Chapter 3 3D-Analysis of the Earth's Surface Crustal Deformation

3.1. General

Deformation of the Earth is a three-dimensional phenomenon in nature. Therefore, the study of this phenomenon in two dimensions by reducing the observations onto the surface of a horizontal datum or, considering it as a pure vertical process may result in an erroneous picture of deformation. Especially, where the deformation rate in the vertical direction is larger or at least comparable to the horizontal ones, the impact of the vertical deformations on the horizontal ones may be quite considerable. To study this effect, two new approaches have been developed and analyzed in this research. One is based on the Lagrangian representation of the problem and the other is based on its Isoparametric form (Flügge, 1972).

Naturally, easier methods whose results are immediately applicable and directly interpretable are more desirable. Two-dimensional methods of deformation analysis benefit from providing numerical results that are immediately applicable and easily interpretable. This is because it is always possible to establish a functional relation between the parameters that characterize the deformation in two-dimensions (strain parameters) and stress, i.e. constitutive equations, for all types of rehologies. Therefore, an attempt is made to extend two of them to three-dimensions.

The process of extending the analysis of deformations from two to three dimensions is not as straightforward as it may seem. Various features such as the network size and the network configuration challenge this process, pose complexities and raise questions that need to be clarified and discussed in detail.

In this chapter, the mathematical framework of this study is to be explained. Through the analytical consideration of the normal matrix in both approaches mentioned above, it shall be shown that the 3D-representation of the Earth's surface crustal deformation is an ill-

conditioned problem. The application of the new approaches to real data sets will be discussed in Chapter 7. There, the two methods will be applied to the test area of this research.

3.2. Methodologies to the 3D-Analysis of Surface Deformations

Among the different methods that can be used for analyzing the surface deformations of the Earth, geodetic approaches are the only ones that can provide us the information on the contemporary state of deformation within the Earth's crust (Jackson and McKenzie, 1988). Other methods can portray the mean state of deformation either averaged over large temporal and spatial scales or vice versa. For example, Paleomagnetic measurements, the only non-geodetic source of information on rigid body rotations of the Earth's crust, are averaged over millions of years (Thatcher, 1995). Seismic methods (Kostrov, 1974; Jackson and McKenzie, 1988; Constantinos et al., 1992; Kiratzi and Papazachos, 1994) use the earthquake moment tensors for computing the averaged distortions of the Earth's crust within a specific volume and over several years of time. Using instruments like tilt and strain meters, deformation parameters are directly measured within a small area and for a short period of time.

According to Berber et al., (2003) the earliest known publication on the application of geodetic techniques for the analysis of the Earth's surface deformations is due to Terada and Miyabe (1929) where strain analysis was used for describing surface seismic deformations. Since then, various methods have been developed and proposed (Frank, 1966; Welsch, 1979; Bibby, 1982; Chen, 1991; Altiner, 1999; Krumm and Grafarend, 2002). Assuming that small vertical deformations would have negligible effect on the horizontal ones (Lichtenegger and Sünkel, 1989), the majority of the existing geodetic techniques systematically ignore the effect of the vertical deformations on the horizontal ones.

From a pure mathematical point of view, the deformation process can be taken as a mapping that transforms a body from its unstrained to a deformed state:

$$\mathbf{X}_2 = \mathbf{F}(\boldsymbol{\Theta})\mathbf{X}_1 \tag{3.1}$$

In this equation, $\mathbf{X}_1$ and $\mathbf{X}_2$ are $3p\times1$ vectors whose elements are the coordinates of the material points of deformable body before and after the deformation, the vector-valued function $\mathbf{F}(\boldsymbol{\Theta})$ is the corresponding mapping function and p is the number of the points that have been taken for discretizing the problem. Deformation of the body is characterized through the mapping function $\mathbf{F}$ and deformation parameters $\boldsymbol{\Theta}$. This approach to the analysis of deformation founds the mathematical basis in the theory of shape analysis (Crosilla, 2003; Dryden and Mardia, 1998) where for example thin-plate spline functions are commonly in use as the mapping function for characterizing the deformation. Methods of geometrical shape analysis fail to support the physics of the deformation process, established through the constitutive equations in the theory of continuum mechanics (see for example Flügge, 1972). Therefore, they are not tailored to the research area where the modeling of the dynamics of deformation is the main concern.

