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# Symmetries and turbulence modeling

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## ABSTRACT

This work applies new insights into turbulent statistics gained by Lie symmetry analysis to the closure problem of turbulence. Founded in the mathematics of partial differential equations, Lie symmetries have helped advances in many fields of modern physics. The main reason for this is their ability to encode important physical principles that are implicitly expressed by governing equations. Newly discovered symmetries of the multi-point correlation equations describing turbulent motion have been shown to encode two central effects of turbulent statistics: intermittency and non-Gaussianity. Moreover, these symmetries play a pivotal role in obtaining turbulent scaling laws such as the logarithmic law of the wall. Evidently, correctly preserving these symmetry properties in a turbulence model would render it capable of accurately predicting important effects of turbulent statistics and turbulent scaling. As these symmetry constraints have so far not been taken into account when devising turbulence models, we present a completely new modeling framework that can yield models fulfilling these conditions. In order to accomplish this, it turns out to be helpful, if not necessary, to introduce an entirely new symmetry-based modeling strategy that allows systematically constructing equations based on symmetry constraints imposed on them. From these considerations, it can be shown that in order to create meaningful turbulence models that fulfill these constraints, it is necessary to introduce a new velocity and pressure field. A possible skeleton of model equations for second moment closure is presented.



## I. INTRODUCTION

Even though turbulence is a phenomenon of central importance in many natural and technical flows, its efficient and accurate numerical treatment remains an open challenge even in the age of supercomputers. Incompressible turbulent flows are governed by the Navier–Stokes equations,

$$\frac{\partial U_i}{\partial x_i} = 0, \tag{1}$$

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + \frac{\partial P}{\partial x_i} - \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} = 0, \tag{2}$$

where $t$ stands for the temporal coordinate, $x_i$ is the spatial coordinate, $U_i$ represents the velocity, $P$ is the pressure divided by the density, and $\nu$ is the kinematic viscosity. Here and throughout the paper, the Einstein summation convention is implied. Even though Eqs. (1) and (2) contain the entire information needed to numerically investigate turbulence, the direct numerical simulation of the Navier–Stokes system turns out to be unfeasible for most technical applications. Due to turbulent velocity and pressure fluctuations on small scales, whose ratio to the largest scales, according to Kolmogorov theory, decreases with $\mathrm{Re}^{-3/4}$ in each spatial direction, an extremely high spatial and temporal numerical resolution would theoretically be required, restricting the application of this approach to low Reynolds number flows. Rather than resolving the randomly fluctuating velocity field and subsequently calculating the averaged velocity field, an equation for the time- or ensemble-averaged velocity is solved directly. The most common procedure to derive such an equation was proposed by Reynolds.[1] First, velocity and pressure are decomposed into a mean and a fluctuating part,

$$U_i = \overline{U}_i + u_i, \quad P = \overline{P} + p. \tag{3}$$

Here, averaged quantities are denoted with a bar, and fluctuating variables are assigned lowercase letters. Second, inserting (3) into the Navier–Stokes equations (1) and (2) and averaging the equations yields the Reynolds-Averaged Navier–Stokes (RANS) equations,

$$\frac{\partial \overline{U}_i}{\partial x_i} = 0, \tag{4}$$

$$\frac{\partial \overline{U}_i}{\partial t} + \overline{U}_j \frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{P}}{\partial x_i} - \nu \frac{\partial^2 \overline{U}_i}{\partial x_j \partial x_j} + \frac{\partial R_{ij}^{(0)}}{\partial x_j} = 0. \tag{5}$$

The main advantage of this statistical representation of turbulence is its similarity to the original system. Apart from the unknown Reynolds stress tensor $R_{ij}^{(0)} = \overline{u_i u_j}$, which can be interpreted as an additional stress specific to turbulent flow, the form of Eqs. (4) and (5) is equivalent to the original Navier–Stokes equations (1) and (2). The "(0)" in the exponent denotes one-point quantities to be defined more precisely below.

Obviously, the existence of a new unknown term at this stage prohibits solving the RANS equations. A transport equation for the unknown Reynolds stress tensor is given by[2]

$$\begin{aligned} &\frac{\partial R_{ij}^{(0)}}{\partial t} + \overline{U}_k \frac{\partial R_{ij}^{(0)}}{\partial x_k} + R_{ik}^{(0)} \frac{\partial \overline{U}_j}{\partial x_k} + R_{jk}^{(0)} \frac{\partial \overline{U}_i}{\partial x_k} \\ &- \overline{p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}^{(0)} + 2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}^{(0)} \\ &+ \frac{\partial}{\partial x_k}\left(R_{ijk}^{(0)} + \overline{(\delta_{jk} u_i + \delta_{ik} u_j) p}^{(0)} - \nu \frac{\partial R_{ij}^{(0)}}{\partial x_k}\right) = 0, \end{aligned} \tag{6}$$

where $\delta_{ij}$ is the Kronecker delta. As no additional physical information is employed in obtaining this equation, it is not surprising that the original problem is not solved. Indeed, Eq. (6) contains multiple new unknown terms, namely, the fifth to the eighth term. Again, equations for these terms can be derived, and the new equations contain further unknown expressions. We are, hence, confronted with an infinite hierarchy of unclosed equations, which constitutes the famous closure problem of turbulence. In order to obtain numerical or analytical solutions of this hierarchy, the infinite chain has to be broken at some point by disregarding the existence of higher level equations. The unknown terms in the highest considered equation are replaced by empirical closure relations relating the higher-order correlations to the lower ones.

Hypothetically, an ideal turbulence model could be used for an arbitrarily complex flow without prior knowledge of flow-specific model parameters. Donaldson and Rosenbaum[3] established the well-known concept of invariant modeling, which states that such a model has to fulfill at least the following four conditions:

(i) the model equations are in correct tensor formulation,
(ii) all equations are dimensionally correct,
(iii) the model is Galilean invariant, and
(iv) all relevant conservation laws are fulfilled.

Note that fulfilling the four conditions does not guarantee accuracy of the model. They only establish a foundation that allows the model to perform equally under a wide range of flow conditions, to be pointed out below.

Even though conditions (i)–(iv) of invariant modeling are very intuitive, it is worthwhile to examine how to justify them rigorously. Correct tensor formulation ensures that the choice of the coordinate system does not affect the form of the equations and, hence, the results obtained with the model. In other words, the model is invariant under a change of the coordinate system. Similarly, dimensional consistency warrants invariance under a change of the unit system. Galilean invariance allows one to arbitrarily choose the frame of reference. For instance, it must not have an effect on measurement results if the entire system is moved at a constant velocity. These seemingly unrelated observations can be unified using the concept of symmetries, which is the common principle in all cases, i.e., a specific variable transformation that leaves the equations unchanged. Transformations that leave objects invariant are known as invariant or symmetry transformations. In colloquial language, the term symmetry typically refers to a geometrical transformation, such as a rotation, that leaves an object, e.g., a circle, unchanged. Extending this concept to algebraic and differential equations has proven extremely useful in mathematics and physics. In turbulence modeling, it has mostly been used implicitly and intuitively, without embracing its full mathematical potential.

The reason for this disregard of symmetry theory becomes apparent when studying the symmetries of the Navier–Stokes equations, which were first obtained by Bytev.[4] It should be noted that obtaining the symmetries of any equation is an algorithmic, albeit possibly very time-consuming task. When analyzing Eqs. (1) and (2), we first consider the limit of vanishing molecular viscosity, i.e., the Euler equations, whose symmetries are given by the transformations

$$T_t: \quad t^* = t + a_T, \quad x_i^* = x_i, \quad U_i^* = U_i, \quad P^* = P; \tag{7}$$

$$T_{\mathrm{rot}_\alpha}: \quad t^* = t, \quad x_i^* = x_j Q_{ij}^{[\alpha]}, \quad U_i^* = U_j Q_{ij}^{[\alpha]}, \quad P^* = P; \tag{8}$$

$$T_{\mathrm{Gal}_i}: \quad t^* = t, \quad x_i^* = x_i + f_{\mathrm{Gal}_i}(t), \quad U_i^* = U_i + f'_{\mathrm{Gal}_i}(t), \quad P^* = P - x_i f''_{\mathrm{Gal}_i}(t); \tag{9}$$

$$T_P: \quad t^* = t, \quad x_i^* = x_i, \quad U_i^* = U_i, \quad P^* = P + f_P(t); \tag{10}$$

$$T_{\mathrm{Sc},I}: \quad t^* = t, \quad x_i^* = x_i e^{a_{\mathrm{Sc},I}}, \quad U_i^* = U_i e^{a_{\mathrm{Sc},I}}, \quad P^* = P e^{2a_{\mathrm{Sc},I}}; \tag{11}$$

$$T_{\mathrm{Sc},II}: \quad t^* = t e^{a_{\mathrm{Sc},II}}, \quad x_i^* = x_i, \quad U_i^* = U_i e^{-a_{\mathrm{Sc},II}}, \quad P^* = P e^{-2a_{\mathrm{Sc},II}}, \tag{12}$$

where the constant rotational matrices $\mathbf{Q}^{[\alpha]}$ are given by

$$\mathbf{Q}^{[1]} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos a_{\mathrm{rot}_1} & -\sin a_{\mathrm{rot}_1} \\ 0 & \sin a_{\mathrm{rot}_1} & \cos a_{\mathrm{rot}_1} \end{pmatrix}, \tag{13}$$

$$\mathbf{Q}^{[2]} = \begin{pmatrix} \cos a_{\mathrm{rot}_2} & 0 & -\sin a_{\mathrm{rot}_2} \\ 0 & 1 & 0 \\ \sin a_{\mathrm{rot}_2} & 0 & \cos a_{\mathrm{rot}_2} \end{pmatrix}, \tag{14}$$

$$\mathbf{Q}^{[3]} = \begin{pmatrix} \cos a_{\mathrm{rot}_3} & \sin a_{\mathrm{rot}_3} & 0 \\ -\sin a_{\mathrm{rot}_3} & \cos a_{\mathrm{rot}_3} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (15)$$

Therein, $a_i$ stand for arbitrary constants, and $f_P(t)$ and $f_{\mathrm{Gal}_i}(t)$ are free functions. In concrete terms, if any of the symmetry transformations (7)–(12) are inserted into (1) and (2) (with $\nu = 0$), the form of these equations does not change when written in the $*$-variables. Hence, symmetries are form invariant transformations.

In the Navier–Stokes case (i.e., $\nu \neq 0$), due to the appearance of the kinematic viscosity, there is only one scaling symmetry, and Eqs. (11) and (12) combine to

$$T_{\mathrm{Sc,ns}}: \; t^* = te^{2a_{\mathrm{Sc,ns}}}, \quad x_i{}^* = x_i e^{a_{\mathrm{Sc,ns}}}, \quad U_i{}^* = U_i e^{-a_{\mathrm{Sc,ns}}}, \quad P^* = Pe^{-2a_{\mathrm{Sc,ns}}}. \quad (16)$$

Anyone familiar with the theory of symmetries might realize that these symmetries are for the most part equivalent to the principles of invariant modeling, which can explain why symmetries have rarely been explicitly considered in turbulence modeling.

The principle of correct tensor formulation is connected with the rotational and translational symmetries given by (8) and (9) with $f'_{\mathrm{Gal}_i}(t) = 0$. The generalized Galilean group can also directly be found in (9). Equations (11) and (12) express the principle of dimensional consistency because changing the unit system effectively amounts to a rescaling of all variables. Furthermore, it should not matter at which point in time a process starts, which is expressed by the symmetry $T_t$, i.e., (7). Finally, in incompressible flow, the absolute value of the pressure is irrelevant, and only pressure differences affect the velocity field. This is the meaning of (10), the symmetry $T_P$.

It has to be emphasized that symmetries offer more than insights into fundamental physical principles. Among other things, they can be used to obtain specific solutions of differential equations. Examples from the field of fluid mechanics include the invariant solutions known for laminar flows, which, if only scaling such as (16) has been used, are referred to as self-similar solutions. Although these solutions were historically obtained by different methods, many of them can be derived directly from the Navier–Stokes symmetries.[5] The explicit usage of symmetries for constructing exact solutions for problems in fluid mechanics are demonstrated in such works as that of Andreev *et al.*[6]

One of the early works to explicitly use symmetry methods in turbulence in the sense of invariant modeling is the contribution by Oberlack,[7] though the focus therein lay mainly on closure for large eddy simulations. Popovich and Bihlo[8] developed a systematical procedure to generate invariant models using the terminology of symmetries. This procedure is completely general; in fact, it is not even restricted to turbulence modeling. Bihlo and Bluman[9] extended this method to also incorporate conservation laws and not only symmetries. A scale invariance criterion that is crucial for large eddy simulation (LES) modeling was presented by Schaefer-Rolffs, Knöpfel, and Becker[10] and applied to geophysical fluids in Schaefer-Rolffs.[11] The authors raise the important point that whereas many symmetries, thanks to their intimate connection to conserved quantities, are generally included into models and numerical schemes, some scaling symmetries are overlooked because breaking them does not lead to a violation of conservation laws. Independent of these relatively recent efforts, symmetries have been intuitively incorporated into turbulence models much earlier, though a very different terminology was used, namely, that of Donaldson and Rosenbaum.[3] As a consequence, little or no use was made of the mathematical tools that the theory of Lie symmetry groups entails.

The first class of turbulence models to fulfill all Navier–Stokes symmetries given by Eqs. (7)–(10) and (16) were two-equation eddy-viscosity models, the most famous representatives of which are the $k$-$\varepsilon$-model[12] and the $k$-$\omega$-model.[13] This is not very surprising as these models were developed with regard to the constraints subsumed under invariant modeling.

Interestingly, the perhaps most prominent shortcoming of eddy-viscosity models, their inaccuracy for rotational and high-streamline curvature flow, is also connected to symmetries. This time, however, the problem is that the Boussinesq[14] approximation, which is used to model the Reynolds stresses in these models, allows too many symmetries: It is invariant under a transformation that amounts to rotating the system at a constant angular velocity, and therefore, the model is unable to correctly account for the effect of the system rotation or streamline curvature on the turbulent stresses.[15] To complicate matters further, a model using the Boussinesq approximation, as a whole, has no additional symmetries. Indeed, such a model does correctly account for the effect of system rotation on the mean velocities, but not for its effect on the Reynolds stresses. This illustrates that assessing whether a model is in agreement with a given set of symmetries can be a somewhat delicate task. The best way to understand the problem is to look at turbulent flows where the Coriolis effects do not appear on the mean momentum equations such as the rotating pipe and channel flows with rotation about the streamwise direction. For these flows, the influence of rotation is only due to the second moments. If, however, these terms are modeled by the Boussinesq approximation, model results are qualitatively completely wrong.[16,17]

Among one-point models, the most reliable and universal ones are of course Reynolds stress models. Founded on the pioneering work of Chou[2] and Rotta,[18] some of the most prominent Reynolds stress models are the LRR model by Launder, Reece, and Rodi[19] and the SSG model.[20] Instead of using the Boussinesq approximation, Reynolds stress models do not introduce a model for the Reynolds stress tensor directly but use Eq. (6). Only for the unknown terms in this equation, model assumptions are made. Their predictive quality is again linked to the observation that, in addition to fulfilling all Navier–Stokes symmetries, they correctly do not introduce any additional symmetries. Unfortunately, the numerical convergence of Reynolds stress models is notoriously difficult, which has given rise to nonlinear eddy-viscosity models.[21] These models exhibit similar numerical convergence as linear eddy-viscosity models and overcome their limitations regarding non-inertial reference frame turbulence by extending the Boussinesq approximation with additional terms that account for system rotation or streamline curvature.

