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Contents

Zusammenfassung

Diese Dissertation befasst sich größtenteils mit verschiedenen Persistenz-Problemen sowie einem Problem aus der Risikotheorie. Beide Themen sind dem Gebiet der Wahrscheinlichkeitstheorie zuzuordnen.

Die Wahrscheinlichkeit, dass ein stochastischer Prozess innerhalb des Zeitintervalls $[0, T]$ oder $[-T, T]$ unterhalb einer Barriere bleibt, bezeichnet man als Persistenz-Wahrscheinlichkeit. Dabei betrachten man den Fall $[0, T]$ bei sogenannten einseitigen stochastischen Prozessen mit nichtnegativem Zeitindex, während man den Fall $[-T, T]$ bei sogenannten zweiseitigen stochastischen Prozessen betrachtet, deren Zeitindex zusätzlich für negative Zeiten definiert ist. In diesem Kontext beginnt die Analyse solcher Persistenz-Wahrscheinlichkeiten häufig mit dem Untersuchen ihres asymptotischen Verhaltens für $T \to \infty$.

In dieser Arbeit beschäftigen wir uns zunächst mit Gaußprozessen in diskreter Zeit, welche zeitdiskrete Analoga der zweiseitigen gebrochenen Brownschen Bewegung beziehungsweise der zweiseitigen integrierten gebrochenen Brownschen Bewegung sind. In beiden Fällen zeigen wir, dass die Persistenz-Wahrscheinlichkeit polynomiell in T fällt und bestimmen die polynomielle Rate. Unsere Beweistechnik im Fall der zeitdiskreten Analoga der zweiseitigen gebrochenen Brownschen Bewegung unterscheidet sich stark von der Beweistechnik im zeitstetigen Fall in [61]. Unser Resultat deckt eine große Klasse zeitdiskreter Prozesse ab und wir erhalten stärkere asymptotische Abschätzungen für die Persistenz-Wahrscheinlichkeit als im zeitstetigen Fall in [61]. Bei den zeitdiskreten Analoga der zweiseitigen integrierten gebrochenen Brownschen Bewegung gehen wir ähnlich wie im zeitstetigen Fall in [62] vor und setzen die Persistenz-Wahrscheinlichkeit mit dem Erwartungswert eines Funktionals in Beziehung, dessen asymptotisches Verhalten wir bestimmen können.

Im Gegensatz zur Untersuchung der Persistenz-Wahrscheinlichkeit der einseitigen gebrochenen Brownschen Bewegung, gab es noch keinen rigorosen Ansatz in der Mathematik-Literatur eine „gebrochene Brownsche Bewegung bedingt darauf positiv zu sein“ zu definieren. Wir betrachten in dieser Arbeit ein leicht abgewandeltes Problem, bei dem die gebrochene Brownsche Bewegung negative Werte annehmen darf, dafür jedoch „bestraft“ wird. Unsere Modifikation lässt sich durch die Beweistechnik in der Arbeit [61]

motivieren, in welcher die Persistenz-Wahrscheinlichkeit der einseitigen gebrochenen Brownschen Bewegung untersucht wird. Im Rahmen der vorliegenden Dissertation konzentrieren wir uns auf den Brownschen Spezialfall. Für diesen leiten wir eine stochastische Differentialgleichung her, die vom Grenzprozess erfüllt wird. Dann vergleichen wir diese mit der stochastischen Differentialgleichung des Brownschen Meanders beziehungsweise der des dreidimensionalen Besselprozesses, welche beide als sinnvolle Grenzprozesse der „Brownschen Bewegung bedingt darauf positiv zu sein" in der Literatur auftauchen.

Anschließend stellen wir zufällige zentrierte Irrfahrten mit endlicher Varianz in den Mittelpunkt. Diese Klasse von stochastischen Prozessen wurde bereits intensiv in der Literatur untersucht und Resultate über die Persistenz-Wahrscheinlichkeiten sowie funktionale Grenzwertsätze sind bekannt. Wir untersuchen Modifikationen dieser klassischen Probleme. Dazu führen wir eine Klasse von Absorptionsmechanismen ein und untersuchen das Verhalten von zufälligen Irrfahrten, die nicht absorbiert werden. Unsere Hauptresultate dienen als eine Art Werkzeugsatz, mit dem leicht Resultate über Persistenz-Wahrscheinlichkeiten sowie funktionale Grenzwertsätze für zufällige Irrfahrten, die nicht absorbiert werden, gewonnen werden können. Um unsere Resultate zu beweisen, verfahren wir teilweise wie in [78] und verknüpfen Sätze der Erneuerungstheorie mit klassischen Persistenz-Resultaten.

Schließlich beschäftigen wir uns mit einem Problem aus der Risikotheorie. Wir betrachten den klassischen Cramér-Lundberg Prozess, welcher die Risikoreserve einer Versicherungsgesellschaft beschreibt. Im Gegensatz zu den vorangegangen Persistenz-Wahrscheinlichkeiten, untersuchen wir nun die sogenannte Ruin-Wahrscheinlichkeit. Dies ist die Wahrscheinlichkeit, dass der betrachtete stochastische Prozess zu irgendeinem Zeitpunkt einen negativen Wert annimmt. Die Ruin-Wahrscheinlichkeit ist unter unseren Annahmen kleiner als 1 und hängt vom Startkapital u der Versicherungsgesellschaft ab. In dieser Arbeit betrachten wir Modelle mit einer abgeschwächten Definition von Ruin. Wenn die Risikoreserve der Versicherungsgesellschaft negativ wird, erklären wir die Versicherungsgesellschaft nicht unmittelbar als insolvent, sondern erlauben, dass sie durch bestimmte Mechanismen weiterhin „überleben" kann. Unter einer allgemeinen Annahme an solche Mechanismen, welche von den meisten solcher modifizierten Modelle aus der Literatur erfüllt wird, untersuchen wir dann das asymptotische Verhalten der modifizierten Ruin-Wahrscheinlichkeit für $u \to \infty$ und vergleichen es mit dem der klassischen Ruin-Wahrscheinlichkeit. Wir nehmen dabei an, dass die Cramér-Bedingung erfüllt ist oder die Verteilung der integrierten Schadenshöhen subexponentiell ist. Unsere Ergebnisse basieren auf Resultaten, welche das Verhalten des stochastischen Prozesses beim Übergang in die negative Halbachse charakterisieren.

Summary

In this thesis, we are concerned with different persistence problems as well as a problem from risk theory.

The probability that a stochastic process does not cross a barrier within the time interval $[0, T]$ or $[-T, T]$ is called persistence probability. The case $[0, T]$ is considered for so-called one-sided processes whose index variable takes only non-negative numbers, while the case $[-T, T]$ is considered for so-called two-sided processes whose index variable takes negative and non-negative numbers. Often in this context, one is first interested in the asymptotic behavior of the persistence probability, as $T \to \infty$.

We first deal with two classes of Gaussian sequences that are discrete-time analogs of two-sided fractional Brownian motion and two-sided integrated fractional Brownian motion, respectively. In both cases, we show that the persistence probability decreases polynomially and determine the polynomial rate. To prove the result for discrete-time analogs of two-sided fractional Brownian motion, we present a new approach, which is completely different from the approach in the continuous-time case in [61]. Our technique applies for a large class of discrete-time processes and leads to stronger asymptotic bounds of the persistence probability than in the continuous-time result in [61]. In the case of discrete-time analogs of two-sided integrated fractional Brownian motion, we proceed similarly as in [62], where the continuous-time case is considered, and relate the persistence probability to the expectation of a functional whose asymptotic behavior can be determined.

While the persistence probability for one-sided fractional Brownian motion has been studied intensively, there has not been a rigorous attempt in the mathematics literature to define a "fractional Brownian motion conditioned to be positive". We consider a slightly modified problem (motivated by the technique in [61]), where the fractional Brownian motion is penalized instead of being killed when becoming negative. We mainly discuss this result in the Brownian context. That is, we give a representation of the limiting process in terms of an explicit stochastic differential equation. Then, we compare it to the stochastic differential equation fulfilled by the Brownian meander and the three-dimensional Bessel process, which both appear in the literature as sensible definitions of a Brownian motion conditioned to be positive.

After that, we consider centered random walks with finite variance. This

class of processes has been studied profoundly, and persistence probability results as well as scaling limit results are available. We study modifications of these classical persistence problems. More precisely, we introduce a class of absorption mechanisms and study the behavior of random walks that do not get absorbed. Our main results serve as a toolkit that allows obtaining persistence probability and scaling limit results for many different examples in this class. To prove the results, we partly proceed as in [78], where a special case of our model is considered, and combine results from renewal theory and classical persistence results.

Finally, we turn our attention to a problem from risk theory. Here, we consider the classical Cramér-Lundberg process that describes the amount of surplus of an insurance portfolio. In contrast to the previously discussed persistence probability, we are now interested in the so-called ruin probability, which is the probability that the process crosses zero at some time (which is a probability smaller than 1 under our assumptions). The ruin probability depends on the initial capital u, and, for a given process, it is often the first goal to study the behavior of the ruin probability, as $u \to \infty$. We study the asymptotics of the ruin probability with a modified notion of ruin. The modification is as follows. If the portfolio becomes negative, the asset is not immediately declared ruined but may survive due to certain mechanisms. Under a rather general assumption on the mechanism, which is satisfied by most of such modified models from the literature, we study the relation of the asymptotics of the modified ruin probability to the classical ruin probability. This is done under the Cramér condition as well as for subexponential integrated claim sizes. Our results are based on results characterizing the distribution of the process at the first zero-crossing time.

Chapter 1

Introduction

Persistence concerns the study of stochastic processes that have a long negative or positive excursion. In the major part of this thesis, we deal with persistence problems for fractional Brownian motion and related processes as well as modified persistence problems for centered random walks with finite variance. We also consider a problem from risk theory. There, the probability that an insurance company goes bankrupt, as the initial capital tends to infinity, is studied. We are concerned with the classical Cramér-Lundberg model but modified notions of ruin.

In this chapter, we state the general mathematical problems, motivate where our interest comes from, and give an overview of the related literature.

1.1 Persistence

1.1.1 The problem

In Chapters 2 - 4, we are concerned with so-called persistence problems: Let $(Z(t))_{t\in\mathbb{T}}$ be a real-valued stochastic process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{T} = \mathbb{N}$ or $\mathbb{T} = \mathbb{Z}$ in discrete time and $\mathbb{T} = \mathbb{R}_{\geq}$ or $\mathbb{T} = \mathbb{R}$ in continuous time. Further, let $Z(0) \geq 0$ a.s. Then, we study the event

$$A_T := \{Z(t) \geq 0 \colon t \in [-T, T] \cap \mathbb{T}\}, \quad \text{as} \quad T \to \infty. \tag{1.1}$$

Often, one is first interested in the asymptotic behavior of the persistence probability $\mathbb{P}(A_T)$, as $T \to \infty$. In many cases of interest, one has

$$\mathbb{P}(A_T) = T^{-\theta+o(1)}, \quad \text{as} \quad T \to \infty, \tag{1.2}$$

and it is the primarily goal to find the persistence exponent θ.

A further task is to define the corresponding process conditioned to be positive. Since in most cases of interest, one has $\mathbb{P}(Z(t) \geq 0 \colon t \in \mathbb{T}) = 0$, it is not obvious how to define such a process. There are two common approaches in the literature of conditioning a stochastic process to be positive: In the first approach, one considers, for fixed $T_0 > 0$, the sequence of measures

$$\mathbb{P}\left((Z(t))_{t\in[-T_0,T_0]\cap\mathbb{T}} \in \cdot \mid A_T\right), \quad \text{as} \quad T \to \infty, \tag{1.3}$$

on an appropriate function space and then determines the limiting measure. In the second approach, one sets $I = [0, 1]$ in the one-sided case ($\mathbb{T} = \mathbb{N}$ or $\mathbb{T} = \mathbb{R}_{\geq}$) and $I = [-1, 1]$ in the two-sided case ($\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$). Then, one considers the sequence of measures

$$\mathbb{P}\left((a_T^{-1} Z(T \cdot t))_{t\in I} \in \cdot \mid A_T\right), \quad \text{as} \quad T \to \infty, \tag{1.4}$$

on $(C[0, 1], \|\cdot\|_\infty)$ and $(C[-1, 1], \|\cdot\|_\infty)$, respectively, where $\|\cdot\|_\infty$ denotes the supremum norm and $(a_T)_{T\in\mathbb{T}}$ denotes a proper scaling sequence. In discrete time, the corresponding linearly interpolated process is considered. Again, it is the goal to determine the limiting measure.

In this thesis, we will only be concerned with the latter approach of conditioning and will sometimes refer to the process distributed according to the resulting limiting measure as scaling limit. Depending on the related literature, we will occasionally assume that the process starts in $(-\infty, 0]$ and study the same questions as above for the event $\{Z(t) \leq 0 \colon t \in [-T, T] \cap \mathbb{T}\}$, as $T \to \infty$.

In the next two subsections, we will specify the persistence problems we are concerned with in this thesis, give an overview of the relevant persistence literature and motivate our interest. For a recent overview on persistence in general, we refer to the surveys [23] and [56] in the physics literature and [15] in the mathematics literature.

1.1.2 Fractional Gaussian processes

Fractional Brownian motion

In Chapter 2 and Chapter 3, we will study different persistence problems for fractional Brownian motion and related processes. For this purpose, let us recall that a fractional Brownian motion (FBM) is a centered Gaussian process $(W_H(t))_{t\in\mathbb{R}}$ with covariance

$$\mathbb{E}\left[W_H(t)W_H(s)\right] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t - s|^{2H}\right), \quad t, s \in \mathbb{R},$$

where $0 < H < 1$ is a constant parameter called Hurst parameter. For $H = 1/2$, this is a usual Brownian motion. For any $0 < H < 1$, the process is H-self-similar and has stationary increments but no independent increments (unless $H = 1/2$).

Related work

Many dynamical systems with long-range dependence from the finance and physics literature involve FBM and related processes. For instance, FBM is used to model the long-range dependence of stock prices and volatility, see [27] and [46]. In the physics literature, a polymer model involving FBM is considered in [80]. For more examples, we refer to [19].

In many cases, certain properties of such models are related to the persistence probabilities of the underlying process, and thus, the study of the persistence probabilities of FBM and related processes has attained substantial relevance in the recent theoretical physics and mathematics literature. For instance, see [63] and [62], where a relation between the Hausdorff dimension of Lagrangian regular points for the inviscid Burgers equation with FBM initial velocity and the persistence probabilities of integrated fractional Brownian motion is established; the interest for it arises from [73] and [75]. Moreover, persistence probabilities of FBM are studied in [61] and results for their discrete-time analogs are obtained in [14] and [55]. Further, in [65], a physical model involving FBM is studied as an extension to the Sinaĭ model, see also [9]. There, persistence probabilities are related to scaling properties of a quantity called steady-state current. Moreover, persistence of non-Markovian processes that are similar to FBM are studied in [26] and [14], confirming results in [68] and [57]. Further, the study of many physical systems with long-range dependence is related to properties of a sensibly defined FBM conditioned to be positive, see e.g. [80] and [58].

While the rigorous study of FBM conditioned to be positive is still unexplored territory in the mathematics literature, many contributions to the study of the persistence probability of FBM and related processes have been made recently. In the following, we give an overview of relevant results in this active field and will further place results of this thesis within this broader context.

Persistence results

Note that $\mathbb{P}(W_H(s) \leq 0 \colon s \in [0,T]) = 0$ for all $T > 0$. In order to exclude such trivial cases in continuous time, we will consider the persistence event $\{W_H(s) \leq 1 \colon s \in [0,T]\}$ (instead of $\{W_H(s) \leq 0 \colon s \in [0,T]\}$) for the continuous-time process $(W_H(t))_{t\in\mathbb{R}}$. Alternatively, one could consider the processes $(W_H(t))_{t\in\mathbb{R}}$ starting in -1 (instead of 0).

Let us first consider the Brownian case ($H = 1/2$). Here, explicit expressions for the persistence probability can be easily derived from the reflection principle. Particularly, using the strong Markov property of Brownian motion, one obtains

$$\mathbb{P}\left(W_{1/2}(s) \leq 1 \colon s \in [0,T]\right) = 2\mathbb{P}\left(|W_{1/2}(T)| \leq 1\right) \sim \sqrt{\frac{2}{\pi}}T^{-1/2},$$

as $T \to \infty$. Here and below, we write $f(T) \sim g(T)$ if $f(T)/g(T) \to 1$, as $T \to \infty$. By the independence of the Brownian motion restricted to positive and negative time, the result in the two-sided case follows immediately. Here, we have

$$\mathbb{P}\left(W_{1/2}(s) \leq 1 \colon s \in [-T,T]\right) = \mathbb{P}\left(W_{1/2}(s) \leq 1 \colon s \in [0,T]\right)^2 \sim \frac{2}{\pi} T^{-1},$$

as $T \to \infty$. The situation for $H \neq 1/2$ is completely different. Due to a lack of properties such as the (strong) Markov property, there is a need for different techniques to obtain persistence results. Here, G. Molchan made a main contribution to the study of the persistence probability of FBM in [61]. He showed that one has in the one-sided case

$$T^{-(1-H)} e^{-k\sqrt{\log(T)}} \leq \mathbb{P}\left(W_H(s) \leq 1 \colon s \in [0,T]\right) \leq T^{-(1-H)} e^{+k\sqrt{\log(T)}},$$

for some $k > 0$ and T large enough. The crucial idea of the proof is to relate the persistence probability to the expectation of the functional $\left(\int_0^T \exp(W_H(s))\,\mathrm{d}s\right)^{-1}$, whose asymptotic rate can be determined. Heuristically, typical paths of FBM contributing to the probability of the persistence event tend to escape to $-\infty$ rather than oscillating around the origin. But these are exactly those paths for which the functional is large. In Chapter 3 of this thesis, we will use this smoother functional (compared to the indicator function of the persistence event) to motivate a modified approach to define FBM conditioned to be positive in the sense of (1.4); see Theorem 3.1 and the following discussion there in the Brownian case.

The result of G. Molchan in [61] could be slightly improved by F. Aurzada in [8]. He showed that a constant $c > 0$ (depending on H) exists such that for T large enough

$$T^{-(1-H)} \log(T)^{-c} \leq \mathbb{P}\left(W_H(s) \leq 1 \colon s \in [0,T]\right) \leq T^{-(1-H)} \log(T)^{c}.$$

Again, the proof relied on the functional that was also used by G. Molchan. A further improvement of the upper bound was obtained in Theorem 1 in [14]. For $H \in (1/2, 1)$, a constant $c > 0$ exists such that

$$c^{-1} T^{-(1-H)} \log(T)^{-1/(2H)} \leq \mathbb{P}\left(W_H(s) \leq 1 \colon s \in [0,T]\right) \leq c T^{-(1-H)}.$$

The lower bound also holds for any $H \in (0,1)$. Recall that a different approach is made to prove the result, where the authors first pass over to a corresponding discrete-time process. Further, [14] contains a persistence probability result for discrete-time analogs of one-sided FBM, which we can improve as a byproduct of our technique in Chapter 2 of this thesis; see Corollary 2.12.

In the two-sided case, G. Molchan showed (Theroem 3 in [61]) that

$$T^{-1} e^{-k\sqrt{\log(T)}} \leq \mathbb{P}\left(W_H(s) \leq 1 \colon s \in [-T,T]\right) \leq T^{-1} e^{+k\sqrt{\log(T)}},$$

for some $k > 0$ and T large enough. We will show a similar result (without lower order terms) for a large class of discrete-time analogs of two-sided FBM in Chapter 2 of this thesis; see Theorem 2.1.

In [62], the two-sided integrated fractional Brownian motion (IFBM) $(I_H(t))_{t\in\mathbb{R}}$, defined by $I_H(t) := \int_0^t W_H(s)\,\mathrm{d}s$, is considered. There, it is shown that

$$\mathbb{P}\left(I_H(s) \le 1\colon s \in [-T,T]\right) = T^{-(1-H)+o(1)}, \quad \text{as} \quad T \to \infty.$$

Again, passing over to a discrete-time process allows the proof to work. The result is then obtained by relating the persistence probability to the expectation of a functional for which the asymptotic behavior can be determined. In Chapter 2 of this thesis, we will consider a large class of discrete-time analogs of two-sided IFBM and will show similarly the corresponding persistence result; see Theorem 2.2.

Finally, we mention that finding the persistence exponent of one-sided IFBM is still an open problem. In [63], computational evidence is produced for the conjecture that the persistence exponent of one-sided IFBM is given by $H(1-H)$.

1.1.3 Random walks

Classical persistence results

In Chapter 4, we will be concerned with modified persistence problems for random walks. For this purpose, let us recall some facts from the classical situation. Let $(S_n)_{n\in\mathbb{N}}$ be a centered random walk with finite variance. Then, a powerful theory for persistence probabilities in (1.1) is available for the process $(S_n)_{n\in\mathbb{N}}$, which goes back to E. Sparre Andersen and B. Rogozin, see e.g. [76], [77], and [70]. In this case, one has

$$\mathbb{P}_x(S_n \ge 0\colon 0 \le n \le N) \sim c_x N^{-1/2}, \quad \text{as} \quad N \to \infty, \tag{1.5}$$

where the probability measure $\mathbb{P}_x$ indicates that the random walk starts in x and c_x denotes a positive constant depending on the distribution of S_1 and x. Moreover, random walks conditioned to be positive in the sense of (1.3) and (1.4), respectively, have been studied profoundly. Let $(W(t))_{t\in[0,1]}$ be a Brownian motion. Then, the Brownian meander $(X^{(me)}(t))_{t\in[0,1]}$ is defined by

$$X^{(me)}(t) := \left|\frac{1}{(1-\tau)^{1/2}} W(\tau + (1-\tau)t)\right|, \quad t \in [0,1], \tag{1.6}$$

where $\tau := \sup\{t \in [0,1]\colon W(t) = 0\}$. E. Bolthausen showed in [22] that the scaling limit in the sense of (1.4) of a linearly interpolated centered random walk with finite variance is a Brownian meander. This result was obtained earlier by D. Iglehart in [47] under the stronger assumption of finite third moments of the random walk. Further, we refer to [17] for a limit result in the sense of (1.3).

Related work

In the mathematics literature, different modifications and extensions of these classical questions for random walks have been considered. For instance, in [49], classical persistence results are generalized to multidimensional random walks, and, in [30], the asymptotic behavior of a multidimensional random walk in a general cone is studied. Moreover, different modifications of the persistence event in (1.1) for one-dimensional random walks are studied in the mathematics literature. For instance, in [51], J. Kemperman considers a model where the random walk can stay a geometrically distributed time below zero instead of immediately getting killed when crossing zero. In [78], random walks that avoid a bounded Borel set with non-empty interior are studied by V. Vysotsky. The latter problem is in turn related to the study of persistence probabilities of iterated random walks. In continuous time, we refer to [34] and [35], where stable processes and Lévy processes with zero mean and finite variance, respectively, that avoid an interval are studied.

Persistence results and methods of proof

In Chapter 4, we present a general approach to treat many different models with modified persistence events, see Theorem 4.1 and Theorem 4.2. Important examples of models in this class are the models considered by J. Kemperman and V. Vysotsky in [51] and [78], respectively. Since their techniques to prove the persistence results are entirely different, we discuss them in more detail in the following.

We begin by sketching the classical approach to obtain persistence results for random walks, which is generalized in [51]. Let $(S_n)_{n\in\mathbb{N}}$ be a random walk starting in $x = 0$. Further, let T_1 denote the first time that the random walk enters the negative half-line, that is

$$T_1 := \inf\{n \in \mathbb{N} \colon S_n < 0\}.$$

Then, Sparre Andersen's formula (see e.g. Theorem XII.7.1 in [40]) states that, for $s \in (-1, 1)$,

$$\sum_{n=0}^{\infty} s^n \mathbb{P}(T_1 > n) = \exp\left(\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbb{P}(S_n \geq 0)\right). \tag{1.7}$$

The desired persistence probabilities are encoded as coefficients of the generating function on the left-hand side in (1.7). Now, the idea is to let $s \nearrow 1$ and to use a Tauberian theorem for monotonic sequences (see Theorem XIII.5.5 in [40]) in order to obtain asymptotic results for the persistence probability. This can be done straightforwardly under Spitzer's condition

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{P}(S_k < 0) = r \in [0, 1],$$

which is equivalent to

$$\lim_{n\to\infty} \mathbb{P}(S_n < 0) = r,$$

see [18] and [32]. One obtains the persistence exponent $\theta = 1 - r$ in (1.2). In the setup of centered random walks with finite variance, we have by the central limit theorem that $\lim_{n\to\infty} \mathbb{P}(S_n < 0) = 1/2$, and thus, $\theta = 1/2$. In fact, even the exact asymptotics in (1.5) can be obtained by this technique. We further mention that many other interesting results from fluctuation theory are based on (1.7) or the more general Spitzer's formula, see e.g. [5] and the references therein. For a recent account of the classical theory in continuous time, we refer to [33].

Now, we consider the more general situation in [51]. Let U be a geometrically distributed random variable with parameter $\rho \in [0, 1)$ which is independent of the random walk $(S_n)_{n\in\mathbb{N}}$. We define

$$\tilde{T}_1 := \inf\{n \in \mathbb{N} \colon \#\{k \leq n \colon S_k < 0\} \geq U\}.$$

Thus, instead of being immediately killed when entering the negative half-line, the random walk is allowed to visit the negative half-line a geometrically distributed number of times before getting killed. In this case, one obtains as a consequence of Theorem 15.1 in [51] that, for $s \in (-1, 1)$,

$$\sum_{n=0}^{\infty} s^n \mathbb{P}(\tilde{T}_1 > n) = \exp\left(\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbb{P}(S_n \geq 0) + \sum_{n=1}^{\infty} \frac{(\rho s)^n}{n} \mathbb{P}(S_n < 0)\right). \quad (1.8)$$

This identity generalizes Sparre Andersen's formula since (1.7) is contained as the special case $\rho = 0$. Again, the same procedure as above yields persistence results. It is not hard to see that the classical and the modified persistence probability differ asymptotically only by a constant factor since the term $\sum_{n=1}^{\infty} \frac{(\rho s)^n}{n} \mathbb{P}(S_n < 0)$ in (1.8) converges to a constant in the application of the Tauberian theorem, as $s \nearrow 1$. We emphasize at this point that the technique of J. Kemperman uses the memorylessness of the geometrical distribution and cannot be applied if U has a different distribution. Our results in Chapter 4 can be applied for arbitrary distributions of U.

Now, we turn our attention to the situation in [78], where centered random walks with finite variance that avoid a bounded Borel set B with non-empty interior are studied. More precisely, the stopping time

$$T_B := \inf\{n \in \mathbb{N} \colon S_n \in B\}$$

is consider, instead of T_1 as in the classical situation. In order to sketch the rough idea of the proof, let us assume that the interval $B = (-d, d)$ is avoided (with d large, so that we need no further assumptions at this stage). Then, due to the fact that the variance of the process is finite, one can show that a constant $\gamma \in (0, 1)$ exists with

$$\mathbb{P}_x(|S_{T_1}| \geq d) \leq \gamma \quad \text{for all } x \geq d. \quad (1.9)$$

Further, using again that the variance of the process is finite, one has

$$\mathbb{E}_x[|S_{T_1}|] = o(x), \quad \text{as} \quad x \to \infty. \tag{1.10}$$

Both, (1.9) and (1.10), can be obtained using results from renewal theory. Further, it is well-known that a constant $c > 0$ exists such that $c_x \sim cx$, as $x \to \infty$, with c_x from (1.5). Combining this with (1.10), one would expect that, once the random walk has attained a high level and it jumps over the interval, it would typically start afresh from a position much closer to the interval and it is harder not to hit the interval in the future. Therefore, intuitively, it is more efficient to survive for the random walk if all jumps occur at the beginning before the random walk has moved far away from the interval. Further, also taking (1.9) into account, one would expect that only a few of such jumps occur. These ideas can be used to prove persistence results rigorously. More precisely, in [78], it is shown that

$$\mathbb{P}_x(T_B > N) \sim c_x N^{-1/2}, \quad \text{as} \quad N \to \infty.$$

Further, a scaling limit result in the sense of (1.4) is proved. For a more detailed discussion of the intuition behind the proofs, we refer to Section 2 in [78]. We also note at this point that our technique in Chapter 4 is inspired by the approach in [78]. Moreover, we mention that [66] contains a further approach by means of potential theory and the results there cover the most important part of the persistence probability result in [78].

1.2 Risk theory

1.2.1 The problem

In Chapter 5, we are concerned with a question from risk theory. Let $(U(t))_{t\geq 0}$ denote a stochastic process describing the amount of surplus of an insurance portfolio indexed by time that starts in u (initial capital). Let $\mathbb{P}_u$ be the corresponding probability measure. In the classical setup, the time of ruin in such a model is defined by $T_1 := \inf\{t > 0 \colon U(t) < 0\}$ with the convention $\inf \emptyset := \infty$. Then, one is often first interested in the analysis of the classical ruin probability

$$\psi_{\text{cl}}(u) := \mathbb{P}_u(T_1 < \infty), \quad \text{as} \quad u \to \infty. \tag{1.11}$$

In contrast to the study of the persistence probabilities in Chapters 2 - 4, one typically has $\mathbb{P}_u(T_1 < \infty) < 1$ and studies the behavior of the ruin probability $\mathbb{P}_u(T_1 < \infty)$, as $u \to \infty$ (instead of considering $\mathbb{P}(T_1 > T)$, as $T \to \infty$).

The aim of Chapter 5 is to study the asymptotics of the ruin probability of a large class of models with a modified notion of ruin. We consider the classical Cramér-Lundberg process and assume that the Cramér condition is satisfied or the integrated claim sizes are subexponential.

1.2.2 Cramér-Lundberg model

The model

We recall that the classical Cramér-Lundberg process $(U(t))_{t\geq 0}$ is given by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0,$$

where $u \geq 0$ denotes the initial capital, $c > 0$ is the constant premium rate, $(N(t))_{t\geq 0}$ is a Poisson process with rate $\lambda > 0$ describing the number of claims until time t, and the sequence of non-negative i.i.d. claim sizes is denoted by $(Y_k)_{k\in\mathbb{N}}$ and is also independent of $(N(t))_{t\geq 0}$. Further, we assume that $\mathbb{E}[Y_1] = \mu > 0$ and that the net profit condition $c > \lambda\mu$ is satisfied. We denote the distribution function of Y_1 by F.

