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Pentru Mak & Tak, cu dragoste

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Abstract

Economic agents are not fully rational machines, but humans with limited capacities, feelings, and subjective perceptions and beliefs. Such less rational aspects of behavior can become extremely important, especially in financial markets. This dissertation aims at quantifying different behavioral aspects of financial decision making.

The first behavioral aspect addressed relates to the implementation of practical trading rules. These rules are mostly simplifying and provide only imperfectly accurate information. Nevertheless, this information may induce sufficient asymmetry among market participants in order to affect prices. This holds in a market where imperfectly informed users of practical trading rules meet perfectly informed traders and uninformed liquidity traders. Both the accuracy of practical trading rules and the number of their users can change the trade conditions, primarily the gap between the two prices set for buying and selling the same asset. Particular trading rules of wide practical use appear to be successful in terms of monetary profits.

Affective states, in particular emotions, represent the second behavioral aspect analyzed. They impact on traders' beliefs and actions and thus are transmitted into prices. We design a particular agent category, the emotional traders, who exclusively follow their affect and intuition in both thinking and acting. They face rational traders, who form beliefs in a traditional way and maximize the expected utility of wealth, as well as randomly acting noise traders. The presence of emotional traders clearly influences market prices. An appropriate rational strategy requires the adaption to the consequent market conditions. Emotional trade does not necessarily impede market stability and efficiency. Certain emotional profiles are apt to survive, and can even outperform, in terms of profits, rational strategies.

Third, we study subjective perceptions of financial risks. These perceptions change the attitude of non-professional investors towards financial investments and the potential losses they generate, as well as their decisions on wealth allocation. Subjective perceptions depend on different behavioral parameters. The past performance of risky investments and the revision frequency of risky portfolios impact investors' perceptions and decisions. Substantial differences result depending on whether or not consumption is regarded as a utility generator, besides financial wealth.

Zusammenfassung

WIRTSCHAFTSAGENTEN sind keine vollständig rationalen Maschinen, sondern Menschen mit begrenzter Leistungsfähigkeit, Gefühlen sowie subjektiven Wahrnehmungen und Meinungen. Solche weniger rationalen Verhaltensaspekte können überaus wichtig werden, vor allem in Finanzmärkten. Diese Dissertation nimmt sich vor, verschiedene behavioristische Aspekte von Finanzmarktentscheidungen zu quantifizieren.

Der zuerst berücksichtigte behavioristische Aspekt bezieht sich auf den Einsatz von praktischen Handelsregeln. Diese Regeln sind meistens vereinfachend und liefern nur unvollständig genaue Informationen. Nichtsdestotrotz können diese Informationen eine ausreichende Asymmetrie unter den Marktteilnehmern erzeugen, um die Preise zu beeinflussen. Dies gilt für einen Markt, in dem unvollständig informierte Anwender praktischer Entscheidungsregeln auf vollständig informierte Händler und zufällig handelnde, uninformierte Agenten treffen. Die Handelsbedingungen, insbesondere die Differenz zwischen den zwei Preisen, die für das Kaufen und Verkaufen desselben Wertpapiers festgesetzt werden, können sich sowohl durch die Genauigkeit praktischer Handelsregeln, als auch durch die Anzahl ihrer Anwender verändern. Spezielle Handelsregeln von weitreichendem praktischem Gebrauch scheinen finanziell erfolgreich zu sein.

Affektive Zustände, insbesondere Emotionen, stellen den zweiten analysierten behavioristische Aspekt dar. Sie wirken sich auf die Meinungen und Aktionen von Finanzagenten aus und werden dadurch auf Preise übertragen. Wir gestalten eine spezielle Agentenkategorie, die emotionalen Händler, welche im Denken und Handeln ausschließlich ihrem Affekt und ihrer Intuition folgen. Sie stehen rationalen Händlern gegenüber, welche sich ihre Meinung auf traditionelle Weise bilden und ihren erwarteten Vermögensnutzen maximieren, sowie auch zufällig handelnden Noise Traders. Die Anwesenheit von emotionalen Händlern hat definitiv einen Einfluss auf die Marktpreise. Eine geeignete rationale Strategie erfordert die Anpassung an die daraus resultierenden Marktbedingungen. Emotionaler Handel behindert die Marktstabilität und -effizienz nicht unbedingt. Bestimmte emotionale Typen erweisen sich als überlebensfähig und können sogar rationale Strategien in finanzieller Sicht übertreffen.

Drittens untersuchen wir die subjektiven Wahrnehmungen von Finanzrisiken. Diese Wahrnehmungen verändern die Einstellung nicht-professioneller Investoren gegenüber Finanzanlagen und den damit verbundenen Verlusten sowie auch ihre Entscheidungen über ihre Vermögensverteilung. Subjektive Wahrnehmungen sind von verschiedenen behavioristischen Parametern abhängig. Die vergangene Wertentwicklung der risiko-betroffenen Anlagen sowie die Überprüfungshäufigkeit ihrer finanziellen Wertentwicklung wirken sich auf die Wahrnehmung und die Entscheidungen der Investoren aus. Wesentliche Unterschiede ergeben sich je nachdem, ob Konsum neben finanziellem Vermögen als Nutzengenerator betrachtet wird.

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Abbreviations

ADF	augmented Dickey-Fuller test
AMEX	American Stock Exchange
ARCH	autoregressive conditional heteroscedasticity
ARMA	autoregressive-moving average
CARA	constant absolute risk aversion
CBOT	Chicago Board of Trade
cdf	cumulative distribution function
CPT	cumulative prospect theory
CRRA	constant relative risk aversion
CVaR	conditional Value-at-Risk
ES	expected shortfall
EUT	expected utility theory
GARCH	generalized autoregressive conditional heteroscedasticity
gRA	global first-order risk aversion
i.i.d.	independent and identically distributed
JB	Jarque-Bera test
LEL	limited expected losses
LSE	London Stock Exchange
mLA	myopic loss aversion
NASDAQ	National Association of Securities Dealers Automated Quotations
NYSE	New York Stock Exchange
pdf	probability distribution function
PT	prospect theory
R-FORA	non-expected recursive utility with first-order risk aversion
S&P 500	Standard & Poors 500-stock index
TCE	tail conditional expectation
VaR	Value-at-Risk
VaR*	individual loss level (individual Value-at-Risk)
VaR^{ex}	exogenous Value-at-Risk
WCE	worst conditional expectation

Introductive word

In spite of the theoreticians' belief that everything can be pictured by neat, easily tractable models, the real world proves, on each occasion, to be substantially more complex and hence less predictable than their equations. Being part of this reality (even the most evolved one) human beings are highly intricate constructs. Until now, we – theoreticians inclusively – have not been able to fully understand how human minds function, either to explain and predict the outcome of our minds, i.e. human decisions.

However, one thing can be easily told by simple observation: Human decisions fail exasperatingly often to follow the spotless logic of rationality, as imagined and put on paper by economists. This occurs especially in complex environments, for instance in very dynamic, uncertain, or informationally dense ones. Financial markets are obviously a good example of such environments. It is no wonder that the evolution of financial markets remains a puzzle even to highly trained experts, since the tools by which we attempt to describe and predict decisions of financial agents simplify their behavior to such an extent that they may fail to account for essential aspects.

The most frequent fallacy of (classic) economic models is to assume that all economic agents – and especially traders and investors – are rational machines, able to process the tremendous quantity of information available nowadays to everyone at almost no cost, to compute and weigh out all possible risks, and to make decisions in the blink of an eye. What such assumptions rule out is exactly the "humanity" of human decisions; in other words, our own inability to form one-to-one mental mirrors of the real world, which carry within the entire information surrounding us, and to further process all this information at maximal speed. At the same time, the assumption of rationality also omits the human gift to cope with such "humanity-due deficiencies", for instance by retaining only some, but relevant elements of information or by finding less exact, but faster and sufficiently good solutions to complex problems; Omitted is also the fact that some aspects of the human personality such as affective reactions might be helpful in making use of this gift and making "right" decisions. In more technical terms, economic models mostly neglect what we can call *behavioral aspects* of human decision making. Taken into account, such aspects could improve our theoretical representation and hence the understanding of real environments, in particular of financial markets.

In this context, the present work attempts to capture better the way in which financial decisions appear to be often made in practice: neither fully nor always rationally. The adopted perspective is an economic one, aimed at capturing in measurable (and thus predictable) structures, different behavioral aspects, such as heuristics, emotions, subjective perceptions, etc. It is interesting to note that other sciences, such as psychology and neurobiology, have recognized for a long time already not only the existence, but also the potentially positive effect of behavioral aspects. However, economists have only lately become aware of these results and considered integrating them in their work.

The following chapters enlarge and combine existent settings of financial and behavioral economics in order to account for important, and possibly additional, behavioral elements. Moreover, they also propose new measures and even develop frameworks that are – at least in part – new, for quantifying such elements. Our contribution is first theoretical. Due to the lack of appropriate data sets, which is a frequent problem in the research of individual behaviors, we are not able to perform direct empirical estimations of behavioral variables. Yet, we underpin and extend our theoretical settings by means of numerical simulations that account for various constellations of behavioral parameters that can be considered to be plausible in practice.

Each chapter of this dissertation places the accent on a certain facet of financial decision making and adopts a particular perspective. For this reason, we chose to organize every chapter as a work on its own. At the beginning of each chapter, we introduce the specific problem, the studied constructs, and the applied tools, in the form of reviews of the main theoretical and empirical results obtained so far in the respective field of research. Our own contribution is subsequently detailed. It mostly consists of a theoretical part followed by an applicative one which is aimed at exemplifying and testing the theoretical findings.

The first chapter analyzes the role of imperfect information in financial markets and, as a particularization, of simple trading rules of wide practical use. Such rules are examples of heuristics. We are interested in observing how they affect prices and what are the chances for their users to make profits and survive in the market. We propose an extension of classic, information-based settings of market microstructure, where all traders remain fully rational, but the accuracy of their information is better nuanced. Specifically, we introduce a new category of traders who dispose of imperfect information and face perfectly informed and uninformed agents. Practical decision rules stand for a potential source of imperfect information. The emphasis is on the impact of imperfectly informed traders on the price formation and evolution, which results from the information asymmetry generated by their presence in the market. This information asymmetry becomes manifest through the bid-ask spread set by the market maker between the buy and the sell price of the same asset. The spread is always positive and becomes larger when the imperfectly informed trade intensifies, either due to an increased accuracy or due to a wider use of imperfect information.

An interesting fact suggested by the applicative part of this first setting is that simple trading rules are not necessarily arbitraged out by full information. This holds, among others, under the assumption that all traders behave rationally, in the sense that they form beliefs following the Bayes rule. One could then ask what happens if some traders not only apply simplifying rules, but also think in a less rational manner. The second chapter goes a step further in the behavioral field, in the attempt to investigate on the substrate of human (and possibly not fully rational) behavior and the mechanisms of its transmission into actions. In particular, we concentrate on the influence of emotions on financial decision making. We advance a theoretical model of thinking processes that accounts for the difference between reason and emotion, and further develop on belief formation, and on how these beliefs shape actions and flow into market prices. Again, we work with three trader categories, which are this time designed to comply with the assumptions of rational, emotional, and random behavior. Accounting for different possible manifestations of emotions, we observe that certain emotionally driven agents have the chance to survive in financial markets, and even to outperform their rational peers. Markets are not necessarily destabilized by the presence of emotional traders. This is fostered by the fact that the best rational strategy appears to be the adaption to the market conditions generated by emotional traders.

The third and final chapter changes the focus from pure trading aspects to investment decisions. Now we consider non-professional investors, in particular their attitudes towards financial risks and the resulting decisions on wealth allocation. The main behavioral aspect analyzed here consists of subjective perceptions and loss attitudes. We combine established models stemming from different fields, such as financial models of capital allocation and behavioral models of perception, and introduce new variables that describe loss attitudes. Our non-professional investors are first assumed to be exclusively concerned with financial investments and hence to aim at splitting their wealth between risky and risk-free assets. In a second step, they derive utility from both consumption and financial wealth and allocate wealth between consumption and financial assets in total. The past performance of risky assets and the evaluation frequency of risky performance appear to play a determinant role in wealth allocation. With investment-based utility, we are able to make a recommendation for an “optimal” frequency at which risky portfolios should be revised in order to maximize risky holdings. When consumption is accounted for, loss attitudes and wealth allocation depend on the theoretical framework used, specifically if investors maximize expected or non-expected utility. Our loss attitude measure appears to be appropriate for describing real behaviors.

CHAPTER 1

Imperfect Information, Practical Trading Rules, and Asset Prices

"The more perfect a thing is, the more susceptible to good and bad treatment it is."
DANTE ALIGHIERI.

THIS chapter studies the effect of imperfect information, such as that derived from practical trading rules, on market prices. This effect mainly originates in the information asymmetry generated by imperfect information. We commence by reviewing the most important theoretical and empirical results of market microstructure settings that deal with the bid-ask spread. The spread represents the gap between the two distinct prices at which an asset can be simultaneously bought or sold. One main determinant of the spread is, besides inventory and order-processing costs, the information asymmetry among market participants. Numerous settings develop on information asymmetries and the adverse selection problem they induce.

Our contribution consists of modeling a sequential trading environment where the information asymmetry is better nuanced than in previous models. In particular, we add to the commonly considered categories of (perfectly) informed and uninformed traders, a third one: imperfectly informed traders. Imperfect information can stem from practical trading rules, such as technical or fundamental analysis, and we explicitly model its accuracy. The impact of imperfect information on prices is twofold: first qualitative, resulting from the precision of this information, and second quantitative, given by the proportion of partially informed trades to the totality of trades. We theoretically show that the more intense this impact is, either in the qualitative or in the quantitative sense, the more pronounced is the information asymmetry and thus the higher is the spread set by the market maker. These conclusions are graphically exemplified for different market configurations. Moreover, we study the price evolution when simple trading strategies of chartist and fundamentalist origins are applied for deriving imperfect information. Some of these strategies can, under certain market conditions, entail not only positive, but also highest profits in the market.

1.1 Theoretical overview

We commence this chapter by a theoretical overview of the literature on market microstructure, aimed at introducing our own setting from Section 1.2 and facilitating its understanding. The emphasis is on the bid-ask spread and its determinants, among which we pay special attention to information asymmetries.

1.1.1 Market microstructure

Market microstructure studies the process of price formation in financial markets. Particular attention is given to how investor beliefs flow into prices[1], to the contribution of specific trading mechanisms[2] to the price formation, and to the price evolution in time[3]. Explaining how exchanging assets takes place and affects prices, market microstructure gives thus an insight within the "black box" of financial markets. It therefore enriches the traditional economic approaches that concentrate on the existence of a market equilibrium without paying attention to how this equilibrium is reached.

Madhavan (2000) distinguishes among four main streams of research in market microstructure: price formation and discovery, market structure and design[4], disclosure of information[5], and applications to other areas of finance[6]. In this section, we are particularly concerned with the first area, the *price formation and discovery*, and the role of information asymmetries. According to the same author, two dimensions of the price formation process have been considered in the literature so far: a *static* one (i.e. the trading costs) and a *dynamic* one (how information is impounded into prices).

A first intuition for the existence of *trading costs* is advanced in Demsetz (1968). Accordingly, they can occur in consequence of the time dimension of the trading process,

[1] The price formation is often – and especially in earlier works – modeled in the spirit of the classic economic theory: Market participants form rational beliefs (expectations); These beliefs shape demands and supplies; The aggregation of traders' demands and supplies entails equilibrium prices, where numerous papers consider that this process is conducted by a Walrasian auctioneer. Thus, markets clear at the equilibrium price.

[2] According to O'Hara (1995), trading mechanisms include: the market players (e.g. customers, brokers, dealers, market makers, etc.; see Harris (2003) for a practice-oriented description of their function), the trading location (physical or virtual), and the trading rules (that differ substantially among exchanges).

[3] And thus to how prices reflect information.

[4] The focus is here on liquidity and market quality, more specifically on: market type (such as continuous or single-auction, auction or order-driven, floor or automatic), order types (market, limit, stop orders, etc.), protocols (tick size, trading halts, circuit breakers, etc.), etc.

[5] In other words transparency (specifically, pre- and post-trade, anonymity, voluntary disclosure, etc.).

[6] Such as corporate finance, asset pricing, international finance, etc.

as traders who are willing to transact as soon as possible are also disposed to pay a price for this trade immediacy.[7]

Two manifestations of trading costs are the bid-ask *spread* and the market impact of the trade, in particular how the market *liquidity* is affected.[8] The dynamic aspect of information incorporation into prices is related to the question of *market efficiency*.

We revise the most important theoretical and empirical results concerning the spread, as our model in Section 1.2 elaborates further on this issue. Conclusions related to the liquidity problem and to the information incorporation into prices are underlined throughout the exposition.

On real markets, there are often not one but two different prices for buying and selling the same asset: the *ask* and the *bid*, respectively. The positive gap between them is denoted as **spread**. The spread formation has received various theoretical support so far. Three main determinants of the spread have been isolated: the *order processing costs*, the *inventory costs*, and the *adverse information costs*. We subsequently review some results with respect to the two latter categories. The role of the order processing costs is detailed in Roll (1984).[9] Our interest lies in the adverse information costs, that constitute a basic argument of our model in Section 1.2. All presented issues are to be understood in the context of markets with *market makers* as price-fixing instances.[10]

Inventory models

Inventory models represent one of the first microstructural attempts to explain the bid-ask spread. This is considered to be determined by inventory risks, i.e. by uncertainties that occur due to random fluctuations of the asset inventories accumulated by market makers in consequence of the accepted trades. As such fluctuations are not permanent[11],

[7] Note that Demsetz (1968) views market makers as passive providers of immediacy.

[8] According to Lee, Mucklow, and Ready (1993), the market liquidity has two dimensions: the spread and the depth, where the latter is defined as the number of shares available at each quoted price.

[9] Roll (1984) is also the first to suggest a measure of the effective bid-ask spread, under the assumptions of efficient markets and stationary return distributions. This measure equals twice the squared root of the serial covariance of price changes taken with negative sign. Using this measure, the average percentage bid-ask spread across all stocks listed at NYSE and AMEX between 1963-1982 is estimated at 0.298% (1.74%) from daily (weekly) returns.

[10] In general, market makers are those who accommodate trading needs of other market participants. They are responsible for the price discovery and stabilization, as well as for ensuring the fairness, liquidity, and continuity of the market. See Madhavan (2000). Under the denomination of market makers we include the NYSE specialists, the NASDAQ and LSE dealers, the Tokyo saitori, etc., where a sharper distinction among these different categories is to be found in Harris (2003).

[11] Specifically, there is an inventory mean reversion: Market makers tend to set quotes in order to bring inventories to their preferred positions. See Biais, Glosten, and Spatt (2002) for a concentrated explanation.

inventory risks can manifest only in the short-run. In the long-run, prices should revert to their "true" value[12], so that markets become efficient. Consequently, prices exhibit negative serial correlation.[13] Note that in this context, the role of the market makers is to balance supply and demand *across time*, by means of the inventory.[14]

The first paper that comments on the behavior of market makers (specialists) at the NYSE suggesting the existence of preferred inventory positions is due to Smidt (1971). Departing from his ideas, a first group of inventory models analyzes the **nature of the order flow**. An important assumption of these models is the risk neutrality of the market maker. They advance different interpretations of the spread: In Garman (1976), the spread represents the defence mechanism of the market maker against failure (i.e. running out of cash).[15] In the extension by Amihud and Mendelson (1980), the spread becomes a consequence of the market power of the market maker and serves for reaching a preferred inventory position.[16]

A second group of inventory models concentrates on the **optimization problem of the market maker**, which is this time considered to be risk averse.[17] The spread forms as a consequence of this risk aversion. Specifically, in the setting of Stoll (1978) the spread reflects the costs of the exposure to different types of risk (such as holding costs, order costs, and information costs).[18] The multiperiod extension by Ho and Stoll (1981) considers additional transactions uncertainty and shows that the spread can be

[12] That corresponds to the balanced order flows.

[13] Note that order processing costs also yield, ceteris paribus, to negative serial correlation in returns, as demonstrated in Roll (1984).

[14] As underlined in Madhavan (2000). In the terms of Demsetz (1968), the market maker provides for immediacy of execution. However, market makers in inventory models are more than passive suppliers of immediacy, but active participants in price setting.

[15] Specifically, the market maker is obliged to ensure trading continuity when market buy and sell orders arrive stochastically. She does this by carrying stock inventories. Each inventory-independent strategy leads to failure, so that the market maker is obliged to relate prices to inventories in order to avoid failure.

[16] Specifically, the preferred inventory is the level at which the market maker's profits per unit of time are maximal. It is independent of the true asset value, the ask and bid prices decrease monotonically in the inventory, and the spread is always positive. The optimal pricing policy is consistent with the efficient market hypothesis in the sense that no (uninformed) trader can make profits by speculating in the market.

[17] Specifically, the market maker attempts to maximize the risk-return profile of the portfolio that she is forced to hold as a liquidity provider. This yields minimal inventory costs. The main function of prices is to adjust the inventory to the desired position, i.e. the one that maximizes her expected wealth utility.

[18] In particular, holding costs depend on transaction value, return variance, dealer wealth and risk-aversion, etc. Under the mere consideration of holding costs, the inventory itself does not affect the size, but only the placement of the spread. Similar conclusions hold in Ho and Stoll (1981), where the spread remains independent of the inventory level, but varies with the market maker's time horizon. Furthermore, in Stoll (1978) order costs are fixed transaction costs that decrease with the transaction size. Information costs emerge in consequence of the adverse selection from dealing with superiorly informed traders.

formally split into a risk neutral component and a risk premium.[19] The same twofold spread decomposition holds in the discrete-time setting of O'Hara and Oldfield (1986) that allows for both market and limit orders[20], as well as for multiple uncertainty.[21]

A last group of inventory models considers **multiple market makers**, the main role of which is to provide liquidity. For instance in the setting of Cohen, Maier, Schwartz, and Whitcomb (1981) the spread represents a consequence of the fact that it is suboptimal to trade continuously.[22] Moreover, Ho and Stoll (1983) show that inter-dealer competition reduces the spread.[23]

As resumed by Coughenour and Shastri (1999), inventory models reach the conclusion that the spread widens in the presence of higher prices or of elevated risk, but shrinks for higher trading volumes or for an increased number of market makers.

Empirical evidence supports only in part the above theoretical results on the inventory importance. For instance, Smidt (1971) finds that daily closing inventory positions in NYSE stocks have both a contemporaneous and a subsequent price impact. Ho and Macris (1984) confirm the theoretical findings in Ho and Stoll (1981) for two types of AMEX options on two stocks during August-September 1981: The percentage spread of these options depends significantly on the risk aversion of the market maker, and her inventory affects the transaction prices (as well as the timing and direction of transactions).

Several further papers corroborate with the idea that market makers hold preferred inventory positions and tend to fix prices in such a way as to encourage those transactions that correct their inventory imbalances. In particular, investigating 16 NYSE-listed stocks in the period February-December 1987, Madhavan and Smidt (1993) find that inventory fluctuations lead to opposite changes in the quoted prices. Also, inventories revert to long-

[19] The risk-neutral spread maximizes expected profits. The risk premium consists of first- and second-order components, both of which depend on risk aversion, transaction value, and return variance.

[20] While market orders are to be executed at the prevailing market price, limit orders specify a limit price (specifically, a minimum sell and a maximum buy price) and a quantity. See Harris (2003).

[21] Note that this twofold decomposition concerns the market orders. In particular, the spread contains a component for known limit orders, a risk-free adjustment for expected market orders and a risk premium for market orders and inventory uncertainty. Moreover, as a consequence of this multiple uncertainty, the inventory position impacts now on both the placement and the magnitude of the spread.

[22] In particular, transaction costs entail jumps in the execution probability of limit orders in the neighborhood of the market bid and ask prices. These jumps underlie the so-called "gravitational effect" that make traders jump their own limit price schedules and prefer the execution via sure market orders. This generates positive spreads, where the spread size depends on the movements between market and limit orders that rely on the profits from immediacy. Jumps are more probable for thinner (i.e. less actively traded) securities and thus equilibrium spreads are wider for such securities.

[23] For the case with two dealers and two stocks, asks and bids are proven to be only second-best prices, while spreads are positive in equilibrium. In general, spreads are always positive but depend on the number of dealers.

term targets (which can shift with the stock risk-profile), but this process is very slow.[24] An explanation for these weak inventory effects is found in a later paper by Madhavan and Sofianos (1998) based on 1993 NYSE data: NYSE specialists appear to control their inventories rather by timing the trade direction and size than through the quoted prices.[25]

Other markets are also investigated: Lyons (1995) reveals the existence of strong inventory effects in the spot DM/$-foreign-exchange market during 1992, August 3-7.[26] Considering the futures-trading activity in the first half of 1992, Manaster and Mann (1996) further acknowledge an aggressive inventory management at CME.[27] In spite of the negative correlation between inventory and trade direction, FX-market makers appear to quote relatively high ask (low bid) prices when their position is long (short), such that inventory correlates positively with reservation prices.

Information-based models

Information-based models constitute another category of market microstructure settings, that draw explicitly upon the information asymmetries among market participants and the emerging adverse selection.[28] The *adverse selection* problem faced by the market maker can be resumed as follows[29]: The market maker recognizes that trading with superiorly informed traders results in losses.[30] She consequently sets not one but two different prices for buying and selling the same asset, which are the ask and the bid, respectively. These prices – and in particular the spread that forms between them – are fixed in order to recover potential losses from transactions with uninformed traders. Therefore, one important result is that positive spreads may arise in the absence of any

[24]Specifically, it takes over 49 days for an inventory imbalance to be reduced by a half, but after controlling for shifts in desired inventories this time reduces to 7.3 days.

[25]In particular, specialists trade against moving prices, i.e. participate more actively as sellers (buyers) when holding long (short) inventory positions. In addition, they trade more for smaller trade sizes and when spreads are wider.

[26]This result is surprising since FX-dealers intensively use additional inventory-control instruments, such as direct and indirect trade through brokers or laying off inventory at other dealers' prices. Relative to the equity or futures specialists, these instruments should allow for a better inventory control.

[27]Specifically, market makers appear to be very active profit seekers and inventory adjustment is much faster than in equity markets.

[28]Note that symmetric information markets, in the absence of frictions, are efficient. See Madhavan (2000).

[29]In the original interpretation of Akerlof (1970), the adverse selection points to the following problem: When buyers cannot precisely infer the quality of the products in a market, the average quality of the entire supply deteriorates. Accordingly, market makers cannot accurately evaluate the information degree of their counter-parties, they account for the possibility that some trades will be informed. This worsens the transaction terms in general (for all traders).

[30]The general intuition of information-based models is that trading on information can entail substantial gains.

transaction or inventory costs and even in competitive markets, only in consequence of adverse information costs. Note that by contrast to the transient impact of the inventory, the adverse selection puts a permanent mark on prices. Markets are efficient in the limit,but the price-convergence speed depends on various factors. Moreover, there is no serial covariance on prices induced by adverse selection.[31]

A first series of information-based models, which we refer to as **competitive-behavior models**, assume that market participants act competitively. Within this category, Copeland and Galai (1983) are the first to show that, in a single-period setting, information (asymmetry) alone is sufficient in order to generate positive spreads, even with competitive risk neutral market makers.[32] Glosten and Milgrom (1985) and Easley and O'Hara (1987) go a step further analyzing multiple rounds of trade where the market maker learns the information of the informed traders from the order flow and prices eventually converge to the true asset value. The seminal paper of Glosten and Milgrom (1985) represents the cornerstone of a wider class of models that assume competitive behavior and sequential trade execution. It also constitutes the theoretical support of our model in Section 1.2 and its main assumptions and results are presented in the sequel.

Glosten and Milgrom (1985) design a market with three types of players: informed traders (who trade on their superior information), uninformed traders (who trade for information-exogenous reasons, such as liquidity needs), and the market maker. All these players are assumed to be risk neutral and competitive. At each trading time, traders can buy or sell one unit of an asset at prices already set by the market maker. The true value of the traded asset represents a random variable known only by the informed traders. Traders are probabilistically chosen to trade from the population of traders, trade takes place sequentially, and the market maker is always confronted with the same population. Prices are competitively set, so that the expected profit of the market maker from any trade is nil (the zero-profit condition). In particular, they equal the market maker's expectation conditional on the asset value given the type of trade that occurs. – These prices are said to be "regret-free", i.e. fair given the occurring trade. – As (the types of) trades carry valuable information, the market maker revises her beliefs (in other words she learns) at each trade and adjusts prices accordingly. The belief revision follows

[31]See Glosten and Harris (1988) among others. Note that the same conclusion holds in pure order-processing cost models, according to Stoll (1989).

[32]Moreover, they show that the spread depends positively on prices, on the return variance, and on the number of informed traders, and negatively on volume, depth, continuity, and competition intensity.

the Bayes rule.

Glosten and Milgrom (1985) show that the asymmetric information alone entails a positive spread.[33] This spread depends on the number of informed traders, the uninformed supply and demand elasticities, and the nature of the underlying information. Prices form Martingales thus being semi-strong form efficient, i.e. they reflect all public information. In addition, adverse information costs induce no serial correlation in prices.[34] Finally, when the adverse selection is extremely elevated, spreads grow so large that markets shut down.

The role of trade size is extensively analyzed in Easley and O'Hara (1987). This setting is similar to Glosten and Milgrom (1985) in that informed and uninformed traders arrive to trade sequentially, in random order, and that all market participants are competitive and risk neutral. However, two new aspects are introduced: First, variable quantities – specifically, either small or large – of the risky asset can be traded. This induces adverse selection since large orders are a sign of informed trading.[35] Second, it is possible that no information events occur in one period (information-event uncertainty). Thus, the market maker has to infer from the trades not only the direction but also the existence of new information. It is shown that two equilibria may hold in this setting: a separating one, where informed (uninformed) trade only large (small) quantities and there is a positive spread only at large quantities; and a pooling one, where informed and uninformed trade either small or large quantities and there is a spread at all quantities.[36] Prices are shown to depend on the trade sequence and to converge to the true asset value with a speed that is sensitive to different factors, such as the market size, depth, volume, variance, etc.[37]

A further category of models develops the findings in Easley and O'Hara (1987) that the price sequence and/or the volume can provide additional information with respect to

[33] As the transaction and inventory costs are set to zero, the arising spread – that is proven to always be positive – is due only to adverse information costs.

[34] By contrast to the negative dependency generated by inventory costs, the risk aversion or the market power of the market maker addressed by inventory models.

[35] Being profit-maximizers, informed traders will prefer to trade more.

[36] The separating (pooling) equilibrium predominates in markets with sufficient width (in narrow markets) or with few (many) informed traders. The small-trade spread is yet lower than the large-trade one. Therefore, prices depend on the trade size and the spread is not an accurate indicator of the market goodness. A pooling equilibrium appears to hold for the stock followed in the empirical analysis of Easley, Kiefer, and O'Hara (1997b), where volume carries no further information than that in the trades.

[37] In particular, prices are Martingales with respect to public information, but not Markov and hence not only the aggregate volume, but also the entire trade sequence becomes informative. The importance of depth is reinforced by the results in Easley, Kiefer, O'Hara, and Paperman (1996). By contrast, in the setting of Easley, Kiefer, and O'Hara (1997b) trade size provides no additional information with respect to the transactions themselves, while in Easley, Kiefer, and O'Hara (1997a) the information content of different trade sizes varies across stocks.

single prices. They are known under the name of **noisy rational-expectations models** and are mainly concerned with how information is impounded into prices.

The main insights of these models refers to the fact that the price adjustment can be correlated with *time* (in particular with the presence or absence of trade) and *volume*. For instance, in Diamond and Verrecchia (1987) the absence of trades becomes a signal for bad news and makes both bid and ask prices fall and traders postpone their trading to later date.[38] By contrast, in Easley and O'Hara (1992) the trade absence points to a lower probability of information-based trading. Thus, the occurrence of transactions reveals the existence of new information, while the trade type gives an account of the direction of information. The spread narrows for longer time intervals between trades and grows with the overall volume of past transactions[39] and hence the volume affects not only the prices but also their speed of adjustment to new information.[40] As prices are Martingales but not Markov, their sequence is informative. The same result that the price sequence can be more informative than single prices holds when there are multiple sources of uncertainty, as demonstrated in Brown and Jennings (1989) and Grundy and McNichols (1989) among others.[41] Due to information asymmetries, prices not only clear the market, but also aggregate information.[42]

In the same context, Easley, Kiefer, O'Hara, and Paperman (1996) investigate the impact of liquidity on the spread size. They theoretically show and empirically verify for a sample of 90 NYSE stocks during October-December 1990 that spreads increase with the probability of informed trades, which is lower for the high volume stocks.[43] Based on

[38] Specifically, Diamond and Verrecchia (1987) analyze the impact of short-sale constraints on prices. In their setting, risk-neutral informed and uninformed traders face three situations: costless constrains, proceeds restrictions, and short-prohibitions. The total prohibition of short sales renders the spread wider and makes prices incorporate new information more slowly. More relaxed short-sale constraints attain the opposite effect with respect to the price adjustment, but exhibit ambiguous impact on the spread. Specifically, constraints applied merely to uninformed trades improve informational efficiency, especially with respect to bad news.

[39] Specifically, the time between trades carries relevant information, as the probability assigned to the occurrence of informational events increases in the trading frequency. In addition, the volume is related to the no-trade outcomes and hence to the probability of informational events. The unexpected volume affects both price levels and volatility.

[40] This prediction is supported empirically by Lee, Mucklow, and Ready (1993), who report a strong positive (negative) relation between volume and spreads (depths) at NYSE.

[41] In particular, both papers analyze two-period economies where investors receive noisy private signals at the first trade and public ones at the second one. They both demonstrate the existence of linear (noisy rational expectations) equilibria where, although single prices are not revealing, their combination is. The difference is that Brown and Jennings (1989) consider random exogenous supply, while Grundy and McNichols (1989) model the supply uncertainty. Moreover, Brown and Jennings (1989) conclude, on the basis of numerical analysis of 1978 CBOT data, that technical analysis can be valuable even in a rational-investor economy, but such a market is inefficient in Fama's sense.

[42] Thus, uninformed traders can learn from simply observing the price sequence.

[43] They design a discrete-and-continuous time, sequential trade setting, with a risk-neutral and compet-

six stocks from the same NYSE sample, Easley, Kiefer, and O'Hara (1997a) find that the information content of large and small trades can be distinct and that uninformed trades are history dependent. Reversals in order flows and the absence of trades are found to be most informative.

The role of the trading volume under asymmetric and noisy information is also analyzed in the noisy rational-expectations settings. Wang (1994) shows that the transaction volume decreases for a more pronounced information asymmetry among market participants[44], being positively correlated with absolute excess returns and also with the arrival of public information. Blume, Easley, and O'Hara (1994) argue that the volume statistic provides valuable information that cannot be deduced from prices, facilitating learning and hence the adjustment of prices to information.[45] In addition, Lee, Mucklow, and Ready (1993) underline the importance of market depths – in combination with spreads – in characterizing market liquidity.

Before closing this subsection, let us recall several **empirical findings** that support the idea that the adverse selection represents a significant determinant of the spread. Considering a sample of 20 NYSE stocks during December 1981-January 1983, Glosten and Harris (1988) suggest a twofold spread decomposition: A permanent component (due to adverse selection) is estimated at around 80% of the spread, and a transitory one (due to inventory costs, specialist market power, clearing costs) amounts to the remaining 20%.[46] Lee, Mucklow, and Ready (1993) conclude based on 1988 intraday data that spreads in the same market widen for large transactions and in anticipation of earning

itive market maker, informed and uninformed traders. A new method for assessing the effect of informed trading from the market maker's beliefs is suggested in this context. In particular, the probabilities of the occurrence and the direction of information events, as well as the arrival rates of informed and uninformed traders are estimated by maximizing the likelihood of observing a certain order sequence.

[44]As less-informed traders face higher chances of losing in front of the better informed.

[45]Specifically, two groups of myopic traders disposing of different information – i.e. with identical mean and a common error term, but differently distributed idiosyncratic errors – trade one risky and one riskless asset. By contrast to Brown and Jennings (1989) and Grundy and McNichols (1989), the aggregate supply is now fixed and the source of noise is the information precision. The equilibrium price is shown to be non-revealing and one group looks for more information concerning the signal quality in the volume. For a fixed precision, the volume is a V-shaped function of the equilibrium price. The steepness and dispersion of this curve depend on the precision and dispersion of information. Thus, the volume offers supplementary information with respect to prices, as it represents a basis for disentangling between the quality and the direction of information. Moreover, while prices eventually converge to the full-information value, but volume does not shrink to zero, in other words the trade does not cease as beliefs converge. The paper explains the emergence of trading patterns and also underlines the utility of technical analysis as a tool for learning uncertainty.

[46]These components interact, but while the former depends on the trade size, the latter does not. Explicitly accounting for price discreetness renders the permanent (transitory) component to be 35% (65%).

announcements, as the adverse selection is more pronounced in both cases.[47] Huang and Stoll (1994) also confirm the significant impact of large trades on bid and ask prices.[48] More generally, French and Roll (1986) underpin the importance of information trading in the formation of prices, by observing that the volatility of stock returns is mainly caused by the information impounded into prices when exchanges open.[49] Finally, Manaster and Mann (1996) verify the effect of the informational content of order flows on prices.[50]

Strategic-trader models

Within the same category of information-based settings, strategic-trader models assume that some traders are able to determine not only the optimal size, but also the optimal time of their trades. In other words, they act strategically, i.e. account for the impact of their trades on prices. These models put emphasis not on the spread, but on the incorporation of information into prices, which, as mentioned at the beginning of this section, is a dynamic aspect of price formation.

In early models, strategic behavior is considered to be an exclusive attribute of **informed traders**. The cornerstone setting of this category is due to Kyle (1985) and its most important assumptions and results are revised in the sequel.

By contrast with the sequential trading in Glosten and Milgrom (1985), Kyle (1985) assumes a batched order execution. The market participants are a single informed trader, uninformed liquidity (or noise) traders and a market maker. All of them are again risk neutral and revise beliefs according to the Bayes rule. However, unlike in Glosten and Milgrom (1985), traders can exchange unbounded quantities of an asset[51] during a finite number of trade rounds. In so doing, they simultaneously submit market orders which are batched together, so that the market maker can merely observe the total order flow. A single market-clearing price is set at each trading time, namely after orders are placed.[52]

[47] At the same time, depths fall. They note that only the combination of spreads and depths is relevant for inferring changes in market liquidity, and not on merely one of these two variables. Market makers appear to manage information-asymmetry risks actively, by means of spreads and depths.

[48] See also the comments below related to the spread decomposition.

[49] Specifically, the large difference of over 70% in volatility between trading and non-trading hours observed for all NYSE- and AMEX-listed stocks from January 1963 through December 1982 appears to be driven in proportion of 88-96% by differences in the flow of (private) information.

[50] Note that, however, they cannot exclude the inventory influence. Specifically, changes in prices in response to order flows depend on inventory levels. See also the comments above with respect to empirical findings related to inventory models.

[51] The asset has a normally-distributed true value.

[52] Again, following the zero-profit condition, this price equals the expected asset value, conditional on the total order flow.

The informed trader acts strategically in that she accounts for the effect of the own trades on the equilibrium price. Her goal is to maximize future trading profits, up to the end of the entire trading interval. Uninformed traders transaction random (normally distributed) asset amounts.

Kyle (1985) analyzes two settings: a single-auction and a multiple-trading one.[53] He shows that in both of them there is a unique linear equilibrium, where the optimal asset quantity traded by the informed is linear in the true asset value and depends on the variance of the uninformed trader flow. The larger this variance is, the better can the informed trader hide her identity from the market maker and the higher are her profits.[54] Equilibrium prices are linear in the aggregated volume of the current trades and their adjustment depends on the amount of noise to informative trading. As the informed trader always trades proportionally to the uninformed order size, the price behavior is independent of volume. Information is gradually incorporated into prices, which eventually converge to their full-information values. In the single-period setting, half of the information of the active informed trader is revealed in the price at each trade. With multiple periods, prices have constant volatility and are efficient (i.e. Martingales).[55]

In spite of the apparent disagreement in the designs of Glosten and Milgrom (1985) and Kyle (1985), both models draw upon the idea that adverse selection – that the market maker has to face in trading with better informed traders – generates costs. In Glosten and Milgrom (1985) these costs become manifest in the form of the bid-ask spread, while in Kyle (1985) they influence the market liquidity. Back and Baruch (2004) show that both settings are consistent: Under the assumptions of small trade sizes and frequent uninformed trades in Glosten and Milgrom (1985) and of small time intervals and low variance of the uninformed trades in Kyle (1985), the two markets reach the same equilibrium.[56]

There are numerous extensions of the Kyle (1985) model. First, Back (1992) allows for more general distributions of the true asset value and shows that it is optimal for

[53] The analysis in the multiple-trading setting is undertaken in continuous time.

[54] In other words, profits of the informed traders result from the variation of order sizes.

[55] Note however that prices are only semi-strong, but not strong form efficient, due to noise trading that renders impossible the full inference of informed signals from prices.

[56] In particular, Back and Baruch (2004) consider a version of the model in Glosten and Milgrom (1985), where the occurrence of uninformed trades is probabilistic, but the single informed trader can chose trading times. The spread amounts in this version to a quantity that is approximatively twice the order size multiplied by the inverse-liquidity parameter in Kyle (1985). Moreover, the equilibrium in the considered modification of Glosten and Milgrom (1985) assumes mixed strategies and thus allows for randomization. This implies the possibility of trading contrary to the received information, which is denoted as bluffing.

the informed trader to set the same order flow distribution as the uninformed flow (i.e. Brownian motion).[57] Prices incorporate all information at the end of the trading, almost surely and change proportionally to cumulative order sizes.

Further extensions incorporate multiple informed traders. For instance, the endogenous number of informed traders affects their strategy in the two-period setting of Kyle (1984). Prices become more informative both for an increased amount of noise trading and when there is more private information.[58]

These conclusions may change dramatically when the assumption of risk-neutral informed traders is dropped. Subrahmanyam (1991) points out a dual effect of the risk aversion in a one-period setting: Risk-averse informed investors trade less than their risk neutral peers, but their risk tolerance is also affected by the total number of trades. Therefore the market liquidity can even decrease when more informed traders enter the market.[59]

Holden and Subrahmanyam (1992) analyze further how quick prices incorporate (long-lived) information when multiple risk-averse informed traders compete with each other.[60] Their conclusions differ radically from those in Kyle (1985) (where a single informed trader was active in the market): There exists as well a unique linear equilibrium (which moreover resembles the one in Kyle (1985), but where all parameters depend on the number of informed traders), but, since informed traders trade more aggressively in early periods, prices become efficient much faster.[61] Furthermore, Blume and Easley (1990) show that there is no standard rational-expectations (i.e. Walrasian) equilibrium when each informed trader disposes of her own information, independently of the number of traders. Kyle (1989) considers more complex order strategies of both strategically acting informed and uninformed traders and demonstrates the existence of an imperfectly competitive rational-expectation equilibrium, where prices reveal less information than in the

[57]Note that Back (1992) extends the time-continuous version of Kyle (1985), which implies that markets are infinitely tight. The solution in Kyle (1985) is replicated when true asset values are normally distributed. For log-normal distributions, the equilibrium prices amount to a geometric Brownian motion.

[58]Individual profits depend on how total profits split among informed traders. They consequently reduce at higher amounts of noise trading. In addition, the size of the individual trades varies with the total number of traders. The informed activity shrinks for an enhanced amount of private information.

[59]Specifically, liquidity is nomonotonic in the number of informed traders, their risk aversion, and the precision of their information, and, with endogenous information, also in the variance of liquidity trades. Prices become less efficient for an intensified liquidity trading.

[60]And the informed traders are assumed to act identically.

[61]In fact, with an infinite number of auctions – or equivalently with an infinitesimal interval between auctions – information is almost immediately incorporated into prices. The same occurs when the number of informed traders grows to infinity (perfect competition). Such a market is perfect in the sense that it is infinitely deep, tight, and resilient.

competitive equilibrium.[62]

A series of further theoretical settings allows both **informed and uninformed traders** to behave strategically. This model relaxation often entails multiple and/or non-linear equilibria.

Admati and Pfleiderer (1988) develop the basis setting in this category of strategic-trader models. It considers two categories of uninformed traders, both of which trade for exogenous liquidity reasons and cannot split the traded amount. The difference between them draws upon the choice of transaction time: While nondiscretionary traders must transact at fixed times, discretionary traders can freely opt for a trading moment. The other actors of this strategic play are the informed traders and a competitive market maker. All market participants are risk neutral.[63] Discretionary liquidity traders appear to concentrate their trades in equilibrium.[64] When they can allocate trades across time, this concentration is higher at times closer to the realization of their demands. In this situation, multiple equilibria become possible. During periods of discretionary concentration, informed trading is also more intense[65] and, with endogenous information, prices are more informative. Madhavan, Richardson, and Roomans (1997) show that NYSE-trading concentrates at the beginning of the day.

Foster and Viswanathan (1990) extend the model of Admati and Pfleiderer (1988) for long-lived information. There is a single informed trader active in the market and the discretionary uninformed traders can delay their transactions for at most one day. While private information arrives every day, public information is known only at the end of the trading days. Thus, there is a substantial informational advantage of the informed trader at the openings, in particular on Mondays, and it is more pronounced the longer the markets were closed.[66] Foster and Viswanathan (1990) show that multiple equilibria

[62]Specifically, the designed market prices, unique for buys and sells, are set by a Walrasian auctioneer. The traders are: noise traders, motivated by exogenous reasons; and informed and uninformed traders which can submit more complicated demand schedules, are risk averse, and maximize the expectations of negative exponential utilities. The symmetric linear equilibrium exists under certain conditions, such as the existence of sufficient informed traders with homogenous information and risk averse, and normally-distributed random variables. Equilibrium prices never reveal more than one-half of the private precision of informed speculators.

[63]In particular, different settings are analyzed. First, when discretionary traders are forced to trade only once, the number of informed traders is fixed and their information homogenous. Second, when traders can become informed at certain costs. Third, when information is heterogenous and, finally, when discretionary traders are allowed to split trades across periods.

[64]Where the optimal trading time corresponds to the minimum cost period offered by the market maker.

[65]Which improves trading terms for liquidity traders and enhances the volume and returns patterns induced by discretionary traders.

[66]In the absence of public information, the optimal behavior of the single informed trader is to delay

are possible but, in each of them, there is a clear Monday-pattern: On Mondays, trading volume is low and trading costs and return variance differ with respect to other days of the week. This result offers a new perspective compared to the sequential trade models and the strategic investor models where prices are Martingales and markets efficient.

Spiegel and Subrahmanyam (1992) extend the model in Kyle (1984) suggesting an endogenous motivation for uninformed trading: risk hedging.[67] A linear equilibrium exists only if the risk-averse (uninformed) hedgers are not dominated by the risk-neutral informed traders. Market liquidity and price efficiency are nonmonotonic in the number of hedgers.[68]

Finally, Admati and Pfleiderer (1989) extend their earlier setting considering the implications of bid and ask commissions.[69] The exchanged quantities at a certain trading time are set to one unit of stock and the commissions are made public at the beginning of the trade interval. One possible equilibrium is characterized through the concentration of the trades of discretionary buyers and sellers, each in one period. This concentration reduces commissions but consequently attracts more informed traders. The two concentration periods for discretionary buys and sells are necessarily distinct with endogenous information. In this case, prices deviate from their expected values at the two moments of concentration and equal these values at all other times.

Note that, in essence, both inventory and information-based models consider the effect of the order flow on prices. As observed by Madhavan (2000), they both predict that prices move in the direction of the order flow[70], but for different reasons: according to the former,

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trades as much as possible. This renders the price variance constant across the days of the week. By contrast, with an informative daily public signal the informed trader transacts more intensely at the beginning of the week and hence the price variance declines through the week. A second setting assumes that some liquidity traders (called discretionary) are allowed to delay transactions for a maximum of one day. With no public information, the strategy of the informed trader aims at keeping constant the trading costs of the discretionary traders. Hence, postponing transactions by these traders is not optimal. Under valuable public information, the equilibrium trading pattern depends on the number of discretionary liquidity traders. As the public information is more accurate, discretionary traders pool their trades on two days of the week, but do not trade on Mondays, while for less accurate public signals discretionary trades concentrate on Fridays.

[67] Specifically, uninformed trade for the reason of sharing endowment risk. They act strategically, attempting to maximize expected utility.

[68] Specifically, liquidity can decrease in the number of hedgers, when their risk aversion is high. Moreover, for the number of informed traders approaching an upper bound markets become illiquid and the linear equilibrium disappears. In addition, the individual welfare of uninformed traders decreases with the number of their informed peers, while the informed profits may also shrink when more hedgers are active in the market.

[69] The ask commission is defined as the difference between the ask price and the expected asset value, while the bid commission stands for the difference between the expected value and the bid price.

[70] Specifically, that market makers rise (reduce) prices in consequence of buyer-initiated (seller-initiated) trades.

the order flow determines changes in the inventory of market makers; By contrast, in the information-based models the order flow carries valuable information and entails belief revisions.

Spread components

In reality, all three main effects that have been theoretically insolated (inventory costs, adverse information costs, and order-processing costs) become simultaneously manifest. Several theoretical and empirical works attempt to disentangle and assess the magnitude of the the corresponding spread components.

Thus, Hasbrouck (1988) develops a VAR-method for comparing the importance of inventory and information effects. These effects are both found to be significant for NYSE-listed stocks during March-April 1985, where the latter dominates the former one.[71] These results are qualitatively replicated by other authors for different data and using different techniques. For instance, Madhavan and Smidt (1991) develop a Bayesian model for intraday pricing that incorporates fixed transaction costs, inventory costs, and asymmetric information costs. They also provide a measure of information asymmetry, which is the weight placed by the market maker on prior beliefs.[72] Tests of the model for NYSE-specialist market data from February to December 1987, specifically for 16 stocks traded by a single specialist firm, confirm the major role of the information asymmetry as perceived by the market maker and the existence of only weak inventory effects. Also, both new information and trading volume move prices.[73] This model is extended by Lyons (1995) and applied on the spot foreign exchange DM/$-market, where both inventory and adverse-information effects appear to be significant and strong.[74] Moreover, Madhavan and Smidt (1993) use the same data set as in their earlier paper for estimating a time-series model with random level shifts that captures both inventory and adverse-information effects on prices. Their results point to weak inventory effects and positive correlations of quote revisions and unexpected order flows.[75] Using the VAR-method, Hasbrouck and

[71]Moreover, negative autocorrelation related to inventory effects is significant only for low-volume but not for high-volume stocks.

[72]Which is very small for high asymmetries, as the market maker overweights the importance of the current order flow. Accordingly, the change in price is due to the innovation in the order flow and not to its aggregated value.

[73]Specifically, spreads are higher during the October 1987 market crash, trades in more active stocks have smaller price impact compared to less active stocks, large blocks have more impact than small ones, and buyer-initiated move prices more than seller-initiated trades.

[74]In particular, for each net open position of $10 million, the inventory effect is of three-quarters of a pip and the information effect of one pip.

[75]The model considers a market maker which acts both as active investor and dealer and an informed

Sofianos (1993) find evidence for both inventory and asymmetric-information effects for a NYSE-stock sample from November 1990 to January 1991, where the considered firms and the data set length vary across analyzes. In addition, inventory adjustments are found to be slow (from days to weeks) which reinforces the above presented empirical conclusions related to the inventory models.

The two-equation model in Huang and Stoll (1994) proves the ability of microstructure variables to predict the very short-run price behavior. In particular, the simultaneous existence of all three spread effects – adverse selection, inventory, and order processing effects – is confirmed for price and quotes data (in five-minutes interval) for 20 stocks of the Major Market Index and S&P futures from 1988.[76] Huang and Stoll (1997) formally identify and empirically estimate two spread components by means of the serial covariance of price series for 19 stocks in the Major Market Index in 1992. According to their two-way decomposition of the spread – in order processing and pooled inventory and information costs – order processing costs play the main role in the spread determination at NYSE (with a weight of almost 89%). According to a three-way spread decomposition, the order-processing cost component remains the highest (with 62.7%), followed by the inventory component (28.7%), while information asymmetry costs are much lower (9.6%) and also significantly smaller for large trades. Nevertheless, empirical investigations performed on different data sets and for other markets, such as Stoll (1989) for over 750 NASDAQ firms between October-December 1994, reach qualitatively contrary conclusions: The information component is now estimated at 43% of the spread, while the inventory component reaches only 10%, the rest of 47% being due to order processing costs.

Moreover, Lin, Sanger, and Booth (1995) perform estimations of the spread components based on 1988 transaction data of 150 NYSE firms and approximate the overall adverse information component to 35% and the order processing component to 33%. Both components depend on the trade size. Moreover, spreads follow a U-shaped pattern during the day, where the adverse information component decrease, while order processing costs are inversely U-shaped.[77] Madhavan, Richardson, and Roomans (1997) develop

investor, both of which behave strategically. For more comments related to the inventory effect, see the beginning of this section, under inventory models. As far as the information effects are concerned, large trades appear to convey little information.

[76]For instance, expected quote (transaction) returns are positively (negatively) correlated with the deviation between the transaction price and the quote midpoint, which is consistent with both adverse selection and order processing theories. Moreover, the negative serial dependence in quote returns, the reaction of transaction prices to volume – for instance the price reaction to large trades – underpin the inventory theories.

[77]Estimation takes place in a one-period setting and spread is decomposed into two elements. The

a structural model of intraday price formation. Estimations performed on the basis of NYSE 1990 transaction data for 274 firms assess the information component at approximatively 43% on average and the transaction costs to be the remaining 57%. They also observe that, while information asymmetry diminishes over the day, transaction costs (in the main due to inventory holdings) grow. This provides an explanation for the U-shaped patterns in spreads and volatility and the decrease of ask-price variance. In addition, the fraction of the spread due to asymmetric information is found to fall from 51% to 36% during the day.[78]

George, Kaul, and Nimalendran (1991) design a new procedure for estimating the spread components from both transaction prices and bid and ask quotes.[79] Estimation results using daily and weekly data from AMEX/NYSE (during January 1963-December 1985) and NASDAQ (January 1983-December 1985) suggest that the adverse selection part of the spread is only 8-13% (daily-weekly data), the inventory costs are negligible and hence order processing costs play the leading role (with 87-92% of the spread). Finally, Prucyk (2005) develops on the estimation of the three spread components derived by Huang and Stoll (1997). For a sample of 30 NYSE stocks with options at CBOT during 1993-1994, they show that it is the inventory component – and not the adverse selection one – that increases for very liquid stocks when volatility changes are high in absolute value.[80]

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adverse information (order-processing) component depends positively (negatively) on trade size and increases (decreases) from 18.9% to 62.6% (46% to 20.1%). The order persistence amounts to around 66% and decreases with trade size. For trades at NASDAQ or in regional markets, the adverse selection costs are not significantly different from zero. Order processing costs are higher than at NYSE but do not vary with trade size.

[78]The used estimation method is GMM. The estimated autocorrelation of the order flow is positive. The estimated spread lies between 14-15 cents and is high at the beginning (end) of the day due to high information asymmetries (inventory costs), while the effective spread – that accounts also for possible order execution within the quotes – increases monotonically during the day. Furthermore, the intraday volatility is decomposed into components due to public information shocks and trading frictions which are further attributed to price discreetness, asymmetric information and transaction costs and their interaction. The asymmetric-information impact on volatility is low, but transactions costs play a much more important role in this context.

[79]They argue that previous estimators, based on the assumption that inventory costs are the only source of return autocorrelation, are biased due to the time variation of returns. The new spread measure relies on the serial covariance of the difference between transaction returns and returns based on bid prices and is robust with respect to the data frequency. Estimations rely on the idea that the spread is independent of the trade size, so that the obtained values are valid for small transactions (or equivalently can be interpreted as lower-bounds for large trades).

[80]Specifically, they consider return volatility to be an insufficient risk measure. Changes in volatility are as well important, as they point to arrival of new information and its direction. Thus, quoted spreads widen (depths shrink) for higher volatility, but also when volatility increases and decreases. This reaction cannot be significantly attributed to the concerns of market makers with adverse selection, but to inventory adjustments, since market makers appear to be forced to be more active in periods of higher volatility changes which rises inventory holding costs.

There are also some studies concerned with inferring the magnitude of the spread components in order-driven markets. Thus, Brockman and Chung (1999) estimate the (median) adverse-information and order-processing components on the Hong Kong SE at around 33% and 45%, respectively.[81] In the Taiwanese futures markets TAIFEX (electronic screen-based call trading), Huang (2004) finds a much lower information component than at SGX-DT (continuous floor trading), where he uses intraday data between January-September 2001.[82]

[81]The adverse information component is inversely related to the transaction volume in dollars and to the firm size, while order processing costs appear to be insensitive to both volume and firm size. A third spread component accounting for order persistence amounts to 60,7%, increases in volume and in firm size and points to the tendency of buys (sells) to follow buys (sells).

[82]Specifically, the asymmetric information component is approximatively 26.5% (51.1%) at TAIFEX (SGX-DT). The corresponding order processing component amounts to 73.5% (48.9%). Moreover, both components exhibit U-shaped patterns, depend significantly on volatility and information, but not on the number of trades. These results hold for both exchanges.

1.2 Imperfect information, practical trading rules, and asset prices

1.2.1 Introduction

For the last thirty years the academic community has been engaged in a considerable debate about the efficiency of financial markets, but has not yet reached a satisfactory conclusion. In the meantime, the trader community has kept investing according to different practical trading rules, such as fundamental indicators (e.g. book-to-market values, price-earnings or price-cash-flows ratios) and technical patterns (derived from the systematical observation of prices and trading volumes). Despite the growing interest with regard to the application of such empirical methods, their efficiency as a support in decision making as well as their consequences still remain controversial and unsatisfactory when investigated from a theoretical viewpoint.

In our view, the essence of practical trading rules can be expressed in terms of information: The purpose of these rules is to provide information, but the accuracy of this information can fluctuate in dependency of different factors (such as the method itself, the way in which it is applied, the extent of their dissemination, the environment, etc.). Our goal is to quantify the impact of imperfect information, specifically of that obtained by means of practical decision rules, on market prices.

Our model extends the sequential-trading setting of Glosten and Milgrom (1985) by introducing a supplementary category of informed traders. These traders merely dispose of inaccurate information regarding the true value of the risky asset and hence we refer to them as *imperfectly informed traders*. They coexist with two categories commonly considered by previous market microstructure settings: the *perfectly informed traders* and the uninformed *liquidity traders*. As already mentioned, we can think of the information of our imperfectly informed traders as stemming from systematic observations of market data, performed by means of methods of wide practical use, such as fundamental or technical analysis.

In line with Easley and O'Hara (1987), we further assume that information (both perfect and imperfect) reaches the market stochastically, so that both the trading itself and its direction carry informational value. The informed traders follow their signals and hence their actions are entirely dictated by the information they receive or derive. Prices are set by a market maker who, as all traders, is competitive, risk neutral, and forms Bayesian beliefs. At each trade, two distinct prices are fixed for buying and selling the same risky asset and the gap between them represents the spread.

We show how the accuracy of imperfect information (the *qualitative effect*) and the dimension of the imperfectly informed group (the *quantitative effect*) reflect on prices and point out some differences relative to the respective impact of perfect information. Although information is now present in the market in different degrees of accuracy, the spread is always positive and prices remain semi-strong form efficient in the short run and strong form efficient in the long run, where efficiency is defined in line with Fama (1970). In particular, prices at each trade represent market maker's expectations of the true risky value, given the public information, and in the long term and considered as a whole they follow a Martingale. However, the single transaction prices are not Markovian when imperfect information relies on past market evolutions, as it is often the case when imperfectly informed traders apply trading rules commonly used in practice. In other words, the sequence of each of the double prices set simultaneously for buying and selling the risky asset becomes informative.