All models discussed so far are based on the RANS equations (4) and (5). However, different approaches do exist. The Navier–Stokes-alpha model[22] performs the averaging on the level of Lagrangian trajectories before adopting an Eulerian description. Unsurprisingly, the resulting equations take a rather unusual form; in particular, the convective term looks very different compared to its counterpart in Eq. (5). It turns out that the symmetry-based modeling approach presented in this work leads to models bearing some loose resemblance to the Navier–Stokes-alpha model.

This paper is structured as follows: In Sec. II, we discuss a number of alternatives to the RANS formulation, including the one that is most convenient for the present discussion. Section III gives an introduction to the necessary background of Lie symmetry theory and discusses recent results that have inspired this contribution. The main result of the present work, a general method to systematically construct equations from given symmetries and a RANS modeling strategy based on this method, is presented in Sec. IV.

## II. ONE- AND TWO-POINT TURBULENCE DESCRIPTION

Although the RANS formulation expressed by Eqs. (4) and (5) is by far the most common way of describing turbulent flows, numerous other approaches exist. The most important advantage of Reynolds' formulation is probably its intuitive interpretation, with the only difference from the equations describing laminar flow being one additional term in the momentum equation. On the other hand, the equations for higher-order moments such as Eq. (6) contain a relatively high number of terms, many of which are unknown. Investigations extensively dealing with higher-moment equations are, therefore, more conveniently conducted in alternative approaches.

The problem of nonlinearity present in the conventional formulation can be circumvented simply by omitting the decomposition of velocity and pressure into a mean and a fluctuating part as expressed in Eq. (3). The averaged momentum equation (5), then, takes the form

$$\frac{\partial \overline{U}_i}{\partial t}+\frac{\partial H_{ij}^{(0)}}{\partial x_j}+\frac{\partial \overline{P}}{\partial x_i}-\nu\frac{\partial^2\overline{U}_i}{\partial x_j\partial x_j}=0. \tag{17}$$

The new unknown term here is the second statistical moment based on instantaneous velocities $H_{ij}^{(0)}=\overline{U_iU_j}$. Remarkably, this system is linear, with the nonlinearity of the Navier–Stokes equations fully absorbed into the new unknown variable. This linearity, which also extends to the higher-moment equations, is the main advantage of this approach. The fluctuation formulation of the momentum equation given by (5) is easily recovered from (17) using $H_{ij}^{(0)}=\overline{U}_i\overline{U}_j+R_{ij}^{(0)}$, which follows from averaging rules. The new unknown variable is obviously quite difficult to grasp intuitively because it is affected by both mean velocities and turbulent fluctuations, which is why this approach is less frequently used in turbulence research. Mathematically, both approaches presented in this section, which subsequently are denoted as the fluctuation and the instantaneous approach, are of course fully equivalent.

The instantaneous approach can also be applied in the context of a two-point formulation, with, e.g., the second instantaneous two-point moment defined as $H_{ij}=\overline{U_i(\boldsymbol{x})U_j(\boldsymbol{y})}$. The transport equations for the one-point correlations in instantaneous formulation read

$$\frac{\partial H_{ij}^{(0)}}{\partial t}+\frac{\partial H_{ijk}^{(0)}}{\partial x_k}+\frac{\partial \overline{PU_i}^{(0)}}{\partial x_j}+\frac{\partial \overline{PU_j}^{(0)}}{\partial x_i}-\overline{P\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)}^{(0)}$$
$$-\nu\frac{\partial^2 H_{ij}^{(0)}}{\partial x_k\partial x_k}+2\nu\overline{\frac{\partial U_i}{\partial x_k}\frac{\partial U_j}{\partial x_k}}^{(0)}=0, \tag{18}$$

and in the two-point approach, the rather concise equation

$$\frac{\partial H_{ij}}{\partial t}+\frac{\partial H_{(ik)j}}{\partial x_k}+\frac{\partial H_{i(jk)}}{\partial y_k}+\frac{\partial \overline{PU_j}}{\partial x_i}+\frac{\partial \overline{U_iP}}{\partial y_j}$$
$$-\nu\frac{\partial^2 H_{ij}}{\partial x_k\partial x_k}-\nu\frac{\partial^2 H_{ij}}{\partial y_k\partial y_k}=0 \tag{19}$$

is obtained,[23] with $\overline{U_iP}=\overline{U_i(\boldsymbol{x})P(\boldsymbol{y})}$. Note that we have adopted Rotta's notation[24] here, with parentheses around indices indicating that the corresponding velocities are evaluated at the same point in space. The disadvantage is that the variables in which the equations are formulated are not as straightforward to interpret intuitively.

Oberlack and Rosteck[25,26] conducted a symmetry analysis based on the instantaneous system, i.e., the averaged Navier–Stokes system (4) and (17) extended by (19) and an infinite hierarchy of higher-order equations, but the results can easily be expressed in any other of the presented formulations. All symmetries of the unaveraged Navier–Stokes system, which we refer to as classical symmetries from here on, are preserved in the averaged system. Formulating these symmetries in terms of statistical variables is easily done by averaging the original transformations. Employing the same order of symmetries as in Eqs. (7)–(12) and (16), this leads to

$$T_t:\ t^*=t+a_T,\ \ x_i^*=x_i,\ \ \overline{U}_i^{\,*}=\overline{U}_i,\ \ H_{ij}^{(0)^*}=H_{ij}^{(0)},\ \ \overline{P}^*=\overline{P}; \tag{20}$$

$$\begin{aligned}T_{\mathrm{rot}_\alpha}:\ & t^*=t,\quad x_i^*=x_jQ_{ij}^{[\alpha]},\quad \overline{U}_i^{\,*}=\overline{U}_jQ_{ij}^{[\alpha]},\\ & H_{ij}^{(0)^*}=H_{kl}^{(0)}Q_{ik}^{[\alpha]}Q_{jl}^{[\alpha]},\quad \overline{P}^*=\overline{P};\end{aligned} \tag{21}$$

$$\begin{aligned}T_{\mathrm{Gal}_i}:\ & t^*=t,\quad x_i^*=x_i+f_{\mathrm{Gal}_i}(t),\quad \overline{U}_i^{\,*}=\overline{U}_i+f'_{\mathrm{Gal}_i}(t),\\ & H_{ij}^{(0)^*}=H_{ij}^{(0)}+f'_{\mathrm{Gal}_i}(t)\overline{U}_j+f'_{\mathrm{Gal}_j}(t)\overline{U}_i+f'_{\mathrm{Gal}_i}(t)f'_{\mathrm{Gal}_j}(t),\\ & \overline{P}^*=\overline{P}-x_if''_{\mathrm{Gal}_i}(t);\end{aligned} \tag{22}$$

$$T_P:\ t^*=t,\ \ x_i^*=x_i,\ \ \overline{U}_i^{\,*}=\overline{U}_i,\ \ H_{ij}^{(0)^*}=H_{ij}^{(0)},\ \ \overline{P}^*=\overline{P}+f_P(t); \tag{23}$$

$$\begin{aligned}T_{\mathrm{Sc},I}:\ & t^*=t,\quad x_i^*=x_ie^{a_{\mathrm{Sc},I}},\quad \overline{U}_i^{\,*}=\overline{U}_ie^{a_{\mathrm{Sc},I}},\\ & H_{ij}^{(0)^*}=H_{ij}^{(0)}e^{2a_{\mathrm{Sc},I}},\quad \overline{P}^*=\overline{P}e^{2a_{\mathrm{Sc},I}};\end{aligned} \tag{24}$$

$$\begin{aligned}T_{\mathrm{Sc},II}:\ & t^*=te^{a_{\mathrm{Sc},II}},\quad x_i^*=x_i,\quad \overline{U}_i^{\,*}=\overline{U}_ie^{-a_{\mathrm{Sc},II}},\\ & H_{ij}^{(0)^*}=H_{ij}^{(0)}e^{-2a_{\mathrm{Sc},II}},\quad \overline{P}^*=\overline{P}e^{-2a_{\mathrm{Sc},II}}.\end{aligned} \tag{25}$$

In the Navier–Stokes case, the scaling symmetries (24) and (25) again combine to

$$\begin{aligned}T_{\mathrm{Sc,ns}}:\ & t^*=te^{2a_{\mathrm{Sc,ns}}},\quad x_i^*=x_ie^{a_{\mathrm{Sc,ns}}},\ \overline{U}_i^{\,*}=\overline{U}_ie^{-a_{\mathrm{Sc,ns}}},\\ & H_{ij}^{(0)^*}=H_{ij}^{(0)}e^{-2a_{\mathrm{Sc,ns}}},\quad \overline{P}^*=\overline{P}e^{-2a_{\mathrm{Sc,ns}}}.\end{aligned} \tag{26}$$

For simplicity, only second velocity moments and no velocity-pressure correlations are explicitly written out. As the application of any classical symmetry transformation commutes with averaging, it is a straightforward matter to obtain the transformations of any statistical moment. General expressions valid for all statistical moments require a more advanced notation and are given in the work of Rosteck.[23]

The seminal finding by Oberlack and Rosteck[23,26] is that additional symmetries are present in the averaged system, which we call statistical symmetries. Note that this term may mislead one into asserting that the symmetries themselves are in some way statistical, which is of course not the case—instead, their name stems from the fact that they are specific to a statistical description of turbulence. Before presenting and discussing them, some background on the mathematical theory is given.

We also mention here that in (19) and all higher-moment equations in the two-point formulation, the right-hand side terms can be written as a divergence expression. Therefore, each of these equations is connected with its own statistical conservation law, which (in connection with "classical" conservation laws) may also be useful for developing turbulence models. However, this line of thought is not within the scope of the current investigation.

## III. SYMMETRIES AND STATISTICAL SYMMETRIES IN TURBULENCE

A short introduction to Lie group theory, which lays the foundation for the present investigation, is given in the following. The reader interested in a more extensive discussion is referred to the textbook by Bluman, Cheviakov, and Anco.[27]

Given a set of independent variables $\boldsymbol{x}$ and dependent variables $\boldsymbol{z}$, the transformation

$$T : \boldsymbol{x}^* = \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{z}; a), \quad \boldsymbol{z}^* = \boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{z}; a) \tag{27}$$

is called a symmetry of equation $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{z}_{,1}, \ldots, \boldsymbol{z}_{,n}) = 0$ if

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{z}_{,1}, \ldots, \boldsymbol{z}_{,n}) = 0 \Longleftrightarrow \boldsymbol{F}(\boldsymbol{x}^*, \boldsymbol{z}^*, \boldsymbol{z}^*_{,1}, \ldots, \boldsymbol{z}^*_{,n}) = 0 \tag{28}$$

holds, where $\boldsymbol{z}_{,n}$ refers to the $n$th derivative of $\boldsymbol{z}$ and $a$ is an arbitrary constant, i.e., $a \in \mathbb{R}$, referred to as the group parameter. In the present context, $\boldsymbol{F}$ in (28), e.g., refers to the Navier–Stokes equations (1) and (2), which are invariant under (7)–(10) and (16), which are instances of the general transformation (27). Note that there exists an alternative but equivalent notation using mappings,[28] where, instead of (27), one would write

$$T : \boldsymbol{x} \to \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{z}; a), \quad \boldsymbol{z} \to \boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{z}; a). \tag{29}$$

We select to use the first notation in this work.

Assuming that $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ admit group properties,[27] we can expand them into a Taylor series around $a = 0$, i.e.,

$$\begin{aligned} \boldsymbol{x}^* &= \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{z}, a = 0) + a\frac{\partial \boldsymbol{\phi}}{\partial a}\bigg|_{a=0} + O(a^2) \\ &= \boldsymbol{x} + a\boldsymbol{\xi} + O(a^2), \end{aligned} \tag{30}$$

$$\begin{aligned} \boldsymbol{z}^* &= \boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{z}, a = 0) + a\frac{\partial \boldsymbol{\psi}}{\partial a}\bigg|_{a=0} + O(a^2) \\ &= \boldsymbol{z} + a\boldsymbol{\eta} + O(a^2). \end{aligned} \tag{31}$$

According to Lie's first theorem, the term linear in $a$ is sufficient to represent the effect of the entire transformation. The proof of this theorem heavily relies on the prerequisite that the transformation (27) is not arbitrary but admits group properties.[27]

The possibility to represent any transformation group in linearized form greatly simplifies the theory, especially when concerned with nonlinear equations such as the Navier–Stokes system. This motivates the definition of the infinitesimal generator

$$X = \xi_i(\boldsymbol{x}, \boldsymbol{z}) \frac{\partial}{\partial x_i} + \eta_j(\boldsymbol{x}, \boldsymbol{z}) \frac{\partial}{\partial z_j}$$

with

$$\xi_i = \frac{\partial \boldsymbol{\phi}_i}{\partial a}\bigg|_{a=0}, \quad \eta_j = \frac{\partial \boldsymbol{\psi}_j}{\partial a}\bigg|_{a=0}, \tag{32}$$

which is much more convenient to use than the global transformation (27). With (30)–(32), the invariance of equation $\boldsymbol{F}$ as defined in (27) and (28) may be expressed by the fully equivalent invariance condition

$$X\boldsymbol{F}|_{\boldsymbol{F}=0} = 0. \tag{33}$$

At this point, a key outcome of symmetry theory should be mentioned because it is pivotal for the subsequent theory. It concerns the combination of symmetries admitted by an equation, which is also a symmetry and usually referred to as a multi-parameter symmetry group. The proof of this fact is simple and may be directly transferred to the infinitesimal form of transformations. In this notation according to (30)–(32), the combination of symmetries is a simple linear combination of the corresponding infinitesimals. In other words, the admitted symmetries form a linear vector space.

Differential equations are handled by interpreting appearing derivatives as new variables, thus conceptually turning the differential equation into an algebraic equation with more unknowns. The variable space containing all independent and dependent variables and their derivatives is commonly referred to as jet space. In the spirit of a clear notation, derivatives that are viewed as new variables in the jet space are denoted using a comma notation, e.g., $\partial \overline{U}_i/\partial t$ is written as $\overline{U}_{i,t}$. This bears the potential of confusion because these are two different notations for essentially the same thing. However, it is necessary to distinguish between the differentiation appearing, e.g., in (32) and the derivation in physical space found, e.g., in the continuity equation (1). In order to deduce the transformation behavior of jet variables, the so-called prolongations of the infinitesimal generator have to be obtained, which is nothing else than the chain rule of differentiation written in infinitesimal form and further discussed in Appendix A.

