x,t)\, dt, \qquad \omega \in \mathbb{C}^+, \tag{5.13}$$

where $\mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Due to (5.11), $\tilde{\varphi}(\cdot,\omega)$ are H^1-valued analytic functions of $\omega \in \mathbb{C}^+$. Equation (5.10) implies that $\tilde{\varphi}$ satisfies

$$-\omega^2 \tilde{\varphi}(x,\omega) = \partial_x^2 \tilde{\varphi}(x,\omega) - m^2 \tilde{\varphi}(x,\omega) + \sum_{I=1}^{N} \delta(x - X_I) \tilde{f}_I(\omega), \qquad \omega \in \mathbb{C}^+. \tag{5.14}$$

The fundamental solutions $G_\pm(x,\omega) = \dfrac{e^{\pm ik(\omega)|x|}}{\pm 2ik(\omega)}$ satisfy

$$G_\pm''(x,\omega) + (\omega^2 - m^2) G_\pm(x,\omega) = \delta(x), \qquad \omega \in \mathbb{C}^+.$$

The solution $\tilde{\varphi}(x,\omega)$ could be written as a linear combination of these fundamental solutions. Arguing as before (4.24), we see that, as the matter of fact, $\tilde{\varphi}(x,\omega)$ is expressed in terms of G_+ only:

$$\tilde{\varphi}(x,\omega) = -\sum_{I=1}^{N} G_+(x - X_I, \omega) \tilde{f}_I(\omega) = -\sum_{I=1}^{N} \frac{e^{ik(\omega)|x - X_I|}}{2ik(\omega)} \tilde{f}_I(\omega), \qquad \omega \in \mathbb{C}^+. \tag{5.15}$$

Lemma 5.2. *(1)*

$$\tilde{f}_I(\omega) = \lim_{\epsilon\to 0+} \tilde{f}_I(\omega + i\epsilon), \qquad \omega \in \mathbb{R}, \quad 1 \le I \le N,$$

with the convergence in $\mathscr{S}'(\mathbb{R})$.

(2)

$$\tilde{\varphi}(x,\omega) = \lim_{\epsilon\to 0+} \tilde{\varphi}(x, \omega + i\epsilon), \qquad \omega \in \mathbb{R},$$

with the convergence in $\mathscr{S}'(\mathbb{R}, H^1(\mathbb{R}))$.

The proof repeats the proof of Lemma 4.4.

The formula (5.12) follows from taking the limit $\operatorname{Im}\omega \to 0+$ in both the left- and right-hand sides of the relation (5.15) and using Lemma 5.2. Note that the exponential factors $e^{ik(\omega+i0)|x-X_I|}$ are not smooth as functions of ω, and therefore are not multipliers in the space of tempered distributions. To take the limit $\operatorname{Im}\omega \to 0+$ in (5.15), we take into account that $\tilde{\varphi}(X_I,\omega)$ and $\tilde{f}_I(\omega)$ are quasimeasures, while the exponential factors are multiplicators in the space of quasimeasures. For more details, see Appendix D, Lemma D.6.

□

Denote

$$\varphi_I(t) := \varphi(X_I, t), \qquad 1 \le I \le N. \tag{5.16}$$

Lemma 5.3. *For* $x \le X_1$ *and* $x \ge X_N$*, one has:*

$$\tilde{\varphi}(x,\omega) = \begin{cases} e^{-ik_+(\omega)(x-X_1)}\tilde{\varphi}_1(\omega), & x \le X_1, \\ e^{ik_+(\omega)(x-X_N)}\tilde{\varphi}_N(\omega), & x \ge X_N, \end{cases} \qquad \omega \in \mathbb{R}.$$

Proof. For $x \le X_1$, Proposition 5.1 yields

$$\tilde{\varphi}(x,\omega) = -\sum_{I=1}^{N} \frac{e^{-ik(\omega)(x-X_I)}}{2ik(\omega)} \tilde{f}_I(\omega) = -e^{-ik(\omega)(x-X_1)} \sum_{I=1}^{N} \frac{e^{-ik(\omega)(X_1-X_I)}}{2ik(\omega)} \tilde{f}_I(\omega),$$

hence

$$\tilde{\varphi}(x,\omega) = e^{-ik(\omega)(x-X_1)}\tilde{\varphi}(X_1,\omega), \quad x \le X_1, \quad \omega \in \mathbb{R}.$$

Similarly, for $x \ge X_N$, the relation (5.12) yields

$$\tilde{\varphi}(x,\omega) = e^{ik(\omega)(x-X_N)}\tilde{\varphi}(X_N,\omega), \qquad x \ge X_N, \quad \omega \in \mathbb{R}.$$

□

Lemma 5.4. *For any* I*,* $1 \le I < N-1$*,*

$$\tilde{\varphi}_{I+1}(\omega) = e^{-i(X_{I+1}-X_I)k(\omega)}\tilde{\varphi}_I(\omega) - \sum_{J\le I} e^{ik(\omega)(X_I-X_J)} \frac{\sin\big(k(\omega)(X_{I+1}-X_I)\big)}{k(\omega)} \tilde{f}_J(\omega),$$

where $\omega \in \mathbb{R}$.

Proof. We have:

$$\tilde{\varphi}_I(\omega) := \tilde{\varphi}(X_I,\omega) = -\sum_{J=1}^{N} \frac{e^{ik(\omega)|X_I-X_J|}}{2ik(\omega)} \tilde{f}_J(\omega), \qquad \omega \in \mathbb{R},$$

$$\tilde{\varphi}_{I+1}(\omega) := \tilde{\varphi}(X_{I+1}, \omega) = -\sum_{J=1}^{N} \frac{e^{ik(\omega)|X_{I+1}-X_J|}}{2ik(\omega)} \tilde{f}_J(\omega), \qquad \omega \in \mathbb{R}.$$

Therefore, there is the following relation:

$$\begin{aligned} &\tilde{\varphi}_{I+1}(\omega) - e^{-i(X_{I+1}-X_I)k(\omega)}\tilde{\varphi}_I(\omega) \\ &= -\sum_{J=1}^{N} \frac{e^{ik(\omega)|X_{I+1}-X_J|} - e^{ik(\omega)|X_I-X_J|}e^{-i(X_{I+1}-X_I)k(\omega)}}{2ik(\omega)} \tilde{f}_J(\omega) \\ &= -\sum_{J\le I} \frac{e^{ik(\omega)(X_{I+1}-X_J)} - e^{ik(\omega)(2X_I-X_{I+1}-X_J)}}{2ik(\omega)} \tilde{f}_J(\omega) \\ &= -\sum_{J\le I} e^{ik(\omega)(X_I-X_J)} \frac{\sin\big(k(\omega)(X_{I+1}-X_I)\big)}{k(\omega)} \tilde{f}_J(\omega), \qquad \omega \in \mathbb{R}. \end{aligned} \tag{5.17}$$

Let us note that the terms with $J > I$ disappeared from the summation due to the ordering $X_1 < X_2 < \cdots < X_N$. □

5.3 Absolute continuity for large frequencies

Lemma 5.5. *The distributions $\tilde{\varphi}_1(\omega)$, $\tilde{\varphi}_N(\omega)$ are absolutely continuous for $\omega \in \mathbb{R}\backslash[-m,m]$, and moreover*

$$\int_{\omega\in\mathbb{R}\backslash[-m,m]} \big[|\tilde{\varphi}_1(\omega)|^2 + |\tilde{\varphi}_N(\omega)|^2\big] \frac{k_+(\omega)}{\omega}\, d\omega < \infty, \tag{5.18}$$

where $k_+(\omega)/\omega > 0$ for $\omega \in \mathbb{R}\backslash[-m,m]$ by (4.16).

The bound for each of $\tilde{\varphi}_1(\omega)$, $\tilde{\varphi}_N(\omega)$ is obtained by applying the proof of Proposition 4.7 and using the representation for $\tilde{\varphi}(x,\omega)$ for $x \le X_1$ and $x \ge X_N$ from Lemma 5.3.

5.4 Spectral analysis of omega-limit trajectories

The Fourier transform of β in time, $\tilde{\beta}(x,\omega)$, is a continuous function of $x \in \mathbb{R}$ with values in tempered distributions of $\omega \in \mathbb{R}$. By (5.6), it satisfies the equation

$$-\omega^2\tilde{\beta}(x,\omega) = \tilde{\beta}''(x,\omega) - m^2\tilde{\beta}(x,\omega) + \sum_{I=1}^{N} \delta(x-X_I)\tilde{g}_I(\omega), \qquad (x,\omega) \in \mathbb{R}^2, \tag{5.19}$$

valid in the sense of tempered distributions of $(x,\omega) \in \mathbb{R}^2$, where $\tilde{g}_I(\omega)$ are the Fourier transforms of the functions

$$g_I(t) := F_I(\beta(X_I,t)), \qquad 1 \le I \le N. \tag{5.20}$$

We also denote

$$\beta_I(t) := \beta(X_I,t), \qquad \Sigma_I := \operatorname{supp}\tilde{\beta}_I, \qquad 1 \le I \le N. \tag{5.21}$$

Lemma 5.6. *There is the following relation:*

$$\tilde{\beta}_{I+1}(\omega) = e^{-i(X_{I+1}-X_I)k(\omega)}\tilde{\beta}_I(\omega) - \sum_{J\le I} e^{ik(\omega)(X_I-X_J)} \frac{\sin\big(k(\omega)(X_{I+1}-X_I)\big)}{k(\omega)} \tilde{g}_J(\omega),$$

where $\omega \in \mathbb{R}$.

Proof. The convergence (5.5) implies that for all $1 \le I \le N$,

$$f_I(t_{j_r}+t) := F(\psi(X_I, t_{j_r}+t)) \xrightarrow{\mathscr{S}'} F(\beta(X_I,t)) =: g_I(t), \qquad r\to\infty,$$

and hence

$$e^{-i\omega t_{j_r}}\tilde f_I(\omega) \xrightarrow{\mathscr{S}'} \tilde g_I(\omega), \qquad r\to\infty. \tag{5.22}$$

Corollary C.2 implies that, for any $T>0$, $S_{t_j}(\chi,\dot\chi) \xrightarrow{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} (0,0)$ as $j\to\infty$, hence the convergence (5.5) leads to

$$S_{t_{j_r}}(\varphi,\dot\varphi) \xrightarrow[r\to\infty]{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} (\beta,\dot\beta).$$

This, together with the continuity of the projection

$$\Pi : \mathscr{Y}^{-\varepsilon} \to C_b([-X,X]), \qquad \Pi :\ (\psi,\pi)\mapsto\psi, \qquad \forall X>0$$

(see Lemma B.7 in Appendix B), implies that, for any $x\in\mathbb{R}$, there is the convergence

$$\varphi(x,t_{j_r}+t) \xrightarrow{C_b([-T,T])} \beta(x,t), \qquad r\to\infty,$$

for any $T>0$, hence

$$e^{-i\omega t_{j_r}}\tilde\varphi(x,\omega) \xrightarrow{\mathscr{S}'} \tilde\beta(x,\omega), \qquad r\to\infty. \tag{5.23}$$

To finish the proof, we apply (5.22) and (5.23) to the representation from Lemma 5.4. Note that the factor $\frac{\sin\big(k(\omega)(X_{I+1}-X_I)\big)}{k(\omega)}$ is a smooth function of ω and defines a multiplicator in the space of tempered distributions. The factors $e^{ik(\omega)(X_I-X_J)}$ are not smooth at $\omega=\pm m$; still, since $\tilde\varphi_I$, $\tilde f_I$ belong to the space of quasimeasures and $e^{ik(\omega)(X_I-X_J)}$ are multiplicators in this space (see Appendix D, Lemma D.6), the proof follows. □

Lemma 5.7. *For $I=1$ and $I=N$, one has $\Sigma_I := \operatorname{supp}\tilde\beta_I \subset [-m,m]$.*

Proof. This follows from Lemma 4.10, applied to the convergence (5.23), and using the fact that $\tilde\varphi_1$ and $\tilde\varphi_N$ are locally L^2 for $\omega\in\mathbb{R}\backslash[-m,m]$ (see Lemma 5.5). □

Proposition 5.8. *Any omega-limit trajectory $\beta(x,t)$ is a solitary wave, i.e. $\beta(x,t)=\phi(x)e^{-i\omega_\star t}$ with $\omega_\star\in[-m,m]$ and $\phi(x)\in H^1(\mathbb{R})$.*

Proof. The proof is based on the following lemmas.

Lemma 5.9. *If $\Sigma_1=\emptyset$, then $\beta(x,t)\equiv 0$.*

Proof. The condition $\Sigma_1=\emptyset$ is equivalent to $\beta_1(t)\equiv 0$. This implies that $g_1(t) := F_1(\beta_1(t))\equiv 0$. By Lemma 5.6, $\beta_2(t)\equiv 0$. By induction, $\tilde\beta_I(\omega)\equiv 0$, $\tilde g_I(\omega)\equiv 0$ for $1\le I\le N$. □

Now we consider the case $\Sigma_1\ne\emptyset$.

Lemma 5.10. *If $\Sigma_1\ne\emptyset$, then $\Sigma_1=\{\omega_\star\}$ for some $\omega_\star\in[-m,m]$.*

Proof. By Lemma 5.7, we know that $\Sigma_1\subset[-m,m]$. To show that Σ_1 consists of a single point, we assume that, on the contrary, $\inf\Sigma_1<\sup\Sigma_1$. By (2.20), the Fourier transform $\tilde g_1(\omega)$ of $g_1(t) := F_1(\beta(X_1,t))$ is given by

$$\tilde g_1 = -\sum_{l=1}^{p_1} 2l\, u_{1,l}\, \underbrace{(\tilde\beta_1 * \tilde{\bar\beta}_1) * \ \dots\ * (\tilde\beta_1 * \tilde{\bar\beta}_1)}_{l-1} * \tilde\beta_1. \tag{5.24}$$

Applying the Titchmarsh Convolution Theorem (see Appendix E) to the convolutions in (5.24), we obtain the following equalities:

$$\begin{aligned} \inf\operatorname{supp}\tilde g_1 &= \inf\operatorname{supp}\tilde{\upbeta}_1 + (p_1-1)\inf\operatorname{supp}(\tilde{\upbeta}_1 * \tilde{\bar{\upbeta}}_1) \\ &= \inf\Sigma_1 + (p_1-1)(\inf\Sigma_1 - \sup\Sigma_1), \qquad (5.25) \\ \sup\operatorname{supp}\tilde g_1 &= \sup\operatorname{supp}\tilde{\upbeta}_1 + (p_1-1)\sup\operatorname{supp}(\tilde{\upbeta}_1 * \tilde{\bar{\upbeta}}_1) \\ &= \sup\Sigma_1 + (p_1-1)(\sup\Sigma_1 - \inf\Sigma_1), \qquad (5.26) \end{aligned}$$

where we used the relations

$$\inf\operatorname{supp}\tilde{\bar{\upbeta}}_1 = -\sup\operatorname{supp}\tilde{\upbeta}_1, \qquad \sup\operatorname{supp}\tilde{\bar{\upbeta}}_1 = -\inf\operatorname{supp}\tilde{\upbeta}_1.$$

Note that the Titchmarsh Theorem is applicable since $\operatorname{supp}\tilde{\upbeta}_1$ is compact ($\operatorname{supp}\tilde{\upbeta}_1 \subset [-m,m]$ by Lemma 5.7). Note that, because of $\inf\Sigma_1 \ge -m$ and $\sup\Sigma_1 \le m$, one has

$$\operatorname{supp}\tilde g_1 \subset [-(2p_1-1)m, (2p_1-1)m]. \qquad (5.27)$$

Since we assumed that $\inf\Sigma_1 < \sup\Sigma_1$, (5.25) and (5.26) imply that $\inf\operatorname{supp}\tilde g_1 < \inf\Sigma_1$, $\sup\operatorname{supp}\tilde g_1 > \sup\Sigma_1$. The ratio $\sin(k_+(\omega)(X_2-X_1))/k_+(\omega)$ could only vanish at the points $\omega = \pm\omega_{1,n}$, where

$$\omega_{I,n} := \sqrt{\frac{\pi^2 n^2}{|X_{I+1}-X_I|^2} + m^2}, \qquad 1 \le I \le N-1, \quad n \in \mathbb{N}.$$

Using the condition (2.21) and the inclusion (5.27), we conclude that $\operatorname{supp}\tilde g_1 \cap \{\pm\omega_{1,n}\colon n \in \mathbb{N}\} = \emptyset$. Therefore, $\sin(k_+(\omega)(X_2-X_1))/k_+(\omega)$ does not vanish on $\operatorname{supp}\tilde g_1$, and Lemma 5.6 with $I = 1$ implies that

$$\inf\Sigma_2 := \inf\operatorname{supp}\tilde{\upbeta}_2 = \inf\operatorname{supp}\tilde g_1 < \inf\Sigma_1,$$
$$\sup\Sigma_2 := \sup\operatorname{supp}\tilde{\upbeta}_2 = \sup\operatorname{supp}\tilde g_1 > \sup\Sigma_1.$$

We proceed by induction, proving that

$$\inf\Sigma_1 > \inf\Sigma_2 > \cdots > \inf\Sigma_N, \qquad \sup\Sigma_1 < \sup\Sigma_2 < \cdots < \sup\Sigma_N. \qquad (5.28)$$

It then follows that $\inf\Sigma_N < \sup\Sigma_N$. Starting from $I = N$ and going to the left, we could as well prove the opposite inequalities:

$$\inf\Sigma_1 < \inf\Sigma_2 < \cdots < \inf\Sigma_N, \qquad \sup\Sigma_1 > \sup\Sigma_2 > \cdots > \sup\Sigma_N. \qquad (5.29)$$

The contradiction of (5.28) and (5.29) shows that our assumption that $\inf\Sigma_1 < \sup\Sigma_1$ was false, hence $\Sigma_1 \subset \{\omega_\star\}$ for some $\omega_\star \in [-m,m]$. □

Thus, $\operatorname{supp}\tilde\beta_1(\omega) = \Sigma_1 \subset \{\omega_\star\}$, with $\omega_\star \in [-m,m]$. Therefore,

$$\tilde\beta_1(\omega) = a_1\delta(\omega-\omega_\star), \qquad \text{with some } \ a_1 \in \mathbb{C}. \qquad (5.30)$$

Note that the derivatives $\delta^{(k)}(\omega-\omega_\star)$, $k \ge 1$ do not enter the expression for $\tilde\beta_1(\omega) = \mathcal{F}_{t\to\omega}[\beta(X_1,t)]$ since $\beta(x,t)$ is a bounded continuous function of $(x,t) \in \mathbb{R}^2$ due to the bound (5.7).

Lemma 5.11. *$\tilde\beta(x,\omega) = 2\pi\phi(x)\delta(\omega-\omega_\star)$, where $\phi \in H^1(\mathbb{R})$.*

Proof. It suffices to notice that if $\operatorname{supp}\tilde{\upbeta}_1 = \{\omega_\star\}$, then also $\operatorname{supp}\tilde g_1 \subset \{\omega_\star\}$, and by the induction argument applied to Lemma 5.6 one has the inclusions $\operatorname{supp}\tilde{\upbeta}_I \subset \{\omega_\star\}$ for all $1 \le I \le N$. □

Now we can finish the proof of Proposition 5.8. Lemma 5.11 implies that $\beta(x,t) = \phi(x)e^{-i\omega_\star t}$, where $\phi \in H^1(\mathbb{R})$ by (5.7). This finishes the proof of Proposition 5.8. Note that $\omega = \pm m$ could only correspond to the zero solution (see Remark A.2). □

According to Lemma 3.14, Proposition 5.8 completes the proof of Theorem 2.8.

Chapter 6

Klein-Gordon with mean field interaction

In this chapter, we are going to prove Theorem 2.15, which states the convergence to the set of solitary waves for all finite energy solutions to the complex Klein-Gordon field $\psi(x,t)$ with the mean field self-interaction at $N \in \mathbb{N}$ points:

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - m^2\psi(x,t) + \sum_{I=1}^{N} \rho_I(x) F_I(\langle \rho_I, \psi(\cdot,t)\rangle), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}. \tag{6.1}$$

Above, $\rho_I(x) = \rho(x - X_I)$, with $X_I \in \mathbb{R}^n$, $1 \le I \le N$, and ρ a smooth coupling function from the Schwartz class: $\rho \in \mathscr{S}(\mathbb{R}^n)$, $\rho \not\equiv 0$.

We assume that the dimension is $n \ge 3$.

