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CASED

A Multivariate Signature Scheme with a Partially Cyclic Public Key

Albrecht Petzoldt, Stanislav Bulygin and Johannes Buchmann
TU Darmstadt, CASED

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Motivation (1)

Multivariate Public Key Cryptography

- Secure against attacks with quantum computers
- Suitable for the use on low cost devices
- Fast and efficient
- But: large keys

Motivation (2)

Key size reduction

Reduction of the private key

Use sparse central polynomials (e.g. TTS)

Reduction of the public key

The Oil and Vinegar Signature Scheme

- Choose two integers o and v, set $n = o + v$.
- Set $V = \{1, \ldots, v\}$ and $O = \{v+1, \ldots, n\}$.
- Choose randomly o quadratic polynomials $q^{(1)}, \ldots, q^{(o)}$ of the form

$$q^{(k)}(x_1, \ldots, x_n) = \sum_{i,j \in V, i \le j} q_{ij}^{(k)} x_i x_j + \sum_{i \in V, j \in O} q_{ij}^{(k)} x_i x_j + \sum_{i \in V \cup O} q_i^{(k)} x_i + q_0^{(k)}, \quad (k = 1, \ldots, o)$$

- Define the central map Q as $Q = (q^{(1)}, \ldots, q^{(o)})$
- Choose randomly an affine invertible map $T = (M_T, c_T)$.
- **public key**: $P = Q \circ T$
- **Private key**: Q, T

For security reasons it is necessary to choose $v \ge 2o$.

Oil and Vinegar (2)

Signature generation

- Compute $h = H(m) \in K^o$
- Compute one preimage y of h under Q
  - Choose the vinegar variables $x_1, \ldots, x_v$ at random
  - Solve the resulting linear system for the oil variables
- Compute $z = T^{-1}(y) \in K^n$

Signature verification

- Compute $h = H(m)$ and $w = P(z)$

If $w = h$ holds, the signature is accepted, otherwise rejected.

Construction (1)

$$P = \sum_{i=1}^{n}\sum_{j=i}^{n} p_{ij}^{(1)} x_i x_j + \sum_{i=1}^{n} p_i^{(1)} x_i + p_0^{(1)},$$

$$\ldots,$$

$$\sum_{i=1}^{n}\sum_{j=i}^{n} p_{ij}^{(o)} x_i x_j + \sum_{i=1}^{n} p_i^{(o)} x_i + p_0^{(o)}$$

↓ graded lexicographical ordering

$$M_P = \begin{pmatrix} p_{11}^{(1)} & p_{12}^{(1)} & \ldots & p_{nn}^{(1)} & p_1^{(1)} & \ldots & p_n^{(1)} & p_0^{(1)} \\ \vdots & & & & & & & \vdots \\ p_{11}^{(o)} & p_{12}^{(o)} & \ldots & p_{nn}^{(o)} & p_1^{(o)} & \ldots & p_n^{(o)} & p_0^{(o)} \end{pmatrix} = \begin{pmatrix} \pi_{11} & \ldots & \pi_{1d} \\ \vdots & & \vdots \\ \pi_{o1} & \ldots & \pi_{od} \end{pmatrix}$$

where $d = \frac{(n+1)\cdot(n+2)}{2}$.

We want $M_P = (B \mid C),$

where B is a partially circulant matrix.

Construction (2)

Due to the relation $P = Q \circ T$ we get the following equations between the quadratic coefficients of P and Q :

$$p_{ij}^{(k)} = \sum_{r=1}^{n} \sum_{s=r}^{n} \alpha_{ij}^{rs} \cdot q_{rs}^{(k)} = \sum_{r=1}^{v} \sum_{s=r}^{n} \alpha_{ij}^{rs} \cdot q_{rs}^{(k)} \qquad (1)$$

with

$$\alpha_{ij}^{rs} = \begin{cases} t_{ri} \cdot t_{si} & (i = j) \\ t_{ri} \cdot t_{sj} + t_{rj} \cdot t_{si} & \text{otherwise} \end{cases}$$

Assign the coefficients of T some elements of K.

→ We get linear relations between the $p_{ij}^{(k)}$ and the $q_{rs}^{(k)}$.

Construction (3)

For $i = 1, \ldots, o$ we define two $D = \frac{v \cdot (v+1)}{2} + o \cdot v$-vectors

$$v_P^{(i)} = (p_{kl}^{(i)} \mid 1 \le k \le v, k \le l \le n)$$

and

$$v_Q^{(i)} = (q_{kl}^{(i)} \mid 1 \le k \le v, k \le l \le n)$$

containing the first D coefficients of the i-th public and central polynomials.