When the mapping $\mathbf{F}$ in Equation (3.1) is taken as an affine transformation, deformation parameters benefit from direct physical implications: they characterize homogeneous deformation of a deformable body (Sokolnikoff, 1956). Moreover, the mapping function can be written as the sum of a symmetric matrix ($\mathbf{S}$) and a skew-symmetric matrix ($\mathbf{R}$). Based on this idea, Brunner (1979) proposed a three-dimensional approach for the 3d-analysis of the overall deformations of the Earth’s crust.

3.3. 3D-Isoparametric Representation of Deformation

3.3.1. Infinitesimal Deformations

According to the theory of elasticity (Love, 1944; Jaeger, 1969), the continuous deformations of a deformable body can be formulated either in terms of the relative changes in the distances of a point and its surrounding points (Isoparametric representation of deformation) or in terms of the changes in their relative positions (Lagrangian representation of deformation).

The basic assumption in both approaches is that those points that contribute in estimating the parameters of deformation at one point (hereafter are called contribution

points) should fall in a small vicinity of the point at which the deformation parameters are to be estimated (hereafter is called computation point).

Consider two material points *A* and *B* of a deformable volume *V*. The two points are located at a distance *r* of each other before the body deforms. If (x, y, z) are the coordinates of the particle *A* in its un-deformed state, the coordinates of the material point *B* can be written as $(x + rl, y + rm, z + rn)$ where (l, m, n) are the direction cosines of the line which joins the two points in the corresponding coordinate system before it deforms. Due to some deformation, particle *A* moves to the new position $(x + u_A, y + v_A, z + w_A)$ in which $u_A = u(x, y, z)$ $v_A = v(x, y, z)$ and $w_A = w(x, y, z)$ are displacement functions that characterize the deformation at this point. Similarly, point *B* takes up the new position $(x + rl + u_B, y + rm + v_B, z + rn + w_B)$ where:

$$u_B = u(x + rl, y + rm, z + rm) \tag{3.2}$$

$$v_B = v(x + rl, y + rm, z + rm) \tag{3.3}$$

$$w_B = w(x + rl, y + rm, z + rm) \tag{3.4}$$

Linearizing the displacements of particle *B* about the displacements at point *A* gives:

$$u_B = u_A + r\left(l\frac{\partial u}{\partial x} + m\frac{\partial u}{\partial y} + n\frac{\partial u}{\partial z}\right) + \sum_{j=2}^{\infty}\frac{1}{j!}\left(rl\frac{\partial}{\partial x} + rm\frac{\partial}{\partial y} + rn\frac{\partial}{\partial z}\right)^j u \tag{3.5a}$$

$$v_B = v_A + r\left(l\frac{\partial v}{\partial x} + m\frac{\partial v}{\partial y} + n\frac{\partial v}{\partial z}\right) + \sum_{j=2}^{\infty}\frac{1}{j!}\left(rl\frac{\partial}{\partial x} + rm\frac{\partial}{\partial y} + rn\frac{\partial}{\partial z}\right)^j v \tag{3.5b}$$

$$w_B = w_A + r\left(l\frac{\partial w}{\partial x} + m\frac{\partial w}{\partial y} + n\frac{\partial w}{\partial z}\right) + \sum_{j=2}^{\infty}\frac{1}{j!}\left(rl\frac{\partial}{\partial x} + rm\frac{\partial}{\partial y} + rn\frac{\partial}{\partial z}\right)^j w \tag{3.5c}$$

If point *B* is close enough to particle *A* such that the square and higher order terms of the distance r are negligible, these equations can be further simplified to:

$$
\begin{aligned}
x_B &= x + rl + u_A + r(l\frac{\partial u}{\partial x} + m\frac{\partial u}{\partial y} + n\frac{\partial u}{\partial z}) \\
y_B &= y + rl + v_A + r(l\frac{\partial v}{\partial x} + m\frac{\partial v}{\partial y} + n\frac{\partial v}{\partial z}) \\
z_B &= z + rl + w_A + r(l\frac{\partial w}{\partial x} + m\frac{\partial w}{\partial y} + n\frac{\partial w}{\partial z})
\end{aligned}
\tag{3.6}
$$