The symmetries of the averaged inviscid system given by Eqs. (20)–(25) (with $\nu = 0$) in infinitesimal form read

$$X_t = \frac{\partial}{\partial t}, \tag{34}$$

$$X_{\text{rot}_\alpha} = \varepsilon_{jk\alpha} x_j \frac{\partial}{\partial x_k} + \varepsilon_{jk\alpha} \overline{U}_j \frac{\partial}{\partial \overline{U}_k} + \left(\varepsilon_{ki\alpha} H^{(0)}_{kj} + \varepsilon_{kj\alpha} H^{(0)}_{ik}\right) \frac{\partial}{\partial H^{(0)}_{ij}}, \tag{35}$$

$$X_P = f_P(t) \frac{\partial}{\partial \overline{P}}, \tag{36}$$

$$X_{\mathrm{Gal}} = f_{\mathrm{Gal}_i}(t)\frac{\partial}{\partial x_i} + f'_{\mathrm{Gal}_i}(t)\frac{\partial}{\partial \overline{U}_i} - x_i f''_{\mathrm{Gal}_i}(t)\frac{\partial}{\partial \overline{P}} + \left(f'_{\mathrm{Gal}_i}(t)\overline{U}_j + f'_{\mathrm{Gal}_j}(t)\overline{U}_i\right)\frac{\partial}{\partial H_{ij}^{(0)}}, \tag{37}$$

$$X_{\mathrm{Sc},I} = x_i\frac{\partial}{\partial x_i} + \overline{U}_i\frac{\partial}{\partial \overline{U}_i} + 2\overline{P}\frac{\partial}{\partial \overline{P}} + 2H_{ij}^{(0)}\frac{\partial}{\partial H_{ij}^{(0)}}, \tag{38}$$

$$X_{\mathrm{Sc},II} = t\frac{\partial}{\partial t} - \overline{U}_i\frac{\partial}{\partial \overline{U}_i} - 2\overline{P}\frac{\partial}{\partial \overline{P}} - 2H_{ij}^{(0)}\frac{\partial}{\partial H_{ij}^{(0)}}, \tag{39}$$

and the Navier–Stokes scaling symmetry (26) becomes

$$X_{\mathrm{Sc,ns}} = 2t\frac{\partial}{\partial t} + x_i\frac{\partial}{\partial x_i} - \overline{U}_i\frac{\partial}{\partial \overline{U}_i} - 2\overline{P}\frac{\partial}{\partial \overline{P}} - 2H_{ij}^{(0)}\frac{\partial}{\partial H_{ij}^{(0)}}. \tag{40}$$

The infinitesimal form of the symmetries of the unaveraged Euler and Navier–Stokes equations (1) and (2) [introduced in global form in (7)–(12) and (16)] can be found in Appendix B 1.

As mentioned before, additional symmetries were found by Oberlack and Rosteck in the averaged system given by (4), (17), and (19) and an infinite set of higher-order correlation equations. The first one is a statistical scaling symmetry, which amounts to a multiplication of all statistical moments by the same factor. Its global form reads

$$T_{\mathrm{Sc,stat}}: \; t^* = t, \quad x_i{}^* = x_i, \quad \overline{U}_i{}^* = \overline{U}_i e^{a_{\mathrm{Sc,stat}}}, \quad \overline{P}^* = \overline{P}e^{a_{\mathrm{Sc,stat}}}, \quad H_{ij}{}^* = H_{ij}e^{a_{\mathrm{Sc,stat}}}, \quad \overline{PU}_i^{(0)^*} = \overline{PU}_i^{(0)}e^{a_{\mathrm{Sc,stat}}}, \ldots. \tag{41}$$

In the instantaneous mean Navier–Stokes equations (4) and (17) as well as in the two-point approach given by Eq. (19), this is easily verified. After inserting the transformation (41), all terms contain the same factor of $e^{a_{\mathrm{Sc,stat}}}$, which cancels out. Second, any statistical moment in instantaneous form can be translated, i.e.,

$$T_{\mathrm{Tr,stat,1}}: \; t^* = t, \quad x_i{}^* = x_i, \quad \overline{U}_i{}^* = \overline{U}_i + a_{\mathrm{Tr,stat},I,i}, \quad H_{ij}{}^* = H_{ij}, \quad \overline{PU}_i^{(0)^*} = \overline{PU}_i^{(0)}; \tag{42}$$

$$T_{\mathrm{Tr,stat,2}}: \; t^* = t, \quad x_i{}^* = x_i, \quad \overline{U}_i{}^* = \overline{U}_i, \quad H_{ij}{}^* = H_{ij} + a_{\mathrm{Tr,stat},II,ij}, \quad \overline{PU}_i^{(0)^*} = \overline{PU}_i^{(0)}; \tag{43}$$

$$T_{\mathrm{Tr,stat,3}}: \; t^* = t, \quad x_i{}^* = x_i, \quad \overline{U}_i{}^* = \overline{U}_i, \quad H_{ij}{}^* = H_{ij}, \quad \overline{PU}_i^{(0)^*} = \overline{PU}_i^{(0)} + a_{\mathrm{Tr,stat},III,i}, \quad \ldots \tag{44}$$

which is due to the fact that all terms in (4) and (5) appear under a divergence. The infinitesimal generators associated with (41)–(44) will be needed later and read as

$$X_{\mathrm{Sc,stat}} = \overline{U}_i\frac{\partial}{\partial \overline{U}_i} + \overline{P}\frac{\partial}{\partial \overline{P}} + H_{ij}^{(0)}\frac{\partial}{\partial H_{ij}^{(0)}} + \cdots, \tag{45}$$

$$X_{\mathrm{Tr,stat,1}} = \frac{\partial}{\partial \overline{U}_i}, \tag{46}$$

$$X_{\mathrm{Tr,stat,2}} = \frac{\partial}{\partial H_{ij}^{(0)}}, \tag{47}$$

$$X_{\mathrm{Tr,stat,3}} = \frac{\partial}{\partial \overline{PU}_i^{(0)}}. \quad \ldots \tag{48}$$

This is a collection of infinitely many symmetries because each statistical moment can be translated individually, even higher ones such as the triple velocity correlation that we do not consider here. Similarly, the scaling symmetry (41) [or (45)] scales all appearing statistical moments including higher ones not directly considered here. In order to keep the notation simple, we only write them out for first and second moments.

It is to be noted that Galilean invariance and the statistical symmetry (46) have a certain similarity. Obviously, the Galilean symmetry (22) has been incorporated into essentially all modern turbulence models. However, Galilean invariance is only observed if the full unsteady equations (5) are employed. In stationary flows, in which $\partial\overline{U}_i/\partial t$ in (5) vanishes, the Galilean symmetry is broken. Therefore, any modeling effort focused on stationary flow, which investigations of wall-bounded shear flows are commonly concerned with, cannot make use of this symmetry. This has led to symmetry (42) frequently being used in turbulence modeling due to its similarity with the Galilean transformation. Any model mainly concerned with stationary turbulence which was meant to make use of the Galilean group, hence, essentially included the statistical symmetry (42). In fact, the symmetry (42) is essential in any turbulence model to mimic the well-known logarithmic law of the wall, as is discussed in the work of Oberlack.[29]

Compared to the classical symmetries, the physical meaning of these statistical symmetries perhaps seems a bit more opaque because due to the fact that they act on the variable space of statistical moments, they cannot directly be interpreted as a physical transformation. However, their relevance can be clearly perceived through the invariant solutions, or scaling laws, in obtaining which they play a pivotal role, as has been mentioned above for the logarithmic law of the wall for the near-wall turbulent velocity profile. This method has been successfully applied in the work of Oberlack;[30] Oberlack, Wenzel, and Peters;[31] Oberlack and Rosteck;[25] Rosteck;[23] Oberlack *et al.*;[32] and Sadeghi, Oberlack, and Gauding.[33]

Moreover, the perception of the statistical symmetries (45)–(47) may be understood by considering its meaning if written in probability density function (PDF) formulation. Doing so, it turns out that (45) refers to the intermittency, and (42)–(44) extended by the higher moments depict the non-Gaussianity of turbulence.[34] In turn, by incorporating the above statistical symmetries into a turbulence model, these key effects can immediately be mimicked.

In order to bridge the gap between these findings and the more familiar notation of turbulence theory, it proves helpful to rewrite the statistical symmetries in terms of one-point fluctuation moments using Sec. II. As this process involves a recursion, it is not feasible to give general results describing arbitrarily high order statistical moments. We, therefore, restrict ourselves to the Reynolds stress tensor and ignore the existence of any higher velocity or velocity-pressure moments for now. This makes the analysis much easier while still providing enough information to discuss, e.g., eddy-viscosity models. In the commonly used variables, the statistical scaling symmetry (41) reads

$$T_{\text{Sc,stat}}: \; t^* = t, \quad {x_i}^* = x_i, \quad \overline{U}_i^* = \overline{U}_i e^{a_{\text{Sc,stat}}}, \quad \overline{P}^* = \overline{P} e^{a_{\text{Sc,stat}}}, \\ R_{ij}^{(0)*} = \left(R_{ij}^{(0)} + \overline{U}_i \overline{U}_j\right) e^{a_{\text{Sc,stat}}} - \overline{U}_i \overline{U}_j e^{2a_{\text{Sc,stat}}}, \tag{49}$$

and the translation symmetries (42) and (43) become

$$T_{\text{Tr,stat,1}}: \; t^* = t, \; {x_i}^* = x_i, \; \overline{U}_i^* = \overline{U}_i + a_{\text{Tr,stat},I,i}, \; \overline{P}^* = \overline{P}, \\ R_{ij}^{(0)*} = R_{ij}^{(0)} - \overline{U}_j a_{\text{Tr,stat},I,i} - \overline{U}_i a_{\text{Tr,stat},I,j} - a_{\text{Tr,stat},I,i} a_{\text{Tr,stat},I,j}, \tag{50}$$

$$T_{\text{Tr,stat,2}}: \; t^* = t, \quad {x_i}^* = x_i, \quad \overline{U}_i^* = \overline{U}_i, \quad \overline{P}^* = \overline{P}, \\ R_{ij}^{(0)*} = R_{ij}^{(0)} + a_{\text{Tr,stat},IIij}. \tag{51}$$

The infinitesimal form of these symmetries is given in Appendix B 2. To the authors' knowledge, there is currently no turbulence model that generally incorporates these statistical symmetries. However, for some special flows, it is possible that the symmetry-breaking terms of an existing model vanish, allowing the model to fulfill the statistical symmetries only in that case. An example of such a special flow configuration is parallel shear flow and, in particular, the logarithmic law of the wall already mentioned above, which inherently relies on (46) [or the equivalent (50)].[29] As many turbulence models are calibrated using flows of this category, it is apparent that at least some information included in these symmetries has already found its way into the existing turbulence models. Of course, it is doubtful whether that information persists when the model is applied to more complicated flows. A logical step to further improve the predictive quality of turbulence models is, hence, to develop a model that naturally includes these additional symmetries. This challenge is tackled in Sec. IV.

### A. Consequences of symmetry breaking of turbulence models

In the Introduction, some comments have already been made about the consequences of turbulence models violating symmetries. We will now revisit this issue.

Even though it may seem that way from the wording, a violation of symmetries—even of fundamental classical ones—does not immediately render the model useless. In fact, some early models such as Prandtl's mixing length model do not fulfill the requirements of the rotational and scaling symmetries [(34)–(37) and (40)]. In practice, this leads to the well-recognized fact that these models are not fully general but are only useful, e.g., in parallel shear flow.

A similar pattern is observed with linear eddy-viscosity models admitting a symmetry corresponding to time-dependent rotation, producing inaccurate results in rotational and high-streamline curvature flows. This symmetry violation does not necessarily affect their ability to predict flows in which these effects are not important.

In the same sense, the fact that essentially all existing RANS models violate the statistical symmetries (45)–(48) does not contradict the fact that many of them perform very well throughout a wide range of flows. However, we advocate that by incorporating the statistical symmetries, an even greater degree of generality could be achieved (see Sec. IV E).

## IV. EXTENDED SYMMETRY INVARIANT MODELING

The requirement that a turbulence model must fulfill all classical symmetries of the Navier–Stokes equations is obviously a well-established principle, even though it was stated in a different form by Donaldson and Rosenbaum. Recalling the history of turbulence modeling with respect to symmetries as is done in Sec. I, symmetry has been a guiding principle for decades, though very little explicit use was made. The present work is aimed at extending the invariant modeling approach by not only explicitly considering the classical symmetries but also requiring invariance of the model with respect to the statistical symmetries. Such a statistically invariant model could be expected to comprise information describing the statistical behavior of turbulence, including intermittency and non-Gaussianity,[34] and, hence, extending well beyond general properties of classical physics found in the flow (see Sec. IV E).

The classical procedure of turbulence modeling can be shown to be not sufficient in this context, forcing the development of a completely new strategy. We can, to a great extent, rely on the recent work of Popovich and Bihlo,[8] who established a theoretical foundation for algorithmically obtaining invariant models. Even though the theory has been developed with regard to only the classical symmetries, the framework can be extended to cope with the new challenges introduced by the statistical symmetries. It must be highlighted that creating new model equations from symmetries is rather similar to creating symmetry invariant solutions. In both cases, invariant functions are calculated, though when deriving equations, physical derivatives are treated as ordinary variables.

The problem of constructing equations from symmetries is also known as the inverse problem on group classification, for which, apart from the method presented in the following, there also exists the method of equivariant moving frames pioneered by Olver.[35,36] For an application of this method to turbulence modeling, see the work of Bihlo, Dos Santos Cardoso-Bihlo, and Popovych.[37]

### A. Introductory example: Constructing the Euler equations from their symmetries

Before applying the algorithm that generates equations from symmetries to the closure problem of turbulence, we consider as a simple example the Euler equations, i.e., Eqs. (1) and (2) with $\nu = 0$, which can be almost uniquely derived from their symmetries. To understand the method, we pretend that we do not know the exact form of the Euler equations but only that they are some general function,

$$\boldsymbol{F}\left(t; x_i; U_i; P; U_{i,t}; U_{i,x_j}; P_{,t}; P_{,x_i}\right) = 0. \tag{52}$$

The use of the jet notation serves to highlight that the derivatives are not treated differently than the other variables in the method presented here. In fact, this allows applying the same well-established techniques that enable one to construct invariant solutions to the challenge of constructing equations.

It may seem arbitrary to restrict the list of variables to only first derivatives. After all, in the only slightly different case of non-vanishing viscosity (i.e., the Navier–Stokes case), the second spatial derivative of the velocity appears in the equations. However, it is better in this context to think about Eq. (2) as a first-order system consisting of momentum conservation and a material law. That way, an extension to the Navier–Stokes case would simply be achieved by adding the molecular stress tensor (and its first derivatives) to the list of variables, rather than arbitrarily increasing the order of derivatives to two. Moreover, the material law for Newtonian Fluids is quite different in nature than the axiom of momentum conservation, and subsequently, their symmetries are fundamentally different. Treating these equations separately allows for a natural incorporation of these differences. Hence, in general, first derivatives as arguments for $\boldsymbol{F}$ in (52) are always sufficient to derive equations, though the choice of variables obviously affects the physical processes that can be represented by the resulting equations.

Apart from the dependencies in Eq. (52), we also assume to know the symmetries of the Euler equations, namely, symmetries (7)–(12), so we can use as a constraint on $\boldsymbol{F}$ in (52) that it has to be invariant under all of these symmetries at the same time. Here, we may point out the main two differences between invariant solutions and invariant modeling. The most striking difference is that for invariant solutions, usually only one or a few symmetries are employed, leading to free parameters in the final solution. For invariant modeling purposes, the equations to be derived have to be invariant under all symmetries separately, which obviously is a much more severe constraint.