Pick

$$(\psi_0, \pi_0) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n). \tag{6.2}$$

According to Theorem B.16 *(1)* there exists a global solution to (6.1), which we denote $\psi(x,t)$, with the initial data

$$(\psi, \dot{\psi})|_{t=0} = (\psi_0, \pi_0). \tag{6.3}$$

By Theorem B.1 *(4)*,

$$(\psi, \dot{\psi}) \in C_b(\mathbb{R}, \mathscr{X}). \tag{6.4}$$

6.1 Compactness

Fix any $\varepsilon > 0$. Proposition 3.8 applied to equation (6.1) (see Remark 3.2) states that, for any sequence $t_j \to +\infty$, there exists a subsequence t_{j_r}, $r \in \mathbb{N}$, such that, for any $T > 0$,

$$S_{t_{j_r}}(\psi, \dot{\psi}) \xrightarrow[r\to\infty]{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} (\beta, \dot{\beta}), \tag{6.5}$$

for some $\beta \in C(\mathbb{R}, H^1(\mathbb{R}^n))$ such that $\dot{\beta} \in C(\mathbb{R}, L^2(\mathbb{R}^n))$. The function $\beta(x,t)$ satisfies equation (6.1),

$$\ddot{\beta}(x,t) = \Delta\beta(x,t) - m^2\beta(x,t) + \sum_{I=1}^{N} \rho_I(x) F_I(\langle \rho_I, \beta\rangle), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{6.6}$$

which is understood in the sense of distributions, and obeys the following bound:

$$\sup_{t\in\mathbb{R}} \|(\beta, \dot{\beta})|_t\|_{\mathscr{X}} < \infty. \tag{6.7}$$

By Lemma 3.14, the proof of Theorem 2.15 will follow if we check that every omega-limit trajectory $\beta(x,t)$ belongs to the set of solitary waves:

$$\beta(x,t)=\phi_{\omega_\star}(x)e^{-i\omega_\star t},\qquad x\in\mathbb{R}^n,\quad t\in\mathbb{R},\tag{6.8}$$

with some $\omega_\star\in\mathbb{R}$.

Define $\chi(x,t)$ as the solution to the following Cauchy problem:

$$\ddot\chi(x,t)=\Delta\chi(x,t)-m^2\chi(x,t),\qquad (\chi,\dot\chi)\big|_{t=0}=(\psi_0,\pi_0),\tag{6.9}$$

where (ψ_0,π_0) is the initial data from (6.2). Define $\varphi(x,t)$ by

$$\varphi(x,t)=\begin{cases}0, & t<0,\\ \psi(x,t)-\chi(x,t), & t\ge 0.\end{cases}\tag{6.10}$$

Then $\varphi(x,t)$ satisfies

$$\ddot\varphi(x,t)=\Delta\varphi(x,t)-m^2\varphi(x,t)+\sum_{I=1}^N\rho_I(x)f_I(t),\quad (\varphi,\dot\varphi)\big|_{t\le 0}=(0,0),\tag{6.11}$$

where

$$f_I(t):=\Theta(t)F_I(\langle\rho_I,\psi(\cdot,t)\rangle).$$

Note that $\langle\rho_I,\psi(\cdot,t)\rangle$ belongs to $C_b(\mathbb{R})$ by (6.4). Hence,

$$f_I(\cdot)\in C_b(\mathbb{R}).\tag{6.12}$$

On the other hand, since $\chi(t)$ is a finite energy solution to the free Klein-Gordon equation, we also have

$$(\chi,\dot\chi)\in C_b(\mathbb{R},\mathscr{X}).\tag{6.13}$$

Hence, the function $\varphi(t)=\Theta(t)(\psi(t)-\chi(t))$ also satisfies

$$(\varphi,\dot\varphi)\in C_b(\mathbb{R},\mathscr{X}),\qquad t\in\mathbb{R}.\tag{6.14}$$

6.2 Spectral representation

Let us consider the complex Fourier transform of $\varphi(x,t)$:

$$\tilde\varphi(x,\omega)=\mathcal{F}_{t\to\omega}[\varphi(x,t)]:=\int_0^\infty e^{i\omega t}\varphi(x,t)\,dt,\quad \omega\in\mathbb{C}^+,\quad x\in\mathbb{R}^n,\tag{6.15}$$

where $\mathbb{C}^+:=\{z\in\mathbb{C}:\ \operatorname{Im}z>0\}$. Due to (6.14), $\tilde\varphi(\cdot,\omega)$ is an H^1-valued analytic function of $\omega\in\mathbb{C}^+$. Equation (6.11) for φ implies that

$$-\omega^2\tilde\varphi(x,\omega)=\Delta\tilde\varphi(x,\omega)-m^2\tilde\varphi(x,\omega)+\sum_{I=1}^N\rho_I(x)\tilde f_I(\omega),\quad \omega\in\mathbb{C}^+,\quad x\in\mathbb{R}^n,$$

where

$$\tilde f_I(\omega)=\int_0^\infty e^{i\omega t}f_I(t)\,dt,\qquad \omega\in\mathbb{C}^+,\tag{6.16}$$

is the complex Fourier transform of $f_I(t)$. The solution $\tilde\varphi(x,\omega)$ is analytic for $\omega\in\mathbb{C}^+$ and can be represented by

$$\tilde\varphi(x,\omega)=\sum_{I=1}^N\mathit{\Sigma}_I(x,\omega)\tilde f_I(\omega),\qquad \omega\in\mathbb{C}^+.\tag{6.17}$$

Lemma 6.1. *(1)*

$$\tilde{f}_I(\omega) = \lim_{\epsilon\to 0+} \tilde{f}_I(\omega + i\epsilon), \qquad \omega \in \mathbb{R}, \quad 1 \le I \le N, \tag{6.18}$$

with the convergence in $\mathscr{S}'(\mathbb{R})$.

(2)

$$\tilde{\varphi}(x,\omega) = \lim_{\epsilon\to 0+} \tilde{\varphi}(x,\omega + i\epsilon), \qquad \omega \in \mathbb{R}, \tag{6.19}$$

with the convergence in $\mathscr{S}'(\mathbb{R}, H^1(\mathbb{R}^n))$.

The proof repeats the proof of Lemma 4.4 and is based on the convergence

$$e^{-\epsilon t} f_I(t) \xrightarrow[\epsilon\to 0+]{\mathscr{S}'} f_I(t), \qquad e^{-\epsilon t}\varphi(x,t) \xrightarrow[\epsilon\to 0+]{\mathscr{S}'(\mathbb{R},H^1(\mathbb{R}^n))} \varphi(x,t)$$

which follows from $f_I|_{t<0} = 0$, $\varphi|_{t<0} = 0$, and the bounds (6.12) and (6.14).

Now we can justify the representation (6.17) for $\omega \in \mathbb{R}$, if the multiplication in (6.17) is understood in the sense of distributions.

Proposition 6.2. *There is the following identity, understood in the sense of distributions:*

$$\tilde{\varphi}(x,\omega) = \sum_{I=1}^{N} \varSigma_I(x,\omega)\tilde{f}_I(\omega), \qquad \omega \in \mathbb{R}. \tag{6.20}$$

Proof. Since we assume that $n \ge 3$, for each $x \in \mathbb{R}^n$,

$$\varSigma_I(x,\omega) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{i\xi\cdot x} e^{-i\xi\cdot X_I}\hat{\rho}(\xi)\,d\xi}{\xi^2 + m^2 - (\omega + i0)^2} \tag{6.21}$$

is a smooth function of $\omega \in \mathbb{R}$, and hence is a multiplicator in the space of tempered distributions in the variable ω. The rest of the proof is based on the relation (6.17) and the convergence stated in Lemma 6.1. □

6.3 Absolute continuity for large frequencies

Let $k(\omega)$ denote the branch of $\sqrt{\omega^2 - m^2}$ such that $\operatorname{Im}\sqrt{\omega^2 - m^2} \ge 0$ for $\omega \in \mathbb{C}^+$; see (4.14). The function $k(\omega)$ is analytic for $\omega \in \mathbb{C}^+$. We extend it to $\omega \in \overline{\mathbb{C}^+}$ by continuity.

We write the Fourier transform of (6.20) as follows:

$$\hat{\tilde{\varphi}}(\xi,\omega) = \sum_{I=1}^{N} \hat{\varSigma}_I(\xi,\omega)\tilde{f}_I(\omega) = \hat{\varSigma}(\xi,\omega)\sum_{I=1}^{N} e^{-i\xi\cdot X_I}\tilde{f}_I(\omega), \qquad \omega \in \mathbb{R}, \tag{6.22}$$

where

$$\hat{\varSigma}(\xi,\omega) = \frac{\hat{\rho}(\xi)}{\xi^2 + m^2 - (\omega + i0)^2}. \tag{6.23}$$

Proposition 6.3. *For any finite open interval W such that $\overline{W} \cap ([-m,m] \cup Z_\rho) = \emptyset$ there is a constant $C_W > 0$ such that*

$$\int_{\mathbb{S}^{n-1}\times W} \left| \sum_{I=1}^{N} e^{-ik(\omega)\boldsymbol{\theta}\cdot X_I}\tilde{f}_I(\omega) \right|^2 d\Omega_{\boldsymbol{\theta}}\,d\omega \le C_W. \tag{6.24}$$

Remark 6.4. By (6.18), $\tilde f_I(\omega) = \tilde f_I(\omega + i0)$.

Proof. The Parseval identity applied to

$$\tilde\varphi(x, \omega + i\epsilon) = \int_0^\infty \varphi(x,t) e^{i\omega t - \epsilon t}\, dt, \qquad \epsilon > 0 \tag{6.25}$$

leads to

$$\int_{-\infty}^{\infty} \|\tilde\varphi(\cdot, \omega + i\epsilon)\|_{L^2}^2 \, d\omega = 2\pi \int_0^\infty \|\varphi(\cdot,t)\|_{L^2}^2 e^{-2\epsilon t}\, dt.$$

Since $\sup_{t\ge 0} \|\varphi(\cdot,t)\|_{H^1} < \infty$ by (6.14), we may bound the right-hand side by C_1/ϵ, with some $C_1 > 0$.
Taking into account (6.17), we arrive at the key inequality

$$\int_{-\infty}^{\infty} \|\sum_{I=1}^N \Sigma_I(\cdot, \omega + i\epsilon)\tilde f_I(\omega + i\epsilon)\|_{L^2}^2 \, d\omega \le \frac{C_1}{\epsilon}. \tag{6.26}$$

Noting that $\hat\Sigma_I(\xi, \omega + i\epsilon) = e^{-i\xi\cdot X_I}\hat\Sigma(\xi, \omega + i\epsilon)$, with $\hat\Sigma(\xi, \omega + i\epsilon)$ from (6.23), we rewrite (6.26) as

$$\int_{\mathbb{R}} \epsilon \left\| \sum_{I=1}^N \Sigma_I(\cdot, \omega + i\epsilon)\tilde f_I(\omega + i\epsilon) \right\|_{L^2}^2 d\omega$$
$$= \int_{\mathbb{R}^n \times \mathbb{R}} \epsilon |\hat\Sigma(\xi, \omega + i\epsilon)|^2 \left| \sum_{I=1}^N e^{-i\xi\cdot X_I} \tilde f_I(\omega + i\epsilon) \right|^2 \frac{d^n\xi}{(2\pi)^n}\, d\omega \le C_1. \tag{6.27}$$

Fix a finite open interval W such that $\overline{W} \cap ([-m, m] \cup Z_\rho) = \emptyset$. Denote

$$\boldsymbol{W}^\epsilon := \{(\xi, \omega)\colon \omega \in W,\ |\omega - \sqrt{\xi^2 + m^2}| < \epsilon\} \subset \mathbb{R}^n \times \mathbb{R}, \tag{6.28}$$

as on Fig. 6.1. Due to the inequality (6.27), the following weaker inequality also takes place:

$$\int_{\boldsymbol{W}^\epsilon} \epsilon |\hat\Sigma(\xi, \omega + i\epsilon)|^2 \left| \sum_{I=1}^N e^{-i\xi\cdot X_I} \tilde f_I(\omega + i\epsilon) \right|^2 \frac{d^n\xi}{(2\pi)^n}\, d\omega \le C_1. \tag{6.29}$$

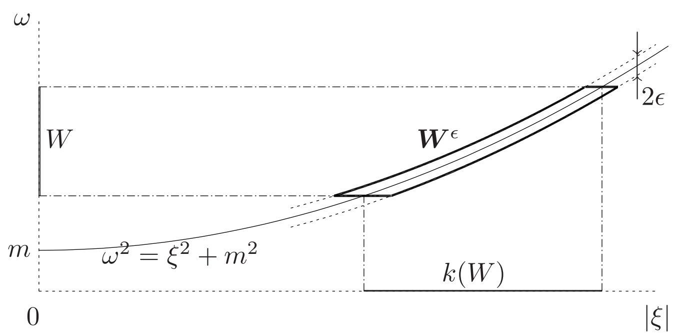

Figure 6.1: Domain $\boldsymbol{W}^\epsilon$ and intervals W and $k(W)$.

Lemma 6.5. *There exists a constant $C_2 > 0$ such that for any $\epsilon \in (0, 1)$ there is the inequality*

$$\int_{\boldsymbol{W}^\epsilon} \epsilon |\hat\Sigma(\xi, \omega + i\epsilon)|^2 \left| \sum_{I=1}^N e^{-ik(\omega)\boldsymbol{\theta}_\xi\cdot X_I} \tilde f_I(\omega + i\epsilon) \right|^2 \frac{d^n\xi}{(2\pi)^n}\, d\omega \le C_2, \tag{6.30}$$

where $\boldsymbol{\theta}_\xi = \frac{\xi}{|\xi|}$.

Proof. For brevity, denote $\mathrm{f}_I = \tilde{f}_I(\omega + i\epsilon)$. We are done if we can prove that the difference between the left-hand sides of (6.29) and (6.30) is bounded by a constant which depends on W but not on $\epsilon \in (0,1)$. By the triangle inequality,

$$\begin{aligned}
&\int\limits_{\boldsymbol{W}^\epsilon} \epsilon|\hat{\Sigma}(\xi,\omega+i\epsilon)|^2 \left| \left|\sum_{I=1}^N e^{-i\xi\cdot X_I}\mathrm{f}_I\right|^2 - \left|\sum_{I=1}^N e^{-ik(\omega)\boldsymbol{\theta}_\xi\cdot X_I}\mathrm{f}_I\right|^2 \right| \frac{d^n\xi}{(2\pi)^n}\,d\omega \\
&\le \int\limits_{\boldsymbol{W}^\epsilon} \epsilon|\hat{\Sigma}(\xi,\omega+i\epsilon)|^2 \left|\sum_{I=1}^N e^{-i\xi\cdot X_I}\mathrm{f}_I - \sum_{I=1}^N e^{-ik(\omega)\boldsymbol{\theta}_\xi\cdot X_I}\mathrm{f}_I\right|^2 \frac{d^n\xi}{(2\pi)^n}\,d\omega \\
&\le \int\limits_{\boldsymbol{W}^\epsilon} \epsilon|\hat{\Sigma}(\xi,\omega+i\epsilon)|^2 \left|\sum_{I=1}^N e^{-ik(\omega)\boldsymbol{\theta}_\xi\cdot X_I}(e^{i(k(\omega)\boldsymbol{\theta}_\xi-\xi)\cdot X_I}-1)\mathrm{f}_I\right|^2 \frac{d^n\xi}{(2\pi)^n}\,d\omega. \qquad (6.31)
\end{aligned}$$

According to (6.4), $|f_I(t)| = |F_I(\langle\rho_I,\psi(\cdot,t)\rangle)|$ is bounded uniformly in time. By (6.16), we know that $|\mathrm{f}_I| = |\tilde{f}_I(\omega+i\epsilon)| \le C\epsilon^{-1}$. We also have $|e^{i(k(\omega)\boldsymbol{\theta}_\xi-\xi)\cdot X_I}-1| \le C\epsilon$ for $(\xi,\omega) \in \boldsymbol{W}^\epsilon$, with some $C \in \mathbb{R}$ independent on $\epsilon \in (0,1)$. Therefore, (6.31) is bounded by

$$\int_{\boldsymbol{W}^\epsilon} \epsilon|\hat{\Sigma}(\xi,\omega+i\epsilon)|^2 \frac{d^n\xi}{(2\pi)^n}\,d\omega \le \int_{\boldsymbol{W}^\epsilon} \epsilon\frac{|\hat{\rho}(\xi)|^2}{4m^2\epsilon^2}\frac{d^n\xi}{(2\pi)^n}\,d\omega \le \int_{\mathbb{R}^n} \epsilon\frac{|\hat{\rho}(\xi)|^2}{4m^2\epsilon^2}\frac{d^n\xi}{(2\pi)^n}\,2\epsilon \le \text{const}, \qquad (6.32)$$

where const depends on W but not on ϵ. Above, we used the expression $\hat{\Sigma}(\xi,\omega+i\epsilon) = \frac{\hat{\rho}(\xi)}{\xi^2+m^2-(\omega+i\epsilon)^2}$ (see (6.23)) and the bound

$$|\xi^2+m^2-(\omega+i\epsilon)^2|^2 \ge |\mathrm{Im}\,(\xi^2+m^2-(\omega+i\epsilon)^2)|^2 \ge 4m^2\epsilon^2, \qquad (\xi,\omega) \in \boldsymbol{W}^\epsilon.$$

The integration in ω contributed 2ϵ, which is the thickness of $\boldsymbol{W}^\epsilon$ in the ω-direction (see Fig. 6.1).

It follows that the right-hand side in (6.31) is bounded by a constant independent on $\epsilon \in (0,1)$. This finishes the proof. □

Lemma 6.6. *There exist $\epsilon_W \in (0,1)$ and $C_3 > 0$ such that*

$$\int_{\boldsymbol{W}^\epsilon\cap(\mathbb{R}^n\times\{\omega\})} \epsilon|\hat{\Sigma}(\xi,\omega+i\epsilon)|^2 \frac{d^n\xi}{(2\pi)^n} \ge C_3, \qquad \omega \in W, \quad 0 < \epsilon \le \epsilon_W.$$

Proof. First, we note that $k(W)$ is a finite open interval bounded away from 0; see Fig. 6.1. Since the function

$$h(\eta) := \frac{1}{(2\pi)^n}\int_{|\xi|=\eta} |\hat{\rho}(\xi)|^2\,d^{n-1}S_\xi$$

is smooth and strictly positive for $\eta \in \overline{k(W)}$, there exist $\epsilon_W > 0$ and $c_W > 0$ such that $h(\eta) \ge c_W$ for all η such that $(\xi,\omega) \in \boldsymbol{W}^\epsilon$ for $|\xi| = \eta$, $\epsilon \in (0,\epsilon_W)$. Hence, using (6.23),

$$\begin{aligned}
&\int_{\boldsymbol{W}^\epsilon\cap(\mathbb{R}^n\times\{\omega\})} \epsilon|\hat{\Sigma}(\xi,\omega+i\epsilon)|^2 \frac{d^n\xi}{(2\pi)^n} \\
&\ge \int_{\boldsymbol{W}^\epsilon\cap(\mathbb{R}^n\times\{\omega\})} \epsilon\left|\frac{\hat{\rho}(\xi)}{\xi^2+m^2-(\omega+i\epsilon)^2}\right|^2 \frac{d^n\xi}{(2\pi)^n} \\
&\ge c_W \int\limits_{\eta>0,\ |\omega-\sqrt{\eta^2+m^2}|<\epsilon} \frac{\epsilon\,d\eta}{|\eta^2+m^2-(\omega+i\epsilon)^2|^2}, \qquad (6.33)
\end{aligned}$$

where we took into account the definition (6.28). Pick $\delta_W < |k(W)|/2$; then, for $\eta_0 \in k(W)$, either $[\eta_0-\delta_W,\eta_0] \subset k(W)$, or $[\eta_0,\eta_0+\delta_W] \subset k(W)$, or both. Therefore, the integration in η is over an interval of length at least $\frac{\epsilon}{2}\min_{\omega\in\overline{W}}|k'(\omega)|$. Moreover, for $|\omega-\sqrt{\eta^2+m^2}|<\epsilon$, the magnitude of the denominator is bounded from above:

$$\begin{aligned}|\eta^2+m^2-(\omega+i\epsilon)^2|^2 &= (\eta^2+m^2-\omega^2+\epsilon^2)^2+4\omega^2\epsilon^2\\ &\le (\sqrt{\eta^2+m^2}-\omega)^2(\sqrt{\eta^2+m^2}+\omega)^2+\mathrm{const}\epsilon^2\le \mathrm{const}\epsilon^2,\end{aligned}$$

where the constant in the right-hand side depends on W but not on ϵ. □

Combining Lemmas 6.5 and 6.6, we get:

$$\int_{\mathbb{S}^{n-1}\times W}\left|\sum_{I=1}^{N}e^{-ik(\omega)\boldsymbol{\theta}\cdot X_I}\tilde f_I(\omega+i\epsilon)\right|^2 d\Omega_{\boldsymbol{\theta}}\,d\omega\le C_2/C_3,\qquad 0<\epsilon\le\epsilon_W.$$

We conclude that the set of functions

$$g_{W,\epsilon}(\boldsymbol{\theta},\omega)=\sum_{I=1}^{N}e^{-ik(\omega)\boldsymbol{\theta}\cdot X_I}\tilde f_I(\omega+i\epsilon),\qquad 0<\epsilon\le\epsilon_W,$$

defined for $\boldsymbol{\theta}\in\mathbb{S}^{n-1}$, $\omega\in W$, is bounded in the Hilbert space $L^2(\mathbb{S}^{n-1}\times W)$, and hence is weakly compact. The convergence of the distributions (6.18) implies the weak convergence $g_{W,\epsilon}\underset{\epsilon\to0+}{\rightharpoonup}g_W$ in the Hilbert space $L^2(\mathbb{S}^{n-1}\times W)$. The limit function $g_W\in L^2(\mathbb{S}^{n-1}\times W)$. coincides with the distribution $\sum_{I=1}^{N}e^{-ik(\omega)\boldsymbol{\theta}\cdot X_I}\tilde f_I(\omega)$, on $\mathbb{S}^{n-1}\times W$. This proves the bound (6.24). □

Proposition 6.7. *The distributions $\tilde f_I(\omega+i0)$, $1\le I\le N$, are locally L^2 for $\omega\in\mathbb{R}\backslash([-m,m]\cup Z_\rho)$.*

Proof. We split the proof into four lemmas.