Therefore, the distance r_1 between the two points in its strained state is:

$$
\begin{aligned}
r_1^2 = &\left(rl + r(l\frac{\partial u}{\partial x} + m\frac{\partial u}{\partial y} + n\frac{\partial u}{\partial z}) \right)^2 + \left(rm + r(l\frac{\partial v}{\partial x} + m\frac{\partial v}{\partial y} + n\frac{\partial v}{\partial z}) \right)^2 \\
&+ \left(rn + r(l\frac{\partial w}{\partial x} + m\frac{\partial w}{\partial y} + n\frac{\partial w}{\partial z}) \right)^2
\end{aligned}
\tag{3.7}
$$

Moreover, if the deformations are so small that the squares and products of the quantities like $\partial u / \partial x$ can be safely ignored, Equation (3.7) can be further reduced to:

$$
q = e_{xx} l^2 + e_{yy} m^2 + e_{zz} n^2 + e_{xy} lm + e_{xz} nl + e_{yz} mn \tag{3.8a}
$$

where $q = (r_1 - r)/r$ is known as linear elongation and:

$$
e_{xx} = \frac{\partial u}{\partial x}, e_{yy} = \frac{\partial v}{\partial y}, e_{zz} = \frac{\partial w}{\partial z}, e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, e_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, e_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{3.8b}
$$

are by definition the components of the second-rank infinitesimal tensor of strain. Here it should be noted that since quantities like $\partial u / \partial x$ characterize the relative displacements of the points *A* and *B* (see Equations (3.5)), we can equivalently assume that the relative displacements of the two particles are small enough to neglect their squares and products.

3.3.2. Finite Deformations

When deformations are large, it is not possible to ignore the products of the quantities like $\partial u / \partial x$, etc. Therefore, for finite deformations these quantities should be kept in Equation (3.7). Few steps would lead us to the following equivalent quadratic form:

$$(Q=)\frac{r_1^2}{r^2} = (1+2E_{xx})l^2 + (1+2E_{yy})m^2 + (1+2E_{zz})n^2 + 2E_{xy}lm + 2E_{xz}nl + 2E_{yz}mn \quad (3.9a)$$

where:

$$\begin{aligned}
E_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\} \\
E_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} \\
E_{zz} &= \frac{\partial w}{\partial z} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} \\
E_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z} \\
E_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z} \\
E_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}
\end{aligned} \quad (3.9b)$$

are (by definition) known as the finite components of strain.

Since in geodetic networks stations are normally far from each other, practically the application of the Equations (3.8) and (3.9) is only limited to deformation fields in which the linear approximation in Equations (3.5) is conceivable. A diagnosis method has been formulated in this research by which the deformation parameters that are sensitive to this approximation can be discriminated from the others. The method will be discussed in detail in the next chapter of this thesis.

Since the displacements are only known at the network stations, strains should be estimated at the location of the network stations too. For this purpose, the relative change in the length of the vectors between each station and the surrounding contributing points can be

used. If totally p points are incorporated in the computation of deformation at computation point k, it is easy to show that the coefficient (design) matrix **A**, for the system of observation equations (3.8a), in a local coordinate system is given by the equation:

$$\mathbf{A} = \begin{bmatrix} \frac{\Delta E_{k1}^2}{L_{k1}^2} & \frac{\Delta N_{k1}^2}{L_{k1}^2} & \frac{\Delta E_{k1}\Delta N_{k1}}{L_{k1}^2} & \frac{\Delta E_{k1}\Delta U_{k1}}{L_{k1}^2} & \frac{\Delta N_{k1}\Delta U_{k1}}{L_{k1}^2} & \frac{\Delta U_{k1}^2}{L_{k1}^2} \\ \frac{\Delta E_{k2}^2}{L_{k2}^2} & \frac{\Delta N_{k2}^2}{L_{k2}^2} & \frac{\Delta E_{k2}\Delta N_{k2}}{L_{k2}^2} & \frac{\Delta E_{k2}\Delta U_{k2}}{L_{k2}^2} & \frac{\Delta N_{k2}\Delta U_{k2}}{L_{k2}^2} & \frac{\Delta U_{k2}^2}{L_{k2}^2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\Delta E_{kp}^2}{L_{kp}^2} & \frac{\Delta N_{kp}^2}{L_{kp}^2} & \frac{\Delta E_{kp}\Delta N_{kp}}{L_{kp}^2} & \frac{\Delta E_{kp}\Delta U_{kp}}{L_{kp}^2} & \frac{\Delta N_{kp}\Delta U_{kp}}{L_{kp}^2} & \frac{\Delta U_{kp}^2}{L_{kp}^2} \end{bmatrix}_{p\times 6} \tag{3.10}$$

In this equation ΔE_{ki}, ΔN_{ki} and ΔU_{ki} ($i = 1, 2, \ldots, \mathrm{p}$) are the components of the position vectors $\overrightarrow{ki}$ in the local *ENU*-coordinate system and L_{ki} is the length of vector $\overrightarrow{ki}$ (see Figure 3.1).

