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Zusammenfassung

In dieser Dissertation beschäftigen wir uns mit der numerischen Approximation von parabolisch-elliptischen Interfaceproblemen. Zur Lösung werden verschiedene Varianten der nichtsymmetrischen Kopplungsmethode von MacCamy und Suri aus [Quart. Appl. Math., 44 (1987), S. 675–690] verwendet. Im Speziellen betrachten wir die Kopplung der Finiten Elemente Methode (FEM) mit der Randelementemethode (BEM) für ein einfaches Modellproblem. Wir zeigen die Wohlgestelltheit des Problems und, dass wir eine quasi-optimale Lösung erhalten, auch wenn der Rand des Gebietes nicht glatt ist. Hieraus können Fehlerabschätzungen optimaler Ordnung abgeleitet werden. Des Weiteren betrachten wir die darauf folgende Zeitdiskretisierung durch eine Variante der impliziten Eulermethode. Genauso wie für die Semidiskretisierung können wir für die Volldiskretisierung Wohlgestelltheit und Quasioptimalität unter minimalen Regularitätsanforderungen zeigen. Hieraus können wieder Fehlerabschätzungen optimaler Ordnung abgeleitet werden.

Die Klasse der parabolisch-elliptischen Interfaceprobleme umfasst auch konvektionsdominierte Diffusions-Konvektions-Reaktions-Probleme. Dies stellt die Lösungsmethode vor weitere Herausforderungen, da zum Beispiel die Finite Elemente Methode keine stabile Lösung für konvektionsdominierte Probleme berechnen kann. Zwei Möglichkeiten stabile Lösungen zu erhalten, sind die Finite Volumen Methode (FVM) mit einer Upwindstabiliserung sowie die Streamline Upwind Petrov Galerkin Methode (SUPG). Die FVM erhält zusätzlich noch die numerischen Flüsse. Die SUPG Methode hingegen ist eine einfache Erweiterung der FEM. Daher betrachten wir auch die FVM-BEM sowie die SUPG-BEM Kopplung für die Semidiskretisierung des parabolisch-elliptischen Problems. Für die Zeitdiskretisierung verwenden wir wieder die Variante der impliziten Eulermethode. Hierdurch können wir die Semi- sowie die Volldiskretisierung unter minimalen Regularitätsanforderungen analysieren.

Zuletzt zeigen wir einige numerische Beispiele, welche die theoretischen Resultate illustrieren und einige mögliche praktische Anwendungen aufzeigen, wie zum Beispiel Wirbelstromprobleme oder Probleme aus der Fluidmechanik.

Abstract

In this thesis, we consider the numerical approximation of parabolic-elliptic interface problems with variants of the non-symmetric coupling method of MacCamy and Suri [Quart. Appl. Math., 44 (1987), pp. 675–690]. In particular, we look at the coupling of the Finite Element Method (FEM) and the Boundary Element Method (BEM) for a basic model problem and establish well-posedness and quasi-optimality of this formulation for problems with non-smooth interfaces. From this, error estimates with optimal order can be deduced. Moreover, we investigate the subsequent discretisation in time by a variant of the implicit Euler method. As for the semi-discretisation, we establish well-posedness and quasi-optimality for the fully discrete scheme under minimal regularity assumptions on the solution. Error estimates with optimal order follow again directly.

The class of parabolic-elliptic interface problems also includes convection-dominated diffusion-convection-reaction problems. This poses a certain challenge to the solving method, as for example the Finite Element Method cannot stably solve convection-dominated problems. A possible remedy to guarantee stable solutions is the use of the vertex-centred Finite Volume Method (FVM) with an upwind stabilisation option or the Streamline Upwind Petrov Galerkin method (SUPG). The FVM has the additional advantage of the conservation of the numerical fluxes, whereas the SUPG is a simple extension of FEM. Thus, we also look at an FVM-BEM and SUPG-BEM coupling for a semi-discretisation of the underlying problem. The subsequent time-discretisation will again be achieved by the variant of the implicit Euler method. This allows us to develop an analysis under minimal regularity assumptions, not only for the semi-discrete systems but also for the fully-discrete systems.

Lastly, we show some numerical examples to illustrate our theoretical results and to give an outlook to possible practical applications, such as eddy current problems or fluid mechanics problems.

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Contents

"The beginner should not be discouraged if he finds he does not have the prerequisites for reading the prerequisites."

Paul Halmos

Introduction

IN THIS THESIS we will look at a certain system of partial differential equations, namely at a parabolic problem that is coupled with an elliptic problem via an interface. We will then go on to investigate several numerical methods to solve such a problem.

In the spirit of the opening quote of this chapter, we will start with some (informal) descriptions and a historical overview of what has been done in the realm of parabolic-elliptic interface problems. The exact model problem will be introduced in Chapter 2. For now, it is enough to know that we look at a parabolic (and thus time-dependent) PDE in a bounded domain that is coupled to an elliptic PDE in an unbounded domain. They are coupled at an interface and fulfil some transmission conditions that connect the interior to the exterior solution. A motivation for this setup comes from so-called eddy current problems in electromagnetics, see e.g. [MS87]. In this setup, we have a conducting material in an electric field which is surrounded by air. We can set a boundary around the conducting material and model this (interior) part with a parabolic equation. The rest, which now is the exterior, is unbounded and here the electric field in air can be modelled by an elliptic equation. The coupling of a parabolic problem to an exterior problem can also be seen as a replacement of (maybe) unknown Dirichlet and/or Neumann data, see also [Era12, Remark 2.1].

A numerical approach to tackle this kind of problem without having to truncate the domain is to use the Boundary Element Method (BEM), which reduces the exterior problem to a problem on the boundary (the interface). Then we can use another method in the interior domain and use the transmission conditions to couple the two methods. An overview of state-of-the-art couplings with BEM for the elliptic-elliptic coupling – thus for steady-state equations – is given in [Aur+13]. For the elliptic-elliptic coupling Johnson and Nédélec introduced the non-symmetric coupling of BEM and the Finite Element Method (FEM) in [JN80]. This was later extended by MacCamy and Suri to the parabolic-elliptic case in [MS87], where they established the well-posedness of our model problem with a Galerkin approximation in space. Their analysis is based on the compactness of an integral operator, called the double layer operator, which relies on the assumption that the boundary is smooth, see [Cos88b]. As a by-product of their analysis, the authors also proved quasi-optimal error estimates in the energy norm for general Galerkin approximations under mild assumptions on the approximation spaces. This also means that a discretisation by appropriate finite and boundary elements directly leads to error estimates with optimal order for the resulting semi-discrete schemes.

The Galerkin approximation in space leads to an ordinary differential equation in time which, consequently, has to be discretised as well. This could not be done in the paper [MS87] and remained an open question.

Another approach to solving elliptic-elliptic interface problems is the symmetric coupling of FEM and BEM proposed in [Cos88a], which was later extended to parabolic-elliptic problems by Costabel, Ervin, and Stephan in [CES90]. This different approach also allowed them to prove well-posedness and quasi-optimality of Galerkin approximations without the restrictive smoothness assumption on the domain. In addition, they investigated the subsequent time discretisation by the Crank-Nicolson method and established error estimates for the resulting fully-discrete scheme. Still, it would be desirable to use the non-symmetric coupling instead, as the symmetric coupling involves more boundary integral operators and thus is computationally more expensive.

Their analysis of the fully-discrete coupling was based on the introduction of an elliptic projection with corresponding error estimates in L^2, which relied on classical duality arguments (see e.g. [Whe73]). Note that these duality arguments usually require the domain to be convex. This approach cannot be carried over to the non-symmetric coupling of MacCamy and Suri, as it lacks adjoint consistency. Therefore, "an analysis of a fully discretized version of their coupling scheme is not available and will be difficult", as argued in [CES90].

But in the meantime things have changed. In 2009 a groundbreaking paper by Sayas, [Say09], about the non-symmetric FEM-BEM coupling was published, which was the first one to show the well-posedness of the method. Only then the coercivity of the coupled bilinear form could be deduced in [Ste11] (and was later extended in [OS13]). An equivalent stabilised coercive formulation has then been introduced in [Aur+13]. These results allow us to close the gaps and analyse the non-symmetric coupling for parabolic-elliptic interface problems in the natural energy norm of the problem.

To close the gaps for the fully-discrete case, we choose to use a variant of the implicit Euler method, similar to the one in [Tan14], which allows us to show a quasi-optimality result by writing the fully-discrete problem consistently with the semi-discrete problem. For the quasi-optimality in the semi- and the fully-discrete case we make use of the L^2–projection onto the discrete space. Moreover, we do not need to assume a convex domain. Regarding the standalone FEM for parabolic problems, [CH02] gives quasi-optimality in the semi-discrete case by using the L^2–projection, in contrast to the more classical approaches (see [DD70; Whe73; Cha+06]). The role of the stability of the L^2–projection for the quasi-optimality has later been investigated in [TV16].

The eddy current problem motivating the parabolic-elliptic coupling in [MS87] can then be solved with the non-symmetric method. We give an example with the derivation of the problem in Section 4.1.2.

For most parabolic-elliptic interface problems, this FEM-BEM coupling is a sufficient choice. However, this class of problems also includes diffusion-convection-reaction equations in the interior domain which can be dominated by convection and thus pose some challenges to the numerical method. In the convection-dominated case, the FEM-BEM coupling is no longer stable for reasonable mesh sizes and yields unwanted oscillations. Furthermore, for fluid mechanics applications – which can be modelled by such diffusion-convection-reaction equations – we would like to have a method that conserves the numerical fluxes. The classical FEM, however, is not conservative.

The study [Aug+11] compares different stable discretisation methods for convection-diffusion equations with dominating convection. The work concludes that the Streamline Upwind Petrov Galerkin (SUPG) method or the Finite Volume Method (FVM) with upwind stabilisation are often sufficient, although they are the easiest approaches. When we speak of FVM, we always mean the vertex-centred FVM. The other schemes studied were mostly non-linear and thus had a higher computational cost, which was not justified by the results.

SUPG is based on a stabilised Finite Element Method by adding so-called artificial diffusion in the direction of the streamlines, steered by a stabilisation parameter. It is important to note that the parameter has to be chosen appropriately depending on the specific example and an optimal parameter cannot be given. The FVM approach, on the other hand, is based on a reformulation of the problem as a balance equation and allows a natural upwind stabilisation. Additionally, it preserves conservation of numerical fluxes and on certain grids it fulfils the maximum principle. Therefore, FVM is often the method of choice for fluid mechanics applications. Although numerical fluxes are not conservative for SUPG, the comparison in [Aug+11] showed that SUPG creates sharper layers than FVM with upwinding but does not completely avoid spurious oscillations. Additionally, SUPG can easily be generalised to higher order polynomials, whereas the classical FVM is purely a first-order approach.

This motivates us to consider the coupling of the vertex-centred FVM with BEM and the coupling of SUPG with BEM. The coupling of FVM and BEM has already been considered in the stationary case for different ways to couple. The so-called three-field coupling has been analysed in [Era12, vertex-centred FVM-BEM] and [Era13a, cell-centred FVM-BEM]. The non-symmetric coupling of FVM and BEM has then later been analysed in [EOS17], which will be the main reference for the FVM-BEM coupling in this thesis. For the SUPG-BEM coupling, no literature is available so far.

For the discretisation in time, we will again use the variant of the implicit Euler we introduced for the FEM-BEM coupling. In the FVM-BEM case, we will additionally look at the classical implicit Euler method, which only differs in the right-hand side from the variant. The variant is computationally more expensive but allows us to state quasi-optimality results under minimal regularity assumptions also for the FVM-BEM and SUPG-BEM coupling. For the classical Euler scheme, however, we need standard regularity conditions due to Taylor expansion techniques. In contrast to the analysis of the FEM-BEM coupling, it is not possible to perform the analysis of the FVM-BEM coupling and the SUPG-BEM coupling in the full energy norm, i.e., we have to omit the dual norm of the time derivative. Moreover, both FVM-BEM and SUPG-BEM do not have a classical "global" Galerkin orthogonality. Hence we will also have to handle some extra terms concerning the model input data. However, the analysis still holds under minimal regularity requirements on the solution.

Main Contributions

In this work, we close the gap in the analysis of the non-symmetric coupling method for parabolic-elliptic interface problems. We also analyse the coupling with FVM and SUPG. The main contributions and some of the contents have already been published in [EES18; ES19b; ES19a]. Our main results can be summarised as follows:

- Based on the coercivity of the variational formulation ([Say09; Ste11; Aur+13] and [EOS17] for a more general problem) we can extend the results of [MS87; CES90] to the non-symmetric coupling method on non-smooth domains. In particular, we establish well-posedness of this formulation and prove quasi-optimal error estimates for Galerkin approximations.

- In the second step of this analysis, we also consider the time discretisation of the semi-discrete scheme of [MS87] by a variant of the implicit Euler method. We utilise a formulation that is fully consistent with the continuous variational formulation and does not require additional smoothness of the solution or the data; see [Tan14] for a related approach in the context of parabolic problems. This allows us to establish well-posedness and quasi-optimal approximation properties with respect to the energy norm under minimal smoothness assumptions on the solution.

- We formulate the non-symmetric coupling of the Finite Volume Method with the Boundary Element Method which leads to the semi-discretisation of the model problem. This is based on the ideas of the stationary problem in [EOS17].

- We show the convergence of the semi-discrete scheme under minimal regularity requirements on the solution and provide error estimates with optimal rates.

- For the full discretisation with the variant of the backward Euler scheme, we provide convergence under minimal regularity assumptions on the solution and provide error estimates with optimal rates. If we use the classical Euler scheme for time discretisation, the usual regularity assumptions for the time component lead to first-order error estimates.

- It is important to note that the analysis still holds if we use an upwind stabilisation. It also holds if we consider the model problem in three dimensions, although the analysis is carried out for a two-dimensional problem.

- We can apply the analysis in this work also for standalone FVM, i.e., one has Dirichlet and/or inflow/outflow Neumann boundary conditions instead of the coupling conditions. Note that our results also improve results in the literature, e.g., [ELL02; CLT04].

- Lastly, we also formulate the non-symmetric coupling of the Streamline Upwind Petrov Galerkin Method with the Boundary Element Method and show the convergence of the semi-discrete scheme (in a special case) and the fully-discrete scheme under minimal regularity requirements on the solution and provide error estimates with optimal rates.

Outline

In Chapter 1 we state the basic notation. We introduce the basics of BEM, namely how to reduce a problem on an unbounded domain to a boundary integral equation. Afterwards, we introduce the triangulation and discrete spaces we need for the discretisation of the interior and exterior problems.

Chapter 2 states the exact model problem and its variational formulation. Then a semi-discretisation with an abstract Galerkin approach is analysed for well-posedness and quasi-optimal approximation properties. A subsequent time discretisation by a variant of the implicit Euler method then yields quasi-optimal error estimates. This is exemplified with a particular FEM-BEM discretisation afterwards. An excursion shows what we can do if we have a domain with a smooth boundary that cannot be discretised directly.

In Chapter 3 we extend these ideas to convection-dominated problems. First, we look at the coupling of BEM with FVM and analyse the semi- and fully-discrete formulation. Here we also look at some classical estimates by using the classical backward Euler method. Then we look at the coupling with SUPG and again obtain similar results for the semi- and fully-discrete formulation.

The theoretical results are accompanied by some numerical experiments in Chapter 4. These support the convergence results and also show some possible applications of the methods. Lastly, we end with some concluding remarks and an outlook about possible future work.

1

Basic Notation and Preliminaries

BEFORE WE BEGIN to investigate parabolic-elliptic interface problems, we have to fix the basic notation we will use throughout the thesis. Furthermore, important basics and results from the literature are stated here for easy reference.

Throughout the thesis we assume $\Omega \subset \mathbb{R}^2$ to be an open bounded Lipschitz domain with $\operatorname{diam}(\Omega) < 1$. The latter assumption is needed later for the coercivity of the single layer operator. Note that $\operatorname{diam}(\Omega) < 1$ can always be achieved by scaling. All results can be transferred to three dimensions as well where this assumption is not necessary. The boundary of Ω will be denoted by Γ and the exterior space by $\Omega_e = \mathbb{R}^2 \setminus \overline{\Omega}$.

To denote the dependence of constants, we will use the notation $C = C(\Omega, \dots)$ which means that the constant C depends in particular on the domain Ω, and so on. We also use $a \lesssim b$ to denote $a \leq C \cdot b$ when we want to omit the constant C, which is independent of b.

1.1 Sobolev Spaces

We write $L^p(\cdot)$, $p \in \mathbb{N}$, for the usual Lebesgue spaces. The space $W^{1,\infty}(\Omega)$ is the space of all Lipschitz continuous functions. The Sobolev spaces $H^s(\Omega)$, $s \in \mathbb{R}$, can be introduced via the notion of weak derivatives, a comprehensive overview can be found in [AF03]. The Sobolev spaces on the boundary $H^s(\Gamma)$, $s \in \mathbb{R}$, on the other hand, can be introduced by means of local parametrisations, see [Ste08, Section 2.5]. The corresponding norms will be denoted by $\|\cdot\|_{L^p(\cdot)}$ and $\|\cdot\|_{H^s(\cdot)}$. The space of all traces of functions from $H^s(\Omega)$ is $H^{s-1/2}(\Gamma)$ for $s > 1/2$, see [Eva10; McL00] for details. We denote the L^2 scalar product for $\omega \subset \Omega$ by $(\cdot\,,\cdot)_\omega$ and duality between $H^s(\Gamma)$ and $H^{-s}(\Gamma)$ is given by the extended L^2-scalar product $\langle\cdot\,,\cdot\rangle_\Gamma$. Similarly, we denote the duality product between $H^1(\Omega)$ and $H^1(\Omega)'$ by $\langle\cdot\,,\cdot\rangle_\Omega$.

To shorten the notation, we will adopt the notation of [CES90] and use

$$H = H^1(\Omega) \qquad \text{and} \qquad B = H^{-1/2}(\Gamma)$$

for the main function spaces, which are natural to parabolic-elliptic problems. Furthermore, we denote by

$$H_T = L^2(0, T; H) \qquad \text{and} \qquad B_T = L^2(0, T; B)$$

the corresponding Bochner-Sobolev spaces of functions on $[0, T]$ with values in H and B, respectively (see [Eva10]). The associated dual spaces are given by

$$H' = H^1(\Omega)' \qquad \text{and} \qquad B' = H^{1/2}(\Gamma),$$

as well as

$$H'_T = L^2(0, T; H') \qquad \text{and} \qquad B'_T = L^2(0, T; B').$$

We also abbreviate the spaces $L^2(0, T; L^2(\Omega))$ and $L^2(0, T; L^2(\Gamma))$ by

$$L^2_{T,\Omega} = L^2(0, T; L^2(\Omega)) \qquad \text{and} \qquad L^2_{T,\Gamma} = L^2(0, T; L^2(\Gamma)),$$

respectively. All spaces above are Hilbert spaces if equipped with their natural norms, i.e.,

$$\|v\|^2_{H_T} = \int_0^T \|v(t)\|^2_H \,\mathrm{d}t.$$

We further use

$$Q_T = \left\{ v \in H_T \,:\, \partial_t v \in H'_T \text{ and } v(0) = 0 \right\}$$

to denote the natural energy space for a parabolic problem on Ω with the norm

$$\|u\|^2_{Q_T} := \|u\|^2_{H_T} + \|\partial_t u\|^2_{H'_T}.$$

This space, again, is complete. It is well-known that the space Q_T is continuously embedded in $C([0, T]; L^2(\Omega))$; see, e.g., [Eva10]. This also allows us to impose initial values for weak functions as in the function space Q_T. We use the notation $C(\Omega)$ for the space of continuous functions and $C^k(\Omega)$ for the space of k-times differentiable functions. With $C^\infty_c(\Omega)$ we denote the smooth functions with compact support.

To simplify notation we also use a product space and norm notation, i.e., we equip the space $\mathcal{H} := H \times B = H^1(\Omega) \times H^{-1/2}(\Gamma)$ with the norm

$$\|\mathbf{v}\|^2_{\mathcal{H}} := \|v\|^2_{H^1(\Omega)} + \|\psi\|^2_{H^{-1/2}(\Gamma)}$$

for $\mathbf{v} = (v, \psi) \in \mathcal{H}$.

1.2 Boundary Integral Operators

Later we will need several boundary integral operators (which are the basis of the Boundary Element Method). The main references here are [McL00; Ste08; SS04; Cos88b].

We will look at the operators associated with the Laplace problem. We denote the Laplace operator (or Laplacian) by $\Delta f = \operatorname{div}(\nabla f)$.

First, we need to introduce the trace operator. The trace generalises the concept of the restriction of a function to the boundary in the Sobolev setting. Foremost we will need the trace inequality, see [McL00, Theorem 3.37 and Theorem 3.38] and the notion of the interior and the exterior trace operator, see [SS04, Theorem 2.6.8].

Lemma 1.1 (Trace inequality). *Define the operator* $\gamma_0 \colon C_c^\infty(\overline{\Omega}) \to C_c^\infty(\Gamma)$, $\gamma_0(u) = u|_\Gamma$. *Then for a Lipschitz domain* Ω *and* $\frac{1}{2} < s < \frac{3}{2}$ *there exists a unique extension to a bounded operator*

$$\gamma_0 \colon H^s(\Omega) \to H^{s-1/2}(\Gamma), \tag{1.1}$$

called the trace. This also means that the trace inequality holds:

$$\|\gamma_0 u\|_{H^{s-1/2}(\Gamma)} \le C_{tr} \|u\|_{H^s(\Omega)}, \tag{1.2}$$

with a constant $C_{tr} = C_{tr}(\Omega) > 0$. *The interior trace will be denoted by* γ_0^{int} *and the exterior trace – coming from the outside* Ω_e *– will be denoted by* γ_0^{ext}.

From the trace inequality, we can deduce that

$$\langle \psi, v \rangle_\Gamma \le \|\psi\|_{H^{-1/2}(\Gamma)} \|v\|_{H^{1/2}(\Gamma)} \le C_{tr} \|\psi\|_{H^{-1/2}(\Gamma)} \|v\|_{H^1(\Omega)}$$

for all $\psi \in H^{-1/2}(\Gamma)$ and $v \in H^1(\Omega)$. In the first and second statement, one should formally write $\gamma_0 v$ instead of v, but in the rest of the document we skip the explicit notation of the trace operator since the meaning is clear from the context.

As an analogon to the trace operator for the normal derivative, we can define the *conormal derivative* for the Laplace operator by

$$\gamma_1(u) := \gamma_0(\nabla u) \cdot \mathbf{n}.$$

Again, we write γ_1^{int} for the interior conormal derivative and γ_1^{ext} for the exterior conormal derivative.

For the Laplace operator we can define a *fundamental solution*, which can informally be described as an inverse of the differential operator. In two dimensions the fundamental solution of the Laplace operator is given by

$$G(x, y) = -\frac{1}{2\pi} \log |x - y|,$$

see [Ste08, Section 5.1]. With this fundamental solution, we can define the single layer potential

$$(\widetilde{\mathscr{V}} v)(x) = \int_\Gamma G(x, y) v(y) \, \mathrm{d}s_y,$$

and the double layer potential

$$(\widetilde{\mathscr{K}} v)(x) = \int_\Gamma \partial_{\mathbf{n}_y} G(x, y) v(y) \, \mathrm{d}s_y.$$

Here, the conormal derivative $\partial_\mathbf{n} u = \nabla u \cdot \mathbf{n}|_\Gamma$ is taken in direction of the unit normal vector $\mathbf{n}$ on Γ pointing outward with respect to Ω. Because we assumed the domain to be Lipschitz, Rademacher's theorem guarantees the existence of the outer normal vector, see [SS04, Theorem 2.7.1].

If we assume an appropriate radiation condition at infinity, we can formulate a representation formula for a Laplace problem on an unbounded exterior domain Ω_e. This is sometimes also called Green's third identity.

Theorem 1.2 (Representation formula)**.** *For a function u_e that fulfils $-\Delta u_e = 0$ in Ω_e and the radiation condition $u_e(x) = a \log|x| + \mathcal{O}(|x|^{-1})$ for $|x| \to \infty$ with a constant a, we have the following representation formula*

$$\begin{aligned} u_e(x) &= \int_\Gamma \partial_{\mathbf{n}_y} G(x,y) u_e(y)|_\Gamma \, \mathrm{d}s_y - \int_\Gamma G(x,y) \partial_{\mathbf{n}} u_e(y)|_\Gamma \, \mathrm{d}s_y \\ &= \widetilde{\mathcal{K}} u_e|_\Gamma(x) - \widetilde{\mathcal{V}} \partial_{\mathbf{n}} u_e|_\Gamma(x). \end{aligned} \tag{1.3}$$

Proof. This is a combination of [McL00, Theorem 7.12] and [McL00, Theorem 8.9]. □

The boundary potentials can be extended to continuous linear operators and fulfil certain jump relations.

Lemma 1.3. *The single layer potential $\widetilde{\mathcal{V}}$ can be uniquely extended to a bounded linear operator, such that*

$$\begin{aligned} \widetilde{\mathcal{V}} &: H^{-1/2}(\Gamma) \to H^1_{loc}(\Omega_e), \\ \gamma_0 \widetilde{\mathcal{V}} &: H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma). \end{aligned}$$

The double layer potential $\widetilde{\mathcal{K}}$ can be uniquely extended to a bounded linear operator, such that

$$\begin{aligned} \widetilde{\mathcal{K}} &: H^{1/2}(\Gamma) \to H^1_{loc}(\Omega_e), \\ \gamma_0 \widetilde{\mathcal{K}} &: H^{1/2}(\Gamma) \to H^{1/2}(\Gamma). \end{aligned}$$

Furthermore, the surface potentials fulfil the following jump relations:

$$\begin{aligned} [\![\gamma_0 \widetilde{\mathcal{V}} \psi]\!] &= \gamma_0^{ext} \widetilde{\mathcal{V}} \psi - \gamma_0^{int} \widetilde{\mathcal{V}} \psi = 0 \quad \text{for all } \psi \in H^{-1/2}(\Gamma), \\ [\![\gamma_0 \widetilde{\mathcal{K}} v]\!] &= \gamma_0^{ext} \widetilde{\mathcal{K}} v - \gamma_0^{int} \widetilde{\mathcal{K}} v = v \quad \text{for all } v \in H^{1/2}(\Gamma), \end{aligned}$$

where $[\![\cdot]\!]$ denotes the jump at Γ.

Proof. See [Ste08, Lemma 6.6] for the single layer potential, [Ste08, Lemma 6.10] for the double layer potential and [McL00, Theorem 6.11] for the jump relations. □

Using the above properties and by taking the trace of the boundary layer potentials, we obtain the *single layer operator* and the *double layer operator*

$$\mathcal{V} := \gamma_0^{int} \widetilde{\mathcal{V}} \qquad \text{and} \qquad \mathcal{K} := \frac{1}{2} + \gamma_0^{int} \widetilde{\mathcal{K}}.$$

The fraction in front of the double layer potential is $\frac{1}{2}$ almost everywhere on Γ, see e.g. [SS04, Corollary 3.3.12].

One can show that these operators are continuous, see [Cos88b, Theorem 1] and [Ste08, Theorem 6.34].

Lemma 1.4. *Let $s \in [-1/2, 1/2]$ and Γ be the boundary of a Lipschitz domain Ω. Then the boundary integral operators are bounded, i.e., they are continuous linear operators, such that*

$$\mathscr{V} : H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma),$$
$$\mathscr{K} : H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma).$$

Not only is the single layer operator bounded but it is also coercive, given that we have $\operatorname{diam}(\Omega) < 1$. In three dimensions it is coercive without this restriction, see [Ste08, Theorem 6.22 and Theorem 6.23].

Lemma 1.5. *Let* $\operatorname{diam}(\Omega) < 1$, *then $\mathscr{V}$ is coercive on $H^{-1/2}(\Gamma)$, i.e.,*

$$\langle \mathscr{V}\psi, \psi\rangle_\Gamma \geq C_{\mathscr{V}} \|\psi\|^2_{H^{-1/2}(\Gamma)} \qquad \text{for all } \psi \in H^{-1/2}(\Gamma),$$

with some $C_{\mathscr{V}} > 0$ independent of ψ.

Furthermore, $\mathscr{V}$ is symmetric and with the $H^{-1/2}(\Gamma)$-coercivity we can deduce that

$$\|\cdot\|^2_{\mathscr{V}} := \langle \mathscr{V}\cdot, \cdot\rangle_\Gamma$$

defines a norm in $H^{-1/2}(\Gamma)$ which is equivalent to $\|\cdot\|_{H^{-1/2}(\Gamma)}$.

Similarly, we can define the norm $\|\cdot\|^2_{\mathscr{V}^{-1}} := \langle \mathscr{V}^{-1}\cdot, \cdot\rangle_\Gamma$. In this norm the double layer operator fulfils the following contraction property, see [Ste08, Corollary 6.27].

Lemma 1.6. *For $u \in H^{1/2}(\Gamma)$ there holds*

$$\|(1/2 + \mathscr{K})u\|_{\mathscr{V}^{-1}} \leq C_{\mathscr{K}} \|u\|_{\mathscr{V}^{-1}},$$

with a constant $C_{\mathscr{K}} \in [1/2, 1)$.

The contraction constant will appear in the analysis of the non-symmetric coupling with a matrix-valued diffusion coefficient.

Now we can get back to the representation formula to derive an integral equation which will be used in the coupling formulation later to solve the exterior Laplace problem. Taking the exterior trace of the representation formula (1.3) leads to

$$\gamma_0^{ext} u_e = \gamma_0^{\text{ext}}(\widetilde{\mathscr{K}}(\gamma_0^{ext} u_e)) - \gamma_0^{ext}(\widetilde{\mathscr{V}}(\gamma_1^{ext} u_e)).$$

Now we can use the jump relations and the definitions of the boundary layer operators, in particular

$$\gamma_0^{ext} \widetilde{\mathscr{V}}(\gamma_1^{ext} u_e) = \gamma_0^{int} \widetilde{\mathscr{V}}(\gamma_1^{ext} u_e) = \mathscr{V}(\gamma_1^{ext} u_e),$$

and

$$\gamma_0^{ext} \widetilde{\mathscr{K}}(\gamma_0^{ext} u_e) = \left(\mathbf{I} + \gamma_0^{int} \widetilde{\mathscr{K}}\right)(\gamma_0^{ext} u_e) = \left(\frac{1}{2}\mathbf{I} + \mathscr{K}\right)(\gamma_0^{ext} u_e).$$

This results in

$$\gamma_0^{ext} u_e = (1/2 + \mathcal{K})(\gamma_0^{ext} u_e) - \mathcal{V}(\gamma_1^{ext} u_e).$$

When we bring over $\gamma_0^{ext} u_e$, rearrange slightly and write $\phi = \gamma_1^{ext} u_e$ we obtain

$$\mathcal{V}\phi + (1/2 - \mathcal{K})\gamma_0^{ext} u_e = 0. \tag{1.4}$$

This is called *Symm's equation* and will later be used in the coupling process to solve the exterior problem. After we have solved this equation, we can plug the solution back into the representation formula to obtain a solution of the original problem. Note that for a classical Dirichlet or Neumann problem we would know either the trace or the normal derivative of our solution on the boundary and thus the other one would be our unknown.

For sufficiently smooth functions, we can write the boundary layer operators again as integrals.

Lemma 1.7 (Integral representations)**.** *Let $\psi \in L^\infty(\Gamma)$ and $v \in H^{1/2}(\Gamma)$, then the operators have the following integral representation:*

$$(\mathcal{V}\psi)(x) = \int_\Gamma G(x,y)\psi(y)\,\mathrm{d}s_y \quad \textit{for all } x \in \Gamma,$$
$$(\mathcal{K}v)(x) = \lim_{\varepsilon \to 0} \int_{y \in \Gamma : |y-x| \geq \varepsilon} \gamma_{1,y}^{int} G(x,y) v(y)\,\mathrm{d}s_y \quad \textit{for all } x \in \Gamma.$$

Proof. See [Ste08, Lemma 6.7 and 6.11]. □

This is most certainly fulfilled for the discretisation spaces we will later use.

1.3 Triangulations

For the discretisation of the interior problem, we need to triangulate the interior domain Ω. On the boundary, we also need a subdivision which can in general be independent of the interior triangulation. Because we will need a dual mesh for the Finite Volume Method, the triangulation for the Finite Element Method will be called primal mesh.

Primal Mesh

Let $\mathcal{T}$ denote a triangulation or primal mesh of Ω consisting of non-degenerate closed triangles denoted by $K \in \mathcal{T}$. We assume that the triangulation is regular in the sense of Ciarlet [Cia78]. The corresponding sets of nodes and edges are denoted by $\mathcal{N}$ and $\mathcal{E}$, respectively. We write $h_K := \sup_{x,y \in K} |x-y|$ for the Euclidean diameter of $K \in \mathcal{T}$ and h_E for the length of an edge $E \in \mathcal{E}$. The (global) mesh-size is $h := \max_{K \in \mathcal{T}} h_K$. As usual we denote by ρ_K the inner circle radius of the triangle $K \in \mathcal{T}$. Furthermore, we denote by $\mathcal{E}_K \subset \mathcal{E}$ the set of all edges of K, i.e., $\mathcal{E}_K := \{E \in \mathcal{E} : E \subset \partial K\}$ and by $\mathcal{E}_\Gamma := \{E \in \mathcal{E} : E \subset \Gamma\}$ the set of

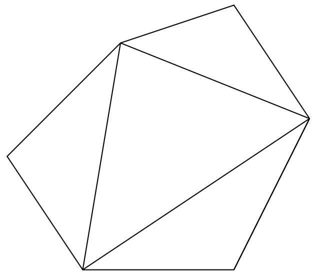


Figure 1.1: The primal mesh $\mathscr{T}$.

all edges on the boundary Γ. Then we assume that the partition $(\mathscr{T}, \mathscr{E}_\Gamma)$ is η-quasi-uniform with $\eta > 0$, i.e.,

$$\eta h \le \rho_T \le h_K \le h \qquad \text{and} \qquad \eta h \le h_E \le h \qquad \text{for all } K \in \mathscr{T},\ E \in \mathscr{E}_\Gamma.$$

We will use $\mathscr{E}_\Gamma$ as a segmentation of the boundary of $\mathscr{T}$, but note that we could theoretically use a surface mesh $\mathscr{E}_\Gamma$ that is decoupled from the mesh $\mathscr{T}|_\Gamma$. This does not hold true for the coupling with the Finite Volume Method, where the surface mesh cannot be chosen completely independent of the interior mesh, see [Era13b, Remark 3.1]. Also note that these conditions particularly imply that Γ is a polygon. An example primal mesh is visualised in Figure 1.1.

Dual Mesh

We build boxes, called control volumes, by connecting the barycentre of an element $K \in \mathscr{T}$ with the midpoint of the edges $E \in \mathscr{E}_K$. These control volumes constitute a new mesh $\mathscr{T}^*$ of Ω whose elements are non-degenerate and closed because of the non-degeneracy of the elements of the primal mesh $\mathscr{T}$. For every vertex $a_i \in \mathscr{N}$ of $\mathscr{T}$ $(i = 1 \dots \#\mathscr{N})$ we associate a unique box $V_i \in \mathscr{T}^*$ containing a_i. The dual mesh is visualised in Figure 1.2.

Approximation Spaces

Throughout the thesis we will use the approximation spaces

$$\mathscr{S}^1(\mathscr{T}) := \left\{ u \in C(\Omega) : u|_T \in \mathscr{P}^1(K) \text{ for all } K \in \mathscr{T} \right\} \qquad \text{and} \tag{1.5}$$

$$\mathscr{P}^0(\mathscr{E}_\Gamma) := \left\{ \phi \in L^2(\Gamma) : \phi|_E \in \mathscr{P}^0(E) \text{ for all } E \in \mathscr{E}_\Gamma \right\}, \tag{1.6}$$

consisting of globally continuous and piecewise linear functions over $\mathscr{T}$ and piecewise constant functions over $\mathscr{E}_\Gamma$, respectively. Furthermore, we introduce a discretisation space on the dual mesh

$$\mathscr{P}^0(\mathscr{T}^*) := \left\{ v \in L^2(\Omega) : v|_V \in \mathscr{P}^0(V) \text{ for all } V \in \mathscr{T}^* \right\},$$

consisting of piecewise constant functions on $\mathscr{T}^*$.

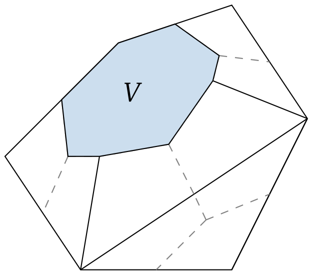


Figure 1.2: The dual mesh $\mathscr{T}^*$ corresponding to the primal mesh $\mathscr{T}$ of Figure 1.1 with a control volume V.

Additionally, we introduce the broken Sobolev space

$$H^s(\mathscr{E}_\Gamma) := \begin{cases} \{\phi \in L^2(\Gamma) : \phi|_E \in H^s(E) \text{ for all } E \in \mathscr{E}_\Gamma\} & \text{for } \quad 0 < s \leq 1, \\ H^s(\Gamma) & \text{for } -1 \leq s \leq 0, \end{cases}$$

with the corresponding norms

$$\|\phi\|^2_{H^s(\mathscr{E}_\Gamma)} := \begin{cases} \sum_{E \in \mathscr{E}_\Gamma} \|\phi\|^2_{H^s(E)} & \text{for } \quad 0 < s \leq 1, \\ \|\phi\|^2_{H^s(\Gamma)} & \text{for } -1 \leq s \leq 0; \end{cases}$$

see, e.g., [Ste08, Ch. 2].

For the approximation spaces we use, we have an inverse inequality (see [Cia78, Theorem 3.2.6] and [GHS00, Remark 3.6]).

Lemma 1.8 (Inverse inequality). *For a family of regular triangulations $(\mathscr{T}_h)_h$ we have for all $v_h \in \mathscr{S}^1(\mathscr{T}_h)$ and $0 \leq l \leq k$*

$$\|v_h\|_{H^k(\Omega)} \leq C_{inv} h^{l-k} \|v_h\|_{H^l(\Omega)},$$

with a constant $C_{inv} > 0$. There also holds for $0 \leq s \leq 1$

$$\|v_h\|_{L^2(\Omega)} \leq C_{inv} h^{-s} \|v_h\|_{H^{-s}(\Omega)}.$$

By means of the characteristic function χ_i^* over the volume V_i associated with $a_i \in \mathscr{N}$ we write $v_h^* \in \mathscr{P}^0(\mathscr{T}^*)$ as

$$v_h^* = \sum_{a_i \in \mathscr{N}} v_i^* \chi_i^*,$$

with $v_i^* \in \mathbb{R}$. In that sense we define the $\mathscr{T}^*$-piecewise constant interpolation operator

$$\mathscr{I}_h^* : C(\overline{\Omega}) \to \mathscr{P}^0(\mathscr{T}^*), \quad (\mathscr{I}_h^* v)(x) := \sum_{a_i \in \mathscr{N}} v(a_i) \chi_i^*(x),$$

which has the following properties.

Lemma 1.9. *Let $K \in \mathcal{T}$ and $E \in \mathcal{E}_K$. For $v_h \in \mathcal{S}^1(\mathcal{T})$ there holds*

$$\int_E (v_h - \mathcal{I}_h^* v_h)\,\mathrm{d}s = 0, \tag{1.7}$$
$$\|v_h - \mathcal{I}_h^* v_h\|_{L^2(K)} \le h_K \|\nabla v_h\|_{L^2(K)}, \tag{1.8}$$
$$\|v_h - \mathcal{I}_h^* v_h\|_{L^2(E)} \le C h_E^{1/2} \|\nabla v_h\|_{L^2(K)}, \tag{1.9}$$
$$\|\mathcal{I}_h^* v_h\|_{L^2(\Omega)} \le C \|v_h\|_{L^2(\Omega)}. \tag{1.10}$$

The constant $C > 0$ depends only on the domain.

Proof. The estimates (1.7)–(1.9) are well known, see e.g. [EOS17, Lemma 3]. The stability (1.10) follows from (1.8) and the inverse inequality from Lemma 1.8, i.e.,

$$\begin{aligned}\|\mathcal{I}_h^* v_h\|_{L^2(\Omega)} &\le \|\mathcal{I}_h^* v_h - v_h\|_{L^2(\Omega)} + \|v_h\|_{L^2(\Omega)}\\ &\le Ch\|\nabla v_h\|_{L^2(\Omega)} + \|v_h\|_{L^2(\Omega)}\\ &\le C\|v_h\|_{L^2(\Omega)}.\end{aligned}$$

□

Lemma 1.10 ([CQ99, Lemma 2.2])**.** *The operator $\mathcal{I}_h^*$ is self-adjoint in the L^2 scalar product, which means that for all $v_h, w_h \in \mathcal{S}^1(\mathcal{T})$*

$$(w_h\,, \mathcal{I}_h^* v_h)_\Omega = (v_h\,, \mathcal{I}_h^* w_h)_\Omega. \tag{1.11}$$

This allows us to define the norm

$$\|w_h\|_\chi := (w_h\,, \mathcal{I}_h^* w_h)_\Omega^{1/2}, \tag{1.12}$$

which is equivalent to $\|w_h\|_{L^2(\Omega)}$.

1.4 Useful Estimates

Two inequalities which we will frequently use are the Cauchy-Schwarz and Young's inequality. Although they are standard, we give a short reminder.

