Fairness Considerations in Cooperative Games

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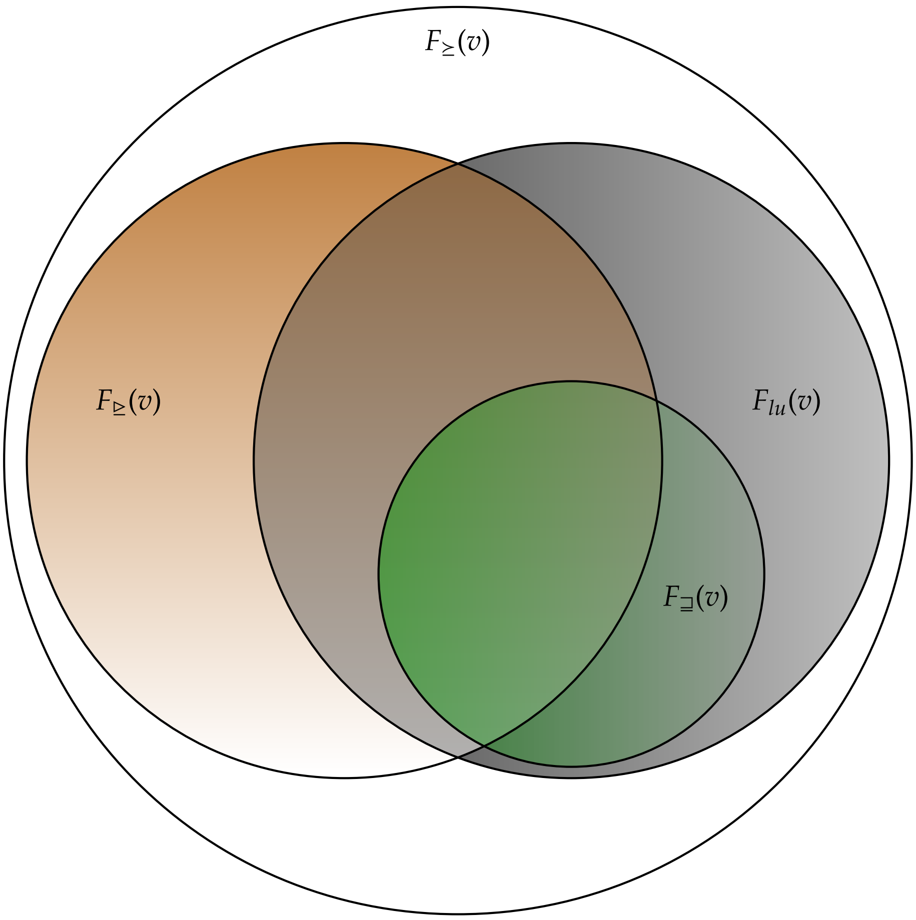

Family of *according to desirability* fairness predicates (see Section 3).

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Deutsche Zusammenfassung

Die vorgelegte Arbeit motiviert und entwickelt ein Modell, das Fairness-Erwägungen in die Theorie der kooperativen Spiele integriert. Dabei kann Fairness kein fester, kulturell unabhängiger Begriff sein. Schon allein in der deutschen politischen Debatte meint mancher Bedarfsgerechtigkeit, ein anderer Leistungsgerechtigkeit und ein dritter Chancengleichheit, wenn von Fairness gesprochen wird.

Für jedes Spiel entscheidet eine Fairness-Bedingung darüber, welche Imputationen als fair und welche als unfair gelten. Sowohl etablierte als auch neue Fairness-Bedingungen werden im Rahmen dieser Theorie formalisiert und untersucht.

Als ein Kriterium für die Adäquatheit von Fairnessforderungen wird der Begriff der *Erfüllbarkeit im Core (engl: satisfiability within the core)* entwickelt. Darüber hinaus werden spezifische Klassen von Spielen genauer untersucht, insbesondere den Bankrottspielen gilt ein längerer Abschnitt.

Im dritten Teil der Arbeit wird der Tatsache Rechnung getragen, dass die verschiedenen Spieler eines kooperativen Spiels im Normalfall unterschiedliche Vorstellungen davon haben, was unter dem Begriff Fairness überhaupt zu verstehen ist. Eine modifizierte Stabilitätsbedingung ersetzt in diesem Setting die Core-Ungleichungen, ein Beispiel zeigt auf, wie in speziellen Situationen die kulturelle Diskrepanz zwischen einzelnen Spielern für die Gesamtsituation auch stabilisierend wirken kann. Unter einigen Bedingungen lässt sich schlie§lich beweisen, dass es für die Spieler optimal ist, ihr persönliches Fairness-Empfinden unverfälscht offenzulegen.

Introduction

This thesis motivates and introduces a way to model fairness considerations in cooperative game theory. Fairness can not be hoped to be modeled as a fixed concept, independent of personas or cultures. Even if one restricts attention to a rather narrow field, like the political debate in Germany, the word fairness carries a variety of meanings ranging from *equal merits for equal achievement* or *meritocracy* to *equal distribution of chances* or even, sometimes, *distribution according to needs*.

Therefore the thesis does not introduce a definition of *fairness*, but rather shows how to model fairness concepts as predicates on the imputation space. For each game, each fairness concept will label imputations as either fair, or unfair. A range of traditional and new fairness concepts are developed within this setting, and studied.

As a benchmark for the feasibility of a fairness concept, the concept of *satisfiability within the core* is introduced. The rationale behind this is that it would be very disadvantageous if all players had the same concept of fairness and this fairness culture would still prevent stable cooperation in some situations where the core is non-empty.

The thesis consists of three parts. The first part starts out, of course, with an introduction of important concepts and examples of cooperative game theory. While the most relevant traditional examples are included, a number of examples are proposed for the first time. Then a model of fairness and a rationality-of-fairness argument based on satisfiability within the core are developed.

The second part deals with more specific classes of games. A certain emphasis is put on the case of bankruptcy games, but some results are also developed for convex games, the positive cone of unanimity games and for 1-convex bankruptcy games. For convex games a conjecture is given as an open problem.

The third part addresses the fact that different players might have different cultures of fairness – perceiving different imputations as fair, or unfair. An adapted form of the core inequalities gives the modified stability condition resulting in this setting. An example shows, how cooperation can become stable, interestingly enough, when certain players are *culturally incompatible*. A first strategy-proof-reporting result is also given on the reporting of individual fairness notions in a somewhat restricted setting.

To a higher degree than in the other parts, the results of part three invite new questions and point towards open problems and opportunities for future work in this field.

Part I

Fairness in Cooperative Games

I want a fair share.
because I know I can get one.
and so can you.

Overview of Part I

Part I of the thesis treats why and how to model fairness expectations in cooperative game theory. The first section sets up the stage by recalling the basic definitions of and key results on cooperative games with transferable utility. A whole range of examples, both traditional ones and others that are first introduced in this thesis, are also included. A (or rather: some) treatment of fairness in the theory is motivated and an overview of the classical approaches is given in section 2. We proceed to introduce fairness concepts as predicates on the imputation space in section 3, where a range of such predicates are also introduced and discussed, and wrap up Part I in section 4, where we introduce a viability benchmark for fairness concepts and revisit the concepts from section 3.
We start by recalling the definitions that are the foundation of the following work. We generally follow the notation that Krabs gives in the German textbook [Krab 05].

1 Cooperative Games with Transferable Utility

Given a set of players, a cooperative game with transferable utility is defined via a characteristic function v from the set of coalitions to a set of payments, assigning a value (hence the v) to each coalition (group) of players. These games are, in literature, also referred to as *games in characteristic function form* (see for example [Drie 88]).
The players' actions are (i) to choose which coalitions to form, and (ii) to distribute the joint profits among their coalition.
Cooperative games have been defined and studied already in the monumental book of von Neumann and Morgenstern [Neum 44] and applications range from engineering and information technology to political science and purely economical questions.
In this thesis, the term *cooperative game* means *cooperative game with transferable utility* throughout.
Where no confusion can arise, we will, in a slight abuse of notation, omit brackets and commas, thus $v(A \cup i) = v(A \cup \{i\})$, $v(ijk) = v(\{i, j, k\})$ etc.
For $x \in \mathbb{R}^n$ and $J \subseteq \{1, ..., n\}$ we also denote the sum of x_j, $j \in J$ by $x(J)$, thus for $N = \{1, ..., n\}$ we have $x(N) = \|x\|_1$.

1.1 Definition and Examples

In the following, cooperative games and certain important classes of games are defined and a wide range of examples is introduced. Some of these are paradigmatic for certain fairness considerations.

1.1. Definition. A **cooperative game** is a tuple (N, v), where N is a set of players and $v : \mathfrak{P}(N) \longrightarrow \mathbb{R}$, $v(\varnothing) = 0$ is a function that we call **characteristic function** for coalitions.
For $|N| = n$ we call (N, v) a **cooperative n-person game**. The class of cooperative n-person games is denoted by $\mathbf{G}^n$.

We expect the players of a n-person game to be numbered serially and identify $N = \{1, \ldots n\}$. The function v is interpreted to give the value of a **coalition** in the sense that for $A \subseteq N$ the players in coalition A can obtain a total payoff of $v(A)$ by skillful cooperation (regardless of what other players do).
According to standard literature, this approach to a game is sensible whenever the players are able to write and enforce binding agreements at low or zero cost. This is the essential premise necessary for them to act as coalitions efficiently.
Where externalities are present and important between a cooperating group A of players and the other players, it is more adequate to adopt a concept of partition function replacing the characteristic function, the payoff to group A then also depends on the coalitions that the players in $N - A$ form, or do not form, see for example [Ray 07]. We develop a theory of fair division in the easier setting of the classic characteristic function though, since we will mostly study games where the grand coalition emerges, thus no externalities are present.
A **subgame** of a cooperative game (N, v) is given by a subset $S \subset N$ of the players and the restricted characteristic function $v|_S : \mathfrak{P}(S) \longrightarrow \mathbb{R}$.
We write $MC_i(A)$ for the **marginal contribution** of player i to coalition A that is $MC_i(A) := v(A) - v(A - i)$. Obviously the marginal contribution of a player to coalitions that he is not a member of is zero.
It is quite common to consider the following classes of games:

(i) The class of **super-additive** n-player games $\mathbf{SA}^n$, that consists of the games where the inequality $v(A) + v(B) \leq v(A \cup B)$ holds for all coalitions $A, B \subseteq N$ with $A \cap B = \varnothing$ (see [Krab 05]). Super-additivity seems natural when one assumes that players that cooperate do so efficiently and retain the ability to act as if they did not cooperate. Models without super-additivity might still be adequate in specific situations, large groups often work less efficient for a variety of reasons.

   In the examples we will be working with, however, super-additivity is a very natural assumption.

(ii) The class of **convex** n-player games $\mathbf{C}^n$, that consists of those games,

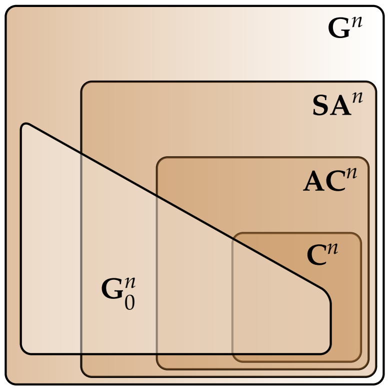

Figure 1: important classes of games

where the inequality $v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$ holds for all coalitions $A, B \subseteq N$ (see [Krab 05]).

This is equivalent to: $MC_i(A) \leq MC_i(B)$ for each player i and for all $A \subseteq B$ with $i \in A$ [Drie 88, Proposition V.1.1].

Intuitively, convex games are those where cooperation is rewarded most.

(iii) The class of **average-convex** n-player games $\mathbf{AC}^n$, that consists of the games where $\sum_{i \in A} MC_i(A) \leq \sum_{i \in A} MC_i(B)$ holds for all coalitions $A \subseteq B \subseteq N$ (see [Iñar 93]). While convexity ensures that every player's marginal contribution grows with the coalition he joins, average convexity can be understood to mean that the marginal contributions of players increase *on average* when the coalition grows.

(iv) A game is called **zero-normalized** if for each player $i \in N$ the individual payoff $v(i)$ equals zero. The class of zero-normalized n-player games is denoted by $\mathbf{G}_0^n$ and meets all of the above classes.

Convexity implies average-convexity, which does impliy super-additivity.
A zero-normalized game can be thought of as a purely cooperative game – every value generated in such a game is generated through cooperation and cooperation only.
Given an arbitrary game (N, v) one can always split it into a **trivial component**, that is a payoff vector $s = (v(1), v(2), ..., v(n))^t$ that assigns to each player her individual payoff and a **purely cooperative component**, that is the zero-normalized game (N, v_0) where $v_0(A) = v(A) - \sum_{a \in A} v(a)$, i.e. $v_0 = v - s$, with appropriate definition of the sum of a n-vector and a n-player game.
The reasoning behind this is that, if players receive payoff without cooperating, it should not change the situation much to implement these payoffs first and then play the remaining cooperative game.

Outside this typical hierarchy of classes, Driessen defines k-convex games for $k \in \mathbb{N}$ using the notion of core cover. A special case that we will discuss in part two of this thesis are the 1-convex games.

1.2. Definition. A cooperative n-person game (N, v) is called **1-convex game**, if for all coalitions $S \subseteq N$ the inequality

$$v(S) \leq v(N) - b(N - S) \tag{1}$$

holds. The set of 1-convex n player games is denoted by $\mathbf{C}_\mathbf{1}^n$.

An equivalent formulation of (1) in terms of the gap function, that we will introduce later on, when discussing the τ-value, is that $0 \leq g(N) \leq g(S)$ for all coalitions S, which means $\lambda_i = g(N)$ for all players i.
For now it suffices to note, that 1-convex games are usually not also convex. If they are, then they are very special bankruptcy games that are treated in section 7.
We now introduce several paradigmatic examples that will be revisited again at different points later on. Several of these are used abundantly in the literature on cooperative game theory. The others are defined specifically to illustrate different aspects of fairness.

1.3. Example: Production Economy. Consider a production economy in which several landless peasants and one landowner are involved. The particular production model has already been studied in [Shap 67] and in [Chet 76].
For this economic model we suppose that the peasants have nothing to contribute but their labour and are all of the same type. The landowner hires peasants to cultivate her land. If t peasants are hired, the monetary value of the crop of the land cultivated by these t peasants is denoted by $f(t) \in \mathbb{R}$. The function $f : \{0, 1, ..., m\} \longrightarrow \mathbb{R}$ is called the **production function** where m is the total number of peasants ($m \geq 1$). Throughout the following, it is required that the production function f satisfies the next two conditions:

(i) A landowner by himself can not produce anything, i.e. $f(0) = 0$.

(ii) The function f is nondecreasing, i.e. $f(t + 1) \geq f(t)$ for $t = 0, \ldots, m - 1$.

The two conditions imply that f is a nonnegative function.
We regard the landowner as player 1 and the peasants as players 2, ..., $m + 1$. Then this situation can be modeled as a cooperative $(m + 1)$-person

game (N, v), where its player set is $N = \{1, ..., m+1\}$ and the payoff function $v : \mathfrak{P}(N) \longrightarrow \mathbb{R}$ is given by

$$v(S) = \begin{cases} 0 & 1 \notin S, \\ f(|S| - 1) & 1 \in S. \end{cases}$$

The worth of any coalition consisting of only peasants equals zero because the peasants do not own any land. Further, the worth of any coalition containing the landowner equals the monetary value of the crop of the land cultivated by the peasants in the involved coalition. Note that $v(i) = 0$ for all $i \in N$.
If f is strictly convex, i.e. if $f(t+1) - f(t) > f(t) - f(t-1)$ for all $1 \leq t \leq m - q$, then $(N, v) \in \mathbf{C}^n$ (see [Drie 88, V.2]).

1.4. Example: Over Employment and Investor Game. Consider a production economy with two peasants and one land-owner. Now let $f(1) = 6$ and $f(2) = 8$. This super-additive, but not average-convex, game is called **over employment game**. The name is due to the fact that the marginal value of employing both peasants versus only one is positive, but small. Value is mostly created where the land-owner employs a single peasant. We will use this example to show how certain exaggerated fairness demands of the peasants might subject cooperation to instability, if not making it outright impossible to cooperate.

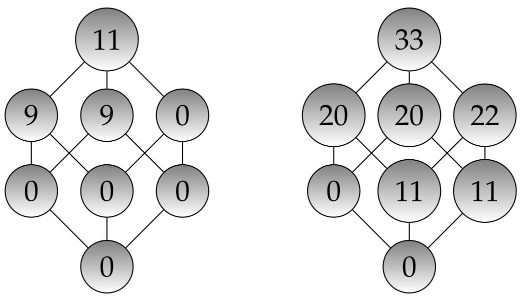

Figure 2: over employment and game (left) and investor game (right)

The **investor game** v_+ is a translation of the over employment game, where each peasants receives an additional payoff of 11 regardless of cooperation, thus we have $v_+(1) = 0$, $v_+(2) = v_+(3) = 11$, $v_+(12) = v(13) = 20$, $v_+(23) = 22$ and $v_+(123) = 33$. We should change the players' names in order to have a convincing story for the game: Let us call player 1 the entrepreneur, who has a good business idea but lacks funding, and players 2 and 3 are investors who hold a fortune of 11 units each. Here the key point is for the entrepreneur to find at least one investor. If both investors invest, the entrepreneur will create some additional value, e.g. by being able to

invest stronger into initial marketing, but the value created when the second investor joins is only 2, while the coalitions $\{1,2\}$ or $\{1,3\}$ create an extra value of 9 through cooperation.
The investor game is again super-additive, but not average-convex, since it is only a translation of the over employment game. It will be interesting where we note that the desirability of players switches, when one zero-normalizes the game and pays the 11 units to players 2 and 3 before the game starts (one then, again, plays the over employment game, as the purely cooperative component).

1.5. Example: Exchange Economy. We consider an example of an exchange economy (see [Drie 88] or [Krab 05]). In this example the payoff funtion $v : \mathfrak{P}(N) \longrightarrow \mathbb{R}$ of the game (N, v) is given by

$$v(K) = \min\{|K \cap P|, \alpha|K \cap Q|\} \text{ for all } K \subseteq N, \tag{2}$$

where $P \cup Q = N, P \cap Q = \emptyset, \frac{1}{2} \leq \alpha \leq 1$ are given. An interpretation of the payoff function is the following: the players in P and Q respectively own two different types of goods which can be used to generate value, when brought together in the relation $\alpha^{-1} : 1$.
We assume $n = 3, P = \{1\}, Q = \{2, 3\}$ and obtain:

$$v(N) = 1, \quad v(\{1,2\}) = v(\{1,3\}) = \alpha, \quad v(\{2,3\}) = 0,$$
$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0.$$

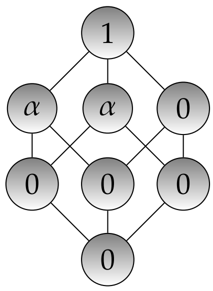

Figure 3: 3 player exchange economy

Note that for $\alpha = \frac{9}{11}$ this is again a scaled version of the over employment game.

1.6. Example: Bankruptcy and Simple Bankruptcy Games.
Around the year 1140 A.D., rabbi Ibn Ezra gave the next problem:

> Jacob died and each of his four sons Reuben, Simeon, Levi and Judah respectively produced a deed that Jacob willed to him his entire estate, half, one third, one quarter of his estate on his death. All deeds bear the same date and the total estate is 120 units.

Rabbi Ibbn Ezra's problem belongs to the class of bankruptcy problems, as do several similar problems from the Talmud, which are obviously even older by far. Aristotle, too, has already considered this kind of problem and proposed a fair division rule.

A **general bankruptcy problem** is defined as an ordered Pair (E, d), where $E \in \mathbb{R}$ and $d = (d_1, d_2, ..., d_n)^t \in \mathbb{R}^n$ such that $d_i \geq 0$ for all $1 \leq i \leq n$ and $0 \leq E \leq d(N)$.

The reader may think of a person who dies, leaving the debts $d_1, d_2, ..., d_n$, totaling at least as much as his estate E. The problem is that the debts are mutually inconsistent in that the estate is insufficient to meet all of the debts.

A general bankruptcy problem is called **simple bankruptcy problem** if $E \geq \max\{d_1, d_2, ..., d_n\}$.

The game theoretic approach to the bankruptcy problem goes back to [ONei 80], where the corresponding **bankruptcy game** $(N, v_{E,d})$ is defined by $N = \{1, 2, ..., n\}$ and

$$v_{E,d}(S) = \max\Big\{0, \quad E - d(N - S)\Big\} \quad \text{for all } S \subseteq N. \tag{3}$$

So, the player set N consists of the n creditors (or heirs) and the worth of coalition S equals either zero or what is left of the estate after each member j of the complementary coalition $N - S$ is paid his associated claim d_i.