Additionally, we define a $D \times D$ matrix A containing the coefficients of the equations (1).

$$A = \left(\alpha_{ij}^{rs} \middle| \begin{array}{l} 1 \le i \le v, i \le j \le n \text{ for the columns} \\ 1 \le r \le v, r \le s \le n \text{ for the rows} \end{array} \right) = \begin{pmatrix} \alpha_{11}^{11} & \alpha_{12}^{11} & \ldots & \alpha_{vn}^{11} \\ \alpha_{11}^{12} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \alpha_{11}^{vn} & \ldots & \ldots & \alpha_{vn}^{vn} \end{pmatrix}$$

Construction (4)

We obtain for $i = 1, \ldots, o$

$$v_P^{(i)} = v_Q^{(i)} \cdot A \quad (2)$$

We can use these relations to reduce the public key size of UOV by a large factor.

Of 1000 matrices A over GF(256) were invertible

(o,v)	(2,4)	(5,10)	(10,20)	(15,30)	(20,40)
# invertible	993	996	997	995	994

Assuming that A is invertible, we can prove the following theorem:

Construction (5)

Theorem: For every $s \leq \frac{v \cdot (v+1)}{2} + ov$, $\vec{b} = (b_1, \ldots, b_s) \in_R K^s$ and invertible affine map $T = (M_T, c_T) : K^n \to K^n$ there exist two quadratic maps $P, Q : K^n \to K^o$ such that

1. Q is a UOV map
2. We have $P = Q \circ T$ as a composition of mappings
3. The entries of the matrix M_P fulfill

$$\pi_{ij} = b_{(j-i \bmod s)+1} \quad (1 \leq i \leq o, 1 \leq j \leq s)$$

The Scheme (1)

- Key Generation

1. For $s \leq D$ choose a vector $\vec{b} = (b_1, \ldots, b_s) \in_R K^s$.
2. Choose an affine map $T = (M_T, c_T) : K^n \to K^n$ at random. If M_T is not invertible, choose again.
3. Compute for T the corresponding matrix A. If A is not invertible, go back to step 2.
4. For $i = 1, \ldots, o$ set

$$v_P^{(i)} = S^{i-1}(\vec{b}),$$

where $S(\vec{b})$ is the cyclic right shift of the vector $\vec{b}$.

The Scheme (2)

5. Solve for $i = 1, \ldots, o$ the linear systems given by equation (2) to get the vectors $v_Q^{(i)}$ and therewith the quadratic coefficients of the central polynomials.
6. Choose the coefficients of the linear terms of the central polynomials at random.
7. Compute the public key as $P = Q \circ T$.

The Scheme (3)

- The public key consists of the vector $\vec{b}$ and the last $\frac{(n+1)\cdot(n+2)}{2} - s$ columns of M_P .
- The private key consists of the maps Q and T .
- **public key size:**

$$s + o \cdot \left(\frac{(n+1)\cdot(n+2)}{2} - s \right) = o \cdot \frac{(n+1)\cdot(n+2)}{2} - s \cdot (o-1)$$ field elements

- **private key size:**

$$o \cdot \left(\frac{v\cdot(v+1)}{2} + o \cdot v + n + 1 \right) + n \cdot (n+1)$$ field elements

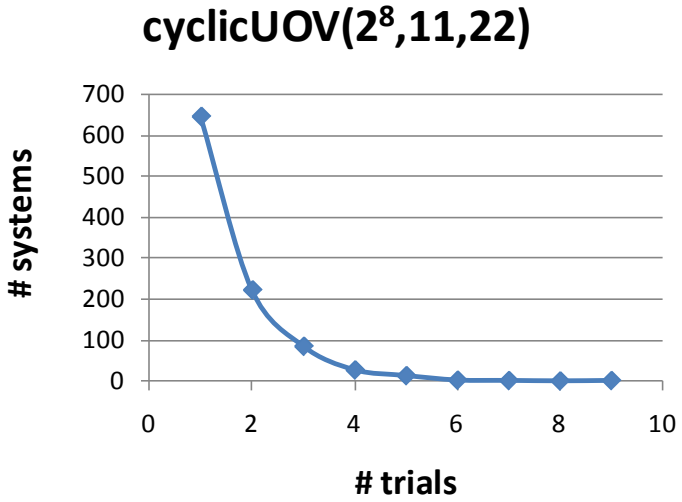
- The signature generation and verification works as in the standard UOV scheme.