Lemma 1.11 (Cauchy-Schwarz inequality)**.** *Let H be a Hilbert space with scalar product $(\cdot\,,\cdot)_H$. Then for $x, y \in H$*

$$(x\,, y)_H \le \|x\|_H \|y\|_H.$$

Lemma 1.12 (Young's inequality)**.** *Let $a, b \in \mathbb{R}_+$, then there holds for $\varepsilon > 0$*

$$ab \le \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}.$$

Later, in the analysis of the implicit Euler scheme, we will use the following estimate derived from a Taylor expansion.

Lemma 1.13 (Taylor estimate). *Let $g \in H^1([0,T])$ with $T > 0$ and let $0 = t^0 < t^1 < \ldots < t^N = T$, $N \in \mathbb{N}$ be a partition of the time interval $[0,T]$. Then*

$$\sum_{n=1}^{N} \tau^n (g^n)^2 = \sum_{n=1}^{N} \tau^n g(t^n)^2 \lesssim \|g\|^2_{L^2(0,T)} + \tau^2 \|g'\|^2_{L^2(0,T)}. \tag{1.13}$$

Proof. For $t^{n-1} \leq t \leq t^n$ we see with Taylor expansion that

$$g(t^n) = g(t) + \int_t^{t^n} g'(s)\,\mathrm{d}s.$$

Then, we see with the Cauchy-Schwarz inequality that

$$g(t^n)^2 \leq 2\Big[g^2(t) + \Big(\int_t^{t^n} g'(s)\,\mathrm{d}s\Big)^2\Big] \leq 2\Big[g^2(t) + \tau^n \int_{t^{n-1}}^{t^n} (g'(s))^2\,\mathrm{d}s\Big].$$

Integration over $[t^{n-1}, t^n]$ and summing over $n = 1, \ldots, N$ leads to the assertion. □

2

Numerical Treatment of Parabolic-Elliptic Interface Problems

IN THIS CHAPTER, we will look at a basic parabolic-elliptic interface problem and how we can solve it numerically by using a coupling of the Finite Element Method and the Boundary Element Method. In particular, our model problem consists of the heat equation in the interior domain and the Laplace equation in the exterior (unbounded) domain. This model problem is inspired by the problem used in the analysis of eddy currents in the magneto-quasistatic regime. Some physical considerations leading to the model can be found in [MS87, Section 7]. In Section 4.1.2 an eddy current example is computed numerically. The main idea of this problem is to have an electromagnetic field on the whole space and then add some metallic obstacles to interfere with the field. The resulting field can then be computed by such a parabolic-elliptic interface problem. To obtain this formulation, we have to assume a special setup to simplify the underlying Maxwell's equations.

For this model problem, we will use the non-symmetric coupling of the Finite Element Method and the Boundary Element Method, which will be analysed in the semi-discrete case and in the fully-discrete case. For the full discretisation, we use a variant of the implicit Euler method. This variant allows us to write the fully-discrete problem consistently with the semi-discrete problem and thus the analysis carries over. Furthermore, we analyse the problem in its natural energy norm, given by the Bochner-Sobolev space the solution lies in. In the previous work [MS87] only the semi-discrete case was analysed and the boundary was required to be smooth. In [CES90] the symmetric coupling was analysed for both semi- and full discretisation, but the results for the full discretisation were not optimal with respect to the regularity requirements on the solution.

To obtain the main results, we first establish some stability results also called *energy estimates*. Then we prove the quasi-optimality results for the semi- and the fully-discrete non-symmetric coupling on Lipschitz domains in the natural energy norm. Lastly, we look at a particular FEM-BEM discretisation and its convergence behaviour. These results have already been published in the paper [EES18]. In the last section of the chapter, we will have a short look at how to handle smooth boundaries numerically, as we have to assume polygonal boundaries for the analysis of the FEM-BEM coupling.

2.1 Model Problem

The first model problem we will look at is a simple version of the one we will eventually analyse. In general, the method of this chapter carries over to the more complex model problem as well, but to simplify the notation (and also to avoid some technicalities) we will first look at the following problem.

We will state the needed regularity to solve this problem later. For now we search for $u\colon \Omega \to \mathbb{R}$ and $u_e\colon \Omega_e \to \mathbb{R}$ such that

$$\partial_t u - \Delta u = f \qquad \text{in } \Omega \times (0,T), \tag{2.1}$$

$$-\Delta u_e = 0 \qquad \text{in } \Omega_e \times (0,T), \tag{2.2}$$

with coupling conditions across the interface given by

$$u = u_e + g_1 \qquad \text{on } \Gamma \times (0,T), \tag{2.3}$$

$$\partial_{\mathbf{n}} u = \partial_{\mathbf{n}} u_e + g_2 \qquad \text{on } \Gamma \times (0,T). \tag{2.4}$$

The domain has been introduced before, but to recapitulate: we need a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ with $\operatorname{diam}(\Omega) < 1$. The results hold for two and three dimensions, but we will only present the two-dimensional case here. The coupling boundary is denoted by $\Gamma := \partial\Omega$ and the complement of Ω by $\Omega_e = \mathbb{R}^2 \backslash \overline{\Omega}$. Furthermore, our starting time will always be 0 and the end time is denoted by $T > 0$. The model setup is depicted in Figure 2.1.

The *model input data* for the model are f, g_1, and g_2. To ensure the uniqueness of the solution, we additionally require the initial and radiation conditions

$$u(\cdot,0) = 0 \qquad \text{on } \Omega, \tag{2.5}$$

$$u_e(x,t) = a(t)\log|x| + \mathcal{O}(|x|^{-1}) \qquad |x| \to \infty. \tag{2.6}$$

The function $a(t) : [0,T] \to \mathbb{R}$ is unknown and automatically determined in the solving process, see Remark 2.2. For a model problem in three dimensions, we only have to replace the radiation condition (2.6) by $u_e(x,t) = \mathcal{O}(|x|^{-1})$, $|x| \to \infty$. In our model problem, we might also allow inhomogeneous initial data and extra Dirichlet or Neumann boundaries in the interior domain. Then we have to modify our analysis in the respective places, the results however still hold and only change in obvious ways.

2.2 Weak Formulation

Before we start to look at any type of discretisation, we have to bring our system into a weak form. Then we can state stability and unique solvability which will be useful for the forthcoming discretisation as well.

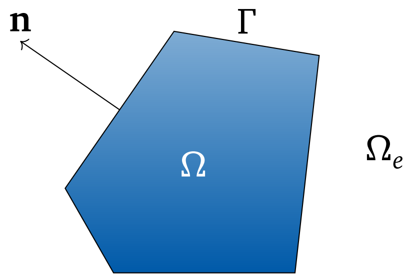


Figure 2.1: The setup of the model problem.

Let (u, u_e) denote a sufficiently smooth solution of problem (2.1)–(2.6) and let t be fixed but arbitrary. Multiplying equation (2.1) with a test function $v \in H^1(\Omega)$, integrating over Ω, and using integration by parts formally leads to

$$\int_\Omega \partial_t u(t) v \,\mathrm{d}x + \int_\Omega \nabla u(t) \cdot \nabla v \,\mathrm{d}x - \int_\Gamma \partial_{\mathbf{n}} u|_\Gamma(t) v \,\mathrm{d}s = \int_\Omega f(t) v \,\mathrm{d}x.$$

Now we use equation (2.4) with $\phi := \partial_{\mathbf{n}} u_e|_\Gamma$ to replace the interior conormal derivative.

$$\int_\Omega \partial_t u(t) v \,\mathrm{d}x + \int_\Omega \nabla u(t) \cdot \nabla v \,\mathrm{d}x - \int_\Gamma \phi(t) v \,\mathrm{d}s = \int_\Omega f(t) v \,\mathrm{d}x + \int_\Gamma g_2(t) v \,\mathrm{d}s.$$

For the right-hand side, we will use the shorthand notation

$$\langle f_B(t), v\rangle_\Omega := \int_\Omega f(t) v \,\mathrm{d}x + \int_\Gamma g_2(t) v \,\mathrm{d}s \tag{2.7}$$

and write $f_B \in H_T' = L^2(0, T; H^1(\Omega)')$.

For the exterior problem, we use an integral formulation based on the representation formula (see Section 1.2). Then Symm's equation (1.4) leads us to the following problem

$$\mathcal{V}\phi(t) + (1/2 - \mathcal{K}) u_e|_\Gamma(t) = 0.$$

Using the coupling condition (2.3) to replace $u_e|_\Gamma(t)$ by $u|_\Gamma(t)$ we obtain

$$\mathcal{V}\phi(t) + (1/2 - \mathcal{K}) u|_\Gamma(t) = (1/2 - \mathcal{K}) g_1(t) =: g_{\mathcal{K}}(t). \tag{2.8}$$

A combination of the above formulas leads to the following weak formulation, which will be the starting point of our analysis. We shortly repeat the abbreviations of the used spaces. The spaces on the boundary are defined as $B_T = L^2(0, T; H^{-1/2}(\Gamma))$ and $B_T' = L^2(0, T; H^{1/2}(\Gamma))$, and the energy space is defined as $Q_T = \left\{u \in H_T : \partial_t u \in H_T' \text{ and } u(0) = 0\right\}$ with $H_T = L^2(0, T; H^1(\Omega))$.

Problem 2.1 (Variational problem)**.** Given $f_B \in H_T'$ and $g_{\mathcal{K}} \in B_T'$, find $u \in Q_T$ and $\phi \in B_T$ such that

$$\langle \partial_t u(t), v\rangle_\Omega + (\nabla u(t), \nabla v)_\Omega - \langle \phi(t), v\rangle_\Gamma = \langle f_B(t), v\rangle_\Omega, \tag{2.9}$$

$$\langle (1/2 - \mathcal{K}) u(t)|_\Gamma, \psi\rangle_\Gamma + \langle \mathcal{V}\phi(t), \psi\rangle_\Gamma = \langle g_{\mathcal{K}}(t), \psi\rangle_\Gamma, \tag{2.10}$$

for all test functions $v \in H = H^1(\Omega)$ and $\psi \in B = H^{-1/2}(\Gamma)$, and for a.e. $t \in [0, T]$.

Remark 2.2. Any sufficiently smooth solution of (2.1)–(2.6) also solves (2.9)–(2.10) with $\langle f_B(t), v\rangle_\Omega = \langle f(t), v\rangle_\Omega + \langle g_2(t), v\rangle_\Gamma$ and $\langle g_{\mathscr{K}}(t), \psi\rangle_\Gamma = \langle (1/2 - \mathscr{K})g_1(t), \psi\rangle_\Gamma$ and, vice versa, any regular solution (u, ϕ) of (2.9)–(2.10) is a classical solution of (2.1)–(2.6). We note that $a(t)$ in (2.6) can be expressed directly in terms of the field u_e, once the solution (u, ϕ) of (2.9)–(2.10) is known, i.e., $a(t) = \frac{1}{2\pi}\int_\Gamma \phi(t)\,\mathrm{d}s$, where $\phi(t) = \partial_{\mathbf{n}} u_e|_\Gamma(t)$, see, e.g. [GH92].

The standard approach for parabolic problems whose leading term is not elliptic is to make use of the zero-order term with the time derivative. As we have exactly this case with the heat equation we could use this approach, but this would also introduce a dependence on $e^{\lambda T}$ into the constants. This is further discussed for the extended problem in Remark 3.1. But for the non-symmetric (later FEM-BEM) coupling there exists another way to obtain a coercive bilinear form. Similar as in [Aur+13] or [EOS17, Section 2] we can define a stabilised bilinear form

$$\mathscr{b}_{stab}((u,\phi);(v,\psi)) := \mathscr{b}((u,\phi);(v,\psi)) + ((1/2 - \mathscr{K})u + \mathscr{V}\phi\,,1)_\Gamma((1/2 - \mathscr{K})v + \mathscr{V}\psi\,,1)_\Gamma,$$

with the coupled bilinear form involving all terms on the left hand side of (2.9)–(2.10) except the time derivative, defined by

$$\mathscr{b}((u,\phi);(v,\psi)) := (\nabla u\,, \nabla v)_\Omega - \langle \phi\,, v\rangle_\Gamma + \langle (1/2 - \mathscr{K})u\,, \psi\rangle_\Gamma + \langle \mathscr{V}\phi\,, \psi\rangle_\Gamma.$$

The right-hand side also has to be stabilised, which results in

$$\begin{aligned} F_{stab}((v,\psi);t) := {} & \langle f(t), v\rangle_\Omega + \langle g_2(t), v\rangle_\Gamma + \langle (1/2 - \mathscr{K})g_1(t), \psi\rangle_\Gamma \\ & + ((1/2 - \mathscr{K})g_1(t), 1)_\Gamma((1/2 - \mathscr{K})v + \mathscr{V}\psi\,, 1)_\Gamma. \end{aligned}$$

The problem given by the stabilised bilinear form, in fact, yields an equivalent problem, see [EOS17, Lemma 2].

Lemma 2.3. *The variational formulation Problem 2.1 is equivalent to the stabilised problem given by: find* $\mathbf{u} = (u, \phi) \in Q_T \times B_T$ *such that*

$$(\partial_t u_h(t), v_h)_\Omega + \mathscr{b}_{stab}(\mathbf{u}(t);\mathbf{v}) = F_{stab}(\mathbf{v}; t),$$

for all $\mathbf{v} = (v, \psi) \in H \times B$ *and for a.e.* $t \in [0, T]$.

Then, for the analysis of Problem 2.1, we need the following auxiliary result which states that the (stabilised) bilinear form is, in fact, coercive and continuous.

Lemma 2.4. *The bilinear form* $\mathscr{b}_{stab}$ *is coercive on* $H^1(\Omega) \times H^{-1/2}(\Gamma)$ *and is continuous, i.e.,*

$$\mathscr{b}_{stab}((v,\psi);(v,\psi)) \geq \alpha_{stab}\big(\|v\|^2_{H^1(\Omega)} + \|\psi\|^2_{H^{-1/2}(\Gamma)}\big) \qquad \text{and} \tag{2.11}$$

$$\mathscr{b}_{stab}((u,\phi);(v,\psi)) \leq C_c(\|u\|_{H^1(\Omega)} + \|\phi\|_{H^{-1/2}(\Gamma)})(\|v\|_{H^1(\Omega)} + \|\psi\|_{H^{-1/2}(\Gamma)}), \tag{2.12}$$

with $\alpha_{stab} > 0$ *and* $C_c > 0$ *independent of the functions* $u, v \in H^1(\Omega)$ *and* $\phi, \psi \in H^{-1/2}(\Gamma)$.

Proof. The coercivity estimate for the bilinear form $\mathscr{b}_{stab}(\cdot;\cdot)$, follows directly by applying [EOS17, Theorem 1] with $\mathbf{A} = \mathscr{I}$, $C_{\mathbf{bc}} = 1$, and $\beta = 1$. The continuity follows from [EOS17, Lemma 1] and the mapping properties of the integral operators in Lemma 1.4. $\square$

Remark 2.5. If we allow a diffusion tensor $\mathbf{A}$ in (2.1), the coercivity (2.11) only holds if $\lambda_{\min}(\mathbf{A}) > C_{\mathscr{K}}/4$, where $\lambda_{\min}(\mathbf{A})$ denotes the minimal eigenvalue of $\mathbf{A}$ and $C_{\mathscr{K}} \in [1/2, 1)$ is the contraction constant of the double layer operator $\mathscr{K}$ of Lemma 1.6. More details can be found in [Aur+13; EOS17]. This also means that we can extend this coercivity estimate to the more complex problem of Chapter 3.

Using the coercivity of Lemma 2.4 and Lemma 1.4, we now prove the well-posedness of Problem 2.1.

Theorem 2.6 (Well-posedness of the variational problem)**.** *For any $f_B \in H_T'$ and $g_{\mathscr{K}} \in B_T'$, Problem 2.1 admits a unique weak solution $(u, \phi) \in Q_T \times B_T$ and*

$$\|u\|_{Q_T} + \|\phi\|_{B_T} \leq C(\|f_B\|_{H_T'} + \|g_{\mathscr{K}}\|_{B_T'}),$$

with a constant $C = C(\alpha_{stab}, \Omega) > 0$ that is independent of the data f_B and $g_{\mathscr{K}}$.

Proof. The proof is mainly standard and the techniques can be found in [Eva10, Part II, Sec. 7.1.2] or [DL00, Chapter XVIII, Sec. 3]. The estimate for $\|\partial_t u\|_{H_T'}$ can also be found there, see also Lemma 2.13, but we will present the rest of the proof for the non-symmetric coupling. Testing the weak formulation Problem 2.1 with $\mathbf{v} = (v, \psi) = \mathbf{u} = (u, \phi) \in Q_T \times B_T$ and using that

$$\langle \partial_t u(t), u(t)\rangle_\Omega = \frac{1}{2}\Big((\partial_t u(t), u(t))_\Omega + (u(t), \partial_t u(t))_\Omega\Big) = \frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)},$$

lead to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|u(t)\|^2_{L^2(\Omega)} + \mathscr{b}_{stab}(\mathbf{u}(t);\mathbf{u}(t))\\ &= \langle f_B(t), u(t)\rangle_\Omega + \langle g_{\mathscr{K}}(t), \phi(t)\rangle_\Gamma + (g_{\mathscr{K}}(t), 1)_\Gamma((1/2 - \mathscr{K})u(t) + \mathscr{V}\phi(t), 1)_\Gamma.\end{aligned}$$

Now we can use the coercivity (2.11), the continuity of the boundary integral operators, the Cauchy-Schwarz inequality and the trace inequality to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} &+ \alpha_{stab}\Big(\|u(t)\|^2_H + \|\phi(t)\|^2_B\Big)\\ &\leq C\Big(\|f_B(t)\|_{H'} + \|g_{\mathscr{K}}(t)\|_{B'}\Big)\Big(\|u(t)\|_H + \|\phi(t)\|_B\Big).\end{aligned}$$

Young's inequality then leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} &+ \alpha_{stab}\Big(\|u(t)\|^2_H + \|\phi(t)\|^2_B\Big)\\ &\leq \frac{1}{2\alpha_{stab}}C\Big(\|f_B(t)\|^2_{H'} + \|g_{\mathscr{K}}(t)\|^2_{B'}\Big) + \frac{\alpha_{stab}}{2}\Big(\|u(t)\|^2_H + \|\phi(t)\|^2_B\Big).\end{aligned}$$

Now, bringing over the last term, integrating over t from 0 to T and using $\|u(0)\|_{L^2(\Omega)} = 0$ shows the energy estimate with a constant C which only depends on the constant α_{stab} of (2.11), Ω, and C_{tr}, but in particular not on T. Lastly, existence and uniqueness follow from standard results such as [Eva10, Part II, Sec. 7.1.2]. □

Remark 2.7. To prove this energy estimate we could also use the Steklov-Poincaré Operator $\mathscr{S}$ (also known as Dirichlet-to-Neumann Map) and (2.10) to express the exterior solution by $\phi(t) = \mathscr{S}u(t) + \mathscr{R}g_{\mathscr{K}}(t)$ with $-\mathscr{S} = \mathscr{V}^{-1}(1/2 - \mathscr{K})$ and $\mathscr{R} = \mathscr{V}^{-1}$. This is possible because $\mathscr{V}$ is coercive, see Lemma 1.5, and thus invertible. This approach has been taken in [EES18, Theorem 4] as in this case we can work with a Gårding Inequality for the bilinear form and thus we do not have to use the stabilisation (but again have the dependence of the constant in the energy estimate on T). With this, we can reduce (2.9)–(2.10) to

$$\langle \partial_t u(t), v\rangle_\Omega + \tilde{b}(u(t), v) = \langle f_B(t), v\rangle_\Omega + \langle \mathscr{R}g_{\mathscr{K}}(t), v\rangle_\Gamma, \tag{2.13}$$

with the bilinear form $\tilde{b}(u, v) := (\nabla u, \nabla v)_\Omega + \langle -\mathscr{S}u, v\rangle_\Gamma$. Now we can show that $\tilde{b}(u, v)$ also fulfils a Gårding Inequality (on $H^1(\Omega)$). That constitutes that the reduced problem (2.13) is uniformly parabolic and we can use standard results about variational evolution problems, see, e.g., [Eva10, Part II, Sec. 7.1.2] to obtain the bounds for u. To bound the second solution component ϕ we can use (2.10) and the ellipticity of $\mathscr{V}$ which gives

$$C_{\mathscr{V}}\|\phi(t)\|^2_{H^{-1/2}(\Gamma)} \le \langle \mathscr{V}\phi(t), \phi(t)\rangle_\Gamma = -\langle (1/2 - \mathscr{K})u(t), \phi(t)\rangle_\Gamma + \langle g_{\mathscr{K}}(t), \phi(t)\rangle_\Gamma \tag{2.14}$$

$$\le \big((1/2 + C_{\mathscr{K}cont})C_{tr}\|u(t)\|_{H^1(\Omega)} + \|g_{\mathscr{K}}(t)\|_{H^{1/2}(\Gamma)}\big)\|\phi(t)\|_{H^{-1/2}(\Gamma)}. \tag{2.15}$$

In the last step, we used the trace inequality and the boundedness of $\mathscr{K}$.

In this case, we could also use that the exterior Steklov-Poincaré Operator $-\mathscr{S} = V^{-1}(1/2 - \mathscr{K})$ is coercive on $H^{1/2}(\Gamma)$, see, e.g., [CS95] for the equivalent symmetric representation. Then the bilinear form $\tilde{b}(u, v)$ is coercive by itself, so we would not need to enforce the Gårding Inequality and the constant in the would be independent of T. However, the argument involving the Gårding Inequality is more general and carries over verbatim also to the discretisation and more general problems (which we look at in the next chapter). Still, we will use the stabilisation, as it gives an easy way to obtain a coercive bilinear form and lets the constants be independent of T.

Now we can reformulate the energy estimate in terms of the model input data without using the shorthand notations.

Corollary 2.8. *For $f \in H'_T$, $g_1 \in B'_T$, and $g_2 \in B_T$ our model problem (2.1)–(2.6) admits a unique weak solution $(u, \phi) \in Q_T \times B_T$ and*

$$\|u\|_{Q_T} + \|\phi\|_{B_T} \le C(\|f\|_{H'_T} + \|g_1\|_{B'_T} + \|g_2\|_{B_T}), \tag{2.16}$$

with a constant $C = C(\alpha_{stab}, \Omega) > 0$ that is independent of the data f, g_1, and g_2.

Proof. This follows directly from Theorem 2.6 with (2.7) and (2.8). □

2.3 Galerkin Approximation

Now we want to look at an abstract Galerkin discretisation of our (weak) model problem. Therefore let $H^h \subset H^1(\Omega)$ and $B^h \subset H^{-1/2}(\Omega)$ be finite dimensional subspaces. Similar to the continuous case, we define corresponding Bochner spaces

$$
\begin{aligned}
H_T^h &= L^2(0,T;H^h),\\
B_T^h &= L^2(0,T;B^h),
\end{aligned}
$$

and the energy space

$$
Q_T^h = \left\{ v_h \in H^1(0,T;H^h) \,:\, v_h(0) = 0 \right\}.
$$

Then we consider the following Galerkin approximation of Problem 2.1.

Problem 2.9 (Semi-discrete problem). Given $f_B \in H'_T$ and $g_{\mathscr{K}} \in B'_T$, find $u_h \in Q_T^h$ and $\phi_h \in B_T^h$ such that

$$
(\partial_t u_h(t), v_h)_\Omega + (\nabla u_h(t), \nabla v_h)_\Omega - (\phi_h(t), v_h)_\Gamma = \langle f_B(t), v_h \rangle_\Omega, \tag{2.17}
$$

$$
((1/2 - \mathscr{K}) u_h(t), \psi_h)_\Gamma + (\mathscr{V} \phi_h(t), \psi_h)_\Gamma = (g_{\mathscr{K}}(t), \psi_h)_\Gamma, \tag{2.18}
$$

for all test functions $v_h \in H^h$ and $\psi_h \in B^h$, and for a.e. $t \in [0,T]$.

As we need the coercivity of the overall system in the analysis, we also make use of the bilinear form $\mathscr{b}_{stab}$ here. Again, the stabilised formulation is equivalent to the semi-discrete problem. A modified version of [EOS17, Lemma 4] yields the following lemma.

Lemma 2.10. *Let $1 \in B^h$, then Problem 2.9 is equivalent to the stabilised problem given by: find $\mathbf{u}_h = (u_h, \phi_h) \in Q_T^h \times B_T^h$ such that*

$$
(\partial_t u_h(t), v_h)_\Omega + \mathscr{b}_{stab}(\mathbf{u}_h(t); \mathbf{v}_h) = F_{stab}(\mathbf{v}_h; t),
$$

for all $\mathbf{v}_h = (v_h, \psi_h) \in H^h \times B^h$ and for a.e. $t \in [0,T]$.

The analysis of this Galerkin approximation can be carried out with similar arguments as used in [CES90] and [MS87]. Here, we make use of Lemma 2.4 to get rid of the smoothness assumption on Γ imposed in [MS87].

We will still briefly state the main results and sketch the basic ideas of their proofs. Due to Lemma 2.4, the well-posedness of the above problem follows again by standard energy arguments.

Lemma 2.11. *For any $f_B \in H'_T$ and $g_{\mathscr{K}} \in B'_T$, Problem 2.9 admits a unique solution $(u_h, \phi_h) \in Q_T^h \times B_T^h$. Moreover,*

$$
\|u_h\|_{H_T} + \|\phi_h\|_{B_T} \leq C\big(\|f_B\|_{H'_T} + \|g_{\mathscr{K}}\|_{B'_T}\big), \tag{2.19}
$$

with a constant $C = C(\alpha_{stab}, \Omega) > 0$ that is independent of the data f_B, $g_{\mathscr{K}}$, and the approximation spaces H^h, B^h.

Proof. We can prove this fairly analogous to Theorem 2.6. Testing the weak formulation Problem 2.1 with $(v_h, \psi_h) = (u_h, \phi_h) \in Q_T^h \times B_T^h$, using the coercivity (2.11) and the Cauchy-Schwarz inequality results in

$$\frac{1}{2}\frac{d}{dt}\|u_h(t)\|^2_{L^2(\Omega)} + \alpha_{stab}\big(\|u_h(t)\|_H^2 + \|\phi_h(t)\|_B^2\big) \\ \leq C\big(\|f_B(t)\|_{H'} + \|g_{\mathcal{K}}(t)\|_{B'}\big)\big(\|u_h(t)\|_H + \|\phi_h(t)\|_B\big).$$

Again, we can now employ Young's inequality, integrate over t from 0 to T and use that $\|u_h(0)\|_{L^2(\Omega)} = 0$ to obtain the energy estimate. Existence and uniqueness also follow from standard results in [Eva10, Part II, Sec. 7.1.2]. □

Remark 2.12. We could also prove the energy estimate with a discrete analogon of the Steklov-Poincaré Operator which we used in Remark 2.7. In this case $\mathcal{S}_h : H^h \to B^h$ can be defined by

$$\langle -\mathcal{V}\mathcal{S}_h u_h\,, \psi_h\rangle_\Gamma = \langle (1/2 - \mathcal{K})u_h\,, \psi_h\rangle_\Gamma \qquad \text{for all } \psi_h \in B^h,$$

and $\mathcal{R}_h : H^{1/2}(\Gamma) \to B^h$ can be defined by

$$\langle \mathcal{V}\mathcal{R}_h g_{\mathcal{K}}\,, \psi_h\rangle_\Gamma = \langle g_{\mathcal{K}}\,, \psi_h\rangle_\Gamma \qquad \text{for all } \psi_h \in B^h.$$

Accordingly, we can use this to express $\phi_h(t) = \mathcal{S}_h u_h(t) + \mathcal{R}_h g_{\mathcal{K}}(t)$ and use the same arguments as in the continuous case. For more details, see [EES18, Lemma 9].

Now we also want to obtain a uniform estimate for the time derivative $\partial_t u_h$, which is not included in (2.19). Therefore we proceed with similar arguments as in [MS87; CES90].

Let $P_h : L^2(\Omega) \to H^h$ denote the L^2–orthogonal projection defined by

$$(P_h v\,, w_h)_\Omega = (v\,, w_h)_\Omega \qquad \text{for all } w_h \in H^h. \tag{2.20}$$

We say that the L^2–projection P_h is stable in $H^1(\Omega)$ when there exists a constant $C_P > 0$ such that

$$\|P_h v\|_{H^1(\Omega)} \leq C_P \|v\|_{H^1(\Omega)} \qquad \text{for all } v \in H^1(\Omega). \tag{2.21}$$

This imposes a mild condition on the approximation space H^h, which is not very restrictive in practice; see Section 2.5 for an example and further discussion. So from now on, we assume that our spaces are chosen such that the L^2–projection is stable in $H^1(\Omega)$.

This property and equation (2.17) can now be used to deduce a uniform bound for the norm $\|\partial_t u_h\|_{H^1(\Omega)'}$ of the time derivative and the following energy estimate.

Lemma 2.13 (Discrete energy estimate)**.** *Let P_h be $H^1(\Omega)$-stable. Then for any $f_B \in H_T'$ and $g_{\mathcal{K}} \in B_T'$ there holds for the solution $(u_h, \phi_h) \in Q_T^h \times B_T^h$ to Problem 2.9 that*

$$\|u_h\|_{Q_T} + \|\phi_h\|_{B_T} \leq C\big(\|f_B\|_{H_T'} + \|g_{\mathcal{K}}\|_{B_T'}\big),$$

with a constant $C = C(\alpha_{stab}, \Omega, C_P) > 0$ that is independent of $f_B, g_{\mathcal{K}}$ and the approximation spaces H^h and B^h.

Proof. By definition of the dual norm and the L^2–projection, we obtain

$$\|\partial_t u_h(t)\|_{H^1(\Omega)'} = \sup_{0\neq v\in H^1(\Omega)} \frac{(\partial_t u_h(t), v)_\Omega}{\|v\|_{H^1(\Omega)}} = \sup_{0\neq v\in H^1(\Omega)} \frac{(\partial_t u_h(t), P_h v)_\Omega}{\|v\|_{H^1(\Omega)}}. \tag{2.22}$$

Using equation (2.17), the Cauchy-Schwarz inequality, and the trace inequality, one can further estimate

$$(\partial_t u_h(t), P_h v)_\Omega \leq \big(\|u_h(t)\|_{H^1(\Omega)} + C_{tr}\|\phi_h(t)\|_{H^{-1/2}(\Omega)} + \|f_B(t)\|_{H^1(\Omega)'}\big)\|P_h v\|_{H^1(\Omega)}.$$

Therefore, the stability of P_h yields

$$\|\partial_t u_h(t)\|_{H^1(\Omega)'} \leq C\big(\|u_h(t)\|_{H^1(\Omega)} + \|\phi_h(t)\|_{H^{-1/2}(\Gamma)} + \|f_B(t)\|_{H^1(\Omega)'}\big).$$

Then the assertion of the lemma follows by integration over time and combination with the estimate (2.19) for $\|u_h\|_{H_T}$ and $\|\phi_h\|_{B_T}$ stated in Lemma 2.11. □

By the combination of the previous lemmata and the variational problems defining the continuous and the semi-discrete solution, we obtain the following quasi-optimality result.

Theorem 2.14 (Quasi-optimal-approximation, [EES18, Theorem 12]). *Let P_h be $H^1(\Omega)$-stable. Furthermore let $(u,\phi) \in Q_T \times B_T$ and $(u_h,\phi_h) \in Q_T^h \times B_T^h$ denote the solutions of Problem 2.1 and Problem 2.9, respectively. Then there holds that*

$$\|u - u_h\|_{Q_T} + \|\phi - \phi_h\|_{B_T} \leq C\big(\|u - \tilde{u}_h\|_{Q_T} + \|\phi - \tilde{\phi}_h\|_{B_T}\big),$$

for all functions $\tilde{u}_h \in Q_T^h$ and $\tilde{\phi}_h \in B_T^h$ with a constant $C = C(\alpha_{stab}, \Omega, C_P) > 0$ which is independent of the model input data $f_B, g_{\mathcal{K}}$ and of the spaces H^h and B^h.

Proof. This type of result has first been proved in [CES90] for the symmetric coupling method. Using Lemma 2.4, their proof can be adapted to the non-symmetric coupling as well. Let $\tilde{u}_h \in Q_T^h$ and $\tilde{\phi}_h \in B_T^h$ be arbitrary. By

$$\begin{aligned} \|u - u_h\|_{Q_T} &\leq \|u - \tilde{u}_h\|_{Q_T} + \|\tilde{u}_h - u_h\|_{Q_T} \qquad \text{and} \\ \|\phi - \phi_h\|_{B_T} &\leq \|\phi - \tilde{\phi}_h\|_{B_T} + \|\tilde{\phi}_h - \phi_h\|_{B_T}, \end{aligned}$$

we split the error into an *approximation error* and a *discrete error* component. The first part already appears in the final estimate. To estimate the discrete error components, we note that the discrete problem (2.17)–(2.18) is consistent with the continuous problem (2.9)–(2.10). Hence, due to Galerkin orthogonality, we may write the discrete error components $w_h = \tilde{u}_h - u_h$ and $\varphi_h = \tilde{\phi}_h - \phi_h$ as the solution of the system

$$(\partial_t w_h(t), v_h)_\Omega + (\nabla w_h(t), \nabla v_h)_\Omega - (\varphi_h(t), v_h)_\Gamma = \langle F(t), v_h\rangle_\Omega, \tag{2.23}$$

$$((1/2 - \mathcal{K})w_h(t), \psi_h)_\Gamma + (\mathcal{V}\varphi_h(t), \psi_h)_\Gamma = (G(t), \psi_h)_\Gamma, \tag{2.24}$$

for all $v_h \in H^h$ and $\psi_h \in B^h$ with the right-hand sides $F(t)$ and $G(t)$ defined by

$$\begin{aligned} \langle F(t), v\rangle_\Omega &:= \langle \partial_t \tilde{u}_h(t) - \partial_t u(t), v\rangle_\Omega + (\nabla \tilde{u}_h(t) - \nabla u(t), \nabla v)_\Omega - \langle \tilde{\phi}_h(t) - \phi(t), v\rangle_\Gamma, \\ \langle G(t), \psi\rangle_\Gamma &:= \langle (1/2 - \mathcal{K})(\tilde{u}_h(t) - u(t)), \psi\rangle_\Gamma + \langle \mathcal{V}(\tilde{\phi}_h(t) - \phi(t)), \psi\rangle_\Gamma, \end{aligned}$$

for all $v \in H$ and $\psi \in B$. With the bounds from the integral and trace operators, the Cauchy-Schwarz inequality, and integrating with respect to time, one can see that

$$\|F\|_{L^2(0,T;H^1(\Omega)')} \leq C\big(\|u-\tilde{u}_h\|_{Q_T} + \|\phi - \tilde{\phi}_h\|_{B_T}\big),$$
$$\|G\|_{L^2(0,T;H^{1/2}(\Gamma))} \leq C\big(\|u-\tilde{u}_h\|_{H_T} + \|\phi - \tilde{\phi}_h\|_{B_T}\big).$$

Note that the system (2.23)–(2.24) with the right-hand sides F and G has the same form as (2.17)–(2.18). Therefore, Lemma 2.13 can be applied and finally shows that

$$\|\tilde{u}_h - u_h\|_{Q_T} + \|\tilde{\phi}_h - \phi_h\|_{B_T} \leq C\big(\|u-\tilde{u}_h\|_{Q_T} + \|\phi - \tilde{\phi}_h\|_{B_T}\big).$$

Together with the error splitting, this completes the proof. □

Remark 2.15. As a direct consequence of Theorem 2.14, we also obtain

$$\|u-u_h\|_{Q_T} + \|\phi - \phi_h\|_{B_T} \leq C\big(\|u-P_h u\|_{Q_T} + \|\phi - \Pi_h \phi\|_{B_T}\big),$$

where P_h is the $L^2(\Omega)$ projection operator introduced in (2.20), $\Pi_h : H^{-1/2}(\Gamma) \to B_h$ is the $H^{-1/2}(\Gamma)$–projection operator, and $C = C(\alpha_{stab}, \Omega, C_P) > 0$. This allows us to obtain explicit error bounds for particular choices of approximation spaces by using interpolation error estimates in the energy spaces. This will be concretised in Section 2.5.

2.4 Fully-Discrete Formulation

The previous section transformed our continuous problem into an ordinary differential equation by using a Galerkin discretisation in space. For the discretisation in time, we consider a particular one-step method that allows us to establish quasi-optimality of the fully-discrete scheme under minimal regularity assumptions. A similar approach was used in [Tan14, Sec. 4.1.] for the discretisation of a parabolic problem. Also, the analysis in this section can be found in [EES18]. First of all, we introduce some notation which we need to formulate our time discretisation scheme. Let $0 = t^0 < t^1 < \ldots < t^N = T$, $N \in \mathbb{N}$ be a partition of the time interval $[0, T]$. Further, we denote by $\tau^n = t^n - t^{n-1}$ the local time step sizes and set $\tau := \max_{n=1,\ldots,N} \tau^n$.

In this section we search for approximations $u_{h,\tau} \in Q_T^{h,\tau}$ and $\phi_{h,\tau} \in B_T^{h,\tau}$ with

$$Q_T^{h,\tau} := \big\{u \in C(0,T;H^h) : u(0) = 0, u|_{[t^{n-1},t^n]} \text{ is linear in } t\big\} \qquad \text{and}$$
$$B_T^{h,\tau} := \big\{\phi \in L^2(0,T;B^h) : \phi|_{(t^{n-1},t^n]} \text{ is constant in } t\big\}.$$

This means that our functions are piecewise linear (or piecewise constant) in time. So this approach is comparable to a (discontinuous) Galerkin approximation in time. Furthermore, for sufficiently regular functions with respect to t, we denote by $v^n = v(t^n)$ the values at the grid points. For $u_{h,\tau} \in Q_T^{h,\tau}$ the operator ∂_t has to be understood piecewisely with respect to the time mesh, in particular,

$$\partial_t u_{h,\tau}|_{(t^{n-1},t^n)} = d_\tau u_{h,\tau}^n, \qquad \text{with} \qquad d_\tau u_{h,\tau}^n := \frac{1}{\tau^n}(u_{h,\tau}^n - u_{h,\tau}^{n-1}). \tag{2.25}$$

Now we introduce the centrepiece of this section, which is a weighted averaging operator

$$\widehat{v}^n = \frac{1}{\tau^n}\int_{t^{n-1}}^{t^n} v(t)\omega^n(t)\,\mathrm{d}t, \qquad \text{with} \qquad \omega^n(t) = \frac{6t - 2t^n - 4t^{n-1}}{\tau^n}, \tag{2.26}$$

and define our fully-discrete system as follows.

Problem 2.16 (Full discretisation)**.** Find $u_{h,\tau} \in Q_T^{h,\tau}$ and $\phi_{h,\tau} \in B_T^{h,\tau}$ such that

$$(\widehat{\partial_t u}_{h,\tau}^n, v_h)_\Omega + (\widehat{\nabla u}_{h,\tau}^n, \nabla v_h)_\Omega - (\widehat{\phi}_{h,\tau}^n, v_h)_\Gamma = \langle \widehat{f}_B^n, v_h\rangle_\Omega, \tag{2.27}$$

$$((1/2 - \mathscr{K})\widehat{u}_{h,\tau}^n, \psi_h)_\Gamma + (\mathscr{V}\widehat{\phi}_{h,\tau}^n, \psi_h)_\Gamma = (\widehat{g}_{\mathscr{K}}^n, \psi_h)_\Gamma, \tag{2.28}$$

for all $v_h \in H^h \subset H^1(\Omega)$ and $\psi_h \in B^h \subset H^{-1/2}(\Gamma)$ and for all $1 \le n \le N$.

We have chosen the piecewise linear weight function $\omega^n(t)$ in (2.26) such that the weighting operator fulfils the following identities. This also emphasises the interpretation of Problem 2.16 as a variant of a classical implicit Euler time discretisation.

Lemma 2.17. *The following identities,*

$$\widehat{u}_{h,\tau}^n = u_{h,\tau}^n, \qquad \widehat{\partial_t u}_{h,\tau}^n = d_\tau u_{h,\tau}^n = \frac{1}{\tau^n}(u_{h,\tau}^n - u_{h,\tau}^{n-1}), \qquad \text{and} \qquad \widehat{\phi}_{h,\tau}^n = \phi_{h,\tau}^n,$$

hold for all $n \in \mathbb{N}$, $u_{h,\tau} \in Q_T^{h,\tau}$, and $\phi_{h,\tau} \in B_T^{h,\tau}$.