However, it guarantees that invariant solutions such as the log law or an extended jet flow scaling law such as that found in the work of Sadeghi, Oberlack, and Gauding[33] can be generated using any combination of the symmetries considered. Furthermore, and rather apparently, for invariant modeling purposes, the sought-for functions such as $\boldsymbol{F}$ in (52) contain derivatives, which obviously are not useful in the context of invariant solutions at least in most cases.

For the present example, it must be pointed out that we are not considering the averaged form of the Euler equations, and therefore, we do not demand invariance with respect to the statistical symmetries (45)–(48). In concrete terms, this implies that the wanted invariant equation (33) has to hold for all symmetries, i.e.,

$$X_1\boldsymbol{F}|_{\boldsymbol{F}=0} = 0 \wedge X_2\boldsymbol{F}|_{\boldsymbol{F}=0} = 0 \wedge \cdots \wedge X_6\boldsymbol{F}|_{\boldsymbol{F}=0} = 0, \qquad (53)$$

where for simplicity of notation, $X_1 \cdots X_6$ refer to the symmetries of the Euler equations in infinitesimal form (B1)–(B6). Equation (53) is a system of partial differential equations (PDEs) whose solution is the most general form of an equation in the arguments of $\boldsymbol{F}$ that is invariant under the symmetries (B1)–(B6).

Rather than giving this solution directly, we demonstrate the solution steps in order to reveal the constraints implied by each individual symmetry. In order to simplify the following discussion, it makes sense to begin with what we subjectively consider the simpler symmetries. Therefore, we first demand the invariance of $\boldsymbol{F} = 0$ with respect to the translation symmetry in time given by Eq. (B1), leading to

$$X_t\boldsymbol{F} = \frac{\partial \boldsymbol{F}}{\partial t} = 0, \qquad (54)$$

where we omit the arguments of $\boldsymbol{F}(t; x_i; U_i; P; U_{i,t}; U_{i,x_j}; P_{,t}; P_{,x_i})$ for brevity. Equation (54) implies that $\boldsymbol{F}$ cannot depend on $t$ directly, i.e., $t$ has to be removed from the list of possible arguments. The form of the Euler equation can, thus, be narrowed down to

$$\boldsymbol{F}\big(x_i; U_i; P; U_{i,t}; U_{i,x_j}; P_{,t}; P_{,x_i}\big) = 0. \qquad (55)$$

Note that temporal derivatives are not affected and are correctly allowed to appear in the equation. Next, we consider the translation symmetry in space, which is a special case of the generalized Galilean group (B3) with $f'_{\mathrm{Gal}_i}(t) = 0$, yielding

$$X_{x_i}\boldsymbol{F} = \frac{\partial \boldsymbol{F}}{\partial x_i} = 0, \qquad (56)$$

which removes $x_i$ from the list of variables. Similarly, the pressure translation symmetry (B4) for $f'_P(t) = 0$ removes $P$ from the list of possible variables, and the same symmetry with $f''_P(t) = 0$ eliminates $P_{,t}$. At this point, the form of the Euler equations has been constrained to

$$\boldsymbol{F}\big(U_i; U_{i,t}; U_{i,x_j}; P_{,x_i}\big) = 0, \qquad (57)$$

and any equation written in terms of these variables is guaranteed to fulfill the symmetries invoked up to this point.

The next step is to require invariance with respect to the generalized Galilean symmetry (B3) for arbitrary free functions $f_{\mathrm{Gal}_i}(t)$. Calculating the first prolongation using Eq. (A1) leads to the PDE system

$$X^{(1)}_{\mathrm{Gal}}\boldsymbol{F} = f'_{\mathrm{Gal}_i}(t)\frac{\partial \boldsymbol{F}}{\partial U_i} + \Big(f''_{\mathrm{Gal}_i}(t) - U_{i,x_j}f'_{\mathrm{Gal}_j}(t)\Big)\frac{\partial \boldsymbol{F}}{\partial U_{i,t}} - f''_{\mathrm{Gal}_i}(t)\frac{\partial \boldsymbol{F}}{\partial P_{,x_i}} = 0, \qquad (58)$$

which further reduces the form of $\boldsymbol{F}$ to

$$\boldsymbol{F}\big(U_{i,t} + U_jU_{i,x_j} + P_{,x_i}; U_{i,x_j}\big) = 0. \qquad (59)$$

It is apparent that this already closely resembles the familiar form of the Euler equations. An additional constraint can be inferred from the rotational symmetry (B2), leading to

$$U_{i,t} + U_jU_{i,x_j} + P_{,x_i} = 0, \qquad (60)$$

$$\boldsymbol{F}\big(U_{i,x_i}; U_{i,x_j}U_{j,x_i}; U_{i,x_j}U_{j,x_k}U_{k,x_i}\big) = 0. \qquad (61)$$

If we exclude the second and third tensor invariants of the velocity gradient, i.e., $U_{i,x_j}U_{j,x_i}$ and $U_{i,x_j}U_{j,x_k}U_{k,x_i}$ (this is the only step in this derivation we cannot justify based on symmetries), the scaling symmetry (B6) constrains (61) to the continuity equation for incompressible flows, i.e.,

$$U_{i,x_i} = 0. \qquad (62)$$

The Euler equations have, thus, been constructed from their symmetries.

This shows how tightly the Euler equations are constrained by their symmetries. Indeed, it appears that most (if not all) of the physical information contained in the Euler equations can be found in their symmetries. Although it might not be more than an interesting exercise to derive from symmetries the equations that are already known, the method can be extremely powerful when applied to modeling challenges.

In the context of turbulence modeling, the following rationale may be put forward: (i) Given the infinite hierarchy of moment equations, we may calculate the symmetries found in this exact description of turbulence. (ii) In order to construct a turbulence model, we select a finite set of variables that the model should depend on and (iii) derive the general form of equations in these variables that is invariant under the selected symmetries of the exact, infinite-dimensional equation system. Model equations constructed in this fashion are guaranteed to contain all of the physical information found in the symmetries of the exact system. Obviously, step (ii) entails a great degree of freedom for the modeler, which is why this rationale does not lead to only one concrete model but rather to a new class of models. As turns out, if invariance with respect to the statistical symmetries (41)–(44) is required, the resulting models exhibit significant differences compared to the existing models.

It must be pointed out that to some narrow degree, this rationale is already being followed by most turbulence modelers. Employing the invariant modeling approach (see Sec. I) ensures that a number of symmetry-related properties of the exact system (namely, dimensional and tensorial correctness, and Galilean invariance) are preserved in the model equations. Failure to do so would yield a model that behaves very differently compared to the exact equations and, thus, classical mechanics. In Secs. IV B–IV E, we take the invariant modeling approach a step further by (i) incorporating the statistical symmetries (41)–(44) in addition to the symmetries of the Euler and Navier–Stokes equations, i.e., (20)–(26), and (ii) generating equations using a formal method instead of a heuristic trial-and-error approach.

## B. Failure of classical approach

In classical modeling, the set of unclosed equations that are to be modeled is naturally divided into a closed part and an unclosed part. For example, in the context of eddy-viscosity turbulence modeling, Eq. (4) and all but the last term of Eq. (5) constitute the closed part, and the last term of Eq. (5), i.e., the Reynolds stress tensor, is the unclosed part.

As a side note, this can lead to some ambiguity about what is meant by the term turbulence model. Sometimes, only the closure relations, i.e., the empirical equations that replace the unclosed part, are viewed as the turbulence model. We, therefore, emphasize that the entire set of equations to be solved, e.g., in a numerical simulation, constitute the turbulence model.

Classical invariant modeling only focuses on the unclosed part, ensuring that the inserted model terms do not violate any symmetries. This implicitly relies on the condition that the closed part by itself fulfills all symmetries. Only then an isolated treatment of the unclosed part is possible. In the context of turbulence modeling based on the classical symmetries given by Eqs. (20)–(26), this condition is fulfilled: The closed part of Eqs. (4) and (5) is essentially equivalent to the unaveraged Navier–Stokes Eqs. (1) and (2) and, therefore, exhibits all classical symmetries given by Eqs. (20)–(26). This greatly simplifies the modeling procedure. As long as the terms replacing the unclosed part are in agreement with these classical symmetries, the model as a whole preserves them as well.

This situation stands in contrast to the present modeling challenge, which differs from the classical situation in that we also wish to incorporate the statistical symmetries (41)–(43). The closed part of the averaged Navier–Stokes equations (4) and (5) by itself violates these statistical symmetries. Only the combination of the closed and unclosed part fulfills these symmetries. This means that the unclosed part cannot be treated independently of the closed part, but instead the equation has to be modeled as a whole. These considerations also extend to Reynolds stress models.

To demonstrate how severe the restrictions imposed by the statistical symmetries (41)–(43) in combination with the classical symmetries defined by Eqs. (20)–(26) are, it is shown that without any additional model variables, no meaningful model equations are possible.

In the following, we will attempt to derive a Reynolds stress transport model that is invariant under the classical and the statistical symmetries (34)–(37), (40), and (45)–(48). The complicated situation detailed above motivates the idea to employ the formal modeling algorithm demonstrated in Sec. IV A rather than a heuristic approach. To that end, we start with a completely generic equation of the form

$$\boldsymbol{F}\Big(t; x_i; \overline{U}_i; \overline{P}; \overline{U}_{i,t}; \overline{U}_{i,x_j}; \overline{P}_{,t}; \overline{P}_{,x_i}; H_{ij}^{(0)}; H_{ij,t}^{(0)}; H_{ij,x_k}^{(0)};\\ \overline{U_iP_{,x_j}}^{(0)}; \overline{U_iP_{,x_j,t}}^{(0)}; \overline{U_iP_{,x_j,x_k}}^{(0)}\Big) = 0. \tag{63}$$

For this simple demonstration, we ignore viscous effects and, hence, omit the molecular stress tensor and dissipation (though incorporating these effects would not entail any problems). Requiring invariance with respect to the same symmetries as before, the set of variables is again reduced. The translation symmetries, both classical and statistical, have a similar effect as before, condensing the set of possible variables to

$$\boldsymbol{F}\Big(\overline{U}_{i,t}; \overline{U}_{i,x_j}; \overline{P}_{,x_i}; H_{ij,t}^{(0)}; H_{ij,x_k}^{(0)}; \overline{U_iP_{,x_j}}^{(0)}; \overline{U_iP_{,x_j,x_k}}^{(0)}\Big) = 0. \tag{64}$$

The Galilean symmetry (37) again has to be prolongated, and this time also its action on the second moments and their derivatives has to be accounted for. Its generator, therefore, becomes

$$X_{\mathrm{Gal}}^{(1)} = (f''_{\mathrm{Gal}_i}(t) - \overline{U}_{i,x_j} f'_{\mathrm{Gal}_j}(t))\frac{\partial}{\partial \overline{U}_{i,t}} - f''_{\mathrm{Gal}_i}(t)\frac{\partial}{\partial \overline{P}_{,x_i}} + (\overline{U}_{i,t} f'_{\mathrm{Gal}_j}(t)\\ + \overline{U}_{j,t} f'_{\mathrm{Gal}_i}(t) + \overline{U}_i f''_{\mathrm{Gal}_j}(t) + \overline{U}_j f''_{\mathrm{Gal}_i}(t) - H_{ij,x_k}^{(0)} f'_{\mathrm{Gal}_k}(t))\frac{\partial}{\partial H_{ij,t}^{(0)}}\\ + (\overline{U}_{i,x_k} f'_{\mathrm{Gal}_j}(t) + \overline{U}_{j,x_k} f'_{\mathrm{Gal}_i}(t))\frac{\partial}{\partial H_{ij,x_k}^{(0)}} + (\overline{P}_{,x_j} f'_{\mathrm{Gal}_i}(t) - \overline{U}_i f''_{\mathrm{Gal}_j}(t))\\ \times \frac{\partial}{\partial \overline{U_iP_{,x_j}}^{(0)}} + (\overline{P}_{,x_jx_k} f'_{\mathrm{Gal}_i}(t) - \overline{U}_{i,x_k} f''_{\mathrm{Gal}_j}(t))\frac{\partial}{\partial \overline{U_iP_{,x_j,x_k}}^{(0)}}. \tag{65}$$

Evaluating (65) on (64), i.e., $X_{\mathrm{Gal}}^{(1)}\boldsymbol{F}$, yields the quite complicated PDE system

$$(f''_{\mathrm{Gal}_i}(t) - \overline{U}_{i,x_j} f'_{\mathrm{Gal}_j}(t))\frac{\partial F}{\partial \overline{U}_{i,t}} - f''_{\mathrm{Gal}_i}(t)\frac{\partial F}{\partial \overline{P}_{,x_i}} + (\overline{U}_{i,t} f'_{\mathrm{Gal}_j}(t)\\ + \overline{U}_{j,t} f'_{\mathrm{Gal}_i}(t) + \overline{U}_i f''_{\mathrm{Gal}_j}(t) + \overline{U}_j f''_{\mathrm{Gal}_i}(t) - H_{ij,x_k}^{(0)} f'_{\mathrm{Gal}_k}(t))\frac{\partial F}{\partial H_{ij,t}^{(0)}}\\ + (\overline{U}_{i,x_k} f'_{\mathrm{Gal}_j}(t) + \overline{U}_{j,x_k} f'_{\mathrm{Gal}_i}(t))\frac{\partial F}{\partial H_{ij,x_k}^{(0)}} + (\overline{P}_{,x_j} f'_{\mathrm{Gal}_i}(t) - \overline{U}_i f''_{\mathrm{Gal}_j}(t))\\ \times \frac{\partial F}{\partial \overline{U_iP_{,x_j}}^{(0)}} + (\overline{P}_{,x_jx_k} f'_{\mathrm{Gal}_i}(t) - \overline{U}_{i,x_k} f''_{\mathrm{Gal}_j}(t))\frac{\partial F}{\partial \overline{U_iP_{,x_j,x_k}}^{(0)}} = 0. \tag{66}$$

The solution procedure of this system is detailed in Appendix C 1 and leads to

$$\boldsymbol{F}\Big(\overline{U}_{i,t}+H^{(0)}_{ij,x_j}+\overline{P}_{,x_i};\overline{U}_{i,x_j};H^{(0)}_{ij,t}+H^{(0)}_{ij,x_k}\gamma_k+\overline{U_iP_{,x_j}}^{(0)}+\overline{U_jP_{,x_i}}^{(0)}\Big)=0, \tag{67}$$

where $\gamma_i$ is implicitly defined by

$$\overline{U}_{i,t}+\overline{U}_{i,x_j}\gamma_j+\overline{P}_{,x_i}=0. \tag{68}$$

Ten equations have to be constructed from the variables given by Eq. (67) to fully constrain the three velocity components, the pressure, and the six independent components of the Reynolds stress tensor. Additionally, closure for the pressure-velocity correlation has to be provided. It is easily seen that the exact (inviscid) RANS momentum equation (5) is obtained from uncoupling the first variable and setting it to zero. The averaged continuity equation (4) is obtained by multiplying the second variable with $\delta_{ij}$. Moreover, it appears that the last variable can serve as a reasonable foundation for a second moment model equation. Its first two terms closely resemble the typical form of a material derivative, except that in the place of the velocity, a variable denoted here with $\gamma_i$ appears.