Numerical simulations support these results and also serve to extend them in some particular cases. Specifically, we analyze the evolution of prices and trader profits when imperfect information is derived from three simple trading rules. These strategies can be considered as examples of heuristics, which represent a broad category of simplified decision rules commonly employed in making decisions and are detailed in Section 2.1.2. They also belong to categories of trading methods of wide use in practice: technical analysis (also known as chart analysis) and fundamental analysis. In particular, two of our practical trading rules are of chartist type and hence derive recommendations from past price movements. The third one is of fundamentalist type and compares the intrinsic value of the risky asset (i.e. the true value) with its market value (i.e. its price). When imperfectly informed traders use the chartist strategies, prices converge to the true asset values. This not always the case for the fundamentalist rule. Moreover, one of the chartist rules (based on the observation of successive trades of the same type) yields not only positive, but also highest group-profits in the market. Thus, users of certain practical and simple trade rules appear to be fit for survival, even when faced with perfect information.

The remainder of the section is organized as follows: We first introduce the theoretical setting in Section 1.2.2, starting with the main variables and assumptions, continuing with the belief updating process of the market maker, and finally deriving the expressions of the buy and sell prices. Section 1.2.3 details the results. We first examine the general price evolution in the short and the long run. Second, we concentrate on the impact of each imperfect and perfect information on market conditions. Up to this point, our analysis is theoretical and mainly considers ceteris paribus variations of the spread in dependency of different model variables. Section 1.2.3 makes use of numerical simulations in order to exemplify the theoretical results and investigates three simple trading rules. The focus is extended in order to encompass not only the price evolution, but also the trader profits. The final Section 1.2.4 summarizes the main findings. Mathematical proofs, graphs, and further results are included in Appendix A.1.

1.2.2 Theoretical model

Variables and assumptions

Our setting takes on the main assumptions in Glosten and Milgrom (1985) and Easley and O'Hara (1987). In particular, it provides for two main categories of market participants: *traders* who periodically subscribe buy and sell orders; and a *market maker* whose principal task consists in setting prices and maintaining a fair, orderly, liquid and efficient market.[83] All market participants are considered to be risk neutral[84] and myopically competitive.[85] Also, they use the Bayes rule in order to form expectations of the economy type.

The trade takes place at each time $t = 1, \dots, T$. The trade object consists in one risky asset with the (true) value V. This value represents an indicator of the economy

[83]As discussed in Section 1.1.1, there is no clear and unitary assignment of these attributes to a certain market participant in financial market terminology. At the NYSE or AMEX, the so-called specialists fully undertake these tasks, while the market makers act sooner as brokers/dealers. On the floor of the Frankfurt Stock Exchange the so called Amtlicher Kursmakler is responsible for price setting and maintaining of proper market conditions. In our view, the name "market maker" best reflects the principal task of such an agent, namely "to make the market" (i.e. to bring together the traders' demand and supply). As she accommodates but does not initiate trades, the market maker can be considered to be passive. This conforms with reality, as, according to Harris (2003), market makers are unable to control the timing of their trades.

[84]Their utility functions exhibit a linear evolution subject to their wealth.

[85]The term "myopia" is used here in the sense of short-term behavior (and *not* in the sense defined in Section 3.1.3). Thus, the agents do not account for the effects of their actions on subsequent prices. Their trading horizon is restricted to one period. Hence, their periodical trading decisions are independent, as underlined in O'Hara (1995). The survey by Taylor and Allen (1992) makes the case for an appropriate application of practical trading rules in a market with myopic traders. The use of technical analysis is ascertained to be greater for short-, than for long-term decisions. Brown and Jennings (1989) demonstrate furthermore, that the technical analysis exhibits a certain value in a linear two-period rational expectations equilibrium of a market with myopic traders.

type, which is considered to be fixed during the whole trade. It is formally modelled as a binary random variable with two possible values: high and low, $V \in \{V^H, V^L\}$, where $V^H \geq V^L$. The value V becomes public information only at the end time T.[86] Some traders – specifically the informed[87] – stochastically receive information about V during the trade.

The trade unfolds as follows: At each time t, the market maker fixes the current buy and sell prices and a trader is randomly chosen to trade.[88] She can refuse or accept trading. In the latter case, she submits either a buy or a sell order for one unit of risky asset.[89]

The *public information* set at t consists of the past prices and the current quotes for buying or selling the risky asset (i.e. the so called bid and ask) and can be written as $h_t \equiv \{X_1, \ldots, X_{t-1}, X_t^B, X_t^S\}$, where $h_0 \equiv \emptyset$.[90] For reasons of notational simplicity, we henceforth omit the notation h_t for all conditional probabilities and expectations at time t, and replace it with the subscript $_t$.

We denote the probability of a high risky value, assessed on the basis of the public information at time t by:[91]

$$p_t = P_t(V = V^H) = 1 - P_t(V = V^L), \tag{1.1}$$

and refer to it as *public beliefs*.

As stressed in the introduction, a particular assumption of our model is the existence

[86]Thus, all market participants become equally informed at T. In principle, we can assume that the whole trading period consists of many uninterrupted episodes of the form $t = 1, \ldots, T$ that correspond in practice to the trading days, months or years and are naturally separated by intervening periods. At the beginning ($t = 0$) and at the end ($t = T$) of such an interval, the agents dispose of homogenous information.

[87]See the definition of informed traders below.

[88]In other words, the trade is sequential and anonymous. The market maker is assumed to know the probabilistic structure of the arrival process, which, according to Glosten and Milgrom (1985), allows her to make correct inferences from the observed data.

[89]Our traders can submit only market orders that are immediately executed. The restrictions upon the order size and the trading frequency should help avoid a premature trade cessation. Otherwise, the perfectly informed traders could already trade at $t = 1$ as much as possible, thereby making the price fully informative from the beginning and eliminating every trade incentive.

[90]Prices are formally defined below.

[91]Thus, p_t reflects the public opinion. For instance, at $t = 0$, all traders are aware of the prior probability $p_0 = P_0(V = V^H) = 1 - P_0(V = V^L)$ derived from the results of the previous trade intervals of length T. For instance, when the risky value disclosed at the end of the previous interval was high (low), p_0 increases (decreases). During the coming trade interval traders gather further decision-relevant information, namely on the basis of the collective experience or also with the help of new periodical information.

of three *trader groups*: the *perfectly informed* ($\mathbf{a}$), the *imperfectly informed* ($\mathbf{b}$),[92] and the *liquidity traders* ($\mathbf{c}$). These groups are all non-overlapping and homogenous,[93] but differ in the degree and the accuracy of the received information as well as in the employed trade strategies. The market maker ascribes constant probabilities to the fact that a trader belongs to a certain group. Specifically, these probabilities are identical to the proportion of each group to the totality of traders in the market $n_{\mathbf{g}}$, where $\mathbf{g} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\sum_{\mathbf{g}} n_{\mathbf{g}} = 1$.[94] In addition, we assume that there are always some liquidity traders active in the market $n_{\mathbf{c}} > 0$.[95]

Concerning the *degree and accuracy of information* at their disposal, traders separate in two distinct categories: informed and uninformed.[96] First, liquidity traders are driven by trade exogenous reasons, such as the need of liquidity, and do not obtain any particular information on the risky value. Consequently, their information set exclusively consists of public information $h_{\mathbf{c}t} \equiv h_t$ and they can be considered to be *uninformed.*

Second, both perfectly and imperfectly informed agents may receive – or derive – information about the risky asset value and we refer to them as *informed traders.* This information can be zero (no information), positive or negative (corresponding to the high and the low value of the risky asset, respectively), that is $s_{\mathbf{g}t} \in \{0, -1, 1\}$, where $\mathbf{g} \in \{\mathbf{a}, \mathbf{b}\}$. Furthermore, the probabilities of information arrival for the two informed types are denoted by:

$$\begin{aligned} \alpha_t &= P_t(s_{\mathbf{a}t} \neq 0) \\ \beta_t &= P_t(s_{\mathbf{b}t} \neq 0). \end{aligned} \tag{1.2}$$

Thus, the information set of the informed at time t yields $h_{\mathbf{g}t} \equiv \{h_t, s_{\mathbf{g}t}\}$, for $\mathbf{g} \in \{\mathbf{a}, \mathbf{b}\}$.

However, the perfectly informed traders (group $\mathbf{a}$) receive completely accurate information which allows them to recognize exactly the economic situation and to choose the

[92] Our interest is in the case when the imperfectly informed traders apply simple trading rules, of wide-spread use in practice, such as methods of fundamental or technical analysis.

[93] Here we refer to informational homogeneity: all members of a group dispose of the same information at the same time and interpret it in the same way.

[94] Easley, Kiefer, and O'Hara (1997b) estimate the fraction of informed traders in the US-market at about 17%. The ZEW survey of Rebitzky (2004) assesses a fraction of technical analysts in the German Foreign Exchange Market of approximatively 30%, while the share of fundamentalists participating in trading is roughly 60%.

[95] As underlined in see Easley and O'Hara (1987), as long as the number of liquidity traders remains strictly positive, the market maker can compensate the losses from doing business with informed agents by the gains from transactions with the uninformed ones. Otherwise, the excessive and repeated losses resulting from buying or selling exclusively from or to the informed traders could cause a definitive trade cessation.

[96] Received information is here considered to be private signals and/or signals derived as a result of systematical analysis of the market data.

appropriate trade alternative:

$$\begin{aligned} P_t(s_{\mathbf{at}} = 1 | s_{\mathbf{at}} \neq 0, V = V^H) &= 1 - P_t(s_{\mathbf{at}} = -1 | s_{\mathbf{at}} \neq 0, V = V^H) = 1 \\ P_t(s_{\mathbf{at}} = 1 | s_{\mathbf{at}} \neq 0, V = V^L) &= 1 - P_t(s_{\mathbf{at}} = -1 | s_{\mathbf{at}} \neq 0, V = V^H) = 0. \end{aligned} \tag{1.3}$$

In contrast, the imperfectly informed (group $\mathbf{b}$) acquire only inaccurate signals:

$$\left.\begin{aligned} P_t(s_{\mathbf{bt}} = 1 | s_{\mathbf{bt}} \neq 0, V = V^H) &= 1 - P_t(s_{\mathbf{bt}} = -1 | s_{\mathbf{bt}} \neq 0, V = V^H) = q_t^H \\ P_t(s_{\mathbf{bt}} = 1 | s_{\mathbf{bt}} \neq 0, V = V^L) &= 1 - P_t(s_{\mathbf{bt}} = -1 | s_{\mathbf{bt}} \neq 0, V = V^H) = q_t^L \end{aligned}\right. \quad \text{s.t. } q_t^H \geq q_t^L. \tag{1.4}$$

The condition $q_t^H \geq q_t^L$ stands for the fact that the market maker believes that the imperfectly informed traders receive no misleading information, i.e. no positive (negative) information in bad (good) economic states. This amounts to the belief that these traders can be considered to be informed – or to have a sufficiently good sense of the market evolution – even though the precision of their signals may not be perfect.[97]

In particular, perfect information can be insider information. Imperfect information could be *derived* from (public) market data by means of specific methods, such as technical or fundamental analysis. The different trading rules pooled together by these two method categories are widely used by individual traders in practice and have been shown to achieve positive performance.[98] As public information is freely available to all traders, the users of practical trading rules should be able to perform their analysis and hence derive new information at each time t. Thus, the probability of imperfect information β_t should be high, specifically close to 1. On the other hand, the probability of perfect information α_t may be closer to 0, as the occurrence of private information cannot represent a common event. Moreover, practical trading rules can be considered as examples of simplifying rules, which are termed in psychology as *heuristics*.[99] Note that, since in the present setting all traders use the Bayes rule in order to form beliefs, we can consider that such heuristics are rationally employed.[100] At the end of Section 1.2.3, we will observe the

[97]Of course, only in this case considering imperfectly informed traders as a separated category of *informed* traders makes economic sense.

[98]Most models that account for technical or fundamental analysis assume two groups of traders: informed (often called sophisticated or rational traders) and uninformed (mostly designated as liquidity or noise traders). The fundamentalists are mostly integrated into the first group, while the technical analysts are viewed as being uninformed, such as in De Long, Shleifer, Summers, and Waldmann (1990). We consider that neither technical nor fundamental rule can deliver perfect information, so that they both amount to particular cases of imperfectly informed strategies. More details in this respect are deferred to the final applicative part of Section 1.2.3.

[99]For more details on heuristics, see the subsequent Section 2.1.

[100]Please refer to Chapter 2 for an example of non-rational employment of heuristics.

market evolution for some simple trading rules of both fundamentalist or technical type.

Thei *trading strategies* are formulated as follows: Informed traders can either buy or sell one unit of risky asset, or simply do nothing at each time t, i.e. $x_{\mathbf{gt}} \in \{B, S, \emptyset\}$, for $\mathbf{g} \in \{\mathbf{a}, \mathbf{b}\}$. There are no budget constraints that can restrict trader decisions. Moreover, we consider that both informed groups fully trust their signals, irrespective of the true risky value. Hence they buy, sell, or do nothing when they receive positive, negative, and no information, respectively:[101]

$$\begin{aligned}
&P_t(x_{\mathbf{gt}} = B | s_{\mathbf{gt}} = 1, V = V^H) = 1 - P_t(x_{\mathbf{gt}} = S | s_{\mathbf{gt}} = 1, V = V^H) \\
&\qquad = P_t(x_{\mathbf{gt}} = B | s_{\mathbf{gt}} = 1, V = V^L) = 1 - P_t(x_{\mathbf{gt}} = S | s_{\mathbf{gt}} = 1, V = V^L) =: 1 \\
&P_t(x_{\mathbf{gt}} = S | s_{\mathbf{gt}} = -1, V = V^H) = 1 - P_t(x_{\mathbf{gt}} = B | s_{\mathbf{gt}} = -1, V = V^H) \\
&\qquad = P_t(x_{\mathbf{gt}} = S | s_{\mathbf{gt}} = -1, V = V^L) = 1 - P_t(x_{\mathbf{gt}} = B | s_{\mathbf{gt}} = -1, V = V^L) =: 1 \\
&P_t(x_{\mathbf{gt}} = \emptyset | s_{\mathbf{gt}} = 0, V = V^H) = P_t(x_{\mathbf{gt}} = \emptyset | s_{\mathbf{gt}} = 0, V = V^L) =: 1.
\end{aligned} \tag{1.5}$$

In other words, the market maker considers that all (perfectly and imperfectly) informed traders trust their information, as they cannot directly observe the actual risky value during the trade.

As liquidity traders do not dispose of more than the public information, they are assumed to trade at random by either buying or selling one unit of risky asset, irrespective of the economic situation, with identical probabilities:[102]

$$\begin{aligned}
&P_t(x_{\mathbf{ct}} = B | V = V^H) = P_t(x_{\mathbf{ct}} = S | V = V^H) \\
&\qquad = P_t(x_{\mathbf{ct}} = B | V = V^L) = P_t(x_{\mathbf{gt}} = S | V = V^L) = 0.5.
\end{aligned} \tag{1.6}$$

The market maker accommodates the buy and sell orders issued by the traders and executes them at the currently quoted buy and sell prices, that are also referred to as *ask* X_t^B and *bid* X_t^S, respectively.[103] These prices are competitively set,[104] such that the

[101]The conditions from Equations (1.5) result in:

$$\begin{aligned}
&P_t(x_{\mathbf{gt}} = B | s_{\mathbf{gt}} = 1) = 1 - P_t(x_{\mathbf{gt}} = S | s_{\mathbf{gt}} = 1) = 1 \\
&P_t(x_{\mathbf{gt}} = S | s_{\mathbf{gt}} = -1) = 1 - P_t(x_{\mathbf{gt}} = B | s_{\mathbf{gt}} = -1) = 1 \\
&P_t(x_{\mathbf{gt}} = \emptyset | s_{\mathbf{gt}} = 0) = 1.
\end{aligned}$$

[102]Obviously, Equation (1.6) results in identical probabilities of buying and selling, independently of the risky value $P_t(x_{\mathbf{ct}} = B) = P_t(x_{\mathbf{ct}} = S) = 0.5$.

[103]We assume, that all transactions take place at exactly the quoted prices. In reality, dealers can preferentially treat some traders by offering them price reductions. See Harris (2003).

[104]The monopolistic power of the single market maker is constrained by the duty to set fair and efficient

market maker gains no profit from any of the buys and sells undertaken.[105] They are calculated on the basis of market maker's assessments regarding the type of economy. The market maker is subsequently committed to fulfill all the received traders' orders at these prices. The ask price normally exceeds the bid price by the amount of the *bid-ask spread* S_t. As discussed in Section 1.1.1, the spread should cover the order processing costs, the inventory costs, and the costs of adverse selection that arise during trading and represents the only source of earnings for the market maker. We expect a *constant* fraction of the spread to be responsible for the first two cost-generating factors. We also require that all transactions are processed by the market maker, i.e. there is no direct transaction among traders.[106]

Probability assessments of the market maker

According to the above assumptions, at each time t the trading is initiated by the market maker who fixes the current prices for buying and selling the asset. Then, the randomly chosen trader i observes these prices and acts, in that she either issues a market order for buying or for selling one unit risky asset or does nothing. The market maker commits herself to accepting and executing the submitted orders. She is aware of the fact that some traders may be better informed, in which case the order execution results in a loss for the market maker. Consequently, the ask and the bid are set so that contingent losses suffered in facing the informed are covered by gains from trade with uninformed, and the total expected trade result is nil. In other words, the two distinct prices for buy and sell protect the market maker against costs originating in informational disadvantages or, in other words, against the *adverse selection* generated by asymmetric market information.

This section presents the assessments of the market maker regarding the true risky value. The final prices are based on these assessments and hence result from the market maker's view over the economy type, the trading process, and the traders' strategies. In particular, the market maker attempts to ascertain the probabilities that the chosen trader is informed and prefers to buy, sell, or do nothing. Recalling that she applies the

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prices. This analysis can therefore be viewed as a marginal case of the general situation with many competing market makers. Some real markets (e.g. NYSE) do indeed function with only one market maker per traded asset. According to Demsetz (1968) and Harris (2003), even in such markets there are factors accounting for competition, such as competing markets with lower bid-ask spreads, limit orders, other specialists, floor traders, etc.

[105] The competitive price setting represents an application of the zero expected profit-condition. Accordingly, the prices equal the expectations of the market maker regarding the value of the risky asset, conditional upon the available information.

[106] Thus, our market functions as a quote-driven system, as defined in Demarchi and Foucault (1998).

Bayes rule in order to update probabilities, her assessments with respect to the informed traders $\mathbf{g} \in \{\mathbf{a}, \mathbf{b}\}$ yield:

$$\begin{aligned}
P_t(x_{it} = B | i \in \mathbf{g}, V = V^H) &= P_t(x_{it} = B | i \in \mathbf{g}, V = V^H, s_{\mathbf{g}t} = 0) P_t(s_{\mathbf{g}t} = 0 | V = V^H) \\
&\quad + P_t(x_{it} = B | i \in \mathbf{g}, V = V^H, s_{\mathbf{g}t} = 1) P_t(s_{\mathbf{g}t} = 1 | V = V^H) \\
&\quad + P_t(x_{it} = B | i \in \mathbf{g}, V = V^H, s_{\mathbf{g}t} = -1) P_t(s_{\mathbf{g}t} = -1 | V = V^H) \\
P_t(x_{it} = S | i \in \mathbf{g}, V = V^H) &= \sum_{s_{\mathbf{g}t} \in \{-1;0;1\}} P_t(x_{it} = S | i \in \mathbf{g}, V = V^H, s_{\mathbf{g}t}) P_t(s_{\mathbf{g}t} | V = V^H) \\
P_t(x_{it} = B | i \in \mathbf{g}, V = V^L) &= \sum_{s_{\mathbf{g}t} \in \{-1;0;1\}} P_t(x_{it} = B | i \in \mathbf{g}, V = V^L, s_{\mathbf{g}t}) P_t(s_{\mathbf{g}t} | V = V^L) \\
P_t(x_{it} = S | i \in \mathbf{g}, V = V^L) &= \sum_{s_{\mathbf{g}t} \in \{-1;0;1\}} P_t(x_{it} = S | i \in \mathbf{g}, V = V^L, s_{\mathbf{g}t}) P_t(s_{\mathbf{g}t} | V = V^L).
\end{aligned}$$

According to our model assumptions, this results for the perfectly informed traders in:

$$\begin{aligned}
P_t(x_{it} = B | i \in \mathbf{a}, V = V^H) &= \alpha_t \\
P_t(x_{it} = S | i \in \mathbf{a}, V = V^H) &= 0 \\
P_t(x_{it} = B | i \in \mathbf{a}, V = V^L) &= 0 \\
P_t(x_{it} = S | i \in \mathbf{a}, V = V^L) &= \alpha_t,
\end{aligned} \tag{1.7}$$

and for the imperfectly informed in:

$$\begin{aligned}
P_t(x_{it} = B | i \in \mathbf{b}, V = V^H) &= \beta_t q_t^H \\
P_t(x_{it} = S | i \in \mathbf{b}, V = V^H) &= \beta_t (1 - q_t^H) \\
P_t(x_{it} = B | i \in \mathbf{b}, V = V^L) &= \beta_t q_t^L \\
P_t(x_{it} = S | i \in \mathbf{b}, V = V^L) &= \beta_t (1 - q_t^L),
\end{aligned} \tag{1.8}$$

while liquidity traders buy and sell with equal probability as assumed in Equations (1.6).

Price formation

As mentioned above, the market maker fixes the periodical ask and bid prices competitively, so that no profits result from accommodating the received buy and sell orders. The prices should thus equate the expected values of the risky asset, conditional on the market maker's information set.

The *ask* X_t^B (*bid* X_t^S) represents the price at which traders can buy (sell) the risky asset and is derived as:

$$\begin{aligned}
X_t^B = E_t[V|x_{it} = B] &= V^H P_t(V = V^H|x_{it} = B) + V^L P_t(V = V^L|x_{it} = B) \\
&= V^L + (V^H - V^L) P_t(V = V^H|x_{it} = B) \\
X_t^S = E_t[V|x_{it} = S] &= V^H P_t(V = V^H|x_{it} = S) + V^L P_t(V = V^L|x_{it} = S) \\
&= V^L + (V^H - V^L) P_t(V = V^H|x_{it} = S).
\end{aligned} \tag{1.9}$$

The probability of a high asset value when the arriving trader i buys (sells) can be computed by means of the Bayes formula and yields:

$$\begin{aligned}
P_t(V = V^H|x_{it} = B) &= \frac{P_t(x_{it} = B|V = V^H) P_t(V = V^H)}{P_t(x_{it} = B|V = V^H) P_t(V = V^H) + P_t(x_{it} = B|V = V^L) P_t(V = V^L)} \\
P_t(V = V^H|x_{it} = S) &= \frac{P_t(x_{it} = S|V = V^H) P_t(V = V^H)}{P_t(x_{it} = S|V = V^H) P_t(V = V^H) + P_t(x_{it} = S|V = V^L) P_t(V = V^L)},
\end{aligned} \tag{1.10}$$

where the following holds according to our model assumptions:

$$\begin{aligned}
P_t(x_{it} = B|V = V^H) &= \sum_{\mathbf{g} \in \{\mathbf{a},\mathbf{b},\mathbf{c}\}} n_{\mathbf{g}} P_t(x_{it} = B|i \in \mathbf{g}, V = V^H) \\
&= \frac{1}{2}\Big((1 + (2\alpha_t - 1)n_{\mathbf{a}} + (2\beta_t q_t^H - 1)n_{\mathbf{b}}\Big) \\
P_t(x_{it} = S|V = V^H) &= \sum_{\mathbf{g} \in \{\mathbf{a},\mathbf{b},\mathbf{c}\}} n_{\mathbf{g}} P_t(x_{it} = S|i \in \mathbf{g}, V = V^H) \\
&= \frac{1}{2}\Big((1 - n_{\mathbf{a}} + (2\beta_t(1 - q_t^H) - 1)n_{\mathbf{b}}\Big) \\
P_t(x_{it} = B|V = V^L) &= \sum_{\mathbf{g} \in \{\mathbf{a},\mathbf{b},\mathbf{c}\}} n_{\mathbf{g}} P_t(x_{it} = B|i \in \mathbf{g}, V = V^L) \\
&= \frac{1}{2}\Big((1 - n_{\mathbf{a}} + (2\beta_t q_t^L - 1)n_{\mathbf{b}}\Big) \\
P_t(x_{it} = S|V = V^L) &= \sum_{\mathbf{g} \in \{\mathbf{a},\mathbf{b},\mathbf{c}\}} n_{\mathbf{g}} P_t(x_{it} = S|i \in \mathbf{g}, V = V^L) \\
&= \frac{1}{2}\Big((1 + (2\alpha_t - 1)n_{\mathbf{a}} + (2\beta_t(1 - q_t^L) - 1)n_{\mathbf{b}}\Big).
\end{aligned} \tag{1.11}$$

Thus, the ask and the bid result in:[107]

$$\begin{aligned}
X_t^B &\overset{(1)}{=} V^L + (V^H - V^L)p_t \frac{1 + (2\alpha_t - 1)n_{\mathbf{a}} + (2\beta_t q_t^H - 1)n_{\mathbf{b}}}{1 + (2\alpha_t p_t - 1)n_{\mathbf{a}} + (2\beta_t q_t^H p_t + 2\beta_t q_t^L (1 - p_t) - 1)n_{\mathbf{b}}} \\
&\overset{(2)}{=} V^H + (V^L - V^H)(1 - p_t) \frac{1 - n_{\mathbf{a}} + (2\beta_t q_t^L - 1)n_{\mathbf{b}}}{1 + (2\alpha_t p_t - 1)n_{\mathbf{a}} + (2\beta_t q_t^H p_t + 2\beta_t q_t^L (1 - p_t) - 1)n_{\mathbf{b}}}
\end{aligned} \tag{1.12a}$$

$$\begin{aligned}
X_t^S &\overset{(1)}{=} V^L + (V^H - V^L)p_t \frac{1 - n_{\mathbf{a}} + (2\beta_t(1 - q_t^H) - 1)n_{\mathbf{b}}}{1 + (2\alpha_t(1 - p_t) - 1)n_{\mathbf{a}} + (2\beta_t(1 - q_t^H)p_t + 2\beta_t(1 - q_t^L)(1 - p_t) - 1)n_{\mathbf{b}}} \\
&\overset{(2)}{=} V^H + \frac{(V^L - V^H)(1 - p_t)\Big(1 + (2\alpha_t - 1)n_{\mathbf{a}} + (2\beta_t(1 - q_t^L) - 1)n_{\mathbf{b}}\Big)}{1 + (2\alpha_t(1 - p_t) - 1)n_{\mathbf{a}} + (2\beta_t(1 - q_t^H)p_t + 2\beta_t(1 - q_t^L)(1 - p_t) - 1)n_{\mathbf{b}}}.
\end{aligned} \tag{1.12b}$$

Consequently, the price formation process is characterized by a double price setting in every trade period. The difference between the two simultaneously set ask and bid prices is denoted as the *bid-ask spread* $S_t = X_t^B - X_t^S$. According to Equations (1.12), the spread is always positive and can be expressed as follows:

$$\begin{aligned}
S_t &= 4(V^H - V^L)p_t(1 - p_t)\frac{(1 - n_{\mathbf{a}} - n_{\mathbf{b}} + \alpha_t n_{\mathbf{a}} + \beta_t n_{\mathbf{b}})(\alpha_t n_{\mathbf{a}} + \beta_t(q_t^H - q_t^L)n_{\mathbf{b}})}{\Pi_t}, \text{ where} \\
\Pi_t &= \Big(1 + (2\alpha_t p_t - 1)n_{\mathbf{a}} + (2\beta_t q_t^H p_t + 2\beta_t q_t^L(1 - p_t) - 1)n_{\mathbf{b}}\Big) \\
&\quad \cdot \Big(1 + (2\alpha_t(1 - p_t) - 1)n_{\mathbf{a}} + (2\beta_t(1 - q_t^H)p_t + 2\beta_t(1 - q_t^L)(1 - p_t) - 1)n_{\mathbf{b}}\Big).
\end{aligned} \tag{1.13}$$

1.2.3 Results

In this section, we first present some general price and spread properties. Second, we discuss how the ask, the bid, and the spread vary subject to group specific features of the two categories of informed traders, such as the informational probabilities and the proportion to the totality of traders, other things being equal. Unless otherwise specified, all proofs are included in Appendix A.1.1.

For reasons of analytical simplicity, we focus on ceteris paribus price variations, i.e. in dependency of each variable of interest (one at a time), such as the time, the proportions of different trader groups, the information probabilities, etc. Joint effects of these variables are addressed graphically. We focus on the following parameter combination: a good

[107]For each the ask and the bid, the formulations (1) and (2) are equivalent, and we make use of them in the proofs of the propositions in Section 1.2.3.

economy $V = V^H$, proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$,[108] public beliefs $p_t = 0.50$, a probability of perfect information $\alpha_t = 0.20$ and accuracies of imperfect information in good and bad economies of $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$, respectively. Note that at each step of our analysis, all variables other than the argument of the considered ceteris paribus function are fixed.[109] Concerning the price evolution in time, we also present results for $V = V^L$, random $\alpha_t \sim U[0,1]$, and random $\beta_t q_t^H, \beta_t q_t^L \sim U[0,1]$.

General results

The general results presented here replicate the findings of previous market microstructure models that assume the existence of only two trader categories, informed and uninformed, such as Glosten and Milgrom (1985) and Easley and O'Hara (1987). We show that, even with better nuanced informational asymmetries – in particular with our third category of imperfectly informed traders – the price formation process continues to be governed by the same rules.

Let us first consider prices in a general sense, that is with respect to all possible transactions that may occur at time t. These prices, which we henceforth refer to as "general prices" or "prices as a whole" and denote by X_t, represent formally expectations of the true risky value given the current public information and yield:

$$\begin{aligned} X_t &= E_t[V] \\ &= E_t[V|x_{it} = B] \cdot \mathbf{1}_{\{x_{it}=B\}} + E_t[V|x_{it} = S] \cdot \mathbf{1}_{\{x_{it}=S\}} + E_t[V|x_{it} = \emptyset] \cdot \Big(1 - \mathbf{1}_{\{x_{it}=B\}} - \mathbf{1}_{\{x_{it}=S\}}\Big), \end{aligned}$$

where $\mathbf{1}$ stands for the indicator function.

Our first general result stresses that prices as a whole are efficient in the sense that public information offers no profit opportunities with respect to them:

Proposition 1. *The sequence of general transaction prices X_t forms a Martingale relative to the public information h_t.*

In consequence, first differences of general prices are serially uncorrelated.

[108]This comes in line with Easley, Kiefer, and O'Hara (1997b), who find a proportion of informed trades of around 17%. Moreover, for $n_{\mathbf{b}} = 75\%$, the number of remaining uninformed trades yields $n_{\mathbf{c}} = 5\%$, which can be considered as a reasonable minimal proportion of exogenously motivated trades.

[109]For instance, dependencies on a certain variable other than time (for instance on α_t) represent evolutions of the time-t prices as function of that variable when the remaining variables are "frozen" at time t (e.g. $X_t^B(\alpha_t)$ and $X_t^S(\alpha_t)$). Thus, we have chosen to keep the time-subscripts $_t$, since we do not work with values that are constant over all $t = 1, \ldots, T$, but *at* a fixed time t.

However, detailing the analysis for the double price setting process at each time t offers new insights: The two distinct but simultaneous prices set for trading the risky asset, namely the ask X_t^B and the bid X_t^S, may depend on history. Specifically, this occurs when the imperfect information is derived from the observation of past prices. In this case, the sequence of each ask and bid prices becomes informative and thus offers additional information with respect to the single transaction prices.

Proposition 2. *When imperfect information is history dependent, the transaction prices X_t^B and X_t^S are not Markovian.*

Several further conclusions address general, previously shown properties of prices at time t. First, under the assumptions of our model, information asymmetries result in a positive gap between the buy and sell prices:

Proposition 3. *At time t, the spread S_t is positive and the ask and bid prices X_t^B and X_t^S are bounded by the high and the low value of the risky asset $V^L \leq X_t^S \leq X_t^B \leq V^H$.*

Recall that our model assumes constant costs of order processing and inventory. Consequently, the sole source of this positive spread resides in the adverse selection costs that the market maker has to pay when she trades with better informed agents. Augmenting the ask and lowering the bid thus turns into a defence mechanism, that allows the market maker to recover contingent losses from trades with the informed at the expense of the uninformed traders. Naturally, this price setting procedure affects all agents in the market.

These ideas are summarized by the following corollary:

Corrolary 3.1. *The positive bid-ask spread S_t at a given trade time t represents the market maker's reaction to the adverse selection problem generated by the informational asymmetries present in the market.*

The spread expansion in consequence of the increasing adverse selection is consistent with the evidence provided by other authors and summarized in Section 1.1.1. The result expressed in the above Corollary thus speaks for the validity of our model.

Moreover, we can show that the market maker indeed expects to lose money to perfectly informed traders, as well as to imperfectly informed who dispose of highly accurate information, but makes profits from the trades with uninformed traders. The expected group specific profits $E_t[G_{\mathbf{g}t}|x_{\mathbf{g}t}]$, for $\mathbf{g} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, can be calculated on the basis of the differences between the expected value of the risky asset and the price of the undertaken action:

$$\begin{aligned} E_t[G_{\mathbf{gt}}|x_{\mathbf{gt}} = B] &= E_t[V|x_{\mathbf{gt}} = B] - X_t^B \\ &= V^H P_t(V = V^H|x_{\mathbf{gt}} = B) + V^L P_t(V = V^L|x_{\mathbf{gt}} = B) - X_t^B \\ E_t[G_{\mathbf{gt}}|x_{\mathbf{gt}} = S] &= X_t^S - E_t[V|x_{\mathbf{gt}} = B] \\ &= X_t^S - V^H P_t(V = V^H|x_{\mathbf{gt}} = S) - V^L P_t(V = V^L|x_{\mathbf{gt}} = S), \end{aligned} \tag{1.14}$$

and, thus, should be positive for the perfectly informed, negative for the liquidity traders, and variable for the imperfectly informed.

Proposition 4. *At time t, perfect information always generates positive expected profits $E_t[G_{\mathbf{at}}] \geq 0$, while liquidity traders are expected to lose money constantly $E_t[G_{\mathbf{ct}}] \leq 0$. Imperfect information of high (low) accuracy is expected to result in gains (losses).*

Remark 4.1 Specifically, the market maker expects highly accurate imperfect information under a current buy or sell when $(q_t^H - q_t^L)(1 - n_{\mathbf{a}} - n_{\mathbf{b}}) \geq 2\alpha_t q_t^L n_{\mathbf{a}}$ and $(q_t^H - q_t^L)(1 - n_{\mathbf{a}} - n_{\mathbf{b}}) \geq 2\alpha_t(1 - q_t^H)n_{\mathbf{a}}$, respectively.

Furthermore, it is interesting to analyze the price evolution subject to the prior probability p_t ascribed by the market maker to a positive evolution of the economy at large and known to all market participants at time t.

Proposition 5. *Both ask and bid prices at time t increase with the prior probability of a good economy p_t, other things being equal. The curvature of the ask X_t^B (bid X_t^S) is concave (convex) and hence the spread S_t is concave. The spread is zero when the true risky value is common knowledge $p_t \in \{0; 1\}$ and attains its maximum for a middle-range prior probability of a high risky value p_t^{max}.*

The exact formula of the spread-maximizing probability $p_t^{\max}$ is given in the proof of this proposition in Appendix A.1.1. It depends on the probability of perfect information α_t and on the accuracy of imperfect information in good and bad economies $\beta_t q_t^H$ and $\beta_t q_t^L$, in sum on the "quality" of the information that arrives in the market. In general, the spread is higher at middle-range values of the probability scale, as these values stand for higher uncertainty about the economy type.

The evolution of the ask and bid prices, as well as that of the spread, at the fixed time t and in ceteris paribus dependency on the public beliefs p_t at the same moment, is illustrated in Figure 1.1, in both good economies $V = V^H$ and bad economies $V = V^L$, and for the usual proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, perfect information probability $\alpha_t = 0.20$, and imperfect information accuracies $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$. The resulting spread-maximizing probability is $p_t^{\max} = 0.50$.[110]

Finally, we return to the market evolution in time and refer how public beliefs change at each trade.

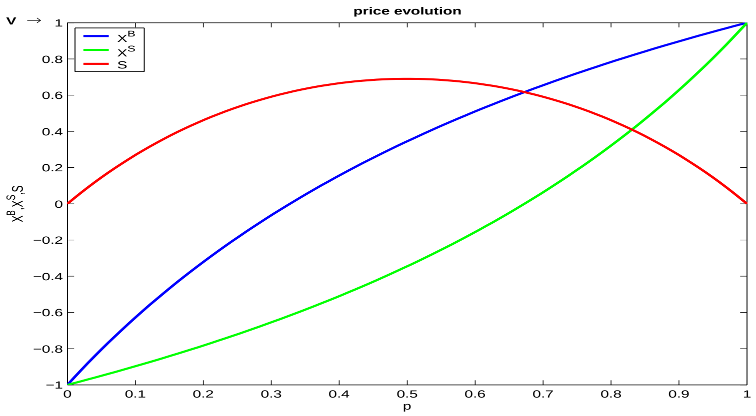
Proposition 6. *In time, the prior probability of a good economy p_t incorporates the information gathered by the market maker from the undertaken trades and converges towards 1 (0) for a high (low) risky value $V = V^H$ ($V = V^L$).*

Remark 6.1 Formally, the ratio $p_{t+1}/(1-p_{t+1})$ of the probabilities of high and low risky values at time $t+1$ equals the product of the same ratio one period before $p_t/(1-p_t)$, with the ratio of the prior probabilities $P_t(X_{t+1}^B, X_{t+1}^S|V^H)/P_t(X_{t+1}^B, X_{t+1}^S|V^L)$ that the market maker quotes at $t+1$ certain prices subject to the public information at t and to a certain (good vs. bad) economy.

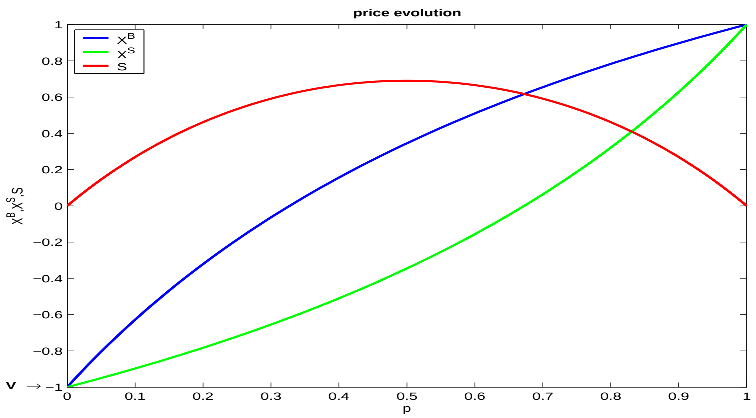
Exact formulas are included in the proof of this proposition in Appendix A.1.1. Thus, all market participants should be able to infer the real risky value after a sufficient number of transactions. We simulate the course of p_t and the corresponding ask and bid prices for both fixed and random values of the probabilities of perfect and imperfect information α_t and β_t, and for random variations of the ratio $P_t(X_{t+1}^B, X_{t+1}^S|V^H)/P_t(X_{t+1}^B, X_{t+1}^S|V^L)$, where all random variables are uniformly distributed in $[0, 1]$.

The evolution of prices and prior beliefs in time, for proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, and in good economies $V = V^H$ (bad economies $V = V^L$),

[110] We have considered an entire range of alternative parameter values and observed that the spread maximizing probability $p_t^{\max}$ varies in the direction of the true risky value. Specifically, for high risky value $V = V^H$, keeping the accuracy in bad economies $\beta_t q_t^L$ constant and increasing the accuracy in good economies $\beta_t q_t^H$ yields increasing $p_t^{\max}$. In addition, for low risky value $V = V^L$, $p_t^{\max}$ drops for fixed $\beta_t q_t^H$ and decreasing $\beta_t q_t^L$. Some numerical examples are the following: For $V = V^H$ and $\beta_t q_t^L = 0.33$, $\beta_t q_t^H = 0.75$ results in $p_t^{\max} = 0.5191$, $\beta_t q_t^H = 0.90$ yields $p_t^{\max} = 0.5917$, and $\beta_t q_t^H = 1$ gives $p_t^{\max} = 0.7341$. For $V = V^L$ and $\beta_t q_t^H = 0.67$, $\beta_t q_t^L = 0.25$ yields $p_t^{\max} = 0.4809$, $\beta_t q_t^L = 0.10$ results in $p_t^{\max} = 0.4083$, and $\beta_t q_t^L = 0$ gives $p_t^{\max} = 0.2659$. This reaction can be explained if we recall that $p_t = P_t(V|V = V^H)$ is the probability of a *good* economic development and that $\beta_t q_t^H$ ($\beta_t q_t^L$) reflects the information accuracy in good (bad) states. Thus, the more accurate the imperfect information becomes – i.e. the higher $\beta_t q_t^H$ is, when $V = V^H$, or the lower $\beta_t q_t^L$ is, when $V = V^L$ – the faster it should be incorporated into prices. The public beliefs (which are also the market maker's beliefs) p_t should hence point more and more to the true risky value, or, equivalently, lie closer to the right end of the probability scale for $V = V^H$ and the low end for $V = V^L$. We can also observe that $p_t^{\max}$ represents the point of maximal reluctance of the market maker towards meeting better informed investors and hence towards making losses in good economies. Then, $\beta_t q_t^H = 0.67$, $\beta_t q_t^L = 0.33$ describe for both $V = V^H$ and $V = V^L$ a situation in which the maximal loss-reluctance of the market maker $p_t^{\max}$ corresponds to the highest uncertainty with respect to the true economy 0.50.



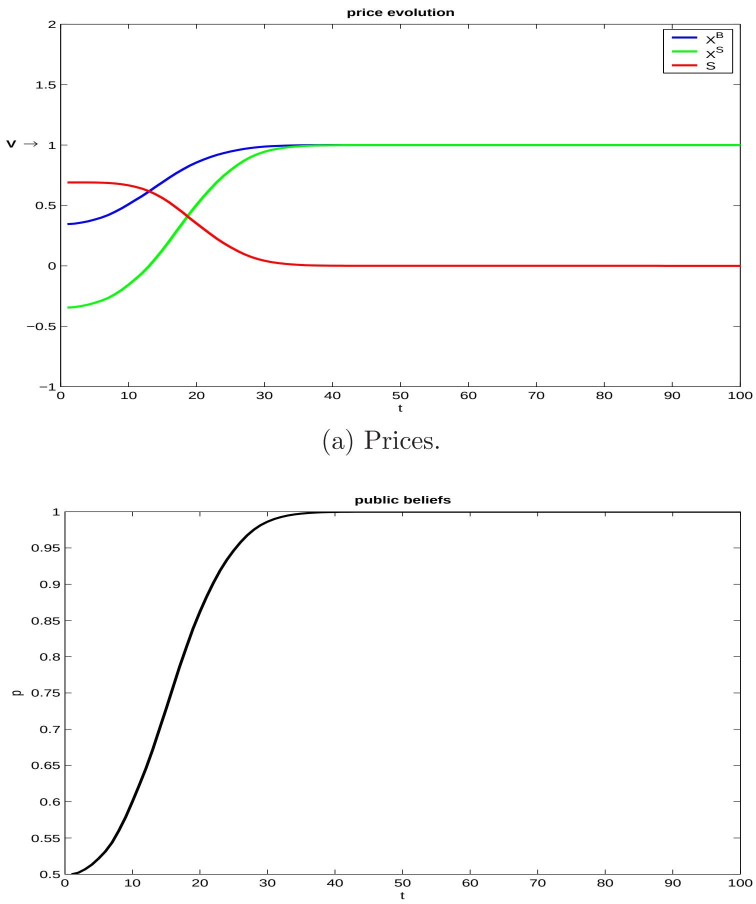
(a) Good economy $V = V^H$.



(b) Bad economy $V = V^L$.

Figure 1.1: The evolution of time-t prices subject to the public beliefs p_t, in different economies, for proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, a probability of perfect information $\alpha_t = 0.20$, and accuracies of imperfect information $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$.

are depicted in Figure 1.2 (Figure A.1 in Appendix A.1.2) for fixed perfect information probability $\alpha_t = 0.20$ and imperfect information accuracies $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$, and in Figure 1.3 (Figure A.2 in Appendix A.1.2) for random probabilities α_t, $\beta_t q_t^H$, and $\beta_t q_t^L$. We observe that, in time, the information is incorporated in prices and the public beliefs p_t converge to their true value. This process is noisier for random probabilities of (perfect and imperfect) information.



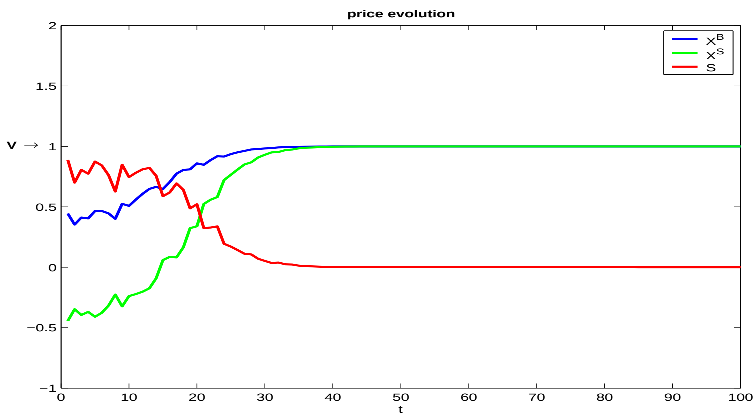
(a) Prices.

(b) Public beliefs.

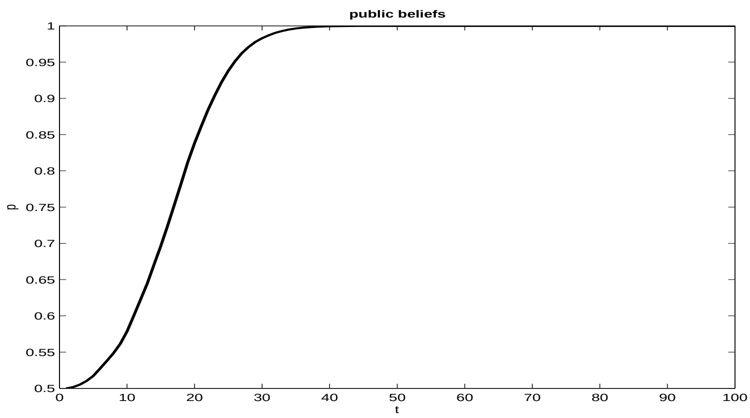
Figure 1.2: The evolution of prices and public beliefs in time, in a good economy $V = V^H$, for proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, a probability of perfect information $\alpha_t = 0.20$, and accuracies of imperfect information $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$.

The impact of imperfect information on prices

In the sequel, we focus on the impact of imperfect information on prices, according to the view – in particular, to the beliefs – of the market maker. This impact is twofold: On the one hand, a *qualitative influence* originates in the accuracy of imperfect information, measured by the probabilities that such information consists of positive signals in good



(a) Prices.



(b) Public beliefs.

Figure 1.3: Evolution of prices and public beliefs in time, in a good economy $V = V^H$, for proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, random probability of perfect information α_t, and random probabilities of imperfect information βq^H and βq^L of middle accuracy 0.50.

and bad economies $\beta_t q_t^H$ and $\beta_t q_t^L$, respectively. On the other hand, a *quantitative effect* is generated by the proportion of imperfectly informed traders $n_{\mathbf{b}}$ to the totality of traders.

Recall that we focus on ceteris paribus variations of prices in dependency of each variable of interest, while joint effects are graphically depicted for our particular cases.

A general conclusion can be formulated as follows:

Proposition 7. *The intensification, either in the qualitative or in the quantitative sense, of the trade activity of imperfectly informed traders at time t, other things being equal, entails the deterioration of trading terms for all market participants.*

The "deterioration of trading terms" here stands for the simultaneous augmentation of the ask and diminution of the bid, which results in a spread enlargement. The intuitive motivation of the effect addressed in Proposition 7 is straightforward: The market maker, although unaware of the identity of a potential trading counterparty, is acquainted with the fact that doing business with better informed agents generates losses. Thus, she faces adverse selection. Her defence mechanism consists in enhancing the spread, and she does so to a greater extent when the probability of facing better informed traders is higher.

In order to demonstrate Proposition 7, we rephrase it by means of several more specific statements focussing on variations in each of the qualitative and quantitative variables of interest.

First, we concentrate on the qualitative impact of imperfect information on prices. In this context, the variable $\beta_t q_t^H$ ($\beta_t q_t^L$) plays an important role when the economy is in a good (bad) state, as in this case the accuracy of imperfect information depends positively (negatively) on it.[111] The effect of the ceteris paribus variation of the probabilities of imperfect information that conforms with the true risky value on prices can be summarized as follows:

Proposition 7.1. *An increase of the accuracy of imperfect information in a good economy $\beta_t q_t^H$, other things being equal, makes the ask X_t^B (bid X_t^S) at time t to grow (drop) following a concave course. Subject to the decrease of the accuracy of imperfect information in a bad economy $\beta_t q_t^L$, prices exhibit similar trends but convex curvatures. In both situations, the spread S_t increases.*

We assumed that the market maker believes that informed traders follow their signals. Thus, the probability of imperfectly informed buys during a positive (negative)

[111]Recall that a decrease of $\beta_t q_t^L$ is equivalent to an increase of the probability of receiving negative signals in a bad economy $\beta_t(1 - q_t^L)$.

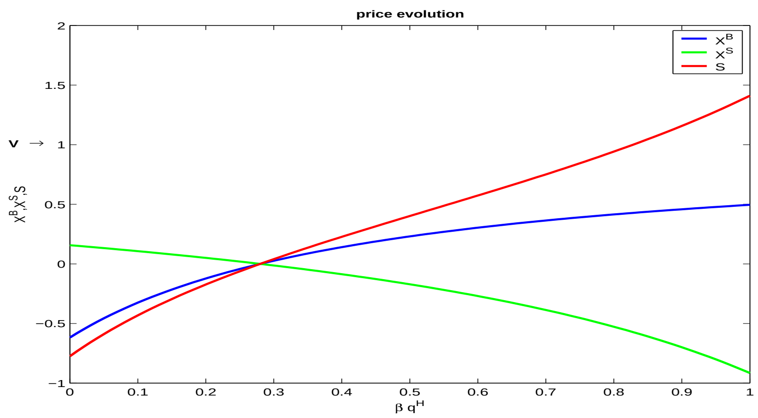
economic development increases with the probability that imperfectly informed traders receive positive (negative) signals, other things being equal. At the same time, the probability of imperfectly informed sells diminishes. Consequently, the market maker reckons on a higher loss probability and widens the bid-ask spread preventively. Interestingly, the spread increases at lower (higher) speed with the accuracy of imperfect information in good (bad) economies, suggesting that the intensity of the market maker reaction to better information depends on the type of economy.

Figure 1.4 depicts the evolution of prices and of the spread as a function of the accuracy of practical rules at time t, in a good economy $V = V^H$ (bad economy $V = V^L$), for proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, a probability of perfect information $\alpha_t = 0.20$, the accuracy of imperfect information in the opposite economy $\beta_t q_t^L = 0.33$ ($\beta_t q_t^H = 0.67$), and public beliefs $p_t = 0.50$. Note that the spread takes negative – and hence implausible – values as long as the imperfect information is excessively inaccurate $q_t^H < q_t^L$. This case is ruled out by our model, as required in Equation (1.4). Also, the spread exhibits an inflexion point which lies for the considered parameter values at $\beta_t q_t^H = 0.5329$ for the curve in panel a and at $\beta_t q_t^L = 0.4671$ for the curve in panel b. The spread at t as function of $\beta_t q_t^H$ is concave on the left and convex on the right of the inflexion point, while the opposite holds for the evolution of the spread as a function of $\beta_t q_t^L$.

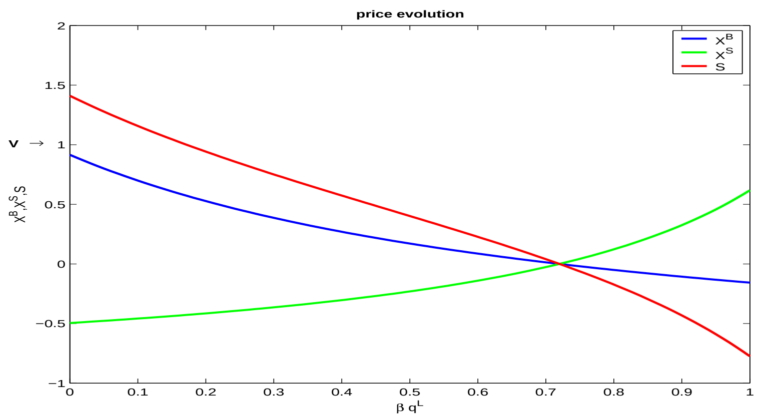
For the same values of the remaining parameters, Figure A.3 in Appendix A.1.2 shows the price and spread evolution subject to the joint variation of $\beta_t q_t^H$ and $\beta_t q_t^L$. We observe that the spread grows dramatically for higher informational accuracy in both good and bad economies, i.e. for high $\beta_t q_t^H$ and low $\beta_t q_t^L$. Its curvature changes across the different combinations of $\beta_t q_t^H$ and $\beta_t q_t^L$. The spread would be negative again for $q_t^H < q_t^L$, a case which is irrelevant for our analysis.

Second, we address the quantitative influence of imperfect information on prices, which can be traced back to the fraction of traders disposing of such information $n_{\mathbf{b}}$. Based on the price expressions in Equations (1.12), we can stress the following:

Proposition 7.2. *When imperfect information is sufficiently accurate, the ask X_t^B (bid X_t^S) at time t depends positively (negatively) on the proportion of imperfectly informed traders active in the market $n_{\mathbf{b}}$, other things being equal. The corresponding spread S_t increases thus for higher values of this proportion. Moreover, in this case the price curvatures vary with the probabilities of positive imperfect information at time t, irrespective*



(a) Time-t accuracy in good economy for $\beta_t q_t^L = 0.33$.



(b) Time-t accuracy in bad economy for $\beta_t q_t^H = 0.67$.

Figure 1.4: The evolution of time-t prices subject to the accuracies of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$, in a good economy $V = V^H$, for proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, public beliefs $p_t = 0.50$, and a probability of perfect information $\alpha_t = 0.20$.

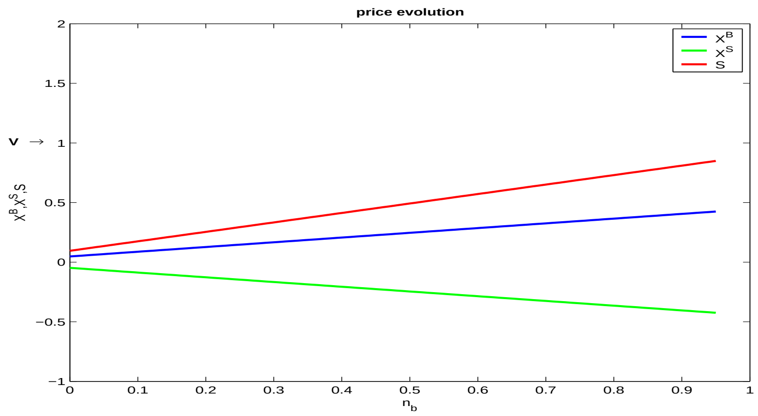
of the economy type.

Remark 7.2.1 At time t, the imperfect information is sufficiently accurate with respect to the ask X_t^B if the following holds: $\beta_t(q_t^H - q_t^L)(1-n_{\mathbf{a}}) + \alpha_t(1-2\beta_t q_t^L)n_{\mathbf{a}} \geq 0$. With respect to the bid X_t^S, the sufficient accuracy condition yields $\beta_t(q_t^H - q_t^L)(1-n_{\mathbf{a}}) + \alpha_t(1-2\beta_t(1-q_t^H))n_{\mathbf{a}} \geq 0$.
A sufficient condition regarding the information accuracy that makes the ask X_t^B increase with $n_{\mathbf{b}}$, other things being equal, is that the probability of deriving positive signals in a bad economy is low, that is $P_t(s_{\mathbf{b}t} = 1|V = V^L) = \beta_t q_t^L \leq 0.50$. For the bid, sufficiently accurate information is attained when negative signals in good economies are sooner improbable, that is $P_t(s_{\mathbf{b}t} = -1|V = V^H) = \beta_t(1-q_t^H) \leq 0.50$, a case when the bid decreases with $n_{\mathbf{b}}$.

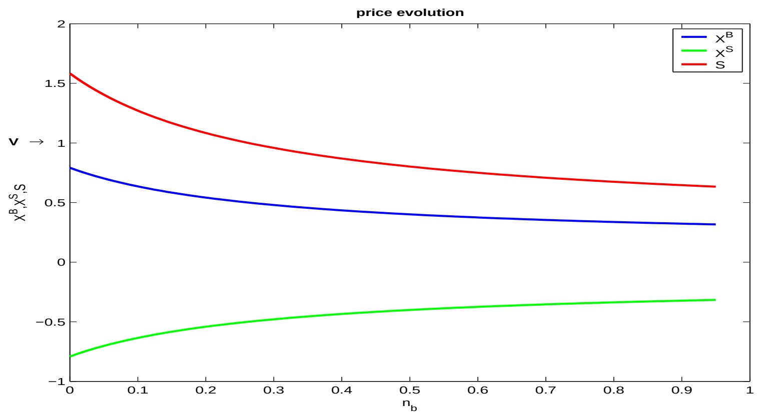
Remark 7.2.2 Specifically, with sufficiently accurate imperfect information, the ask X_t^B price evolves convexly when the derivation of positive imperfect information at time t is less probable relative to negative or no information $P_t(s_{\mathbf{b}t} = 1) = \beta_t q_t^H p_t + \beta_t q_t^L(1-p_t) \leq 0.50$. Otherwise the ask is concave. In contrast, the bid X_t^S is convex when it is less probable that negative information is obtained compared to positive or no information $P_t(s_{\mathbf{b}t} = -1) = \beta_t(1-q_t^H)p_t + \beta_t(1-q_t^L)(1-p_t) \leq 0.50$. Otherwise the bid is concave.

Proposition 7.2 suggests that, after isolating the (first and second-order ceteris paribus) influence of the number of imperfectly informed traders, the market evolution continues to depend on the accuracy of imperfect information. In other words, the quantitative impact cannot be fully disentangled from the qualitative one. The qualitative component, given by the information accuracy perceived by the market maker, appears to play a particularly important role both when the market maker concentrates on the number and on the precision of the imperfectly informed trades. It is possible that the market maker fears possible information asymmetries more than a higher number of trades that are possibly unadvantageous but not in a high degree. However, the joint effect of numerous and imperfectly, but yet sufficiently well informed, traders gives rise to intense adverse selection.

Figure 1.5 presents the prices and the spread at time t, as functions of the proportion of imperfectly informed traders $n_{\mathbf{b}}$, in a good economy $V = V^H$, public beliefs $p_t = 0.50$, either for a fixed proportion of either fully informed traders $n_{\mathbf{a}} = 20\%$ or liquidity traders $n_{\mathbf{c}} = 5\%$, for a probability of perfect information $\alpha_t = 0.20$, and imperfect information accuracies $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$. Indeed, for a fixed fraction of perfectly informed traders, the conditions in Remarks 7.2.1 and 7.2.2 are fulfilled, the spread increases and both the prices and the spread evolve linearly in dependency of $n_{\mathbf{b}}$. In contrast, fixing $n_{\mathbf{c}} = 5\%$ and varying $n_{\mathbf{b}}$ violates the conditions in Remark 7.2.1, while those in in Remarks 7.2.2 are still fulfilled and hence the spread decreases convexly. Intuitively, with (more imperfectly informed traders and implicitly) less perfectly informed traders active in the market, the market maker fears less extremely high losses caused by very accurate information and thus reduces the barriers meant to cover such losses, in particular the spread.