Related work

For a comprehensive overview of the classical theory in the Cramér-Lundberg model, we refer to [31] and [38]. Besides the study of the classical ruin probability in (1.11), different ruin related quantities have attracted much attention in the literature. For instance, see the well-cited work of H. Gerber and E. Shiu [41] and the vast number of papers that followed. Moreover, many extensions and modifications of the classical Cramér-Lundberg model have been established. Again, in many situations, one is first interested in the corresponding questions from the classical setup. In the recent literature, modified definitions of ruin are considered. For instance, in [25], a model is studied where the insurance company can borrow money at a certain debit interest when $U(t)$ is negative. Further, the concept of Parisian ruin has been much discussed in the literature. Here, the surplus process is allowed to stay negative for a continuous time interval of a fixed or random length, see [29], [28], [54], [52] and for the cumulative situation [44]. In omega models, the insurance company goes bankrupt at a random time at some surplus dependent bankruptcy rate when $U(t)$ is negative, see [3], [42], and [4]. This model is in turn linked to models where the insurance company can just go bankrupt at random observation times, see [1] and [2].

Classical results and methods of proof

In the following, we recall the approaches to obtain asymptotic results for the classical ruin probability if the Cramér condition is fulfilled or the integrated claim sizes are subexponential.

We first recall that the Cramér condition holds if $\lambda\mathbb{E}[\exp(RY_1) - 1] = cR$ for a constant $R > 0$. Then,

$$\psi_{\text{cl}}(u) \sim ke^{-Ru}, \quad \text{as} \quad u \to \infty,$$

with

$$k = \left[\frac{\lambda R}{c - \lambda\mu} \int_0^\infty x e^{Rx}(1 - F(x))\,\mathrm{d}x\right]^{-1},$$

see e.g. Theorem 1.2.2 in [38]. In a first step of the proof, using the renewal structure of the process, the defective renewal equation

$$\psi_{\mathrm{cl}}(u) = \frac{\mu\lambda}{c}(1 - F_I(u)) + \frac{\mu\lambda}{c}\int_0^u \psi_{\mathrm{cl}}(u - x) F_I(\,\mathrm{d}x) \tag{1.12}$$

is deduced. Here, F_I denotes the distribution function which is defined by $F_I(t) := \frac{1}{\mu}\int_0^t (1 - F(s))\,\mathrm{d}s$ for $t \geq 0$. Then, using (1.12), the standard renewal equation

$$e^{Ru}\psi_{\mathrm{cl}}(u) = e^{Ru}\frac{\mu\lambda}{c}(1 - F_I(u)) + \frac{\mu\lambda}{c}\int_0^u e^{R(u-x)}\psi_{\mathrm{cl}}(u - x) e^{Rx} F_I(\,\mathrm{d}x) \tag{1.13}$$

can be obtained in a second step. In this step, the Cramér condition is needed to show that $\frac{\mu\lambda}{c}\int_0^\infty e^{Rx} F_I(\,\mathrm{d}x) = 1$, and thus, (1.13) is actually a standard renewal equation. Now, the claim follows straightforwardly by applying a classical result from renewal theory, see Theorem A4.3(b) in [38].

Let us now consider the so-called heavy-tailed case. In preparation, we note that the identity

$$\psi_{\mathrm{cl}}(u) = (1 - \frac{\lambda\mu}{c})\sum_{n=0}^\infty \left(\frac{\lambda\mu}{c}\right)^n (1 - F_I^{(*n)}(u))$$

can be obtained from (1.12), see e.g. the proof of Theorem 1.2.2 in [38]. We assume that, for all $n \in \mathbb{N}$, one has

$$(1 - F_I^{(*n)}(u)) \sim n \cdot (1 - F_I(u)), \quad \text{as} \quad u \to \infty.$$

In this case, F_I is called subexponential. Now, one is tempted to conclude that

$$\begin{aligned}
\psi_{\mathrm{cl}}(u) &= (1 - \frac{\lambda\mu}{c})\sum_{n=0}^\infty \left(\frac{\lambda\mu}{c}\right)^n (1 - F_I^{(*n)}(u)) \\
&\sim (1 - F_I(u)) \cdot (1 - \frac{\lambda\mu}{c})\sum_{n=0}^\infty \left(\frac{\lambda\mu}{c}\right)^n n \\
&= (1 - F_I(u)) \cdot \frac{\lambda\mu}{c - \lambda\mu},
\end{aligned}$$

as $u \to \infty$. Of course, the second step needs some further justification (to apply the dominated convergence theorem there). Here, we refer to Section 1.3 in [38] for a rigorous proof.

Both proofs heavily rely on the specific definition of the Cramér-Lundberg model, including particularly the definition of ruin. Thus, usually, the above techniques cannot be adapted in an obvious way to models with a different notion of ruin. In Chapter 5, we choose therefore a different approach which is based on limit results for the distribution of the Cramér-Lundberg process at the (classical) time of ruin.

1.3 Outline

In Chapter 2, we study the persistence probability of discrete-time analogs of two-sided fractional Brownian motion and two-sided integrated fractional Brownian motion, respectively. In Section 2.1, we introduce the considered processes and state our main results of the chapter, Theorem 2.1 and Theorem 2.2. Then, in Section 2.2, we collect some basic properties of the considered processes, discuss a modification of Theorem 2.2 (see Corollary 2.11), and provide some tools for our proofs. In particular, we present some results concerning the reproducing kernel Hilbert spaces of the considered processes, which are neede to get our change of measure arguments to work. As a byproduct, we use them to improve Theorem 11 in [14], where the persistence problem for the discrete-time analogs of one-sided fractional Brownian motion is considered. Finally, in Section 2.3, we prove the main results.

In Chapter 3, we present a modification of the approach in (1.4) to define a fractional Brownian motion conditioned to be positive. After motivating our approach in Section 3.1, we state the main result (Theorem 3.1) in Subsection 3.2.1. Then, we discuss the result in the Brownian case in terms of stochastic differential equations in Subsection 3.2.2. Afterward, in Section 3.3, we give the main idea of the proof of Theorem 3.1 and the full proofs of the results from Subsection 3.2.2.

Chapter 4 deals with a class of absorption mechanisms that generalize the classical persistence problem for random walks. In Section 4.1, we introduce the absorption model and state our main results of the chapter, namely a persistence probability result (Theorem 4.1) and a scaling limit result (Theorem 4.2). In Section 4.2, we provide several examples of absorption mechanisms and apply our theorems. Auxiliary statements can be found in Section 4.3. After setting up notation in Subsection 4.3.1, we collect results that do not use assumptions from the absorption model in Subsection 4.3.2. Results that are based on the absorption model can be found in Subsection 4.3.3. Finally, we prove our main results in Section 4.4.

In Chapter 5, we study the asymptotics of the ruin probability in the Cramér-Lundberg model with a modified notion of ruin. In Section 5.1, we introduce our model. Afterward, we state and prove our main results in Section 5.2. Finally, in Section 5.3, we apply our results to a bunch of examples and give a short outlook.

Remark. This thesis is based on the articles [10], [11], [12], and [24]; see also the remarks at the end of each chapter.

Chapter 2

Persistence probabilities of fractional Gaussian sequences

In this chapter, we analyze the persistence probability of the discrete-time analogs of two-sided fractional Brownian motion (FBM) and two-sided integrated fractional Brownian motion (IFBM), respectively. Our study extends continuous-time results in [61] and [62] to a wide class of discrete-time processes.

2.1 Introduction

2.1.1 Discrete-time analogs of FBM and IFBM

Let $(W_H(t))_{t\in\mathbb{R}}$ denote a two-sided FBM with Hurst parameter $0 < H < 1$. Further, let $(I_H(t))_{t\in\mathbb{R}}$ denote a two-sided IFBM. Recall that the persistence probabilities of two-sided FBM and two-sided IFBM have been studied in [61] and [62], respectively. In this chapter, we will examine the discrete-time case. For a more in-depth discussion on the related literature in this context, we refer to Section 1.1.2.

In order to define the discrete-time analogs of these processes, we let $(\xi_n)_{n\in\mathbb{Z}}$ be a real-valued stationary centered Gaussian sequence such that

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbb{E}\xi_j\xi_k \sim n^{2H}\ell(n), \quad \text{as} \quad n\to\infty, \tag{2.1}$$

with $0 < H < 1$ and ℓ slowly varying at infinity. A function ℓ is called slowly varying at infinity if $\ell(ax)/\ell(x) \to 1$, as $x\to\infty$, for all $a > 0$. In this case, (2.1) implies the weak convergence result

$$\left(\frac{1}{n^H\ell(n)^{1/2}}\sum_{k=1}^{\lfloor nt\rfloor}\xi_k\right)_{t\geq 0} \Rightarrow (W_H(t))_{t\geq 0} \tag{2.2}$$

with fractional Brownian motion $(W_H(t))_{t\geq 0}$, see e.g. Theorem 4.6.1 in [79]. For this reason, it is natural to consider the stationary increments sequence $(S_n)_{n\in\mathbb{Z}}$ given by

$$S_n - S_{n-1} := \xi_n \quad \text{for } n \in \mathbb{Z} \text{ and } S_0 := 0 \tag{2.3}$$

as a discrete-time analog of FBM.

Now, we will define the discrete-time analog of IFBM such that symmetry properties like in the continuous-time setting are satisfied. With this in mind, a natural discrete-time analog is given by

$$I_n - I_{n-1} := \tilde{S}_n := (S_n + S_{n-1})/2 \quad \text{for } n \in \mathbb{Z} \text{ and } I_0 := 0. \tag{2.4}$$

In Section 2.2, we discuss relations to the process with increments $(S_n)_{n\in\mathbb{Z}}$ (instead of $(\tilde{S}_n)_{n\in\mathbb{Z}}$), which may also seem natural but for which our method of proof does not apply directly due to a lack of symmetry.

2.1.2 Results

We recall that in [61] the persistence probability of two-sided FBM is studied. There, it is shown that

$$T^{-1}e^{-k\sqrt{\log(T)}} \leq \mathbb{P}\left(W_H(s) \leq 1\colon s \in [-T,T]\right) \leq T^{-1}e^{+k\sqrt{\log(T)}},$$

for some $k > 0$ and T large enough. Our first result treats the discrete-time analog. The technique we use to prove the theorem is entirely different from the one in [61]. We further emphasize that, unlike in the continuous-time result above, our discrete-time result gives polynomial bounds without lower order terms.

Theorem 2.1. *Let $(\xi_n)_{n\in\mathbb{Z}}$ be a real-valued stationary centered Gaussian sequence such that (2.1) holds. Then there is a constant $c > 0$ such that, for every $N \geq 1$,*

$$c^{-1}N^{-1} \leq \mathbb{P}\left(S_n \leq 0\colon -N \leq n \leq N\right) \leq N^{-1}.$$

In order to prove the corresponding result for the process $(I_n)_{n\in\mathbb{Z}}$, we will use a change of measure argument. This argument requires an additional assumption. For this purpose, let μ denote the spectral measure of the sequence $(\xi_n)_{n\in\mathbb{Z}}$, i.e.,

$$\mathbb{E}\xi_j\xi_k =: \int_{(-\pi,\pi]} e^{i|j-k|u}\,\mathrm{d}\mu(u).$$

The spectral measure μ has a (possibly vanishing) component that is absolutely continuous with respect to the Lebesgue measure. Let us denote by p its density, i.e., $\mathrm{d}\mu(u) =: p(u)\,\mathrm{d}u + \mathrm{d}\mu_s(u)$. We will assume that p satisfies

$$p(u) \sim \ell(1/u)|u|^{1-2H}, \quad \text{as} \quad u \to 0, \tag{2.5}$$

where ℓ is a slowly varying function at infinity. It is well-known that (2.5) implies (2.1) and thus (2.2).

To understand the nature of this assumption, we consider the fractional Gaussian noise process $(\xi_n^{\text{FGN}})_{n\in\mathbb{Z}}$, defined by $\xi_n^{\text{FGN}} := W_H(n) - W_H(n-1)$. This stationary centered Gaussian sequence has an absolutely continuous spectral measure with density function p_{FGN} that satisfies (see e.g. [71])

$$p_{\text{FGN}}(u) \sim m_H |u|^{1-2H}, \quad \text{as} \quad u \to 0,$$

where $m_H = \Gamma(2H+1)\sin(\pi H)/2\pi$. So, we assume that the density of the absolutely continuous part of the spectral measure of the stationary process $(\xi_n)_{n\in\mathbb{Z}}$ is comparable to the spectral density of fractional Gaussian noise, up to the slowly varying function ℓ.

We are now ready to state our second main result.

Theorem 2.2. *Let $(\xi_n)_{n\in\mathbb{Z}}$ be a real-valued stationary centered Gaussian sequence such that (2.5) holds. Then,*

$$\mathbb{P}\left(I_n \le 0\colon\ -N \le n \le N\right) = N^{-(1-H)+o(1)}.$$

We recall that [62] considers the continuous-time case. There, it is shown that

$$\mathbb{P}\left(I_H(s) \le 1\colon s \in [-T, T]\right) = T^{-(1-H)+o(1)}.$$

Many arguments from that paper can be adapted to our setup. However, for instance, arguments using self-similarity need to be replaced by new ideas. Furthermore, new results concerning the change of measure are needed and may be of independent interest.

For example, as a byproduct of the change of measure techniques, we can improve Theorem 11 in [14], where the persistence problem of the one-sided discrete-time analog of FBM is considered. There, it is shown that for every real-valued stationary centered Gaussian sequence $(\xi_n)_{n\in\mathbb{N}}$ such that (2.1) holds and every $a > 0$, there is some constant $c > 0$ such that

$$\begin{aligned} c^{-1} N^{-(1-H)} \frac{\sqrt{\ell(N)}}{\sqrt{\log(N)}} \le \mathbb{P}\left(S_n < 0\colon 1 \le n \le N\right) \quad \text{and} \\ \mathbb{P}\left(S_n < -a\colon 1 \le n \le N\right) \le c N^{-(1-H)} \sqrt{\ell(N)}. \end{aligned} \tag{2.6}$$

Thus, one has a lower bound for the probability $\mathbb{P}\left(S_n < b\colon 1 \le n \le N\right)$, if b is non-negative, and an upper bound, if b is negative. In order to get both, a lower estimate and an upper estimate, for any arbitrary $b \in \mathbb{R}$, [14] uses a change of measure argument. To get this argument to work, a strong assumption on the covariance function of $(S_n)_{n\in\mathbb{N}}$ is made; namely $\inf_{n\ge 1} \mathbb{E} S_1 S_n > 0$ (see also our Remark 2.13 below). We are able to prove upper and lower bounds whenever (2.5) is satisfied. We state this result as Corollary 2.12 below.

2.2 Preliminaries

2.2.1 Basic properties of FBM and IFBM

Let $(W_H(t))_{t\in\mathbb{R}}$ be a FBM with Hurst parameter $0 < H < 1$ and $(I_H(t))_{t\in\mathbb{R}}$ an IFBM. We recall that the process $(W_H(t))_{t\in\mathbb{R}}$ has stationary increments. But, unlike $(W_H(t))_{t\in\mathbb{R}}$, the process $(I_H(t))_{t\in\mathbb{R}}$ does not have stationary increments. Instead, the process satisfies for all $t_0 \in \mathbb{R}$

$$(I_H(t+t_0) - I_H(t_0) - tW_H(t_0))_{t\in\mathbb{R}} \stackrel{d}{=} (I_H(t))_{t\in\mathbb{R}} .$$

In the discrete-time setup, we have analogous properties, which we state in the following lemma.

Lemma 2.3. *Let $(\xi_n)_{n\in\mathbb{Z}}$ be a real-valued stationary sequence, then*

$$(S_{n_0+n} - S_{n_0})_{n\in\mathbb{Z}} \stackrel{d}{=} (S_n)_{n\in\mathbb{Z}} \quad \textit{for } n_0 \in \mathbb{Z}. \tag{2.7}$$

Further, one has

$$(I_{n_0+n} - I_{n_0} - nS_{n_0})_{n\in\mathbb{Z}} \stackrel{d}{=} (I_n)_{n\in\mathbb{Z}} \quad \textit{for } n_0 \in \mathbb{Z}. \tag{2.8}$$

Proof. By the definition of the process $(S_n)_{n\in\mathbb{Z}}$ in (2.3), we have for $n_0 \in \mathbb{Z}$

$$S_{n_0+n} - S_{n_0} = \begin{cases} \sum_{k=1}^{n} \xi_{n_0+k}, & \text{if } n > 0, \\ -\sum_{k=n+1}^{0} \xi_{n_0+k}, & \text{if } n < 0, \\ 0, & \text{if } n = 0. \end{cases}$$

In particular, we have

$$S_n = \begin{cases} \sum_{k=1}^{n} \xi_k, & \text{if } n > 0, \\ -\sum_{k=n+1}^{0} \xi_k, & \text{if } n < 0, \\ 0, & \text{if } n = 0. \end{cases} \tag{2.9}$$

Now, using that $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary sequence, (2.7) follows. Further, by the definition of the process $(I_n)_{n\in\mathbb{Z}}$ in (2.4), we obtain for $n_0 \in \mathbb{Z}$ that

$$\begin{aligned} I_{n_0+n} - I_{n_0} - nS_{n_0} &= \begin{cases} \sum_{k=1}^{n} \frac{S_{n_0+k}+S_{n_0+k-1}}{2} - nS_{n_0}, & \text{if } n > 0, \\ -\sum_{k=n+1}^{0} \frac{S_{n_0+k}+S_{n_0+k-1}}{2} - nS_{n_0}, & \text{if } n < 0, \\ 0, & \text{if } n = 0. \end{cases} \\ &= \begin{cases} \sum_{k=1}^{n} \frac{(S_{n_0+k}-S_{n_0})+(S_{n_0+k-1}-S_{n_0})}{2}, & \text{if } n > 0, \\ -\sum_{k=n+1}^{0} \frac{(S_{n_0+k}-S_{n_0})+(S_{n_0+k-1}-S_{n_0})}{2}, & \text{if } n < 0, \\ 0, & \text{if } n = 0. \end{cases} \end{aligned}$$

Here, we used that $\sum_{k=n+1}^{0} 1 = |n|$ for $n < 0$. Since $S_0 = 0$, one has particularly

$$I_n = \begin{cases} \sum_{k=1}^{n} \frac{S_k+S_{k-1}}{2}, & \text{if } n > 0, \\ -\sum_{k=n+1}^{0} \frac{S_k+S_{k-1}}{2}, & \text{if } n < 0, \\ 0, & \text{if } n = 0. \end{cases} \tag{2.10}$$

Thus, using (2.7), (2.8) follows. □

In the next lemma, we recall another useful identity for stationary increments sequences.

Lemma 2.4. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a real-valued stationary sequence, then*

$$\mathbb{E}S_jS_k = \frac{1}{2}\left(\mathbb{E}S_j^2 + \mathbb{E}S_k^2 - \mathbb{E}S_{|j-k|}^2\right) \quad \textit{for} \quad j,k \in \mathbb{N}.$$

Proof. We have, for $j < k$,

$$\begin{aligned} S_jS_k &= (S_k - (S_k - S_j))\, S_k = S_k^2 - (S_k - S_j)S_k \\ &= S_k^2 - (S_k - S_j)\,((S_k - S_j) + S_j) = S_k^2 - (S_k - S_j)^2 - S_kS_j + S_j^2. \end{aligned}$$

Thus, using (2.7) in the second step, we obtain

$$2\mathbb{E}S_jS_k = \mathbb{E}S_k^2 + \mathbb{E}S_j^2 - \mathbb{E}(S_k - S_j)^2 = \mathbb{E}S_j^2 + \mathbb{E}S_k^2 - \mathbb{E}S_{|j-k|}^2,$$

which shows the claim. □

2.2.2 Inequalities for centered Gaussian processes

Now, we will collect some useful inequalities for centered Gaussian processes. One of them is based on Slepian's lemma which is usually stated as follows (see e.g. Theorem 2.4.8 in [43]):

Let $Y = (Y(1), \ldots, Y(d))$ and $Z = (Z(1), \ldots, Z(d))$ be centered jointly normal vectors in $\mathbb{R}^d$, respectively, such that, for $t_1, t_2 \in \mathbb{T}$,

$$\mathbb{E}\left[Y(t_1)Y(t_2)\right] \le \mathbb{E}\left[Z(t_1)Z(t_2)\right] \quad \text{and} \quad \mathbb{E}\left[Y(t_1)^2\right] = \mathbb{E}\left[Z(t_1)^2\right]. \tag{2.11}$$

Then, one has

$$\mathbb{P}\left(Z(t) \le f(t) \colon t \in \mathbb{T}\right) \ge \mathbb{P}\left(Y(t) \le f(t) \colon t \in \mathbb{T}\right). \tag{2.12}$$

Lemma 2.5. *Let $(X(t))_{t\in\mathbb{T}}$ be some centered Gaussian process with $\mathbb{T}$ being countable and $\mathbb{E}[X(t_1)X(t_2)] \ge 0$ for $t_1, t_2 \in \mathbb{T}$. Let $f\colon \mathbb{T} \to \mathbb{R}$ be a function and $I \subset \mathbb{T}$. Then,*

$$\mathbb{P}\left(X(t) \le f(t) \colon t \in \mathbb{T}\right) \ge \mathbb{P}\left(X(t) \le f(t) \colon t \in I\right) \cdot \mathbb{P}\left(X(t) \le f(t) \colon t \in \mathbb{T} \setminus I\right).$$

Proof. First, let us consider the case $\mathbb{T} = \{1, \dots, d\}$. Now, we define the process $(Z(t))_{t\in\mathbb{T}} := (X(t))_{t\in\mathbb{T}}$. Further, let $(X'(t))_{t\in\mathbb{T}}$ be an independent copy of $(X(t))_{t\in\mathbb{T}}$. Then, we define $(Y(t))_{t\in\mathbb{T}}$ by

$$Y(t) := \begin{cases} X(t), & \text{if } t \in I, \\ X'(t), & \text{if } t \in \mathbb{T} \setminus I. \end{cases}$$

Using the non-negative correlations, the inequality in (2.11) is satisfied by $(Y(t))_{t\in\mathbb{T}}$ and $(Z(t))_{t\in\mathbb{T}}$. By the definition of $(Y(t))_{t\in\mathbb{T}}$ and $(Z(t))_{t\in\mathbb{T}}$, the equation in (2.11) holds. Thus, by (2.12) and the independence of $(X(t))_{t\in\mathbb{T}}$ and $(X'(t))_{t\in\mathbb{T}}$, it follows that

$$\begin{aligned} &\mathbb{P}\left(X(t) \le f(t) \colon t \in \mathbb{T}\right) \\ &\qquad \ge \mathbb{P}\left(\{X(t) \le f(t) \colon t \in I\} \cap \{X'(t) \le f(t) \colon t \in \mathbb{T} \setminus I\}\right) \\ &\qquad = \mathbb{P}\left(X(t) \le f(t) \colon t \in I\right) \cdot \mathbb{P}\left(X(t) \le f(t) \colon t \in \mathbb{T} \setminus I\right). \end{aligned}$$

Now, the statement of the lemma follows by the continuity property of the probability measure $\mathbb{P}$. □

Let us now recall the definition of the reproducing kernel Hilbert space (RKHS) of a centered Gaussian process $(X(t))_{t\in\mathbb{T}}$. For this purpose, let $\mathbb{H}$ denote the L^2-closure of the set $\operatorname{span}\{X(t) : t \in \mathbb{T}\}$. Then the RKHS $\mathcal{H}$ of $(X(t))_{t\in\mathbb{T}}$ is the Hilbert space of functions

$$\mathbb{T} \ni t \mapsto \mathbb{E}\left[X(t)h\right], \quad h \in \mathbb{H},$$

with inner product $\langle \mathbb{E}[Xh_1], \mathbb{E}[Xh_2] \rangle_{\mathcal{H}} = \mathbb{E}[h_1 h_2]$.

The following version of Proposition 1.6 in [13], will be an important tool throughout this chapter.

Proposition 2.6. *Let $(X(t))_{t\in\mathbb{T}}$ be some centered Gaussian process with RKHS $\mathcal{H}$ and $\mathbb{T}$ being countable. Denote by $\|\cdot\|_{\mathcal{H}}$ the norm in $\mathcal{H}$. Then, for each $f \in \mathcal{H}$ and each measurable S such that $\mathbb{P}(X \in S) \in (0,1)$, we have*

$$e^{-\sqrt{2\|f\|_{\mathcal{H}}^2 \log(1/\mathbb{P}(X\in S))} - \|f\|_{\mathcal{H}}^2/2}\, \mathbb{P}(X \in S) \le \mathbb{P}(X + f \in S). \tag{2.13}$$

If $\|f\|^2 < 2\log(1/\mathbb{P}(X \in S))$, we have in addition

$$\mathbb{P}(X + f \in S) \le e^{\sqrt{2\|f\|_{\mathcal{H}}^2 \log(1/\mathbb{P}(X\in S))} - \|f\|_{\mathcal{H}}^2/2}\, \mathbb{P}(X \in S). \tag{2.14}$$

Remark 2.7. Proposition 1.6 in [13] is more general than Proposition 2.6 since there centered Gaussian processes attaining values in a general Banach space are considered. Due to the assumption that $\mathbb{T}$ is countable, Proposition 2.6 can be applied without verifying that the codomain of the process is a Banach space. Further, we mention that the proof of Proposition 1.6 in [13] fails if $\|f\|^2 \ge 2\log(1/\mathbb{P}(X \in S))$. Thus, unlike in [13], we have excluded this case here. In the applications of this proposition that we know of, the function $f \in \mathcal{H}$ is fixed and one is interested in the asymptotic behavior of the probabilities $\mathbb{P}\left(X \in S^{(N)}\right)$, as $N \to \infty$, where $(S^{(N)})_{N\in\mathbb{N}}$ is a sequence of measurable sets such that $\lim_{N\to\infty} \mathbb{P}\left(X \in S^{(N)}\right) = 0$. In this case, the condition is satisfied for N large enough. Hence, Proposition 1.6 in [13] can be applied in the same way as before.

Sketch of proof of Proposition 2.6. We prove the statement along the same lines as in the proof of Proposition 1.6 in [13]. The main difference is that, since $\mathbb{T}$ is countable here, we can restrict ourselves first to the finite-dimensional case. Hence, the present approach can be easily understood without knowledge on RKHSs, and we do not need to apply the general Cameron-Martin formula with its technical conditions (see e.g. Theorem 9.3 in [53]).

Thus, let us first assume that $|\mathbb{T}| < \infty$. Without loss of generality, we assume that $\mathbb{T} = \{1, 2, \dots, d\}$ and X is a centered d-dimensional non-degenerate normal distribution with covariance matrix K. Then, X has density

$$\frac{1}{\sqrt{(2\pi)^d \det(K)}} \exp\left(-\frac{1}{2} x^{\mathrm{T}} K^{-1} x\right).$$

Let $f \in \mathcal{H}$, then we can regard f as a vector in $\mathbb{R}^d$. Now, we choose a vector $\lambda \in \mathbb{R}^d$ such that $f = K\lambda$. Then, one has

$$f(t) = \sum_{k=1}^{d} K_{t,k}\lambda_k = \sum_{k=1}^{d} \mathbb{E}[X_t X_k]\lambda_k = \mathbb{E}[X_t h],$$

where $h = \lambda^{\mathrm{T}} X = \sum_{k=1}^{d} \lambda_k X_k$. Further, h is a centered normally distributed random variable and

$$\|f\|_{\mathcal{H}}^2 = \|h\|_2^2 = \left\|\sum_{k=1}^{n} \lambda_k X_k\right\|_2^2 = \sum_{i,j=1}^{d} \lambda_i \lambda_j \mathbb{E}[X_i X_j] = \sum_{i,j=1}^{d} \lambda_i \lambda_j K_{i,j} = \lambda^{\mathrm{T}} K \lambda.$$

Let S be a Borel set in $\mathbb{R}^d$. Then, using that $\lambda = K^{-1} f$, we obtain

$$\begin{aligned}
&\mathbb{P}(X + f \in S) \\
&= \int_{S-f} \frac{1}{\sqrt{(2\pi)^d \det(K)}} \exp\left(-\frac{1}{2} x^{\mathrm{T}} K^{-1} x\right) \mathrm{d}x \\
&= \int_{S} \frac{1}{\sqrt{(2\pi)^d \det(K)}} \exp\left(-\frac{1}{2} (x-f)^{\mathrm{T}} K^{-1} (x-f)\right) \mathrm{d}x \\
&= \int_{S} \frac{1}{\sqrt{(2\pi)^d \det(K)}} \exp\left(-\frac{1}{2} x^{\mathrm{T}} K^{-1} x + \lambda^{\mathrm{T}} x\right) \cdot \exp\left(-\frac{1}{2} \lambda^{\mathrm{T}} K \lambda\right) \mathrm{d}x \\
&= \mathbb{E}\left[\mathbb{1}_{X\in S} \exp\left(\lambda^{\mathrm{T}} X\right)\right] \exp\left(-\frac{1}{2} \|f\|_{\mathcal{H}}^2\right).
\end{aligned}$$

Now, using Hölder's inequality, we obtain for any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$
\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{X \in S} \exp\left(\lambda^{\mathrm{T}} X\right)\right] &\leq \mathbb{P}\left(X \in S\right)^{\frac{1}{p}} \mathbb{E}\left[e^{q\lambda^{\mathrm{T}} X}\right]^{\frac{1}{q}} \\
&= \mathbb{P}\left(X \in S\right)^{\frac{1}{p}} \mathbb{E}\left[e^{qh}\right]^{\frac{1}{q}} \\
&= \mathbb{P}\left(X \in S\right)^{\frac{1}{p}} e^{\frac{1}{2}q\|h\|_2^2} \\
&= \mathbb{P}\left(X \in S\right)^{\frac{1}{p}} e^{\frac{1}{2}q\|f\|_{\mathcal{H}}^2} \\
&= \mathbb{P}\left(X \in S\right) e^{\frac{1}{q}\log(1/\mathbb{P}(X \in S)) + \frac{1}{2}q\|f\|_{\mathcal{H}}^2}. \qquad (2.15)
\end{aligned}
$$

A simple calculation yields that the choice $q = \sqrt{2\log(1/\mathbb{P}(X \in S))/\|f\|_{\mathcal{H}}^2}$ minimizes the upper bound in (2.15). (We emphasize that, since we require $q > 1$, the condition $2\log(1/\mathbb{P}(X \in S))/\|f\|_{\mathcal{H}}^2 > 1$ must be satisfied. The proof of Proposition 1.6 in [13] fails at this point, see Remark 2.7.) Altogether, we thus obtain

$$
\begin{aligned}
&\mathbb{P}\left(X + f \in S\right) \\
&\leq \mathbb{P}\left(X \in S\right) e^{\frac{\log(1/\mathbb{P}(X \in S))}{\sqrt{2\log(1/\mathbb{P}(X \in S))/\|f\|_{\mathcal{H}}^2}} + \frac{1}{2}\sqrt{2\log(1/\mathbb{P}(X \in S))/\|f\|_{\mathcal{H}}^2}\|f\|_{\mathcal{H}}^2} e^{-\|f\|_{\mathcal{H}}^2/2} \\
&= \mathbb{P}\left(X \in S\right) e^{\sqrt{2\|f\|_{\mathcal{H}}^2 \log(1/\mathbb{P}(X \in S))} - \|f\|_{\mathcal{H}}^2/2}.
\end{aligned}
$$

Analogously, using the reverse Hölder inequality, one gets the lower bound.