Particularly, it follows immediately from this definition that the characteristic function of the bankruptcy game satisfies $v_{E,d}(N) = E$ and, for all $i \in N$, $v_{E,d}(i) \leq d_i$, with equality only for the uninteresting cases of $E = d(N)$ or $d(i) = 0$.

Bankruptcy games are always convex [Drie 88] and a bankruptcy game is zero-normalized if $d(N) \geq \max\{d_1, d_2, ..., d_n\} + E$.

The bankruptcy game corresponding to a *simple* bankruptcy problem is called a **simple bankruptcy game**.

A **division rule** is a nonnegative function f that assigns to any general bankruptcy problem (E, d) an efficient payoff vector $f(E, d) \in \mathbb{R}^n_+$, efficient meaning that the whole estate is distributed, i.e.$\|f_j(E, d)\|_1 = E$.

Revisiting the example from rabbi Ibn Ezra we have a bankruptcy game with four players, estate $E = 120$ and the four claims $d_1 = 30$, $d_2 = 40$,

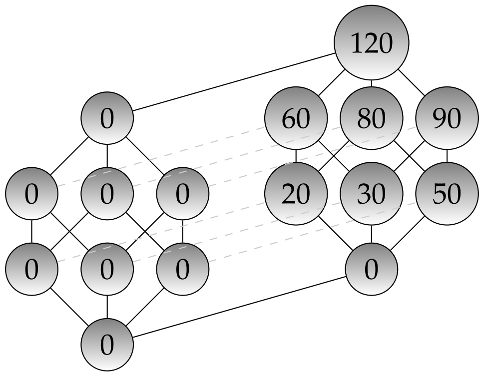

Figure 4: Bankruptcy game for Ibn Ezra's problem

$d_3 = 60$, $d_4 = 120$, that we model via the characteristic function (v) given by

$$\begin{array}{lll} v(14) = 20 & v(24) = 30 & v(34) = 50 \\ v(124) = 60 & v(134) = 80 & v(234) = 90 \\ v(N) = 120 & v(S) = 0 & \text{for all other } S \subseteq N. \end{array}$$

We will study division rules, game theoretic solutions of the cooperative game and fairness in section 6.

1.7. Example: Finding a Diamond.
This following example will be useful to motivate fairness considerations and show the limitations of core-rationality to model rational human behavior.
Think of two persons finding a diamond that is attached to a basketball hoop with a short strap of string. Neither of the two however is able to jump high enough and untie the knot in midair. Only if they cooperate and one of them lifts the other up, they can secure the giant unappropriated diamond, which has a market value of exactly 10,000 Dollars. We do not care who lifts whom.
The mathematical model of the game is (N, v) with $N = \{1, 2\}$, $v(\emptyset) = v(1) = v(2) = 0$ and $v(12) = 10,000$.
In this situation, cooperation is highly lucrative and—according to homo economics behavior—players should cooperate even if they are only payed a small part of the total gain, at least if one player gets a chance to credibly proclaim that he will not cooperate unless almost all of the benefit goes to him. If one player by chance is able to do so first, the other should, in this rationale, agree to a split of 1: 9,999 rather than not cooperating. However the symmetry of the situation will make it unlikely that the two persons would agree to a split of 1 : 9,999 in real life.
This is a convex game and (except for scaling) also a simple unanimity game, as we will see.

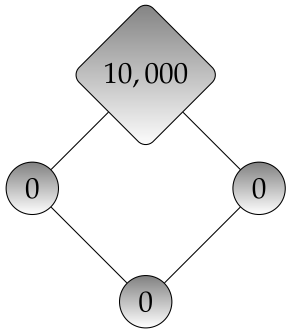

Figure 5: finding a diamond

1.8. Example: Simple Games.
Game theory can be used to describe the abstract power of a voter in voting systems because of the invention of simple games in [Neum 44]. These simple games are completely characterized by the fact that the coalitions in the game can be divided into two types: winning (powerful) and losing (powerless) coalitions. An n-person game v is said to be a **simple game** if $v(S) \in \{0, 1\}$ for all $S \subseteq N$, $v(N) = 1$, and $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$.
The last condition is known as the **monotonicity** condition for the n-person game v. The class of simple n-person games is denoted by $\mathbf{S}^n$. The players whose absence gives rise only to losing coalitions are of the utmost importance and are called **veto players**. We denote the set of all veto players in a simple n-person game v by $J(v) := \left\{ i \in N \,\middle|\, v(N - i) = 0 \right\}$.
An example of a simple game is as follows. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$ be such that $1 \leq m < n$. Consider a council consisting of n members including one chairman (player 1). In order to pass a bill, at least m votes are needed including the vote of the chairman. The simple n-person game v which corresponds to this voting situation is given by $v(S) = 1$ if $1 \in S$ and $|S| \geq m$ and $v(S) = 0$ otherwise. Note that $J(v) = \{1\}$.

1.9. Example: Unanimity Game.
Let $m = 1$, then the simple game with the chairman, as above, is called the unanimity game with respect to player 1. Winning or losing is equivalent to having the omni-potent chairman in the coalition. In general, the unanimity game with respect to a nonempty coalition represents the voting system in which the members of the involved coalition play a prominent part because their votes are needed in order to pass a bill.
That is, for any $T \subseteq N, T \neq \emptyset$, the **unanimity game** $u_T \in \mathbf{G}^n$ is defined by $u_T(S) = 1$ if $S \supseteq T$, $u_T(S) = 0$ otherwise.
Thus a coalition is powerful (powerless respectively) in the simple unanimity game u_T if it does (not) include all members of T and its associated worth is put to one (zero).

From this, it follows that the members of T are precisely the veto players in the unanimity game u_T, i.e. $J(u_T) = T$.
It is proved in [Drie 88] that the unanimity games form a basis for the cone $\mathbf{G}^n$, i.e. every $v \in \mathbf{G}^n$ can be written as a linear combination of n player unanimity games. A corollary of this is Driessen's Theorem 1.2., that the dimension of the polyhedral cone $\mathbf{C}^n$ in $\mathbf{G}^n$ is full, i.e. $\dim \mathbf{C}^n = 2^n - 1$.

1.2 Imputations and the Core

In most games the grand coalition N of all players has the largest value, namely $v(N)$. The question arises, whether this coalition is also worthwhile for all players, whether it is rationally sensible for everybody to cooperate. Criteria for this question are practically relevant.
In order to answer this question, one has to ask how the total payoff can be distributed in a way that satisfies all players.

1.10. Definition. Let $(N, v) \in \mathbf{G}^n$. An **imputation** of v is a vector $x \in \mathbb{R}^n$ with $x(N) = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in \{1, \ldots, n\}$.
A vector that only satisfies the first condition is called **pre-imputation**. The second condition is called **individual rationality** for obvious reasons.
An imputation is thus a distribution of payoff that grants each player at least the amount that he can gain by playing solo.
The sets of imputations and pre-imputations of (N, v) are denoted by $I(v)$ and $I^*(v)$.

A cooperative n-person game (N, v) has at least one imputation whenever $\sum_{i=1}^n v(\{i\}) \leq v(N)$, and it has infinitely many imputations when the inequality is strict.
In the latter case it is necessary for the players to agree on one specific imputation. Several criteria for the feasibility of imputations have been introduced, the most prominent one being the *core porperty* first defined in modern form by Gillies in [Gill 59].

1.11. Definition. The **core** $C(v)$ of the cooperative n-person game (N, v) is the set of those imputations x that satisfy the following **core property**:

$$\Big(\forall A \subseteq N\Big)\; v(A) \leq x(A). \tag{4}$$

That is: Any coalition A is granted a payoff that is at least as big as the coalition's value.
Note that by definition any game's core is a convex and bounded polytope.
The **excess of the coalition** A **with respect to the pre-imputation** x is defined to be

$$e^v(A,x) := v(A) - x(A). \tag{5}$$

A nonnegative excess of A at x in the game v represents the gain to the coalition A if its members withdraw from the payoff vector x in order to form their own coalition. We often write $e(A,x)$ instead of $e^v(A,x)$.
In these terms, the core consists of imputations without positive excess for any nonempty coalition.
We now give the cores of several of the above examples. Readers who are not familiar with the theory are encouraged to check the correctness of these results as an exercise, since the core is the single most important rationality constraint on imputations in use today.
First let us revisit the over employment game that is both a special case of the production economy and the exchange economy (Example 1.4), its characteristic function is shown on the left side of figure 6.

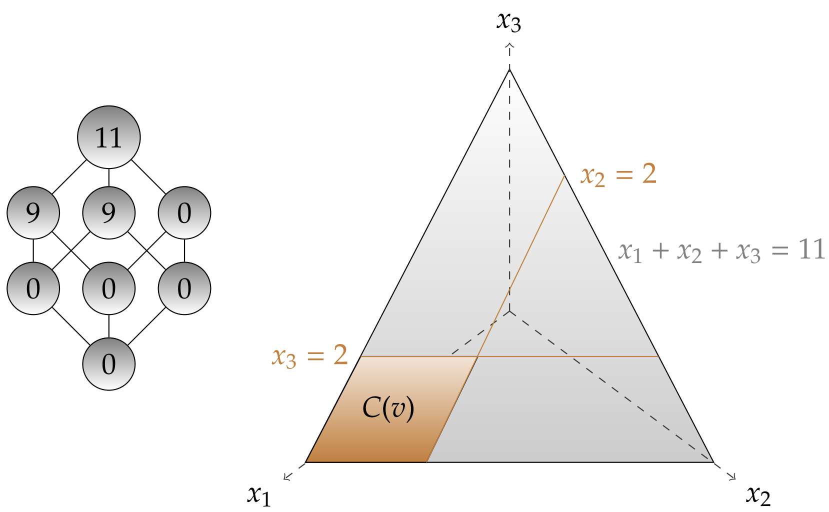

Figure 6: over employment game with core

The **core of the over employment game** is given by all those imputations where players 2 and 3 receive no more than 2 payoff.
As a subset of the 2-simplex $11 \cdot \mathcal{S}_2 = \left\{ 11 \cdot x \in \mathbb{R}^3 \,\middle|\, x_1 + x_2 + x_3 = 1 \right\}$ it is shown on the right side of figure 6
Core imputations of the game are for example given by $(8,0,0)^t$, $(4,2,2)^t$ or $(6,2,0)^t$. The imputation $(4,3,1)^t$ is not a member of the core, because

players 1 and 3 are a blocking coalition—they can play without player 2 and generate a a profit of $6 > 5 = 4 + 1$.
The **core of the bankruptcy game for Ibn Ezra's problem** (Example 1.6), like every zero-normalized bankruptcy game, consists of all imputations where no player receives a payoff that is larger than his claim. So, with the vector of claims being $(30, 40, 60, 120)^t$, a payoff like $(30, 20, 60, 10)^t$ might seem surprising, but is still in the core.
The **core of finding a diamond** (Example 1.7) contains every imputation. That is all the payoff going to player 1 is in the core, as is the imputation that assigns everything to player 2 or a $50 - 50$ split.
The following observation will be useful later:

1.12. Theorem. *Let $(N, v) \in \mathbf{G}^n$ be a cooperative game, s be the vector of individual payoffs $s = (v(1), ..., v(n))^t$ and $v_0 = v - s$ be its zero-normalized, purely cooperative, component.*
Then $I^*(v) = I^*(v_0) + s$, $I(v) = I(v_0) + s$ and $C(v) = C(v_0) + s$.

Proof. Since $v_0(A) = v(A) - s(A)$ we have $v(N) = v_0(N) + S(N)$, thus $I^*(v) = I^*(v_0)+s$. The Imputations of v_0 consist of all positive pre-imputations of v_0. Adding s we obtain the pre-imputations of v, where $x_i \geq s_i$ for all i, which are exactly the imputations of v.
The core property for an imputation x and coalition A in the game v_0 is given by $v_0(A) \leq x(A)$, which is equivalent to $v(A) \leq (x - s)(A)$, the core property for the imputation $x - s$ and coalition A in the game v. □

1.3 Solution Concepts From Literature

Since the core of a game will often also have infinitely many elements, and—like in *finding a diamond*—these can be very different to each other in nature, the question of choice of a single imputation remains and is, in fact, the driving question of cooperative game theory. In this section we will introduce several solution concepts for this problem. We will use the term **one-point solution concept** for mappings that take each game (N, v) to a singleton subset of $I(v)$ and **partial one-point solution concept**, when the image of this mapping is either a singleton or empty.

Strong ε-Core and Least Core

The core of a game is given by imputations with nonpositive excess for each coalition. However, it is unclear if every player and coalition would choose

to cooperate on a grand scale, if its added value from doing so is zero, or close to zero.
Thus, for some $\varepsilon \in \mathbb{R}$ we define the **strong ε-core** $C_\varepsilon(v)$ to consist of those imputations $x \in I(v)$ where the excess of every nonempty coalition A is at most ε. Note that

(i) $C_0(v) = C(v)$,

(ii) for $\varepsilon < 0$ the strong ε-core consists of the imputations where every coalition A would lose at least $|\varepsilon|$ units of transferable utility when leaving the great coalition,

(iii) for very small values of ε, say $\varepsilon = -v(N)$, we have $C_\varepsilon(v) = \emptyset$,

(iv) for positive ε, the strong ε-core models core-stability under the assumption that breaking away from the grand coalition N is associated to a cost of ε for any coalition A. This cost might be enforced by contracts or model the cost of reorganization,

(v) for large values of ε, like $\varepsilon = v(N)$, we have $C_\varepsilon(v) = I(v)$.

1.13. Definition. Let (N, v) be a cooperative n-person game. The **least core** $C_l(v)$ of the game is the intersection of all non-empty strong ε-cores of v, which is identical to that strong ε-core with $l = \min\left\{\varepsilon \in \mathbb{R} \;\middle|\; C_\varepsilon \neq \emptyset\right\}$.

Let us illustrate the least core by giving the **least core of the over employment game**. Figure 7 shows the game's core and strong (-0.5)-core.
We find that the strong (-1)-core is given by imputations where

(i) each player receives at least 1

(ii) coalitions $A = \{1, 2\}$ and $B = \{1, 3\}$ receive at least 10,

(iii) players 2 and 3 receive a total of at least 1—which is no new restriction.

It follows that $C_{-1}(v)$ is a singleton and that for $\varepsilon < -1$ the set $C_{-1}(v)$ is empty. Hence we find that $C_l(v) = C_{-1}(v) = \{(9, 1, 1)^t\}$.
Simple calculations yield that the **least core of the bankruptcy game for Ibn Ezra's problem** (Example 1.6, figure 4), is given by $C_{-15}(v)$, which contains those imputations $x \in I(v)$, where $x_1 = 15$, $15 \leq x_2 \leq 35$, $15 \leq x_3 \leq 45$, $15 \leq x_4 \leq 75$ and that the **least core of finding a diamond** is a singleton set that contains only the imputation $(5,000, 5,000)^t$.
Note that the the core of finding a diamond was a line, while its least core was a point, core of this bankruptcy game was a 3-simplex, i.e. the convex

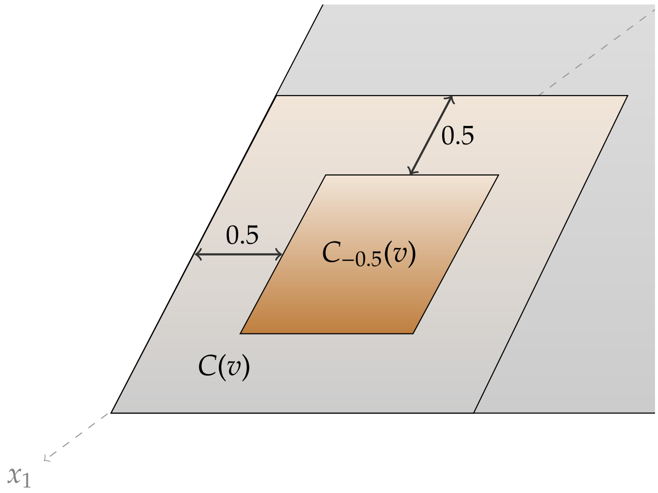

Figure 7: core and strong (−0.5)-core of the over employment game

hull of 4 points in space, i.e. the image of a tetrahedron under an invertible linear map, whereas the least core is a 2-simplex, a triangle. The core of the over employment game had 3 dimensions, whereas its least core is a single point, i.e. a 0-simplex of no dimension. It is quite natural for the least-core to have a dimension that is smaller than the dimension of the core, which will often be $n - 1$.

A **dummy player** is a player $i \in N$ who's marginal contribution to every coalition A (where $i \in A$) equals his individual value, i.e. $MC_i(A) = v(i)$ for all $A \subset N, i \in A$.

Obviously a dummy player can never receive a payoff different from $v(i)$ in any core imputation, since larger payoffs to i in imputation x yield positive $e^v(N - i, x)$ and if i receives less than $v(i)$ in a pre-imputation p, then p is not individually rational, i.e. not even an imputation.

We conclude the following

1.14. Proposition. *The core of a game with dummy players does not have full dimension and every strong ε-core of such a game is empty for negative ε.*

The Egalitarian Core and $LS(v)$

Egalitarianism is the strife of a community to spread the total wealth as equally as possible among its members, while satisfying certain stability requirements of the allocation. The notion of egalitarianism is frequently used outside the theory of games. See for example [Thom 94] for applications in bargaining theory.

Moulin, Dutta and Ray first introduced the so-called egalitarian allocation in [Dutt 89a], as a solution concept that combines (recursively defined)

stability and egalitarianism. For convex games the egalitarian allocation always exists and it is an element of the core.
Arin and Iñarra use another definition of egalitarian allocations [Arin 01]. They use the same notion of egalitarianism—the widely accepted Lorenz criterion—but as a notion of stability they use the game's core. As a consequence, the latter type of egalitarian allocation exists for a given game precisely when the core of the game is not empty.
In the class of convex games both notions coincide (see again [Arin 01]). This, together with the guarantee of existence for a relatively large and manageable class of games, makes the notion of Arin and Iñarra an interesting alternative. The most prominent drawback of their definition is that more than one allocation may be egalitarian.
This motivates Arin and Kuipers to introduce several solution concepts that assign exactly one egalitarian allocation to each game with non-empty core in [Arin 08].
From the latter article we take the following definitions:

1.15. Definition. For two players i and j in a game (N, v), an allocation $x \in I(v)$, and a real number $\alpha > 0$, we say that (i, j, x, α) is an **equalizing bilateral transfer** (of size α from i to j with respect to x) if $x_i - \alpha \geq x_j + \alpha$.
Now an imputation $x \in C(v)$ is called **egalitarian** if no core allocation y is the result of an equalizing bilateral transfer with respect to x. A core allocation x is **strongly egalitarian** if no core allocation y is the result of a finite sequence of equalizing bilateral transfers starting from x.
We write $C_e(v)$ for the set of egalitarian core allocations and $C_{se}(v)$ for strongly egalitarian core allocations.

1.16. Definition. For a balanced game (N, v), the least squares Solution $LS(v)$ is defined as the unique allocation x in $C(v)$ for which $||x||_2 < ||y||_2$ for all $y \in C(v)$.
As usual, $||x||_2$ denotes the Euclidean length $\sqrt{\sum_{i \in N} x_i^2}$ of x.

The existence of an allocation minimizing the Euclidean length is obvious, since it is the solution of an optimization problem with continuous (even quadratic) objective on a compact set (with linear constraints). Uniqueness is shown in [Arin 08].
Arin et al. prove, that LS is indeed strongly egalitarian for all games with non-empty core.

Note that the egalitarian Core C_e and LS both do not split, as we see from the following example.
Let, as in Example 1.4 and figure 2, v be the three player over employment game and v_+ be the investor game. Then the imputation sets $I(v)$ and $I(v_+)$ and the cores $C(v)$ and $C(v_+)$ are of the same shape, the difference being a translation by $(0, 11, 11)^t$. The only element of $C_e(v)$ and $C_e(v_+)$ are, however, $LS(v) = (7, 2, 2)^t$ and $LS(v_+) = (11, 11, 11)^t$ respectively and these are in opposing corners of $C(v)$ and $C(v_+)$, as figure 8 shows.