Since $u_{h,\tau}$ and $\phi_{h,\tau}$ are piecewise linear and constant, respectively, we easily see that

$$\|u_{h,\tau}\|_{H_T}^2 \le \frac{4}{3}\sum_{n=1}^N \tau^n \|u_{h,\tau}^n\|_{H^1(\Omega)}^2 \quad \text{and} \quad \|\phi_{h,\tau}\|_{B_T}^2 \le \sum_{n=1}^N \tau^n \|\phi_{h,\tau}^n\|_{H^{-1/2}(\Omega)}^2. \tag{2.29}$$

Furthermore, for any $v \in L^2(0,T;X)$ with values in some Hilbert space X, we have

$$\sum_{n=1}^N \tau^n \|\widehat{v}^n\|_X^2 \le 4\|v\|_{L^2(0,T;X)}^2. \tag{2.30}$$

Proof. Note that $u_{h,\tau}$ and ω^n are linear in t in the interval $[t^{n-1}, t^n]$. With standard hat ansatz functions, we easily get a system of linear equations for $\widehat{u}_{h,\tau}^n = u_{h,\tau}^n$. Then some basic calculations result in the ω^n of (2.26). Note that $\partial_t u_{h,\tau}$ and $\phi_{h,\tau}$ are constant in (t^{n-1}, t^n) and the identities follow directly from the first one. The last estimate follows directly by the Cauchy-Schwarz inequality and $\|\omega^n(t)\|_{L^2(t^{n-1},t^n)}^2 = 4\tau^n$. □

Remark 2.18. With the above identities, the discrete system of Problem 2.16 can be equivalently written as

$$(d_\tau u_{h,\tau}^n, v_h)_\Omega + (\nabla u_{h,\tau}^n, \nabla v_h)_\Omega - (\phi_{h,\tau}^n, v_h)_\Gamma = \langle \widehat{f}_B^n, v_h\rangle_\Omega, \tag{2.31}$$

$$((1/2 - \mathscr{K})u_{h,\tau}^n, \psi_h)_\Gamma + (\mathscr{V}\phi_{h,\tau}^n, \psi_h)_\Gamma = (\widehat{g}_{\mathscr{K}}^n, \psi_h)_\Gamma, \tag{2.32}$$

for all $v_h \in H^h \subset H^1(\Omega)$ and $\psi_h \in B^h \subset H^{-1/2}(\Gamma)$, and for all $1 \le n \le N$. Hence, the fully-discrete Problem 2.16 amounts to a discretisation of Problem 2.9 in time by a variant of the implicit Euler method, i.e., it differs only in the right-hand side, which is treated in a special way in order to reduce the regularity requirements on the data. An error analysis of the coupling with the classical implicit Euler scheme and other time discretisation schemes in the natural energy norm is also possible. However, one needs the usual Taylor expansions and therefore some regularity requirements on the data $\tilde{f}$, $\tilde{g}$, $\tilde{h}$, and the solution. This is carried out for the FVM-BEM coupling in Section 3.2.4.

Remark 2.19. By testing (2.9)–(2.10) with $v = v_h$ and $\psi = \psi_h$, multiplying with the weight function ω^n, and integrating over the time interval $[t^{n-1}, t^n]$, one can see that

$$\begin{aligned}\langle \widehat{\partial_t u}^n, v_h\rangle_\Omega + (\widehat{\nabla u}^n, \nabla v_h)_\Omega - \langle \widehat{\phi}^n, v_h\rangle_\Gamma &= \langle \widehat{f_B}^n, v_h\rangle_\Omega,\\ ((1/2-\mathscr{K})\widehat{u}^n, \psi_h)_\Gamma + (\mathscr{V}\widehat{\phi}^n, \psi_h)_\Gamma &= (\widehat{g}^n_{\mathscr{K}}, \psi_h)_\Gamma,\end{aligned}$$

for all $v_h \in H^h$, $\psi_h \in B^h$, and all $1 \le n \le N$. This shows that the fully-discrete scheme (2.27)–(2.28) is a Petrov-Galerkin approximation and is thus consistent with the variational problem (2.9)–(2.10).

For the fully-discrete formulation, we can again use the stabilised bilinear form that we used in the analysis of the semi-discrete formulation to obtain coercivity. We will therefore swap $\mathscr{b}$ with $\mathscr{b}_{stab}$ in the proofs.

In the following, we derive error estimates for the fully-discrete scheme in the energy norm by an extension of our arguments for the analysis of the Galerkin semi-discretisation. Let us start with establishing the corresponding fully-discrete energy estimate.

Lemma 2.20 (Well-posedness). *For any $f_B \in H_T'$ and $g_{\mathscr{K}} \in B_T'$, Problem 2.16 admits a unique solution $(u_{h,\tau}, \phi_{h,\tau}) \in Q_T^{h,\tau} \times B_T^{h,\tau}$ and*

$$\|u_{h,\tau}\|_{H_T} + \|\phi_{h,\tau}\|_{B_T} \le C\big(\|f_B\|_{H_T'} + \|g_{\mathscr{K}}\|_{B_T'}\big), \tag{2.33}$$

with a constant $C = C(\alpha_{stab}, \Omega) > 0$ that is independent of the data f_B, $g_{\mathscr{K}}$, and the approximation spaces H^h and B^h.

Proof. First, we write the stabilised formulation of Problem 2.16 with $\mathbf{u}_{h,\tau} = (u_{h,\tau}, \phi_{h,\tau}) \in Q_T^{h,\tau} \times B_T^{h,\tau}$ and $\mathbf{v}_h = (v_h, \psi_h) \in H^h \times B^h$ as

$$\begin{aligned}&\frac{1}{\tau^n}(u^n_{h,\tau} - u^{n-1}_{h,\tau}, v_h)_\Omega + \mathscr{b}_{stab}(\mathbf{u}_{h,\tau}; \mathbf{v}_h)\\ &\qquad\qquad = \langle \widehat{f_B}^n, v_h\rangle_\Omega + (\widehat{g}^n_{\mathscr{K}}, \psi_h)_\Gamma + (\widehat{g}^n_{\mathscr{K}}, 1)_\Gamma((1/2-\mathscr{K})v_h + \mathscr{V}\psi_h, 1)_\Gamma.\end{aligned}$$

Here, we already used the properties of the weighted averaging of Lemma 2.17. By testing with $(v_h, \psi_h) = (u^n_{h,\tau}, \phi^n_{h,\tau})$, using the relation

$$\begin{aligned}(u^n_{h,\tau} - u^{n-1}_{h,\tau}, u^n_{h,\tau})_\Omega &= \frac{1}{2}(u^n_{h,\tau} - u^{n-1}_{h,\tau}, u^n_{h,\tau} + u^{n-1}_{h,\tau})_\Omega + \frac{1}{2}(u^n_{h,\tau} - u^{n-1}_{h,\tau}, u^n_{h,\tau} - u^{n-1}_{h,\tau})_\Omega\\ &= \frac{1}{2}\Big((u^n_{h,\tau}, u^n_{h,\tau})_\Omega - (u^{n-1}_{h,\tau}, u^{n-1}_{h,\tau})_\Omega + (u^n_{h,\tau} - u^{n-1}_{h,\tau}, u^n_{h,\tau} - u^{n-1}_{h,\tau})_\Omega\Big)\\ &= \frac{1}{2}\Big(\|u^n_{h,\tau}\|^2_{L^2(\Omega)} - \|u^{n-1}_{h,\tau}\|^2_{L^2(\Omega)} + \|u^n_{h,\tau} - u^{n-1}_{h,\tau}\|^2_{L^2(\Omega)}\Big),\end{aligned}$$

applying the Cauchy-Schwarz, trace, and Young's inequality as well as the coercivity (2.11) of the bilinear form $\mathscr{b}_{stab}$ we get

$$\begin{aligned}\frac{1}{2\tau^n}\Big(\|u_{h,\tau}^n\|_{L^2(\Omega)}^2 - \|u_{h,\tau}^{n-1}\|_{L^2(\Omega)}^2 + \|u_{h,\tau}^n - u_{h,\tau}^{n-1}\|_{L^2(\Omega)}^2\Big) + \alpha_{stab}\Big(\|u_{h,\tau}^n\|_{H^1(\Omega)}^2 + \|\phi_{h,\tau}^n\|_{H^{-1/2}(\Gamma)}^2\Big)\\ \le \frac{\alpha_{stab}}{2}\Big(\|u_{h,\tau}^n\|_{H^1(\Omega)}^2 + \|\phi_{h,\tau}^n\|_{H^{-1/2}(\Gamma)}^2\Big) + \frac{1}{\alpha_{stab}}\|\widehat{f}_B^n\|_{H^1(\Omega)'}^2 + \frac{1}{\alpha_{stab}}\|\widehat{g}_{\mathscr{K}}^n\|_{H^{1/2}(\Gamma)}^2.\end{aligned}$$

This shows that the problems are uniquely solvable at every time step.

Multiplying with $2\tau^n$, bringing the terms with $u_{h,\tau}^n$ and $\phi_{h,\tau}^n$ on the right-hand side over to the left-hand side and summing over n from 1 to N leads to

$$\begin{aligned}\|u_{h,\tau}^N\|_{L^2(\Omega)}^2 + \alpha_{stab}\sum_{n=1}^N \tau^n\Big(\|u_{h,\tau}^n\|_{H^1(\Omega)}^2 + \|\phi_{h,\tau}^n\|_{H^{-1/2}(\Gamma)}^2\Big)\\ \le C\sum_{n=1}^N \tau^n\Big(\|\widehat{f}_B^n\|_{H^1(\Omega)'}^2 + \|\widehat{g}_{\mathscr{K}}^n\|_{H^{1/2}(\Gamma)}^2\Big),\end{aligned} \tag{2.34}$$

with a constant $C > 0$. For the left-hand side, we can use (2.29) of Lemma 2.17. For the right-hand side of (2.34) we can use (2.30) of Lemma 2.17 to see that

$$\sum_{n=1}^N \tau^n\|\widehat{f}_B^n\|_{H^1(\Omega)'}^2 \le 4\|f_B\|_{H_T'}^2 \qquad \text{and} \qquad \sum_{n=1}^N \tau^n\|\widehat{g}_{\mathscr{K}}^n\|_{H^{1/2}(\Gamma)}^2 \le 4\|g_{\mathscr{K}}\|_{B_T'}^2. \tag{2.35}$$

Now the energy estimate (2.33) follows from (2.34)–(2.35). □

Remark 2.21. Again, we could also employ the argument with the Gårding inequality for the fully-discrete formulation. The main drawback of this argument is the constant factor $e^{2N\tau}$ that will come up in the constant of the energy estimate. This constant depends in particular on the quasi-uniformity of the time mesh. We would also have to assume that $\tau \le 1/4$ (but this is not very restrictive in practice).

With similar arguments as used for the analysis on the semi-discrete level, we also obtain a bound for the time derivative $\partial_t u_{h,\tau}$ of the discrete solution.

Lemma 2.22 (Energy estimate). *Let P_h be $H^1(\Omega)$-stable. Then, for any $f_B \in H_T'$ and $g_{\mathscr{K}} \in B_T'$, the solution $(u_{h,\tau}, \phi_{h,\tau}) \in Q_T^{h,\tau} \times B_T^{h,\tau}$ of Problem 2.16 satisfies*

$$\|u_{h,\tau}\|_{Q_T} + \|\phi_{h,\tau}\|_{B_T} \le C\big(\|f_B\|_{H_T'} + \|g_{\mathscr{K}}\|_{B_T'}\big), \tag{2.36}$$

with a constant $C = C(\alpha_{stab}, \Omega, C_P) > 0$ that is independent of the data f_B, $g_{\mathscr{K}}$, and the approximation spaces H^h and B^h.

Proof. In view of Lemma 2.20, we only have to estimate

$$\|\partial_t u_{h,\tau}\|_{H_T'}^2 = \sum_{n=1}^N \tau^n\|d_\tau u_{h,\tau}^n\|_{H^1(\Omega)'}^2.$$

With similar reasoning as in Lemma 2.13, we obtain

$$\|d_\tau u_{h,\tau}^n\|_{H^1(\Omega)'} = \sup_{0\neq v\in H^1(\Omega)} \frac{(d_\tau u_{h,\tau}^n, v)_\Omega}{\|v\|_{H^1(\Omega)}} = \sup_{0\neq v\in H^1(\Omega)} \frac{(d_\tau u_{h,\tau}^n, P_h v)_\Omega}{\|v\|_{H^1(\Omega)}}. \tag{2.37}$$

By equation (2.31) and the Cauchy-Schwarz inequality, we further get

$$(d_\tau u_{h,\tau}^n, P_h v)_\Omega \leq \big(\|u_{h,\tau}^n\|_{H^1(\Omega)} + C_{tr}\|\phi_{h,\tau}^n\|_{H^{-1/2}(\Gamma)} + \|\widehat{f}_B^n\|_{H^1(\Omega)'}\big)\|P_h v\|_{H^1(\Omega)}.$$

The H^1-stability of P_h therefore yields for (2.37)

$$\|d_\tau u_{h,\tau}^n\|_{H^1(\Omega)'} \leq C\big(\|u_{h,\tau}^n\|_{H^1(\Omega)} + \|\phi_{h,\tau}^n\|_{H^{-1/2}(\Gamma)} + \|\widehat{f}_B^n\|_{H^1(\Omega)'}\big).$$

The assertion now follows by squaring this estimate, multiplying with τ^n, summing from 1 to N over n, and using the estimates (2.33) and (2.35). □

Using these energy estimates, we can proceed as in the semi-discrete case and can also prove a quasi-optimality result also for the fully-discrete scheme.

Theorem 2.23 (Quasi-optimality of the fully-discrete scheme, [EES18, Theorem 20]). *Let P_h be $H^1(\Omega)$-stable, furthermore let $(u,\phi)\in Q_T\times B_T$ and $(u_{h,\tau},\phi_{h,\tau})\in Q_T^{h,\tau}\times B_T^{h,\tau}$ denote the solutions of Problem 2.1 and Problem 2.16, respectively. Then*

$$\|u-u_{h,\tau}\|_{Q_T} + \|\phi-\phi_{h,\tau}\|_{B_T} \leq C\big(\|u-\tilde{u}_{h,\tau}\|_{Q_T} + \|\phi-\tilde{\phi}_{h,\tau}\|_{B_T}\big), \tag{2.38}$$

for all functions $\tilde{u}_{h,\tau}\in Q_T^{h,\tau}$ and $\tilde{\phi}_{h,\tau}\in B_T^{h,\tau}$, and a constant $C = C(\alpha_{stab},\Omega,C_P) > 0$ that is independent of the data f_B, $g_{\mathscr{K}}$, and the approximation spaces H^h and B^h.

Proof. The result follows with similar arguments as used in the proof of Theorem 2.14. Let $\tilde{u}_{h,\tau}\in Q_T^{h,\tau}$ and $\tilde{\phi}_{h,\tau}\in B_T^{h,\tau}$ be arbitrary. Then we split the error

$$\begin{aligned}\|u-u_{h,\tau}\|_{Q_T} &\leq \|u-\tilde{u}_{h,\tau}\|_{Q_T} + \|\tilde{u}_{h,\tau}-u_{h,\tau}\|_{Q_T},\\ \|\phi-\phi_{h,\tau}\|_{B_T} &\leq \|\phi-\tilde{\phi}_{h,\tau}\|_{B_T} + \|\tilde{\phi}_{h,\tau}-\phi_{h,\tau}\|_{B_T}.\end{aligned}$$

To estimate the discrete error, we recall the consistency of the fully-discrete scheme (2.27)–(2.28) with the variational problem (2.9)–(2.10), see Remark 2.19. Hence, the discrete error components $w_{h,\tau} := \tilde{u}_{h,\tau} - u_{h,\tau}$ and $\varphi_{h,\tau} := \tilde{\phi}_{h,\tau} - \phi_{h,\tau}$ fulfil the system

$$(\widehat{\partial_t w}_{h,\tau}^n, v_h)_\Omega + (\widehat{\nabla w}_{h,\tau}^n, \nabla v_h)_\Omega - (\widehat{\varphi}_{h,\tau}^n, v_h)_\Gamma = \langle \widehat{F}^n, v_h\rangle_\Omega, \tag{2.39}$$

$$((1/2-\mathscr{K})\widehat{w}_{h,\tau}^n, \psi_h)_\Gamma + (\mathscr{V}\widehat{\varphi}_{h,\tau}^n, \psi_h)_\Gamma = (\widehat{G}^n, \psi_h)_\Gamma, \tag{2.40}$$

for all $v_h\in H^h$, $\psi_h\in B^h$, and all $1\leq n\leq N$ with the averaged right-hand sides $\widehat{F}$ and $\widehat{G}$ obtained from

$$\begin{aligned}\langle F(t), v\rangle_\Omega &:= \langle \partial_t\tilde{u}_{h,\tau}(t) - \partial_t u(t), v\rangle_\Omega + (\nabla\tilde{u}_{h,\tau}(t) - \nabla u(t), \nabla v)_\Omega - \langle \tilde{\phi}_{h,\tau}(t) - \phi(t), v\rangle_\Gamma,\\ \langle G(t), \psi\rangle_\Gamma &:= \langle (1/2-\mathscr{K})(\tilde{u}_{h,\tau}(t) - u(t)), \psi\rangle_\Gamma + \langle \mathscr{V}(\tilde{\phi}_{h,\tau}(t) - \phi(t)), \psi\rangle_\Gamma,\end{aligned}$$

for all $v \in H$ and $\psi \in B$. Note that the system (2.39)–(2.40) has the same form as (2.27)–(2.28) with the right-hand sides $\widehat{F}^n$ and $\widehat{G}^n$

$$\langle \widehat{F}^n, v\rangle_\Omega := \langle \widehat{\partial_t \tilde{u}}^n_{h,\tau} - \widehat{\partial_t u}^n, v\rangle_\Omega + (\widehat{\nabla \tilde{u}}^n_{h,\tau} - \widehat{\nabla u}^n, \nabla v)_\Omega - \langle \widehat{\tilde{\phi}}^n_{h,\tau} - \widehat{\phi}^n, v\rangle_\Gamma,$$
$$\langle \widehat{G}^n, \psi\rangle_\Gamma := \langle (1/2 - \mathscr{K})(\widehat{\tilde{u}}^n_{h,\tau} - \widehat{u}^n), \psi\rangle_\Gamma + \langle \mathscr{V}(\widehat{\tilde{\phi}}^n_{h,\tau} - \widehat{\phi}^n), \psi\rangle_\Gamma.$$

Thus we can apply the energy estimate (2.36) of Lemma 2.22. The estimates

$$\|F\|_{H_T'} \le C\big(\|u - \tilde{u}_{h,\tau}\|_{Q_T} + \|\phi - \tilde{\phi}_{h,\tau}\|_{B_T}\big),$$
$$\|G\|_{B_T'} \le C\big(\|u - \tilde{u}_{h,\tau}\|_{H_T} + \|\phi - \tilde{\phi}_{h,\tau}\|_{B_T}\big),$$

and the error splitting complete the proof for (2.38). □

Remark 2.24. The time discretisation strategy can also be applied directly to the continuous variational problem (2.9)–(2.10). Let us denote by

$$Q_T^\tau = \big\{u \in Q_T \,:\, u|_{[t^{n-1},t^n]} \text{ is linear in } t\big\} \qquad \text{and}$$
$$B_T^\tau = \big\{\phi \in B_T \,:\, \phi|_{(t^{n-1},t^n]} \text{ is constant in } t\big\}$$

the corresponding function spaces and let $(u_\tau, \phi_\tau) \in Q_T^\tau \times B_T^\tau$ be the respective solution obtained by time discretisation of the continuous variational problem. The well-posedness of this time-discretised problem follows by simply setting $Q_T^{h,\tau} = Q_T^\tau$ and $B_T^{h,\tau} = B_T^\tau$ in the above results. As a consequence, we also obtain the quasi-optimal error bound

$$\|u - u_\tau\|_{Q_T} + \|\phi - \phi_\tau\|_{Q_T} \le C\big(\|u - \tilde{u}_\tau\|_{Q_T} + \|\phi - \tilde{\phi}_\tau\|_{B_T}\big),$$

for all $\tilde{u}_\tau \in Q_T^\tau$ and $\tilde{\phi}_\tau \in B_T^\tau$ with a constant C being independent of u, ϕ and the temporal grid. We do not need the L^2–projection to be H^1–stable for this result to hold true.

Remark 2.25. Explicit error bounds for the time discretisation of the continuous and the semi-discrete variational problem can also be obtained via the usual Taylor estimates under some regularity assumptions on the solution. As mentioned before, this is carried out in Section 3.2.4 for the classical Euler method. In the next section, we will use the properties of the L^2–projections to obtain linear convergence with respect to τ and independent of the spatial approximation.

Furthermore, other time discretisation schemes are possible here, e.g., choose $w^n(t) = 1$ in (2.26). Then the identities of Lemma 2.17 are

$$\widehat{u}^n_{h,\tau} = (u^n_{h,\tau} + u^{n-1}_{h,\tau})/2, \qquad \widehat{\partial_t u}^n_{h,\tau} = d_\tau u^n_{h,\tau} = \frac{1}{\tau^n}(u^n_{h,\tau} - u^{n-1}_{h,\tau}), \quad \text{and} \quad \widehat{\phi}^n_{h,\tau} = \phi^n_{h,\tau},$$

and the discrete system Problem 2.16 becomes a variant of the Crank-Nicolson time discretisation. The analysis, however, has to be changed to obtain the second-order convergence of the time stepping scheme.

2.5 FEM-BEM Coupling

In this section, we discuss a particular space discretisation with lowest order finite and boundary elements. Together with the time discretisation of the previous section, this yields a fully-discrete method which converges uniformly and exhibits order optimal convergence rates under minimal regularity assumptions on the solution. Note that one is not restricted to this choice of approximation spaces, it is merely an example to show practical convergence rates.

In the following, we assume that we have a triangulation $(\mathscr{T}, \mathscr{E}_\Gamma)$ of Ω and Γ as introduced in Section 1.3. For the Galerkin approximation in space, we will utilise the spaces also introduced in Section 1.3 and set them as $H^h = \mathscr{S}^1(\mathscr{T})$ and $B^h = \mathscr{P}^0(\mathscr{E}_\Gamma)$.

Remark 2.26. The assumptions on the triangulation also mean that Γ is a polygon. It is also possible to handle curved boundaries, but in this case we either have to approximate the boundary by a polygon or use curved finite/boundary elements. This is investigated further in Section 2.6.

For this choice of approximation spaces, we can state some approximation properties and the stability of the L^2–projection P_h and the $H^{-1/2}$–projection Π_h.

Lemma 2.27 ([EES18, Lemma 24])**.** *For the above choice of approximation spaces, the L^2–projection is stable in $H^1(\Omega)$, i.e. (2.21) is valid with a constant C_P independent of the mesh-size. Moreover, the operator P_h can be extended to a bounded linear operator on $H^1(\Omega)'$. Hence, for all $0 \le s \le 1$ and $0 \le s_e \le 3/2$ we have*

$$\begin{aligned}
\|u - P_h u\|_{H^1(\Omega)} &\le C h^s \|u\|_{H^{1+s}(\Omega)}, && u \in H^{1+s}(\Omega),\\
\|u - P_h u\|_{H^1(\Omega)'} &\le C h^s \|u\|_{H^{1-s}(\Omega)'}, && u \in H^{1-s}(\Omega)',\\
\|\phi - \Pi_h \phi\|_{H^{-1/2}(\Gamma)} &\le C h^{s_e} \|\phi\|_{H^{-1/2+s_e}(\mathscr{E}_\Gamma)}, && \phi \in H^{-1/2+s_e}(\mathscr{E}_\Gamma).
\end{aligned}$$

The constant $C = C(\eta) > 0$ is independent of the particular size of the triangulation.

Proof. The assertion about ϕ follows from [Ste08, Th. 10.4, (10.10)]. Validity of the stability (2.21) for these particular function spaces has been shown in [CES90] via an inverse inequality. Now we turn to the remaining estimates: Let $P_h^1 : H^1(\Omega) \to H^h$ be the H^1–orthogonal projection defined by

$$(P_h^1 u, v_h)_{H^1(\Omega)} = (u, v_h)_{H^1(\Omega)} \qquad \text{for all } v_h \in H^h,$$

and recall that $\|u - P_h^1 u\|_{H^1(\Omega)} \le C' h^s \|u\|_{H^{1+s}(\Omega)}$ for $0 \le s \le 1$; see, e.g., [BS08; Ste08]. Then

$$\begin{aligned}
\|u - P_h u\|_{H^1(\Omega)} &\le \|u - P_h P_h^1 u\|_{H^1(\Omega)} + \|P_h(u - P_h^1 u)\|_{H^1(\Omega)}\\
&\le (1 + C_P)\|u - P_h^1 u\|_{H^1(\Omega)} \le (1 + C_P) C' h^s \|u\|_{H^{1+s}(\Omega)},
\end{aligned}$$

where we used the projection property of P_h, the stability (2.21), and the approximation properties of P_h^1 in the last two steps. By definition of the dual norm, we further have for a function $u \in L^2(\Omega)$:

$$\begin{aligned}\|u - P_h u\|_{H^1(\Omega)'} &= \sup_{0 \neq v \in H^1(\Omega)} \frac{(u - P_h u, v)_\Omega}{\|v\|_{H^1(\Omega)}} \\ &= \sup_{0 \neq v \in H^1(\Omega)} \frac{(u, v - P_h v)_\Omega}{\|v\|_{H^1(\Omega)}} \leq Ch\|u\|_{L^2(\Omega)}.\end{aligned}$$

Here we used the standard estimate $\|v - P_h v\|_{L^2(\Omega)} \leq Ch\|v\|_{H^1(\Omega)}$ for the L^2–projection in the last step. With a similar duality argument and the stability (2.21), one can further see that $\|P_h u\|_{H^1(\Omega)'} \leq C_P \|u\|_{H^1(\Omega)'}$ for all functions in $L^2(\Omega)$. By density of $L^2(\Omega)$ in $H^1(\Omega)'$, we can extend P_h to a bounded linear operator on $H^1(\Omega)'$, and obtain for $u \in H^1(\Omega)'$

$$\|u - P_h u\|_{H^1(\Omega)'} \leq (1 + C_P)\|u\|_{H^1(\Omega)'}.$$

Noting that $L^2(\Omega) = H^0(\Omega) = H^0(\Omega)'$ and interpolating the two latter bounds now allows us to establish the second estimate for u, which completes the proof. □

Remark 2.28. Due to the results of [BPS01; BY14; GHS16], Lemma 2.27 also holds true on rather general shape-regular meshes under a mild growth condition on the local mesh-size. Hence the stability of the L^2–projection can still be guaranteed for adaptive mesh refinement. In this thesis, however, we will only look at uniform refinements (see also the examples in Chapter 4). With standard arguments, these estimates can also be generalised to polynomial approximations of higher order. All results that are presented below can thus be extended to such more general situations.

As a consequence of these approximation error bounds and the quasi-optimality of the semi-discretisation, we obtain the following quantitative error estimate.

Theorem 2.29. *Let (u, ϕ) and $(u_h, \phi_h) \in Q_T^h \times B_T^h$ denote the solutions of Problem 2.1 and Problem 2.9, respectively. Then*

$$\begin{aligned}&\|u - u_h\|_{Q_T} + \|\phi - \phi_h\|_{B_T} \\ &\quad \leq Ch^s\big(\|u\|_{L^2(0,T;H^{1+s}(\Omega))} + \|\partial_t u\|_{L^2(0,T;H^{1-s}(\Omega)')} + \|\phi\|_{L^2(0,T;H^{-1/2+s}(\mathscr{E}_\Gamma))}\big)\end{aligned}$$

for all $0 \leq s \leq 1$, $u(t) \in H^{1+s}(\Omega)$, $\partial_t u \in H^{1-s}(\Omega)'$, $\phi(t) \in H^{-1/2+s}(\mathscr{E}_\Gamma)$, and for a.e. $t \in [0, T]$ with $C = C(\alpha, \Omega, C_P, \eta) > 0$ that is independent of the data f_B, $g_{\mathscr{K}}$, and the approximation spaces H^h and B^h.

Proof. The result follows directly from Theorem 2.14 and Lemma 2.27. □

Remark 2.30. Let us further emphasise that the estimate of the theorem is optimal with respect to both the approximation properties of the spaces Q_T^h and B_T^h and the smoothness requirements on the solution. Furthermore, the method even converges without any smoothness assumptions on the solution, i.e., for all $u \in Q_T$ and $\phi \in B_T$.

For the full discretisation, we will also need the L^2–projection in time, i.e., operators $P^\tau : Q_T \to Q_T^\tau$ and $\Pi^\tau : B_T \to B_T^\tau$. These satisfy the following approximation properties, which can be proved with similar reasoning as for the spatial projections.

Lemma 2.31. *For $0 \le r \le 1$, the L^2–projections in time fulfil*

$$\begin{aligned}\|u - P^\tau u\|_{Q_T} &\le C\tau^r\big(\|\partial_t u\|_{H^r(0,T;H^1(\Omega)')} + \|u\|_{H^r(0,T;H^1(\Omega))}\big),\\ \|\phi - \Pi^\tau \phi\|_{B_T} &\le C\tau^r\|\phi\|_{H^r(0,T;H^{-1/2}(\Gamma))},\end{aligned}$$

with $u \in H^r(0,T;H^1(\Omega))$, $\partial_t u \in H^r(0,T;H^1(\Omega)')$, and $\phi \in H^r(0,T;H^{-1/2}(\Gamma))$.

Now we can put everything together and obtain the following convergence result for the fully-discrete FEM-BEM coupling. If we fulfil all the regularity requirements, we get first-order convergence in both, space and time. This is again optimal with respect to the needed regularity and the approximation properties of the spaces. If we do not meet the regularity requirements, we still obtain a convergence result, but with a reduced order of convergence.

Theorem 2.32 ([EES18, Theorem 28])**.** *Let (u,ϕ) and $(u_{h,\tau},\phi_{h,\tau}) \in Q_T^{h,\tau} \times B_T^{h,\tau}$ denote the solutions of Problem 2.1 and Problem 2.16, respectively. Then there holds that*

$$\begin{aligned}&\|u - u_{h,\tau}\|_{Q_T} + \|\phi - \phi_{h,\tau}\|_{B_T}\\ &\quad\le C_1 h^s\big(\|u\|_{L^2(0,T;H^{1+s}(\Omega))} + \|\partial_t u\|_{L^2(0,T;H^{1-s}(\Omega)')} + \|\phi\|_{L^2(0,T;H^{-1/2+s}(\mathscr{E}_\Gamma))}\big)\\ &\qquad\quad + C_2\tau^r\big(\|\partial_t u\|_{H^r(0,T;H^1(\Omega)')} + \|u\|_{H^r(0,T;H^1(\Omega))} + \|\phi\|_{H^r(0,T;H^{-1/2}(\Gamma))}\big),\end{aligned}$$

for all $0 \le s \le 1$ and for all $0 \le r \le 1$ with $u \in H^r(0,T;H^1(\Omega)) \cap L^2(0,T;H^{1+s}(\Omega))$, $\partial_t u \in H^r(0,T;H^1(\Omega)') \cap L^2(0,T;H^{1-s}(\Omega)')$, and $\phi \in H^r(0,T;H^{-1/2}(\mathscr{E}_\Gamma)) \cap L^2(0,T;H^{-1/2+s}(\mathscr{E}_\Gamma))$ with $C_1 = C_1(\alpha,\Omega,\eta,C_P) > 0$ and $C_2 = C_2(\alpha,\Omega,C_P) > 0$ that are independent of the data f_B, $g_{\mathscr{K}}$, and the approximation spaces H^h and B^h.

Proof. By the triangle inequality, we obtain

$$\begin{aligned}\|u - P^\tau P_h u\|_{Q_T} &\le \|u - P_h u\|_{Q_T} + \|P_h u - P^\tau P_h u\|_{Q_T},\\ \|\phi - \Pi^\tau \Pi_h \phi\|_{B_T} &\le \|\phi - \Pi_h \phi\|_{B_T} + \|\Pi_h \phi - \Pi^\tau \Pi_h \phi\|_{B_T}.\end{aligned}$$

Since the projection operators commute (as they operate in different domains), we can change their order in the second term in each line. Then we use the stability of the spatial projection operators guaranteed by Lemma 2.27 and the approximation properties of the time projections P^τ. We obtain

$$\begin{aligned}\|P_h u - P^\tau P_h u\|_{Q_T} &\le C\tau^r\big(\|\partial_t u\|_{H^r(0,T;H^1(\Omega)')} + \|u\|_{H^r(0,T;H^1(\Omega))}\big),\\ \|\Pi_h \phi - \Pi^\tau \Pi_h \phi\|_{B_T} &\le C\tau^r\|\phi\|_{H^r(0,T;H^{-1/2}(\Gamma))}.\end{aligned}$$

Now we apply Theorem 2.23 with $\tilde{u}_{h,\tau} = P^\tau P_h u$ and $\tilde{\phi}_{h,\tau} = \Pi^\tau \Pi_h \phi$. The estimates from Lemma 2.27 for the approximation errors lead to the assertion. □

Remark 2.33. From the previous result, we also obtain a corresponding estimate

$$\begin{aligned}&\|u - u_\tau\|_{Q_T} + \|\phi - \phi_\tau\|_{B_T}\\ &\quad\le C\tau^r\big(\|\partial_t u\|_{H^r(0,T;H^1(\Omega)')} + \|u\|_{H^r(0,T;H^1(\Omega))} + \|\phi\|_{H^r(0,T;H^{-1/2}(\Gamma))}\big)\end{aligned}$$

with $C = C(\Omega) > 0$ for the approximation (u_τ, ϕ_τ), obtained by the time discretisation scheme without additional Galerkin approximation in space. The proof of this result simply follows by setting $Q_T^h = Q_T$, $B_T^h = B_T$ and $Q_T^{h,\tau} = Q_T^\tau$, $B_T^{h,\tau} = B_T^\tau$ in the previous theorem. Note that we do not need the assumptions on the space triangulation in this case.

To summarise: In this chapter we introduced a non-symmetric coupling of FEM and BEM for parabolic-elliptic interface problems. We could show that this coupling, paired with a variant of the implicit Euler method, is well-posed and yields a quasi-optimal discretisation of our model problem.

2.6 Excursion: Approximation of a Smooth Domain

Until now, we always assumed that the boundary is polygonal. Usually, this is sufficient, especially if we take into account that most state-of-the-art modelling software used in the engineering sciences yields polygonal meshes anyway. Still, if we think about the real world, we often encounter things that have a curved boundary. For this reason, we pose the question of what happens when our approximation has to handle a curved boundary in this section. An early version of the results of this section has already been published in [ES19a]. We note that currently a new trend, isogeometric methods, is replacing such approximation methods. Thus, this section merely gives one idea how to handle curved boundaries.

2.6.1 A Simple Domain Approximation for Domains with Smooth Boundaries

The main idea of this section is to approximate the domain with a curved boundary by a polygonal triangulation. Then we are essentially in the same setting as before, but we introduced an additional approximation error which has to be analysed consequently. Unfortunately, the analysis used before cannot be applied directly, but with some modifications, we can still obtain a quasi-optimality result for the semi-discretisation and a comparable result for the full discretisation. To obtain this, we consider the approximating bilinear form to be a perturbation of the original bilinear form and analyse the order of the error. This type of analysis is standard in the field of domain approximation and the results used here can be found in [Ber89; ER13] for the finite element part and [JN80; LR77] for the boundary element and coupling part. The same type of analysis will be used for the coupling of FVM and BEM in the next chapter. Because this case is more important to us, we will only present shortened proofs in this section.

Another approach, which we will not investigate further here, is considered in [Gon06]. There, a symmetric FEM-BEM coupling in two dimensions for a time-dependent problem with general Lipschitz boundaries is analysed. The technique consists of introducing an artificial smooth boundary with a parametrisation that is used to formulate the weak form and afterwards discretise using curved finite elements, introduced for the FEM-BEM coupling for a stationary problem in [MGP00].

Now we assume that the boundary Γ of Ω is in $\mathscr{C}^3$. We also assume that the jumps fulfil $g_1 = g_2 = 0$ to simplify the notation. Let Ω_h be a polygonal approximation of Ω with a quasi-

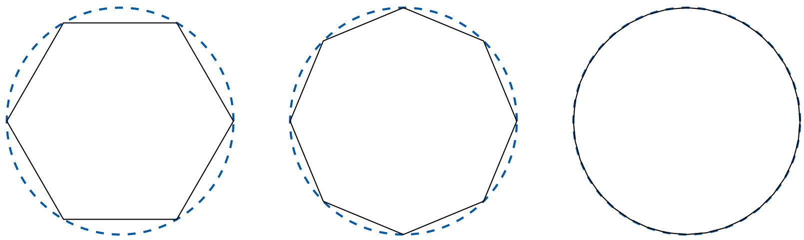


Figure 2.2: A depiction of a polygonal approximation of a smooth domain (in this case a circle, drawn with a dashed line) with 6, 8 and 36 vertices.

uniform triangulation $\mathscr{T} = \{K\}$ such that the vertices of $\Gamma_h = \partial\Omega_h$ also lie on Γ. The induced boundary triangulation will be denoted by $\mathscr{E}_\Gamma = \{E\}$. Such a polygonal approximation is depicted in Figure 2.2. Furthermore, we will need an exact triangulation $\widetilde{\mathscr{T}} = \{\widetilde{K}\}$ of Ω consisting additionally of curved triangles such that $\overline{\Omega} = \bigcup_{\widetilde{K}\in\widetilde{\mathscr{T}}} \widetilde{K}$. Note that the notation introduced (or altered) in this section will not be used in the rest of the document anymore.

Then we can define several mappings that transform the elements of the triangulation. There exist $A_K \in \mathbb{R}^{2\times 2}$ and $b_K \in \mathbb{R}^2$, such that

$$F_K : \hat{K} \to K, \quad F_K(\hat{x}) = A_K\hat{x} + b_K,$$

transforms the reference triangle $\hat{K}$ to a triangle K. The transformation $\widetilde{F}_K := F_K + \Phi_K$ maps the reference triangle onto a curved triangle $\widetilde{K}$, where $\Phi_K \in \mathscr{C}^3(\hat{K}, \mathbb{R}^2)$, a construction can be found in [Ber89, Section 6.1]. By composing the two functions we can define a mapping from $K \in \mathscr{T}$ to $\widetilde{K} \in \widetilde{\mathscr{T}}$ by:

$$G_h : \Omega_h \to \Omega, \quad G_h|_K := \widetilde{F}_K \circ F_K^{-1},$$

with $G_h(x) = x$ for all $x \in K$ with $G_h(K) \cap \Gamma = \emptyset$ and Jacobi matrix DG_h. Do not confuse this with G, Green's function of the Laplacian. The mapping has the property that for all $K \in \mathscr{T}$ (see also [ER13, Proposition 4.7]):

$$\|\det(DG_h) - 1\|_{L^\infty(K)} \le Ch_K, \tag{2.41}$$

with $h_K = \operatorname{diam}(K)$ (and $h = \max_{K\in\mathscr{T}} h_K$). The constant C is independent of h_K.

By setting $g_h := G_h|_{\Gamma_h}$, we obtain $g_h : \Gamma_h \to \Gamma$. For the determinants, we introduce abbreviations adopted from [ER13]:

$$J_h := \det(DG_h), \quad \mu_h := \det(Dg_h).$$

2.6.2 Analysis of the Semi-Discrete System

For the discretisation in space, we will use a Galerkin ansatz with the space of continuous piecewise linear functions $\mathscr{S}^1(\mathscr{T})$ and the space of piecewise constants $\mathscr{P}^0(\mathscr{E}_\Gamma)$ as in the previous section. Then our semi-discrete problem reads as follows.

Problem 2.34 (Semi-discrete problem). Find $u_h \in Q_T^h$ and $\phi_h \in B_T^h$ such that

$$(\partial_t u_h(t), v_h)_{\Omega_h} + \mathscr{b}_h((u_h(t), \phi_h(t)); (v_h, \psi_h)) = (f_h(t), v_h)_{\Omega_h}, \tag{2.42}$$

for all $\mathbf{v} = (v_h, \psi_h) \in \mathscr{S}^1(\mathscr{T}) \times \mathscr{P}^0(\mathscr{E}_\Gamma)$ and a.e. $t \in [0, T]$. The function f_h is defined by $f_h = (f \circ G_h) \cdot J_h$ and $\mathscr{b}_h$ is defined, similar to the FEM-BEM coupling, by

$$\begin{aligned}\mathscr{b}_h((u_h(t), \phi_h(t)); (v_h, \psi_h)) =& (\nabla u_h(t), \nabla v_h)_{\Omega_h} - (\phi_h(t), v_h)_{\Gamma_h} \\ &+ ((1/2 - \mathscr{K}) u_h(t), \psi_h)_{\Gamma_h} + (\mathscr{V} \phi_h(t), \psi_h)_{\Gamma_h}.\end{aligned}$$

Remark 2.35. This bilinear form is a perturbed version of the original bilinear form. Such a perturbation could be analysed with the so-called Strang Lemma, but because of several special terms, the analysis is done explicitly.

In Section 2.2 we introduced a stabilised equivalent bilinear form that is coercive (see Lemma 2.4) and also noted that there are various ways to obtain a coercive bilinear form. Therefore we will make a slight abuse of notation at this point and assume that the bilinear form $\mathscr{b}_h$ is coercive to make the forthcoming proofs more readable. From now on we will use the abbreviations $\mathscr{H} := H^1(\Omega) \times H^{-1/2}(\Gamma)$, $\mathscr{H}_T := L^2(0, T; H^1(\Omega)) \times L^2(0, T; H^{-1/2}(\Gamma))$ and for the discretisation spaces $\mathscr{H}^h := \mathscr{S}^1(\mathscr{T}) \times \mathscr{P}^0(\mathscr{E}_\Gamma)$ and $\mathscr{H}_T^h := Q_T^h \times B_T^h$.

To be able to compare Problem 2.1 and Problem 2.34, we introduce the notion of so-called *lifted functions*:

$$\widetilde{v}_h : \Omega \to \mathbb{R}^2, \quad \widetilde{v}_h := v_h \circ G_h^{-1} \quad \text{for } v_h : \Omega_h \to \mathbb{R}^2, \tag{2.43}$$

$$\widetilde{\psi}_h : \Gamma \to \mathbb{R}, \quad \widetilde{\psi}_h := \psi_h \circ g_h^{-1} \quad \text{for } \psi_h : \Gamma_h \to \mathbb{R}. \tag{2.44}$$

By using the change of variables formula and the definition of f_h, we can see that

$$(f_h, v_h)_{\Omega_h} = (f, \widetilde{v}_h)_\Omega,$$

for all $v_h \in \mathscr{S}^1(\mathscr{T})$. That means that the altered right-hand side of (2.42) is *consistent* with the weak formulation Problem 2.1.