Equation (3.10) can further help us to deepen our understanding of 3D-Isoparametric representation of deformations. The study of some special configurations of computation and contributing points can clarify this argument:

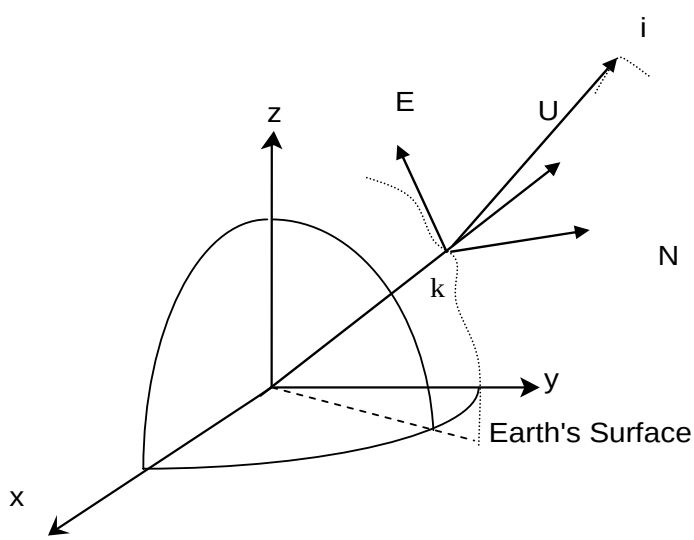

Figure 3.1: Local versus geocentric Cartesian coordinate systems

1) One of the three baseline components is much smaller than its length.

A good example would be the case in which there is no height difference between the computation and the contributing points. In that case: $\Delta U_{\mathrm{ki}} = 0$ ($\forall i$, $i = 1, 2, ..., \mathrm{p}$) and thereby, $rank\,(\mathbf{A}) = 3 < 6$. In practice, small height differences between the computation and contributing points are more probable. In this case, the rank deficient problem above changes to an ill-conditioned problem. It shall be shown in the next chapter that such a badly conditioned problem is highly sensitive to the perturbations of the input parameters. Another example for such a problem is that if contribution points are all aligned along one of the local coordinate axes, say the East-West coordinate axis; $\Delta N_{\mathrm{ki}} = 0$ ($\forall i$, $i = 1, 2, ..., \mathrm{p}$) and therefore, $rank\,(\mathbf{A}) = 3 < 6$. If the condition is only approximately fulfilled, the rank deficient problem again changes to an ill-conditioned problem.

2) Symmetry with respect to the computation point.

Perfect symmetry in the configuration of at least two contributing points with respect to the computation point again renders the problem to a rank deficient one. When the condition is approximately fulfilled, the problem will be ill-conditioned.

The aforementioned hindrances can be avoided when a network is to be designed for monitoring the 3D-deformations of the Earth's crust. The diagnosis method of Chapter 4 can also be efficiently used for pre-analyzing the sensitivity of the deformation tensor elements to the configuration of the network that is to be designed.

3.4. 3D-Lagrangian Representation of Deformation

The Isoparametric representation of deformation is based on the relative changes in the baselines' lengths. Since a vector length is coordinate invariant, this approach to the analysis of deformation is also invariant with respect to the coordinate systems in which the input coordinates are given. In contrary, the Lagrangian approach is based on the relative coordinate changes of the network stations and hence it is coordinate variant.

To account for the datum problem, different methods have been proposed (Frank, 1966; Welsch, 1979; Segal and Mathews, 1988; Prescott, 1981; Snay and Cline, 1980; Prescott, 1981; Bibby, 1982; Bock, 1983; Guohua and Prescott, 1986; Dermanis and

Grafarend, 1993; Kaniuth, 1997; Altiner, 1999; Krumm and Grafarend, 2002). To avoid assigning and interpreting the possible changes of the reference frame to the deformation of a body, the coordinates of all epochs should refer to the same reference frame (see for example Bibby, 1982, Becker et al., 2002). Today, the IGS stations are routinely used as fiducial points to constrain the datum defect in the system of normal equations (e.g. Reilinger et. al., 1997a; McClusy et. al. 2000).Thereby, deformation parameters that are estimated from them are referred to the same reference frame.