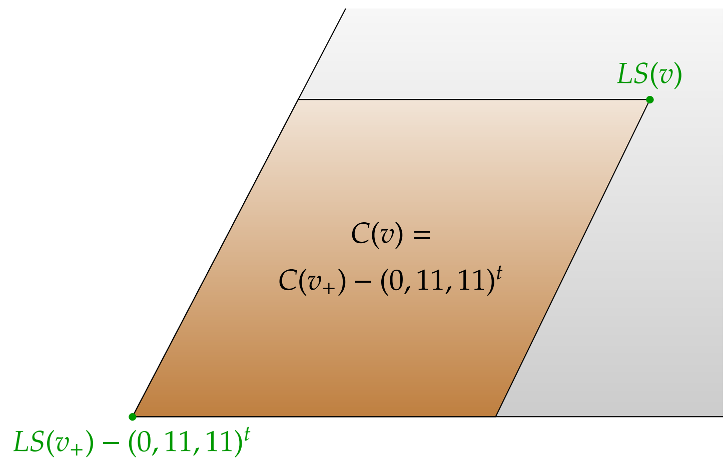

Figure 8: egalitarian solutions $LS(v)$ and $LS(v_+)$

Bargaining Set

The concepts that we treat throughout the thesis neglect the bargaining process that will actually take place when a cooperative game is played. The various bargaining sets, introduced in [Auma 64a], are more closely tied to the bargaining process since they take account of the possible threats and counter-threats made by coalitions. We introduce the bargaining set which is obtained by taking account of objections and counter-objections made by single players.

1.17. Definition. Let $(N, v) \in \mathbf{G}^n$ and $x \in I(v)$. An **objection** of player i against another player j with respect to the imputation x in the game v is a pair (y, S), where $i \in S$, $j \notin S$ and $y = (y_k)_{k \in S}$ is a $|S|$-tuple of real numbers satisfying $y(S) = v(S)$ and $y_k > x_k$ for $k \in S$.

A **counter-objection** to the above objection (y, S) is a pair (z, T), where $j \in T, i \notin T$ and $z = (z_k)_{k \in T}$ is a $|T|$-tuple of real numbers satisfying $z(T) = v(T)$, $z_k \geq x_k$ for $k \in T$ and $z_k > y_k$ for $k \in T \cap S$.

Thus, an objection of i against j at an imputation consists of a coalition S containing player i but not player j, and a feasible payoff vector for S that is preferred to the given imputation by every member of the coalition S. Note that a coalition S with $i \in S$, $j \notin S$ can be used for an objection of i against j only if the corresponding excess $e^v(S, x)$ is positive.
A counter-objection to this objection consists of another coalition T containing player j but not player i, and a feasible payoff vector for T that is weakly preferred to the above payoff vector for S by every member of $T \cap S$ and that is also weakly preferred to the given imputation by every member of $T - S$. Notice that the excess $e^v(T, x)$ of any coalition T which is used for the counter-objection, must be nonnegative.

1.18. Definition. Let $(N, v) \in \mathbf{G}^n$. An imputation $x \in I(v)$ is said to belong to the **bargaining set** $M(v)$ of the game v if for any objection of one player against another player with respect to the imputation x in the game v, there exists a counter-objection.

These observations imply that $C(v) \subseteq M(v)$ for every game. We follow [Drie 88] writing $M(v)$ or M instead of the conventional notation $M_1^{(i)}(v)$.
As a matter of fact, the bargaining set M of any game is nonempty. Davis and Maschler [Davi 63] presented a direct proof of this statement by considering some properties of the bargaining set M, whereas Peleg [Pele 63, Pele 67] proved this statement in an indirect way by using Brouwer's fixed point theorem. An alternative proof of the nonemptiness of the bargaining set is the existence of the Nucleolus introduced below.
The bargaining set M is a finite union of closed convex polyhedra (see [Masc 66a]). By this result, the bargaining set M itself is also closed, but it is in general not a convex set.
Maschler, Peleg and Shapley established that the bargaining set M of a convex game coincides with the core of the game [Masc 72].
Today, bargaining theory is a whole discipline within the theory of cooperative games. A model for proposal-response-bargaining games, more general than for example Rubinstein-Ståhl bargaining and Baron-Ferejohn bargaining, is given in [Ray 07], where Ray gives a broad overview over the theory. We will come back to the characteristic function - approach of

cooperative theory though via the kernel and nucleolus. Both are aspects of games in characteristic form that are closely related to the bargaining set M.

Kernel and Nucleolus

Let $v \in \mathbf{G}^n$ and $x \in I^*(v)$. The **maximum surplus of player** i **over another player** j **with respect to the pre-imputation** x **in the game** v is given by

$$s^v_{ij}(x) := \max\left\{e^v(A,x) \;\middle|\; A \subseteq N, i \in A, j \notin A\right\}. \tag{6}$$

A nonnegative maximum surplus of i over j at a pre-imputation x represents the maximal amount that player i can gain without the cooperation of player j by withdrawing from the payoff vector x and forming a coalition not containing player j, on the understanding that the other members of the formed coalition are satisfied with the amount they had according to the payoff vector x. Thus, the maximum surplus $s^v_{ij}(x)$ can be regarded as a measure of the power of player i to threaten player j with respect to the pre-imputation x.
In case the pre-imputation x is also individually rational, player j is immune to threats whenever $x_j = v(j)$ because player j can get the amount $v(j)$ by going alone. We say player i outweighs player j with respect to the imputation $x \in I(v)$ if $x_j > v(j)$ and $s^v_{ij}(x) > s^v_{ji}(x)$.
The kernel is defined as the set of all imputations for which no player outweighs another player. The prekernel consists of pre-imputations for which any two players are equally powerful concerning their mutual threats. Formally, both the kernel and the prekernel are described as follows

1.19. Definition. Let $v \in \mathbf{G}^n$.
The **kernel** $K(v)$ of the game v is the set of all imputations $x \in I(v)$ satisfying for all $i, j \in N, i \neq j$,

$$\left(s^v_{ij}(x) - s^v_{ji}(x)\right)\left(x_j - v(j)\right) \leq 0 \quad \text{and} \tag{7}$$

$$\left(s^v_{ij}(x) - s^v_{ji}(x)\right)(x_i - v(i)) \leq 0. \tag{8}$$

The **prekernel** $K^*(v)$ of the game v is the set of all pre-imputations $x \in I^*(v)$ satisfying for all $i, j \in N, i \neq j$,

$$s^v_{ij}(x) = s^v_{ji}(x). \tag{9}$$

By the above definition, the kernel is a finite union of closed convex polyhedra because it is determined by a system of inequalities. The relevant

polyhedra are studied in [Masc 66b] in order to give an algebraic existence proof of the kernel[1].
Some important properties of the kernel are:

1.20. Proposition.

(i) *The kernel is always a subset of the bargaining set $M(v)$*

(ii) *For n-player games v and w where $C(v) = C(w)$ we have $K(v) \cap C(v) = K(w) \cap K(w)$.*

(iii) *Every imputation in the kernel possesses the* **dummy player property**, *i.e. if i is a dummy player and $x \in K(v)$, then $x_i = v(i)$ (for $x \in K^*(v)$ one still has $x_i \leq v(i)$).*

(iv) *Let players i and j be* **substitutes** *in the game v, i.e. $v(S \cup i) = v(S \cup j)$ for all $S \subseteq (N - \{i, j\})$. Then for $x \in K(v)$ or $x \in K^*(v)$ we always have $x_i = x_j$.*

(v) *Let player i be more* **desirable** *than player j in the game v, i.e. $v(S \cup i) \geq v(S \cup j)$ for all $S \subseteq (N - \{i, j\})$. Then for $x \in K(v)$ or $x \in K^*(v)$ we always have $x_i \geq x_j$.*

Driessen proves (i), (ii), (iii) and (iv) in [Drie 88], while (v) is shown by Maschler and Peleg in [Masc 66b].
Recalling the 3 player exchange economy (Example 1.5, figure 3), the proposition guarantees that the (pre)kernel of the game only contains pre-imputations of the form $x = (1 - 2\beta, \beta, \beta)$ where $\beta \in \mathbb{R}$. Then, with $s_{12}(x) = \max(2\beta - 1, \alpha - 1 + \beta)$ and $s_{21} = \max(-\beta, \beta)$ one concludes that $s_{12}(v) = s_{21}(v)$ if and only if $\beta = \frac{1}{2}(1 - \alpha)$. Considering that $s_{23}(x) = s_{32}(x)$, $s_{12}(x) = s_{13}(x)$ and $s_{21}(x) = s_{31}(x)$, one obtains by straightforward calculations that

$$K(v) = K^*(v) = \left\{ \frac{1}{2}(2\alpha, 1 - \alpha, 1 - \alpha)^t \right\}. \tag{10}$$

This imputation occupies the central position within the kernel and in fact, it represents the so-called nucleolus of the game v.
An algebraic proof of the non-emptiness of the kernel has been presented in [Masc 66b]. The same method of proof was used there to show that the kernel $K(v)$ of a game v intersects any nonempty set $C_\varepsilon(v)$. For the well-known algebraic representation result, see their paper or, for example [Drie 88].
Pursuing this result in [Schm 69], Schmeidler was led to the discovery of the **nucleolus**.

[1]an indirect existence proof of the kernel using Brouwer's fixed point theorem had been given by Davis and Maschler in [Davi 63]

Let $v \in \mathbf{G}^n$. For any n-tuple $x \in \mathbb{R}^n$, let $\theta(x)$ be the 2^n-tuple whose components are the excesses $e^v(A, x)$, $A \subseteq N$, arranged in non-increasing order. Thus, $\theta_i(x) \geq \theta_j(x)$ where ever $i \leq j$.

The involved excesses are usually nonpositive (e.g., at core-elements) and therefore, the excesses are regarded as losses or complaints, while the vectors $\theta(x), x \in \mathbb{R}^n$, are interpreted as complaint vectors.

Now the lexicographic order $\leq_L$ on $\mathbb{R}^{2^n}$ is used to order the complaint vectors by taking into account their largest complaint or, if these are equal, their second largest complaint and so on. For all $x, y \in \mathbb{R}^n$ we write $\theta(x) <_L \theta(x)$ if there exists an integer k such that for all $i < k$ $\theta_i(x) = \theta_i(y)$ and $\theta_k(x) < \theta_k(y)$, whereas $x \leq_L y$ means that either $x <_L y$ or $x = y$.

1.21. Definition. The **nucleolus** $N(v)$ of a game $v \in \mathbf{G}^n$ is the set of all imputations $x \in I(v)$ satisfying $\theta(x) <_L \theta(y)$ for all $y \in I(v)$. When $N(v)$ is a singleton (and we will see that this is always the case), the only element is denoted by $n(v)$.

Thus, the nucleolus consists of imputations that minimize the complaint function $\theta(x)$ in the lexicographic order over the nonempty compact convex imputation set. Schmeidler himself gave both a topological and an algebraic existence proof of the nucleolus in terms of continuous functions and nonempty compact sets (see again [Schm 69]). An alternative existence (and in addition, a uniqueness) proof of the nucleolus concept is based on the geometric characterization of the nucleolus presented in [Masc 78].

Informally, their constructive proof proceeds as follows. First of all, we determine the nonempty compact convex set of all imputations that minimize the maximum excess over the nontrivial coalitions. Then we remove the coalitions whose excess with respect to the imputations f this set can not be further reduced. Secondly, we begin over again to minimize the maximum excess over the remaining coalitions. The resulting compact convex subset of the previous set of imputations is in general nonempty and once again, we remove the coalitions whose excess with respect to the imputations of this second set can not be further lowered. This procedure continues, but it stops whenever all nontrivial coalitions are removed. Formally, the procedure is described as follows.

Let $v \in \mathbf{G}^n$ where $n \geq 2$. We define $X^0 := I(v)$, $\Sigma^0 := \left\{ A \subseteq N \mid A \neq N, \emptyset \right\}$ and for $j = 1, 2, ..., \kappa$ we define recursively:

$$\varepsilon^j := \min_{x \in X^{j-1}} \max_{A \in \Sigma^{j-1}} e^v(A, x) \tag{11}$$

$$X^j := \left\{ x \in X^{j-1} \,\middle|\, \max_{A \in \Sigma^{j-1}} e^v(A, x) = \varepsilon^j \right\} \tag{12}$$

$$\Sigma_j := \left\{ A \in \Sigma^{j-1} \,\middle|\, e^v(A, x = \varepsilon^j \text{ for all } x \in X^j \right\} \tag{13}$$

$$\Sigma^j := \Sigma^{j-1} - \Sigma_j \tag{14}$$

where $\kappa := \min\left\{ j \,\middle|\, j \geq 1, \Sigma^j = \emptyset \right\}$.
We apply the above procedure to the three-person game v of figure 3, where $0.5 < \alpha < 1$. Then $I(v) = \mathcal{S}_2 = \left\{ x \in \mathbb{R}^3 \,\middle|\, x_1 + x_2 + x_3 = 1 \right\}$ and also

$$\varepsilon^1 = \min_{x \in I(v)} \max\{-x_1, -x_2, -x_3, x_3 + \alpha - 1, x_2 + \alpha - 1\}. \tag{15}$$

Note that $-x_3 \leq \frac{1}{2}(\alpha - 1)$ if and only if $x_3 + \alpha - 1 \geq \frac{1}{2}(\alpha - 1)$. From this equivalence and the recursive definition above, we obtain

$$\varepsilon_1 = \frac{1}{2}(\alpha - 1), \quad X^1 = \{\frac{1}{2}(2\alpha, 1 - \alpha, 1 - \alpha)\}, \quad \Sigma^1 = \{\{1\}, \{2, 3\}\}, \tag{16}$$

$$\varepsilon_2 = \alpha - 1, \quad X^2 = X^1, \quad \Sigma^2 = \{\{1\}\}, \tag{17}$$

$$\varepsilon_3 = -\alpha, \quad X^3 = X^1, \quad \Sigma^2 = \emptyset. \tag{18}$$

It turns out that κ is always well defined, X^κ is always a singleton and is identical to the Nucleolus of the game (see for example [Drie 88]).
It follows that $N(v)$ is always the singleton set $N(v) = \{n(v)\}$. The following results also follow (see again [Drie 88]):

1.22. Proposition. *Let (N, v) be a cooperative n-person game. Then*

(i) *$n(v)$ is an element of $C_\varepsilon(v) \cap I(v)$, whenever this set is not empty.*

(ii) *In particular $n(v) \in C(v)$ whenever $C(v) \neq \emptyset$.*

(iii) *$n(v) \in K(v)$, i.e. the nucleolus is in the intersection of the core and the kernel.*

As a corollary, it is clear that whenever a game has non-empty core, the intersection of its core and its kernel is not empty.
Remembering the results on the kernel, we find that the nucleolus possesses the substitution property and the dummy player property and reflects player desirability in the sense that more desirable players get a larger payoff.
Let us calculate the **nucleolus and the kernel of the over employment game** (Example 1.4). Recall its characteristic function shown in figure 2 and that the core consists of all imputations where $x_2, x_3 \leq 2$.

Proposition 1.20 guarantees that the (pre)kernel of the game only contains pre-imputations of the form $x = (11 - 2\beta, \beta, \beta)$ where $\beta \in \mathbb{R}$. Then, with $s_{12}(x) = \max(2\beta - 11, \beta)$ and $s_{21} = \max(-\beta, \beta)$ one concludes that $s_{12}(v) = s_{21}(v)$ if and only if $\beta = 1$. Considering that $s_{23}(x) = s_{32}(x)$, $s_{12}(x) = s_{13}(x)$ and $s_{21}(x) = s_{31}(x)$, one obtains by straightforward calculations that

$$K(v) = K^*(v) = \left\{(9, 1, 1)^t\right\}. \tag{19}$$

and, since the nucleolus is contained in the kernel, $n(v) = (9, 1, 1)^t$ as well.
Looking at Ibn Ezra's 4-player bankruptcy game (Example 1.6, figure 4), we calculate the nucleolus and get $n(b) = (15, 20, 30, 55)^t$.
The kernel of this game is a singleton, consisting only of the nucleolus, which can be checked either by a lengthy calculation or using the fact that this game is convex and that for all convex games $K^*(b) = K(b) = \{n(b)\}$ (see, for example, [Drie 88, Theorem V.7.3])

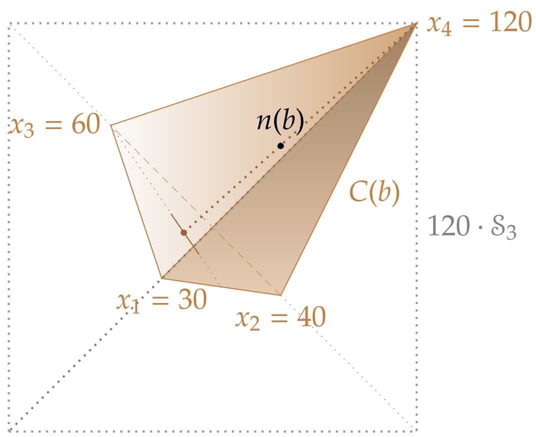

Figure 9: core and kernel of the bankruptcy game for Ibn Ezra's problem

The kernel of the unanimity game d of finding a diamond is also a singleton: $K(d) = \{n(d)\} = \{(5,000, 5,000)^t\}$.

The Shapley Value

The Shapley value of a cooperative n-person game is a one-point solution concept defined on the class of super-additive games introduced in [Shap 53]. While the Shapley value can be (and has been) calculated also for games that fail to be super-additive, it is individually rational, i.e. an imputation, whenever the game is super-additive.

1.23. Definition. Let (N, v) be a super-additive cooperative n-person game, then the **Shapley value** $\phi(v) \in \mathbb{R}^n$ is defined via

$$\phi_i(v) := \sum_{\substack{K \subseteq N \\ i \in K}} (\gamma_n(K) \cdot MC_i(K)) \text{ for all } i = 1, \ldots, n, \tag{20}$$

where $\gamma_n(K) := (n!)^{-1} \cdot (|K| - 1)! \cdot (n - |K|)!$ for all $K \subseteq N,\ K \neq N$.

For each fixed $i \in N$ the function $\gamma_n : (N) \longrightarrow [0, 1]$, is a probability distribution over the collection of subsets of N that contain player i. Notice that this probability distribution arises from the assumption that the coalition, to which player i joins, is equally likely to be of any size t, $0 \leq t \leq n-1$, and that all such coalitions of the same size t are equally likely.
If, for each $K \subseteq N$, $i \in K$, the value $\gamma_n(K)$ is interpreted as the probability that the player i joins the coalition $K - i$ and the marginal contribution $v(K) - v(K - i)$ is payed to player i for joining the coalition, then the Shapley value $\phi_i(v)$ is simply the **expected payoff** to player i in the game (N, v).
The Shapley value is a possible distribution of the total payoff, that is: $\phi(N) = v(N)$. Proof for this property can be found in [Drie 88].
If we assume the game (N, v) to be super-additive, then for all K with $i \notin K$ we have $v(K \cup i) - v(K) \geq v(i)$, a property that extends to the weighted sum, hence the Shapley value is an imputation.

1.24. Proposition. *For every average-convex game v, the Shapley value $\phi(v)$ is a core imputation, i.e. $\phi(v) \in C(v)$.*

Proof of the Proposition (and of an even stronger result) is given, for example, in [Drie 98].
The following axiomatic characterization of the Shapley value is well known and shown already in [Shap 53].

1.25. Proposition. *The Value $\phi : \mathbf{G}^n \longrightarrow \mathbb{R}^n$ satisfies, and is the only value that satisfies the following axioms:*

(i) *efficiency, i.e. for every game v, $\phi(N) = v(N)$,*

(ii) *additivity, i.e. for two games v and w, $\phi(v + w) = \phi(v) + \phi(w)$,*

(iii) *symmetry, i.e. for two players i, j that are substitutes in v, $\phi_i(v) = \phi_j(v)$,*

(iv) *null player axiom, i.e. if i is a null player of game v, then $\phi_i(v) = 0$.*

In addition, the Shapley value of a super-additive game is an imputation, i.e. individually rational.

Neyman in [Neym 88] shows an even stronger result, namely that even if one restricts the domain to the additive group $G(v)$ generated by a game $v \in \mathbf{G}^n$ and all of its subgames, every efficient, additive, symmetric mapping $\Psi : G(v) \longrightarrow \mathbb{R}^n$ must be the Shapley value.
Let $(\{1,2,3\}, v)$ be the over employment game from Example 1.4, figure 2, $(\{1,2,3,4\}, b)$ be the bankruptcy game from Ibn Ezra (Example 1.6, figure 4) and $(\{1,2\}, w)$ be the game of finding a diamond (Example 1.7).
Then $\phi(v) = \frac{1}{6}(40,13,13)^t$, $\phi(b) = \frac{1}{6}(40,55,115,175)^t$ and $\phi(w) = 5,000(1,1)^t$.
Note that $\phi(v) \notin C(v)$, so the over employment game is not average-convex, whereas the convex games b and w obviously are, thus their Shapley value is contained in their respective core.