(a) Fixed proportion of perfectly informed traders $n_{\mathbf{a}} = 20\%$.



(b) Fixed proportion of liquidity traders $n_{\mathbf{c}} = 5\%$.

Figure 1.5: The evolution of time-t prices subject to the proportion of imperfectly informed traders $n_{\mathbf{b}}$, in a good economy $V = V^H$, for public beliefs $p_t = 0.50$, a probability of perfect information $\alpha_t = 0.20$, and accuracies of imperfect information $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$..

An intuition for the joint influence of qualitative and quantitative factors is provided in Figures A.4-A.7 in Appendix A.1.2, for the same particular cases. As expected, the depreciation of the trading conditions is more acute for jointly higher accuracy of imperfect information and more market participants who have access to this information, i.e. for higher $\beta_t q_t^H$ (lower $\beta_t q_t^L$, respectively) and higher $n_{\mathbf{b}}$, as long as the number of perfectly informed traders is fixed. For fixed $n_{\mathbf{c}}$, the improvement in accuracy is somewhat counterbalanced by the increase in the proportion of the imperfectly informed traders (that implies a decrease of the number of perfectly informed). The obvious motivation is the increased reluctance of the market maker towards losses from transactions with better informed traders, that become more probable in the first situation and less probable in the second one.

Remark 7.2.3 Prices evolve as described in Proposition 7.2 during an interval with no perfectly informed trades, specifically either due to the fact that there are no fully informed traders active in the market $n_{\mathbf{a}} = 0$, or because no perfect information reaches the market at the time of informed trades $\alpha_t = 0$.

The proof is evident from Equations (1.12). The two cases in Remark 7.2.3 represent manifestations of the same situation: the absence of perfect information. Either this information cannot be accessed at all during the trade (formally, there is no group of traders who have access to private information) or at the trade time t (such that the group **a** does not trade at t). Then the impact of the dimension of the single informed trader group that is active, namely the group **b**, becomes apparent and can be fully separated from the qualitative influence.

Finally, the joint impact of the prior probability of a high risky value p_t and the accuracy of imperfect information $\beta_t q_t^H$ ($\beta_t q_t^L$) on prices at time t is depicted in Figure A.8 (Figure A.9), again in a good economy $V = V^H$, for proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, and fixed probability of perfect information $\alpha_t = 0.20$, public beliefs $p_t = 0.50$, and imperfect information accuracy $\beta_t q_t^L = 0.33$ ($\beta_t q_t^H = 0.67$). Of note is that a more accurate imperfect information enhances the impact of public beliefs on prices, i.e. the spread increases and changes curvature. As expected, in case of more pronounced information asymmetries, the market maker becomes more wary in formulating periodical prices and conforms more closely with the available information provided by the prior probability of a good economy.

When imperfect information is derived by a larger number of traders (i.e. for higher $n_{\mathbf{b}}$) and at the same time public beliefs reflect high uncertainty (i.e. for middle-range p_t), the spread grows (falls) for fixed $n_{\mathbf{a}}$ ($n_{\mathbf{c}}$). This conclusion is illustrated in Figure A.10 (Figure A.11) for similar values of the remaining parameters and should be understood in the light of the same idea: As the market maker should fear informed traders more than uninformed but fully informed more than imperfectly informed, the spread increases with the ratio of the proportions of fully and partially informed traders. This reaction is thus consistent with the idea that adverse selection increases for both more informed traders and more accurate information.

The impact of perfect information on prices

In this section we replicate the analysis of the qualitative and quantitative impacts with respect to perfect information.

We commence again by formulating a general conclusion that is in the sequel analyzed separately for qualitative and quantitative aspects:

Proposition 8. *The intensification, either in the qualitative or in the quantitative sense, of the trade activity of perfectly informed traders at time t, other things being equal, entails the deterioration of trading terms for all market participants.*

In spite of the similarity of Propositions 7 and 8, there are several differences regarding the impacts of perfect and imperfect information on prices, on which we comment in the sequel.

Similarly to the above section dedicated to the imperfectly informed traders, Proposition 8 is demonstrated in two steps: At first, we investigate the role played in the price evolution by the probability α_t that perfectly informed traders receive information at time t. We again denote this effect as *qualitative* and summarize it as follows:

Proposition 8.1. *An increase of the probability of perfect information α_t, other things being equal, entails a corresponding convex increase (concave decrease) of the ask X_t^B (bid X_t^S) at time t, thus a concave enhancement of the corresponding spread S_t.*

As often demonstrated in the literature, the market maker is confronted with more serious adverse selection when it is more probable that perfectly accurate signals reach the market. Her reaction consists of widening the spread. Note however that, while for the imperfectly informed the qualitative effect on the spread curvature was ambiguous,[112] the spread clearly increases faster for lower values of the full information probability α_t. It appears that perfectly informed traders are already perceived as a menace for low probabilities that they receive and trade on their information.

[112] The spread curvature depends on the information accuracy in different economies. See the proof of Proposition 7.1 in Appendix A.1.2.

The time-t price course dictated by the probability α_t that perfect information reaches the insider group, other things being equal, is depicted in Figure 1.6, for our usual particular case with good economy $V = V^H$, proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, public beliefs $p_t = 0.50$, and imperfect information accuracies $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$.

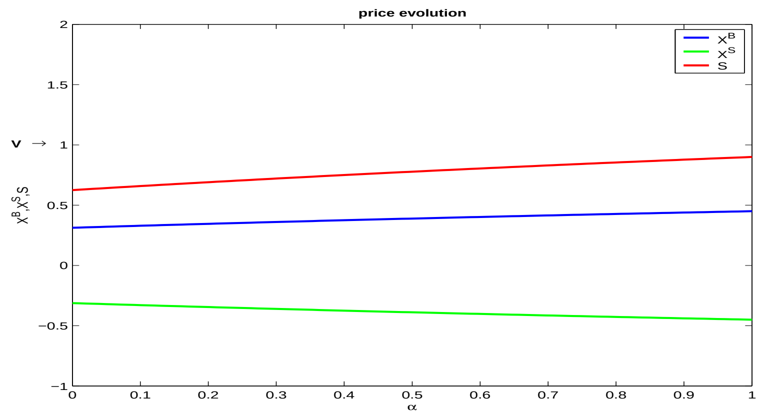


Figure 1.6: The evolution of time-t prices subject to the probability of perfect information α_t, in a good economy $V = V^H$, for proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, public beliefs $p_t = 0.50$, and accuracies of imperfect information $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$.

The following statement formalizes what we denote as the *quantitative* influence of perfect information on prices:

Proposition 8.2. *An increase of the proportion of fully informed traders $n_{\mathbf{a}}$, other things being equal, yields an enhancement (a reduction) of the ask X_t^B (bid X_t^S) at time t, thus an increase of the corresponding spread S_t. The curvature of prices depends on the probability that informed traders receive information α_t and on the prior probability of the economic state p_t.*

Remark 8.2.1 Specifically, the ask X_t^B and the bid X_t^S are convex if $\alpha_t p_t \leq 0.50$ and $\alpha_t(1 - p_t) \leq 0.50$, respectively, and concave otherwise.
In particular, if $\alpha_t p_t \geq 0.50$ the ask is concave and the bid convex, so that the spread S_t evolves concavely, while for $\alpha_t(1 - p_t) \geq 0.50$ the opposite holds.

Obviously, when the market maker anticipates that more fully informed traders are present in the market, the probability ascribed to making losses is higher. This renders the ask (bid) higher (lower). Of course, the variation speed of the spread depends on the probability that these traders receive information and hence proceed to trade. For instance, when perfectly informed traders receive information with a low probability α_t, the ask (bid) grows (drops) at an increasing (decreasing) speed, i.e. convexly (concavely), subject to a higher proportion of fully informed traders $n_{\mathbf{a}}$. In other words, in this case informed traders are considered to be dangerous only if they are numerous, because otherwise the probability that they trade is small.

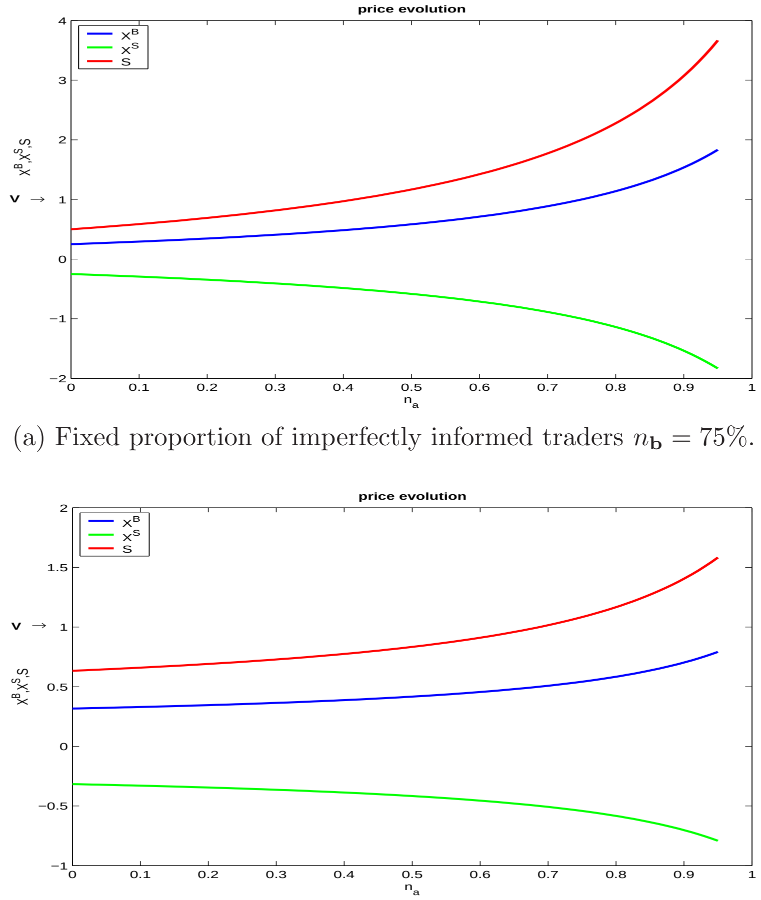
Likewise with respect to imperfect information, the quantitative effect of perfect information cannot be entirely separated from the qualitative one. However, this interdependency is now weaker, specifically only of second-order degree, thus affecting not the trend, but only the variation speed of prices. Note that the second-order dependency now also implies the prior beliefs p_t.

The evolution of time-t prices subject to the proportion of perfectly informed traders is presented in Figures 1.7, for fixed proportions of imperfectly informed traders $n_{\mathbf{b}} = 75\%$ and of liquidity traders $n_{\mathbf{c}} = 5\%$, in good economies $V = V^H$, perfect information probability $\alpha_t = 0.20$, public beliefs $p_t = 0.50$, and imperfect information accuracies $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$. We can observe that in both cases, the conditions in Remark 8.2.1 are fulfilled, so that the ask is convex, the bid concave, and hence the spread increases convexly. Together with the above results illustrated in Figure 1.5, we can clearly observe that the proportion of perfectly informed traders is the one that dictates the price evolution.

The joint effect of $n_{\mathbf{a}}$ and α_t on prices is illustrated in Figures A.12 and A.13 in Appendix A.1.2, for identical values of the remaining parameters. Other things being equal, the spread changes from extremely convex (concave) and increasing (decreasing) to linearly increasing in $n_{\mathbf{a}}$, for higher probabilities α_t that the perfect information reaches the market, when the proportion of imperfectly informed traders (liquidity traders) is fixed at $n_{\mathbf{b}} = 75\%$ ($n_{\mathbf{c}} = 5\%$). Naturally, when α_t is low, it takes a very numerous presence of potential insiders in the market for the market maker to take protective measures against possible losses.

Imperfect information from practical trading rules: some particular cases

As stressed above, imperfect information can be obtained applying common trading methods, such as the technical or fundamental analysis. The *technical analysis* attempts to forecast future evolutions by examining past movements in price and trade volume. In contrast, the *fundamental analysis* investigates the company, market, and global environments in order to determine possible causes for these movements.



(a) Fixed proportion of imperfectly informed traders $n_\mathbf{b} = 75\%$.

(b) Fixed proportion of liquidity traders $n_\mathbf{c} = 5\%$.

Figure 1.7: The evolution of time-t prices subject to the proportion of perfectly informed traders $n_\mathbf{a}$, in a good economy $V = V^H$, for public beliefs $p_t = 0.50$, a probability of perfect information $\alpha_t = 0.20$, and accuracies of imperfect information $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$.

Technical and fundamental analysis The technical analysis considers the market as the sole "instance" that values an asset. Technical analysts resort, among others, to the visual examination of the charts of past price and volume series. Thus, the core field of technical analysis methods is also known under the name of *chart analysis* or *chartism*. The purpose of this optical inspection is to recognize repetitive patterns in prices. Technical analysts believe that prices follow general movement directions – referred to as *trends* – and that price history is recurrent. Thus, repetitive patterns in past prices are reckoned to give valuable indications with respect to future market tendencies. In particular, different patterns suggest different action courses to by taken by traders, and manifold chart techniques have been developed in order to recognize them in due time for making profits.[113]

In contrast, the fundamental analysis argues for the existence of so-called *fundamental values* of financial assets. These values reflect the real, intrinsic value of the company that issued the respective asset, and not how this is priced by traders. Fundamental values are thus independent of transitory market phenomena. Their estimation necessitates complex studies involving not only financial aspects, but also broader economic interconnections, and even political, social, and ethical factors. As long as the fundamental value of an asset is lower (higher) than its market price, the asset is considered to be undervalued (overvalued) and the resultant recommendation of fundamentalists is to buy (sell).[114]

Both technical and fundamental analysis represent sets of rules that are, in the main, handy and easy to understand by common traders. This motivates their wide use in practice. In spite of the fact that initially, economists have refuted the chances of success of all not (fully-)rational strategies, more recent theoretical and empirical research speaks for the fact that technical and fundamental analysis can offer valuable information. For instance, several models, such as De Long, Shleifer, Summers, and Waldmann (1990), con-

[113]These patterns correspond optically to different shapes, such as spikes, wedges, triangles, etc., or to the combination of these simple shapes into more complicated ones, denoted as hedge-and-shoulders, double and triple tops and bottoms, etc. The numerous patterns considered by chartists to be relevant can be grouped into several main categories: trend, reversal, or continuation patterns. As reflected by the respective names, these patterns indicate the beginning, reversal, or continuation of price trends. Please refer to Murphy (1999) and Edwards, Magee, and Bassetti (2001) for more details.
Strictly speaking, when our imperfectly informed traders apply technical analysis methods, the probabilities $\beta_t q_t^H$ and $\beta_t q_t^L$ should be functions of both price and volume histories at time t. As our model does not account for the influence of trading volume, $\beta_t q_t^H$ and $\beta_t q_t^L$ will be based merely on the public price history h_t.

[114]According to Murphy (1999), fundamental values are derived from the analysis of the global, market, and firm-specific situation. When our imperfectly informed traders make use of fundamental analysis, the probabilities $\beta_t q_t^H$ and $\beta_t q_t^L$ should depend both on the on the current public information h_t and on the fundamental value of the traded asset.

sider fundamentalists as belonging to the category of informed traders. Informed traders are traditionally proven to outperform uninformed ones. Several further approaches corroborate with the relevance of technical information (as a combination of the past prices and volumes) or of some technical methods (such as moving average) for obtaining positive excess returns in the stock and foreign exchange market. Specifically, Treynor and Ferguson (1985), Brown and Jennings (1989), and Blume, Easley, and O'Hara (1994) account from a theoretical point of view for the rationality of the technical analysis and the effectiveness of using a price sequence instead of single prices. They also emphasize the benefits of the combination of past prices and volumes for choosing optimal trading strategies. Using real market data, Brock, Lakonishok, and Le Baron (1992), Allen and Karjalainen (1999), Neely, Weller, and Dittmar (1997), and Lo, Mamaysky, and Wang (2000) consider the success of different technical methods to be, in the main, positive.

We consider practical trading rules – in particular technical and fundamental analysis – as examples of heuristics, i.e. rules-of-thumb meant to facilitate and speed up the decisional process.[115] These rules provide information about future market evolutions, but the accuracy of this information varies across methods and in dependence on the specific market conditions. Thus, technical and fundamental analysts are potential partially-informed traders in the sense of our model. It is then interesting to observe how prices evolve in our setting when imperfect information is generated from certain practical trading rules of chartist or fundamentalist type; And also which are the profit chances offered by these rules to their users, recalling that they are rationally employed and coexist with full information.

[115] Please refer to Section 2.1.2 for more details on heuristics.

Simple trading strategies and further assumptions To this end, we concentrate on three simple strategies of both chartist and fundamentalist origins. In order to introduce them formally, we avail ourselves of the following variables:

- the price midquote: $X_t^{\text{mid}} = \dfrac{X_t^B + X_t^S}{2}$ (1.15a)

- the five-day moving-average line: $\text{MA}_t^{(5)} = \dfrac{1}{5}\displaystyle\sum_{i=1}^{5} X_{t-i}^{\text{mid}}$ (1.15b)

- the subjective expectations of the true risky value (the fundamental value):

$$
\begin{aligned}
E_{\mathbf{b}t}[V] &= V^H P_t(V = V^H | s_{\mathbf{b}t}) + V^L P_t(V = V^L | s_{\mathbf{b}t}) = V^L + (V^H - V^L) P_t(V = V^H | s_{\mathbf{b}t}) \\
&= V^L + (V^H - V^L) \frac{P_t(s_{\mathbf{b}t} | V = V^H) p_t}{P_t(s_{\mathbf{b}t})} \\
&= \begin{cases} V^L + (V^H - V^L) p_t \dfrac{\beta_t q_t^H}{\beta_t q_t^H p_t + \beta_t q_t^L (1 - p_t)}, & \text{for } s_{\mathbf{b}t} = 1 \\ V^L + (V^H - V^L) p_t, & \text{for } s_{\mathbf{b}t} = 0 \\ V^L + (V^H - V^L) p_t \dfrac{\beta_t (1 - q_t^H)}{\beta_t (1 - q_t^H) p_t + \beta_t (1 - q_t^L)(1 - p_t)}, & \text{for } s_{\mathbf{b}t} = -1. \end{cases}
\end{aligned}
\tag{1.15c}
$$

Then, our practical trading rules can be formulated as follows:

(TA-1) (momentum strategy): The imperfectly informed traders buy (sell) after observing two successive buys (sells). Otherwise, they do nothing.

(TA-2) (moving-average strategy): The imperfectly informed traders buy (sell) if the current midquote X_t^{mid} crosses from below (from above) the five-days moving-average line $\text{MA}_t^{(5)}$. In case of equality, no order is submitted.

(FA) (fundamentalist strategy): The imperfectly informed traders buy (sell) if their expectation with respect to the true risky value – or, equivalently, the fundamental value – $E_{\mathbf{b}t}[V]$ lies above (below) the current midquote X_t^{mid}. In case of equality, no action is undertaken.

Note that, although stylized, the first two strategies belong to the broad category of chartist methods. In technical terms, they stand for momentum and moving-average techniques, respectively. In essence, momentum strategies make use of the existing trend. Momentum traders buy (sell) in rising (falling) markets. This is precisely what our (TA-1)-traders do, specifically in a simplified form where trends are assessed from the last two past trades. Moving-average methods are based on the comparison of current prices with the market trend, where the latter is obtained from averaged past evolutions. Our strategy (TA-2) proceeds similarly and considers averages over the last five days of trade. Finally, the third strategy (FA) relies on the comparison of group specific expectations of the true risky value to market prices and hence is of fundamentalist type.

In the sequel, we simulate the price evolution resulting from our model equations, when imperfectly informed traders use one of the above simple trading strategies. In so doing, we consider various parameter constellations and market configurations. In order to keep the exposition as clear as possible, we will present only a part of the results, which is also representative across all analyzed cases. Unless otherwise specified, the subsequent comments rely on the following assumptions: The proportion of perfectly informed traders is set at $n_{\mathbf{a}} = 20\%$. Imperfectly informed traders are active in the market in high, middle, or low proportions, that is $n_{\mathbf{b}} \in \{25; 50; 75\}\%$, respectively. The rest of trades is automatically considered to be uninformed.[116] Trade starts with neutral beliefs $p_0 = 0.50$ and unfolds over $T = 100$ periods. At each time t, an agent is chosen to trade with replacement from the trader pit.[117] The probability α_t that perfectly informed traders receive information is either random (in particular, uniformly distributed in $[0, 1]$) or fixed (i.e. $\alpha_t = \alpha$ over all trading times $t = 1, \ldots, T$, where our subsequent comments are based on $\alpha = 0.20$).

Note that all three simple strategies assume that imperfectly informed traders *always* derive information, that is $\beta_t = 1, \forall t = 1, \ldots, T$. Consequently, the accuracy of imperfect information is given merely by the probability q_t^H (q_t^L) in good (bad) economies. For reasons of consistency, we continue to refer to $\beta_t q_t^H$ and $\beta_t q_t^L$ as the accuracies of imperfect information. Let us now detail the choice of these probabilities.

Our simulations are run taking $\beta_t q_t^H$ and $\beta_t q_t^L$ to vary randomly around three distinct accuracy thresholds in the set $\{0.25; 0.50; 0.75\}$. Specifically, $\beta_t q_t^H$ ($\beta_t q_t^L$) lies above (below) the accuracy threshold with a quantity that is uniformly distributed in $[0, 1]$. In so doing, we attempt to study market evolutions subject to different information accuracies. It is yet important to distinguish between the chartist and the fundamentalist strategies: The chartist strategies rely on a constantly enlarging data set (since at each trade past

[116] We also run simulations for a fixed $n_{\mathbf{c}} = 5\%$. The corresponding results will be, in part, addressed below.

[117] The results obtained for the case when traders were chosen to trade without replacement are similar to those presented below.

data series incorporate the current prices), so that $\beta_t q_t^H$ and $\beta_t q_t^L$ change at each time t. In contrast, the fundamentalist strategy (FA) assumes that traders derive the information about the economic situation (or, equivalently, the fundamental value) from analyzing fundamental factors, i.e. independently of the trade process. Therefore, the information probabilities $\beta_t q_t^H$ and $\beta_t q_t^L$ of the (FA)-traders are fixed at the beginning of trade $t = 0$, and remain constant over all $t = 1, \ldots, T$. We emphasize this fact by dropping the time subscripts $_t$ when the (FA)-strategy is applied and hence work with the notations βq^H and βq^L.

More explanations have to be given on how exactly the information accuracy is set for each of our three simple strategies. The momentum strategy (TA-1) implies that, after two successive buys, the imperfectly informed traders are also buying the risky asset, which can be formally written as $x_{\mathbf{b}t} = B$. Since traders actions perfectly mirror their information, we also have $\beta_t q_t^H = \beta_t q_t^L = 1$. Similarly, after two successive sells, (TA-1)-traders should sell and thus $x_{\mathbf{b}t} = S$, which is equivalent to $\beta_t(1 - q_t^H) = \beta_t(1 - q_t^L) = 1$ and hence $\beta_t q_t^H = \beta_t q_t^L = 0$.[118] Otherwise, no action is undertaken, which equivalently yields $\beta_t = 0$.[119]

The second chartist strategy (TA-2) is similarly designed and follows the moving-average condition specified above: A buy $x_{\mathbf{b}t} = B$ (sell $x_{\mathbf{b}t} = S$) and hence perfect accuracy $\beta_t q_t^H = \beta_t q_t^L = 1$ ($\beta_t q_t^H = \beta_t q_t^L = 0$) results when the price midquote crosses the five-day MA-line from below (from above). Otherwise, $x_{\mathbf{b}t} = \emptyset$ and hence $\beta_t = 0$.

For the fundamentalist strategy (FA), βq^H and βq^L are uniquely determined *before* the beginning of trade, but in a similar manner: As explained above, they vary around the same three accuracy thresholds in $\{0.25; 0.50; 0.75\}$ by a random quantity that is uniformly distributed in $[0, 1]$. (FA)-traders compute the expected risky value according to Equation (1.15c), the last expression of which has three branches. Their signal $s_{\mathbf{b}t}$ will reflect the most probable – in terms of the accuracies βq^H and βq^L– of these three values.[120]

[118] In the strict sense, (TA-1) requires that the probability of buying after a series of two successive buys is 1, that is $P_t(x_{\mathbf{b}t} = B) = \beta_t q_t^H p_t + \beta_t q_t^L(1 - p_t) = 1$. Analogously, after two successive sells we have $P_t(x_{\mathbf{b}t} = S) = \beta_t(1 - q_t^H)p_t + \beta_t(1 - q_t^L)(1 - p_t) = 1$. For showing how this yields the conditions stressed in the text, let us consider the following equation $xp + y(1 - p) = 1$, where all $p, x, y \in [0, 1]$. We can easily derive $y = (1 - xp)/(1 - p) \leq 1$ and hence $x = 1$. Then, $y = 1$ as well. Then, this result applies for both cases, namely with $x = \beta_t q_t^H, y = \beta_t q_t^L$ after two successive buys and $x = \beta_t(1 - q_t^H), y = \beta_t(1 - q_t^L)$ after two successive sells. In other words, a buy (sell) becomes sure after a series of two consecutive buys (sells), *irrespective* of the economy type.

[119] More exactly, $\beta_t q_t^H = \beta_t q_t^L = 0$ and $\beta_t(1 - q_t^H) = \beta_t(1 - q_t^L) = 0$, which implied $\beta_t = 0$.

[120] For instance, if in a good economy $V = V^H$ the probability of a positive signal $P_t(s_{\mathbf{b}t} = 1|V = V^H) = \beta_t q_t^H p_t + \beta_t q_t^L(1 - p_t)$ is higher than each the probability of no signal $P_t(s_{\mathbf{b}t} = 0|V = V^H) = 1 - \beta_t$ and

The public beliefs p_{t+1} at the beginning of time $t+1$ are derived as posterior probabilities at t, conditional on the new information set $h_t \equiv h_{t-1} \bigcup \{x_t, X_t^B, X_t^S\}$ and based on the Bayes rule. This yields:

$$p_{t+1} = P(V = V^H | x_t, h_t \setminus \{x_t\}) = \frac{P_t(x_t | V = V^H) p_t}{P_t(x_t)}$$
$$= \begin{cases} \dfrac{p_t\Big(1 + (2\alpha_t - 1)n_{\mathbf{a}} + (2\beta_t q_t^H - 1)n_{\mathbf{b}}\Big)}{1 + (2\alpha_t p_t - 1)n_{\mathbf{a}} + (2\beta_t q_t^H p_t + 2\beta_t q_t^L (1 - p_t) - 1)n_{\mathbf{b}}}, \text{ for } x_t = B \\ \dfrac{p_t\Big(1 - n_{\mathbf{a}} + (2\beta_t(1 - q_t^H) - 1)n_{\mathbf{b}}\Big)}{1 + (2\alpha_t(1 - p_t) - 1)n_{\mathbf{a}} + (2\beta_t(1 - q_t^H)p_t + 2\beta_t(1 - q_t^L)(1 - p_t) - 1)n_{\mathbf{b}}}, \text{ for } x_t = S \\ p_t, \text{ for } x_t = \emptyset. \end{cases} \tag{1.16}$$

Results Under the above assumptions, we can now detail the estimation results. First, we address the price convergence, then the qualitative and quantitative impacts of imperfect information, and finally the trader gains.

Prices converge towards the true risky value, as long as the information asymmetries in the market are not too high and the trading lasts sufficiently long. Specifically, the convergence manifests for both TA-strategies, even with relatively numerous imperfectly-informed traders $n_{\mathbf{b}} = 75\%$, and irrespective of the way in which full information reaches the market (specifically for fixed $\alpha = 0.20$ or random $\alpha_t \sim U[0, 1]$). This can be observed in panels a and b of Figures 1.8 and 1.9 for good economies $V = V^H$. The moving-average strategy (TA-2) appears to induce somewhat faster convergence, especially when the probability of perfect information α_t is random. Qualitatively similar results were obtained for bad economic states $V = V^L$, across all considered combinations of parameters. An example for the same $n_{\mathbf{a}} = 20\%$, $n_{\mathbf{b}} = 75\%$, and random α_t is illustrated in Figure 1.10. Consequently, we henceforth focus on the case with $V = V^H$.

Yet, when imperfectly informed fundamentalists (FA) are present in high proportions in the market, prices do not converge within the interval of $T = 100$ trades, as shown by Panels c of Figures 1.8, 1.9, and 1.10. In particular, spreads are strictly positive, which is due to the following fact: Although the price corresponding to the "right" action, dictated by the true economy type – i.e. the ask X_t^B (the bid X_t^S) for $V = V^H$ ($V = V^L$) – comes fast close to the true risky value V, the other price bounces below (above) this value in

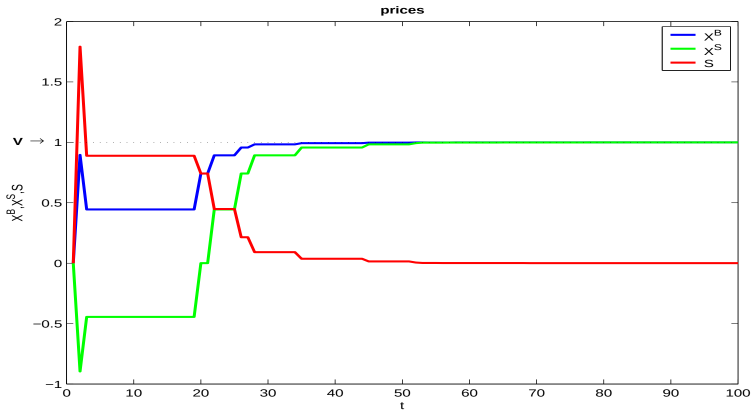
the probability of low signals $P_t(s_{\mathbf{b}t} = -1 | V = V^H) = \beta_t(1 - q_t^H)p_t + \beta_t(1 - q_t^L)(1 - p_t)$, then we take $s_{\mathbf{b}t} = 1$. Thus, $E_{\mathbf{b}t}[V]$ is derived according to the first branch of Equation (1.15c).

good (bad) economies.[121] Apparently, the (FA)-strategy does not allow the market maker to infer (sufficiently fast) the true economic state from the trader actions.

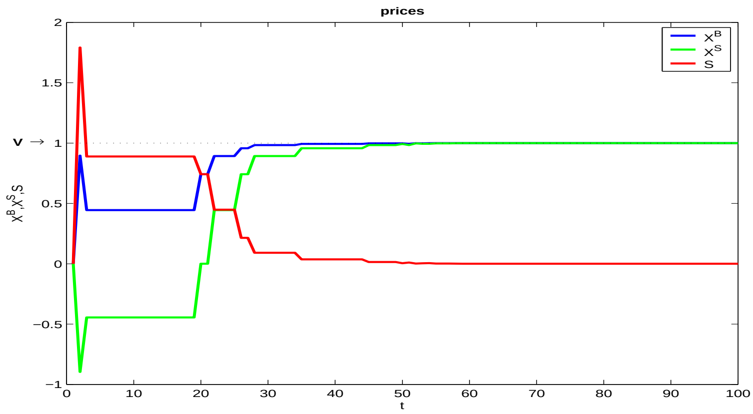
Moreover, with (FA) as an imperfectly informed strategy, the price convergence is ensured only for sufficiently high proportions of informed trades and/or for sufficiently high information accuracy. Thus, we find that prices attain the true risky value within the $T = 100$ trades merely for low proportions of fundamentalists $n_{\mathbf{b}} \leq 13\%$ and in our usual case with $n_{\mathbf{a}} = 20\%$, $V = V^H$, and random $\alpha_t \sim U[0,1]$ and $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50. For $n_{\mathbf{b}} = 15\%$ (and the same $n_{\mathbf{a}} = 20\%$, $V = V^H$, and random α_t), it takes high accuracies $\beta_t q_t^H$ and $\beta_t q_t^L$, specifically around a threshold of at least 0.90, in order to ensure price convergence within 100 trades.[122] If we fix the proportion of liquidity traders at $n_{\mathbf{c}} = 5\%$, then $V = V^H$, random α_t, random $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50 entail convergence within $T = 100$ trades for $n_{\mathbf{b}} \leq 43\%$ (and, equivalently, for $n_{\mathbf{a}} \geq 52\%$). With $n_{\mathbf{c}} = 5\%$ and $n_{\mathbf{b}} = 50\%$, it takes around 2000 trades for that market stability is attained.

[121] Where the variations are higher for random α_t.

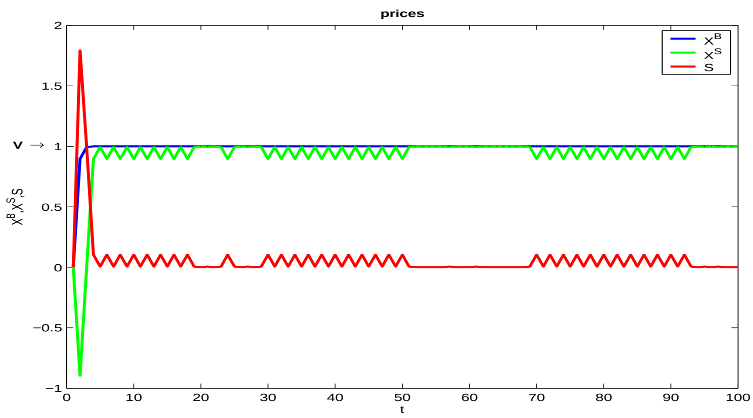
[122] The variations with the other parameters are also interesting. Following results were obtained for $n_{\mathbf{b}} = 13\%$: With a fixed $\alpha = 0.20$, around $T = 50$ trades are necessary for that prices come sufficiently close to the true value $V = V^H$. In contrast, the trade should unfold over at least $T = 140$ moments in order to ensure convergence for random $\alpha_t \sim U[0,1]$ and in bad economy $V = V^L$. Taking a high accuracy threshold of 0.9 for βq^H and βq^L ensures price convergence already within $T = 80$ for $V = V^L$ and random α_t.



(a) Momentum strategy (TA-1), for random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.

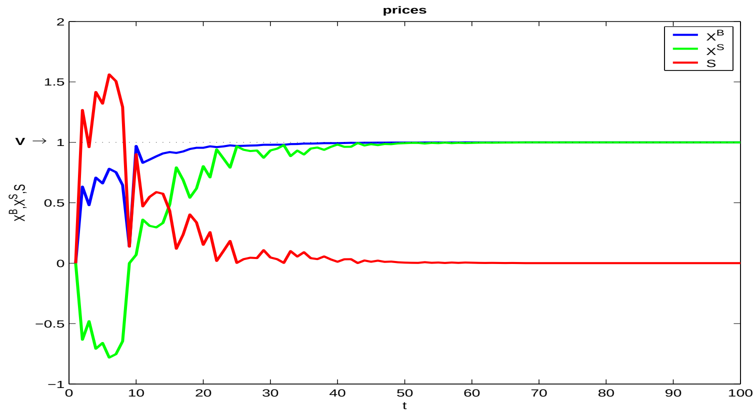


(b) Moving-average strategy (TA-2), for random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.

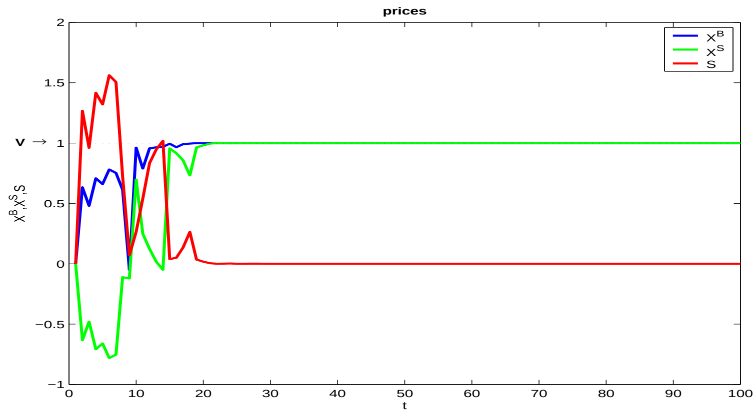


(c) Fundamentalist strategy (FA), for random probabilities of imperfect information βq^H and βq^L of middle accuracy 0.50.

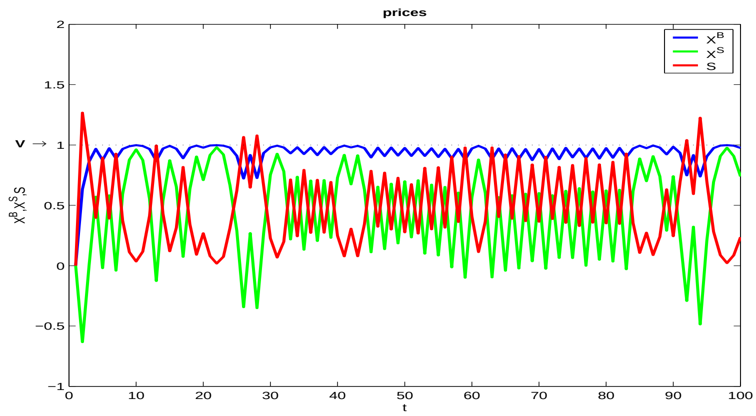
Figure 1.8: Price evolution for all three imperfectly informed strategies, in a good economy $V = V^H$, for a fixed probability of perfect information $\alpha = 0.20$, and proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$.



(a) Momentum strategy (TA-1), for random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.

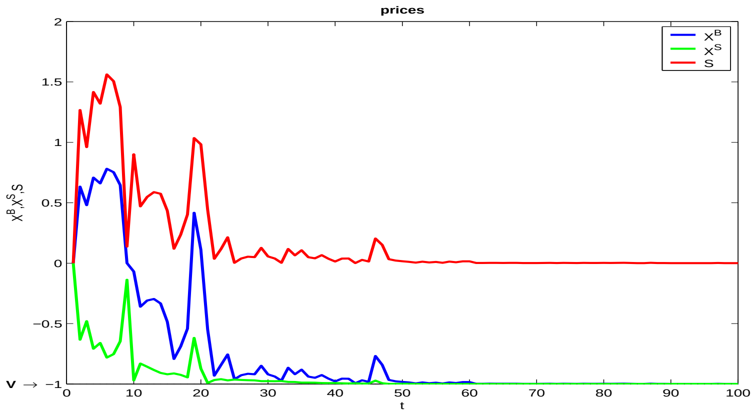


(b) Moving-average strategy (TA-2), for random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.

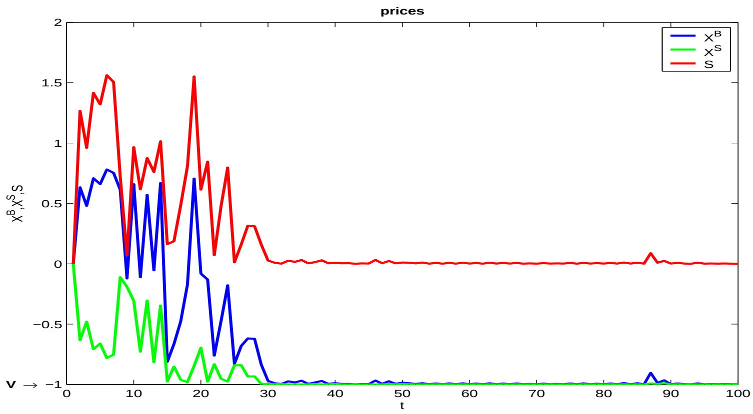


(c) Fundamentalist strategy (FA), for random probabilities of imperfect information βq^H and βq^L of middle accuracy 0.50.

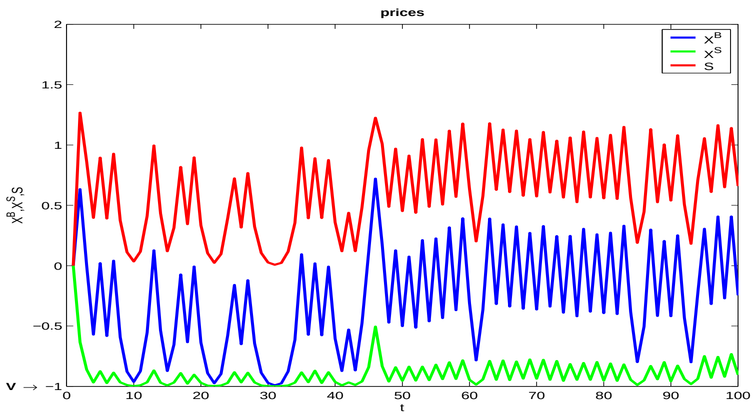
Figure 1.9: Price evolution for all three imperfectly informed strategies, in good economy $V = V^H$, for a random probability of perfect information $\alpha_t \sim U[0,1]$, and proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$.



(a) Momentum strategy (TA-1), for random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.



(b) Moving-average strategy (TA-2), for random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.



(c) Fundamentalist strategy (FA), for random probabilities of imperfect information βq^H and βq^L of middle accuracy 0.50.

Figure 1.10: Price evolution for all three imperfectly informed strategies, in a bad economy $V = V^L$, for a random probability of perfect information $\alpha_t \sim U[0,1]$, and proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$.

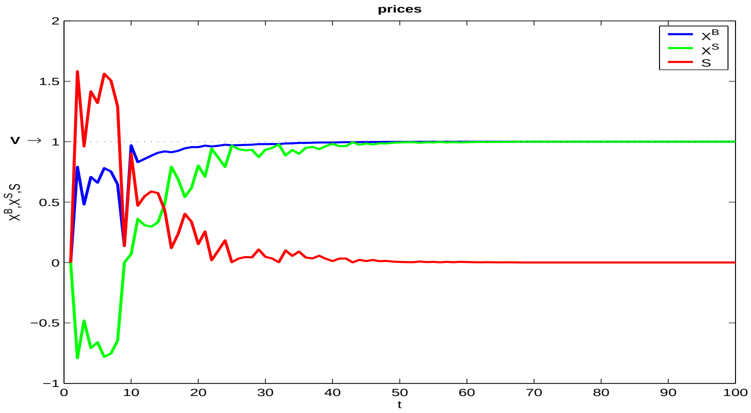
Let us further analyze the twofold influence of the users of practical trading rules on prices. Recall that the accuracy of their information generates the qualitative impact, while the trade participation intensity underlies the quantitative influence. Note that the two chartist strategies (TA-1) and (TA-2) yield similar results, from which we subsequently illustrate only those obtained under (TA-1).

Figure 1.11 illustrates the *qualitative effect* of the (TA-1)-strategy on prices for our usual case with good economies $V = V^H$, random $\alpha_t \sim U[0,1]$, and proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$. Our theoretical model in Section 1.2.3 predicts that spreads lower for more accurate imperfect information. Indeed, the spread in Figure 1.11 appears to vary (slightly) less for higher accuracy thresholds. However, there is no clear difference among the three panels of this figure. This is due to the fact that for our chartist strategies (TA-1) and (TA-2) the information probabilities $\beta_t q_t^H$ and $\beta_t q_t^L$ turn into binary variables. They take the value 1 when a certain, strategy-specific condition is met – for instance, after two successive actions of the same type for (TA-1), or when prices cross the moving-average line for (TA-2) – and 0 otherwise.

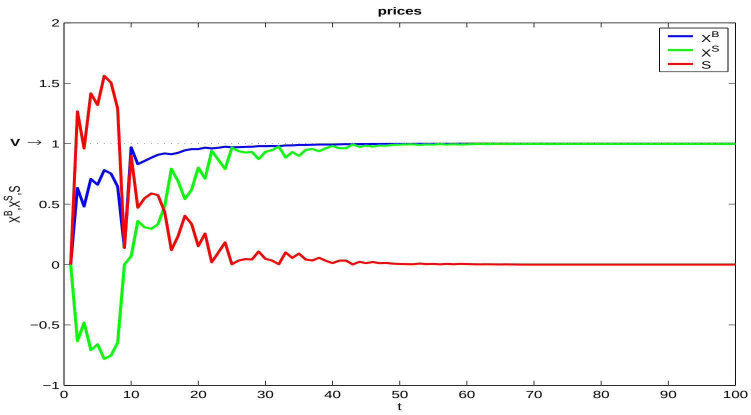
In essence, the same problem occurs for the fundamentalist strategy (FA), where the information probabilities βq^H and βq^L take either the value 1, when the deviation of the subjective expectation from market prices has a certain sign, or 0 otherwise. The qualitative effect of the fundamentalist strategy is somewhat easier to recognize in Figure 1.12, where the spread decreases and prices bounce less for higher accuracy thresholds of imperfect information.

The *quantitative effect* of imperfect information on prices can also be investigated. As Proposition 7.2 predicts, the spread is higher for larger proportions $n_{\mathbf{b}}$ of users of practical trading rules. This affirmation is supported for the momentum strategy (TA-1), as apparent in Figure 1.13, and for the fundamentalist strategy (FA) in Figure 1.14. However, note that these figures depict a situation that is similar but not identical to the one studied in the theoretical section. There we have considered *ceteris paribus* variations of prices with $n_{\mathbf{b}}$; In contrast, the present application assumes that other variables also change, such as the public beliefs p_t (see Equation 1.16). Figures 1.13 and 1.14 indicate that smaller $n_{\mathbf{b}}$ yield a less pronounced price variation and render the price convergence towards true asset values more probable. With (TA-1) the convergence is not necessarily faster. This is plausible since less momentum traders $n_{\mathbf{b}}$ coupled with a constant proportion of perfectly informed traders $n_{\mathbf{a}}$ result in a larger participation of liquidity traders $n_{\mathbf{c}}$. This reduces the amount of information to be incorporated into prices.

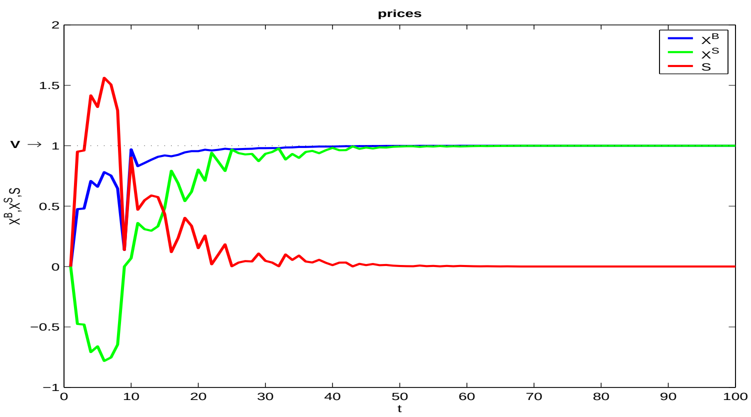
We also consider a market where the proportion of liquidity traders is fixed at $n_{\mathbf{c}} = 5\%$ and hence the variation of $n_{\mathbf{b}}$ entails changes in the perfectly informed trades $n_{\mathbf{a}}$. The pure qualitative and quantitative effects analyzed in the theoretical part are now contaminated by the variation of both $n_{\mathbf{a}}$ and p_t. Thus, the qualitative effect becomes optically indistinguishable. As for the quantitative one, Figures A.15 and A.16 in Appendix A.1.2 show that prices attain the true risky value for lower proportions of imperfectly informed traders $n_{\mathbf{b}}$ (in particular of momentum traders or fundamentalists, respectively). The moving average strategy (TA-2) delivers again results similar to (TA-1). Obviously, with less users of practical trading rules and, consequently, with more perfectly-informed traders, information should be faster impounded into prices.



(a) High accuracy of imperfect information (threshold 0.75).

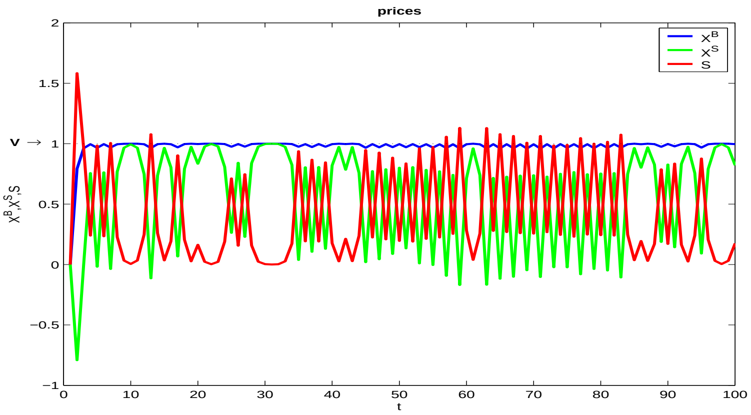


(b) Middle accuracy of imperfect information (threshold 0.50).

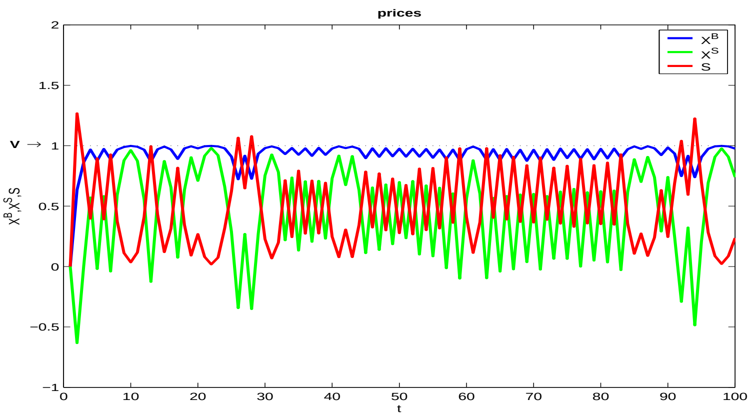


(c) Low accuracy of imperfect information (threshold 0.25).

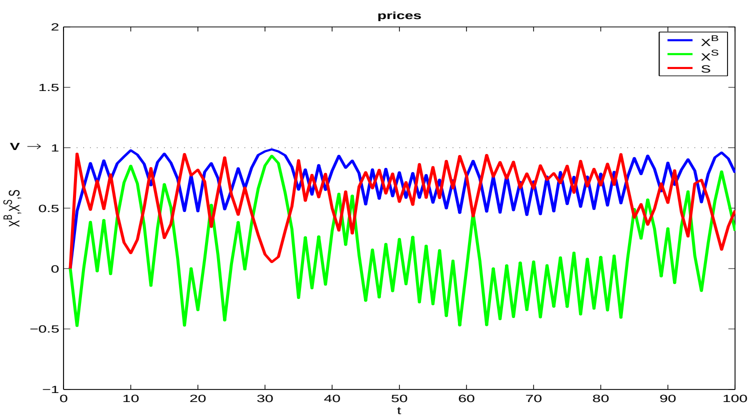
Figure 1.11: Price evolution for the momentum strategy (TA-1) of different accuracies, $T = 100$, in a good economy $V = V^H$, for random probability of perfect information $\alpha_t \sim U[0,1]$, and proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$.



(a) High accuracy of imperfect information (threshold 0.75).

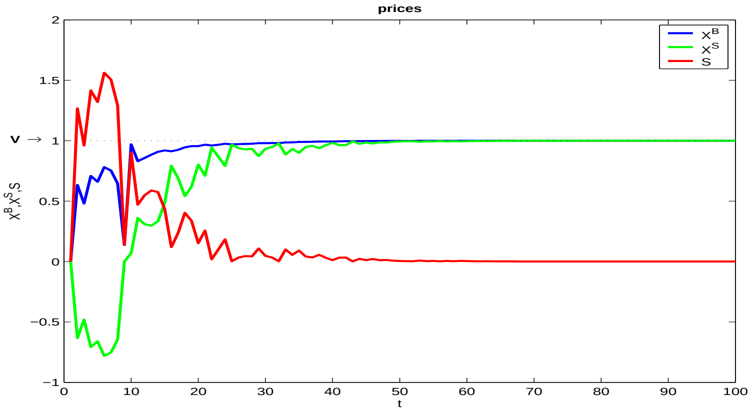


(b) Middle accuracy of imperfect information (threshold 0.50).

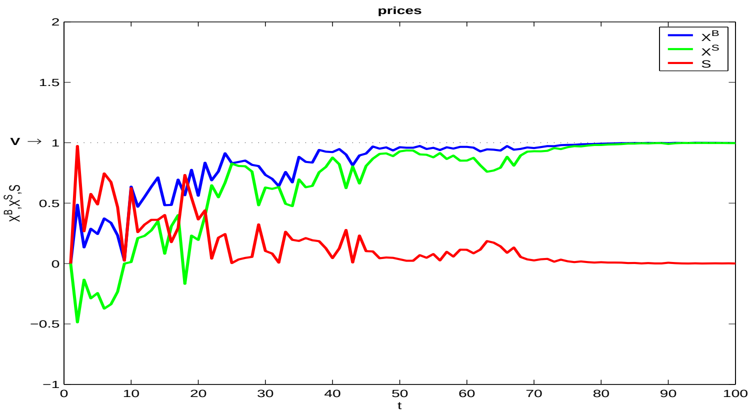


(c) Low accuracy of imperfect information (threshold 0.25).

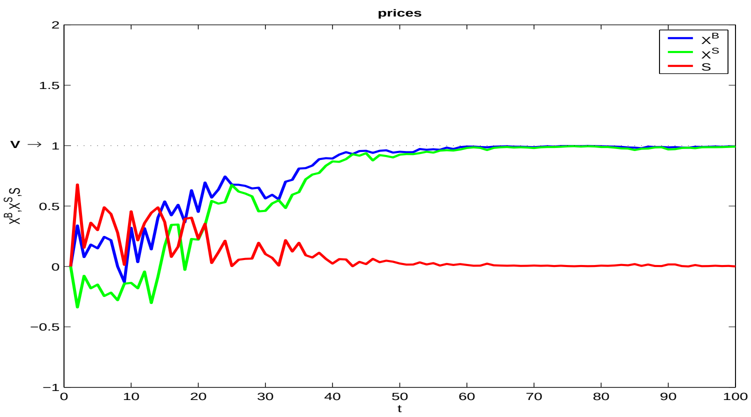
Figure 1.12: Price evolution for the fundamentalist strategy (FA) of different accuracies, $T = 100$, in good economy $V = V^H$, for random probability of perfect information $\alpha_t \sim U[0,1]$, and proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$.



(a) High proportion of imperfectly informed traders $n_{\mathbf{b}} = 75\%$.

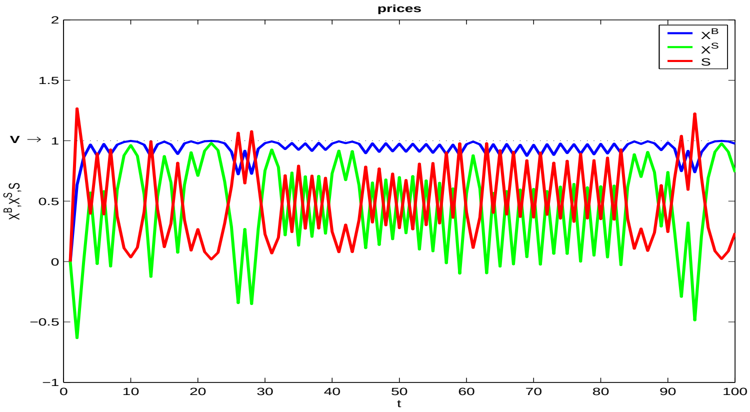


(b) Middle proportion of imperfectly informed traders $n_{\mathbf{b}} = 50\%$.

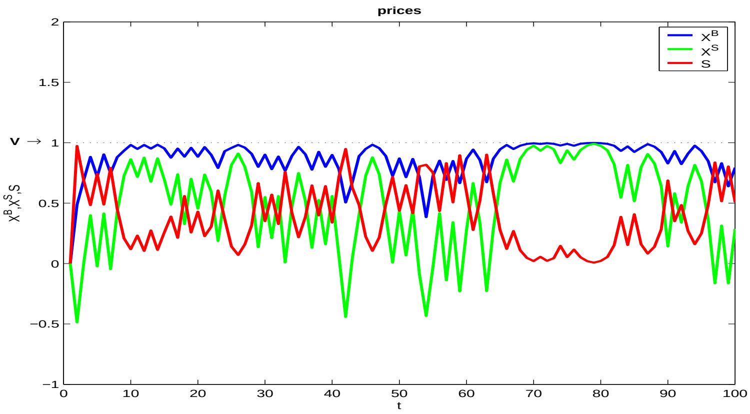


(c) Low proportion of imperfectly informed traders $n_{\mathbf{b}} = 25\%$.

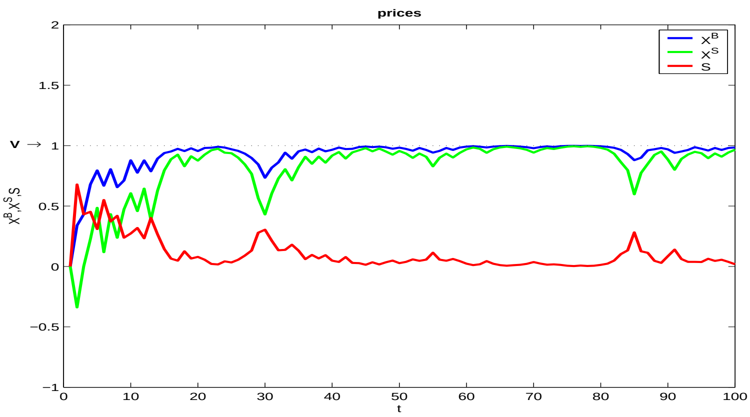
Figure 1.13: Price evolution for the momentum strategy (TA-1) for different proportions $n_{\mathbf{b}}$ of imperfectly informed traders, in a good economy $V = V^H$, for a random probability of perfect information $\alpha_t \sim U[0,1]$, a proportion of perfectly informed traders $n_{\mathbf{a}} = 20\%$, and random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.



(a) High proportion of imperfectly informed traders $n_{\mathbf{b}} = 75\%$.



(b) Middle proportion of imperfectly informed traders $n_{\mathbf{b}} = 50\%$.



(c) Low proportion of imperfectly informed traders $n_{\mathbf{b}} = 25\%$.

Figure 1.14: Price evolution for the fundamentalist strategy (FA) for different proportions $n_{\mathbf{b}}$ of imperfectly informed traders, in a good economy $V = V^H$, for a random probability of perfect information $\alpha_t \sim U[0,1]$, a proportion of perfectly informed traders $n_{\mathbf{a}} = 20\%$, and probabilities of imperfect information βq^H and βq^L of middle accuracy 0.50.

Furthermore, we compute the cumulated monetary *gains* of each trader group.[123] The background intuition for doing this is that observing monetary results we can gain an insight in the survival chances of the traders.[124] We would expect that perfectly informed traders not only always gain money from their trades, but also make the highest profits in the market.

Indeed – as illustrated in Figure 1.15 (Figure 1.16) for $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, a partial-information accuracy threshold of 0.50, and random α_t (fixed $\alpha = 0.20$) – the cumulated individual profits of perfectly informed traders are always positive. Simulations performed for further combinations of parameters show that the individual profits of **a**-traders are also the highest in the market across all analyzed cases. However, this does not necessarily hold with respect to group profits, with respect to which the group dimension plays a decisive role. Thus, the users of practical decision rules can earn more money, as a group, than their perfectly informed peers, as it is the case in panel a of Figure 1.17 for the (TA-1)-strategy. Note also that individual profits are on average higher with random probability of perfect information α_t. Interestingly, the momentum traders (TA-1) always make positive profits, both individually and as a group. In particular, this occurs for all considered information accuracies, i.e. $\beta_t q_t^H$ and $\beta_t q_t^L$ varying randomly around the thresholds in $\{0.25; 0.50; 0.75\}$. Individual profits also remain positive, irrespective of the trader participation $n_{\mathbf{b}}$. Moreover, for a sufficiently high proportion $n_{\mathbf{b}}$, (TA-1) yields for its users *as a group* even higher monetary gains than those obtained from perfect information. The corresponding individual gains remain comparable with those of perfectly informed traders as long as the probability of perfect information α_t is random. This situation is exemplified for $n_{\mathbf{b}} = 75\%$ and in good economic states $V = V^H$ in panels a of Figures 1.15 and 1.16.