Now, let $(X(t))_{t \in \mathbb{T}}$ be a centered Gaussian process with arbitrary (countable) $\mathbb{T} = \{t_1, t_2, \ldots\}$ and let $f \in \mathcal{H}$ be a function in its RKHS. Then, there is a function $h \in \mathbb{H}$ such that

$$f(t) = \mathbb{E}[X(t)h], \quad t \in \mathbb{T}.$$

Since $\mathbb{H}$ is the L^2-closure of the set $\operatorname{span}\{X(t) : t \in \mathbb{T}\}$, we can choose a sequence $(h_n)_{n \in \mathbb{N}} \subset \operatorname{span}\{X(t) : t \in \mathbb{T}\}$ with $\lim_{n \to \infty} \|h_n - h\|_2 = 0$. Let $f_n \in \mathcal{H}$ denote the function in the RKHS of $(X(t))_{t \in \mathbb{T}}$ corresponding to h_n. Then, by the Cauchy-Schwarz inequality, we have, for all $t \in \mathbb{T}$,

$$|f(t) - f_n(t)|^2 = |\mathbb{E}[X(t)(h - h_n)]|^2 \leq \mathbb{E}[X(t)^2]\|h - h_n\|_2^2 \to 0, \qquad (2.16)$$

as $n \to \infty$. Thus, for a set

$$S = \{x = (x_{t_1}, x_{t_2}, \ldots) \in \mathbb{R}^{\mathbb{T}} \colon (x_{t_1}, \ldots, x_{t_d}) \in B\} \qquad (2.17)$$

with $B \in \mathcal{B}(\mathbb{R}^d)$ being open, we obtain, only using (2.16), that

$$\lim_{n \to \infty} \mathbb{P}\left(X + f_n \in S\right) = \mathbb{P}\left(X + f \in S\right).$$

Applying (2.13) and (2.14) for f_n in the already shown finite dimensional case and using then $\lim_{n\to\infty} \|f_n\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}$, we obtain the statement for all $f \in \mathcal{H}$. Further, the statement holds for all sets S in (2.17) with $B \in \mathcal{B}(\mathbb{R}^d)$ since such sets can be approximated from above by open sets. Now, let S be a set in the generated product σ-algebra of $(X(t))_{t\in\mathbb{T}}$. Then, we consider the sets

$$S^{(n)} := \{x = (x_{t_1}, x_{t_2}, \dots) \in \mathbb{R}^{\mathbb{T}} \colon (x_{t_1}, \dots, x_{t_n}) \in S\}$$

and let $n \to \infty$. By the continuity property of $\mathbb{P}$, the statement follows. □

In the following, we show the existence of functions in the RKHSs of the considered processes with certain asymptotic behavior. These functions will be used in several applications of Proposition 2.6.

Proposition 2.8. *Let $H \in (0,1)$, $\rho \in (-1, H-1)$ and let $\mathcal{H}_H(\xi)$ denote the RKHS of the process $(\xi_n)_{n\in\mathbb{Z}}$. Then, if (2.5) is satisfied, there is an even function $f_\xi \in \mathcal{H}_H(\xi)$ such that $f_\xi > 0$ and $f_\xi(n) \sim n^\rho$, as $n \to \infty$.*

Proof. Recall that $f_\xi \in \mathcal{H}_H(\xi)$ if and only if there is a function $\varphi \in L^2(\mu)$ with $f_\xi(n) = \int_{(-\pi,\pi]} \varphi(u) e^{-inu} \,\mathrm{d}\mu(u)$, see e.g. Comment 2.2.2 (c) in [6]. In order to prove the proposition, we will first consider a function $\varphi_1 \in L^2(\mu)$ such that the corresponding function $f_1 \in \mathcal{H}_H(\xi)$ has the correct asymptotic behavior. This function can attain non-positive values at finitely many times. To fix this, we will construct afterward another function $\varphi_2 \in L^2(\mu)$ such that the corresponding function $f_2 \in \mathcal{H}_H(\xi)$ is non-negative, takes positive values when f_1 takes non-positive values and decays faster than f_1. Then, for suitable constants $c_1, c_2 > 0$, the function $f_\xi := c_1 f_1 + c_2 f_2$ has the required properties.

Construction of f_1: Due to (2.5), there is a function $\tilde{\ell}$ and a constant $u_0 > 0$ such that $p(u) = \tilde{\ell}(u)|u|^{1-2H}$ for $u \in [-u_0, u_0]$ and $\tilde{\ell}$ is slowly varying at zero. By Potter's theorem, see Theorem 1.5.6 in [21], u_0 can be chosen such that $\tilde{\ell}(u_0)/\tilde{\ell}(u) \le A\left(\frac{|u|}{u_0}\right)^{-\delta}$ for $|u| < u_0$, fixed $A > 1$, and fixed $0 < \delta < 2(H-1-\rho)$ with $\rho \in (-1, H-1)$. We set

$$\varphi_1(u) := \begin{cases} |u|^{2H-2-\rho}/\tilde{\ell}(u), & u \in [-u_0, u_0] \cap \operatorname{supp}(\mu_s)^C, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\varphi_1 \in L^2(\mu)$ because

$$\begin{aligned}\int_{(-\pi,\pi]} |\varphi_1(u)|^2 \,\mathrm{d}\mu(u) &= \int_{-u_0}^{u_0} \frac{|u|^{4H-4-2\rho}}{\tilde{\ell}^2(u)} \tilde{\ell}(u)|u|^{1-2H} \,\mathrm{d}u \\ &= \int_{-u_0}^{u_0} \frac{|u|^{2H-3-2\rho}}{\tilde{\ell}(u)} \,\mathrm{d}u \\ &= \frac{1}{\tilde{\ell}(u_0)} \int_{-u_0}^{u_0} \frac{\tilde{\ell}(u_0)}{\tilde{\ell}(u)} |u|^{2H-3-2\rho} \,\mathrm{d}u \\ &\le \frac{A}{\tilde{\ell}(u_0)} \int_{-u_0}^{u_0} \left(\frac{|u|}{u_0}\right)^{-\delta} |u|^{2H-3-2\rho} \,\mathrm{d}u < \infty.\end{aligned}$$

Here we used that $2H - 3 - 2\rho - \delta > -1$. Moreover,

$$\begin{aligned}\int_{(-\pi,\pi]} \cos(nu)\varphi_1(u) \,\mathrm{d}\mu(u) &= \int_{-u_0}^{u_0} \cos(nu) \frac{|u|^{2H-2-\rho}}{\tilde{\ell}(u)} \tilde{\ell}(u)|u|^{1-2H} \,\mathrm{d}u \\ &= \int_{-u_0}^{u_0} \cos(nu)|u|^{-\rho-1} \,\mathrm{d}u \\ &= n^{\rho} \int_{-nu_0}^{nu_0} \cos(v)|v|^{-\rho-1} \,\mathrm{d}v \\ &= 2n^{\rho} \int_{0}^{nu_0} \cos(v)|v|^{-\rho-1} \,\mathrm{d}v.\end{aligned}$$

Since $-\rho - 1 < 0$, it is easy to show, using the Leibniz criterion and the concavity of $(\cdot)^{-\rho-1}$, that the latter integral converges to a constant $c/2 > 0$, as $n \to \infty$. Thus,

$$f_1(n) = \int_{(-\pi,\pi]} \varphi_1(u) e^{-inu} \,\mathrm{d}\mu(u) \sim cn^{\rho}, \quad \text{as} \quad n \to \infty.$$

Construction of f_2: Choose n_0 such that f_1 attains only positive values for $|n| > n_0$. Let $g \in C^1$ be an even real-valued function with support contained in $[-u_0/2, u_0/2]$ such that its Fourier coefficients for $|n| \le n_0$ do not vanish, e.g. take any smooth even function g with $g(u) > 0$ for $|u| < \min(u_0/2, \pi/(2n_0))$ and $g(u) = 0$ otherwise. Then, the function G, given by $G(u) := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v)\overline{g}(u - v) \,\mathrm{d}v$, has Fourier coefficients $\hat{G}_n = |\hat{g}_n|^2$. In particular $\hat{G}_n > 0$ for $|n| \le n_0$. Moreover, $G \in C^2$ because G is the convolution of two differentiable functions. Thus, we have

$$0 \le \hat{G}_n = \frac{1}{(in)^2} (\widehat{G''})_n \le \frac{\sup_{x\in(-\pi,\pi]} |G''(x)|}{|n|^2} \quad \text{for } n \in \mathbb{Z} \setminus \{0\}.$$

Now, we consider the function

$$\varphi_2(u) := \begin{cases} \frac{G(u)}{|u|^{1-2H}\tilde{\ell}(u)}, & u \in [-u_0, u_0] \cap \operatorname{supp}(\mu_s)^C, \\ 0, & \text{otherwise.} \end{cases}$$

Let M denote the maximum of G, then

$$\begin{aligned}\int_{(-\pi,\pi]} |\varphi_2(u)|^2 \,\mathrm{d}\mu(u) &= \int_{-u_0}^{u_0} \frac{G(u)^2}{|u|^{2-4H}\tilde{\ell}(u)} \tilde{\ell}(u)|u|^{1-2H}\,\mathrm{d}u \\ &\le \int_{-u_0}^{u_0} \frac{M^2}{|u|^{1-2H}\tilde{\ell}(u)}\,\mathrm{d}u \\ &= \frac{1}{\tilde{\ell}(u_0)} \int_{-u_0}^{u_0} \frac{\tilde{\ell}(u_0)}{\tilde{\ell}(u)} \frac{M^2}{|u|^{1-2H}}\,\mathrm{d}u \\ &\le \frac{A}{\tilde{\ell}(u_0)} \int_{-u_0}^{u_0} \left(\frac{|u|}{u_0}\right)^{-\delta} \frac{M^2}{|u|^{1-2H}}\,\mathrm{d}u < \infty,\end{aligned}$$

since $2H-1-\delta > -1$. Furthermore, we have by the construction of φ_2 that

$$f_2(n) = \int_{(-\pi,\pi]} \varphi_2(u)e^{-inu}\,\mathrm{d}\mu(u) = \int_{-\pi}^{\pi} G(u)e^{-inu}\,\mathrm{d}u = \hat{G}_n.$$

□

As a corollary of Proposition 2.8, we show the existence of functions with certain asymptotic behavior in the RKHSs of $(S_n)_{n\in\mathbb{Z}}$ and $(I_n)_{n\in\mathbb{Z}}$.

Corollary 2.9. *Let $H \in (0,1)$ and $\rho \in (-1, H-1)$. Let $\mathcal{H}_H(S)$ and $\mathcal{H}_H(I)$ denote the RKHSs of the processes $(S_n)_{n\in\mathbb{Z}}$ and $(I_n)_{n\in\mathbb{Z}}$, respectively. Then, if (2.5) is satisfied, there is a function $f_S \in \mathcal{H}_H(S)$ such that $f_S(n) > 0$, $f_S(-n) < 0$ for $n > 0$, $f_S(0) = 0$, and*

$$-f(-n) \sim f_S(n) \sim n^{\rho+1}, \quad \text{as} \quad n \to \infty.$$

Further, there is a function $f_I \in \mathcal{H}_H(I)$ such that $f_I(n) > 0$ for $n \in \mathbb{Z}\setminus\{0\}$, $f_I(0) = 0$, and

$$f_I(-n) \sim f_I(n) \sim n^{\rho+2}, \quad \text{as} \quad n \to \infty.$$

Proof. Let $f_\xi \in \mathcal{H}_H(\xi)$ be the positive and even function in Proposition 2.8 with $f_\xi(n) \sim n^\rho$, as $n \to \infty$. Then, by the definition of the RKHS, there is a random variable h in the L^2-closure of the set $\operatorname{span}\{\xi_n : n \in \mathbb{Z}\}$ such that $f_\xi(n) = \mathbb{E}[\xi_n h]$. Now, let f_S and f_I be given by $f_S(n) = (\rho+1)\mathbb{E}[S_n h]$ and $f_I(n) = (\rho+1)(\rho+2)\mathbb{E}[I_n h]$, respectively. Since the sets $\operatorname{span}\{\xi_n : n \in \mathbb{Z}\}$, $\operatorname{span}\{S_n : n \in \mathbb{Z}\}$, and $\operatorname{span}\{I_n : n \in \mathbb{Z}\}$ coincide, we have $f_S \in \mathcal{H}_H(S)$ and $f_I \in \mathcal{H}_H(I)$. By $f_\xi(n) \sim n^\rho$, as $n \to \infty$, and (2.9), we have

$$f_S(n) = (\rho+1)\sum_{k=1}^{n} f_\xi(k) \sim n^{\rho+1}, \quad \text{as} \quad n \to \infty.$$

For $n < 0$, we proceed analogously. Thus, by the same argument and (2.10), we obtain further that

$$f_I(-n) \sim f_I(n) = (\rho+1)(\rho+2)\left(\sum_{k=1}^{n-1} \mathbb{E}\left[S_k h\right] + \mathbb{E}\left[S_n h\right]/2\right) \sim n^{\rho+2},$$

as $n \to \infty$. $\square$

Remark 2.10. In Corollary 2.9, the functions $f_S \in \mathcal{H}_H(S)$ and $f_I \in \mathcal{H}_H(I)$ can be chosen such that f_S is odd with $f_S(n) > 0$ for $n > 0$ and $f_S(n) \sim n^{\rho+1}$, as $n \to \infty$, and f_I is even and positive on $\mathbb{Z} \setminus \{0\}$ with $f_I(n) \sim n^{\rho+2}$, as $n \to \infty$. To obtain this slightly stronger result, let h be the random variable in the L^2-closure of the set $\operatorname{span}\{\xi_n \colon n \in \mathbb{Z}\}$ such that $f_\xi(n) = \mathbb{E}\left[\xi_n h\right]$. Clearly, one can also choose a random variable h' in this L^2-closure such that $f_\xi(n-1) = \mathbb{E}\left[\xi_n h'\right]$. Now, considering the random variable $(h+h')/2$ instead of h in the proof of Corollary 2.9, the existence of an odd function f_S with the desired properties follows. Analogously, one obtains the result for f_I.

2.2.3 Examples and comments

As a first application of Corollary 2.9, we compare the persistence probabilities of $(I_n)_{n\in\mathbb{Z}}$ to a closely related process. Let $(\bar{I}_n)_{n\in\mathbb{Z}}$ be the sequence given by $\bar{I}_n - \bar{I}_{n-1} := S_n$ for $n \in \mathbb{Z}$ and $\bar{I}_0 := 0$. This process is related to the process $(I_n)_{n\in\mathbb{Z}}$ by the identity $\bar{I}_n = I_n + S_n/2$ for $n \in \mathbb{Z}$. Both processes are defined as integrals of stationary increments sequences that have FBM as scaling limit. In our context, the major difference between these processes is that $(I_n)_{n\in\mathbb{Z}}$ vanishes only at 0 and satisfies $(I_n)_{n\in\mathbb{Z}} \stackrel{d}{=} (I_{-n})_{n\in\mathbb{Z}}$, whereas $\bar{I}_{-1} = \bar{I}_0 = 0$ and $\bar{I}_1$ does not vanish. The symmetry property of $(I_n)_{n\in\mathbb{Z}}$ resembles the continuous-time case and is needed in the proof of Theorem 2.2. In the following corollary, we relate the persistence probabilities of both processes.

Corollary 2.11. *Let $(\xi_n)_{n\in\mathbb{Z}}$ be a real-valued stationary centered Gaussian sequence such that (2.5) holds. Then,*

$$\mathbb{P}\left(\bar{I}_n \le 0\colon\ -N-1 \le n \le N\right) \le \mathbb{P}\left(I_n \le 0\colon |n| \le N\right).$$

If in addition $\mathbb{E}\left[\bar{I}_j \bar{I}_k\right] \ge 0$ for all $j,k \in \mathbb{Z}$, then one has

$$\mathbb{P}\left(I_n \le 0\colon |n| \le N\right) \le \mathbb{P}\left(\bar{I}_n \le 0\colon |n| \le N\right) \ell_0(N),$$

where ℓ_0 denotes a slowly varying function at infinity.

Proof. The first inequality follows directly from the definitions of the processes since one has $I_n = (\bar{I}_n + \bar{I}_{n-1})/2$ for all $n \in \mathbb{Z}$. Using Slepian's lemma (see Lemma 2.5) and the additional assumption on the correlations of $(\bar{I}_n)_{n\in\mathbb{Z}}$, we obtain

$$\begin{aligned}\mathbb{P}\left(\bar{I}_n \le 0\colon |n| \le N\right) \ge & \,\mathbb{P}\left(\bar{I}_n \le 0\colon |n| \le \log(N)\right) \\ & \cdot \mathbb{P}\left(\bar{I}_n \le 0\colon \log(N) < |n| \le N\right).\end{aligned} \tag{2.18}$$

By the same argument in the first step and Theorem 2.1 in the third step, we have

$$\begin{aligned}\mathbb{P}\left(\bar{I}_n \le 0\colon |n| \le \log(N)\right) \ge & \,\mathbb{P}\left(\bar{I}_n \le 0\colon 0 \le n \le \log(N)\right) \\ & \cdot \mathbb{P}\left(\bar{I}_n \le 0\colon -\log(N) \le n < 0\right) \\ \ge & \,\mathbb{P}\left(S_n \le 0\colon 0 \le n \le \log(N)\right) \\ & \cdot \mathbb{P}\left(S_n \ge 0\colon -\log(N) \le n < 0\right) \\ \ge & \,c^{-2}\log(N)^{-2}.\end{aligned}$$

Thus, the first factor on the right-hand side in (2.18) can be estimated by a slowly varying function at infinity. It remains to relate the second factor on the right-hand side in (2.18) to the probability $\mathbb{P}\left(I_n \le 0\colon |n| \le N\right)$.

By Corollary 2.9, for $\varepsilon \in (0, 1/4)$, a function $f \in \mathcal{H}_H(I)$ exists such that $f(n) \ge |n|^{1+H-\varepsilon}$ for all $n \in \mathbb{Z}$. Obviously, we have

$$\begin{aligned}&\mathbb{P}\left(I_n \le -n^{1+H-\varepsilon}\colon \log(N) < |n| \le N\right) \\ &\quad \le \mathbb{P}\left(\bar{I}_n \le 0\colon \log(N) < |n| \le N\right) \\ &\quad\quad + \mathbb{P}\left(\exists n\colon \bar{I}_n - I_n > n^{1+H-\varepsilon},\ \log(N) < |n| \le N\right).\end{aligned} \tag{2.19}$$

We will see that the second term on the right-hand side is of lower order, while the term on the left-hand side can be related to $\mathbb{P}\left(I_n \le 0\colon |n| \le N\right)$. For this purpose, let X denote a standard normal random variable. Then, by using that $\bar{I}_n - I_n = S_n/2$ in the first step and by using in the second step that, due to (2.1), $\mathbb{V}(S_n) \le n^{2(H+\varepsilon)}$, for n large enough, we obtain, for N large enough,

$$\begin{aligned}&\mathbb{P}\left(\exists n\colon \bar{I}_n - I_n > n^{1+H-\varepsilon},\ \log(N) < |n| \le N\right) \\ &\quad \le 2 \sum_{n=\lceil \log(N)\rceil}^{N} \mathbb{P}\left(S_n/2 > n^{1+H-\varepsilon}\right) \\ &\quad \le 2 \sum_{n=\lceil \log(N)\rceil}^{N} \mathbb{P}\left(n^{H+\varepsilon}X > n^{1+H-\varepsilon}\right) \\ &\quad \le 2N\mathbb{P}\left(X > \log(N)^{1-2\varepsilon}\right) \\ &\quad \le 2Ne^{-(\log(N))^{2-4\varepsilon}/2} \\ &\quad \le 2N^{-2}.\end{aligned} \tag{2.20}$$

In the fourth step above, we used the standard estimate $\mathbb{P}(X > x) \leq e^{-x^2/2}$ for $x \geq 1$. Finally, using Proposition 2.6, we obtain for N large enough

$$\begin{aligned}\mathbb{P}(I_n \leq 0 \colon |n| \leq N) &\leq \mathbb{P}(I_n \leq 0 \colon \log(N) < |n| \leq N) \\ &\leq \mathbb{P}(I_n \leq -f(n) \colon \log(N) < |n| \leq N) \\ &\quad \cdot e^{\sqrt{2\|f\|_{\mathcal{H}}^2 \log(1/\mathbb{P}(I_n \leq 0\colon |n|\leq N))} - \|f\|_{\mathcal{H}}^2/2} \\ &\leq \mathbb{P}\left(I_n \leq -n^{1+H-\varepsilon} \colon \log(N) < |n| \leq N\right) \\ &\quad \cdot e^{\sqrt{2\|f\|_{\mathcal{H}}^2 \log(1/\mathbb{P}(I_n \leq 0\colon |n|\leq N))} - \|f\|_{\mathcal{H}}^2/2}.\end{aligned}$$

This, together with (2.19), (2.20), and Theorem 2.2, finishes the proof. □

As another application of Corollary 2.9, we can give an improvement of Theorem 11 in [14]:

Corollary 2.12. *Let* $(\xi_n)_{n\in\mathbb{N}}$ *be a real-valued stationary centered Gaussian sequence such that* (2.5) *holds. Then, for every* $b \in \mathbb{R}$*, there is some constant* $c > 0$ *such that*

$$\begin{aligned}N^{-(1-H)}\sqrt{\ell(N)}e^{-c\sqrt{\log(N)}} &\leq \mathbb{P}\left(\max_{1\leq n\leq N} S_n \leq b\right) \\ &\leq N^{-(1-H)}\sqrt{\ell(N)}e^{c\sqrt{\log(N)}} \quad \text{for all } N \in \mathbb{N}.\end{aligned}$$

Proof. Let $b > 0$. By Corollary 2.9, there is a function $f \in \mathcal{H}_H(S)$ with $f(n) \geq 2b$ for all $n \geq 1$. Further, using the lower estimate in (2.6), we have for N large enough

$$N^{-1} \leq \mathbb{P}(S_n \leq b \colon 1 \leq n \leq N).$$

This together with Proposition 2.6 yields for N large enough

$$\begin{aligned}&\mathbb{P}(S_n \leq -b \colon 1 \leq n \leq N) \\ &\qquad = \mathbb{P}(S_n + f(n) \leq -b + f(n) \colon 1 \leq n \leq N) \\ &\qquad \geq \mathbb{P}(S_n + f(n) \leq b \colon 1 \leq n \leq N) \\ &\qquad \geq \mathbb{P}(S_n \leq b \colon 1 \leq n \leq N)\, e^{-\sqrt{2\|f\|_{\mathcal{H}}^2 \log(N)} - \|f\|_{\mathcal{H}}^2/2}.\end{aligned}$$

Combining this with (2.6) finishes the proof. □

Remark 2.13. In Theorem 11 in [14], the authors assume $\inf_{n\geq 1} \mathbb{E}S_nS_1 > 0$ to get the change of measure argument to work. For instance, the fractional Gaussian noise process $(\xi_n^{\text{FGN}})_{n\in\mathbb{N}}$ satisfies this assumption. This can be easily verified by using Lemma 2.4 and $\mathbb{E}S_n^2 = n^{2H}$. In general, this does not remain true if one only has (2.1). For example, consider the case where $\sum_{j=1}^n \sum_{k=1}^n \mathbb{E}\xi_j\xi_k = n^{2H}\ell(n)$ with $\ell(x) = 1 + \cos(\pi x)/\log(x)$ in (2.1). Then, one has $\sum_{j=1}^n \sum_{k=1}^n \mathbb{E}\xi_j\xi_k \sim n^{2H}$ but the function $\mathbb{E}S_nS_1$ attains infinitely often positive and negative values since, by Lemma 2.4,

$$\begin{aligned}\mathbb{E}S_nS_1 &= \frac{1}{2}\left(\mathbb{E}S_n^2 - \mathbb{E}S_{n-1}^2 + \mathbb{E}S_1^2\right)\\ &= \frac{1}{2}\left(n^{2H}\left(1-\frac{(-1)^n}{\log(n)}\right) - (n-1)^{2H}\left(1-\frac{(-1)^{n-1}}{\log(n-1)}\right) + \mathbb{E}S_1^2\right)\\ &\sim \frac{1}{2}\left(2Hn^{2H-1} - n^{2H}\frac{(-1)^n}{\log(n)} + \mathbb{E}S_1^2\right), \quad \text{as} \quad n\to\infty.\end{aligned}$$

Remark 2.14. Consider the function $f\colon \mathbb{N}\to\mathbb{R}$ with $f(n) = \mathbb{1}_{n=1}$. Clearly, f is in the RKHS of the process $(\xi_n)_{n\geq 1}$ if and only if $\xi_1 \notin \mathbb{H}_2$, where $\mathbb{H}_2$ denotes the L^2-closure of the set $\operatorname{span}\{\xi_n : n\geq 2\}$. It is well-known that this condition is equivalent to the Kolmogorov condition

$$\int_{-\pi}^{\pi} \log(p(u))\,\mathrm{d}u > -\infty, \tag{2.21}$$

where p denotes the density of the component of the spectral measure of $(\xi_n)_{n\geq 1}$ that is absolutely continuous with respect to the Lebesgue measure, see e.g. Theorem 2.5.4 in [6]. In this case, all constant functions are in the RKHS of the process $(S_n)_{n\geq 1}$. Hence, the proof of Corollary 2.12 still works if we replace condition (2.5) by (2.1) and (2.21).

2.3 Proofs

2.3.1 Proof of Theorem 2.1

Upper bound

Let T_N denote the time where the process $(S_n)_{n\in\mathbb{Z}}$ attains its maximum on $\{0,1,\ldots,N\}$. Since $(S_n)_{n\in\mathbb{Z}}$ has stationary increments (see (2.7)) and $\mathbb{P}(S_j = S_k) = 0$ for $j\neq k$, the upper bound follows from

$$\begin{aligned}N\cdot\mathbb{P}(S_n\leq 0\colon -N\leq n\leq N) &\leq \sum_{k=1}^N \mathbb{P}(S_n\leq 0\colon -k\leq n\leq N-k)\\ &= \sum_{k=1}^N \mathbb{P}(S_n\leq S_k\colon 0\leq n\leq N)\\ &= \sum_{k=1}^N \mathbb{P}(T_N = k)\\ &\leq 1.\end{aligned}$$

Lower bound

Using again the stationary increments of $(S_n)_{n\in\mathbb{Z}}$, we obtain

$$\begin{aligned}
(N+1)\cdot \mathbb{P}\left(S_n \le 0\colon\ -N \le n \le N\right) \\
&\ge \sum_{k=0}^{N} \mathbb{P}\left(S_n \le 0\colon\ -N-k \le n \le 2N-k\right) \\
&= \sum_{k=0}^{N} \mathbb{P}\left(S_n \le S_{N+k}\colon 0 \le n \le 3N\right) \qquad (2.22) \\
&= \sum_{k=0}^{N} \mathbb{P}\left(T_{3N} = N+k\right) \\
&= \mathbb{P}\left(T_{3N} \in [N, 2N]\right).
\end{aligned}$$

Now, we consider the continuous functional $F\colon (D[0,1], \|\cdot\|_\infty) \to (\mathbb{R}, |\cdot|)$ given by

$$F(g) = \left(\sup_{x\in\left(\frac{1}{3},\frac{2}{3}\right)} g(x) - \sup_{x\in\left(0,\frac{1}{3}\right)\cup\left(\frac{2}{3},1\right)} g(x)\right)_+ \wedge 1,$$

where $(x)_+ := \max(x,0)$ for $x \in \mathbb{R}$ and $D[0,1]$ denotes the set of all càdlàg functions on $[0,1]$. We set

$$Y_N(t) = \frac{1}{N^H \ell(N)^{1/2}} \sum_{k=1}^{\lfloor Nt \rfloor} \xi_k, \quad t \in [0,1].$$

Due to (2.2), it follows that

$$\mathbb{P}\left(T_{3N} \in [N, 2N]\right) = \mathbb{E}\left[\mathbb{1}_{T_{3N}\in[N,2N]}\right] \ge \mathbb{E}F\left(Y_{3N}\right) \to c_0 > 0,$$

as $N \to \infty$. This and (2.22) show the lower bound.

2.3.2 Proof of Theorem 2.2

The proof is structured as follows: We first consider the functional

$$F_N := \sum_{k=1}^{N-1} \left(\gamma^-_{k,k} - \gamma^+_{k,N-k}\right)_+, \qquad (2.23)$$

where for $k \in \mathbb{Z}$ and $m \in \mathbb{N}$

$$\gamma^-_{k,m} := \min_{1\le n\le m} \frac{I_k - I_{k-n}}{n} \quad \text{and} \quad \gamma^+_{k,m} := \max_{1\le n\le m} \frac{I_{k+n} - I_k}{n},$$

and determine the polynomial order of $\mathbb{E}F_N$, as $N \to \infty$. Then, we relate the quantity $\mathbb{E}F_N$ to the probability

$$\tilde{p}_N := \mathbb{P}\left(I_n + |n| \leq 0\colon |n| \leq N\right). \tag{2.24}$$

Finally, we obtain the asymptotic order of

$$p_N := \mathbb{P}\left(I_n \leq 0\colon |n| \leq N\right) \tag{2.25}$$

from (2.24) by using a change of measure argument (Proposition 2.6 and Corollary 2.9).

We stress that our proof mostly follows the arguments in continuous time by G. Molchan [62]. However, we have to digress from that line of proof whenever dealing with change of measure arguments and when self-similarity is used. In particular, we mention a non-trivial replacement of self-similarity (see arguments between (2.42) and (2.43)), which may be of independent interest.