The τ-Value

We now define the τ-value of a cooperative n-person game, a one-point solution concept introduced by Stef Tijs in [Tijs 81].
For $i \in N$ we call $b_i := MC_i(N) = v(N) - v(N - i)$ the **utopia value** of player i. In a core imputation no player can ever get a payoff that exceeds his utopia value. Therefore the **utopia vector** b is an upper bound for core imputations. Generally every player will end up getting less than his utopia value, because for all interesting games $v(N) \leq b_1 + \ldots + b_n$. The **gap function** g measures the amount of *disappointment* the players that unite in coalition A have to put up with:

$$\begin{aligned} g : \mathfrak{P}(N) &\longrightarrow \mathbb{R} \\ A &\longmapsto b(A) - v(A). \end{aligned}$$

Games where $g(N) = 0$ have a trivial core imputation b and are the only games that are both convex and 1-convex.
For every player i the **concession value** λ_i is defined via:

$$\lambda_i := \min\left\{g(K) \;\middle|\; K \subseteq N, i \in K\right\} \tag{21}$$

1.26. Proposition. *For core elements $x \in C(v)$ we find*

$$\left(\forall i \in N\right) b_i - \lambda_i \leq x_i \leq b_i. \tag{22}$$

A proof for this Proposition is given, for example, in [Krab 05].
If $\lambda(N) \geq g(N)$ holds and $g(K)$ is never negative, the game (N, v) is called **quasi-balanced**.
The class of quasi-balanced n-player games is denoted by $\mathbf{QB}^n$. Every game with non-empty core is indeed quasi-balanced (see for example [Krab 05]). The τ**-value** can then be defined.

1.27. Definition. The τ-value of a quasi-balanced cooperative n-person game (N, v) is given by:

$$\tau(v) := b - \frac{g(N)}{\lambda(N)} \cdot \lambda, \tag{23}$$

for $g(N) > 0$ and $\tau(v) := b$, if $g(N) = 0$.

1.28. Proposition. *A necessary and sufficient criterium for $\tau(v) \in C(v)$ is that the following implication holds for all coalitions A:*

$$g(A) > 0 \quad \wedge \quad 2 \leq |A| \leq n-2 \Longrightarrow \frac{\lambda(N)}{g(N)} \geq \frac{\lambda(A)}{g(A)}. \tag{24}$$

Again, proof for the Proposition can be found in [Krab 05].
A super-additive example of a game where $C(v) \neq \emptyset$, but $\tau(v) \notin C(v)$ is the four player game where $v(12) = v(13) = v(23) = v(123) = 2$, $v(N) = 3$ and $v(S) = 0$ otherwise (See figure 10).

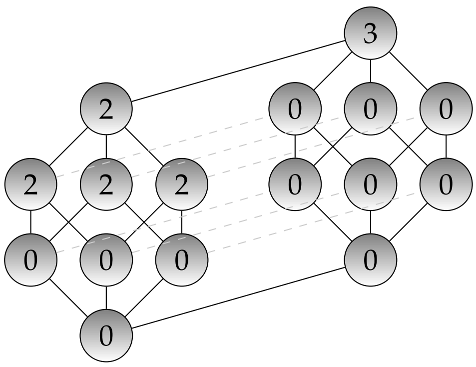

Figure 10: 4 player game, where $\tau(v) \notin C(v)$

Here $C(v)$ consists only of the imputation $(1, 1, 1, 0)^t$, while the τ-value is given by $\tau(v) = (0.9, 0.9, 0.9, 0.3)^t$. Note however that the super-additive hull of v, that is w with $w(124) = w(134) = w(234) = 2$ and $w(S) = v(S)$ otherwise, has the same core and for that game $\tau(w) = (1, 1, 1, 0)^t \in C(w)$.
As a convex example of a game where $\tau(v) \notin C(v)$, a five player airport cost game is given in [Drie 88, p. 133].
Let us calculate some τ-values for the examples from Subsection 1.1.
Let again $(\{1, 2, 3\}, v)$ be the over employment game from Example 1.4, figure 2, $(\{1, 2, 3, 4\}, b)$ be the bankruptcy game for Ibn Ezra's problem (Example 1.6, figure 4) and $(\{1, 2\}, w)$ be the game of finding a diamond (Example 1.7). Then $\tau(v) = (9, 1, 1)^t$, $\tau(b) = (14.4, 19, 2, 28.8, 57.6)^t$ and $\tau(w) = 5,000(1, 1)^t$.

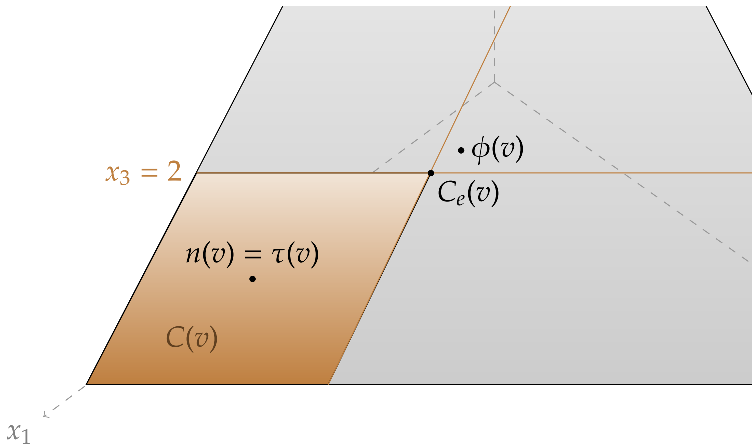

Figure 11: solutions to the over employment game

2 Approaches to fairness (and their limitations)

This section gives an overview of the usual approaches to model fairness in game theoretic literature. But let us first of all take note of why fairness considerations should not be disregarded in the theory.

2.1 Two motivating examples

Let us first look at examples in cooperative game theory and see how fairness, whatever that might mean, ought to play a role in these games. Start out by remembering the two player Example 1.7 of finding a diamond. Obviously each imputation is a core element, because no coalition save N gets any payoff at all. Thus according to the rationality of core solutions, each of the two friends would agree to cooperate even for a marginal share of 1 USD. However, it seems pretty obvious that such proposals will not lead to cooperation, at least in the case where negotiations between the players are not costly. Think, for example, of Rubinstein-Ståhl Bargaining with δ=1, that yields rejections of every offer other than a 50-50 split.[2]
What makes the 50-50 split so attractive is that the game is symmetric so there can be hardly any just reason for either player to get more then the other.
Next let us have a look at a simple cake-devision-problem with three players, shown in figure 12.
While in the previous example there were too many core imputations, here there are none at all. The game is totally symmetric, nonetheless, bargaining will usually lead to asymmetric payoffs.

[2]for a good overview of bargaining theory see for example [Ray 07]

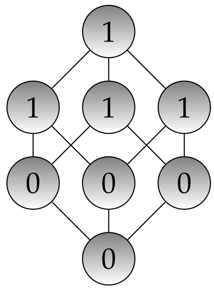

Figure 12: three player majority cake division

This punishes one player for being unlucky in the bargaining protocol. If players are even slightly risk-averse and have a chance to write contracts before the characteristic function of the game is even known to them, they might rule out something like this happening by, for example, agreeing that they use the Shapley value of whatever game they are about to play.
Thus, we can hope that if players can agree on certain common concepts of fairness and we allow them to enforce these, that might even help players to cooperate and share their profit in games with an empty core.
While in this example no value is destroyed if only two players cooperate, one could also set $v(N) = 1.4$ and still have a game with an empty core. Bargaining could then lead to an efficient (but again asymmetric) distribution over time, when agreements are reversible, but not instantly (see again [Ray 07]).

2.2 Motivational data

Let us now have a look at the research on the topic of fairness that psychologists and economists have been doing empirically with human test subjects.
A game similar to Example 1.7 (finding A diamond) has been experimented with on humans by economists (see [Henr 04], [Oost 04]). In the **ultimatum game** one player is given the opportunity to propose a distribution of USD 100 between himself and a partner. The partner can then accept the proposal and take his share, or he can veto—in which case neither of the players gets anything. There is no discussion or communication, the second player can only accept or decline; the game is not repeated. This protocol destroys the symmetry of Example 1.7 and the only Nash equilibrium of the players would be to offer a minimal share for the offerer and to accept for the acceptor.
Behavioral data tells a different story: when offered a share of USD 25, most players decline the offer.
It has been hypothesized (e.g. by James Surowiecki) that very unequal

allocations are rejected only because the absolute amount of the offer is low. The concept here is that if the amount to be split were ten million dollars a 90:10 split would probably be accepted rather than spurning a million dollar offer. Essentially, this explanation says that the absolute amount of the endowment is not significant enough to produce strategically optimal behavior. However, many experiments have been performed where the amount offered was substantial: studies by Cameron and Hoffman et al. have found that the higher the stakes are the closer offers approach an even split, even in a USD 100 game played in Indonesia, where average 1995 per-capita income was USD 670. Rejections are reportedly independent of the stakes as this level, with USD 30 offers being turned down in Indonesia, as in the United States [Came 95].
The **homo economicus** in his ideal form would follow the logic of core imputations, but he does not seem to be a common species. The homo sapiens acts altruistically and unlike primates he has a sense of fairness and acts upon it, even if this means a direct personal loss of money. On a side note: The prospect of having more than someone else is also way more important to people than the actual amount of their belongings. When questioned, many subjects would prefer an annual salary of USD 50,000 for themselves and 40,000 for a second person doing the same job over the option to get USD 70,000 , while the co-worker gets 80,000.
Another study by Fehr and his colleague Helen Bernard yields the following result: If three-year-olds are asked whether they would prefer to get 1 bag of sweets while another kid gets 2 bags, or whether both kids should get 1 bag each rather, they are indifferent—50 percent choose each. They simply lack interest in the other child's belongings. With eight-year-olds however, 90% choose the second option. The tendency to begrudge each other seems to be human nature, developing in early age.
After all this motivation, we move on to examine the way that fairness is presently modeled in cooperative game theory literature.

2.3 Egalitarian Core as a Fairness Concept

The egalitarian core is a strong fairness concept that has a natural ring to it. No difference in player's payoff of an egalitarian value can be considered unfair, since the game's *rules*, i.e. the function v does not allow for an equalizing bilateral transfer, less the sanity condition of the core be violated. Let us consider, however a 2-player game, where $v(1) = 1$, $v(2) = 0$ and $v(12) = 2$. The egalitarian solution $C_e(v) = \left\{(1, 1)^t\right\}$ of this example is not satisfactory. Why would player 1 cooperate, if all the profit goes to player 2?
The split of $\tau(v) = n(v) = \phi(v) = (1.5, 0.5)^t$ imposes itself as more natural.
Since it does not split, i.e. $C_e(v) \neq C_e(v_0) + s$ for $v = v_0 + s$, one might feel that the egalitarian core has a bias for the weaker players, in a sense it

is *overdoing* fairness—especially when large non-cooperative components s are involved.

2.4 Inequity Aversion & Envy Freeness

Inequity Aversion is an approach to the Homo Economicus question that has been around for about 30 years. The most recent and dominant formulation is due to [Fehr 99]. It is to say that player i is willing to trade α_i units of payoff in order to reduce another player's advantage over him by 1 unit.
Then α is the vector of personal inequity aversion for the players and a player's IA-utility in the imputation x can be defined as

$$u_i(x) = x_i - \alpha_i \cdot \sum_{j \neq i} \max\{0, x_j - x_i\}. \tag{25}$$

To be even more general, one also can model aversion to inequitable situations where the player himself is the one *better of*, scaled by another vector, usually called β.

$$u_i(x) = x_i - \alpha_i \cdot \sum_{j \neq i} \max\{0, x_j - x_i\} - \beta_i \cdot \sum_{j \neq i} \max\{0, x_i - x_j\}. \tag{26}$$

Since the players utility is not exactly their payoff, we are venturing in the field of nontransferable utility (NTU) games, but for now let us stay with TU games and look at those core imputations that would give no player any incentive to break away in order to increase his utility because of his personal α and β values.
As a first consideration take a two player game where $v(1) = a$, $v(2) = b$, $v(1, 2) = a + b + c$, let α be given and β be zero. Suppose without loss of generality that $a \leq b$.
Before any cooperation occurs, an inequity of $b - a$ is already in place. If we now distribute $(a + c_a, b + c_b)$, where $c_a + c_b = c$, the inequity has grown (or diminished, if negative) by $c_b - c_a = c - 2c_a$. Player one would stop cooperating if $c_a < \alpha_1 \cdot (c - 2c_a)$.
In literature α_1 is often something greater or equal to 0.25, so let us consider $\alpha_1 = 0.25$.
We get:

$$\begin{aligned} \text{cooperation} &\iff c_a \geq 0.25c - 0.5c_a \\ &\iff 1.5c_a \geq 0.25c \\ &\iff 6c_a \geq c \quad \iff \quad \frac{c_a}{c} \geq \frac{1}{6} \end{aligned}$$

That is a player with α of 0.25 would demand at least a sixth of the cooperation profit. Analogously a player with $\alpha = 1$ would demand at least a

third of the cooperation profit (or a third of his utopia value, which is the same here).
When α_1 approaches infinity, the minimum acceptable share for the player approaches $\frac{c}{2}$ asymptotically.
This is an example of the often times adequate behavior of the inequity aversion model, although the game was not zero-normalized.
Let us now return to the over employment game of Example 1.4, it's characteristic function repeated in figure 13.

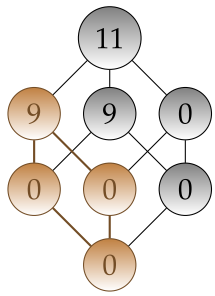

Figure 13: over employment game and two player subgame

For any value of $\alpha > 0.4$ players will prefer losing a payoff of 2 to accepting somebody else to get a payoff that surpasses their own by 5, thus a split of 2 : 7 is not something they would accept.
So even in the egalitarian solution $(7, 2, 2)^t$ of the game, the inequity-averse utility of players 2 and 3 is smaller than zero. Together the coalition $\{2, 3\}$ has negative utility in every core imputation and thus could write a binding agreement to not cooperate with player 1 over an infinite (or simply very long) time horizon.
Player 1 could offer player 2 to work in a mutually beneficial two-player coalition, dividing $(4.5, 4.5)^t$, which is a core element (and the nucleolus) of the two player subgame, but this is not stable with respect to the original game. Player three would want to replace player two in this coalition, offering a smaller split and competition between these would drive them to the point of zero gain, since they are identical players and can do nothing but offer their work for less to the land-owner, player 1. Thus, being farsighted, they should stick to their agreement and inequity aversion will eliminate cooperation in this example.
While the concept of inequity aversion is intuitive and explains observed behavior of test subjects in a range of different games (e.g. Public Goods Games), it does not allow for stronger players to receive larger pay-offs without the inequity averse players *feeling bad about it*.
The simple reason for this is, that the theory of inequity aversion has been developed from examples in which all players are substitutes, so with all

due respect to its feasibility and explanatory strength in such games, it is not a satisfactory model for less symmetrical games.

Analogously, in the case of non transferable utility when we discuss the distribution of a set of inseparable goods and players hold private utility functions for sets of the goods, the notion of fairness usually considered in literature on allocation mechanism design is **envy freeness**. Envy freeness is the property that requires each player to prefer the share allocated to her over any share allocated to another player. An envy free division of the item set $\{book, baseball\}$ between two players is possible, if one of the players prefers the book and the other would rather have the baseball.

Very much like concepts of inequity aversion, requiring envy freeness only makes perfect sense when every player is equally entitled to his share of the goods. Suppose that the three players are friends who have been sharing a flat for a couple of years. During this time, player 1 has bought a dishwasher, a TV set and a kitchen table and they have all been using these in their shared living room. When player 1 at some point in time moves out of the flat, it would be absurd for the three players to start thinking about an envy-free division of these three items (at least when no side payments are discussed).

2.5 The values ϕ, n and τ

The axioms of the Shapley value (see Proposition 1.25) make it a natural choice when looking for a fair distribution and in fact it is widely used.

Several authors have decided to use the τ-value as their solution concept, when addressing matters of fairness in the past (see, for example, [Zele 08] or [Bran 02]).

Both values, however, are not necessarily contained in the core of every game, which, in a way, restricts their feasibility to the classes of games where they are.

The nucleolus point is consistent with the desirability relation of players and is contained in the least-core, granting equal and maximal cooperation benefits to all members and coalitions.

All these values are one-point solution concepts however and will, for many games, be different from each other. Picking one of these as a notion of fairness therefore rules out the others, although they, too, are sensible distributions of the game's outcome. Players with too particular notions of fairness will encounter difficulties cooperating with players who have different notions of fairness—we will elaborate on this observation in Part III of this thesis.

Also, both the Shapley value and the τ-value are no coherent division rules for bankruptcy problems, as we will see in section 6.

3 Fairness Predicates

We will introduce different fairness concepts as *predicates* on the imputation space, so let us start out by defining this notion.

3.1. Definition. A **predicate on the imputation space** of a cooperative n-person game is a mapping P that assigns every game (N, v) a subset $P(v) \subseteq I(v)$.

Note that predicates are of the same type as solution concepts. However we will be interested in whether or not certain (one-point or set valued) solution concepts satisfy certain predicates in the sense of the solution concept being a subset of the predicate. The dummy player property on its own does not make much of a solution concept, it is a property that solutions might fulfill or violate, this is why it provides clarity of nomenclature calling the mappings used in this sense predicates on $I(v)$ rather than solution concepts. Some predicates that we have seen so far in game theory are the following

- The **dummy player predicate** DP rules out those imputations with positive payoffs for players that contribute nothing.
- A (partial) one-point solution concept P satisfies **anonymity** if for any permutation σ of the player set N we have $P(v)_i = P(\sigma(v))_{\sigma(i)}$.
- A (partial) one-point solution concept P is **additive** if for two cooperative n-person games (N, v) and (N, w) (where P(v) and P(w) are both non-empty) the equation $P(v + w) = P(v) + P(w)$ holds.

With the splitting of games into a trivial and a zero-normalized component in mind, we can also define predicates that split.

3.2. Definition. A predicate P on the imputation space of cooperative n-person games is said to **split** if for all (N, v) we have $P(v_0)+s(v) = P(v)$.

In the following subsections we will introduce a range of fairness predicates that are more or less intuitive and/or closely related to standard literature. We will later (when we study their satisfiability) see, how some of these predicates can be motivated, but for now we just introduce them and hope that the reader will find them natural enough to consider.

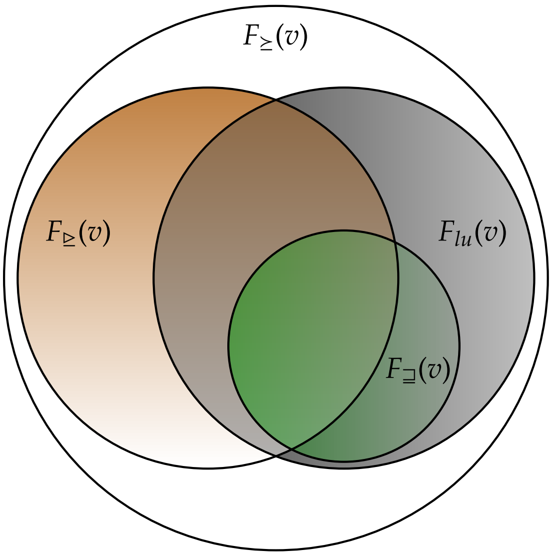

Figure 14: family of *according to desirability* fairness concepts

3.1 Payoff According to Desirability

From a fairness point of view, it seems natural to ask, whether players that are more desirable than others, i.e. players that contribute more, receive greater or (at least) equal payoff than their inferiors. In fact, this is so natural a consideration that the dummy player property, the substitution property and the question, whether imputations reflect the desirability relation on the player set, have been studied for more than 40 years, that is long before experiments on behavior have suggested that fairness considerations are quite important to humans, in contradiction to the homo economicus image of man [Masc 66b].

We will therefore introduce different relations on N and on $\mathfrak{P}(N)$ and develop a first family of fairness predicates when we call those imputations fair, that reflect these relations.

Desirability of Players, $F_{\geq}(v)$

The first desirability relation we define is the desirability of players that was studied by Maschler and Peleg in [Masc 66b] and that is reflected in all kernel imputations (see proposition 1.20 above).