As $\gamma_i$ is defined in terms of velocity and pressure derivatives as expressed by Eq. (68), it is straightforward to calculate its behavior under the considered symmetry transformations. Under all classical symmetries, it transforms precisely like a velocity but remains invariant under the statistical symmetries. For example, when applying the Galilean transformation (9) to $\gamma_i$, it is transformed to $\gamma_i+f'_{\mathrm{Gal}_i}(t)$. As the first statistical translation symmetry (42) prohibits an explicit appearance of the velocity, we can view $\gamma_i$ as its replacement.

Even though this in theory constitutes a rather remarkable result, from a practical point of view, $\gamma_i$ in (68) has a number of highly problematic features. The most striking one is that it is only defined where the inverse of the velocity gradient exists, which is generally not the case. This renders any equation in which $\gamma_i$ appears incapable of reliably describing the physics of turbulence. We, thus, have to conclude that from the set of variables assumed in Eq. (63), no meaningful equation for the second moments can be derived.

### C. New invariant modeling approach

Having demonstrated that the assumed set of variables in the previous section, i.e., Sec. IV B, is too restrictive, the idea is now to introduce a new model variable in order to obtain additional freedom. We are encouraged by the results in Sec. IV B that we need a model variable that behaves like a velocity under the classical symmetries but is not transformed by the statistical symmetries. Even though this is not yet a very concrete concept, it is sufficient information to carry out the algorithm presented in Secs. IV A and IV B.

Suggested by $\gamma_i$ in (68), we extend the generic form assumed in Eq. (63) by the new model variable denoted here with $\hat{U}_i$ and a corresponding pressure $\hat{P}$, which, respectively, transform like $\overline{U}_i$ and $\overline{P}$ under all classical symmetries but are invariant under the statistical symmetries, leading to

$$\boldsymbol{F}\Big(x_i,t,\overline{U}_i,\overline{P},H^{(0)}_{ij},\hat{U}_i,\hat{P},\overline{U}_{i,x_j},\overline{U}_{i,t},\overline{P}_{,x_i},\overline{P}_{,t},\hat{U}_{i,x_j},\hat{U}_{i,t},\hat{P}_{,x_j},\hat{P}_{,t},H^{(0)}_{ij,x_k},H^{(0)}_{ij,t},\overline{U_iP_{,x_j}}^{(0)},\overline{U_iP_{,x_j,x_k}}^{(0)},\overline{U_iP_{,x_j,t}}^{(0)}\Big)=0. \tag{69}$$

It is not necessary to specify anything about these new variables apart from their behavior under the symmetry transformations we are going to use.

Demanding invariance with regard to the classical and statistical translation symmetries, as before in the context of deriving Eq. (64), reduces the list of variables to

$$F\Big(\hat{U}_i,\overline{U}_{i,x_j},\overline{U}_{i,t},\overline{P}_{,x_i},\hat{U}_{i,x_j},\hat{U}_{i,t},\hat{P}_{,x_j},H^{(0)}_{ij,x_k},H^{(0)}_{ij,t},\overline{U_iP_{,x_j}}^{(0)},\overline{U_iP_{,x_j,x_k}}^{(0)}\Big)=0. \tag{70}$$

Next, invariance with respect to the Galilean symmetry, which now reads in prolongated form as

$$\begin{aligned}X^{(1)}_{\mathrm{Gal}}=&f'_{\mathrm{Gal}_i}(t)\frac{\partial}{\partial\hat{U}_i}+(f''_{\mathrm{Gal}_i}(t)-\overline{U}_{i,x_j}f'_{\mathrm{Gal}_j}(t))\frac{\partial}{\partial\overline{U}_{i,t}}+(f''_{\mathrm{Gal}_i}(t)\\&-\hat{U}_{i,x_j}f'_{\mathrm{Gal}_j}(t)\frac{\partial}{\partial\hat{U}_{i,t}}-f''_{\mathrm{Gal}_i}(t)\frac{\partial}{\partial\overline{P}_{,x_i}}-f''_{\mathrm{Gal}_i}(t)\frac{\partial}{\partial\hat{P}_{,x_i}}+(\overline{U}_{i,t}f'_{\mathrm{Gal}_j}(t)\\&+\overline{U}_{j,t}f'_{\mathrm{Gal}_i}(t))+\overline{U}_if''_{\mathrm{Gal}_j}(t)+\overline{U}_jf''_{\mathrm{Gal}_i}(t)-H^{(0)}_{ij,x_k}f'_{\mathrm{Gal}_k}(t))\frac{\partial}{\partial H^{(0)}_{ij,t}}\\&+(\overline{U}_{i,x_k}f'_{\mathrm{Gal}_j}(t)+\overline{U}_{j,x_k}f'_{\mathrm{Gal}_i}(t))\frac{\partial}{\partial H^{(0)}_{ij,x_k}}+(\overline{P}_{,x_j}f'_{\mathrm{Gal}_i}(t)-\overline{U}_if''_{\mathrm{Gal}_j}(t))\\&\times\frac{\partial}{\partial\overline{U_iP_{,x_j}}^{(0)}}+(\overline{P}_{,x_jx_k}f'_{\mathrm{Gal}_i}(t)-\overline{U}_{i,x_k}f''_{\mathrm{Gal}_j}(t))\frac{\partial}{\partial\overline{U_iP_{,x_j,x_k}}^{(0)}},\end{aligned} \tag{71}$$

is invoked. The steps to solving the system of partial differential equations

$$X^{(1)}_{\mathrm{Gal}}F=0 \tag{72}$$

are detailed in Appendix C 2. The result is that Eqs. (4) and (17) (without viscous terms) are reobtained (again invoking the rotational and scaling symmetries), and in addition, for the newly introduced model variables $\hat{U}_i$ and $\hat{P}$, we find

$$\hat{U}_{i,t}+\hat{U}_j\hat{U}_{i,x_j}+\hat{P}_{,x_i}=0, \tag{73}$$

$$\hat{U}_{i,x_i}=0. \tag{74}$$

Furthermore, for the second velocity moments, we find the transport equation

$$\frac{\partial H^{(0)}_{ij}}{\partial t}+\hat{U}_k\frac{\partial H^{(0)}_{ij}}{\partial x_k}+\overline{U_iP_{,x_j}}^{(0)}+\overline{U_jP_{,x_i}}^{(0)}+\hat{U}_i\frac{\partial H^{(0)}_{jk}}{\partial x_k}+\hat{U}_j\frac{\partial H^{(0)}_{ik}}{\partial x_k}-\frac{\partial\overline{U}_i}{\partial x_k}\hat{U}_j\hat{U}_k-\frac{\partial\overline{U}_j}{\partial x_k}\hat{U}_i\hat{U}_k=0, \tag{75}$$

where, interestingly enough, also an equation for the velocity–pressure gradient terms is obtained,

$$\frac{\partial\overline{U_iP_{,x_j}}^{(0)}}{\partial x_k}-\hat{U}_i\frac{\partial^2\overline{P}}{\partial x_j\partial x_k}-\frac{\partial\hat{P}}{\partial x_j}\frac{\partial\overline{U}_i}{\partial x_k}=0. \tag{76}$$

Note that (76) is an independent equation and cannot be inferred from (75). The equations shown here are minimal in the sense that they only contain the terms necessary to fulfill the symmetries. In order to accurately predict turbulent flows, additional terms, including viscous terms but also model terms, are needed. Nonetheless, apart from exemplifying the structure that is required to fulfill the statistical symmetries (41)–(44), the equations derived here may also be valuable as a foundation for future model development.

In order to facilitate the comparison to the existing models, Eq. (75) can be rewritten in fluctuation formulation. The model equation for the Reynolds stress tensor then reads

$$\frac{\partial R_{ij}^{(0)}}{\partial t}+\overline{U}_k\frac{\partial R_{ij}^{(0)}}{\partial x_k}=\frac{\partial \overline{U}_j}{\partial x_k}(\overline{U}_i-\hat{U}_i)(\overline{U}_k-\hat{U}_k)+\frac{\partial \overline{U}_i}{\partial x_k}(\overline{U}_j-\hat{U}_j)(\overline{U}_k-\hat{U}_k)+\frac{\partial R_{ik}^{(0)}}{\partial x_k}(\overline{U}_j-\hat{U}_j)+\frac{\partial R_{jk}^{(0)}}{\partial x_k}(\overline{U}_i-\hat{U}_i)-\frac{\partial R_{ij}^{(0)}}{\partial x_k}(\overline{U}_k-\hat{U}_k)-\overline{u_i\frac{\partial p}{\partial x_j}}^{(0)}-\overline{u_j\frac{\partial p}{\partial x_i}}^{(0)}. \tag{77}$$

The left hand side is identical to the original transport equation (6), and the first group of terms on the right-hand side resemble production terms.

The equations for the velocity–pressure gradient correlation (76) in fluctuation form become

$$\frac{\partial\left(\overline{u_i\frac{\partial p}{\partial x_j}}^{(0)}\right)}{\partial x_k}+\frac{\partial^2\overline{P}}{\partial x_j\partial x_k}(\overline{U}_i-\hat{U}_i)+\frac{\partial \overline{U}_j}{\partial x_k}\left(\frac{\partial \overline{P}}{\partial x_j}-\frac{\partial \hat{P}}{\partial x_j}\right)=0. \tag{78}$$

The most important question at this point seems to be the interpretation of $\hat{U}_i$ and $\hat{P}$, and, by extension, the frequently appearing velocity difference $\overline{U}_i-\hat{U}_i$, which we address in Sec. IV D.

## D. Interpretation of the new model variables

The central question at this stage is what physical meaning can be attached to the new model variables $\hat{U}_i$ and $\hat{P}$. Essentially, we are looking for physical velocity and pressure that behave like the mean velocity and pressure under all classical symmetries but are invariant under the statistical symmetries.

An obvious example for such variables is the velocity and pressure fields obtained from a classical turbulence model, whose equations fulfill the classical symmetries while violating the statistical ones. Rather than interpreting these fields as the final results, a model developed in the present framework would consider them as the variables $\hat{U}_i$ and $\hat{P}$ and, then, yield an improved solution for the "real" velocity and pressure fields. If this line of thinking is followed, the present framework can be viewed as a correction algorithm that improves the results of the existing models in cases where the statistical symmetries are important.

Examples for such cases are discussed in Sec. IV E.

## E. Expected benefits of the new models

Symmetry invariant modeling ensures consistent physical behavior for arbitrarily complicated flow configurations. The expectation that a model developed within the presented framework will, indeed, have an improved predictive quality partly relies on the great significance of the statistical symmetries for predicting the scaling of turbulent flows, as discussed in previous publications such as those of Oberlack;[30] Oberlack, Wenzel, and Peters;[31] Oberlack and Rosteck;[25] Rosteck;[23] Oberlack *et al.*;[32] and Sadeghi, Oberlack, and Gauding.[33]

In concrete terms, it is expected that a model that is invariant under all classical and statistical symmetries can not only predict the scaling of the mean velocities correctly, which is at least partially accomplished by the existing models, but also accurately describe the scaling of the Reynolds stresses.

In the context of jet and wake type flows, the symmetry properties of the existing models should generally suffice in order to predict the mean velocities in classical self-similar flow. However, as has been discussed in Refs. 38–41, a deviation from classical self-similar behavior is frequently observed and could potentially be described by assuming an influence of the statistical scaling symmetry (49). This in turn allows obtaining a more general scaling law for the Reynolds stresses. Using the methods explained in such contributions as Ref. 33, one obtains the scaling law,

$$\overline{U}_i=\widetilde{U}_i(\widetilde{x})x_1^{-\frac{1}{2}}, \tag{79}$$

$$R_{ij}^{(0)}=\widetilde{H}_{ij}(\widetilde{x})x_1^{-A_2}-\widetilde{U}_i\widetilde{U}_j(\widetilde{x})x_1^{-1}, \tag{80}$$

where $\widetilde{x}$, $\widetilde{U}_i$, and $\widetilde{H}_{ij}$ are the constants of integration and $A_2$ is a parameter whose value could be inferred from the experimental data. If $A_2=1$, the classical self-similar behavior is recovered. The derivation of (79) and (80) is detailed in Appendix D 1. Such a behavior cannot be represented by classical Reynolds stress transport models because they do not exhibit this statistical scaling behavior. To see this, we consider a Reynolds Stress model of the form

$$\frac{\partial R_{ij}^{(0)}}{\partial t}+\overline{U}_k\frac{\partial R_{ij}^{(0)}}{\partial x_k}=-R_{ik}^{(0)}\frac{\partial \overline{U}_j}{\partial x_k}-R_{jk}^{(0)}\frac{\partial \overline{U}_i}{\partial x_k}+\cdots, \tag{81}$$

where we assume that closure relations for the unknown terms have been inserted, but we do not need to specify these closure relations at this point. Using the usual assumptions for jet type flow, Eq. (81) reduces to

$$\overline{U}_1\frac{\partial R_{ij}^{(0)}}{\partial x_1}+\overline{U}_2\frac{\partial R_{ij}^{(0)}}{\partial x_2}=-R_{i1}^{(0)}\frac{\partial \overline{U}_j}{\partial x_1}-R_{i2}^{(0)}\frac{\partial \overline{U}_j}{\partial x_2}-R_{j1}^{(0)}\frac{\partial \overline{U}_i}{\partial x_1}-R_{j2}^{(0)}\frac{\partial \overline{U}_i}{\partial x_2}+\cdots. \tag{82}$$

By now inserting the scaling laws for the mean velocity and the generalized scaling for the Reynolds stress, i.e., Eqs. (79) and (80), respectively, and the constraint (D7), Eq. (82) can only be self-similar if $A_2=1$, i.e., the classical scaling for the Reynolds stresses is recovered, and the generalized scaling, i.e., Eq. (80) with arbitrary $A_2$, cannot be represented.