Upper bound for $\mathbb{E}F_N$

In the following, we fix N and write $\gamma_k^- = \gamma_{k,k}^-$ and $\gamma_k^+ = \gamma_{k,N-k}^+$ to ease notation. Let $C_N\colon [0,N] \to \mathbb{R}$ denote the concave majorant of I_n on $[0,N]$, i.e., C_N is the smallest concave function with $I_n \leq C_N(n)$. Obviously, C_N is a piecewise linear function and we denote by $\{k_1, k_2, \ldots\}$ (depending on N) its nodal points. Note that at these points the slope on the left is $\gamma_{k_i}^-$ and the slope on the right is $\gamma_{k_i}^+$. Further, we note that $\gamma_k^- - \gamma_k^+ \geq 0$ if and only if k is a nodal point of C_N. In that case, one has $\gamma_{k_i}^+ = \gamma_{k_{i+1}}^-$. Thus,

$$F_N = \sum_{k=1}^{N-1} \left(\gamma_k^- - \gamma_k^+\right)_+ = \sum_i \left(\gamma_{k_i}^- - \gamma_{k_{i+1}}^-\right) = \gamma_0^+ - \gamma_N^-. \tag{2.26}$$

Using $\mathbb{E}S_N = 0$ in the second step, identity (2.8) in the third step, and $(I_n)_{n\in\mathbb{Z}} \overset{d}{=} (I_{-n})_{n\in\mathbb{Z}}$ in fourth step, we obtain

$$\begin{aligned}\mathbb{E}\left[-\gamma_N^-\right] &= \mathbb{E}\left[-\min_{1\leq n\leq N} \frac{I_N - I_{N-n}}{n}\right] \\ &= \mathbb{E}\left[\max_{1\leq n\leq N} \frac{I_{N-n} - I_N - (-n)S_N}{n}\right] \\ &= \mathbb{E}\left[\max_{1\leq n\leq N} \frac{I_{-n}}{n}\right] = \mathbb{E}\left[\max_{1\leq n\leq N} \frac{I_n}{n}\right] = \mathbb{E}\gamma_0^+.\end{aligned}$$

Therefore, we get from (2.26) that

$$\mathbb{E}F_N = 2\mathbb{E}\gamma_0^+. \tag{2.27}$$

Due to (2.27), one obtains the upper estimate

$$\begin{aligned}\mathbb{E}F_N = 2\mathbb{E}\gamma_0^+ &= 2\mathbb{E}\left[\max_{1\le n\le N}\frac{I_n}{n}\right] = 2\mathbb{E}\left[\max_{1\le n\le N}\frac{\sum_{k=1}^n \tilde{S}_k}{n}\right]\\ &\le 2\mathbb{E}\left[\max_{1\le n\le N}\frac{\sum_{k=1}^n \max_{1\le j\le N}\tilde{S}_j}{n}\right] = 2\mathbb{E}\left[\max_{1\le j\le N}\tilde{S}_j\right].\end{aligned}$$

It can be obtained from (2.2) that

$$\frac{1}{N^H\ell(N)^{1/2}}\mathbb{E}\left[\max_{1\le n\le N}\tilde{S}_n\right]\to\mathbb{E}\left[\sup_{t\in[0,1]}W_H(t)\right]\in(0,\infty),$$

as $N\to\infty$, where $(W_H(t))_{t\in[0,1]}$ is a fractional Brownian motion, see e.g. the proof of Theorem 11 in [14]. Thus, there is a positive constant c such that for all N

$$\mathbb{E}F_N \le c\,\ell(N)^{1/2}N^H. \tag{2.28}$$

In the following, c will denote a varying positive constant independent of N for ease of notation.

Lower bound for $\mathbb{E}F_N$

Since $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary process, we have by Lemma 2.4 that

$$\mathbb{E}S_jS_k = \frac{1}{2}\left(\mathbb{E}S_j^2+\mathbb{E}S_k^2-\mathbb{E}S_{|j-k|}^2\right).$$

Consequently, using (2.10), we have

$$\begin{aligned}\mathbb{E}\left(I_N+\frac{S_N}{2}\right)^2 &= \mathbb{E}\left(\sum_{k=1}^N S_k\right)^2 = \sum_{j=1}^N\sum_{k=1}^N \mathbb{E}S_jS_k\\ &= \frac{1}{2}\sum_{j=1}^N\sum_{k=1}^N\left(\mathbb{E}S_j^2+\mathbb{E}S_k^2-\mathbb{E}S_{|j-k|}^2\right).\end{aligned}$$

Counting how often $\mathbb{E}S_k^2$ is added yields

$$\begin{aligned}\mathbb{E}\left(I_N+\frac{S_N}{2}\right)^2 &= \frac{1}{2}\sum_{k=1}^N N\mathbb{E}S_k^2+\frac{1}{2}\sum_{k=1}^N N\mathbb{E}S_k^2-\sum_{k=1}^N(N-k)\mathbb{E}S_k^2\\ &= \sum_{k=1}^N k\mathbb{E}S_k^2.\end{aligned}$$

Since, by (2.1), $\mathbb{E}S_n^2\sim n^{2H}\ell(n)$, we can apply Proposition 1.5.8 in [21] to obtain

$$\mathbb{E}\left(I_N+\frac{S_N}{2}\right)^2 \sim N^{2H+2}\ell(N)/(2H+2). \tag{2.29}$$

Now, using the Cauchy-Schwarz inequality, we have $|\mathbb{E}S_N I_N| \leq \sqrt{\mathbb{E}S_N^2 \mathbb{E}I_N^2}$ and we can thus conclude from (2.29) and $\mathbb{E}S_n^2 \sim n^{2H}\ell(n)$ that

$$\mathbb{E}I_N^2 \sim N^{2H+2}\ell(N)/(2H+2). \tag{2.30}$$

In the following, we let $\|\cdot\|_2$ denote the norm $\|X\|_2 = \mathbb{E}\left[|X|^2\right]^{1/2}$ and use again the notation $(x)_+ := \max(x, 0)$ for $x \in \mathbb{R}$. Moreover, we recall the identity $\mathbb{E}X_+ = (2\pi)^{-1/2}\|X\|_2$ for a centered normal random variable X. Now, we can give a lower bound for $\mathbb{E}F_N$. By (2.27) and $\mathbb{E}I_1 = 0$, we have

$$\begin{aligned}
\mathbb{E}F_N &= 2\mathbb{E}\left[\max_{1\leq n\leq N} \frac{I_n}{n}\right] = 2\mathbb{E}\left[\max_{1\leq n\leq N} \frac{I_n}{n} - I_1\right] \\
&= 2\mathbb{E}\left(\max_{1\leq n\leq N} \frac{I_n}{n} - I_1\right)_+ \geq 2\mathbb{E}\left(\frac{I_N}{N} - I_1\right)_+ \\
&= \sqrt{2/\pi}\left\|\frac{I_N}{N} - I_1\right\|_2 \geq \sqrt{2/\pi}\left(\left\|\frac{I_N}{N}\right\|_2 - \|I_1\|_2\right),
\end{aligned}$$

where we used the reverse triangle inequality in the last step. Thus, by (2.30), we have

$$\mathbb{E}F_N \geq c^{-1}\ell(N)^{1/2}N^H. \tag{2.31}$$

Upper bound for $\tilde{p}_N$

In order to get an upper bound for the probability in (2.24), it is convenient to consider the random variable

$$\vartheta_N := \left(\gamma_{0,N}^- - \gamma_{0,N}^+\right)_+$$

and to use the inequality

$$\mathbb{E}\left(\gamma_k^- - \gamma_k^+\right)_+ \geq \mathbb{E}\vartheta_N. \tag{2.32}$$

To see (2.32), note that by using (2.8), we obtain

$$\begin{aligned}
\gamma_{k,N}^- - \gamma_{k,N}^+ &= \min_{1\leq n\leq N} -\frac{I_{k-n} - I_k}{n} - \max_{1\leq n\leq N} \frac{I_{k+n} - I_k}{n} \\
&= \min_{1\leq n\leq N} -\frac{I_{k-n} - I_k - (-n)S_k}{n} - \max_{1\leq n\leq N} \frac{I_{k+n} - I_k - nS_k}{n} \\
&\overset{d}{=} \min_{1\leq n\leq N} -\frac{I_{-n}}{n} - \max_{1\leq n\leq N} \frac{I_n}{n} = \gamma_{0,N}^- - \gamma_{0,N}^+.
\end{aligned} \tag{2.33}$$

Further, one has

$$\begin{aligned}
\gamma_k^- - \gamma_k^+ &= \min_{1\leq n\leq k} -\frac{I_{k-n} - I_k}{n} - \max_{1\leq n\leq N-k} \frac{I_{k+n} - I_k}{n} \\
&\geq \min_{1\leq n\leq N} -\frac{I_{k-n} - I_k}{n} - \max_{1\leq n\leq N} \frac{I_{k+n} - I_k}{n} = \gamma_{k,N}^- - \gamma_{k,N}^+.
\end{aligned} \tag{2.34}$$

Combining the last two identities, we get (2.32). Now, applying Markov's inequality, we see that

$$\mathbb{E}\vartheta_N \geq 2\mathbb{P}\left(\vartheta_N \geq 2\right). \tag{2.35}$$

Using (2.28), (2.32), and (2.35), we thus obtain

$$\begin{aligned} c\,\ell(N)^{1/2}N^H \geq \mathbb{E}F_N &= \sum_{k=1}^{N-1} \mathbb{E}\left(\gamma_k^- - \gamma_k^+\right)_+ \\ &\geq (N-1)\,\mathbb{E}\vartheta_N \geq 2(N-1)\,\mathbb{P}\left(\vartheta_N \geq 2\right) \\ &\geq 2(N-1)\,\mathbb{P}\left(\gamma_{0,N}^- \geq 1, \gamma_{0,N}^+ \leq -1\right) \\ &= 2(N-1)\,\mathbb{P}\left(\min_{1\leq n\leq N} -I_{-n}/n \geq 1,\ \max_{1\leq n\leq N} I_n/n \leq -1\right) \\ &= 2(N-1)\,\mathbb{P}\left(I_n + |n| \leq 0\colon |n| \leq N\right). \end{aligned} \tag{2.36}$$

Hence, we have for any N

$$\mathbb{P}\left(I_n + |n| \leq 0\colon |n| \leq N\right) \leq c\,\ell(N)^{1/2}N^{-(1-H)}. \tag{2.37}$$

Lower bound for $\tilde{p}_N$

Along the lines of the proof of (2.32), one gets an analogous estimate when replacing N by $\tilde{k} := \min(k, N-k)$ in (2.33) and (2.34); namely

$$\mathbb{E}\left(\gamma_k^- - \gamma_k^+\right)_+ \leq \mathbb{E}\vartheta_{\tilde{k}}. \tag{2.38}$$

Now, let $1 - H < \alpha < 1$. Noting that

$$\vartheta_N = \left(\gamma_{0,N}^- - \gamma_{0,N}^+\right)_+ = \left(\min_{1\leq n\leq N} -\frac{I_{-n}}{n} - \max_{1\leq n\leq N}\frac{I_n}{n}\right)_+$$

is monotonically decreasing in N, we have for $N^\alpha \leq k \leq N - N^\alpha$

$$\mathbb{E}\vartheta_{\tilde{k}} \leq \mathbb{E}\vartheta_{\lceil N^\alpha\rceil}. \tag{2.39}$$

Thus, using (2.38) and (2.39), we obtain

$$\begin{aligned} \mathbb{E}F_N &= \sum_{k=1}^{N-1} \mathbb{E}\left(\gamma_k^- - \gamma_k^+\right)_+ \\ &\leq \left(N - 2\left\lfloor N^\alpha\right\rfloor\right)\mathbb{E}\vartheta_{\lceil N^\alpha\rceil} + 2\sum_{k=1}^{\lfloor N^\alpha\rfloor} \mathbb{E}\vartheta_k. \end{aligned}$$

Moreover, we know from (2.36), that we have for all k

$$\mathbb{E}\vartheta_k \leq c\,\ell(k)^{1/2}k^{H-1}. \tag{2.40}$$

Hence, by (2.40) and Proposition 1.5.8 in [21], we obtain

$$\sum_{k=1}^{\lfloor N^\alpha \rfloor} \mathbb{E}\vartheta_k \le c\,\ell(\lfloor N^\alpha \rfloor)^{1/2} N^{\alpha H}.$$

Thus, we have

$$c^{-1}\ell(N)^{1/2}N^H \le \mathbb{E}F_N \le N\mathbb{E}\vartheta_{\lceil N^\alpha \rceil} + c\,\ell(\lfloor N^\alpha \rfloor)^{1/2} N^{\alpha H},$$

where the first inequality is due to (2.31). Since $\alpha < 1$, we obtain

$$c^{-1}\ell(N)^{1/2}N^{H-1} \le \mathbb{E}\vartheta_{\lceil N^\alpha \rceil}.$$

Replacing $\lceil N^{1/\alpha} \rceil$ by N yields

$$\ell_1(N)N^{-(1-H)/\alpha} \le \mathbb{E}\vartheta_N,$$

where ℓ_1 is a slowly varying function at infinity.

Fix $q > 1$ to be chosen later and let $\|\cdot\|_q$ denote the norm $\mathbb{E}\left[|X|^q\right]^{1/q}$ for some random variable X. Then, using again $\vartheta_N \le \vartheta_1$ and using Hölder's inequality, we obtain

$$\mathbb{E}\vartheta_N = \mathbb{E}\vartheta_N \mathbb{1}_{\vartheta_N > 0} \le \mathbb{E}\vartheta_1 \mathbb{1}_{\vartheta_N > 0} \le \|\vartheta_1\|_q \mathbb{P}\left(\vartheta_N > 0\right)^{1-1/q}.$$

Further,

$$\|\vartheta_1\|_q \le \|(I_{-1} - I_1)_+\|_q \le \|I_{-1} - I_1\|_q \le c\sqrt{q},$$

using that $I_{-1} - I_1$ is a Gaussian random variable. So, we have

$$\frac{\ell_1(N)}{c\sqrt{q}} N^{-(1-H)/\alpha} \le \mathbb{P}\left(\vartheta_N > 0\right)^{1-1/q}.$$

Now, setting $q := \log(N) + 1$ yields

$$\ell_2(N)N^{-(1-H)/\alpha} \le \mathbb{P}\left(\vartheta_N > 0\right), \tag{2.41}$$

where ℓ_2 is a slowly varying function at infinity.

In the following, we will relate the probability $\mathbb{P}\left(\vartheta_N > 0\right)$ to the probability in (2.24). Due to the obvious lack of self-similarity in discrete time, we need to replace the arguments in continuous time in [62] by a new technique that essentially uses the symmetry of the process. This is divided into four steps.

Step 1: We start with a change of measure argument. By Corollary 2.9, we can find a function $f \in \mathcal{H}_H(I)$ such that $f(n) \ge \frac{3}{2}|n|$ for all $n \in \mathbb{Z}$. Then, using (2.13) from Proposition 2.6, we obtain

$$\begin{aligned} &e^{-\sqrt{2\|f\|^2\log(1/\mathbb{P}(\vartheta_N>0))}-\|f\|^2/2}\,\mathbb{P}\left(\vartheta_N>0\right)\\ &\qquad\le \mathbb{P}\left(\min_{-N\le n\le -1}\frac{I_n+f(n)}{n}-\max_{1\le n\le N}\frac{I_n+f(n)}{n}>0\right)\\ &\qquad\le \mathbb{P}\left(\min_{-N\le n\le -1}\frac{I_n+\frac{3}{2}|n|}{n}-\max_{1\le n\le N}\frac{I_n+\frac{3}{2}|n|}{n}>0\right)\\ &\qquad= \mathbb{P}\left(\min_{-N\le n\le -1}\frac{I_n}{n}-\max_{1\le n\le N}\frac{I_n}{n}>3\right)\\ &\qquad= \mathbb{P}\left(\vartheta_N>3\right). \end{aligned} \tag{2.42}$$

Step 2: Let

$$A_0^{(N)} := \left\{(x_{-N},\dots,x_{-1},x_1,\dots,x_N)\in\mathbb{R}^{2N} \colon x_n\le -|n|, 1\le |n|\le N\right\},$$

and let

$$A_m^{(N)} := A_0^{(N)} + mb^{(N)}, \quad \text{where } b^{(N)} := (-N,\dots,-1,1,\dots,N).$$

We write $I\in A_m^{(N)}$ instead of $(I_{-N},\dots,I_{-1},I_1,\dots,I_N)\in A_m^{(N)}$ in the following to ease notation. We will show that $\{\vartheta_N>3\}\subseteq \cup_{m\in\mathbb{Z}}\{I\in A_m^{(N)}\}$. For this purpose, let $m^{(N)}$ be defined as the integer-valued random variable such that $\min_{-N\le n\le -1}\frac{I_n}{n}\in[m^{(N)}+1, m^{(N)}+2)$. Now, assuming that $\vartheta_N>3$, and thus, $\min_{-N\le n\le 1} I_n/n - \max_{1\le n\le N} I_n/n>3$, we can conclude that

$$\max_{1\le n\le N}\frac{I_n}{n} < \min_{-N\le n\le -1}\frac{I_n}{n}-3 < m^{(N)}-1.$$

Thus, we have $I\in A_{m^{(N)}}^{(N)}$.

Step 3: We show that $\mathbb{P}\left(I\in A_0^{(N)}\right)\ge \mathbb{P}\left(I\in A_m^{(N)}\right)$. For this purpose, we make use of an argument that is commonly used to prove Anderson's inequality. It is well-known that for any convex subsets $A,B\subseteq\mathbb{R}^{2N}$ and $0<\lambda<1$, one has

$$\mu\left(\lambda A+(1-\lambda)B\right)\ge \mu\left(A\right)^{\lambda}\mu\left(B\right)^{1-\lambda},$$

where μ is a centered Gaussian measure on $\mathbb{R}^{2N}$, see e.g. Theorem 2 in [67]. Since $(I_{-N},\dots,I_{-1},I_1,\dots,I_N)$ is a centered Gaussian random variable, by setting $\lambda=1/2$, we obtain

$$\begin{aligned} \mathbb{P}\left(I\in A_0^{(N)}\right) &= \mathbb{P}\left(I\in \frac{1}{2}A_{-m}^{(N)}+\frac{1}{2}A_m^{(N)}\right)\\ &\ge \mathbb{P}\left(I\in A_{-m}^{(N)}\right)^{1/2}\mathbb{P}\left(I\in A_m^{(N)}\right)^{1/2} = \mathbb{P}\left(I\in A_m^{(N)}\right). \end{aligned}$$

Here, we used that $A_0^{(N)} = \frac{1}{2}A_{-m}^{(N)} + \frac{1}{2}A_m^{(N)}$ and that, by the symmetry of the process $(I_n)_{n\in\mathbb{Z}}$, $\mathbb{P}\left(I \in A_{-m}^{(N)}\right) = \mathbb{P}\left(I \in A_m^{(N)}\right)$.

Step 4: Now, we relate the quantities $\mathbb{P}(\vartheta_N > 3)$ and $\tilde{p}_N$. Recall that $\mathbb{P}(X > x) \leq e^{-x^2/2}$ for $x \geq 1$ and X being a standard normal random variable. Thus, since I_{-1} is a centered Gaussian random variable, we can choose a constant c_0 such that $\mathbb{P}(I_{-1} \leq -(a_N + 1)) = o(N^{-1})$ for $a_N = \sqrt{c_0 \log(N)}$. Further, by $\mathbb{P}(\cup_{m \geq a_N} A_m) \leq \mathbb{P}(I_{-1} < -(a_N + 1))$ and the symmetry of the process $(I_n)_{n\in\mathbb{Z}}$, we get

$$\mathbb{P}\left(\cup_{|m|\geq a_N} A_m\right) = o(N^{-1}).$$

Altogether we thus obtain

$$\begin{aligned} \mathbb{P}(\vartheta_N > 3) &\leq \mathbb{P}(\cup_{m\in\mathbb{Z}} A_m) \\ &\leq \sum_{|m|<a_N} \mathbb{P}(A_m) + \mathbb{P}\left(\cup_{|m|\geq a_N} A_m\right) \\ &\leq 2a_N \mathbb{P}(A_0) + 2\mathbb{P}(I_{-1} \leq -(a_N + 1)) \\ &= 2a_N \mathbb{P}(I_n + |n| \leq 0\colon |n| \leq N) + o(N^{-1}). \end{aligned} \tag{2.43}$$

Putting this together with (2.41) and (2.42), we get

$$\ell_3(N) N^{-(1-H)/\alpha} \leq \mathbb{P}(I_n + |n| \leq 0\colon |n| \leq N), \tag{2.44}$$

where ℓ_3 denotes a slowly varying function at infinity.

Polynomial rate of p_N

Clearly, we have from (2.44)

$$\begin{aligned} \ell_3(N) N^{-(1-H)/\alpha} &\leq \mathbb{P}(I_n + |n| \leq 0\colon |n| \leq N) \\ &\leq \mathbb{P}(I_n \leq 0\colon |n| \leq N) = p_N. \end{aligned} \tag{2.45}$$

In particular, $p_N \geq c^{-1}N^{-1}$ for some suitable constant c. This estimate will be used in the following change of measure argument. Due to Corollary 2.9, we can choose a function $f \in \mathcal{H}_H(I)$ with $f(n) \geq |n|$ for all $n \in \mathbb{Z}$. Then, by (2.37) and Proposition 2.6, we obtain

$$\begin{aligned} c\,\ell(N)^{1/2} N^{-(1-H)} &\geq \mathbb{P}(I_n + |n| \leq 0\colon |n| \leq N) \\ &\geq \mathbb{P}(I_n + f(n) \leq 0\colon |n| \leq N) \\ &\geq \mathbb{P}(I_n \leq 0\colon |n| \leq N)\, e^{-\sqrt{2\|f\|^2 \log(1/p_N)} - \|f\|^2/2} \\ &\geq \mathbb{P}(I_n \leq 0\colon |n| \leq N)\, e^{-\sqrt{2\|f\|^2 \log(cN)} - \|f\|^2/2}. \end{aligned} \tag{2.46}$$

Finally, we take log in (2.45), and (2.46) and divide by $\log(N)$. Then, taking $\limsup_N$ and $\liminf_N$, respectively, and letting $\alpha \nearrow 1$ yields

$$\lim_{N\to\infty} \frac{\log\left(\mathbb{P}\left(I_n \leq 0\colon |n| \leq N\right)\right)}{\log(N)} = H - 1.$$

This finishes the proof.

Remark. Most parts of Chapter 2 appeared in the journal *Journal of Statistical Physics* in the article entitled *Persistence Probabilities of Two-Sided (Integrated) Sums of Correlated Stationary Gaussian Sequences* (see [10]).

Chapter 3

Penalizing fractional Brownian motion for being negative

In this chapter, we present a first contribution to the rigorous study of fractional Brownian motion (FBM) conditioned to be positive. More precisely, we consider a modification of the approach in (1.4), where the process is penalized instead of being immediately killed when becoming negative. Moreover, we discuss the result in the Brownian case in terms of stochastic differential equations.

3.1 Introduction

Let $(W_H(t))_{t\geq 0}$ be a one-sided FBM with Hurst parameter $0 < H < 1$. As discussed in Section 1.1.1, there are two common approaches in the literature to condition a stochastic process to be positive. In our situation, these are the following: For every $T_0 > 0$, one can study the sequence of measures

$$\begin{aligned}&\mathbb{P}\big((W_H(s))_{s\in[0,T_0]} \in \cdot \;\big|\; W_H(s) \geq -1\,\forall s \in [0,T]\big)\\&= \mathbb{E}\left[\mathbb{1}_{(W_H(s))_{s\in[0,T_0]}\in\cdot} \cdot \frac{\mathbb{1}_{W_H(s)\geq -1\,\forall s\in[0,T]}}{\mathbb{P}(W_H(s) \geq -1\,\forall s \in [0,T])}\right],\end{aligned} \tag{3.1}$$

on $C[0,T_0]$, as $T \to \infty$. This corresponds to the approach in (1.3). Alternatively, following the approach in (1.4), one can study the sequence of measures

$$\begin{aligned}&\mathbb{P}\big((T^{-H}W_H(Tt))_{t\in[0,1]} \in \cdot \;\big|\; W_H(s) \geq -1\,\forall s \in [0,T]\big)\\&= \mathbb{E}\left[\mathbb{1}_{(T^{-H}W_H(Tt))_{t\in[0,1]}\in\cdot} \cdot \frac{\mathbb{1}_{W_H(s)\geq -1\,\forall s\in[0,T]}}{\mathbb{P}(W_H(s) \geq -1\,\forall s \in [0,T])}\right],\end{aligned} \tag{3.2}$$

on $C[0,1]$, as $T \to \infty$, in order to define a FBM conditioned to be positive. Instead of considering the sequence in (3.2), we will consider the following modification.

We will study the sequence

$$\mathbb{E}\left[\mathbb{1}_{(T^{-H}W_H(Tt))_{t\in[0,1]}\in\,\cdot}\;\cdot\;\frac{\left(\int_0^T \exp(-W_H(s))\,\mathrm{d}s\right)^{-1}}{I(T)}\right], \tag{3.3}$$

as $T\to\infty$, where

$$I(T) := \mathbb{E}\left[\left(\int_0^T \exp(-W_H(s))\,\mathrm{d}s\right)^{-1}\right].$$

This is motivated by the technique used by G. Molchan to determine the persistence exponent of one-sided FBM, see Statement 1 in [61]. There, the indicator function in

$$\mathbb{P}(W_H(s)\geq -1, s\in[0,T]) = \mathbb{E}\left[\mathbb{1}_{W_H(s)\geq -1\,\forall s\in[0,T]}\right]$$

is replaced by the smoother functional

$$\left(\int_0^T \exp(-W_H(s))\,\mathrm{d}s\right)^{-1},$$

see our earlier discussion in Section 1.1.2.

After stating a weak convergence result for the sequence in (3.3), we will investigate the Brownian case ($H=1/2$) in more detail and show that the resulting limiting process satisfies a concrete stochastic differential equation (SDE). In the Brownian setup, the sequence in (3.2) weakly converges to the law of the Brownian meander (see [37] for the Brownian motion or [22] and [47] for the corresponding discrete-time analogs), whereas the weak limit of the sequence in (3.1) is given by the law of the three-dimensional Bessel process on $[0,T_0]$ (see e.g. [60]). Therefore, we will compare the SDE which is fulfilled by our limiting process with the one of the Brownian meander and the one of the three-dimensional Bessel process.

3.2 Results

3.2.1 Weak convergence result

We begin by stating the main weak convergence result, which is Theorem 2.1 in [12].

Theorem 3.1. *Let $(W_H(t))_{t\geq 0}$ be a FBM with Hurst parameter $H\in(0,1)$ and, for every $T\geq 1$, let $(X_{H,T}(t))_{t\in[0,1]}$ be a process whose distribution is given by* (3.3)*, i. e.*

$$\mathbb{P}((X_{H,T}(t))_{t\in[0,1]}\in\,\cdot\,) = \mathbb{E}\left[\mathbb{1}_{(T^{-H}W_H(Tt))_{t\in[0,1]}\in\,\cdot}\;\cdot\;\frac{\left(\int_0^T \exp(-W_H(s))\,\mathrm{d}s\right)^{-1}}{I(T)}\right].$$

Then, for $T \to \infty$,

$$(X_{H,T}(t))_{t\in[0,1]} \Rightarrow (X_H(t))_{t\in[0,1]}\,,$$

on $(C[0,1], \|\cdot\|_\infty)$, where $(X_H(t))_{t\in[0,1]}$ is a process whose law is given by

$$\mathbb{P}((X_H(t))_{t\in[0,1]} \in \,\cdot\,) := \mathbb{E}\left[\mathbb{1}_{(W_H(t))_{t\in[0,1]}\in\,\cdot} \cdot \frac{W_H(1) - M_H(1)}{\mathbb{E}[-M_H(1)]}\right]$$

with $M_H(t) := \min_{s\in[0,t]} W_H(s),\ t \geq 0$, being the running minimum of W_H.

Note that the distribution of the limiting process $(X_H(t))_{t\in[0,1]}$ under $\mathbb{P}$ is equal to the law of $(W_H(t))_{t\in[0,1]}$ under the probability measure

$$\mathbb{Q}(\,\cdot\,) := \int_{\cdot} \frac{W_H(1) - M_H(1)}{\mathbb{E}[-M_H(1)]}\,\mathrm{d}\mathbb{P}. \tag{3.4}$$

As one would expect, the density $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ rewards the paths that tend to escape to $+\infty$ since, in this case, $W_H(1)$ becomes large and $M_H(1)$ stays near to zero. However, contrary to a possible limit of the distributions of the original problem in (3.2), the limit distribution $\mathbb{P}((X_H(t))_{t\in[0,1]} \in \,\cdot\,)$ is not concentrated on paths staying positive. Nevertheless, we are able to compare our process and the one we are primarily interested in at least in the Brownian case by means of SDEs.

3.2.2 Explicit SDE in the Brownian case

We consider the Brownian case in this subsection, that is, we consider the case $H = 1/2$. To simplify notation, we abbreviate $(X_{1/2}(t))_{t\in[0,1]}$ to $(X(t))_{t\in[0,1]}$ and set $M_X(t) := \min_{s\in[0,t]} X(s)$. Further, let Φ_{1-t} denote the distribution function of the $\mathcal{N}(0, 1-t)$-distribution.

Proposition 3.2. *Let $c\colon [0,1]\times[0,\infty) \to [0,\infty)$ be defined as*

$$c(t,x) := \frac{2\,\Phi_{1-t}(x) - 1}{x + 2\int_x^\infty (1 - \Phi_{1-t}(s))\,\mathrm{d}s}.$$

Then, a Brownian motion $(\tilde{W}(t))_{t\in[0,1]}$ exists such that $(X(t))_{t\in[0,1]}$ satisfies the SDE

$$\mathrm{d}X(t) = c(t, X(t) - M_X(t))\,\mathrm{d}t + \mathrm{d}\tilde{W}(t).$$

In the following, we will discuss the nature of the process $(X(t))_{t\in[0,1]}$ and we will compare it (in terms of SDEs) to the limiting processes of (3.1) and (3.2). For this purpose, we recall that the limiting process in (3.1) is a three-dimensional Bessel process starting at zero $(X^{(be)}(t))_{t\geq 0}$, whose law is given by the law of the absolute value of a three-dimensional Brownion motion. Further, it is well-known that a Brownian motion $(\tilde{W}(t))_{t\geq 0}$ exists such that the SDE

$$\mathrm{d}X^{(be)}(t) = c^{(be)}(X^{(be)}(t))\,\mathrm{d}t + \mathrm{d}\tilde{W}(t) \tag{3.5}$$

with

$$c^{(be)}(x) := c^{(be)}(t,x) := \frac{1}{x}$$

is satisfied, see e.g. Proposition 3.21 in [50]. The following lemma contains a SDE that is satisfied by a Brownian meander (see (1.6) for the definition), which is the limiting process in (3.2).