It has been introduced in the context of that proposition, but we repeat the definition and introduce a notation:

3.3. Definition. Of two players $i, j \in N$, player i is called more desirable than player j, if for all $A \subseteq N - \{i, j\}$ the inequality $v(A \cup i) \geq v(A \cup j)$ holds. This **player desirability relation** is denoted by $i \succeq j$.
The imputations $x \in I(v)$ where for all $i, j \in N$ the implication $i \succeq j \Rightarrow x_i \geq x_j$ holds are called **player desirability-fair imputation**, the set of all these imputations is denoted by $F_{\succeq}(v)$.
If both $i \succeq j$ and $j \succeq i$, then players i and j are substitutes. In that case we write $i \sim j$.

As we have seen, the kernel is always made of player desirability-fair imputations, i.e. $K(v) \subseteq F_{\succeq}(v)$ for every game, which implies that $n(v) \in F_{\succeq}(v)$ and that $F_{\succeq}(v) \neq \emptyset$ for all games.
Figure 15 shows the fairness predicate $F_{\succeq}(v)$ in the case of the over employment game.

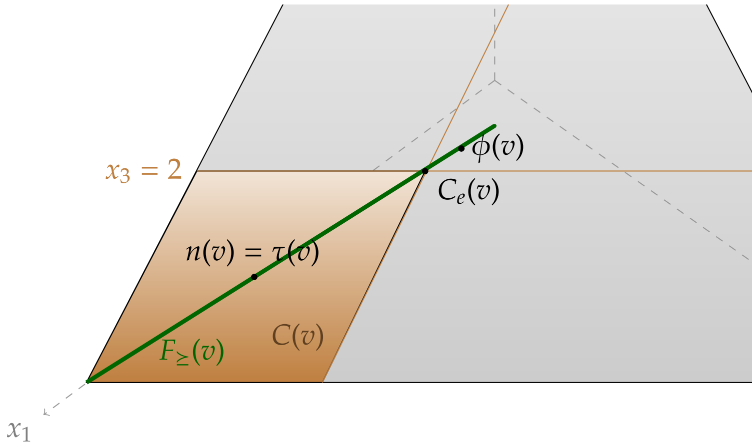

Figure 15: fairness predicate $F_{\succeq}(v)$ of the over employment game

3.4. Lemma. *Let $c \in C(v)$ be a core imputation that is unfair in the sense that $i \succeq j$, but $c_i < c_j$. Let further $d \in \mathbb{R}^n$ with $d_i = c_j$, $d_j = c_i$ and $d_k = c_k$ for all $k \neq i, j$. Then d is a core imputation as well.*

Proof. Since $v(i) \geq v(j)$ and c with $c(i) < c(j)$ was an imputation, it follows that d is an imputation.
It remains to show, that all inequalities of the form $v(A) \leq d(A)$ hold. For $i, j \notin A$ and for $i, j \in A$ we still find that $v(A) \leq c(A) = d(A)$.
For $i \in A, j \notin A$ we have $v(A) \leq c(A) = c_i + c(A - i) < c_j + c(A - i) = d(A)$,

while for $i \notin A, j \in A$ we find that $v(A) = v((A-j)\cup j) \leq c((A-j)\cup j) < c((A-j)\cup i) = d(A)$, which concludes the proof of the Lemma. □

3.5. Lemma. *Let $c \in C(v)$ be a core imputation and $S = \{s_1, ..., s_k\} = [s_1]_\sim \subseteq N$, that is the k players in S are substitutes.*
Define d to be the imputation that pools the c-payoffs to the players in S and divides these equally amongst them, i.e. $d \in \mathbb{R}^n$ with $d_k := c_k$ for $k \notin S$ and $d_{s_i} := \frac{1}{k}c(S)$ for all $i = 1, \ldots, k$. Then d is also a core imputation.

Proof. Since $d(N) = c(N)$ and $c(N) = v(N)$, obviously d is efficient.
It remains to see that for all $K \subseteq N$ we have $d(K) \geq v(K)$.
Think of the elements of S ordered according to their payoff in c, so that $S = \{s_1, \ldots, s_k\} = \{e_1, \ldots, e_k\}$ with $c(e_1) \leq c(e_2) \leq ... \leq c(e_k)$. Let $S_j = \{e_1, \ldots, e_j\}$ and write

$$\mathrm{avg}(S_j) := \tfrac{1}{j}\sum_{i=1}^{j} c(e_i). \tag{27}$$

From the ordering of the $c(e_i)$ we conclude that $c(e_{i+1}) \geq \mathrm{avg}(S_i)$ for all i and thus $(j+1)\,\mathrm{avg}(S_{j+1}) = \sum_{i=1}^{j+1} c(e_i) = c(e_{j+1}) + \sum_{i=1}^{j} c(e_i) = j\,\mathrm{avg}(S_j) + c(e_{j+1}) \geq j\,\mathrm{avg}(S_j) + \mathrm{avg}(S_j) = (j+1)\,\mathrm{avg}(S_j)$. This implies $\mathrm{avg}(S_1) \leq \mathrm{avg}(S_2) \leq \ldots \leq \mathrm{avg}(S_k)$, which is to say $d(S_i) \geq c(S_i)$ for all $i = 1, ..., k$.
Now, let $K \subseteq N$ and divide K into $A := K \cap I$ and $B := K - A$. Note that $B \subseteq (N - S)$ and thus

$$c(B) = d(B). \tag{28}$$

Let further $a = |A|$ and $\mathcal{A} = S_a$, then we have $|A| = |\mathcal{A}|$, thus

$$d(A) = d(\mathcal{A}).,\tag{29}$$

and furthermore, since $\mathcal{A} = S_a$,

$$d(\mathcal{A}) \geq c(\mathcal{A}). \tag{30}$$

Therefore we have $d(K) = d(A) + d(B) =^{(29)} d(\mathcal{A}) + d(B) =^{(28)} d(\mathcal{A}) + c(B) \geq^{(30)} c(\mathcal{A}) + c(B) \geq v(\mathcal{A} \cup B)$, since $c \in C(v)$.
But since A and $\mathcal{A}$ are both subsets of I and $|A| = |\mathcal{A}|$ and all players in I are substitutes, there is a bijection of the members in $A - \mathcal{A}$ and the members of $\mathcal{A} - A$ and these players can successively be replaced without changing the value v of the coalition, thus $v(\mathcal{A} \cup B) = v(A \cup B) = v(K)$, so continuing from $d(K) \geq v(\mathcal{A} \cup B)$ we get $d(K) \geq v(\mathcal{A} \cup B) = v(A \cup B) = v(K)$. □

Lemma 3.4 and Lemma 3.5 imply that $C(v) \neq \emptyset \implies C(v) \cap F_{\geq}(v) \neq \emptyset$, which is a satisfiability result that we will come back to in section 4.

3.6. Theorem. *Let* $(N, v) \in \mathbf{G}^n$.

(i) *If* v *is quasi-balanced, the* τ*-value* $\tau(v)$ *is defined and is an element of* $F_{\geq}(v)$.

(ii) *If the game is super-additive and* $v(i) \geq 0$ *for all* i*, the Shapley-value* $\phi(v)$ *is an imputation and it is also an element of* $F_{\geq}(v)$.

(iii) *If* $C(v) \neq \emptyset$*, then the egalitarian core* $C_e(v)$ *is non-empty and is contained in* $F_{\geq}(v)$.

Proof.

(i) Let us assume that $g(N) > 0$ and hence $\lambda(N) > 0$. In the case $g(N) = 0$, where the τ-value assigns each player his utopia value, the τ-value obviously respects $\geq$.

The τ-value is defined for all quasi-balanced games and it is an imputation. To check that it is an element of $F_{\geq}(v)$ we have to check that for $i \geq j$ we have $\tau_i(v) \geq \tau_j(v)$.

Indeed for $i \geq j$ we have $b_i \geq b_j$. The definition of λ_j is $g(K \cup j)$, $j \notin K$ for some K that minimizes this term, while λ_i is $g(A)$ of some minimal-gap coalition with $i \in A$.

In the case that $i \in K$ we find

$$\begin{aligned}\tau_i(v) = b_i - \frac{g(N)}{\lambda(N)} g(A) &\geq b_i - \frac{g(N)}{\lambda(N)} g(K \cup j) \\ &\geq b_j - \frac{g(N)}{\lambda(N)} g(K \cup j) = \tau_j(v).\end{aligned}$$

Otherwise (i.e. $i \notin K$), we know that $g(K \cup i)$ is larger than or equal to $g(A)$. It follows that

$$\begin{aligned}\tau_i(v) &\geq b_i - \frac{g(N)}{\lambda(N)} g(K \cup i) \\ &= b_i - \frac{g(N)}{\lambda(N)} (b(K) + b_i - v(K \cup i)) \\ &= \left(1 - \frac{g(N)}{\lambda(N)}\right) b_i - \frac{g(N)}{\lambda(N)} (b(K)) + \frac{g(N)}{\lambda(N)} \underbrace{v(K \cup i)}_{\geq v(K \cup j)} \\ &\geq \left(1 - \frac{g(N)}{\lambda(N)}\right) b_j - \frac{g(N)}{\lambda(N)} (b(K)) + \frac{g(N)}{\lambda(N)} v(K \cup j) \\ &= \tau_j(v).\end{aligned}$$

So in both cases the τ-value respects player desirability relations.

(ii) For a proof that the Shapley value is an imputation under these conditions, see [Drie 88].

Now, notice that the Shapley value can be calculated by taking the sum over all coalitions:

$$\phi_i(v) = \sum_{S \subseteq N} \gamma_n(S) \cdot (v(S) - v(S - i))$$

or, for that reason, by taking the sum over any set $\mathcal{Y}$ of coalitions, as long as every $A \in \mathfrak{P}(N)$ with $i \in A$ is contained in $\mathcal{Y}$.

Now let $i \succeq j$. We will now see that $\phi_i(v) \geq \phi_j(v)$. For this purpose we divide the powerset of N into three types of coalitions. Let $\mathcal{I}$ be coalitions that contain player i but not player j and $\mathcal{J}$ be the coalitions that contain player j, but not player i. Let $\mathcal{K}$ be all remaining coalitions, that is those that contain either both i and j or that contain neither.

For players i and j we can pick $\mathcal{Y} = \mathcal{K} \cup \mathcal{I}$ and $\mathcal{Y} = \mathcal{K} \cup \mathcal{J}$ respectively as index for the sum in order to calculate the Shapley value, according to the remark above.

Note that for $A \in \mathcal{K}$ we have $v(A - i) \leq v(A - j)$, because $i \succeq j$. Furthermore $\mathcal{I}$ and $\mathcal{J}$ are related bijectively by:

$$\begin{aligned} t_{i,j} &: \mathcal{I} \longrightarrow \mathcal{J} \\ A \cup \{i\} &\longmapsto A \cup \{j\}\,, \text{ for all appropriate } A. \end{aligned}$$

This map $t_{i,j}$ is decreasing in the sense that for $A \in \mathcal{I}$ we have $v(A) \geq v(t_{i,j}(A))$ and hence, with $A - i = t_{i,j}(A) - j$,

$$v(A) - v(A - i) \geq v(t_{i,j}(A)) - v(t_{i,j}(A) - j). \tag{31}$$

Furthermore $\gamma_n(A) = \gamma_n(t_{i,j}(A))$, because $t_{i,j}$ preserves cardinality.

Note also that for $A \in \mathcal{K}$ we have

$$v(A) - v(A - i) \geq v(A) - v(A - j), \tag{32}$$

since either $i, j \notin A$, and both terms equal zero, or $\{i, j\} \subseteq A$, in which case with $B := A - \{i, j\}$ we have $v(A - i) = v(B + j) \leq v(B + i) = v(A - j)$.

Now looking at $\phi_i(v)$ and $\phi_j(v)$ we find that

$$\begin{aligned}
\phi_i(v) &= \sum_{A\in\mathcal{K}} \gamma_n(A)\cdot(v(A)-v(A-i)) + \sum_{A\in\mathcal{I}} \gamma_n(A)\cdot(v(A)-v(A-i)) \\
&\overset{(32)}{\geq} \sum_{A\in\mathcal{K}} \gamma_n(A)\cdot(v(A)-v(A-j)) + \sum_{A\in\mathcal{I}} \gamma_n(A)\cdot(v(A)-v(A-i)) \\
&\overset{(31)}{\geq} \sum_{A\in\mathcal{K}} \gamma_n(A)\cdot(v(A)-v(A-j)) + \\
&\qquad \sum_{A\in\mathcal{I}} \gamma_n(t_{i,j}(A))\cdot(v(t_{i,j}(A))-v(t_{i,j}(A)-j)) \\
&= \sum_{A\in\mathcal{K}} \gamma_n(A)\cdot(v(A)-v(A-j)) + \sum_{A\in\mathcal{J}} \gamma_n(A)\cdot(v(A)-v(A-j)) \\
&= \phi_j(v).
\end{aligned}$$

(iii) Now we assume that $C_e(v) \not\subseteq F_{\geq}(v)$. Then let $x \in C_e(v) - F_{\geq}(v)$. So there are two players i and j with $i \geq j$ and $x_j = x_i + 2\alpha$ for some positive α.

Let x' be the result of the equalizing bilateral transfer of size α from player j to player i.

Then, since $x \in C(v)$ and (with Lemma 3.4) the imputation d with $d_i = x_j$ and $d_j = x_i$ is also a core element, with the convexity of the core it follows that $x' \in C(v)$.

For any $x \in C_e(v)$ however there are no core imputations that are the result of equalizing bilateral transfers, thus the assumption leads to a contradiction and indeed have $C_e(v) \subseteq F_{\geq}(v)$,

which completes the Theorem's proof. □

Weak Desirability of Players, $F_{\trianglerighteq}(v)$

The powerset of N has 2^n elements, so in games with many players it is, computationally, increasingly difficult to check each players contribution to all the coalitions versus each other players contribution.
While even for games with several hundred players this is of course negligible when one has access to computers, it remains doubtful, whether people playing the game would go through the effort of all that in order to get a feeling about their own desirability in the game, in order to compare, which players are equal, which are more powerful and which are less powerful than themselves.
Should they base their fairness notions on such comparison, it might be conceivable that all they compare are the individual payoffs and the utopia values of players, i.e. their marginal contributions to the grand coalition.

3.7. Definition. On the player set N of a cooperative n-person gamewe define the **weak desirability relation** $\trianglerighteq$ by means of

$$i \trianglerighteq j \text{ if and only if } v(i) \geq v(j) \text{ and } v(N-i) \leq v(N-j). \tag{33}$$

The imputations $x \in I(v)$ where for all $i, j \in N$ the implication $i \trianglerighteq j \Rightarrow x_i \geq x_j$ holds are called $weakplayerdesirability - fairimputation$, the set of all these imputations is denoted by $F_{\trianglerighteq}(v)$.

Obviously $i \succeq j$ implies $i \trianglerighteq j$ and thus $F_{\trianglerighteq}(v) \subseteq F_{\succeq}(v)$ for each cooperative game v.

3.8. Theorem. *Let the cooperative n-person game(N, v) be quasi-balanced and 1-convex, i.e. $0 \leq g(N) \leq g(K)$ for all $K \subseteq N$. Then $\tau(v) \in F_{\trianglerighteq}(v) \cap C(v)$.*

Proof. The τ-value is an imputation for all quasi-balanced games.
It follows from the 1-convexity, that $\lambda_i = g(N)$ for all $i = 1, \ldots, n$. Therefore the τ-value is given by

$$\tau_i(v) = b_i - \frac{1}{n} \cdot g(N) \quad \text{for all } i = 1, \ldots, n. \tag{34}$$

Furthermore it follows that

$$\frac{\lambda(N)}{g(N)} \geq \frac{\lambda(K)}{g(K)} \text{ for all } K \subsetneqq N \text{, if } g(N) > 0. \tag{35}$$

which implies that $\tau(v) \in C(v)$ (see Proposition 1.28).[3]
To check that it reflects the weak desirability relation, we have to check that for $i \trianglerighteq j$ we always have $\tau_i(v) \geq \tau_j(v)$.
For this it suffices to show that $b_i \geq b_j$. This is, however, a direct consequence of $v(N-i) \leq v(N-j)$. □

3.9. Theorem. *Let $(N, v) \in \mathbf{G}^n$ be quasi-balanced and $0 \leq \lambda_i = g(i)$ for all i. Then $\tau(v) \in F_{\trianglerighteq}(v)$.*

[3] The case $g(N) = 0$ is uninteresting. Of corse the Theorem will still hold.

Proof. Let $i \trianglerighteq j$, that is $b_i \geq b_j$ and $v(i) \geq v(j)$.
Since $\lambda_i = g(i) = b(i) - v(i)$ and all $g(k) \geq 0$ we have $b_k \geq v(k)$ and thus $\sum_{k \in N} v(k) \leq v(N)$, which implies $g(N) = u(N) - v(N) \leq \lambda(N) = u(N) - \sum_{k \in N} v(k)$. Therefore $0 \leq Q := \frac{g(N)}{\lambda(N)} \leq 1$.[4]
Now $\tau_i(v) = b_i - Q(b_i - v(i)) = b_i(1-Q) + v(i) \cdot Q$, and likewise $\tau_j(v) = b_j(1-Q) + v(j) \cdot Q$, which is not larger.

□

3.10. Corollary. *The τ-value reflects weak player desirability for all convex games, since for these games $g(S+j) - g(S) = b(S+j) - v(S+j) - b(S) + v(S) = MC_j(N) - MC_j(S+j) > 0$, i.e. the gap function is monotonous, which implies $\lambda_i = g(i) \geq 0$ for all i.*

Driessen also shows this in [**?**]Driessen
Note that the Shapley Value is in general **not** an element of $F_{\trianglerighteq}(v)$. Consider, as a counter example, the convex four person game given in the figure 16.

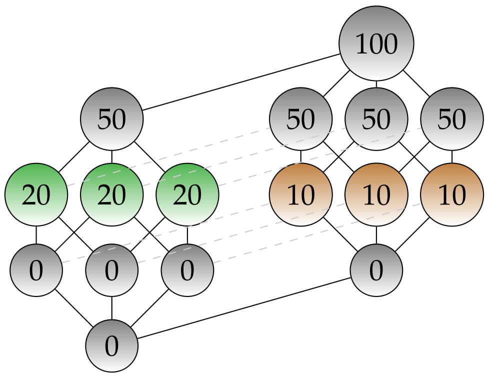

Figure 16: Shapley value does not reflect weak desirability of players

Here $\phi(v) = \frac{1}{6}(155, 155, 155, 135)^t \neq \frac{100}{4}(1, 1, 1, 1)^t$, which is the only element of $F_{\trianglerighteq}(v)$, since $1 \trianglerighteq 2 \trianglerighteq 3 \trianglerighteq 4 \trianglerighteq 1$.
For games with 2 or 3 players, weak desirability is the same as desirability, so that for these games, $F_{\geq}(v) = F_{\trianglerighteq}(v)$ and thus the τ-value, nucleolus and the Shapley-value are of course strongly fair, in the sense that they reflect weak desirability of players.

Desirability of Coalitions, $F_{\sqsupseteq}(v)$

Note that in the Exchange Economy game from Example 1.5 the imputation $(1, 0, 0)^t$ respects player desirability. Player 1 is the dominant player of the

[4]it has been pointed out to the author that this follows directly from the fact that the game is quasi-balanced, which one needs to assume, so that the τ-value be defined.

game, thus the fairness concept $F_{\geq}(v)$ does not prevent him from taking all the profit.
Players 2 and 3 of the game might reason that *together* they are worth no less than player 1 and demand equal shares between him and the two of them. This fairness consideration can be modeled in the following way, naturally lifting the desirability relation from N to the powerset $\mathfrak{P}(N)$.

3.11. Definition. We define the **desirability relation on coalitions** via

$$A \sqsupseteq B \quad :\Longleftrightarrow \quad \big(\forall C \subseteq (N - (A \cup B))\big)\ v(C \cup A) \geq v(C \cup B). \tag{36}$$

This relation $\sqsupseteq$ is a pre-order on the powerset lattice of N.
The **desirability of coalitions fairness** concept is given by the predicate $F_{\sqsupseteq}(v) \subseteq I(v)$ where $x \in F_{\sqsupseteq}(v)$, if we find that $x(A) \geq x(B)$ holds for each $A \sqsupseteq B$, that is, if the distribution of the payoff reflects the relation of coalition desirability.