Lemma 6.8. *Let $k>0$. Assume that the vectors $X_J\in\mathbb{R}^n$, $1\le J\le N$, are pairwise different. Then there exist vectors $\boldsymbol{\theta}_I\in\mathbb{S}^{n-1}$, $1\le I\le N$, such that*

$$\det_{1\le I,J\le N}e^{-ik\boldsymbol{\theta}_I\cdot X_J}\ne0.$$

Proof. Let us choose a (two-dimensional) plane $\boldsymbol{A}$ through the origin in $\mathbb{R}^n$ such that the orthogonal projections of X_J onto $\boldsymbol{A}$, which we denote by $Y_J=P_{\boldsymbol{A}}(X_J)$, are pairwise different. It suffices to show that we can choose $\boldsymbol{\theta}_J\in\mathbb{S}^{n-1}\cap\boldsymbol{A}$ such that

$$\det_{1\le I,J\le N}e^{-ik\boldsymbol{\theta}_I\cdot Y_J}\ne0. \tag{6.34}$$

It is enough to consider the case when all Y_J are pairwise linearly independent and have different lengths. Indeed, since Y_J are pairwise different, there exists $Y_0\in\boldsymbol{A}$ such that Y_0+Y_J are pairwise linearly independent and have different lengths; at the same time,

$$\det_{1\le I,J\le N}e^{-ik\boldsymbol{\theta}_I\cdot(Y_0+Y_J)}=\Big(\prod_{I=1}^{N}e^{-ik\boldsymbol{\theta}_I\cdot Y_0}\Big)\det_{1\le I,J\le N}e^{-ik\boldsymbol{\theta}_I\cdot Y_J},$$

with the factor $\prod_I e^{-ik\boldsymbol{\theta}_I\cdot Y_0}$ different from zero.

We will prove (6.34) by induction in N, assuming that the vertices X_J are numbered so that

$$|Y_1|<|Y_2|<\cdots<|Y_N|. \tag{6.35}$$

The claim is true for $N=1$ since $e^{-ik\boldsymbol{\theta}_1\cdot Y_1}\ne0$ for any $\boldsymbol{\theta}_1\in\mathbb{S}^{n-1}\cap\boldsymbol{A}$. Assume that the statement is true for some $M\ge1$, $M<N$: there exist vectors $\boldsymbol{\theta}_I\in\mathbb{S}^{n-1}\cap\boldsymbol{A}$, $1\le I\le M$, such that

$$\det_{1\le I,J\le M}e^{-ik\boldsymbol{\theta}_I\cdot Y_J}\ne0. \tag{6.36}$$

Then we need to check that the statement is also true for $M+1$. That is, we need to show that there exists $\boldsymbol{\theta}_{M+1} \in \mathbb{S}^{n-1} \cap \boldsymbol{A}$ such that

$$\det_{1\le I,J\le M+1} e^{-ik\boldsymbol{\theta}_I\cdot Y_J} \ne 0. \tag{6.37}$$

According to (6.36), there is a unique set of numbers $a_J \in \mathbb{C}$, $1 \le J \le M$, such that

$$\sum_{J=1}^{M} a_J \begin{bmatrix} e^{-ik\boldsymbol{\theta}_1\cdot Y_J} \\ \vdots \\ e^{-ik\boldsymbol{\theta}_M\cdot Y_J} \end{bmatrix} + \begin{bmatrix} e^{-ik\boldsymbol{\theta}_1\cdot Y_{M+1}} \\ \vdots \\ e^{-ik\boldsymbol{\theta}_M\cdot Y_{M+1}} \end{bmatrix} = 0. \tag{6.38}$$

To prove (6.37), we need to show that the relation

$$\sum_{J=1}^{M} a_J e^{-ik\boldsymbol{\theta}\cdot Y_J} + e^{-ik\boldsymbol{\theta}\cdot Y_{M+1}} = 0 \tag{6.39}$$

can not be valid for all $\boldsymbol{\theta} \in \mathbb{S}^{n-1} \cap \boldsymbol{A}$; this, in turn, will imply that there exists $\boldsymbol{\theta}_{M+1} \in \mathbb{S}^{n-1} \cap \boldsymbol{A}$ such that the columns $\begin{bmatrix} e^{-ik\boldsymbol{\theta}_1\cdot Y_J} \\ \vdots \\ e^{-ik\boldsymbol{\theta}_{M+1}\cdot Y_J} \end{bmatrix}$, $1 \le J \le M+1$, are linearly independent, leading to (6.37).

We parametrize $\boldsymbol{\theta} \in \mathbb{S}^{n-1} \cap \boldsymbol{A} = \mathbb{S}^1$ by the angle $\vartheta \in [0, 2\pi)$. Let $\gamma_J \in [0, 2\pi)$, $1 \le J \le M+1$, be the angles corresponding to the directions $Y_J/|Y_J| \in \mathbb{S}^1$. Note that since Y_J are pairwise linearly independent, all the angles γ_J are different. The relation (6.39) takes the form

$$f(\vartheta) = 0, \tag{6.40}$$

where

$$f(\vartheta) = \sum_{J=1}^{M} a_J e^{-ik|Y_J|\cos(\vartheta-\gamma_J)} + e^{-ik|Y_{M+1}|\cos(\vartheta-\gamma_{M+1})}. \tag{6.41}$$

For $\vartheta \in \mathbb{C}$, the formula (6.41) defines an entire function; let us show that f is not identically zero. Let $\vartheta = u + iv$, where $u, v \in \mathbb{R}$. Since

$$\cos(\vartheta-\gamma) = \cos(u+iv-\gamma) = \cos(u-\gamma)\cosh v - i\sin(u-\gamma)\sinh v, \qquad \gamma \in \mathbb{R},$$

the definition (6.41) takes the form

$$f(\vartheta) = \sum_{J=1}^{M} a_J e^{-k|Y_J|(i\cos(u-\gamma_J)\cosh v + \sin(u-\gamma_J)\sinh v)} \\ + e^{-k|Y_{M+1}|(i\cos(u-\gamma_{M+1})\cosh v + \sin(u-\gamma_{M+1})\sinh v)}. \tag{6.42}$$

Taking into account (6.35), we derive the following asymptotics along the line $\operatorname{Re}\vartheta = \gamma_{M+1} - \frac{\pi}{2}$ (which means that $u = \gamma_{M+1} - \frac{\pi}{2}$ and $v \in \mathbb{R}$):

$$f\Big(\gamma_{M+1} - \frac{\pi}{2} + iv\Big) \sim e^{k|Y_{M+1}|\sinh v}, \qquad v \to +\infty. \tag{6.43}$$

It follows that $f(\vartheta)$ is an entire function which is not identically equal to zero. Therefore, (6.40) can hold at no more than finitely many values $\vartheta \in [0, 2\pi)$. We pick $\boldsymbol{\theta}_{M+1}$ so that the corresponding angle ϑ is not a root of (6.40). With this particular value of $\boldsymbol{\theta}_{M+1}$, (6.37) is satisfied. This finishes the induction argument. □

Lemma 6.9. *For any $\omega \in \mathbb{R}\backslash[-m,m]$, there is an open neighborhood $W \subset \mathbb{R}\backslash[-m,m]$ such that there is a family of vectors $\boldsymbol{\theta}_I(\omega,\boldsymbol{\tau}) \in \mathbb{S}^{n-1}$, $1 \le I \le N$, smoothly parametrized by $\omega \in \overline{W}$ and $\boldsymbol{\tau} \in \mathbb{B}^{n-1} \subset \mathbb{R}^{n-1}$, with $\mathbb{B}^{n-1}$ a unit ball in $\mathbb{R}^{n-1}$, so that*

$$\det_{1\le I,J\le N} e^{-ik(\omega)\boldsymbol{\theta}_I(\omega,\boldsymbol{\tau})\cdot X_J} \ne 0$$

for all $\omega \in \overline{W}$, $\boldsymbol{\tau} \in \mathbb{B}^{n-1}$, and so that for each $1 \le I \le N$ and $\omega \in \overline{W}$ the map

$$\boldsymbol{\tau} \mapsto \boldsymbol{\theta}_I(\omega,\boldsymbol{\tau}), \qquad \boldsymbol{\tau} \in \mathbb{B}^{n-1}, \tag{6.44}$$

is a diffeomorphism.

Proof. The proof immediately follows from Lemma 6.8. □

Fix $\omega \in \mathbb{R}\backslash([-m,m] \cup Z_\rho)$, and let W be an open neighborhood of ω as in Lemma 6.9. We assume that W is small enough, so that

$$\overline{W} \cap ([-m,m] \cup Z_\rho) = \emptyset. \tag{6.45}$$

Let the matrix

$$R_{IJ}(\omega,\boldsymbol{\tau}), \qquad \omega \in \overline{W}, \quad \boldsymbol{\tau} \in \mathbb{B}^{n-1},$$

be the inverse to $A_{IJ}(\omega,\boldsymbol{\tau}) = e^{-ik(\omega)\boldsymbol{\theta}_I(\omega,\boldsymbol{\tau})\cdot X_J}$. Pick a function $\varsigma \in C_0^\infty(\mathbb{B}^{n-1})$ such that $\int_{\mathbb{B}^{n-1}} \varsigma(\boldsymbol{\tau})\,d\boldsymbol{\tau} = 1$. Denote

$$R_I(\omega,\boldsymbol{\theta}) = \int_{\mathbb{B}^{n-1}} \sum_{J=1}^N R_{IJ}(\omega,\boldsymbol{\tau})\delta_{\boldsymbol{\theta}_J(\omega,\boldsymbol{\tau})}(\boldsymbol{\theta})\,\varsigma(\boldsymbol{\tau})\,d\boldsymbol{\tau}, \tag{6.46}$$

where $\delta_{\boldsymbol{\theta}_0}(\boldsymbol{\theta})$ is a delta-function on $\mathbb{S}^{n-1}$ supported at $\boldsymbol{\theta}_0 \in \mathbb{S}^{n-1}$.

Lemma 6.10. *For each $1 \le I \le N$, the operator*

$$\mathscr{R}_I :\ u(\omega,\boldsymbol{\theta}) \mapsto \mathscr{R}_I u(\omega) := \int R_I(\omega,\boldsymbol{\theta})u(\omega,\boldsymbol{\theta})\,d\Omega_{\boldsymbol{\theta}}$$

acts continuously from $L^2(W \times \mathbb{S}^{n-1})$ to $L^2(W)$.

Proof. For a given value $\omega \in W$, let $\boldsymbol{\tau}_I(\omega,\boldsymbol{\theta})$ be the inverse function to $\boldsymbol{\theta}_I(\omega,\boldsymbol{\tau})$ which exists for $\boldsymbol{\theta} \in \{\boldsymbol{\theta}_I(\omega,\boldsymbol{\tau})\colon \boldsymbol{\tau} \in \mathbb{B}^{n-1}\}$. It suffices to notice that the function $R_I(\omega,\boldsymbol{\theta})$ defined in (6.46) is smooth, since

$$\delta_{\boldsymbol{\theta}_I(\omega,\boldsymbol{\tau})}(\boldsymbol{\theta})\,\varsigma(\boldsymbol{\tau}) = \frac{\delta(\boldsymbol{\tau} - \boldsymbol{\tau}_I(\omega,\boldsymbol{\theta}))}{\left|\det \frac{\partial\boldsymbol{\theta}_I(\omega,\boldsymbol{\tau})}{\partial\boldsymbol{\tau}}\right|}\,\varsigma(\boldsymbol{\tau}).$$

□

Lemma 6.11. *For any functions $\tilde f_I \in L^2_{loc}(\mathbb{R})$, $1 \le I \le N$, there is the identity*

$$\mathscr{R}_I\Big(\sum_{K=1}^N e^{-ik(\omega)\boldsymbol{\theta}\cdot X_K}\tilde f_K(\omega)\Big)\Big|_W = \tilde f_I|_W.$$

Proof.

$$\mathscr{R}_I\Big(\sum_{K=1}^{N} e^{-ik(\omega)\boldsymbol{\theta}\cdot X_K}\tilde{f}_K(\omega)\Big) = \sum_{K=1}^{N}\int_{\mathbb{S}^{n-1}} R_I(\boldsymbol{\theta},\omega)e^{-ik(\omega)\boldsymbol{\theta}\cdot X_K}\tilde{f}_K(\omega)\,d\Omega_{\boldsymbol{\theta}}$$
$$= \sum_{K=1}^{N}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{B}^{n-1}}\sum_{J=1}^{N} R_{IJ}(\omega,\boldsymbol{\tau})\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_J(\omega,\boldsymbol{\tau}))\,\varsigma(\boldsymbol{\tau})e^{-ik(\omega)\boldsymbol{\theta}\cdot X_K}\tilde{f}_K(\omega)\,d\boldsymbol{\tau}\,d\Omega_{\boldsymbol{\theta}}$$
$$= \sum_{K=1}^{N}\int_{\mathbb{B}^{n-1}}\sum_{J=1}^{N} R_{IJ}(\omega,\boldsymbol{\tau})\,\varsigma(\boldsymbol{\tau})e^{-ik(\omega)\boldsymbol{\theta}_J(\omega,\boldsymbol{\tau})\cdot X_K}\tilde{f}_K(\omega)\,d\boldsymbol{\tau}$$
$$= \sum_{K=1}^{N}\int_{\mathbb{B}^{n-1}}\delta_{IK}\,\varsigma(\boldsymbol{\tau})\tilde{f}_K(\omega)\,d\boldsymbol{\tau} = \tilde{f}_I(\omega).$$

□

By Proposition 6.3 and (6.45),

$$\sum_{K=1}^{N} e^{-ik(\omega)\boldsymbol{\theta}\cdot X_K}\tilde{f}_K(\omega) \in L^2(W\times\mathbb{S}^{n-1}).$$

Since $\mathscr{R}_I$ is continuous from $L^2(W\times\mathbb{S}^{n-1})$ to $L^2(W)$ by Lemma 6.10, Lemma 6.11 proves that $\tilde{f}_I \in L^2(W)$. This finishes the proof of Proposition 6.7. □

6.4 Spectral analysis of omega-limit trajectories

For a particular omega-limit trajectory $\beta(x,t)$ which appears in (6.5), we denote

$$\beta_I(t) = \langle\rho_I,\beta(\cdot,t)\rangle, \qquad g_I(t) = F_I(\langle\rho_I,\beta(\cdot,t)\rangle). \tag{6.47}$$

According to the convergence (6.5), for any $T>0$ and any $1\le I\le N$,

$$f_J(t_{j_r}+t) = F_I(\langle\rho_I,\psi(\cdot,t_{j_r}+t)\rangle) \xrightarrow[r\to\infty]{C_b([-T,T])} F_I(\langle\rho_I,\beta(\cdot,t)\rangle) = g_I(t). \tag{6.48}$$

Also, by (6.5) and Corollary C.2,

$$S_{t_{j_r}}(\varphi,\dot{\varphi}) \xrightarrow[r\to\infty]{C_b([-T,T],\mathscr{Y}^{-\varepsilon})} (\beta,\dot{\beta}). \tag{6.49}$$

Using (6.20), (6.48), (6.49), and taking into account that $\varSigma_I(x,\omega)$ is smooth (hence a multiplicator in $\mathscr{S}'$), we obtain the following relation which holds in the sense of distributions:

$$\tilde{\beta}(x,\omega) = \sum_{I=1}^{N}\varSigma_I(x,\omega)\tilde{g}_I(\omega), \qquad x\in\mathbb{R}^n, \qquad \omega\in\mathbb{R}. \tag{6.50}$$

Proposition 6.12. $\operatorname{supp}\tilde{g}_I \subset [-m,m]\cup Z_\rho$*, where* Z_ρ *is defined in* (2.27).

Proof. By (6.48) and the continuity of the Fourier transform in the space of tempered distributions,

$$\tilde{f}_I(\omega)e^{-i\omega t_{j_r}} \xrightarrow[r\to\infty]{\mathscr{S}'} \tilde{g}_I(\omega), \qquad 1\le I\le N.$$

Since $\tilde{f}_I(\omega)$ is locally L^2 for $\omega \in \mathbb{R}\backslash([-m,m]\cup Z_\rho)$ by Proposition 6.7, the proof follows from Lemma 4.10. □

From (6.50), we deduce that

$$\tilde{\beta}_I(\omega) = \sum_{J=1}^{N} \sigma_{IJ}(\omega)\tilde{g}_J(\omega), \qquad \omega \in \mathbb{R}. \tag{6.51}$$

Proposition 6.13. *There exists* $\omega_\star \in Z_\rho \cup [-m,m]$ *such that* $\operatorname{supp}\tilde{\beta}_I \subset \{\omega_\star\}$, $1 \le I \le N$.

Proof. Denote

$$\omega^- = \min_I\{\inf\operatorname{supp}\tilde{\beta}_I\}, \quad \omega^+ = \max_I\{\sup\operatorname{supp}\tilde{\beta}_I\}. \tag{6.52}$$

We claim that the assumption $\omega^- < \omega^+$ leads to a contradiction.

Lemma 6.14. $\operatorname{supp}\tilde{\beta}_I \subset [-m,m]\cup Z_\rho, \quad \omega^\pm \in [-m,m]\cup Z_\rho$.

Proof. This follows from Proposition 6.12, the relation (6.51), and the definition (6.52). □

Lemma 6.15. *For all* $1 \le I \le N$,

$$\sup\operatorname{supp}\tilde{\beta}_I + (p-1)(\sup\operatorname{supp}\tilde{\beta}_I - \inf\operatorname{supp}\tilde{\beta}_I) \le \omega^+,$$
$$\inf\operatorname{supp}\tilde{\beta}_I - (p-1)(\sup\operatorname{supp}\tilde{\beta}_I - \inf\operatorname{supp}\tilde{\beta}_I) \ge \omega^-.$$

Remark 6.16. In particular, Lemma 6.15 states that if $\sup\operatorname{supp}\tilde{\beta}_I = \omega^+$, then $\operatorname{supp}\tilde{\beta}_I = \{\omega^+\}$; if $\inf\operatorname{supp}\tilde{\beta}_I = \omega^-$, then $\operatorname{supp}\tilde{\beta}_I = \{\omega^-\}$.

Proof. Let us assume that, on the contrary, there is I, $1 \le I \le N$, such that

$$\sup\operatorname{supp}\tilde{\beta}_I + (p-1)(\sup\operatorname{supp}\tilde{\beta}_I - \inf\operatorname{supp}\tilde{\beta}_I) > \omega^+. \tag{6.53}$$

By the Titchmarsh Convolution Theorem applied to (2.32), we have:

$$\sup\operatorname{supp}\tilde{g}_I = \sup\operatorname{supp}\tilde{\beta}_I + (p-1)(\sup\operatorname{supp}\tilde{\beta}_I - \inf\operatorname{supp}\tilde{\beta}_I). \tag{6.54}$$

By (6.53) and (6.54), $\sup\operatorname{supp}\tilde{g}_I > \omega^+$. Then the right-hand side of (6.54) is strictly greater than $\sup\operatorname{supp}\tilde{\beta}_I$, hence there exists

$$\omega > \omega^+ := \max_{1\le J\le N}(\sup\operatorname{supp}\tilde{\beta}_J) \tag{6.55}$$

such that $\omega \in \operatorname{supp}\tilde{g}_I$. By Proposition 6.12 and (6.51),

$$\operatorname{supp}\tilde{\beta}_I \subset Z_\rho \cup [-m,m], \qquad \operatorname{supp}\tilde{g}_I \subset Z_\rho \cup [-m,m].$$

By Assumption 2.12, $\det\limits_{1\le I,J\le N}\sigma_{IJ}(\omega) \ne 0$ for $\omega \in \cup_I\operatorname{supp}\tilde{g}_I$, hence (6.51) implies that $\omega \in \cup_I\operatorname{supp}\tilde{\beta}_I$, contradicting (6.55). Thus, our assumption (6.53) can not be true. □

Lemma 6.17. *The points* $\omega = \omega^-$ *and* $\omega = \omega^+$ *are isolated points of the supports of* $\tilde{\beta}_I(\omega)$ *and* $\tilde{g}_I(\omega)$.

Proof. For the support of $\tilde{\beta}_I$, the statement of the lemma follows from Lemma 6.15 (either $\omega^\pm \notin \operatorname{supp}\tilde{\beta}_I$ or $\operatorname{supp}\tilde{\beta}_I \subset \{\omega^\pm\}$). Now one can make the desired conclusion on the support of $\tilde{g}_I$ using (6.51) and the property that $\det\limits_{1\le I,J\le N}\sigma_{IJ}(\omega)$ vanishes at $\omega \in Z_\sigma^N$, which is a discrete set of points by Assumption 2.12. □

By Lemma 6.17, there exist open neighborhoods $\mathcal{O}^-$ and $\mathcal{O}^+$ of $\omega = \omega^-$ and $\omega = \omega^+$, respectively, so that for any $1 \le I \le N$,

$$\mathcal{O}^\pm \cap \operatorname{supp}\tilde{\beta}_I \subset \{\omega^\pm\}, \quad \mathcal{O}^\pm \cap \operatorname{supp}\tilde{g}_I \subset \{\omega^\pm\}.$$

Let $\zeta^\pm \in C_0^\infty(\mathbb{R})$ be such that $\operatorname{supp}\zeta^\pm \subset \mathcal{O}^\pm$.