To analyse the error, we transfer our discrete system to Ω. Firstly, we reformulate the term with the time derivative

$$(\partial_t u_h, v_h)_{\Omega_h} = (\frac{1}{J_h} \partial_t \widetilde{u}_h, \widetilde{v}_h)_\Omega.$$

To reformulate the bilinear form, we use the identity:

$$\nabla(\widetilde{w}_h) = \nabla(w_h \circ G_h^{-1}) = DG_h^{-T}(\widetilde{\nabla w_h}) = DG_h^{-T}(\nabla w_h) \circ G_h^{-1},$$

where $DG_h^{-T}=(DG_h^{-1})^T$. So we obtain for $(u_h,\phi_h)\in\mathscr{H}_T^h$ and $(v_h,\psi_h)\in\mathscr{H}^h$:

$$\begin{aligned}
\mathscr{b}_h((u_h,\phi_h);(v_h,\psi_h)) &= \int_{\Omega_h}\nabla u_h\nabla v_h\,\mathrm{d}x-\int_{\Gamma_h}\phi_h v_h\,\mathrm{d}s\\
&\quad+\frac{1}{2}\int_{\Gamma_h}u_h\psi_h\,\mathrm{d}s-\int_{\Gamma_h}\int_{\Gamma_h}\partial_n G u_h\,\mathrm{d}s_y\psi_h\,\mathrm{d}s_x\\
&\quad+\int_{\Gamma_h}\int_{\Gamma_h}G\phi_h\,\mathrm{d}s_y\psi_h\,\mathrm{d}s_x\\
&=\int_{\Omega}DG_h^T\nabla\widetilde{u}_h DG_h^T\nabla\widetilde{v}_h\frac{1}{J_h}\,\mathrm{d}x-\int_{\Gamma}\widetilde{\phi}_h\widetilde{v}_h\frac{1}{\mu_h}\,\mathrm{d}s\\
&\quad+\frac{1}{2}\int_{\Gamma}\widetilde{u}_h\widetilde{\psi}_h\frac{1}{\mu_h}\,\mathrm{d}s-\int_{\Gamma}\int_{\Gamma}\partial_n\widetilde{G}\widetilde{u}_h\frac{1}{\mu_h}\,\mathrm{d}s_y\widetilde{\psi}_h\frac{1}{\mu_h}\,\mathrm{d}s_x\\
&\quad+\int_{\Gamma}\int_{\Gamma}\widetilde{G}\widetilde{\phi}_h\frac{1}{\mu_h}\,\mathrm{d}s_y\widetilde{\psi}_h\frac{1}{\mu_h}\,\mathrm{d}s_x=:\widetilde{\mathscr{b}}_h((\widetilde{u}_h,\widetilde{\phi}_h);(\widetilde{v}_h,\widetilde{\psi}_h)),
\end{aligned}$$

with $\widetilde{G}(x,y)=G(g_h^{-1}(x),g_h^{-1}(y))$.

Using the results from [ER13, Lemma 6.2] and [JN80, Example 1] (which is based on [LR77, Proposition 3.2]), we obtain:

Lemma 2.36 (Error in perturbed bilinear form). *Let $(u_h,\phi_h)\in\mathscr{S}^1(\mathscr{T})\times\mathscr{P}^0(\mathscr{E}_\Gamma)$ and $(v_h,\psi_h)\in\mathscr{S}^1(\mathscr{T})\times\mathscr{P}^0(\mathscr{E}_\Gamma)$. Then*

$$|\mathscr{b}((\widetilde{u}_h,\widetilde{\phi}_h);(\widetilde{v}_h,\widetilde{\psi}_h))-\widetilde{\mathscr{b}}_h((\widetilde{u}_h,\widetilde{\phi}_h);(\widetilde{v}_h,\widetilde{\psi}_h))|\le Ch\|(\widetilde{u}_h,\widetilde{\phi}_h)\|_{\mathscr{H}}\|(\widetilde{v}_h,\widetilde{\psi}_h)\|_{\mathscr{H}},$$

with a constant $C=C(\Omega)>0$ independent of h.

Finally, we can show the convergence of the semi-discrete scheme. The proof is a variation of the proof of the convergence of the FVM-BEM scheme, which will be presented in the next chapter.

Theorem 2.37 (Quasi-optimality of the semi-discrete scheme). *Let $(u,\phi)\in Q_T\times B_T$ be the solution to Problem 2.1 and $(u_h,v_h)\in Q_T^h\times B_T^h$ the solution of Problem 2.34. Then we get*

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathscr{H}_T(\Omega_h\times\Gamma_h)}\le\left(\|\partial_t u-\partial_t\widetilde{v}_h\|_{H'_T}+\|\mathbf{u}-\widetilde{\mathbf{v}}_h\|_{\mathscr{H}_T}+h\|\partial_t u\|_{H'_T}+h\|\widetilde{\mathbf{v}}_h\|_{\mathscr{H}_T}\right),$$

for all $\mathbf{v}_h=(v_h,\psi_h)\in\mathscr{H}_T^h$ with $C=C(\alpha_{stab},\Omega)>0$.

Proof. Let $\mathbf{v}_h=(v_h,\psi_h)\in\mathscr{H}_T^h$ be arbitrary. First, we split the error into an approximation error and a discrete error component, where we write $\mathscr{H}(\Omega_h\times\Gamma_h)$ for $H^1(\Omega_h)\times H^{-1/2}(\Gamma_h)$;

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathscr{H}(\Omega_h\times\Gamma_h)}\le\|\mathbf{u}-\mathbf{v}_h\|_{\mathscr{H}(\Omega_h\times\Gamma_h)}+\|\mathbf{u}_h-\mathbf{v}_h\|_{\mathscr{H}(\Omega_h\times\Gamma_h)}. \tag{2.45}$$

Hence, we only have to estimate the norms of the discrete error $\mathbf{w}_h=(w_h,\varphi_h):=\mathbf{u}_h-\mathbf{v}_h\in\mathscr{H}_T^h$. First, we need to introduce $\|w\|_{J_h}=(\frac{1}{J_h}w,w)_\Omega^{1/2}$. Then we can use $(\frac{1}{J_h}\partial_t v,v)_\Omega=$

$\frac{1}{2}\partial_t\|v\|^2_{J_h}$, the coercivity of $\widetilde{b}_h$ and plugging in $\mathbf{w}_h$ as trial and test functions into (2.42) to obtain:

$$\frac{1}{2}\partial_t\|\widetilde{w}_h\|^2_{J_h} + \|\mathbf{w}_h\|^2_{\mathcal{H}(\Omega_h\times\Gamma_h)} \lesssim \langle\frac{1}{J_h}\partial_t\widetilde{w}_h, \widetilde{w}_h\rangle_\Omega + \widetilde{b}_h(\widetilde{\mathbf{w}}_h;\widetilde{\mathbf{w}}_h).$$

Now we continue with using the semi-discrete formulation and adding the weak form, resulting in

$$\begin{aligned}\frac{1}{2}\partial_t\|\widetilde{w}_h\|^2_{J_h} + \|\mathbf{w}_h\|^2_{\mathcal{H}(\Omega_h\times\Gamma_h)} &\le \langle\frac{1}{J_h}\partial_t\widetilde{u}_h, \widetilde{w}_h\rangle_\Omega - \langle\frac{1}{J_h}\partial_t\widetilde{v}_h, \widetilde{w}_h\rangle_\Omega \\ &\quad + \widetilde{b}_h(\widetilde{\mathbf{u}}_h;\widetilde{\mathbf{w}}_h) - \widetilde{b}_h(\widetilde{\mathbf{v}}_h;\widetilde{\mathbf{w}}_h) \\ &= \langle\partial_t u, \widetilde{w}_h\rangle_\Omega - \langle\frac{1}{J_h}\partial_t\widetilde{v}_h, \widetilde{w}_h\rangle_\Omega + \langle f, \widetilde{w}_h\rangle_\Omega \\ &\quad + b(\widetilde{\mathbf{v}}_h;\widetilde{\mathbf{w}}_h) - \widetilde{b}_h(\widetilde{\mathbf{v}}_h;\widetilde{\mathbf{w}}_h) + b(\mathbf{u}-\widetilde{\mathbf{v}}_h;\widetilde{\mathbf{w}}_h) - \langle f, \widetilde{w}_h\rangle_\Omega.\end{aligned}$$

The terms concerning the time derivative can further be estimated by

$$\begin{aligned}&\langle\partial_t u, \widetilde{w}_h\rangle_\Omega - \langle\frac{1}{J_h}\partial_t\widetilde{v}_h, \widetilde{w}_h\rangle_\Omega \\ &\qquad = \langle\partial_t u - \frac{1}{J_h}\partial_t u, \widetilde{w}_h\rangle_\Omega + \langle\frac{1}{J_h}(\partial_t u - \partial_t\widetilde{v}_h), \widetilde{w}_h\rangle_\Omega \\ &\qquad \le Ch\|\partial_t u\|_{H^1(\Omega)'}\|\widetilde{w}_h\|_{H^1(\Omega)} + \|\frac{1}{J_h}\|_{L^\infty(\Omega)}\|\partial_t u - \partial_t\widetilde{v}_h\|_{H^1(\Omega)'}\|\widetilde{w}_h\|_{H^1(\Omega)}.\end{aligned}$$

Because $\frac{1}{J_h}$ is bounded in $L^\infty(\Omega)$, we can treat it as a constant.

The other terms can be bounded by Lemma 2.36 and the continuity of the bilinear form b. All in all, we arrive at

$$\begin{aligned}&\frac{1}{2}\partial_t\|w_h\|^2_{J_h} + \|\mathbf{w}_h\|^2_{\mathcal{H}(\Omega_h\times\Gamma_h)} \\ &\qquad \le C\left(\|\partial_t u - \partial_t\widetilde{v}_h\|_{H^1(\Omega)'} + h\|\partial_t u\|_{H^1(\Omega)'} + h\|\widetilde{\mathbf{v}}_h\|_{\mathcal{H}} + \|\mathbf{u}-\widetilde{\mathbf{v}}_h\|_{\mathcal{H}}\right)\|\widetilde{\mathbf{w}}_h\|_{\mathcal{H}}.\end{aligned}$$

Using Young's inequality, the error splitting and integrating over t from 0 to T lead to the final estimate. □

With the usual approximation results (for example via the L^2–projection, see Lemma 2.27), we obtain the following convergence result.

Corollary 2.38 (First-order convergence of the semi-discrete scheme). *Let $(u,\phi) \in Q_T \times B_T$ be the solution to Problem 2.1 and $(u_h, v_h) \in \mathcal{H}^h_T$ the solution of Problem 2.34. Then we get*

$$\begin{aligned}&\|u_h - u\|_{L^2(0,T;H^1(\Omega_h))} + \|\phi_h - \phi\|_{L^2(0,T;H^{-1/2}(\Gamma_h))} \\ &\qquad \le C\left[h^s\left(\|u\|_{L^2(0,T;H^{1+s}(\Omega))} + \|\partial_t u\|_{L^2(0,T;H^{1-s}\Omega)')} + \|\phi\|_{L^2(0,T;H^{s-1/2}(\Gamma))}\right)\right],\end{aligned}$$

for $C = C(\alpha_{stab}, \Omega, \eta) > 0$ and all $0 \le s \le 1$, $u(t) \in H^{1+s}(\Omega)$, $\partial_t u(t) \in H^{1-s}(\Omega)'$, $\phi(t) \in H^{-1/2+s}(\Gamma)$, and for a.e. $t \in [0,T]$.

2.6.3 Analysis of the Fully-Discrete System

For the full discretisation, we restrict ourselves to the implicit Euler method, although the variation introduced in Section 2.4 is also feasible.

We use the same division of the time interval as introduced there with the local time step $\tau^n = t^n - t^{n-1}$ and recall the definition of the discrete derivative $d_\tau u^{n+1} := (u^{n+1} - u^n)/\tau^n$.

Then the system reads:

Problem 2.39 (Fully-discrete formulation). Find $u_h^n \in \mathcal{S}^1(\mathcal{T})$, $\phi_h^n \in \mathcal{P}^0(\mathcal{E}_\Gamma)$ for $n = 0, \dots, N$, such that (u_h^n, ϕ_h^n) fulfil for all $(v_h, \psi_h) \in \mathcal{S}^1(\mathcal{T}) \times \mathcal{P}^0(\mathcal{E}_\Gamma)$

$$(d_\tau u_h^n, v_h)_{\Omega_h} + b_h\left((u_h^n, \phi_h^n); (v_h, \psi_h)\right) = (f_h^n, v_h)_{\Omega_h}, \tag{2.46}$$

for $1 \le n \le N$ and $u_h^0 = 0$, where $f_h^n(x) := f_h(x, t^n)$.

The convergence of the fully-discrete scheme now follows with standard techniques (again similar to the analysis of the FVM-BEM coupling).

Theorem 2.40 (Convergence of the fully-discrete scheme). *Let $(u, \phi) \in Q_T \times B_T$ be a sufficiently smooth solution to Problem 2.1 and let $(u_h^n, \phi_h^n) \in \mathcal{H}^h$ be the solution of Problem 2.39. Then we obtain*

$$\begin{aligned}
&\left(\sum_{n=1}^{N} \tau^n \left(\|u_h^n - u(t^n)\|^2_{H^1(\Omega_h)} + \|\phi_h^n - \phi(t^n)\|^2_{H^{-1/2}(\Gamma_h)}\right)\right)^{1/2} \\
&\qquad \le C\left(\|\partial_t u - \partial_t \widetilde{v}_h\|_{H_T'} + \|\mathbf{u} - \widetilde{\mathbf{v}}_h\|_{\mathcal{H}_T} + h(\|\partial_t \widetilde{v}_h\|_{H_T'}) + h\|\widetilde{\mathbf{v}}_h\|_{\mathcal{H}_T})\right. \\
&\qquad\quad \left. + \tau(\|\partial_{tt} u\|_{H_T'} + \|\partial_t \mathbf{u} - \widetilde{\partial}_t \mathbf{v}_h\|_{\mathcal{H}_T})\right),
\end{aligned}$$

for all $\mathbf{v}_h = (v_h, \psi_h) \in \mathcal{H}_T^h$ with the constant $C = C(\alpha_{stab}, \Omega) > 0$ being independent of h.

Proof. As in the semi-discrete case, we use an error splitting, i.e., let $\mathbf{v}_h = (v_h, \psi_h) \in H^1(0, T; \mathcal{S}^1(\mathcal{T})) \times H^1(0, T; \mathcal{P}^0(\mathcal{E}_\Gamma))$ be arbitrary with $v_h(0) = 0$, then we split for $n = 1, \dots, N$:

$$\|\mathbf{u}(t^n) - \mathbf{u}_h^n\|_{\mathcal{H}(\Omega_h \times \Gamma_h)} \le \|\mathbf{u}(t^n) - \mathbf{v}_h^n\|_{\mathcal{H}(\Omega_h \times \Gamma_h)} + \|\mathbf{u}_h^n - \mathbf{v}_h^n\|_{\mathcal{H}(\Omega_h \times \Gamma_h)}. \tag{2.47}$$

Now we estimate the discrete error and define $\mathbf{w}_h^n := (w_h^n, \varphi_h^n) := \mathbf{u}_h^n - \mathbf{v}_h^n \in \mathcal{H}^h$. With this we can retrace the steps of the proof of Theorem 2.37, use

$$\left(\frac{1}{J_h} d_\tau \widetilde{w}_h^n, \widetilde{w}_h^n\right)_\Omega \ge \frac{1}{2\tau^n}(\|\widetilde{w}_h^n\|^2_{J_h} - \|\widetilde{w}_h^{n-1}\|^2_{J_h}),$$

and arrive at

$$\begin{aligned}
&\frac{1}{2\tau^n}\left(\|\widetilde{w}_h^n\|^2_{J_h} - \|\widetilde{w}_h^{n-1}\|^2_{J_h}\right) + \|\mathbf{w}_h^n\|^2_{\mathcal{H}(\Omega_h \times \Gamma_h)} \\
&\qquad \lesssim \langle \partial_t u(t^n), \widetilde{w}_h^n\rangle_\Omega - \langle \frac{1}{J_h} d_\tau \widetilde{v}_h^n, \widetilde{w}_h^n\rangle_\Omega \\
&\qquad\quad + b(\widetilde{\mathbf{v}}_h^n; \widetilde{\mathbf{w}}_h^n) - \widetilde{b}_h(\widetilde{\mathbf{v}}_h^n; \widetilde{\mathbf{w}}_h^n) + b(\mathbf{u}(t^n) - \widetilde{\mathbf{v}}_h^n; \widetilde{\mathbf{w}}_h^n).
\end{aligned}$$

The terms with the time derivative can further be bounded by using the Cauchy-Schwarz inequality

$$
\begin{aligned}
&\langle \partial_t u(t^n), \widetilde{w}_h^n \rangle_\Omega - \langle \frac{1}{J_h} d_\tau \widetilde{v}_h^n, \widetilde{w}_h^n \rangle_\Omega \\
&= \langle \partial_t u(t^n) - d_\tau u(t^n), \widetilde{w}_h^n \rangle_\Omega + \langle \partial_t u(t^n) - d_\tau \widetilde{v}_h^n, \widetilde{w}_h^n \rangle_\Omega + \langle d_\tau \widetilde{v}_h^n - \frac{1}{J_h} d_\tau \widetilde{v}_h^n, \widetilde{w}_h^n \rangle_\Omega \\
&\lesssim \Big(\|\partial_t u(t^n) - d_\tau u(t^n)\|_{H^1(\Omega)'} + \|\partial_t u(t^n) - d_\tau \widetilde{v}_h^n\|_{H^1(\Omega)'} + \|(1 - \frac{1}{J_h}) d_\tau \widetilde{v}_h^n\|_{H^1(\Omega)'} \Big) \|\widetilde{w}_h^n\|_{H^1(\Omega)}.
\end{aligned}
$$

The other terms can be bounded as before, using Lemma 2.36, (2.41) and the continuity of the bilinear form $\mathscr{b}$. Then, multiplying by τ^n and summing over n from 1 to N, we get

$$
\begin{aligned}
&\frac{1}{2}\|\widetilde{w}_h^N\|_{J_h}^2 + \sum_{n=1}^N \tau^n \|\mathbf{w}_h^n\|^2_{\mathscr{H}(\Omega_h \times \Gamma_h)} \\
&\lesssim \sum_{n=1}^N \tau^n \Big(\|\partial_t u(t^n) - d_\tau u(t^n)\|^2_{H^1(\Omega)'} + \|\partial_t u(t^n) - d_\tau \widetilde{v}_h^n\|^2_{H^1(\Omega)'} \\
&\qquad + h^2 \|d_\tau \widetilde{v}_h^n\|^2_{H^1(\Omega)'} + h^2 \|\widetilde{v}_h^n\|^2_{H^1(\Omega)} + \|\mathbf{u}(t^n) - \widetilde{\mathbf{v}}_h^n\|^2_{\mathscr{H}} \Big).
\end{aligned}
$$

With classical Taylor series, i.e., with the integral form of the remainder, we estimate

$$
\begin{aligned}
\sum_{n=1}^N \tau^n \|\partial_t u(t^n) - d_\tau u(t^n)\|^2_{H^1(\Omega)'} &\le \tau^2 \|\partial_{tt} u\|^2_{H_T'}, \\
\sum_{n=1}^N \tau^n \|d_\tau u(t^n) - d_\tau \widetilde{v}_h^n\|^2_{H^1(\Omega)'} &\le \|\partial_t u - \partial_t \widetilde{v}_h\|^2_{H_T'}, \\
\sum_{n=1}^N \tau^n h^2 \|d_\tau \widetilde{v}_h^n\|^2_{H^1(\Omega)'} &\le h^2 \|\partial_t v_h\|^2_{H_T'}.
\end{aligned}
$$

For all the other terms we use Lemma 1.13. Then we use the error splitting (2.47) to prove the assertion. □

Here we can also use the usual approximation results to obtain first-order convergence.

Corollary 2.41 (First-order convergence of the fully-discrete scheme). *Let $(u, \phi) \in Q_T \times B_T$ be the solution to Problem 2.1 and $(u_h, v_h) \in \mathscr{H}^h$ the solution of Problem 2.34. Then we get*

$$
\Big[\sum_{n=1}^N \tau^n \Big(\|u(t^n) - u_h^n\|^2_{H^1(\Omega_h)} + \|\phi(t^n) - \phi_h^n\|^2_{H^{-1/2}(\Gamma_h)} \Big) \Big]^{1/2} = \mathscr{O}(h + \tau),
$$

for all $0 \le s \le 1$, $u \in H^1(0,T; H^2(\Omega))$, $\partial_t u \in H^1(0,T; L^2(\Omega))$, $\partial_{tt} u \in L^2(0,T; H^1(\Omega)')$, and $\phi \in H^1(0,T; H^{1/2}(\Gamma))$.

These results show that the boundary approximation does not affect the first-order convergence of the scheme (because the order of the approximation is also of first-order). Thus we can also handle domains with non-polygonal boundaries.

3

The Convection-Dominated Case

To treat more complex problems, we can extend our model problem to diffusion-convection-reaction problems by adding a first-order term (the convection) and a zero-order term (the reaction). Furthermore, we assume that the problem is convection-dominated, which means that the convection coefficient is somewhat larger (e.g. its norm is larger) than the diffusion coefficient. This extended model problem can be seen as the time-dependent prototype of transport and flow of a substance in a porous medium coupled to a diffusion process in a surrounding unbounded domain. A derivation and motivation of the model problem can be found in [Era12].

Convection-dominated problems pose new challenges to the solving method as the big difference in the scales of the coefficients can lead to unwanted (spurious) oscillations in the computed solution on meshes of practicable size. This happens when we use, for example, the Finite Element Method which we used in the previous chapter. As this is also the case for coupled parabolic-elliptic problems, we have to face the same challenges and use a method that produces stable solutions for convection-dominated problems. For this reason, we will look at both the Finite Volume Method with upwind stabilisation (FVM) as well as at the Streamline Upwind Petrov Galerkin method (SUPG). We will investigate the well-posedness of the schemes, as well as the convergence behaviour. For the full discretisation we will use the variant of the implicit Euler we used for the FEM-BEM coupling as well in Section 2.4. For the FVM we also look at the classical implicit Euler method.

3.1 An Extended Model Problem and Variational Formulation

First, we extend the model problem introduced in Section 2.1. Namely, we introduce a matrix-valued coefficient for the diffusion and convection and reaction parts.

The domain itself remains unchanged, but we additionally need the (known) *model parameters*: a symmetric diffusion matrix $\mathbf{A}$, a possibly dominating velocity field $\mathbf{b}$, and a reaction coefficient c. Furthermore, the coupling boundary Γ is divided in an inflow and outflow part, namely $\Gamma^{in} := \{x \in \Gamma : \mathbf{b}(x) \cdot \mathbf{n}(x) < 0\}$ and $\Gamma^{out} := \{x \in \Gamma : \mathbf{b}(x) \cdot \mathbf{n}(x) \geq 0\}$, respectively.

Then we can formulate our (now extended) model problem: Find u and u_e such that

$$\partial_t u + \operatorname{div}(-\mathbf{A}\nabla u + \mathbf{b}u) + cu = f \qquad \text{in } \Omega \times (0,T), \tag{3.1}$$
$$-\Delta u_e = 0 \qquad \text{in } \Omega_e \times (0,T), \tag{3.2}$$

with coupling conditions across the interface given by

$$u = u_e + g_1 \qquad \text{on } \Gamma \times (0,T), \tag{3.3}$$
$$(\mathbf{A}\nabla u - \mathbf{b}u)\cdot\mathbf{n} = \partial_{\mathbf{n}} u_e + g_2 \qquad \text{on } \Gamma^{in} \times (0,T), \tag{3.4}$$
$$(\mathbf{A}\nabla u)\cdot\mathbf{n} = \partial_{\mathbf{n}} u_e + g_2 \qquad \text{on } \Gamma^{out} \times (0,T), \tag{3.5}$$

with a fixed time $T > 0$. The radiation condition remains unchanged, but we also look at non-homogeneous initial values here

$$u(\cdot,0) = q \qquad \text{on } \Omega, \tag{3.6}$$
$$u_e(x,t) = a(t)\log|x| + \mathcal{O}(|x|^{-1}) \qquad |x| \to \infty. \tag{3.7}$$

The *model input data* are q, f, g_1, and g_2.

For the *model parameters*, we assume the following regularity conditions: The diffusion matrix $\mathbf{A} : \Omega \to \mathbb{R}^{2\times 2}$ has piecewise Lipschitz continuous entries; i.e., entries in $W^{1,\infty}(K)$ for every $K \in \mathcal{T}$. Additionally, $\mathbf{A}$ is bounded, symmetric, and uniformly positive definite. The minimum eigenvalue of $\mathbf{A}$ is denoted by $\lambda_{\min}(\mathbf{A})$. Furthermore, $\mathbf{b} \in W^{1,\infty}(\Omega)^2$ and $c \in L^\infty(\Omega)$ fulfil

$$\frac{1}{2}\operatorname{div}\mathbf{b}(x) + c(x) =: \gamma > 0, \quad \text{for almost every } x \in \Omega.$$

Hence, the system is indeed parabolic-elliptic. For the *model input data*, we allow $q \in L^2(\Omega)$, $f \in H_T'$, $g_1 \in B_T'$ and $g_2 \in B_T$.

Remark 3.1. To handle the case $\frac{1}{2}\operatorname{div}\mathbf{b} + c = 0$, we could use a standard transformation of the whole system as mentioned before, i.e., multiplying by $e^{-\lambda t}$ with $\lambda > 0$ leads to a system (3.1)–(3.7) in the variables $u_\lambda = ue^{-\lambda t}$ and $u_{e,\lambda} = u_e e^{-\lambda t}$ with an additional term λu_λ in (3.1). Hence, we obtain a transformed system that fulfils $\frac{1}{2}\operatorname{div}\mathbf{b} + c + \lambda > 0$ and we are in the situation above. However, the constants of the energy estimates and the resulting estimates also depend on $e^{\lambda T}$. Another option is to use the implicit stabilisation for the non-symmetric coupling to obtain a coercive bilinear form of the whole system as we did in Chapter 2. Then the constants in the estimates do not depend on $e^{\lambda T}$.

Note that the model parameters $\mathbf{A}$, $\mathbf{b}$, and c are time-independent. For the FVM-BEM coupling, we will also devise an analysis of the fully-discrete system with the classical Euler scheme for time discretisation. This analysis can easily be transferred to time-dependent parameters. For the time discretisation with the variant of the implicit Euler, however, the extension is an open question.

By some easy modifications, we can extend the variational formulation Problem 2.1 to our extended model problem. One of the modifications concerns the definition of Q_T, which we redefine as

$$Q_T = \left\{ v \in H_T \,:\, \partial_t v \in H_T' \text{ and } v(0) = q \right\}.$$

Problem 3.2 (Extended variational problem)**.** Given $f \in H_T'$, $g_1 \in B_T'$, and $g_2 \in B_T$, find $u \in Q_T$ and $\phi \in B_T$ such that

$$\langle \partial_t u(t), v\rangle_\Omega + \mathscr{A}(u(t), v) - \langle \phi(t), v\rangle_\Gamma = \langle f(t), v\rangle_\Omega + \langle g_2(t), v\rangle_\Gamma, \tag{3.8}$$

$$\langle (1/2 - \mathscr{K})u(t)|_\Gamma, \psi\rangle_\Gamma + \langle \mathscr{V}\phi(t), \psi\rangle_\Gamma = \langle (1/2 - \mathscr{K})g_1(t), \psi\rangle_\Gamma, \tag{3.9}$$

for all test functions $v \in H$ and $\psi \in B$ and for a.e. $t \in [0, T]$. The bilinear form $\mathscr{A}(\cdot, \cdot)$ is defined by

$$\mathscr{A}(u(t), v) := (\mathbf{A}\nabla u(t) - \mathbf{b}u(t), \nabla v)_\Omega + (cu(t), v)_\Omega + \langle \mathbf{b}\cdot\mathbf{n}\, u(t), v\rangle_{\Gamma^{out}}.$$

For convenience we write the system (3.8)–(3.9) in a more compact form. With the product space $\mathscr{H} = H \times B$, we introduce the continuous bilinear form $\mathscr{B} : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ by

$$\begin{aligned} \mathscr{B}((u, \phi); (v, \psi)) :=& \mathscr{A}(u, v) - \langle \phi, v\rangle_\Gamma \\ &+ \langle (1/2 - \mathscr{K})u|_\Gamma, \psi\rangle_\Gamma + \langle \mathscr{V}\phi, \psi\rangle_\Gamma, \end{aligned} \tag{3.10}$$

and the linear functional $F : \mathscr{H} \to \mathbb{R}$ by

$$F((v, \psi); t) := \langle f(t), v\rangle_\Omega + \langle g_2(t), v\rangle_\Gamma + \langle (1/2 - \mathscr{K})g_1(t), \psi\rangle_\Gamma. \tag{3.11}$$

Then (3.8)–(3.9) is equivalent to:

Problem 3.3. Find $\mathbf{u} = (u, \phi) \in \mathscr{H}_T = Q_T \times B_T$ such that

$$\langle \partial_t u(t), v\rangle_\Omega + \mathscr{B}(\mathbf{u}(t); \mathbf{v}) = F(\mathbf{v}; t) \quad \text{for all } \mathbf{v} = (v, \psi) \in \mathscr{H} \text{ and a.e. } t \in [0, T]. \tag{3.12}$$

This problem is again well-posed.

Theorem 3.4 (Well-posedness of the model problem)**.** *Let $\lambda_{\min}(\mathbf{A}) - \frac{1}{4}C_{\mathscr{K}} > 0$ with the contraction constant of the double layer operator $C_{\mathscr{K}} \in [1/2, 1)$. The weak solution $\mathbf{u} = (u, \phi) \in Q_T \times B_T$ of the model problem (3.8)–(3.9) or (3.12) exists and is unique. Furthermore, there holds*

$$\|\partial_t u\|_{H_T'} + \|u\|_{H_T} + \|\phi\|_{B_T} \le C\big(\|f\|_{H_T'} + \|q\|_{L^2(\Omega)} + \|g_1\|_{B_T'} + \|g_2\|_{B_T}\big),$$

with a constant $C = C(\Omega) > 0$.

Proof. The bilinear form $\mathscr{B}(\cdot;\cdot)$ is $\mathscr{H}$-coercive and continuous, see [EOS17, Theorem 1 and Remark 2]. Hence, the proof of Theorem 2.6 is applicable with the additional term $\|q\|_{L^2(\Omega)}$ coming up when integrating the inequality. □

Remark 3.5. The condition $\lambda_{\min}(\mathbf{A}) - C_{\mathscr{K}}/4 > 0$ results from our non-symmetric coupling approach. In general, this is not necessary for the well-posedness of the model problem (3.1)–(3.7), e.g., if one uses the symmetric coupling approach. Note that this can pose a restriction to the scaling of our domain or coefficients (as we assumed that $\operatorname{diam}(\Omega) < 1$) and this excludes some choices of coefficients.

3.2 Coupling with the Finite Volume Method

Where we used FEM to solve the problem in the interior domain in the previous chapter, we will now employ FVM instead. Since FVM is based on a balance equation, it naturally conserves numerical fluxes. Furthermore, an (optional) upwinding strategy also guarantees the stability of the numerical scheme for convection dominated problems, but with retention of numerical flux conservation. A good overview of the properties of FVM (global conservativity, inverse monotonicity) can be found in [KA03, Chapter 6]. An early (if not first) mathematical analysis of the vertex-centred FVM can be found in [BR87] and [Hac89]. These ideas have been extended in [Mic96]. Later works put the method into a more modern framework, see, e.g., [ELL02] or [CLT04] for parabolic problems, or a Céa-type estimate for a general second order elliptic PDE in [EP16; EP17]. The contents of this section are already available in [ES19b].

3.2.1 Finite Volume Bilinear Form and Upwind Stabilisation

In the following, we omit the dependence on t in the notation, all expressions hold for a.e. $t \in [0, T]$ and we assume everything is sufficiently smooth. A Finite Volume Method is based on the reformulation of the differential equation as a conservation law, i.e., a balance equation through the boundary of some cells. These cells are given by the dual mesh $\mathcal{T}^*$, which has been introduced in Section 1.3. We achieve this reformulation if we formally integrate our interior equation (3.1) over the control volumes $V \in \mathcal{T}^*$ and use the Gaussian divergence theorem to rewrite it:

$$\begin{aligned}\int_V f \,\mathrm{d}x = \int_V \partial_t u \,\mathrm{d}x + \int_{\partial V\setminus\Gamma} (-\mathbf{A}\nabla u + \mathbf{b}u)\cdot \mathbf{n}\,\mathrm{d}s \\ + \int_{\partial V\cap\Gamma} -(\mathbf{A}\nabla u - \mathbf{b}u)\cdot \mathbf{n}\,\mathrm{d}s + \int_V cu\,\mathrm{d}x.\end{aligned}$$

Now we make use of the jump relations (3.4)–(3.5) on the boundary. If we additionally replace u by $u_h \in \mathcal{S}^1(\mathcal{T})$ and $\phi = \partial_{\mathbf{n}} u_e|_\Gamma$ by $\phi_h \in \mathcal{P}^0(\mathcal{E}_\Gamma)$, we get

$$\begin{aligned}\int_V \partial_t u \,\mathrm{d}x + \int_{\partial V\setminus\Gamma} (-\mathbf{A}\nabla u_h + \mathbf{b}u_h)\cdot \mathbf{n}\,\mathrm{d}s + \int_{\partial V\cap\Gamma^{\mathrm{out}}} \mathbf{b}\cdot\mathbf{n}u_h\,\mathrm{d}s \\ - \int_{\partial V\cap\Gamma} \phi_h\,\mathrm{d}s + \int_V cu_h\,\mathrm{d}x = \int_V f\,\mathrm{d}x + \int_{\partial V\cap\Gamma} g_2\,\mathrm{d}s\end{aligned}$$

for all $V \in \mathcal{T}^*$. By testing the equation with a piecewise constant function on the dual mesh $\mathcal{T}^*$, the system can be understood as a Petrov-Galerkin method. This means that the trial and the test space are different. Recalling the piecewise constant interpolation $\mathcal{I}_h^*$, we can indeed rewrite the FVM with $\mathcal{I}_h^* v_h \in \mathcal{P}^0(\mathcal{T}^*)$ for all $v_h \in \mathcal{S}^1(\mathcal{T})$ as

$$(\partial_t u_h, \mathcal{I}_h^* v_h)_\Omega + \mathcal{A}_V(u_h, v_h) - \langle \phi_h, \mathcal{I}_h^* v_h\rangle_\Gamma = (f, \mathcal{I}_h^* v_h)_\Omega + \langle g_2, \mathcal{I}_h^* v_h\rangle_\Gamma, \tag{3.13}$$

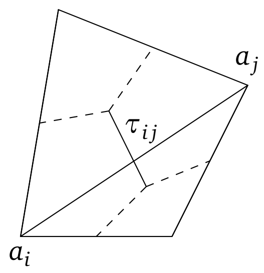


Figure 3.1: The intersection τ_{ij} with the nodes a_i and a_j, where we replace u_h for the upwind stabilisation.

with the finite volume bilinear form $\mathscr{A}_V : \mathscr{S}^1(\mathscr{T}) \times \mathscr{S}^1(\mathscr{T}) \to \mathbb{R}$ defined by

$$\mathscr{A}_V(u_h, v_h) := \sum_{a_i \in \mathscr{N}} v_h(a_i) \Bigg(\int_{\partial V_i \setminus \Gamma} (-\mathbf{A}\nabla u_h + \mathbf{b} u_h) \cdot \mathbf{n} \,\mathrm{d}s \\ + \int_{\partial V_i \cap \Gamma^{\text{out}}} \mathbf{b} \cdot \mathbf{n} u_h \,\mathrm{d}s + \int_{V_i} c u_h \,\mathrm{d}x \Bigg). \tag{3.14}$$

Remark 3.6. Under certain conditions, e.g., if $\mathbf{A}$ is only $\mathscr{T}$-piecewise constant and $\mathbf{b} = 0, c = 0$, the matrix generated by the FVM bilinear form $\mathscr{A}_V(\cdot, \cdot)$ coincides with the matrix generated by the FEM bilinear form. Then the FEM and FVM only differ in the right-hand sides. See also [Hac89, Sections 3.1 and 3.2].

To get a stable solution for convection-dominated problems, we can stabilise FVM through the use of an upwind scheme. To define an upwind stabilisation for FVM [RST08, Part II, Section 3.1 and Part IV, Section 2], we simply replace the terms with $\mathbf{b}u_h$ on the interior edges of the dual mesh by a convex combination of the nodal values depending on the direction of the convectional flux. On the intersection $\tau_{ij} = V_i \cap V_j \neq \emptyset$ of two neighbouring cells, we replace u_h by

$$u_{h,ij} := \lambda_{ij} u_h(a_i) + (1 - \lambda_{ij}) u_h(a_j). \tag{3.15}$$

The setup with the intersection τ_{ij} is visualised in Figure 3.1. The parameter λ_{ij} is computed in the following way: first we compute the average of the convection over the τ_{ij}, i.e.,

$$\beta_{ij} := \frac{1}{|\tau_{ij}|} \int_{\tau_{ij}} \mathbf{b} \cdot \mathbf{n}_i \,\mathrm{d}s,$$

where $\mathbf{n}_i$ is the unit outer normal with respect to V_i, and the average of the diffusion

$$\mathbf{A}_{ij} := \frac{1}{|\tau_{ij}|} \int_{\tau_{ij}} \mathbf{A} \,\mathrm{d}s.$$

Then λ_{ij} is defined by

$$\lambda_{ij} := \Phi(\beta_{ij}|\tau_{ij}|/\|\mathbf{A}_{ij}\|_\infty),$$

with a weight function $\Phi\colon \mathbb{R} \to [0,1]$ determined by the used upwind scheme. The argument of this weight function is the local Péclet number, which describes the ratio of the convection to the diffusion locally.

The easiest scheme is the full upwind scheme with $\Phi(t) := (\operatorname{sign}(t)+1)/2$, which leads to $u_{h,ij} = u_h(a_i)$ for $\beta_{ij} \geq 0$ and $u_{h,ij} = u_h(a_j)$ otherwise. Since the full upwind scheme is very diffusive, another option is the steerable upwinding defined by

$$\Phi(t) := \begin{cases} \min(2|t|^{-1}, 1)/2, & \text{for } t < 0, \\ 1 - \min(2|t|^{-1}, 1)/2, & \text{for } t \geq 0. \end{cases}$$

Replacing the respective term in the original finite volume bilinear form (3.14) by (3.15) leads to the upwind bilinear form :

$$\begin{aligned} \mathscr{A}_V^{up}(u_h, v_h) := \sum_{a_i \in \mathscr{N}} v_h(a_i) \Bigg(\int_{\partial V_i \setminus \Gamma} -\mathbf{A}\nabla u_h \cdot \mathbf{n} \, \mathrm{d}s + \int_{V_i} c u_h \, \mathrm{d}x \\ + \sum_{j \in \mathscr{N}_i} \int_{\tau_{ij}} \mathbf{b} \cdot \mathbf{n} u_{h,ij} \, \mathrm{d}s + \int_{\partial V_i \cap \Gamma^{\text{out}}} \mathbf{b} \cdot \mathbf{n} u_h \, \mathrm{d}s \Bigg), \end{aligned} \tag{3.16}$$

where $\mathscr{N}_i$ is the set of neighbouring nodes of $a_i \in \mathscr{N}$.

3.2.2 Semi-Discrete FVM-BEM

After we have introduced the FVM, we can define the coupling with BEM similar to the FEM-BEM coupling of the previous chapter. More precisely, we replace the bilinear form in the first equation (3.8) by the finite volume bilinear form and change the test space as seen in (3.13). Then we will establish a convergence result for the semi-discrete problem via energy estimates. From now on we assume the following regularities for the model input data: $q \in L^2(\Omega)$, $f \in L^2_{T,\Omega}$, $g_1 \in B'_T$, and $g_2 \in L^2_{T,\Gamma}$.

As discretisation spaces we will use $H_T^h = L^2(0,T;\mathscr{S}^1(\mathscr{T}))$, $B_T^h = L^2(0,T;\mathscr{P}^0(\mathscr{E}_\Gamma))$, and the energy space $Q_T^h = \{v_h \in H^1(0,T;\mathscr{S}^1(\mathscr{T})) : v_h(0) = P_h q\}$, with the L^2–orthogonal projection $P_h : L^2(\Omega) \to \mathscr{S}^1(\mathscr{T})$. This results in the following semi-discrete problem.

Problem 3.7. Find $u_h \in Q_T^h$ and $\phi_h \in B_T^h$ such that

$$\begin{aligned} (\partial_t u_h(t), \mathscr{I}_h^* v_h)_\Omega + \mathscr{A}_V(u_h(t), v_h) \\ -(\phi_h(t), \mathscr{I}_h^* v_h)_\Gamma = (f(t), \mathscr{I}_h^* v)_\Omega + (g_2(t), \mathscr{I}_h^* v)_\Gamma, \end{aligned} \tag{3.17}$$

$$((1/2-\mathscr{K})u_h(t), \psi_h)_\Gamma + (\mathscr{V}\phi_h(t), \psi_h)_\Gamma = ((1/2-\mathscr{K})g_1(t), \psi_h)_\Gamma, \tag{3.18}$$

for all $v_h \in \mathscr{S}^1(\mathscr{T}), \psi_h \in \mathscr{P}^0(\mathscr{E}_\Gamma)$ and a.e. $t \in [0,T]$. Obviously, the bilinear form $\mathscr{A}_V$ can be replaced by the upwind bilinear form $\mathscr{A}_V^{up}$.