Obviously $i \geq j \iff \{i\} \sqsupseteq \{j\}$ and thus $F_{\sqsupseteq}(v) \subseteq F_{\geq}(v)$ for all games.
Recall the over employment game (see the left side of figure 17), it turns out, that for this super-additive game, $F_{\sqsupseteq}(v) \cap C(v) = \emptyset$ and that $\tau(v)$ and $\phi(v)$ both do not respect coalition desirability. To see this observe that any imputation that pays more than 2 to either worker is not element of this game's core, since $b_2 = b_3 = 2$. The Shapley value of the game is given by $\phi(v) = \frac{1}{6}(40, 13, 13)^t$ and the τ-value is given by $\tau(v) = (9, 1, 1)^t$. The coalition of both peasants is however just as desirable as the coalition consisting of the land owner alone, as figure 17 also shows, thus $F_{\sqsupseteq}(v) = \left\{\frac{1}{4}(22, 11, 11)^t\right\}$.

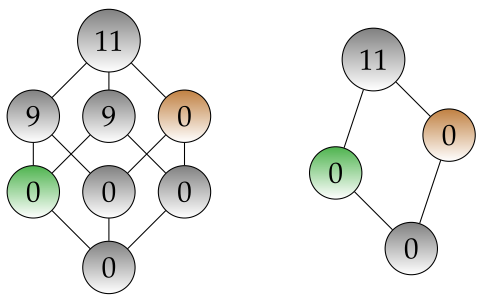

Figure 17: equally desirable coalitions in the over employment and game

This example can be generalized to arbitrary production economies with $n - 1$ peasants and 1 land-owner. The coalition of all peasants will always be just as desirable as the landowner is and thus the only imputation in

$F_{\sqsupseteq}(v)$ will be the one that assigns $\frac{v(N)}{2}$ to the land-owner and $\frac{v(N)}{2(n-1)}$ to each peasant.
As a corollary we immediately get that the Shapley value, the τ-value, the nucleolus and the egalitarian values need not be coalitional-desirability-fair, even for convex games.
To see this take $n = 4$ and let a coalition consisting of the land-owner and $1, 2$ or 3 peasants produce $1, 2$ and 4 units of utility respectively, i.e. the production function is given by $f(x) = 2^{x-1}$, which is a strictly convex function. For the corresponding production game (N, v) the Shapley value is given by $\phi(v) = (1.75, 0.75, 0.75, 0.75)^t$, the τ-value is given by $\tau(v) = (1.6, 0.8, 0.8, 0.8)^t$ and the nucleolus can be easily calculated to be the only egalitarian solution of the game, i.e. $n(v) = (1, 1, 1, 1)^t$. All these values fail to meet the requirement of $x_1 = x_2 + x_3 + x_4$.
To make things worse, we can easily give a convex game in $F_{\sqsupseteq}^{-1}(\emptyset)$. Take $N = \{1, 2, 3, 4\}$, let $T = N - 4$ and consider the four player unanimity game u_T. While the Shapley value, the nucleolus, the τ-value, the only egalitarian value and the only $F_{\succeq}$-fair value is given by $\frac{1}{3}(1, 1, 1, 0)^t$, we have $\{1, 2\} \sim_{\sqsupseteq} \{3\}$ and $\{1\} \sim_{\sqsupseteq} \{2, 3\}$. We conclude that in a $F_{\sqsupseteq}$-fair imputation x we would have $x_1 + x_2 = x_3$ and also $x_1 = x_2 + x_3$, so the payoff to player 2 would be zero. And – with the same argument, the payoff to everyone else would be zero too – but that is not possible, so $F_{\sqsupseteq}(u_T) = \emptyset$.
It is, however, not unrealistic to find solutions that reflect the desirability of coalitions for specific games or classes of games. As we will see in section 6, the Aristotelian proportional division rule for bankruptcy games is an example that does reflect coalitional desirability.

Desirability of Equivalence Classes (Labor Union), $F_{lu}(v)$

We have defined weak player desirability under the impression that players are not expected to compare large numbers of marginal contributions in order to get an impression of their personal desirability in a game.
With the same argument, it seems sensible that players should not compare all 2^N possible coalitions against each other for desirability. It is also doubtful, whether players would care to identify with each of the coalitions that they could partake in the sense that they would stop cooperating, demanding a coalition desirability fair share, when coalition desirability is violated for some arbitrary cryptic coalitions.
Many of the games that appear throughout literature regularly, however, have large numbers of players that are substitutes. Recall, for example, the production economy or the exchange economy where there are only two types of players.
If players observe that they are substitutes and agree to share a fairness notion that guarantees them equal payoffs, it is quite natural to assume that they would act as a sort of labor union, trying to maximize their individual

outcome by increasing the outcome of this labor union. This is why labor unions of substitute players seem more natural than mixed labor unions. Also, we have seen that $F_{\sqsupseteq}$ is usually empty, even for convex four player games. The predicate of labor union fairness we are about to define does not have such adverse properties.

3.12. Definition. The **labor union fairness** concept is given by the predicate $F_{lu}(v) \subseteq I(v)$ where $x \in F_{lu}(v)$, if

(i) we find that $x(K) \geq x_l$ holds for each $K \sqsupseteq \{l\}$, where K is a class of substitute players and l is a single player and

(ii) $x \in F_{\geq}(v)$.

Obviously we have $\mathcal{F}_{\sqsupseteq}(v) \subseteq \mathcal{F}_{lu}(v) \subseteq \mathcal{F}_{\geq}(v)$. While labor union fairness does not compare any set of likely or unlikely coalitions it ensures that a labor union of substitutes do not receive less in sum than any individual player *(manager)* does, where the labor union, as a total, is more desirable than the manager.
Where the Shapley value, the τ-value and the nucleolus did not yield equal payoffs for the set of peasants and the land-owner in the exponential production economy with production function $f(x) = 2^{x-1}$ above and they failed to meet desirability-of-coalitions fairness, they all gave to the *labor union* of peasants more than they did grant the land-owner, thus satisfying labor union fairness.
We will see that $\mathcal{F}_{lu}$ contains the egalitarian core for all convex games, we first need a Lemma which is pretty intuitive geometrically (see figure 18):

3.13. Lemma. *Let (N, v) be a convex cooperative n-person game and $I = \{i_1, i_2, \ldots, i_r\}$ be a set of substitute players. Let $x \in \mathcal{F}_{\geq}(v) \cap C(v)$ be a respect player desirability fair core element.*
Let further $K_0 \subseteq N - I, K_1 = K_0 \cup i_1, K_2 = K_1 \cup i_2, \ldots, K_r = K_0 \cup I$.
If there exists an $a \in \{1, \ldots, r-1\}$ with $v(K_a) = x(K_a)$, it follows that $v(K_s) = x(K_s)$ for all $s \in \{0, \ldots, r\}$.

Proof. We will first see that for all $s > a$ the equation holds.
Assume that there is some $s \geq a$ with $v(K_s) < x(K_s)$. In that case let $l := \min\left\{l' \in \{1, \ldots r-a\} \,\middle|\, v(K_{a+l'}) < x(K_{a+l'})\right\}$.
Then obviously $MC_{i_{a+l}}(K_{a+l-1}) < x_{i_{a+l}} = x_{i_a}$.
But also, since $x \in C(v)$, we have $v(K_{a-1}) \leq x(K_{a-1})$, thus $MC_{i_a}(K_{a-1}) \geq x_{i_a}$.
Convexity ensures that $MC_{i_a}(K_{a-1}) \leq MC_{i_a}(K_{a+l} - \{i_a\}) = MC_{i_{a+l}}(K_{a+l-1})$.
Concluding we have $x_{i_a} \leq MC_{i_a}(K_{a-1}) = MC_{i_{a+l}}(K_{a+l-1}) < x_{i_a}$, which is a contradiction.

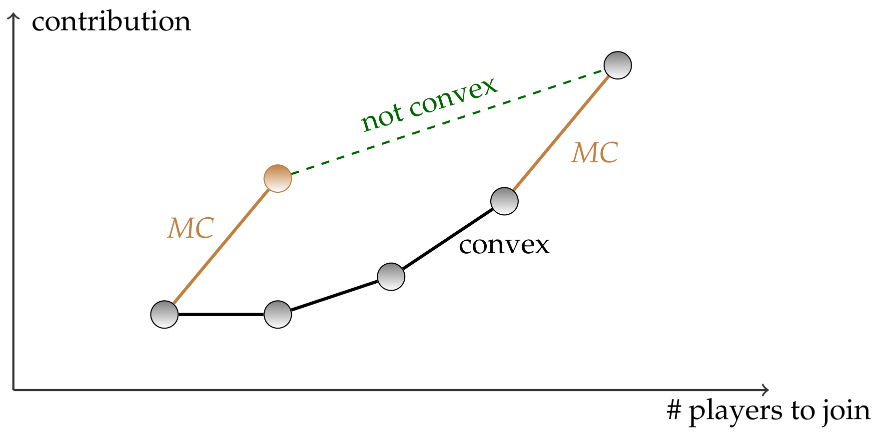

Figure 18: geometric intuition behind Lemma 3.13. If the convex function grows by the maximal *MC* on the left, it must be a line.

The proof for all $s < a$ is the same. □

We now prove the Theorem.

3.14. Theorem. *$C_e(v) \subseteq \mathcal{F}_{lu}(v)$ for convex games (N, v).*

Proof. Let $[i]_\sim = \{i_1, ..., i_r\}$ and suppose $[i]_\sim \sqsupseteq \{j\}$.
Let further $x \in C_e(v)$.
Suppose that $r \cdot x_i \leq x_j$. Then for sure $x_i < x_j$ and thus (since x is egalitarian) there exists no equalizing bilateral transfer of any size from j to i without leaving the core. In other words, every transfer leaves the core and thus there is a coalition K with $i \notin K$, $j \in K$ and $v(K) = x(K)$.
Then, with the Lemma 3.13, we can assume that K does not meet $[i]_\sim$.
Now we have $v(K \cup [i]_\sim - j) \geq v(K) = x(K)$ and since $rx_i \leq x_j$ we also have $x(K) \geq x(K \cup [i]_\sim - j)$.
On the other hand $x \in C(v)$, thus $v(K \cup [i]_\sim - j) \leq x(K \cup [i]_\sim - j)$, thus $v(K \cup [i]_\sim - j) = x(K) = x(K \cup [i]_\sim - j)$.
It follows that $rx_i = x_j$. □

We have, so far, introduced Fairness predicates $F_{(-)}$ that were of the same type. Each expected the imputation to reflect a certain relation on the player set or on the set of coalitions.
Let us now also include different, albeit related fairness considerations.

3.2 Egalitarianism, $F_e(v)$

As mentioned above, despite its doubtful behavior for non-zero-normalized games with significant non-cooperative component s, egalitarianism must be regarded as a well known and widely spread notion of fairness. We therefore include the egalitarian core in the family of fairness concepts simply by by writing $F_e(v) = C_e(v)$.

We have already seen (remember figure 11), that the nucleolus, the τ-value and the Shapley value are, in general, not fair in the sense of egalitarianism.

3.3 Zero-normalization, $F^0_{(-)}$

The idea of zero-normalization is widely spread in cooperative game literature and, although the author holds no experimental data on this, he believes that players of a game will, in general, not disagree, when offered to receive their individual value $v(i)$ beforehand and play the purely cooperative component v_0 of the game at hand afterwards.

Players will receive at least their individual value in every imputation, for $v = v_0 + s$ we have $\tau(v) = \tau(v_0) + s$, $\phi(v) = \phi(v_0) + s$, $n(v) = n(v_0) + s$, $C(v) = C(v_0) + s$ ... indeed, the general inequality $C_e(v) \neq C(v_0) + s$ is a key reason why we find the egalitarian approach to be less feasible for games with non-cooperative component s than for those without.

The fairness concepts above do, however, not split. Thus for any predicate on the imputation space $P : v \longmapsto P(v) \subseteq I(v)$ we define the **zero-refinement of the predicate** via $P^0(v) := P(v) \cap (P(v_0) + s)$, where v is split into trivial and zero-normalized components $v = v_0 + s$.

Therefore, for a given fairness predicate $F_{(-)}(v)$ and an imputation $x \in I(v)$ we can check, whether both $x \in F_{(-)}(v)$ and also $(x - s) \in F_{(-)}(v_0)$, that is, whether the imputation still looks fair when considering the cooperative part of the imputation in the zero-normalized game.

For example $F^0_{\geq}(v)$ is the set of those imputations x where for all pairs i, j with $i \geq_v j$ we have $x_i \geq x_j$ and also for all pairs k, l with $k \geq_{v_0} l$ we have $x_k - s_k \geq x_l - s_l$.

In the special case of $F^0_{\geq}(v)$, we prove the following Lemma in order to arrive at Theorem 3.16 that allows us to construct elements of $F^0_{\geq}(v)$ for every game with non-empty core.

3.15. Lemma. *Let (N, v) be a cooperative n-person game and ε be a positive real number. Suppose that $i + \varepsilon \geq j$, in the sense that for every $K \subseteq N - \{i, j\}$ we have $v(K \cup \{i\}) + \varepsilon \geq v(K \cup \{j\})$. Let further $x \in C(v)$ with $x_j > x_i + \varepsilon$.*

We define $\alpha := x_j - x_i - \varepsilon$ *and the imputation* x' *through*

$$(x')_j = x_j - \frac{\alpha}{2} \tag{37}$$

$$(x')_i = x_i + \frac{\alpha}{2} \tag{38}$$

$$(x')_k = x_k \qquad \text{for } k \notin \{i, j\}. \tag{39}$$

In these circumstances we have $x' \in C(v)$.

Proof. For $K \subseteq N, \{i, j\} \subseteq K$ we have $x_K = (x')_K$ and $x \in C(v)$, so the vector x' does not violate the core condition for coalition K. The same holds fot $K \subseteq N - \{i, j\}$.
For $i \in K, j \notin K$ we have $(x')_K \geq x_K \geq v(K)$.
Now for $i \notin K, j \in K$ we have

$$(x')_K = (x')_{K-\{j\}} + (x')_j \geq (x')_{K-\{j\}} + \varepsilon + (x')_i \geq \varepsilon + v(K - \{j\} \cup \{i\}) \geq v(K). \tag{40}$$

The last inequality holds since $i + \varepsilon \geq j$ and the one before that is true since the core condition holds for $j \notin K$, as we have seen. □

3.16. Theorem. *Let* (N, v) *be a cooperative n-person game with non-empty core and* $v_0 = v - s$ *be its purely cooperative zero-normalization. Let further* e_0 *be any element of* $C_e(v_0)$, *i.e. any egalitarian solution of the zero-normalized game.*
Then $e := s + e_0 \in F^0_{\geq}(v)$.

Proof. According to the last part of Theorem 3.6, we know that $C_e(v_0) \subseteq F_{\geq}(v_0)$, so it remains to be seen that for some $e_0 \in C_e(v_0)$ the core imputation $e = e_0 + s$ is always a member of $F_{\geq}(v)$.
Now let $i \geq j$ and assume that we have $(e_0 + s)_i \leq (e_0 + s)_j$. It follows from $i \geq j$ that $s_i \geq s_j$. Define $\sigma := s_i - s_j$.
Let us look at the game (N, v_0) and note that in this game we have $i + \sigma \geq j$ (using the notation we introduced in Lemma 3.15).
Since $(e_0)_i \leq (e_0)_j + \sigma$, it follows from that Lemma that the result of an equalizing bilateral transfer of size $0.5 \cdot (x_j - x_i - \sigma)$ from player j to player i would be element of the core $C(v_0)$, which obviously contradicts $x \in C_e(v_0)$.
Thus it cannot occur that $(e_0 + s)_i \leq (e_0 + s)_j$ and we have $i \geq j \Longrightarrow e_i \geq e_j$.
□

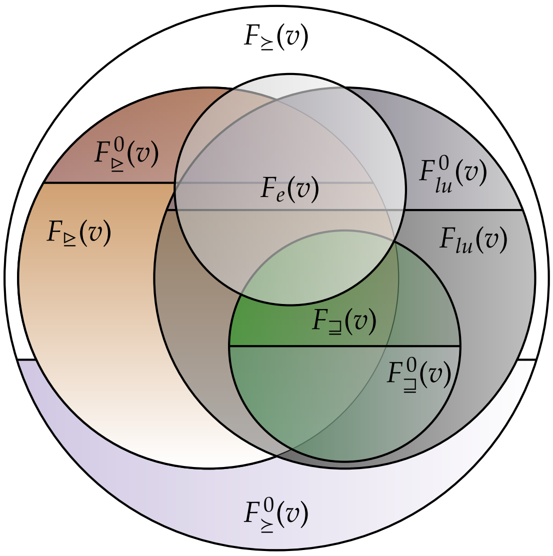

Figure 19: collection of 9 fairness concepts, not including $F^0_e(v)$, which is almost inevitably empty.

4 Core-Satisfiability

It is not practical to work with (or propagate) fairness concepts that are too strong. The one-point solution according to a "Every Player Must Get The Same"-fairness principle, for example, would (for certain games) produce vectors that violate individual rationality and thus those vectors are not even imputations. As we have seen, F^0_e is usually empty and $F^0_{\sqsupseteq}$ is often empty, although there is a whole class of games (the bankruptcy games), where it is not. Obviously emptiness is not a desirable property for a property that determines a set of acceptable imputations.

The standard we impose for fairness predicates is slightly stronger still, we will check whether they are satisfiable within the core—ideally for any game, possibly for a class of games.

4.1. Definition. A predicate P is **satisfiable within the core** in a class $\mathcal{G}$ of games if for each game $(N, v) \in \mathcal{G}$ the implication $C(v) \neq \emptyset \Longrightarrow P(v) \cap C(v) \neq \emptyset$ holds.

It is possible to define satisfiability within other rationality constraints than the core, but we will for now stick to the core rationale and hence mean *satisfiable within the core*, whenever we say *satisfiable*.

One can argue that it is, for a player, **rational to demand a *fair* share** with respect to some satisfiable fairness predicate, because *he knows he can*. If the players can, ex ante, agree on a notion of fairness that is satisfiable in all games that they are about to play, they can agree to choose fair imputations even before the game rules and their roles in the game are known to them, that is while they are still equals. Once the game rules become apparent, they might be expected to stick to their fairness notion, ex post, since it produces core imputations and there is no sub-coalition that can gain from leaving the grand coalition under these circumstances.
This benchmark for fairness notions is what the author had in mind, beginning this part of the thesis with the words:

I want a fair share.
because I know I can get one.
and so can you.

Using Lemma 3.4 and Lemma 3.5 and re-invoking the Theorems proved throughout the thesis, we establish our satisfiability results:

4.2. Theorem.

(i) *$F_{\geq}$ is satisfiable for all $(N, v) \in \mathbf{G}^n$. That is to say: For every game with nonempty core, there exists (at least) one core element that reflects the desirability of players.*

(ii) *$F^0_{\geq}$ is satisfiable within the core of every game $(N, v) \in \mathbf{G}^n$. If the core of $v = v_0 + s$ is non-empty, then the imputations $C_e(v_0) + s$ are elements of $F^0_{\geq}(v)$.*

(iii) *$F_{\trianglerighteq}$ and $F^0_{\trianglerighteq}$ are satisfiable for 1-convex games, as the τ-value reflects the weak desirability relation $\trianglerighteq$ and is a core element for all $(N, v) \in \mathbf{AC}^n$.*

(iv) *$F_{\trianglerighteq}$ and $F^0_{\trianglerighteq}$ are satisfiable for convex games, if the τ-value of these games belongs to the core.*

(v) *$F_{\trianglerighteq}$ is not satisfiable for in $\mathbf{SA}^n$.*

(vi) *$F_{lu}(v)$ is satisfiable in $\mathbf{C}^n$, as every egalitarian value is labor union fair in a convex game.*

(vii) *$F_{lu}(v)$ is not satisfiable in $\mathbf{SA}^n$.*

Proof.

(i) Starting with any core element x we can construct a core element that respects desirability in a few steps by solving all the violations of fairness, for which we use the two Lemmata.

Our strategy for this is:

First of all we look at the pairs of players a and b where $a \succ b$, i.e. $a \succeq b$, but not $b \succeq a$. For these pairs we ensure that $x_a \geq x_b$ holds by sorting individual payoffs according to Lemma 3.4. This is possible and also easy to do; every computer scientist knows a whole range of algorithms which do just that.

For chains of substitute players we now use Lemma 3.5 to reach fairness. The order we created relative to other non-substitute players in the first step is not destroyed in the process, because convex combinations of several real numbers are always below the largest and above the smallest of these numbers.

This first part of the Theorem is also a direct corollary of the next, however this proof yields a new way to construct fair core imputations starting from arbitrary ones.

(ii) Theorem 3.16 establishes the results.

(iii) That the τ-value is a core element and reflects weak desirability for these games is established in Theorem 3.8, since $\tau(v) = \tau(v_0) + s$, the satisfiability of $F^0_{\trianglerighteq}$ follows as a corollary.