Lemma 6.18. *There exist* $B_I^{\pm}$, $G_I^{\pm} \in \mathbb{C}$, $1 \le I \le N$, *such that*

$$\zeta^{\pm}(\omega)\tilde{\beta}_I(\omega) = 2\pi B_I^{\pm}\delta(\omega - \omega^{\pm}), \qquad \zeta^{\pm}(\omega)\tilde{g}_I(\omega) = 2\pi G_I^{\pm}\delta(\omega - \omega^{\pm}), \qquad 1 \le I \le N.$$

Proof. One uses the inclusions $\operatorname{supp}\zeta^{\pm}\tilde{\beta}_I \subset \{\omega^{\pm}\}$, $\operatorname{supp}\zeta^{\pm}\tilde{g}_I \subset \{\omega^{\pm}\}$, and argues that the expressions for $\zeta^{\pm}(\omega)\tilde{\beta}_I(\omega)$ and $\zeta^{\pm}(\omega)\tilde{g}_I(\omega)$ in terms of $\delta(\omega - \omega^{\pm})$ and its derivatives can not contain terms with $\delta^{(k)}(\omega - \omega^{\pm})$, $k \ge 1$, due to the boundedness of $(\check{\zeta}^{\pm} * \beta_I)(t)$ and $(\check{\zeta}^{\pm} * g_I)(t)$, where $\check{\zeta}^{\pm}(t)$ is the inverse Fourier transform of $\zeta^{\pm}$. This boundedness takes place in view of the definition (6.47) and the bound (6.7). □

Now let us finish the proof of Proposition 6.13. Introduce the sets

$$\mathcal{I}^- = \{I\colon\ \operatorname{supp}\tilde{\beta}_I = \{\omega^-\}\} \subset \mathbb{N}, \quad \mathcal{I}^+ = \{I\colon\ \operatorname{supp}\tilde{\beta}_I = \{\omega^+\}\} \subset \mathbb{N}. \tag{6.56}$$

Let us assume that $|\mathcal{I}^-| \le |\mathcal{I}^+|$ (the other case is treated similarly). Multiplying (6.51) by $\zeta^-(\omega)$ (and factoring out $\delta(\omega - \omega^-)$), we obtain the following relations:

$$B_I^- = \sum_{J=1}^{N} \sigma_{IJ}(\omega^-)G_J^-. \tag{6.57}$$

For $I \in \mathcal{I}^+$, one has $\operatorname{supp}\tilde{\beta}_I = \{\omega^+\}$, $\operatorname{supp}\tilde{g}_I \subset \{\omega^+\}$, hence $B_I^- = G_I^- = 0$. Therefore, (6.57) yields

$$B_I^- = \sum_{J=1}^{N} \sigma_{IJ}(\omega^-)G_J^- = \sum_{J\in\{1,\,\dots,\,N\}\setminus\mathcal{I}^+} \sigma_{IJ}(\omega^-)G_J^-. \tag{6.58}$$

Since $|\mathcal{I}^-| \le |\mathcal{I}^+|$, we can pick $\mathcal{I}_1 \subset \{1, \dots, N\}$ such that $\mathcal{I}_1 \cap \mathcal{I}^- = \emptyset$ and $|\mathcal{I}_1| = N - |\mathcal{I}^+|$. Therefore, considering the relations (6.58) with $I \in \mathcal{I}_1$ (when $B_I^- = 0$ due to $\mathcal{I}_1 \cap \mathcal{I}^- = \emptyset$), we conclude that

$$0 = B_I^- = \sum_{J\in\{1,\,\dots,\,N\}\setminus\mathcal{I}^+} \sigma_{IJ}(\omega^-)G_J^-, \qquad I \in \mathcal{I}_1. \tag{6.59}$$

We know from (6.57) that not all G_I^-, $I \in \{1, \dots, N\}\setminus\mathcal{I}^+$, are equal to zero; hence, (6.59) implies that

$$\det_{I\in\mathcal{I}_1,J\in\{1,\,\dots,\,N\}\setminus\mathcal{I}^+} \sigma_{IJ}(\omega^-) = 0. \tag{6.60}$$

According to Definition 2.11, $\omega^- \in Z_\sigma^*$. On the other hand, by Lemma 6.14, $\omega^- \in [-m, m] \cap Z_\rho$, contradicting Assumption 2.12.

The case $|\mathcal{I}^-| \ge |\mathcal{I}^+|$ is treated similarly; in that case, the conclusion is that $\omega^+ \in Z_\sigma^*$, again leading to a contradiction with Assumption 2.12.

It follows that there exists $\omega_\star$ such that $\operatorname{Spec}\beta_J \subset \{\omega_\star\}$ for $1 \le J \le N$, finishing the proof of Proposition 6.13. □

By Proposition 6.13, $\beta(x,t)$ is a solitary wave. According to Lemma 3.14, this finishes the proof of Theorem 2.15.

Chapter 7

Multifrequency solitary waves

We will show that when the assumptions of Theorem 2.8 are not satisfied, then the attractor could be more complicated because the equation admits multifrequency solitary wave solutions.

7.1 Klein-Gordon with several oscillators

7.1.1 Linear degeneration

Let us consider equation (2.16) with $N = 2$, when some of F_J are linear, satisfying $F_J(\psi) = c_J\psi$, with $c_J \in \mathbb{R}$, $c_J \neq 0$ (in (2.20), some of p_J are equal to 1).

Proposition 7.1. *If the condition $p_J \geq 2$ in (2.20) is violated, so that $F_J(\psi) = c\psi$ for some J, with a constant $c \in \mathbb{R}$, $c \neq 0$, then the conclusion of Theorem 2.8 may no longer be correct.*

Proof. We are going to construct the multifrequency solitary waves. Consider the equation

$$\ddot{\psi} = \psi'' - m^2\psi + \delta(x)F_1(\psi) + \delta(x-L)F_2(\psi), \tag{7.1}$$

where

$$F_1(\psi) = \alpha\psi + \beta|\psi|^2\psi, \qquad F_2(\psi) = \gamma\psi, \qquad \alpha,\ \beta,\ \gamma \in \mathbb{R}. \tag{7.2}$$

Note that the function F_2 is linear, failing to satisfy (2.20) (where one now has $p_2 = 1$).

We denote

$$\kappa(\omega) := -ik_+(\omega), \qquad \omega \in \mathbb{R}, \tag{7.3}$$

where $k_+(\omega)$ was introduced in (4.15). We then have $\operatorname{Re}\kappa(\omega) \geq 0$, and also

$$\kappa(\omega) = \sqrt{\omega^2 - m^2} > 0 \quad \text{for} \quad -m < \omega < m.$$

The function

$$\psi(x,t) = \begin{cases} (A+B)e^{\kappa(\omega)x}\sin\omega t, & x \leq 0, \\ \big(Ae^{-\kappa(\omega)x} + Be^{\kappa(\omega)x}\big)\sin\omega t + C\sinh(\kappa(3\omega)x)\sin 3\omega t, & x \in [0,L], \\ \big(Ae^{-\kappa(\omega)} + Be^{\kappa(\omega)(2L-x)}\big)\sin\omega t + \frac{Ce^{-\kappa(3\omega)(x-L)}\sin 3\omega t}{\sinh(\kappa(3\omega)L)}, & x \geq L, \end{cases}$$

where $\omega \in (0, m/3)$, will be a solution if the jump conditions are satisfied at $x = 0$ and at $x = L$:

$$-\psi'(0+,t) + \psi'(0-,t) = \alpha\psi(0,t) + \beta\psi^3(0,t), \tag{7.4}$$

$$-\psi'(L+,t) + \psi'(L-,t) = \alpha\psi(L,t) + \beta\psi^3(L,t). \tag{7.5}$$

Using the identity

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta, \tag{7.6}$$

we see that

$$\alpha(A+B)\sin\omega t + \beta((A+B)\sin\omega t)^3$$
$$= \Big(\alpha(A+B) + \beta\frac{3(A+B)^3}{4}\Big)\sin\omega t - \beta\frac{(A+B)^3}{4}\sin 3\omega t.$$

Collecting the terms at $\sin\omega t$ and at $\sin 3\omega t$, we write the condition (7.4) as the following system of equations:

$$2\kappa(\omega)A = \Big(\alpha(A+B) + \beta\frac{3(A+B)^3}{4}\Big), \tag{7.7}$$

$$-\kappa(3\omega)C = -\beta\frac{(A+B)^3}{4}. \tag{7.8}$$

Similarly, the condition (7.5) is equivalent to the following two equations:

$$2B\kappa(\omega)e^{\kappa(\omega)L} = \gamma(Ae^{-\kappa(\omega)L} + Be^{\kappa(\omega)L}), \tag{7.9}$$

$$\frac{\kappa(3\omega)C}{\sinh(\kappa(3\omega)L)} + \kappa(3\omega)C\cosh(\kappa(3\omega)L) = \gamma C\sinh(\kappa(3\omega)L). \tag{7.10}$$

Equations (7.7), (7.8), (7.9), and (7.10) could be satisfied for arbitrary $L > 0$. Namely, for any $\omega \in (0, m/3)$, one uses (7.10) to determine γ. For any $\beta \neq 0$, there is always a solution (A, B) to the nonlinear system (7.7), (7.9). Finally, C is obtained from (7.8). □

7.1.2 Wide gaps

Let us consider equation (2.16) with $N = 2$ when (2.20) is satisfied.

Proposition 7.2. *If the inequality* (2.21) *is violated, then the conclusion of Theorem 2.8 may no longer be correct.*

Proof. We will show that if $L := X_2 - X_1$ is sufficiently large, then one can take $F_1(\psi)$ and $F_2(\psi)$ satisfying (2.20) such that the trajectory attractor $\mathfrak{A}$ of the equation contains the multifrequency solutions which are not in the set $\mathfrak{S}$ of solitary waves defined in (2.5). For our convenience, we assume that $X_1 = 0$, $X_2 = L$. We consider the model (2.16) with

$$F_1(\psi) = F_2(\psi) = F(\psi), \qquad \text{where} \quad F(\psi) = \alpha\psi + \beta|\psi|^2\psi, \qquad \alpha,\, \beta \in \mathbb{R}. \tag{7.11}$$

In terms of the condition (2.20), $p_1 = p_2 = 2$. We take L to be large enough:

$$L > \frac{\pi}{2^{3/2}m}. \tag{7.12}$$

Consider the function

$$\psi(x,t) = A(e^{-\kappa(\omega)|x|} + e^{-\kappa(\omega)|x-L|})\sin\omega t + B\chi_{[0,L]}(x)\sin(k(3\omega)x)\,\sin 3\omega t, \tag{7.13}$$

where A, $B \in \mathbb{C}$. Then $\psi(x,t)$ solves (2.16) for x away from the points X_J. We require that

$$k(3\omega) = \frac{\pi}{L}, \tag{7.14}$$

so that $\psi(x,t)$ is continuous in $x \in \mathbb{R}$ and symmetric with respect to $x = L/2$:

$$\psi(x,t) = \psi(\frac{L}{2} - x, t), \qquad x \in \mathbb{R}.$$

We need $|\omega| < m$ to have $\kappa(\omega) > 0$, and $3\omega \in \mathbb{R}\backslash[-m, m]$ to have $k(3\omega) \in \mathbb{R}$. We take $\omega > 0$, and thus $m < 3\omega < 3m$. By (7.14), this means that we need

$$m < \sqrt{\frac{\pi^2}{L^2} + m^2} < 3m.$$

The second inequality is satisfied by (7.12).

Due to the symmetry of $\psi(x,t)$ with respect to $x = L/2$, the jump conditions both at $x = X_1 = 0$ and at $x = X_2 = L$ take the following identical form:

$$2A\kappa(\omega)\sin\omega t - Bk(3\omega)\sin 3\omega t = F\big(A(1 + e^{-\kappa(\omega)L})\sin\omega t\big). \tag{7.15}$$

We use the following relation which follows from (7.6):

$$\begin{aligned} &F\Big(A(1 + e^{-\kappa(\omega)L})\sin\omega t\Big) \\ &= \Big(\alpha A(1 + e^{-\kappa(\omega)L}) + \frac{3}{4}\beta|A|^2 A(1 + e^{-\kappa(\omega)L})^3\Big)\sin\omega t \\ &\quad -\frac{1}{4}\beta|A|^2 A(1 + e^{-\kappa(\omega)L})^3 \sin 3\omega t. \end{aligned} \tag{7.16}$$

Collecting in (7.15) the terms at $\sin\omega t$ and at $\sin 3\omega t$, we obtain the following system:

$$\begin{cases} 2A\kappa(\omega) = \alpha A(1 + e^{-\kappa(\omega)L}) + \frac{3}{4}\beta|A|^2 A(1 + e^{-\kappa(\omega)L})^3, \\ Bk(3\omega) = \frac{1}{4}\beta|A|^2 A(1 + e^{-\kappa(\omega)L})^3. \end{cases} \tag{7.17}$$

Assuming that $A \neq 0$, we divide the first equation by A:

$$2\kappa(\omega) = \alpha(1 + e^{-\kappa(\omega)L}) + \frac{3}{4}\beta|A|^2(1 + e^{-\kappa(\omega)L})^3. \tag{7.18}$$

The condition for the existence of a solution $A \neq 0$ is

$$\Big(\frac{2\kappa(\omega)}{1 + e^{-\kappa(\omega)L}} - \alpha\Big)\beta > 0. \tag{7.19}$$

Once we found A, the second equation in (7.17) can be used to express B in terms of A.

Remark 7.3. Condition (7.19) shows that we can choose $\beta < 0$ taking large $\alpha > 0$. The corresponding potential $U(\psi) = -\alpha|\psi|^2/2 - \beta|\psi|^4/4$ satisfies (2.20).

□

7.2 Klein-Gordon with mean field interaction

Now we consider the model (2.23) in the situation when Assumption 2.12 is violated. We will only consider (2.23) with $N = 1$. In this case, we show that there could exist multifrequency solutions, indicating that the set of all (one-frequency) solitary waves is only a subset of the global attractor.

Fix $\omega_1 \in (m, 3m)$. Set $\omega_0 = \omega_1/3$ and pick $\rho \in \mathscr{S}(\mathbb{R}^n)$ such that the following conditions are satisfied:

$$\hat\rho\big|_{|\xi|=\sqrt{\omega_1^2 - m^2}} = 0, \tag{7.20}$$

$$\sigma(\omega_1) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{|\hat\rho(\xi)|^2\, d^n\xi}{\xi^2 + m^2 - \omega_1^2} = 0. \tag{7.21}$$

These two equalities imply that $\sigma(\omega)$ vanishes at a certain point of Z_ρ, violating Assumption 2.12.

Lemma 7.4. *There exist $a \in \mathbb{R}$, $b < 0$ so that equation (2.23) with the nonlinearity*

$$F(z) = az + b|z|^2 z, \qquad z \in \mathbb{C},$$

admits multifrequency solutions $\psi \in C(\mathbb{R}, H^1)$ of the form

$$\psi(x,t) = \phi_0(x)\sin\omega_0 t + \phi_1(x)\sin\omega_1 t, \qquad \omega_0 = \frac{\omega_1}{3}, \quad \phi_0,\, \phi_1 \in H^1(\mathbb{R}^n),$$

with both ϕ_0 and ϕ_1 nonzero.

Proof. To make sure that the nonlinearity does not produce higher frequencies, we assume that

$$\langle \rho, \phi_1 \rangle = 0. \tag{7.22}$$

Due to this assumption,

$$F(\langle \rho, \psi\rangle) = F(\langle \rho, \phi_0\rangle \sin\omega_0 t) = a\langle \rho, \phi_0\rangle \sin\omega_0 t + b\langle \rho, \phi_0\rangle^3 \frac{3\sin\omega_0 t - \sin 3\omega_0 t}{4}.$$

Collecting the terms with the factors of $\sin\omega_0 t$ and $\sin\omega_1 t = \sin 3\omega_0 t$, we rewrite the equation $\ddot{\psi} = \Delta\psi - m^2\psi + \rho F(\langle \rho, \psi\rangle)$ as two following equalities:

$$-\omega_0^2\phi_0 = \Delta\phi_0 - m^2\phi_0 + \rho(x)\Big(a\langle \rho, \phi_0\rangle + \frac{3b\langle \rho, \phi_0\rangle^3}{4}\Big), \tag{7.23}$$

$$-\omega_1^2\phi_1 = \Delta\phi_1 - m^2\phi_1 - \rho(x)\frac{b\langle \rho, \phi_0\rangle^3}{4}. \tag{7.24}$$

We define $\phi_0(x)$ by $\hat{\phi}_0(\xi) = \frac{\hat{\rho}(\xi)}{\xi^2 + m^2 - \omega_0^2}$. Since $m^2 - \omega_0^2 > 0$, there is the inclusion $\phi_1 \in H^1(\mathbb{R}^n)$. Moreover,

$$\langle \rho, \phi_0\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{|\hat{\rho}(\xi)|^2\, d^n\xi}{\xi^2 + m^2 - \omega_0^2} = \sigma(\omega_0) > 0,$$

due to strict positivity of $\sigma(\omega)$ for $|\omega| < m$ (see (2.30)). Hence, for any b (we take $b < 0$ so that the potential satisfies $\inf U > -\infty$, complying with (2.32)), we may pick a such that (7.23) is satisfied. We then use (7.24) to define the function $\phi_1(x)$ by

$$\hat{\phi}_1(\xi) = -\frac{b\langle \rho, \phi_0\rangle^3}{4}\frac{\hat{\rho}(\xi)}{\xi^2 + m^2 - \omega_1^2} = -\frac{b\sigma(\omega_0)^3}{4}\frac{\hat{\rho}(\xi)}{\xi^2 + m^2 - \omega_1^2}.$$

Due to (7.20), $\phi_1 \in H^1(\mathbb{R}^n)$. We are left to check that ϕ_0 satisfies the assumption (7.22). Indeed, due to (7.21),

$$\langle \rho, \phi_1\rangle = -\frac{b\sigma(\omega_0)^3}{4}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{|\hat{\rho}(\xi)|^2\, d^n\xi}{\xi^2 + m^2 - \omega_1^2} = 0.$$

□

Appendix A

Existence of solitary waves

A.1 Solitary waves for Klein-Gordon with N oscillators

The following proposition provides a concise description of all solitary waves.

Proposition A.1. *Assume that $F_J(\psi)$, $1 \le J \le N$, satisfy* (2.2)*. Then the set of all solitary wave solutions* (2.5) *of equation* (2.16) *consists of solutions $\psi(x,t) = \phi_\omega(x)e^{-i\omega t}$ with*

$$\phi_\omega(x) = \sum_{J=1}^{N} C_J e^{-\kappa(\omega)|x-X_J|}, \qquad \kappa(\omega) = \sqrt{m^2-\omega^2}, \tag{A.1}$$

where $\omega \in [-m,m]$ and $C_J \in \mathbb{C}$, $1 \le J \le N$, satisfy the following relations:

$$2\kappa(\omega)C_J = F_J\Big(\sum_{K=1}^{N} C_K e^{-\kappa(\omega)|X_J-X_K|}\Big). \tag{A.2}$$

Remark A.2. By (A.1), $\omega = \pm m$ can only correspond to zero solution.

Remark A.3. In the case $N = 1$, the conditions of Proposition A.1 for the existence of a nonzero solitary wave with some $\omega \in (-m,m)$ only require having $C \ne 0$ such that $2C\sqrt{m^2-\omega^2} = F(C)$. Therefore, we can state the following necessary and sufficient condition for having nonzero solitary waves:

$$\exists C \in \mathbb{C}\backslash 0 \quad \text{so that} \quad 0 < \frac{F(C)}{2C} \le m.$$

The case $F(C)/(2C) = m$ corresponds to the solitary wave with $\omega = 0$, which is a stationary solution to (2.1) given by $\psi(x,t) = Ce^{-m|x|}$.

Proof. We will only prove this Proposition A.1 for the case $N = 1$.

Substituting $\phi_\omega(x)e^{-i\omega t}$ into (2.1), we get the equation

$$-\omega^2\phi_\omega(x)e^{-i\omega t} = \phi_\omega''(x)e^{-i\omega t} - m^2\phi_\omega(x)e^{-i\omega t} + \delta(x)F(e^{-i\omega t}\phi_\omega(0)), \tag{A.3}$$

where $(x,t) \in \mathbb{R}\times\mathbb{R}$. We can assume that $\phi_\omega(0) \ne 0$. Indeed, if $\phi_\omega(0) = 0$, then (A.3) turns into a homogeneous second-order linear differential equation, which together with the inclusion $\phi_\omega \in H^1(\mathbb{R})$ results in $\phi_\omega(x) \equiv 0$.