In contrast, the moving-average strategy (TA-2) always results in losses, irrespective of the number of its supporters $n_{\mathbf{b}}$ and of the accuracy of their information $\beta_t q_t^H$ and $\beta_t q_t^L$. One possible illustration is given in panels b of Figures 1.15 and 1.17.

Panels c of the same figures show that fundamentalists (FA) who derive information in good (bad) economies with an accuracy around a middle-range threshold of 0.50, can also come best off in terms of their individual profits. This occurs for shorter or longer time intervals, at the beginning of the trade. This finding is replicated for numerous other combinations of parameters, such as for bad economies $V = V^L$, different information

[123] Note that we refer here to the *realized* gains, and not to the expected gains analyzed in Proposition 4.
[124] For more details on survival, please refer to Section 2.1.2.

accuracy thresholds, or different proportions of fundamentalists $n_{\mathbf{b}}$. Moreover, with low proportions $n_{\mathbf{b}}$ – in particular, comparable with those of perfectly informed traders $n_{\mathbf{a}}$ – and a fixed $\alpha = 0.20$, both individual and group profits of the fundamentalists are positive and the highest in the market during the entire trade interval. This result is illustrated in Figure A.17 in Appendix A.1.2 for $n_{\mathbf{b}} = 25\%$ and $V = V^H$ and also holds qualitatively in bad economies $V = V^L$, for different accuracy thresholds, and even for different values of the fixed perfect-information probability α.

More intuition for the success or failure of our simple trading strategies can be gained by analyzing the number of buys and sells effectively executed by the respective imperfectly informed traders. Mostly, momentum traders (TA-1) act in a "correct" way, i.e., they buy the risky asset in good economies $V = V^H$ and sell it in bad ones $V = V^L$. The rate of "correct guesses" is maximal (i.e. 100% of the cases) for high proportions $n_{\mathbf{b}} = 75\%$ and decreases when momentum traders become less numerous. Since, by the essence of their strategy, (TA-1)-traders are following the trend of the market (i.e. mimicking previous trades), their trades reinforce the price movements. If these traders have the chance to observe (a sufficient number of) previous informed trades, their actions should speed up the price convergence towards the true risky value and thus increase the market efficiency. Otherwise, they can only enhance the market noise.[125]

In contrast, the number of imperfectly informed buys and sells resulting from the moving-average strategy (TA-2) is better balanced. Under the market conditions considered here, this strategy entails actions that are similar to the random liquidity trades. This renders plausible our finding that it cannot generate positive profits.[126]

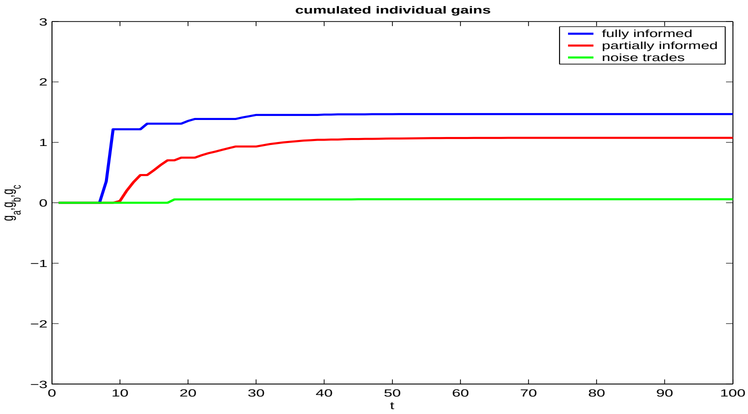
The fundamentalist strategy (FA) entails more frequent sells (buys) in good (bad) economies. Thus, it pushes the market in the "wrong" direction, i.e. opposed to the true risky value.[127] For this reason, prices converge much more slowly or not at all when

[125]Out of $T = 100$ trades, for random α_t and $V = V^H$ ($V = V^L$), momentum-traders buy 72 times and never sell (never buy, but sell 25 times). When $\alpha = 0.20$ and $V = V^H$ ($V = V^L$), the number of buys yields 31 (0) and that of sells 0 (32). For low $n_{\mathbf{b}} = 25\%$, a random α_t and $V = V^H$ ($V = V^L$), we observe 7 (2) buys and 3 (6) sells. Finally, a fixed $\alpha = 0.20$ and $V = V^H$ ($V = V^L$) generate 9 (4) buys and 7 (15) sells.

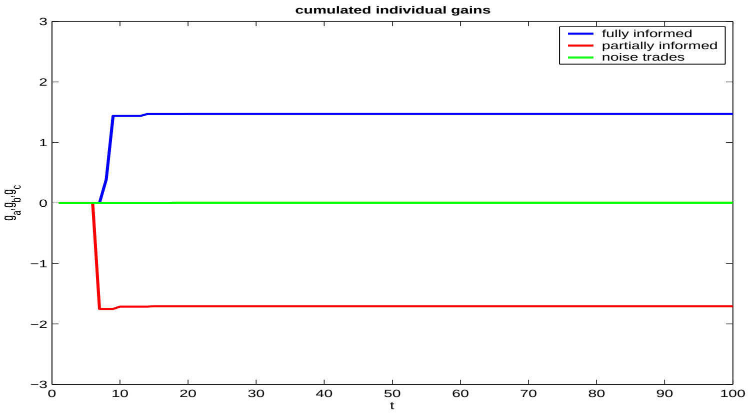
[126]For $V = V^H$ ($V = V^L$), (TA-2) results in 19 (28) buys and 21 (26) sells out of 100 trades when α_t is random, and in 1 (8) buys and 1 (10) sells with fixed $\alpha = 0.20$. When moving-average traders are less numerous $n_{\mathbf{b}} = 25\%$, the corresponding numbers of buys and sells for $V = V^H$ ($V = V^L$) and random α_t are 3 (5) and 1 (2), and for fixed $\alpha = 0.20$ respectively 3 (5) and 4 (3).

[127]The fundamentalist strategy amounts to 29 (44) buys and 49 (34) sells out of 100 trades for $V = V^H$ ($V = V^L$) and random α_t, and to 28 (45) buys and 47 (30) sells for fixed $\alpha = 0.20$. For low $n_{\mathbf{b}} = 25\%$, $V = V^H$ ($V = V^L$) and random α_t generate 4 (15) fundamentalists' buys and 18 (7) sells, while a fixed $\alpha = 0.20$ yields 11 (18) buys and 16 (9) sells. Middle proportions $n_{\mathbf{b}} = 50\%$ entail qualitatively similar results.

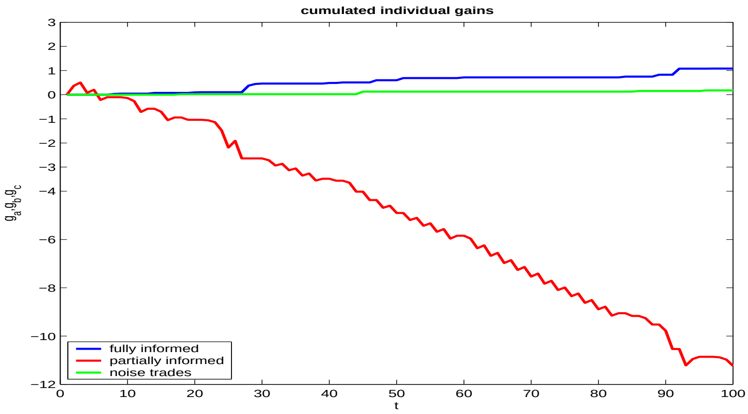
imperfect information is derived by means of this strategy, as observed above, which lowers the market efficiency. However, this does not imply that the fundamentalists are constantly loosing money. As illustrated in panels c of Figures 1.14 and A.17, these traders are able to generate sufficient noise in order to benefit from it and to outperform their fully informed peers.



(a) Momentum strategy (TA-1), for random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.

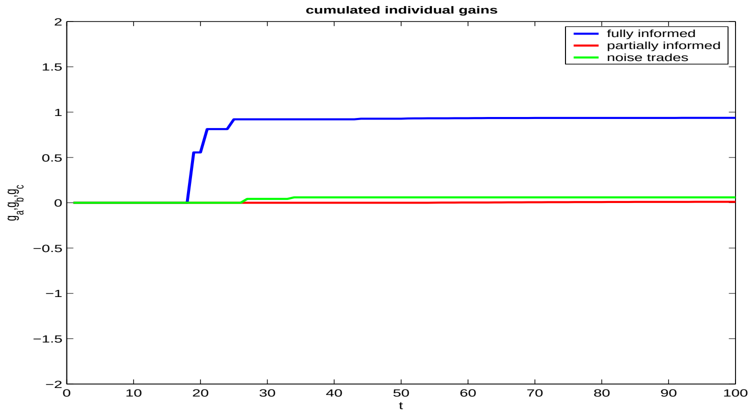


(b) Moving-average strategy (TA-2), for random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.

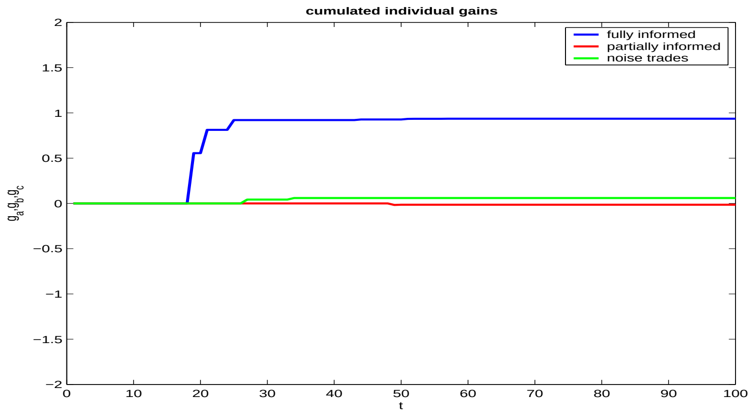


(c) Fundamentalist strategy (FA), for random probabilities of imperfect information βq^H and βq^L of middle accuracy 0.50.

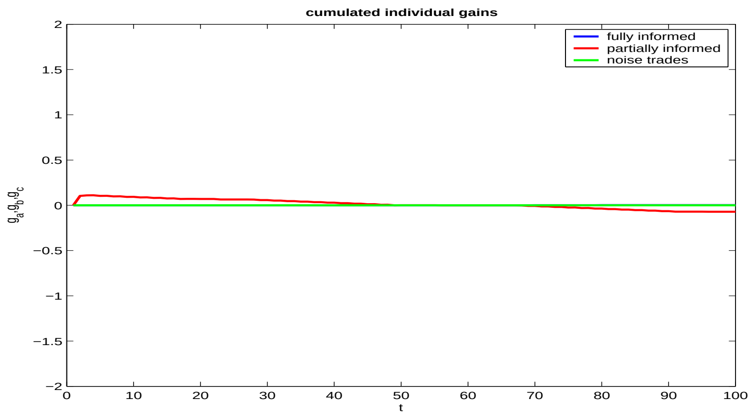
Figure 1.15: The evolution of individual cumulated gains for all three imperfectly informed strategies, in a good economy $V = V^H$, for a random probability of perfect information $\alpha_t \sim U[0,1]$, and proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$.



(a) Momentum strategy (TA-1), for random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.

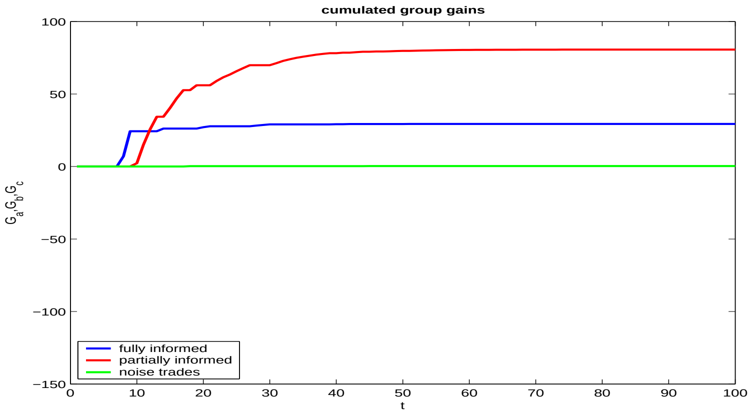


(b) Moving-average strategy (TA-2), for random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.

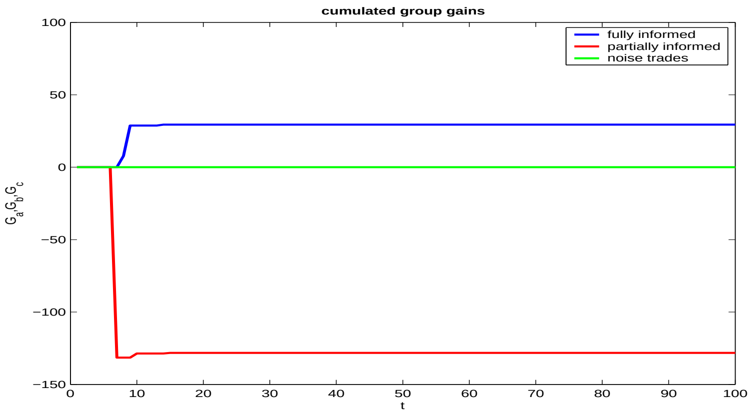


(c) Fundamentalist strategy (FA), for random probabilities of imperfect information βq^H and βq^L of middle accuracy 0.50.

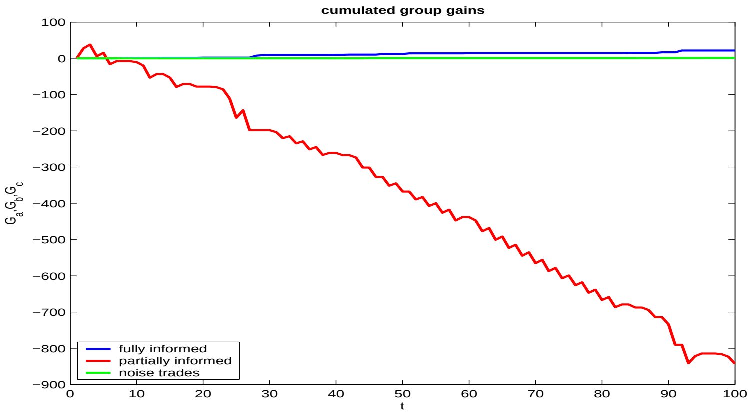
Figure 1.16: The evolution of individual cumulated gains for all three imperfectly informed strategies, in a good economy $V = V^H$, for a fixed probability of perfect information $\alpha = 0.20$, and proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$.



(a) Momentum strategy (TA-1), for random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.



(b) Moving-average strategy (TA-2), for random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.



(c) Fundamentalist strategy (FA), for random probabilities of imperfect information βq^H and βq^L of middle accuracy 0.50.

Figure 1.17: The evolution of cumulated group gains for all three imperfectly informed strategies, in a good economy $V = V^H$, for a random probability of perfect information $\alpha_t \sim U[0,1]$, and proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$.

In sum, practical trading rules appear to offer the possibility of realizing positive profits, both individually and as a group. Moreover, depending on the specific market conditions, their users can even come best off in the market, during shorter or longer time periods. Specifically, our users of simple momentum and fundamentalists strategies can gain more than all other traders, but this is not the case for our moving-average traders. Possible motivations are that users of practical rules simply mimic previously observed "correct" actions and thus reinforce price convergence; Yet, they could also increase sluggishness by adopting "incorrect" actions and profit from the emerging noise.[128]

1.2.4 Summary and conclusions

Our paper extends the model of Glosten and Milgrom (1985) in order to capture better nuanced information asymmetries. In particular, we are interested in the impact of imperfect information on prices, when traders acting on both perfect and no information are also active in the market. We rely on the same main assumptions as in Glosten and Milgrom (1985): The trade takes place sequentially; At each time, a trader is stochastically chosen from the totality of traders and can submit (only) market orders for buying or selling one single unit of risky asset; The true value of the risky asset remains unknown to the market until the end of the trade, but some traders may receive information about it already during the trade interval; Submitted orders are executed by a competitive and risk-neutral market maker; Prices are competitively set, so that the market maker expects zero total profits; Inventory and order processing costs are considered to be constant. Moreover, as in Easley and O'Hara (1987), information arrives randomly in the market.

Our contribution consists in refining the view over information asymmetries, by introducing a supplementary category of informed traders. Specifically, instead of the common, but somewhat coarse, twofold categorization of traders in informed and uninformed, we consider *three* trader groups: the usual informed traders (who dispose here of perfect information about the true risky value); the usual uninformed traders (motivated by exogenous trade reasons, such as liquidity needs); and a new group of imperfectly informed traders who derive some information, the accuracy of which is explicitly modeled. We consider that our threefold trader-type setting resembles better real markets, where, besides liquidity traders and insiders which have access to perfect information, there is also

[128]In an evolutionary environment, where traders can switch between groups, we expect that more uninformed traders start using practical rules. This would render group **b** even more numerous and thus reinforce the impact on prices.

a relatively high proportion of users of practical trading methods, such as technical or fundamental analysis. The users of practical rules derive information from the systematical analysis, that employs specific tools, of market-specific and/or market-exogenous data. This information is imperfect – more exactly imperfectly accurate – in the sense that it can be more or less valuable, depending on the deployed method and the skills of the respective traders. However, this information is used in a rational manner, since all traders forming expectations in line with the Bayes rule. We are interested in how this imperfect information affects prices.

Our "threefold" setting replicates the main theoretical results of the settings with only two categories of traders: In consequence of the information asymmetry, the market maker is confronted with adverse selection. Therefore, she sets two different prices for buying and selling the risky asset at each trading time, namely the ask and the bid, respectively. We theoretically show that a positive gap, denoted as spread, forms between these prices on the condition that the imperfect information is somewhat accurate. General prices are Martingales, but, when imperfect information is derived from past market data, the single prices are not Markovian. In other words, prices as a whole are efficient while the ask and bid sequences can be informative relative to single transaction prices. We also analyze how prices evolve with respect to public beliefs and determine the public belief level at which the spread is maximized.

We subsequently detail the impact of imperfectly information on prices. In so doing, we distinguish a twofold effect: qualitative, i.e. given by the accuracy of imperfect information, and quantitative, i.e. due to the dimension of the group of imperfectly informed traders. These two influences are studied separately, by considering first- and second-order effects on prices of all variables that characterize the imperfectly informed trade. Our main finding is that an intensification of the imperfectly informed trade, either in the qualitative or in the quantitative sense, entails worse overall trading conditions, i.e. higher spreads. The same conclusion holds with respect to the impact of perfect information on prices, although several differences manifest with respect to second-order effects.

Finally, we analyze how prices evolve in time when three simple trading strategies are employed, one at a time, for generating imperfect information. All these strategies belong to groups of heuristical methods of wide practical use, such as the technical and fundamental analysis. In particular, we account for the simplified forms of two chartist strategies, namely of a momentum and a moving average one, and of one fundamentalist strategy. When imperfectly informed traders use the fundamentalist strategy, prices do not always converge within the considered trading interval or, as the case may be, the convergence necessitates longer intervals. When prices do converge, they constantly reach the true risky value. Yet, for both chartist strategies, we observe price convergence across all considered cases (with different proportions of chartists and different chartist information accuracy). Moreover, the momentum strategy always ensures positive profits, the moving-average one yields merely losses for its supporters, and the fundamentalist strategy makes money only during shorter time intervals or under specific market conditions. In sum, (some) practical trading rules appear to be able to provide, at least under certain circumstances and when they are employed in a rational manner, the opportunity of making profits; and, in line with Lo (2004), of surviving in the market.[129] This supports the claim of the same author that survival can be reached, among others, through heuristics.

[129]Lo (2004) argues that in financial markets the most fit for survival are the richest.

CHAPTER 2

Emotions and Financial Decision Making

"All learning has an emotional base."
PLATO.

THIS CHAPTER deals with the role of affect and emotions in financial decision making. We commence by a review of the main findings on emotions stemming from other sciences such as psychology and neurobiology. Accordingly, emotions are indissolubly related to all human decisions and unavoidable to asset trading. Emotions can even improve human decisions, especially in situations of high uncertainty and time pressure such as those specific to financial markets.

We contribute to the theoretical study of the emotions' role in financial markets by developing a model in which three categories of traders face each other: rational, emotional, and noise traders. Each of these categories is guided by different perceptions of information and different action principles. We suggest a method for quantifying belief formation, which conforms to the Bayesian logic for the rational traders and complies with affective intuition for the emotional ones. Rational traders maximize expected wealth utility, demands develop proportionally to subjective beliefs, and prices are linear in the total order flow. Under these assumptions, the best rational strategy in equilibrium is to perfectly adapt to the market conditions created by the other market participants. In spite of their apparently simplistic demand strategy and distorted revision of beliefs, emotional traders not only decisively influence prices, but can even make more money than their rational peers. Thus, prices in (certain) financial markets may represent a thermometer of the market mood and emotions, rather than informative signals as stated in traditional financial theory.[1]

[1]The main ideas of this chapter are based on joint work with Diego Salzman. The current version has been developed with the help of Michel Baes.

2.1 Theoretical overview

This section introduces the notion of emotion and addresses its effects on decision making. As (traditional) Economics mostly abstracts from the existence of emotions, we commence by resuming the view of neurobiologists and psychologists about what emotions mean at a psychophysiological level. In the context of the traditional economic assumption regarding the rationality of economic decisions, we subsequently discuss the relation between emotions and cognition and why emotions are not only indispensable but can even be beneficial to decision making.

2.1.1 Definition of emotions and related notions

Emotions are defined by Damasio (1996) (p. 139) as "the combination of a mental evaluative process, simple or complex, with dispositional responses to that process, mostly toward the *body* proper, resulting in an emotional body state, but also toward the *brain* itself [...] resulting in additional mental changes." In other words, emotions generate both a behavior as a reaction to the inducing situation, and a change in internal state which prepares the organism for that particular behavior.

In order to better understand this definition and how emotions emerge, we rely on a classification suggested by the same author. This classification uses the closely related notion of *feelings* that, according to Damasio (1996), designate the experience of the changes induced by an emotion, in juxtaposition to the mental images that has initiated them. Thus, feelings represent perceived emotions, in other words, emotions that became conscious.[2] Feelings imply the (subjective) perception of the stimulus itself, as well as of the engendered body state and changes in mental processes. Therefore, feelings rely on cognition,[3] specifically the cognition of our body state and of how it modifies under the influence of the stimulus.

Note however that most authors do not discriminate between emotions and feelings.[4] Even Damasio (1996), underlines that both emotions and feelings rely on the same basic processes.[5] As this distinction is of no importance for our model in Section 2.2, after

[2]Damasio (1996) notes that we can speak about "feelings of emotions".

[3]This already emphasizes the close interrelation of affect and cognition discussed in Section 2.1.2.

[4]For instance, Oatley and Jenkins (1996), talk about emotion as a feeling, namely of what is going on inside the body. Also, Frijda, Manstead, and Bem (2000b) (p. 5) define emotions as "states that comprise feelings [...]".

[5]He stresses that these basic processes are two, namely the view of a body state jointly with the mental images that triggered it and a certain evolution of the accompanying mental processes.

giving a brief account on the classification of emotions and feelings in Damasio (1996), we employ the terms of "emotions" and "feelings" interchangeably.

Damasio (1996) classifies emotions and feelings in three main categories.

- The *primary* (or *innate*) *emotions* are specific reactions to particular stimuli, that rely on innate and inflexible mechanisms. The associated primary feelings are happiness, sadness, anger, fear, and disgust. At physiological level, these emotions are automatically triggered when the limbic system – especially the amygdala – recognizes the respective stimulus. Hence, stimuli entail responses before and/or without necessarily becoming conscious. This type of emotions is shaped to ensure that the organism survives. It is not specific to humans. Yet in humans, primary emotions are mostly followed by the corresponding feelings, so that, becoming conscious, they can be predicted and controlled.

- The *secondary emotions* can be felt as variations of primary feelings, such as euphoria, melancholy, panic, shyness, remorse, embarrassment, or vindication. Secondary emotions are acquired as, in the course of individual experience, the human mind systematically associates primary emotions to a broader range of encountered stimuli and situations. At physiological level, secondary emotions need not only the support of the limbic system but also the participation of prefrontal and somatosensory cortices.

- Apart from the feelings that stem from emotions, there is another category of feelings relying on background body states. Therefore, they are denoted as *background feelings* and are mostly generated by body states that prevail between emotional states. Thus, background feelings are of low intensity. The collection of background feelings that persist over a longer period (such as hours or days) is denoted as *mood*.

This classification of emotions already points out further terminology nuances, this time in terms of duration. According to Oatley and Jenkins (1996), *emotions* last for short intervals of time, up to several hours. In contrast, *moods* stretch over more days or weeks, while *personal traits* accompany us over years or the whole lifespan. Again, we do not differentiate among these technical terms, as they all denote emotional states of different length and intensity, but which may become manifest in similar ways in our model. Finally, the *affect* is a more general term that includes emotions, moods, and feelings. According to Damasio (1999), it refers to how we emotionally experience certain stimuli and situations and this every day of our lives, independently of the current mood. We employ the term "affect" in the same general sense, and underline that it always includes emotions.[6]

Clearly, emotions are not simple mental processes, but complex and varied collections of responses that imply a whole range of mental and physical transformations. Oatley and Jenkins (1996) distinguish among several – tightly linked – *stages of emotions*. They correspond, in essence, to the above definition of emotions.

1. Accordingly, emotions start by the evaluation of the trigger event. This is the *appraisal* phase that consists of recognizing the relevance of the event itself.[7]

2. In the subsequent phase of *context evaluation*, thoughts about the context serve to the evaluation of the context of the trigger event, such as the meaning of the event and how to cope with it.

3. Emotions can then be identified with the *action readiness*,[8] the setting of priorities, and the prompting of plans. Different types of action readiness describe distinct emotions.

4. Finally, emotions get around to be experienced, namely as mental states that are often accompanied by *physiological changes*, *expressions*, and/or *actions*. Although these collateral manifestations may let us recognize the respective emotion, there is mostly no one-to-one correspondence system.

Note that further definitions and classifications of emotions can be found in the literature. Although they focus on slightly different aspects, all of them include the same main characteristics of emotions that emerge from the definitions and classifications presented above, i.e. the tendency to act, the resulting bodily changes, and the valence. There are yet several interesting views to note. Thus, Frijda (1986) implicitly connects emotions to cognitive antecedents (i.e. beliefs) (see the subsequent Section 2.1.2 for a more detailed

[6]A last term occurring in the literature but unused in the present work is *sentiment*. In line with Frijda and Mesquita (2000), sentiments are emotions that turned into long-term beliefs.

[7]Note that this recognition is not necessarily conscious. According to the same authors, appraisal profiles explicitly characterize different emotions and are shaped according to several appraisal features. Such features are: the *goal relevance*, according to which the event is categorized as emotional or non-emotional (and discarded as irrelevant in the latter case), the *goal congruence* that entails positive emotions (such as happiness, love, pride) or negative emotions (such as anger, fear, sadness), and the type of *ego involvement* in the event (such as neutral, enhancing, or damaging self-esteem).

[8]The readiness to act constitutes, according to Frijda (1986), the core of an emotion.

discussion on the interdependency between affect and cognition).[9] Oatley and Jenkins (1996) divide emotions into basic categories, underlining the fact that they can be both conscious and unconscious.[10] Finally, Loewenstein and Lerner (2003) rely on the moment when emotions are experienced in order to distinguish between expected and immediate emotions.[11]

2.1.2 Emotions and decision making

Primarily, economists either negated any significant influence of emotions on decision making or reckoned emotions as undesirable due to their exclusively negative consequences.[12] According to the latter view, emotions would distort pure rational behavior, which was assumed as the comportment norm of economic agents. Even research in psychology and neurobiology has initially focussed on the cognitive aspects of decision making. For instance, the original *heuristics-and-biases program* of Kahneman and Tversky – a cornerstone of behavioral finance – accounts for the failures (i.e. fallacies) resulting from the employment of simple decision rules, called heuristics,[13] on the decision rationality.[14] The analyzed heuristics were mainly cognitive,[15] yet emotions were considered to enlarge the panoply of causes for such faulty behavior patterns.

[9] Specifically, Frijda (1986) and Elster (2003) define emotions by means of six features that serve to distinguish them from other visceral factors, such as hunger, pain, etc. (For a more detailed analysis on visceral factors and their effects on economic decisions, see Loewenstein (2000).) These features are: *cognitive antecedents*, which account for the fact that emotions are triggered by beliefs, *intentional objects*, pointing out that emotions have an object (this contradicts yet the claim that emotions are not necessarily conscious; see the classification in Oatley and Jenkins (1996) below), *physiological arousal*, that refers to the bodily changes induced by emotions, *physiological expressions*, that sometimes allow us to recognize the respective emotion, the *valence*, that describes the location of emotions on the pleasure-pain scale, and *action tendencies*, with the character of urges or impulses.

[10] Specifically, emotions are classified into nine basic functional families. All of them are based in the limbic system, but each of them sets the brain into different modes of organization (see Section 2.1.2 for further details on this cognitive function of emotions). Thus, emotions can be *free-floating*, that is, unintentional and unconscious. Happiness, sadness, anger and fear belong to this category. There are yet emotions that always have an *object*, that is, they are felt towards a certain goal. For example attachment love, caregiving love, sexual love, disgust, and contempt.

[11] Accordingly, *experienced emotions* represent expectations of the emotions that will be experienced by choosing a certain decision alternative. They have been incorporated in recent economic models of decision making. *Immediate emotions* are experienced at the time of decision making and can drive behavior in directions opposed to the long-run evaluation of the decision problem.

[12] See Smith (1759) and Peters and Slovic (2000). As noted in Frijda, Manstead, and Bem (2000b) and Loewenstein and Lerner (2003), the negative view of emotions is common to philosophy, literature, and even law.

[13] For more details on heuristics see subsequent comments in this section.

[14] See Gilovich, Griffin, and Kahneman (2002) for a summary of the main work in this respect.

[15] The three main heuristics considered in the original paper of Tversky and Kahneman (1974) are the representativeness, the availability, and the adjustment and anchoring. Kahneman and Frederick (2002) reformulate them as the attribute substitution and the affect heuristic.

However, psychologists and neurobiologists are the first to recognize the fact that, desirable or not, emotions always impact on human decisions. Thus, emotions are described as "the very center of human mental life" (Oatley and Jenkins (1996), p. 122) and regarded as our link to the outside world.

In this framework, emotions are proved to show both disruptive (i.e. negative) and functional (i.e. positive) effects on decision making. According to Damasio (1996) and Frijda, Manstead, and Bem (2000b), the *negative effects* occur for example when emotions have no object and can be misattributed to false causes, entailing biases against objective facts or interfering with working memory. Loewenstein and Lerner (2003) mention several main reasons for such negative effects: First, emotions that have evolved to solve certain decision problems may be incompatible with the current environment; Second, the emotions experienced at the time of decision making (the immediate emotions) are extremely sensitive to non-normative guiding factors of behavior and also distort evaluations of the probability and value of choice alternatives. In the same spirit, Simon (1967) observes that emotions may become disruptive when the triggering stimuli are intense and persistent, and repeatedly interrupt and hence prevent an organized behavioral response.[16]

The more recent perspective over emotions has however changed. (See Elster (1999).) Damasio (1996) suggests that emotions turn into *suitable decision tools* when the quality of decisions is measured by the survival in a given environment. The main reason is that, as underlined in Oatley and Jenkins (1996), emotions serve to focus on certain events and to discard irrelevant decisions alternatives. Since deciding well means very often – especially under time pressure, uncertainty, and limited resources – deciding *fast and well enough*, emotions represent survival-oriented (i.e. adaptive) instruments developed in the course of our biological evolution. As emphasized in Elster (2003), emotions improve decision making in twofold respect: first by avoiding the delay of (vital) decisions. In other words, emotions help us to make *some* decision; And second by improving the decision quality that could be achieved exclusively by rational deliberation.

Numerous recent studies detail the motivation and implications of emotions as beneficial decision making tools. We commence by supporting evidence from psychology and neurobiology. This evidence serves the purpose of better understanding the psychophysiological mechanisms of emotions, and hence the origins and the manifestations of their

[16]He also notes that emotional behavior can be learned. This can both enhance and reduce the "emotionality" of responses, the former when occuring emotions stop the process of fulfilling real-time needs and the latter by adaptive improvement of the reaction programs.

positive features. Also, it helps to better define the relation between emotions and rationality.

In this context, the *somatic markers hypothesis* of Damasio (1996) is one of the main theories meant to explain how emotions occur, to which extent they imply the body and/or the brain, and which is the valence of their impact on decision making. *Somatic markers* are defined as instances of feelings connected – by learning and experience accumulated during education and socialization – to secondary emotions. Their function is to "label" decision alternatives as favorable or dangerous.[17] A first general consequence of this labelling is underlined in Slovic, Finucane, Peters, and MacGregor (2002): Since they mark the mental representations of reality (as positive or negative), somatic markers – hence emotions – *direct the decision making.* Second, Damasio (1996) argues that the choice alternatives marked as dangerous are automatically eliminated. Thus, somatic markers provide for reducing the number of choice options and hence for *fastening decisions.* This is a crucial feature in particular in dynamic environments and/or when decisions need to be made under time pressure, as it is the case on financial markets. Slovic, Finucane, Peters, and MacGregor (2002) stress as well that, in spite of the importance of the deliberation in certain decision making situations, affect and emotions provide a faster, easier, and more efficient way to cope with the environmental complexity and uncertainty.[18] The somatic markers hypothesis explains in detail a fact apparent from the definition but with important implications for decision making: Emotions can be rooted in both the body *and* the brain. The usual inducers of emotions are representations of objects or situations that can come either from the *outside* world and elicit bodily reactions or from *inside* of the organism, being entirely triggered in the brain. Consequently, emotional reactions do not necessarily respond to real stimuli, in which case they are even faster and may be more difficult to control.[19] We can thus conclude that emotions play an *adaptive* – hence necessary and positive – role in decision making.

The beneficial aspects of emotions for decision making can be further motivated in

[17] In other words, somatic markers provide criteria for ranking the choice alternatives.

[18] They even speak about an "affective rationality", as the optimal behavior may be achieved by means of the affect.

[19] Damasio (1996) offers more details on the functioning of somatic markers. They are mainly experienced as bodily responses to external stimuli and many feelings stem indeed from changes of the bodily state. This mechanism is denoted as the *body loop.* However, our brains appear to have evolved in order to minimize the reaction time and the energy consumption during the response to certain stimuli. Certain neural devices, that help us to feel "as if" bodily changes would occur, have been developed. The trigger signals are now entirely processed by the brain and do not necessitate the intervention of the body. This corresponds to the so-called *as-if loop.* Of course, the core condition for such "bypass devices" to start functioning is that the process implying actual bodily reactions had run at least once.

view of the relation between *rationality and emotions.* It is not possible to draw a clear separation line between these phenomena, either with respect to their development at physiological level or to their impact on decision making. In effect, the mechanism of behavior is based on the *interaction* of rational and emotional processes. Yet, as all cognitive processes not only rely on emotions but are also framed by them, “emotions may be an indispensable foundation for rationality” (Damasio (1996), p. 164). In particular, since emotions are indissolubly related to the body (see the definition in Section 2.1.1), they guide all posterior mental processes, hence all cognitive processes which are slower and arise subsequently. Moreover, emotions themselves require the intermediation of both the brain core (viewed as the center of affect) and the cortex (viewed as the center of cognition).[20] Furthermore, Frijda, Manstead, and Bem (2000b) consider that beliefs – hence cognition – are not sufficient to initiate action, they need the support of emotional impulses. Accordingly, “emotions can awaken, intrude into, and shape beliefs [...]” (p. 5). This occurs in that emotions can either create new beliefs or change the strength of existent ones (i.e. amplify or alter them, and/or increase their resistance to change).[21] In essence, emotions appear to *guide cognition.*[22] Damasio (1996) explains that somatic markers boost the other two supporting mechanisms of reasoning, i.e. the attention and the working memory, throughout the cognitive system.[23] This idea is reinforced by Oatley and Jenkins (1996), who argue that emotions have two main cognitive properties. The first is the *management of action* and is intermediated by the so-called informational signals carried by emotions. These signals carry information about events that caused the emotion and commands to specific destinations. Thus, emotions change the readiness to act. The second cognitive function of emotions is to set the cognitive system into distinct *modes of organization.* This is achieved through specific signals that control brain organization but have no informational content. The effects of this brain structuring are to modify perception, to direct attention, to give preferential access to certain memories, and to bias thinking. Thus, emotions guide the cognitive search for possible plans. Note that

[20]On the one hand, emotions can be characterized as concrete, cognitive and neural. On the other hand, although rational processes occur in the newest part of the human brain, i.e. the neocortex, rationality actually implies the collaboration of neocortical and subcortical regions. It is intrinsically related to the biological regulation that occurs in the older brain areas, such as the brain stem and the limbic system that is the center of emotions.

[21]In addition, Frijda and Mesquita (2000) argue that emotions include the formation of beliefs and stimulate their elaboration.

[22]Or, as stressed in Loewenstein and Lerner (2003), emotions serve an essential function in coordinating cognition and behavior.

[23]This claim is equivalent to the assertion in Frijda, Manstead, and Bem (2000b) that emotions provide information and guide attention.

control signals may be accompanied by informational ones (which gives rise to feelings), but although this is often the case, it is not absolutely necessary.[24]

In the same context of the necessary and positive contribution of emotions to decision making, researchers have also focused on economical aspects, such as the decision optimality measured in terms of opportunity costs. They show that pure rationality is merely a theoretical abstraction and emotions can foster decision optimality in practical situations. As already emphasized in Simon (1967), the human knowledge and information processing capacities are limited. These limitations turn into constraints for real decision problems and make a fully rational search through all possible alternatives – as self-evident according to the traditional economic theory of rational agents – virtually impossible. In practice, the search for the best decision alternative stops either when the goal has been achieved, or when a certain amount of time has elapsed, or, in the majority of cases, at a partial solution that is found to be satisfactory. The latter is denoted in Simon (1967) as *satisficing*.[25]

Search and stopping rules are assessed in real life by means of "shortcut rules" called *heuristics*.[26] As already mentioned, the heuristics-and-biases program of Kahneman and Tversky develops this idea focussing on cognitive heuristics and on their negative impact on decision making. In contrast, Gigerenzer, Todd, and the ABC research group (eds.) (1999) introduce the concept of *fast and frugal heuristics*. As models of bounded rationality, they are meant to replace the theoretical idealization of decision making referred to as *unbounded rationality*. Accordingly, fast and frugal heuristics guide search in twofold manner: First, they define easily computable stopping rules and hence constitute one possible form of satisficing. Second, they provide facile decision rules, that represent a trade-off between generality and specificity. The fast and frugal heuristics are consequently rules of thumb that can lead to accurate and useful decisions. They are considered to be a part of the set of specialized cognitive mechanisms developed in the course of the human evolution that are shaped to cope with the environmental challenges.[27] Moreover, fast and frugal heuristics perform well when their structure is adapted to the environment.[28] As

[24] This occurs for instance in case of free-floating emotions.

[25] For a more detailed description of satisficing, see Simon (1982).

[26] According to the Merriam-Webster dictionary at www.merriam-webster.com, the term "heuristic" comes from the Greek "heuriskein" and means "to discover".

[27] This set is denoted by Gigerenzer and Todd (1999) as the "adaptive toolbox". For more details hereon and on the fast-and-frugal heuristics program see Gigerenzer and Selten (1999).

[28] Gigerenzer and Todd (1999) denote such a situation as "ecological rationality". Fast and frugal heuristics are ecologically rational in environments with noncompensatory information, scarce information, T-shaped distributions, or decreasing populations. Moreover, fast and frugal heuristics are consid-

they consist of a small set of simple rules, fast and frugal heuristics are mostly robust to environmental changes.[29] Beside cognitive heuristics, emotions are considered to belong to this type of heuristic principles. As Slovic, Finucane, Peters, and MacGregor (2002) mention, rationality of decisions should rather be understood in terms of "deciding in the best interests". Since interests depend on individual and environmental constraints, heuristics – in particular, emotions – ensure adaption and can provide the best solution to the constrained decision problems addressed above.

Moreover, Oatley and Jenkins (1996) recall two further reasons that can additionally impede full rationality: First, the same individual usually envisages multiple and often contradictory goals. Second, the achievement of these goals mostly involves other people and hence the coordination of their own and others' actions. Since emotions provide guidance in such ambiguous and complex situations, they become necessary for *complementing reasoning*.[30] In essence, emotions serve as *judgment and decision making heuristics*, especially when the analyzed situation is complex and the mental resources limited (see Slovic, Finucane, Peters, and MacGregor (2002)). They provide for the ability of deciding when the corresponding rational decision faculty cannot act (at full power) given the internal and external constraints. Thus, they supplement and enhance rationality, as argued in Elster (2003).

The two different views of heuristics as biases (Kahneman and Tversky) and as adaptive tools (Gigerenzer and colleagues) are reconciliated in Gilovich, Griffin, and Kahneman (2002). The simultaneous occurrence of positive and negative effects of emotions on decision making can be accordingly explained in terms of *dual processes*. These processes also shed a new light on the controversy regarding rationality and emotions and complete the explanations given above. According to Kahneman and Frederick (2002), two systems for judgment and choice operate in parallel and interact in generating final human behaviors.[31] They can be described following Sloman (2002), Stanovich and West (2002), and Slovic, Finucane, Peters, and MacGregor (2002). Thus, *System 1* (also called the *associative* or the *experiential system*) relies on temporal and similarity relations, in one word on *perception* (or *intuition*). It maps the reality into images with affective load. Moreover,

ered as "socially rational", i.e. they guide behavior in fast changing environments, where decisions have to be coordinated with other individuals.

[29]The main classes of fast and frugal heuristics studied in Gigerenzer and Todd (1999) are ignorance-based heuristics, one-reason decision making, elimination heuristics, and satisficing heuristics.

[30]Oatley and Jenkins (1996), argue that this guidance especially concerns our social relations. It is the result of making available a set of action alternatives already stored in the brain.

[31]See also Kahneman (2003) for an overview over the two-system model.

it mostly involves automatic (thus fast), heuristic-based, spontaneous, intuitive, and effortless processes and entails highly contextualized, personalized, and socialized inferences and predictions. In contrast, *System 2* (also the *rule-based* or the *rational system*) is based on a set of abstract variables and rules of logic and evidence. It is deliberative, effortful, slow, analytical, and strategic. Put in a different way, it acts in accordance with *reason* (or *knowledge*). It mainly involves controlled (thus slow) processes and yields more general mental representations. The interaction of these two systems complies with the principle of *collaboration*, yet their responses may strongly diverge from each other. In such cases, System 1 features *primacy*, since its automatic response is faster and effortless, while System 2 *controls* – and can overrule – the associative reaction. Also, the two systems are intrinsically related and support each other. For instance, rule-based reasoning is needed to develop associative structures, whereas associative reasoning becomes necessary when rules are inaccessible – such as in new situations or environments – and is helpful when knowledge has been already integrated in our mental landscape.

Lo (2004) develops these ideas even further in his *adaptive markets hypothesis*. This hypothesis attempts to reconciliate market efficiency with behavioral evidence that has been originally interpreted as a counterexample of rationality and thus of efficiency. The adaptive markets hypothesis builds on evolutionary principles, stressing that individuals act in self-interest, make mistakes, learn and adapt. Thus, their behavior is not necessarily intrinsic and exogenous, but evolves by natural selection and depends on the particular environment in which the selection occurs. In this context, individuals are organisms that attempt to maximize the survival of their genetic material (in other words, of their species). In financial markets, survival of the fittest becomes survival of the richest. The sole objective and the organizing principle of markets is survival, while the maximization of profits, utility, etc. remains just an aspect of market ecology. Survival is reached through satisficing, and stopping rules are determined through trial and error. This implies that choices rely on past experience and best guesses of the optimum, and learning occurs by receiving positive/negative reinforcement from the choice outcomes. Moreover, decision rules (e.g. trading or investment strategies) consist of heuristics that provide the best adaptation to the environment. Naturally, as the environment changes, the heuristics employed might not be suitable anymore, so they entail "irrational" behavior. Therefore, strategies follow cycles of profitability and loss in response to the variation of the market conditions.[32]

2.1.3 Emotional traders

As most of the findings presented above are obtained in an experimental context, the main challenge remains to establish to what extent and how emotions become manifest in real settings, in particular in financial markets.

Lo and Repin (2002) conduct a first field study with ten professional traders during their trading activity.[33] Using financial real-time data, they define different type of events, such as deviations from the mean, trend reversals, or volatility events. Standard neurobiological measurement of psychophysiological responses[34] allows for the quantification of the *real-time* traders' emotions during these events. The results point out significant differences in mean psychophysiological responses of *all* traders during transient events[35] and high volatility phases.[36] Although the more experienced traders exhibit weaker emotional reactions, it is clear that emotions *always* become manifest. This even occurs to professional traders who can be considered as the most "rational" market participants (due to their training and the resources at their disposal). Lo and Repin (2002) conclude that emotions are significant determinants of the evolutionary fitness of financial traders.

However, this study has not allowed for measuring the impact of emotions on performance, as the earning recordings were confidential data. To this end, Lo, Repin, and Steenbarger (2002) perform another field-experiment, this time with 80 participants in a day-trading program between July-August 2002. The emotional responses of the subjects are measured during a month, this time indirectly, by means of a questionnaire that became standard in psychology in studying emotional responses.[37] The main finding is that extreme emotional responses are contra-productive for the trading performance.[38]

[32]Lo (2004) concludes that market efficiency is highly contextual and dynamic, and that the market equilibrium should rather be understood as a steady-state limit under constant environmental conditions. However, markets are mostly *not* in equilibrium and, since evolution depends on the environmental conditions, there is no specific direction of evolution.

[33]The traders are from a major financial institution in Boston.

[34]I.e. the responses of the autonomous nervous system, specifically the skin conductance, the blood pressure, the heart rate, electromyographic data, the respiration rate, and the body temperature.

[35]Which become manifest by an increased electrodermal response.

[36]Where the cardiovascular response is significantly enhanced.

[37]I.e. the UWIST-MACL (University of Wales Institute of Science and Technology Mood Adjective Checklist). This questionnaire allows to evaluate the two major dimensions of emotions: valence and arousal. To this end, all emotions are defined after the combinations of pleasant/unpleasant and activated/deactivated states. The measurements are performed before, during, and after the trade.

[38]The overall results are yet less clear: Pleasant (unpleasant) states appear to increase (decrease) performance, but not in a significant way. Arousal is somewhat less significant as valence and top

Furthermore, in a controversial experiment Shiv, Loewenstein, Bechara, Damasio, and Damasio (2000) find that emotionally impaired subjects are more willing to gamble for high stakes than non-impaired people, as they do not experience the unpleasant feeling of loss. Even more surprising is the fact that people with brain damages generally make better financial decisions than those with normal IQs.

In sum, these studies, although somewhat scarce, reinforce the experimental evidence and speak for the manifestation of emotions in real financial markets. Our model in Section 2.2 adopts a theoretical perspective and attempts to show the important role that emotions can play in financial decision making. In particular, we are interested in the use of emotions as an analytical toolbox employed to establish trading strategies that appear to be as good as – if not better than – rational ones, in order to ensure survival in competitive environments.

To this end, we design a market where three main categories of traders confront with each other by trading one risky asset. Two of these categories are common to market microstructure models: the *rational traders* and the *noise traders*. The former form beliefs by combining prior and current information in a balanced Bayesian way, and attempt to maximize expected utility of wealth. The latter act randomly, driven by exogenous reasons such as liquidity needs. In addition, we introduce a new category of market participants, denoted as *emotional traders*, who follow their intuition in both forming beliefs with respect to future price evolutions and formulating periodical demands. Our rational and emotional traders typify the two-system logic addressed above in this section: While rational traders are characterized by the deliberate rule-based System 2, the associative and intuitional System 1 governs emotional decisions.

Although our setting is theoretical and could merely be tested by means of numerical simulations, we believe that it does not lie so far from reality. First, the three trader categories resemble real markets; There we can find professional traders who dispose of sufficient resources and motivation in order to make decisions in a way approaching the rational type, as well as trades impelled by exogenous reasons, similar to the random actions of the noise traders in our model. Moreover, some market participants may speculate on public information in an intuitive and affect-driven way. The motivation for the existence of a trader category such as our emotional traders relies on the evidence

performers seem less emotionally affected. In addition, no correlation between trader profiles and certain personality traits (such as extraversion, agreeableness, conscientiousness, neuroticism, openness) could be detected.

presented at the beginning of this section.

Second, further assumptions of our model, such as the belief formation and the general "logic" followed by emotional traders in shaping their demands (i.e. affect and intuition), also receive support from neurobiology and psychology. For instance, the fact that *beliefs* form by the superposition of past and current evidence is not only a theoretical idealization that underlies the Bayes rule. Damasio (1996) argues that the acquisition of somatic-marker signaling – that we know from the above comments in this section to guide decision making – occurs in the prefrontal cortices. This area receives signals regarding existing and incoming knowledge of external world, innate biological preferences, and the changes of body states as a consequence of this knowledge and of these preferences. In other words, prior and current information stemming from both outside and inside world is combined here and generates somatic markers. Recall however that our mind does not work on real information, but on individual representations of reality. They form exactly in these prefrontal cortices and are categorized in the perspective of personal relevance. In addition, Damasio (1996) considers a matter of individuality the extent to which decision making depends on real somatic states or symbols of them. Hence, it is plausible to assume that some individuals rely more than others on their mind images of reality in forming beliefs. Also, triggering activities from the brain – specifically from "as if" body states generated in emotional areas – can unconsciously bias the cognitive processes. According to the same author this biasing can yet be for the better, as the chances of potentially negative decisions are reduced and deliberation time gained. Damasio (1996) denotes this covert mechanism as *intuition* and it becomes the guiding principle of our emotional traders.

Third, in the same context of belief formation, the possibility that, in reality, some traders may overemphasize the *current* evidence on the account of their affect-driven behavior is further supported by the remark in Slovic, Finucane, Peters, and MacGregor (2002) that the precision of the affective meaning of a stimulus influences the ability to use information and hence its evaluability. Thus, impressions with a more intense affective load receive a higher weight in impression formation, judgment, and decision making. Also, as previously mentioned in this section, emotions represent the key ingredient in the formation and change of beliefs.[39] Finally, the case when emotional traders can put excessive weight on *past* elements of beliefs is confirmed – among others – by the remark

[39]The relation between emotions and beliefs is exhaustively studied in Frijda, Manstead, and Bem (2000a).

in Sloman (2002) that reasoning can be affected by the so called "belief bias". Specifically, a-priori formed beliefs can inhibit logic responses that account for current evidence. These two opposed tendencies to over- or underweight prior relative to current evidence correspond further to the heuristics of *representativeness* and *conservatism*, respectively.[40] They both become frequently manifest in real decision situations.[41] Note that, since representativeness stands for an extensive concept with multiple manifestations, we will use in Section 2.2 the denomination of *impulsiveness* for the specific manifestation of representativeness which is the neglect of prior relative to current information. Also, as noted by Kirchler and Maciejovsky (2002), both heuristics can be used simultaneously in real decision situations, depending on the framing of the decision problem (see Hoffrage (2004)). In other words, putting the same situation in a different light can make the same person to act either overconfidently or conservatively.

As discussed by Slovic, Finucane, Peters, and MacGregor (2002), relying on emotions can have *negative* consequences. This occurs either in response to unknown stimuli, for which the experiential system is not prepared, or when other people try to manipulate affective reactions. The former situation receives support in our hypothetical setting: When an excessive emphasis is put on new evidence, emotional traders are mostly worse off than their rational peers (and even than pure noise traders). This is reasonable to expect when, for instance, emotional traders are confronted with totally new situations, for which they have developed no intuition, so that "following their noses" cannot be very helpful. However, we do not account for manipulative actions, as our rational traders do not attempt to change perceptions; They simply adapt to them, while the rest of the traders manifest no concern with other groups' actions.

[40] The representativeness is one of the three basic heuristics of Kahneman and Tversky's program. According to Tversky and Kahneman (1992), it occurs when judgments of probability are replaced by judgments of similarity, so that objects are ascribed to categories on the basis of the correspondence in terms of few – and often irrelevant – features of the respective category. In particular, Tversky and Kahneman (1974) argue that representativeness entails insensitivity to prior outcomes, to sample size, to predictability, misconception of chance, of regression, and illusion of validity.

[41] Teigen (2004) notes that the representativeness heuristic can be easily employed due to its minimal requirements concerning the amount of cognitive resources. Moreover, it applies to a wide range of situations where the objective probabilities cannot be calculated, and mostly offers the correct result for simple problems. Therefore, in practice, people often tend to rely on representativeness instead of probability judgements. In contrast, Wallsten (1972) argues that conservative people tend to misinterpret information, to aggregate it in a wrong way or to manifest response biases against the use of very small or very large probabilities.

2.2 Emotions and financial decision making

2.2.1 Introduction

For many years, Economics has relied on the fallacy that people exclusively apply rational calculations to economic decisions and rule their lives by economic models. A commonly considered feature of rationality is the capacity of forming and revising beliefs by combining prior information with new evidence. This process is formally described by the so-called Bayes rule.[42] It is important to note that, according to this rule, identical weights are ascribed to the past and current elements of information that are pooled together in the new belief.

In Economics and more specifically in Finance, the traditional theory relies on the assumption that traders are able to process the relevant information at their disposal and to form unbiased probability judgments based on the Bayes rule.[43] As the entire market information thus becomes part of agents' beliefs and hence of prices, markets are efficient and prices informative in the traditional framework (Samuelson 1965). This concept is known as the *efficient market hypothesis* (Fama 1970). Consequently, this assumption rules out every opportunity to make profits by forecasting future prices.

However, psychologists have wondered if the Bayes rule truly describes how people revise their beliefs. Following Birnbaum (2004), we can classify their opinions in three periods: An early one that supports the Bayesian rule as a rough descriptive model of how humans combine and update evidence; A second period dominated by Kahneman and Tversky's assertion that people do not use base rates (i.e. prior information) or respond to differences in validity of sources of evidence; And a more recent period showing that people indeed rely on base rates and source credibility, but combine this information by means of an averaging model which is not consistent with the Bayes rule.[44]

Another common assumption of classic economic models is that rational agents are maximizers of utility, which is derived, for instance, from consumption or wealth. Their

[42]Rational decision making under uncertainty has always been associated with Bayesian inference. This concept, named after the reverend and mathematician Thomas Bayes and first extended by Laplace (1774), can be seen as the cornerstone of economic decision theory. In a nutshell, the *Bayes rule* tells how to revise beliefs (rationally) by combining prior and new information. These two pieces of information are described respectively by two random variable distributions, where the second is conditional on the first one.

[43]See, for instance, the notes on the expected utility theory from Section 3.1.2, especially Footnote 87.

[44]The distinctive feature of the averaging model is that the directional impact of information depends on the relation between the new evidence and the current opinion, as stressed in Birnbaum and Stegner (1979) and Anderson (1981).

actions are therefore guided by a (somewhat) strategic view over the course of action to be taken and how this affects the entire market evolution.

However, numerous empirical studies have emphasized the perpetuated existence of traders who employ a small number of simple and quick rules of thumb in order to make decisions under uncertainty. These traders are referred to as “not fully” or “quasi rational” and their action rules as *heuristics*. Heuristics appear to be based on strong emotional components. They can sometimes lead to systematic mistakes, denoted as *biases*. Nevertheless, heuristics have proved to be useful in practice, especially when decisions have to be made under time constraints, uncertainty, or huge amounts of information. More details on what heuristics are and on their role in decision making can be found in the above Section 2.1.2.

Within the last years, a new paradigm which tries to integrate the classical financial theory with the behavioral perspective has been developed in Lo (2004) under the name of the *adaptive markets hypothesis*. As stressed in the same Section 2.1.2, it is based on Darwin’s theory of evolution and considers individuals as organisms that try to maximize the survival of their species. During the decision making process, they develop heuristics in order to maximize the efficiency of their responses to uncertainty. However, since the environment is constantly changing, we can observe behavioral biases given the maladaptation to new circumstances.

The natural question arising in this context is if traders who follow their intuition and affect – in particular their emotions – and use heuristics can survive when confronted with rational traders and if so, under which conditions. This section attempts to further explore this question and hence to determine if the rational decision making represents the unique path towards survival in uncertain environments such as financial markets. To this end, we rely on market microstructure features and incorporate the role of emotions in financial decision making.

Specifically, we design a market with three groups of traders: *rational traders*, *emotional traders*, and *noise traders*. These categories – which can be viewed in an evolutionary perspective, as species – differ in the way they form beliefs and make decisions.

In essence, the belief formation is based on how information is perceived. In this context, rational traders consider both prior and current information to be equally important in order to infer price evolutions. They perform what we denote as a *balanced* combination of elements of information, in the Bayesian spirit. In addition, rational traders account for the existence of other traders with specific beliefs and their possible influence on prices. In contrast, emotional traders remain unconcerned with the existence of other traders and tend to under- or over-weight the prior relative to the current information. Thus, they act either *impulsively* or *conservatively*, being guided by their affective personality traits.[45] Such behaviors represent manifestations of frequent thinking heuristics. Note that, as noise traders are assumed to trade randomly, we are not interested in their beliefs.[46]

Emotional traders differ from their rational peers also with respect to the way in which they act, in particular to their demand strategy. In the vein of classical Economics, rational traders maximize the expected utility of wealth. Emotional traders simply follow their intuition. Specifically, they trade in accordance with their subjective beliefs in the price evolution. The emotional demand strategy can be considered as an example of action heuristics. As mentioned above, noise traders act randomly. Further details on the theoretical foundation, the intuition, and the implications of these assumptions can be found in Section 2.1.3.

We show that emotional traders exert an important influence on prices. The market equilibrium can be reached if rational traders adapt to the conditions created by emotional traders and their noise peers. In other words, the best rational strategy is to take into account the existence of other strategies. This appears to be indeed the most "rational" way, both in the classic sense (since thereby the goal of expected utility maximization is reached) and in an evolutionary sense (since adaption should ensure survival). However, rational traders are not necessarily accumulating the highest wealth in the market. They can be worse off than emotional traders and even systematically lose money. We theoretically derive the conditions under which such a situation occurs. In addition, we empirically explore possible parameter constellations that facilitate an emotional lead over rational traders. For instance, rational traders perform better than impulsive emotional traders but worse than conservative ones. In sum, our results support the possibility that emotional traders can survive – and even dominate the market – in spite of their apparently simplistic strategy and "distorted" belief formation processes.

Moreover, our simulations indicate that market evolutions are not necessarily per-

[45]In essence, emotional traders can be considered to be quite young and self confident and having no formal education concerning financial markets. Thus, they do not analyze the market development using sophisticated tools but instead rely on their own intuition and experience in the market.

[46]Since these beliefs cannot affect their actions.

turbed by the emotional presence. Our markets are always stable in response to singular shocks (in particular, log-returns are stationary). They even come close to efficiency when emotional traders perceive past evidence to be at least as important as new one, in other words when they think conservatively. A too elevated presence of emotional traders in the market can yet destabilize the trade of insufficiently liquid assets.

The remainder of the section is organized as follows: Section 2.2.2 presents our theoretical model following the main steps of the price emergence. In particular, we first formalize the formation of group specific beliefs by combination of prior and current elements of information. Second, we show how traders formulate their demands based on these subjective beliefs. To this end, we detail the price setting rule and subsequently the demand strategies of each trader group. The price formation is thirdly addressed. Fourthly, we derive the individual wealth of the different trader categories. Finally, we study an extension of the model in a particular case with dynamic belief updating. The applicative Section 2.2.3 comments on the simulation results – in particular log-returns, individual demands, and individual wealth and wealth growth – obtained under various parameter constellations. The final Section 2.2.4 summarizes the main findings. Intermediate mathematical proofs and further graphical results are included in Appendix A.2.