(iv) This is seen in Corollary 3.10. Again, since $\tau(v) = \tau(v_0) + s$, the satisfiability of $F^0_{\trianglerighteq}$ follows alongside.

(v) Consider the super-additive zero-normalized four player game of figure 20 with $v(N) = 5$, $v(N - i) = 3$ for all players i, $v(12) = 3$ and $v(A) = 0$ for all other $A \subset N$. The core contains $(1.5, 1.5, 1, 1)^t$, so it is not empty. On the other hand, $1 \trianglerighteq 2 \trianglerighteq 3 \trianglerighteq 4 \trianglerighteq 1$, so the only $F_{\trianglerighteq}$-fair imputation is the equal division $(1.25, 1.25, 1.25, 1.25)^t$, which has positive excess for the coalition consisting of the first two players: $e^v\left(\{1, 2\}, (1.25, 1.25, 1.25, 1.25)^t\right) = \frac{1}{2}$.

(vi) Theorem 3.14 establishes this result.

(vii) As we have seen in Subsection 3.1, the over employment game is a counter example.

Thus, we have shown the entire theorem. □

That $F_e(v)$ is satisfiable within the core of every game is self-evident. F^0_e is usually not. Egalitarian solutions are, in a way, opposed to the idea that the distribution of $v_0(N)$ should not depend on the trivial component s.

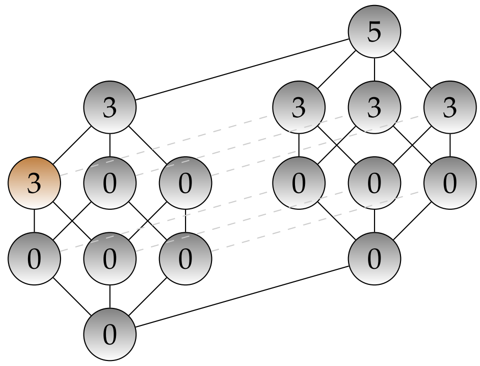

Figure 20: four player counter example for Theorem 4.2 (iv)

We note that $F_{\sqsupseteq}(v)$ is not satisfiable within the core for all games, as we have seen in the case of the over employment game in figure 17.
We also cannot hope to satisfy $F_{\trianglerighteq}(v)$ within the core, as the following convex counter-example shows.

4.3. Example. *Let $A = \{1, 2, 3\}$, $B = \{4, 5\}$ and v be the five player game that is the sum of the two unanimity games u_A and u_B. That is $v(S) = 0$, if S contains neither A nor B, $v(S) = 1$ if either A or B are contained in S and $v(N) = 2$. Being a positive sum of unanimity games, v is obviously convex.*
Since all players have the same utopia value and the game is zero-normalized, we have $1 \trianglerighteq 2 \trianglerighteq 3 \trianglerighteq 4 \trianglerighteq 5 \trianglerighteq 1$ and thus for $x \in F_{\trianglerighteq}(v)$ we necessarily have $x = \tau(v) = \frac{2}{5}(1, 1, 1, 1, 1)^t$.
But then $x(B) = \frac{4}{5} < 1 = v(B)$, thus weak desirability of players is not satisfiable for this convex game.

Part II

Fairness in Specific Classes of Games

when you have a hammer,
everything starts looking like a nail.
take time to understand the problem at hand
and proceed using the right tool for the job.

Overview of Part II

Part II of the thesis treats fairness in special classes of games that allow for stronger results, since the extension of these results is smaller. Section 5 will examine convex games, and - even more specifically - the positive cone of unanimity games that naturally arises when studying the basis of unanimity games for the cone of all convex games (not every linear combination of unanimity games is convex, every positive combination, however, is — and these satisfy some additional useful properties, especially in the context of the Shapley value).
Section 6 treats the case of bankruptcy games, where considerable prior work has been done on the topic of fair division — not only by Balinski, Young, Maschler and Aumann, but already by Aristotle and the writers of the Talmud. The classic literature on this field is summarized and related to the fairness-concepts that were introduced in the first part of the thesis.
In a rather short section 7, the games that are both convex and 1-convex are identified and studied.
Let us shift our attention now to the widely established class of convex games.

5 Convex Games and the Positive Cone of Unanimity Games

Convex games have especially strong incentives to cooperate. The core of these games is usually quite large and we can hope to find an imputation, within the stability of the core, that satisfies a whole range of desirable fairness properties at once.
Driessen shows that the core of a convex game can be easily described as the convex hull of it's **marginal worth vectors**.

5.1. Definition. Let $(N, v) \in \mathbf{G}^n$ and θ be a permutation of N and let Θ^n be the set of all such permutations.
The **marginal worth vector** $x^\theta(v) \in \mathbb{R}^n$ with respect to the ordering θ in the game v is given by

$$x_i^\theta(v) := v(P_i^\theta + i) - v(P_i^\theta), \tag{41}$$

where $P_i^\theta := \left\{ j \in N \,\middle|\, \theta(j) < \theta(i) \right\}$.

A payoff according to a marginal worth vector therefore is constructed if the players join the grand coalition in some arbitrary order and are each

paid their marginal contribution at the time that they join.
The following Theorem is given by [Drie 88, Theorem V.3.7.]:

5.2. Theorem. *The following four statements are equivalent.*

(i) (N, v) *is a convex game.*

(ii) $x^\theta(v) \in C(v)$ *for all* $\theta \in \Theta^n$.

(iii) $C(v) = \text{conv}\left\{x^\theta(v) \mid \theta \in \Theta^n\right\}$.

(iv) $\text{ext}\, C(v) = \left\{x^\theta(v) \mid \theta \in \Theta^n\right\}$.

Furthermore, we know that for convex games, labor union fairness is satisfiable and satisfied, for example, by the egalitarian solution $LS(v)$. Also, under the assumption that the τ-value is a member of the core, weak player desirability can be satisfied... It is of interest, if the intersection of $F_{lu} \cap F_{\trianglerighteq}$ is satisfiable, or if even $(F_{lu} \cap F_{\trianglerighteq} \cap F_e)$ is satisfiable in the class $\mathbf{C}^n$ of convex games.
We state, without proof, the following conjecture as an **open problem**:

5.3. Conjecture. *Let* $(N, v) \in \mathbf{C}^n$ *be a game that satisfies (24), so that the* τ*-value is a member of the core. Then the* τ*-value is labor union fair, i.e.* $\tau(v) \in F_{lu}(v)$.

If this conjecture can be proved to be correct, it will follow, as a corollary, that $F_{lu} \cap F_{\trianglerighteq}$ is satisfiable for all convex games where the τ-value is a core element.
As we have mentioned in Example 1.9, it is well known (and easy to see) that the unanimity games form a linear basis for the cone of games, and – being convex – also of the cone of convex games.
Therefore it will be helpful to have the following results in place:

5.4. Observation. *Let* (N, u_T) *be the n-player unanimity game for sub-coalition* $T \subseteq N$ *and let* $S = N - T$.
Then

(i) *the Shapley value* $\phi(u_T)$, τ*-value* $\tau(u_T)$, *egalitarian least squares solution* $LS(u_T)$ *and nucleolus* $n(u_T)$ *all agree and assign* 0 *to each player of* S *and* $\frac{1}{|T|}$ *to each player in* T.

(ii) *all players in* S *are substitutes (hence also equals w.r.t.* $\trianglerighteq$*):* $s_1 \sim s_2$ *and* $s_1 \sim_{\trianglerighteq} s_2$ *for all* $s_1, s_2 \in S$.

(iii) *all players in* T *are equally desirable, and more desirable than the players in* S*, i.e. for all* $t_1, t_2 \in T$ *and all* $s \in S$ *we have* $t_1 \sim t_2 \succeq s$ *and thus also* $t_1 \sim_{\trianglerighteq} t_2 \trianglerighteq s$.

(iv) *Two disjoint coalitions are equally desirable when both do or do not meet T and when both meet T. If one of them is contained in S and the other is not, then the other is more desirable. That is for all* A_1, A_2 *with* $T \cap A_1 \neq \emptyset$ *and* $T \cap A_2 \neq \emptyset$ *and all* $B_1, B_2 \subseteq S$ *we have* $A_1 \sim_{\sqsupseteq} A_2 \sqsupset B1 \sim_{\sqsupseteq} B_2$.

Add to these observations the fact that the Shapley value is additive, i.e. for $T_1, ..., T_m \subseteq N$ and $v = \sum_{i=1}^{m} r_i u_{T_i}$ we have $\phi(v) = \sum_{i=1}^{m} r_i \phi(u_{T_i})$.
Since we have seen in Subsection 3.1 that we cannot expect $F_{\sqsupseteq}$ to be non-empty for convex games (or even for unanimity games), we investigate the exact nature of F_{lu}.
Let, as above, $T_1, ..., T_m \subseteq N$ and $v = \sum_{i=1}^{m} r_i u_{T_i}$. Without loss of generality let all r_i be non-zero. Two players j_1 and j_2 are substitutes in this game if for each T_i, $i = 1, ..., m$ we have $j_1 \in T_i \iff j_2 \in T_i$. While it is obvious that this will make them substitutes, the condition is also necessary, because if there were coalitions T_a, T_b, ... that contain only one of the players, there would also need to be such coalitions of minimal size.
Now if T_a is a coalition with $i \in T_a$ and $j \notin T_a$ and all coalitions in $\{T_1, ..., T_m\}$ of size smaller than T_a contain either both players or none of the two, then we find that

$$v(j_1 \cup (T_a - j_1)) = v(j_2 \cup (T_a - j_1)) + \underbrace{r_a}_{\neq 0}, \tag{42}$$

thus players j_1 and j_2 cannot be substitutes if there are $T_a \in \{T_1, ..., T_m\}$ that contain exactly one of the two players.
So far we have used the convexity of the game. For the following considerations we will need a stronger assumption, namely that the game is a positive linear combination of unanimity games.

5.5. Definition. The **positive cone of unanimity games** between n players, $\mathbf{PCU}^n$ consists of all linear combinations of n-player unanimity games with only positive coefficients.

Obviously $\mathbf{PCU}^n \subseteq \mathbf{C}^n$.
So let, as above, $T_1, ..., T_m \subseteq N$ and $v = \sum_{i=1}^{m} r_i u_{T_i}$, but now all $r_i > 0$. Then if $J = [j]_{\sim}$ is a class (labor union) of substitute players and $J \sqsupseteq \{k\}$.
Now, similar to the argument above, we find that all $T_i \in \{T_1, ..., T_m\}$ that contain k also contain all of J, since if that weren't the case, then there would have to be a T_a of minimal size that contains k, but does not meet J. But then $v((T_a - k) \cup k)$ would be greater than $v((T_a - k) \cup J)$, which contradicts $J \sqsupseteq \{k\}$.
The following result follows immediately:

5.6. Theorem. *If* $(N, v) \in \mathbf{PCU}^n$ *then* $\phi(v) \in F_{lu}(v)$.

6 Bankruptcy Games

When we consider a general bankruptcy problem (E, d), the corresponding bankruptcy game v has certain properties that allow for special treatments. The most prominent is that the characteristic function v depends only on the utopia vector $b = (MC_1(N), MC_2(N), ..., MC_n(N))^t$. Also every bankruptcy game is convex, so everything that was said about convex games in section 5 applies.
Balinski and Young introduce the notion of *coherence* in the context of fair representation in democratic voting [Bali 82], [Bali 01] and Michael Balinski uses the same notion as fairness concept for bankruptcy problems [Bali 03], [Bali 04], [Bali 05].
Two ancient division rules are given by Aristotle and the Talmud.
The latter one is known as the **contested garment division rule** or the **rule of contested garment** and given in the Mishna (Baba Metzia, Babylonian Talmud):

> Two hold a garment; one claims it all, the other claims half of it. Then the former receives three quarters and the latter receives one quarter.

In another place, the Talmud gives a solution for a bankruptcy problem between three wives left as heirs by the dying husband. Aumann and Maschler have clarified the rule in [Auma 85].
Aristotle argues that the only fair division is one proportional to the sizes of the claims, i.e. the **Aristotlelian division rule** implements equal distribution of proportional losses, the solution proposed by the Talmud on the other hand implements equal distribution of absolute losses. Both these rules are coherent in the following sense.
A division rule is **coherent**, if the distribution of n units between player a and player b solely depends on the size of their claims—d_a and d_b respectively—and not on the claims of the other players.
Given a function $g(e, d_a, d_b) : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow (x, e - x) \in \mathbb{R}^2_+$ that provides a division of $e \leq d_a + d_b$ units between players a and b there is exactly one

coherent division rule $f(E, d) = x$ with $\left(x_i, x_j\right) = g\left(x_i + x_j, d_i, d_j\right)$ for each pair i, j of players.
A coherent division rule is therefore determined by a function g where the first component g_1 is increasing in the parameter d_a and the second component is increasing in d_b, g is continuous and satisfies $x = g_1(x, d_a, d_b) + g_2(x, d_a, d_b)$ for all positive pairs d_a, d_b with $d_a + d_b \geq x$.[5]
For convenience let $d_a > d_b$, the ancient rules of division mentioned above are constructed based on:

$$g_{\text{Talmud}}(x, d_a, d_b) = \begin{cases} (\frac{x}{2}, \frac{x}{2}) & \text{if } x < d_b, \\ \left(x - d_b + \frac{d_b}{2}, \frac{d_b}{2}\right) & \text{if } d_b < x < d_a - \frac{d_b}{2}, \\ \left(d_a - \frac{d_a+d_b-x}{2}, d_b - \frac{d_a+d_b-x}{2}\right) & \text{else.} \end{cases} \tag{43}$$

$$g_{\text{Aristotle}}(x, d_a, d_b) = \frac{x}{d_a + d_b}\left(d_a, d_b\right). \tag{44}$$

Figure 21 shows how the point $g(1, d_a, d_b)$ on the simplex $\mathcal{S}_1$ is determined in both methods. Note that debts greater than x behave like debts of x in the Talmudic division rule, while this is not the case for the Aristotelian division rule. The points $(d_a|d_b)$ are thus, in the Talmud procedure, first mapped to $(\max\{d_a, 1\}|\max\{d_b, 1\})$ and then from there they are projected orthogonally onto $\mathcal{S}_1$, while in the Aristotelian procedure they are projected onto the simplex along the line through $(0|0)$ and $(d_a|d_b)$.
On of the points, $(d_a|d_b)$ in figure 21 violates the imposed norm that $d_a \geq d_b$ obviously, but the figure is less cluttered this way and there are no other consequences.
The question remains, how to find f, given g. In the Proportional rule advocated by Aristotle this is trivial, for the Talmudic rule, one follows the following procedure:
First, order the claims from the smallest d_1 to the largest d_n. We have two cases:

(i) Case 1: $d(N)/2 \geq E$ (Total is less than half of the claims).

We will distribute the asset in small increments.

(a) Divide the first increment equally. Continue in a similar way until either the first claimant receives half of his/her claim $d_1/2$ or the asset runs out, whichever happens first.

(b) After this, divide each additional increment equally among the claimants 2 through n until the second claimant receives the halfway mark $d_2/2$ or the asset runs out.

[5]The assumptions that g be increasing and continuous might be relaxed in theory, but they should both be common sense.

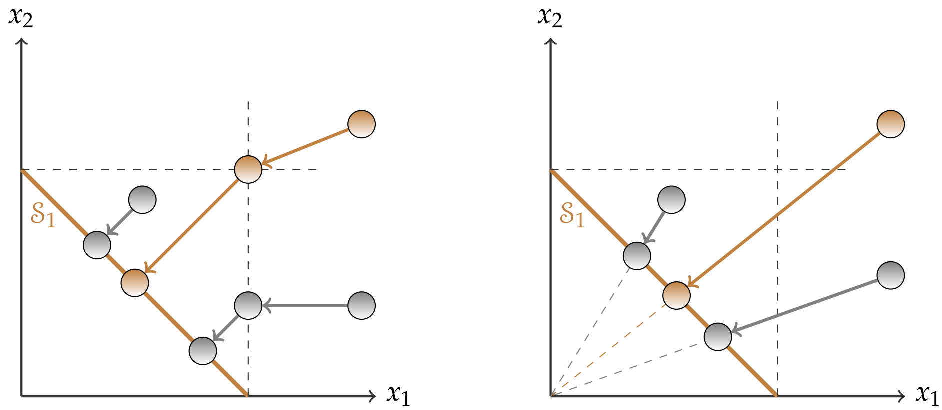

Figure 21: division according to Talmud (left) and Aristotle (right)

(c) Then divide the next increment equally among claimants 3 through n, and so on.

(ii) Case 2: $d(N)/2 < E$ (Total is more than half of the claims).

In this case we divide the total deficit (i.e. $d(N) - E$) between the agents as in Case 1.

If we want a coherent division rule to reflect player desirability, i.e. be $F_{\geq}(v)$-fair, it is necessary and sufficient that for each $d_a \geq d_b$, the underlying two player division rule g satisfies $g_1(x, d_a, d_b) \geq g_2(x, d_a, d_b)$ as well.
This immediately implies that if $d_a = d_b$, i.e. $(d_a|d_b)$ lies on the line of identity, then the point $g(x, d_a, d_b)$ needs to be on the line of identity too.
Figure 22 shows the graph of $f(x, d_a, d_b)$ for $x = 1, ..., d_a + d_b$ for both division rules. Other (usually: bi-monotonous and continuous) graphs connecting $(0|0)$ and $(d_a|d_b)$ gives rise to another coherent division rule.
The division rule will respect the desirability relation on the player set, as long as the graph does not venture into the red territory above the line of identity.
The two player division rule indicated by the red line in figure 22 follows a logic of serving the strong first – here the players who has the smaller claim gets close to nothing, unless the player with the larger claim is satisfied. We will not study this example further and thus we give no division function g for it, we just want to note that it would be $F_{\geq}$-fair, which illustrates how this fairness concept gives a lower, but no upper bound for the merits of being a player that performs very well.
Therefore, if it seems weird that the distribution of payoff should invert the order of the players' rightful claims, i.e. g should not venture into the

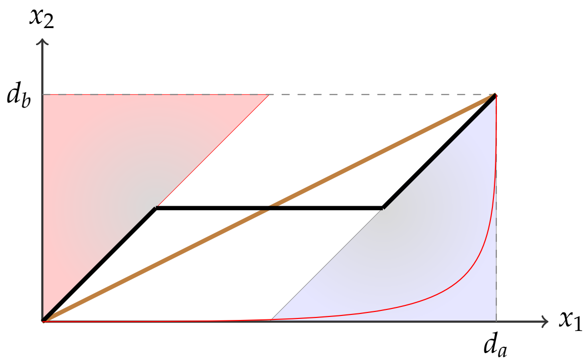

Figure 22: Talmudic (black) and Aristotelian (brown) division, payoff perverting desirability (red area), "serve-strong-first" division rule (red line)

red triangle, a symmetric argument can rule out those paths g that venture into the blue-grey triangle on the other side, since here the distribution of absolute losses does not respect the order of the players' claims, that is the weaker player has to carry the larger *absolute* loss.

6.1. Definition. We call every coherent division rule based on a two player division function g that satisfies

$$x - g(x, d_a, d_b)_2 \geq g(x, d_a, d_b)_1 \geq g(x, d_a, d_b)_2 \text{ for all } d_a \geq d_b \qquad (45)$$

a **fair coherent division rule**.