Let us assume that the coordinates of all epochs are referred to the same reference frame. Consider two material points *A* and *B* in a deformable body. Particle *B* is assumed to be in a small vicinity of the point *A*. If (x_A, y_A, z_A) are the coordinates of the point *A* before deformation and $(\Delta x, \Delta y, \Delta z)$ are the projections of the distance between the two points onto the geocentric Cartesian coordinate axes, the coordinates of the particle *B* will be $(x_A + \Delta x, y_A + \Delta y, z_A + \Delta z)$. If the displacement functions $u = u(x, y, z)$, $v = v(x, y, z)$ and $w = w(x, y, z)$ characterize the continuous deformation of the body, then the displacement of point *B* relative to *A* can be written:

$$
\begin{aligned}
\Delta u &= \Delta x \frac{\partial u}{\partial x} + \Delta y \frac{\partial u}{\partial y} + \Delta z \frac{\partial u}{\partial z} + \sum_{j=2}^{\infty} \frac{1}{j!} \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} + \Delta z \frac{\partial}{\partial z} \right)^j u \\
\Delta v &= \Delta x \frac{\partial v}{\partial x} + \Delta y \frac{\partial v}{\partial y} + \Delta z \frac{\partial v}{\partial z} + \sum_{j=2}^{\infty} \frac{1}{j!} \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} + \Delta z \frac{\partial}{\partial z} \right)^j v \\
\Delta w &= \Delta x \frac{\partial w}{\partial x} + \Delta y \frac{\partial w}{\partial y} + \Delta z \frac{\partial w}{\partial z} + \sum_{j=2}^{\infty} \frac{1}{j!} \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} + \Delta z \frac{\partial}{\partial z} \right)^j w
\end{aligned}
\tag{3.11}
$$

When the two particles are close enough to each other, terms of power two and more in the relative coordinates of the two points (Δx, Δy and Δz) can be ignored. This, reduces these equations to:

$$\begin{aligned}
\Delta u &= \Delta x\,\frac{\partial u}{\partial x} + \Delta y\,\frac{\partial u}{\partial y} + \Delta z\,\frac{\partial u}{\partial z} \\
\Delta v &= \Delta x\,\frac{\partial v}{\partial x} + \Delta y\,\frac{\partial v}{\partial y} + \Delta z\,\frac{\partial v}{\partial z} \\
\Delta w &= \Delta x\,\frac{\partial w}{\partial x} + \Delta y\,\frac{\partial w}{\partial y} + \Delta z\,\frac{\partial w}{\partial z}
\end{aligned} \tag{3.12}$$

Equations above can be rewritten in terms of strain parameters using their definition in terms of displacement gradients and the following additional conventional terms:

$$2\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},\ 2\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},\ 2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{3.13}$$

This will lead us to the following set of simultaneous equations for computing the deformation parameters:

$$\begin{aligned}
\Delta u &= e_{xx}\Delta x + \frac{1}{2}e_{xy}\Delta y + \frac{1}{2}e_{zx}\Delta z - \omega_z\Delta y + \omega_y\Delta z \\
\Delta v &= \frac{1}{2}e_{xy}\Delta x + e_{yy}\Delta y + \frac{1}{2}e_{yz}\Delta z - \omega_x\Delta z + \omega_z\Delta x \\
\Delta w &= \frac{1}{2}e_{zx}\Delta x + \frac{1}{2}e_{yz}\Delta y + e_{zz}\Delta z - \omega_y\Delta x + \omega_x\Delta y
\end{aligned} \tag{3.14}$$

When ω_x, ω_y and ω_z are small, the terms like $-\omega_z y + \omega_y z$ represent rigid body rotations of magnitude $\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$ about an axis in the direction $\left(\omega_x : \omega_y : \omega_z\right)$ (Love, 1944).