2.2.2 Theoretical model

This section shows how the subjective beliefs of different market participants can be translated into prices. After defining the main trader categories denoted as *trader groups*, we first model their subjective beliefs about the current price evolution. These subjective opinions originate in the mindset specific to each trader group and shape the trader actions. Thus, we can subsequently formulate the demand strategy of each trader group. In so doing, we rely on the assumption that prices develop proportionally to the current total order flow received by the market maker. Finally, we show how prices result from demands, considering that they are set by the market maker to be proportional to the current total order flow. As a result, we can quantify the influence of different trader groups – specifically, of their beliefs, demand strategies, and number – on market prices and assess the evolution of their wealth. At the end of this section, we extend our perspective by analyzing a setting with dynamic belief updating, where current subjective beliefs serve as prior beliefs at the subsequent trade.

We consider that our population of traders is divided into three groups: *rational* *traders*, *emotional traders*, and *noise traders*. Formally, these groups are referred to by the superscripts ^{r}, ^{e}, and ^{n}, respectively. We assume that each group is homogenous with respect to thinking and actions.

In particular, the rational traders follow the basic principles of rational trading: They form beliefs by combining prior and current information in a balanced Bayesian way and attempt to maximize expected utility of wealth. In contrast, emotional traders *merely* follow their intuition. This occurs first in evaluating the importance of different sources of information accessed in order to form opinions, and second in translating these opinions into periodical demands. Noise traders do not follow any particular strategy. They act randomly, being driven by exogenous reasons, such as liquidity needs. Therefore, their opinions play no role with respect to the price evolution.

Throughout this paper, we assume that the time is discretised. The trade develops over $t = 1, \ldots, T$ times. The trade object is a single risky asset. At each trade t and simultaneously, the traders submit market orders for buying or selling this asset, according to their own strategies and hence based on their subjective beliefs. These orders are received and (fully) executed by a market maker, whose (sole) function is to set prices.[47] All traders, as well as the market maker, are risk neutral and competitive. From a microstructural point of view, this setting resembles to Kyle (1985), as described in Section 1.1.1. We are not interested in the bid-ask spread, but in how market liquidity fluctuates in consequence of trader actions.

Belief formation

In this subsection, we are primarily interested in how traders perceive information that they subsequently incorporate in their trading strategies. In particular, we focus on the mental processes developed by rational and emotional traders in order to create and revise their beliefs. Since noise traders act randomly, we do not need to consider the way in which they perceive information.

At each time t, every market participant has access to the *public information set*, denoted as $\mathcal{F}_{t-1}$. This set consists of past prices and past total order flows. Note that all variables employed in the subsequent calculations at date t are functions of the public information, although we frequently omit $\mathcal{F}_{t-1}$ in our notation for the sake of simplicity.

[47] In our model, the market maker is not necessarily a person, but can be also an electronic system that pools together the trader orders and uses a certain rule to fix prices, in particular that current prices are proportional to the current total order flow.

We denote by P_t the asset price at time t. The corresponding log-price is written $p_t = \log(P_t)$, the gross return $R_t = P_t/P_{t-1}$, and the log-return $r_t = p_t - p_{t-1}$.

To the beliefs of each rational and emotional traders we associate a probability density function denoted by f^r and f^e, respectively. These functions depend only on the public information set $\mathcal{F}_{t-1}$ and describe how each trader group perceives the distribution of the current gross log-returns r_t. The random variables that correspond to the log-returns in the view of the rational, emotional, and noise traders, are denoted by r^r_t, r^e_t, and r^n_t, respectively.[48] For rational traders, the price of the asset should then be the random variable $P^r_t = P_{t-1}\exp(r^r_t)$ and its density function can easily be computed from f^r. Emotional traders rather expect the price to be distributed as the random variable $P^e_t = P_{t-1}\exp(r^e_t)$.

Henceforth, we denote the random variable r^r_t as the *rational subjective log-return* at time t. Similarly, the random variable r^e_t is referred to as the *emotional subjective log-return* at time t. In other words, the random variables r^r_t and r^e_t shall be understood as *subjective beliefs* (or *subjective opinions*) over log-returns. Let us now focus on the construction of their probability densities.

As underlined above, trader beliefs exclusively rely on the information publicly available at time t which consists, in fact, of the prices and total order flows up to $t-1$ inclusively. Thus, the discrepancy in beliefs originates in the different interpretations of the *same* public information. These beliefs can be decomposed in *prior* and *current* elements.[49]

Prior beliefs exclusively originate in the available past information. This information serves to infer so-called *a-priori* views concerning current log-returns. The a-priori opinions originate in the past decisions of the different traders and are specific to each trader groups. They are modelled here as random variables that we denote as $r^{r,p}_t$, $r^{e,p}_t$, and $r^{n,p}_t$ where the first superscript refers to the corresponding group of traders and the second one ^{p} indicates their past nature. Traders can – but are not compelled to – estimate the probability density functions of each prior belief variable, on the basis of past moves and decisions (of their own and other groups). In other words, it is possible that some traders formulate a guess about the beliefs of other market participants. We will see that this is

[48]Recall that merely r^r_t and r^e_t have a meaningful interpretation, as only rational and emotional traders have demand strategies that are shaped according to their beliefs. In contrast, noise traders act randomly and hence r^n_t is rather a formal notation meant to facilitate the model tractability.

[49]More exactly, pieces of *perceived* information.

the case for rational traders, while their affect-driven emotional peers plunge in their own world and account exclusively for their own opinions.

Rational and emotional traders use only those a-priori views that, in their opinion, can have a potential impact on current prices. Formally, the prior beliefs of the trader group i are summarized by the probability density function φ^i, where $i \in \{r; e\}$. Rational traders can be considered as being aware of the existence of different strategies in the market. Hence, they are likely to incorporate in their subjective beliefs the a-priori opinions of all other market participants. In contrast, emotional beliefs are driven by affect. Thus, this category of traders is probably not concerned with how other agents form expectations. It is more reasonable to think that they exclusively employ the own variable $r_t^{e,p}$ in the formation of their opinions. We denote by $\Re_t^i$ the set of a-priori views that are considered by a group i to be relevant for the evolution of current prices. In essence, $\Re_t^i \subset \{r_t^{e,p}, r_t^{e,p}, r_t^{n,p}\}$ and the densities φ^i represent estimations of the joint probability density function of the relevant a-priori opinions over the group specific set $\Re_t^i$.

Current beliefs illustrate the view of each trader group concerning the development of current prices in consequence of a-priori opinions. Formally, the function $g^i(r, \Re_t^i)$ describes how the trader group i evaluates the relative importance of the relevant a-priori opinions in $\Re_t^i$ and their impact on current log-returns r_t. In other words, the value $g^i(r, y)$ indicates how likely the trader group i considers that $r_t \equiv r$ when the relevant prior beliefs are identical to y. In order to simplify the subsequent mathematical manipulations, we assume that the functions g^i are nonnegative and normalize them such that their integral over r equals 1.

The subjective probability densities that describe the group specific beliefs are obtained by integrating the current beliefs over the set of relevant a-priori opinions. The employed measure is the probability density function of the respective prior beliefs. For the trader group i, this results in the integral:

$$f^i(r) = \int_{\mathbb{R}^{k_i}} \varphi^i(y) g^i(r, y) dy, \tag{2.1}$$

where the integer k_i represents the number of a-priori opinions that are relevant for the group i (or the dimension of the set $\Re_t^i$). Note that the normalization of g^i ensures that the function f^i is a probability density function.

This preliminary model complies with the traditional Bayesian framework in the following sense: We consider that for rational traders, both prior and current elements of beliefs are equally important for ascertaining market evolutions. The judgements of emotional traders are yet under the influence of their affective profile. They rely more on their intuition or feeling about past and new information in order to make choices. Specifically, emotional traders can under- or over-estimate the importance of the past with respect to new evidence. In our model, this yields to putting distinct weights a and b – that we refer to as the *belief weights* – on the perceived prior and current elements of information φ and g, respectively.[50] Accordingly, we modify the preliminary Equation (2.1) as follows:

$$f^i(r) = K^i_{ab} \int_{\mathbb{R}^{k_i}} \Big(\varphi^i(y)\Big)^a \Big(g^i(r,y)\Big)^b dy, \tag{2.2}$$

where the real parameters $a, b > 0$ allow us to formalize the idea of affect-driven belief formation.[51] The coefficient K^i_{ab} is defined as:

$$K^i_{ab} = \left(\int_{\mathbb{R}^{k_i} \times \mathbb{R}} \Big(\varphi^i(y)\Big)^a \Big(g^i(r,y)\Big)^b dydr\right)^{-1}. \tag{2.3}$$

and ensures that f^i remains a probability density function. Rational traders $i = r$ combine past and current elements of information in a *balanced* way, so that $a = b = 1$. Concerning the emotional group $i = e$, $b > a$ points to what we call an *impulsive* reaction to current market events, since the impact of new information prevails. In contrast, when $b < a$, emotional traders consider new evidence as less important than subjective beliefs formed in the past and act *conservatively*.[52] As discussed in Section 2.1.3, impulsiveness (as a type of representativeness) and conservatism represent two frequent biases of thinking and judgment.

We subsequently specify the particular form of the density functions φ^i and g^i that underlies our theoretical analysis, focusing on two special cases. They will serve as an example of an economy where emotional traders may perform better than their rational

[50] The emotional belief formation rule is inspired by the averaging model of Birnbaum and Stegner (1979) and is formally defined in the style of Grether (1980) and Shefrin (2005). Note that we account only for the possibility that the beliefs weights a, b stand for a psychological trait of emotional traders that does not change over the trade interval. The case with variable a_t, b_t is a topic of further research.

[51] The marginal cases $a = 0$ and $b = 0$ are only of mathematical interest. They appear to be irrelevant from an economical perspective, as situations such as with completely uninformative prior or current information are hardly to be expected in conjunction with real decision problems. Shefrin (2005) suggests $b > 1$ and $0 < a < 1$ as more realistic values of the belief weights.

[52] For a better understanding of how power weights can affect a pdf, see Figure A.18 in Appendix A.2.

peers.

As a consequence of the discrepancies in the views over market evolutions, rational and emotional traders assign different distribution laws to prior subjective beliefs. For reasons of theoretical simplicity, we work with parameterized normal distributions. As mentioned above, emotional traders guide their beliefs by means of their current affective state. This leads them to focus on their own opinions, without considering those of other trader groups. Thus, their prior beliefs exclusively depend on the random variable $r_t^{e,p}$, and φ^e is the probability density function of the following distribution:

$$N\Big(\tilde{r}_{t-1}^e, (\sigma^e)^2\Big). \tag{2.4}$$

We consider yet that rational traders take into account the a-priori opinions of *all* market participants. Specifically, they estimate the a-priori random variables of the different trader groups to be normally distributed:

$$r_t^{r,p} \sim N\Big(\tilde{r}_{t-1}^r, (\sigma^r)^2\Big), \quad r_t^{e,p} \sim N\Big(\tilde{r}_{t-1}^e, (\sigma^e)^2\Big), \quad r_t^{n,p} \sim N\Big(0, (\sigma^n)^2\Big). \tag{2.5}$$

We make here the simplifying assumption that rational traders are able to accurately guess the distribution of $r_t^{e,p}$. Given that noise traders are known to act randomly, it is natural to consider that the estimated mean of their prior opinions reduces to zero.[53] Moreover, we assume that rational traders consider their own a-priori opinions $r_t^{r,p}$ to be independent of others. Indeed, they know that their thinking is sufficiently sophisticated so that it cannot be guessed by other traders. These considerations entail rational prior beliefs with the probability density function φ^r that corresponds to the following distribution:

$$N\left(\begin{pmatrix} \tilde{r}_{t-1}^r \\ \tilde{r}_{t-1}^e \\ 0 \end{pmatrix}, \begin{pmatrix} (\sigma^r)^2 & 0 & 0 \\ 0 & (\sigma^e)^2 & \sigma^{en} \\ 0 & \sigma^{en} & (\sigma^n)^2 \end{pmatrix}\right). \tag{2.6}$$

Accordingly, rational traders also assume that the prior beliefs of the remaining participants can be related to each other, as indicated by the parameter σ^{en}.

Let us now focus on the form of the density functions of current beliefs g^i. These

[53]For instance, prior noise trader opinions can be centered on the return of risk-free assets, in other words on what they could gain by putting the money with the bank. Without loss of generality, we can normalize the risk-free rate to be $R_f = 1$ which yields a zero mean in log-terms.

functions describe how the corresponding trader group evaluates the relative importance of the relevant a-priori views – the own views and those of other traders – with respect to the market state at previous time $t-1$. We suppose that the evaluation occurs through *linear combination* of these ingredients. For simplicity reasons, we assume that the function $g^i(r, \Re_t^i)$ is a normal probability density and encompasses an exogenous noise component corresponding to possible exogenous market shocks. Naturally, its mean results from the aforementioned linear combination in the view of each group i.

For the emotional traders, the function $g^e(r, x)$ can then be formulated as the the pdf of the following normal distribution:

$$N\Big(k_{t-1} + k^e x, \sigma^2\Big), \tag{2.7}$$

where x represents the value of the random variable $r_t^{e,p}$, k_{t-1} indicates the perceived level of the past market evolution, and k^e expresses the intensity of the influence of emotional trader views on market prices, as estimated by themselves. The parameter σ refers to the standard deviation of the exogenous noise component.

For rational traders, the function $g^r(r, x_1, x_2, x_3)$ corresponds to the distribution:

$$N\Big(c_{t-1} + c^r x_1 + c^e x_2 + c^n x_3, \sigma^2\Big), \tag{2.8}$$

where the variables x_1, x_2, and x_3 refer to the a-priori views of the rational, emotional, and noise traders, respectively, as seen by the rational traders. The parameters c^r, c^e, and c^n reflect the relative importance of these various influences, and we will be able to determine their exact form in the framework of our simplified model. In a more general setting, we could assume that they are proportional to the sizes of the respective groups, or to the amount of trades that they initiate.

At the cost of several elementary algebraic manipulations (see the proof in Appendix A.2) and provided that the variance matrix of φ^r is invertible, we can determine the subjective-return densities of the emotional and rational traders which are distributed as follows:

$$r_t^r \sim N\Big(c_{t-1} + c^r \tilde{r}_{t-1}^r + c^e \tilde{r}_{t-1}^e, \sigma^2 + (c^r\sigma^r)^2 + (c^e\sigma^e)^2 + (c^n\sigma^n)^2 + 2c^e c^n \sigma^{en}\Big) \tag{2.9a}$$

$$r_t^e \sim N\Big(k_{t-1} + k^e \tilde{r}_{t-1}^e, \frac{\sigma^2}{b} + \frac{(k^e\sigma^e)^2}{a}\Big). \tag{2.9b}$$

Equations (2.9) reveal the discrepancy between rational and emotional views over the log-return distribution: Rational traders allow for the existence of different opinions in the market and combine past and new elements of information in a balanced way. In contrast, emotional traders guide their beliefs by means of their affective profile. This leads to a focus on the own price expectations and an impulsive over- or under-weighting of one of the sources of information considered in order to revise beliefs. Thus, the belief weights a and b impact the variance of the emotionally perceived log-returns and therefore on the perceived volatility - that is, the risk - of the emotional holdings of risky assets.

In our theoretical discussion, we explore two hypotheses about how rational traders perceive the link between the a-priori views of emotional and noise traders:

1. First, we consider that rational traders regard emotional and noise trader beliefs as independent and hence set $\sigma^{en} = 0$. The rational subjective log-returns from Equation (2.9a) yield:

$$r_t^r \sim N\Big(c_{t-1} + c^r \tilde{r}_{t-1}^r + c^e \tilde{r}_{t-1}^e, \sigma^2 + (c^r\sigma^r)^2 + (c^e\sigma^e)^2 + (c^n\sigma^n)^2\Big).$$

   This expression can be rewritten considering the following independent random variables:

$$\zeta_t \sim N\Big(0, \sigma^2\Big) \tag{2.10a}$$
$$\epsilon_t^r \sim N\Big(0, (\sigma^r)^2\Big) \tag{2.10b}$$
$$\epsilon_t^e \sim N\Big(0, (\sigma^e)^2\Big) \tag{2.10c}$$
$$\epsilon_t^n \sim N\Big(0, (\sigma^n)^2\Big). \tag{2.10d}$$

   The first of them ζ is denominated as *exogenous noise*, and the subsequent ones as *endogenous noises*. The remaining ones are group specific and refer in particular to the rational noise ϵ^r, the emotional noise ϵ^e, and the noise trader noise ϵ^n.

   Thus, the subjective log-returns of all three trader groups yield:

$$r_t^r = c_{t-1} + c^r \tilde{r}_{t-1}^r + c^e \tilde{r}_{t-1}^e + \zeta_t + c^r \epsilon_t^r + c^e \epsilon_t^e + c^n \epsilon_t^n \tag{2.11a}$$
$$r_t^e = k_{t-1} + k^e \tilde{r}_{t-1}^e + \frac{1}{\sqrt{b}}\zeta_t + \frac{k^e}{\sqrt{a}}\epsilon_t^e \tag{2.11b}$$
$$r_t^n = \epsilon_t^n. \tag{2.11c}$$

Recall that r_t^n is merely a formal notation, as we are not interested in the beliefs of the noise traders.

2. In the second setting, rational agents consider that the a-priori emotional beliefs $r_t^{e,p}$ contain a noise component *identical* to that of noise traders. In mathematical terms, this amounts to assuming that:

$$r_t^{e,p} = \tilde{r}_{t-1}^e + r_t^{n,p}, \tag{2.12}$$

and, thus, the standard deviations of the emotional and noise trader noises are identical $\sigma^e = \sigma^n = \sqrt{\sigma^{en}}$.

In other words, the emotional prior beliefs are considered to be related – in terms of their variance – to the noise generated by noise traders. However, there is a difference between emotional and noise traders: Emotional agents follow a certain strategy which entails the deterministic parameter $\tilde{r}_{t-1}^e$. Laxly put, the prior beliefs of emotional traders are regarded as non-arbitrary, but yet fairly "inefficient" (as very close to noise trading).

Formally, the prior beliefs of rational traders now depend on only two independent random variables, so that the distribution φ^r corresponds to:

$$N\left(\begin{pmatrix} \tilde{r}_{t-1}^r \\ \tilde{r}_{t-1}^e \end{pmatrix}, \begin{pmatrix} (\sigma^r)^2 & 0 \\ 0 & (\sigma^e)^2 \end{pmatrix}\right),$$

and the current beliefs g^r to the distribution:

$$N\Big(c_{t-1} - c^n \tilde{r}_{t-1}^e + c^r x_1 + (c^e + c^n) x_2, \sigma^2\Big).$$

Using the same technique as for deriving Equations (2.11), we deduce easily that the rational subjective log-return is normally distributed with the following parameters:

$$r_t^r \sim N\Big(c_{t-1} + c^r \tilde{r}_{t-1}^r + c^e \tilde{r}_{t-1}^e, \sigma^2 + (c^r \sigma^r)^2 + ((c^e + c^n)\sigma^e)^2\Big).$$

Again, we consider three independent and normally distributed noise terms, namely:

$$\zeta_t \sim N\Big(0, \sigma^2\Big) \tag{2.13a}$$

$$\eta_t^r \sim N\Big(0, (\sigma^r)^2\Big) \tag{2.13b}$$

$$\eta_t^e \sim N\Big(0, (\sigma^e)^2\Big), \tag{2.13c}$$

where ζ_t stands for the *exogenous noise*, while the random variables η_t^r and η_t^e represent the *endogenous noise* originating respectively from the rational traders and from their emotional peers.

Then, the subjective log-returns of all trader groups can be rewritten as follows:

$$r_t^r = c_{t-1} + c^r \tilde{r}_{t-1}^r + c^e \tilde{r}_{t-1}^e + \zeta_t + c^r \eta_t^r + (c^e + c^n)\eta_t^e \tag{2.14a}$$

$$r_t^e = k_{t-1} + k^e \tilde{r}_{t-1}^e + \frac{1}{\sqrt{b}}\zeta_t + \frac{k^e}{\sqrt{a}}\eta_t^e. \tag{2.14b}$$

$$r_t^n = \eta_t^e. \tag{2.14c}$$

Note that the assumptions made in this section are meant to keep the mathematical representation of the rational and emotional thinking processes as simple as possible. Consequently, they yield a somewhat simplistic logic in a rather static framework. We attempt to overcome this problem first at the end of this theoretical section by accounting for a dynamic formation of prior beliefs from past posterior ones and, second, in the applicative part by considering different possibilities of belief updating, such as quasi-dynamic updating of the mean priors or full updating of the priors.

Price setting mechanism

Once traders have formed – respectively revised – beliefs about the price evolution, they use them in order to decide how much of the risky asset they want to trade at the current time t.

In order to formally develop on this decision and the resulting demands of each trader group, it is first necessary to clarify how prices emerge in our market setting. As mentioned in the introduction, the price fixing instance in our model is a risk neutral and competitive market maker. We assume that, at each trade t, she fixes current prices P_t

to be proportional with the current total order flow Q_t advanced for execution by the traders, specifically as:

$$P_t = P_{t-1} + \lambda Q_t, \tag{2.15}$$

where $\lambda > 0$ and, as in Kyle (1985), $1/\lambda$ measures the market depth.[54]

The total order flow Q_t represents the sum of all buy and sell orders issued by the three trader groups, which is:

$$Q_t = N^r Q_t^r + N^e Q_t^e + N^n Q_t^n, \tag{2.16}$$

where Q_t^i stands for the individual demand of a trader from group i and N^i for the dimension of group i, with $i \in \{r; e; n\}$, so that $N^r + N^e + N^n = N$ is the total number of traders in the market.

Note finally that we consider a fixed supply of risky asset $Q > 0$ and do not impose any constraint on the quantity of risky asset traded at each time t (for instance, short-selling is possible).

Asset demands

Equipped with these formal tools, we can now return to the traders and discuss how they determine the direction (buy or sell) and the dimension (how much) of their current orders. The decision thereupon follows the specific logic of each trader group, that we henceforth refer to as *demand strategy*.

Similarly to the belief formation process described above, we design the rational and emotional demand strategies to be distinct and to comply with the two-system approach proposed by psychologists and described in Section 2.1: Rational traders act in accordance with the traditional principles of maximization of wealth. Thus, rational decisions can be considered as being dominated by *reasoning*. In contrast, emotional traders are solely guided by intuition in devising their demands, so that their cognitive processes are dominated by *affect*. Finally, the trading strategy of noise traders is mostly driven by *exogenous reasons*, such as the need of liquidity, and therefore consists of purely random actions.

[54] The market depth gives the order size necessary in order to move the market by a given amount. According to Hasbrouck (2007), depths is – besides breadth and resiliency – a manifestation form of market liquidity. It is thus closely related to the intensity of trading a certain asset. As this intensity is based on the interest of the traders in the respective asset, which can change in time but yet not at a very high speed, it is plausible to assume that λ is constant over the entire trade interval.

We commence by presenting the rational demand strategy which aims at maximizing expected-wealth utility. Due to the assumed risk neutrality of our market participants, this amounts to maximizing currently expected wealth.[55] Moreover, rational traders believe that the risky asset has a liquidation value $V \geq 0$ that should become public information at the end of the trade. Henceforth, we refer to V as the *true value* of the risky asset, as it exists only in the opinion of rational traders.[56] The expected wealth of a (single) rational agent W_t^r results from the stream of the quantities Q_t^r traded at each time $t = 1, \ldots, T$ (i.e. the demands) valued at the difference between the true value V and the price payed P_t. Then, the rational decision problem at each trade t, the result of which is the current demand, can be formulated as follows:

$$E_t[U(W_t^r)] = E_t[W_t^r] = W_{t-1}^r + Q_t^r(V - P_t) \quad \underset{Q_t^r}{\longrightarrow} \quad \max. \tag{2.17}$$

From Equation (2.17) it is apparent that the rational demand depends on the current price P_t. Since prices are set by the market maker according to the pricing rule in Equation (2.15), the optimal rational demand at time t can be written as the solution of the following optimization problem:[57]

$$\max_x \quad W_t(x) = W_{t-1}^r + x\Big(V - \lambda(N^r x + N^e Q_t^e + N^n Q_t^n)\Big), \tag{2.18}$$

which yields:

$$x^* = \frac{V}{2\lambda N^r} - \frac{N^e Q_t^e + N^n Q_t^n}{2N^r}, \tag{2.19}$$

under the condition – already fulfilled by the assumption in Section 2.2.2 – of a positive inverse market depth $\lambda > 0$.

Thus, as rational traders are the single category concerned with the impact of their

[55]Specifically, risk neutral agents have a linear utility function $U(W) = W$.

[56]The use of a true value can be regarded as being similar to a fundamentalist analysis. Chapter 1 considered fundamentalist strategies as examples or heuristics. This does not exclude a rational manner of employing them. Thus, even when the rational traders in the present setting use fundamentalist methods, they remain fully rational with respect to belief formation and continue to maximize expected wealth utility. Moreover, it is plausible to think that traders who are concerned with what other traders do and with the general market evolution, attempt to determine an intrinsic value of the traded assets. For instance, Section 1.1.1 mentions several important market-microstructure settings which assume trader rationality but also work with a "true value" and even define market efficiency with respect it.

[57]In practice, certain constraints may be imposed on the market, such as the fact that traders cannot infinitely lose money. For instance, they may be allowed to borrow money from the bank in order to invest it in the risky asset, but only up to a certain limit $\underline{W}$. This limits then becomes a lower bound for the current wealth. For rational traders, this can be written as $W_t^r \geq \underline{W}$. Note that augmenting the optimization problem in Equation (2.18) by such a wealth constraint yields the same solution in Equation (2.19).

trades on prices,[58] they maximize their expected wealth only if they shape their own views to best fit the current market conditions. In other words, rational traders must be sufficiently smart in order to *adapt* to the presence of further traders with different views over price evolutions. In an evolutionary sense, this strategy is indeed the most rational one and should provide the highest chances of survival.[59]

We now attempt to find out what exactly this adaption means, specifically how rational beliefs need to cope with the market conditions generated by the activity of other trader groups in order to generate a maximum of wealth. Formally, we let the rational subjective log-returns r_t^r from the above section on belief formation flow into the formula of the optimal rational demand $Q_t^r = x^*$ from Equations (2.19) that depends on the demands of emotional and noise traders Q_t^e and Q_t^n. In so doing, we assume that the rational demand linearly follows the rational subjective gross returns $R_t^r = \exp(r_t^r)$ and define:

$$Q_t^r = \beta + \beta^r (R_t^r - 1) P_{t-1}, \tag{2.20}$$

where $\beta^r \neq 0$ represents the sensitivity of the rational demand to rational price expectations and β an arbitrary constant showing the trade availability with invariant prices. In equilibrium, i.e. provided that Equation (2.18) is satisfied, we obtain the following expression for the rational subjective gross returns:

$$R_t^r = 1 - \frac{\beta}{\beta^r P_{t-1}} + \frac{x^*}{\beta^r P_{t-1}} = 1 - \frac{\beta}{\beta^r P_{t-1}} + \frac{V}{2\lambda N^r \beta^r P_{t-1}} - \frac{N^e Q_t^e + N^n Q_t^n}{2N^r \beta^r P_{t-1}}. \tag{2.21}$$

Let us now detail the demands of emotional and noise traders. Similarly to the way they form beliefs, emotional traders exclusively follow their intuition also in formulating asset demands. In so doing, they do not deliberately account for changes in the market environment generated by the presence of different strategies – as it was the case for rational traders – but blindly "follow their noses".

We assume that emotional agents trade proportionally to their beliefs, without being concerned with wealth maximization. This can be considered as a "rule-of-thumb" or, in technical terms, a heuristic. In this section, we are interested in the effect of applying

[58]Recall that this assumption is also valid with respect to the belief formation: Rational traders are the only ones who account for the existence of other market participants with different beliefs.

[59]Note that the adaption of rational traders *cannot* be related to the adaption function of emotions discussed in Section 2.1.2. While emotions *intrinsically* (i.e. unconsciously) facilitate adaption, our rational traders chose *deliberately* (i.e. consciously) to shape their actions in order to fit the market conditions.

heuristics, this time not in a rational way as in Section 1.2, but in an affect-driven way. The emotional demand can be formulated as follows:

$$Q_t^e = \beta^e(P_t^e - P_{t-1}) = \beta^e(R_t^e - 1)P_{t-1}, \tag{2.22}$$

where $\beta^e \neq 0$ stands for the emotional demand sensitivity to changes in the emotional price expectations. Recall that $R_t^e = \exp(r_t^e)$ is the emotional subjective gross return expectation which is, according to the assumptions from the first part of this section, log-normally distributed. Thus, the emotional demand is also log-normally distributed.

Noise traders trade randomly and, for symmetry reasons, we define their current demand as follows:

$$Q_t^n = P_t^n - P_{t-1} = (R_t^n - 1)P_{t-1}, \tag{2.23}$$

where – in merely formal notation as noise traders do not form beliefs – $R_t^n = \exp(r_t^n)$ is a log-normally distributed variable.

With the assumed expressions of demands of emotional and noise traders in Equations (2.22) and (2.23), the rational subjective gross returns from Equation (2.21) yield:

$$R_t^r = 1 - \frac{\beta}{\beta^r P_{t-1}} + \frac{V}{2\lambda N^r \beta^r P_{t-1}} - \frac{N^e \beta^e}{2N^r \beta^r}(R_t^e - 1) - \frac{N^n}{2N^r \beta^r}(R_t^n - 1). \tag{2.24}$$

As $R_t^i = \exp(r_t^i)$, where $i \in \{r; e; n\}$ and r_t^i are normally distributed, we can infer plausible values for the parameters of the rational beliefs in Equation (2.9a). Specifically, Equation (2.24) must hold in moments. Thus, we can think of the rational strategy as of a mixed one, where the optimal behavior consists in choosing randomly between possible moves, so that the conditions created by the demands of emotional and noise traders are satisfied in means or in the first moments, for instance. Since Equation (2.24) also holds in variance, rational traders are able to absorb the noise generated by other traders, which should contribute to market stability.

However, it is not possible to derive an analytical solution for the rational belief parameters c_{t-1}, c^r, c^e, c^n in Equation (2.9a) from the moments of Equation (2.24). Therefore, we further present an approximative solution, that holds for sufficiently low values of the subjective log-returns and is based on the approximation $R_t^i - 1 \approx r_t^i$, where $i \in \{r; e; n\}$. Thus, we have:

$$2N^r \beta^r r_t^r + 2N^r \frac{\beta}{P_{t-1}} + N^e \beta^e r_t^e + N^n r_t^n \approx \frac{V}{\lambda P_{t-1}}, \tag{2.25}$$

where r_t^i follow Equations (2.11) and (2.14), respectively, for the two employed scenarios with independent and identical emotional and noise trader noise. Then the following must hold *approximatively* for that rational traders maximize current wealth:

1. When rational traders regard emotional and noise trader beliefs as independent $\sigma^{en} = 0$:

$$\begin{aligned}
\beta^r &= -\frac{N^e \beta^e}{2N^r \sqrt{b}} \\
c^r &= 0 \\
c^e &= k^e \sqrt{\frac{b}{a}} \\
c^n &= \frac{N^n \sqrt{b}}{N^e \beta^e} \\
c_{t-1} &= \frac{\sqrt{b}}{N^e \beta^e P_{t-1}} \left(2N^r \beta - \frac{V}{\lambda} \right) + \left(k_{t-1} + k^e \left(1 - \frac{1}{\sqrt{a}} \right) \tilde{r}^e_{t-1} \right) \sqrt{b},
\end{aligned} \tag{2.26}$$

   and thus the rational subjective log-returns yield:

$$r_t^r = \frac{\sqrt{b}}{N^e \beta^e P_{t-1}} \left(2N^r \beta - \frac{V}{\lambda} \right) + k_{t-1}\sqrt{b} + k^e \sqrt{b} \tilde{r}^e_{t-1} + \zeta_t + k^e \sqrt{\frac{b}{a}} \epsilon_t^e + \frac{N^n \sqrt{b}}{N^e \beta^e} \epsilon_t^n. \tag{2.27}$$

2. When rational traders regard emotional and noise trader beliefs as dependent on each other and take $\sigma^e = \sigma^n = \sqrt{\sigma^{en}}$:

$$\begin{aligned}
\beta^r &= -\frac{N^e \beta^e}{2N^r \sqrt{b}} \\
c^r &= 0 \\
c^n &= -c^e + k^e \sqrt{\frac{b}{a}} + \frac{N^n \sqrt{b}}{N^e \beta^e} \\
c_{t-1} &= \frac{\sqrt{b}}{N^e \beta^e P_{t-1}} \left(2N^r \beta - \frac{V}{\lambda} \right) + \left(k_{t-1} + \left(k^e - \frac{c^e}{\sqrt{b}} \right) \tilde{r}^e_{t-1} \right) \sqrt{b},
\end{aligned} \tag{2.28}$$

   and thus the rational subjective log-returns yield:

$$r_t^r = \frac{\sqrt{b}}{N^e \beta^e P_{t-1}} \left(2N^r \beta - \frac{V}{\lambda} \right) + k_{t-1}\sqrt{b} + k^e \sqrt{b} \tilde{r}^e_{t-1} + \zeta_t + \left(\frac{k^e}{\sqrt{a}} + \frac{N^n}{N^e \beta^e} \right) \sqrt{b} \eta_t^e. \tag{2.29}$$

In other words, rational traders can maximize their current wealth if they are able to correctly infer the actions of other traders. Specifically, they have to estimate the proportions N^e, N^n and the behavioral emotional profile given by the emotional belief parameters k_{t-1}, k^e, a, b and the emotional demand sensitivity β^e. Note that rational traders who perfectly adapt to the environmental conditions should discard their own prior opinions $\tilde{r}^r_{t-1}$ in forming current beliefs (and replace them by other's beliefs). This occurs by setting $c^r = 0$ so that the rational strategy does not induce any further specific noise in returns (of variance $(\sigma^r)^2$). Also, the rational demand should change in the opposite direction with respect to the demands of other traders. Specifically, we obtain as a result that the sensitivity to changes in own beliefs β^r has the opposite sign to the corresponding emotional sensitivity β^e. However, the rational demand constant β can be freely chosen. Finally, note that the weight b put on current emotional elements of belief inflates all terms of the rational subjective expectations, apart from the exogenous noise. Thus, the more impulsively emotional traders act, the higher should be the price differences expected by rational traders and consequently the higher the rational demand of the risky asset.

At first, the conditions in Equations (2.26) and (2.28) may appear as impossible to meet in real settings where behavior changes over time. However, these equations merely require that rational traders infer time-invariable features of the emotional group, excepting k_{t-1}. In the static world of our setting, where traders do not enter or leave the market and do not change strategy, assessing such features should not necessitate extremely high skills. It may for example be carried out by repeated observation of the behavior of market participants during prior trading intervals. Moreover, it is plausible to assume that emotional traders treat k_{t-1} as constant in time $k_{t-1} = k$, given that possible past variations that can affect their expectations in Equations (2.11b) and (2.14b) are already encompassed by the prior mean belief $\tilde{r}^e_{t-1}$.

Market prices

When rational traders maximize their expected wealth, the total order flow results as:

$$Q_t = \frac{V}{2\lambda} + \frac{N^e Q_t^e + N^n Q^n}{2}. \tag{2.30}$$

With the demands of emotional and noise traders in Equations (2.22) and (2.23), the equilibrium gross returns yield:

$$R_t = \frac{P_t}{P_{t-1}} = 1+\frac{V}{2P_{t-1}}+\frac{\lambda}{2P_{t-1}}(N^e Q_t^e+N^n Q_t^n) = 1+\frac{V}{2P_{t-1}}+\frac{\lambda}{2}\Big(N^e\beta^e(R_t^e-1)+N^n(R_t^n-1)\Big). \tag{2.31}$$

Using the same approximation as above, we can rewrite Equation (2.31) in terms of log-returns:

$$r_t \approx \frac{V}{2P_{t-1}} + \frac{\lambda}{2}(N^e\beta^e r_t^e + N^n r_t^n). \tag{2.32}$$

In other words, the returns in equilibrium result in the sum of subjective returns of emotional and noise traders, weighted by the respective group dimensions.

Equation (2.32) already reveals the influence of different trader categories on prices: Emotional and noise traders impact log-returns directly proportional to their presence in the market, specifically to their numbers N^e and N^n, respectively. Moreover, when emotional traders expect, for instance, increasing prices $r_t^e > 0$, their subjective opinions will directly translate into proportionally increasing log-returns r_t, where the strength of this relation depends on the intensity at which emotional traders incorporate beliefs into their actions β^e. Both emotional and noise trader influences are controlled by the asset liquidity, being in particular reduced with respect to more liquid assets, i.e. for a lower λ. Finally, rational subjective beliefs cannot directly affect prices, since they are shaped to agree with that what other traders are supposed to do. The single trace of rational presence is incorporated by the true risky value V and prices increase from one trading to the other with $V/2$, so that too high V-values should render markets instable.

We can further develop on the approximative expression of the market log-returns considering the two noise scenarios. Thus, we yield the following:

1. When rational traders regard emotional and noise trader beliefs as independent $\sigma^{en} = 0$:

$$r_t \approx \frac{V}{2P_{t-1}} + \frac{\lambda}{2}\left(N^e\beta^e k_{t-1} + N^e\beta^e k^e \tilde{r}^e_{t-1} + \frac{N^e\beta^e}{\sqrt{b}}\zeta_t + \frac{N^e\beta^e k^e}{\sqrt{a}}\epsilon^e_t + N^n\epsilon^n_t\right). \quad (2.33)$$

Thus, the distribution of the log-returns can be approximated by the following normal distribution:

$$N\Big(\frac{V}{2P_{t-1}} + \frac{\lambda N^e\beta^e}{2}(k_{t-1} + k^e\tilde{r}^e_{t-1}), \frac{(N^e\beta^e\sigma)^2}{b} + \frac{(N^e\beta^e k^e\sigma^e)^2}{a} + (N^n\sigma^n)^2\Big).$$

2. When rational traders regard emotional and noise trader beliefs as dependent on each other and take $\sigma^e = \sigma^n = \sqrt{\sigma^{en}}$:

$$r_t \approx \frac{V}{2P_{t-1}} + \frac{\lambda}{2}\left(N^e\beta^e k_{t-1} + N^e\beta^e k^e \tilde{r}^e_{t-1} + \frac{N^e\beta^e}{\sqrt{b}}\zeta_t + \left(\frac{N^e\beta^e k^e}{\sqrt{a}} + N^n\right)\eta^e_t\right). \quad (2.34)$$

Thus, the log-returns are approximatively distributed as follows:

$$N\Big(\frac{V}{2P_{t-1}} + \frac{\lambda N^e\beta^e}{2}(k_{t-1} + k^e\tilde{r}^e_{t-1}), \frac{(N^e\beta^e\sigma)^2}{b} + \left(\frac{N^e\beta^e k^e}{\sqrt{a}} + N^n\right)^2(\sigma^e)^2\Big).$$

The approximative expressions of log-returns in Equations (2.33) and (2.34) allow us to analyze in more detail the emotional impact. As expected, an increased participation of emotional traders in the market, i.e. a higher N^e, yields higher but also more volatile returns. Thus, not only the chances of making more money from the trade, but also the risks to lose more money, are increased. The same increase in both mean and volatility of market returns occurs for higher emotional demand sensitivities β^e. Interestingly, the more unbalanced the belief formation of emotional traders is, i.e. the higher the belief weights a or b are, the lower should be the market volatility, other things being equal. In addition, negative values of the emotional demand sensitivity β^e or of any of the emotional belief parameters k_{t-1} and k^e could even induce price decreases between subsequent trades. We denote such emotional thinking processes – with negative β^e, k_{t-1}, or k^e – as of *contrarian type*. They could, for instance, counterbalance the inflating influence of a higher value V.

Note that the approximative expression of log-returns has constant variance in both scenarios. Its mean is also constant. Thus, the approximated log-returns from Equation

(2.32) are stationary if the following holds:

$$k_{t-1} + k^e \tilde{r}^e_{t-1} = \kappa - \frac{V}{\lambda N^e \beta^e P_{t-1}},$$

with κ an arbitrary constant. At first, it appears rather improbable that emotional traders – who are neither concerned with beliefs of other traders nor with fundamental aspects such as the true value of the risky asset – can form beliefs such that the above condition is met. However, note that V is fixed and hence so is also the quantity $V/(\lambda N^e \beta^e)$. Thus, this condition could be fulfilled if the mean emotional prior $\tilde{r}^e_{t-1}$ depends on the price P_{t-1} or, in a very general sense, if emotional prior beliefs are based on (a combination of past) market returns. Our applicative results from Section 2.2.3 show that with this type of emotional belief formation, log-returns – not only in their approximative, but also in their exact formulation – are stationary.

Trader wealth

In the sequel, we turn our attention to the question if and which traders could survive in the market described by our model. As noted by Lo (2004) in his adaptive markets hypothesis (see Section 2.1.2), in financial markets survival of the fittest becomes survival of the richest. Therefore, the accumulated wealth can be regarded as a measure of the survival capacity of our traders. Another such measure is the wealth growth. This section derives and compares the individual wealth and its growth for rational, emotional, and noise traders. Note that we work here with the *realized* wealth (and not with the expected one, as used, for instance, by rational traders in the optimization problem in Equation (2.17)).

The current *individual wealth* of a trader from group $i \in \{r; e; n\}$ can be written as:

$$W^i_t = W^i_{t-1} + Q^i_t(P_t - P_{t-1}) = W^i_{t-1} + Q^i_t \lambda (N^r Q^r_t + N^e Q^e_t + N^n Q^n_t), \tag{2.35}$$

where the last equality results from Equation (2.15). The *group wealth* is easily obtained by multiplying the expression in Equation (2.35) by the corresponding group dimension N^i.

In order to theoretically compare the individual wealth of the different trader categories, it is easier to work with the variations from one trade to the other. Formally, the current *change in individual wealth* yields:

$$\Delta W_t^i = W_t^i - W_{t-1}^i = Q_t^i \lambda (N^r Q_t^r + N^e Q_t^e + N^n Q_t^n), \tag{2.36}$$

where the total order flow follows Equation (2.30) in the linear equilibrium defined above.

Thus, the rational individual wealth changes between successive trades with the following quantity:

$$\Delta W_t^r = \frac{\lambda}{4N^r} \left(\left(\frac{V}{\lambda} \right)^2 - (N^e Q_t^e + N^n Q_t^n)^2 \right). \tag{2.37}$$

Note that the change in rational individual wealth is positive $\Delta W_t^r \geq 0$, that is the wealth of rational individual increases, as long as rational traders ascribe a sufficiently high true value to the risky asset $V \geq \lambda(N^e Q_t^e + N^n Q_t^n)$. Specifically, this value should exceed the price impact of the total order flow generated by emotional and noise traders. Otherwise, $\Delta W_t^r < 0$ and rational traders lose money from their trades. As V is fixed, it is possible to find values for which the negativity condition holds at each trade and thus rational traders always register losses. This is a first interesting result: Although the strategy adopted by our rational traders preserves important features of rationality – both in the sense of classical Economics (rational traders maximized expected wealth utility) and in an evolutionary sense (they adapt to the market conditions) – we can theoretically prove that rational individual wealth does not necessarily increase at each trade in the "emotionally noisy" environment of our model. In other words, the rational strategy does not guarantee survival.

It is also interesting to note that the wealth change of rational individuals is inversely proportional to the number of rational traders N^r active in the market. In essence, rational traders split profits (and losses) among themselves. Thus, profit chances of rational individuals are higher when the rational presence is less pronounced.

The change of the emotional individual wealth between successive trades can be formulated as follows:

$$\Delta W_t^e = \frac{\lambda}{2} Q_t^e \left(\frac{V}{\lambda} + N^e Q_t^e + N^n Q_t^n \right). \tag{2.38}$$

Obviously, as long as $Q_t^e(V/\lambda + N^e Q_t^e + N^n Q_t^n) \geq 0$, emotional traders also earn money from trading. Thus, a trading strategy guided by affective factors does not necessarily entail permanent loses.

A similar result holds for noise traders, the wealth of which changes at each trade with the following quantity:

$$\Delta W_t^n = \frac{\lambda}{2} Q_t^n \left(\frac{V}{\lambda} + N^e Q_t^e + N^n Q_t^n \right). \tag{2.39}$$

Another interesting conclusion is reached comparing the changes of individual wealth: Assuming that both rational and emotional traders are gaining money between successive trades, the rational gains do not have to be higher than the emotional ones. Specifically, if:

$$N^e Q_t^e \in \left(-\infty, -\frac{V}{\lambda} - N^n Q_t^n \right] \bigcup \left[\frac{N^e}{2N^r + N^e} \left(\frac{V}{\lambda} - N^n Q_t^n \right), \infty \right), \tag{2.40}$$

the emotional wealth varies more than the rational one $\Delta W_t^e \geq \Delta W_t^r$. Hence, when $\Delta W_t^e \geq 0$, emotional traders earn more money on the current trade than their rational peers.

Due to the symmetry, analogous conditions hold for noise traders, that is:

$$N^n Q_t^n \in \left(-\infty, -\frac{V}{\lambda} - N^e Q_t^e \right] \bigcup \left[\frac{N^n}{2N^r + N^n} \left(\frac{V}{\lambda} - N^e Q_t^e \right), \infty \right), \tag{2.41}$$

and the current change in wealth of noise traders is higher than that of rational traders $\Delta W_t^n \geq \Delta W_t^r$.

Note that, when emotional and noise traders are for instance selling a sufficiently high amount of the risky asset $N^e Q_t^e + N^n Q_t^n \leq V/\lambda$, then each of these trader categories gain more money than the rational ones, since, first, $V \geq \lambda(N^e Q_t^e + N^n Q_t^n)$ and hence $\Delta W_t^r \geq 0$, and, second, both Equations (2.40) and (2.41) are fulfilled, so that both $\Delta W_t^e, \Delta W_t^n \geq \Delta W_t^r$. Moreover, the group which trades more than the other will realize the highest profits, that is either the emotional traders if $Q_t^e \geq Q_t^n$ or the noise traders if, in contrast, $Q_t^e \geq Q_t^n$. In other words, it is possible that rational traders earn less money than every other trader in the market, and even that noise traders are dominating in terms of individual wealth.

In sum, we were able to find theoretical support of the possibility that rational strategies do not have to be the fittest under any circumstance.

Before closing this section and following Blume and Easley (1992), we introduce an additional measure of the traders' fitness and their survival chances. This is the *growth* *of individual wealth* and we define it as follows:

$$\frac{W_t^i}{W_t^{tot}} - \frac{W_{t-1}^i}{W_{t-1}^{tot}}, \tag{2.42}$$

where $W_t^{tot} = \sum_{l=1}^{N} W_t^l$ is the total wealth of traders at time t. We will analyze the evolution of the growth of individual wealth in the applicative Section 2.2.3.

An extension: dynamic belief updating

This section attempts to extend the above model in order to allow for dynamic belief updating. In so doing, we follow the logic described in the above section on belief formation and preserve most of the respective assumptions.

In particular, the specification of our dynamic structure can be described as follows: We merely develop on the first scenario, in which emotional and noise trader noises are independent of each other. At each time t, the *prior beliefs* (of rational and emotional traders) are set identical to the posterior beliefs at the previous trade. In the formal notation used in the first part of Section 2.2.2, $r_t^{i,p} = r_{t-1}^i$, for $i \in \{r; e\}$, and hence prior beliefs have time-changing mean and variance.

Let us first formalize the dynamic formation of prior beliefs in the case of emotional traders. Emotional agents start trading with fixed initial beliefs which we continue to assume to be normally distributed and formally write them as:

$$r_0^e \sim N(r^e, (\sigma^e)^2). \tag{2.43}$$

As mentioned above, the posterior beliefs at $t-1$ serve as prior information at t to emotional traders. Thus, the prior emotional beliefs are normally distributed with the following mean and variance:

$$r_t^{e,p} = r_{t-1}^e \sim N\Big(\tilde{r}_{t-1}^e, (\sigma_{t-1}^e)^2\Big). \tag{2.44}$$

As far as the rational traders are concerned, their belief formation follows a more complex logic, as already underlined above. In particular, rational beliefs arise from the combination of different pieces of information: first, the rational own prior opinions $r_t^{r,p}$ of the price evolution. Such opinions can be, for instance, the result of a more complex analysis, that possibly includes market exogenous information and is performed before trading, conducted by rational traders in order to determine the true risky value V. Thus, the own prior opinions are rather independent of the market evolution and can be written as:

$$r_t^{r,p} \sim N(r^v, (\sigma^r)^2), \tag{2.45}$$

where r^v stands for the log-return on the true risky value (e.g. V valued at the risk-free rate R_f) and σ^r for the standard deviation of the prior rational beliefs. The second element of information entering the rational beliefs consists of an estimation of how emotional traders expect prices to evolve. Similarly to the first part of Section 2.2.2, we assume that rational traders are able to correctly guess the emotional subjective prior $r_t^{e,p}$, but recall that this prior is now the result of the dynamic updating process described by the above Equation (2.44). Third, rational agents also account for the influence of noise traders. As they are certain that noise traders act randomly, the noise trader a-priori opinions can be formally rendered as:

$$r_t^{n,p} \sim N(0, (\sigma^n)^2). \tag{2.46}$$

Furthermore, as $r_t^{r,p}$ most probably represents information that has nothing in common with how the market evaluates the risky asset, we can consider this random variable to be independent of the priors of emotional and noise traders $r_t^{e,p}$ and $r_t^{n,p}$. All priors $r_t^{i,p}$, where $i \in \{r; e; n\}$, are assumed to be i.i.d. and, according to the first noise scenario, $r_t^{e,p}$ and $r_t^{n,p}$ are independent of each other.

Concerning the *current beliefs*, we work with the same Equations (2.8) and (2.7). Also, the formation of *posterior beliefs* continues to follow Equation (2.2), where $a = b = 1$ for rational traders, but different belief weights a and b can be ascribed to the prior and current elements of belief by the emotional traders.

Thus, the emotional subjective log-returns yield the following distribution:

$$r_t^e \sim N\Big(k_{t-1} + k^e \tilde{r}_{t-1}^e, \frac{\sigma^2}{b} + \frac{(k^e \sigma_{t-1}^e)^2}{a}\Big),$$

from which we obtain, through iterate replacement of the mean and variance expressions, the following:

$$r_t^e \sim N\Big(k_{t-1} \sum_{s=0}^{t-1} (k^e)^s + (k^e)^t r^e, \sum_{s=0}^{t-1} \left(\frac{(k^e)^2}{a}\right)^s \frac{\sigma^2}{b} + \left(\frac{(k^e)^2}{a}\right)^t (\sigma^e)^2\Big). \tag{2.47}$$

Similarly, the rational subjective log-returns have the following distribution:

$$r_t^r \sim N\Big(c_{t-1} + c^r r^v + c^e \tilde{r}_{t-1}^e, \sigma^2 + (c^r\sigma^r)^2 + (c^e\sigma_{t-1}^e)^2 + (c^n\sigma^n)^2\Big),$$

which further yields:

$$\begin{aligned} r_t^r \sim N\Bigg(& c_{t-1} + c^r r^v + c^e\left(k_{t-1}\sum_{s=0}^{t-1}(k^e)^s + (k^e)^t r^e\right), \\ & (c^r\sigma^r)^2 + \left(1 + \frac{(c^e)^2}{b}\sum_{s=0}^{t-1}\left(\frac{(k^e)^2}{a}\right)^s\right)\sigma^2 + \left(\frac{(k^e)^2}{a}\right)^t (c^e\sigma^e)^2 + (c^n\sigma^n)^2\Bigg). \end{aligned} \tag{2.48}$$

For finite trade intervals $t = 1, \ldots, T$ and $|k^e| \in \mathbb{R}\backslash\{1; \sqrt{a}\}$, we obtain the dynamical equivalent of Equations (2.9) as:

$$\begin{aligned} r_t^r \sim N\Bigg(& c_{t-1} + c^r r^v + c^e k_{t-1}\frac{1-(k^e)^t}{1-k^e} + c^e(k^e)^t r^e, \\ & (c^r\sigma^r)^2 + \left(1 + \frac{(c^e)^2}{b}\frac{1-\left(\frac{(k^e)^2}{a}\right)^t}{1-\frac{(k^e)^2}{a}}\right)\sigma^2 + \left(\frac{(k^e)^2}{a}\right)^t (c^e\sigma^e)^2 + (c^n\sigma^n)^2\Bigg) \end{aligned} \tag{2.49a}$$

$$r_t^e \sim N\left(k_{t-1}\frac{(k^e)^t - 1}{k^e - 1} + (k^e)^t r^e, \frac{\left(1-\frac{(k^e)^2}{a}\right)^t}{1-\frac{(k^e)^2}{a}}\frac{\sigma^2}{b} + \left(\frac{(k^e)^2}{a}\right)^t (\sigma^e)^2\right). \tag{2.49b}$$

Since all above further considerations regarding trader demands and price formation continue to hold, the market returns in Equation (2.32) can be approximated, under dynamic belief updating, as follows:

$$r_t \approx \frac{V}{2P_{t-1}} + \frac{\lambda}{2}N^e\beta^e\left(k_{t-1}\frac{1-(k^e)^t}{1-k^e} + (k^e)^t r^e\right) + \frac{\lambda}{2}\sqrt{\frac{1-\left(\frac{(k^e)^2}{a}\right)^t}{b\left(1-\frac{(k^e)^2}{a}\right)}}\zeta_t + \frac{\lambda}{2}\left(\frac{(k^e)^2}{a}\right)^{\frac{t}{2}}\epsilon_t^e + \frac{\lambda N^n}{2}\epsilon_t^n. \tag{2.50}$$

Recall that the random variable ζ_t stands for *exogenous noise*, ϵ_t^e for *emotional noise* and

ϵ_t^e for *noise trader noise*. These variables are independently distributed as follows:

$$\begin{aligned}\zeta_t &\sim N\Big(0,\sigma^2\Big)\\ \epsilon_t^e &\sim N\Big(0,(\sigma^e)^2\Big)\\ \epsilon_t^n &\sim N\Big(0,(\sigma^n)^2\Big).\end{aligned}$$

The ratio V/P_{t-1} can be further understood as the return on the true value. Thus, denoting the log-return on the true value by r_{t-1}^v, we can approximate it by $V/P_{t-1} \approx r_{t-1}^v + 1$. Then, the distribution of the market log-returns can be approximated by the following normal distribution:

$$N\Bigg(\frac{r_{t-1}^v+1}{2}+\frac{\lambda N^e\beta^e k_{t-1}\Big(1-(k^e)^t\Big)}{2(1-k^e)}+\frac{\lambda N^e\beta^e(k^e)^t}{2}r^e,\\ \frac{(\lambda N^e\beta^e)^2}{4}\left(\frac{1-\left(\frac{(k^e)^2}{a}\right)^t}{1-\frac{(k^e)^2}{a}}\frac{\sigma^2}{b}+\frac{(\lambda N^e\beta^e)^2}{4}\left(\frac{(k^e)^2}{a}\right)^{\frac{t}{2}}(\sigma^e)^2+\frac{(\lambda N^n)^2}{4}(\sigma^n)^2\right)\Bigg). \tag{2.51}$$

Recall that rational traders maximize expected wealth and the market is in equilibrium if Equation (2.25) holds. Thus, the rational strategy amounts to adapting to the market conditions created by other traders. This is possible if rational traders are able to infer, with sufficient accuracy, the impact of emotional agents and noise traders on prices.

With dynamic belief updating and for finite T, this occurs only if the emotional agents limit their actions in a somewhat drastic way. One possibility is that emotional traders continue to account for their own mean priors in updating beliefs and hence $k^e \neq 0$. In this case, the only solution to Equation (2.25) requires that $\beta^e = 0$ (and implicitly $\beta^r = 0$). In other words, emotional traders do not take part in the trade since, according to Equation (2.22), their demand is always nil $Q_t^e = 0$. Moreover, by Equation (2.20), rational traders can trade only constant quantities $Q_t^r = \beta$. Eventually, this implies that also the noise trader demand Q_t^n is constant in equilibrium, according to Equation (2.19). Obviously, this is not a realistic assumption.

The other situation in which rational traders succeed in maximizing expected wealth requires that their emotional peers ignore their own mean prior information in updating opinions, i.e. $k^e = 0$. Then, the rational belief parameters must satisfy according to

Equation (2.26) the following conditions:

$$\begin{aligned}
\beta^r &= -\frac{N^e\beta^e}{2N^r\sqrt{b+(c^e)^2}} \\
c^r &= 0 \\
c^n &= \frac{N^n}{N^e\beta^e}\sqrt{b+(c^e)^2} \\
c_{t-1} &= k_{t-1}(\sqrt{b+(c^e)^2}-c^e) - \frac{\sqrt{b+(c^e)^2}}{N^e\beta^e P_{t-1}}\left(\frac{V}{\lambda}-2N^r\beta\right),
\end{aligned} \tag{2.52}$$

while c^e can be freely chosen. Consequently, there exists not a single, but multiple strategies that can be adopted by rational traders in order to adapt and hence maximize expected wealth. With $k^e = 0$, the rational and emotional subjective log-returns entail:

$$r_t^r = k_{t-1}\sqrt{b+(c^e)^2} + \sqrt{\frac{b+(c^e)^2}{b}}\zeta_t + c^n\epsilon_t^n \tag{2.53a}$$

$$r_t^e = k_{t-1} + \frac{1}{\sqrt{b}}\zeta_t, \tag{2.53b}$$

and the market log-returns can be approximated as follows:

$$r_t \approx \frac{r_{t-1}^v+1}{2} + \frac{\lambda N^e\beta^e k_{t-1}}{2} + \frac{\lambda N^e\beta^e}{2\sqrt{b}}\zeta_t + \frac{\lambda N^n}{2}\epsilon_t^n. \tag{2.54}$$

Note that in this case the emotional impact on prices is not anymore carried out by means of the specific noise ϵ_t^e, and the weight of prior elements of belief a plays no further role in this context. The emotional impact reduces to the influence of the constant k_{t-1}, of the demand sensitivity β^e, and of the number of emotional traders N^e.

Increasing the length of the trade interval $T \to \infty$ yields – for sufficiently small values of the emotional parameter k^e, more exactly if $|k^e| < \min\{1; \sqrt{a}\}$ – the following limit-distributions of the rational and emotional subjective log-returns:

$$r_t^r \xrightarrow{d} N\left(c_{t-1} + c^r r^v + \frac{c^e k_{t-1}}{1-k^e}, (c^r\sigma^r)^2 + \left(1 + \frac{(c^e)^2}{b\left(1-\frac{(k^e)^2}{a}\right)}\right)\sigma^2 + (c^n\sigma^n)^2\right) \tag{2.55a}$$

$$r_t^e \xrightarrow{d} N\left(\frac{k_{t-1}}{1-k^e}, \frac{\sigma^2}{b\left(1-\frac{(k^e)^2}{a}\right)}\right). \tag{2.55b}$$

Note that, at the limit, the influence of the emotional noise – of variance $(\sigma^e)^2$ – is entirely eliminated. Moreover, both limit-distributions of the rational and emotional subjective log-returns exhibit time-constant variances, while their means depend on time-varying parameters such as c_{t-1} and k_{t-1}. Nevertheless, it is plausible that at least the emotional traders do not account for the "constants" in their linear combination of information that serves as support for their beliefs, so that $k_{t-1} = 0$. In such a case, the emotional log-returns become stationary at the limit.[60]

In the same limit-situation, the market log-returns are approximatively distributed as follows:

$$N\Big(\frac{r^v_{t-1}+1}{2} + \frac{\lambda N^e \beta^e k_{t-1}}{2(1-k^e)}, \frac{(\lambda N^e \beta^e)^2}{4b\left(1-\dfrac{(k^e)^2}{a}\right)}\sigma^2 + \frac{(\lambda N^n)^2}{4}(\sigma^n)^2\Big), \tag{2.56}$$

which is stationary if:

$$\frac{\lambda N^e \beta^e k_{t-1}}{1-k^e} = \kappa - (r^v_{t-1}+1), \tag{2.57}$$

with κ an arbitrary constant.