Lemma 3.3. *Let $(X^{(me)}(t))_{t\in[0,1]}$ be a Brownian meander and let the function $c\colon [0,1]\times(0,\infty)\to(0,\infty)$ be defined as*

$$c^{(me)}(t,x) := \frac{\exp\left(-\frac{x^2}{2(1-t)}\right)}{\int_0^x \exp\left(-\frac{y^2}{2(1-t)}\right)\mathrm{d}y}.$$

Then, a Brownian motion $(\tilde{W}(t))_{t\in[0,1]}$ exists such that $(X^{(me)}(t))_{t\in[0,1]}$ satisfies the SDE

$$\mathrm{d}X^{(me)}(t) = c^{(me)}(t, X^{(me)}(t))\,\mathrm{d}t + \mathrm{d}\tilde{W}(t).$$

Remark 3.4. After proving Lemma 3.3, the author was made aware of the work [16], where the result is included in Lemma 2, as well as Exercise 3.6 in [59], where it is left to the reader to deduce the statement as a special case of the generalized meander. However, the given proof in Section 3.3 of this thesis is different since it is essentially based on the transition density of the Brownian meander, which is not used in [16] and [59].

In the following, we compare our limiting process $(X(t))_{t\in[0,1]}$ and the Brownian meander $(X^{(me)}(t))_{t\in[0,1]}$ in terms of SDEs. Additionally, the limiting process $(X^{(be)}(t))_{t\geq 0}$ in (3.1) turns out to be useful for comparisons, but recall that the approach of conditioning in (3.1) is different to the approach where one gets $(X(t))_{t\in[0,1]}$ and $(X^{(me)}(t))_{t\in[0,1]}$ as limiting processes. Further, we emphasize that, in the situation of Proposition 3.2, one would think of $c(t,x)$ as the drift away from the former minimum of the process at time t, whereas $c^{(me)}(t,x)$ can be thought of as the drift away from 0 at time t. Similarly, $c^{(be)}(t,x)$ can be seen as the drift away from 0 at time t of $(X^{(be)}(t))_{t\geq 0}$.

Let us now examine the time dependence of the drift terms. While the drift effect of the terms $c(t,x)$ and $c^{(me)}(t,x)$ changes in time, there is no time dependence in $c^{(be)}(t,x)$. This is rather unsurprising since the three-dimensional Bessel process can be thought of as a Brownian motion that is conditioned to stay positive for the infinite future, whereas the conditions for the processes $(X(t))_{t\in[0,1]}$ and $(X^{(me)}(t))_{t\in[0,1]}$ just take the future until time 1 into account. The drift effect for the Brownian meander decreases in time because with progressing time, it becomes easier to stay above zero until time 1. In contrast, the drift $c(t,x)$ increases in time. That is, it becomes more unfavorable to attain a new minimum before maximizing the difference $X(1)-M_X(1)$. The latter quantity comes from the distribution in (3.4) that favors paths with a large difference $X(1)-M_X(1)$. Moreover, we note that $c(t,x)\to c^{(be)}(t,x)$, as $t\to 1$.

Now, we fix t and vary x. Then, one has $c(t,x)\sim c^{(be)}(x)=1/x$, as $x\to\infty$. Thus, for large values of x, the drift that maximizes the difference $X(1)-M_X(1)$ in the situation of Proposition 3.2 and the drift that prevents the process from attaining negative values in the three-dimensional Bessel case coincide. But we see a completely different behavior at 0. The drift term vanishes when the process gets close to its former minimum, i.e. $c(t,x)\to 0$, as $x\to 0$, whereas $c^{(be)}(x)=1/x$. This seems to be natural since once the process is close to its former minimum, it is not as expensive anymore to take first a new minimum and maximize the difference $X(1)-M_X(1)$ afterward. In contrast, we can observe sort of the opposite behavior for the Brownian meander. Here, we have $c^{(me)}(t,x)\sim C\exp(-x^2/2(1-t))$ for some appropriate constant $C>0$, as $x\to\infty$. Thus, the drift term decays much faster, which is due to the finite time horizon in which the process has to stay positive (in contrast to the infinite time horizon in the three-dimensional Bessel case). For $x\to 0$, we have $c^{(me)}(t,x)\sim c^{(be)}(t,x)=1/x$. This seems to be natural again since being close at 0, only the drift that pushes the process away from 0 becomes relevant.

3.3 Proofs

3.3.1 Method of proof of Theorem 3.1

For the proof of Theorem 3.1, we refer to [12]. The crucial idea of the proof is to adopt the technique that G. Molchan used to prove Statement 1 in [61]. There, the persistence probability $\mathbb{P}(W_H(s)\ge -1, s\le T)$ was related to the functional

$$I(T)=\mathbb{E}\left[\left(\int_0^T \exp(-W_H(s))\,\mathrm{d}s\right)^{-1}\right],$$

whose asymptotic behavior could be determined.

Similarly, for $t_1,\dots,t_d\in[0,1]$, the convergence of the finite-dimensional distributions

$$\begin{aligned}&\mathbb{P}((X_{H,T}(t))_{t\in\{t_1,\dots,t_d\}}\in\,\cdot\,)\\&=\frac{1}{I(T)}\mathbb{E}\left[\mathbb{1}_{(T^{-H}W_H(Tt))_{t\in\{t_1,\dots,t_d\}}\in\,\cdot}\cdot\left(\int_0^T \exp(-W_H(s))\,\mathrm{d}s\right)^{-1}\right],\end{aligned}$$

as $T \to \infty$, could be proved in [12], as it turns out that the additional indicator function in the expectation causes no serious technical problems. In a further step, using again similar techniques as key ingredients, the tightness of the sequence of probability measures is shown.

3.3.2 Proof of Proposition 3.2

In the following, we abbreviate the process $(W_{1/2}(t))_{t\in[0,1]}$ to $(W(t))_{t\in[0,1]}$ and the process $(M_{1/2}(t))_{t\in[0,1]}$ to $(M(t))_{t\in[0,1]}$ to be consistent with the notation in Proposition 3.2. Let us recall that, by Girsanov's theorem, we have in our situation the following (see Theorem IV.38.5 in [69]):

Since the probability measures $\mathbb{Q}$ and $\mathbb{P}$ are equivalent, there exists a previsible process $(c(t))_{t\in[0,1]}$ such that

$$Z(t) := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp\bigg(\int_0^t c(s)\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t c(s)^2\,\mathrm{d}s\bigg) \tag{3.6}$$

and, under $\mathbb{Q}$,

$$\tilde{W}(t) = W(t) - \int_0^t c(s)\,\mathrm{d}s,\ t \in [0,1],$$

is a Brownian motion. Here, $(\mathcal{F}_t)_{t\in[0,1]}$ denotes the natural filtration of $(W(t))_{t\in[0,1]}$.

Hence, an absolutely continuous change of measure corresponds to a change of drift. This is the key observation in the proof of the proposition, and therefore, it is the main task to show that the process $(\tilde{W}(t))_{t\in[0,1]}$ is given by

$$\tilde{W}(t) = W(t) - \int_0^t c(s, W(s) - M(s))\,\mathrm{d}s,\ t \in [0,1],$$

where

$$c(t,x) = \frac{2\,\Phi_{1-t}(x) - 1}{x + 2\int_x^\infty (1 - \Phi_{1-t}(s))\,\mathrm{d}s}, \quad t \in [0,1],\ x \in [0,\infty).$$

To do so, we will derive an explicit expression of $c(t)$ in (3.6) and show that it coincides with $c(t, W(t) - M(t))$. Then, recalling that under $\mathbb{Q}$ the process $(W(t))_{t\in[0,1]}$ has the same distribution as the process $(X(t))_{t\in[0,1]}$ under $\mathbb{P}$, the claim follows.

First, we note that $(Z(t))_{t\in[0,1]}$ is the unique solution of the SDE

$$Z(t) = 1 + \int_0^t Z(s)c(s)\,\mathrm{d}W(s), \tag{3.7}$$

see e.g. Example 3.3.9 in [50]. Further, by Theorem 1 in [74], $M(1)$ has the stochastic integral representation

$$M(1) = \mathbb{E}[M(1)] - 2\int_0^1 (\Phi_{1-s}(W(s) - M(s)) - 1)\,\mathrm{d}W(s).$$

We thus obtain, together with (3.4) and the fact that $(Z(t))_{t\in[0,1]}$ is a martingale under $\mathbb{P}$,

$$\begin{aligned}
Z(t) &= \mathbb{E}[Z(1) \mid \mathcal{F}_t] \\
&= \mathbb{E}\left[\frac{W(1) - M(1)}{\mathbb{E}[-M(1)]} \,\middle|\, \mathcal{F}_t \right] \\
&= \mathbb{E}\left[1 + \frac{\int_0^1 (2\,\Phi_{1-s}(W(s) - M(s)) - 1)\,\mathrm{d}W(s)}{\mathbb{E}[-M(1)]} \,\middle|\, \mathcal{F}_t \right] \\
&= 1 + \frac{\int_0^t (2\,\Phi_{1-s}(W(s) - M(s)) - 1)\,\mathrm{d}W(s)}{\mathbb{E}[-M(1)]}, \quad t \in [0,1].
\end{aligned}$$

Consequently, we can conclude from (3.7) that

$$Z(s)c(s) = \frac{2\,\Phi_{1-s}(W(s) - M(s)) - 1}{\mathbb{E}[-M(1)]}, \quad s \in [0,1]. \tag{3.8}$$

Now, using again that $(Z(t))_{t\in[0,1]}$ is a $\mathbb{P}$-martingale, and thus, one has $Z(t) = \mathbb{E}\left[\frac{W(1)-M(1)}{\mathbb{E}[-M(1)]} \,\middle|\, \mathcal{F}_t \right]$, the computation of $c(t)$ reduces to a computation of the conditional expectation $\mathbb{E}[W(1) - M(1) \mid \mathcal{F}_t]$. To do this, we split the process at time t and then define $W^{(t)} := W(1) - W(t)$ and $M^{(t)} := \min_{x\in[t,1]} W(x) - W(t)$ to simplify notation. Note that

$$\begin{aligned}
|M^{(t)}| > W(t) - M(t) \quad &\Leftrightarrow \quad W(t) - \min_{x\in[t,1]} W(x) > W(t) - M(t) \\
&\Leftrightarrow \quad \min_{x\in[t,1]} W(x) < M(t) \\
&\Leftrightarrow \quad M(1) < M(t),
\end{aligned}$$

and thus,

$$\begin{aligned}
W(1) - M(1) &= W(t) + W^{(t)} - (M^{(t)} + W(t)) \\
&= W^{(t)} - M^{(t)} \quad \text{if} \quad |M^{(t)}| > W(t) - M(t).
\end{aligned} \tag{3.9}$$

Accordingly, one has

$$\begin{aligned}
|M^{(t)}| \leq W(t) - M(t) \quad &\Leftrightarrow \quad W(t) - \min_{x\in[t,1]} W(x) \leq W(t) - M(t) \\
&\Leftrightarrow \quad \min_{x\in[t,1]} W(x) \geq M(t) \\
&\Leftrightarrow \quad M(1) = M(t),
\end{aligned}$$

and thus,

$$W(1) - M(1) = W(t) + W^{(t)} - M(t) \quad \text{if} \quad |M^{(t)}| \leq W(t) - M(t). \tag{3.10}$$

Now, using (3.9) and (3.10) in the first step, we obtain

$$
\begin{aligned}
\mathbb{E}[W(1) - M(1) \mid \mathcal{F}_t] &= \mathbb{E}\Big[\mathbb{1}_{|M^{(t)}|>W(t)-M(t)}(W^{(t)} - M^{(t)}) \mid \mathcal{F}_t\Big] \\
&\quad + \mathbb{E}\Big[\mathbb{1}_{|M^{(t)}|\le W(t)-M(t)}(W(t) + W^{(t)} - M(t)) \mid \mathcal{F}_t\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_{|M^{(t)}|>W(t)-M(t)}|M^{(t)}| \mid \mathcal{F}_t\Big] \\
&\quad + \mathbb{E}\Big[\mathbb{1}_{|M^{(t)}|\le W(t)-M(t)}(W(t) - M(t)) \mid \mathcal{F}_t\Big] \\
&\quad + \mathbb{E}[W^{(t)}] \\
&= \int_0^\infty \mathbb{P}\Big(\mathbb{1}_{|M^{(t)}|>W(t)-M(t)}|M^{(t)}| > x \mid \mathcal{F}_t\Big)\,\mathrm{d}x \\
&\quad + (W(t) - M(t)) \cdot \mathbb{P}\Big(|M^{(t)}| \le W(t) - M(t) \mid \mathcal{F}_t\Big) \\
&\quad + 0 \\
&= (W(t) - M(t)) \cdot \mathbb{P}\Big(|M^{(t)}| > W(t) - M(t) \mid \mathcal{F}_t\Big) \\
&\quad + \int_{W(t)-M(t)}^\infty \mathbb{P}(|M^{(t)}| > x)\,\mathrm{d}x \\
&\quad + (W(t) - M(t)) \cdot \mathbb{P}\Big(|M^{(t)}| \le W(t) - M(t) \mid \mathcal{F}_t\Big) \\
&= (W(t) - M(t)) + \int_{W(t)-M(t)}^\infty \mathbb{P}(|M^{(t)}| > x)\,\mathrm{d}x.
\end{aligned}
\tag{3.11}
$$

Using $\mathbb{P}(|M^{(t)}| > x) = \mathbb{P}(|M(1-t)| > x) = 2\,(1 - \Phi_{1-t}(x))$, which is a simple conclusion from the reflection principle, the claim follows from (3.8) and (3.11) since

$$
\begin{aligned}
c(t) &= \frac{Z(t)c(t)}{Z(t)} \\
&= \frac{2\,\Phi_{1-t}(W(t) - M(t)) - 1}{(W(t) - M(t)) + \int_{W(t)-M(t)}^\infty \mathbb{P}(|M^{(t)}| > s)\,\mathrm{d}s} \\
&= \frac{2\,\Phi_{1-t}(W(t) - M(t)) - 1}{(W(t) - M(t)) + 2\int_{W(t)-M(t)}^\infty (1 - \Phi_{1-t}(s))\,\mathrm{d}s} \\
&= c(t, W(t) - M(t)).
\end{aligned}
$$

3.3.3 Proof of Lemma 3.3

We first note that, by (3.5),

$$
\tilde{W}(t) = X^{(be)}(t) - \int_0^t \frac{1}{X^{(be)}(s)}\,\mathrm{d}s, \quad t \in [0,1],
$$

is a Brownian motion. Further, it is well-known that the distributions of $(X^{(be)}(t))_{t\in[0,1]}$ and $(X^{(me)}(t))_{t\in[0,1]}$ are equivalent, see e.g. Section 4 in [48]. Therefore, similarly to the proof of Proposition 3.2, we can conclude that, by Girsanov's theorem, passing over to the equivalent measure $\mathbb{Q}$ given by $\mathrm{d}\mathbb{Q}(\omega) := \frac{\mathrm{d}\mathbb{P}_{X^{(me)}}}{\mathrm{d}\mathbb{P}_{X^{(be)}}}(X^{(be)}(\omega))\,\mathrm{d}\mathbb{P}(\omega)$ causes a change of drift of the Brownian motion $(\tilde{W}(t))_{t\in[0,1]}$. That is, a previsible process $(\tilde{c}(t))_{t\in[0,1]}$ exists such that

$$\tilde{W}(t) - \int_0^t \tilde{c}(s)\,\mathrm{d}s = X^{(be)}(t) - \int_0^t \left(\frac{1}{X^{(be)}(s)} + \tilde{c}(s)\right)\mathrm{d}s, \quad t\in[0,1],$$

is a Brownian motion under the new measure. Moreover, $(X^{(be)}(t))_{t\in[0,1]}$ has the distribution of a Brownian meander under the new measure. Consequently, a Brownian motion $(\bar{W}(t))_{t\in[0,1]}$ and a previsible process $(\bar{c}(t))_{t\in[0,1]}$ exist such that the process $(X^{(me)}(t))_{t\in[0,1]}$ satisfies the SDE

$$\mathrm{d}X^{(me)}(t) = \bar{c}(t)\,\mathrm{d}t + \mathrm{d}\bar{W}(t).$$

Thus, it remains to show that $\bar{c}(t) = c^{(me)}(t, X^{(me)}(t))$. For this purpose, we recall that $(X^{(me)}(t))_{t\in[0,1]}$ is a non-homogeneous Markov process with transition density given by

$$\begin{aligned}
&\mathbb{P}\Big(X^{(me)}(t+s)\in\,\mathrm{d}y\;\Big|\;X^{(me)}(t)=x\Big)\\
&= (\varphi_s(y-x)-\varphi_s(y+x))\,\frac{\Phi_{1-t-s}(y)-\frac{1}{2}}{\Phi_{1-t}(x)-\frac{1}{2}}\,\mathrm{d}y,
\end{aligned} \tag{3.12}$$

where φ_s is the density of the $\mathcal{N}(0,s)$-distribution, see e.g. Section 1 in [37]. Thus, it remains to show that

$$c^{(me)}(t,x) = \lim_{s\searrow 0}\frac{1}{s}\,\mathbb{E}\Big[X^{(me)}(t+s)-X^{(me)}(t)\;\Big|\;X^{(me)}(t)=x\Big],$$

see e.g. Section 11 in [64]. In order to compute the limit on the right-hand side, we recall that $\mathbb{P}(Y > x) \le \exp(-x^2/2)/x$ for $x > 0$ and Y being a standard normal random variable, and thus, by the Cauchy-Schwarz inequality, a constant $C_k > 0$ exists for all $k\in\mathbb{N}$ such that, for $y > 0$,

$$\begin{aligned}
\int_y^\infty u^k\varphi_s(u)\,\mathrm{d}u &\le \mathbb{E}\Big[(\sqrt{s}\,Y)^{2k}\Big]^{\frac{1}{2}}\;\mathbb{P}(\sqrt{s}\,Y > y)^{\frac{1}{2}}\\
&\le C_k s^{\frac{2k+1}{4}}y^{-\frac{1}{2}}\exp\Big(-\frac{y^2}{4s}\Big).
\end{aligned} \tag{3.13}$$

Thus, for any $0 < \delta < x$, we obtain

$$\begin{aligned}
&\lim_{s\searrow 0} \frac{1}{s}\,\mathbb{E}\Big[X^{(me)}(t+s) - X^{(me)}(t) \;\Big|\; X^{(me)}(t) = x\Big] \\
&= \lim_{s\searrow 0} \frac{1}{s}\,\mathbb{E}\Big[X^{(me)}(t+s) - x \;\Big|\; X^{(me)}(t) = x\Big] \\
&= \lim_{s\searrow 0} \frac{1}{s} \int_0^{\infty} (y-x)\left(\varphi_s(y-x) - \varphi_s(y+x)\right) \frac{\Phi_{1-t-s}(y) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}}\,\mathrm{d}y \\
&= \lim_{s\searrow 0} \frac{1}{s} \int_{-x}^{\infty} u\left(\varphi_s(u) - \varphi_s(u+2x)\right) \frac{\Phi_{1-t-s}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}}\,\mathrm{d}u \\
&= \lim_{s\searrow 0} \frac{1}{s} \int_{-\delta}^{\delta} u\varphi_s(u) \frac{\Phi_{1-t-s}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}}\,\mathrm{d}u \\
&\quad + \lim_{s\searrow 0} \frac{1}{s} \int_{-x}^{-\delta} u\varphi_s(u) \frac{\Phi_{1-t-s}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}}\,\mathrm{d}u \\
&\quad + \lim_{s\searrow 0} \frac{1}{s} \int_{\delta}^{\infty} u\varphi_s(u) \frac{\Phi_{1-t-s}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}}\,\mathrm{d}u \\
&\quad - \lim_{s\searrow 0} \frac{1}{s} \int_{-x}^{\infty} u\varphi_s(u+2x) \frac{\Phi_{1-t-s}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}}\,\mathrm{d}u \\
&= \lim_{s\searrow 0} \frac{1}{s} \int_{-\delta}^{\delta} u\,\varphi_s(u) \frac{\Phi_{1-t-s}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}}\,\mathrm{d}u. \qquad (3.14)
\end{aligned}$$

Here, we used that $\frac{\Phi_{1-t-s}(u+x)-\frac{1}{2}}{\Phi_{1-t}(x)-\frac{1}{2}}$ is bounded in u for fixed x, t, and s and (3.13) in the last step. Considering the first order Taylor approximation of $\frac{\Phi_{1-t}(u+x)-\frac{1}{2}}{\Phi_{1-t}(x)-\frac{1}{2}}$ at $u = 0$, we have, for $\varepsilon > 0$ and a small enough choice of $\delta > 0$,

$$\begin{aligned}
&\left| \frac{\Phi_{1-t}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}} - 1 - \frac{\exp\left(-\frac{x^2}{2(1-t)}\right)}{\int_0^x \exp\left(-\frac{y^2}{2(1-t)}\right)\mathrm{d}y}\, u \right| \\
&= \left| \frac{\Phi_{1-t}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}} - 1 - c^{(me)}(t,x)\, u \right| \le \varepsilon\, |u|, \quad u \in [-\delta, \delta], \qquad (3.15)
\end{aligned}$$

and, for some appropriate choice of $C > 0$,

$$\begin{aligned}
|\Phi_{1-t-s}(u+x) - \Phi_{1-t}(u+x)| &= \mathbb{P}\left(\frac{u+x}{\sqrt{1-t}} \le Y \le \frac{u+x}{\sqrt{1-t-s}} \right) \qquad (3.16) \\
&\le Cs, \quad u \in [-\delta, \delta].
\end{aligned}$$

Thus, we obtain

$$
\begin{aligned}
&\left| \frac{1}{s} \int_{-\delta}^{\delta} u\,\varphi_s(u) \left(\frac{\Phi_{1-t-s}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}} - \frac{\Phi_{1-t}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}} \right) \mathrm{d}u \right| \\
&\leq \frac{1}{s} \int_{-\delta}^{\delta} \left| u\,\varphi_s(u) \frac{\Phi_{1-t-s}(u+x) - \Phi_{1-t}(u+x)}{\Phi_{1-t}(x) - \frac{1}{2}} \right| \mathrm{d}u \\
&\leq C \int_{-\delta}^{\delta} \left| u\,\varphi_s(u) \frac{1}{\Phi_{1-t}(x) - \frac{1}{2}} \right| \mathrm{d}u \\
&\to 0,
\end{aligned}
$$

as $s \searrow 0$, where we used (3.16) in the second step. Now, using (3.15), we obtain further

$$
\begin{aligned}
&\limsup_{s\searrow 0} \left| \frac{1}{s} \int_{-\delta}^{\delta} u\,\varphi_s(u) \left(\frac{\Phi_{1-t-s}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}} - 1 - c^{(me)}(t,x)\,u \right) \mathrm{d}u \right| \\
&\leq \limsup_{s\searrow 0} \frac{1}{s} \int_{-\delta}^{\delta} \left| u\,\varphi_s(u) \left(\frac{\Phi_{1-t}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}} - 1 - c^{(me)}(t,x)\,u \right) \right| \mathrm{d}u \\
&\quad + \limsup_{s\searrow 0} \frac{1}{s} \int_{-\delta}^{\delta} \left| u\,\varphi_s(u) \left(\frac{\Phi_{1-t-s}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}} - \frac{\Phi_{1-t}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}} \right) \right| \mathrm{d}u \\
&\leq \varepsilon \lim_{s\searrow 0} \frac{1}{s} \int_{-\delta}^{\delta} u^2 \varphi_s(u)\,\mathrm{d}u + 0 \\
&= \varepsilon. \qquad (3.17)
\end{aligned}
$$

Thus, using (3.17) in the first step, we can continue the computation in (3.14) and obtain

$$
\begin{aligned}
&\lim_{s\searrow 0} \frac{1}{s} \int_{-\delta}^{\delta} u\,\varphi_s(u) \frac{\Phi_{1-t-s}(u+x) - \frac{1}{2}}{\Phi_{1-t}(x) - \frac{1}{2}} \,\mathrm{d}u \\
&= \lim_{s\searrow 0} \frac{1}{s} \int_{-\delta}^{\delta} u\,\varphi_s(u)\,(1 + c^{(me)}(t,x)\,u)\,\mathrm{d}u \\
&= 0 + c^{(me)}(t,x) \lim_{s\searrow 0} \frac{1}{s} \int_{-\delta}^{\delta} u^2 \varphi_s(u)\,\mathrm{d}u \\
&= c^{(me)}(t,x),
\end{aligned}
$$

which shows the claim.

Remark 3.5. It is worth pointing out that $(X^{(me)}(t))_{t\in[0,1]}$ can further be characterized as a Doob h-transform of Brownian motion: Using the explicitly known transition density of $(X^{(me)}(t))_{t\in[0,1]}$ in (3.12), it is just a simple observation that this transition density can be obtained by the Doob h-transform, for the harmonic function

$$h(t,x) = \mathbb{P}_x(B(s) > 0\colon s \in [0,1-t]) = 2\Phi_{1-t}(x) - 1$$

of Brownian motion on $[0,\infty)$, with killing at 0. The last equality follows straightforwardly from the reflection principle. Similarly, the transition density of Brownian motion on $[0,\infty)$, with killing at 0, can be obtained from the refection principle. The explicit expression of h can be used to obtain the SDE in Lemma 3.3 directly, see Section IV.39 in [69].

Remark. Most parts of this chapter are also available on *arXiv* in the preprint entitled *Penalizing fractional Brownian motion for being negative* (see [12]). The content of [12] was created in collaboration with F. Aurzada and M. Kilian, and only parts of this work are presented in this thesis. The remaining parts, including particularly the proof of Theorem 3.1, will almost surely appear in a doctoral thesis of M. Kilian.

Chapter 4

Limit theorems for random walks with absorption

In this chapter, we introduce a class of absorption mechanisms that generalize the classical persistence problem for random walks. More precisely, we consider centered random walks with finite variance and study the behavior of such random walks that do not get absorbed. Particularly, we prove persistence probability results as well as scaling limit results in this context. Further, we will see that our model brings different situations from the literature together and can be applied to many more examples.

4.1 Introduction

4.1.1 Absorption model

Let $(S_n)_{n\in\mathbb{N}}$ be a real-valued random walk with i.i.d. increments $X_1, X_2, \ldots$ starting in $S_0 = x \in \mathbb{R}$ and let $\mathbb{P}_x$ be the corresponding probability measure. We assume that $\mathbb{E}[X_1] = 0$ and $\sigma^2 := \mathbb{V}[X_1] \in (0, \infty)$. Then, our absorption model is defined as follows:

We denote by T_k the time of the k-th zero-crossing, so we set $T_0 := 0$ and

$$T_{k+1} := \inf\{n > T_k \colon S_n < 0,\ S_{T_k} \geq 0 \text{ or } S_n \geq 0,\ S_{T_k} < 0\}.$$

Further, we let U denote either a real-valued random variable or a sequence $U^{(0)}, U^{(1)}, \ldots$ of (not necessarily independent) real-valued random variables. Let $U_0, U_1, \ldots$ be independent copies of U that are also independent of $(S_n)_{n\in\mathbb{N}}$. Moreover, let $K_i \colon \mathbb{R} \times \mathbb{R} \to \{0, 1\}$ or, respectively, $K_i \colon \mathbb{R}^{\mathbb{N}} \times \mathbb{R} \to \{0, 1\}$ be measurable functions for $i \in \mathbb{N}$. Then, we define the time of absorption as

$$\tau := \inf\{n \colon \exists k \geq 0 \text{ such that } T_k \leq n < T_{k+1} \text{ and } K_{n-T_k}(U_k, S_n) = 1\}.$$

The family of functions $(K_i)_{i\in\mathbb{N}}$ describes the mechanism how the random walk gets absorbed. This mechanism depends on the passed time since the last zero-crossing, some random input, and the position of the random walk.

To get a better understanding of this model, let us consider several examples. In the classical situation ($\tau = T_1$), we do not need a random input and set $K_i(u,x) = 1$ if $x < 0$ and $K_i(u,x) = 0$ otherwise. We recall that this case is well understood, see Section 1.1.3, and one has

$$\mathbb{P}_x(\tau > N) = \mathbb{P}_x(S_n \geq 0\colon n \leq N) \sim c_x N^{-1/2}, \quad \text{as} \quad N \to \infty, \tag{4.1}$$

where $c_x > 0$ denotes a constant depending on the distribution of X_1 and x.

Let us also consider the situations in [51] and [78]. In [51], the random walk can stay a geometrically distributed time below zero instead of immediately getting killed when crossing zero. If we want to model this situation, we choose U geometrically distributed with parameter $q \in (0,1)$ and $K_i(u,x) = 1$ if $x < 0$, $i \geq u$ and $K_i(u,x) = 0$ otherwise. Then, the random walk gets absorbed when it is negative and the time spent below zero exceeds an independent geometrically distributed input. We emphasize at this point that our results also cover the situation with arbitrary distributed U. In contrast, it was crucial in [51] that U is geometrically distributed so that the Markovian structure is preserved.

In [78], random walks that avoid a bounded Borel set are studied. Modeling this situation is again very simple since no random input U is needed. Let B be the Borel set that is avoided by the random walk. Then, we set $K_i(u,x) = 1$ if $x \in B$ and $K_i(u,x) = 0$ otherwise.

These models will be important examples in this chapter; see also Section 1.1.3 for a more detailed discussion of these examples.

4.1.2 Results

Let us first fix some notation, which we will use to state our results. We define

$$\mathbb{E}[X; A] := \mathbb{E}[X \mathbb{1}_A],$$

where $\mathbb{1}_A$ denotes the indicator function of the measurable set A. We will fix a sequence $(a_n)_{n\in\mathbb{N}}$ of positive real numbers with $a_n = o(1)$ such that $a_n n^{1/2} \nearrow \infty$. Further, for a sequence $(A_n)_{n\in\mathbb{N}}$ of non-empty subsets of $\mathbb{R}$ and positive sequences $(f_x(n))_{n\in\mathbb{N}}$, $(g_x(n))_{n\in\mathbb{N}}$ depending on $x \in \mathbb{R}$, we say $f_x(n) \sim g_x(n)$ uniformly in A_n if $\sup_{x\in A_n}\left|\frac{f_x(n)}{g_x(n)} - 1\right| \to 0$, as $n \to \infty$.

Our main results will reduce persistence probability and scaling limit problems to related problems corresponding to the stopping time $\min(T_1, \tau)$ instead of the stopping time τ. These problems in turn can be reduced in many cases of interest to very well understood classical problems, as we will see later in this subsection and Section 4.2.