Similar to the Isoparametric formulation of the problem, due to the large distances between the stations of a geodetic network; practically the application of Equations (3.14) is only limited to deformations in which linear approximation in Equations (3.11) is conceivable. The diagnosis method of Chapter 4 can also be used here for identifying the deformation parameters that are sensitive to this approximation. Moreover, it is not difficult to prove that for homogeneous deformations of an arbitrary degree *p*, the nonlinear terms in Equations (3.5) and (3.11) are identically zero, for example:

$$\sum_{j=2}^{\infty} \frac{1}{j!}\left(rl\,\frac{\partial}{\partial x} + rm\,\frac{\partial}{\partial y} + rn\,\frac{\partial}{\partial z} \right)^{j} u\left(x, y, z\right) = 0$$
$$\sum_{j=2}^{\infty} \frac{1}{j!}\left(\Delta x\,\frac{\partial}{\partial x} + \Delta y\,\frac{\partial}{\partial y} + \Delta z\,\frac{\partial}{\partial z} \right)^{j} u\left(x, y, z\right) = 0 \tag{3.15}$$

Similar to the Isoparametric representation of deformation, if totally p points are incorporated in the computation of deformation at computation point k, it is easy to show that in a local coordinate system the coefficient (design) matrix **A**, for the system of observations equations (3.14), takes the following form:

$$\mathbf{A} = \begin{bmatrix} \Delta E_{k1} & 0 & \frac{1}{2}\Delta N_{k1} & 0 & \frac{1}{2}\Delta U_{k1} & 0 & 0 & \Delta U_{k1} & -\frac{1}{2}\Delta N_{k1} \\ 0 & \Delta N_{k1} & \frac{1}{2}\Delta E_{k1} & 0 & 0 & \frac{1}{2}\Delta U_{k1} & -\Delta U_{k1} & 0 & -\Delta E_{k1} \\ 0 & 0 & 0 & \Delta U_{k1} & \frac{1}{2}\Delta E_{k1} & \frac{1}{2}\Delta N_{k1} & \Delta N_{k1} & -\Delta E_{k1} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \Delta E_{kp} & 0 & \frac{1}{2}\Delta N_{kp} & 0 & \frac{1}{2}\Delta U_{kp} & 0 & 0 & \Delta U_{kp} & -\frac{1}{2}\Delta N_{kp} \\ 0 & \Delta N_{kp} & \frac{1}{2}\Delta E_{kp} & 0 & 0 & \frac{1}{2}\Delta U_{kp} & -\Delta U_{kp} & 0 & -\Delta E_{kp} \\ 0 & 0 & 0 & \Delta U_{kp} & \frac{1}{2}\Delta E_{kp} & \frac{1}{2}\Delta N_{kp} & \Delta N_{kp} & -\Delta E_{kp} & 0 \end{bmatrix}_{3p\times 9} \tag{3.16}$$

This equation can further help us to deepen our understanding of the characteristics of the 3D-Lagrangian approach in comparison to the Isoparametric one. To clarify this argument, the same configurations of computation and contributing points are discussed here again:

1) One of the three baseline components is zero or much smaller than the others.

If either of the baseline components diminishes to zero, for example: $\Delta U_{\mathrm{ki}} = 0$ ($\forall i$, $i = 1, 2, \ldots, \mathrm{p}$), the 3D-Lagrangian representation of deformation will be also a singular problem ($rank\,(\mathbf{A}) = 8 < 9$). Nevertheless, the singularity of the problem is in a better condition compared to 3D-Isoparametric approach. Also the conditioning of matrix **A** in Equation (3.16) is better than the conditioning of matrix **A** in Equation (3.10). To clarify this argu-

ment let us assume that one of the baseline components, e.g. ΔN_{ki} $(i = 1, 2, ..., p)$, is much smaller than the others. According to equation (3.16) only one of the columns of the design matrix, the second column, in the 3D-Lagrangian representation of deformation will be close to zero. Equation (3.10) shows that in a similar situation three columns of matrix **A**, in this case the second, third and fifth columns, in the 3D-Isoparametric representation of deformation will be close to zero. This is because according to our assumption:

$$\Delta N_{ki} \ll \Delta E_{ki}\ ,\ \Delta N_{ki} \ll \Delta U_{ki} \text{ and } \Delta N_{ki} \ll L_{ki}\ .$$

Also obviously:

$$\Delta E_{ki} / L_{ki} < 1,\ \ \Delta U_{ki} / L_{ki} < 1 \text{ and } \Delta N_{ki} / L_{ki} \ll 1.$$

Consequently the products: $\Delta N_{ki}^2 / L_{ki}^2$, $\Delta E_{ki} \Delta N_{ki} / L_{ki}^2$ and $\Delta U_{ki} \Delta N_{ki} / L_{ki}^2$ will be very close to zero. Therefore, a better conditioning is expected for the Lagrangian approach compared to the Isoparametric one when either of the baseline components is much smaller than the others.