Letting $T \to \infty$ should give to the rational traders sufficient time to learn the emotional behavior and adapt to it. For this adaptation to be successful, the following must hold:

$$\begin{aligned}
\beta^r &= -\frac{N^e\beta^e}{2N^r}\left(b\left(1-\frac{(k^e)^2}{a}\right)+(c^e)^2\right)^{-\frac{1}{2}} \\
c^r &= 0 \\
c^n &= -\frac{N^n}{2N^r\beta^r} = \frac{N^n}{N^e\beta^e}\sqrt{b\left(1-\frac{(k^e)^2}{a}\right)+(c^e)^2} \\
c_{t-1} &= \frac{k_{t-1}}{1-k^e}\left(\sqrt{b\left(1-\frac{(k^e)^2}{a}\right)+(c^e)^2}-c^e\right) \\
&\quad -\frac{1}{N^e\beta^e P_{t-1}}\sqrt{b\left(1-\frac{(k^e)^2}{a}\right)+(c^e)^2}\left(\frac{V}{\lambda}-2N^r\beta\right).
\end{aligned} \tag{2.58}$$

Again, the choice of the weight accorded to emotional elements of belief c^e is free. The

[60] Since we are working with normal distribution, the constance of the mean and variance ensures not only weak, but also strong stationarity.

rational and emotional subjective log-returns result then in:

$$r_t^r = \sqrt{b\left(1-\frac{(k^e)^2}{a}\right)+(c^e)^2}\left(\frac{k_{t-1}}{1-k^e}-\frac{\dfrac{V}{\lambda}-2N^r\beta}{N^e\beta^e P_{t-1}}\right)+\sqrt{\frac{b\left(1-\dfrac{(k^e)^2}{a}\right)+(c^e)^2}{b\left(1-\dfrac{(k^e)^2}{a}\right)}}\zeta_t + c^n\epsilon_t^n \tag{2.59a}$$

$$r_t^e = \frac{k_{t-1}}{1-k^e}+\frac{1}{\sqrt{b\left(1-\dfrac{(k^e)^2}{a}\right)}}\zeta_t, \tag{2.59b}$$

while the market log-returns can be approximated by:

$$r_t \approx \frac{r_{t-1}^v+1}{2}+\frac{\lambda N^e\beta^e k_{t-1}}{2(1-k^e)}+\frac{\lambda N^e\beta^e}{2\sqrt{b\left(1-\dfrac{(k^e)^2}{a}\right)}}\zeta_t+\frac{\lambda N^n}{2}\epsilon_t^n. \tag{2.60}$$

Note that the results obtained for finite T in Equations (2.52)-(2.54) are identical to the results obtained at the limit from Equations (2.58)-(2.60), when taking $k^e = 0$.

2.2.3 Application

This section analyzes, by means of simulation techniques and for various parameter constellations, the evolution of market returns and of the rational and emotional wealth that results from the theoretical setting developed in Section 2.2.2.

In order to keep the exposition as clear as possible, we present here only the most important results. The particular assumptions and values – more exactly, value ranges – of the behavioral parameters that underlie these results are enumerated first. All further deviations from these assumptions and values are explicitly indicated in the text.

The subsequent findings map *average* values obtained over $n = 10$ rounds of each $T = 100$ trade times. Results are made comparable by setting the initial seed to the same value for all cases.

Our population consists of $N = 100$ traders, out of which the number of noise traders is fixed at $N^n = 5$, while that of rational and emotional traders can vary. In particular, we refer to three different *cases* denoted as low, middle, and high proportions of emotional traders: $N^e \in \{25; 50; 75\}$, respectively. The remaining $N^r = N - N^e - N^n$ traders are rational.

Furthermore, our *individual* agents start trading (at time $t = 0$) with identical endowments of risky asset $Q_0^r = Q_0^e = Q_0^n = Q/N$, as well as with identical wealth $W_0^r = W_0^e = W_0^n = 1$.[61] We account for two situations of starting the trade of each parallel round $j = 1, \ldots, n$: either from identical initial values, namely unitary (subjective and market) prices and gross returns, and nil log-returns, which is equivalent to having independent rounds, or from the values at which the last round $j-1$ closed. In the latter situation, trade starts at $j = 1$ again with unitary prices and gross returns, and with nil log-returns.

In line with Hasbrouck (2005), we fix the inverse market liquidity at $\lambda = 0.08$. As this is not sufficient for Cases B and C, we will also work with $\lambda = 0.008$. The standard deviations of the exogenous, rational, emotional, and noise trader noise are taken to be respectively $\sigma = 0.01, \sigma^r = 0.02, \sigma^e = 0.03, \sigma^n = 0.03$.

The three *types* of emotional weighting of beliefs mentioned in Section 2.2.2 – impulsive, balanced, and conservative – are accounted for by considering adequate belief weight ratios a/b, namely over-unitary, unitary, and sub-unitary. Specifically, we fix the weight of prior elements of belief to be $a = 1$ and let the weight put on current elements of belief vary in the set $b \in \{100; 1; 0.01\}$. In addition, we work with the following values of the emotional belief parameters: The sensitivity of emotional demand to subjective-return changes is $\beta^e = 1$, the belief constant $k_{t-1} = 0$ at each trade, and the weight of prior beliefs $k^e = N^e/N$. We also include some further results for other values of these parameters. According to the last part of Section 2.2.2, we set $k^e = 0$ for the case with dynamic belief updating and finite rounds of trade $T = 100$.

The rational belief parameters c_{t-1}, c^r, c^e, c^n and the rational demand sensitivity β^r are approximated depending on the assumed noise *scenario*, with independent or identical emotional and noise trader noise. Specifically, they result from Equations (2.26) and (2.28) for our initial setting, and from Equations (2.52) and (2.58) for the dynamic setting. Recall that the dynamic setting merely accounts for the first scenario, with independent emotional and noise trader noise. When the rounds of trade are independent of each other, the rational demand constant is set at $\beta = 0.5$. For continuing rounds of trade, the rational demand constant β is fixed at 0.5 in the first round $j = 1$, while in the subsequent ones $1 < j \leq n$, we estimate β based on the previous round and in line with the following idea: At the end of trade T, rational traders expect the price to equal the true risky value

[61] We have also considered the possibility that the initial endowment and the wealth is identical among trader-groups. The result are qualitatively similar.

$P_T^r = V$. Therefore, their demand at T should equal zero $Q_T^r = 0$. Then, according to Equation (2.20), the rational demand constant β yields:

$$\beta = -\beta^r (V - P_{T-1}),$$

where P_{T-1} stands for the last but one price of the current trading round. It is plausible to assume that rational traders have already formed an opinion on P_{T-1} from the previous rounds of trade in which they took part. For instance, they may set P_{T-1} to be equal with the last price of the previous round, which is the case when the trade is continued from one round to the other.

In addition, beliefs can be updated according to different *rules*. First, we work in our initial setting and consider that rational and emotional traders update merely the mean prior beliefs $\tilde{r}_{t-1}^r$ and $\tilde{r}_{t-1}^e$, respectively. This setting denoted as "quasi-dynamic". In this context, we subsequently refer to the following two possibilities:
(a) The mean prior beliefs are inferred from previous market returns. Specifically, we set the prior means to be:

$$\tilde{r}_{t-1}^r = \text{mean}[r_{t-s}], \forall s = 1, \ldots, t-1$$

$$\tilde{r}_{t-1}^e = \text{mean}[r_{t-s}], \forall s = S_1, \ldots, S_2, \text{ where } \begin{cases} S_1 = t - \left\lfloor \dfrac{t-1}{2} \right\rfloor, S_2 = t-1, \text{ for Type i} \\ S_1 = 1, S_2 = t-1, \text{ for Type b} \\ S_1 = \left\lfloor \dfrac{t-1}{2} \right\rfloor, S_2 = t-1, \text{ for Type c }, \end{cases}$$

and thus the emotional prior mean depends on the emotional type. Obviously, for the balanced Type b, the rational and emotional prior means are identical $\tilde{r}_{t-1}^r = \tilde{r}_{t-1}^e$.
(b) The mean priors are obtained from what we denote as past demands, in particular from past order flows and past demands. Specifically, $\tilde{r}_{t-1}^r$ results from an approximation of Equation (2.30) that is $Q^r(t) \approx P_{t-1}(\beta + \beta^r r_t^r)$, where Q_t^r takes the form in Equation (2.20). Moreover, we approximate $\tilde{r}_{t-1}^e$ from the emotional demand Q_t^e in Equation (2.22), which is linear in the log-normal variable R_t^e. Thus, we obtain:

$$\begin{aligned} \tilde{r}_{t-1}^r &= \frac{1}{N^r \beta^r P_{t-2}} \left(\frac{V}{\lambda} - Q_{t-1} \right) - \frac{\beta}{\beta^r} \\ \tilde{r}_{t-1}^e &= \log \left(1 + \frac{Q_{t-1}^e}{\beta^e P_{t-2}} \right). \end{aligned}$$

The rationale for these assumptions is that rational traders are deeply concerned with what other market participants are doing, so that they might reformulate prior beliefs in dependency on the newest public information (i.e. on the last total order flow). In addition, emotional traders account only for their own actions, so that we can imagine a situation when they update their beliefs on average, considering the last quantity they demanded. In essence, the rule for emotional traders is equivalent to $\tilde{r}^e_{t-1} = r^e_{t-1}$.[62] Second, beliefs can be fully updated in the sense considered in the last part of Section 2.2.2. In this context, we work with the corresponding formulas and account for the following two rules that we denote as "dynamic": with finite $T = 100$ and with a very high $T = 10000$. The latter is aimed at better resembling the limit-case with infinitely long trading rounds. In so doing, we set the initial mean priors of rational and emotional traders in Equations (2.44) and (2.45) to $r^v = 1$ and $r^e = 0$, respectively. We chose the same $c^e = N^e/N$ and $\sigma = 0.01, \sigma^r = 0.02, \sigma^e = 0.03, \sigma^n = 0.03$. Recall also that the dynamic updating with finite rounds of trade assumes $k^e = 0$.

In sum, we resolve for analyzing the following:

- Three cases describing the composition of the trader population:

Case A: Low proportion of emotional traders $N^e/N = 25\%$.

Case B: Middle proportion of emotional traders $N^e/N = 50\%$.

Case C: High proportion of emotional traders $N^e/N = 75\%$.

- Two scenarios regarding the correlation of noise terms:

Scenario 1: Independent emotional and noise trader noise $\sigma^{en} = 0$.

Scenario 2: Identical emotional and noise trader noise with $\sigma^e = \sigma^n = \sqrt{\sigma^{en}}$.

- Two streams of belief-updating rules:
    - Quasi-dynamic, i.e. updating of mean priors:

      Rule qd-1: From previous market log-returns r_s, where $0 < s < t$.

      Rule qd-2: From current demands (in particular from the current total order flow Q_t for the rational traders and from the current subjective log-return r^e_t for the emotional traders).

[62] We have also employed further belief rules. Accordingly, both rational and emotional mean priors of the next period are obtained: from current demands, as expectations or as averages of the respective subjective log-returns at the current trade, or from cumulated past returns. The results are qualitatively similar.

- Dynamic:

  Rule d-1: With finite rounds of trade $T = 100$ (and $k^e = 0$).

  Rule d-2: With very long rounds of trade $T = 10000$ (at the limit).

- Three emotional trader types with respect to belief formation:

  Type i: Impulsive $b/a = 100$.

  Type b: Balanced $b/a = 1$.

  Type c: Conservative $b/a = 0.01$.

- Two manners of organizing the trade:

  - Independent parallel rounds (and β fixed).
  - Continuing parallel rounds (and β based on the previous round).

We commence by generating the normally distributed noise terms with the considered standard deviations $\sigma, \sigma^r, \sigma^e, \sigma^n$. Using the above parameter values, we subsequently derive the emotional subjective returns both in logarithmic form r_t^e and as gross returns R_t^e. To this end, we employ Equations (2.11b) and (2.14b) for the quasi-dynamic updating, and Equations (2.53b) and (2.59b) for the dynamic updating. Then, Equations (2.11c) and (2.14c) deliver estimated of r_t^n and R_t^n. The rational subjective returns in equilibrium r_t^r and R_t^r result from the adaptation conditions in Equations (2.26) and (2.28) for the quasi-dynamic belief rule, and from Equations (2.52) and (2.58) for the dynamic belief updating. After computing the current subjective returns, the demands of each trader group are calculated according to Equations (2.19), (2.22), and (2.23). Equation (2.31) delivers values for the equilibrium gross returns. We also compute the approximative log-returns according to the adequate Equation (2.33-2.34), (2.54), or (2.60). The mean priors are then ascertained according to the different belief rules. Finally, the wealth of individual traders from each group $i \in \{r; e; n\}$ is derived from Equation (2.35) and the corresponding group wealth by multiplying the individual wealth by the corresponding group dimension N^i. We also compute the wealth growth according to Equation (2.42). Wealth and its growth can be regarded as measures of trader survival.

In the sequel, we focus on Scenario 1, for which Rules (qd-1) and (d-1) can easily be compared. Moreover, we present only the results obtained for a true risky value $V = 1$, an initial rational demand constant $\beta = 0.5$, and emotional belief parameters $k_{t-1} = 0$,

$k^e = N^e/N$, and $\beta^e = 1$. We exemplify how trade evolves when performing independent and continuing parallel rounds of trade for the same Scenario 1 and Rule (qd-1). The differences among the Cases A, B, and C are detailed for each of the Rules (qd-1) and (d-1). The corresponding figures for Scenario 2, as well as other figures referred to in the text, can be found in Appendix A.2. Log-returns and demands are depicted with means and confidence bands.

Case A: Low proportion of emotional traders $N^e/N = 25\%$.

Rule (qd-1): Quasi-dynamic belief updating from previous market returns.

Let us start the exposition of our findings for Case A. Relative to Cases B and C, the dimension of the emotional group is smallest, so that the influence of emotional traders on market evolutions should be minimal. As mentioned above, we focus on Scenario 1 and hence assume independent emotional and noise trader noises, and on the quasi-dynamic belief updating Rule (qd-1). We work with the usual values of the rational and emotional parameters $V = 1$, $\beta = 0.5$, $k_{t-1} = 0$, $k^e = N^e/N$, and $\beta^e = 1$, and consider independent parallel rounds of trade. This combination of cases, scenarios, rules, and parameter values is henceforth referred to as our *benchmark*.

The evolution of the log-returns, individual demands, individual wealth, and the growth of individual wealth in Case A, under Scenario 1, Rule (qd-1), and when the $n = 10$ rounds of trade are independent of each other, are illustrated in the following Figures 2.9-2.12.

As observed in the theoretical part, the market volatility should be lower when the asymmetry in the emotional way of combining past and current elements of belief is more pronounced. This is what Figure 2.9 shows under the assumption that $a = 1$: Log-returns become more volatile for lower ratios b/a, in other words when emotional traders for beliefs conservatively.[63] This occurs as higher weights are assigned to the exogenous noise term ζ_t in the final log-return expression approximated by Equation (2.33).

Several further statistical investigations are performed in order to analyze the stability and efficiency of a market where log-returns evolve as in the same Figure 2.9. For all three emotional types, the augmented Dickey-Fuller test (abbr. ADF) rejects the hypothesis of unit roots, already at 1% confidence, thus speaking for stationarity.[64] Log-returns exhibit

[63] Specifically, Type i yields a standard deviation of log-returns of 0.066083, Type b of 0.067460, and Type c of 0.07666.

[64] In particular, ADF=-7.682851 for Type i, ADF=-4.756128 for Type b, and ADF=-6.232654 for Type c, where the test is based on the Schwartz information criterion with maximal 12 lags and the values are

yet positive autocorrelation and hence are predictable. The time interval on which the serial dependency stretches appears yet to be substantially shorter when emotional traders behave conservatively.[65] Moreover, the hypothesis of normally distributed log-returns is dismissed by the Jarque-Bera test (abbr. JB) for all emotional types.

In very general terms, we can conclude that markets where an emotionally guided activity induces specific conditions to which rational traders adapt, appear to be stable in front of non-recurring shocks but rather inefficient, in the sense that prices are predictable to a certain degree. A conservative belief formation on the part of the emotional traders may reduce the inertia of price movements and hence the predictability.

The demands of individuals belonging to each trader group are depicted in Figure 2.10. The emotional demand Q_t^e is on average positive, pointing out the fact that emotional traders mostly prefer to buy the risky asset. This demand becomes more volatile and follows a more pronounced up-sloping path for lower b/a-ratios. Note that the individual activity of rational traders remains low in all cases compared to that of the emotional or noise trading individuals. Indeed, the rational *group* has to face an increased total order flow issued by the other traders, but it is at the same time sufficiently numerous (recall that in Case A rational agents form the majority of traders) for the individual participation of each rational individuum to remain at low levels.

Since profit chances – as well as loss dangers – are higher in more volatile markets, individual traders from all groups accumulate higher wealth when emotional beliefs are conservative $b/a = 0.01$, as apparent in Figure 2.11. Note that in the analyzed Case A, under Scenario 1, and with Rule (qd-1), and irrespective of the emotional type, the emotional individual gain is highest. Eventually, rational individuals even lose money. Unreported results confirm that the same holds also with respect to the group wealth, although the rational group is almost three times as numerous as the emotional one.

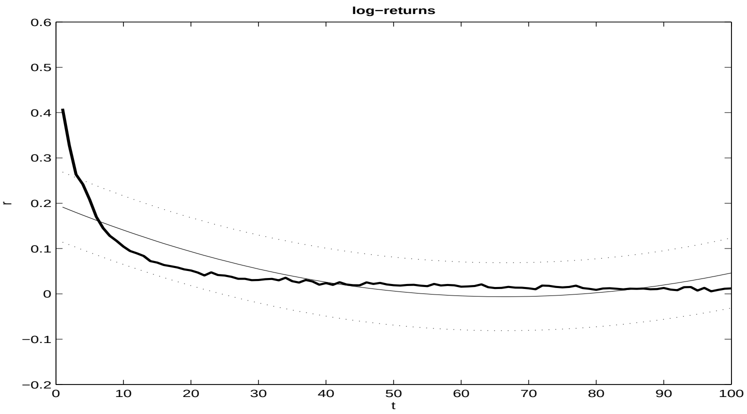
The dominance of emotional traders is reinforced by the growth of individual wealth from Figure 2.12. The growth ratios start from negative values for emotional traders, but pass fast into the positive domain and remain on average above zero, while the evolution of the rational wealth growth follows the opposite course. However, the discrepancies between rational and emotional traders in terms of growth of their individual wealth

significant at all levels.

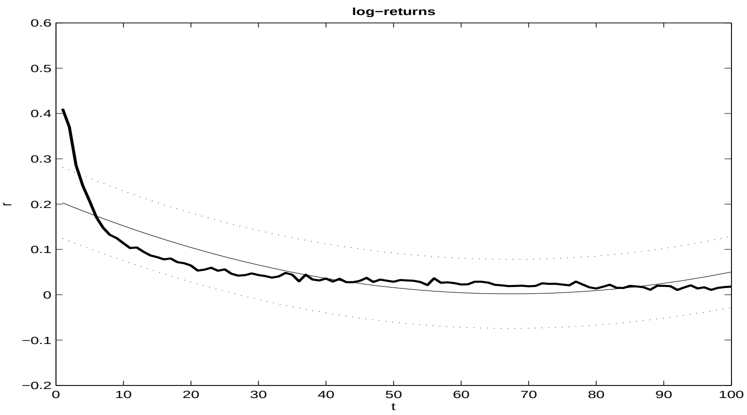
[65]Specifically, log-returns for Types i, b, and c are sufficiently well described by respectively ARMA(5,1), ARMA(6,1), and ARMA(1,1) processes. The first order autocorrelation coefficients AC(1) are always positive and a positive constant appears to significantly contribute to each of these specifications.

reduce towards the end of the trade.

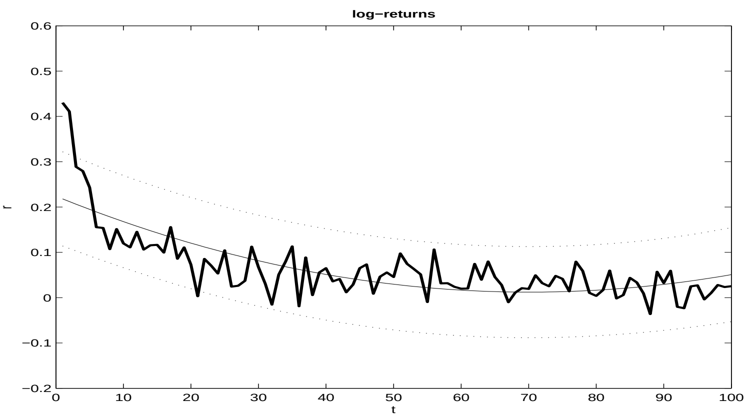
Thus, it appears that emotional traders can make money from their trades in markets similar to those modeled in our setting. Moreover, in the short run they can even come best off in terms of wealth – individually and group specifically – as well as with respect to the growth of individual wealth. In the long run, their chances of survival, measured by the growth of individual wealth, become yet comparable to those of their rational peers.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

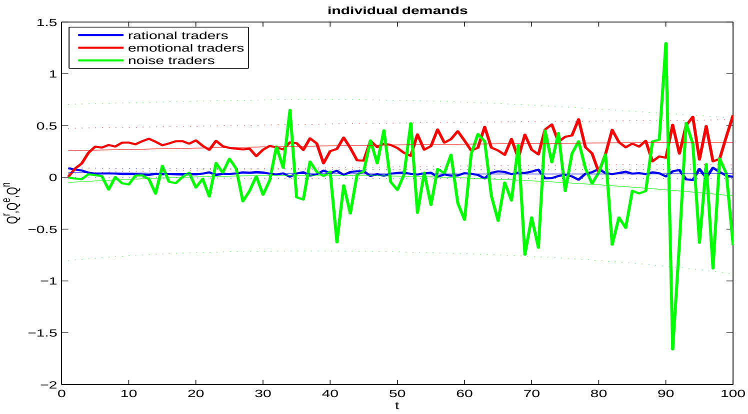


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

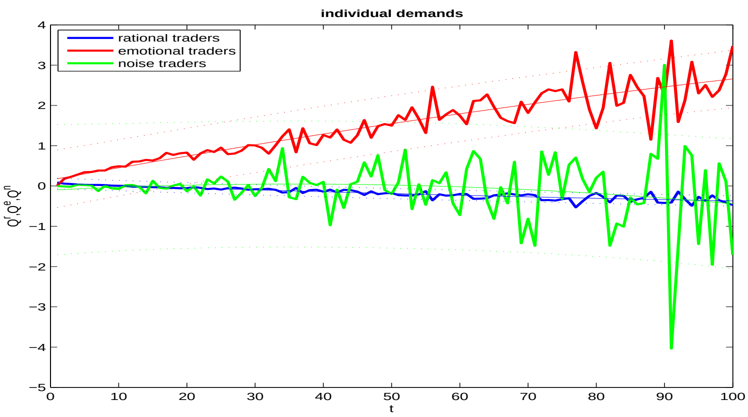


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

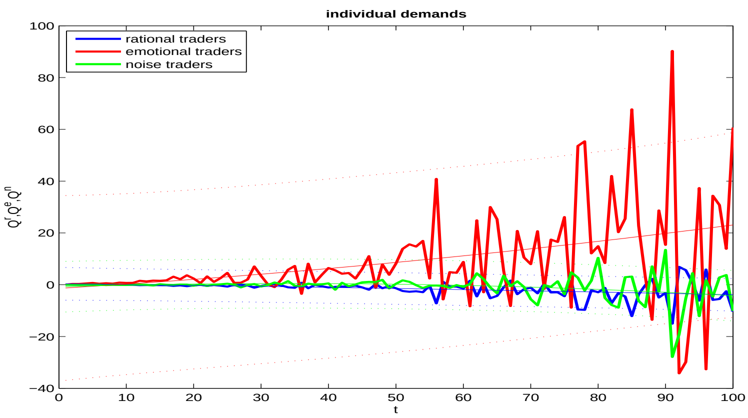
Figure 2.1: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

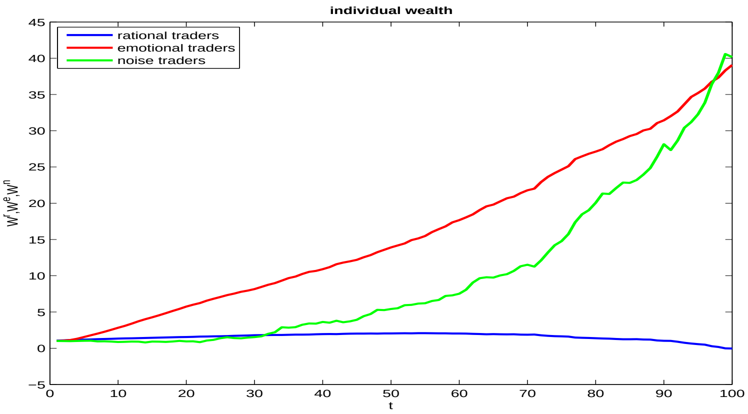


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

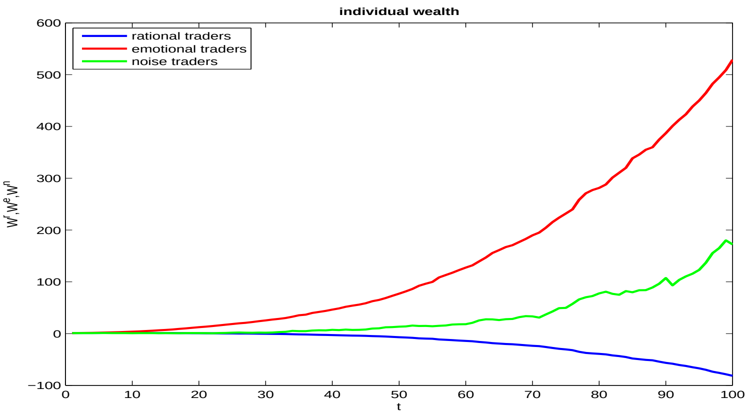


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

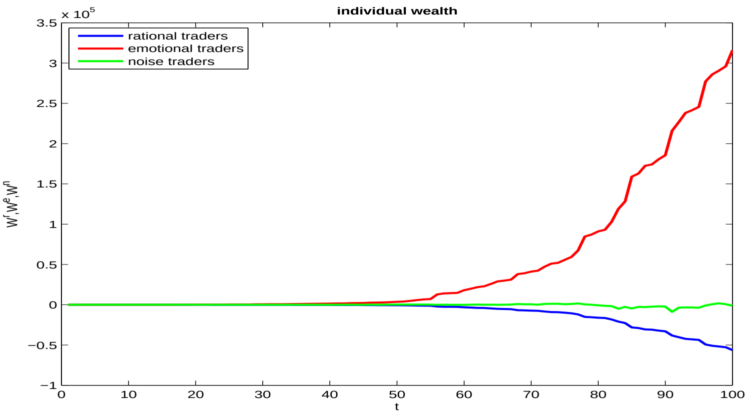
Figure 2.2: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

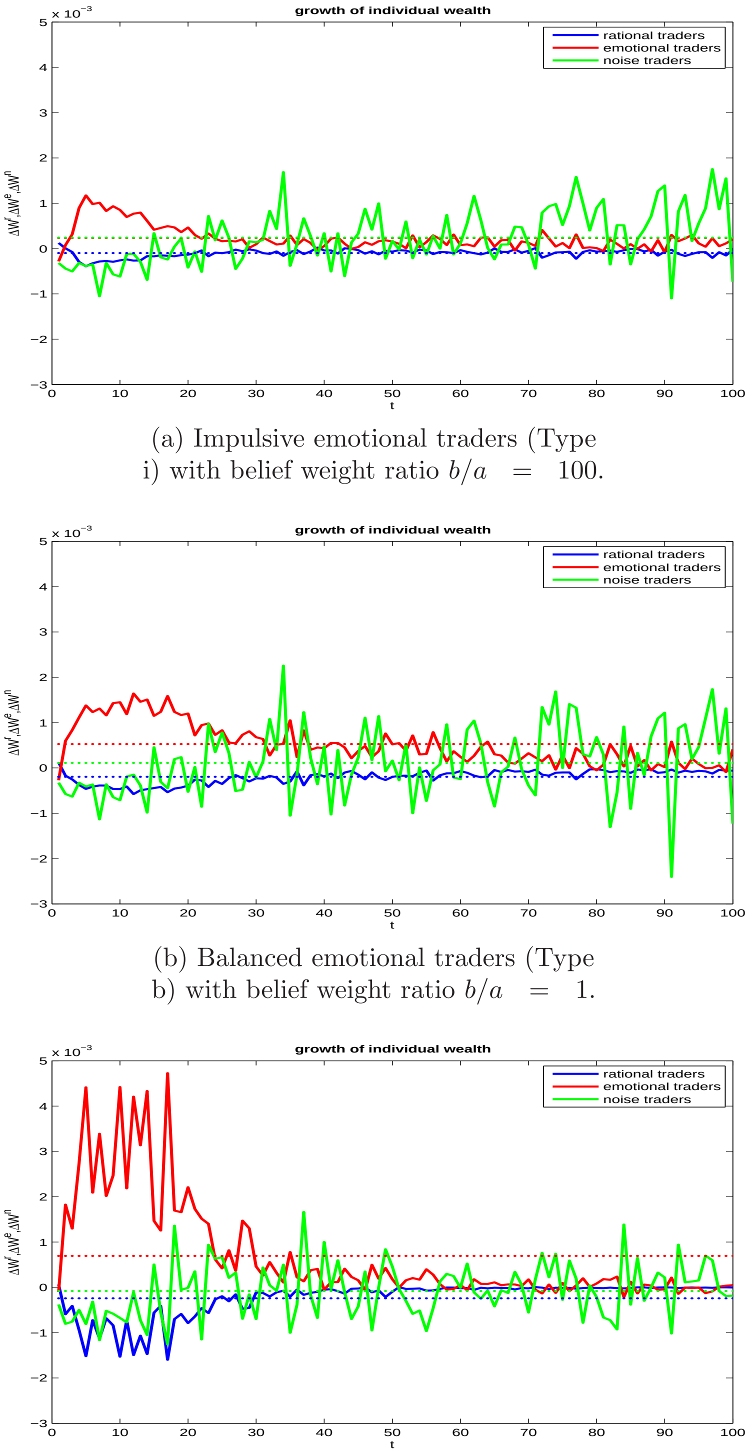


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.



(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure 2.3: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure 2.4: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.

In the sequel, we continue to analyze Case A but turn our attention, one at a time, to the following alternative market conditions: Scenario 2, continuing parallel rounds of trade, further belief updating rules such as Rules (qd-1), (d-1), and (d-2), and different values of the behavioral parameters β^e, k^e, k_{t-1}, β, and V. These situations are characterized by the change of a single element – scenario, rule, parameter, assumption, etc. – with respect to our benchmark, other things being equal.

Scenario 2: Identical emotional and noise trader noise with $\sigma^e = \sigma^n = \sqrt{\sigma^{en}}$.

The corresponding evolutions of prices, demands, and wealth under Scenario 2 are presented in Figures A.19-A.22 in Appendix A.2. The similarity between emotional and noise traders in terms of noise appears to exhibit a low influence on market and individual evolutions and is almost unnoticeable on optical inspection.

Independent vs. continuing rounds of trade.

Let us now compare our benchmark case (with $n = 10$ independent rounds of trade starting from identical conditions) with the situation when the trade starts, in round 1, from the same initial conditions, but continues from one round to the other in the remaining rounds. Rational traders adjust their demand constant β from one round to the other. The trade continuation is, in essence, equivalent to much longer trading rounds, so that traders should have the opportunity to "learn" from past rounds, increasing efficiency. As usual, we will *average* the values of log-returns, demands, and wealth across all n rounds.

Indeed, continuing rounds reduce both the market volatility[66] and the time span over which serial dependencies in log-returns stretch.[67] It also accentuates the discrepancies engendered by the different emotional types. For instance, the increase in log-return volatility with the importance ascribed by emotional traders to prior relative to current beliefs is more substantial, as illustrated in Figure 2.5. Moreover, although log-returns remain stationary for all emotional types, the conservative behavior appears to foster the market efficiency even more evidently: For Type c, and under both scenarios, we can

[66]Specifically, the standard deviations of log-returns for Types i, b, and c are respectively 0.007599, 0.00894, and 0.034093.

[67]For Types i and b, log-returns are sufficiently well described by ARMA(1,1) processes. They remain yet non-normally distributed (JB). The usual ADF-test based on the Schwartz information criterion with maximal 12 lags delivers the following test values: ADF=-6.492826 for Type i, ADF=-6.162888 for Type b, and ADF=-9.424551 for Type c. Thus, the hypothesis of no units roots cannot be rejected at any significance level. For Type c, log-returns do not significantly serially correlate, nor deviate from normality.

detect no significant serial correlation in log-returns (neither in their mean nor in their variance), neither departures from normality.

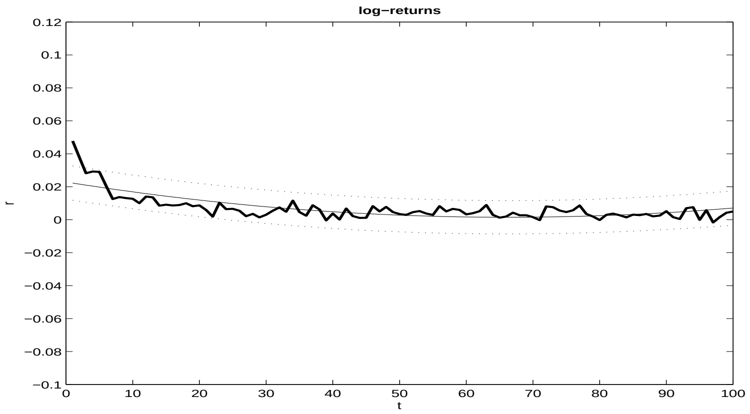
Regarding the average demands, emotional traders mostly buy the risky asset, while rational ones sell it, and the average demands of rational and emotional traders are no longer diverging from each other within the $T = 100$ trades.

Finally, the wealth of emotional individuals in Figure 2.7 remains higher than the rational one, but it is the highest in the market only as long as the emotional profile is of Type c. When emotional traders think impulsively, the market is dominated by noise traders. In terms of growth of individual wealth, as illustrated in Figure 2.8, emotional traders appear to be better off more frequently than rational ones, but this relation changes periodically. In general, the wealth growth of all trader categories converges to common values soon after the trade starts, and the convergence speed increases from Type i to Type c.

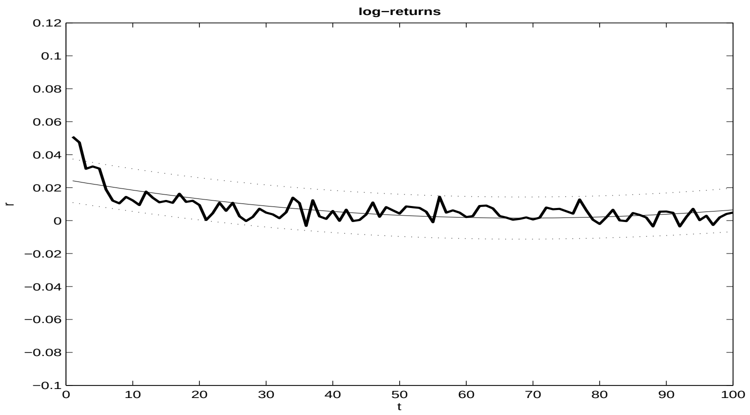
Relative to our benchmark case, Scenario 2 appears to generate somewhat more noticeable changes when the rounds of trade are continuing, as apparent in Figures A.23-A.26 in Appendix A.2. The evolution patterns observed for Scenario 1 are conserved, but for Type c the mean log-returns attain only half of those observed under Scenario 1,[68] and the growth of individual wealth of emotional traders is highest at the beginning of the trade.

In sum, the trade continuation appears to foster the market efficiency and stability. This might rely, among others, on the increased accuracy of the rational adaptation, given by the fact that rational traders can adjust one more parameter from one round to the other, specifically their demand constant β. Thus, the market returns are less volatile in general and even unpredictable when emotional traders think conservatively. Conservatism also represents the sole emotional profile that guarantees the highest individual wealth. In terms of individual wealth, rational traders remain worst off in the market, but in terms of growth of individual wealth, all traders obtain identical results by the end of the trade. The convergence to common values of the growth of individual wealth appears to be much faster for conservative emotional profiles.

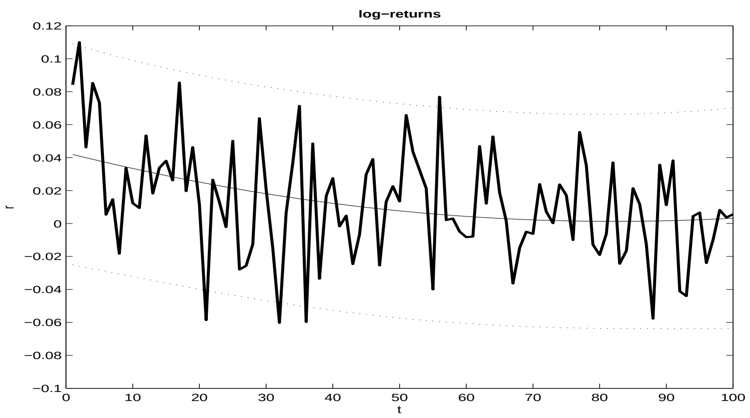
[68]Specifically, the mean log-returns for Type c under Scenario 1 (Scenario 2) is 0.012616 (0.006633).



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

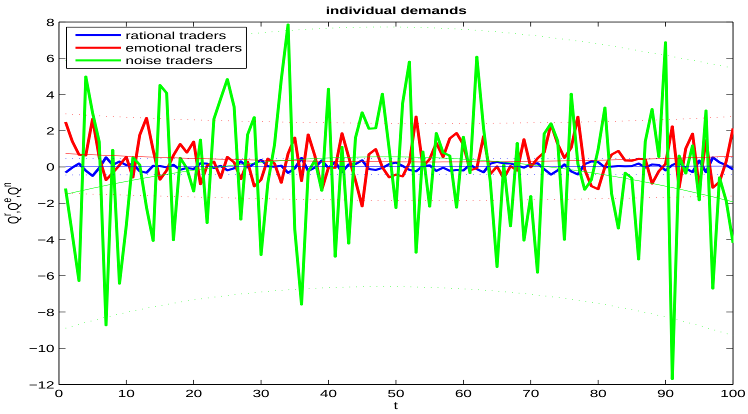


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

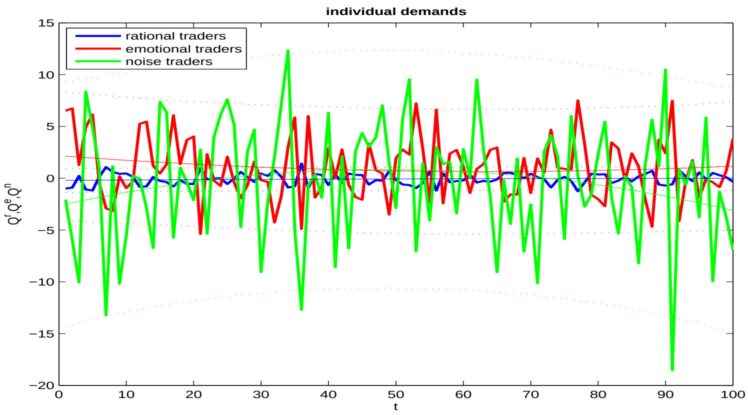


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

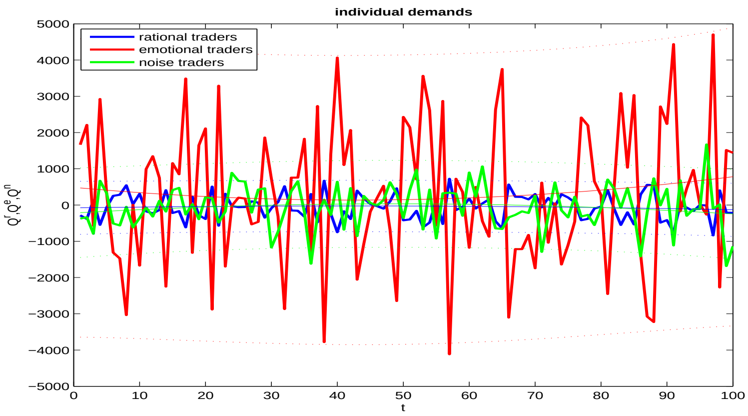
Figure 2.5: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ continuing parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

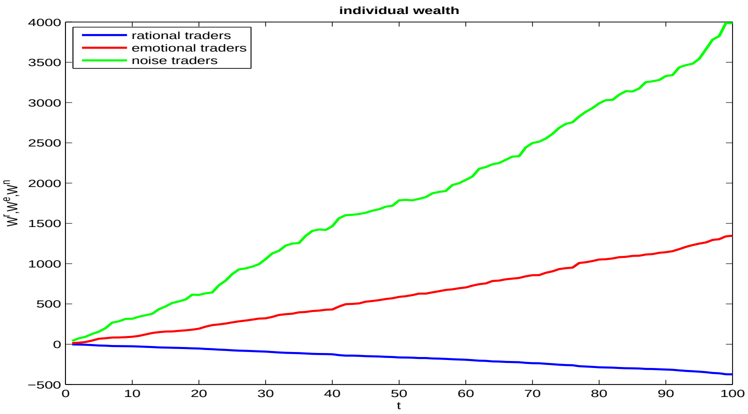


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

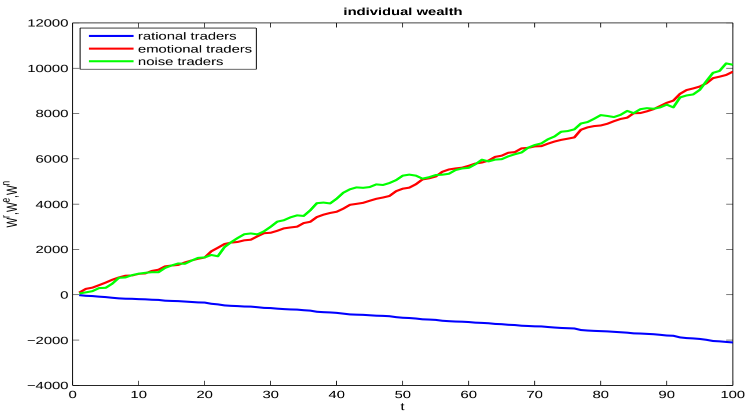


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

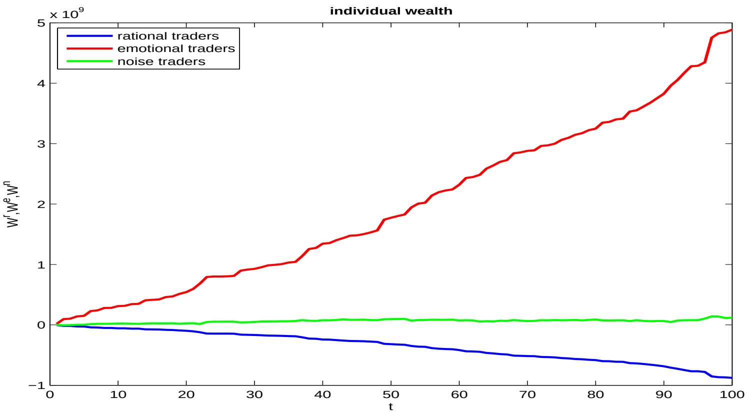
Figure 2.6: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ continuing parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

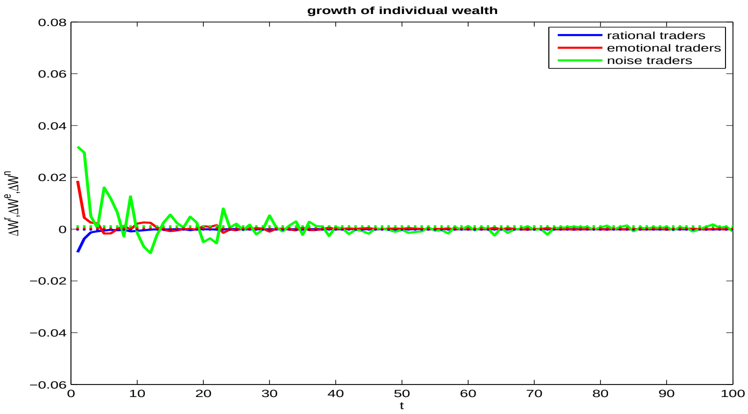


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

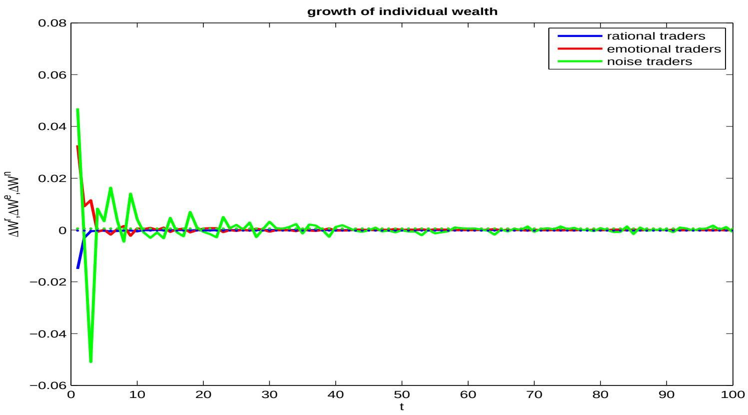


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

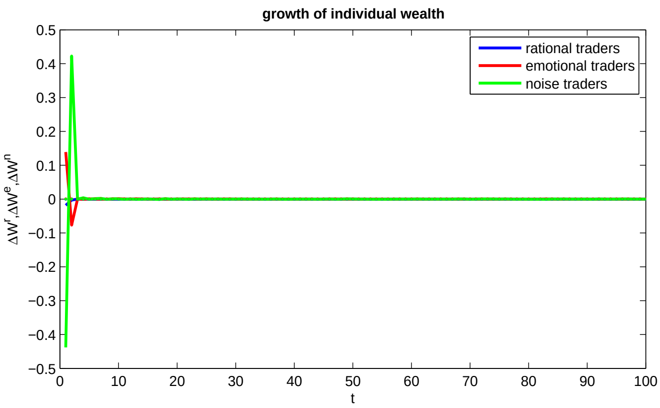
Figure 2.7: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ continuing parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.



(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.



(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure 2.8: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ continuing parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.

Rule (qd-2): Quasi-dynamic updating of mean prior beliefs from current demands.

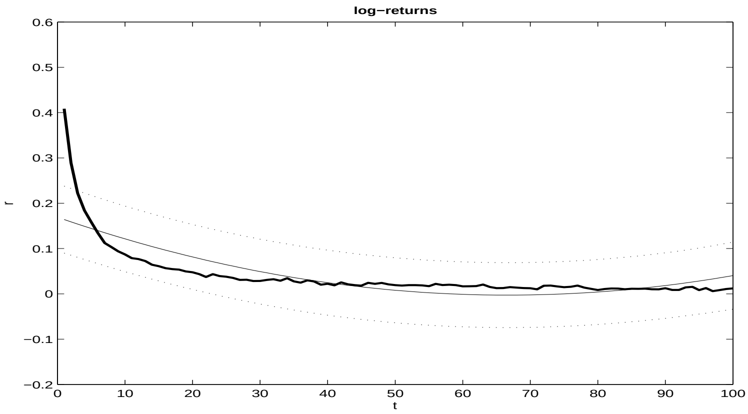
The following Figures 2.9-2.12 present the evolutions of log-returns, and of individual demands and wealth when rational and emotional beliefs are quasi-dynamically updated from current demands according to Rule (qd-2).

Rule (qd-2) does not entail noticeable differences in log-returns relative to our benchmark Rule (qd-1).[69] More substantial changes are yet to be noticed with respect to individual demands: Rational traders are buying on average the risky asset, while the mean demand of emotional traders unfolds in both positive and negative domains. The corresponding changes in the mean demand of individual emotional traders are easier to recognize for Type c.

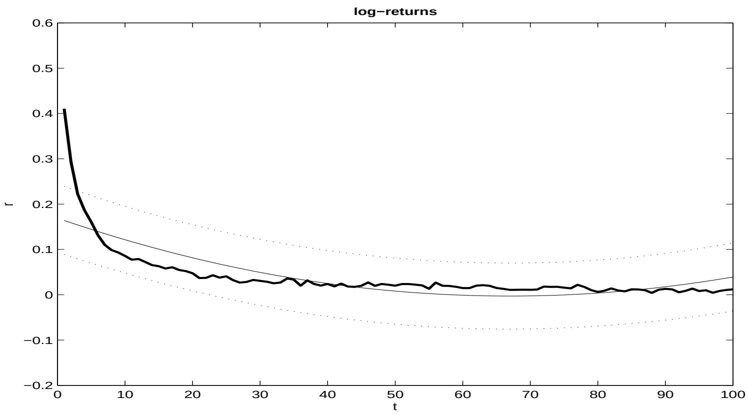
Consequently, emotional traders accumulate the highest individual wealth during the entire round of trade only when they behave conservatively. For impulsive and even balanced emotional belief formation, rational traders dominate the market, at least over the first half of the trade. The individual emotional wealth grows constantly, but, again, this growth is always positive only for Type c. Similar patterns emerge under Scenario 2 and are illustrated in Figures A.27-A.30 in Appendix A.2.

In sum, the updating of mean beliefs based on current demands appears to entail no relevant changes in terms of efficiency with respect to our benchmark case. Nevertheless, this is not the case as far as financial success is concerned: In the short run and as long as emotional traders think either impulsively or in a balanced way, the rational strategy now offers the best profit chances in the market. Yet, in the longer run, or for conservative emotional profiles, not the rational, but the emotional traders perform best.

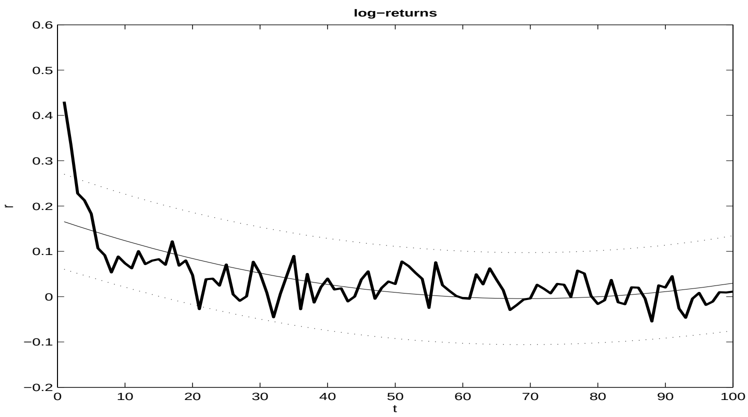
[69]Specifically, Types i, b, and c yield the following average standard deviations of log-returns: 0.058195, 0.056342, and 0.069264, which are somewhat lower than under Rule (qd-1). They are sufficiently well described by respectively ARMA(5,1), ARMA(6,1), and AR(5) processes. They are all stationary, as the ADF-test values, based on the Schwartz information criterion with maximal 12 lags, are significant at all levels. Specifically, these values yield ADF=-6.797712 for Type i, ADF=-4.527510 for Type b, and ADF=-7.244705 for Type c. Moreover, log-returns are not normally distributed, according to the JB-test.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

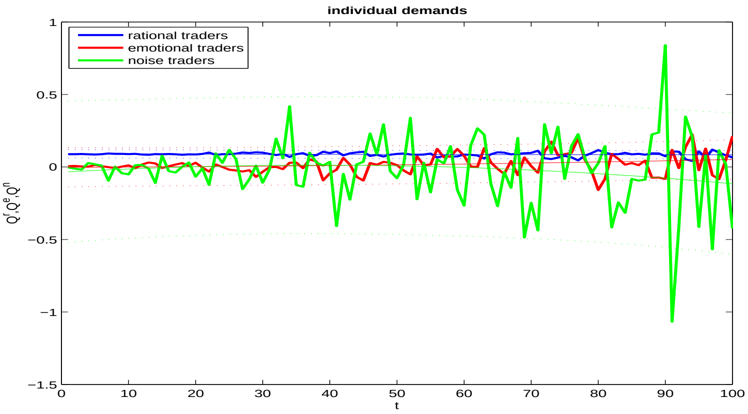


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

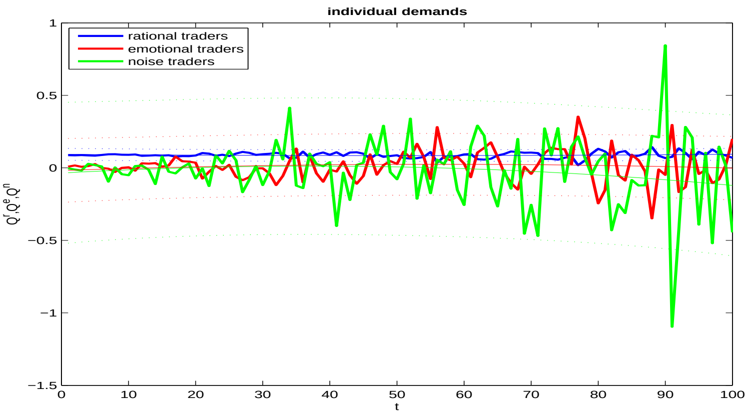


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

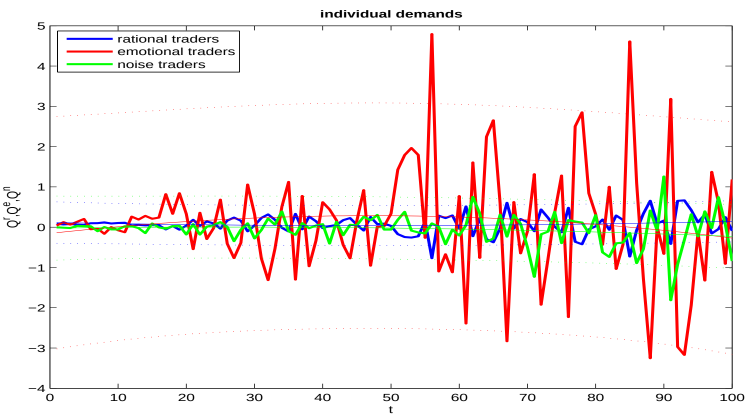
Figure 2.9: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from current demands (Rule qd-2), over $n = 10$ independent parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

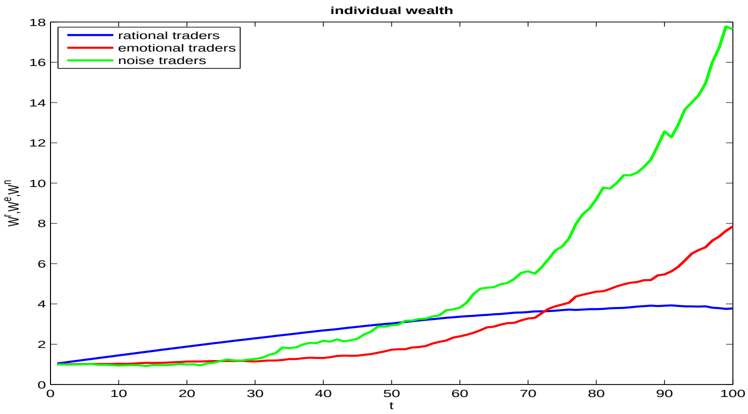


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

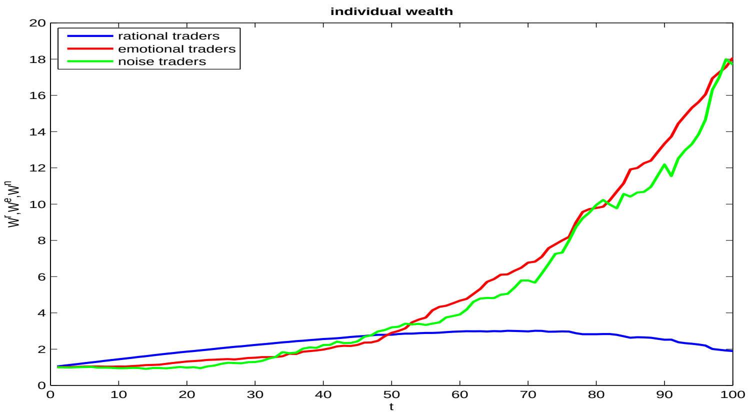


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

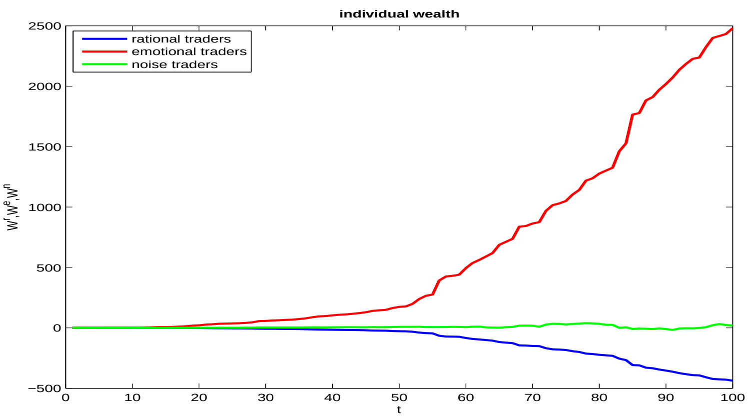
Figure 2.10: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from current demands (Rule qd-2), over $n = 10$ independent parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

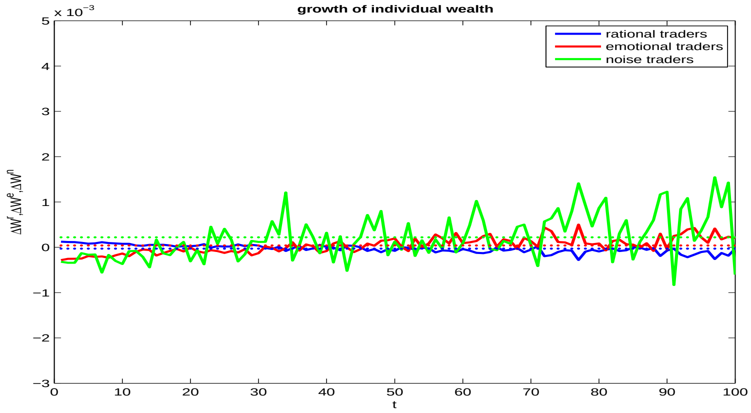


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

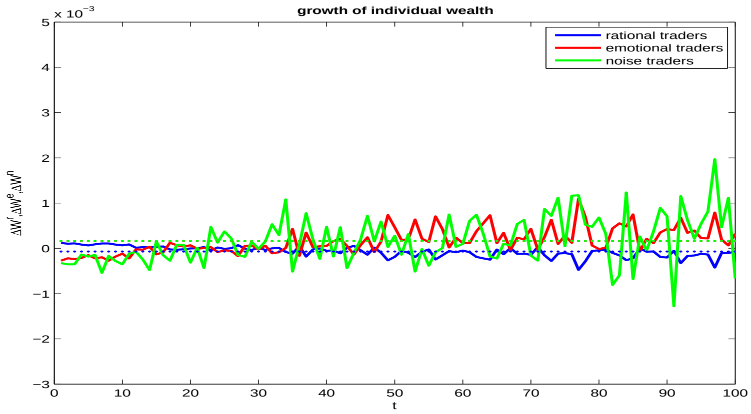


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

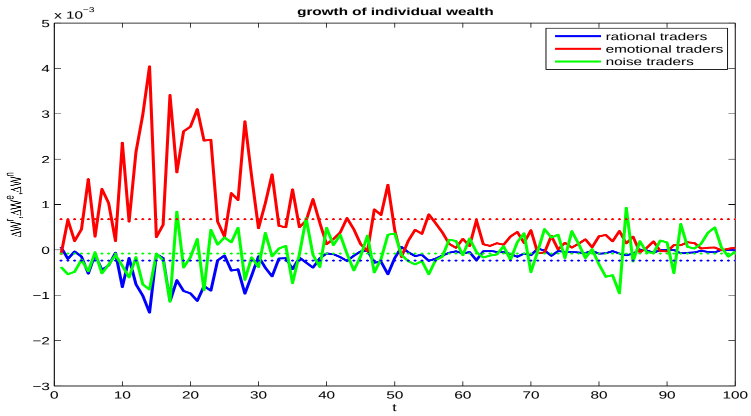
Figure 2.11: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from current demands (Rule qd-2), over $n = 10$ independent parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.