However, in the exact equations, the more general scaling behavior given by Eqs. (79) and (80) can occur, as is best seen in the instantaneous formulation (18). The new model equation (75) for the presently considered flows reduces to

$$\hat{U}_1\frac{\partial H_{ij}^{(0)}}{\partial x_1}+\hat{U}_2\frac{\partial H_{ij}^{(0)}}{\partial x_2}+\hat{U}_i\frac{\partial H_{j1}^{(0)}}{\partial x_1}+\hat{U}_i\frac{\partial H_{j2}^{(0)}}{\partial x_2}+\hat{U}_j\frac{\partial H_{i1}^{(0)}}{\partial x_1}+\hat{U}_j\frac{\partial H_{i2}^{(0)}}{\partial x_2}+\overline{U_iP_{,x_j}}^{(0)}+\overline{U_jP_{,x_i}}^{(0)}-\frac{\partial\overline{U}_i}{\partial x_1}\hat{U}_j\hat{U}_1-\frac{\partial\overline{U}_i}{\partial x_2}\hat{U}_j\hat{U}_2-\frac{\partial\overline{U}_j}{\partial x_1}\hat{U}_i\hat{U}_1-\frac{\partial\overline{U}_j}{\partial x_2}\hat{U}_i\hat{U}_2=0. \quad (83)$$

Inserting the scaling laws, i.e., Eqs. (79) and (80), and a scaling law for $\hat{U}_i$, which we assume to be of the form

$$\hat{U}_i=\tilde{\hat{U}}_i x_2^{-\hat{A}_1}, \quad (84)$$

where $\hat{A}_1$ is a flow-dependent parameter whose value will, in practice, be controlled by the precise form of the equations used to determine $\hat{U}_i$, leads to an equation of the form

$$J(\widetilde{U}_i,\widetilde{H}_{ij},\tilde{\hat{U}}_i)x_1^{-\hat{A}_1-A_2-1}+K(\widetilde{U}_i,\widetilde{H}_{ij},\tilde{\hat{U}}_i)x_1^{-2\hat{A}_1-A_1-1}, \quad (85)$$

where $J$ and $K$ are expressions whose exact form is not relevant for this analysis. The requirement that the two exponents must be equal together with the constraint on the mean velocity resulting from boundary conditions (D7) leads to the requirement

$$A_2=\hat{A}_1+A_1=\hat{A}_1+\frac{1}{2}. \quad (86)$$

For $\hat{A}_1=1/2$, the classical behavior is recovered; however, it is apparent that the new model equation, unlike classical models, does not strictly require this behavior but instead depends on the scaling of $\hat{U}_i$. Thus, the new model behaves more like the exact equations, which also do not directly require the classical self-similar results.

It must be noted that it is presently not the purpose to find numerical or experimental examples for a flow exhibiting this behavior but to show that the new model equations behave more like the exact equations by also allowing for the generalized self-similar scaling given by (80), rendering the new model more general and, thus, reliable than the existing ones.

One might object that the symmetry properties of solutions such as those shown above are not enough to justify the use of symmetries to constrain turbulence models. After all, it is well-known that solutions may in general violate one or more symmetries of the underlying equations. However, it is empirically known that in turbulence, invariant solutions such as the log law, for which these additional symmetries are necessary, play an important role even in complicated flows. Therefore, preserving these symmetries in a turbulence model can lead to better model predictions. We note that in addition to symmetry constraints, it may also be worthwhile to include conservation laws in the modeling procedure. An advantage they have compared to symmetries is that they directly impose constraints on the solution; thus, the positive effect of including conservation laws can be more directly appreciated in a model.

Generally speaking, the statistical symmetries (41)–(44) encode important principles of turbulent statistics, namely, intermittency and non-Gaussianity. A model fulfilling these symmetries will, therefore, be able to more accurately account for these important effects.

The development presented in this section constitutes the final primary contribution of the current work. It is important to point out that the presented equations are not a complete turbulence model but instead form a skeleton of a new model that results from employing the presented framework. We stress that the method presented here does not replace the existing modeling strategies but rather complements them, much like dimensional analysis has supported turbulence modeling. Therefore, it is not reasonable to expect a single concrete turbulence model as a result from this procedure. Based on the form of the presented equations, it is, however, apparent that any model resulting from the modeling framework employed here will look quite different from the existing models.

## V. CONCLUSION

The present work revolves around three important points. First, the important role that symmetries have already played in turbulence research as an underlying and implicitly used concept is discussed while also hinting at the underused potential of the concept that we seek to exploit in this work. Examples for widely used concepts that are special instances of symmetry methods include tensor invariant theory and dimensional analysis. Furthermore, the well-known observation that linear eddy-viscosity models are inadequate for rotating and high-streamline curvature flows can be explained by the Boussinesq approximation admitting too many symmetries. In particular, it is invariant under a time-dependent rotation, falsely rendering it insensitive to system rotation.[15] Second, it is discussed that the statistical symmetries are in fact pivotal in order to accurately and reliably describe turbulent flows due to their relevance for turbulent scaling. In particular, scaling of second moments is captured. Third, a procedure allowing the development of model equations based on symmetries and a resulting method to extend the existing turbulence models to also fulfill the statistical symmetries of the averaged Navier–Stokes equations are presented.

The findings presented here lay the foundation for developing a completely new class of turbulence models, with the possibility to base them on the already existing modeling strategies. Future work will be aimed at implementing, validating, and further improving such models.

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## APPENDIX A: PROLONGATION OF INFINITESIMAL GENERATORS

For some infinitesimal generator (32), the prolongation reads

$$X^{(n)}=\xi_i\frac{\partial}{\partial x_i}+\eta_i\frac{\partial}{\partial y_i}+\eta_{i;j_1}\frac{\partial}{\partial y_{i,j_1}}+\eta_{i;j_1j_2}\frac{\partial}{\partial y_{i,j_1j_2}}+\cdots. \quad (A1)$$

The infinitesimals for first-order derivatives are

$$\eta_{i;j} = \frac{D\eta_i}{Dx_j} - y_{i,k}\frac{D\xi_k}{Dx_j}, \tag{A2}$$

and the additional infinitesimals can be calculated recursively using

$$\eta_{i;j_1\cdots j_s} = \frac{D\eta_{i;j_1\cdots j_{s-1}}}{Dx_{j_s}} - y_{i,kj_1\ldots j_{s-1}}\frac{D\xi_k}{Dx_{j_s}}, \qquad \frac{D}{Dx_j} = \frac{\partial}{\partial x_j} + y_{i,j}\frac{\partial}{\partial y_i} + y_{i,jk}\frac{\partial}{\partial y_{i,k}} + \cdots, \tag{A3}$$

where $D/Dx_j$ is referred to as the total differential operator.

## APPENDIX B: ADDITIONAL INFINITESIMAL GENERATORS

### 1. Classical symmetries in unaveraged variables

The infinitesimal form of the Navier–Stokes symmetries as given in global form by Eqs. (7)–(16) read

$$X_t = \frac{\partial}{\partial t}, \tag{B1}$$

$$X_{\mathrm{rot}_\alpha} = \varepsilon_{jk\alpha}x_j\frac{\partial}{\partial x_k} + \varepsilon_{jk\alpha}U_j\frac{\partial}{\partial U_k}, \tag{B2}$$

$$X_{\mathrm{Gal}} = f_{\mathrm{Gal}_i}(t)\frac{\partial}{\partial x_i} + f'_{\mathrm{Gal}_i}(t)\frac{\partial}{\partial U_i} - x_i f''_{\mathrm{Gal}_i}(t)\frac{\partial}{\partial P}, \tag{B3}$$

$$X_P = f_P(t)\frac{\partial}{\partial P}, \tag{B4}$$

$$X_{\mathrm{Sc},I} = x_i\frac{\partial}{\partial x_i} + U_i\frac{\partial}{\partial U_i} + 2P\frac{\partial}{\partial P}, \tag{B5}$$

$$X_{\mathrm{Sc},II} = t\frac{\partial}{\partial t} - U_i\frac{\partial}{\partial U_i} - 2P\frac{\partial}{\partial P}, \tag{B6}$$

$$X_{\mathrm{Sc,ns}} = 2t\frac{\partial}{\partial t} + x_i\frac{\partial}{\partial x_i} - U_i\frac{\partial}{\partial U_i} - 2P\frac{\partial}{\partial P}, \tag{B7}$$

though, strictly speaking, the Navier–Stokes equations are only invariant under (B1)–(B4) and (B7), while the Euler equations are invariant under (B1)–(B6).

### 2. Statistical symmetries in fluctuation approach

The statistical symmetries (41)–(43) written for the fluctuating variables read

$$X_{\mathrm{Tr,stat},I,i} = \frac{\partial}{\partial \overline{U}_i} + \left(\delta_{ij}\overline{U}_k + \delta_{ik}\overline{U}_j\right)\frac{\partial}{\partial R^{(0)}_{jk}}, \tag{B8}$$

$$X_{\mathrm{Tr,stat}II,ij} = \frac{\partial}{\partial R^{(0)}_{ij}}, \tag{B9}$$

$$X_{\mathrm{Sc,stat}} = \overline{U}_i\frac{\partial}{\partial \overline{U}_i} + \overline{P}\frac{\partial}{\partial \overline{P}} + \left(R^{(0)}_{ij} - \overline{U}_i\overline{U}_j\right)\frac{\partial}{\partial R^{(0)}_{ij}}. \tag{B10}$$

## APPENDIX C: DERIVING A REYNOLDS STRESS TRANSPORT MODEL FROM SYMMETRIES

### 1. Using no additional model variables

We assume the model equations to be of the general form

$$\boldsymbol{F}\Big(x_i, t, \overline{U}_i, \overline{P}, H^{(0)}_{ij}, \overline{U_iP_{,x_j}}^{(0)}, \overline{U}_{i,x_j}, \overline{U}_{i,t}, \overline{P}_{,x_i}, \overline{P}_{,t}, H^{(0)}_{ij,x_k}, H^{(0)}_{ij,t}, \overline{U_iP_{,x_j,x_k}}^{(0)}, \overline{U_iP_{,x_j,t}}^{(0)}\Big) = 0, \tag{C1}$$

where we do not consider viscous effects for now.

The classical symmetries of the Euler equations are given by the infinitesimal generators (35)–(39) and the statistical symmetries by (45)–(47). Demanding invariance of (C1) with respect to all the symmetries given by Eqs. (34)–(47) leads to the linear system of partial differential equations,

$$X_1\boldsymbol{F}|_{\boldsymbol{F}=0} = 0 \wedge X_2\boldsymbol{F}|_{\boldsymbol{F}=0} = 0 \wedge \cdots \wedge X_9\boldsymbol{F}|_{\boldsymbol{F}=0} = 0. \tag{C2}$$

This system could be fed as a whole into a computer algebra system to be solved automatically. However, for humans, it is much simpler to consider one symmetry at a time. We start with the translation symmetry in time (34),

$$X_t\boldsymbol{F} = \frac{\partial \boldsymbol{F}}{\partial t} = 0. \tag{C3}$$

Thus, the time must not appear explicitly in (C1), which is, therefore, reduced to

$$\boldsymbol{F}\big(x_i, \overline{U}_i, \overline{P}, H^{(0)}_{ij}, \overline{U_iP_{,x_j}}^{(0)}, \overline{U}_{i,x_j}, \overline{U}_{i,t}, \overline{P}_{,x_i}, \overline{P}_{,t}, H^{(0)}_{ij,x_k}, H^{(0)}_{ij,t}, \overline{U_iP_{,x_j,x_k}}^{(0)}, \overline{U_iP_{,x_j,t}}^{(0)}\Big) = 0. \tag{C4}$$

The other translation symmetries each yield a similar result: The Galilean symmetry (37) with a constant $f_{\mathrm{Gal}_i}(t)$, i.e., translation invariance in space, eliminates $x_i$, the pressure translation symmetry (36) with $f_P(t) = a_P$ eliminates $\overline{P}$, and the same symmetry with $f'_P(t) = a_P$ eliminates $\overline{P}_{,t}$ and $\overline{U_iP_{,x_j,t}}^{(0)}$. The statistical translation symmetries (46)–(48) eliminate $\overline{U}_i, H^{(0)}_{ij}$ and $\overline{U_iP_{,x_j}}^{(0)}$. This leaves us with the general form

$$\boldsymbol{F}\Big(\overline{U_iP_{,x_j}}^{(0)}, \overline{U}_{i,x_j}, \overline{U}_{i,t}, \overline{P}_{,x_i}, H^{(0)}_{ij,x_k}, H^{(0)}_{ij,t}, \overline{U_iP_{,x_j,x_k}}^{(0)}\Big) = 0, \tag{C5}$$

which is invariant under the symmetries (34), (36), (46), (47), and (37) with the free function set to a constant. Next, we demand invariance with respect to the full Galilean symmetry (37) for arbitrary functions $f_{\mathrm{Gal}_i}(t)$. Evaluating the invariance condition $X^{(1)}_{\mathrm{Gal}}\boldsymbol{F} = 0$ leads to the scalar PDE

$$\begin{aligned}&(f''_{\mathrm{Gal}_i}(t) - \overline{U}_{i,x_j}f'_{\mathrm{Gal}_j}(t))\frac{\partial \boldsymbol{F}}{\partial \overline{U}_{i,t}} - f''_{\mathrm{Gal}_i}(t)\frac{\partial \boldsymbol{F}}{\partial \overline{P}_{,x_i}}\\ &+(\overline{U}_{i,t}f'_{\mathrm{Gal}_j}(t) + \overline{U}_{j,t}f'_{\mathrm{Gal}_i}(t) + \overline{U}_i f''_{\mathrm{Gal}_j}(t) + \overline{U}_j f''_{\mathrm{Gal}_i}(t)\\ &-H^{(0)}_{ij,x_k}f'_{\mathrm{Gal}_k}(t))\frac{\partial \boldsymbol{F}}{\partial H^{(0)}_{ij,t}} + (\overline{U}_{i,x_k}f'_{\mathrm{Gal}_j}(t) + \overline{U}_{j,x_k}f'_{\mathrm{Gal}_i}(t))\frac{\partial \boldsymbol{F}}{\partial H^{(0)}_{ij,x_k}}\\ &+(\overline{P}_{,x_j}f'_{\mathrm{Gal}_i}(t) - \overline{U}_i f''_{\mathrm{Gal}_j}(t))\frac{\partial \boldsymbol{F}}{\partial \overline{U_iP_{,x_j}}^{(0)}}\\ &+(\overline{P}_{,x_jx_k}f'_{\mathrm{Gal}_i}(t) - \overline{U}_{i,x_k}f''_{\mathrm{Gal}_j}(t))\frac{\partial \boldsymbol{F}}{\partial \overline{U_iP_{,x_j\ ,x_k}}^{(0)}} = 0.\end{aligned} \tag{C6}$$

Using the method of characteristics, this system can be transformed into the ordinary differential equation (ODE) system

$$\frac{d\boldsymbol{F}}{d\tau}=0, \tag{C7}$$

$$\frac{d\overline{U}_{i,x_j}}{d\tau}=0, \tag{C8}$$

$$\frac{d\overline{U}_{i,t}}{d\tau}=f''_{\mathrm{Gal}_i}(t)-\overline{U}_{i,x_j}f'_{\mathrm{Gal}_j}(t), \tag{C9}$$

$$\frac{d\overline{P}_{,x_i}}{d\tau}=-f''_{\mathrm{Gal}_i}(t), \tag{C10}$$

$$\frac{dH^{(0)}_{ij,t}}{d\tau}=\overline{U}_{i,t}f'_{\mathrm{Gal}_j}(t)+\overline{U}_{j,t}f'_{\mathrm{Gal}_i}(t)+\overline{U}_if''_{\mathrm{Gal}_j}(t)+\overline{U}_jf''_{\mathrm{Gal}_i}(t)-H^{(0)}_{ij,x_k}f'_{\mathrm{Gal}_k}(t), \tag{C11}$$

$$\frac{dH^{(0)}_{ij,x_k}}{d\tau}=\overline{U}_{i,x_k}f'_{\mathrm{Gal}_j}(t)+\overline{U}_{j,x_k}f'_{\mathrm{Gal}_i}(t), \tag{C12}$$

$$\frac{d\overline{U_iP_{,x_j}}^{(0)}}{d\tau}=\overline{P}_{,x_j}f'_{\mathrm{Gal}_i}(t)-\overline{U}_if''_{\mathrm{Gal}_j}(t), \tag{C13}$$

$$\frac{d\overline{U_iP_{,x_j,x_k}}^{(0)}}{d\tau}=\overline{P}_{,x_jx_k}f'_{\mathrm{Gal}_i}(t)-\overline{U}_{i,x_k}f''_{\mathrm{Gal}_j}(t). \tag{C14}$$