With $\mathcal{H}^h = \mathcal{S}^1(\mathcal{T}) \times \mathcal{P}^0(\mathcal{E}_\Gamma)$, we define the more compact bilinear form $\mathcal{B}_V : \mathcal{H}^h \times \mathcal{H}^h \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{B}_V((u_h,\phi_h);(v_h,\psi_h)) :=& \mathcal{A}_V(u_h,v_h) - (\phi_h\,,\mathcal{I}_h^* v_h)_\Gamma \\ &+ ((1/2-\mathcal{K})u_h\,,\psi_h)_\Gamma + (\mathcal{V}\phi_h\,,\psi_h)_\Gamma,\end{aligned} \tag{3.19}$$

and the linear functional $F_V : \mathcal{H}^h \to \mathbb{R}$ by

$$F_V((v_h,\psi_h);t) := (f(t)\,,\mathcal{I}_h^* v_h)_\Omega + (g_2(t)\,,\mathcal{I}_h^* v_h)_\Gamma + ((1/2-\mathcal{K})g_1(t)\,,\psi_h)_\Gamma. \tag{3.20}$$

Hence, the system (3.17)–(3.18) is equivalent to:

Problem 3.8. Find $\mathbf{u}_h = (u_h,\phi_h) \in Q_T^h \times B_T^h$ such that

$$(\partial_t u_h(t)\,,\mathcal{I}_h^* v_h)_\Omega + \mathcal{B}_V(\mathbf{u}_h(t);\mathbf{v}_h) = F_V(\mathbf{v}_h;t) \tag{3.21}$$

for all $\mathbf{v}_h = (v_h,\psi_h) \in \mathcal{H}^h$ and a.e. $t \in [0,T]$, where we can replace $\mathcal{A}_V$ by $\mathcal{A}_V^{up}$ in $\mathcal{B}_V$.

For the analysis of the system (3.21), we employ some results from the stationary FVM-BEM coupling in [EOS17]. The main idea is to measure the discrete difference between the right-hand sides and the bilinear forms (3.10) and (3.19):

Lemma 3.9 ([EOS17, Lemma 5]). *For $\mathbf{w}_h = (w_h,\varphi_h) \in \mathcal{H}^h$ and an arbitrary but fixed t there holds*

$$\begin{aligned}|F(\mathbf{w}_h;t) - F_V(\mathbf{w}_h;t)| \le C\Big(&\sum_{K\in\mathcal{T}} h_K \|f(t)\|_{L^2(K)} \|\nabla w_h\|_{L^2(K)} \\ &+ \sum_{E\in\mathcal{E}_\Gamma} h_E^{1/2} \|g_2(t) - \overline{g}_2(t)\|_{L^2(E)} \|\nabla w_h\|_{L^2(K_E)}\Big),\end{aligned}$$

with a constant $C = C(\Omega) > 0$ independent of h. Here, $\overline{g}_2(t)$ is the $\mathcal{E}_\Gamma$-piecewise integral mean of $g_2(t) \in L^2(\Gamma)$ and $K_E \in \mathcal{T}$ the element associated with E.

Lemma 3.10 ([EOS17, Lemma 7]). *For $\mathbf{v}_h = (v_h,\psi_h) \in \mathcal{H}^h$ and $\mathbf{w}_h = (w_h,\varphi_h) \in \mathcal{H}^h$ there holds*

$$|\mathcal{B}(\mathbf{v}_h;\mathbf{w}_h) - \mathcal{B}_V(\mathbf{v}_h;\mathbf{w}_h)| \le C \sum_{K\in\mathcal{T}} \big(h_K \|v_h\|_{H^1(K)} \|w_h\|_{H^1(K)}\big),$$

with a constant $C = C(\mathbf{A},\mathbf{b},c,\Omega) > 0$ independent of h. The result still holds if we replace $\mathcal{A}_V$ by $\mathcal{A}_V^{up}$ in the corresponding bilinear forms.

Remark 3.11. The restriction $\mathbf{b}\cdot\mathbf{n} \in \mathcal{P}^0(\mathcal{E}_\Gamma^{in})$ in [EOS17, Lemma 7], where $\mathcal{E}_\Gamma^{in}$ denotes the set of all edges on the inflow boundary Γ^{in}, results from the estimate [EOS17, Lemma 6]. However, this is not necessary. In fact, we can estimate the last term of [EOS17, eq. (38)] in the following way: let $v_h, w_h \in \mathcal{S}^1(\mathcal{T})$ and let $\overline{v}_h \in \mathcal{P}^0(\mathcal{E}_\Gamma)$ be the best $L^2(\Gamma)$ approximation of v_h. We see with the property (1.7) of $\mathcal{I}_h^*$ that

$$\begin{aligned}-\sum_{E\in\mathcal{E}_\Gamma^{in}} (\mathbf{b}\cdot\mathbf{n}\, v_h\,, w_h - \mathcal{I}_h^* w_h)_E &\le C \sum_{E\in\mathcal{E}_\Gamma^{in}} \|\mathbf{b}\|_{L^\infty(E)} |(v_h - \overline{v}_h\,, w_h - \mathcal{I}_h^* w_h)_E| \\ &\le C \sum_{E\in\mathcal{E}_\Gamma^{in}} \|\mathbf{b}\|_{L^\infty(E)} h_{K_E} \|v_h\|_{H^1(K_E)} \|w_h\|_{H^1(K_E)},\end{aligned}$$

where $K_E \in \mathcal{T}$ is the element associated with E. For the last estimate we used the Cauchy-Schwarz inequality, $\|v_h - \overline{v}_h\|_{L^2(E)} \leq C h_{K_E}^{1/2} \|\nabla v_h\|_{L^2(K_E)}$, and the approximation property (1.9) of $\mathcal{I}_h^*$. We can also use this for the stabilised FVM-BEM coupling versions with $\mathcal{A}_V^{up}$ and it enhances the result in [Era12, Lemma 5.2 and Theorem 5.3] for the three-field FVM-BEM coupling.

An extension of Lemma 2.4 shows that $\mathcal{B}(\cdot;\cdot)$ is coercive (see [EOS17, Theorem 1] and Remark 2.5). Using this and Lemma 3.10, it can be shown that the finite volume bilinear form is also coercive.

Lemma 3.12 ([EOS17, Theorem 2]). *For h small enough, let $\lambda_{\min}(\mathbf{A}) - \frac{1}{4}C_{\mathcal{K}} > 0$ with $C_{\mathcal{K}} \in [1/2, 1)$. Then there holds for all $\mathbf{v}_h = (v_h, \psi_h) \in \mathcal{H}^h$*

$$\mathcal{B}_V(\mathbf{v}_h; \mathbf{v}_h) \geq \alpha_{Vstab} \|\mathbf{v}_h\|_{\mathcal{H}}^2 = \alpha_{Vstab} \Big(\|v_h\|_{H^1(\Omega)}^2 + \|\psi_h\|_{H^{-1/2}(\Gamma)}^2 \Big). \tag{3.22}$$

The constant $\alpha_{Vstab} > 0$ depends on the model data $\mathbf{A}$, $\mathbf{b}$, c and on $C_{\mathcal{K}}$. The coercivity still holds if we replace $\mathcal{A}_V$ by $\mathcal{A}_V^{up}$ in (3.19). Furthermore, the bilinear form is continuous.

Remark 3.13. The semi-discrete systems (3.17)–(3.18) and (3.21) lead to a system of ordinary differential equations

$$M U_h'(t) + B \begin{pmatrix} U_h(t) \\ \Phi_h(t) \end{pmatrix} = F(t).$$

Here, $U_h(t) \in \mathbb{R}^{n_1}$, $\Phi_h(t) \in \mathbb{R}^{n_2}$, and $F(t) \in \mathbb{R}^{n_1+n_2}$ for some $n_1, n_2 \in \mathbb{N}$ and a fixed but arbitrary t. The matrix B is non-symmetric and positive definite, which follows directly from Lemma 3.12. The mass matrix M, resulting from $(\partial_t u_h, \mathcal{I}_h^* v_h)_\Omega$, is positive definite as well; see, e.g., [CLT04, Section 3]. Therefore, the ODE-system and thus also the semi-discrete system are uniquely solvable by the theorem of Picard-Lindelöf.

Additional to the unique solvability, we also establish an energy estimate for the semi-discretisation, which is similar to the result for the continuous problem. Note that we cannot show an energy estimate in the full energy norm as we cannot use the duality argument we used in Lemma 2.13 in the FVM-BEM case.

Lemma 3.14 (Well-posedness of the semi-discrete FVM-BEM). *For h small enough, let $\lambda_{\min}(\mathbf{A}) - \frac{1}{4}C_{\mathcal{K}} > 0$, $C_{\mathcal{K}} \in [1/2, 1)$. The solution $(u_h, \phi_h) \in Q_T^h \times B_T^h$ of (3.21) fulfils*

$$\|u_h\|_{H_T} + \|\phi_h\|_{B_T} \leq C \left(\|f\|_{L^2_{T,\Omega}} + \|q\|_{L^2(\Omega)} + \|g_1\|_{B_T'} + \|g_2\|_{L^2_{T,\Gamma}} \right),$$

with a constant $C = C(\alpha_{Vstab}, \Omega) > 0$.

Proof. In (3.21) we choose $\mathbf{u}_h(t) = (u_h(t), \phi_h(t)) \in \mathcal{H}^h$ as the test function for a fixed but arbitrary t. Since $\mathcal{I}_h^*$ is self-adjoint (1.11) and defines a norm (1.12) (see Lemma 1.10), we see that

$$\frac{1}{2} \frac{d}{dt} \|u_h(t)\|_\chi^2 = (\partial_t u_h(t), \mathcal{I}_h^* u_h(t))_\Omega.$$

Then we can use the coercivity (3.22) of $\mathscr{B}_V$ to see that

$$\frac{1}{2}\frac{d}{dt}\|u_h(t)\|_\chi^2 + \|\mathbf{u}_h(t)\|_{\mathscr{H}}^2 \lesssim (\partial_t u_h(t), \mathscr{I}_h^* u_h(t))_\Omega + \mathscr{B}_V(\mathbf{u}_h(t); \mathbf{u}_h(t)).$$

With the stability $\|\mathscr{I}_h^* u_h(t)\|_{L^2(\Omega)} \le C\|u_h(t)\|_{L^2(\Omega)}$, see (1.10), the result follows from standard calculations as in Theorem 2.6. □

The main result of this section is the following convergence of the semi-discrete scheme.

Theorem 3.15 (Convergence of the semi-discrete FVM-BEM, [ES19b, Theorem 19]). *There exists $h_{\max} > 0$ such that for $\mathscr{T}$ sufficiently fine, i.e., $h < h_{\max}$, the following statement holds: Let $\lambda_{\min}(\mathbf{A}) - \frac{1}{4}C_{\mathscr{K}} > 0$, $C_{\mathscr{K}} \in [1/2, 1)$. The discrete solution $\mathbf{u}_h = (u_h, \phi_h) \in Q_T^h \times B_T^h$ of (3.21) converges to the weak solution $\mathbf{u} = (u, \phi) \in Q_T \times B_T$ of (3.12), i.e., there holds*

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|_{\mathscr{H}_T} \le C\Big(h\|f\|_{L^2_{T,\Omega}} + h^{1/2}\|g_2 - \overline{g}_2\|_{L^2_{T,\Gamma}} + h\|v_h\|_{H_T} \\ + h\|\partial_t v_h\|_{L^2_{T,\Omega}} + \|\partial_t u - \partial_t v_h\|_{H_T'} + \|\mathbf{u} - \mathbf{v}_h\|_{\mathscr{H}_T}\Big)\end{aligned}$$

for all $\mathbf{v}_h = (v_h, \psi_h) \in Q_T^h \times B_T^h$. Here $\overline{g}_2$ is the $\mathscr{E}_\Gamma$-piecewise integral mean of the normal derivative jump $g_2 \in L^2_{T,\Gamma}$. and $\|\cdot\|_{\mathscr{H}_T}^2 = \|\cdot\|_{H_T}^2 + \|\cdot\|_{B_T}^2$. The constant $C = C(\alpha_{V stab}, \mathbf{A}, \mathbf{b}, c, \Omega) > 0$ is independent of h. The result still holds if we replace $\mathscr{A}_V$ by $\mathscr{A}_V^{up}$ in the corresponding bilinear forms.

Proof. Let $\mathbf{v}_h = (v_h, \psi_h) \in Q_T^h \times B_T^h$ be arbitrary. First, we split the error into an approximation error and a discrete error component;

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathscr{H}_T} \le \|\mathbf{u} - \mathbf{v}_h\|_{\mathscr{H}_T} + \|\mathbf{u}_h - \mathbf{v}_h\|_{\mathscr{H}_T}. \tag{3.23}$$

Hence, we only have to estimate the norms of the discrete error $\mathbf{w}_h = (w_h, \varphi_h) := \mathbf{u}_h - \mathbf{v}_h \in Q_T^h \times B_T^h$. Using that $\frac{1}{2}\frac{d}{dt}\|w_h\|_\chi^2 = (\partial_t w_h, \mathscr{I}_h^* w_h)_\Omega$ and the coercivity (3.22) of the finite volume bilinear form $\mathscr{B}_V(\cdot;\cdot)$ leads to

$$\frac{1}{2}\frac{d}{dt}\|w_h\|_\chi^2 + \|\mathbf{w}_h\|_{\mathscr{H}}^2 \lesssim (\partial_t w_h, \mathscr{I}_h^* w_h)_\Omega + \mathscr{B}_V(\mathbf{w}_h; \mathbf{w}_h).$$

Using the discrete FVM-BEM scheme (3.21) and adding the weak form (3.12), we see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w_h\|_\chi^2 + \|\mathbf{w}_h\|_{\mathscr{H}}^2 \lesssim \langle \partial_t u, w_h\rangle_\Omega - (\partial_t v_h, \mathscr{I}_h^* w_h)_\Omega + F_V(\mathbf{w}_h; t) - F(\mathbf{w}_h; t) \\ + \mathscr{B}(\mathbf{v}_h; \mathbf{w}_h) - \mathscr{B}_V(\mathbf{v}_h; \mathbf{w}_h) + \mathscr{B}(\mathbf{u} - \mathbf{v}_h; \mathbf{w}_h).\end{aligned}$$

To estimate the terms with the time derivatives we apply (1.8):

$$\begin{aligned}&\langle \partial_t u, w_h\rangle_\Omega - (\partial_t v_h, \mathscr{I}_h^* w_h)_\Omega \\ &\qquad = (\partial_t v_h, w_h - \mathscr{I}_h^* w_h)_\Omega + \langle \partial_t u - \partial_t v_h, w_h\rangle_\Omega \\ &\qquad \lesssim h\|\partial_t v_h\|_{L^2(\Omega)}\|\nabla w_h\|_{L^2(\Omega)} + \|\partial_t u - \partial_t v_h\|_{H^1(\Omega)'}\|w_h\|_{H^1(\Omega)}.\end{aligned} \tag{3.24}$$

We estimate the other terms by Lemma 3.9, Lemma 3.10, and the continuity of the bilinear form $\mathscr{B}$. Thus we get

$$\frac{1}{2}\frac{d}{dt}\|w_h\|_\chi^2 + \|\mathbf{w}_h\|_{\mathscr{H}}^2 \lesssim \Big(\|\partial_t u - \partial_t v_h\|_{H^1(\Omega)'} + h\|\partial_t v_h\|_{L^2(\Omega)} + h\|f\|_{L^2(\Omega)} + h^{1/2}\|g_2 - \overline{g}_2\|_{L^2(\Gamma)} + h\|v_h\|_{H^1(\Omega)} + \|\mathbf{u}-\mathbf{v}_h\|_{\mathscr{H}}\Big)\|\mathbf{w}_h\|_{\mathscr{H}}.$$

Young's inequality with $\varepsilon > 0$, integration over t from 0 to T, and the fact that

$$\int_0^T \frac{1}{2}\frac{d}{dt}\|w_h\|_\chi^2 \,\mathrm{d}t = \underbrace{\frac{1}{2}\|w_h(T)\|_\chi^2}_{\geq 0} - \underbrace{\frac{1}{2}\|w_h(0)\|_\chi^2}_{=0} \geq 0,$$

lead to

$$\|\mathbf{w}_h\|_{\mathscr{H}_T}^2 \leq C\frac{1}{\varepsilon}\Big[\|\partial_t u - \partial_t v_h\|_{H_T'}^2 + \|\mathbf{u}-\mathbf{v}_h\|_{\mathscr{H}_T}^2 + h\|g_2 - \overline{g}_2\|_{L^2_{T,\Gamma}}^2 + h^2\Big(\|f\|_{L^2_{T,\Omega}}^2 + \|v_h\|_{H_T}^2 + \|\partial_t v_h\|_{L^2_{T,\Omega}}^2\Big)\Big] + C\varepsilon\|\mathbf{w}_h\|_{\mathscr{H}_T}^2.$$

We consequently choose $\varepsilon > 0$ such that $C\varepsilon \leq 1/2$ and conclude the assertion with $\mathbf{w}_h = \mathbf{u}_h - \mathbf{v}_h$ and the error splitting (3.23). For the stabilised FVM-BEM coupling version with $\mathscr{A}_V^{up}$, the proof is the same. □

With the approximation results of Lemma 2.27, we obtain the following a priori estimate.

Corollary 3.16 (Convergence rates of the semi-discrete FVM-BEM)**.** *Let P_h be H^1-stable, e.g., $\mathscr{T}$ is quasi-uniform. With the assumptions of Theorem 3.15 we obtain*

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathscr{H}_T} \leq C\Big[h^s\Big(\|u\|_{L^2(0,T;H^{1+s}(\Omega))} + \|\partial_t u\|_{L^2(0,T;H^{1-s}(\Omega)')} + \|\phi\|_{L^2(0,T;H^{s-1/2}(\Gamma))} + \|g_2\|_{L^2(0,T;L^2(\Gamma)\cap H^{s-1/2}(\Gamma))}\Big) + h\Big(\|f\|_{L^2_{T,\Omega}} + \|u\|_{H_T} + \|\partial_t u\|_{L^2_{T,\Omega}}\Big)\Big] = \mathscr{O}(h^s).$$

for all $0 \leq s \leq 1$, $u(t) \in H^{1+s}(\Omega)$, $\partial_t u(t) \in L^2(\Omega)$, $\phi(t) \in H^{-1/2+s}(\Gamma)$, $g_2(t) \in L^2(\Gamma) \cap H^{-1/2+s}(\Gamma)$, and for a.e. $t \in [0,T]$. The constant $C = C(\alpha_{V stab}, \mathbf{A}, \mathbf{b}, c, \Omega, \eta) > 0$ is independent of h.

Proof. The result follows directly from Theorem 3.15 and Lemma 2.27 with $v_h = P_h u$ and $\psi_h = \Pi_h \phi$. □

Remark 3.17. Due to Lemma 2.27, it is enough to demand $\phi(t) \in H^{-1/2+s}(\mathscr{E}_\Gamma)$ and $g_2 \in H^{-1/2+s}(\mathscr{E}_\Gamma)$ if $s > 1/2$ in Corollary 3.16.

3.2.3 Full Discretisation with a Variant of the Implicit Euler Scheme

The discretisation in space of the model problem (3.1)–(3.7) in Section 3.2.2 leads to a stiff system of ordinary differential equations, see Remark 3.13. The advantage of this *method of lines* approach is that we can choose between several time discretisation schemes. As in the FEM-BEM case, we apply an implicit scheme for the subsequent time discretisation. First, we will use the variant of the implicit Euler scheme introduced in Section 2.4. In the next subsection, we define a fully-discrete system with the aid of the classical implicit Euler scheme for time discretisation where we demand the usual regularity for the time component of the model data and solution. Again, we will investigate a priori convergence estimates.

Hence, we search for functions $u_{h,\tau} \in Q_T^{h,\tau}$ and $\phi_{h,\tau} \in B_T^{h,\tau}$ with

$$\begin{aligned} Q_T^{h,\tau} &:= \left\{ v \in C(0,T;\mathscr{S}^1(\mathscr{T})) \,:\, v(0) = P_h q,\, v|_{[t^{n-1},t^n]} \text{ is linear in } t \right\} \qquad \text{and} \\ B_T^{h,\tau} &:= \left\{ \psi \in L^2(0,T;\mathscr{P}^0(\mathscr{E}_\Gamma)) \,:\, \psi|_{(t^{n-1},t^n]} \text{ is constant in } t \right\}, \end{aligned}$$

and, recalling the definitions of Section 2.4, define our fully-discrete system as follows.

Problem 3.18 (VarIE-FVM-BEM). Find $u_{h,\tau} \in Q_T^{h,\tau}$ and $\phi_{h,\tau} \in B_T^{h,\tau}$ such that

$$(\widehat{\partial_t u}_{h,\tau}^{\,n}, \mathscr{I}_h^* v_h)_\Omega + \mathscr{A}_V(\widehat{u}_{h,\tau}^n, v_h) - (\widehat{\phi}_{h,\tau}^n, \mathscr{I}_h^* v_h)_\Gamma = (\widehat{f}^n, \mathscr{I}_h^* v_h)_\Omega + (\widehat{g}_2^n, \mathscr{I}_h^* v_h)_\Gamma, \tag{3.25}$$

$$((1/2-\mathscr{K})\widehat{u}_{h,\tau}^n, \psi_h)_\Gamma + (\mathscr{V}\widehat{\phi}_{h,\tau}^n, \psi_h)_\Gamma = ((1/2-\mathscr{K})\widehat{g}_1^n, \psi_h)_\Gamma \tag{3.26}$$

for all $v_h \in \mathscr{S}^1(\mathscr{T}) \subset H^1(\Omega)$ and $\psi_h \in \mathscr{P}^0(\mathscr{T}) \subset H^{-1/2}(\Gamma)$ and for all $1 \le n \le N$.
In compact notation: Find $\mathbf{u}_{h,\tau} = (u_{h,\tau}, \phi_{h,\tau}) \in \mathscr{H}_T^{h,\tau}$ such that

$$(\widehat{\partial_t u}_{h,\tau}^{\,n}, \mathscr{I}_h^* v_h)_\Omega + \mathscr{B}_V(\widehat{\mathbf{u}}_{h,\tau}^n; \mathbf{v}_h) = \widehat{F}_V(\mathbf{v}_h; t^n) \tag{3.27}$$

for all $\mathbf{v}_h = (v_h, \psi_h) \in \mathscr{H}^h$. Here, $\widehat{F}_V$ is the ω-weighted average (2.26) of F_V defined in (3.20). In (3.25) and in (3.27) we can replace $\mathscr{A}_V$ by $\mathscr{A}_V^{up}$.

Remark 3.19. With the identities of Lemma 2.17, the discrete system (3.25)–(3.26) is equivalent to

$$\begin{aligned} (d_\tau u_{h,\tau}^n, \mathscr{I}_h^* v_h)_\Omega &+ \mathscr{A}_V(u_{h,\tau}^n, \mathscr{I}_h^* v_h) \\ &- (\phi_{h,\tau}^n, \mathscr{I}_h^* v_h)_\Gamma = (\widehat{f}^n, \mathscr{I}_h^* v_h)_\Omega + (\widehat{g}_2^n, \mathscr{I}_h^* v_h)_\Gamma, \end{aligned} \tag{3.28}$$

$$((1/2-\mathscr{K})u_{h,\tau}^n, \psi_h)_\Gamma + (\mathscr{V}\phi_{h,\tau}^n, \psi_h)_\Gamma = ((1/2-\mathscr{K})\widehat{g}_1^n, \psi_h)_\Gamma \tag{3.29}$$

for all $v_h \in \mathscr{S}^1(\mathscr{T}) \subset H^1(\Omega)$ and $\psi_h \in \mathscr{P}^0(\mathscr{E}_\Gamma) \subset H^{-1/2}(\Gamma)$, and for all $1 \le n \le N$. The same holds if we replace $\mathscr{A}_V$ by $\mathscr{A}_V^{up}$. This system differs from a time discretisation by a classical implicit Euler only in the right-hand side, cp. (3.33)–(3.34) in Section 3.2.4.

As in Section 2.4, we rewrite the variational form (2.9)–(2.10) to see that the fully-discrete system (3.25)–(3.26) is consistent. More precisely, by testing (2.9)–(2.10) with $v = v_h$ and

$\psi = \psi_h$, multiplying with the weight function ω^n, and integrating over the time interval $[t^{n-1}, t^n]$, we see that

$$\begin{aligned}
&\langle \widehat{\partial_t u}^n, v_h\rangle_\Omega + \mathscr{A}(\widehat{u}^n, v_h) - \langle \widehat{\phi}^n, v_h\rangle_\Gamma = \langle \widehat{f}^n, v_h\rangle_\Omega + \langle \widehat{g}_2^n, v_h\rangle_\Gamma,\\
&((1/2 - \mathscr{K})\widehat{u}^n, \psi_h)_\Gamma + (\mathscr{V}\widehat{\phi}^n, \psi_h)_\Gamma = ((1/2-\mathscr{K})\widehat{g}_1^n, \psi_h)_\Gamma
\end{aligned}$$

for all $v_h \in \mathscr{S}^1(\mathscr{T})$, $\psi_h \in \mathscr{P}^0(\mathscr{E}_\Gamma)$. We write this system in the compact form with $\mathbf{u} = (u, \phi) \in \mathscr{H}_T$

$$\langle \widehat{\partial_t u}^n, v_h\rangle_\Omega + \mathscr{B}(\widehat{\mathbf{u}}^n; \mathbf{v}_h) = \widehat{F}(\mathbf{v}_h; t^n), \tag{3.30}$$

for all $\mathbf{v}_h \in \mathscr{H}^h$, where $\widehat{F}$ is the ω-weighted averaged (2.26) of F defined in (3.11).

Lemma 3.20 (Well-posedness of the fully-discrete system VarIE-FVM-BEM). *For h small enough, let $\lambda_{\min}(\mathbf{A}) - \frac{1}{4}C_{\mathscr{K}} > 0$, $C_{\mathscr{K}} \in [1/2, 1)$. The solution $\mathbf{u}_{h,\tau} = (u_{h,\tau}, \phi_{h,\tau}) \in \mathscr{H}_T^{h,\tau} = Q_T^{h,\tau} \times B_T^{h,\tau}$ of (3.27) fulfils*

$$\|\mathbf{u}_{h,\tau}\|_{\mathscr{H}_T} \le C\left(\|f\|_{L^2_{T,\Omega}} + \|q\|_{L^2(\Omega)} + \|g_1\|_{B_T'} + \|g_2\|_{L^2_{T,\Gamma}}\right),$$

with $C = C(\alpha_{Vstab}, \mathbf{A}, \mathbf{b}, c, \Omega) > 0$.

Proof. For a time t^n, we estimate

$$\begin{aligned}
(d_\tau u_{h,\tau}^n, \mathscr{I}_h^* u_{h,\tau}^n)_\Omega &\ge \frac{1}{\tau^n}(u_{h,\tau}^n - u_{h,\tau}^{n-1}, \mathscr{I}_h^* u_{h,\tau}^n)_\Omega - \frac{1}{2\tau^n}\|u_{h,\tau}^n - u_{h,\tau}^{n-1}\|_\chi^2\\
&= \frac{1}{2\tau^n}(u_{h,\tau}^n - u_{h,\tau}^{n-1}, \mathscr{I}_h^*(u_{h,\tau}^n + u_{h,\tau}^{n-1}))_\Omega\\
&= \frac{1}{2\tau^n}(\|u_{h,\tau}^n\|_\chi^2 - \|u_{h,\tau}^{n-1}\|_\chi^2),
\end{aligned} \tag{3.31}$$

where we used the fact that $\mathscr{I}_h^*$ is self-adjoint (1.11). The assertion follows with standard arguments from the equivalent discrete system (3.28)–(3.29) with $v_h = u_{h,\tau}^n$, (3.31), the coercivity of $\mathscr{B}_V$, $\|\mathscr{I}_h^* u_{h,\tau}^n\|_{L^2(\Omega)} \le C\|u_{h,\tau}^n\|_{L^2(\Omega)}$, and Lemma 2.17. ☐

Theorem 3.21 (Convergence of the fully-discrete system VarIE-FVM-BEM, [ES19b, Theorem 28]). *There exists $h_{\max} > 0$ such that for $\mathscr{T}$ sufficiently fine, i.e., $h < h_{\max}$, the following statement holds: Let $\lambda_{\min}(\mathbf{A}) - \frac{1}{4}C_{\mathscr{K}} > 0$. For the solution $\mathbf{u} = (u, \phi) \in \mathscr{H}_T = Q_T \times B_T$ of our model problem (3.12) and the discrete solution $\mathbf{u}_{h,\tau} = (u_{h,\tau}, \phi_{h,\tau}) \in \mathscr{H}_T^{h,\tau} = Q_T^{h,\tau} \times B_T^{h,\tau}$ of our fully-discrete system (3.27), there holds*

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_{h,\tau}\|_{\mathscr{H}_T} \le C\Big(&\|\mathbf{u} - \mathbf{v}_{h,\tau}\|_{\mathscr{H}_T} + \|\partial_t u - \partial_t v_{h,\tau}\|_{H_T'} + h\|\partial_t v_{h,\tau}\|_{L^2_{T,\Omega}}\\
&+ h\|v_{h,\tau}\|_{H_T} + h\|f\|_{L^2_{T,\Omega}} + h^{1/2}\|g_2 - \overline{g}_2\|_{L^2_{T,\Gamma}}\Big),
\end{aligned}$$

for all $\mathbf{v}_{h,\tau} = (v_{h,\tau}, \psi_h) \in \mathscr{H}_T^{h,\tau}$, where $\overline{g}_2 \in L^2(0,T;\mathscr{P}^0(\mathscr{E}_\Gamma))$ is the $\mathscr{E}_\Gamma$-piecewise integral mean of $g_2 \in L^2_{T,\Gamma}$, $\|\cdot\|_{\mathscr{H}_T}^2 = \|\cdot\|_{H_T}^2 + \|\cdot\|_{B_T}^2$ and $C = C(\alpha_{Vstab}, \mathbf{A}, \mathbf{b}, c, \Omega) > 0$. This result also holds if we replace $\mathscr{A}_V$ by the upwind version $\mathscr{A}_V^{up}$.

Proof. The proof utilises results and techniques from the proofs of Theorem 3.15 and Lemma 2.20. First, we split the error into an approximation error and a discrete error component, i.e., for arbitrary $\mathbf{v}_{h,\tau} = (v_{h,\tau}, \psi_{h,\tau}) \in \mathscr{H}_T^{h,\tau} = Q_T^{h,\tau} \times B_T^{h,\tau}$

$$\|\mathbf{u} - \mathbf{u}_{h,\tau}\|_{\mathscr{H}_T} \le \|\mathbf{u} - \mathbf{v}_{h,\tau}\|_{\mathscr{H}_T} + \|\mathbf{u}_{h,\tau} - \mathbf{v}_{h,\tau}\|_{\mathscr{H}_T}. \tag{3.32}$$

We only have to estimate the discrete error part. By using the notation $\mathbf{w}_{h,\tau} = (w_{h,\tau}, \varphi_{h,\tau}) := \mathbf{u}_{h,\tau} - \mathbf{v}_{h,\tau} \in \mathscr{H}_T^{h,\tau}$, we estimate for a time t^n as in (3.31)

$$(d_\tau w_{h,\tau}^n, \mathscr{I}_h^* w_{h,\tau}^n)_\Omega \ge \frac{1}{2\tau^n}(\|w_{h,\tau}^n\|_\chi^2 - \|w_{h,\tau}^{n-1}\|_\chi^2).$$

With this estimate and the coercivity (3.22) of $\mathscr{B}_V$, we get with similar steps as in the proof of Theorem 3.15:

$$\begin{aligned}
&\frac{1}{2\tau^n}(\|w_{h,\tau}^n\|_\chi^2 - \|w_{h,\tau}^{n-1}\|_\chi^2) + \|\mathbf{w}_{h,\tau}^n\|_{\mathscr{H}}^2 \\
&\quad\lesssim (d_\tau w_{h,\tau}^n, \mathscr{I}_h^* w_{h,\tau}^n)_\Omega + \mathscr{B}_V(\mathbf{w}_{h,\tau}^n; \mathbf{w}_{h,\tau}^n) \\
&\quad\lesssim \langle \widehat{\partial_t u}^n, w_{h,\tau}^n\rangle_\Omega - (d_\tau v_{h,\tau}^n, \mathscr{I}_h^* w_{h,\tau}^n)_\Omega + \widehat{F}_V(\mathbf{w}_{h,\tau}^n; t^n) - \widehat{F}(\mathbf{w}_{h,\tau}^n; t^n) \\
&\qquad + \mathscr{B}(\mathbf{v}_{h,\tau}^n; \mathbf{w}_{h,\tau}^n) - \mathscr{B}_V(\mathbf{v}_{h,\tau}^n; \mathbf{w}_{h,\tau}^n) + \mathscr{B}(\widehat{\mathbf{u}}^n - \mathbf{v}_{h,\tau}^n; \mathbf{w}_{h,\tau}^n) \\
&\quad\lesssim \Big(\|\widehat{\partial_t u}^n - d_\tau v_{h,\tau}^n\|_{H^1(\Omega)'} + h\|d_\tau v_{h,\tau}^n\|_{L^2(\Omega)} + h\|\widehat{f}^n\|_{L^2(\Omega)} \\
&\qquad\quad + h^{1/2}\|\widehat{g}_2^n - \widehat{\overline{g}}_2^n\|_{L^2(\Gamma)} + h\|v_{h,\tau}^n\|_{H^1(\Omega)} + \|\widehat{\mathbf{u}}^n - \mathbf{v}_{h,\tau}^n\|_{\mathscr{H}}\Big)\|\mathbf{w}_{h,\tau}^n\|_{\mathscr{H}},
\end{aligned}$$

where we used the discrete system (3.27) with $\widehat{\mathbf{u}}_{h,\tau}^n = \mathbf{u}_{h,\tau}^n$ and the ω-weighted variational form (3.30). In the last step we used (3.24), Lemma 3.9, Lemma 3.10, and the continuity of the bilinear form $\mathscr{B}$. Young's inequality with $\epsilon > 0$, multiplying the whole inequality with τ^n and summing over n lead to

$$\begin{aligned}
&\frac{1}{2}\|w_{h,\tau}^N\|_\chi^2 + \sum_{n=1}^N \tau^n \|\mathbf{w}_{h,\tau}^n\|_{\mathscr{H}}^2 \\
&\quad\lesssim \sum_{n=1}^N \tau^n \Big(\|\widehat{\partial_t u}^n - d_\tau v_{h,\tau}^n\|_{H^1(\Omega)'}^2 + h^2\|d_\tau v_{h,\tau}^n\|_{L^2(\Omega)}^2 + h^2\|\widehat{f}^n\|_{L^2(\Omega)}^2 \\
&\qquad\qquad + h\|\widehat{g}_2^n - \widehat{\overline{g}}_2^n\|_{L^2(\Gamma)}^2 + h^2\|v_{h,\tau}^n\|_{H^1(\Omega)}^2 + \|\widehat{\mathbf{u}}^n - \mathbf{v}_{h,\tau}^n\|_{\mathscr{H}}^2\Big).
\end{aligned}$$

Finally, we estimate with $d_\tau v_{h,\tau}^n = \widehat{\partial_t v_{h,\tau}}^n$ and $v_{h,\tau}^n = \widehat{v}_{h,\tau}^n$ and the inequalities (2.29)–(2.30) from Lemma 2.17

$$\begin{aligned}
\|\mathbf{w}_{h,\tau}\|_{\mathscr{H}_T}^2 \lesssim{}& \|\partial_t u - \partial_t v_{h,\tau}\|_{H_T'}^2 + h^2\|\partial_t v_{h,\tau}\|_{L^2_{T,\Omega}}^2 + h^2\|f\|_{L^2_{T,\Omega}}^2 \\
&+ h\|g_2 - \overline{g}_2\|_{L^2_{T,\Gamma}}^2 + h^2\|v_{h,\tau}\|_{H_T}^2 + \|\mathbf{u} - \mathbf{v}_{h,\tau}\|_{\mathscr{H}_T}^2
\end{aligned}$$

With $\mathbf{w}_{h,\tau} = \mathbf{u}_{h,\tau} - \mathbf{v}_{h,\tau}$ and (3.32) we prove the assertion. □

With the estimates for the projection P_h and Π_h in Lemma 2.27 and the estimates for the L^2–projection in time P^τ and Π^τ in Lemma 2.31, the following corollary is valid if P_h is H^1-stable, see (2.21).

Corollary 3.22 (A priori estimate for the fully-discrete system VarIE-FVM-BEM)**.** *Let P_h be H^1-stable, e.g., $\mathcal{T}$ is quasi-uniform. With the assumptions of Theorem 3.21, there holds*

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_{h,\tau}\|_{\mathcal{H}_T} \lesssim h^s\Big(&\|u\|_{L^2(0,T;H^{1+s}(\Omega))} + \|\partial_t u\|_{L^2(0,T;H^{1-s}(\Omega)')} \\ &+ \|\phi\|_{L^2(0,T;H^{s-1/2}(\Gamma))} + \|g_2\|_{L^2(0,T;L^2(\Gamma)\cap H^{-1/2+s}(\Gamma))}\Big) \\ &+ h\|\partial_t u\|_{L^2_{T,\Omega}} + h\|u\|_{H_T} + h\|f\|_{L^2_{T,\Omega}} \\ &+ \tau^r\Big(\|\partial_t u\|_{H^r(0,T;H^1(\Omega)')} + \|u\|_{H^r(0,T;H^1(\Omega))} + \|\phi\|_{H^r(0,T;H^{-1/2}(\Gamma))}\Big) \\ = \mathcal{O}(&h^s + \tau^r),\end{aligned}$$

for all $0 \le s \le 1$ and $0 \le r \le 1$ with $u \in H^r(0,T;H^{1+s}(\Omega))$, $\partial_t u \in H^r(0,T;L^2(\Omega))$, and $\phi \in H^r(0,T;H^{-1/2+s}(\Gamma))$, and $g_2(t) \in L^2(0,T;L^2(\Gamma)\cap H^{-1/2+s}(\Gamma))$.

Proof. The proof follows the lines of the proof of Theorem 2.32 and uses Theorem 3.21 with $v_{h,\tau} = P^\tau P_h u$ and $\psi_{h,\tau} = \Pi^\tau \Pi_h \phi$. □

Remark 3.23. In Corollary 3.22 it is also enough to demand $\phi \in H^r(0,T;H^{-1/2+s}(\mathcal{E}_\Gamma))$ and $g_2 \in L^2(0,T;H^{-1/2+s}(\mathcal{E}_\Gamma))$ if $s > 1/2$, see Remark 3.17.

3.2.4 Full Discretisation with the Classical Implicit Euler Scheme

In the following, we define a classical implicit Euler approach for the time discretisation of the semi-discrete system (3.17)–(3.18) or (3.21). In contrast to Section 3.2.3, we require more regularity in the time component for some model input data, namely, $q \in L^2(\Omega)$, $f \in H^1(0,T;L^2(\Omega))$, $g_1 \in H^1(0,T;H^{1/2}(\Gamma))$, and $g_2 \in H^1(0,T;L^2(\Gamma))$. With the notation introduced at the beginning of Section 2.4, the fully-discrete system reads:

Problem 3.24 (ClaIE-FVM-BEM)**.** Set $u_h^0 = P_h q \in \mathcal{S}^1(\mathcal{T})$. Find sequences $(u_h^n) \subset \mathcal{S}^1(\mathcal{T})$ and $(\phi_h^n) \subset \mathcal{P}^0(\mathcal{E}_\Gamma)$ for $n = 1,\dots,N$ such that

$$(d_\tau u_h^n, \mathcal{I}_h^* v_h)_\Omega + \mathcal{A}_V(u_h^n, \mathcal{I}_h^* v_h) - (\phi_h^n, \mathcal{I}_h^* v_h)_\Gamma = (f^n, \mathcal{I}_h^* v_h)_\Omega + (g_2^n, \mathcal{I}_h^* v_h)_\Gamma, \tag{3.33}$$

$$((1/2-\mathcal{K})u_h^n, \psi_h)_\Gamma + (\mathcal{V}\phi_h^n, \psi_h)_\Gamma = ((1/2-\mathcal{K})g_1^n, \psi_h)_\Gamma, \tag{3.34}$$

for all $v_h \in \mathcal{S}^1(\mathcal{T}) \subset H^1(\Omega)$ and $\psi_h \in \mathcal{P}^0(\mathcal{T}) \subset H^{-1/2}(\Gamma)$.
In compact notation: Find the sequence $\mathbf{u}_h^n = (u_h^n, \phi_h^n) \in \mathcal{H}^h = \mathcal{S}^1(\mathcal{T}) \times \mathcal{P}^0(\mathcal{E}_\Gamma)$ for $n = 1,\dots,N$ with $u_h^0 = P_h q$ such that

$$(d_\tau u_h^n, \mathcal{I}_h^* v_h)_\Omega + \mathcal{B}_V(\mathbf{u}_h^n; \mathbf{v}_h), = F_V(\mathbf{v}_h, t^n), \tag{3.35}$$

for all $\mathbf{v}_h = (v_h, \psi_h) \in \mathcal{H}^h$, where $F_V(\mathbf{v}_h, t^n)$ is defined in (3.20).

Remark 3.25. In fact, the system ClaIE-FVM-BEM (3.33)–(3.34) only differs from the variant VarIE-FVM-BEM (3.25)–(3.26) in the right-hand side, see also Remark 3.19. The advantage of this approach is the less expensive evaluation of the right-hand side. Numerical experiments have shown that the two approaches do not seem to differ in terms of the error.