(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.



(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure 2.12: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from current demands (Rule qd-2), over $n = 10$ independent parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.

Rule (d-1): Dynamic belief updating.

Let us now compare the quasi-dynamic and the dynamic belief updating exemplified by Rules (qd-1) and (d-1). This is interesting since, in our opinion, the dynamic updating gives a better description of real situations. We remain at analyzing our benchmark Case A, under Scenario 1, and with independent parallel rounds of trade. Recall however that now, according to the theoretical considerations, emotional traders do not account for their own mean prior beliefs $k^e = 0$. In order to ensure the direct comparability, we run simulations under Rule (qd-1) taking $k^e = 0$, the results of which can be observed in Figures A.31-A.34 in Appendix A.2. The main findings for Rule (d-1) are illustrated in the subsequent Figures 2.13-2.16.

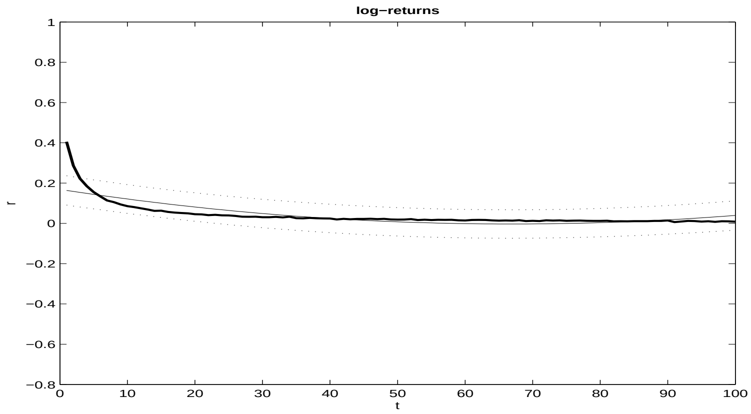
Contrary to the above discussed continuation of the rounds of trade – which is another situation lying closer to reality – Rule (d-1) appears to deteriorate stability in terms of volatility, but to improve efficiency as far as log-return predictability is concerned. For Types i and b, the standard deviation of log-returns is comparable with that obtained under Rule (qd-1) and $k^e = 0$, but for Type c it attains considerably higher values.[70] Yet, for the latter Type c we cannot find any significant evidence for departures from stationarity or from normality of log-returns, the less for serial correlation. Also for Types i and b log-returns remain stationary,[71] but their distributions significantly differ from normal and serial dependence can be detected at different lags.[72]

The trade activity of the impulsive emotional traders is very low and rational traders appear to buy on average the risky asset, irrespective of the emotional profile. All traders earn money from the trade, but now the impulsive emotional strategy is clearly the worst in the market, and the conservative one, as usual, the best. Balanced emotional traders outperform their rational peers in terms of individual wealth in the second half of the trade, but gain less than the pure noise traders. These evolutions are confirmed also by the growth of individual wealth and are comparable to those under Rule (qd-1) and $k^e = 0$.

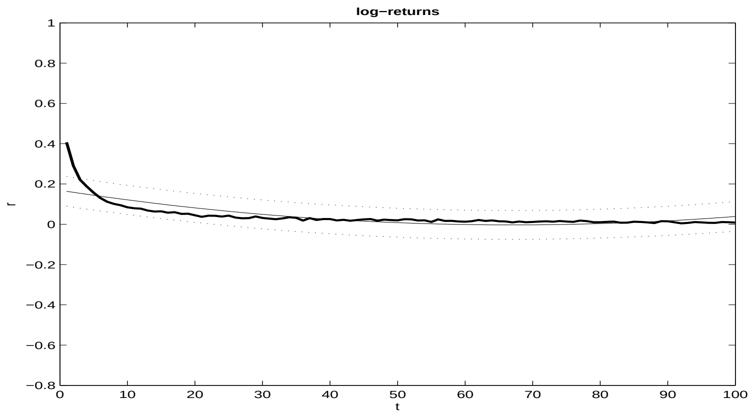
[70] Specifically, Type i yields with Rule (d-1) (Rule (qd-1) for $k^e = 0$) a standard deviation of log-returns of 0.057898 (0.057929), Type b of 0.058299 (0.058299), and Type c of 0.307146 (0.68114).

[71] In particular, ADF=-6.153836 for Type i, ADF=-4.896707 for Type b, and ADF=-10.20728 for Type c, where the test is based on the Schwartz information criterion with maximal 12 lags and are significant at all levels.

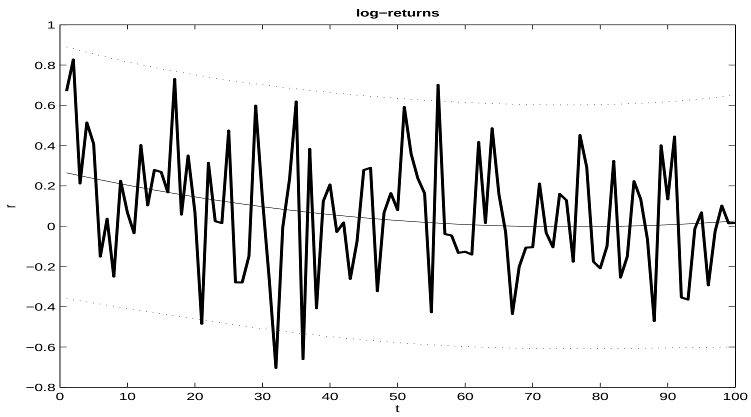
[72] Specifically, we can fit ARMA(6,1)-specifications for each emotional type.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

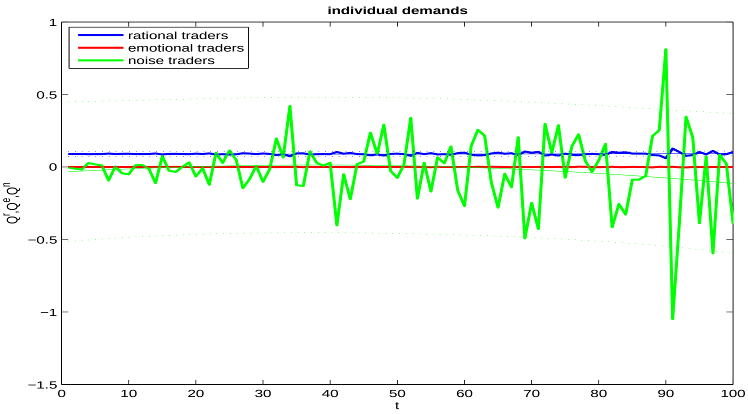


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

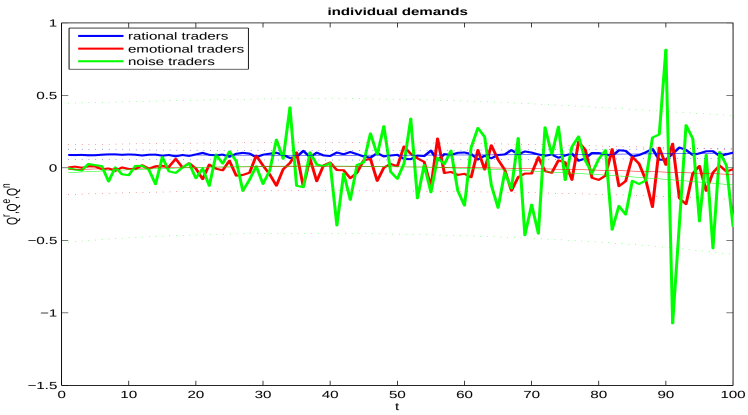


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

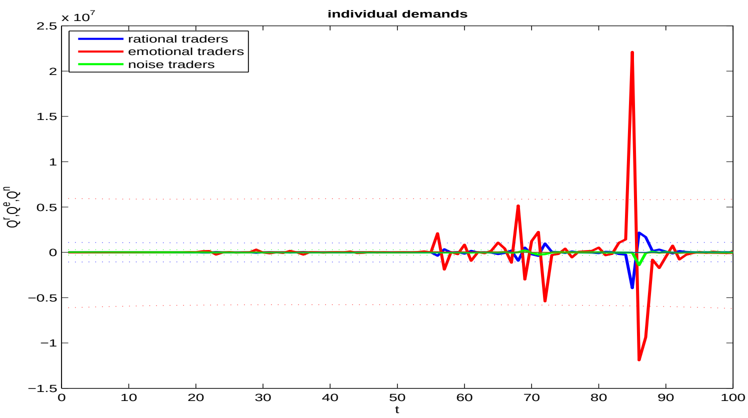
Figure 2.13: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), dynamic belief updating (Rule d-1), over $n = 10$ independent parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = 0, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

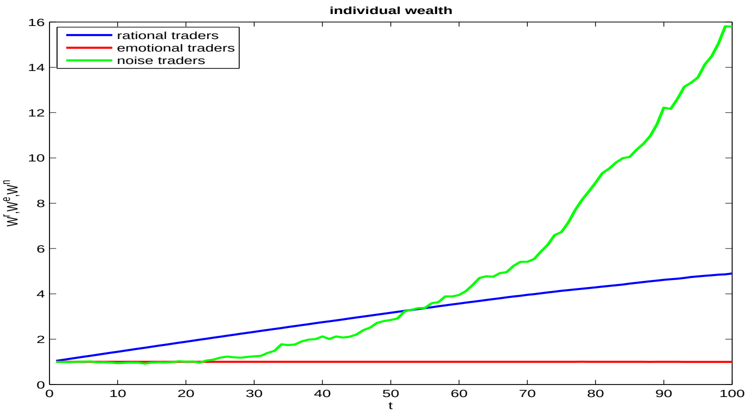


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

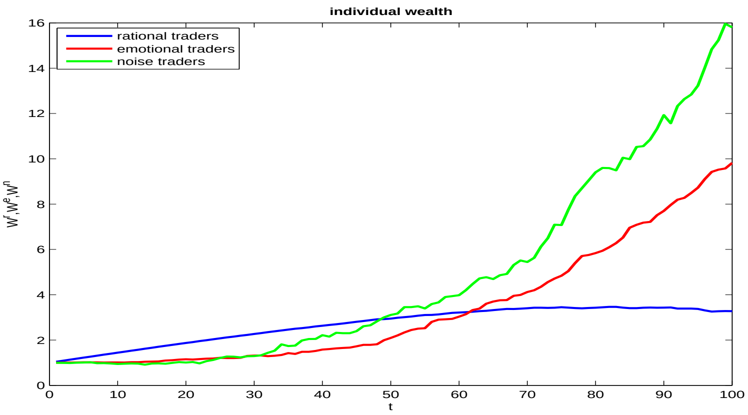


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

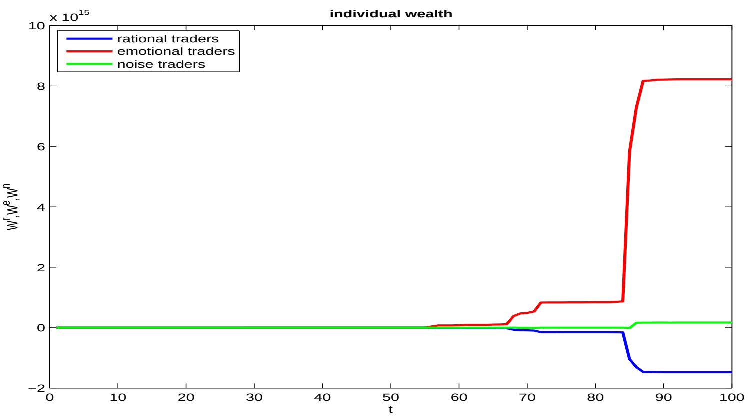
Figure 2.14: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), dynamic belief updating (Rule d-1), over $n = 10$ independent parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = 0, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

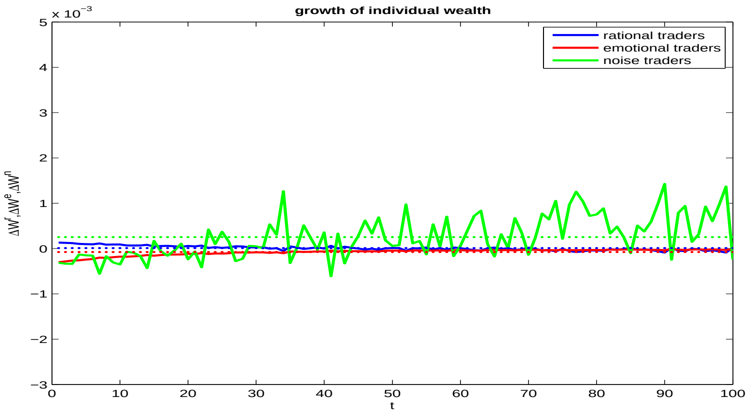


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

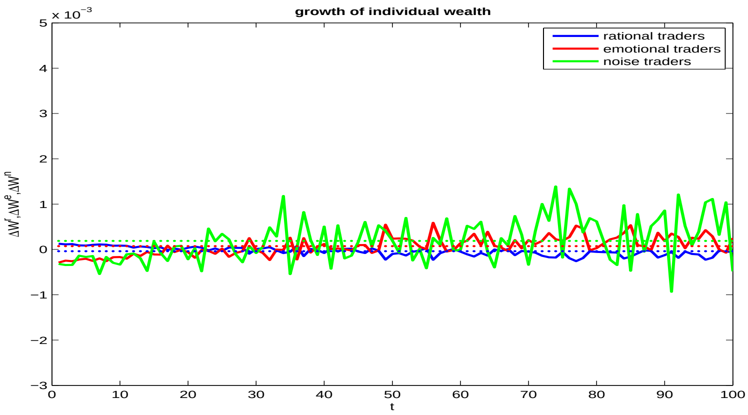


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

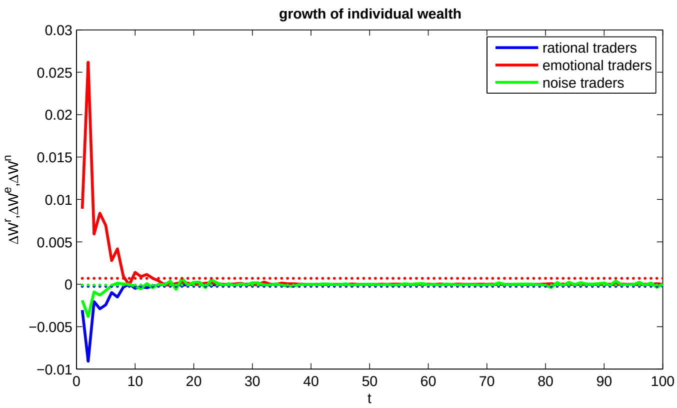
Figure 2.15: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), dynamic belief updating (Rule d-1), over $n = 10$ independent parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = 0, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.



(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.



(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure 2.16: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), dynamic belief updating (Rule d-1), over $n = 10$ independent parallel rounds of trade, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = 0, \beta^e = 1$.

Rule (d-2): Dynamic belief updating in the very long run.

The main qualitative patterns in the evolution of log-returns, individual demands, and individual wealth, observed for the previous cases are preserved for very high lengths of a round of trade. The results for $T = 10000$ are depicted in Figures A.35-A.38 in Appendix A.2.

A more detailed analysis of log-returns show that they remain stationary,[73] not-normally distributed, but are considerably less volatile and exhibit more complex patterns of serial dependency. In particular, for Types i and b we can detect serial correlation in variance, while Type c is characterized by very small positive autocorrelations in means.[74]

With impulsive emotional agents, both rational and emotional traders are losing money from their trades. In terms of individual wealth, the winners of the course for survival are the noise traders. They also make the highest profits when emotional traders of Type b are active in the market. Such emotional traders are better off than their rational peers, who constantly register losses, and accumulate positive wealth. As usual, the conservative emotional traders are clearly dominating the market. After few trades, the growth of the individual wealth of noise traders exhibits the highest fluctuation in the market.

In the long run, markets where beliefs are dynamically updated according to Rule (d-2) appear to offer better profit chances to those traders who act arbitrarily.

Rule (qd-1) and *different behavioral parameters*.

Before presenting Cases B and C, we consider how the market evolution changes for different values of the behavioral parameters, such as the (initial) rational demand constant β, the true risky value V in the view of rational traders, the emotional demand sensitivity β^e, and the emotional belief parameters k_{t-1} and k^e. We refer again to Scenario 1, Rule (qd-1), and independent rounds of trade.

Our previous results for our benchmark Case A, Scenario 1, and Rule (qd-1) appear to be almost invariant to changes in the rational demand constant β. Therefore, we do not further report on them.

According to Equation (2.31), prices should increase for higher risky values in the

[73]In particular, ADF-tests based on the Schwartz information criterion with maximal 12 lags deliver the following test values: ADF=-97.75469 for Type i, ADF=-30.42287 for Type b, and ADF=-20.03438 for Type c. These values are significant at all levels.

[74]Specifically, Types i, b, and c yield an average standard deviation of log-returns of 0.007251, 0.007920, and 0.033171, respectively. The log-returns can be described by the following processes: ARMA(1,1) in mean and ARCH(2) in variance for Types i and b, and a simple ARMA(1,1) for Type c, while the autocorrelation coefficients for the row log-return series are significant but already very low (under 3.5%) for Type c.

opinion of rational traders V. This is indeed what Figures A.39-A.42 in Appendix A.2 show for a value of $V = 10$. Demands and wealth courses are similar in shape, but amplified, relative to the case with $V = 1$, so that emotional traders succeed in dominating the market in terms of individual wealth, irrespective of their psychological profile. The growth of individual wealth attains much faster common levels among all trader categories.

We can yet note important qualitative changes in results when emotional traders think or act in what we denoted to be a *contrarian* way, in particular when they employ either negative belief parameters k^e or negative demand sensitivities β^e. Such reactions prove to be disadvantageous for impulsive or balanced emotional traders, who constantly register loses. Although the rational strategy creates positive capital, the individual wealth of rational traders is eventually exceeded by that of pure noise traders. In terms of the growth of individual wealth, emotional traders appear yet to recover in time and get to dominate their rational peers after almost two-thirds of the trade. Conservative emotional traders preserve their lead upon other traders, both with respect to the level as well as to the growth of their individual wealth. These results are depicted in Figures A.43-A.50 in Appendix A.2 for negative $\beta^e = -1$ and $k^e = -N^e/N$.

Note also that for $\beta^e = -1$, the average level of all analyzed variables lowers which points to a reduction of the trading activity. As expected, emotional traders are now mostly selling the risky asset. Unreported results for $\beta^e = 2$ suggest further that an increase in the emotional demand sensitivity renders more pronounced the patterns in returns, demands, and wealth observed for $\beta^e = 1$. Emotional traders clearly dominate the market in terms of individual wealth, irrespective of their psychological profile. The same occurs for higher emotional belief constants k_{t-1}, the overall influence of which is more substantial than that of β^e.

Case B: Middle proportion of emotional traders $N^e/N = 50\%$.

Let us now consider the Case B, where half of the traders are emotional. The first consequence of this increased activity of emotional traders is that the market with $\lambda = 0.08$ becomes instable. Therefore, we resolve for analyzing more liquid assets and henceforth work with $\lambda = 0.008$. For the purpose of quantitative comparability, we include the results for case A and $\lambda = 0.008$ in Appendix A.2.

The market evolution in Case B is depicted in Figures 2.17-2.20 for Scenario 1, and in Figures A.59-A.62 in Appendix A.2 for Scenario 2. The main variation patterns underlined above are preserved: When emotional beliefs are of the conservative Type c, log-returns are most volatile but exhibit the smallest serial correlation, and the emotional strategy is the most fruitful.[75] Log-returns remain stationary but not normally distributed for all emotional types.[76] Again, the change in scenario appears to entail no noticeable influence on market evolutions.

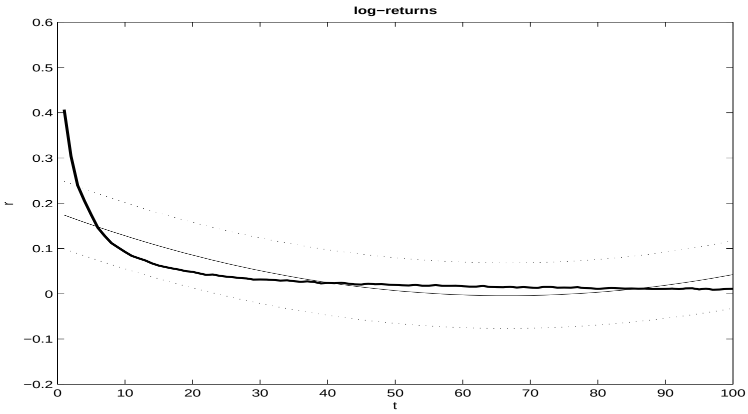
Several differences are yet worth to be noted: First, both rational and emotional traders are sooner buying the risky asset. Rational traders are always buying higher average quantities only for emotional profiles of Type i. For Types b and c, the emotional demand starts lower but exceeds the rational one in less than one-fifth of the trade interval.

Second, the monetary success of the different strategies appears to change to a certain extent. Thus, when emotional traders act impulsively, rational traders dominate the market in terms of individual wealth. When rational traders resolve to adapt to the presence of emotional traders of Type b, they succeed in being better off than their emotional peers only during the first one-third of the trade. Finally, with emotional belief formation of Type c, rational traders lose money from their trades and emotional ones dominate the market.[77] Merely during very short intervals (of less than 5 trades) at the beginning of the trade, rational traders outperform emotional ones in terms of growth of individual wealth. Afterwards, the individual wealth of all three trader categories converges towards identical values. This convergence is faster for Types i and b, but not full for Type c. The individual wealth of conservative emotional traders grows at highest soon after the trade starts, while the wealth of their rational peers fluctuates by negative amounts.

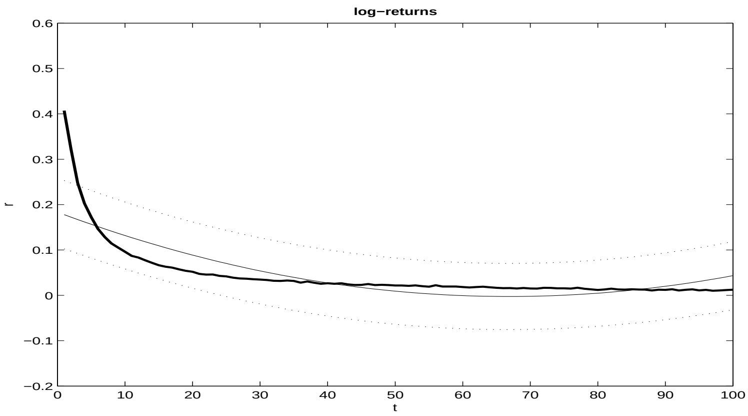
[75] Specifically, the standard deviations of log-returns for Types i, b, and c are respectively 0.060899, 0.061688, and 0.063002. The corresponding processes that fit sufficiently well the data are ARMA (5,1), ARMA(6,1), and ARMA(1,1) in mean and ARCH(1) in variance.

[76] The ADF-test based on the Schwartz information criterion with maximal 12 lags delivers the following test values for Types i, b, and c, respectively: -10.61510, -5.259028, and -6.084258. They are significant at all levels.

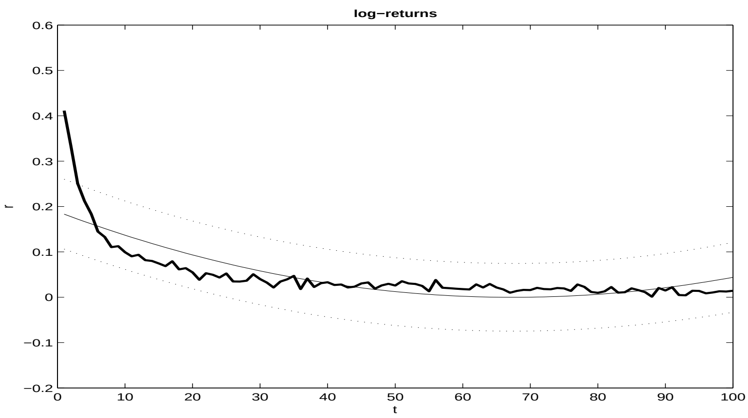
[77] Unreported results confirm all these findings also with respect to the group wealth.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

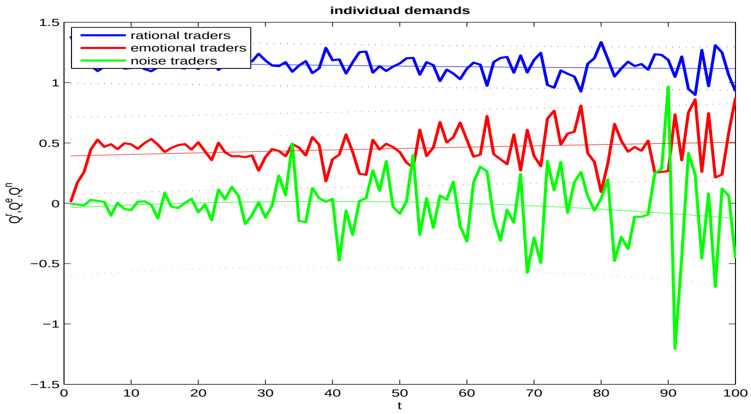


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

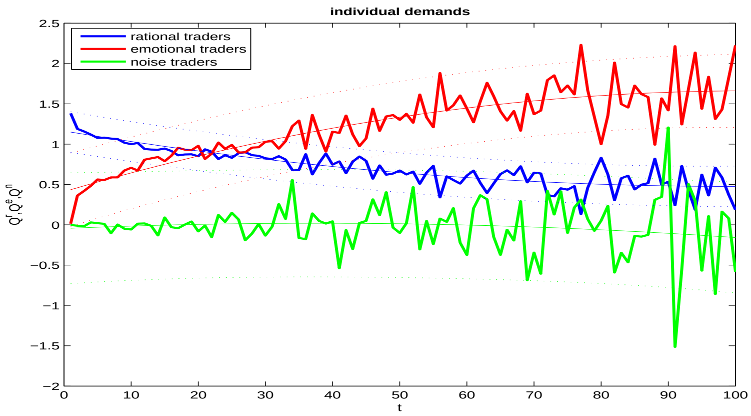


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

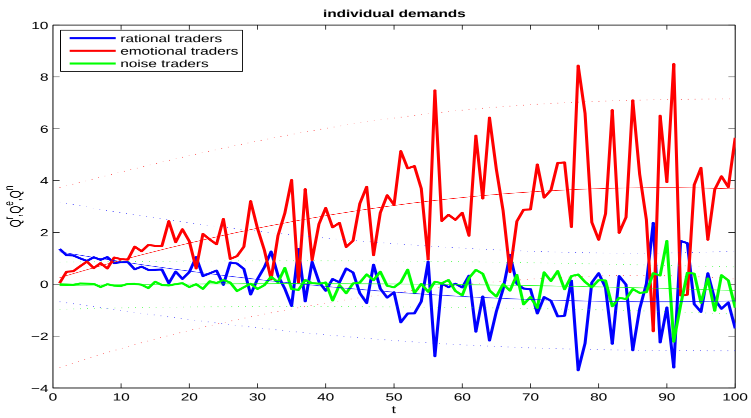
Figure 2.17: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 45\%, N^e/N = 50\%, N^n/N = 5\%$ (Case B), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel rounds of trade, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

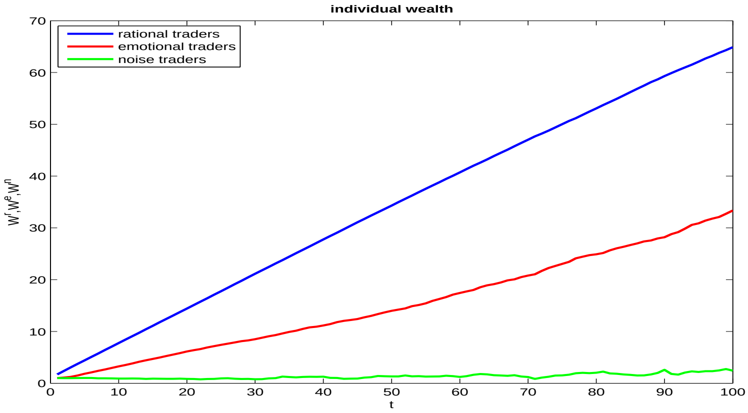


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

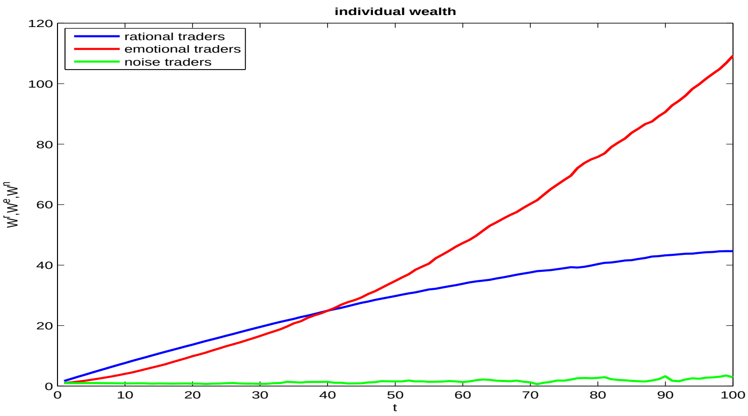


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

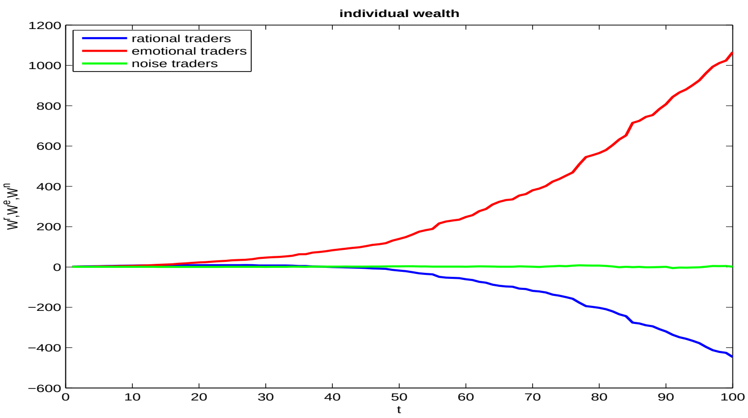
Figure 2.18: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 45\%, N^e/N = 50\%, N^n/N = 5\%$ (Case B), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel rounds of trade, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

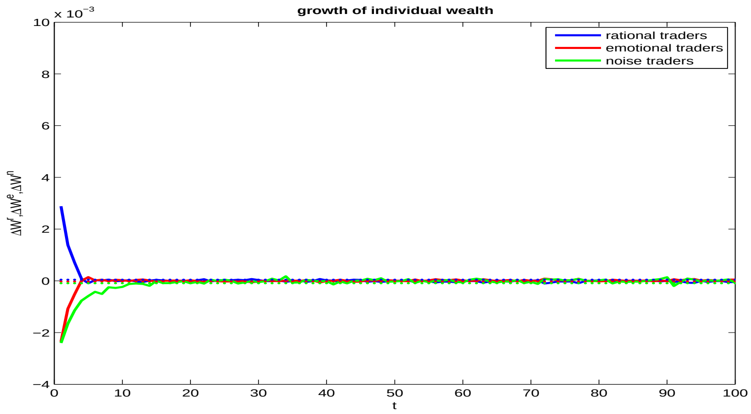


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

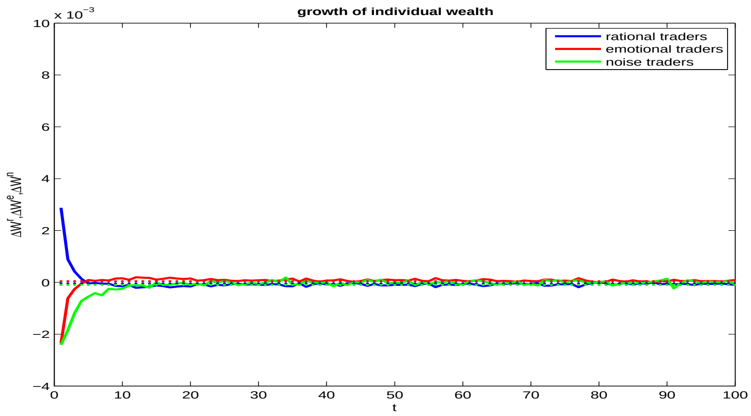


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

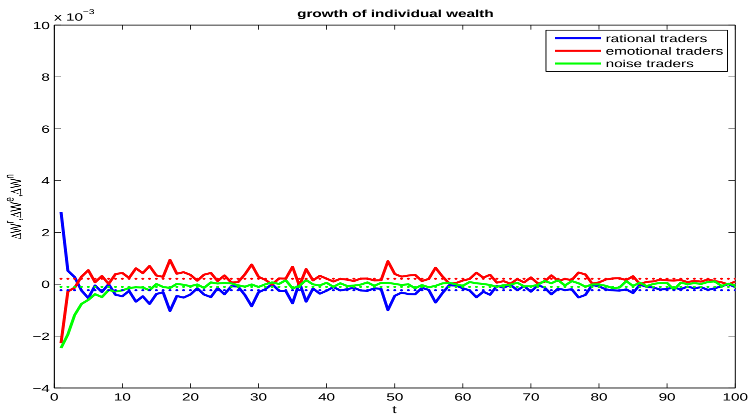
Figure 2.19: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 45\%, N^e/N = 50\%, N^n/N = 5\%$ (Case B), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel rounds of trade, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.



(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.



(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure 2.20: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 45\%, N^e/N = 50\%, N^n/N = 5\%$ (Case B), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel rounds of trade, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.

Case C: High proportion of emotional traders $N^e/N = 75\%$.

The evolutions of prices, demands, and wealth for the highest participation level of emotional traders in the market of 75% and for the same benchmark Rule (qd-1), and an increased liquidity $\lambda = 0.008$ are depicted in Figures 2.21-2.24. The corresponding graphical results for Scenario 2 can be observed in Figures A.63-A.66 in Appendix A.2.

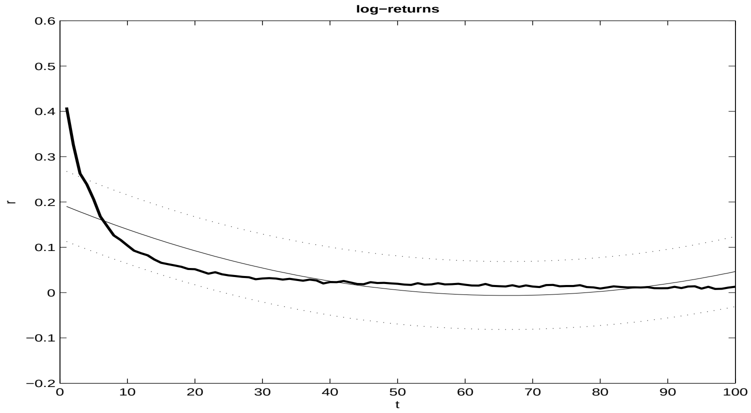
As expected, log-returns are somewhat more volatile than in Case B, but remain stationary and non-normally distributed, which has been so far the case for Rule (qd-1).[78] Individual rational and emotional demands clearly diverge for other emotional profiles than Type i, where emotional traders mostly buy and rational ones mostly sell the risky asset.

Emotional traders clearly impose their dominance in terms of wealth within a round of trade. This holds over the entire trading round for Types b and c. Balanced or conservative emotional traders always accumulate higher individual wealth than their rational peers, the individual wealth of which decreases. Emotional traders of Type i get to earn higher profits than rational traders after around two-thirds of the trade.[79] The growth of individual wealth of all trader categories converge to zero, in particular faster for Type i and the slowest for Type c. After the first few trades, the wealth growth of emotional individuals becomes the highest in the market.

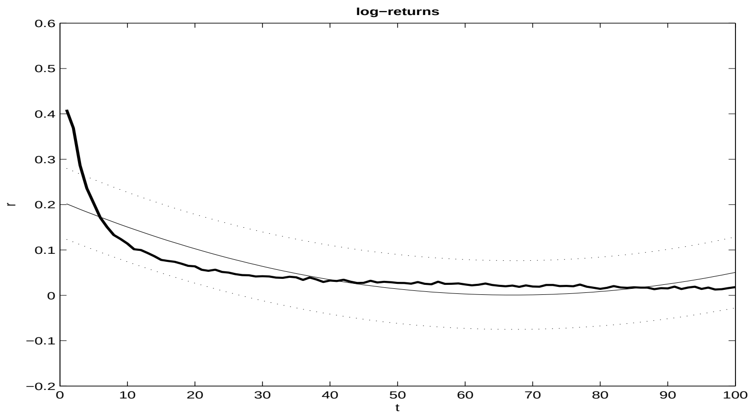
In sum, the increased presence of emotional traders in the market appears to preserve the general price-related conditions, but to change the situation as far as survival – in terms of both individual and group wealth – is concerned. As noted in the theoretical section, the profit chances of rational individuals are fostered by a lower participation N^r, which indeed occurs when going from Case A to Case B and from Case B to Case C. Emotional traders lose the advantage of being the most numerous in the market in front of their rational peers only if they ignore the importance of prior information. In particular, impulsive emotional traders gain less than the rational ones during the entire trade. For balanced emotional profiles, this handicap persists only during the first half of the trade. Finally, conservative emotional traders dominate the market as measured by either individual wealth, or group wealth, or growth of individual wealth.

[78] Specifically, the standard deviations of log-returns for Types i, b, and c are respectively 0.065751, 0.067286, and 0.068738. The corresponding ADF-test values are: -6.873652, -3.653596, and -7.450117, where the second one is significant only at 5%-level and the other two at all levels. The corresponding processes that fit the data sufficiently well are ARMA (5,1) for Type i, ARMA(5,1) for Type b, and ARMA(1,1) for Type c.

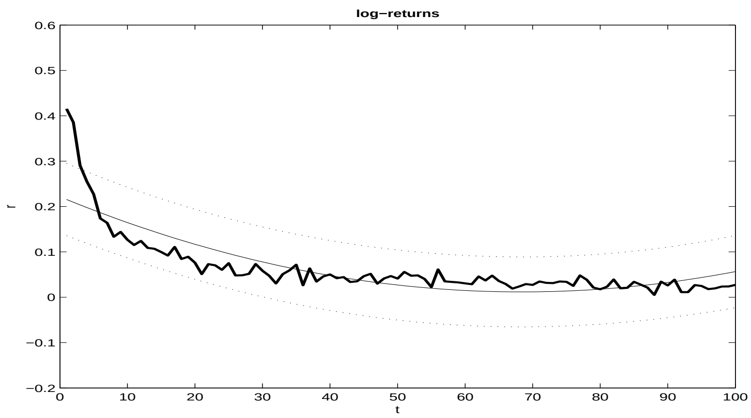
[79] Identical results are obtained with respect to the group wealth.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

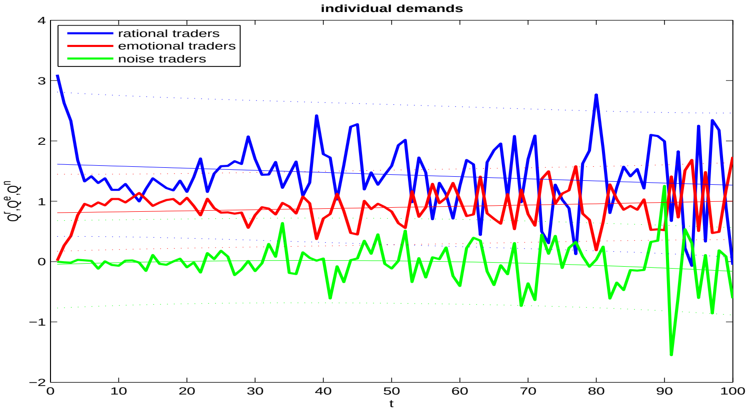


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

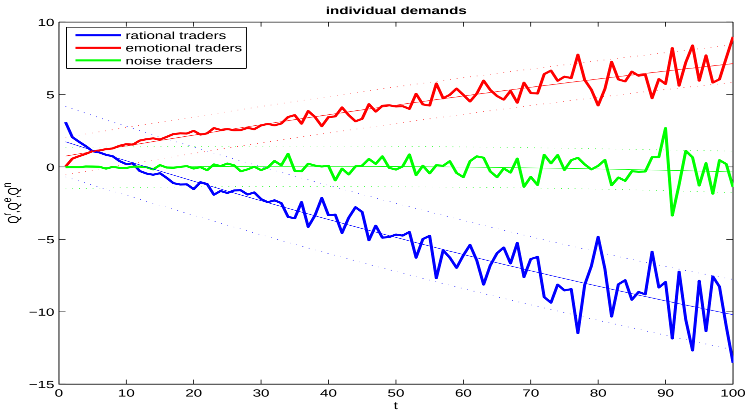


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

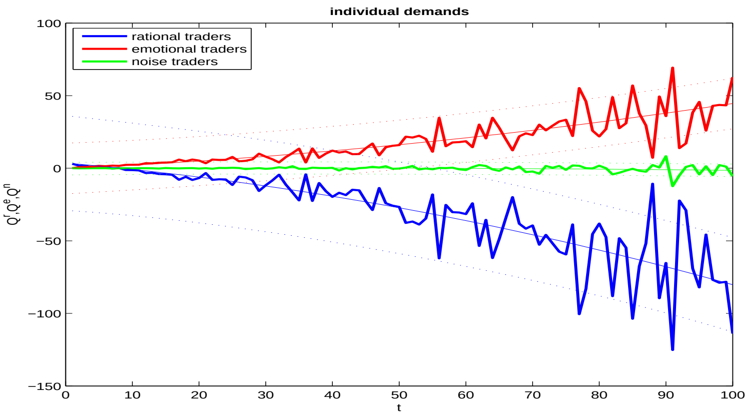
Figure 2.21: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 20\%, N^e/N = 75\%, N^n/N = 5\%$ (Case C), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel rounds of trade, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

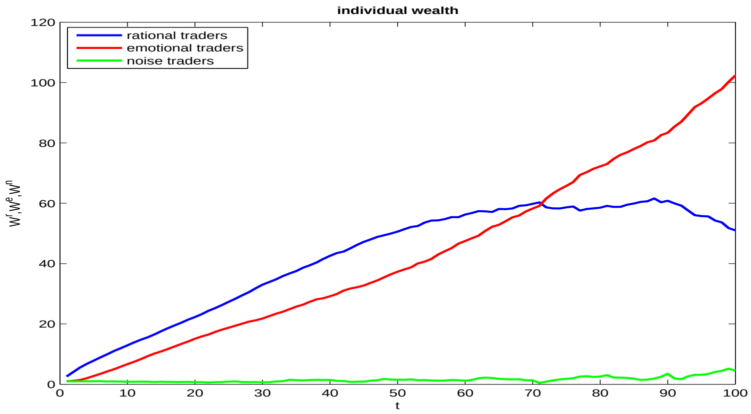


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

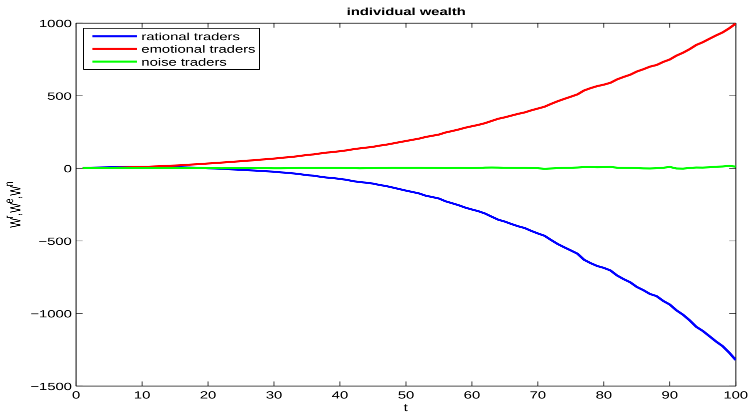


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

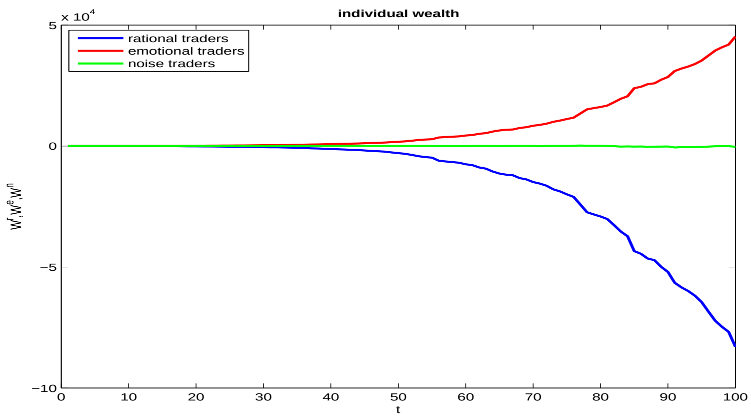
Figure 2.22: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 20\%, N^e/N = 75\%, N^n/N = 5\%$ (Case C), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel rounds of trade, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

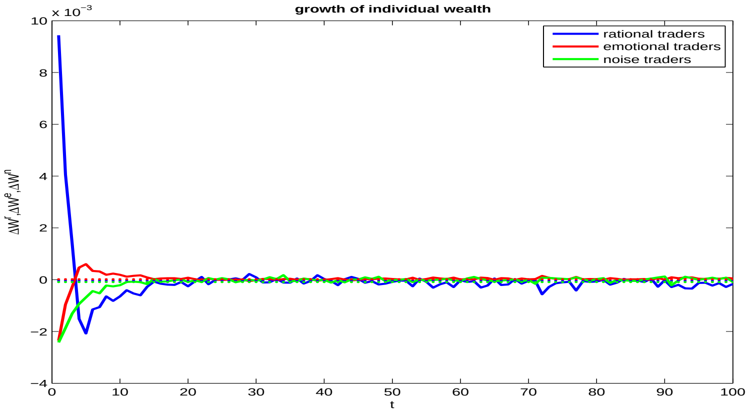


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

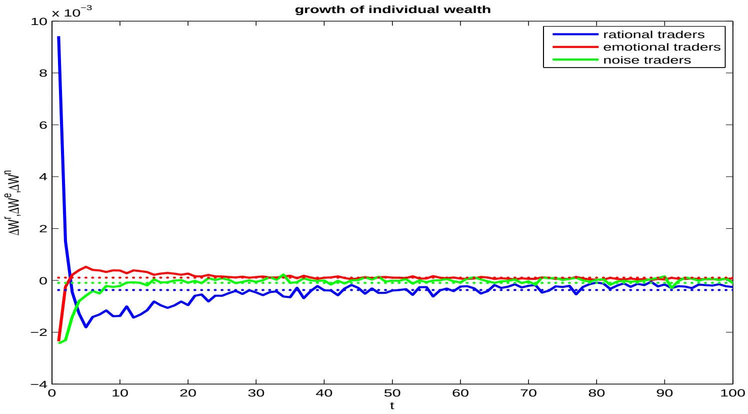


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

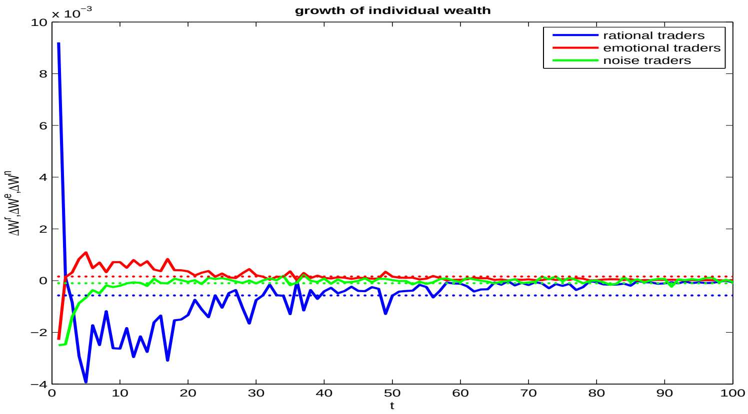
Figure 2.23: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 20\%, N^e/N = 75\%, N^n/N = 5\%$ (Case C), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel rounds of trade, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.



(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.



(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure 2.24: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 20\%, N^e/N = 75\%, N^n/N = 5\%$ (Case C), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel rounds of trade, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.

In sum, our simulations suggest that markets where emotional agents represent at least one-fourth of the traders, can exhibit a stable evolution. Moreover, under certain circumstances, this evolution can even be close to efficient. Emotional strategies are not necessarily ruled out by rational ones and emotional traders can make even higher profits than their rational peers.

In unexpectedly many situations, emotional traders are better off than their rational peers in terms of individual wealth. The "key to success" appears to be *conservatism*, which predisposes emotional traders to ascribe higher weights to the prior information relative to the current one, in forming beliefs. The individual wealth of conservative emotional traders is always the highest in the market and grows at the most elevated rate. For balanced emotional traders, this holds only on certain conditions and exclusively in the last part of the trade. This is an interesting result since such balanced emotional traders are the closest to rational ones in terms of belief formation,[80] so that we might expect that they have the best chances of success.[81] In the presence of impulsive emotional traders, all trader categories have the opportunity to accumulate positive wealth in most of the cases, although impulsive traders mostly lose money in front of their rational peers. Irrespective of the emotional type, in the long run the individual wealth of all traders tends to grow at identical rates and hence offers similar survival chances to all rational, emotional, and even noise traders.

Moreover, the emotional mark on the price evolution is more pronounced in each of the following situations, other things being equal: when the trade continues from one round to each other, when rational and emotional beliefs are dynamically updated, when emotional traders are more numerous, or when they are more sensitive to their own beliefs in thinking (i.e. for higher k^e) or acting (higher β^e). Moreover, the conservative type appears to foster the market efficiency as well. For Type c, the serial dependency patterns of prices are less complex or even not existent.

Therefore, the most important factor that impacts general and individual evolutions –

[80]Specifically, the balanced way of forming beliefs is identical to the rational one, since both trader groups put identical weights on prior and current information. However, the relevant informational set of emotional traders merely includes the own beliefs. Also, the emotional demand strategy remains radically different of the rational one, as it does not result from any optimization problem, but simply evolves proportionally to the emotional subjective beliefs.

[81]However, it is true that in markets with Type b, the monetary results of emotional and rational traders are somewhat better balanced. Specifically, these emotional traders start by being worse off but end up better off than their rational peers. Furthermore, it appears reasonable to assume that conservative behaviors may positively contribute to improving market stability and efficiency and that emotional impulsiveness should be arbitraged out easier by rational strategies.

namely, prices and monetary results – appears to be the *psychological profile* of emotional traders. It is even more important than the number of these traders, since an increased emotional participation to trading does not necessarily improve their profits if, for instance, they think impulsively. The dependency between the emotional and noise trader noise (i.e. the chosen scenario) also plays a secondary role. It becomes almost irrelevant when, for instance, the mean rational and emotional beliefs are updated from past market returns according to Rule (qd-1).

2.2.4 Summary and conclusions

The goal of this section is to model the role of emotions in financial decision making. To this end, we advance a market setting where three distinct categories of economic agents trade a unique risky asset: rational traders, emotional traders, and noise traders. Prices are set by a competitive market maker to be proportional to the current total order flow. We formally represent and study the formation of individual beliefs, the trader demand strategies, their individual and group wealth, and finally the price formation process.

What makes the distinction among our trader categories is, first, the way in which they interpret public information in order to form subjective beliefs, and, second, the strategy they pursue in order to determine the risky-asset amount to be traded. In particular, rational traders combine past and current information in a traditional Bayesian manner. In so doing, they account for the presence of other trade strategies and for their impact on prices. Moreover, rational demands are shaped to maximize expected utility of wealth. In contrast, emotional traders form beliefs in an unbalanced manner, putting different weights on diverse information sources. Thus, they are prone to thinking heuristics commonly met in practice, such as representativeness and conservatism. Also, emotional traders are not concerned with the existence of other agents. Furthermore, emotional traders blindly follow their beliefs in formulating demands and hence make use of heuristics (in the sense of simplifying rules) also in acting. Noise traders act randomly.

The belief formation relies on the interpretation of information. Our interest is in the role of emotional traders as a distinct type of traders driven by affect and intuition. We suggest a way to quantify the emotional process of belief formation, showing how emotional traders may balance between past information and new evidence in contrast to the traditional updating employed by rational traders.

Once formed in the trader minds, beliefs flow into demands. In particular, trade strategies are shaped in linear dependence on subjective opinions. Prices are set to be proportional to the current total order flow and depend thus on the demands of each trader group. We show how the distinct beliefs of different market participants are directly reflected in the informational content of prices. Given the rational strategy and the price setting rule, we derive the equilibrium condition of the market: Rational traders should be able to anticipate the total order flow emanating from the other traders, in other words to adapt to the conditions created by other traders, especially by the emotional ones. Such a strategy appears to be rational not only in classic, but also in an evolutionary sense.

Moreover, we measure the survival chances of our different trader categories by their individual and group wealth, the wealth variation between successive trades, and the growth of individual wealth as a part of the total trader wealth. We infer theoretical conditions on which rational wealth decreases and fluctuates less than the emotional one. This corresponds to the possibilities that rational traders lose money or make lower profits than their emotional peers, respectively.

The somewhat static setting described sofar is subsequently extended to a particular case of dynamic belief updating. We derive the corresponding market prices both in the short and the long run.

Finally, our model is tested by numerical simulations for different parameter constellations and in different market settings. Specifically, we examine the evolution of log-returns, trader demands, and trader wealth for various proportions of rational and emotional traders, distinct behavioral profiles of emotional traders, different relations between the actions of emotional and noise traders, various belief updating rules, and several trade organization possibilities.

Our simulations suggest that markets in which emotionally driven agents are active, can reach stable and closely efficient states. A necessary premise appears to be the existence of other traders, who rationally maximize expected utility of wealth and, in consequence, adapt to the conditions created by emotional trades. Rational traders of this type commit to absorbing the uncertainty generated by other non-strategic trades. Therefore, their presence guarantees – or at least facilitates – the market stability.

In spite of their simplistic thinking processes and action strategies, emotional traders appear to be not only able to survive, but even to dominate such markets. They are best off – in terms of individual and group wealth, as well as of the growth of individual wealth – when they think conservatively, ascribing a higher importance to past evolutions than to current information. Markets with conservative emotional traders are closest to efficient, and rational agents constantly lose money. However, when emotional traders are impulsive (and hence more similar to noise traders), rational traders make higher profits but all traders are able to accumulated positive wealth from the trade. With balanced emotional traders (who come thus closest to the rational way of thinking, although in a more simplistic manner), rational traders dominate the market only in the short run but are eventually overtaken by their emotional peers.

These findings support the idea that, under certain circumstances, emotional traders have high chances of continued existence while rational traders might be even forced to quit the market in consequence of too high loses. This comes to contradict the traditional conviction that rational traders can be the sole survivors in financial markets.

CHAPTER 3

How Investors Face Financial Risk: Loss Aversion and Wealth Allocation

"[...] it is not the loss itself, but the estimate of the loss that troubles us."
SENECA.

THIS CHAPTER focuses on the attitude of non-professional investors towards financial losses and their decisions on wealth allocation, and how these change subject to behavioral factors. We first revise relevant findings related to possible quantifications of risk as a main constraint of capital allocation, as well as to modelling investors' perceptions and attitudes.

Our contribution concerns the integration of behavioral elements into the classic portfolio optimization. We extend a VaR-portfolio model in order to account for individually perceived risk. Individual perceptions are modeled according to an extended prospect-theory framework: Losses loom larger than gains of the same size (loss aversion) and the past risky-portfolio performance changes the subjective valuation of risky investments. The utility of financial investments is overemphasized (myopia). The portfolio model with individual VaR delivers an optimal wealth assignment between risky and risk-free assets.

We proceed in two steps: First, we assume that non-professional investors derive utility exclusively from financial wealth fluctuations. In consequence, they are interested in splitting their wealth between risky and risk-free assets. We analyze how the past performance and the evaluation frequency of risky performance impact non-professional investors' behavior. Estimations based on real market data suggest that myopic loss aversion holds at different evaluation frequencies. One year is the optimal evaluation horizon at which, under practical constraints, risky holdings are maximized. Classic settings using standard VaR-significance levels may underestimate the loss aversion of individual investors.

Second, utility is derived from a twofold source: financial wealth fluctuations and consumption. Wealth has to be split now between consumption and financial assets in total. The aggregate market equilibrium is obtained in two distinct settings: under the maximization of expected and of non-expected utility. Our estimations indicate that the non-expected utility setting is more robust and describes better the behavioral profile of non-professional investors. Compared to their peers in the expected-utility setting, the maximizers of non-expected utility are more averse towards financial losses, and allocate lower percentages of their total wealth to financial investments in total, but higher ones to risky assets in particular. With two-dimensional utility, myopic loss aversion is mostly rejected.[1]

[1] This chapter is based on joint work with Erick Rengifo.

3.1 Theoretical overview

This section introduces theoretical notions – in particular, the Value-at-Risk, the prospective value, and the myopic loss aversion – that underlie our models from Sections 3.2 and 3.3. As measure of market risk, the former notion has gained its importance in theoretical Finance, among other things, in the context of capital allocation decisions. In spite of the wide practical use of VaR, the formal complexity of this construct and its (possibly) problematic implications in terms of accuracy require a more deeper understanding. The latter two notions, the prospective value and the myopic loss aversion, describe behavioral aspects of financial decisions that are directly related to subjective perceptions. They rely, at least in part, on theoretical instruments which are external to the field of Economics.

3.1.1 Value-at-Risk

We commence by explaining the notion of Value-at-Risk (abbr. VaR) as a measure of market risk and showing its importance and its wide-spread use in practice. Subsequently, we present various methods of computing VaR. The focus lies thereby on asset portfolios. These methods can be mainly divided in parametric, non-parametric, and semi-parametric, and exhibit specific advantages and weaknesses. Further, we address the use of VaR in portfolio optimization, namely as a risk constraint, then turn to the drawbacks of VaR and the need to develop further related risk measures. The section closes by a summary of several papers applying VaR to the portfolio optimization, the results of which come closely to our model in Sections 3.2 and 3.3.

Overview and definition

The **Value at Risk** (abbr. VaR) is a highly aggregated measure of market risk that was developed in response to numerous financial disasters in the early 90s.[2] The ensuing reaction of the private sector and the regulators attempted to better keep financial risk under control, turning VaR into a veritable industry standard and a part of the industrial regulatory mechanism.[3] Thus, the 1993 land-mark report dedicated to derivative dealers and end-users issued by the so-called Group of Thirty (G30) recommends the use of VaR

[2]These numerous scandals involved big corporations from almost all industrial sectors, as well as banks, local governments, etc. and were mostly caused by derivative deals. Please refer to Jorion (2001) for a summary and to Partnoy (2003) for a more detailed presentation.