The relation (A.3) turns into the following eigenvalue problem:

$$-\omega^2\phi_\omega(x) = \phi_\omega''(x) - m^2\phi_\omega(x) + \delta(x)F(\phi_\omega(x)), \quad x \in \mathbb{R}. \tag{A.4}$$

The phase factor $e^{-i\omega t}$ can be canceled out due to (2.2). Equation (A.4) implies that away from the origin we have

$$\phi_\omega''(x) = (m^2 - \omega^2)\phi_\omega(x), \qquad x \neq 0,$$

hence $\phi_\omega(x) = C_\pm e^{-\kappa_\pm |x|}$ for $\pm x > 0$, where $\kappa_\pm$ satisfy $\kappa_\pm^2 = m^2 - \omega^2$. Since $\phi_\omega(x) \in H^1$, it is imperative that $\kappa_\pm > 0$; we conclude that $|\omega| < m$ and that $\kappa_\pm = \sqrt{m^2 - \omega^2} > 0$. Moreover, since the function $\phi_\omega(x)$ is continuous, $C_- = C_+ = C \neq 0$ (since we are looking for nonzero solitary waves). We see that

$$\phi_\omega(x) = Ce^{-\kappa|x|}, \qquad C \neq 0, \qquad \kappa \equiv \sqrt{m^2 - \omega^2} > 0. \tag{A.5}$$

Equation (A.4) implies the following gluing condition at $x = 0$:

$$0 = \phi_\omega'(0+) - \phi_\omega'(0-) + F(\phi_\omega(0)). \tag{A.6}$$

This condition and (A.5) lead to the equation $2\kappa C = F(C)$ which is equivalent to (A.2) with $N = 1$. □

A.2 Solitary waves for Klein-Gordon with mean field interaction

Proposition A.4. *Let $\rho \in \mathscr{S}(\mathbb{R}^n)$, $\rho \not\equiv 0$. Assume that $\det_{I,J} \sigma_{IJ}(\omega) \neq 0$ for $\omega \in [-m, m] \cup Z_\rho$. There may only be nonzero solitary wave solutions to* (2.23) *for $\omega \in [-m, m] \cup Z_\rho$, where Z_ρ is defined in* (2.27)*. The profiles of solitary waves are given by*

$$\hat\phi_\omega(\xi) = \frac{\sum_{J=1}^N C_J \hat\rho_J(\xi)}{\xi^2 + m^2 - \omega^2}, \tag{A.7}$$

where $C_J \in \mathbb{C}$, $1 \le J \le N$, satisfy

$$F_J\Big(\sum_{K=1}^N \sigma_{JK}(\omega) C_K\Big) = C_J, \tag{A.8}$$

with $\sigma_{IJ}(\omega)$ from (2.30)*. The existence of such solutions $\{C_J\colon 1 \le J \le N\}$ is a necessary condition for* (A.7) *to represent a solitary wave.*

The condition (A.8) *is also sufficient for* (A.7) *to represent a (finite energy) solitary wave in the case $n \ge 5$, and also in the case $|\omega| \neq m$, $n \ge 2$.*

For $|\omega| = m$, $n \le 4$, the following additional condition is needed for (A.7) *to represent a finite energy solitary wave:*

$$\int_{\mathbb{R}^n} \frac{|\hat\rho(\xi)|^2}{\xi^4}\, d^n\xi < \infty. \tag{A.9}$$

Proof. Substituting the ansatz $\phi_\omega(x)e^{-i\omega t}$ into (2.23), we get the following equation on ϕ_ω:

$$-\omega^2\phi_\omega(x) = \Delta\phi_\omega(x) - m^2\phi_\omega(x) + \sum_{J=1}^N \rho_J(x) F_J(\langle\rho_J, \phi_\omega\rangle), \qquad x \in \mathbb{R}^n.$$

Therefore, all solitary waves satisfy the relation

$$(\xi^2 + m^2 - \omega^2)\hat\phi_\omega(\xi) = \sum_{J=1}^N \hat\rho_J(\xi) F_J(\langle\rho_J, \phi_\omega\rangle). \tag{A.10}$$

For $\omega \in \mathbb{R}\backslash([-m, m] \cup Z_\rho)$ the relation (A.10) leads to $\phi_\omega \notin L^2(\mathbb{R}^n)$ (unless $\phi_\omega \equiv 0$). We conclude that there are no nonzero solitary waves for $\omega \in \mathbb{R}\backslash([-m, m] \cup Z_\rho)$.

Let us consider the case $\omega \in [-m, m] \cup Z_\rho$. From (A.10), we see that

$$\hat{\phi}_\omega(\xi) = \sum_{J=1}^{N} \frac{\hat{\rho}_J(\xi)}{\xi^2 + m^2 - \omega^2} F_J(\langle \rho_J, \phi_\omega \rangle). \tag{A.11}$$

Using the functions $\Sigma_J(x, \omega)$ defined in (2.28), we can write $\phi_\omega(x) = \sum_{J=1}^{N} C_J \Sigma_J(x, \omega)$, with $C_J \in \mathbb{C}$. Substituting this ansatz into (A.11), we can write the condition on C_J in the following form:

$$\sum_{J=1}^{N} \sigma_{IJ}(\omega) F_J\Big(\sum_{K=1}^{N} \sigma_{JK}(\omega) C_K \Big) = \sum_{J=1}^{N} \sigma_{IJ}(\omega) C_J, \tag{A.12}$$

where $\sigma_{IJ}(\omega)$ is defined in (2.30). Since $\sigma_{IJ}(\omega)$ is nondegenerate for $\omega \in [-m, m] \cup Z_\rho$, we can rewrite (A.12) in the form (A.8).

For $n \le 4$, the finiteness of the energy of solitons corresponding to $\omega = \pm m$ is guaranteed by the condition (A.9). □

Appendix B

Global well-posedness

B.1 Klein-Gordon with one oscillator

Here we state and prove the well-posedness result for the equation

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \sum_{J=1}^{N} \delta(x - X_J) F_J(\psi(X_J,t)), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{B.1}$$

which describes the Klein-Gordon field interacting with oscillators F_J, $1 \le J \le N$, located at the points $X_J \in \mathbb{R}$.

We formulate the result for N oscillators, but only give a detailed proof for the case of one oscillator, $N = 1$ (this corresponds to equation (2.1)). The proof for the case $N > 1$ requires minor modifications.

Global well-posedness

Theorem B.1. *Assume that* $F_J(\psi) = -\nabla U_J(\psi)$, $1 \le J \le N$, *where*

$$U_J(\psi) = u_J(|\psi|^2), \qquad u_J(\cdot) \in C^2(\mathbb{R}).$$

Assume that

$$\inf_{\psi \in \mathbb{C}} U_J(\psi) > -\infty, \qquad 1 \le J \le N. \tag{B.2}$$

Then:

(1) For every $(\psi_0, \pi_0) \in \mathscr{X}$ *the Cauchy problem*

$$\begin{cases} \ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \sum_{J=1}^{N} \delta(x - X_J) F_J(\psi(X_J,t)), & x \in \mathbb{R}, \\ (\psi, \dot{\psi})\big|_{t=0} = (\psi_0, \pi_0), \end{cases}$$

where $m > 0$, *has unique solution* $\psi(t)$, $t \in \mathbb{R}$, *such that* $(\psi, \dot{\psi}) \in C(\mathbb{R}, \mathscr{X})$.

(2) The map $W(t) : (\psi_0, \pi_0) \mapsto (\psi(t), \dot{\psi}(t))$ *is continuous in* $\mathscr{X}$ *for each* $t \in \mathbb{R}$.

(3) The values of the energy and charge functionals are conserved along the trajectory:

$$\mathcal{H}(\psi(t), \dot{\psi}(t)) = \text{const}, \qquad \mathcal{Q}(\psi(t), \dot{\psi}(t)) = \text{const}, \qquad t \in \mathbb{R}.$$

(4) The following a priori *bound holds:*

$$\|(\psi(t), \dot{\psi}(t))\|_{\mathscr{X}} \le C(\psi_0, \pi_0), \qquad t \in \mathbb{R}. \tag{B.3}$$

(5) *For any* $0 \le \varepsilon < 1/2$, $E \in \mathbb{R}$, *and* $T > 0$, *the map*

$$W(t):\ \mathscr{X}_E \to \mathscr{X}_E, \qquad W(t):\ (\psi_0, \pi_0) \mapsto (\psi(t), \dot\psi(t)),$$

is continuous in the topology of $\mathscr{Y}^{-\varepsilon}$, *uniformly in* $t \in [-T, T]$. *Above,* $\mathscr{X}_E$ *is defined by*

$$\mathscr{X}_E = \{\Psi \in \mathscr{X} \colon \mathcal{H}(\Psi) \le E\}. \tag{B.4}$$

Remark B.2. In Theorem B.1 *(5)*, we need $\varepsilon < 1/2$ to have the embedding $H^{1-\varepsilon}(\mathbb{R}) \subset C(\mathbb{R})$.

Remark B.3. The condition (B.2) is satisfied if the assumption (2.20) of Theorem 2.8 holds.

Preparation

To simplify formulas, we will only prove Theorem B.1 for $N = 1$, assuming that the oscillator is at the origin ($X_1 = 0$); the corresponding equation is (2.1). The case $N > 1$ requires minor modifications.

Without loss of generality, the condition (B.2) could be substituted by

$$U_J(\psi) > 0, \qquad \psi \in \mathbb{C}, \quad 1 \le J \le N. \tag{B.5}$$

The condition (B.5) is satisfied by U_J from (2.20).

We first need to adjust the nonlinearity F so that it becomes bounded, together with its derivatives. Define

$$\lambda_0 = \sqrt{\frac{\mathcal{H}(\psi_0, \pi_0)}{m}}, \tag{B.6}$$

where the Hamiltonian $\mathcal{H}(\Psi)$ is defined in (2.9) and $(\psi_0, \pi_0) \in \mathscr{X}$ is the initial data from Theorem B.1. We may pick a modified potential function $\widetilde U \in C^2(\mathbb{C}, \mathbb{R})$, $\widetilde U(\psi) = \widetilde U(|\psi|)$, so that

$$\widetilde U(\psi) = U(\psi) \qquad \text{for } |\psi| \le \lambda_0, \tag{B.7}$$

$$\widetilde U(\psi) \ge 0, \quad \psi \in \mathbb{C}, \tag{B.8}$$

and so that $|\widetilde U(\psi)|$, $|\widetilde U'(\psi)|$, and $|\widetilde U''(\psi)|$ are bounded for all $\psi \ge 0$. We define

$$\widetilde F(\psi) = -\nabla \widetilde U(\psi), \qquad \psi \in \mathbb{C}, \tag{B.9}$$

where ∇ stands for the gradient with respect to $\operatorname{Re}\psi$, $\operatorname{Im}\psi$; Then $\widetilde F(e^{is}\psi) = e^{is}\widetilde F(\psi)$ for any $\psi \in \mathbb{C}$, $s \in \mathbb{R}$.

We consider the Cauchy problem for equation (2.1) with the modified nonlinearity,

$$\begin{cases} \ddot\psi(x,t) = \psi''(x,t) - m^2\psi(x,t) + \delta(x)\widetilde F(\psi(0,t)), & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ (\psi, \dot\psi)\big|_{t=0} = (\psi_0, \pi_0), \end{cases} \tag{B.10}$$

which we rewrite in the vector form in terms of $\Psi = \begin{bmatrix} \psi(x,t) \\ \pi(x,t) \end{bmatrix}$:

$$\dot\Psi = \begin{bmatrix} 0 & 1 \\ \partial_x^2 - m^2 & 0 \end{bmatrix} \Psi + \delta(x) \begin{bmatrix} 0 \\ \widetilde F(\psi) \end{bmatrix}, \qquad \Psi\big|_{t=0} = \Psi_0, \tag{B.11}$$

where $\Psi_0 = (\psi_0(x), \pi_0(x))$. Equation (B.10) can formally be written as the following Hamiltonian system (Cf. (2.8)):

$$\dot\Psi(t) = \mathcal{J}\widetilde{\mathcal{H}}'(\Psi), \qquad \mathcal{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{B.12}$$

where $\widetilde{\mathcal{H}}'$ is the variational derivative of the Hamilton functional

$$\widetilde{\mathcal{H}}(\Psi) = \int\limits_{\mathbb{R}} \left(|\pi|^2 + |\nabla\psi|^2 + m^2|\psi|^2\right) dx + \widetilde{U}(\psi(0,t)), \quad \Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix} \in \mathscr{X} \tag{B.13}$$

with respect to $(\operatorname{Re}\psi, \operatorname{Im}\psi)$ and $(\operatorname{Re}\pi, \operatorname{Im}\pi)$. Note that $\widetilde{\mathcal{H}}(\Psi)$ is Fréchet differentiable in the space $\mathscr{X} = H^1 \times L^2$. By the Sobolev embedding theorem, $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, and there is the following inequality:

$$\|\psi\|^2_{L^\infty} \le \frac{1}{2m}(\|\psi'\|^2_{L^2} + m^2\|\psi\|^2_{L^2}) \le \frac{1}{2m}\|\Psi\|^2_{\mathscr{X}}. \tag{B.14}$$

Taking into account (B.8) and (B.13), we obtain the inequality

$$\|\Psi\|^2_{\mathscr{X}} = 2\widetilde{\mathcal{H}}(\Psi) - 2\widetilde{U}(\psi(0)) \le 2\widetilde{\mathcal{H}}(\Psi), \tag{B.15}$$

for any $\Psi \in \mathscr{X}$.

Lemma B.4. *(1) There is the identity $\widetilde{\mathcal{H}}(\Psi_0) = \mathcal{H}(\Psi_0)$.*

(2) If $\Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix} \in \mathscr{X}$ satisfies $\widetilde{\mathcal{H}}(\Psi) \le \widetilde{\mathcal{H}}(\Psi_0)$, then $\widetilde{U}(\psi(x)) = U(\psi(x))$ for any $x \in \mathbb{R}$.

Proof. According to (B.15), the Sobolev embedding (B.14), and the choice of λ_0 in (B.6),

$$\|\psi_0\|^2_{L^\infty} \le \frac{1}{2m}\|\Psi_0\|^2_{\mathscr{X}} \le \frac{\mathcal{H}(\Psi_0)}{m} = \lambda_0^2. \tag{B.16}$$

Thus, by (B.7), $\widetilde{U}(\psi_0(x)) = U(\psi_0(x))$ for all $x \in \mathbb{R}$. This proves (i).

By (B.14), the relation (B.15), the condition $\widetilde{\mathcal{H}}(\Psi) \le \widetilde{\mathcal{H}}(\Psi_0)$, and part (i) of the Lemma, we have:

$$\|\psi\|^2_{L^\infty} \le \frac{1}{2m}\|\Psi\|^2_{\mathscr{X}} \le \frac{\widetilde{\mathcal{H}}(\Psi)}{m} \le \frac{\widetilde{\mathcal{H}}(\Psi_0)}{m} = \frac{\mathcal{H}(\Psi_0)}{m} = \lambda_0^2.$$

Now the statement (ii) follows by (B.7). □

If $\Psi(t)$ solves (B.12), then, by the energy conservation, one has $\widetilde{\mathcal{H}}(\Psi(t)) = \widetilde{\mathcal{H}}(\Psi_0)$. By Lemma B.4 *(2)*, $\widetilde{U}(\psi(x,t)) = U(\psi(x,t))$ for all $x \in \mathbb{R}$, $t \in \mathbb{R}$. Hence, one also has $\widetilde{F}(\psi(x,t)) = F(\psi(x,t))$ for all $x \in \mathbb{R}$, $t \ge 0$, allowing us to conclude that $\psi(t)$ solves (2.1) as well as (B.10).

B.1.1 Local well-posedness

The solution to the Cauchy problem

$$\dot{\Xi} = \begin{bmatrix} 0 & 1 \\ \partial_x^2 - m^2 & 0 \end{bmatrix} \Xi, \qquad \Xi(x,0) = \psi_0(x) = \begin{bmatrix} \psi_0(x) \\ \pi_0(x) \end{bmatrix} \tag{B.17}$$

is represented by

$$\Xi(x,t) = W_0(t)\Xi_0 = \int_{\mathbb{R}} \begin{bmatrix} \partial_t G(x-y,t) & G(x-y,t) \\ \partial_t^2 G(x-y,t) & \partial_t G(x-y,t) \end{bmatrix} \begin{bmatrix} \psi_0(y) \\ \pi_0(y) \end{bmatrix} dy, \tag{B.18}$$

where $G(x,t)$ is the forward fundamental solution to the Klein-Gordon equation,

$$G(x,t) = \frac{1}{2}\Theta(t - |x|)J_0(m\sqrt{t^2 - x^2}), \tag{B.19}$$

with $\Theta(t)$ the Heaviside step function and J_0 the Bessel function (see e.g. [Kom94]). Then the solution to the Cauchy problem (B.11) can be represented by

$$\Psi(x,t) = W_0(t)\Psi_0 + Z[\psi(0,\cdot)](t), \tag{B.20}$$

where

$$Z[\psi(0,\cdot)](t) := \int_0^t W_0(t-s) \begin{bmatrix} 0 \\ \delta(\cdot)\widetilde{F}(\psi(0,s)) \end{bmatrix} ds.$$

Lemma B.5. *For any nonnegative integers j and k there is a constant $C_{j,k} > 0$ such that*

$$|\partial_x^j \partial_t^k J_0(m\sqrt{t^2 - x^2})| \le C_{j,k}(1+t)^{j+k}, \qquad |x| < t. \tag{B.21}$$

Proof. The proof immediately follows from the observation that all the derivatives of the Bessel function $J_0(z)$ are bounded for $z \in \mathbb{R}$, and that $J_0(z)$ can be expanded into an absolutely converging Taylor series in even powers of z. Hence, all derivatives of the function $J_0(\sqrt{r})$ in r are bounded for $r \ge 0$. □

The next lemma establishes the contraction principle for the integral equation (B.20).

Lemma B.6. *There exists a constant $C < \infty$ such that for any two functions*

$$\Psi_j(\cdot,t) = \begin{bmatrix} \psi_j(\cdot,t) \\ \pi_j(\cdot,t) \end{bmatrix} \in C([0,1], \mathscr{X}), \qquad j = 1,\, 2,$$

we have:

$$\|Z[\psi_1(0,\cdot)](t) - Z[\psi_2(0,\cdot)](t)\|_{\mathscr{X}} \le Ct^{1/2} \sup_{s\in[0,t]} \|\Psi_1(\cdot,s) - \Psi_2(\cdot,s)\|_{\mathscr{X}}, \tag{B.22}$$

for $0 \le t \le 1$.

Moreover, for any $0 \le \varepsilon < 1/2$, there is $C_\varepsilon < \infty$ such that

$$\|Z[\psi_1(0,\cdot)](t) - Z[\psi_2(0,\cdot)](t)\|_{\mathscr{X}} \le C_\varepsilon t^{1/2} \sup_{s\in[0,t]} \|\Psi_1(\cdot,s) - \Psi_2(\cdot,s)\|_{\mathscr{Y}^{-\varepsilon}}, \tag{B.23}$$

for $0 \le t \le 1$.

Proof. According to (B.18) and (B.20),

$$Z[\psi_1(0,\cdot)](t) - Z[\psi_2(0,\cdot)](t) = \begin{bmatrix} I(x,t) \\ \partial_t I(x,t) \end{bmatrix}, \tag{B.24}$$

where

$$I(x,t) := \int_0^t G(x,t-s)\left(\widetilde{F}(\psi_1(0,s)) - \widetilde{F}(\psi_2(0,s))\right) ds.$$

First we derive the L^2 estimate for $I(x,t)$:

$$\begin{aligned} \|I(\cdot,t)\|_{L^2} &\le \text{const} \left\| \int_0^t \Theta(t-s-|x|) |\widetilde{F}(\psi_1(0,s)) - \widetilde{F}(\psi_2(0,s))|\, ds \right\|_{L^2} \\ &\le \text{const} \sup_{z\in\mathbb{C}} |\nabla \widetilde{F}(z)| \left\| \int_0^t \Theta(t-s-|x|)\, ds \right\|_{L^2} \sup_{s\in[0,t]} |\psi_1(0,s) - \psi_2(0,s)| \\ &\le \text{const}\, t^{3/2} \sup_{s\in[0,t]} |\psi_1(0,s) - \psi_2(0,s)|, \end{aligned} \tag{B.25}$$

where we took into account that $|\nabla \widetilde{F}(z)|$ is bounded due to the choice of $\widetilde{U}$.

Similarly, we derive the L^2 estimate for the derivative $\partial_t I(x,t)$. We first analyze

$$\partial_t G(x,t-s) = \frac{1}{2}\Theta(t-s-|x|)\partial_t J_0(m\sqrt{(t-s)^2-x^2}) + \frac{1}{2}\,\delta(t-s-|x|).$$

By Lemma B.5, we have $|\partial_t J_0(m\sqrt{(t-s)^2-x^2})| \le 2C_{0,1}$ for $|x| \le |t-s| \le 1$. We conclude that

$$\begin{aligned}
&\|\partial_t I(\cdot,t)\|_{L^2} \\
&\quad\le \left\| \int_0^t \left[2C_{0,1}\Theta(t-s-|x|) + \frac{\delta(t-s-|x|)}{2} \right] ds \right\|_{L^2} \sup_{s\in[0,t]} \left| \widetilde{F}(\psi_1(0,s)) - \widetilde{F}(\psi_2(0,s)) \right| \\
&\qquad + \|G(\cdot,0+)\|_{L^2} \left| \widetilde{F}(\psi_1(0,t)) - \widetilde{F}(\psi_2(0,t)) \right| \\
&\quad\le \mathrm{const}\, \|\Theta(t-|x|)\|_{L^2} \sup_{s\in[0,t]} |\psi_1(0,s)-\psi_2(0,s)| \\
&\quad\le \mathrm{const}\, t^{1/2} \sup_{s\in[0,t]} |\psi_1(0,s)-\psi_2(0,s)|. \qquad \text{(B.26)}
\end{aligned}$$

Note that, by (B.19), $\|G(\cdot,0+)\|_{L^2} = 0$.