We consider the solutions u_h^n and ϕ_h^n of (3.35) to be approximations for $u(t^n)$ and $\phi(t^n)$, respectively. First, we state the unique solvability of our fully-discrete system:

Lemma 3.26 (Well-posedness and discrete energy estimate)**.** *Let $\lambda_{\min}(\mathbf{A}) - \frac{1}{4}C_{\mathscr{K}} > 0$. Then the solution $(u_h^n, \phi_h^n) \in \mathscr{H}^h$ for $n = 1, \dots, N$ of (3.35) is unique and fulfils*

$$\sum_{n=1}^{N} \tau^n \big(\|u_h^n\|^2_{H^1(\Omega} + \|\phi_h^n\|^2_{H^{-1/2}(\Gamma)} \big) \\ \leq C\big(\|q\|^2_{L^2(\Omega)} + \|f\|^2_{H^1(0,T;L^2(\Omega))} + \|g_1\|^2_{H^1(0,T;H^{1/2}(\Gamma))} + \|g_2\|^2_{H^1(0,T;L^2(\Gamma))} \big),$$

with $C = C(\alpha_{Vstab}, \mathbf{A}, \mathbf{b}, c, \Omega) > 0$.

Proof. Testing (3.35) with $\mathbf{v}_h = (v_h, \psi_h) = (u_h^n, \phi_h^n)$, (3.31), the coercivity of $\mathscr{B}_V$, using that $\|\mathscr{I}_h^* u_h^n\|_{L^2(\Omega)} \leq C\|u_h^n\|_{L^2(\Omega)}$, and standard arguments as in Theorem 2.6 lead to

$$\|u_h^N\|^2_{L^2(\Omega)} + \sum_{n=1}^{N} \tau^n \left(\|u_h^n\|^2_{H^1(\Omega)} + \|\phi_h^n\|^2_{H^{-1/2}(\Gamma)} \right) \\ \leq C\Big(\|q\|^2_{L^2(\Omega)} + \sum_{n=1}^{N} \tau^n \big(\|f^n\|^2_{L^2(\Omega)} + \|g_2^n\|^2_{L^2(\Gamma)} + \|g_1^n\|^2_{H^{1/2}(\Gamma)} \big) \Big).$$

Due to the regularity of the model data, we may apply the Taylor approximation estimate of Lemma 1.13 to show the assertion. □

The following theorem provides the convergence of the fully-discrete scheme.

Theorem 3.27 (Convergence of the fully-discrete discrete system ClaIE-FVM-BEM, [ES19b, Theorem 35])**.** *There exists $h_{\max} > 0$ such that for h sufficiently small, i.e., $h < h_{\max}$, the following statement holds: Let $\lambda_{\min}(\mathbf{A}) - \frac{1}{4}C_{\mathscr{K}} > 0$, $C_{\mathscr{K}} \in [1/2, 1)$. Moreover, let u and ϕ and the data be sufficiently smooth. Then the solution $\mathbf{u}_h^n = (u_h^n, \phi_h^n) \in \mathscr{H}^h = \mathscr{S}^1(\mathscr{T}) \times \mathscr{P}^0(\mathscr{E}_\Gamma)$ of (3.35) converges to the weak solution $\mathbf{u} = (u, \phi)$ of (3.12). More precisely: if $u \in H^1(0,T;H^1(\Omega))$, $\partial_t u \in H^1(0,T;H^1(\Omega)')$, $\partial_{tt} u \in L^2(0,T;H^1(\Omega)')$, and $\phi \in H^1(0,T;H^{-1/2}(\Gamma))$, and the model input data $q \in L^2(\Omega)$, $f \in H^1(0,T;L^2(\Omega))$, $g_1 \in H^1(0,T;H^{1/2}(\Gamma))$, and $g_2 \in H^1(0,T;L^2(\Gamma))$, there holds*

$$\Big[\sum_{n=1}^{N} \tau^n \|\mathbf{u}(t^n) - \mathbf{u}_h^n\|^2_{\mathscr{H}} \Big]^{1/2} = \Big[\sum_{n=1}^{N} \tau^n \big(\|u(t^n) - u_h^n\|^2_{H^1(\Omega)} + \|\phi(t^n) - \phi_h^n\|^2_{H^{-1/2}(\Gamma)} \big) \Big]^{1/2} \\ \leq C\Big[\|\mathbf{u} - \mathbf{v}_h\|_{\mathscr{H}_T} + \|\partial_t u - \partial_t v_h\|_{H_T'} + h^{1/2}\big(\|g_2 - \overline{g}_2\|_{L^2_{T,\Gamma}} + \tau \|\partial_t g_2 - \partial_t \overline{g}_2\|_{L^2_{T,\Gamma}} \big) \\ + h\big(\|\partial_t v_h\|_{L^2_{T,\Omega}} + \|f\|_{L^2_{T,\Omega}} + \tau \|\partial_t f\|_{L^2_{T,\Omega}} + \|v_h\|_{H_T} + \tau \|\partial_t v_h\|_{H_T} \big) \\ + \tau\big(\|\partial_{tt} u\|_{H_T'} + \|\partial_t \mathbf{u} - \partial_t \mathbf{v}_h\|_{\mathscr{H}_T} \big) \Big],$$

for all $\mathbf{v}_h = (v_h, \psi_h) \in H^1(0,T;\mathscr{S}^1(\mathscr{T})) \times H^1(0,T;\mathscr{P}^0(\mathscr{E}_\Gamma))$ with $v_h(0) = P_h q$ and $C = C(\alpha_{Vstab}, \mathbf{A}, \mathbf{b}, c, \Omega) > 0$. The statement also holds if we use the upwind stabilised bilinear form with $\mathscr{A}_V^{up}$.

Proof. First, we split the error into an approximation error and a discrete error component, i.e., for arbitrary $\mathbf{v}_h = (v_h, \psi_h) \in H^1(0,T;\mathcal{S}^1(\mathcal{T}) \times H^1(0,T;\mathcal{P}^0(\mathcal{E}_\Gamma))$ with $v_h(0) = P_h q$ we split

$$\|\mathbf{u}(t^n) - \mathbf{u}_h^n\|_{\mathcal{H}} \le \|\mathbf{u}(t^n) - \mathbf{v}_h^n\|_{\mathcal{H}} + \|\mathbf{u}_h^n - \mathbf{v}_h^n\|_{\mathcal{H}}, \tag{3.36}$$

for $n = 1, \dots, N$. Next we estimate the discrete error part and define $\mathbf{w}_h^n := (w_h^n, \varphi_h^n) := \mathbf{u}_h^n - \mathbf{v}_h^n \in \mathcal{H}^h$. Note that $w_h^0 = 0$. Following exactly the lines of the proof for Theorem 3.21 but using (3.12) evaluated in t^n and (3.35), we arrive at

$$\begin{aligned} &\frac{1}{2\tau^n}\big(\|w_h^n\|_\chi^2 - \|w_h^{n-1}\|_\chi^2\big) + \|\mathbf{w}_{h,\tau}^n\|_{\mathcal{H}}^2 \\ &\qquad \lesssim \langle \partial_t u(t^n), w_h^n\rangle_\Omega - (d_\tau v_h^n, \mathcal{I}_h^* w_h^n)_\Omega + F_V(\mathbf{w}_h^n; t^n) - F(\mathbf{w}_h^n; t^n) \\ &\qquad\quad + \mathcal{B}(\mathbf{v}_h^n; \mathbf{w}_h^n) - \mathcal{B}_V(\mathbf{v}_h^n; \mathbf{w}_h^n) + \mathcal{B}(\mathbf{u}(t^n) - \mathbf{v}_h^n; \mathbf{w}_h^n). \end{aligned} \tag{3.37}$$

For the difference of the first two terms on the right-hand side, we see with the Cauchy-Schwarz inequality, and the estimate (1.8) for $\mathcal{I}_h^*$ that

$$\begin{aligned} &\langle \partial_t u(t^n), w_h^n\rangle_\Omega - (d_\tau v_h^n, \mathcal{I}_h^* w_h^n)_\Omega \\ &= \langle \partial_t u(t^n) - d_\tau u(t^n), w_h^n\rangle_\Omega + \langle d_\tau u(t^n) - d_\tau v_h^n, w_h^n\rangle_\Omega + (d_\tau v_h^n, w_h^n - \mathcal{I}_h^* w_h^n)_\Omega \\ &\lesssim \Big(\|\partial_t u(t^n) - d_\tau u(t^n)\|_{H^1(\Omega)'} + \|d_\tau u(t^n) - d_\tau v_h^n\|_{H^1(\Omega)'} + h\|d_\tau v_h^n\|_{L^2(\Omega)}\Big)\|w_h^n\|_{H^1(\Omega)}. \end{aligned}$$

The other terms in (3.37) can be bounded as before, using Lemma 3.9, Lemma 3.10, and the continuity of the bilinear form $\mathcal{B}$. With standard manipulations we estimate

$$\begin{aligned} &\frac{1}{2}\|w_{h,\tau}^N\|_\chi^2 + \sum_{n=1}^N \tau^n \|\mathbf{w}_{h,\tau}^n\|_{\mathcal{H}}^2 \\ &\quad \lesssim \sum_{n=1}^N \tau^n \Big(\|\partial_t u(t^n) - d_\tau u(t^n)\|_{H^1(\Omega)'}^2 + \|d_\tau u(t^n) - d_\tau v_h^n\|_{H^1(\Omega)'}^2 + h^2\|d_\tau v_h^n\|_{L^2(\Omega)}^2 \\ &\qquad\quad + h^2\|f^n\|_{L^2(\Omega)}^2 + h\|g_2^n - \overline{g}_2(t^n)\|_{L^2(\Gamma)}^2 + h^2\|v_h^n\|_{H^1(\Omega)}^2 + \|\mathbf{u}(t^n) - \mathbf{v}_h^n\|_{\mathcal{H}}^2\Big). \end{aligned}$$

With classical Taylor series, i.e., with the integral form of the remainder, we estimate

$$\begin{aligned} \sum_{n=1}^N \tau^n \|\partial_t u(t^n) - d_\tau u(t^n)\|_{H^1(\Omega)'}^2 &\le \tau^2 \|\partial_{tt} u\|_{H_T'}^2, \\ \sum_{n=1}^N \tau^n \|d_\tau u(t^n) - d_\tau v_h^n\|_{H^1(\Omega)'}^2 &\le \|\partial_t u - \partial_t v_h\|_{H_T'}^2, \\ \sum_{n=1}^N \tau^n \|d_\tau v_h^n\|_{L^2(\Omega)}^2 &\le \|\partial_t v_h\|_{L^2_{T,\Omega}}^2. \end{aligned}$$

For all the other terms we use Lemma 1.13 to prove the assertion with the error splitting (3.36). □

For simplicity we only state first-order convergence which follows directly from Theorem 3.27 with the aid of Lemma 2.27.

Corollary 3.28 (First-order convergence of the fully-discrete ClaIE-FVM-BEM). *Let P_h be H^1-stable, e.g., $\mathscr{T}$ is quasi-uniform. Additionally to the assumptions of Theorem 3.27, we require $u \in H^1(0,T;H^2(\Omega))$, $\partial_t u \in H^1(0,T;L^2(\Omega))$, $\partial_{tt} u \in L^2(0,T;H^1(\Omega)')$, and $\phi \in H^1(0,T;H^{1/2}(\Gamma))$, and for the model input data $q \in L^2(\Omega)$, $f \in H^1(0,T;L^2(\Omega))$, $g_1 \in H^1(0,T;H^{1/2}(\Gamma))$, and $g_2 \in H^1(0,T;H^{1/2}(\Gamma))$. Then there holds*

$$\Big[\sum_{n=1}^{N} \tau^n \Big(\|u(t^n) - u_h^n\|^2_{H^1(\Omega)} + \|\phi(t^n) - \phi_h^n\|^2_{H^{-1/2}(\Gamma)} \Big)\Big]^{1/2} = \mathscr{O}(\tau + h).$$

This also holds if we use the upwind stabilised bilinear form $\mathscr{A}_V^{up}$ instead of $\mathscr{A}_V$.

Remark 3.29. The left-hand sides from Theorem 3.27 and Corollary 3.28 are discrete versions of the norm $\|\cdot\|_{\mathscr{H}_T}$. By some linear interpolation and with (2.29), we can state the assertions as for the version with the variant implicit Euler time discretisation scheme in Section 3.2.3. This shows that the results are asymptotically equivalent, but the approach using the variant requires less regularity of the solution. Concerning the needed regularity, note that in Corollary 3.28 it is enough to demand $\phi \in H^1(0,T;H^{1/2}(\mathscr{E}_\Gamma))$ and $g_2 \in H^1(0,T;H^{1/2}(\mathscr{E}_\Gamma))$ if $s > 1/2$, see Remark 3.17.

Another advantage of the analysis of this particular time discretisation is that we can easily adapt the theory to a Crank-Nicolson scheme obtaining convergence of second order in time.

3.3 Coupling with the Streamline Upwind Petrov Galerkin Method

The FVM is not the only method to tackle the problem of obtaining non-oscillatory solutions to convection-dominated problems. A second, easy and linear, option is the SUPG method, which is based on FEM and therefore extends the FEM-BEM coupling of Chapter 2. In this section we only consider piecewise linear finite elements for SUPG. The main idea is to add artificial diffusion in the direction of the streamlines and thus to reduce the effect of the convection. The method is widely used for diffusion-convection-reaction problems and, as concluded in [Aug+11], it is a good alternative to FVM as it is an efficient method that computes solutions with sharp and exact layers, although it does not conserve fluxes. Still, it can produce some spurious oscillations, which FVM with upwinding does not. This is mainly based on the stabilisation parameter which is used in SUPG to steer the amount of artificial diffusion; but in general, an optimal parameter cannot be determined exactly.

The existing literature concerned with the analysis of the SUPG method (for time-dependent problems) cannot be transferred directly to the analysis of the SUPG-BEM coupling. The method itself has been introduced in [BH82]. In [Bur11] a semi-discrete version of SUPG is analysed, which is achieved by obtaining control over the so-called material derivative (in this case defined by $D_t v := \partial_t v + \mathbf{b} \cdot \nabla v$). There we would need the assumption that $\operatorname{div} \mathbf{b} = 0$ and furthermore some boundary terms would not vanish due to the missing (homogeneous) Dirichlet boundary conditions in our coupling approach. In [Bur10] this approach is also applied to a fully-discrete SUPG method, but for a transport problem (without any diffusion). Stability of a fully-discrete SUPG method has then been investigated in [BGS04]. Another technique is used [JN11] (and also in [FGAN16] for localised estimates). There, classical energy estimates lead to a convergence result. For a stabilisation parameter not depending on the length of the time step, an analysis via the control of the material derivative is also included, but also requires a divergence-free convectional velocity and Dirichlet boundary conditions. Therefore, we will use the same technique as for the FVM-BEM coupling. In the following we assume that the initial value $q = 0$ to ease the notation.

We introduce the stabilisation parameter $\delta|_K := \delta_K$ for $K \in \mathscr{T}$ via

$$\delta_K = \begin{cases} \frac{\delta_1 h_K}{\|\mathbf{b}\|_{L^\infty(K)}} & \text{for } Pe > 1, \\ \frac{\delta_2 h_K^2}{\|\mathbf{b}\|_{L^\infty(K)}} & \text{for } Pe \leq 1, \end{cases} \tag{3.38}$$

with constants $\delta_1 \in \mathbb{R}_+$ and $\delta_2 \in \mathbb{R}_+$ chosen appropriately. The (mesh) Péclet number is defined by

$$Pe = \frac{\|\mathbf{b}\|_{L^\infty(\Omega)} h}{\|\mathbf{A}\|_{L^\infty(\Omega)}}.$$

The interesting case is $Pe > 1$, when the problem is convection-dominated. In the other case, a stabilisation would not be necessary but can still be applied. In the following, we will use δ as an abbreviation for the piecewisely defined parameter. For a discussion of the choice of the stabilisation parameter (in the steady-state case), see [Kno08].

To add the artificial diffusion, we look at the original equation (3.1) of the model problem (without the time derivative, this will be added to the right-hand side) in the discrete space $\mathscr{S}^1(\mathscr{T})$ and test it with $\mathbf{b} \cdot \nabla v_h$. Writing this element-wise and multiplying the resulting term with the stabilisation parameter δ_K yields

$$\sum_{K \in \mathscr{T}} \delta_K(\operatorname{div}(-\mathbf{A}\nabla u_h(t) + \mathbf{b}u_h(t)) + cu_h(t), \mathbf{b} \cdot \nabla v_h)_K.$$

The divergence has to be understood piecewisely for discrete functions. Then we can add this to the usual weak Galerkin formulation and obtain the SUPG bilinear form

$$\begin{aligned} \mathscr{A}_{SUPG}(u_h(t), v_h) := (\mathbf{A}\nabla u_h(t) - \mathbf{b}u_h(t), \nabla v_h)_\Omega + (cu_h(t), v_h)_\Omega + (\mathbf{b} \cdot \mathbf{n}\, u_h(t), v_h)_{\Gamma^{out}} \\ + \sum_{K \in \mathscr{T}} \delta_K(\operatorname{div}(-\mathbf{A}\nabla u_h(t) + \mathbf{b}u_h(t)) + cu_h(t), \mathbf{b} \cdot \nabla v_h)_K, \end{aligned}$$

for $(u_h, \phi_h) \in H^1(0, T; \mathscr{S}^1(\mathscr{T})) \times L^2(0, T; \mathscr{P}^0(\mathscr{E}_\Gamma))$ and $(v_h, \psi_h) \in \mathscr{H}^h = \mathscr{S}^1(\mathscr{T}) \times \mathscr{P}^0(\mathscr{E}_\Gamma)$. The right-hand side has to be modified appropriately.

Most of the estimates will be given in a weaker norm, tailored to the SUPG method. The SUPG norm is defined by

$$\|u_h\|_{SUPG} := \left(\lambda_{min}(\mathbf{A}) \|\nabla u_h\|^2_{L^2(\Omega)} + \sum_{K \in \mathscr{T}} \delta_K \|\mathbf{b} \cdot \nabla u_h\|^2_{L^2(K)} + \left\| \left(\frac{1}{2} \operatorname{div} \mathbf{b} + c \right)^{1/2} u_h \right\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Analogous to the Bochner-Sobolev norms, we define the norm $L^2(0,T;SUPG)$ with the SUPG norm.

With the definition of the SUPG norm, one can show that the SUPG norm is equivalent to the usual H^1-norm (cp. [JN11, Section 5]).

Lemma 3.30. *The SUPG norm fulfils for $v \in H^1(\Omega)$:*

$$C_1\|v\|_{SUPG} \leq \|v\|_{H^1(\Omega)} \leq C_2\|v\|_{SUPG},$$

with $C_1, C_2 > 0$ being independent of h.

3.3.1 Semi-Discrete SUPG-BEM

Changing the bilinear form to $\mathscr{A}_{SUPG}$ in the FEM-BEM coupling and using the definition of $g_{\mathscr{K}}$ from Section 2.2, we can formulate the semi-discrete problem as

Problem 3.31 (Semi-discrete SUPG-BEM)**.** Given $f \in L^2_{T,\Omega}$, $g_2 \in B_T$, and $g_{\mathscr{K}} \in B'_T$, find $u_h \in Q^h_T$ and $\phi_h \in B^h_T$ such that

$$\begin{aligned}(\partial_t u_h(t), v_h)_\Omega + \mathscr{A}_{SUPG}(u_h(t), v_h) - (\phi_h(t), v_h)_\Gamma = (f(t), v_h)_\Omega + (g_2(t), v_h)_\Gamma,& \\ + \sum_{K\in\mathscr{T}} \delta_K(f(t) - \partial_t u_h(t), \mathbf{b}\cdot\nabla v_h)_K,& \end{aligned} \tag{3.39}$$

$$((1/2 - \mathscr{K})u_h(t), \psi_h)_\Gamma + (\mathscr{V}\phi_h(t), \psi_h)_\Gamma = (g_{\mathscr{K}}(t), \psi_h)_\Gamma, \tag{3.40}$$

for all test functions $v_h \in \mathscr{S}^1(\mathscr{T})$ and $\psi_h \in \mathscr{P}^0(\mathscr{E}_\Gamma)$, and for a.e. $t \in [0,T]$. In short hand notation, we can write this as: find $\mathbf{u}_h = (u_h, \phi_h) \in Q^h_T \times B^h_T$, such that

$$\begin{aligned}(\partial_t u_h(t), v_h)_\Omega + \mathscr{B}_{SUPG}(\mathbf{u}_h(t); \mathbf{v}_h) = F_{SUPG}(\mathbf{v}_h, t)& \\ - \sum_{K\in\mathscr{T}} \delta_K(\partial_t u_h(t), \mathbf{b}\cdot\nabla v_h)_K,&\end{aligned}$$

for all test functions $\mathbf{v}_h = (v_h, \psi_h) \in \mathscr{H}^h$ and a.e. $t \in [0,T]$.

The added SUPG-BEM bilinear form has the form

$$\begin{aligned}\mathscr{B}_{SUPG}((u_h(t), \phi_h(t)); (v_h, \psi_h)) := \mathscr{A}_{SUPG}(u_h(t), v_h) - (\phi_h(t), v_h)_\Gamma& \\ + ((1/2 - \mathscr{K})u_h(t), \psi_h)_\Gamma + (\mathscr{V}\phi_h(t), \psi_h)_\Gamma,&\end{aligned}$$

and the added right-hand side

$$\begin{aligned}F_{SUPG}((v_h, \psi_h), t) := (f(t), v_h)_\Omega + (g_2(t), v_h)_\Gamma& \\ + \sum_{K\in\mathscr{T}} \delta_K(f(t), \mathbf{b}\cdot\nabla v_h)_K + ((1/2 - \mathscr{K})g_1(t), \psi_h)_\Gamma.&\end{aligned} \tag{3.41}$$

Before we begin the analysis of the SUPG-BEM method, we need some preliminary lemmata concerning some properties of the bilinear form. First, we state the coercivity of the SUPG bilinear form.

Lemma 3.32 (Coercivity of the SUPG and SUPG-BEM bilinear forms). *If the SUPG parameters are chosen such that*

$$0 \leq \delta_K \leq \frac{1}{2} \min \left\{ \frac{\gamma}{\| \operatorname{div}(\mathbf{b}) + c\|^2_{L^\infty(K)}}, \frac{h_K^2}{\lambda_{min}(\mathbf{A}) C_{inv}^2} \right\}, \tag{3.42}$$

with the constant C_{inv} from the inverse inequality of Lemma 1.8, then the bilinear form $\mathscr{A}_{SUPG}$ is coercive, i.e., it fulfils for all $u_h \in \mathscr{S}^1(\mathscr{T})$

$$\mathscr{A}_{SUPG}(u_h, u_h) \geq \frac{1}{2} \|u_h\|^2_{SUPG}.$$

Furthermore, if $\lambda_{min}(\mathbf{A}) - \frac{1}{2} C_{\mathscr{K}} > 0$ with $C_{\mathscr{K}} \in [1/2, 1)$, then the bilinear form $\mathscr{B}_{SUPG}$ is also coercive, i.e., it fulfils for all $(u_h, \phi_h) \in \mathscr{H}^h = \mathscr{S}^1(\mathscr{T}) \times \mathscr{P}^0(\mathscr{E}_\Gamma)$

$$\mathscr{B}_{SUPG}((u_h, \phi_h); (u_h, \phi_h)) \geq \alpha_{SUPG} \left(\|u_h\|^2_{SUPG} + \|\phi_h\|^2_{H^{-1/2}(\Gamma)} \right),$$

with a constant $\alpha_{SUPG} > 0$ being defined as

$$\alpha_{SUPG} = \min \left\{ 1, \frac{1}{2} \left[\frac{1}{2} \lambda_{min}(\mathbf{A}) + 1 - \sqrt{\left(\frac{1}{2} \lambda_{min}(\mathbf{A}) - 1 \right)^2 + C_{\mathscr{K}}} \right] \right\}.$$

Proof. The proof follows from [EOS17, Theorem 1] combined with [Sty05, Lemma 10.3]. □

Hence we choose δ_1 and δ_2 in δ_K of (3.38) such that (3.42) is fulfilled.

Later, we will also need the following results concerning the consistency of the SUPG bilinear form.

Lemma 3.33 (Difference between weak and SUPG bilinear forms). *For $\mathbf{v}_h = (v_h, \psi_h) \in \mathscr{H}^h$ and $\mathbf{w}_h = (w_h, \varphi_h) \in \mathscr{H}^h$, it holds that*

$$|\mathscr{B}(\mathbf{v}_h; \mathbf{w}_h) - \mathscr{B}_{SUPG}(\mathbf{v}_h; \mathbf{w}_h)| \leq C \sum_{K \in \mathscr{T}} \delta_K \|v_h\|_{H^1(K)} \|w_h\|_{H^1(K)},$$

with a constant $C = C(\mathbf{A}, \mathbf{b}, c) > 0$ that depends on the model data. Note that there holds at least $\delta_K = \mathscr{O}(h)$, such that we have a first-order estimate in h.

Proof. In the following, we use that $\Delta v_h = 0$ (for piecewise linear finite elements) and omit the dependency on t:

$$\begin{aligned}
|\mathscr{B}(\mathbf{v}_h; \mathbf{w}_h) - \mathscr{B}_{SUPG}(\mathbf{v}_h; \mathbf{w}_h)| &= \left| \sum_{K \in \mathscr{T}} \delta_K (\operatorname{div}(-\mathbf{A} \nabla v_h + \mathbf{b} v_h) + c v_h \, , \mathbf{b} \cdot \nabla w_h)_K \right| \\
&= \left| \sum_{K \in \mathscr{T}} \delta_K (-\operatorname{div}(\mathbf{A}) \nabla v_h + \operatorname{div}(\mathbf{b}) v_h + \mathbf{b} \cdot \nabla v_h + c v_h \, , \mathbf{b} \cdot \nabla w_h)_K \right| \\
&\leq \sum_{K \in \mathscr{T}} \delta_K \Big(\| \operatorname{div}(\mathbf{A})\|_{L^\infty(K)} \|\nabla v_h\|_{L^2(K)} \|\mathbf{b} \cdot \nabla w_h\|_{L^2(K)} + \|\mathbf{b}\|_{L^\infty(K)} \|\nabla v_h\|_{L^2(K)} \|\mathbf{b} \cdot \nabla w_h\|_{L^2(K)} \\
&\qquad + \| \operatorname{div}(\mathbf{b}) + c\|_{L^\infty(K)} \|v_h\|_{L^2(K)} \|\mathbf{b} \cdot \nabla w_h\|_{L^2(K)} \Big) \\
&\leq C \sum_{K \in \mathscr{T}} \delta_K \|v_h\|_{H^1(K)} \|w_h\|_{H^1(K)}.
\end{aligned}$$

□

Unfortunately, due to the term with the time derivative on the right-hand side of the formulation (3.39)–(3.40), it is not directly possible to obtain the well-posedness and an energy estimate of the semi-discrete scheme. In the next section we will see that we obtain a well-posed problem for the full discretisation. Still, under some special assumptions (e.g. $\delta = \mathscr{O}(h^2)$, which is fulfilled, for example, when $Pe \leq 1$), we can show the convergence and an error estimate of the semi-discrete SUPG-BEM coupling scheme. This result is suboptimal also in terms of the dependency of the occurring constants on the model data. However, the result shows that in this case SUPG has the same asymptotic behaviour as the FEM-BEM coupling.

Theorem 3.34 (Convergence of the semi-discrete SUPG-BEM)**.** *Let h be small enough and let $\lambda_{\min}(\mathbf{A}) - \frac{1}{2}C_{\mathscr{K}} > 0$. Furthermore let $(u,\phi) \in Q_T \times B_T$ and $(u_h, \phi_h) \in Q_T^h \times B_T^h$ denote the solutions of Problem 3.2 and Problem 3.31, respectively. Furthermore, let $\delta = \mathscr{O}(h^2)$ and P_h be H^1–stable. Then, for $\partial_t u \in L^2_{T,\Omega}$, there holds that*

$$
\begin{aligned}
&\|\partial_t u - \partial_t u_h\|_{H_T'} + \|u - u_h\|_{L^2(0,T;SUPG)} + \|\phi - \phi_h\|_{B_T} \\
&\leq C\big(\|u - v_h\|_{Q_T} + \|\phi - \psi_h\|_{B_T} + \delta\|\partial_t u - \partial_t v_h\|_{L^2_{T,\Omega}} + \delta\|v_h\|_{H_T} + \delta\|\partial_t v_h\|_{L^2_{T,\Omega}} + \delta\|f\|_{L^2_{T,\Omega}}\big),
\end{aligned}
$$

for all functions $v_h \in Q_T^h$ and $\psi_h \in B_T^h$ with a constant $C = C(\alpha_{SUPG}, \mathbf{A}, \mathbf{b}, c, \Omega) > 0$ which is independent of the model input data f, g_2, $g_{\mathscr{K}}$.

Proof. We proceed as in the FVM-BEM case, see Theorem 3.15. Let $\mathbf{v}_h = (v_h, \psi_h) \in Q_T^h \times B_T^h$ be arbitrary. First, we split the error into an approximation error and a discrete error component

$$
\|\mathbf{u} - \mathbf{u}_h\|_{SUPG\times H^{-1/2}(\Gamma)} \leq \|\mathbf{u} - \mathbf{v}_h\|_{SUPG\times H^{-1/2}(\Gamma)} + \|\mathbf{u}_h - \mathbf{v}_h\|_{SUPG\times H^{-1/2}(\Gamma)}, \tag{3.43}
$$

$$
\|\partial_t u(t) - \partial_t u_h(t)\|_{H^1(\Omega)'} \leq \|\partial_t u(t) - \partial_t v_h(t)\|_{H^1(\Omega)'} + \|\partial_t u_h(t) - \partial_t v_h(t)\|_{H^1(\Omega)'}. \tag{3.44}
$$

Then we only have to estimate the norms of the discrete error $\mathbf{w}_h = (w_h, \varphi_h) := \mathbf{u}_h - \mathbf{v}_h \in Q_T^h \times B_T^h$ and use that $\|\mathbf{u} - \mathbf{u}_h\|_{SUPG\times H^{-1/2}(\Gamma)} \leq \|\mathbf{u} - \mathbf{u}_h\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)}$.

Using the coercivity Lemma 3.32 of the SUPG-BEM bilinear form $\mathscr{B}_{SUPG}(\cdot;\cdot)$, we can write for a fixed time t

$$
\frac{1}{2}\frac{d}{dt}\|w_h(t)\|^2_{L^2(\Omega)} + \|\mathbf{w}_h(t)\|^2_{SUPG\times H^{-1/2}(\Gamma)} \lesssim \langle \partial_t w_h(t), w_h(t)\rangle_\Omega + \mathscr{B}_{SUPG}(\mathbf{w}_h(t); \mathbf{w}_h(t)).
$$

Now we can use the formulation of the problem (3.39)–(3.40), add the weak form (3.12) and the Cauchy-Schwarz inequality to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_h(t)\|^2_{L^2(\Omega)} + \|\mathbf{w}_h(t)\|^2_{SUPG\times H^{-1/2}(\Gamma)} \\
&\lesssim \langle \partial_t u(t), w_h(t)\rangle_\Omega - \langle \partial_t v_h(t), w_h(t)\rangle_\Omega + F_{SUPG}(\mathbf{w}_h; t) - F(\mathbf{w}_h; t) \\
&\quad + \mathscr{B}(\mathbf{v}_h(t); \mathbf{w}_h(t)) - \mathscr{B}_{SUPG}(\mathbf{v}_h(t); \mathbf{w}_h(t)) + \mathscr{B}(\mathbf{u}(t) - \mathbf{v}_h(t); \mathbf{w}_h(t)) \\
&\quad - \delta(\partial_t u_h(t), \mathbf{b}\cdot\nabla w_h(t))_\Omega + \delta(\partial_t v_h(t) - \partial_t v_h(t), \mathbf{b}\cdot\nabla w_h(t))_\Omega \\
&\lesssim \big(\|\partial_t u(t) - \partial_t v_h(t)\|_{H^1(\Omega)'} + \delta\|f(t)\|_{L^2(\Omega)} + \delta\|v_h(t)\|_{H^1(\Omega)} \\
&\quad + \|\mathbf{u}(t) - \mathbf{v}_h(t)\|_{\mathscr{H}} + \delta\|\partial_t u_h(t) - \partial_t v_h(t)\|_{L^2(\Omega)} + \delta\|\partial_t v_h(t)\|_{L^2(\Omega)}\big)\|\mathbf{w}_h(t)\|_{\mathscr{H}}.
\end{aligned} \tag{3.45}
$$

Here we also used Lemma 3.33, the continuity of the bilinear form $\mathscr{B}$ and $\mathscr{H} = H^1(\Omega) \times H^{-1/2}(\Gamma)$. Before we move on, we look at an estimate for the error made in the time derivative, where we use the error splitting (3.44). Then we can rewrite the norm

$$\|\partial_t u_h(t) - \partial_t v_h(t)\|_{H^1(\Omega)'} = \sup_{0\neq\vartheta\in H^1(\Omega)} \frac{(\partial_t w_h(t), \vartheta)_\Omega}{\|\vartheta\|_{H^1(\Omega)}} = \sup_{0\neq\vartheta\in H^1(\Omega)} \frac{(\partial_t w_h(t), P_h\vartheta)_\Omega}{\|\vartheta\|_{H^1(\Omega)}}.$$

Then, using (3.8), (3.39), the continuity of $\mathscr{A}$ and $\mathscr{A}_{SUPG}$, Lemma 3.33, inserting some terms and the Cauchy-Schwarz inequality, we can write with $w_h = u_h - v_h$:

$$\begin{aligned}(\partial_t w_h(t), P_h\vartheta)_\Omega &= (\partial_t u(t), P_h\vartheta)_\Omega - (\partial_t v_h(t), P_h\vartheta)_\Omega + \mathscr{A}(u(t), P_h\vartheta) - \mathscr{A}_{SUPG}(u_h(t), P_h\vartheta)\\ &\quad - (\phi(t) - \phi_h(t), P_h\vartheta)_\Gamma + \delta(f(t) - \partial_t u_h(t), \mathbf{b}\cdot\nabla P_h\vartheta)_\Omega\\ &= (\partial_t u(t) - \partial_t v_h(t), P_h\vartheta)_\Omega + \mathscr{A}(v_h(t), P_h\vartheta) - \mathscr{A}_{SUPG}(v_h(t), P_h\vartheta)\\ &\quad - \mathscr{A}_{SUPG}(u_h(t) - v_h(t), P_h\vartheta) + \mathscr{A}(u(t) - v_h(t), P_h\vartheta)\\ &\quad - (\phi(t) - \psi_h(t), P_h\vartheta)_\Gamma - (\phi_h(t) - \psi_h(t), P_h\vartheta)_\Gamma + \delta(f(t), \mathbf{b}\cdot\nabla P_h\vartheta)_\Omega\\ &\quad - \delta(\partial_t u_h(t) - \partial_t v_h(t), \mathbf{b}\cdot\nabla P_h\vartheta)_\Omega - \delta(\partial_t v_h(t), \mathbf{b}\cdot\nabla P_h\vartheta)_\Omega\\ &\lesssim \big(\|\partial_t u(t) - \partial_t v_h(t)\|_{H^1(\Omega)'} + \delta\|v_h(t)\|_{H^1(\Omega)} + \|\mathbf{u}(t) - \mathbf{v}_h(t)\|_{\mathscr{H}}\\ &\quad + \|\phi_h(t) - \psi_h(t)\|_{H^{-1/2}(\Gamma)} + \delta\|f(t)\|_{L^2(\Omega)} + \delta\|\partial_t v_h(t)\|_{L^2(\Omega)}\\ &\quad + \delta\|\partial_t u_h(t) - \partial_t v_h(t)\|_{L^2(\Omega)}\big)\,\|P_h\vartheta\|_{H^1(\Omega)}.\end{aligned}$$

The discrete terms are estimated as for (3.45). Using the H^1–stability of the L^2–projection P_h, we obtain an estimate for $\|\partial_t u_h(t) - \partial_t v_h(t)\|_{H^1(\Omega)'}$. We still have the term $\delta\|\partial_t u_h(t) - \partial_t v_h(t)\|_{L^2(\Omega)}$ on the right-hand side. Using an inverse inequality we can further bound this by

$$\delta\|\partial_t w_h(t)\|_{L^2(\Omega)} \le Ch^{-1}\delta\|\partial_t w_h(t)\|_{H^1(\Omega)'}.$$

Now we can get back to the first estimate (3.45). Using Young's inequality with $\varepsilon > 0$ and adding the estimate for the time derivative, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w_h(t)\|^2_{L^2(\Omega)} + \|\partial_t w_h(t))\|^2_{H^1(\Omega)'} + \|\mathbf{w}_h(t)\|^2_{SUPG\times H^{-1/2}(\Gamma)}\\ &\qquad \le C\frac{1}{\varepsilon}\Big(\|\partial_t u(t) - \partial_t v_h(t)\|^2_{H^1(\Omega)'} + \delta^2\|f(t)\|^2_{L^2(\Omega)} + \delta^2\|v_h(t)\|^2_{H^1(\Omega)}\\ &\qquad\quad + \|\mathbf{u}(t) - \mathbf{v}_h(t)\|^2_{\mathscr{H}} + \delta^2\|\partial_t v_h(t)\|^2_{L^2(\Omega)}\Big)\\ &\qquad\quad + C\varepsilon\Big[h^{-2}\delta^2\|\partial_t w_h(t)\|^2_{H^1(\Omega)'} + \|\mathbf{w}_h(t)\|^2_{SUPG\times H^{-1/2}(\Gamma)}\Big].\end{aligned}$$

Now, choosing $C\varepsilon < 1/2$, we can bring the last two terms over to the other side. Assuming that $\delta = \mathcal{O}(h^2)$ and that h is small enough, the term with time derivative can be incorporated in the existing term.

Integrating over t from 0 to T, and the fact that

$$\int_0^T \frac{1}{2}\frac{d}{dt}\|w_h\|^2_{L^2(\Omega)}\,\mathrm{d}t = \underbrace{\frac{1}{2}\|w_h(T)\|^2_{L^2(\Omega)}}_{\ge 0} - \underbrace{\frac{1}{2}\|w_h(0)\|^2_{L^2(\Omega)}}_{=0} \ge 0,$$

together with the error splitting (3.43)–(3.44) yields the desired result. □

Remark 3.35. The terms involving the stabilisation parameter δ amount for the perturbation introduced by the SUPG method. Because δ scales at least like h, we can interpret these terms as a first-order perturbation, which therefore will not interfere with the optimal convergence rate, which will be stated in the next corollary.

By using the usual approximation arguments of Lemma 2.27, $\delta = \mathcal{O}(h^2)$ and Theorem 3.34 we get:

Corollary 3.36. *Let $(u,\phi) \in Q_T \times B_T$ and $(u_h, \phi_h) \in Q_T^h \times B_T^h$ denote the solutions of Problem 3.2 and Problem 3.31, respectively. Furthermore, let P_h be H^1-stable, $\lambda_{\min}(\mathbf{A}) - \frac{1}{2}C_{\mathcal{K}} > 0$ and $\delta = \mathcal{O}(h^2)$. Then, for h small enough, $u \in L^2(0,T;H^{1+s}(\Omega))$, $\partial_t u \in L^2_{T,\Omega}$ and $\phi \in L^2(0,T;H^{s-1/2}(\Gamma))$, there holds that*

$$\begin{aligned}
&\|\partial_t u - \partial_t u_h\|_{H_T'} + \|u - u_h\|_{L^2(0,T;SUPG)} + \|\phi - \phi_h\|_{B_T} \\
&\leq C h^s \big(\|\partial_t u\|_{L^2_{T,\Omega}} + \|u\|_{L^2(0,T;H^{1+s}(\Omega))} + \|\phi\|_{L^2(0,T;H^{s-1/2}(\Gamma))} + \|f\|_{L^2_{T,\Omega}} \big) = \mathcal{O}(h^s),
\end{aligned}$$

with a constant $C = C(\alpha_{SUPG}, \mathbf{A}, \mathbf{b}, c, \Omega, \eta, C_P) > 0$ and $0 \leq s \leq 1$.

3.3.2 Fully-Discrete SUPG-BEM

For the fully-discrete coupling of SUPG and BEM, we will also use the variant of the implicit Euler method from Section 2.4.

Problem 3.37 (VarIE-SUPG-BEM). Find $u_{h,\tau} \in Q_T^{h,\tau}$ and $\phi_{h,\tau} \in B_T^{h,\tau}$ such that

$$\begin{aligned}
(\widehat{\partial_t u}^n_{h,\tau}, v_h)_\Omega + \mathscr{A}_{SUPG}(\widehat{u}^n_{h,\tau}, v_h) - (\widehat{\phi}^n_{h,\tau}, v_h)_\Gamma &= (\widehat{f}^n, v_h)_\Omega + (\widehat{g}^n_2, v_h)_\Gamma \\
&\quad + \sum_{K \in \mathcal{T}} \delta_K (\widehat{f}^n - \widehat{\partial_t u}^n_{h,\tau}, \mathbf{b} \cdot \nabla v_h)_\Omega, \qquad (3.46)\\
((1/2 - \mathcal{K})\widehat{u}^n_{h,\tau}, \psi_h)_\Gamma + (\mathcal{V}\widehat{\phi}^n_{h,\tau}, \psi_h)_\Gamma &= ((1/2 - \mathcal{K})\widehat{g}^n_1, \psi_h)_\Gamma, \qquad (3.47)
\end{aligned}$$

for all $v_h \in \mathcal{S}^1(\mathcal{T})$ and $\psi_h \in \mathcal{P}^0(\mathcal{T})$ and for all $1 \leq n \leq N$.
We can write this compactly as: Find $\mathbf{u}_{h,\tau} = (u_{h,\tau}, \phi_{h,\tau}) \in Q_T^{h,\tau} \times B_T^{h,\tau}$ such that

$$(\widehat{\partial_t u}^n_{h,\tau}, v_h)_\Omega + \mathscr{B}_{SUPG}(\widehat{\mathbf{u}}^n_{h,\tau}; \mathbf{v}_h) = \widehat{F}^n_{SUPG}(\mathbf{v}_h) - \sum_{K \in \mathcal{T}} \delta_K (\widehat{\partial_t u}^n_{h,\tau}, \mathbf{b} \cdot \nabla v_h)_\Omega, \qquad (3.48)$$

for all $\mathbf{v}_h = (v_h, \psi_h) \in \mathcal{H}^h$, and $1 \leq n \leq N$, where $F_{SUPG}(\mathbf{v}_h)^n = F_{SUPG}(\mathbf{v}_h, t^n)$ is defined in (3.41).