2) Symmetry with respect to the computation point.

Perfect symmetry in the configuration of at least two contributing points with respect to computation point also leads to rank deficiency in the problem. When contributing points are nearly symmetric with respect to computation point, the problem will be again ill-conditioned.

3.5. Finite versus Infinitesimal Strains

The derivation of Equations (3.8) and (3.9) immediately gives rise to the question when the non-linearity of deformations should be taken into account. As was pointed out above, the assumption that the quantities such as $\partial u / \partial x$ are small so that their squares and products are negligible is equivalent to the assumption that relative displacements of neighboring points are small. These relative displacements are characterized by the following equations:

$$\begin{aligned}
\Delta u &= r\left(l\,\frac{\partial u}{\partial x} + m\,\frac{\partial u}{\partial y} + n\,\frac{\partial u}{\partial z} \right) \\
\Delta v &= r\left(l\,\frac{\partial v}{\partial x} + m\,\frac{\partial v}{\partial y} + n\,\frac{\partial v}{\partial z} \right) \\
\Delta w &= r\left(l\,\frac{\partial w}{\partial x} + m\,\frac{\partial w}{\partial y} + n\,\frac{\partial w}{\partial z} \right)
\end{aligned} \tag{3.17}$$

Computing the sum of the squares of these quantities provides us a practical means for evaluating the adequacy of a linear model (infinitesimal theory of strain) for analyzing the deformation of a body:

$$\frac{1}{2}\frac{\Delta u^2 + \Delta v^2 + \Delta w^2}{r^2} = K_{xx} l^2 + K_{yy} m^2 + K_{zz} n^2 + K_{xy} lm + K_{xz} nl + K_{yz} mn \tag{3.18a}$$

where $K_{xx}, K_{yy}, K_{zz}, K_{xy}, K_{xz}$ and K_{yz} are the non-linear terms in the components of finite strains (see Equation (3.9b)). Comparing this equation to Equation (3.8a) reveals that the adequacy of a linear model in a specific problem can be judged through the comparison of the linear elongation q with the ratio:

$$q' = \frac{1}{2}\frac{\Delta u^2 + \Delta v^2 + \Delta w^2}{r^2}$$

When

$$q = \frac{r_1 - r}{r} >> \frac{1}{2}\frac{\Delta u^2 + \Delta v^2 + \Delta w^2}{r^2} \tag{3.18b}$$

the infinitesimal theory of strain can lead to satisfactory results. Otherwise, non-linearity of deformations has a considerable contribution and therefore, a linear model is not adequate. Clearly, this method of qualifying the mathematical model is independent of the preferred form (Isoparametric or Lagrangian) for representing the deformation.

In the finite element approach, deformation parameters are estimated for the centroid of irregularly shaped elements (constrained by the network configuration) whose displacements are unknown (Dermanis and Grafarend, 1993). Therefore, before analyzing the deformation, it is not possible to set up the relative displacements of the elements'

centroid and the elements' nodes in order to assess the qualification process mentioned above.

3.6. Transforming Strains

For illustration and interpretation purposes, it is always desirable to represent computed deformations in a local curvilinear coordinate system. For this purpose, two possibilities exists: One is the direct formulation of the problem in the desired curvilinear coordinate system and the other, is computing these parameters in a geocentric Cartesian coordinate system and transforming them to the desired local coordinates. The latter approach has two clear advantages over the former one:

1) The former approach requires reformulation of the problem when an improvement in approximation (for example from spherical to ellipsoidal) is found to be necessary. While, the latter adds versatility to computations because, once the parameters are computed in a geocentric coordinate system, they can be transformed to the desired local curvilinear coordinates without any further need to reformulating the whole problem.

2) In the former approach mathematical models, and thereby the computational procedure, unnecessarily becomes complicated.