The norm $\|\partial_x I(\cdot,t)\|_{L^2}$ is estimated similarly, with the same result:

$$\|\partial_x I(\cdot,t)\|_{L^2} \le \mathrm{const}\, t^{1/2} \sup_{s\in[0,t]} |\psi_1(0,s)-\psi_2(0,s)|.$$

Using the estimates $\|I(\cdot,t)\|_{L^2}$, $\|\partial_x I(\cdot,t)\|_{L^2}$, and $\|\partial_t I(\cdot,t)\|_{L^2}$ in (B.24), one arrives at

$$\|Z[\psi_1(0,\cdot)](t) - Z[\psi_2(0,\cdot)](t)\|_{\mathscr{X}} \le \mathrm{const}\, t^{1/2} \sup_{s\in[0,t]} |\psi_1(0,s)-\psi_2(0,s)|, \quad 0 \le t \le 1. \qquad \text{(B.27)}$$

Finally, to get (B.22), one uses the estimate

$$|\psi_1(0,s)-\psi_2(0,s)| \le \mathrm{const}\|\psi_1(\cdot,s)-\psi_2(\cdot,s)\|_{H^1} \le \mathrm{const}\|\Psi_1(\cdot,s)-\Psi_2(\cdot,s)\|_{\mathscr{X}}. \qquad \text{(B.28)}$$

Let us obtain the refinement (B.23).

Lemma B.7. *Let $0 \le \varepsilon < 1/2$. For any $X > 0$, the projection*

$$\Pi:\ (\psi,\pi) \mapsto \psi$$

is continuous as a map from $\mathscr{Y}^{-\varepsilon}$ to $C_b([-X,X])$. That is, there exists $C < \infty$ (which depends on ε and X) such that for any $f \in H^{1-\varepsilon}(\mathbb{R})$,

$$|f(x)| \le C \sum_{R\in\mathbb{N}} 2^{-R}\|f\|_{H_R^{1-\varepsilon}}, \qquad |x| \le X.$$

Proof. Recall that the norms in $L^2_R(\mathbb{B}^n_R)$ and in $H^s_R(\mathbb{B}^n_R)$ are defined by duality with $L^2_0(\mathbb{B}^n_R)$ and $H_0^{-s}(\mathbb{B}^n_R)$, where $\mathbb{B}^n_R \subset \mathbb{R}^n$ is the ball of radius R.

Denote

$$s = 1-\varepsilon.$$

Pick $R_0 \in \mathbb{N}$ such that $R_0 > X$. Fix $\rho \in C_0^\infty(-R_0,R_0)$, with $\rho(x) = 1$ for $|x| \le X$. By the Sobolev embedding, there is $C > 0$ so that

$$|f(x)| = |\rho(x)f(x)| \le C\|\rho f\|_{H^s} = C\|\rho f\|_{H^s_{R_0}}, \qquad |x| \le X. \qquad \text{(B.29)}$$

Essentially, the desired bound follows from $H^s(\mathbb{R})$ being an algebra for $s>1/2$; anyways, let us give a detailed proof. Due to the definition of $\|\cdot\|_{H^s_R}$ by duality, we have:

$$\|\rho f\|_{H^s_{R_0}} = \sup_{\substack{\alpha\in C_0^\infty(-R_0,R_0)\\ \|\alpha\|_{H^{-s}}=1}} \langle\alpha,\rho f\rangle \le \sup_{\substack{\alpha\in C_0^\infty(-R_0,R_0)\\ \|\alpha\|_{H^{-s}}=1}} \|\alpha\rho\|_{H^{-s}}\|f\|_{H^s_{R_0}}. \tag{B.30}$$

We claim that $\alpha\mapsto\alpha\rho$ is continuous in $H^{-s}(\mathbb{R})$; it then would follow that $\|\rho f\|_{H^s_{R_0}}\le \mathrm{const}\|f\|_{H^s_{R_0}}$. Alternatively, we may prove that the map $\hat\alpha\mapsto\hat\rho*\hat\alpha$ is continuous in $\widehat{H^{-s}(\mathbb{R})}$, which is isometric to $L^2(\mathbb{R},(1+|\xi|)^{-2s}\,d\xi)$. Thus, we need to prove that the integral operator with the Schwartz kernel $K(\xi,\eta)=(1+|\xi|)^{-s}\hat\rho(\xi-\eta)(1+|\eta|)^s$ is continuous in $L^2(\mathbb{R})$. This continuity, in turn, follows from the bound $|\hat\rho(\xi)|\le C_N/(1+|\xi|)^N$ for some $C_N<\infty$, applying Peetre's inequality, which gives

$$|K(\xi,\eta)|\le\left(\frac{1+|\eta|}{1+|\xi|}\right)^s\frac{C_N}{(1+|\xi-\eta|)^N}\le\frac{C_N}{(1+|\xi-\eta|)^{N-s}},$$

and then using the Schur test, which applies since both

$$\sup_\xi\int_{\mathbb{R}}|K(\xi,\eta)|\,d\eta,\qquad \sup_\eta\int_{\mathbb{R}}|K(\xi,\eta)|\,d\xi$$

are finite when N is sufficiently large. Combining (B.29), (B.30), and the continuity of $\alpha\to\alpha\rho$ in $H^{-s}(\mathbb{R})$, we conclude that there is $C'<\infty$ such that

$$|f(x)|\le C\|\rho f\|_{H^s_{R_0}}\le C'\|f\|_{H^s_{R_0}}\le C'2^{R_0}\sum_{R=1}^\infty 2^{-R}\|f\|_{H^s_R},\qquad |x|\le X.$$

We are done. □

Lemma B.7 and Definition 2.4 *(3)* of the norm $\|\cdot\|_{\mathscr{Y}^{-\varepsilon}}$ show that for each $0\le\varepsilon<1/2$ there exists $C<\infty$ such that

$$|\psi_1(0,s)-\psi_2(0,s)|\le C\|\Psi_1(\cdot,s)-\Psi_2(\cdot,s)\|_{\mathscr{Y}^{-\varepsilon}}, \tag{B.31}$$

for all $\Psi_1=(\psi_1,\dot\psi_1)\in C(\mathbb{R},\mathscr{X})$, $\Psi_2=(\psi_2,\dot\psi_2)\in C(\mathbb{R},\mathscr{X})$. Using this bound in (B.27), we arrive at (B.23). □

For $E\ge0$, define $\mathscr{X}_E\subset\mathscr{X}$ as the set of states of energy not larger than E (see (B.4)).

Corollary B.8. *(1) For any $E>0$ there exists $\tau=\tau(E)>0$ such that for any $\Psi_0\in\mathscr{X}_E$ there is a unique solution $\Psi(x,t)\in C([0,\tau],\mathscr{X})$ to the Cauchy problem* (B.11) *with the initial condition $\Psi(0)=\Psi_0$.*

(2) The map $W(t):\ \Psi_0\mapsto\Psi(t)$, $t\in[0,\tau]$ is a continuous map from $\mathscr{X}_E$ to $\mathscr{X}$.

B.1.2 Smoothness of the solution

In this section, we will study the smoothness of the solution

$$\Psi(x,t)=(\psi(x,t),\pi(x,t))\in C([0,\tau],\mathscr{X})$$

constructed in Corollary B.8 *(1)* assuming that $\psi_0(x),\pi_0(x)\in C_0^\infty(\mathbb{R})$. According to the integral representation (B.20), $\psi(x,t)$, $t\in[0,\tau]$, can be represented as

$$\psi(x,t)=\int_{\mathbb{R}}(\partial_tG(x-y,t)\psi_0(y)+G(x-y,t)\pi_0(y))\,dy+\int_0^t G(x,t-s)\widetilde F(\psi(0,s))\,ds. \tag{B.32}$$

First, let us prove the smoothness of the function $\psi(0,t)$.

Lemma B.9. $\psi(0,t) \in C^\infty([0,\tau])$.

Proof. The integral representation (B.32) implies that, for $t \in [0,\tau]$,

$$\psi(0,t) = \int\limits_{\mathbb{R}} (\partial_t G(y,t)\psi_0(y) + G(y,t)\pi_0(y))\,dy + \frac{1}{2}\int\limits_0^t J_0(m(t-s))\widetilde{F}(\psi(0,s))\,ds. \qquad \text{(B.33)}$$

The first integral is a smooth function. Further, from $\|\psi(\cdot,t)\|_{H^1} \le C < \infty$, $t \in [0,\tau]$, we conclude that $|\psi(0,t)|$ is bounded. Hence, (B.33) implies that $\psi(0,\cdot) \in C([0,\tau])$, and then by induction that $\psi(0,\cdot) \in C^\infty([0,\tau])$ since the Bessel function is smooth. □

Now, from (B.32), we conclude that $\psi(x,t)$ is smooth away from the singularities of $G(x,t)$.

Proposition B.10. *The solution $\psi(x,t)$ is piecewise smooth inside each of the four regions of $[0,\tau]\times\mathbb{R}$ cut off by the lines $x = 0$ and $x = \pm t$.*

Proof. The first integral in the right-hand side of (B.32) is infinitely smooth in x and t for all $x \in \mathbb{R}$, $t \ge 0$. Now let us consider the second integral in the right-hand side of (B.32), which could be written as follows:

$$\frac{\Theta(t-|x|)}{2}\int_0^{t-|x|} J_0(m\sqrt{(t-s)^2-x^2})\widetilde{F}(\psi(0,s))\,ds. \qquad \text{(B.34)}$$

Here the function $\widetilde{F}(\psi(0,s))$ is smooth in $s \in [0,\tau]$ by Lemma B.9. By Lemma B.5, all the partial derivatives of $J_0(m\sqrt{(t-s)^2-x^2})$ in x and t are continuous and uniformly bounded for $|x| < t-s$, $t \le \tau$. Therefore, (B.34) is smooth, with all the derivatives uniformly bounded, in each of the regions $0 \le x \le t$, $-t \le x \le 0$. In the regions $|x| > t$, (B.34) is identically equal to zero. □

Lemma B.11. *For $0 < t \le \tau$,*

$$\lim_{x\to 0-} \dot\psi(x,t) = \lim_{x\to 0+} \dot\psi(x,t). \qquad \text{(B.35)}$$

Proof. We have to analyze only the contribution from the second term in the right-hand side of (B.32), that is,

$$\partial_t \int\limits_0^t G(x,t-s)\widetilde{F}(\psi(0,s))\,ds = G(x,0+)\widetilde{F}(\psi(0,t)) + \int\limits_0^t \partial_t G(x,t-s)\widetilde{F}(\psi(0,s))\,ds.$$

The first term in the right-hand side is equal to zero for $x \ne 0$. The second term is continuous since the Green function $G(x,t-s)$ is smooth at $x = 0$ for $t-s > 0$. □

Lemma B.12. *For $0 < t \le \tau$,*

(1) $\dot\psi(x,t) + \psi'(x,t)$ is continuous across the characteristic $x = t$.

(2) $\dot\psi(x,t) - \psi'(x,t)$ is continuous across the characteristic $x = -t$.

Proof. The proofs for both statements of the Lemma are identical; we will only prove the first statement with $x > 0$. We have to study only the contribution from the second term in the right-hand side of (B.32), i.e.

$$(\partial_t+\partial_x)\int\limits_0^t G(x,t-s)\widetilde{F}(\psi(0,s))\,ds = \int\limits_0^t (\partial_t+\partial_x)G(x,t-s)\widetilde{F}(\psi(0,s))\,ds. \qquad \text{(B.36)}$$

Here we took into account that, as above, $G(x,0+)\widetilde{F}(\psi(0,t)) = 0$ for $x \ne 0$. Next key observation is that, for $x > 0$, the derivative $\partial_t + \partial_x$ applied to $G(x,t)$, does not produce a delta-function:

$$(\partial_t+\partial_x)G(x,t) = \frac{1}{2}\left\{\Theta(t-x)(\partial_t+\partial_x)J_0(m\sqrt{t^2-x^2})\right\}.$$

Hence, the integral (B.36) is continuous in x and t across the line $x = t$, $0 < t \le \tau$ by Lemma B.5. □

B.1.3 Energy conservation and global well-posedness

Lemma B.13. *For the solution to the Cauchy problem* (B.11) *with the initial data* $\Psi_0 \in \mathscr{X}$, *the energy is conserved:* $\widetilde{\mathcal{H}}(\Psi(t)) = \text{const}$, $t \in [0, \tau]$.

Proof. We follow [Kom95]. First, we prove that the energy is conserved for the smooth initial data with compact support: $\Psi_0 = \begin{bmatrix} \psi_0 \\ \pi_0 \end{bmatrix}$, with $\psi_0, \pi_0 \in C_0^\infty(\mathbb{R})$. Consider the norm $\|\cdot\|_{\mathscr{X}}$ introduced in Definition 2.4 *(1)*,

$$\|\Psi(t)\|^2_{\mathscr{X}} = \int_{-\infty}^{\infty} [|\dot\psi|^2 + |\psi'|^2 + m^2|\psi|^2]\,dx, \qquad t \in [0, \tau]. \tag{B.37}$$

We split this integral into four pieces: The integration over $(-\infty, -t)$, $(-t, 0)$, $(0, t)$, and (t, ∞). By Proposition B.10, on the support of each of these integrals $\psi(x, t)$ for $t \in [0, \tau]$ is a smooth function of x and t. Then, differentiating, we may express $\partial_t \|\Psi(t)\|^2_{\mathscr{X}}$ as

$$\partial_t \|\Psi(t)\|^2_{\mathscr{X}} = \Big[|\dot\psi|^2 + |\psi'|^2 + m^2|\psi|^2\Big]_{x=-t-0}^{x=-t+0} \tag{B.38}$$
$$-\Big[|\dot\psi|^2 + |\psi'|^2 + m^2|\psi|^2\Big]_{x=t-0}^{x=t+0} + 2\int_{-\infty}^{\infty} [\dot\psi\ddot\psi + \psi'\dot\psi' + m^2\psi\dot\psi]\,dx, \quad t \in [0, \tau].$$

The terms $m^2|\psi|^2$ could be discarded due to continuity of ψ across the characteristics $x = \pm t$. Integrating by parts the terms $\psi'\dot\psi'$ and using the cancellations of the integrals due to equation (B.10) away from $x = 0$, we get:

$$\begin{aligned}\partial_t \|\Psi(t)\|^2_{\mathscr{X}} &= \Big[|\dot\psi|^2 + |\psi'|^2 - 2\psi'\dot\psi\Big]_{x=-t-0}^{x=-t+0} \\ &\quad -\Big[|\dot\psi|^2 + |\psi'|^2 + 2\psi'\dot\psi\Big]_{x=t-0}^{x=t+0} - 2\Big[\psi'\dot\psi\Big]_{x=0-}^{x=0+} \\ &= \Big[(\dot\psi - \psi')^2\Big]_{x=-t-0}^{x=-t+0} - \Big[(\dot\psi + \psi')^2\Big]_{x=t-0}^{x=t+0} - 2\Big[\psi'\dot\psi\Big]_{x=0-}^{x=0+}.\end{aligned} \tag{B.39}$$

According to Lemma B.12, the first two terms in (B.39) do not give any contribution. Let us compute the contribution of the last term. According to Lemma B.11, $\dot\psi(0\pm, t) = \dot\psi(0, t)$ for $t \in [0, \tau]$, therefore

$$\Big[\psi'\dot\psi\Big]_{x=0-}^{x=0+} = [\psi'(x, t)]_{x=0-}^{x=0+}\,\dot\psi(0, t) = -\widetilde{F}(\psi(0, t))\dot\psi(0, t) = \frac{d}{dt}\widetilde{U}(\psi(0, t)).$$

In the second equality, we computed the jump of ψ' using equation (B.10) and the piecewise smoothness of the solution. We conclude that

$$\frac{d}{dt}\left\{\frac{1}{2}\|\Psi(t)\|^2_{\mathscr{X}} + \widetilde{U}(\psi(0, t))\right\} = 0,$$

and hence the value of the functional $\widetilde{\mathcal{H}}$ defined in (B.13) is conserved.

Since we proved the energy conservation for the initial data that constitute a dense set in $\mathscr{X}$ and since the dynamical group is continuous in $\mathscr{X}$ by Corollary B.8 *(2)*, we conclude that the energy is conserved for arbitrary initial data from $\mathscr{X}$. □

Corollary B.14. *(1) The solution* Ψ *to the Cauchy problem* (B.11) *with the initial data* $\Psi|_{t=0} = \Psi_0 \in \mathscr{X}$ *exists globally:* $\Psi \in C_b(\mathbb{R}, \mathscr{X})$.

(2) The energy is conserved: $\widetilde{\mathcal{H}}(\Psi(t)) = \widetilde{\mathcal{H}}(\Psi_0)$, $\qquad t \geq 0$.

Proof. Corollary B.8 *(1)* yields a solution $\Psi \in L^\infty([0, \tau], \mathscr{X})$ with a positive $\tau = \tau(E)$. However, the value of $\mathcal{H}(\Psi(t))$ is conserved for $t \leq \tau$ by Lemma B.13. Corollary B.8 *(1)* allows us to extend Ψ to the interval $[\tau, 2\tau]$, and eventually to all $t \geq 0$. In the same way we extend the solution $\Psi(t)$ for all $t < 0$. □

B.1.4 Conclusion of the proof of global well-posedness

The trajectory $\Psi = \begin{bmatrix} \psi(x,t) \\ \pi(x,t) \end{bmatrix} \in C(\mathbb{R}, \mathscr{X})$ is a solution to (B.11). Corollary B.14 *(2)* together with Lemma B.4 *(1)* imply the energy and charge conservation. Then, by Lemma B.4 *(2)*, $\widetilde{U}(\psi(0,t)) = U(\psi(0,t))$, for all $t \in \mathbb{R}$. This tells us that $\psi(x,t)$ is a solution to (2.1). The a priori bound (B.3) follows from (B.15) and the conservation of $\mathcal{H}(\Psi(t))$.

This finishes the proof of Parts *(1)*, *(2)*, *(3)*, and *(4)* of Theorem B.1.

B.1.5 Continuous dependence on the initial data in $\mathscr{Y}^{-\varepsilon}$

Let us prove the continuous dependence on the initial data stated in Theorem B.1 *(5)*.

For any Ψ_1, $\Psi_2 \in C(\mathbb{R}, \mathscr{X})$, we have:

$$\|\Psi_1(t) - \Psi_2(t)\|_{\mathscr{Y}^{-\varepsilon}} \le \|W_0(t)(\Psi_1(0) - \Psi_2(0))\|_{\mathscr{Y}^{-\varepsilon}} + \|Z[\Psi_1](t) - Z[\Psi_2](t)\|_{\mathscr{Y}^{-\varepsilon}}.$$

The first term in the right-hand side is bounded by $C\|\Psi_1(0) - \Psi_2(0)\|_{\mathscr{Y}^{-\varepsilon}}$, due to the continuity of W_0 in $\mathscr{X}^{-\varepsilon}$ and the finite speed of propagation.

The second term is bounded due to the bound (B.23) from Lemma B.6, which yields

$$\|Z[\Psi_1](t) - Z[\Psi_2](t)\|_{\mathscr{Y}^{-\varepsilon}} \le C_\varepsilon t^{1/2} \sup_{s\in[0,t]} \|\Psi_1(s) - \Psi_2(s)\|_{\mathscr{Y}^{-\varepsilon}}, \qquad \text{(B.40)}$$

for $0 \le t \le 1$.

Remark B.15. At this point, we utilize the fact that Ψ_1, $\Psi_2 \in C(\mathbb{R}, \mathscr{X})$, since the bound (B.23) was derived under the assumption of finite energy initial data (so that F, F' could be assumed bounded).

As it follows from (B.40), there is $C > 1$ and $t_0 > 0$ such that for $0 \le t \le t_0$ one has

$$\|\Psi_1(t) - \Psi_2(t)\|_{\mathscr{Y}^{-\varepsilon}} \le C\|\Psi_1(0) - \Psi_2(0)\|_{\mathscr{Y}^{-\varepsilon}}.$$

Applying this bound recursively, we prove that

$$\|\Psi_1(t) - \Psi_2(t)\|_{\mathscr{Y}^{-\varepsilon}} \le C^{1+\frac{|t|}{t_0}} \|\Psi_1(0) - \Psi_2(0)\|_{\mathscr{Y}^{-\varepsilon}}, \qquad t \in \mathbb{R}.$$

This proves the continuity stated in Theorem B.1 *(5)*, finishing the proof of Theorem B.1.

B.2 Klein-Gordon with mean field interaction

B.2.1 Global well-posedness

Now we consider the global well-posedness for the Klein-Gordon field with the mean field interaction at N regions, which is described by equation (2.23):

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - m^2\psi(x,t) + \sum_{J=1}^{N} \rho_J(x) F_J(\langle \rho_J, \psi(\cdot,t)\rangle), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}. \qquad \text{(B.41)}$$

We assume that $\rho_J(x) = \rho(x - X_J)$, $\rho \in \mathscr{S}(\mathbb{R}^n)$.