Before we proceed with the convergence analysis, we can establish an energy estimate for the fully-discrete system. From now on, we additionally assume that the stabilisation parameter fulfils (see [JN11])

$$\delta_K \leq \frac{\tau}{4} \quad \forall K \in \mathcal{T}. \qquad (3.49)$$

Lemma 3.38 (Well-posedness and energy estimate). *Let $\lambda_{\min}(\mathbf{A}) - \frac{2}{3}C_{\mathscr{K}} - \frac{1}{2} > 0$. Then for any $f \in L^2_{T,\Omega}$, $g_2 \in B_T$, and $g_{\mathscr{K}} \in B'_T$, Problem 3.37 admits a unique solution $(u_{h,\tau}, \phi_{h,\tau}) \in Q^{h,\tau}_T \times B^{h,\tau}_T$. Moreover,*

$$\|u_{h,\tau}\|_{L^2(0,T;SUPG)} + \|\phi_{h,\tau}\|_{B_T} \leq C\big((1+\delta)\|f\|_{L^2_{T,\Omega}} + \|g_2\|_{B_T} + \|g_{\mathscr{K}}\|_{B'_T}\big), \tag{3.50}$$

with a constant $C = C(\alpha_{SUPG}, \mathbf{A}, \mathbf{b}, c, \Omega) > 0$ that is independent of the data f, g_2, $g_{\mathscr{K}}$.

Proof. Here, we use the operators $\mathscr{S}_h$ and $\mathscr{R}_h$ defined in Remark 2.12 and express $\phi_{h,\tau} = \mathscr{S}_h u_{h,\tau} + \mathscr{R}_h g_{\mathscr{K}}$. Then we use the properties of the weighting operator of Lemma 2.17 to reduce the fully discrete Problem 3.37 to

$$\begin{aligned}(d_\tau u^n_{h,\tau}, v_h)_\Omega + \widetilde{\mathscr{A}}_{SUPG}(u^n_{h,\tau}, v_h) = (\widehat{f}^n, v_h)_\Omega + (\widehat{g}^n_2, v_h)_\Gamma + (\mathscr{R}_h \widehat{g_{\mathscr{K}}}^n, v_h)_\Gamma \\ + \sum_{K\in\mathscr{T}} \delta_K (\widehat{f}^n, \mathbf{b}\cdot\nabla v_h)_K - \sum_{K\in\mathscr{T}} \delta_K (d_\tau u^n_{h,\tau}, \mathbf{b}\cdot\nabla v_h)_K,\end{aligned}$$

with $\widetilde{\mathscr{A}}_{SUPG}(u^n_{h,\tau}, v_h) := \mathscr{A}_{SUPG}(u^n_{h,\tau}, v_h) - \langle \mathscr{S}_h u^n_{h,\tau}, v_h\rangle_\Gamma$, which is coercive with coercivity constant α_{SUPG}. The last two terms on the right-hand side are the important ones (where we mainly differ from the analysis of the FEM-BEM coupling).

Testing with $v_h = u^n_{h,\tau}$ gives us the term $(d_\tau u^n_{h,\tau}, u^n_{h,\tau})_\Omega$, which we can rewrite by

$$\begin{aligned}(d_\tau u^n_{h,\tau}, u^n_{h,\tau})_\Omega &= \frac{1}{\tau^n}(u^n_{h,\tau} - u^{n-1}_{h,\tau}, u^n_{h,\tau})_\Omega \\ &= \frac{1}{2\tau^n}\left(\|u^n_{h,\tau}\|^2_{L^2(\Omega)} - \|u^{n-1}_{h,\tau}\|^2_{L^2(\Omega)} + \|u^n_{h,\tau} - u^{n-1}_{h,\tau}\|^2_{L^2(\Omega)}\right).\end{aligned}$$

Using this and the coercivity of $\widetilde{\mathscr{A}}_{SUPG}$, we obtain

$$\begin{aligned}&\frac{1}{2\tau^n}\left(\|u^n_{h,\tau}\|^2_{L^2(\Omega)} - \|u^{n-1}_{h,\tau}\|^2_{L^2(\Omega)} + \|u^n_{h,\tau} - u^{n-1}_{h,\tau}\|^2_{L^2(\Omega)}\right) + \alpha_{SUPG}\|u^n_{h,\tau}\|^2_{SUPG} \\ &\leq (\widehat{f}^n, u^n_{h,\tau})_\Omega + (\widehat{g}^n_2, u^n_{h,\tau})_\Gamma + (\mathscr{R}_h\widehat{g_{\mathscr{K}}}^n, u^n_{h,\tau})_\Gamma + (\widehat{f}^n, \delta\mathbf{b}\cdot\nabla u^n_{h,\tau})_\Omega - (d_\tau u^n_{h,\tau}, \delta\mathbf{b}\cdot\nabla u^n_{h,\tau})_\Omega.\end{aligned}$$

Slightly rearranging the terms, using the Cauchy-Schwarz and the trace inequality, the properties of the operator $\mathscr{R}_h$ and then Young's inequality, leads to

$$\begin{aligned}&\frac{1}{2\tau^n}\left(\|u^n_{h,\tau}\|^2_{L^2(\Omega)} - \|u^{n-1}_{h,\tau}\|^2_{L^2(\Omega)} + \|u^n_{h,\tau} - u^{n-1}_{h,\tau}\|^2_{L^2(\Omega)}\right) + \alpha_{SUPG}\|u^n_{h,\tau}\|^2_{SUPG} \\ &\leq \|\widehat{f}^n\|_{L^2(\Omega)}\|u^n_{h,\tau}\|_H + \|\widehat{g_2}^n\|_B\|u^n_{h,\tau}\|_{B'} + \delta\|\widehat{f}^n\|_{L^2(\Omega)}\|\nabla u^n_{h,\tau}\|_{L^2(\Omega)} + \|\mathscr{R}_h\widehat{g_{\mathscr{K}}}^n\|_B\|u^n_{h,\tau}\|_{B'} \\ &\qquad + |(d_\tau u^n_{h,\tau}, \delta\mathbf{b}\cdot\nabla u^n_{h,\tau})_\Omega| \\ &\leq C\left(\|\widehat{f}^n\|_{L^2(\Omega)} + \|\widehat{g_2}^n\|_B + \delta\|\widehat{f}^n\|_{L^2(\Omega)} + \|\widehat{g_{\mathscr{K}}}^n\|_{B'}\right)\|u^n_{h,\tau}\|_{SUPG} \\ &\quad + \frac{1}{2\tau^n}\|u^n_{h,\tau} - u^{n-1}_{h,\tau}\|^2_{L^2(\Omega)} + \frac{1}{8}\|u^n_{h,\tau}\|^2_{SUPG} \\ &\leq C\left((1+\delta^2)\|\widehat{f}^n\|^2_{L^2(\Omega)} + \|\widehat{g_2}^n\|^2_B + \|\widehat{g_{\mathscr{K}}}^n\|^2_{B'}\right) + \frac{1}{2\tau^n}\|u^n_{h,\tau} - u^{n-1}_{h,\tau}\|^2_{L^2(\Omega)} + \frac{\alpha_{SUPG}}{2}\|u^n_{h,\tau}\|^2_{SUPG},\end{aligned}$$

which only holds, if $\alpha_{SUPG} > \frac{1}{4}$. Here we also used that we can estimate (cf. [JN11])

$$\begin{aligned}
&\left|(d_\tau u_{h,\tau}^n, \delta \mathbf{b}\cdot\nabla u_{h,\tau}^n)_\Omega\right| \\
&\leq \frac{1}{\tau^n}\delta^{1/2}\|(u_{h,\tau}^n - u_{h,\tau}^{n-1})\|_{L^2(\Omega)}\delta^{1/2}\|\mathbf{b}\cdot\nabla u_{h,\tau}^n\|_{L^2(\Omega)} \\
&\leq \frac{2}{(\tau^n)^2}\delta\|u_{h,\tau}^n - u_{h,\tau}^{n-1}\|_{L^2(\Omega)}^2 + \frac{1}{8}\delta\|\mathbf{b}\cdot\nabla u_{h,\tau}^n\|_{L^2(\Omega)}^2 \\
&\leq \frac{1}{2\tau^n}\|u_{h,\tau}^n - u_{h,\tau}^{n-1}\|_{L^2(\Omega)}^2 + \frac{1}{8}\|u_{h,\tau}^n\|_{SUPG}^2,
\end{aligned}$$

where we employed (3.49) and Young's inequality. Now we can bring the last two terms to the other side to obtain

$$\frac{1}{2\tau^n}\left(\|u_{h,\tau}^n\|_{L^2(\Omega)}^2 - \|u_{h,\tau}^{n-1}\|_{L^2(\Omega)}^2\right) + \|u_{h,\tau}^n\|_{SUPG}^2 \lesssim (1+\delta^2)\|\widehat{f}^n\|_{L^2(\Omega)}^2 + \|\widehat{g_2}^n\|_B^2 + \|\widehat{g_{\mathscr{K}}}^n\|_{B'}^2.$$

Multiplying with τ^n, summing over n from 1 to N, using (2.30) from Lemma 2.17 and that $u_{h,\tau}$ is piecewise linear yields

$$\frac{1}{2}\|u_{h,\tau}^N\|_{L^2(\Omega)}^2 + \|u_{h,\tau}\|_{L^2(0,T;SUPG)}^2 \lesssim (1+\delta^2)\|f\|_{L_{T,\Omega}}^2 + \|g_2\|_{B_T}^2 + \|g_{\mathscr{K}}\|_{B_T'}^2.$$

The part $\|\phi_{h,\tau}\|_{B_T}$ can then be bounded as in Remark 2.7, (2.14) and overall we obtain the energy estimate. □

Remark 3.39. The condition $\lambda_{\min}(\mathbf{A}) - \frac{2}{3}C_{\mathscr{K}} - \frac{1}{2} > 0$ appears instead of the usual condition $\lambda_{\min}(\mathbf{A}) - \frac{1}{2}C_{\mathscr{K}} > 0$ because we additionally had to assume that $\alpha_{SUPG} > \frac{1}{4}$. This is necessary to incorporate the term $\frac{1}{8}\|u_{h,\tau}^n\|_{SUPG}^2$ into the existing term via Young's inequality. But this allows us to exploit the structure of the discrete time derivative to absorb the stabilisation term with the time derivative into the left-hand side. Thus we do not need to assume that h is small enough in the fully-discrete case.

In the fully-discrete case, we are also interested in a bound for the norm of the time derivative. Unfortunately, we will again need to assume that $\delta = \mathcal{O}(h^2)$ to obtain this result.

Lemma 3.40 (Full energy estimate). *Let $\delta = \mathcal{O}(h^2)$, $\lambda_{\min}(\mathbf{A}) - \frac{1}{2}C_{\mathscr{K}} > 0$ and P_h be H^1-stable. Then, for h small enough and for any $f \in L_{T,\Omega}^2$, $g_2 \in B_T$, and $g_{\mathscr{K}} \in B_T'$, the solution $(u_{h,\tau}, \phi_{h,\tau}) \in Q_T^{h,\tau} \times B_T^{h,\tau}$ of Problem 3.37 fulfils*

$$\|\partial_t u_{h,\tau}\|_{H_T'} + \|u_{h,\tau}\|_{L^2(0,T;SUPG)} + \|\phi_{h,\tau}\|_{B_T} \leq C\big((1+\delta)\|f\|_{L_{T,\Omega}^2} + \|g_2\|_{B_T} + \|g_{\mathscr{K}}\|_{B_T'}\big), \tag{3.51}$$

with a constant $C = C(\alpha_{SUPG}, \mathbf{A}, \mathbf{b}, c, \Omega, C_P) > 0$.

Proof. The proof is similar to the proof of Lemma 2.22. So we have to bound the term

$$\|\partial_t u_{h,\tau}\|_{H_T'}^2 = \sum_{n=1}^{N}\tau^n\|d_\tau u_{h,\tau}^n\|_{H^1(\Omega)'}^2.$$

Then we can write

$$\|d_\tau u^n_{h,\tau}\|_{H^1(\Omega)'} = \sup_{0\neq v\in H^1(\Omega)} \frac{(d_\tau u^n_{h,\tau}, v)_\Omega}{\|v\|_{H^1(\Omega)}} = \sup_{0\neq v\in H^1(\Omega)} \frac{(d_\tau u^n_{h,\tau}, P_h v)_\Omega}{\|v\|_{H^1(\Omega)}}.$$

Now we can use (3.46) and Lemma 2.17 to obtain

$$\begin{aligned}(d_\tau u^n_{h,\tau}, P_h v)_\Omega \leq C\Big(&\|u^n_{h,\tau}\|_{H^1(\Omega)} + \|\phi^n_{h,\tau}\|_{H^{-1/2}(\Gamma)} + \|\widehat{f}^n\|_{L^2(\Omega)} + \|\widehat{g_2}^n\|_B \\ &+\delta\|\widehat{f}^n\|_{L^2(\Omega)} + \delta\|d_\tau u^n_{h,\tau}\|_{L^2(\Omega)}\Big)\|P_h v\|_{H^1(\Omega)}.\end{aligned}$$

Then we can use the H^1–stability of the L^2–projection and, again using $\delta = \mathcal{O}(h^2)$ h small enough and an inverse inequality, we can absorb the last term into the left-hand side to obtain the energy estimate. □

The proof of the convergence result for the fully-discrete SUPG-BEM scheme is similar to the semi-discrete case combined with the proof of the energy estimate Lemma 3.38.

Theorem 3.41 (Convergence of the fully-discrete SUPG-BEM). *Let $(u,\phi) \in Q_T \times B_T$ and $(u_{h,\tau}, \phi_{h,\tau}) \in Q_T^{h,\tau} \times B_T^{h,\tau}$ denote the solutions of Problem 3.2 and Problem 3.37, respectively. Then, if $\lambda_{\min}(\mathbf{A}) - \frac{2}{3}C_{\mathcal{K}} - \frac{1}{2} > 0$, there holds that*

$$\begin{aligned}&\|u-u_{h,\tau}\|_{L^2(0,T;SUPG)} + \|\phi-\phi_{h,\tau}\|_{B_T} \\ &\quad\leq C\big(\|u-v_{h,\tau}\|_{Q_T} + \|\phi-\psi_{h,\tau}\|_{B_T} + \delta\|v_{h,\tau}\|_{H_T} + \delta\|\partial_t v_{h,\tau}\|_{L^2_{T,\Omega}} + \delta\|f\|_{L^2_{T,\Omega}}\big),\end{aligned}$$

for all functions $v_{h,\tau} \in Q_T^{h,\tau}$ and $\psi_{h,\tau} \in B_T^{h,\tau}$ with a constant $C = C(\alpha_{SUPG}, \mathbf{A}, \mathbf{b}, c, \Omega) > 0$ which is independent of the model input data f, g_2, $g_{\mathcal{K}}$.

Proof. As in the semi-discrete case Theorem 3.34, let $\mathbf{v}_{h,\tau} = (v_{h,\tau}, \psi_{h,\tau}) \in Q_T^{h,\tau} \times B_T^{h,\tau}$ be arbitrary and split the error into

$$\|\mathbf{u}-\mathbf{u}_{h,\tau}\|_{SUPG\times H^{-1/2}(\Gamma)} \leq \|\mathbf{u}-\mathbf{v}_{h,\tau}\|_{SUPG\times H^{-1/2}(\Gamma)} + \|\mathbf{u}_{h,\tau}-\mathbf{v}_{h,\tau}\|_{SUPG\times H^{-1/2}(\Gamma)},$$

and set $\mathbf{w}_{h,\tau} := (w_{h,\tau}, \varphi_{h,\tau}) := \mathbf{u}_{h,\tau} - \mathbf{v}_{h,\tau}$.

Proceeding similar to Lemma 3.38, we can use the coercivity of $\mathscr{B}_{SUPG}(\cdot;\cdot)$ to obtain

$$\begin{aligned}&\frac{1}{2\tau^n}\Big(\|w^n_{h,\tau}\|^2_{L^2(\Omega)} - \|w^{n-1}_{h,\tau}\|^2_{L^2(\Omega)} + \|w^n_{h,\tau} - w^{n-1}_{h,\tau}\|^2_{L^2(\Omega)}\Big) + \alpha_{SUPG}\|w^n_{h,\tau}\|^2_{SUPG\times H^{-1/2}(\Gamma)} \\ &\leq (d_\tau w^n_{h,\tau}, w^n_{h,\tau})_\Omega + \mathscr{B}_{SUPG}(\mathbf{w}^n_{h,\tau}; \mathbf{w}^n_{h,\tau}).\end{aligned}$$

Now we can insert some new terms and rearrange to get

$$\begin{aligned}&\frac{1}{2\tau^n}\Big(\|w^n_{h,\tau}\|^2_{L^2(\Omega)} - \|w^{n-1}_{h,\tau}\|^2_{L^2(\Omega)} + \|w^n_{h,\tau} - w^{n-1}_{h,\tau}\|^2_{L^2(\Omega)}\Big) + \alpha_{SUPG}\|w^n_{h,\tau}\|^2_{SUPG\times H^{-1/2}(\Gamma)} \\ &\leq \langle\widehat{\partial_t u}^n, w^n_{h,\tau}\rangle_\Omega - (d_\tau v^n_{h,\tau}, w^n_{h,\tau})_\Omega + \widehat{F}^n_{SUPG}(w^n_{h,\tau}) - \widehat{F}^n(w^n_{h,\tau}) \\ &\quad + \mathscr{B}(\mathbf{v}^n_{h,\tau}; \mathbf{w}^n_{h,\tau}) - \mathscr{B}_{SUPG}(\mathbf{v}^n_{h,\tau}; \mathbf{w}^n_{h,\tau}) + \mathscr{B}(\widehat{u}^n - \mathbf{v}^n_{h,\tau}; \mathbf{w}^n_{h,\tau}) \\ &\quad - \delta(d_\tau u^n_{h,\tau} - d_\tau v^n_{h,\tau}, \mathbf{b}\cdot\nabla w^n_{h,\tau})_\Omega - \delta(d_\tau v^n_{h,\tau}, \mathbf{b}\cdot\nabla w^n_{h,\tau})_\Omega.\end{aligned}$$

Now we can proceed as in Lemma 3.38, bounding $\delta(d_\tau w^n_{h,\tau}, \mathbf{b}\cdot\nabla w^n_{h,\tau})_\Omega$ by $\frac{1}{2\tau^n}\|w^n_{h,\tau} - w^{n-1}_{h,\tau}\|^2_{L^2(\Omega)}$ and for everything else using the corresponding estimates, Cauchy-Schwarz inequality and Young's inequality. Bringing everything to the respective side yields

$$\begin{aligned}
&\frac{1}{2\tau^n}\left(\|w^n_{h,\tau}\|^2_{L^2(\Omega)} - \|w^{n-1}_{h,\tau}\|^2_{L^2(\Omega)}\right) + \|w^n_{h,\tau}\|^2_{SUPG\times H^{-1/2}(\Gamma)}\\
&\lesssim \|\widehat{\partial_t u}^n - d_\tau v^n_{h,\tau}\|_{H^1(\Omega)'} + \delta\|\widehat{f}^n\|_{L^2(\Omega)} + \delta\|v^n_{h,\tau}\|_{H^1(\Omega)} + \|\widehat{u}^n - v^n_{h,\tau}\|_{\mathscr{H}} + \delta\|d_\tau v^n_{h,\tau}\|_{L^2(\Omega)}
\end{aligned}$$

Multiplying by τ^n, summing over n from 1 to N, bounding the resulting terms appropriately and using the error splitting leads to the desired result. □

Remark 3.42. This result and also the previous results from this section also show that the extension of the FEM-BEM coupling to the model problem (3.1)–(3.7) is well-posed and yields a quasi-optimal convergence result. To obtain this, we simply have to choose $\delta = 0$.

Remark 3.43. Note that, as it is the case for the FVM-BEM coupling, we could not show the convergence result in the full energy norm and we have some extra terms accounting for the lack of classical Galerkin-orthogonality.

Again, with the usual approximation arguments (see Lemma 2.27), we get the following order of convergence.

Corollary 3.44. *Let $(u,\phi) \in Q_T \times B_T$ and $(u_{h,\tau}, \phi_{h,\tau}) \in Q^h_T \times B^h_T$ denote the solutions of Problem 3.2 and Problem 3.37, respectively. Furthermore, let P_h be H^1-stable and $\lambda_{\min}(\mathbf{A}) - \frac{2}{3}C_{\mathscr{K}} - \frac{1}{2} > 0$. Then for $u \in L^2(0,T;H^{1+s}(\Omega)) \cap H^r(0,T;H^1(\Omega))$, $\partial_t u \in L^2_{T,\Omega} \cap H^r(0,T;H')$ and $\phi \in L^2(0,T;H^{s-1/2}(\Gamma)) \cap H^r(0,T;H^{-1/2}(\Gamma))$, there holds that*

$$\begin{aligned}
&\|u - u_h\|_{L^2(0,T;SUPG)} + \|\phi - \phi_h\|_{B_T}\\
&\quad\le C\Big[h^s\big(\|\partial_t u\|_{L^2_{T,\Omega}} + \|u\|_{L^2(0,T;H^{1+s}(\Omega))} + \|\phi\|_{L^2(0,T;H^{s-1/2}(\Gamma))} + \|f\|_{L^2_{T,\Omega}}\big)\\
&\qquad+\tau^r\left(\|\partial_t u\|_{H^r(0,T;H')} + \|u\|_{H^r(0,T;H^1(\Omega))} + \|\phi\|_{H^r(0,T;H^{-1/2}(\Gamma))}\right)\Big] = \mathscr{O}(h^s + \tau^r),
\end{aligned}$$

with a constant $C = C(\mathbf{A}, \mathbf{b}, c, \alpha_{SUPG}, \Omega, \eta) > 0$ for $0 \le s \le 1$ and $0 \le r \le 1$.

Remark 3.45. Although this is a (first-order) asymptotic convergence result, it is still of limited use. The constant is also dependent on the model parameters and the h on the right-hand side also includes terms with δ, thus it is possible that the constant can become really big. For a more in-depth analysis of the standalone SUPG method, also concerning this problem, see [Bur11; JN11; FGAN16].

4

Numerical Experiments

To illustrate our theoretical findings, we present some numerical examples in two dimensions. The calculations were performed using MATLAB utilising some functions from the HILBERT-package [Aur+14] for assembling the matrices resulting from the integral operators $\mathscr{V}$ and $\mathscr{K}$. Additional code, developed in [Era10], is also used. For the error discussion, we also consider the L^2–projected analytical solutions $\overline{u}_h(t) \in H^h$ of $u(t)$ and $\overline{\phi}_h(t) \in B^h$ of $\phi(t)$ for a fixed but arbitrary t. Note that the prescribed exterior solutions guarantee at least $\phi(t) \in L^2(\Gamma)$. Hence, we may estimate the error as

$$\|u - u_{h,\tau}\|_{Q_T} \leq \|u - \overline{u}_h\|_{Q_T} + \|\overline{u}_h - u_{h,\tau}\|_{Q_T}, \tag{4.1}$$

$$\|\phi - \phi_{h,\tau}\|_{B_T} \leq \|\phi - \overline{\phi}_h\|_{B_T} + \|\overline{\phi}_h - \phi_{h,\tau}\|_{B_T}. \tag{4.2}$$

The convergence orders of $\|u - \overline{u}_h\|_{Q_T}$ and $\|\phi - \overline{\phi}_h\|_{B_T}$ are known a priori. With the discrete error $e_h(t) := \overline{u}_h(t) - u_{h,\tau}(t)$, we can estimate the non computable dual norm $\|\partial_t e_h\|^2_{H'_T} = \int_0^T \|\partial_t e_h\|^2_{H'}$ in the following way: Let $z_h^a \in H^h$ be the solution to the auxiliary problem

$$(\nabla z_h^a, \nabla v_h)_\Omega + (z_h^a, v_h)_\Omega = (\partial_t e_h, v_h)_\Omega,$$

with $v_h = P_h v$ for all $v \in H$ and P_h being the L^2–projection of Section 2.5. Then the H^1-stability of P_h and the definition of the auxiliary problem lead to

$$\begin{aligned}
\|\partial_t e_h\|_{H^1(\Omega)'} &= \sup_{0 \neq v \in H^1(\Omega)} \frac{(\partial_t e_h, v)_\Omega}{\|v\|_{H^1(\Omega)}} \\
&= \sup_{0 \neq v \in H^1(\Omega)} \left(\frac{(\partial_t e_h, v - P_h v)_\Omega}{\|v\|_{H^1(\Omega)}} + \frac{(\partial_t e_h, P_h v)_\Omega}{\|v\|_{H^1(\Omega)}} \right) \\
&\leq \sup_{0 \neq v \in H^1(\Omega)} \frac{\|z_h^a\|_{H^1(\Omega)} \|P_h v\|_{H^1(\Omega)}}{\|v\|_{H^1(\Omega)}} \leq C_P \|z_h^a\|_{H^1(\Omega)},
\end{aligned}$$

with the constant $C_P > 0$. Together with the stability $\|z_h^a\|_{H^1(\Omega)} \leq \|\partial_t e_h\|_{H^1(\Omega)'}$ we see that $\|z_h^a\|_{H_T}$ is an equivalent norm to $\|\partial_t e_h\|_{H'_T}$. The norm $\|\phi(t) - \phi_{h,\tau}(t)\|_{H^{-1/2}(\Gamma)}$ is also not computable. Hence, we may use the equivalent norm

$$\|\phi(t) - \phi_{h,\tau}(t)\|_{H^{-1/2}(\Gamma)} \sim \|\phi(t) - \phi_{h,\tau}(t)\|_{\mathscr{V}} := \langle \mathscr{V}(\phi(t) - \phi_{h,\tau}(t)), \phi(t) - \phi_{h,\tau}(t) \rangle_\Gamma,$$

see [Era10] for details. Thus $\|\phi - \phi_{h,\tau}\|_{L^2(0,T;\mathscr{V})}$ is an equivalent norm to $\|\phi - \phi_{h,\tau}\|_{B_T}$. We approximate all other spatial norms by Gaussian quadrature or with the matrices from the discretisation. The time integral in the Bochner-Sobolev norms is also computed with a Gaussian quadrature. For the energy norm, we therefore present the upper bound

$$\|u-u_{h,\tau}\|_{Q_T} + \|\phi-\phi_{h,\tau}\|_{B_T} \leq (\|u-u_{h,\tau}\|^2_{H_T} + \|z_h^a\|^2_{H_T})^{1/2} + \|\phi-\phi_{h,\tau}\|_{L^2(0,T;\mathcal{V})}.$$

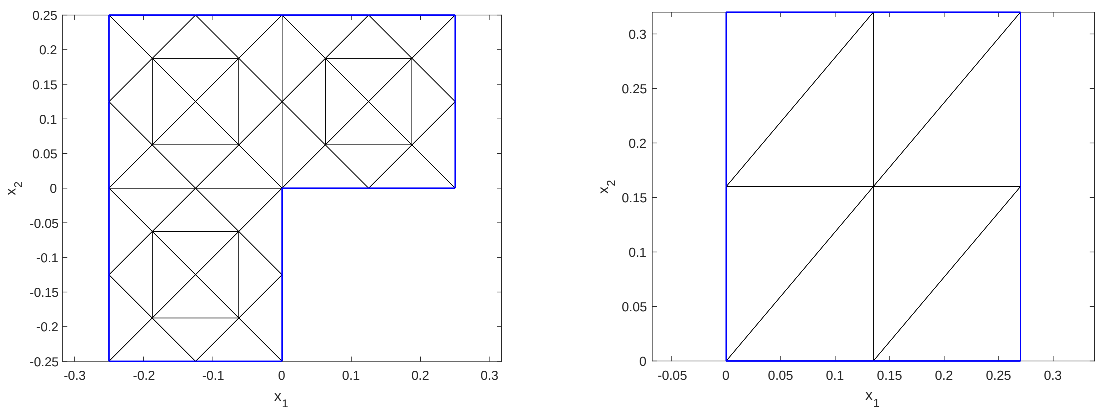
Furthermore, with respect to the error splitting (4.1)–(4.2) we also calculate the error

$$(\|\overline{u}_h - u_{h,\tau}\|^2_{H_T} + \|z_h^a\|^2_{H_T})^{1/2} + \|\overline{\phi}_h - \phi_{h,\tau}\|_{L^2(0,T;\mathcal{V})},$$

with the L^2–projected analytical solutions $\overline{u}_h(t) \in H^h$ of $u(t)$ and $\overline{\phi}_h(t) \in B^h$ of $\phi(t)$.

4.1 FEM-BEM Coupling

For the FEM-BEM coupling, we use the spaces as in Section 2.5. The implementation uses the equivalent system (2.31)–(2.32) instead of Problem 2.16, see Remark 2.18. The right-hand side is built from the model data f, g_1, g_2 with (2.7) and (2.8), and with the aid of the weighted average operator (2.26). For these integrals, we use Gauss quadrature in space and time.



(a) Mesh for Section 4.1.1.

(b) Mesh for Section 4.1.2.

Figure 4.1: The initial triangle meshes for the examples. The bold lines are the coupling boundary (blue).

4.1.1 Tests with Analytical Solutions

In the following, we discuss the convergence behaviour for three examples with analytical solutions. We consider the coupling problem (2.1)–(2.6) on the classical L-shape $\Omega = (-1/4, 1/4)^2 \setminus [0, 1/4] \times [-1/4, 0]$ and the time interval $[0, 1]$.

The uniform initial triangulation (triangles) is plotted in Figure 4.1a with $h = 0.125$. We use uniform time stepping, in particular, we begin with $\tau^n = \tau = 0.05$. The refinement is uniform for both the space and the time grid, and simultaneously. For all three examples, we prescribe the same analytical solution in the exterior domain Ω_e, namely

$$u_e(x_1,x_2,t) = (1-t)\log\sqrt{(x_1+0.125)^2+(x_2-0.125)^2}.$$

Note that this solution is smooth in Ω_e. With the interior solutions given below, we calculate the right-hand side f and the jumps g_1 and g_2 (from $u = u_e + g_1$ and $\partial_{\mathbf{n}} u = \partial_{\mathbf{n}} u_e + g_2$) appropriately.

4.1.1.1 Smooth Solution

For the first example, we use the interior solution

$$u(x_1,x_2,t) = \sin(2\pi t)(1-100x_1^2-100x_2^2)e^{-50(x_1^2+x_2^2)}.$$

Hence, both, u and u_e are smooth and according to Theorem 2.32 we expect the optimal convergence rate $\mathcal{O}(h+\tau)$ which is indeed observed in Figure 4.2. We remark that for stationary problems, we observe in *practical examples* higher convergence rates for $\|\phi-\phi_h\|_{\mathcal{V}}$ and $\|u-u_h\|_{L^2(\Omega)}$, namely $\mathcal{O}(h^{3/2})$ and $\mathcal{O}(h^2)$, see, e.g., [EOS17, Section 5.1.]. Hence the errors $\|\phi-\phi_{h,\tau}\|_{L^2(0,T;\mathcal{V})}$ and $\|u-u_{h,\tau}\|_{L^2(0,T;L^2(\Omega))}$ are dominated from the first-order convergence of the implicit Euler scheme.

4.1.1.2 Generic Singularity at the Reentrant Corner

For the second example, we choose the analytical solution

$$u(x_1,x_2,t) = (1+t^2)r^{2/3}\sin(2\varphi/3),$$

with the polar coordinates $(x_1,x_2) = r(\cos\varphi,\sin\varphi)$, $r\in\mathbb{R}_+$ and $\varphi\in[0,2\pi)$. This solution is a classical test solution in the spatial components and exhibits a generic singularity at the reentrant corner $(0,0)$ of Ω. Note that $\Delta u = 0$ and that the function $u(x_1,x_2,\cdot)$ is only in $H^{1+2/3-\varepsilon}(\Omega)$ for $\varepsilon>0$. The initial data (2.5) in this example is not 0. As analysed in Theorem 2.32 and observed in Figure 4.3 we obtain a reduced convergence rate of $\mathcal{O}(h^{2/3}+\tau)$. Since the conormal derivative of the exterior solution is smooth, the quantities $\|\phi-\phi_{h,\tau}\|_{L^2(0,T;\mathcal{V})}$ and $\|\overline{\phi}_h-\phi_{h,\tau}\|_{L^2(0,T;\mathcal{V})}$ have a higher convergence order.

4.1.1.3 Non-Smooth Function in Time

The third example is less regular in time, but smooth in space, and reads

$$u(x_1,x_2,t) = t^{5/6}(1-100x_1^2-100x_2^2)e^{-50(x_1^2+x_2^2)}.$$

Note that the function $u(x,\cdot)$ is only in $H^{4/3}(0,T)$. According to our analysis, we expect a convergence rate of $\mathcal{O}(h+\tau^{1/3})$. We plot the convergence order with respect to the number of time intervals $(=1/\tau)$ in Figure 4.4. Also note that the energy norm error

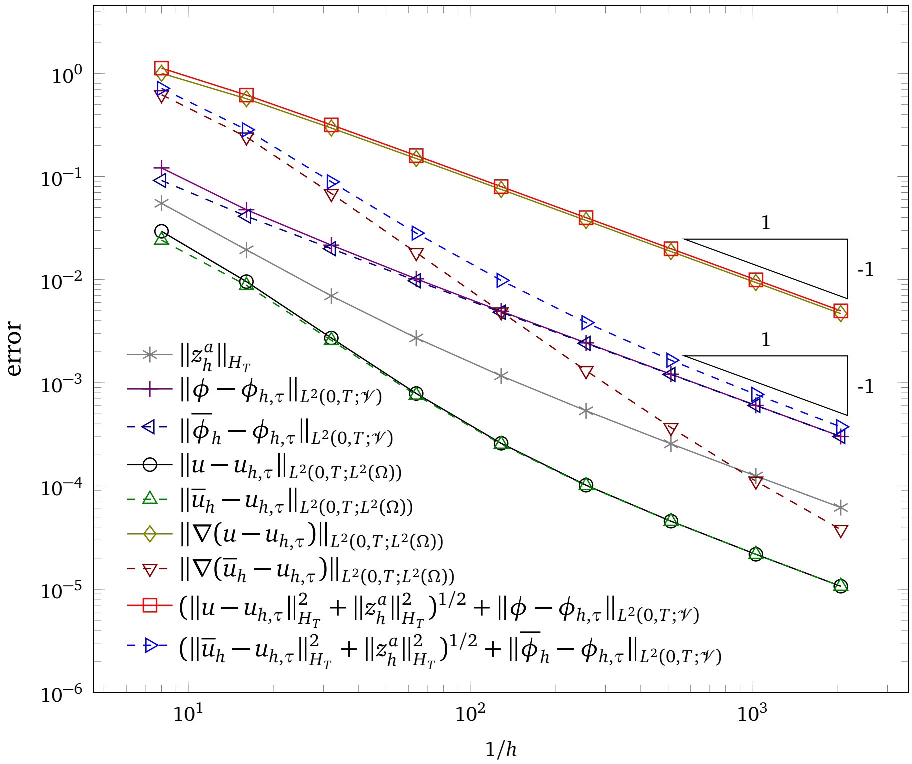


Figure 4.2: The different error components of the solutions $u_{h,\tau}$ and $\phi_{h,\tau}$ of the FEM-BEM coupling for the smooth example in Section 4.1.1.1. The added energy error norms $(\|u - u_{h,\tau}\|_{H_T}^2 + \|z_h^a\|_{H_T}^2)^{1/2} + \|\phi - \phi_{h,\tau}\|_{L^2(0,T;\mathscr{V})}$ and $(\|\overline{u}_h - u_{h,\tau}\|_{H_T}^2 + \|z_h^a\|_{H_T}^2)^{1/2} + \|\overline{\phi}_h - \phi_{h,\tau}\|_{L^2(0,T;\mathscr{V})}$ show the first-order convergence as predicted in Theorem 2.32.

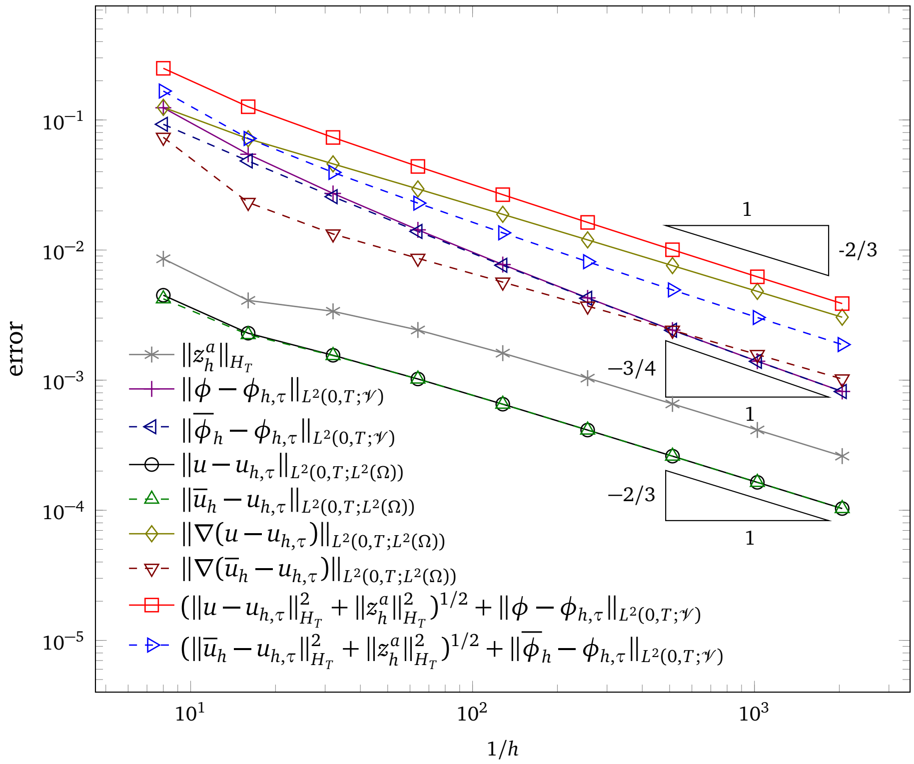


Figure 4.3: The different error components of the solutions $u_{h,\tau}$ and $\phi_{h,\tau}$ of the FEM-BEM coupling for the example with a spatial generic singularity of the interior solution in Section 4.1.1.2. The added energy error norms $(\|u-u_{h,\tau}\|_{H_T}^2+\|z_h^a\|_{H_T}^2)^{1/2}+\|\phi-\phi_{h,\tau}\|_{L^2(0,T;\mathcal{V})}$ and $(\|\overline{u}_h-u_{h,\tau}\|_{H_T}^2+\|z_h^a\|_{H_T}^2)^{1/2}+\|\overline{\phi}_h-\phi_{h,\tau}\|_{L^2(0,T;\mathcal{V})}$ show the reduced convergence order as predicted in Theorem 2.32.

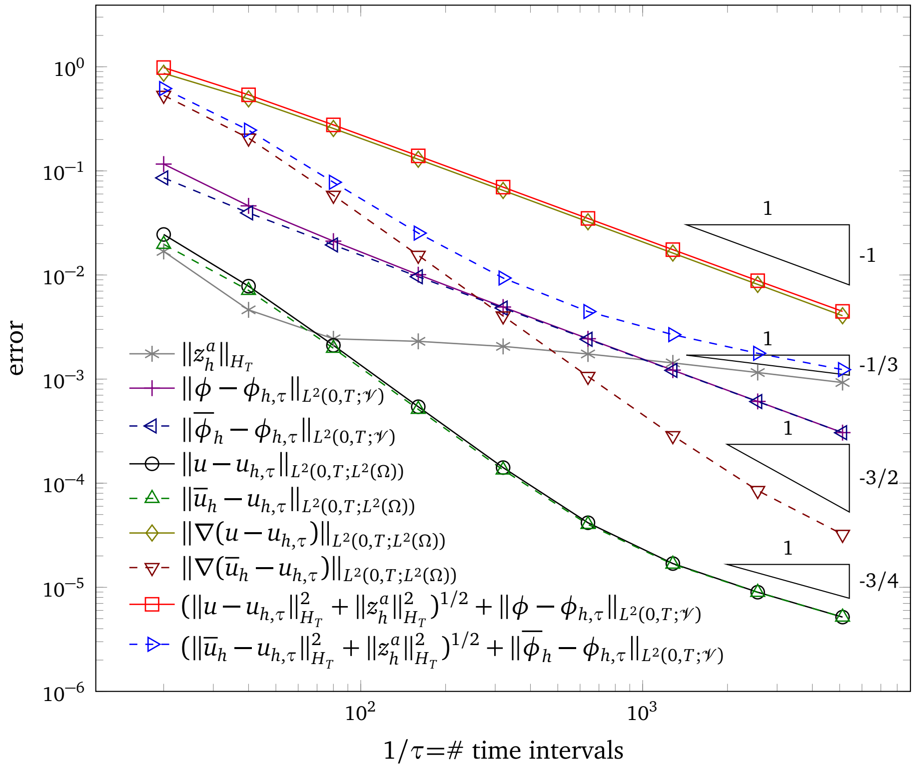


Figure 4.4: The different error components of the solutions $u_{h,\tau}$ and $\phi_{h,\tau}$ of the FEM-BEM coupling for the example with a singularity in the time component of the interior solution in Section 4.1.1.3. The added energy error norms $(\|u-u_{h,\tau}\|_{H_T}^2+\|z_h^a\|_{H_T}^2)^{1/2}+\|\phi-\phi_{h,\tau}\|_{L^2(0,T;\mathscr{V})}$ and $(\|\overline{u}_h-u_{h,\tau}\|_{H_T}^2+\|z_h^a\|_{H_T}^2)^{1/2}+\|\overline{\phi}_h-\phi_{h,\tau}\|_{L^2(0,T;\mathscr{V})}$ show the reduced convergence order as predicted in Theorem 2.32.