Before stating the results, we will take a short look at our assumptions and give a few comments on them. We will assume that there are constants $c > 0$ and $\gamma \in (0,1)$ such that, for all $x \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$\mathbb{P}_x(\tau \geq T_k) \leq c\gamma^k. \tag{C1}$$

We can think of γ as a surviving-fee that is due each time the random walk crosses zero. This assumption is crucial for our results. The following two assumptions encode the relation to the simpler problem corresponding to the stopping time $\min(T_1, \tau)$. We assume that there is a function $u\colon \mathbb{R} \to \mathbb{R}$ such that, for all $y \in \mathbb{R}$,

$$N^{1/2} \cdot \mathbb{P}_y(\tau > N, T_1 > N) \to u(y), \quad \text{as} \quad N \to \infty. \tag{C2}$$

Further, we only study the asymptotic behavior of a random walk in such an absorption model if the random walk starts in a point x such that a constant $c > 0$ can be chosen such that, for all $N \in \mathbb{N}$,

$$\mathbb{P}_x(\tau > N) \geq c^{-1} N^{-1/2}. \tag{C3}$$

The latter assumption guarantees that the probabilities of certain considered events are of the same order as the classical persistence probability in (4.1). In many cases of interest, these conditions can be verified relatively easily. For instance, let us again consider the situation in [51], where the random walk is allowed to stay a geometrically distributed time U below zero. In this case, condition (C1) clearly holds for a suitable choice of c and γ since

$$\mathbb{P}_x(\tau \geq T_k) \leq \mathbb{P}(U > 0)^{(k-1)/2} = (1-q)^{(k-1)/2}.$$

Further, by (4.1), the left term in (C2) converges to c_y for $y \geq 0$, since in this case $\{\tau > N, T_1 > N\} = \{T_1 > N\}$. For $y < 0$, we have

$$N^{1/2} \cdot \mathbb{P}_y(\tau > N, T_1 > N) \leq N^{1/2} \cdot \mathbb{P}(U > N) = N^{1/2} \cdot (1-q)^{N+1} \to 0,$$

as $N \to \infty$. Moreover, by (4.1), condition (C3) holds for $x \geq 0$. For $x < 0$, we note that there is always a positive probability that the random walk reaches the non-negative half-line without getting absorbed after a fixed number of steps. Then, we can use (4.1) again to obtain (C3).

In the situation of [78], where the random walk avoids a bounded Borel set, conditions (C1) - (C3) are fulfilled as well. Since verifying these conditions is slightly more complicated as in the preceding example, we refer to Section 4.2 for a treatment of this situation.

Our first main result in this chapter deals with the asymptotic behavior of the persistence probabilities of a random walk with absorption.

Theorem 4.1. *Assume that* (C1) *and* (C2) *hold. Then, for x satisfying* (C3)*, we have*

$$\mathbb{P}_x(\tau > N) \sim V(x)N^{-1/2}, \quad as \quad N \to \infty, \tag{4.2}$$

where

$$V(x) = \sum_{k=0}^{\infty} \mathbb{E}_x[u(S_{T_k}); \tau \geq T_k].$$

If the uniform condition

$$\sup_{|y| \leq a_N N^{1/2}} \Big| N^{1/2} \cdot \mathbb{P}_y(\tau > N, T_1 > N) - u(y) \Big| = o(1)$$

is fulfilled (instead of condition (C2)*), then statement* (4.2) *holds uniformly in $\{x\colon |x| \leq a'_N N^{1/2}, \mathbb{P}_x(\tau > N) \geq c^{-1}(|x|+1)N^{-1/2}\}$, where $(a'_n)_{n\in\mathbb{N}}$ is a sequence with $a'_n = o(1)$ and $a'_n n^{1/2} \nearrow \infty$.*

Our second main result concerns scaling limits of random walks conditioned not to get absorbed. For this purpose, let $(\hat{S}_n(t))_{t\in[0,1]}$ denote the continuous process with

$$\hat{S}_n(m/n) := S_m/(\sigma n^{1/2}) \quad \text{for } m \in [0,n] \cap \mathbb{N}$$

and which is linearly interpolated elsewhere. As before, σ^2 is the variance of X_1. For an event A, we denote by $Law_y(\hat{S}_n \mid A)$ the probability measure on the space $(C[0,1], \|\cdot\|_\infty)$ corresponding to the process $(\hat{S}_n(t))_{n\in\mathbb{N}}$ starting in y and conditioned on A. Here, $C[0,1]$ denotes the set of continuous functions defined on $[0,1]$ and $\|\cdot\|_\infty$ denotes the supremum norm. Further, for a continuous stochastic process $(X(t))_{t\in[0,1]}$, we denote by $Law(X)$ the corresponding probability measure on $(C[0,1], \|\cdot\|_\infty)$. We will use the symbol $\Rightarrow$ to denote weak convergence of such probability measures.

We assume that there are continuous stochastic processes $(X_+(t))_{t\in[0,1]}$ and $(X_-(t))_{t\in[0,1]}$ such that, for y with $u(y) > 0$,

$$Law_y(\hat{S}_N \mid \tau > N, T_1 > N) \Rightarrow \begin{cases} Law(X_+), & \text{if } y \geq 0, \\ Law(X_-), & \text{if } y < 0. \end{cases} \tag{C4}$$

In many cases of interest, identity (C4) can be easily deduced from classical results. For instance, let us once again consider the situation in [51]. We have already seen that in this situation $u(y) > 0$ if and only if $y \geq 0$. But, for $y \geq 0$, we have $\{\tau > N, T_1 > N\} = \{T_1 > N\}$ and (C4) is covered by classical results with X_+ being a standard Brownian meander; see e.g. [22]. For the situation in [78], we refer again to Section 4.2.

Now, we are ready to state the second main result of this chapter.

Theorem 4.2. *Assume that* (C1), (C2), *and* (C4) *hold. Then, for x satisfying* (C3), *we have*

$$Law_x(\hat{S}_N \mid \tau > N) \Rightarrow Law(\rho X_+ + (1-\rho)X_-),$$

where ρ denotes a random variable that is independent of X_+ and X_- with $\mathbb{P}(\rho = 1) = 1-\mathbb{P}(\rho = 0) = V(x)^{-1}\sum_{k=0}^{\infty}\mathbb{E}_x[u(S_{T_k}); \tau \geq T_k, S_{T_k} \geq 0] \in [0,1]$.

The proofs of our main results are based on the following observation: A random walk that survives a long time in such an absorption model typically crosses zero only a few times at the beginning and also the magnitude of an overshoot at a zero-crossing time is typically small. Once this is formalized and proved, our results can be naturally deduced from (C2) - (C4).

Roughly, two facts contribute to this observation. First, due to inequality (C1), only a few zero-crossings occur. Second, by results from renewal theory, which characterize the sizes of overshoots of a random walk over a fixed level, as well as classical persistence results, facts about the typical zero-crossing behavior of a random walk can be obtained, see Subsection 4.3.2. Combining this with the consequences of (C1) yields the above observation, which is rigorously stated in Lemma 4.6.

While the proof of Theorem 4.1 borrows many arguments from [78], our arguments to prove Theorem 4.2 are completely different from the corresponding ones in [78]. We give a direct proof without using deeper results characterizing tightness of probability measures on function spaces, which makes the proof less technical and the arguments might be easier to adopt in other similar situations.

4.2 Examples

Let us begin by recalling two results from the classical setup. First, we state a uniform version of (4.1) and specify the constant c_x. It holds uniformly in $\{x\colon |x| \leq a_N N^{1/2}\}$ that

$$\mathbb{P}_x(T_1 > N) \sim c_x N^{-1/2}, \tag{4.3}$$

where

$$c_x := \frac{\sqrt{2}|x - \mathbb{E}_x[S_{T_1}]|}{\sigma\sqrt{\pi}};$$

see e.g. Lemma 3 in [78]. Second, as already mentioned, for $x \geq 0$, we have

$$Law_x(\hat{S}_N \mid T_1 > N) \Rightarrow Law(X^{(me)}), \tag{4.4}$$

where $X^{(me)}$ is a standard Brownian meander. For the definition of a standard Brownian meander and a short discussion of its behavior, see (1.6) and Lemma 3.3, respectively. A proof of (4.4) for $x = 0$ can be found in [22].

The result for $x > 0$ follows, for example, with the techniques in the proof of Theorem 4.2.

Now, we will give several examples of absorption mechanisms. In particular, we will show how the situations in [51] and [78] are covered by our results, and thus, how these situations are connected. Since our results can be easily applied in the following examples, we just sketch how conditions (C1) - (C4) can be verified and leave the details to the reader.

First, let us consider two natural generalizations of the situation in [51], where the random walk is allowed to stay a geometrically distributed time U below zero. As earlier discussed in Section 1.1.3, the assumption of U being geometrically distributed is crucial in [51] because the techniques there heavily rely on the Markovian and the homogeneous structure of the process. Our approach does not need this assumption and also covers more sophisticated situations. Further, no scaling limit results have been proved in the literature.

- *Random times below zero:*
  Let U be an arbitrary non-negative random variable. In particular, we allow that $\mathbb{P}(U = \infty) > 0$. We set $K_i(u, x) = 1$ if $i \geq u$, $x < 0$ and $K_i(u, x) = 0$ otherwise. We can think of this model as follows: Every time the random walk enters the negative half-line, it can only survive an independent random time, according to the distribution of U, below zero.

  The case $\mathbb{P}(U = \infty) = 1$ is trivial since, in this case, no absorption occurs. Thus, let us exclude this case in the following. We choose u_0 such that $\mathbb{P}(U > u_0) < 1$. Then, (C1) is fulfilled since

$$\begin{aligned}\mathbb{P}_x(\tau \geq T_k) &\leq \prod_{i=0}^{k-1} \mathbb{P}_x(\{U_i > u_0\} \cup \{T_{i+1} - T_i \leq u_0\} \cup \{S_{T_i} \geq 0\}) \\ &\leq \prod_{i=0}^{k-1} \mathbb{P}(\{U_i > u_0\} \cup \{S_{T_i} \geq 0\}) \leq \mathbb{P}(U > u_0)^{(k-1)/2}.\end{aligned}$$

  For $y \geq 0$, we note that $\{\tau > N, T_1 > N\} = \{T_1 > N\}$, and thus, by (4.3), (C2) holds with $u(y) = c_y$. Further, again by (4.3), we obtain, for $y < 0$, that

$$\begin{aligned}N^{1/2} \cdot \mathbb{P}_y(\tau > N, T_1 > N) &= N^{1/2} \cdot \mathbb{P}_y(T_1 > N)\mathbb{P}(U > N) \\ &\to c_y \mathbb{P}(U = \infty),\end{aligned}$$

  as $N \to \infty$. Thus, (C2) holds with $u(y) = c_y \mathbb{P}(U = \infty)$ for $y < 0$. If U and X_1 are bounded from above, clearly (C3) does not hold for x below a certain negative level. To verify (C3) for all other x, we force the random walk to start with a certain number of positive jumps so that it reaches the non-negative half-line with a positive probability and use (4.3) afterward. Now, we can apply Theorem 4.1. By the same argument, we also obtain the uniform statement in Theorem 4.1. Finally, by (4.4) and an analogous argument as above, we obtain, for y with $u(y) > 0$,

$$Law_y(\hat{S}_N \mid \tau > N, T_1 > N) \Rightarrow \begin{cases} Law(X^{(me)}), & \text{if } y \geq 0, \\ Law(-X^{(me)}), & \text{if } y < 0. \end{cases}$$

Hence, we can apply Theorem 4.2.

- *Inhomogeneous absorption probabilities:*
We start with a measurable function $p\colon \mathbb{R} \to [0,1]$ with $p(x) = 0$ for $x \geq 0$ and $\liminf_{x\to-\infty} p(x) > 0$. Further, we let U be a sequence $U^{(0)}, U^{(1)}, \dots$ of independent random variables which are uniformly distributed on $[0,1]$. Let u denote a sequence $u^{(0)}, u^{(1)}, \dots$ of real numbers in $[0,1]$. Then, we set $K_i(u,x) = 1$ if $p(x) \geq u^{(i)}$ and $K_i(u,x) = 0$ otherwise. We can think of $p(x)$ as the probability of absorption at the point x. Thus, independently in each step, the process gets absorbed with a probability according to its current position.

  Let $y < 0$. Then, since $\liminf_{x\to-\infty} p(x) > 0$, we can consider the event where the random walk starts with a fixed number N_0 of negative jumps so that the probability p_0 of absorption until time N_0 is positive. Thus, it follows that (C1) holds since $\mathbb{P}_x(\tau \geq T_k) \leq (1-p_0)^{(k-1)/2}$. Using the same idea, we obtain that, for $y < 0$,

$$N^{1/2} \cdot \mathbb{P}_y(\tau > N, T_1 > N) \leq N^{1/2} \cdot (1-p_0)^{\lfloor N/N_0 \rfloor} \to 0,$$

  as $N \to \infty$. For $y \geq 0$, we note that $\{\tau > N, T_1 > N\} = \{T_1 > N\}$. Thus, it follows directly from (4.3) that (C2) holds with $u(y) = c_y$. Likewise, it follows directly from (4.4) that, for $y \geq 0$,

$$Law_y(\hat{S}_N \mid \tau > N, T_1 > N) \Rightarrow Law(X^{(me)}).$$

  Now, we can apply Theorem 4.1 and Theorem 4.2. Note that the uniform statement in Theorem 4.1 can be obtained by the same argument. Trivially, by considering $(-S_n)_{n\in\mathbb{N}}$, one obtains results for the case where $\liminf_{x\to\infty} p(x) > 0$ and p vanishes below a given level.

Let us now turn our attention to the situation in [78], where random walks that avoid a bounded Borel set are considered. In the next example, we will see how our results cover this situation. Further, we make the situation in [78] more complex by allowing some additional randomness to demonstrate how robust our toolkit is. Moreover, we consider a converse situation where the random walk is forced to pass an interval when crossing zero.

- *Avoiding random sets:*
  Let U be a discrete random variable on $\mathbb{N}$ and let $B_0, B_1, \ldots$ be a sequence of bounded Borel sets in $(-\infty, 0)$. Further, let us assume that B_0 has non-empty interior, $\mathbb{P}(U = 0) > 0$, and $(S_n)_{n\in\mathbb{N}}$ is non-arithmetic (i.e., $\mathbb{P}(X_1 \in d\mathbb{Z}) < 1$ for all $d > 0$). We set $K_i(u, x) = 1$ if $x \in B_u$ and $K_i(u, x) = 0$ otherwise. Thus, the walk gets absorbed when it hits these randomly chosen sets. If we choose $U = 0$, we are in the situation of [78].

  Using standard results from renewal theory, one can show that (C1) holds. For this purpose, we let the random walk start in the non-negative half-line. Now, the idea is to use the identity

$$\lim_{x\to\infty} \mathbb{P}_x(|S_{T_1}| < K) = \frac{1}{\mathbb{E}_0[|S_{T_1}|]} \int_0^K \mathbb{P}_0(|S_{T_1}| > t)\,\mathrm{d}t,$$

  where $K > 0$ is fixed, see e.g. Theorem III.10.3 (i) in [45]. Noting that the limiting distribution is continuous, we can consider the event where the random walk first exceeds a high enough level and $U = 0$. Then, since B_0 has non-empty interior, the probability of hitting B_0 when jumping over a boundary point of the interior of B_0 is positive. Now, condition (C1) follows from the strong Markov property of $(S_n)_{n\in\mathbb{N}}$. Since the random walk cannot be absorbed in the non-negative half-line, for $y \geq 0$, assumptions (C2) and (C4) are covered by (4.3) and (4.4), respectively. For $y < 0$, let us first fix an index u. Then, an application of our last example (inhomogeneous absorption probabilities) with

$$p(x) = \begin{cases} 1, & \text{if } x \geq 0 \text{ or } x \in B_u, \\ 0, & \text{otherwise,} \end{cases}$$

  yields to persistence probability and scaling limit results for the case that the random walk avoids the set B_u and the non-negative half-line. In particular, we obtain for every $y < 0$ a constant $c_y^{(u)}$ such that

$$N^{1/2} \cdot \mathbb{P}_y(S_n \notin B_u \text{ for } n \leq N,\ T_1 > N) \to c_y^{(u)}, \quad \text{as} \quad N \to \infty.$$

  Therefore, we obtain that (C2) holds with

$$\begin{aligned} &N^{1/2} \cdot \mathbb{P}_y(\tau > N, T_1 > N) \\ &\quad = \sum_{u\in\mathbb{N}} N^{1/2} \cdot \mathbb{P}_y(S_n \notin B_u \text{ for } n \leq N,\ T_1 > N)\mathbb{P}(U = u) \\ &\quad \to \sum_{u\in\mathbb{N}} c_y^{(u)} \mathbb{P}(U = u), \quad \text{as} \quad N \to \infty. \end{aligned}$$

The convergence in the last step follows by (4.3) and the dominated convergence theorem. Hence, we can apply Theorem 4.1 and Theorem 4.2. Note that the uniform statement in Theorem 4.1 can be obtained by the same argument. The d-arithmetic case (a largest constant $d > 0$ with $\mathbb{P}(X_1 \in d\mathbb{Z}) = 1$ exists) can be treated analogously.

- *Crossing zero through an interval:*
  Let I be an open interval containing zero and let $(S_n)_{n\in\mathbb{N}}$ be non-arithmetic. We set $K_i(u, x) = 1$ if $x \notin I$, $i = 0$ and $K_i(u, x) = 0$ otherwise. In this example, the random walk is forced to hit the interval I at zero-crossing times and we have no random input.

  If $\mathbb{P}_x(S_{T_k} \in I) = 1$ for all $k \in \mathbb{N}$, no absorption occurs. Otherwise, using the ideas from the last example (avoiding random sets), the application of our theorems is straightforward. The d-arithmetic case can be treated analogously.

Finally, we will introduce two very simple models and leave it completely to the reader to apply our results.

- *The simplest random example:*
  Let U be Bernoulli distributed. Then, we set $K_i(u, x) = 1$ if $u = 1$, $i = 0$ and $K_i(u, x) = 0$ otherwise. Thus, every time the walk crosses zero, we toss a coin. Depending on the outcome, the walk survives or gets absorbed.

- *Random boundaries:*
  Let U be a non-negative random variable with $\mathbb{P}(U = \infty) < 1$. We set $K_i(u, x) = 1$ if $x \leq -u$ and $K_i(u, x) = 0$ otherwise. Every time the random walk enters the negative half-line, it must stay above a random boundary to survive.

4.3 Auxiliary results

4.3.1 Notation

Let us first fix some notation. For the most part, we will follow the notation in [78]. We set

$$H_k := S_{T_k}.$$

Sometimes it is more convenient to work with one probability measure $\mathbb{P}$ instead of the family of probability measures $(\mathbb{P}_x)_{x\in\mathbb{R}}$. For this reason, we set $\mathbb{P} := \mathbb{P}_0$ and denote by $T_k(x)$ the k-th time where the random walk crosses the level $-x$. More precisely, we set $T_0(x) := 0$ and

$$T_{k+1}(x) := \inf\{n > T_k(x)\colon S_n < -x,\ S_{T_k(x)} \geq -x \text{ or } S_n \geq -x,\ S_{T_k(x)} < -x\}.$$

Accordingly, we set $H_k(x) := S_{T_k(x)} + x$. Further, in some situations, we will use the notation

$$p_y^{(T)}(N) := \mathbb{P}_y(T_1 > N), \quad p_y^{(\tau)}(N) := \mathbb{P}_y(\tau > N),$$

and

$$p_y^{(T,\tau)}(N) := \mathbb{P}_y(T_1 > N, \tau > N).$$

As before, we denote by $(a_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers with $a_n = o(1)$ and $a_n n^{1/2} \nearrow \infty$, as $n \to \infty$. Now, we let $(b_n)_{n\in\mathbb{N}}$ be a sequence of positive integers with $a_n^2 n = o(b_n)$ and $b_n = o(n)$. In particular, it follows that $b_n \to \infty$, as $n \to \infty$. To avoid technical problems, let us choose the sequence $(b_n)_{n\in\mathbb{N}}$ such that the sequences $(b_n)_{n\in\mathbb{N}}$ and $(n - b_n)_{n\in\mathbb{N}}$ are monotonically increasing.

4.3.2 Auxiliary results for random walks

We start by collecting some basic facts about random walks with finite variance, which will be used at several points in this chapter. Let us recall that there is a constant $c > 0$ such that

$$\mathbb{P}_x(T_1 = n) \leq c(|x| + 1)n^{-3/2} \quad \text{for all } x \in \mathbb{R}, \tag{4.5}$$

see Lemma 5 in [39]. Thus,

$$\begin{aligned} \mathbb{P}_x(T_1 > N) &= \sum_{n=N+1}^{\infty} \mathbb{P}_x(T_1 = n) \\ &\leq c(|x| + 1)N^{-1/2} \quad \text{for all } x \in \mathbb{R}. \end{aligned} \tag{4.6}$$

Here and in the following, c denotes a varying positive constant which can change from line to line to ease notation. It is well-known that the ladder heights of the random walk $(S_n)_{n\in\mathbb{N}}$ are integrable if and only if $\mathbb{V}[X_1] < \infty$, see e.g. Theorem 14.4.1 in [5]. Considering the random walk with increments distributed as the positive ladder heights of the random walk $(S_n)_{n\in\mathbb{N}}$, Theorem III.10.2 (iii) in [45] states that $\mathbb{E}_x[H_1] = o(|x|)$, as $x \to -\infty$. Combining this with the corresponding result for the case with increments distributed as the negative ladder heights of $(S_n)_{n\in\mathbb{N}}$, one obtains

$$\mathbb{E}_x[|H_1|] = o(|x|), \quad \text{as} \quad |x| \to \infty. \tag{4.7}$$

Using this fact, it can be obtained that, for any $\alpha \in (0, 1)$, there is a constant K_α such that, for all $x \in \mathbb{R}$,

$$\mathbb{E}_x[|H_1|] \leq \alpha|x| + K_\alpha. \tag{4.8}$$

Next, we show by induction that, for $k \in \mathbb{N}$,

$$\mathbb{E}_x[|H_k|] \leq \alpha^k |x| + \sum_{j=0}^{k-1} \alpha^j K_\alpha.$$

Let us assume that the statement holds for $k-1$, then, by (4.8) and the induction hypothesis, we obtain

$$\begin{aligned}
\mathbb{E}_x[|H_k|] &\leq \int \alpha|y| + K_\alpha \, \mathbb{P}_x(H_{k-1} \in \mathrm{d}y) \\
&\leq \alpha \left(\alpha^{k-1}|x| + \sum_{j=0}^{k-2} \alpha^j K_\alpha \right) + K_\alpha \\
&= \alpha^k |x| + \sum_{j=0}^{k-1} \alpha^j K_\alpha,
\end{aligned}$$

as required. Thus, we can deduce that

$$\mathbb{E}_x[|H_k|] \leq \alpha^k |x| + \sum_{j=0}^{k-1} \alpha^j K_\alpha \leq |x| + K, \tag{4.9}$$

where $K := K_\alpha/(1-\alpha)$.

Lemma 4.3. *Let $k \in \mathbb{N}$ be fixed. Then, as $N \to \infty$, one has*

(a1) $\sup_{|x| \leq a_N N^{1/2}} \mathbb{P}_x(|H_k| > a_N N^{1/2}) = o(1)$,

(a2) $\sup_{|x| \leq a_N N^{1/2}} \mathbb{E}_x[|H_k|; |H_k| > a_N N^{1/2}]/(|x|+1) = o(1)$,

(b1) $\sup_{|x| \leq a_N N^{1/2}} \mathbb{P}_x(T_k > b_N) = o(1)$,

(b2) $\sup_{|x| \leq a_N N^{1/2}} \mathbb{E}_x[|H_k|; T_k > b_N]/(|x|+1) = o(1)$.

Proof. The main ingredients of our proof are identities (4.6), (4.7), (4.9), and the fact that the family of random variables

$$\left\{ \frac{|H_1(x)|}{|x|} : |x| \geq 1 \right\} \tag{4.10}$$

is uniformly integrable, which is Theorem III.10.2 (ii) in [45]. As in (4.7), one considers the random walk with increments distributed as the positive and negative ladder heights of $(S_n)_{n \in \mathbb{N}}$, respectively, to apply the result in [45]. We will prove the different statements in this lemma by induction. Further, note that all statements are trivial for the case $k = 0$.

(a1) First, we will consider the case $k = 1$. Note that, by Markov's inequality and (4.8),

$$\sup_{|x|\le a_N N^{1/2}} \mathbb{P}_x(|H_1| > a_N N^{1/2}) \le \sup_{|x|\le a_N N^{1/2}} \frac{\mathbb{E}_x[|H_1|]}{a_N N^{1/2}}$$
$$\le \sup_{|x|\le a_N N^{1/2}} \frac{\alpha|x| + K_\alpha}{a_N N^{1/2}}$$
$$\le \alpha + \frac{K_\alpha}{a_N N^{1/2}}.$$

Since $a_N N^{1/2} \to \infty$, as $N \to \infty$, and $\alpha \in (0,1)$ was arbitrary, it follows that

$$\sup_{|x|\le a_N N^{1/2}} \mathbb{P}_x(|H_1| > a_N N^{1/2}) = o(1).$$

We will use this fact to prove (a2). Then, the statement for $k \ge 2$ follows from (a2) by Markov's inequality since

$$\sup_{|x|\le a_N N^{1/2}} \mathbb{P}_x(|H_k| > a_N N^{1/2})$$
$$= \sup_{|x|\le a_N N^{1/2}} \mathbb{P}_x(|H_k|\mathbb{1}_{\{|H_k|>a_N N^{1/2}\}} > a_N N^{1/2})$$
$$\le \sup_{|x|\le a_N N^{1/2}} \frac{\mathbb{E}_x[|H_k|; |H_k| > a_N N^{1/2}]}{a_N N^{1/2}}$$
$$= \sup_{|x|\le a_N N^{1/2}} \frac{\mathbb{E}_x[|H_k|; |H_k| > a_N N^{1/2}]}{|x|+1} \cdot \frac{|x|+1}{a_N N^{1/2}}$$
$$\le \sup_{|x|\le a_N N^{1/2}} \frac{\mathbb{E}_x[|H_k|; |H_k| > a_N N^{1/2}]}{|x|+1} \cdot \frac{a_N N^{1/2}+1}{a_N N^{1/2}} = o(1).$$

(a2) We begin by proving the statement for the case $k = 1$. First, let $0 \le x < 1$. Now, if $|H_1(x)| > 1$, we have $S_{T_1(x)} < -1$, and thus, $T_1(x) = T_1(1)$. In this case, it follows that

$$|H_1(x)| = |S_{T_1(x)} + x| = |S_{T_1(1)} + x|$$
$$\le |S_{T_1(1)}| + |x| \le |H_1(1)| + 1.$$

If $|H_1(x)| \le 1$, one trivially has

$$|H_1(x)| \le |H_1(1)| + 1.$$

Hence, for all $0 \le x < 1$, we obtain

$$|H_1(x)| \le |H_1(1)| + 1. \tag{4.11}$$

Thus, for $0 \le x < 1$, we have

$$\begin{aligned}\sup_{0\le x<1} \frac{\mathbb{E}_x[|H_1|; |H_1| > a_N N^{1/2}]}{x+1} &\le \sup_{0\le x<1} \mathbb{E}[|H_1(x)|; |H_1(x)| > a_N N^{1/2}] \\ &\le \mathbb{E}[|H_1(1)| + 1; |H_1(1)| + 1 > a_N N^{1/2}] \\ &= o(1).\end{aligned} \tag{4.12}$$

Further, using (a1) for the case $k = 1$ and using the uniform integrability of (4.10), we can conclude that

$$\begin{aligned}&\sup_{1\le x\le a_N N^{1/2}} \frac{\mathbb{E}_x[|H_1|; |H_1| > a_N N^{1/2}]}{x+1} \\ &\le \sup_{1\le x\le a_N N^{1/2}} \mathbb{E}\left[\frac{|H_1(x)|}{x}; H_1(x) > a_N N^{1/2}\right] = o(1).\end{aligned} \tag{4.13}$$

For negative x, we proceed analogously. Altogether, this shows the claim for $k = 1$.

Now, we proceed by induction. Let us assume that the statement holds for $1 \le j \le k$, and therefore, also (a1) holds for $1 \le j \le k$. We set

$$e^{(j)}(y, a, b) := \mathbb{E}_y[|H_j|; |H_j| > a, T_j > b].$$

Then, for $1 \le j \le k$, by the induction hypothesis, we can choose a sequence $(c_n^{(j)})_{n\in\mathbb{N}}$ with $c_n^{(j)} = o(1)$ such that

$$e^{(j)}(y, a_N N^{1/2}, 0) \le c_N^{(j)}(|y| + 1) \quad \text{and} \quad \mathbb{P}_y(|H_j| > a_N N^{1/2}) \le c_N^{(j)}, \tag{4.14}$$

for all $|y| \le a_N N^{1/2}$. Now, using the strong Markov property and splitting the event of the first jump, we obtain

$$\begin{aligned}&\mathbb{E}_x[|H_{k+1}|; |H_{k+1}| > a_N N^{1/2}] \\ &\quad= \int e^{(k)}(y, a_N N^{1/2}, 0)\, \mathbb{P}_x(H_1 \in \mathrm{d}y) \\ &\quad= \int e^{(k)}(y, a_N N^{1/2}, 0)\, \mathbb{P}_x(H_1 \in \mathrm{d}y, |H_1| > a_N N^{1/2}) \\ &\qquad+ \int e^{(k)}(y, a_N N^{1/2}, 0)\, \mathbb{P}_x(H_1 \in \mathrm{d}y, |H_1| \le a_N N^{1/2}) \\ &\quad\le \int (|y| + K)\, \mathbb{P}_x(H_1 \in \mathrm{d}y, |H_1| > a_N N^{1/2}) \\ &\qquad+ \int c_N^{(k)}(|y| + 1)\, \mathbb{P}_x(H_1 \in \mathrm{d}y, |H_1| \le a_N N^{1/2}) \\ &\quad\le c_N^{(1)}(|x| + K + 1) + c_N^{(k)}(|x| + K + 1),\end{aligned}$$

where we used (4.9), and (4.14) for the cases $j = 1$ and $j = k$. This completes the proof of the statement.

(b1) By (4.6), we have

$$\sup_{|x| \le a_N N^{1/2}} \mathbb{P}_x(T_1 > b_N) \le \sup_{|x| \le a_N N^{1/2}} c(|x|+1)\lfloor b_N \rfloor^{-1/2}$$
$$\le c(a_N N^{1/2} + 1)\lfloor b_N \rfloor^{-1/2} = o(1),$$

since $a_N N^{1/2} = o(b_N^{1/2})$ and $b_N \to \infty$, as $N \to \infty$. Thus, for $k = 1$, the statement holds.

Again, we proceed by induction and assume that the statement holds for $1 \le j \le k$. We have

$$\mathbb{P}_x(T_{k+1} > b_N) \le \mathbb{P}_x(T_k > b_N/2) + \mathbb{P}_x(T_{k+1} - T_k > b_N/2).$$

The first term vanishes uniformly in $\{x\colon |x| \le a_N N^{1/2}\}$ due to the induction hypothesis. For the second term, we obtain

$$\begin{aligned}
&\mathbb{P}_x(T_{k+1} - T_k > \ b_N/2) \\
&= \int p_y^{(T)}(b_N/2)\, \mathbb{P}_x(H_k \in \mathrm{d}y) \\
&= \int p_y^{(T)}(b_N/2)\, \mathbb{P}_x(H_k \in \mathrm{d}y, |H_k| > a_N N^{1/2}) \\
&\quad + \int p_y^{(T)}(b_N/2)\, \mathbb{P}_x(H_k \in \mathrm{d}y, |H_k| \le a_N N^{1/2}) \\
&\le \mathbb{P}_x(|H_k| > a_N N^{1/2}) \\
&\quad + \sup_{|x| \le a_N N^{1/2}} \mathbb{P}_x(T_1 > b_N/2).
\end{aligned}$$

Now, the claim follows from statement (a1) and the induction hypothesis for the case $j = 1$.