Theorem B.16. *Let $\rho \in \mathscr{S}(\mathbb{R}^n)$, $n \ge 1$, and let $F_J(z) = -\nabla U_J(z)$, $1 \le J \le N$, where*

$$U_J(z) = u_J(|z|^2), \qquad u_J \in C^2(\mathbb{R}).$$

Assume that

$$\inf_{z\in\mathbb{C}} U_J(z) > -\infty, \qquad 1 \le J \le N. \qquad \text{(B.42)}$$

Then:

(1) For every $\Psi_0 = (\psi_0, \pi_0) \in \mathscr{X}$, the Cauchy problem

$$\begin{cases} \ddot{\psi}(x,t) = \Delta\psi - m^2\psi + \sum_{J=1}^N \rho_J(x) F_J(\langle \rho_J, \psi(\cdot,t)\rangle), & x \in \mathbb{R}^n, \\ (\psi, \dot{\psi})\big|_{t=0} = (\psi_0, \pi_0), \end{cases} \tag{B.43}$$

where $m > 0$, has a unique global solution $\psi(t)$, $t \in \mathbb{R}$, such that $(\psi, \dot{\psi}) \in C(\mathbb{R}, \mathscr{X})$.

(2) The map $W(t) : (\psi_0, \pi_0) \mapsto (\psi(t), \dot{\psi}(t))$ is continuous in $\mathscr{X}$ for each $t \in \mathbb{R}$.

(3) The values of the energy and charge functionals are conserved along the trajectory:

$$\mathcal{H}(\psi(t), \dot{\psi}(t)) = \text{const}, \qquad \mathcal{Q}(\psi(t), \dot{\psi}(t)) = \text{const}, \qquad t \in \mathbb{R}.$$

(4) The following a priori *bound holds:*

$$\|(\psi(t), \dot{\psi}(t))\|_{\mathscr{X}} \le C(\psi_0, \pi_0) < \infty, \qquad t \in \mathbb{R}. \tag{B.44}$$

(5) For any $\varepsilon \ge 0$, $E \in \mathbb{R}$, and $T > 0$, the map

$$W(t) : \mathscr{X}_E \to \mathscr{X}_E, \qquad W(t) : (\psi_0, \pi_0) \mapsto (\psi(t), \dot{\psi}(t)),$$

is continuous in the topology of $\mathscr{Y}^{-\varepsilon}$, uniformly in $t \in [-T, T]$. Above, $\mathscr{X}_E \subset \mathscr{X}$ is the subset of states of energy not larger than E (see (B.4)*).*

Proof. The proof of this theorem is much simpler than that of Theorem B.1, due to the function ρ in the right-hand side being of Schwartz type. We will only give a sketch.

The local existence stated in Theorem B.1 is obtained by standard arguments from the contraction mapping principle. To achieve this, we use the integral representation for the solutions to the Cauchy problem (B.43) for $t \ge 0$:

$$\Psi(t) = W_0(t)\Psi_0 + Z[\Psi](t),$$

where $\Psi = (\psi, \dot{\psi})$ and

$$Z[\Psi](t) := \sum_{J=1}^N \int_0^t W_0(t-s) \Big[\begin{smallmatrix} 0 \\ \rho_J\, F_J(\langle \rho_J, \psi(\cdot, s)\rangle) \end{smallmatrix} \Big]\, ds.$$

Above, $W_0(t)$ is the dynamical group for the linear Klein-Gordon equation which is a unitary operator in the space $\mathscr{X}^{-\varepsilon}$ for any $\varepsilon \ge 0$. For any $\varepsilon \ge 0$, there exists $C_\varepsilon < \infty$ such that there is a bound

$$\|Z[\Psi_1](t) - Z[\Psi_2](t)\|_{\mathscr{X}^{-\varepsilon}} \le C_\varepsilon |t| \sup_{s \in [0,t]} \|\Psi_1(s) - \Psi_2(s)\|_{\mathscr{X}^{-\varepsilon}}, \quad |t| \le 1, \tag{B.45}$$

which holds for any two functions Ψ_1, $\Psi_2 \in C(\mathbb{R}, \mathscr{X})$. This bound shows that $Z[\psi]$ is a contraction operator in $C_b([0,t], \mathscr{X}^{-\varepsilon})$, $\varepsilon \ge 0$, if $t > 0$ is sufficiently small.

The contraction mapping theorem based on the bound (B.45) on the nonlinear term allows us to prove the existence and uniqueness of a local solution in $\mathscr{X}$, as well as the continuity of the map $W(t)$ (continuity with respect to the initial data). The continuity of $W(t)$ in $\mathscr{Y}^{-\varepsilon}$ follows from its continuity in $\mathscr{X}^{-\varepsilon}$ and the finite speed of propagation.

Now let us discuss the a priori bound stated in Part *(4)* of the Theorem. Adding to $U_J(z)$, $1 \le J \le N$, constants if necessary (this does not change equation (2.23)), we can substitute the condition (B.42) by

$$\inf_{z \in \mathbb{C}} U_J(z) \ge 0. \tag{B.46}$$

The conservation of the values of the energy and charge functionals, $\mathcal{H}$ and $\mathcal{Q}$, is obtained by approximating the initial data in $\mathscr{X}$ with smooth initial data and using the continuity of $W(t)$ in $\mathscr{X}$. The a priori bound (B.44) follows from bounding $\|\Psi\|_{\mathscr{X}}$ in terms of the value of the Hamiltonian (2.26), with the aid of (B.46):

$$\|\Psi\|_{\mathscr{X}}^2 \le 2\,(\mathcal{H}(\Psi)), \qquad \Psi \in \mathscr{X}. \tag{B.47}$$

This bound allows us to extend the existence results for all times, proving the global well-posedness of (B.43) in the energy space.

Finally, the continuity of $W(t)$ in $\mathscr{X}^{-\varepsilon}$ and $\mathscr{Y}^{-\varepsilon}$, $\varepsilon \ge 0$ (Part *(5)* of Theorem B.16) follows from the contraction mapping theorem (based on the bound (B.45)) and the finite speed of propagation. □

Appendix C

Local energy decay

Proposition C.1 (Local energy decay). *For any $n \in \mathbb{N}$ and $m > 0$, if χ solves*

$$\ddot{\chi} = \Delta\chi - m^2\chi, \qquad x \in \mathbb{R}^n, \quad (\chi, \dot{\chi})|_{t=0} = (\psi_0, \pi_0) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n),$$

then, for any $\rho \in \mathscr{S}(\mathbb{R}^n)$,

$$\lim_{t\to\infty} (\|\rho(\cdot)\chi(\cdot,t)\|_{H^1} + \|\rho(\cdot)\dot{\chi}(\cdot,t))\|_{L^2}) = 0.$$

Proposition C.1 immediately yields the decay of the norm of $(\chi, \dot{\chi})$ in the space $\mathscr{Y}^{-\varepsilon}$ introduced in Definition 2.4 *(3)*:

Corollary C.2. *For any $\varepsilon \geq 0$,*

$$\lim_{t\to\infty} \|(\chi, \dot{\chi})|_t\|_{\mathscr{Y}^{-\varepsilon}} = 0.$$

Proof. For the Fourier transform of $\chi(x,t)$ in x, we have:

$$\hat{\chi}(\xi,t) = \hat{\psi}_0(\xi)\cos(\omega(\xi)t) + \hat{\pi}_0(\xi)\frac{\sin(\omega(\xi)t)}{\omega(\xi)},$$

where $\omega(\xi) = \sqrt{m^2 + \xi^2}$.

We will only prove that

$$\lim_{t\to\infty} \|\rho(\cdot)\chi(\cdot,t)\|_{H^1} = 0;$$

the limit $\lim_{t\to\infty} \|\rho(\cdot)\dot{\chi}(\cdot,t)\|_{L^2} = 0$ is computed similarly.

Pick $\epsilon > 0$. We split the initial data ψ_0 and π_0 into $\psi_0 = u_1 + u_2$, $\pi_0 = v_1 + v_2$, so that

$$\|u_1\|_{H^1} + \|v_1\|_{L^2} < \epsilon/2 \tag{C.1}$$

and

$$\hat{u}_2,\ \hat{v}_2 \in \mathscr{S}(\mathbb{R}^n), \quad \operatorname{supp}\hat{u}_2 \cup \operatorname{supp}\hat{v}_2 \subset \{\xi \in \mathbb{R}^n\colon |\xi| \geq \lambda\}, \tag{C.2}$$

for some $\lambda > 0$. Let χ_1 and χ_2 be the solutions to the linear Klein-Gordon equation with the initial data

$$(\chi_1, \dot{\chi}_1)|_{t=0} = (u_1, v_1), \qquad (\chi_2, \dot{\chi}_2)|_{t=0} = (u_2, v_2).$$

Due to (C.1) and the energy conservation, $\|\chi_1(t)\|_{H^1} \leq \epsilon/2$ for $t \in \mathbb{R}$. It suffices to show that

$$\lim_{t\to\infty} \|\rho(\cdot)\chi_2(\cdot,t)\|_{H^1} = 0. \tag{C.3}$$

We have:

$$\|\rho\chi_2(\cdot,t)\|_{L^2}^2 \leq \|\rho\|_{L^2}\|\chi_2(\cdot,t)\|_{L^2}\|\rho\chi_2(\cdot,t)\|_{L^\infty}. \tag{C.4}$$

The first two factors in the right-hand side of (C.4) are bounded uniformly in time. For the last factor in the right-hand side of (C.4), we have:

$$\|\rho(\cdot)\chi_2(\cdot,t)\|_{L^\infty} \le \left\|\hat{\rho} * \left(\hat{u}_2(\cdot)\cos(\omega(\cdot)t) + \hat{v}_2(\cdot)\frac{\sin(\omega(\cdot)t)}{\omega(\cdot)}\right)\right\|_{L^1}. \tag{C.5}$$

Lemma C.3. *Let $f, g \in \mathscr{S}(\mathbb{R}^n)$, and $0 \notin \operatorname{supp} g$. Then, for any $N \in \mathbb{N}$, there is $C_N > 0$ so that*

$$\|f * \big(g(\cdot)e^{i\omega(\cdot)t}\big)\|_{L^1} \le C_N(1+|t|)^{-N}, \qquad t \in \mathbb{R}.$$

Proof. Since $0 \notin \operatorname{supp} g$, $|\nabla_\eta \omega(\eta)|$ is bounded away from zero on the support of g. Therefore, the expression

$$\|f * \big(g(\cdot)e^{i\omega(\cdot)t}\big)\|_{L^1} = \int \left| \int f(\xi-\eta)g(\eta)e^{i\omega(\eta)t}\,d\eta \right| d\xi \tag{C.6}$$

decays faster than any negative power of t due to the stationary phase method. Namely, one can place the operator $L = \frac{1}{i|\nabla\omega(\eta)|^2 t}\nabla_\eta\omega \cdot \nabla_\eta$ in front of the exponential factor $e^{i\omega(\eta)t}$ under the inner integral in the right-hand side of (C.6), and then integrate by parts in η. This gives a factor of t^{-1}. The procedure could be repeated arbitrarily many times. □

Lemma C.3, applied to the right-hand side of (C.5), shows that $\lim\limits_{t\to\infty}\|\rho\chi_2(\cdot,t)\|_{L^\infty} = 0$. This, together with (C.4), yields

$$\lim_{t\to\infty}\|\rho\chi_2\|_{L^2}^2 = 0. \tag{C.7}$$

Similarly, one proves that

$$\lim_{t\to\infty}\|\nabla_x(\rho\chi_2(\cdot,t))\|_{L^2}^2 = 0. \tag{C.8}$$

Each of the terms in the right-hand side of (C.4) could accommodate a derivative in x: $\|\nabla\rho\|_{L^2}$ is bounded, $\|\nabla\chi(\cdot,t)\|_{L^2}$ is bounded uniformly in time, while $\|\nabla(\rho\chi_2(\cdot,t))\|_{L^\infty}$ is bounded by the expression similar to the right-hand side of (C.6), which is dealt with by Lemma C.3.

Using (C.7) and (C.8), we obtain:

$$\lim_{t\to\infty}\|\rho(\cdot)\chi_2(\cdot,t)\|_{H^1} = 0.$$

As we mentioned before, the convergence $\lim\limits_{t\to\infty}\|\rho(\cdot)\dot{\chi}_2(\cdot,t)\|_{L^2} = 0$ is proved similarly.

This finishes the proof. □

Appendix D

Quasimeasures and multiplicators

Here we give the details on quasimeasures from [Kom03] and [KK07a].

D.1 Quasimeasures

Let us denote by $\check{g}$ the inverse Fourier transform of a tempered distribution g:

$$\check{g}(t) = \mathcal{F}^{-1}_{\omega\to t}[g(\omega)].$$

Definition D.1. A tempered distribution $\mu(\omega)$ is a *quasimeasure* if $\check{\mu} \in C_b(\mathbb{R})$.

For example, any function from $L^1(\mathbb{R})$ is a quasimeasure, and so is any finite Borel measure on $\mathbb{R}$.

Lemma D.2. *Let $\mu(\omega)$ be a quasimeasure and $\varphi(\omega)$ be a test function from the Schwartz space $\mathscr{S}(\mathbb{R})$. Then*

$$|\langle\mu(\omega), \varphi(\omega)\rangle| \le C\|\check{\varphi}(t)\|_{L^1(\mathbb{R})}. \tag{D.1}$$

The lemma is a trivial consequence of the Parseval identity:

$$|\langle\mu(\omega), \varphi(\omega)\rangle| = 2\pi|\langle\check{\mu}(t), \check{\varphi}(t)\rangle| \le 2\pi\|\check{\mu}(t)\|_{L^\infty(\mathbb{R})}\|\check{\varphi}(t)\|_{L^1(\mathbb{R})}. \tag{D.2}$$

Definition D.3. $C_{b,F}(\mathbb{R})$ is the vector space of bounded functions $f(t) \in C_b(\mathbb{R})$ endowed with the following convergence: $f_\epsilon(t) \xrightarrow{C_{b,F}} f(t)$, $\epsilon \to 0+$ if and only if

(1) $\forall T > 0, \quad \|f_\epsilon(t) - f(t)\|_{C[-T,T]} \to 0$, $\epsilon \to 0+$;

(2) $\sup\limits_{\epsilon\in(0,1]} \|f_\epsilon(t)\|_{C_b(\mathbb{R})} < \infty$.

This type of convergence coincides with the convergence stated in the Ascoli-Arzelà theorem. Next we introduce the dual class of the "Ascoli-Arzelà quasimeasures".

Definition D.4. $\mathscr{Q}(\mathbb{R})$ is the linear space of all quasimeasures $\mu(\omega)$ endowed with the following convergence:

$$\mu_\epsilon(\omega) \xrightarrow[\epsilon\to 0+]{\mathscr{Q}} \mu(\omega) \quad \text{if and only if} \quad \check{\mu}_\epsilon(t) \xrightarrow[\epsilon\to 0+]{C_{b,F}} \check{\mu}(t).$$

D.2 Multiplicators

Now let us give a simple characterization of multiplicators in $\mathscr{Q}(\mathbb{R})$. Let us consider a continuous function $M(\omega) \in C(\mathbb{R})$. We also denote by M the corresponding operator of multiplication:

$$M: \ \mu(\omega) \mapsto M(\omega)\mu(\omega), \qquad \mu(\omega) \in C_0^\infty(\mathbb{R}).$$

Lemma D.5. *(1) Let $\check{M}(t) \in L^1(\mathbb{R})$. Then the operator M extends to a linear continuous operator in the space of quasimeasures:*

$$M: \ \mathscr{Q}(\mathbb{R}) \to \mathscr{Q}(\mathbb{R}).$$

(2) Let $\mu_\epsilon(\omega) \xrightarrow{\mathscr{Q}} \mu(\omega)$ and $\check{M}_\epsilon(t) \xrightarrow{L^1} \check{M}(t)$ as $\epsilon \to 0+$. Then

$$M_\epsilon(\omega)\mu_\epsilon(\omega) \xrightarrow{\mathscr{Q}} M(\omega)\mu(\omega), \qquad \epsilon \to 0+. \tag{D.3}$$

Proof. First we define $M(\omega)\mu(\omega) := \mathcal{F}_{t\to\omega}[\big(\check{M} * \check{\mu}\big)(t)](\omega)$ that agrees with the case $\mu \in C_0^\infty(\mathbb{R})$. Then (i) follows from (ii) with $M_\epsilon = M$ and $\mu_\epsilon \in C_0^\infty(\mathbb{R})$. To prove (ii), we need to show that

$$\mathcal{F}^{-1}_{\omega\to t}[M_\epsilon(\omega)\mu_\epsilon(\omega)] = \big(\check{M}_\epsilon * \check{\mu}_\epsilon\big)(t) \xrightarrow{C_{b,F}} \big(\check{M} * \check{\mu}\big)(t). \tag{D.4}$$

We have to check both conditions (i) and (ii) of Definition D.3 for the functions

$$f_\epsilon(t) := \mathcal{F}^{-1}_{\omega\to t}[M_\epsilon(\omega)\mu_\epsilon(\omega)] = \big(\check{M}_\epsilon * \check{\mu}_\epsilon\big)(t),$$
$$f(t) := \mathcal{F}^{-1}_{\omega\to t}[M(\omega)\mu(\omega)] = \big(\check{M} * \check{\mu}\big)(t).$$

We have:

$$f_\epsilon(t) - f(t) = \big(\check{M}_\epsilon * \check{\mu}_\epsilon\big)(t) - \big(\check{M} * \check{\mu}\big)(t) = \big((\check{M}_\epsilon - \check{M}) * \check{\mu}_\epsilon\big)(t) + \big(\check{M} * (\check{\mu}_\epsilon - \check{\mu})\big)(t).$$

The first term in the right-hand side converges to zero uniformly in $t \in \mathbb{R}$ since $\check{M}_\epsilon - \check{M} \to 0$ in L^1 while $\check{\mu}_\epsilon \in C_b(\mathbb{R})$ and is bounded uniformly for $\epsilon \in (0,1)$. Let us analyze the second term,

$$\int_{\mathbb{R}} \check{M}(\tau)(\check{\mu}_\epsilon(t-\tau) - \check{\mu}(t-\tau))\, d\tau. \tag{D.5}$$

Since $\check{M} \in L^1$, for any $\delta > 0$ there exists a finite $R > 0$ so that $\int_{|\tau|>R} |\check{M}(\tau)|\, d\tau \le \delta$. On the other hand, for any $T > 0$, the difference $\check{\mu}_\epsilon(t-\tau) - \check{\mu}(t-\tau)$ is uniformly small for $|t| \le T$, $|\tau| < R$ and small ϵ. Therefore, the integral (D.5) converges to zero uniformly in $|t| \le T$ as $\epsilon \to 0+$. Hence, the convergence (i) of Definition D.3 follows.

Finally, the uniform bound (ii) of Definition D.3 for the functions $f_\epsilon(t)$ is obvious. The convergence (D.4) is proved. □

Bounds for multiplicators

Let us justify the properties of the multiplicators which we used in Section 5.1. Recall that we use the notation

$$M_{x,\epsilon}(\omega) := e^{ik(\omega+i\epsilon)|x|}\zeta(\omega), \qquad x \in \mathbb{R}, \quad \epsilon \ge 0,$$

where $\zeta(\omega) \in C_0^\infty(\mathbb{R})$ is a fixed cutoff function, and also the notation

$$N_x(\omega) := (ik(\omega)\operatorname{sgn} x)^j e^{ik(\omega)|x|}(-i\omega)^k \zeta(\omega), \qquad x \in \mathbb{R},$$

where j, k are fixed nonnegative integers.

Lemma D.6. *For any fixed $x \in \mathbb{R}$ we have:*

(1) $\check{M}_{x,\epsilon}(t) \in L^1(\mathbb{R})$ *for any* $\epsilon \ge 0$.

(2) $\check{M}_{x,\epsilon}(t) \xrightarrow{L^1} \check{M}_{x,0}(t), \quad \epsilon \to 0.$

(3) $\check{N}_x \in L^1(\mathbb{R})$, *and for any* $R > 0$ *there exists* $C_{j,k,R} > 0$ *so that*

$$\sup_{|x|\le R} \|\check{N}_x\|_{L^1(\mathbb{R})} \le C_{j,k,R}. \tag{D.6}$$

Proof. For any fixed $x \in \mathbb{R}$, the Puiseux expansion holds:

$$e^{ik(\omega+i\epsilon)|x|} \sim 1 + \sum_{\pm}\sum_{j=1}^{\infty} C_j^{\pm}(x)(\omega + i\epsilon \mp m)^{j/2}, \qquad \omega + i\epsilon \to \pm m, \qquad \epsilon > 0. \tag{D.7}$$

Therefore, the function $\check{M}_{x,\epsilon}(t)$ is smooth and decays at least like $|t|^{-3/2}$ when $t \to \infty$. This finishes the proof of the first statement of the lemma.

The second statement of the lemma follows from (D.7).

The last statement of the lemma follows by the same arguments from the Puiseux expansion for $\check{N}_x(\omega)$ similar to expansion (D.7) with $\epsilon = 0$. □

D.3 Examples of quasimeasures

Let us consider the function

$$f(\omega) = \frac{1}{\omega} e^{\frac{i}{\omega}}. \tag{D.8}$$

Lemma D.7. *(1)* $f(\omega)\,d\omega$ *is not a finite measure.*

(2) $|F(t)| \le \mathrm{const}(1 + |t|^{-1/4})$, *hence* $f \in \mathscr{Q}(\mathbb{R})$.