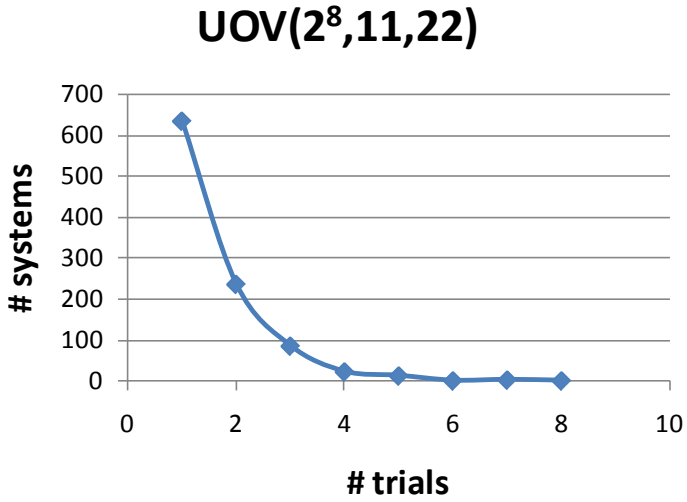
Security (1)

Direct attacks

When attacking the scheme directly, one has to solve a system of o quadratic equations in o+v variables. Therefore, before applying an algorithm like XL or F4, one has to guess at at least v variables to create a determined system.

Does the determined system always have a solution?





Security (2)

Direct attacks (2)

During this guessing process, a major part of the cyclic structure of the original public key gets lost.

We solved determined systems with MAGMA‘s F4 algorithm.

Scheme(2^8,o,v)	10,20	11,22	12,24	13,26	14,28
UOV	54 s	388 s	3141 s	24265 s	173425 s
cyclicUOV	53 s	387 s	3142 s	24258 s	173417 s

Security (3)

UOV Reconciliation

The goal of this attack is to find a basis change of variables which transforms the public key into UOV form

In the first step of the attack one has to solve a system of o quadratic equations in v variables. Again, we have to guess at some of the variables. By doing so, we get a system which looks very similar to the system we got in the direct attack.

Scheme(2^8,o,v)	10,20	11,22	12,24	13,26	14,28
UOV	55 s	386 s	3137 s	24258 s	173419 s
cyclicUOV	54 s	384 s	3136 s	24261 s	173415 s

Security (4)

Rank attacks

Let H_i be the symmetric matrix representing the homogenous quadratic part of the i-th public polynomial.

In the MinRank attack one tries to find a linear combination $H = \sum_{i=1}^{o} \alpha_i H_i$ with $\mathrm{rank}(H) = r < n$.

cyclicUOV	(2^8,o,v,n)	(8,16,24)	(10,20,30)	(12,24,36)	(16,32,48)	(20,40,60)
Rank(H)	n	9964	9957	9959	9964	9958
	n-1	0	0	0	0	0
	n-2	36	43	41	36	42
UOV						
Rank(H)	n	9965	9963	9962	9966	9965
	n-1	35	37	38	34	35
	n-2	0	0	0	0	0

Parameters

Considering our security analysis, we propose the following parameters for our scheme: $q = 256,\ o = 25,\ v = 50$

	Public key size (kB)	Private key size (kB)	Hashsize (bit)	Signature size (bit)
UOV(2^8,24,48)	63.3	61.1	192	576
Rainbow(2^8,17,13,13)	25.7	19.1	208	344
cyclicUOV (2^8,25,50)	12.3	69.1	200	600

→ Reduction of the public key size by **83%**

→ Reduction of the number of multiplications needed for signature verification by **42%**

Future Work (1)

- Extend the security analysis
- Polynomial decomposition
- …

Future Work (2)

We had $P = (B \mid C)$ with a partially circulant matrix B

Try to create UOV-schemes with other forms of the matrix B, for example

- LFSR with a small seed
- PRNG (e.g. AES) with small seed
- Fully random matrix

Future Work (3)

- Extension of the idea to the Rainbow and the enSTS Signature Schemes
- more complex, since $P = S \circ Q \circ T$ is a concatenation of 3 maps
- smaller private key
- smaller signature size
- higher efficiency

Thank you for your attention

Questions ?