$\|u-u_{h,\tau}\|_{Q_T}+\|\phi-\phi_{h,\tau}\|_{B_T}$ represented by $(\|u-u_{h,\tau}\|_{H_T}^2+\|z_h^a\|_{H_T}^2)^{1/2}+\|\phi-\phi_{h,\tau}\|_{L^2(0,T;\mathscr{V})}$ seems to have a misleading convergence order of $\mathscr{O}(\tau)$. The error component $\|z_h^a\|_{H_T}$, representing the dual norm error $\|\partial_t(u-u_{h,\tau})\|_{H_T'}$, has convergence order $\mathscr{O}(\tau^{1/3})$. With respect to $\|\nabla(u-u_{h,\tau})\|_{L^2(0,T;L^2(\Omega))}$, this error component is rather small. Hence the predicted convergence rate $\mathscr{O}(h+\tau^{1/3})$ would be observed asymptotically which cannot be visualised here due to computational restrictions.

4.1.2 A Magnetoquasistatic Problem

In the last example for the FEM-BEM coupling, we want to apply our numerical scheme to a more practical problem. This so-called eddy current problem was also the motivation for the parabolic-elliptic FEM-BEM coupling in the first place (see [MS87]).

Before we arrive at our (slightly modified) model problem (2.1)–(2.6), we have to make some physical considerations. Electromagnetic fields in three dimensions can be described by Maxwell's equations (see e.g. [Jac99, Chapter 6]) which are given by

$$\begin{aligned} \operatorname{div} \mathbf{D} &= \rho, & \operatorname{curl} \mathbf{H} &= \mathbf{J} + \partial_t \mathbf{D}, \\ \operatorname{div} \mathbf{B} &= 0, & \operatorname{curl} \mathbf{E} &= -\partial_t \mathbf{B}. \end{aligned}$$

Here $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, respectively, and ρ is the electric charge density. $\mathbf{D}$ and $\mathbf{B}$ are the electric and magnetic flux densities. Furthermore, we have the following material laws

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{J} = \sigma \mathbf{E} + \mathbf{J}_s, \quad \mathbf{B} = \mu \mathbf{H}.$$

The coefficients are the material parameters electric conductivity $\sigma \geq 0$, electric permittivity $\epsilon > 0$ and magnetic permeability $\mu > 0$. The right-hand side $\mathbf{J}_s$ is defined by $\mathbf{J}_s(x, y, z, t) = \mathbf{J}_0(x, y, z)\sin(2\pi\omega)$ with a frequency ω. Using this, we obtain

$$\begin{aligned} \operatorname{div} \epsilon \mathbf{E} &= \rho, & \operatorname{curl} \mathbf{H} &= \sigma \mathbf{E} + \mathbf{J}_s + \partial_t \mathbf{D}, && (4.3) \\ \operatorname{div} \mu \mathbf{H} &= 0, & \operatorname{curl} \mathbf{E} &= -\partial_t \mu \mathbf{H}. && (4.4) \end{aligned}$$

For the magnetoquasistatic approximation, we assume that

$$\|\partial_t \mathbf{D}\| \ll \|\mathbf{J}_s\|,$$

and thus we can set $\partial_t \mathbf{D} = 0$. This kind of approximation is called the eddy current problem and is analysed, for example, in [SSH08]. Now we make use of a modified magnetic vector potential, called $\mathbf{A}$, that fulfils

$$\mathbf{E} = -\partial_t \mathbf{A},$$

and has been introduced in [ET88]. Plugging this into (4.4), we can deduce that

$$\mu^{-1} \operatorname{curl} \mathbf{A} = \mathbf{H}.$$

Then, inserting this into (4.3) yields

$$\sigma \partial_t \mathbf{A} + \operatorname{curl}(\mu^{-1} \operatorname{curl} \mathbf{A}) = \mathbf{J}_s.$$

This is structurally already close to our model problem, but in three dimensions. Thus we reduce this to a two-dimensional problem by taking a cut through the domain and use that

$$\begin{aligned} \mathbf{J}_s &= (0, 0, J_{s,z}), \\ \mathbf{A} &= (0, 0, A_z). \end{aligned}$$

Renaming A_z to u and $J_{s,z}$ to f we have to change (2.1) to

$$\sigma \partial_t u - \operatorname{div}(\mu^{-1} \nabla u) = f,$$

with piecewise constant coefficients σ and μ^{-1}. The diffusion coefficient does not pose a problem (as we have seen in the analysis in Chapter 3), the coefficient σ also does not

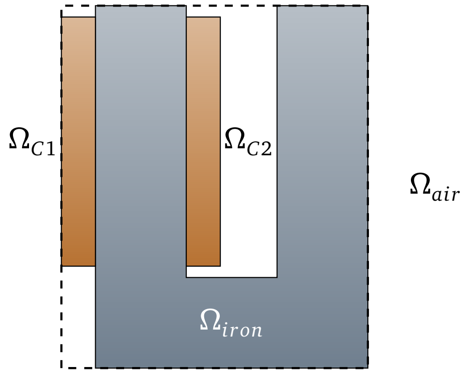


Figure 4.5: The setup of the magnetoquasistatic problem in Section 4.1.2. Everything inside the dashed box is computed with FEM.

alter our analysis as long as it is piecewise constant. In [ET88] it is also stated that for non-conductors we basically obtain the Laplace equation.

In this particular example, we look at an iron core that has a coil wrapped around one side. These different areas are incorporated in a rectangle which is our interior domain, given by $\Omega = [0, 0.27] \times [0, 0.32]$. The subdivision can be seen in Figure 4.5. Everything inside the dashed box will be computed with FEM, the rest (of the air) with BEM. The coefficients are given by

$$\mu^{-1} = \begin{cases} \frac{1}{1000} & \text{in } \Omega_{iron}, \\ 1 & \text{else}, \end{cases}$$

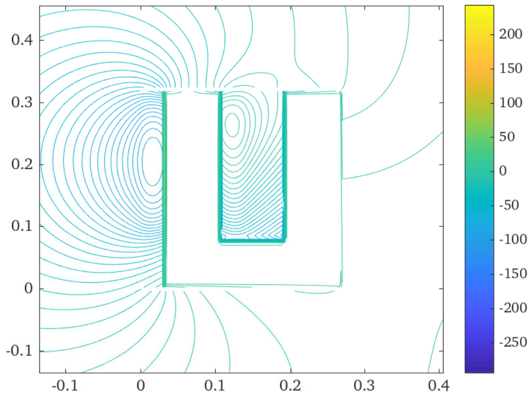
$$\sigma = \begin{cases} 10 & \text{in } \Omega_{iron}, \\ 0 & \text{else}. \end{cases}$$

The right-hand side is given by

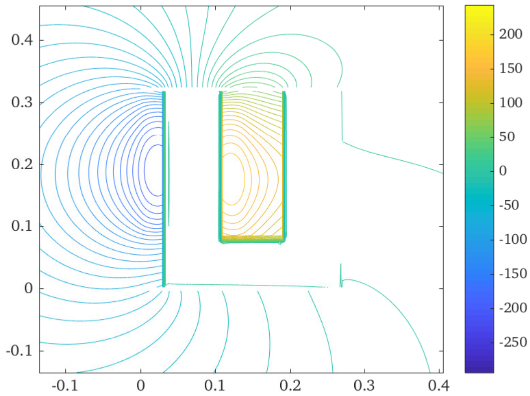
$$f = \begin{cases} \frac{N}{|\Omega_{C1}|} \sin(2\pi\omega t) & \text{in } \Omega_{C1}, \\ -\frac{N}{|\Omega_{C2}|} \sin(2\pi\omega t) & \text{in } \Omega_{C2}, \\ 0 & \text{else}, \end{cases}$$

with a frequency ω of 50Hz and the number of windings $N = 358$. The jumps are set to zero.

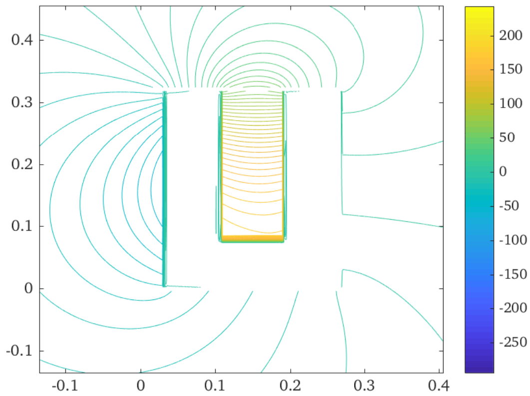
This means that we simulate the parts with the iron core, the coil, and some of the surrounding air with FEM and the unbounded rest of the air with BEM. We started with a zero solution on the initial triangulation shown in Figure 4.1b and refined six times to obtain a solution for the problem. This solution is plotted in Figure 4.6 at three different times. To get the approximation of u_e in Ω_e we use the representation formula (1.3) with the discrete solution $u_{h,\tau}|_\Gamma$ and $\phi_{h,\tau}$. The figure sequence shows how the electric field is building up. We can also see the skin effect, cf. [Jac99, Chapter 5.18]. This effect pushes the electric currents out of the material, which can be seen in Figure 4.6 as the solution is zero in the iron core.



(a) Solution at $t = 0.0125$.



(b) Solution at $t = 0.175$.



(c) Solution at $t = 0.2$.

Figure 4.6: Contour lines of the example in Section 4.1.2 solved with FEM-BEM coupling at different times.

4.2 FVM-BEM Coupling

To test the FVM-BEM coupling, we look at a convection-dominated problem to show the stability of the method. An extension of the problem of Section 4.1.1.2 shows the predicted reduced rate of convergence. The last example for the FVM-BEM coupling is a more practical example that could model the transport of a chemical compound in a (porous) medium.

In all examples we divide Ω into congruent triangles with a mesh-size $h = 0.125$ and divide the time interval $[0,1]$ into uniform time steps with step size $\tau^n = \tau = 0.05$.

4.2.1 Convection-Dominated Diffusion-Convection-Reaction Problem I

The first example has a prescribed smooth analytical solution. In the domain $\Omega = (0,1/2)^2$, we choose

$$u(x_1,x_2) = 0.5(1+t)\left(1-\tanh\left(\frac{0.25-x_1}{0.02}\right)\right),$$

and as the solution in the corresponding exterior domain Ω_e

$$u_e(x_1,x_2) = (1-t)\log\sqrt{(x_1-0.25)^2+(x_2-0.25)^2}.$$

The interior solution has a simulated shock in the middle of the domain, which can pose certain difficulties to the used method. The diffusion $\mathbf{A} = \alpha\mathbf{I}$ has a jump, i.e.,

$$\alpha = \begin{cases} 0.42 & \text{for } x_2 < 0.25, \\ 1 & \text{for } x_2 \geq 0.25. \end{cases}$$

The convectional velocity and the reaction coefficient are set to $\mathbf{b} = (1000x_1, 0)^T$ and $c = 5$, respectively. Furthermore, the jumps g_1, g_2, and the right-hand side f are calculated by means of the analytical solution. Because the problem is convection-dominated we use the full upwind stabilisation $\mathscr{A}_V^{up}$ defined in (3.16). Both the interior and the exterior solution are smooth, thus we expect first-order convergence as predicted by Corollary 3.22. This can be seen in Figure 4.7.

4.2.2 Problem with a Diffusion Matrix on an L-shaped Domain I

The second test shows the reduction of the order of convergence if we do not meet the regularity requirements. This is an extended version of the problem in Section 4.1.1.2. This time we devise a diffusion matrix by

$$\mathbf{A} = \begin{pmatrix} 10+\cos x_1 & 160\,x_1x_2 \\ 160\,x_1x_2 & 10+\sin x_2 \end{pmatrix}.$$

We also have convectional velocity $\mathbf{b} = (1000x_1, 1000x_1)^T$ and a reaction $c = x_2$. To recapitulate the rest of the setup: On the L-shaped domain $\Omega = (-1/4,1/4)^2 \setminus [0,1/4] \times [-1/4,0]$ we prescribe a function with a singularity in the corner $(0,0)$: Let $x = (x_1,x_2) =$

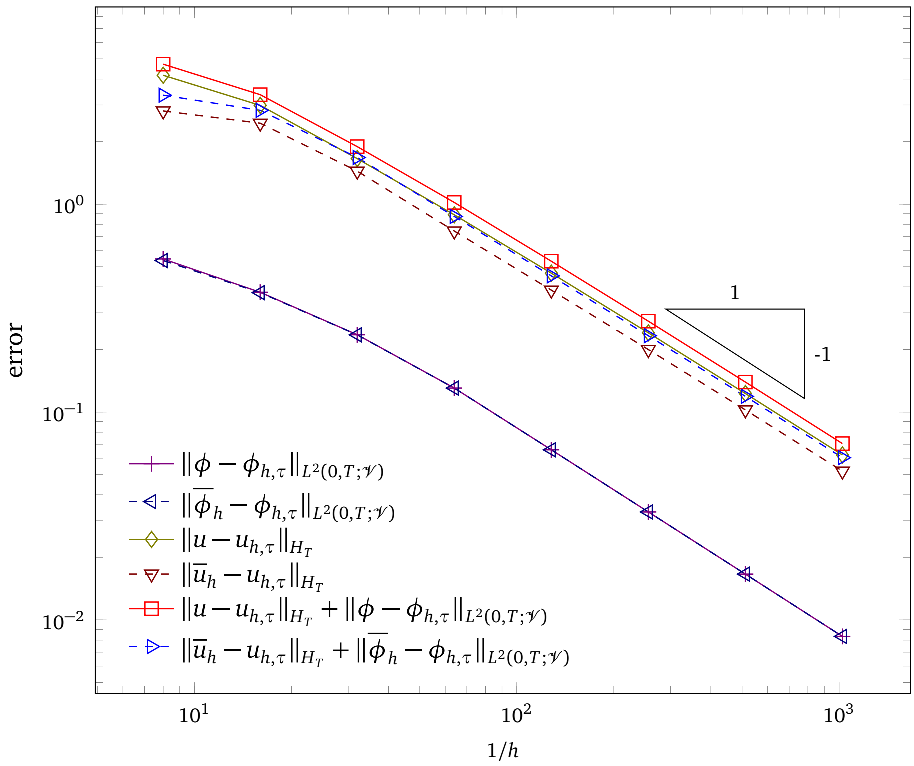


Figure 4.7: The different error components of the solutions $u_{h,\tau}$ and $\phi_{h,\tau}$ of the FVM-BEM coupling for the smooth example in Section 4.2.1. The added energy error norm $\|u-u_{h,\tau}\|_{H_T} + \|\phi - \phi_{h,\tau}\|_{L^2(0,T;\mathcal{V})}$ and $\|\overline{u}_h - u_{h,\tau}\|_{H_T} + \|\overline{\phi}_h - \phi_{h,\tau}\|_{L^2(0,T;\mathcal{V})}$ show first-order convergence.

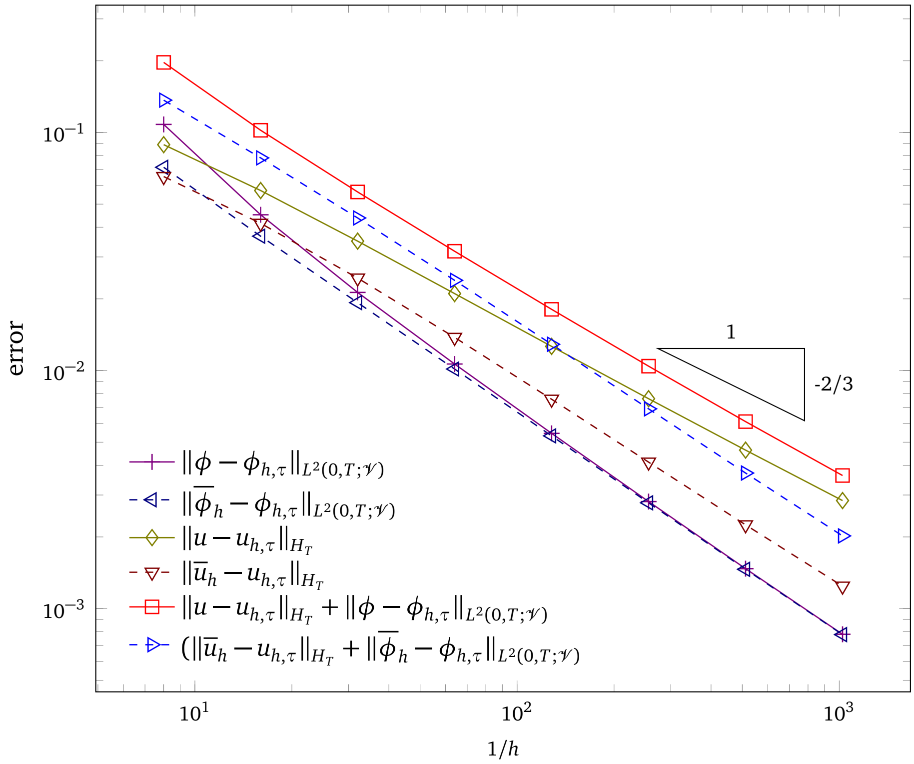


Figure 4.8: The different error components of the solutions $u_{h,\tau}$ and $\phi_{h,\tau}$ of the FVM-BEM coupling for the non-smooth example in space in Section 4.2.2. The added energy error norms $\|u-u_{h,\tau}\|_{H_T} + \|\phi - \phi_{h,\tau}\|_{L^2(0,T;\mathcal{V})}$ and $\|\overline{u}_h - u_{h,\tau}\|_{H_T} + \|\overline{\phi}_h - \phi_{h,\tau}\|_{L^2(0,T;\mathcal{V})}$ show a reduced order of convergence.

$r(\cos\varphi, \sin\varphi)$ with $r \in \mathbb{R}^+$ and $\varphi \in [0, 2\pi)$ be the polar coordinates of a point x, then the analytical solution in the interior reads:

$$u(x_1, x_2) = (1 + t^2) r^{2/3} \sin(2\varphi/3).$$

In the exterior domain Ω_e we choose

$$u_e(x_1, x_2) = (1 - t) \log \sqrt{(x_1 + 0.125)^2 + (x_2 - 0.125)^2}.$$

As before, we compute the jumps g_1, g_2, and the right-hand side f accordingly.

Because the function in the interior has reduced regularity in space (it is only in $H^{1+2/3-\varepsilon}(\Omega)$ for $\varepsilon > 0$), Corollary 3.22 predicts a reduced convergence order of $\mathcal{O}(h^{2/3} + \tau)$, which is indeed observed in the convergence plot Figure 4.8 of our numerical approximation.

4.2.3 A More Practical Problem

In this more practical example, we do not know the analytical solution. Let $\Omega = (-1/4, 1/4)^2$. The diffusion $\mathbf{A} = \alpha\mathbf{I}$ is set to $\alpha = 10^{-3}$, the convection to $\mathbf{b} = (30x_2, -10 + 10x_1)^T$, and the reaction to $c = 1$. The jumps are chosen to be zero and the right-hand side is chosen as

$$f(x_1, x_2, t) = \begin{cases} 5 & \text{for } -0.2 \leq x_1 \leq 0.1, \quad -0.2 \leq x_2 \leq -0.05, \quad t < 0.15, \\ 2.5 & \text{for } -0.2 \leq x_1 \leq 0.1, \quad 0.05 \leq x_2 \leq 0.2, \quad t < 0.4, \\ 0 & \text{else.} \end{cases}$$

This right-hand side may simulate a chemical compound being injected in two areas up to a certain point in time ($t = 0.15$ and $t = 0.4$). Hence, our model problem describes macroscopically the transport of this compound in a (porous) medium. Although in this case the porosity (which would be given by coefficient in front of the time derivative) is uniform and thus we do not have any pores. A reference for the model problem in the interior can be found in [LQF98]. For the exterior, we assume a uniform diffusion process. Due to convection dominance, we apply the full upwind stabilisation $\mathscr{A}_V^{up}$ defined in (3.16). The domain Ω is divided into congruent triangles with a mesh-size $h = 0.125$, and the time step size is set to $\tau^n = \tau = 0.05$. The solution after three refinements is plotted at different times in Figure 4.9. We can see how the solution penetrates the boundary, here we would see oscillations without stabilisation. We investigate this further in Section 4.3.3.

4.3 SUPG-BEM Coupling and Comparison

For the numerical experiments with the SUPG-BEM coupling, we chose the stabilisation parameter

$$\delta_K = \frac{\sqrt{\tau} h_K}{2\|\mathbf{b}\|_2} \left(\coth(Pe) - \frac{1}{Pe} \right),$$

where Pe is the Péclet number. This is a combination of the parameters proposed in [Kno08] and [JN11]. In our examples, this parameter also fulfils the assumptions (3.42) and (3.49) made for the coercivity of the SUPG bilinear form and the well-posedness of the fully discrete scheme.

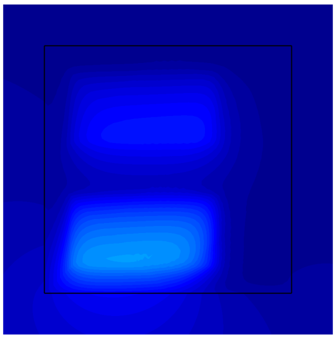
In the following, we will use the same examples as for the FVM-BEM coupling.

4.3.1 Convection-Dominated Diffusion-Convection-Reaction Problem II

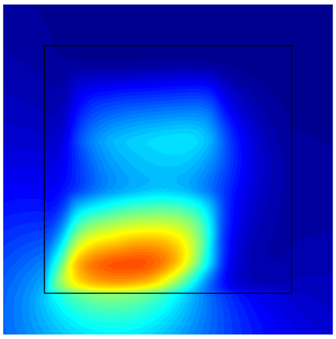
First, we look at the problem of Section 4.2.1. This is a convection-dominated problem, so we need the stabilisation to obtain meaningful solutions. The resulting errors are plotted in Figure 4.10. As the solution is smooth, we can see the first-order convergence as predicted by Corollary 3.44 and the error curves are very similar to the FVM-BEM coupling.

4.3.2 Problem with a Diffusion Matrix on an L-shaped Domain II

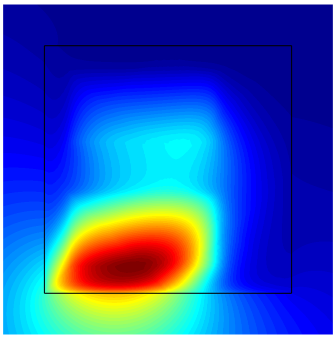
Next, we look at the problem of Section 4.2.2. In this example Corollary 3.44 predicts a reduced order of convergence because the function has a singularity in one corner of the domain. This can be seen in Figure 4.11. Again, the magnitude of the error is comparable to the errors of the FVM-BEM coupling.



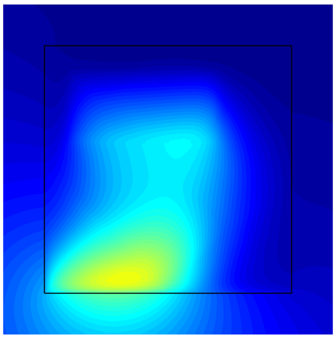
(a) Solution at $t = 0.00625$.



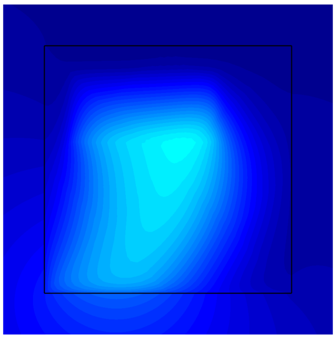
(b) Solution at $t = 0.03125$.



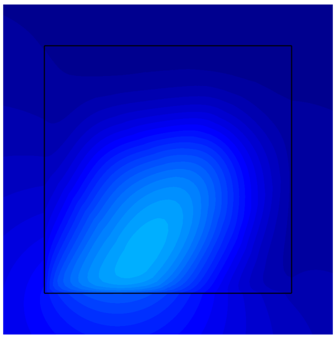
(c) Solution at $t = 0.125$.



(d) Solution at $t = 0.1625$.



(e) Solution at $t = 0.25$.



(f) Solution at $t = 0.4125$.

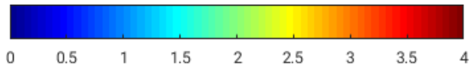


Figure 4.9: Solution of the transport problem in Section 4.2.3 at different times. The source is located at the left-hand side of the domain and stronger in the lower half. We turn off the source in the lower half at $t = 0.15$. At $t = 0.4$ the source is turned off completely.

4.3.3 Comparison of the Methods

The previous examples showed that the convergence behaviour of the different methods is very much comparable. The FEM-BEM coupling should, therefore, be used when appropriate, i.e., for non-convection-dominated problems. For convection-dominated problems, on the other hand, both the FVM-BEM coupling and the SUPG-BEM coupling yield good results. Here we have to mention that the stabilisation given by the SUPG-BEM coupling is strongly dependent on the choice of the stabilisation parameter δ_K. So for this method, the performance depends on the specific example. In [JS08], different choices of the parameter are discussed.

To compare the oscillatory behaviour of the different methods, we look at the problem of Section 4.2.3 again. In Figure 4.12 we can see the solutions of the three methods at time $t = 0.125$. As we can see, FEM-BEM coupling yields the largest oscillations, which are non-existent in the FVM-BEM solution. The SUPG-BEM coupling reduces the oscillations, but cannot completely eliminate them. As mentioned before, this also depends on the choice of δ_K. Thus, the upwind stabilised FVM-BEM is always a good option if it is important to have an oscillation-free solution and flux conservation is mandatory. The SUPG-BEM, on the other hand is a good choice if an implementation of the FEM-BEM coupling is already available, as the modification with the additional stabilisation is easy to implement.

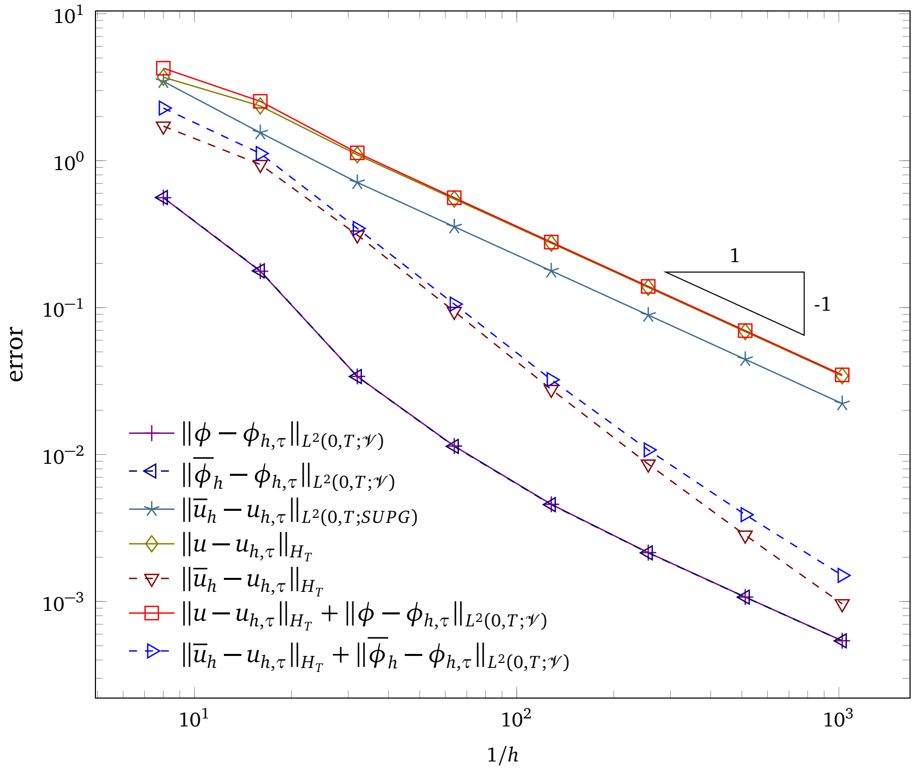


Figure 4.10: The different error components of the solutions $u_{h,\tau}$ and $\phi_{h,\tau}$ of the SUPG-BEM coupling for the smooth example in Section 4.3.1. The added energy error norm $\|u - u_{h,\tau}\|_{H_T} + \|\phi - \phi_{h,\tau}\|_{L^2(0,T;\mathcal{V})}$ shows first-order convergence.

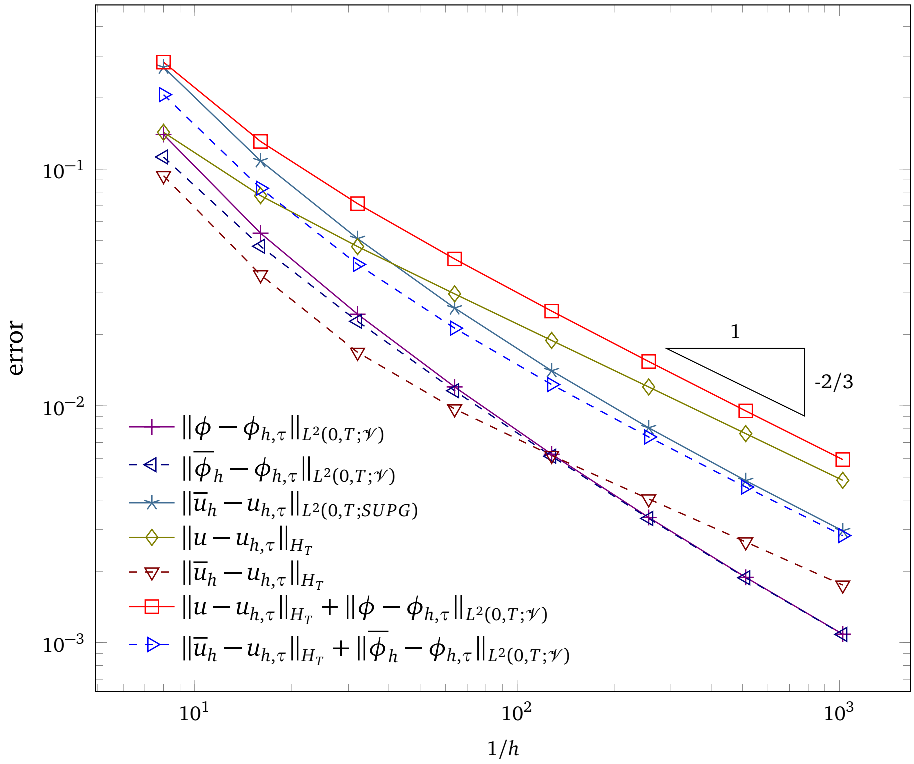
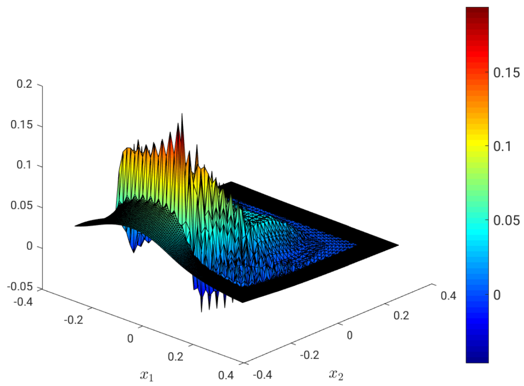
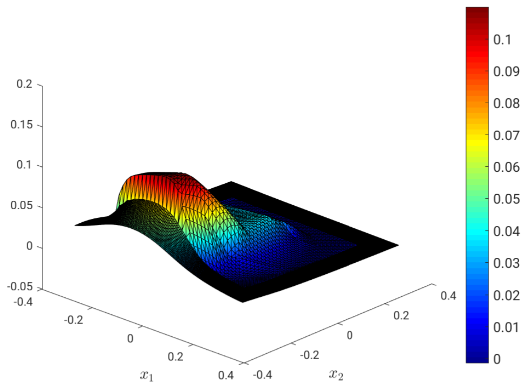


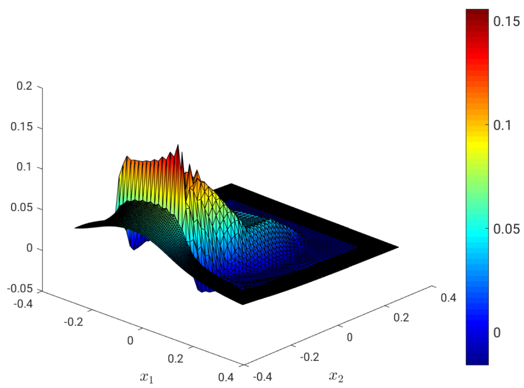
Figure 4.11: The different error components of the solutions $u_{h,\tau}$ and $\phi_{h,\tau}$ of the SUPG-BEM coupling for the smooth example in Section 4.3.2. The added energy error norm $\|u-u_{h,\tau}\|_{H_T}+\|\phi-\phi_{h,\tau}\|_{L^2(0,T;\mathscr{V})}$ shows first-order convergence.



(a) Solution of the FEM-BEM coupling.



(b) Solution of the FVM-BEM coupling.



(c) Solution of the SUPG-BEM coupling.

Figure 4.12: Solution of the transport problem in Section 4.2.3 computed with the three different coupling methods. We can see the oscillations present in the computed solutions.

“The purpose of computing is insight, not numbers.”

Richard Hamming

Conclusions

LASTLY, WE CONCLUDE what we were able to achieve and give a short outlook on what work could be done in the future. In this work, we considered parabolic-elliptic problems, in which the interior problem can be convection-dominated and the exterior domain is unbounded. We provided a refined a priori analysis for the semi-discretisation of the non-symmetric coupling of the Finite Element Method (for the interior problem) and the Boundary Element Method (for the exterior problem). Furthermore, the coupling with the Finite Volume Method (for the interior problem) has been proved to be a good choice for the spatial discretisation of convection-dominated problems, which additionally conserves the numerical fluxes. As a third option, we also considered the coupling with the Streamline Upwind Petrov Galerkin Method, which turned out to be a good choice if you already have an implementation of the FEM-BEM coupling, but yields results with more oscillations than the FVM-BEM coupling.

Moreover, we gave the first a priori analysis for the full discretisation of the different couplings. In the FEM-BEM case, we were able to perform the analysis in terms of the energy norm of the solution space. For all the coupling methods, we were able to show quasi-optimality results for both, the semi- and the full discretisation, using a variant of the implicit Euler method in time. In the case of the couplings with the FVM and the SUPG methods, some perturbation terms have to be taken into account in these results. For the FVM-BEM coupling, we additionally considered a discretisation in time by a classical implicit Euler method.

However, the optimal convergence rate in the L^2 norm, which usually relies on a duality argument, still remains open for all the coupling methods. In case of a non-symmetric approach, adjoint regularity cannot be obtained as easily as in the symmetric case. Thus, our analysis avoided using the elliptic projection and used the L^2–projection instead.

As a side note: our analysis can also be applied for standalone FEM or FVM approximation – we just have to replace coupling conditions by boundary conditions – which improves available results in the literature.

In the numerical experiments, we could verify the theoretical findings. In particular, they showed that our method even converges on non-convex domains with less regular data. We also showed two examples that hint into the direction of practical applications the couplings could be used for.

Most of the results have already been published and can be found in [EES18; ES19b; ES19a].

In the future, one could think of introducing a posteriori error estimators to devise an adaptive algorithm. Such an adaptive procedure could then be used to remedy the loss of convergence rate where the solution does not meet the required regularity. An adaptive non-symmetric coupling has already been stated in the steady-state case in [ES17]. For the time-dependent case, we would also need a process to determine the optimal mesh per time step because it can vary from time to time. This can, for example, be done by also employing a coarsening algorithm additional to a refinement algorithm. Such an adaptive algorithm based on FEM for parabolic problems is described in [EJ91] and [EJ95]. For the symmetric coupling for parabolic-elliptic interface problems, this has already been conducted in [MS97] by using residual a posteriori error estimators. The magnetoquasistatic problem of Section 4.1.2 would also benefit from an adaptive refinement, because the mesh has to be very fine around the boundary of the iron core to treat the skin effect, where the electric currents are pushed to the boundary.

At the moment, there is an ongoing project at the Graduate School of Computational Engineering at TU Darmstadt to investigate isogeometric FEM-BEM coupling for parabolic-elliptic interface problems to practically simulate electrical machines.

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List of Figures

List of Symbols

Common Symbols

$\mathbb{R}$	Real numbers
$\mathbb{N}$	Natural numbers
∇	Gradient
div	Divergence, differential operator
Δ	Laplace operator
∂_t	Partial derivative with respect to t
∂_{tt}	Second partial derivative with respect to t
$\partial_{\mathbf{n}}$	Normal derivative
d_τ	Discrete derivative
C^k	Space of continuous functions
$\mathcal{O}$	Landau symbol for asymptotic upper bound
$\mathbf{I}$	Identity matrix
$\lesssim$	Short-hand notation for $a \leq C \cdot b$ $(a \lesssim b)$

Sobolev Spaces

$L^p(\cdot)$	Lebesgue space of order p
$H^s(\cdot)$	Sobolev space of order s
$W^{1,\infty}(\Omega)$	Space of all Lipschitz continuous functions
$H^s(0,T;X)$	Bochner space of degree s of Hilbert space X
Q_T	Energy space of a parabolic problem
$\|\cdot\|_X$	Norm in space X
$(\cdot,\cdot)_\omega$	Scalar product in $\omega \subset \Omega$
$\langle\cdot,\cdot\rangle_\omega$	Duality product in $\omega \subset \Omega$

Boundary Integral Operators

γ_0	Trace operator
γ_1	Conormal derivative
$\mathscr{V}$	Single layer operator
$\mathscr{K}$	Double layer operator
$\mathscr{S}$	Steklov-Poincaré operator
$\mathscr{R}$	Inverse single layer operator

Model Problem

Ω	Interior domain
Ω_e	Exterior domain
Γ	Boundary of Ω
$\mathbf{n}$	Normal vector
$\mathbf{A}$	Diffusion matrix
$\mathbf{b}$	Convectional velocity
c	Reaction coefficient
γ	Defined as $1/2 \operatorname{div} \mathbf{b} + c$
f	Right-hand side
g_1	Jump function (for the trace)
g_2	Jump function (for the conormal derivative)
a	Constant of the radiation condition
u	Solution of the interior problem
u_e	Solution of the exterior problem
f_B	Combined notation of f and g_2
$g_{\mathscr{K}}$	Short-hand notation for $(1/2 - \mathscr{K})g_1$

Triangulation and Discrete Spaces

$\mathscr{T}$	Primal triangulation
K	Element of the triangulation
$\mathscr{E}$	Edges of the triangulation
$\mathscr{N}$	Nodes of the triangulation
$\mathscr{E}_\Gamma$	Edges on the boundary
$\mathscr{E}_K$	Edges of an element K
$\mathscr{T}^*$	Dual mesh
V	Control volume of dual mesh
τ_{ij}	Intersection between two control volumes
h	Maximum mesh size
$\mathscr{P}^k$	Space of polynomials of degree k
$\mathscr{S}^k$	Space of piecewise continuous polynomials of degree k
$Q_T^{h,\tau}$	Piecewise linear functions in time
$B_T^{h,\tau}$	Piecewise constant functions in time
$\mathscr{I}_h$	Piecewise constant interpolation operator
P_h	L^2-orthogonal projection
Π_h	$H^{-1/2}$-orthogonal projection

Discretisation

$\mathscr{b}(\cdot,\cdot)$	Weak coupling bilinear form (of the reduced problem)
$\mathscr{b}_{stab}(\cdot,\cdot)$	Stabilised version of the bilinear form $\mathscr{b}$
$F_{stab}(\cdot)$	Stabilised added right-hand side
u_h	Discrete interior solution
ϕ_h	Discrete exterior solution
$\widehat{v}^n$	Weighted averaging operator
ω^n	Linear weight of the averaging operator
$\mathscr{A}(\cdot,\cdot)$	Weak bilinear form of the interior problem
$\mathscr{B}(\cdot,\cdot)$	Weak coupling bilinear form (of the extended problem)
$F(\cdot)$	Added right-hand side (of the coupled problem)
$\mathscr{A}_V(\cdot,\cdot)$	Finite Volume bilinear form
$\mathscr{A}_V^{up}(\cdot,\cdot)$	Upwind stabilised Finite Volume bilinear form
$\mathscr{B}_V(\cdot,\cdot)$	FVM-BEM bilinear form
$F_V(\cdot)$	FVM-BEM right-hand side
δ_K	SUPG stabilisation parameter
Pe	Péclet number
$\mathscr{A}_{SUPG}(\cdot,\cdot)$	SUPG bilinear form
$\mathscr{B}_{SUPG}(\cdot,\cdot)$	SUPG-BEM bilinear form
$F_{SUPG}(\cdot)$	SUPG-BEM right-hand side

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