(b2) We proceed similarly as in the proof of (a2). By the monotonicity of $T_1(\,\cdot\,)$ and by (b1) for the case $k = 1$, we obtain by the same arguments used to show (4.12) and (4.13) that

$$\sup_{|x| \le a_N N^{1/2}} \frac{\mathbb{E}_x[|H_1|; T_1 > b_N]}{|x|+1} = o(1),$$

which is the statement for the case $k = 1$.

Let us now assume that the statement holds for $1 \le j \le k$. Then, by the induction hypothesis, we can choose a sequence $(c_n^{(j)})_{n \in \mathbb{N}}$ with $c_n^{(j)} = o(1)$, for $1 \le j \le k$, such that

$$e^{(j)}(y, 0, b_N/2) \le c_N^{(j)}(|y|+1), \quad \mathbb{P}_y(|T_j| > b_N/2) \le c_N^{(j)}, \tag{4.15}$$

and (4.14) hold, for all $|y| \leq a_N N^{1/2}$.

Now, we obtain

$$
\begin{aligned}
\mathbb{E}_x[&|H_{k+1}|; T_{k+1} > b_N] \\
&\leq \int e^{(k)}(y,0,0)\,\mathbb{P}_x(H_1 \in \mathrm{d}y, T_1 > b_N/2) \\
&\quad + \int e^{(k)}(y,0,b_N/2)\,\mathbb{P}_x(H_1 \in \mathrm{d}y, T_1 \leq b_N/2) \\
&\leq \int e^{(k)}(y,0,0)\,\mathbb{P}_x(H_1 \in \mathrm{d}y, T_1 > b_N/2) \\
&\quad + \int e^{(k)}(y,0,b_N/2)\,\mathbb{P}_x(H_1 \in \mathrm{d}y, |H_1| > a_N N^{1/2}) \\
&\quad + \int e^{(k)}(y,0,b_N/2)\,\mathbb{P}_x(H_1 \in \mathrm{d}y, |H_1| \leq a_N N^{1/2}) \\
&\leq \int (|y|+K)\,\mathbb{P}_x(H_1 \in \mathrm{d}y, T_1 > b_N/2) \\
&\quad + \int (|y|+K)\,\mathbb{P}_x(H_1 \in \mathrm{d}y, |H_1| > a_N N^{1/2}) \\
&\quad + \int c_N^{(k)}(|y|+1)\,\mathbb{P}_x(H_1 \in \mathrm{d}y, |H_1| \leq a_N N^{1/2}) \\
&\leq c_N^{(1)}(|x|+K+1) + c_N^{(1)}(|x|+K+1) \\
&\quad + c_N^{(k)}(|x|+K+1).
\end{aligned}
$$

Here, we used (4.9) and (4.15) for $j = k$ in the third step. Further, in the last step, we used (4.14) and (4.15) for the case $j = 1$ and again estimate (4.9). Now, the statement follows.

□

4.3.3 Auxiliary results for random walks with absorption

Lemma 4.4. *Assume* (C1) *holds. Then, $c > 0$ and $\gamma \in (0,1)$ can be chosen such that, for all $x \in \mathbb{R}$ and $k \in \mathbb{N}$,*

$$\mathbb{E}_x[|H_k|; \tau \geq T_k] \leq c\gamma^k(|x|+1).$$

Proof. The proof is divided into three steps.

Step 1: We begin by showing that, for all $\delta > 0$, a constant $K \geq 1$ can be chosen such that

$$\mathbb{P}_x(|H_1| > K) \leq \delta \quad \text{for all } x \in \mathbb{R}. \tag{4.16}$$

Let us first assume that the random walk $(S_n)_{n\in\mathbb{N}}$ is non-arithmetic. Let us further recall, from renewal theory, that in this case

$$\lim_{x\to\infty} \mathbb{P}_x(|H_1| < K) = \frac{1}{\mathbb{E}_0[|H_1|]} \int_0^K \mathbb{P}_0(|H_1| > t)\,\mathrm{d}t, \tag{4.17}$$

see e.g. Theorem III.10.3 (i) in [45]. Since $\mathbb{E}_0[|H_1|] = \int_0^\infty \mathbb{P}_0(|H_1| > t)\,\mathrm{d}t$, it follows directly from (4.17) that, for $\delta > 0$, a constant $x_0 \geq 0$ and K can be chosen such that $\mathbb{P}_x(|H_1| < K) \geq 1 - \delta$ for $x \geq x_0$. For $0 \leq x \leq x_0$, by the same argument that we used to show (4.11), we obtain that

$$|H_1(x)| \leq |H_1(x_0)| + x_0.$$

Thus, K can be chosen such that $\mathbb{P}_x(|H_1| < K) \geq 1 - \delta$ holds for all $x \geq 0$. Analogously, we argue for negative x. In the d-arithmetic case, we use the identity

$$\lim_{n\to\infty} \mathbb{P}_{nd}(|H_1| < kd) = \frac{d}{\mathbb{E}_0[|H_1|]} \sum_{j=0}^{k-1} \mathbb{P}_0(|H_1| > jd),$$

see e.g. Theorem III.10.3 (ii) in [45], and proceed likewise. Hence, in both cases, we obtain (4.16). Moreover, we choose $K \geq 1$ to make our next arguments a little bit smoother.

Step 2: We will now estimate the quantity $\mathbb{E}_x[|H_1|; |H_1| > K]$. First, we consider the case $|x| \geq 1$. Here, since $\delta > 0$ was arbitrary in (4.16), we obtain, due to the uniform integrability of (4.10), that, if K is chosen large enough,

$$\mathbb{E}_x[|H_1|; |H_1| > K] = |x|\,\mathbb{E}\left[\frac{H_1(x)}{|x|}; |H_1| > K\right] \leq \gamma|x| \quad \text{for } |x| \geq 1, \tag{4.18}$$

with γ from (C1). In particular, by an iteration procedure, we obtain that

$$\mathbb{E}_x[|H_k|; |H_j| > K \text{ for } j \leq k] \leq \gamma^k|x| \quad \text{for } |x| \geq 1. \tag{4.19}$$

Further, we obtain by the same argument as in (4.11) and (4.12) that

$$\mathbb{E}_x[|H_1|; |H_1| > K] \leq \gamma \quad \text{for } |x| < 1,$$

if K is chosen large enough. Combing this with (4.18), we obtain

$$\mathbb{E}_x[|H_1|; |H_1| > K] \leq \gamma K \quad \text{for } |x| \leq K. \tag{4.20}$$

Step 3: Now, we are ready to prove the statement of the lemma. For this purpose, we note that

$$\begin{aligned}\{\tau \geq T_k\} \subseteq \bigcup_{j=0}^{k-1} &\{|H_j| \leq K, |H_{j+1}| > K, \ldots, |H_k| > K, \tau \geq T_j\} \\ &\cup \{|H_0| > K, \ldots, |H_k| > K\} \cup \{|H_k| \leq K, \tau > T_k\}.\end{aligned} \tag{4.21}$$

Further, we obtain, for fixed $0 \le j \le k-1$,

$$\begin{aligned}
&\mathbb{E}_x[|H_k|; |H_j| \le K, |H_{j+1}| > K, \dots, |H_k| > K, \tau \ge T_j] \\
&\le \int \gamma^{k-j-1}|y|\, \mathbb{P}_x(H_{j+1} \in \mathrm{d}y, |H_{j+1}| > K, |H_j| \le K, \tau \ge T_j) \\
&\le \int \gamma^{k-j} K\, \mathbb{P}_x(H_j \in \mathrm{d}y, \tau \ge T_j) \\
&\le cK\gamma^k,
\end{aligned}$$

where we used (4.19) in the first step, (4.20) in the second step and (C1) in the last step. Hence, by (4.21), summing over j and using (4.19) and (C1) yields

$$\mathbb{E}_x[|H_k|; \tau \ge T_k] \le cKk\gamma^k + \gamma^k|x| + cK\gamma^k \quad \text{for all } x \in \mathbb{R}.$$

The claim follows now, when we choose (new) suitable constants $c > 0$ and $\gamma \in (0, 1)$. □

Next, we will prove an a priori estimate for the persistence probabilities $\mathbb{P}_x(\tau > N)$, which is of the same type as the classical result in (4.6).

Lemma 4.5. *If* (C1) *holds, then there is a constant* $c > 0$ *such that*

$$\mathbb{P}_x(\tau > N) \le c(|x| + 1)N^{-1/2} \quad \textit{for all } x \in \mathbb{R}.$$

Proof. We first decompose the persistence event as

$$\mathbb{P}_x(\tau > N) = \mathbb{P}_x(\tau > N, T_k > N) + \mathbb{P}_x(\tau > N, T_k \le N), \tag{4.22}$$

where $-\log(N)/\log(\gamma) \le k < -\log(N)/\log(\gamma) + 1$ with γ from (C1). Then, by (C1), the second term can be estimated by

$$\mathbb{P}_x(\tau > T_k) \le c\gamma^k = ce^{k\log(\gamma)} \le ce^{-\frac{\log(N)}{\log(\gamma)}\log(\gamma)} = cN^{-1} = o(N^{-1/2})$$

and thus is negligible. Now, we will estimate the first term on the right-hand side in (4.22). For this purpose, let $d := (1-\gamma)/2$ and note that, for N large enough,

$$\begin{aligned}
\sum_{j=0}^{k-1} \lceil Nd\gamma^j \rceil &\le \sum_{j=0}^{k-1} Nd\gamma^j + k \\
&= Nd\frac{1-\gamma^k}{1-\gamma} + k \\
&= N\frac{1-\gamma}{2} \cdot \frac{1-\gamma^k}{1-\gamma} + k \\
&\le N/2 - \log(N)/\log(\gamma) + 1 \\
&\le N.
\end{aligned}$$

Here, we used that $k < -\log(N)/\log(\gamma) + 1$ in the fourth step. Therefore, by inequality (4.6) in the third step and inequality (C1), and Lemma 4.4 in the fourth step, we obtain, for N large enough,

$$\begin{aligned}
\mathbb{P}_x(\tau > N, T_k > N) &\le \sum_{j=0}^{k-1} \mathbb{P}_x(\tau \ge T_j, T_{j+1} - T_j > \lceil N d\gamma^j \rceil) \\
&= \sum_{j=0}^{k-1} \int p_y^{(T)}(\lceil N d\gamma^j \rceil) \mathbb{P}_x(H_j \in \mathrm{d}y, \tau \ge T_j) \\
&\le \sum_{j=0}^{k-1} \int \frac{c(|y|+1)}{\sqrt{N d\gamma^j}} \mathbb{P}_x(H_j \in \mathrm{d}y, \tau \ge T_j) \\
&\le \sum_{j=0}^{k-1} \frac{c\gamma^j(|x|+1)}{\sqrt{N d\gamma^j}} \\
&\le c(|x|+1)N^{-1/2} \sum_{j=0}^{\infty} (\sqrt{\gamma})^j \\
&\le c(|x|+1)N^{-1/2}.
\end{aligned}$$

Again, the constant c changed from line to line for ease of notation. Choosing c large enough completes the proof. □

The proofs of our main results are based on the next lemma. It allows us to replace persistence events by easier to handle events where all zero-crossings occur at the beginning and overshoots over zero are relatively small.

Lemma 4.6. *Let* $c > 0$. *Assume that* (C1) *holds. Then, we have uniformly in* $\{x \colon |x| \le a_N N^{1/2}, \mathbb{P}_x(\tau > N) \ge c^{-1}(|x|+1)N^{-1/2}\}$ *that*

$$\mathbb{P}_x(\tau > N) \sim \mathbb{P}_x(\exists k \ge 0 \colon |H_k| \le a_N N^{1/2}, T_k \le b_N, T_{k+1} > N, \tau > N).$$

Proof. The proof is organized into two parts. First, we will prove that

$$\sup_{|x| \le a_N N^{1/2}} \frac{\mathbb{P}_x(\exists k \ge 0 \colon T_k \in [b_N, N], \tau > N)}{|x|+1} = o(N^{-1/2}).$$

Then, we will show that

$$\begin{aligned}
&\sup_{|x| \le a_N N^{1/2}} \frac{\mathbb{P}_x(\exists k \ge 0 \colon |H_k| > a_N N^{1/2}, T_k < b_N, T_{k+1} > N, \tau > N)}{|x|+1} \\
&= o(N^{-1/2}).
\end{aligned}$$

Combining both statements, completes the proof.

First part: The first part itself is divided into two steps. Let $\delta \in (0,1)$. We start by considering the case where the process crosses zero in the interval $[b_N, (1-\delta)N]$. By Lemma 4.5, we obtain that

$$\begin{aligned}
\mathbb{P}_x(\exists k \geq 0\colon T_k \in [b_N, (1-\delta)N], \tau > N) \\
&\leq \sum_{k=0}^{\infty} \mathbb{P}_x(T_k \in [b_N, (1-\delta)N], \tau > N) \\
&\leq \sum_{k=0}^{\infty} \int p_y^{(\tau)}(\lceil \delta N \rceil)\, \mathbb{P}_x(H_k \in \mathrm{d}y, T_k > b_N, \tau \geq T_k) \\
&\leq \sum_{k=0}^{\infty} \int \frac{c(|y|+1)}{\sqrt{\delta N}}\, \mathbb{P}_x(H_k \in \mathrm{d}y, T_k > b_N, \tau \geq T_k).
\end{aligned} \tag{4.23}$$

For our next argument, let us recall that, by (b1) and (b2) in Lemma 4.3, we have, for $|x| \leq a_N N^{1/2}$ and some sequence $(c_n^{(k)})_{n\in\mathbb{N}}$ with $c_n^{(k)} = o(1)$,

$$\int (|y|+1)\, \mathbb{P}_x(H_k \in \mathrm{d}y, T_k > b_N) \leq c_N^{(k)}(|x|+1),$$

and that, by Lemma 4.4 and (C1), we have, for all $x \in \mathbb{R}$,

$$\int (|y|+1)\, \mathbb{P}_x(H_k \in \mathrm{d}y, \tau \geq T_k) \leq c\gamma^k(|x|+1).$$

Now, let $\varepsilon > 0$. Let us choose k_0 such that $\sum_{k=k_0}^{\infty} c\gamma^k \leq \varepsilon/2$ and N_0 such that $c_N^{(k)} \leq \varepsilon/(2k_0)$ for $k < k_0$ and $N \geq N_0$. Then, we obtain, for $N \geq N_0$ and $|x| \leq a_N N^{1/2}$,

$$\begin{aligned}
&\sum_{k=0}^{\infty} \int (|y|+1)\, \mathbb{P}_x(H_k \in \mathrm{d}y, T_k > b_N, \tau \geq T_k) \\
&\quad\leq \sum_{k=0}^{k_0-1} \int (|y|+1)\, \mathbb{P}_x(H_k \in \mathrm{d}y, T_k > b_N) \\
&\qquad + \sum_{k=k_0}^{\infty} \int (|y|+1)\, \mathbb{P}_x(H_k \in \mathrm{d}y, \tau \geq T_k) \\
&\quad\leq \varepsilon(|x|+1).
\end{aligned}$$

Since ε was arbitrary, it follows that

$$\sup_{|x|\leq a_N N^{1/2}} \frac{\sum_{k=0}^{\infty} \int (|y|+1)\, \mathbb{P}_x(H_k \in \mathrm{d}y, T_k > b_N, \tau \geq T_k)}{|x|+1} = o(1). \tag{4.24}$$

By (4.23), we thus obtain that

$$\sup_{|x|\leq a_N N^{1/2}} \frac{\mathbb{P}_x(\exists k \geq 0\colon T_k \in [b_N, (1-\delta)N], \tau > N)}{|x|+1} = o((\delta N)^{-1/2}). \tag{4.25}$$

Now, in a second step, we will consider the case of zero-crossings between $(1-\delta)N$ and N (but not in the interval $[b_N,(1-\delta)N]$ as before). Due to inequality (4.5) in the second step and Lemma 4.4, and inequality (C1) in the fourth step, we obtain

$$\begin{aligned}
&\mathbb{P}_x(\exists k\geq 0\colon T_k<b_N,(1-\delta)N<T_{k+1}\leq N,\tau>N)\\
&\quad\leq\sum_{k=0}^{\infty}\mathbb{P}_x(T_k<b_N,(1-\delta)N-b_N\leq T_{k+1}-T_k\leq N,\tau\geq T_k)\\
&\quad\leq\sum_{k=0}^{\infty}\int\sum_{n=\lfloor(1-\delta)N-b_N\rfloor}^{N}\frac{c(|y|+1)}{n^{3/2}}\,\mathbb{P}_x(H_k\in\,\mathrm{d}y,\tau\geq T_k)\\
&\quad\leq\sum_{k=0}^{\infty}\int\frac{c(|y|+1)}{\sqrt{N}}\left(\frac{\sqrt{N}}{\sqrt{\lfloor(1-\delta)N-b_N\rfloor-1}}-1\right)\mathbb{P}_x(H_k\in\,\mathrm{d}y,\tau\geq T_k)\\
&\quad\leq\sum_{k=0}^{\infty}\frac{c\gamma^k(|x|+1)}{\sqrt{N}}\left(\frac{\sqrt{N}}{\sqrt{\lfloor(1-\delta)N-b_N\rfloor-1}}-1\right)\\
&\quad\leq\frac{c(|x|+1)}{\sqrt{N}}\left(\frac{\sqrt{N}}{\sqrt{\lfloor(1-\delta)N-b_N\rfloor-1}}-1\right).
\end{aligned}$$

Again, the constant c changed from line to line for ease of notation. Now, letting $\delta\searrow 0$, as $N\to\infty$, we can conclude that

$$\sup_{x\in\mathbb{R}}\frac{\mathbb{P}_x(\exists k\geq 0\colon T_k<b_N,(1-\delta)N<T_{k+1}\leq N,\tau>N)}{|x|+1}=o(N^{-1/2}).\tag{4.26}$$

Further, if $\delta\searrow 0$ slowly enough, so that (4.25) is in $o(N^{-1/2})$, we can combine (4.25) and (4.26) to obtain that

$$\sup_{|x|\leq a_N N^{1/2}}\frac{\mathbb{P}_x(\exists k\geq 0\colon T_k\in[b_N,N],\tau>N)}{|x|+1}=o(N^{-1/2}).$$

Second part: It remains to control the probability of observing a surviving path with a large overshoot at the last zero-crossing time before N. Using (4.6) in the third step, we obtain

$$\begin{aligned}
&\mathbb{P}_x(\exists k\geq 0\colon |H_k|>a_N N^{1/2},T_k<b_N,T_{k+1}>N,\tau>N)\\
&\quad\leq\sum_{k=0}^{\infty}\mathbb{P}_x(|H_k|>a_N N^{1/2},T_k<b_N,T_{k+1}-T_k>N-b_N,\tau\geq T_k)\\
&\quad\leq\sum_{k=0}^{\infty}p_y^{(T)}(N-b_N)\,\mathbb{P}_x(H_k\in\,\mathrm{d}y,|H_k|>a_N N^{1/2},\tau\geq T_k)\\
&\quad\leq\sum_{k=0}^{\infty}\frac{c(|y|+1)}{\sqrt{N-b_N}}\,\mathbb{P}_x(H_k\in\,\mathrm{d}y,|H_k|>a_N N^{1/2},\tau\geq T_k).
\end{aligned}\tag{4.27}$$

Now, along the same lines as in the proof of (4.24), we obtain, by Lemma 4.4, inequality (C1), and estimates (a1) and (a2) in Lemma 4.3, that

$$\sup_{|x|\le a_N N^{1/2}} \frac{\sum_{k=0}^{\infty}\int(|y|+1)\,\mathbb{P}_x(H_k\in \mathrm{d}y, |H_k|>a_N N^{1/2}, \tau\ge T_k)}{|x|+1} = o(1). \tag{4.28}$$

Thus, combing (4.27) and (4.28), we obtain that

$$\sup_{|x|\le a_N N^{1/2}} \frac{\mathbb{P}_x(\exists k\ge 0\colon |H_k|>a_N N^{1/2}, T_k<b_N, T_{k+1}>N, \tau>N)}{|x|+1}$$
$$= o(N^{-1/2}).$$

□

4.4 Proofs

4.4.1 Proof of Theorem 4.1

We will prove the non-uniform case first and the uniform case afterward. In both cases, the proof is based on the following application of Lemma 4.6. Uniformly in $\{x\colon |x|\le a_N N^{1/2}, \mathbb{P}_x(\tau>N)\ge c^{-1}(|x|+1)N^{-1/2}\}$, we obtain that

$$\begin{aligned}
&N^{1/2}\cdot\mathbb{P}_x(\tau>N)\\
&\quad\sim N^{1/2}\cdot\mathbb{P}_x(\exists k\ge 0\colon |H_k|\le a_N N^{1/2}, T_k\le b_N, T_{k+1}>N, \tau>N)\\
&\quad=\sum_{k=0}^{\infty}\int N^{1/2}\cdot p_y^{(T,\tau)}(N-t)\mathbb{1}_{\{|y|\le a_N N^{1/2},\, t\le b_N\}}\\
&\qquad\qquad\qquad\qquad \mathbb{P}_x(H_k\in \mathrm{d}y, T_k\in \mathrm{d}t, \tau\ge T_k)\\
&\quad\sim\sum_{k=0}^{\infty}\int (N-t)^{1/2}\cdot p_y^{(T,\tau)}(N-t)\mathbb{1}_{\{|y|\le a_N N^{1/2},\, t\le b_N\}}\\
&\qquad\qquad\qquad\qquad \mathbb{P}_x(H_k\in \mathrm{d}y, T_k\in \mathrm{d}t, \tau\ge T_k),
\end{aligned} \tag{4.29}$$

where $(b_n)_{n\in\mathbb{N}}$ is the sequence defined in Subsection 4.3.1. In particular, (4.29) holds for fixed x satisfying (C3). By (C2), we have, for all $y\in\mathbb{R}$,

$$(N-t)^{1/2}\cdot p_y^{(T,\tau)}(N-t)\mathbb{1}_{\{t\le b_N,\, |y|\le a_N N^{1/2}\}}\to u(y), \quad \text{as} \quad N\to\infty.$$

Further, by (4.6), it holds for all $y\in\mathbb{R}$ that

$$N^{1/2}\cdot p_y^{(T,\tau)}(N)\le N^{1/2}\cdot\mathbb{P}_y(T_1>N)\le c(|y|+1).$$

Hence, the statement for fixed x follows immediately from (4.29), by the use of Lemma 4.4, inequality (C1), and the dominated convergence theorem.

Now, we consider the uniform case and thus assume that

$$\sup_{|y|\le a_N N^{1/2}} \left| N^{1/2}\cdot p_y^{(T,\tau)}(N) - u(y) \right| = o(1). \tag{4.30}$$

Let $(a'_n)_{n\in\mathbb{N}}$ be the sequence given by $a'_n := a_{n-b_n}(1-b_n/n)^{1/2}$. Then, since $(b_n)_{n\in\mathbb{N}}$ was chosen such that $(n-b_n)_{n\in\mathbb{N}}$ is monotonically increasing and $a_n n^{1/2} \nearrow \infty$, we have

$$a'_n n^{1/2} = a_{n-b_n}(1-b_n/n)^{1/2} n^{1/2} = a_{n-b_n}(n-b_n)^{1/2} \nearrow \infty.$$

We have further $(a'_n)^2 n \le a_n^2 n = o(b_n)$ and $a'_n n^{1/2} \le a_{n-t}(n-t)^{1/2}$ for $t \le b_n$, by the same argument. Thus, we obtain from (4.30) that

$$\begin{aligned} &\sup_{|y|\le a'_N N^{1/2},\, t\le b_N} \left| (N-t)^{1/2}\cdot p_y^{(T,\tau)}(N-t) - u(y) \right| \\ &\le \sup_{N-b_N\le n\le N}\ \sup_{|y|\le a_n n^{1/2}} \left| n^{1/2}\cdot p_y^{(T,\tau)}(n) - u(y) \right| = o(1). \end{aligned} \tag{4.31}$$

Further, due to the assumption $\mathbb{P}_x(\tau > N) \ge c^{-1}(|x|+1)N^{-1/2}$, we have $N^{1/2}\cdot\mathbb{P}_x(\tau > N) \ge c^{-1}(|x|+1)$. Hence, we obtain that, uniformly in $\{x\colon |x| \le a'_N N^{1/2},\ \mathbb{P}_x(\tau > N) \ge c^{-1}(|x|+1)N^{-1/2}\}$,

$$\begin{aligned} &N^{1/2}\cdot\mathbb{P}_x(\tau > N) \\ &\quad\sim \sum_{k=0}^{\infty} \int u(y)\mathbb{1}_{\{|y|\le a'_N N^{1/2},\, t\le b_N\}}\, \mathbb{P}_x(H_k \in \mathrm{d}y, T_k \in \mathrm{d}t, \tau \ge T_k) \\ &\quad\sim \sum_{k=0}^{\infty} \int u(y)\, \mathbb{P}_x(H_k \in \mathrm{d}y, \tau \ge T_k) \\ &\quad= \sum_{k=0}^{\infty} \mathbb{E}_x[u(H_k); \tau \ge T_k], \end{aligned}$$

where we used (4.29), (4.31), and inequality (C1) in the first step. In the second step, we used inequalities (4.24) and (4.28) combined with the fact that $u(y) \le c(|y|+1)$. This finishes the proof.

4.4.2 Proof of Theorem 4.2

The idea of the proof is based on the observations after the statement of Theorem 4.2: A random walk that survives until time N typically crosses zero only a few times at the beginning and then stays on one side of zero. One would expect that this beginning part should disappear in the scaling limit. Our main tools to show this are Lemma 4.6 from above and Theorem 5.5 in [20], which is an extension of the continuous mapping theorem that allows us to replace the continuous function in the classical theorem by certain sequences of continuous functions. In our situation, it states the following. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of stochastic processes and X be stochastic process in $(C[0,1],\|\cdot\|_\infty)$. Let $\tilde{\Theta}_n\colon (C[0,1],\|\cdot\|_\infty)\to(C[0,1],\|\cdot\|_\infty)$ be measurable functions for $n\in\mathbb{N}$ and assume that

$$\{h\in C[0,1]\colon \exists (h_n)\subseteq C[0,1] \text{ s.t. } \|h_n-h\|_\infty\to 0,\ \|\tilde{\Theta}_n(h_n)-\tilde{\Theta}(h)\|_\infty \not\to 0\}$$

is a null set with respect to $Law(X)$. Then,

$$Law(X_N)\Rightarrow Law(X) \quad \text{implies} \quad Law(\tilde{\Theta}_N(X_N))\Rightarrow Law(\tilde{\Theta}(X)). \tag{4.32}$$

We begin by considering a modified random walk, which is composed of a fixed beginning part and a standard random walk. We will show that the scaling limit of this process does not depend on the beginning part. For this purpose, let us introduce some notation. Let $t_0>0$ and $g\in C[0,t_0]$. For $N\in\mathbb{N}$, we denote by $g^{(N)}\in C[0,t_0/N]$ the rescaled version of g, defined by $g^{(N)}(t):=g(t/N)/(\sigma N^{1/2})$. Further, for $N\geq t_0$ and $h\in C[0,1]$, we denote by $\Theta_N(g^{(N)},h)\in C[0,1]$ the function given by

$$\Theta_N(g^{(N)},h)(t):=\begin{cases} g^{(N)}(t), & \text{if } t<t_0/N,\\ h(t-t_0/N)-h(0)+g^{(N)}(t_0/N), & \text{if } t_0/N\leq t\leq 1.\end{cases}$$

Note that $\Theta_N(g^{(N)},\cdot)$ is continuous on $C[0,1]$ with respect to the supremum norm $\|\cdot\|_\infty$.

Now, we want to study weak limits of the rescaled processes $\Theta_N(g^{(N)},\hat{S}_N)$ in $(C[0,1],\|\cdot\|_\infty)$, as $N\to\infty$. Let $f\colon C[0,1]\to\mathbb{R}$ be a continuous and bounded function. Then, we are interested in the limit of the quantity

$$d_N(f,g^{(N)}):=\frac{\mathbb{E}_y[f(\Theta_N(g^{(N)},\hat{S}_N));T_1>N,\tau>N]}{\mathbb{P}_y(T_1>N,\tau>N)}, \quad \text{as } N\to\infty, \tag{4.33}$$

where $y=g(t_0)$. Let us assume that $u(y)>0$. Then, by (C4), we have

$$\frac{\mathbb{E}_y[f(\hat{S}_N);T_1>N,\tau>N]}{\mathbb{P}_y(T_1>N,\tau>N)}\to\begin{cases}\mathbb{E}[f(X_+)], & \text{if } y\geq 0,\\ \mathbb{E}[f(X_-)], & \text{if } y<0,\end{cases} \quad \text{as} \quad N\to\infty.$$

In the next step, we can thus apply the earlier mentioned extension of the continuous mapping theorem (4.32) with $\tilde{\Theta}_N(\cdot):=\Theta_N(g^{(N)},\cdot)$. We obtain

$$d_N(f,g^{(N)})\to\begin{cases}\mathbb{E}[f(X_+)], & \text{if } y\geq 0,\\ \mathbb{E}[f(X_-)], & \text{if } y<0,\end{cases} \quad \text{as} \quad N\to\infty, \tag{4.34}$$

if $\lim_{N\to\infty}\|\Theta_N(g^{(N)},h_N)-h\|_\infty=0$ for all $h,h_1,h_2,\ldots\in C[0,1]$ with $h(0)=0$ and $\lim_{N\to\infty}\|h-h_N\|_\infty=0$. But this follows already from the following simple calculation. We have

$$\begin{aligned}\|\Theta_N(g^{(N)},h_N)-h\|_\infty &\le \|\Theta_N(g^{(N)},h_N)-\Theta_N(g^{(N)},h)\|_\infty\\ &\quad+\|\Theta_N(g^{(N)},h)-\Theta_N(0,h)\|_\infty+\|\Theta_N(0,h)-h\|_\infty\\ &\le (\|h_N-h\|_\infty+|h_N(0)|)\\ &\quad+\|g^{(N)}\|_\infty+\|\Theta_N(0,h)-h\|_\infty.\end{aligned}$$

Clearly, the first three terms tend to 0, as $N\to\infty$. Since h is uniformly continuous, as a fixed continuous function on a compact interval, also the last term tends to 0, and thus,

$$\|\Theta_N(g^{(N)},h_N)-h\|_\infty\to 0,\quad \text{as}\quad N\to\infty.$$

In the next step, we provide a slightly modified version of Lemma 4.6. Due to (C2) and the dominated convergence theorem, we can replace the integrand in (4.29) by $N^{1/2}\cdot p_y^{(T,\tau)}(N)$ and obtain

$$\mathbb{P}_x(\tau>N)\sim\mathbb{P}_x(\exists k\ge 0\colon T_k\le b_N, T_{k+1}-T_k>N, \tau>N+T_k).$$

Therefore, since

$$\{\exists k\ge 0\colon T_k\le b_N, T_{k+1}-T_k>N, \tau>N+T_k\}\subseteq\{\tau>N\},\tag{4.35}$$

and f is bounded, we obtain

$$\begin{aligned}&\left|\frac{\mathbb{E}[f(\hat S_N);\tau>N]}{\mathbb{P}(\tau>N)}-\frac{\mathbb{E}[f(\hat S_N);\exists k\ge 0\colon T_k\le b_N, T_{k+1}-T_k>N,\tau>N+T_k]}{\mathbb{P}(\tau>N)}\right|\\ &\le\|f\|_\infty\cdot\frac{\mathbb{P}(\{\tau>N\}\setminus\{\exists k\ge 0\colon T_k\le b_N, T_{k+1}-T_k>N,\tau>N+T_k\})}{\mathbb{P}(\tau>N)}\\ &=\|f\|_\infty\cdot\left(1-\frac{\mathbb{P}(\{\exists k\ge 0\colon T_k\le b_N, T_{k+1}-T_k>N,\tau>N+T_k\})}{\mathbb{P}(\tau>N)}\right)\\ &\to 0,\quad\text{as}\quad N\to\infty.\end{aligned}$$

Thus, in the following, we can replace the event $\{\tau>N\}$ by the easier to handle event on the left-hand side in (4.35).