Proof. Consider the intervals

$$I_n = \Big[\frac{1}{2n\pi + 1}, \frac{1}{2n\pi - 1}\Big], \qquad n \in \mathbb{N}. \tag{D.9}$$

Let $\chi_N(\omega) = \sum_{n=1}^{N} \chi_{I_n}(\omega)$. Then

$$\mathrm{Re}\int_{\mathbb{R}} f(\omega)\chi_N(\omega)\,d\omega = \sum_{n=1}^{N}\int_{(2n\pi+1)^{-1}}^{(2n\pi-1)^{-1}} \cos\frac{1}{\omega}\,\frac{d\omega}{\omega} = \sum_{n=1}^{N}\int_{2n\pi-1}^{2n\pi+1} \cos s\,\frac{ds}{s} \ge \sum_{n=1}^{N} \frac{2\cos 1}{2n\pi + 1}$$

could be arbitrarily large when N goes to infinity. This shows that $f(\omega)$ is not a finite measure.

Assume that $t > 0$, and consider

$$F(t) = \int_{\mathbb{R}} e^{i\omega t + \frac{i}{\omega}}\frac{d\omega}{\omega} = 2i\int_0^{\infty} \sin(\omega t + \omega^{-1})\frac{d\omega}{\omega} = 4i\int_{t^{-\frac{1}{2}}}^{\infty} \sin(\omega t + \omega^{-1})\frac{d\omega}{\omega}. \tag{D.10}$$

For $\omega > t^{-1/2}$ the function $\phi(\omega) = \omega t + \omega^{-1}$ satisfies $\phi'(\omega) > 0$, $\phi''(\omega) > 0$. By Lemma D.11, the limit

$$\lim_{\Omega\to\infty}\int_{t^{-1/2}}^{\Omega} \sin(\omega t + \omega^{-1})\frac{d\omega}{\omega} \tag{D.11}$$

exists and is bounded by

$$\int_{t^{-1/2}}^{\omega'} \frac{d\omega}{\omega} \le \frac{\omega' - t^{-1/2}}{t^{-1/2}} = t^{1/2}\omega' - 1,$$

where $\omega' > t^{-1/2}$ is such that $\phi(\omega') = \phi(t^{-1/2}) + \pi$:

$$\omega' t + \frac{1}{\omega'} = 2t^{1/2} + \pi,$$

hence

$$\omega' = \frac{2t^{1/2} + \pi + \sqrt{(2t^{1/2}+\pi)^2 - 4t}}{2t} = t^{-1/2} + \frac{\pi + \sqrt{4\pi t^{1/2} + \pi^2}}{2t}.$$

We conclude that

$$\left| \lim_{\Omega\to\infty} \int_{t^{-1/2}}^{\Omega} \sin(\omega t + \omega^{-1}) \frac{d\omega}{\omega} \right| \le \frac{\sqrt{4t^{1/2}\pi + \pi^2}}{t^{1/2}}.$$

Let us show that (D.11) is also bounded for $0 < t \le 1$. We split the integration into

$$\int\limits_{t^{-1/2}}^{\infty} \sin(\omega t + \omega^{-1}) \frac{d\omega}{\omega} \le \int\limits_{t^{-1/2}}^{2\pi t^{-1}} \sin(\omega t + \omega^{-1}) \frac{d\omega}{\omega} + \int\limits_{2\pi t^{-1}}^{\infty} \sin(\omega t + \omega^{-1}) \frac{d\omega}{\omega}. \qquad \text{(D.12)}$$

As long as $t \le 1$, the absolute value of the first integral in (D.12) could be bounded by

$$\int\limits_{t^{-1/2}}^{2\pi t^{-1}} \Big(|\sin(\omega^{-1})| + \big|\sin(\omega t + \omega^{-1}) - \sin(\omega^{-1})\big| \Big) \frac{d\omega}{\omega} \le C \int\limits_{t^{-1/2}}^{2\pi t^{-1}} (\omega^{-1} + \omega t) \frac{d\omega}{\omega} \le \text{const}.$$

The absolute value of the second integral in (D.12) is bounded with the aid of Lemma D.11:

$$\int_{2\pi t^{-1}}^{\infty} \sin(\omega t + \omega^{-1}) \frac{d\omega}{\omega} \le \frac{\omega' - \omega}{2\pi t^{-1}},$$

where $\omega' > 2\pi t^{-1}$ satisfies $\phi(\omega') = \phi(2\pi t^{-1}) + \pi$, with $\phi(\omega) = \omega t + \omega^{-1}$. Estimating $\omega' - \omega$ by $\pi/\phi'(2\pi t^{-1})$, we get

$$\int_{2\pi t^{-1}}^{\infty} \sin(\omega t + \omega^{-1}) \frac{d\omega}{\omega} \le \frac{\pi}{2\pi t^{-1}(t - (\frac{t}{2\pi})^2)} = \frac{\pi}{2\pi - \frac{t}{2\pi}},$$

This proves the desired bound on (D.12).

The lemma is proved. □

Remark D.8. We have heard from Stas Molchanov, UNC–Charlotte, that a similar example must have been considered by O.S. Ivashov-Musatov.

Remark D.9. We have also heard from Stas Molchanov about the following related result proved by Wiener:

Let $\varphi(t) = \int_{\mathbb{R}} e^{i\omega t} \mu(d\omega)$. Assume that the measure μ is of finite variation and that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(t)|^2\, dt = C.$$

Then $\mu = \sum_j \alpha_j \delta(\omega - \omega_j) + \mu_{a.c.}$, with $C = \sum_j |\alpha_j|^2$.

Remark D.10. It is possible to have $f,\ g \in \mathscr{Q}(\mathbb{R})$ such that $\operatorname{supp} f \subset \mathbb{R}^+$, $\operatorname{supp} f * g \subset \mathbb{R}^+$, while $\operatorname{supp} g \not\subset \mathbb{R}^+$. Here is an example:

$$F(t) = \frac{t-i}{t+i}, \qquad G_0(t) = \frac{t+i}{t-i}, \qquad G(t) = G_0(t) + F(t) = \frac{t+i}{t-i} + \frac{t-i}{t+i}.$$

Then

$$f(\omega) = \delta(\omega) + \Theta(\omega)e^{-\omega}, \qquad g_0(\omega) = \delta(\omega) + \Theta(-\omega)e^{\omega},$$
$$g(\omega) = g_0(\omega) + f(\omega) = 2\delta(\omega) + e^{-|\omega|},$$
$$f * g = f * g_0 + f * f = \delta(\omega) + f * f,$$

with the support in $\mathbb{R}^+$. Recall that $\Theta(\omega)$ is the Heaviside step function.

D.4 Conditionally convergent oscillatory integrals

Let us make precise the elementary observation that the oscillatory integral is approximated by the integral over a half-wave.

Lemma D.11. *Consider*

$$I(\Lambda) = \int_t^\Lambda \sin(\phi(s))g(s)\,ds, \qquad \phi \in C^1(\mathbb{R}), \quad g \in C(\mathbb{R}).$$

Assume that

$$\lim_{s\to\infty} g(s) = 0, \tag{D.13}$$

and that

$$\phi'(s) > 0, \qquad \phi''(s) \geq 0; \qquad g(s) > 0, \qquad g'(s) \leq 0$$

for $s > t$. Then the limit $\lim\limits_{\Lambda\to\infty} I(\Lambda)$ exists, and moreover

$$\Big|\lim_{\Lambda\to\infty} I(\Lambda)\Big| \leq \int_t^{t'} g(s)\,ds,$$

where $t' > t$ is such that $\phi(t') = \phi(t) + \pi$.

Proof. Since ϕ is monotonically increasing for $s > t$, there exist t_j, $j \in \mathbb{N}$, so that $\phi(t_j) = \pi(n+j)$. Define

$$a_0 = \int_t^{t_1} \sin(\phi(s))g(s)\,ds, \qquad a_j = \int_{t_j}^{t_{j+1}} \sin(\phi(s))g(s)\,ds, \quad j \in \mathbb{N}.$$

Due to the assumptions on ϕ and g, $|a_1| \geq |a_2| \geq \dots,\ \lim\limits_{j\to\infty} a_j = 0$, hence the sign-alternating series $\sum_{j=0}^\infty a_j$ is conditionally convergent, proving that $\lim\limits_{\Lambda\to\infty} I(\Lambda)$ exists. Finally,

$$\left|\int_t^\infty \sin(\phi(s))g(s)\,ds\right| = \left|\sum_{j=0}^\infty a_j\right| \leq \max(|a_0|, |a_1|) \leq \int_t^{t'} g(s)\,ds,$$

where $t' > t$ is such that $\phi(t') = \phi(t) + \pi$. □

Appendix E

The Titchmarsh Convolution Theorem

E.1 Statement of the theorem

The Titchmarsh Convolution Theorem has been originally formulated as follows [Tit26]:

> *If $\phi(t)$ and $\psi(t)$ are integrable functions, such that $\int_0^x \phi(t)\psi(x-t)\,dt = 0$ almost everywhere in the interval $0 < x < \kappa$, then $\phi(t) = 0$ almost everywhere in $(0, \lambda)$, and $\psi(t) = 0$ almost everywhere in $(0, \mu)$, where $\lambda + \mu \geq \kappa$.*

The Titchmarsh Convolution Theorem could be restated as the equality

$$\sup \operatorname{supp} \phi * \psi = \sup \operatorname{supp} \phi + \sup \operatorname{supp} \psi, \tag{E.1}$$

which is satisfied if the quantity in its right-hand side is finite. Above, $\phi * \psi$ is the convolution $\phi * \psi(x) = \int_{\mathbb{R}} \phi(x-t)\psi(t)\,dt$. The equality similar to (E.1) takes place for $\inf \operatorname{supp} \phi * \psi$. These equalities imply that the obvious inclusion $\operatorname{supp} \phi * \psi \subseteq \operatorname{supp} \phi + \operatorname{supp} \psi$ is sharp at the boundary if both $\operatorname{supp} \phi$ and $\operatorname{supp} \psi$ are compact. The Titchmarsh Convolution Theorem was originally proved in [Tit26] for functions from L^1, but the statement is easily generalized for compactly supported distributions. The generalization of the Titchmarsh Theorem to higher dimensions can be stated in terms of the convex hulls of the supports [Lio51]:

Theorem E.1 (Titchmarsh Convolution Theorem)**.** *For $f, g \in \mathscr{E}'(\mathbb{R}^n)$,*

$$\text{c.h.} \operatorname{supp} f * g = \text{c.h.} \operatorname{supp} f + \text{c.h.} \operatorname{supp} g. \tag{E.2}$$

Above, $\mathscr{E}'(\mathbb{R})$ is the space of distributions with compact support (dual to the space $\mathscr{E}(\mathbb{R})$ which is $C^\infty(\mathbb{R})$ with the seminorms $\sup_\omega |f^{(k)}(\omega)|$). c.h. denotes the convex hull of the set.

Let us also note that we use the following conventions:

$$\text{For } X, Y \subseteq \mathbb{R}^n, \qquad X + Y = \{x + y, \quad x \in X,\ y \in Y\}; \tag{E.3}$$

$$\text{For } X \subseteq \mathbb{R}^n,\ k \in \mathbb{R}, \qquad kX = \{kx, \quad x \in X\}. \tag{E.4}$$

Different proofs of the Titchmarsh Convolution Theorem are contained in [Hör90, Theorem 4.3.3] (Harmonic Analysis style), [Yos80, Chapter VI] (Real Analysis style), and [Lev96, Lecture 16, Theorem 5] (Complex Analysis style).

Some related results are studied in [Hör63], [Dos67], [BD73, BD75], [Hör79].

E.2 Elementary proof via Paley-Wiener Theorem

We will give an elementary proof based on the Paley-Wiener Theorem. We will consider the one dimension only. Higher dimensional case is proved in the same way, with the higher dimensional version of the Paley-Wiener Theorem and utilizing the concept of the supporting function as in [Hör90].

Titchmarsh Convolution Theorem for $f * f$

Let us first show how to prove of the Titchmarsh Convolution Theorem for $f * f$ using the Paley-Wiener Theorem (see [Yos80, Chapter VI] or [Hör90, Theorem 7.3.1]) which relates the size of the support a distribution f with the growth properties of its Fourier transform,

$$F(\zeta) = \mathcal{F}_{\omega\to\zeta}[f](\zeta) = \int_{\mathbb{R}} e^{-i\zeta\omega} f(\omega)\, d\omega. \tag{E.5}$$

Namely, the Paley-Wiener Theorem states that $f \in \mathscr{D}(\mathbb{R})$, $\operatorname{supp} f \subseteq [-A, A]$, $A \geq 0$ *if and only if* $F(\zeta)$ is an entire function, and for any $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$|F(\zeta)| \leq C_N (1 + |\zeta|)^{-N} e^{A|\operatorname{Im}\zeta|}, \qquad \zeta \in \mathbb{C}. \tag{E.6}$$

The Paley-Wiener Theorem for distributions states that $f \in \mathscr{E}'(\mathbb{R})$, $\operatorname{supp} f \subseteq [-A, A]$ *if an only if* $F(\zeta)$ is an entire function and there exist $C > 0$ and $m \in \mathbb{R}$ so that $F(\zeta)$ satisfies

$$|F(\zeta)| \leq C(1 + |\zeta|)^m e^{A|\operatorname{Im}\zeta|}, \qquad \zeta \in \mathbb{C}. \tag{E.7}$$

Lemma E.2 (Titchmarsh Convolution Theorem for $f * f$)**.** *For any* $f \in \mathscr{E}'(\mathbb{R})$*,*

$$\inf \operatorname{supp}(f * f) = 2 \inf \operatorname{supp} f, \qquad \sup \operatorname{supp}(f * f) = 2 \sup \operatorname{supp} f.$$

We will show that Lemma E.2 is a consequence of the following Lemma.

Lemma E.3.

$$\max\Big(\sup \operatorname{supp}(f * f), -\inf \operatorname{supp}(f * f)\Big) = 2 \max(\sup \operatorname{supp} f, -\inf \operatorname{supp} f).$$

Proof. Denote

$$a = \max(\sup \operatorname{supp} f, -\inf \operatorname{supp} f). \tag{E.8}$$

Assume that

$$\operatorname{supp}(f * f) \subseteq [-2a + \epsilon, 2a - \epsilon] \qquad \text{for some } \epsilon > 0. \tag{E.9}$$

Then, by the Paley-Wiener Theorem, there are $m \geq 0$ and $C > 0$ such that

$$|F(\zeta)|^2 = |\mathcal{F}_{\omega\to\zeta}[f * f](\zeta)| \leq C(1 + |\zeta|)^m e^{(2a-\epsilon)|\operatorname{Im}\zeta|}, \qquad \zeta \in \mathbb{C}. \tag{E.10}$$

It follows that

$$|F(\zeta)| = |F(\zeta)^2|^{\frac{1}{2}} \leq C^{\frac{1}{2}} (1 + |\zeta|)^{\frac{m}{2}} e^{(a-\frac{\epsilon}{2})|\operatorname{Im}\zeta|}, \qquad \zeta \in \mathbb{C}. \tag{E.11}$$

By the Paley-Wiener Theorem, $\operatorname{supp} f \subseteq [-a + \frac{\epsilon}{2}, a - \frac{\epsilon}{2}]$, $\epsilon > 0$, contradicting the assumption of the lemma. Therefore, the inclusion (E.9) is impossible. We are done. □

Proof. [Proof of Lemma E.2] We can shift f so that $\inf \operatorname{supp} f > 0$ and apply Lemma E.3 to the shifted distribution. It follows that $\sup \operatorname{supp}(f * f) = 2 \sup \operatorname{supp} f$. Similarly for inf. □

Titchmarsh Convolution Theorem for $f * g$

Lemma E.4. *Let $f,\, g \in \mathscr{E}'(\mathbb{R})$. Then, for any polynomials α, β,*

$$\inf \operatorname{supp}(\alpha f) * (\beta g) \ge \inf \operatorname{supp} f * g, \qquad \sup \operatorname{supp}(\alpha f) * (\beta g) \le \sup \operatorname{supp} f * g.$$

Proof. It suffices to prove the second inequality, and only for the polynomials $\alpha(\omega) = \omega$, $\beta(\omega) = 1$. Denote

$$f_n(\omega) = \omega^n f(\omega), \qquad g_n(\omega) = \omega^n g(\omega), \qquad A_{mn} := \sup \operatorname{supp} f_m * g_n. \tag{E.12}$$

Let us assume that, contrary to the statement of the Lemma,

$$\sup \operatorname{supp} f_1 * g > \sup \operatorname{supp} f * g. \tag{E.13}$$

This inequality can be rewritten as

$$A_{10} - A_{00} > 0. \tag{E.14}$$

Due to the relation $\omega(f * g)(\omega) = (f_1 * g)(\omega) + (f * g_1)(\omega)$, we have:

$$\sup \operatorname{supp}(f_1 * g + f * g_1) = \sup \operatorname{supp} \omega(f * g)(\omega) \le \sup \operatorname{supp} f * g = A_{00}. \tag{E.15}$$

It follows that

$$\sup \operatorname{supp}(f_1 * g * f_1 * g + f_1 * g * f * g_1) \le \sup \operatorname{supp} f_1 * g + \sup \operatorname{supp}(f_1 * g + f * g_1) \le A_{10} + A_{00}.$$

If we had $\sup \operatorname{supp} f_1 * g * f_1 * g \ne \sup \operatorname{supp} f_1 * g * f * g_1$, then both these quantities would be smaller than or equal to $A_{10} + A_{00}$. By Lemma E.2 and (E.14), this would lead to $\sup \operatorname{supp} f_1 * g \le (A_{10} + A_{00})/2 < A_{10}$, contradicting (E.12). Thus, $\sup \operatorname{supp} f_1 * g * f_1 * g = \sup \operatorname{supp} f_1 * g * f * g_1$, leading to

$$\sup \operatorname{supp} f_1 * g * f_1 * g = \sup \operatorname{supp} f_1 * g * f * g_1 \le \sup \operatorname{supp} f * g + \sup \operatorname{supp} f_1 * g_1. \tag{E.16}$$

If we take into account that $\sup \operatorname{supp} f_1 * g * f_1 * g = 2 \sup \operatorname{supp} f_1 * g$ by Lemma E.2, then (E.16) could be rewritten as

$$2 \sup \operatorname{supp} f_1 * g \le \sup \operatorname{supp} f * g + \sup \operatorname{supp} f_1 * g_1. \tag{E.17}$$

This gives

$$A_{11} - A_{10} \ge A_{10} - A_{00} > 0. \tag{E.18}$$

In the last inequality, we took into account (E.14). The inequalities (E.18) imply that

$$\sup \operatorname{supp} f_1 * g_1 > \sup \operatorname{supp} f_1 * g. \tag{E.19}$$

Just as we derived (E.17) from (E.13), we could use (E.19) to derive

$$2 \sup \operatorname{supp} f_1 * g_1 \le \sup \operatorname{supp} f_1 * g + \sup \operatorname{supp} f_2 * g_1. \tag{E.20}$$

The inequality (E.20) could be written as $A_{21} - A_{11} \ge A_{11} - A_{10}$, and, together with (E.18), this yields

$$A_{21} - A_{11} \ge A_{11} - A_{10} \ge A_{10} - A_{00} > 0.$$

Proceeding by induction, we prove that

$$A_{32} - A_{22} \ge A_{22} - A_{21} \ge A_{21} - A_{11} \ge A_{11} - A_{10} \ge A_{10} - A_{00} > 0,$$

hence

$$A_{nn} \ge A_{00} + 2n(A_{10} - A_{00}). \tag{E.21}$$

At the same time, since $\sup\operatorname{supp} f_n \le \sup\operatorname{supp} f$, $\sup\operatorname{supp} g_n \le \sup\operatorname{supp} g$, we know that

$$\sup\operatorname{supp} f_n * g_n \le \sup\operatorname{supp} f_n + \sup\operatorname{supp} g_n \le \sup\operatorname{supp} f + \sup\operatorname{supp} g.$$

This would be in contradiction with (E.21). Hence, (E.13) is not true. This finishes the proof of the lemma. □

Let us show how to complete the proof of the Titchmarsh theorem for $f*g$. Assume that $\inf\operatorname{supp} f \ge 0$, $\inf\operatorname{supp} g \ge 0$, and that

$$f * g(t) = 0, \qquad 0 \le t \le \kappa. \tag{E.22}$$

This implies that

$$\int_0^t f(t-s)g(s)\,ds = 0, \qquad 0 \le t \le \kappa. \tag{E.23}$$

We may assume that both f and g are continuous. (If not, we consider their antiderivatives $F(t) = \int_{-\infty}^t f(s)\,ds$, $G(t) = \int_{-\infty}^t g(s)\,ds$, which also satisfy $\inf\operatorname{supp} F \ge 0$, $\inf\operatorname{supp} G \ge 0$; integrating (E.23) twice, we obtain $F * G(t) = 0$, $0 \le t \le \kappa$. We may repeat this process until we get functions continuous on $[0,\kappa]$.) By Lemma E.4, (E.23) leads to

$$\int_0^t f(t-s)g(s)s^n\,ds = 0, \qquad n \in \mathbb{N}, \tag{E.24}$$

valid for all $0 \le t \le \kappa$. Since f and g are continuous, Lerch's lemma implies that

$$f(t-s)g(s) = 0, \qquad 0 \le s \le t. \tag{E.25}$$

This in turn implies that there exists $\lambda \ge 0$ such that $f(s) = 0$ for $0 \le s \le \lambda$ and $g(s) = 0$ for $0 \le s \le t-\lambda$.

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