Equations (C7)–(C10) and (C12) can be integrated directly, yielding

$$\boldsymbol{F}=c_1, \tag{C15}$$

$$\overline{U}_{i,x_j}=c_{2\_ij}, \tag{C16}$$

$$\overline{U}_{i,t}=f''_{\mathrm{Gal}_i}(t)\tau-\overline{U}_{i,x_j}f'_{\mathrm{Gal}_j}(t)\tau+c_{3\_i}, \tag{C17}$$

$$\overline{P}_{,x_i}=-f''_{\mathrm{Gal}_i}(t)\tau+c_{4\_i}, \tag{C18}$$

$$H^{(0)}_{ij,x_k}=\overline{U}_{i,x_k}f'_{\mathrm{Gal}_j}(t)\tau+\overline{U}_{j,x_k}f'_{\mathrm{Gal}_i}(t)\tau+c_{5\_ijk}. \tag{C19}$$

All constants of integration are understood to be functions of the other characteristic variable often denoted $s$. For modeling purposes and consistent with the method of characteristics, both $s$ and the constants of integration need to be eliminated from the final equations. Note that it is generally not possible to simply set the constants of integration to zero because doing so can potentially restrict the generality of the so obtained variables. Adding (C17) and (C18) yields

$$\overline{U}_{i,t}+\overline{P}_{,x_i}=-\overline{U}_{i,x_j}f'_{\mathrm{Gal}_j}(t)\tau+c_{3\_i}+c_{4\_i}. \tag{C20}$$

Thus,

$$\overline{U}_{i,x_j}f'_{\mathrm{Gal}_j}(t)\tau=-\overline{U}_{i,t}-\overline{P}_{,x_i}+c_{3\_i}+c_{4\_i}, \tag{C21}$$

and furthermore,

$$f'_{\mathrm{Gal}_i}(t)\tau=\overline{U}^{-1}_{i,x_j}(-\overline{U}_{j,t}-\overline{P}_{,x_j}+c_{3\_j}+c_{4\_j}). \tag{C22}$$

A first result is that we can recover the momentum equation by contracting $j=k$ in (C19), which, using mass conservation, leads to

$$H^{(0)}_{ij,x_j}=\overline{U}_{i,x_j}f'_{\mathrm{Gal}_j}(t)\tau+c_{5\_ijj}. \tag{C23}$$

In turn, adding Eqs. (C21) and (C23) yields

$$\overline{U}_{i,t}+\overline{P}_{,x_i}+H^{(0)}_{ij,x_j}=c_{3\_i}+c_{4\_i}+c_{5\_ijj}. \tag{C24}$$

Any expression that is constant in $\tau$ can be an argument of $\boldsymbol{F}$, which, because of (C15), is also constant in $\tau$. In other words, the expression $\overline{U}_{i,t}+\overline{P}_{,x_i}+H^{(0)}_{ij,x_j}$ is Galilean invariant.

In the context of turbulence modeling, we are of course more interested in obtaining a transport equation for $H^{(0)}_{ij}$. To this end, we add (C11) and (C13), which eliminates $\overline{U}_i$ and $\overline{U}_j$, leading to

$$\frac{dH^{(0)}_{ij,t}}{d\tau}+\frac{d\overline{U_iP_{,x_j}}^{(0)}}{d\tau}+\frac{d\overline{U_jP_{,x_i}}^{(0)}}{d\tau}=\overline{U}_{i,t}f'_{\mathrm{Gal}_j}(t)+\overline{U}_{j,t}f'_{\mathrm{Gal}_i}(t)+\overline{P}_{,x_j}f'_{\mathrm{Gal}_i}(t)+\overline{P}_{,x_i}f'_{\mathrm{Gal}_j}(t)-H^{(0)}_{ij,x_k}f'_{\mathrm{Gal}_k}(t). \tag{C25}$$

Furthermore, employing (C20), we obtain

$$\frac{dH^{(0)}_{ij,t}}{d\tau}+\frac{d\overline{U_iP_{,x_j}}^{(0)}}{d\tau}+\frac{d\overline{U_jP_{,x_i}}^{(0)}}{d\tau}=f'_{\mathrm{Gal}_i}(t)(-\overline{U}_{j,x_k}f'_{\mathrm{Gal}_k}(t)\tau+c_{3\_j}+c_{4\_j})+f'_{\mathrm{Gal}_j}(t)\,(-\overline{U}_{i,x_k}f'_{\mathrm{Gal}_k}(t)\tau+c_{3\_i}+c_{4\_i})-H^{(0)}_{ij,x_k}f'_{\mathrm{Gal}_k}(t). \tag{C26}$$

Before we can integrate this, $H^{(0)}_{ij,x_k}$ has to be replaced using (C19),

$$\begin{aligned}\frac{dH^{(0)}_{ij,t}}{d\tau}+\frac{d\overline{U_iP_{,x_j}}^{(0)}}{d\tau}+\frac{d\overline{U_jP_{,x_i}}^{(0)}}{d\tau}&=f'_{\mathrm{Gal}_i}(t)(-\overline{U}_{j,x_k}f'_{\mathrm{Gal}_k}(t)\tau+c_{3\_j}+c_{4\_j})+f'_{\mathrm{Gal}_j}(t)(-\overline{U}_{i,x_k}f'_{\mathrm{Gal}_k}(t)\tau\\&+c_{3\_i}+c_{4\_i}-\overline{U}_{i,x_k}f'_{\mathrm{Gal}_j}(t)\tau f'_{\mathrm{Gal}_k}(t)-\overline{U}_{j,x_k}f'_{\mathrm{Gal}_i}(t)\tau f'_{\mathrm{Gal}_k}(t)\\&-c_{5\_ijk}f'_{\mathrm{Gal}_k}(t)=2f'_{\mathrm{Gal}_i}(t)(-\overline{U}_{j,x_k}f'_{\mathrm{Gal}_k}(t)\tau)+f'_{\mathrm{Gal}_i}(t)(c_{3\_j}+c_{4\_j})\\&+2f'_{\mathrm{Gal}_j}(t)(-\overline{U}_{i,x_k}f'_{\mathrm{Gal}_k}(t)\tau)+f'_{\mathrm{Gal}_j}(t)(c_{3\_i}+c_{4\_i})-c_{5\_ijk}f'_{\mathrm{Gal}_k}(t).\end{aligned} \tag{C27}$$

Carrying out the integration yields

$$\begin{aligned}H^{(0)}_{ij,t}+\overline{U_iP_{,x_j}}^{(0)}+\overline{U_jP_{,x_i}}^{(0)}&=-f'_{\mathrm{Gal}_i}(t)\tau\overline{U}_{j,x_k}f'_{\mathrm{Gal}_k}(t)\tau+f'_{\mathrm{Gal}_i}(t)\tau(c_{3\_j}+c_{4\_j})\\&-f'_{\mathrm{Gal}_j}(t)\tau\overline{U}_{i,x_k}f'_{\mathrm{Gal}_k}(t)\tau+f'_{\mathrm{Gal}_j}(t)\tau(c_{3\_i}+c_{4\_i})\\&-c_{5\_ijk}f'_{\mathrm{Gal}_k}(t)\tau+c_{6\_ij}.\end{aligned} \tag{C28}$$

This expression can be simplified using Eq. (C19).

Next, we shorten the equations by introducing the constant $c_{7\_i}=c_{3\_i}+c_{4\_i}$. Finally, we use (C21) [and, if necessary, its inverse form (C22)] to eliminate the $f'_{\mathrm{Gal}_i}(t)\tau$-terms. This results in

$$\begin{aligned}H^{(0)}_{ij,t}+\overline{U_iP_{,x_j}}^{(0)}+\overline{U_jP_{,x_i}}^{(0)}&=-H^{(0)}_{ij,x_k}\overline{U}^{-1}_{k,x_l}(-\overline{U}_{n,t}-\overline{P}_{,x_l}+c_{7\_l})\\&-\overline{U}^{-1}_{i,x_k}(-\overline{U}_{k,t}-\overline{P}_{,x_k}+c_{7\_k})c_{7\_j}\\&-\overline{U}^{-1}_{j,x_k}(-\overline{U}_{k,t}-\overline{P}_{,x_k}+c_{7\_k})c_{7\_i}+c_{6\_ij}.\end{aligned} \tag{C29}$$

At this point, we recognize that the expression $\overline{U}_{i,x_k}^{-1}(-\overline{U}_{k,t} - \overline{P}_{,x_k})$ appears in multiple places. We, therefore, introduce the abbreviation

$$\gamma_i = \overline{U}_{i,x_k}^{-1}(-\overline{U}_{k,t} - \overline{P}_{,x_k}). \tag{C30}$$

It is interesting to note that $\gamma_i$ transforms like $\overline{U}_i$ under all classical symmetries (34)–(39) but is invariant under all statistical symmetries.

From the momentum equation, we know that

$$\overline{U}_{i,t} + \overline{P}_{,x_i} = -H^{(0)}_{ik,x_k}. \tag{C31}$$

We can, thus, simplify (C29) to

$$H^{(0)}_{ij,t} + H^{(0)}_{ij,x_k}\gamma_k + \overline{U_i P_{,x_j}}^{(0)} + \overline{U_j P_{,x_i}}^{(0)} = -H^{(0)}_{ij,x_k}\overline{U}_{k,x_n}^{-1}c_{7\_n} - \gamma_i c_{7\_j} - \gamma_j c_{7\_i} + c_{6\_ij}. \tag{C32}$$

By setting $c_{7\_i} = 0$, the left hand side of (C32) becomes Galilean invariant. However, setting $c_{7\_i} = 0$ directly implies that

$$\overline{U}_{i,t} + \gamma_k \overline{U}_{i,x_k} + \overline{P}_{,x_i} = 0, \tag{C33}$$

i.e., the definition of $\gamma_i$ (C30). If $\gamma_i$ is understood to be a model variable on its own, as will be in Appendix C 2, the left hand side of (C32) turns out to be no longer Galilean invariant. In the current analysis, however,

$$H^{(0)}_{ij,t} + H^{(0)}_{ij,x_k}\gamma_k + \overline{U_i P_{,x_j}}^{(0)} + \overline{U_j P_{,x_i}}^{(0)} = 0 \tag{C34}$$

is a possible model equation for the Reynolds stresses. Of course, further invariants could be added to the right-hand side, but (C34) constitutes a minimal example.

We still need an equation to calculate $\overline{U_i P_{,x_j}}^{(0)}$ in (C34). Integrating (C14) and eliminating the $\tau$-terms using (C10) and (C22) leads to

$$\frac{\partial \overline{U_i P_{,x_j}}}{\partial x_k} - \gamma_i \frac{\partial^2 \overline{P}}{\partial x_j \partial x_k} - \frac{\partial \overline{P}}{\partial x_j}\frac{\partial \overline{U}_i}{\partial x_k} = 0. \tag{C35}$$

### 2. Using a new model velocity and pressure

The system of equations derived above already shows the structure of a statistically invariant Reynolds stress model, but it has a number of shortcomings. Most significantly, Eq. (C30) does not uniquely determine $\gamma_i$ depending on the velocity gradient. It is, therefore, necessary to introduce a new model variable we denote $\hat{U}_i$ and to solve a transport equation for it. This model velocity is defined to behave like $\overline{U}_i$ under all classical symmetries (34)–(39) while being invariant under the statistical symmetries (45)–(47). It proves useful to also introduce a corresponding pressure field which we call $\hat{P}$. By definition, it behaves like $\overline{P}$ under the classical symmetries but is also invariant under the statistical ones. The starting point for the derivation is, thus,

$$\boldsymbol{F}\Big(x_i, t, \overline{U}_i, \overline{P}, H^{(0)}_{ij}, \hat{U}_i, \hat{P}, \overline{U}_{i,x_j}, \overline{U}_{i,t}, \overline{P}_{,x_i}, \overline{P}_{,t}, \hat{U}_{i,x_j}, \hat{U}_{i,t}, \hat{P}_{,x_j}, \hat{P}_{,t}, H^{(0)}_{ij,x_k}, H^{(0)}_{ij,t}, \overline{U_i P_{,x_j}}^{(0)}, \overline{U_i P_{,x_j,x_k}}^{(0)}, \overline{U_i P_{,x_j,t}}^{(0)}\Big) = 0. \tag{C36}$$

The translation symmetries invoked above have essentially the same effect here, reducing the set of model variables to

$$\boldsymbol{F}\Big(\hat{U}_i, \overline{U}_{i,x_j}, \overline{U}_{i,t}, \overline{P}_{,x_i}, \hat{U}_{i,x_j}, \hat{U}_{i,t}, \hat{P}_{,x_j}, H^{(0)}_{ij,x_k}, H^{(0)}_{ij,t}, \overline{U_i P_{,x_j}}^{(0)}, \overline{U_i P_{,x_j,x_k}}^{(0)}\Big) = 0. \tag{C37}$$