We recall that, by (4.6),

$$N^{1/2}\cdot p_y^{(T,\tau)}(N)|d_N(f,g^{(N)})|\le c(|y|+1)\|f\|_\infty.\tag{4.36}$$

Further, we set $\hat S_N^{(t)}:=\hat S_N|_{[0,t/N]}\in C[0,t/N]$. Then, by Theorem 4.1 in the second step and by (C2), (4.34), (4.36), and the dominated convergence theorem in the third step, we obtain

$$\begin{aligned}
&\frac{1}{p_x^{(\tau)}(N)}\mathbb{E}_x[f(\hat{S}_N); \exists k \geq 0\colon T_k \leq b_N, T_{k+1} - T_k > N, \tau > N + T_k] \\
&= \frac{1}{p_x^{(\tau)}(N)} \sum_{k=0}^{\infty} \int p_y^{(T,\tau)}(N) d_N(f, g^{(N)}) \mathbb{1}_{\{t \leq b_N\}} \\
&\qquad\qquad\qquad\qquad \mathbb{P}_x(H_k \in \mathrm{d}y, T_k \in \mathrm{d}t, \hat{S}_N^{(t)} \in \mathrm{d}g^{(N)}, \tau \geq T_k) \\
&\sim V(x)^{-1} \sum_{k=0}^{\infty} \int N^{1/2} \cdot p_y^{(T,\tau)}(N) d_N(f, g^{(N)}) \mathbb{1}_{\{t \leq b_N\}} \\
&\qquad\qquad\qquad\qquad \mathbb{P}_x(H_k \in \mathrm{d}y, T_k \in \mathrm{d}t, \hat{S}_N^{(t)} \in \mathrm{d}g^{(N)}, \tau \geq T_k) \\
&\sim V(x)^{-1} \sum_{k=0}^{\infty} \int u(y) \left(\mathbb{E}[f(X_+)] \mathbb{1}_{\{y \geq 0\}} + \mathbb{E}[f(X_-)] \mathbb{1}_{\{y < 0\}} \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{P}_x(H_k \in \mathrm{d}y, \tau \geq T_k) \\
&= \mathbb{P}(\rho = 1)\mathbb{E}[f(X_+)] + \mathbb{P}(\rho = 0)\mathbb{E}[f(X_-)].
\end{aligned}$$

Remark. Most parts of Chapter 4 are to appear in the journal *Journal of Theoretical Probability* in the article entitled *Limit theorems for random walks with absorption* (see [24]). The article has not been allocated to an issue yet but it has already been published online.

Chapter 5

Modified ruin probabilities in the Cramér-Lundberg model

In this chapter, we study the asymptotics of the ruin probability in the Cramér-Lundberg model with a modified notion of ruin. The modification is as follows. If the portfolio becomes negative, the asset is not immediately declared ruined but may survive due to certain mechanisms. Under a rather general assumption on the mechanism – satisfied by most of such modified models from the literature – we study the relation of the asymptotics of the modified ruin probability to the classical ruin probability. This is done under the Cramér condition as well as for subexponential integrated claim sizes.

5.1 Ruin model

Let $(U(t))_{t\geq 0}$ be a classical Cramér-Lundberg process. We recall that

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0,$$

where $u \geq 0$ denotes the initial capital, $c > 0$ is the constant premium rate, $(N(t))_{t\geq 0}$ a Poisson process with rate $\lambda > 0$ describing the number of claims until time t, and the sequence of non-negative i.i.d. claim sizes is denoted by $(Y_k)_{k\in\mathbb{N}}$ and is also independent of $(N(t))_{t\geq 0}$. The process $(U(t))_{t\geq 0}$ describes the amount of surplus of an insurance portfolio indexed by time. Further, we assume that $\mathbb{E}[Y_1] = \mu > 0$ and that the net profit condition $c > \lambda\mu$ is satisfied. We denote the distribution function of Y_1 by F and set $\overline{F}(t) := 1 - F(t)$. Moreover, let us recall that

$$\psi_{\text{cl}}(u) := \mathbb{P}_u(T_1 < \infty)$$

is the classical ruin probability, where, as before, $T_1 := \inf\{t > 0\colon U(t) < 0\}$ and $\inf\emptyset := \infty$. In this context, we refer to T_1 as the time of ruin.

Let us now introduce our more general concept of ruin: Let $\psi\colon \mathbb{R} \to [0,1]$ be a measurable function. Then, $\psi(u)$ is called modified ruin probability, for initial capital u, if

$$\psi(u) = \int_{-\infty}^{0} \psi(y)\, \mathbb{P}_u(U(T_1) \in \mathrm{d}y,\, T_1 < \infty), \quad u \geq 0. \tag{5.1}$$

This assumption expresses that the mechanism that causes ruin gets activated when the process hits the negative half-line. The general form of (5.1) allows us to gather most of the models from the literature as well as many new models under one umbrella. For an overview of models with modified notion of ruin from the literature, we refer to Section 1.2.2 and Section 5.3.

In order to define such a model and to verify (5.1), it is often natural to define a corresponding time of modified ruin τ. Then, we set $\psi(u) := \mathbb{P}_u(\tau < \infty)$. For example, in the situation of cumulative Parisian ruin (at level $r > 0$), the process is allowed to stay negative for the fixed time r in total. In this case, $\tau := \inf\{t > 0\colon \int_0^t \mathbb{1}_{(-\infty,0)}(U_s)\,\mathrm{d}s > r\}$ defines the time of modified ruin. It follows immediately from the strong Markov property that (5.1) is satisfied. Further, note that also every choice of a measurable function $\psi(u)$, for $u < 0$, with values in $[0,1]$, defines such a model via (5.1). Note also that the case $\psi(u) = 1$, for $u < 0$, coincides with the classical case.

5.2 Results

We investigate the two classical situations: Either the Cramér condition is fulfilled, or the integrated claim sizes are subexponential.

We recall, see Section 1.2.2, that the Cramér condition is satisfied for a constant $R > 0$ if

$$\lambda\mathbb{E}[\exp(RY_1) - 1] = cR.$$

In this case, one has

$$\psi_{\mathrm{cl}}(u) \sim k e^{-Ru}, \quad \text{as} \quad u \to \infty,$$

where $k = \left[\frac{\lambda R}{c-\lambda\mu}\int_0^\infty x e^{Rx}(1 - F(x))\,\mathrm{d}x\right]^{-1}$, see e.g. Theorem 1.2.2 in [38].

Futher, we recall that a distribution function F is called subexponential if

$$\lim_{u\to\infty} \frac{\overline{F^{(*n)}}(u)}{\overline{F}(u)} = n \quad \text{for all } n \in \mathbb{N}.$$

(This assumption is equivalent to $\limsup_{u\to\infty} \overline{F^{(*2)}}(u)/\overline{F}(u)) \leq 2$, see e.g. Lemma 1.3.4 in [38].) In this case, we write $F \in \mathcal{S}$. Now, let F_I be the distribution function defined by $F_I(t) := \frac{1}{\mu}\int_0^t \overline{F}(s)\,\mathrm{d}s$ for $t \geq 0$. If $F_I \in \mathcal{S}$, one has

$$\psi_{\mathrm{cl}}(u) \sim \frac{\lambda}{c - \lambda\mu}\int_u^\infty \overline{F}(z)\,\mathrm{d}z = \frac{\lambda\mu}{c-\lambda\mu}\overline{F_I}(u), \quad \text{as} \quad u \to \infty,$$

see e.g. Theorem 1.3.6 in [38]. We refer to this situation as heavy-tailed in the following. For a discussion on subexponential distributions in general, see e.g. [38].

Our main result of this chapter treats the relation of the asymptotics of modified ruin probabilities to the classical ruin probability.

Theorem 5.1. *Let ψ be any measurable function satisfying* (5.1)*.*

1. *Suppose the Cramér condition is fulfilled with parameter $R > 0$. If ψ is continuous or monotone on $(-\infty, 0)$, then $\psi(u) \sim C\psi_{\mathrm{cl}}(u)$, as $u \to \infty$, where $C = \int_{-\infty}^{0} \psi(y)\,\mathbb{P}_\infty(\mathrm{d}y)$ and the probability measure $\mathbb{P}_\infty$ has the distribution function $\frac{\lambda}{c-\lambda\mu}\int_0^\infty (e^{Rz}-1)\overline{F}(z-\cdot)\,\mathrm{d}z$.*

2. *If $F_I \in \mathcal{S}$ and $\lim_{u\to-\infty}\psi(u) = 1$, then $\psi(u) \sim \psi_{\mathrm{cl}}(u)$, as $u \to \infty$.*

Proof. Due to (5.1), we have

$$\begin{aligned}\psi(u) &= \int_{-\infty}^{0} \psi(y)\,\mathbb{P}_u(U(T_1) \in \mathrm{d}y,\, T_1 < \infty)\\ &= \psi_{\mathrm{cl}}(u)\int_{-\infty}^{0} \psi(y)\,\mathbb{P}_u(U(T_1) \in \mathrm{d}y \mid T_1 < \infty),\end{aligned} \tag{5.2}$$

and the analysis of the asymptotic behavior of the modified ruin probabilities reduces to the analysis of the integral $\int_{-\infty}^{0} \psi(y)\,\mathbb{P}_u(U(T_1) \in \mathrm{d}y \mid T_1 < \infty)$, as $u \to \infty$. Our result is based on Theorem 2 in [72], which states that, if the limit

$$\gamma(z) = \lim_{u\to\infty} \frac{\psi_{\mathrm{cl}}(u+z)}{\psi_{\mathrm{cl}}(u)} \tag{5.3}$$

exists, then

$$\begin{aligned}&\lim_{u\to\infty} \mathbb{P}_u(-U(T_1) > x \mid T_1 < \infty)\\ &= \frac{1}{c-\lambda\mu}\left(c\gamma(x) - \lambda\int_0^x \gamma(x-z)\overline{F}(z)\,\mathrm{d}z - \lambda\int_x^\infty \overline{F}(z)\,\mathrm{d}z\right).\end{aligned} \tag{5.4}$$

Let us first assume that the Cramér condition is fulfilled for $R > 0$. Since $\psi_{\mathrm{cl}}(u) \sim ke^{-Ru}$ for some $k > 0$, as $u \to \infty$, the limit in (5.3) exists with $\gamma(z) = e^{-Rz}$. Further, using Fubini's theorem and using the Cramér condition $\lambda\mathbb{E}[\exp(RY_1) - 1] = cR$, one obtains

$$\begin{aligned}
\lambda \int_0^\infty e^{Rz}\overline{F}(z)\,\mathrm{d}z &= \lambda \int_0^\infty \int e^{Rz} \mathbb{1}_{Y_1>z}\,\mathrm{d}\mathbb{P}\,\mathrm{d}z \\
&= \lambda \int \int_0^\infty e^{Rz} \mathbb{1}_{Y_1>z}\,\mathrm{d}z\,\mathrm{d}\mathbb{P} \\
&= \lambda \int \int_0^{Y_1} e^{Rz}\,\mathrm{d}z\,\mathrm{d}\mathbb{P} \\
&= \lambda \int \frac{1}{R}\left(e^{RY_1} - 1\right)\,\mathrm{d}\mathbb{P} \\
&= \frac{\lambda}{R}\mathbb{E}\left[e^{RY_1} - 1\right] = c.
\end{aligned}$$

In particular, one has

$$c - \lambda \int_0^x e^{Rz}\overline{F}(z)\,\mathrm{d}z = \lambda \int_x^\infty e^{Rz}\overline{F}(z)\,\mathrm{d}z.$$

Thus, in this case, the limit in (5.4) can be written as

$$\begin{aligned}
&\frac{1}{c-\lambda\mu}\left(c\gamma(x) - \lambda\int_0^x \gamma(x-z)\overline{F}(z)\,\mathrm{d}z - \lambda\int_x^\infty \overline{F}(z)\,\mathrm{d}z\right) \\
&= \frac{1}{c-\lambda\mu}\left(ce^{-Rx} - \lambda\int_0^x e^{-R(x-z)}\overline{F}(z)\,\mathrm{d}z - \lambda\int_x^\infty \overline{F}(z)\,\mathrm{d}z\right) \\
&= \frac{e^{-Rx}}{c-\lambda\mu}\left(c - \lambda\int_0^x e^{Rz}\overline{F}(z)\,\mathrm{d}z - \lambda\int_x^\infty e^{Rx}\overline{F}(z)\,\mathrm{d}z\right) \\
&= \frac{e^{-Rx}}{c-\lambda\mu}\left(\lambda\int_x^\infty \left(e^{Rz} - e^{Rx}\right)\overline{F}(z)\,\mathrm{d}z\right) \\
&= \frac{\lambda}{c-\lambda\mu}\left(\int_x^\infty \left(e^{R(z-x)} - 1\right)\overline{F}(z)\,\mathrm{d}z\right) \\
&= \frac{\lambda}{c-\lambda\mu}\left(\int_0^\infty \left(e^{Rz} - 1\right)\overline{F}(z+x)\,\mathrm{d}z\right).
\end{aligned}$$

Therefore, $\mathbb{P}_u(U(T_1) \in \cdot \mid T_1 < \infty)$ converges weakly to the probability measure $\mathbb{P}_\infty$ with distribution function

$$x \mapsto \frac{\lambda}{c-\lambda\mu}\int_0^\infty (e^{Rz} - 1)\overline{F}(z-x)\,\mathrm{d}z.$$

If ψ is continuous (and bounded) on $(-\infty, 0)$, the claim follows immediately from (5.2) by the definition of weak convergence. Since the limit distribution is continuous, the claim follows as well if ψ is monotone on $(-\infty, 0)$. For instance, this can be seen by approximating ψ by step functions from above and below. Then, since intervals are continuity sets of the measure $\mathbb{P}_\infty$, an application of the Portmanteau theorem shows the claim.

Now, if $F_I \in \mathcal{S}$, one has $\psi_{\mathrm{cl}}(u) \sim \frac{\lambda}{c-\lambda\mu}\int_u^\infty \overline{F}(z)\,\mathrm{d}z = \frac{\lambda\mu}{c-\lambda\mu}\overline{F_I}(u)$, as $u \to \infty$. Since, by Lemma 1.3.5 in [38], F_I is long-tailed, we have $\gamma(z) = 1$ in (5.3). Therefore, by (5.4), we obtain for all $x \geq 0$ that

$$\begin{aligned}
&\lim_{u\to\infty} \mathbb{P}_u(-U(T_1) > x \mid T_1 < \infty) \\
&= \frac{1}{c-\lambda\mu}\left(c\gamma(x) - \lambda\int_0^x \gamma(x-z)\overline{F}(z)\,\mathrm{d}z - \lambda\int_x^\infty \overline{F}(z)\,\mathrm{d}z\right) \\
&= \frac{1}{c-\lambda\mu}\left(c - \lambda\int_0^\infty \overline{F}(z)\,\mathrm{d}z\right) = \frac{1}{c-\lambda\mu}(c-\lambda\mu) = 1.
\end{aligned}$$

Thus, for any $x \geq 0$,

$$\begin{aligned}
\int_{-\infty}^0 \psi(y)\,\mathbb{P}_u(U(T_1) \in \mathrm{d}y \mid T_1 < \infty) &\geq \inf_{y<-x}\psi(y)\,\mathbb{P}_u(-U(T_1) > x \mid T_1 < \infty) \\
&\to \inf_{y<-x}\psi(y),
\end{aligned}$$

as $u \to \infty$. Since x was arbitrary, we can let $x \to \infty$ and the claim follows. □

Remark 5.2. Theorem 5.1 is particularly useful in the heavy-tailed case. One obtains exact asymptotic results for the modified ruin probabilities without computing $\psi(u)$ explicitly for $u < 0$, as long as $\lim_{u\to-\infty}\psi(u) = 1$. This condition is very natural since, in most situations, it should become impossible to survive with negative surplus u, when $u \to -\infty$. Likewise, without computing $\psi(u)$ explicitly, for $u < 0$, one obtains that modified and classical ruin probabilities differ asymptotically by a constant C if the Cramér condition is fulfilled and ψ is continuous or monotone on $(-\infty, 0)$. In contrast to the heavy-tailed case, in most situations, it is not obvious how the constant C can be computed. Thus, under the Cramér condition, the result can be used primarily to obtain a first classification of the asymptotic behavior of the modified ruin probability $\psi(u)$, as $u \to \infty$. Again, e.g. the monotonicity assumption is very natural since, in most situations, it should become harder to survive when the surplus becomes more negative.

Remark 5.3. The proof of Theorem 5.1 hinges on the limit theorem for the probability measure $\mathbb{P}_u(U(T_1) \in \cdot \mid T_1 < \infty)$, which leads to the asymptotic results. For more precise results, more information about this probability measure is required. For example, explicit results (in terms of $\psi(u)$ for $u < 0$) can be obtained if the claim sizes are phase-type distributed. In this case, the minimum of $(U(t))_{t\geq 0}$ is again phase-type distributed and so is $\mathbb{P}_u(U(T_1) \in \cdot \mid T_1 < \infty)$, see e.g. [36]. The distribution of the minimum of $(U(t))_{t\geq 0}$ determines the classical ruin probability $\psi_{\mathrm{cl}}(u)$. In cases the distribution of $\mathbb{P}_u(U(T_1) \in \cdot \mid T_1 < \infty)$ is known, (5.2) can be used to obtain an explicit expression for $\psi(u)$. For the special case of the exponential distribution, see e.g. [7].

In many modified ruin models from the literature, the process starts renewed after surviving an excursion in the negative half-line. More precisely, in such situations, one has $1-\psi(y) = p_y(1-\psi(0))$ with $p_y := \mathbb{P}_y(T^{(0)} < \tau)$, for $y < 0$, where $T^{(0)} := \inf\{t > 0 \colon U(t) = 0\}$. That means, if the process survives until it reaches zero after becoming negative, the process starts renewed and survives afterward with probability $1-\psi(0)$. In the following proposition, we will give an expression for the modified ruin probability $\psi(u)$ in terms of p_y. Afterward, we will give in Remark 5.6 a new interpretation of the function p_y.

Proposition 5.4. *Let ψ satisfy condition* (5.1). *If $1-\psi(y) = p_y(1-\psi(0))$, for $y < 0$, one has*

$$q_0 := 1-\psi(0) = \frac{1-\psi_{\mathrm{cl}}(0)}{1-p_0}$$

with $p_0 := \mathbb{P}_0(T^{(0)} < \tau, T_1 < \infty)$ and

$$\psi(u) = \psi_{\mathrm{cl}}(u)\left(1 - q_0 \int_{-\infty}^{0} p_y \, \mathbb{P}_u(U(T_1) \in dy \mid T_1 < \infty)\right). \tag{5.5}$$

Proof. By (5.1) and the assumption $1-\psi(y) = p_y(1-\psi(0))$, we obtain

$$\begin{aligned} 1-\psi(u) &= 1 - \int_{-\infty}^{0} \psi(y) \, \mathbb{P}_u(U(T_1) \in \mathrm{d}y,\, T_1 < \infty) \\ &= 1-\psi_{\mathrm{cl}}(u) + \int_{-\infty}^{0} (1-\psi(y)) \, \mathbb{P}_u(U(T_1) \in \mathrm{d}y,\, T_1 < \infty) \\ &= 1-\psi_{\mathrm{cl}}(u) + (1-\psi(0)) \int_{-\infty}^{0} p_y \, \mathbb{P}_u(U(T_1) \in \mathrm{d}y,\, T_1 < \infty). \end{aligned} \tag{5.6}$$

For $u = 0$, it follows that

$$1-\psi(0) = (1-\psi_{\mathrm{cl}}(0)) + (1-\psi(0)) \int_{-\infty}^{0} p_y \, \mathbb{P}_0(U(T_1) \in \mathrm{d}y,\, T_1 < \infty),$$

and thus,

$$q_0 = 1-\psi(0) = \frac{1-\psi_{\mathrm{cl}}(0)}{1-p_0}$$

with $p_0 = \mathbb{P}_0(T^{(0)} < \tau,\, T_1 < \infty) = \int_{-\infty}^{0} p_y \, \mathbb{P}_0(U(T_1) \in \mathrm{d}y,\, T_1 < \infty)$. Now, equation (5.5) follows from (5.6) since

$$\begin{aligned} \psi(u) &= \psi_{\mathrm{cl}}(u) - (1-\psi(0)) \int_{-\infty}^{0} p_y \, \mathbb{P}_u(U(T_1) \in \mathrm{d}y,\, T_1 < \infty) \\ &= \psi_{\mathrm{cl}}(u) - q_0 \psi_{\mathrm{cl}}(u) \int_{-\infty}^{0} p_y \, \mathbb{P}_u(U(T_1) \in \mathrm{d}y \mid T_1 < \infty) \\ &= \psi_{\mathrm{cl}}(u)\left(1 - q_0 \int_{-\infty}^{0} p_y \, \mathbb{P}_u(U(T_1) \in \mathrm{d}y \mid T_1 < \infty)\right). \end{aligned}$$

□

Remark 5.5. Since $\mathbb{P}_0(U(T_1) \in \cdot \mid T_1 < \infty)$ has the distribution function F_I, see e.g. Proposition 8.3.2 in [38], an explicit expression for p_0 in terms of p_y, for $y < 0$, is available. Further, it is well-known that $\psi_{\text{cl}}(0) = \frac{\mu\lambda}{c}$, see e.g. p. 31 in [38]. This together with Proposition 5.4 gives an explicit expression for $\psi(0)$ in terms of p_y for $y < 0$.

Remark 5.6. The formulation of the modified ruin probability in (5.5) in terms of p_y leads to a new perspective: We can think of p_y, for $y < 0$, as the probability of finding an investor when the surplus drops below zero that pays until recovery. This perspective is also a natural starting point to build new models in the sense that any measurable function $p_y \in [0, 1]$ on $(-\infty, 0)$ defines a model with a modified definition of ruin and the preceding interpretation.

Remark 5.7. Proposition 5.4 gives us an exact expression for the modified ruin probability $\psi(u)$ in terms of p_y, for $y < 0$, and the probability measure $\mathbb{P}_u(U(T_1) \in \cdot \mid T_1 < \infty)$. Combining this result with Theorem 5.1, we obtain that, if the Cramér condition is fulfilled and if p_y is continuous or monotone, one has $\psi(u) \sim C\psi_{\text{cl}}(u)$, as $u \to \infty$, with $C = 1 - q_0 \int_{-\infty}^{0} p_y \, \mathbb{P}_\infty(\,\mathrm{d}y)$. The condition $\lim_{u\to-\infty} \psi(u) = 1$ in the second part of Theorem 5.1 translates now into the condition $\lim_{y\to-\infty} p_y = 0$. In this case $\psi(u) \sim \psi_{\text{cl}}(u)$, as $u \to \infty$. Again, we emphasize at this point that the above assumptions are quite natural. For example, it is natural to assume that p_y is monotone since it should be harder to find an investor when the surplus becomes more negative. Similarly, it should become impossible to find an investor with a negative surplus y, as $y \to -\infty$.

5.3 Examples and outlook

We will give examples and show that our results can be applied to many established models from the literature.

Example 5.8. We choose $p_y := p \in [0, 1]$ for $y < 0$. Then, $p_0 = p\frac{\mu\lambda}{c}$, and thus, by (5.5),

$$\psi(u) = \psi_{\text{cl}}(u) \left(1 - p \frac{1 - \frac{\mu\lambda}{c}}{1 - p\frac{\mu\lambda}{c}} \right).$$

This example corresponds to the situation where the probability of finding an investor does not depend on $U(T_1)$.

Example 5.9. If the claim sizes are $\exp(\delta)$-distributed, one obtains straightforwardly from (5.5), using the memorylessness property of the exponential distribution, for arbitrary p_y, that

$$\psi(u) = \psi_{\text{cl}}(u) \frac{1 - c\delta p_0/\lambda}{1 - p_0}, \quad \text{with } p_0 = \frac{\lambda}{c\delta} \int_{-\infty}^{0} p_y \delta e^{\delta y} \, \mathrm{d}y.$$

The situation of phase-type distributed claim sizes is treated analogously.

Next, we will see that our results can be applied to most of the models from the literature with a modified definition of ruin.

Example 5.10. First, let us recall the definitions of Parisian ruin and cumulative Parisian ruin. Let $g_t := \sup\{s \le t \colon U_s \ge 0\}$ be the last time before t where the Cramér-Lundberg process jumped into the negative half-line. Then, the time of Parisian ruin (at level $r > 0$) is defined as

$$\tau := \inf\{t > 0 \colon t - g_t > r\}.$$

That is, ruin occurs when the time of a single excursion in the negative half-line exceeds r. The time of cumulative Parisian ruin (at level $r > 0$) is defined as

$$\tau := \inf\left\{t > 0 \colon \int_0^t \mathbb{1}_{(-\infty,0)}(U_s)\,\mathrm{d}s > r\right\}.$$

Here, the total time spent in the negative half-line is not allowed to exceed r. If the constant r is replaced by an independent exponentially distributed random variable, one obtains the definition of exponential (cumulative) Parisian ruin. In these cases, it is straightforward to verify the assumption on $\psi(u)$ in Theorem 5.1, for $u < 0$. Recall that explicit expressions of the cumulative Parisian ruin probabilities are given in [44] for exponentially distributed claim sizes. Our results extend these results to asymptotic results for more general claim size distributions if the Cramér condition is satisfied or if $F_I \in \mathcal{S}$.

Example 5.11. Our results can be applied to so-called omega models: Let $\omega \colon \mathbb{R} \to \mathbb{R}$ be a monotonically non-increasing function with $\omega(y) = 0$ for $y \ge 0$, and $\omega(y) > 0$ for $y < 0$. Let e_1 be an exp(1)-distributed random variable independent of $(U(t))_{t \ge 0}$. Then, in an omega model, the time of ruin is defined as

$$\tau := \inf\left\{t > 0 \colon \int_0^t \omega(U_s)\,\mathrm{d}s > \mathrm{e}_1\right\}.$$

Thus, ω can be understood as a bankruptcy rate in this model. It is straightforward to verify that ψ is positive and monotone, and that further $\lim_{u \to -\infty} \psi(u) = 1$. Thus, we can apply Theorem 5.1. Particularly, we extend the results in [4] – where the authors restricted themselves to exponentially distributed claim sizes – to asymptotic results if the Cramér condition is fulfilled or if $F_I \in \mathcal{S}$.

Remark 5.12. If the bankruptcy rate in Example 5.11 is constant for $y < 0$, the process can only stay exponential times in the negative half-line. Thus, we have the same situation as in the exponential (cumulative) Parisian ruin model in Example 5.10. Due to the memorylessness of the exponential distribution, this situation coincides further with a model where the insurance company can only go bankrupt after independent exponential times, see [1] and [2]. For this connection and more motivation for omega models, see [4]. Further, if ω is constant for $y < 0$, it is not hard to show that $p_y = e^{\gamma y}$ (in Proposition 5.4), for $y < 0$ and some $\gamma > 0$ depending on the bankruptcy rate and F. Hence, we have four different pictures in this case: Omega model with constant bankruptcy rate, exponential (cumulative) Parisian ruin, random observation times (with exponential times between observations), and a model where the probability of finding an investor decays exponentially.

Example 5.13. In the model considered in [25], the insurance company can borrow money at a fixed debit interest rate when $U(t)$ is negative. Clearly, if the surplus is below a certain negative level, the due interest exceeds the income of the insurance company, and recovery is impossible. Hence, in terms of our model, $\psi(y)$ and p_y take the value 1 and 0, respectively, below this negative level. Thus, Theorem 5.1 can be applied. Moreover, we improve Theorem 4.1 in [25], since we can drop some of the technical assumptions there.

Finally, let us give a short outlook. In this chapter, the Cramér-Lundberg model was considered to demonstrate our technique. We have proved that under the natural assumptions in Theorem 5.1, classical and modified ruin probabilities differ asymptotically by a constant factor if the Cramér condition is satisfied and are asymptotically equivalent if $F_I \in \mathcal{S}$.

There are many ways our results can be generalized. Generally, as soon as limit theorems similar to (5.4) are available for a process, corresponding results can be obtained. For instance, one can involve further quantities that affect the mechanism that causes ruin. For example, the quantity U_{T_1-} can be easily involved, see e.g. [72]. However, there are, to the best of the authors knowledge, no modified ruin definitions in the current literature using this quantity.

Another direction is to consider different types of processes. It seems natural to consider spectrally negative Lévy processes and processes that are perturbed by a Brownian motion. In the latter case, the process does not necessarily enter the negative half-line with a jump, and thus, this event would require additional techniques.

Remark. Most parts of Chapter 5 are to appear in the journal *European Actuarial Journal* in the article entitled *Ruin probabilities in the Cramér-Lundberg model with temporarily negative capital* (see [11]). The article has not been allocated to an issue yet but it has already been published online.

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