Obviously every fair coherent division rule is $F_{\geq}$-fair and, since in bankruptcy games every coalition payoff is determined by the payoffs of the coalitions of size $n-1$, every $F_{\geq}$-fair division rule is also $F_{\trianglerighteq}$-fair.
Also, the proportional division rule of Aristotle is $F_{\sqsupseteq}$-fair, since $A \sqsupseteq B$ if and only if $d(A) \geq d(B)$ and payoffs to coalitions A and B are proportional to these sums.
For three player bankruptcy games where the estate is no more than half the claims, the rule of contested garments is also $F_{\sqsupseteq}$-fair. To see this, it suffices to see that if $d_1 \leq d_2 \leq d_3$ and $d_1 + d_2 \geq d_3$, then also the rule will assign no more than 50 percent of the estate to player 3.
Since $d(N)/2 \geq E$, we know for sure that the claim of player 3 is less than E. Also, in this case, we know that no player receives more than half his claim, so we conclude.
If, on the other hand, $d(N)/2 < E$, even for 3 players we get a counterexample by taking $d = (3, 4, 6)^t$ and $E = 9$, so that the total absolute loss of 4

will be distributed evenly and player 3 will receive $4\frac{2}{3}$, which exceeds $E/2$. So we keep in mind that the rule of contested garment is not $F_{\sqsupseteq}$-fair in every bankruptcy game. We give an example with 6 players, where it is not even labor union fair:
Let $d = (2,2,2,3,3,6)^t$ and $E = 12$. Then the three first players have a total claim of 6, as o players 4 and 5, and as does the last player, thus $\{1,2,3\} \sqsupseteq \{4,5\} \sqsupseteq \{6\} \sqsupseteq \{1,2,3\}$.
However, the rule of contested garment assigns $(1,1,1,2,2,5)^t$, which is not an element of $F_{\sqsupseteq}(v_{E,d})$and is, in fact, not even labor union fair.
Let us come back to the concept of coherence.
Note that for a two player game v with $v(1) = \max(1-b, 0)$, $v(2) = \max(1-a, 0)$ and $v(12) = 1$, the Shapley value, the nucleolus and the τ-value are all identical and given by $n(v) = \tau(v) = \phi(v) = \frac{1}{2}(\max(1-b,0)+a, \max(1-a,0)+b)$, thus the Shapley value, nucleolus and the τ-value of a two player bankruptcy game both equal the Talmudic contested garment rule.
The Shapley value and the τ-value are different from the Talmudic value however in games with more players. As we have calculated in Subsection 1.3, the Shapley value of the four player example of rabbi Ibn Ezra is $\frac{1}{6}(40,55,115,175)^t$ and the τ-value is given by $(14.4, 19, 2, 28.8, 57.6)^t$. The rule of contested garment, on the other hand, assigns $(15,20,30,55)^t$, which is different from both. Since there is only one coherent rule based on g_{Talmud}[6], it immediately follows that **the Shapley value and the τ-value are not coherent**.
Driessen defines the **adjusted proportional rule** AP for bankruptcy problems in [Drie 88, Chapter VI] as the rule that one gets when first zero-normalizing the game by paying $v(i)$ to each player and dividing the remaining sum $E - \sum_{i\in N} v(i)$ according to the Aristotelian proportional division. Driessen also shows that AP coincides with the τ-value, i.e. $AP(E,d) = \tau(v_{E,d})$.
The following Theorem follows directly.

6.2. Theorem. *Let (N,v) be a zero-normalized bankruptcy game, i.e. a bankruptcy game with $g(N) \geq \max\left\{b_i \mid i \in N\right\}$, then the τ-value equals the distribution according to the Aristotelian proportional division and $\tau(v) \in F_{\sqsupseteq}(v)$, that is the τ-value respects the desirability of coalitions.*

For these games, this makes a rather strong case for the Aristotelian proportional division, if we also remember the fact that the Aristotelian division implements desirability of coalitions-fairness. Being fair in the sense of $F_{\sqsupseteq}$

[6] For proof of this fact, see again [Auma 85].

means that a set of players cannot increase their total payoff by redistributing the claims they own amongst each other! That means that there can be no tactical trade with debt obligations among the creditors. This practical consideration, paired with the fact that the division is very intuitive and agrees with the τ-value can seem very convincing.
Another argument, however, backs up the rule of contested garment. Aumann and Maschler show in [Auma 85], that the nucleolus does indeed coincide with the rule of contested garment for any game derived from a bankruptcy problem the usual way:

6.3. Proposition. *Let (E, d) be a bankruptcy problem and $b = v_{E,d}$ be the characteristic function of the corresponding cooperative n-person game. Then the Talmudic rule of contested garment yields the nucleolus $n(v)$ as the division vector.*

The most egalitarian division rule, that is the only coherent rule that produces $f(E, d)$ in the egalitarian core, is shown in figure 23 and given by

$$g(x, d_a, d_b) = \begin{cases} (x/2, x/2) & \text{if } x \leq 2d_b, \\ (x - d_b, d_b) & \text{else.} \end{cases} \quad (\textit{wlog. } d_a \geq d_b) \tag{46}$$

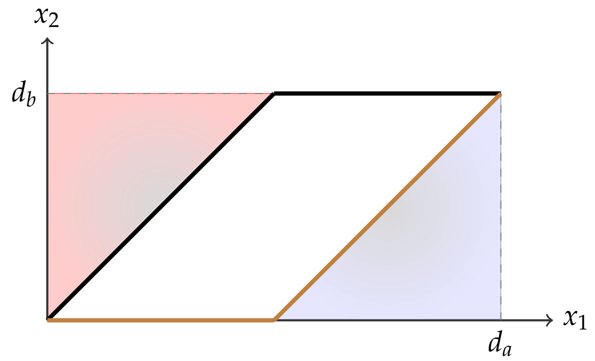

Figure 23: egalitarian division rule (black) and least squares of absolute personal losses (brown)

While it is possible to construct a coherent and $F_{\geq}$-fair solution concept from this, note that this solution always *crawls along the edge of being unfair*. It is biased towards the weaker players in comparison to both the Talmudic and the Aristotelian division rules, the τ-value, the Shapley value, etc.
Another division rule, the **division according to least squares of absolute personal losses** (see, for example [Mesn 08]) is also shown in figure 23. When the vector of absolute personal losses, i.e. $d - x$ is minimized in its Euclidean norm, the solution is biased towards the stronger players in comparison with the contested garment or proportional rule, the τ-value, the Shapley value, etc.

In this light, where the whole white area of figure 23 is viable as a path from $(0|0)$ to $(d_a|d_b)$, the concepts of the egalitarian core and inequity aversion on one side and the least squares of absolute losses solution on the other, represent the two most extreme points of view.

7 1-Convex Bankruptcy Games

A remarkable result on 1-Convex games is that the nucleolus equals the τ-value for these games [Drie 88, Drie 10], which is very helpful, since the nucleolus is, in general, hard to compute. John Nash said, in Stony Brooke 2009, that something he is excited about in the near future of game theory is that it is becoming increasingly easy to access the computational power needed to calculate the nucleolus.
But the τ-value is very easy to compute for these games:

$$n(v) = \tau(v) = b - \tfrac{g(N)}{n}\mathbf{1}^n, \tag{47}$$

where $\mathbf{1}^n$ is the vector $(1, 1, ..., 1)^t \in \mathbb{R}^n$.
For convex games we have $v(S) \geq v(N) - b(N - S)$ for all coalitions S, while for 1-convex games the inequality reads $v(S) \leq v(N) - b(N - S)$, therefore usually a game is not both 1-convex and convex.
For those that are, obviously $v(S) = v(N) - b(N - S)$, which is a term that appears in in the definition of a bankruptcy game (equation 3). To be exact, the characteristic function of a bankruptcy game is given by $v(S) = \max\{v(N) - b(N - S), 0\}$.
These terms are identical, if $v(N) - b(N - S)$ is non-negative for each non-empty $S \subseteq N$, which is to say that for each proper subset $T \subsetneq N$ the sum $b(T)$ of utopia values must be smaller than $v(N)$, which simplifies to the condition that the gap is not larger than the smallest debt, or, in other words, that if the smallest dept is given by d_1, that $v(N) \geq b(N) - d_1$.

7.1. Theorem.

(i) *Let $(E, d_1, ..., d_n)$ be a bankruptcy problem and w.l.o.g. let the debts be ordered by scale, i.e. $d_1 \leq d_2 \leq ... \leq d_n$.*

The corresponding game $v_{E,d}$ is 1-convex if and only if $E > d_1 + d_2 + ... + d_{n-1}$, that is if and only if the gap does not exceed the smallest demand.

$$E - d(N) = g(v_{E,d}) \leq d_1 \iff (N, v_{E,d}) \in \mathbf{C}_1^n. \tag{48}$$

(ii) *Every game $(N, v) \in \mathbf{C}_1^n \cap \mathbf{C}^n$ is a bankruptcy game.*

Bringing together the before-mentioned result on 1-Convex games with Proposition 6.3, we conclude

7.2. Theorem. *Let (N, v) be a 1-convex bankruptcy game, that is a bankruptcy game where $g(N) \leq \min\left\{b_i \mid i \in N\right\}$.*
Then the Talmudic rule of contested garment yields the τ-value, which equals the nucleolus and is given by $b - \frac{g(N)}{n}\mathbf{1}^n$, where $\mathbf{1}^n$ is the vector $(1, 1, ..., 1)^t \in \mathbb{R}^n$.
The same value is obtained by $f_{Aristoteles}(v_0) + s$, where $v = v_0 + s$.

Part III

Individual or Culture Specific Notions of Fairness

those are my principles,
if you don't like them: I have others.

– Groucho Marx

Overview of Part III

Part III of the thesis finally approaches the most natural thing to happen when fair division is discussed — i.e. that different players might not only have conflicting interests when distributing a value, but first of all they will have different ideas on what meaning the word *fairness* carries.
Section 8 gives the fundamentals of this discussion, introducing a model of cooperative games with players that have a certain fairness-cultural background each and developing the modified stability condition for the model.
Section 9 gives a paradigmatic 4 player example for the finding, that imputations that do not belong to the core might be stable due to differences in players' fairness cultures. The opposite case, that players conflicting fairness cultures might prevent cooperation, is obvious and left to the reader.
Section 10 finally gives some ideas on truthful vs. strategic reporting of a player's own culture and closes the thesis.

8 Modified Stability Condition

The idea behind the characteristic function v of a game is that every coalition $A \subseteq N$ can generate its value $v(A)$ by playing a joint strategy, *no matter what players elsewhere do*, at least where v is generated from a game in normal form via the α-characteristic function. It is implicitly assumed, that all players of the complementary set $N - A$ play in a way that minimizes the payoff to the coalition A, when the game is not a constant-sum game.
This problem is taken up explicitly by von Neumann and Morgenstern in their monumental work *Theory of Games and Economic Behavior*, where characteristic functions and cooperative games are first introduced and studied [Neum 44]:

> The desire of the [complementary] coalition $-S$ to harm its opponent, the coalition S, is by no means obvious. Indeed, the natural wish of the coalition $-S$ should not be so much to decrease the expectation value ... of the coalition S as to increase its own expectation value.
>
> ...there may exist an opportunity for genuine increase of productivity, simultaneously in all sectors of society.
>
> ...Indeed, this is more than a mere possibility—the situations to which it refers constitute one of the major subjects with which it refers constitute one of the major subjects with which economic and social theory must deal.
>
> ...In spite of all this, the reader may feel that we have overemphasized the role of threats, compensations, etc., and

> that this may be a one-sidedness of our approach which is likely to vitiate the results in applications. The best answer to this is ... the examination of those applications.

Debraj Ray comments this in [Ray 07] in relation to farsightedness:

> A "deviant group" must be aware that they are potentially vulnerable to *further* deviations by members of their own group.
>
> ...Parallel considerations apply to the notion of blocking in cooperative games: might a block not be subjected to further blocks? (See, e.g. [Auma 64b], [Ray 89], [Dutt 89b], [Dutt 91], [B Du 89], [Mas 89], [Gree 90].)

Considering players that have individual or culture specific notions of fairness, we find it sensible to elaborate on just that: Do cultural differences block a certain imputation, do they (by blocking all viable imputations) prevent cooperation or do certain imputations become viable because coalitions blocking them from a core point of view are not realistic, because they are made of players that could not agree on a fair way to split *their* profits.

8.1. Definition. Let N be the set of players and F_i be a predicate for $i = 1, ..., n$. We understand $F_i(w)$ as the imputations that player i finds acceptable in any game w, any imputation outside of $F_i(w)$ would be regarded as unacceptable by player i so that he would prefer to stop cooperating.
We call the family F_i of predicates the **cultural identification of players**. Every singleton set of players is called **culturally compatible** in any game. A larger set A of players is called **culturally compatible** in a game (N, v) (where $A \subseteq N$), if there is a payoff vector $x \in \mathbb{R}^{|A|}$ that satisfies

(i) $x(A) = v_A(A)$ and

(ii) $x(S) \geq v(S)$ for every culturally compatible subcoalition $S \subseteq A$,

i.e. when x is a payoff vector to A for which the core-condition holds *for culturally compatible coalitions*.

If a set of players is not culturally compatible in a game, they are said to be culturally incompatible.

9 Paradigmatic Example

Let us look at an example. Take four players $N = \{1, 2, 3, 4\}$ and let their cultural identification be given by $F_1 = F_2 = F^0_{\geq}$, $F_3 = F_e$ and $F_4(w) = \phi(w)$.
Players one and two are of a cultural point of view that makes it unacceptable for them that a player who is at least as desirable as another player should receive less. Also they understand how to zero-normalize a game and will not accept distributions that they conceive to be unfair when splitting the game into the zero-normalized and trivial components.
They are mutually culturally compatible in every two player (sub-)game with non-empty core, which follows directly from Theorem 3.16.
Player three is an egalitarian, while player four is, understandably, inspired by the merit of the Shapley value, but arrives at the doubtful conviction that it is the only sensible and fair way to split in any game and is not willing to cooperate on any other basis.
Players $1, 2, 4$ form a culturally compatible set in any zero-normalized three player game v where $\phi(v)$ is a core element (and in most non zero-normalized examples considered so far as well).
Players $1, 2, 3$ form a culturally compatible set in any zero-normalized three player game and players 3 and 4 will be incompatible in most cases.
Consider the 4-player game in figure 24.

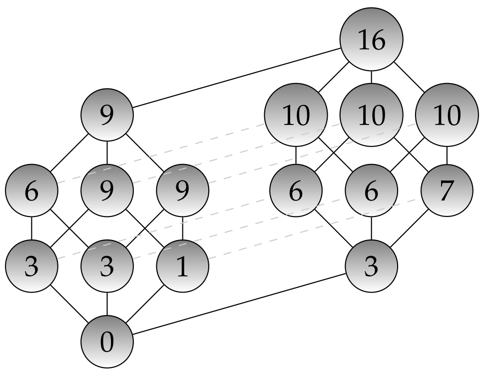

Figure 24: convex four player game

Note that in the sub-game of players $1, 2$ and 3 (as depicted by the left cube) has an empty core. The blocking coalitions $1, 3$ and $2, 3$ however are culturally incompatible, since the only split of $v(13) = 9$ or $v(2, 3) = 9$ between

these players that is acceptable to player 3 would be $(4.5, 4.5)^t$, which is not acceptable to player 1 (or player 2), who considers their two player sub-game to consist of the trivial component $(3, 1)^t$ and a zero-normalized two player game that can create an additional value of 5, so the only acceptable split to player 1 (or player 2) would be $(5.5, 3.5)^t$.
The equal split of $v(123) = 9$ among these three players therefore satisfies the core inequality for every culturally compatible sub-coalition of $\{1, 2, 3\}$. It is obviously egalitarian, satisfying player 3's sense of fairness, and it is an element of $F_{\geq}(w)$, where w is the corresponding 3-player sub-game. If we split $w = w_0 + s = w_0 + (3, 1, 1)^t$, we have (in w_0) $3 \geq 2 \sim 1$ and $(3, 3, 3)^t - (3, 3, 1)^t = (0, 0, 2)^t$ reflects this ordering. Thus the distribution vector $(3, 3, 3)^t$ is acceptable to all three players.
We see that, paradoxically, it is the cultural incompatibility of pairs of players *in their subset* that makes the three players a culturally compatible in the three player (sub-)game.
The values in the right cube have been chosen in such a way that $\phi(v) = (4, 4, 4, 4)^t$, so the Shapley value, which is the only value player 4 would accept is also the egalitarian solution that player 3.
It is, again, a member of $F_{\geq}$ and splits to $(3, 3, 1, 3)^t + (1, 1, 3, 1)^t$. Since player 3 is more desirable than all other players in v_0, the Shapley value is also a member of $F^0_{\geq}$ and therefore a culturally compatible solution of the game.
Again, the coalitions $\{1, 2\}$ and $\{1, 3\}$ would be blocking coalitions, were it not for their cultural incompatibility.
Note that, as mentioned above, if the example were not engineered in such a way, usually players 3 and 4 would not be culturally compatible.

10 Truthful Reporting of own Culture

Let N be a set of players that are about to enter a cooperative game situation, although they do not yet have any information on the game that they are about to play. Suppose that they can write and enforce contracts only after the game itself has been revealed.
Before that happens, imagine (although the first of these assumptions might seem pretty academic)

(i) that each player has an explicit knowledge of his own fairness notions F_i and is able to express it as a function of v in reasonable time,

(ii) the players are serious about their F_i, in the sense that they find any imputation inside their F_i acceptable and any imputation outside of it not acceptable.

(iii) The players are risk averse and have no beliefs about the game that they are about to play.

(iv) in the game to follow, a culturally stable imputation from $\bigcap_{i \in N} F_i$ is chosen by some authority and $v(N)$ is distributed accordingly. If the players turn out to be culturally incompatible in the stage game, no cooperation occurs and the players receive their individual pay-offs.

In this setting, simultaneous reporting of the personal fairness cultures is strategy-proof.

10.1. Theorem. *Under the above assumptions, no player k will have any incentives to report an F_k that is not his own.*

Proof. Firstly, no player will have any benefit from including in his reported $F_k(v)$ of any game v any imputations x that are outside her true $F_k(v)$, since the game v might be played and the imputation x might be chosen, rendering player i unhappy about the outcome that is not according to her personal fairness culture.
Secondly, not including imputation $x \in F_k(v)$ into the reported $F_k(v)$ might render the set of players culturally incompatible, since x might be the only element of $\bigcap_{i \in N} F_i$, thus the untruthful reporting might prevent a worthwhile cooperation that would have been beneficial to player k. Due to her risk aversion and lack of beliefs about the stage game, or the other player's fairness cultures, it makes no sense for player k to exclude x in the $F_k(v)$ that she reports. □

On the other hand, if cultures are reported sequentially, strategic manipulation is obviously an issue. Should the last player to report his culture observe that the intersection of all players fairness cultures in a given game v is a set of multiple elements, he should then chose the element he likes best and report this as his singleton culturally acceptable set of the game.
On the other hand, if a player has beliefs about the other players' notions of fairness, he might use these to strategically manipulate his own reporting of fairness culture, should he be able to announce his fairness notions before them and with them listening. If he is of the conviction, for instance, that all players will always find both the Shapley value and the nucleolus of a certain game acceptable, then he might announce his own fairness notion in a way that rules out one of the both for that game, ideally the one that would assign the smaller payoff to him, personally.
Also, players might benefit from the public appearance that they are culturally compatible with smalls sets other players, where they are really not, since these sub-coalitions would then be considered to be potentially blocking sub-coalitions that must be satisfied by increased pay-off.
I close this thesis by giving, as an **open problem**, to discuss the effects of players that are endowed with a cultural identification that is totally or

partially known on the bargaining processes that might occur between far-sighted players, given the usual range of proposal-answer protocols and in the two settings of (i) permanent contracts and (ii) temporary written agreements.

Appendix

Acknowledgment

First of all, I sincerely thank my PhD mentor Prof. Dr. Werner Krabs for his constant support and because he always encouraged me to address those questions that I find most interesting and challenging personally.
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Erklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Zuhilfenahme der angegebenen Hilfsmittel und Quellen angefertigt habe.

Darmstadt, den 25. Mai 2010

Artus Ph. Rosenbusch

Artus Ph. Rosenbusch

Artus Ph. Rosenbusch was born on Dec 25 1980, in Frankfurt am Main, Germany. He grew up in Dietzenbach as the youngest of four siblings and quickly developed interests in playing and writing music, latin dancing, and oddly—mathematics. Bed-time stories his older brother told him would feature countably many protagonists trying to illustrate the fact that there are really no more integers than there are positive integers.
Artus went to Adolf Reichwein Gymnasium in Heusenstamm, where he picked Mathematics and English as majors and received his Abitur in 1999, being honored for best grades of the year. During this time he started working the piano in bars and on different social events.
The next 8 years were devoted to more music, dance, and the study mathematics and computer science at TU Darmstadt, Germany, and Universidad de Salamanca, Spain, earning his Diplom graded *mit Auszeichnung* (with distinction). His Diplom thesis is titled *Eine direkte Definition der predicate transformer Semantik für nichtdeterministisch-probabilistische Programme*.
During this time he also completed his civil service with Arbeiterwohlfahrt Dietzenbach, which he chose over military service, the alternative for young men in Germany at that time. He also gained exposure to first working experience as an IT developer, as a consultant in the financial services sector, and teaching Linear Algebra, Algebra, Logic and Topology at university.
After some more work in the financial services sector and European Business School (ebs) Finanzakademie, he sought funding from the Foundation of German Businesses (SDW) for his PhD studies, and was, in 2007, granted a full scholarship—in recognition of his track record in academia, in pro bono work and social commitment over the past decade.
In July 2010 Artus joined Booz & Company as an Associate, and has been supporting the senior agenda of clients in the Energy & Utilities, Financial Services and Media & IT sector in Germany, the USA and the Netherlands since. As of today, he enjoys the dynamics of the profession and is looking forward to the next experience.