We now apply the Galilean symmetry (37), which, prolongated and extended by the new model variables, reads

$$\begin{aligned} X^{(1)}_{\mathrm{Gal}} =\,& f'_{\mathrm{Gal}_i}(t)\frac{\partial}{\partial \hat{U}_i} + (f''_{\mathrm{Gal}_i}(t) - \overline{U}_{i,x_j}f'_{\mathrm{Gal}_j}(t))\frac{\partial}{\partial \overline{U}_{i,t}} + (f''_{\mathrm{Gal}_i}(t) \\ &- \hat{U}_{i,x_j}f'_{\mathrm{Gal}_j}(t))\frac{\partial}{\partial \hat{U}_{i,t}} - f''_{\mathrm{Gal}_i}(t)\frac{\partial}{\partial \overline{P}_{,x_i}} - f''_{\mathrm{Gal}_i}(t)\frac{\partial}{\partial \hat{P}_{,x_i}} + (\overline{U}_{i,t}f'_{\mathrm{Gal}_j}(t) \\ &+ \overline{U}_{j,t}f'_{\mathrm{Gal}_i}(t) + \overline{U}_i f''_{\mathrm{Gal}_j}(t) + \overline{U}_j f''_{\mathrm{Gal}_i}(t) - H^{(0)}_{ij,x_k}f'_{\mathrm{Gal}_k}(t))\frac{\partial}{\partial H^{(0)}_{ij,t}} \\ &+ (\overline{U}_{i,x_k}f'_{\mathrm{Gal}_j}(t) + \overline{U}_{j,x_k}f'_{\mathrm{Gal}_i}(t))\frac{\partial}{\partial H^{(0)}_{ij,x_k}} + (\overline{P}_{,x_j}f'_{\mathrm{Gal}_i}(t) - \overline{U}_i f''_{\mathrm{Gal}_j}(t)) \\ &\times \frac{\partial}{\partial \overline{U_i P_{,x_j}}^{(0)}} + (\overline{P}_{,x_j x_k}f'_{\mathrm{Gal}_i}(t) - \overline{U}_{i,x_k}f''_{\mathrm{Gal}_j}(t))\frac{\partial}{\partial \overline{U_i P_{,x_j,x_k}}^{(0)}}, \end{aligned} \tag{C38}$$

to (C37). The invariant surface condition, thus, leads to the ODE system

$$\frac{d\boldsymbol{F}}{d\tau} = 0, \tag{C39}$$

$$\frac{d\hat{U}_i}{d\tau} = f'_{\mathrm{Gal}_i}(t), \tag{C40}$$

$$\frac{d\overline{U}_{i,x_j}}{d\tau} = 0, \tag{C41}$$

$$\frac{d\hat{U}_{i,x_j}}{d\tau} = 0, \tag{C42}$$

$$\frac{d\overline{U}_{i,t}}{d\tau} = f''_{\mathrm{Gal}_i}(t) - \overline{U}_{i,x_j}f'_{\mathrm{Gal}_j}(t), \tag{C43}$$

$$\frac{d\hat{U}_{i,t}}{d\tau} = f''_{\mathrm{Gal}_i}(t) - \hat{U}_{i,x_j}f'_{\mathrm{Gal}_j}(t), \tag{C44}$$

$$\frac{d\overline{P}_{,x_i}}{d\tau} = -f''_{\mathrm{Gal}_i}(t), \tag{C45}$$

$$\frac{d\hat{P}_{,x_i}}{d\tau} = -f''_{\mathrm{Gal}_i}(t), \tag{C46}$$

$$\frac{dH^{(0)}_{ij,t}}{d\tau} = \overline{U}_{i,t}f'_{\mathrm{Gal}_j}(t) + \overline{U}_{j,t}f'_{\mathrm{Gal}_i}(t) + \overline{U}_i f''_{\mathrm{Gal}_j}(t) + \overline{U}_j f''_{\mathrm{Gal}_i}(t) - H^{(0)}_{ij,x_k}f'_{\mathrm{Gal}_k}(t), \tag{C47}$$

$$\frac{dH^{(0)}_{ij,x_k}}{d\tau} = \overline{U}_{i,x_k}f'_{\mathrm{Gal}_j}(t) + \overline{U}_{j,x_k}f'_{\mathrm{Gal}_i}(t), \tag{C48}$$

$$\frac{d\overline{U_i P_{,x_j}}^{(0)}}{d\tau} = \overline{P}_{,x_j}f'_{\mathrm{Gal}_i}(t) - \overline{U}_i f''_{\mathrm{Gal}_j}(t), \tag{C49}$$

$$\frac{d\overline{U_iP_{,x_j}}_{,x_k}^{(0)}}{d\tau} = \overline{P}_{,x_jx_k} f'_{\mathrm{Gal}_i}(t) - \overline{U}_{i,x_k} f''_{\mathrm{Gal}_j}(t). \tag{C50}$$

The combination of (C40), (C42), (C44), and (C46) yields the variable

$$c_{1\_i} = \frac{\partial \hat{U}_i}{\partial t} + \hat{U}_j \frac{\partial \hat{U}_i}{\partial x_j} + \frac{\partial \hat{P}}{\partial x_i}, \tag{C51}$$

from which we derive a transport equation for $\hat{U}_i$,

$$\frac{\partial \hat{U}_i}{\partial t} + \hat{U}_j \frac{\partial \hat{U}_i}{\partial x_j} + \frac{\partial \hat{P}}{\partial x_i} = 0. \tag{C52}$$

As the characteristic system considered here is an extended version of that considered in Appendix C 1, we can follow the same steps as before to integrate (C47), which again leads to

$$\begin{aligned} H_{ij,t}^{(0)} + \overline{U_iP_{,x_j}}^{(0)} + \overline{U_jP_{,x_i}}^{(0)} &= -H_{ij}^{(0)} x_k f'_{\mathrm{Gal}_k}(t)\tau + f'_{\mathrm{Gal}_i}(t)\tau c_{7\_j} \\ &\quad + f'_{\mathrm{Gal}_j}(t)\tau c_{7\_i} + c_{6\_ij}, \end{aligned} \tag{C53}$$

where we are using the same indices for integration constants as in Sec. IV E. Integrating (C40) yields

$$\hat{U}_i = f'_{\mathrm{Gal}_i}(t)\tau + c_{8\_i}. \tag{C54}$$

We also know from (C20) that

$$c_{7\_i} = \overline{U}_{i,t} + \overline{U}_{i,x_k} f'_{\mathrm{Gal}_k}(t)\tau + \overline{P}_{,x_i}. \tag{C55}$$

Combining (C53)–(C55) leads to

$$\begin{aligned} &H_{ij,t}^{(0)} + \hat{U}_k H_{ij,x_k}^{(0)} + \overline{U_iP_{,x_j}}^{(0)} + \overline{U_jP_{,x_i}}^{(0)} \\ &\quad + \hat{U}_i H_{jk,x_k}^{(0)} + \hat{U}_j H_{ik,x_k}^{(0)} - \overline{U}_{i,x_k}\hat{U}_j\hat{U}_k - \overline{U}_{j,x_k}\hat{U}_i\hat{U}_k \\ &\quad = H_{ij,x_k}^{(0)} c_{8\_k} + \hat{U}_i \overline{U}_{j,x_k} c_{8\_k} + \hat{U}_k \overline{U}_{j,x_k} c_{8\_i} + \hat{U}_k \overline{U}_{i,x_k} c_{8\_j} \\ &\quad - H_{jk}^{(0)}, x_k c_{8\_i} - H_{ik,x_k}^{(0)} c_{8\_j} + c_{8\_i} c_{8\_k} \overline{U}_{j,x_k} + c_{6\_ij}. \end{aligned} \tag{C56}$$

This time, we do not need to set $c_{7\_i} = 0$, but, instead, we can set $c_{8\_i} = 0$, which does not lead to any complications. Thus, we obtain as a model equation for the Reynolds stresses

$$\begin{aligned} &\frac{\partial H_{ij}^{(0)}}{\partial t} + \hat{U}_k \frac{\partial H_{ij}^{(0)}}{\partial x_k} + \overline{U_iP_{,x_j}}^{(0)} + \overline{U_jP_{,x_i}}^{(0)} + \hat{U}_i \frac{\partial H_{jk}^{(0)}}{\partial x_k} \\ &\quad + \hat{U}_j \frac{\partial H_{ik}^{(0)}}{\partial x_k} - \frac{\partial \overline{U}_i}{\partial x_k}\hat{U}_j\hat{U}_k - \frac{\partial \overline{U}_j}{\partial x_k}\hat{U}_i\hat{U}_k = 0, \end{aligned} \tag{C57}$$

where the velocity-pressure terms can be obtained from the same equation as before. However, in order to also fulfill the statistical scaling symmetry, it is now possible to replace the mean pressure gradient with the gradient of the newly introduced pressure to yield

$$\frac{\partial \overline{U_iP_{,x_j}}^{(0)}}{\partial x_k} - \hat{U}_i \frac{\partial^2 \overline{P}}{\partial x_j \partial x_k} - \frac{\partial \hat{P}}{\partial x_j}\frac{\partial \overline{U}_i}{\partial x_k} = 0. \tag{C58}$$

Together with the continuity equations for $\overline{U}_i$ and $\hat{U}_i$, and the momentum equation (C24), Eqs. (75), (76), and (C54) constitute a closed set of model equations. This set of equations fulfills all classical and statistical symmetries considered in this work.

## APPENDIX D: INVARIANT SOLUTIONS

The high relevance of statistical symmetries for turbulent flow is perhaps most clearly observed in the context of turbulent scaling laws. Without statistical symmetries, it would not be possible to obtain many well-known scaling laws analytically.[32] Turbulent scaling laws are functions that describe mean velocity profiles in a special flow configuration or region for a wide range of flow conditions. Among the many turbulent scaling laws, the probably most well-known one is the logarithmic law of the wall. Other examples include self-similar solutions for jets, wakes, and homogeneous flows. Historically, these scaling laws were mainly obtained empirically. However, using symmetry theory, they can be calculated systematically.

It is a natural requirement for a turbulence model to reproduce these scaling laws. An early work investigating the scaling behavior of turbulence models was conducted by Guenther and Oberlack.[43] We will return to this point in Sec. IV E.

In mathematics, the concept is commonly referred to as invariant solutions rather than scaling laws, which seems to imply that only scaling symmetries have been invoked, though this is not true for the logarithmic law of the wall as we will show below. Still, in this work, these two terms are used synonymously. The idea is to find functions that are invariant under a subset of symmetries given by Eqs. (34)–(40) and Eqs. (45)–(48). This subset is typically selected based on physical arguments. Works applying this method include those of Oberlack;[30] Oberlack, Wenzel, and Peters;[31] Oberlack and Rosteck;[25] Rosteck;[23] Oberlack *et al.*;[32] and Sadeghi, Oberlack, and Gauding.[33] In Appendix D 1, the method and some of the most important results are briefly presented.

Invariant solutions are functions $\boldsymbol{z} = \boldsymbol{\Theta}(\boldsymbol{x})$ that depend on the independent variables $\boldsymbol{x}$ of the problem and are invariant under a symmetry generator $X$ and at the same time are also a solution of the given differential equation.[27] The invariance of $\boldsymbol{z} = \boldsymbol{\Theta}(\boldsymbol{x})$ under $X$ is equivalent to

$$X\Big(z_i - \Theta_i(\boldsymbol{x})\Big)\big|_{z_i=\Theta_i(\boldsymbol{x})} = 0, \tag{D1}$$

where $\boldsymbol{z}$, as in Sec. III, stands for the dependent variables of the problem. Using the method of characteristics, (D1) is equivalent to the system of ordinary differential equations

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \cdots = \frac{dx_{[n]}}{\xi_{[n]}} = \frac{dz_1}{\eta_1} = \frac{dz_2}{\eta_2} = \frac{dz_{[m]}}{\eta_{[m]}} = \cdots, \tag{D2}$$

where $n$ and $m$ are the number of independent and dependent variables, respectively, and brackets denote that the summation over repeated indices is suppressed. Solving the system (D2) yields functions $\boldsymbol{z} = \boldsymbol{\Theta}(\boldsymbol{x})$ that are invariant under the considered symmetry generator $X$. The constants of integration appearing in the solution of this system appear as similarity variables in the function $\boldsymbol{\Theta}(\boldsymbol{x})$.

### 1. Self-similar jet and wake

An important example to motivate the use of statistical symmetries is turbulent jet flow. We consider a statistically two-dimensional flow in the $x_1$-$x_2$-plane, with the $x_1$-axis pointing in the primary flow direction.

The classical treatment of self-similar flows is discussed in the work of Townsend[44] and has since entered many textbooks. However, the behavior predicted by this theory has not consistently been reproduced in experiments and simulations, which has led to the development of more advanced theories.[38–41] Apparently, under certain circumstances, for reasons that are still under some debate, a deviation from the classical self-similar behavior can be observed. We propose that this deviation can be represented by taking into account the statistical scaling symmetry (49), which is overlooked in the classical theory.

For the jet without a coflow, we consider the linear combination of classical and statistical scaling symmetries,

$$X = a_{\mathrm{Sc},I} X_{\mathrm{Sc},I} + a_{\mathrm{Sc},II} X_{\mathrm{Sc},II} + a_{\mathrm{Sc,stat}} X_{\mathrm{Sc,stat}}. \tag{D3}$$

The resulting invariant surface condition reads

$$\frac{dx_1}{a_{\mathrm{Sc},I} x_1} = \frac{dx_2}{a_{\mathrm{Sc},I} x_2} = \frac{d\overline{U}_{[i]}}{(a_{\mathrm{Sc},I} - a_{\mathrm{Sc},II} + a_{\mathrm{Sc,stat}})\overline{U}_{[i]}} = \frac{dH^{(0)}_{[ij]}}{(2a_{\mathrm{Sc},I} - 2a_{\mathrm{Sc},II} + a_{\mathrm{Sc,stat}})H^{(0)}_{[ij]}}. \tag{D4}$$

The first equation of (D4) is integrated to

$$\widetilde{x} = \frac{x_2}{x_1}, \tag{D5}$$

where $\widetilde{x}$ is a constant of integration, which we can interpret as a similarity variable in the present context. This is equivalent to the classical definition of the similarity variable.[45] Using the first and the third term in Eq. (D4), the scaling law of the mean velocity becomes

$$\overline{U}_i = \widetilde{U}_i x_1^{\frac{a_{\mathrm{Sc},I} - a_{\mathrm{Sc},II} + a_{\mathrm{Sc,stat}}}{a_{\mathrm{Sc},I}}}, \tag{D6}$$

where $\widetilde{U}_i$ is the constant of integration, which is also a new invariant and only depends on $\widetilde{x}$. The boundary condition that the velocity has to be zero as $x_2$ approaches infinity together with the assumption that the jet spreading rate is constant leads to[45]

$$\frac{a_{\mathrm{Sc},I} - a_{\mathrm{Sc},II} + a_{\mathrm{Sc,stat}}}{a_{\mathrm{Sc},I}} = -A_1 = -\frac{1}{2}. \tag{D7}$$

To find the scaling law for the second instantaneous moments, we consider the first and the last term in Eq. (D4) and obtain

$$H^{(0)}_{ij} = \widetilde{H}_{ij} x_1^{\frac{2a_{\mathrm{Sc},I} - 2a_{\mathrm{Sc},II} + a_{\mathrm{Sc,stat}}}{2a_{\mathrm{Sc},I}}}, \tag{D8}$$

where $\widetilde{H}_i$ is another constant of integration, which is also an invariant only depending on $\widetilde{x}$. Expressing this in terms of fluctuation moments yields

$$R^{(0)}_{ij} = \widetilde{H}_{ij} x_1^{\frac{2a_{\mathrm{Sc},I} - 2a_{\mathrm{Sc},II} + a_{\mathrm{Sc,stat}}}{2a_{\mathrm{Sc},I}}} + \widetilde{U}_i \widetilde{U}_j x_1^{2\frac{a_{\mathrm{Sc},I} - a_{\mathrm{Sc},II} + a_{\mathrm{Sc,stat}}}{a_{\mathrm{Sc},I}}}. \tag{D9}$$

Inserting Eq. (D7) leads to

$$R^{(0)}_{ij} = \widetilde{H}_{ij}(\widetilde{x}) x_1^{-A_2} - \widetilde{U}_i \widetilde{U}_j(\widetilde{x}) x_1^{-1}, \tag{D10}$$

where $A_2$ and the constants of integration, i.e., $\widetilde{U}_i$ and $\widetilde{H}_{ij}$, appear as degrees of freedom and could be determined using the experimentally or numerically obtained data. It is apparent that for $A_2 = 1$, the classical scaling laws are recovered from Eqs. (79) and (D10) but that the form derived here is more general and, thus, holds promise to be valid for a wider range of the flow. In particular, form (80), assuming that $A_2$ is slightly larger than 1, can describe the frequently observed behavior that the Reynolds stresses (and higher correlations) take longer to achieve self-preservation (in the classical sense) than the mean velocity.[46]

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