The orientation of the base vectors of spherical and ellipsoidal coordinate systems at a point on their surface with respect to a topocentric Cartesian coordinates whose axes are parallel to the coordinate axes of the geocentric Cartesian coordinate system is given by the following equations (see Appendix A for the derivation details):

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = R_{Spher.} \begin{bmatrix} e \\ n \\ u \end{bmatrix}_{Spher.} = \begin{bmatrix} -\sin\lambda & -\cos\lambda\sin\phi & \cos\lambda\cos\phi \\ \cos\lambda & -\sin\lambda\sin\phi & \sin\lambda\cos\phi \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \begin{bmatrix} e \\ n \\ u \end{bmatrix}_{Spher.} \tag{3.19a}$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = R_{Ellip.} \begin{bmatrix} e \\ n \\ u \end{bmatrix}_{Ellip.} = \begin{bmatrix} -\sin\lambda & \dfrac{-\sin\phi\cos\lambda}{\sqrt{1-2e^2\sin^2\phi}} & \cos\phi\cos\lambda \\ \cos\lambda & \dfrac{-\sin\phi\sin\lambda}{\sqrt{1-2e^2\sin^2\phi}} & \cos\phi\sin\lambda \\ 0 & \dfrac{+\dfrac{b^2}{a^2}\cos\phi}{\sqrt{1-2e^2\sin^2\phi}} & \sin\phi \end{bmatrix} \begin{bmatrix} e \\ n \\ u \end{bmatrix}_{Ellip.} \tag{3.19b}$$

Where, the abbreviations $Spher.$ and $Ellip.$ refer to the local spherical and ellipsoidal coordinate systems respectively.

The transformation of strain tensor is done using strain quadratic. Strain quadratic is an invariant functional of the elements of deformation tensor. In geocentric Cartesian and local coordinate systems, strain quadratic takes up the following forms respectively:

$$f\left(dX, dY, dZ\right) = \begin{bmatrix} dX & dY & dZ \end{bmatrix} \begin{bmatrix} e_{XX} & \frac{1}{2}e_{XY} & \frac{1}{2}e_{XZ} \\ \frac{1}{2}e_{YX} & e_{YY} & \frac{1}{2}e_{YZ} \\ \frac{1}{2}e_{ZX} & \frac{1}{2}e_{ZY} & e_{ZZ} \end{bmatrix} \begin{bmatrix} dX \\ dY \\ dZ \end{bmatrix} \tag{3.20a}$$

$$f\left(de, dn, du\right) = \begin{bmatrix} de & dn & du \end{bmatrix} \begin{bmatrix} e_{ee} & \frac{1}{2}e_{en} & \frac{1}{2}e_{eu} \\ \frac{1}{2}e_{ne} & e_{nn} & \frac{1}{2}e_{nu} \\ \frac{1}{2}e_{ue} & \frac{1}{2}e_{un} & e_{uu} \end{bmatrix} \begin{bmatrix} de \\ dn \\ du \end{bmatrix} \tag{3.20b}$$

Substituting Equation (3.19a or 1.19b) into the Equation (3.20a) gives the following equations:

$$f\left(de,dn,du\right)=\begin{bmatrix}de & dn & du\end{bmatrix}\left(R\begin{bmatrix}e_{XX} & \frac{1}{2}e_{XY} & \frac{1}{2}e_{XZ}\\ \frac{1}{2}e_{YX} & e_{YY} & \frac{1}{2}e_{YZ}\\ \frac{1}{2}e_{ZX} & \frac{1}{2}e_{ZY} & e_{ZZ}\end{bmatrix}R^T\right)\begin{bmatrix}de\\ dn\\ du\end{bmatrix} \tag{3.21a}$$

The comparison of this result with Equation (3.20b) results in the following transformation equation for transforming the strain tensor between a local and a geocentric coordinate system:

$$\begin{bmatrix}e_{\lambda\lambda} & \frac{1}{2}e_{\lambda\phi} & \frac{1}{2}e_{\lambda r}\\ \frac{1}{2}e_{\phi\lambda} & e_{\phi\phi} & \frac{1}{2}e_{\phi r}\\ \frac{1}{2}e_{r\lambda} & \frac{1}{2}e_{r\phi} & e_{rr}\end{bmatrix}=R^T\begin{bmatrix}e_{XX} & \frac{1}{2}e_{XY} & \frac{1}{2}e_{XZ}\\ \frac{1}{2}e_{YX} & e_{YY} & \frac{1}{2}e_{YZ}\\ \frac{1}{2}e_{ZX} & \frac{1}{2}e_{ZY} & e_{ZZ}\end{bmatrix}R \tag{3.21b}$$

where $R=R_{Spher.}$ and $R=R_{Ellip.}$ in spherical and ellipsoidal approximations respectively.