[3]See Jorion (2001) for an overview.

for assessing market risks.[4] In 1995, six major Wall Street companies that are members of the Derivatives Policy Group commit to evaluating and reporting risk in relation to capital on the basis of a 99% VaR over two weeks.[5] Along with this initiative comes the RiskMetrics service of J.P. Morgan. The corresponding database[6] is made freely available at the end of 1994 and serves as support to institutional clients for the assessment of their own VaR.[7] At the same time, the regulators[8] react by recommending a more strict control as well as transparent and pertinent reporting on the balance sheet of derivative deals. Moreover, in 1997 the Securities and Exchange Commission (SEC) introduces rules that stipulate the quantitative disclosure of market risks in the form of either tabular presentations, or of sensitivity analyzes, or of VaR measures.[9] Finally, the requirements of the Basel Committee on Banking Supervision issued in 1996 adopt VaR as the major determinant of the capital that banks are obliged to subscribe in order to cover potential losses from carried risks.[10]

The applications of VaR are multiple, especially since its initially passive function as a reporting tool extended fast to more elaborate control and allocation purposes, as noted in Jorion (2001). Also, VaR was adapted for quantifying and preventing not only market risk – which was its original purpose – but also other types of financial risks, such as credit, liquidity, operational, or legal risk.[11]

Henceforth, we concentrate on the main application field of VaR that refers to the quantification of market risk. While traditional measures – such as the standard deviation – merely capture the risk itself,[12] VaR accounts for the *combined* effect of market risk and the exposure to it.[13] These two elements reveal the twofold origin of financial losses:

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[4] G30 is an international consultative body of financiers, bankers, and academics from leading industrial nations. The report can be found in Group of Thirty (1995).

[5] See Derivatives Policy Group (1995).

[6] This data base contains mainly variance-covariance matrices of risk and correlation measures at different points in time.

[7] See J.P. Morgan (1995). In 1997, the RiskMetrics Group develops the CreditMetrics system dedicated to credit risk measurements in a portfolio framework, while the CorporateMetrics issued in 1999 adapts the original RiskMetrics approach to the needs of non-financial corporations by extending it for longer time spans.

[8] Such as the General Accounting Office (GAO) or the Financial Accounting Standards Board (FASB).

[9] See Securities and Commission (1995).

[10] See Basel Committee on Banking Supervision (2004).

[11] According to Manganelli and Engle (2001), the *market risk* results from uncertain future earnings as a consequence of changing market conditions. *Credit risk* arises when partners do not meet their obligations, while *liquidity risk* is related to large negative cash-flows over short periods, generated by disadvantageous sells of illiquid assets. Finally, *operational risk* emerges when performing market operations, such as instructing payments or settling transactions.

[12] Moreover, the standard deviation accounts for symmetric risk profiles, while VaR measures the downside risk (i.e. losses).

[13] The exposure comes from the positions in different assets summarized in the company portfolio.

from general factors (i.e. the market volatility) and from factors that can be controlled by companies (i.e. the exposure). According to Jorion (2001), the main sources of market risk can be generated by movements in equities (systematic risk), in interest rates (duration), in duration itself (second order movements or gamma risk), in the underlying assets of derivatives (delta), etc. VaR can be used for quantifying each of these market risk categories.

Specifically, VaR can be defined following Duffie and Pan (1997) and Jorion (2001), as *the maximum (or worst) expected loss from a financial investment over a given time horizon and at a given confidence level.*[14] In other words, it provides a measure for downside risks.[15] In general, regulators require that VaR represents a fraction of the available capital. Due to the easiness of use, its apparent precision, and the forward-looking approach it provides, VaR became a risk measurement tool widely accepted in practice.[16]

In mathematical terms, VaR is obtained by setting the probability that the change in the investment value during the indicated period T is lower than the maximum expected loss, identical to the required confidence $1-\alpha$ (or, equivalently, to the significance α):[17]

$$P(W_0(R_T-\mu_T) \leq -\mathrm{VaR}_\alpha) = \alpha \Leftrightarrow \mathrm{VaR}_\alpha = -F^{-1}(\alpha) = -q_\alpha, \tag{3.1}$$

where W_0 stands for the initial investment, R_T for the returns after the fixed period T and μ_T for the corresponding return mean. Moreover, F represents the cdf of the investment value W and q_α its α-quantile.[18] Thus, VaR is set to be equal to the α-quantile of the

[14] This definition implicitly assumes stable market conditions over the designed interval. As noted in Jorion (2001), the length of the time horizon and the level of confidence depend on the purpose of the VaR's use: as a measure for potential losses, as a benchmark for comparing risks across markets, as a criterium for setting the equity capital, or for model-validity checks (backtesting). For instance, the Basel Committee Derivatives Policy Group (1995) imposes a time horizon of two weeks (corresponding to ten business days) and a confidence level of 99% for the calculation of VaR. The minimum capital requirement is obtained by multiplying the obtained VaR by a safety factor of 3. This factor should account for additional risk that cannot be captured by VaR. In contrast, in order to increase the test power, the Basel Committee applies for backtesting the limits of one day and 95% confidence. Moreover, J.P. Morgan's RiskMetrics require computing VaR with 95% confidence over one single day.

[15] As noted in Huschens (2000), VaR should be understood as the loss *threshold* that can be crossed with probability α. Yet the definition of VaR as a *maximum* loss is intuitively more appealing, especially for laymen such as non-professional investors. It is hence commonly adopted in practice and can generate a false perception of VaR.

[16] Smithson and Minton (1996) report various surveys showing a strong increase in the use of VaR as a risk measure among dealers, non-financial end-users, and institutional investors.

[17] Usually, the investment stretches across different assets grouped in a portfolio. Alternatively, $\mathrm{VaR}_\alpha(x) = -\mathrm{VaR}_{1-\alpha}(-x)$ as demonstrated in Pflug (2000).

[18] Specifically, Equation (3.1) defines the so-called *relative* VaR. Analogously, the *absolute* VaR relates to absolute changes in the investment value, i.e. changes are calculated with respect to zero instead of

excess return distribution taken with negative sign.[19] For a given α, the goal is to minimize VaR.

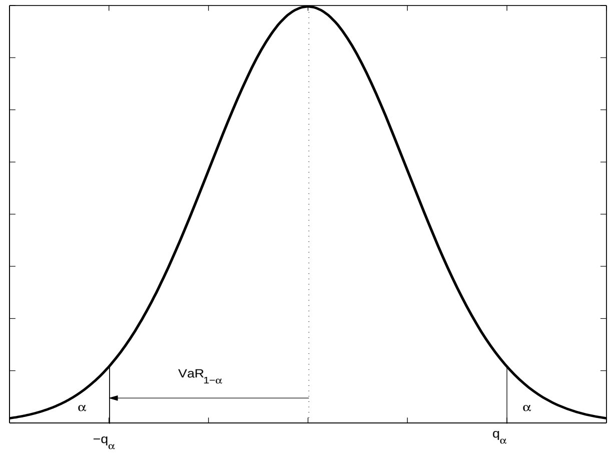

Figure 3.1: VaR for symmetric distributions.

Computing VaR

Due to the complexity of financial market evolutions, the computation of VaR represents a difficult task in practice. To this end, several techniques have been developed. In the following, we present the most widespread ones, which also comply with the different industry standards in risk management in force nowadays.

One important category of VaR-computation methods is represented by **parametric approaches**. They consider that a certain parametric distribution underlies the observed sample. The most common assumption in this vein is that of (log-)normally distributed returns. Accordingly, VaR yields proportional to the return standard deviation over the horizon T, σ_T:

$$\mathrm{VaR}_\alpha = -W_0 \Phi^{-1}(\alpha)\sigma_T, \tag{3.2}$$

where Φ stands for the normal cdf.[20] For i.i.d. returns, the parameter σ_T can be approximated as $\sigma\sqrt{T}$, where σ is assessed by the empirical mean of returns computed over one

the mean value. In line with Jorion (2001), this amounts to $P(W_0 R_T \leq -\mathrm{VaR}_\alpha^{\mathrm{abs}}) = \alpha$. For short time horizons (over which the expected portfolio value $E[W] \approx 0$), the absolute VaR can be considered to be an acceptable approximation.

[19] Note that the definition in Equation (3.1) does not work for ambiguous quantiles, such as those of non-injective or discontinuous cdf. In this case, there exists no inverse F^{-1} in the strict sense. According to Huschens (2000), quantiles can be more generally defined as $P(W < q_\alpha) \leq \alpha \leq P(W \leq q_\alpha)$. Equivalently, F^{-1} can be viewed as the generalized inverse of F, i.e. $F^{-1}(\alpha) = \inf\{x|F(x) \geq \alpha\}$. Thus, VaR results in the lowest α-quantile of the loss distribution taken with negative sign, $\mathrm{VaR}_\alpha = \inf\{-F^{-1}(\alpha)\}$.

[20] Note that Equation (3.2) is equivalent to Equation (3.3), where the quantiles q_α can now be easily derived for the normal distribution.

unit of time.[21] It is yet a well established fact that, in practice, returns often follow other distributions than normal.[22] In this case, Equation (3.2) does no longer hold and more complicated methods involving suitable quantile estimation are required.[23]

As a summary measure of risk, VaR found wide application in **portfolio management**. The *portfolio VaR* expresses the combined risks affecting the portfolio components. The easiest way to compute the portfolio VaR relies on the assumptions of linearity and normal distribution.[24] Denoting the vector of portfolio weights by w and the portfolio variance-covariance matrix by Σ, the portfolio VaR results in:[25]

$$\mathrm{VaR}_\alpha = -W_0 q_\alpha \sqrt{w'\Sigma w}, \tag{3.3}$$

where W_0 is now the initial portfolio value and q_α the α-quantile of the component return vector. Obviously, the portfolio VaR is smaller than the sum of the single-component VaRs. From Equation (3.3), it is apparent that VaR entails a multiple of the portfolio standard deviation which is the measure of risk in the traditional mean-variance framework.[26]

According to Jorion (2001), the portfolio VaR is primarily used for quantifying market risks, while further measures – such as the marginal,[27] the incremental,[28] or the component VaR[29] – can support the active risk management.

[21] Usually, the time unit is taken to be one day. Note that this formula for time scaling cannot be applied when returns are not i.i.d., such as for options.

[22] Returns exhibit positive kurtosis, that is their pdf shows higher peaks at the mean coupled with fatter tails relative to the normal pdf. Positive kurtosis can originate in jumps or stochastic volatility.

[23] Butler and Schachter (1998) develop an estimation procedure that combines smoothing techniques and quantile estimation from historical returns. A more sophisticated technique that accounts for variation in volatility includes GARCH models, as suggested in Manganelli and Engle (2001).

[24] Asset payoffs are assumed to be linear functions of the risk factors, which are considered to be normally distributed.

[25] Note that when the vector of component returns has the mean μ and the variance-covariance matrix Σ, the portfolio returns have $E[R] = w'\mu$ and $Var[R] = w'\Sigma w$. Henceforth, we omit the superscript ^{p} when referring to the portfolio returns and VaR, but add a subscript $_i$ for denoting the returns and the VaR of the i-th asset in the portfolio.

[26] This framework was introduced in Markowitz (1952) and developed by Markowitz (1959).

[27] The marginal VaR that quantifies changes in the portfolio VaR induced by incremental changes of the exposure in component i. According to Jorion (2001), p. 154, it is defined as $\Delta\mathrm{VaR}_{\alpha i} = \dfrac{\partial \mathrm{VaR}_\alpha}{\partial w_i W_0} = q_\alpha \dfrac{Cov(R_i, R)}{\sqrt{w'\Sigma w}}$. Moreover, it is indissolubly related to the measure β of systematic risk in CAPM.

[28] According to Jorion (2001),the incremental VaR expresses the change in VaR due to a new position $\mathrm{iVaR} = \mathrm{VaR}_\alpha^{\text{new pos.}} - \mathrm{VaR}_\alpha$. By Taylor expansion, it is approximatively proportional to the vector of the marginal VaRs, where the proportionality factor is given by the new position. In the context of active risk management, it allows to compute the variance-minimizing position to be adopted (the so-called "best hedge").

[29] The component VaR is defined in Jorion (2001) as the approximate change in portfolio VaR if

With direct regard on portfolios, parametric methods enable the derivation of the portfolio VaR when the portfolio instruments are non-linear in the market variables or when the latter are not normally distributed.[30] The goal is to infer analytically, by means of local derivatives, possible future movements of the portfolio value.[31] This is facilitated by assumptions on the return distribution, the most common of which being normality.

The easiest and most used parametric portfolio method is the *Delta valuation*. It merely considers first derivatives of the portfolio value W with respect to the price vector (or, equivalently, to the return vector R) in order to compute potential losses. These derivatives are referred to as the portfolio-Delta and yield at time $t = 0$ the value $\Delta_0 = \frac{\partial V}{\partial R}\Big|_{t=0}$. For normally distributed returns, the delta valuation represents a direct application of the traditional mean-variance approach in Markowitz (1952).[32] The Delta valuation simplifies calculations by assuming linear exposure in a limited number of risk factors. As noted in Duffie and Pan (1997), it can be considered to be satisfactory only over short time intervals and in the absence of jumps in prices. It can be extended in order to allow for multiple sources of risk by adding the second derivative of the portfolio value with respect to prices $\Gamma = \frac{\partial^2 W}{\partial P^2}$, which amounts to the use of the so-called *Delta-Gamma method*.[33] Further refinements consider either the first derivative over time, *Theta*, which is $\Theta = \frac{\partial W}{\partial t}$ and accounts for the time decay of option values, or a changing volatility σ_t.[34]

A second category of VaR-computation techniques consists of **non-parametric** (or **sampling**) **approaches**. In essence, they rely on historical and/or simulated data, from which empirical return distributions can be derived. VaR is then directly obtained from the corresponding sample quantiles. The apparent advantage of such methods is that no distributional assumptions are needed, hence the respective model errors can be avoided. However, the approximation error increases for more extreme quantiles.[35]

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component i is eliminated $\text{cVaR}_i = \Delta\text{VaR}_{\alpha i} w_i W_0 = \text{VaR}_{\alpha i}\beta_i w_i = \text{VaR}_{\alpha i}\rho_i, \quad \text{s.t.} \quad \sum_i \text{cVaR}_i = \text{VaR}_\alpha$, where ρ_i stands for the correlation of the component i returns with the portfolio returns.

[30] In such cases, there exists no exact VaR-formula such as that in Equation (3.3).

[31] To this end, the Taylor-expansion of the portfolio value provides a linear approximation to possible non-linearities in portfolio returns as a function of market variables.

[32] According to Equation (3.3), VaR represents the product of the α-quantile and the standard deviation of value changes, hence can be delta-approximated by $\text{VaR}_\alpha = -|\Delta_0| q_\alpha \sigma R_0$. The portfolio delta yields to the sum of the component deltas. For portfolios of derivatives, σ represents the standard deviation of the underlying spot return vector.

[33] Note that the portfolio gamma encompasses the component gammas *and* the cross derivatives over each pair of components.

[34] In general, the Cornish-Fisher expansion can be used when one has to compute more than two moments. This expansion allows for the transformation of moments in quantiles and hence provides a good approximation of quantiles in most practical cases, according to Jaschke (2001).

[35] According to Kendall (1994), the asymptotic variance of the quantile estimator $\hat{q}(\alpha)$ is $\alpha(1 -$

Concerning portfolios, the non-parametric methods facilitate the derivation of the portfolio VaR from the empirical distribution of portfolio values obtained by simulation for different scenarios. Thus, the *historical method* generates hypothetical price paths from historical returns.[36] In contrast, the *Monte Carlo simulation* relies on a parametric stochastic process. According to Hager (2006), the exposure to multiple risk factors can be handled in the context of the historical simulation either by the so-called factor approach (that computes VaR-levels for each factor and then aggregates them into a total VaR according to historical correlations) or by the portfolio approach (that reevaluates the entire portfolio for certain past values of risk factors). One way to simulate correlated risk factors within the Monte Carlo framework relies on the transformation of independent into correlated random numbers. The hybrid method designed by Boudoukh, Richardson, and Whitelaw (1998) enables stochastic volatility and represents a variation of historical simulation with weighted returns, where weights depend on the time interval from observation to current date.[37]

Another classification of the methods for computing the portfolio VaR is proposed in Jorion (2001) and concentrates on how the portfolio value is evaluated.[38] Thus, **local-valuation methods** are characterized by a unique portfolio valuation at the initial position. Approaches belonging to this category are the Delta method and its above mentioned variants that include higher order derivatives. In contrast, **full-valuation methods** entirely reprice the portfolio over various scenarios,[39] thus applying non-parametric approaches.[40]

$\alpha)/(Tf^2(q))$, where f stands for the return pdf.

[36]According to Duffie and Pan (1997) and Hull and White (1998a,b), the historical return distribution can be updated for changing volatility and correlations, in that scaling market factors are scaled by means of estimated past and current covariances.

[37]Accordingly, the return quantiles are estimated from historical data using declining exponential weights. Thus, the method gets around the necessity of employing large data sets specific to historical simulation and the normality assumption of exponential approaches. It captures at the same time the volatility pattern. Due to minimal assumptions and flexible model specification, this method is considered an improvement by Manganelli and Engle (2001). However, Hull and White (1998a) stress that the hybrid method inefficiently incorporates stochastic volatility.

[38]Our classification comes in line with Gaivoronski and Pflug (2005) and Manganelli and Engle (2001). It complies with the categories of methods for assessing VaR in general. Jorion (2001)'s categorization refers strictly to the portfolio VaR.

[39]The scenarios can be more or less complex, in the sense that they account for different kinds of exposure (such as the exposure to multiple risk factors or the cross-market exposure) and/or for simultaneous changes in several risk factors. See Duffie and Pan (1997). According to Jorion (2001), full-valuation considers the variation of the portfolio value $dV = V(P_1) - V(P_0)$ for a wide range of price levels, where P_1 represents the future price vector and can be simulated according to different methods.

[40]Engle and Manganelli (2004) suggest a somewhat similar classification in *factor models* (where VaR is considered to be proportional to the portfolio standard deviation that depends on volatilities and correlations of different factors) and *portfolio models* (where VaR is parametrically constructed for instance

Finally, various **semi-parametric approaches** to VaR are possible, such as the combination of local-valuation with the simulation of risk factors, as undertaken in the context of the partial-simulation approach of RiskMetrics.[41] Also, Hull and White (1998b) develop a conditional-distribution approach that allows to close the gap between model- and simulation-based methods.[42] Hull and White (1998a) by incorporating volatility updating into historical simulation.[43] A further semi-parametric approach is the conditional autoregressive value at risk (CAViaR) of Engle and Manganelli (1999) that directly models the time-evolution of quantiles and not the entire distribution of portfolio returns.[44] Also, the quasi-maximum likelihood (QML) GARCH developed by Bollerslev and Wooldridge (1992) represents a combination of GARCH returns and historically simulated residuals.[45] Moreover, Kaplanski (2005) proposes an analytical method for constructing VaR for general (i.e. non-normal) distributions.[46]

Each of these methods has its strong and weak points. In general, parametric methods are prone to model misspecification errors.[47] According to Jorion (2001), the Delta-normal method is efficient for large portfolios without substantial option components[48] and few sources of risk. This is due to the fact that the delta-valuation is computationally fast but does not account for fat tails or asymmetries in the return distribution. With various sources of risk, the delta-gamma approach performs better. The return asymmetry be-

from historical data and the quantiles are estimated by means of rolling windows, GARCH, or extreme values theory).

[41]Local-valuation methods are employed in order to generate an analytical approximation of VaR, while returns and covariances are obtained by simulation. For this simulation, RiskMetrics employs an IGARCH(1,1)-model for returns and assume residuals to be normally distributed. Specifically, it takes $\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)\epsilon_{t-1}^2$, where $\lambda = 0.94$. The use of the analytical formula saves on computations by avoiding the portfolio reevaluation on each trial, as mentioned in Hull and White (1998b).

[42]They propose mapping the residual distribution – obtained by either model assumptions or simulation - into a multivariate normal one. This procedure enables higher order return moments to be reflected in VaR. This approach is also applied to the Riskmetrics database.

[43]The main assumption of this model is that market factors scaled by their volatility, that is measured by the standard deviation, are stationary.

[44]CAViaR models the portfolio quantiles as an autoregressive process that also depends on lagged returns. The specific estimator accounts for volatility clustering and is shown to be consistent and asymptotically normal. A consistent variance-covariance estimator is also provided. In particular, four different model specifications are estimated using the regression quantile method and checked by means of an especially developed test, the so-called dynamic quantile test, for performance in- and out-of-sample.

[45]A brief survey over semi-parametric methods can be found in Manganelli and Engle (2001).

[46]This method is based on the conditional distributions of the portfolio components (that has thus to be known). It has a straightforward application to problems involving few assets but of high practical interest, such as optimizing portfolios for shares and bonds, analyzing the impact of adding an asset to an existent portfolio, or minimizing the portfolio VaR with a put option.

[47]E.g. deriving from the (conditional) multivariate normality assumption that usually does not hold in practice.

[48]E.g. Pearson (2002) advances that the delta approximation underestimates (overestimates) VaR for portfolios of written (purchased) options.

comes a serious problem in case of non-linear instruments such as options. Consequently, for option portfolios full valuation may be necessary, given that it enables the derivation of the complete multivariate distribution. Finally, note that the time adjustment suggested for the estimation of Equation (3.2) (e.g. from daily to ten-days volatility) is more complex in the case of options. Specifically, it does not reduce to the multiplication with the squared root of the time interval.[49] Hence, VaR has to be directly computed over the desired time horizon and not scaled out of the daily measure.

In spite of their accuracy, full-valuation methods are much more costly in terms of computations and, due to the limited number of replications, can also be subject to sampling variation. Although historical simulations exhibit the advantage of accurately reflecting the historical distribution of market variables, they require an adequate amount of past data[50] and become tedious for large portfolios. In addition, the historical method works with fixed rolling windows, which reduces to assuming i.i.d. returns and that past evolutions represent the unique set of future-relevant alternatives.[51] As it cannot account for volatility changes (i.e. for the theta risk), this method is in itself not well-suited to portfolios of derivatives. Besides, it can deliver VaR estimates that exhibit jumps. Moreover, the Monte Carlo simulation captures well complex risks that cannot be covered by other methods, but involves itself a model risk, i.e. the risk of specifying a false stochastic process from which hypothetical returns are generated.

Furthermore, semi-parametric methods retain some of the disadvantages of the procedures on which they rely (e.g. of the parametric assumptions or of the non-parametric choice problems).

Further risk measures

In spite of the large scale use of VaR in practice, researchers have signalled several drawbacks concerning its appropriateness as a risk measure.[52] First, Jorion (2001) notes that

[49]According to Duffie and Pan (1997), this is an acceptable approximation as long as there is no significant variation in standard deviations, no correlation in price changes over the considered time period, neither non-linear dependence of derivative prices on the underlying market prices.

[50]As specified in Hager (2006), representative and sufficiently accurate results necessitate rather large data sets, while the relevance of current evolutions cannot be ensured for observations that lie too far away in the past. In effect, choosing the length of the rolling window is the most difficult problem posed in the context of historical simulation, as emphasized in Manganelli and Engle (2001).

[51]Thus, it does not account for potentially important events that have not occurred in the past, as argued in Manganelli and Engle (2001).

[52]However, Pfingsten, Wagner, and Wolferink (2004) reach the conclusion that the application of different (downside) risk measures – such as VaR and ES – to real trading-book data does not entail high discrepancies in the ranking of risky distributions (as measured by the Spearman-correlation coefficient

according to the VaR-definition, the investment (portfolio) is considered to be "frozen" over the specified time horizon, i.e. the market conditions are taken as constant. However, this rarely holds in practice, as the portfolio composition constantly changes. Second, VaR does not consider, by construction, the impact of extreme events, i.e. those located in the tails of the return distribution.[53] Such events may be rare but are of acute importance. Third, VaR is not a coherent risk measure in the sense of Artzner, Delbaen, Eber, and Heath (1999), in particular it does not recognize the addition and the concentration of risks, and hence can even prevent diversification.[54] Also, Szegö (2002) claims that the non-convexity of VaR renders the implementation of usual – and very efficient – convex optimization settings impossible and the existence of many local extremes makes the VaR-ranking unstable. However, Pflug (2000) proves that the VaR-optimization problem can be reformulated as a fixpoint problem of solutions of linear optimization problems and Gaivoronski and Pflug (2005) suggest feasible algorithms for computing mean-VaR efficient portfolios based on historical data.[55]

According to Pearson (2002), some of these shortcomings can be counterbalanced by applying *stress tests*. Such tests simulate portfolio performance for various scenarios reflecting particular changes in market conditions.[56] Stress testing should compensate for the lack of information on the magnitude of losses occurring when VaR is exceeded, on the direction of the risk exposure (i.e. whether the loss quantified by VaR emerges in a rising or falling market), or on risks encompassed in other factors than the ones considered to be relevant.

Another possibility is to replace VaR by further risk measures as amply proposed in the literature. On the one hand, extended-tail measures – and note that VaR itself is one of them – such as the *tail conditional expectation* (TCE) or the *worst conditional expectation*

of the resulting rankings).

[53]In mathematical terms, VaR is not smooth. Events with a probability lower than α remain unconsidered as long as the significance level is higher than α, a situation that immediately changes by choosing another confidence level, as shown in Rau-Bredow (2002). According to Rockafellar and Uryasev (2001), VaR represents only the lowest loss bound being biased towards optimism.

[54]In mathematical terms, VaR is not subadditive. According to Szegö (2002), the only exception is the case with an elliptic joint distribution – such as the normal distribution – of the addition terms, when the VaR-minimizing portfolio is identical to the Markowitz variance-minimizing solution. In practice, the non-subadditivity implies that the global risk can be higher than the sum of the component risks. In such a case, it would be possible to reduce risk by splitting the investment (e.g. the portfolio) or the business with respect to different risk sources (e.g. components).

[55]Please refer to Footnote 69 for further details on the algorithm in Gaivoronski and Pflug (2005).

[56]The scenarios can be based on past market events, on hypothetical shocks applied to certain market risk factors (that can be further extended by considering the correlations of these core factors with other factors), or on implications of possible events for more major markets.

(WCE), perform somewhat better.[57] As shown in Acerbi and Tasche (2002), they are yet non-continuous in α and hence cannot be applied to non-continuous distributions.[58] On the other hand, there is one alternative measure that proves to be coherent for all distributions, namely the *expected shortfall* (ES). It represents the limit in probability of the expected losses in $\alpha\%$ worst cases. For continuous distribution, ES coincides with the so-called *conditional VaR* (CVaR) and yields:[59]

$$\mathrm{ES}_\alpha = \mathrm{CVar}^\alpha = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_y(W)dy, \tag{3.4}$$

in line with the notation from our above Equation (3.3). For normally distributed portfolio returns, De Giorgi (2002) shows that ES can be written as:

$$\mathrm{ES}_\alpha = \frac{\phi(w)}{\alpha}\sqrt{w'\Sigma w} - w'\mu, \tag{3.5}$$

where ϕ stands for the standard normal pdf. When applied to portfolio optimization, ES entails identical results to VaR for normal distributions, according to Rockafellar and Uryasev (2001). Gaivoronski and Pflug (2005) argue that, for general distributions, ES can be converted to the target variable of a convex optimization problem for which efficient algorithms exist.[60] In addition, it can be expressed in a minimal form that saves computations, with respect to which Rockafellar and Uryasev (2000, 2001) show how

[57] Please refer to Acerbi and Tasche (2002) for the mathematical definitions of TCE and WCE.

[58] These denominations are alternately and often misleadingly used in the literature. We follow here the suggestions in Acerbi and Tasche (2002), though our definition of VaR corresponds to the lower quantile in their Definition 2.1. Specifically, the lower and upper quantile VaR can be defined as $\mathrm{VaR}_\alpha = -q_\alpha = -\inf\{w|P(W \leq w) \geq \alpha\}$ and $\mathrm{VaR}^\alpha = -q^\alpha = -\inf\{w|P(W \leq w) > \alpha\}$, respectively. The lower and upper tail conditional expectations stand for the average loss in the worst $\alpha\%$ cases and amount to $\mathrm{TCE}_\alpha = -E[W|W \leq -\mathrm{VaR}_\alpha]$ and $\mathrm{TCE}^\alpha = -E[W|W \leq -\mathrm{VaR}^\alpha]$, respectively. The worst conditional expectation introduced in Artzner, Delbaen, Eber, and Heath (1999) can be formally stated as $\mathrm{WCE}_\alpha = -\inf\{E[W|A]|P(a) > \alpha\}$.

[59] According to Acerbi and Tasche (2002), the conditional VaR is mostly defined as $\mathrm{CVaR}^\alpha = \inf\{E[(W-s)^-]/\alpha - s|s \in \mathbb{R}\}$, where $(x)^-$ denotes the negative part of a number. A general definition of the expected shortfall (or tail mean) is $\mathrm{ES}_\alpha = -\mathrm{TM}_\alpha = \{E[W\mathbf{1}_{W\leq -\mathrm{VaR}_\alpha}] - \mathrm{VaR}_\alpha[\alpha - P(w \leq -\mathrm{VaR}_\alpha)]\}/\alpha$. For non-continuous distributions, Pflug (2000) shows that $\mathrm{ES}_\alpha = \mathrm{TCE}_\alpha + (\beta-1)(\mathrm{TCE}_\alpha - \mathrm{VaR}_\alpha)$, where $\beta = P(W \leq -\mathrm{VaR}_\alpha(W))/\alpha$. As demonstrated in De Giorgi (2002), for continuous distributions ES is identical to TCE. Also, Acerbi and Tasche (2002) prove that, in general, ES yields the maximum of WCE and the two measures are identical for continuous distributions. Moreover, the following identity holds for integrable random variables: $\mathrm{ES}_\alpha = \mathrm{CVaR}^\alpha = -\{E[W\mathbf{1}_{W\leq s}] + s[\alpha - P(W \leq s)]\}/\alpha, s \in [q_\alpha, q^\alpha]$.

[60] The conversion is possible due to the fact that ES represents the smallest convex majorant of VaR as shown in Pflug (2000). Hence according to Krokhmal, Palmquist, and Uryasev (2001), the minimization of ES - or, equivalently, of a continuously distributed CVaR – leads to near optimal solutions for VaR as well. In essence, replacing VaR by ES (or CVaR) strengthens capital requirements. The same representation as a convex optimization problem can be found in the classical mean-variance framework, where the employed risk measure is the standard deviation of portfolio returns. This is not the case of VaR, of which the optimization is non-convex and hence can lead to multiple local minima.

ES can be computed without first having to calculate VaR[61] and for distributions with jumps. Also, Manganelli and Engle (2001) demonstrate that ES can be derived according to different semi-parametric approaches, as soon as it is stated in terms of standardized residuals. In spite of these more desirable properties, neither ES nor CVaR have yet replaced VaR as a standard measure in the financial industry.[62]

Portfolio optimization under VaR-constraints

Another application of VaR consists of portfolio optimization where market risk is measured by VaR. In essence, it follows the same steps as the classic mean-variance optimization procedure. In particular, the risk measure – now VaR – has first to be minimized for various expected portfolio returns. This secondly allows the derivation of the mean-VaR feasible set. Finally, the mean-VaR efficient frontier can be inferred.[63] Comparisons of mean-variance and mean-VaR efficient frontiers performed in Gaivoronski and Pflug (2005) show that the latter approximates quite well the former one and entails a better equivalent VaR.[64]

However, the portfolio optimization with VaR proves usually to be more difficult with respect to the classic procedure of Markowitz which considers the standard deviation as a risk measure. The motivation is that finding the optimal portfolio implies working with the multivariate distribution of portfolio returns. As noted in De Giorgi (2002), this distribution cannot be analytically formulated for other cases than normally distributed component returns. Numerical approximation becomes hence necessary. In addition, Gaivoronski and Pflug (2005) stress that the VaR-optimization can yield non-convex problems of high computational complexity.[65] Various **optimization algorithms** proposed in the literature address these problems and facilitate the work with non-normal multivariate distributions, and can be found in the papers of Rockafellar and Uryasev (2000),[66]

[61] This would be necessary according to the definition in Equation (3.4).

[62] According to Embrechts, Klüppelberg, and Mikosch (1999), CVaR is however widely accepted in the insurance industry.

[63] Krokhmal, Palmquist, and Uryasev (2001) apply an equivalent formulation of the problem of finding the efficient frontier. This formulation relies on fixing the risk and maximizing returns.

[64] Specifically, the VaR of mean-VaR efficient portfolios provides an improvement of over 10% over the VaR computed in the mean-variance framework in more than 50% of the cases studied in Gaivoronski and Pflug (2005).

[65] There is a vast literature on solving convex optimization problems. See Rockafellar and Uryasev (2000) for further references. The goal of most of papers is to reduce the mean-VaR problem to a convex one that can be efficiently solved. (Note that the mean-variance problem is convex.) However, Rockafellar and Uryasev (2000) underline the fact that, for hedges with relatively few instruments, non-smooth optimization techniques can compete with linear programming.

[66] Specifically, they design a method to optimize ES (see Equation (3.4) and the comments at the end

Krokhmal, Palmquist, and Uryasev (2001),[67] Rockafellar and Uryasev (2001),[68] or Gaivoronski and Pflug (2005)[69]. Alternatively, the univariate distributions of the single portfolio assets can first be considered separately and then integrated through a copula function, as demonstrated in Embrechts, Höing, and Juri (2003).[70] Another – and more general – difficulty with the portfolio calculations resides in the identification of the relevant market factors of which the changes generate risks. Pearson (2002) advises the use of factor models for this purpose.

Several approaches attempt to solve the problem of portfolio optimization under VaR – or VaR-related – constraints, that is also addressed in Sections 3.2 and 3.3. Campbell, Huisman, and Koedijk (2001) are among the first to develop a portfolio selection model with VaR-constraints.[71] They find that the optimal portfolio composition[72] derived from historical returns[73] under the constraint of a 95%-VaR reaches from 36% in stocks for daily data to 90% for monthly data (the rest being allocated to bonds). When the future returns are assumed to follow a normal (Student-t) distribution, these percentages change to 38.22% (40.38%) stocks for daily data.[74] In addition, a performance index similar to

of this section). This method has as a byproduct the value of VaR for continuous distributions. To this end, both risk measures are reformulated in terms of a convex and continuously differentiable function, the minimization of which is equivalent to the minimization of the ES. In essence, ES amounts to the minimum value of this function and the VaR to the left point of the argument set of minima. Finally, Rockafellar and Uryasev (2000) apply this approach to portfolio optimization and compare among the solutions of optimization problems with VaR, ES, and standard deviation as objective functions, as well as to hedging.

[67] They extend the approach in Rockafellar and Uryasev (2000) is extended to optimization problems with ES constraints. See the end of this section for more details on this model. A case study shows that the algorithm is stable, efficient, and flexible, and is able to handle a large number of instruments and scenarios.

[68] In particular, the approach in Rockafellar and Uryasev (2000) is now extended for general distributions, e.g. with jumps, as it it often the case in scenario models. Rockafellar and Uryasev (2001) restate ES as weighted average of VaR and what they define as the upper ES. This procedure is well suited to optimization problems with ES both as an objective function and as a constraint.

[69] In essence, their method is based on the observation that VaR can be split into two components: a first smooth and convex one that can hence be efficiently minimized, and a second irregular and non-differentiable one that can thus entail multiple local minima. The proposed algorithm proceeds by first filtering out the smooth component, then convexly optimizing for it by standard procedures, and finally improving the so obtained approximation.

[70] They propose a general method for constructing optimal bounds for risk measures that relies on the theory of copulae. (A *copula* represents a distribution function that captures the dependence structure among marginals and the corresponding multidimensional distribution. Thus, knowing the copula and the marginal distributions allows inferring the joint distribution. This is yet not the case for most practical applications. Hence, giving bounds for the joint distribution becomes an issue of great interest.) This method is exemplified for VaR, in which case it helps avoiding the lack of subadditivity of this measure and hence determining the value at risk of a joint position from the VaR-s of the marginal positions.

[71] This model underlies our theoretical framework and will be referred in more detail in Section 3.2.2.

[72] In their setting, investors can merely chose between stocks and bonds.

[73] The data set encompasses prices of US bonds, treasury bills, and the S&P 500 index between 1990-1998.

[74] The VaR is estimated from historical returns for different confidence levels ranging from 95-99%.

the Sharpe-ratio is defined in order to comply with the VaR-framework and allows for the comparison of the optimal VaR- and mean-variance portfolios in case of normally distributed returns.[75]

Basak and Shapiro (2001) solve a similar portfolio optimization problem under VaR-constraints. They observe that, when investors dispose of low monetary resources (i.e. in low-wealth states), VaR entails lower wealth levels with respect to the benchmark without risk-constraints. The use of VaR in such situations increases the credit and solvency risk.[76] Basak and Shapiro (2001) consequently propose and evaluate an alternative risk measure aiming at limiting expected losses denoted as *limited expected losses* (LEL).[77] Since LEL results in higher wealth (hence smaller losses) in bad states of the world, LEL-oriented managers never take extreme leveraged positions in such states.[78] The analysis is subsequently extended to a general equilibrium setting where VaR-restricted agents and non-risk agents are faced to each other. The goal is to assess the impact of VaR-constraints on prices. Relative to normal agents, VaR-agents are shown to weight consumption before and after the VaR planning horizon differently. Before the VaR-horizon, values and volatilities of VaR-agents' portfolios are higher than the corresponding ones for non-risk agents. In fact, higher volatilities occur in transition from intermediate to bad states (i.e. in falling stock markets). Thus, VaR-agents take more risk than normal agents do, and this exactly during these bad periods, which yields an increased market volatility.

Further interesting conclusions concerning the VaR-efficient portfolio set are reached by De Giorgi (2002). When portfolio returns are multivariate-normally distributed and

Daily, bi-weekly, and monthly data, as well as different expected return distributions (historical, normal, and Student-t) serve for analyzing the evolution of the optimal portfolio composition and determining efficient portfolio frontiers.

[75]Specifically, this index equals the ratio of the return premium and a measure of risk calculated as the difference between the initial portfolio valuated at the risk-free rate and VaR. Further conclusions refer to the fact that larger deviations from normality entail increased risk underestimation, as higher VaR-confidence levels are chosen. The use of the Student-t distribution yields lower (higher) portfolio VaR for lower (higher) confidence, where these changes are more pronounced for shorter time horizons and are maximal for one day.

[76]Specifically, agents are assumed to be maximizers of (CRRA) expected utility over a certain horizon and to comply with VaR-constraints imposed on that horizon. Their optimal wealth at the end of the planning horizon decomposes into three domains, namely good states of the world (where VaR-agents behave like the benchmark with no constraints), intermediate states (against which VaR-agents fully insure), and bad states (that are left uninsured).

[77]The reason for the VaR-underperfomance would be that it concentrates on probabilities instead of the magnitude of losses. The authors claim that LEL is subadditive, as well as positively homogenous and monotonic, but not translation-invariant, as required in Artzner, Delbaen, Eber, and Heath (1999) for coherent risk measures.

[78]With LEL instead of VaR as a risk measure, agents partially insure also in the bad states and the pdf of the terminal wealth becomes continuous.

no risk-free asset is present in the market, the set of mean-VaR efficient portfolios forms a subset of the efficient portfolios under ES. This latter set is itself contained in the set of classic mean-variance efficient portfolios. In addition, when both risky and risk-free assets can be traded all these three efficient portfolio sets become identical.[79] Consequently, the *Tobin* (or *two fund*) *separation theorem* – which stresses that the efficient portfolio is a combination of the risk-free asset and the tangency portfolio[80] – holds also when investors use VaR or ES for quantifying market risk. As in the long-run returns appear to be normally distributed, De Giorgi (2002) concludes that there is no significant improvement in using one of these more sophisticated risk measures instead of the standard deviation.[81]

In the same spirit, Krokhmal, Palmquist, and Uryasev (2001) construct the efficient frontier of a portfolio of stocks in the SP100 index and the risk-free asset, subject to various constraints on CVaR[82] and generating scenarios from ten-day historical returns. This frontier is subsequently compared to the equivalent mean-variance frontier of the standard Markowitz-approach.[83] The results show that for non-normal return distributions the mean-variance and mean-CVaR frameworks entail distinct results, where the discrepancy increases subject to the chosen confidence level.[84]

The alarm signals pulled by scholars in showing the drawbacks of VaR face yet massive difficulties in getting through to the real financial world. On the one hand, the legislation in force and the plain business practice compel to employ VaR as a risk measure. The bottom line is that VaR has already been established as a widely accepted and implemented standard in risk management. This turns it into a veritable mental anchor for portfolio managers. On the other hand, the lack of sufficient competencies necessary for understanding and tackling all mathematical subtleties of more refined risk measures can only encourage the use of easy-to-implement measures such as VaR. (Recall that, as long

[79] Of course, this holds unless one of the sets is not empty. Emptiness can occur for e.g. too high α, as first shown in Rockafellar and Uryasev (2000).

[80] The tangency portfolio represents a portfolio that is optimal under both situations with and without risk-free assets. When investors hold homogenous probability beliefs, it yields to the normalized optimal portfolio with risk-free assets.

[81] Furthermore, he develops a linear programming algorithm for approximating efficient mean-ES portfolios for other distributions than normal.

[82] In addition to the usual risk constraint –where the measure of risk in this case is CVaR – Krokhmal, Palmquist, and Uryasev (2001) also incorporate constraints on transaction costs and on individual values and positions.

[83] The equivalency is ensured by using identical historical returns.

[84] In principle, the discrepancy is due to the fact that the variance represents a symmetric measure of risk that equally accounts for high gains and high losses. In contrast, VaR and CVaR refer only to the left tail of the distribution corresponding to losses. This suggests a close connection to the distinct subjective perception of gains and losses modelled in the prospect theory.

as the normal distribution can be considered as an acceptable approximation of the relevant risk factors, the use of VaR for selecting optimal portfolios is not incorrect, although it can unnecessarily complicate the optimization procedure. However, if this is not the case, managers should be aware of the biases in assessing risk that follow from the use of VaR and replace VaR by other more appropriate measures of risk.)

Our analysis from Sections 3.2 and 3.3 is based on the portfolio selection under VaR, since this appears to remain the perspective actually adopted by most managers in practice. Yet, we are sooner concerned with the attitude and decisions of *non-professional* investors (and not of professional managers). As non-professional investors are not bound by any written or unwritten laws of business practice, one could ask why such investors should also work with VaR. The motivation resides in a sort of “contamination”: As explained in Section 3.2, most non-professional investors – who are, in essence, non-experts – hire professional managers for providing technical assistance concerning investment decisions. Accustomed to think in terms of VaR, managers ask their non-professional clients to indicate a maximum acceptable level of risk (which further enters the portfolio optimization procedure in the form of the usual risk constraint).[85] Thus, non-professional investors are forced to adopt a VaR perspective and made part of the “VaR-chain”. Of course, non-professional investors may face even higher difficulties in rigorously understanding the concept of VaR and its implications. For them, the subjective perceptions are susceptible to play a determinant important role in this respect, as we will discuss later. Our work focusses on the non-specialist individuals’ perceptions of the maximum acceptable loss, and on their further impact on the final capital allocation decision.

3.1.2 Prospect theory

Before presenting our main model where VaR serves non-professional investors to allocate money among different utility sources, it is important to address the problem of perception. The motivation resides in the fact that these usually non-trained investors perceive financial market risks – and hence VaR – in a subjective way which may deviate substantially from the formal definitions summarized in Section 3.1.1. We formalize the perceptions of non-professional investors in terms of the prospect theory, the main issues of which are addressed in this section.

[85] It is also plausible to think that mangers explain to their clients some details of the procedure used for reaching the optimal asset composition. Since VaR is a key variable of this procedure, non-professional investors will be learning even more about the meaning and use of VaR.

The **prospect theory** (abbr. PT) of Kahneman and Tversky constitutes a cornerstone of behavioral finance. Introduced in Kahneman and Tversky (1979) and extended in Tversky and Kahneman (1992),[86] as well as in many further papers by different authors, PT rebuts the basic principles of the *expected utility theory* (abbr. EUT), the major approach regarding decision making under risk in neoclassical Economics.[87] The need to change the traditional framework came in consequence of the fact that deviations from theoretically prescribed behaviors were repeatedly observed in real decision situations (above all in financial markets).[88]

Specifically, experiments and empirical observations reveal different effects that violate the basic normative axioms imposed by EUT on human preferences and choices.[89] Tversky and Kahneman (1992) summarize them as framing effects,[90] non-linear preferences,[91] source dependence,[92] risk seeking,[93] and loss aversion.[94][95]

[86]The extension is referred to as the cumulative prospect theory (abbr. CPT) and exhibits additional features, such as the usage of cumulative instead of separable decision weights, the coverage of decision problems under both risk and uncertainty and with any number of outcomes, the formulation of distinct weighting functions for gains and losses, and the introduction of diminishing sensitivity and loss aversion.

[87]EUT assumes that investors behave as perfectly rational maximizers of the expected utility of wealth. They are able to process the entire information at their disposal and form unbiased judgments. Thus, they assess the expected utility as a linear combination of final states (or outcomes), weighted by the probabilities of the corresponding events with these outcomes. These probabilities are mostly updated by means of the Bayes rule and must be well-known to the investors. The utility function is taken to be unique for all possible outcomes. Moreover, investors exhibit consistent preferences and risk-averse behaviors, so that the utility function is concave in wealth.
Specifically, EUT postulates the valuation of a future situation – i.e. choice alternative or prospect – as the mathematical expectation of its monetary values. This expectation is defined as the weighted sum of the outcome utilities (*reference dependence*). The gains and losses are evaluated symmetrically (*symmetry of valuation*) and proportionally to the accordant expectations (*non-proportional marginal sensitivity*). The weights represent outcome probabilities that sum up to 1 and are independent of the origin of uncertainty. Accordingly, the utility function depends only on final states. The risk is captured by means of a unique and constant risk coefficient. This renders the entire utility function linear for risk-neutral subjects, concave for risk-averse, and convex for risk-seeking ones, respectively.

[88]Thaler (1985) devises the so-called *behavioral decision research* by enunciating and refuting 15 principles of the classic utility theory: the choice dependence on outcomes and generally on final states, the formulation of decision weights as outcome probabilities and their independence of the source of uncertainty, the independence of preferences of their representation, the preference of dominating alternatives, the equivalence of opportunity and out-of-pocket costs, the optimal search, the influence of sunk costs on decisions, the exclusive dependence of preferences on the qualities of an alternative and not on its perceived merits, the consistency and lack of bias (i.e. the full rationality) of probabilistic judgments, and the Bayesian learning.

[89]These axioms refer to: completeness (that also implies reflexivity), transitivity, the Archimedean property, and independence. The fulfillment of the first two is denoted as rationality.

[90]Different presentations (framing) of the choice problem can entail distinct preferences. This contradicts the assumption of description invariance in the rational theory.

[91]Specifically, utility is non-linear in the outcome probabilities.

[92]Not only the degree but also the source of uncertainty can influence preferences. One classic example is the paradox by Ellsberg (1961).

[93]Namely for losses and for low probable high gains relative to the expected value.

[94]That is, the asymmetry in the perception of gains and losses. More details are given in the text below.

[95]The original Kahneman and Tversky (1979) paper addresses more specifical effects, such as the cer-

PT attempts to incorporate these psychological aspects in the decision making process that is considered to rely on the individual perception of reality. In essence, PT was not intended to be a normative theory based on axioms as EUT, but rather a descriptive approach attempting to capture empirically observed behaviors.[96] Thus, PT stresses that decisions are in fact rarely based on final states (outcomes), but sooner on subjectively perceived changes in welfare generated by these outcomes. Perceptions are formulated relative to a subjective reference point,[97] so that deciders distinguish between positive and negative wealth changes (i.e. gains and losses).

According to the original Kahneman and Tversky (1979) paper, the human choice process develops in two stages. The first one refers to *editing* of choice alternatives (or prospects) and entails a mental representation of them. It implies different operations, such as coding,[98] combination (of probabilities of prospects with identical outcomes), segregation (of the risk-free component from the risky one), cancellation (of components or of outcome-probability pairs that are common among prospects), simplification of prospects (e.g. by probability or outcome rounding), and detection of dominance (where the dominated alternatives are rejected). Naturally, the sequence of these editing operations influences the final edited prospect, hence the preference order.[99] The second stage consists of the *evaluation* of the edited prospects and of the final choice (which is the prospect with the highest ascribed value).

The evaluation phase implies the assessment of an overall value of each choice alternative, denoted as **prospective value**. Formally, it represents the weighted sum of the values subjectively assigned by each individual to the possible outcomes, where outcomes are separately treated as gains (henceforth denoted by a symbol $+$) or losses (denoted by $-$

tainty effect (i.e. the preference of certain smaller to uncertain higher gains which violates the substitution axiom and relates to the well-known Allais (1953) paradox), the reflection effect (i.e. the fact that high risky losses are preferred to certain smaller ones), the probabilistic insurance (i.e. the preference for contingent insurances that provides certain coverage vs. probabilistic insurances, that is due to the fact that different formulations entail distinct preferences over prospects with identical outcomes and probabilities), the isolation effect (according to which prospects appear to be often decomposed and people focus merely on the components that distinguish choice alternatives; as different decompositions are possible, preferences can revert and become inconsistent which violates the completeness and transitivity).

[96]Axiomatizations of CPT for decisions under uncertainty and risk are provided in Wakker and Tversky (1993) and Chateauneuf and Wakker (1999), respectively, and extended in Schmidt (2003) in order to capture the impact of shifting reference points.

[97]As underlined in Kahneman and Tversky (1979), the reference point can be the status quo (e.g. the current asset value) but also an aspiration level. Its shift is possible and affects the preference order.

[98]This assumes defining the reference point and perceptually separate outcomes in gains and losses with respect to it.

[99]This phenomenon is also known as the *framing* of the problem.

).[100] According to Tversky and Kahneman (1992), the prospective value V of outcome i, where $i = 1 \ldots n$, yields:

$$V_i = V_i^+ + V_i^- = \sum_{x_i^+} \pi_i^+ v(x_i^+) + \sum_{x_i^-} \pi_i^- v(x_i^-),$$

where v stands for the value function and π for the decision weights, both of which being defined below. Finally, x_i denotes the possible project outcomes $i = 1, \ldots n$ and x_i^+ (x_i^-) indicates the domain of gains (losses).

The subjective value of outcomes is encompassed by the so-called **value function** which exhibits several particular features. First, it addresses the perceptional segmentation into *two domains* with different evolutions. These domains correspond to gains and losses, so that the value function is asymmetric. The delimitation of the loss and gain domains takes place with respect to a subjective *reference point*.[101] Moreover, the value function exhibits *diminishing sensitivity* in both domains, i.e. its variation decreases with the magnitude of gains and losses, respectively. In addition, as people appear to be more reluctant to incur losses compared to gains of the same size – a property denoted as *loss aversion* -, the value function presents a *kink* at the origin. Consequently, the value function has to be zero at the reference point, steeper for losses than for gains, as well as concave in the domain of gains and convex in the domain of losses (in sum, s-shaped).[102] The CPT of Tversky and Kahneman (1992) formulates the value function v of an individual (investor) k. For simplicity reasons, we henceforth omit the subscript $_k$. This convention will apply to all variables except for the outcome x_i. The CPT-value function yields:

$$v(x_i) = \begin{cases} A(x_i - x_i^0)^\alpha, & \text{if}\, x_i \geq x_i^0 \quad \equiv (x_i^+) \\ -B(x_i^0 - x_i)^\beta, & \text{if}\, x_i < x_i^0 \quad \equiv (x_i^-), \end{cases}$$

where x_i^0 stands for the subjective reference point. In addition, $0 < \alpha, \beta \leq 1$ are specifical

[100]The prospective value is the counterpart of the expected utility function in EUT. Thus, the concept of "utility", defined in EUT in terms of net wealth is replaced by the one of "value", defined in PT in terms of relative wealth.

[101]Formally, the value function is a function of two variables: the reference point and the magnitude of changes with respect to this reference. CPT considers the reference point to be fixed, e.g. corresponding to the status quo or initial endowment. Schmidt (2003) develops an extended axiomatization for variable references, that accounts for the derivation of both value function and decision weights. Davies (2005) extends the notation in Schmidt (2003) in order to allow for the independence of the reference point and hence of the initial endowment and thus provides a basis for the unification of the frameworks for risky and risk-free choices.

[102]Norsworthy, Gorener, Schuler, Morgan, and Li (2004) find empirical evidence on the US-market for reference dependence, asymmetric valuation of gains and losses and diminishing sensitivity.

parameters describing risk aversion,[103] and $B \geq A > 0$ point to loss aversion. Often, the above defined value function is "normalized" by taking $x_i^0 = 0$, $A = 1$ and changing the notation B to λ that is denoted as the *loss aversion coefficient*. Thus, the value function yields:

$$v(x_i) = \begin{cases} x_i^\alpha, & \text{for } x_i \geq 0 \\ -\lambda(-x_i)^\beta, & \text{for } x_i < 0. \end{cases} \tag{3.6}$$

The specific parameters of the value function are estimated in Tversky and Kahneman (1992) to be $\alpha = \beta = 0.88$ and $\lambda = 2.25$.[104] The corresponding course is illustrated in Figure 3.2.

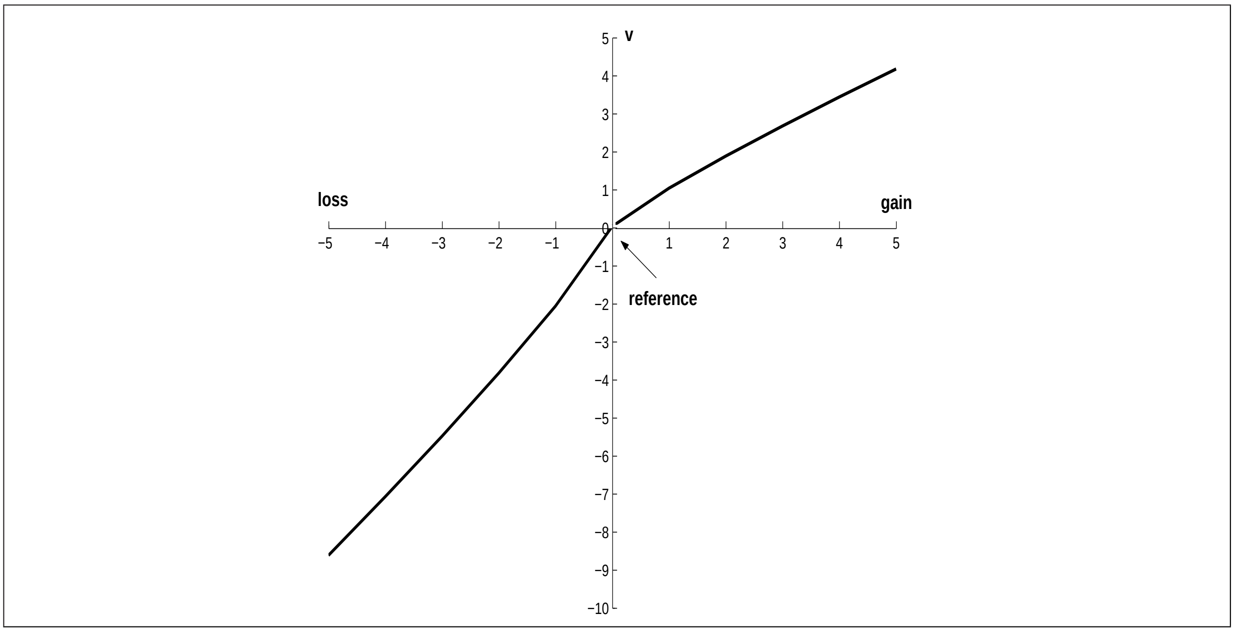


Figure 3.2: The PT-value function, for $\lambda = 2.25$, $\alpha = \beta = 0.88$

According to PT, the outcomes that enter the prospective value may not be weighted by their simple probabilities as in the standard EUT, but by the subjectively perceived counterparts of these probabilities. The latter are denoted as (cumulative) **probability weighting functions** (or **decision weights**) and represent complex, non-linear functions of probabilities. This idea originates in the findings of various psychological ex-

[103] The risk aversion is captured through the concave-convex form of the value function which points to an increased perceived value of every change in wealth due to the risky investment. This change is distinct in the two domains of gains and losses. In mathematical terms, this form is obtained for sub-unitary α and β. Accordingly, the Arrow-Pratt coefficient of relative risk aversion defined in Arrow (1965) $-xv''(x)/v'(x)$ yields $1 - \alpha$ $(1 - \beta)$ for gains (losses).

[104] The result that $\alpha, \beta < 1$ confirms the diminishing sensitivity of the value function in both domains. Further estimates of the curvature parameters are computed in Wu and Gonzales (1996) both for their own data set ($\alpha = 0.52$) and for the data set in Camerer and Ho (1994) ($\alpha = 0.37$). The estimation of λ conforms to median responses obtained in an experiment concerning two-outcome prospects with monetary outcomes and numerical probabilities conducted in Tversky and Kahneman (1992).

periments showing that individuals manifest the tendency to overweight (underweight) small (moderate to large) probabilities. Formally, the weighting functions are designed to be *increasing in probability* and to exhibit *diminishing sensitivity* (i.e. steeper evolution) at (both of) the endpoints of the probability scale, i.e. in 0 and 1.[105] This yields the specifical *inverted S-shape* (concave for small and convex for large probabilities), with an inflection point between $0.30 - 0.40$.[106] Also, the decision weights are mostly sub-certain (i.e. add up to a value less than 1).[107] CPT describes the weighting function π of the outcome i as a difference in functions of the outcome probabilities $w(p)$:[108]

$$\begin{aligned}
\pi_i^+ &= w^+(p_1 + \ldots + p_i) - w^+(p_1 + \ldots + p_{i-1}), \ \ \text{for } 1 \leq i \leq I \\
\pi_i^- &= w^-(p_i + \ldots + p_n) - w^-(p_{i+1} + \ldots + p_n), \ \ \text{for } I \leq i \leq n \\
\pi_1^+ &= w^+(p_1), \quad \pi_n^- = w^-(p_n),
\end{aligned} \tag{3.7}$$

where w^+ and w^- are strictly increasing on $[0, 1]$ and:

$$\begin{aligned}
&w^+(0) = w^-(0) = 0, \quad w^+(1) = w^-(1) = 1 \\
&w^+(p) = \frac{p^\gamma}{[p^\gamma + (1-p)^\gamma]^{1/\gamma}} \\
&w^-(p) = \frac{p^\delta}{[p^\delta + (1-p)^\delta]^{1/\delta}}.
\end{aligned} \tag{3.8}$$

The curvature parameters in Equations (3.8) are estimated to be $\gamma = 0.61$ and $\delta = 0.69$.[109]

[105] The introduction of decision weights of this form is considered to be necessary by Tversky and Kahneman (1992), as the monotonic transformation of outcome probabilities initially suggested in Kahneman and Tversky (1979) does not hold for choice problems with more than two outcomes. This transformation should be replaced by a broader applicable one which can be used for the entire cumulative distribution function, should account separately for gains and losses, and should satisfy stochastic dominance.

[106] Specifically, Wu and Gonzales (1996) find a value of around 0.40 using non-parametric estimation, while in Prelec (1998), parametric estimates of the inflexion point yield 0.37 for the own data set and 0.34 (0.38) for gains (losses) of the data in Tversky and Kahneman (1992). Also, Camerer and Ho (1994) assess the inflexion probability to be 0.30 for the functional form in Equation (3.8).

[107] Prelec (1998) summarizes the properties of the weighting function: it is regressive (which generates the four-fold pattern of risk attitudes: risk-seeking for small-probability gains and large-probability losses and risk-averse for high-probability gains and small-probability losses), asymmetric (which increases risk-aversion for gains and risk-seeking for losses), s-shaped (which enhances the impact of changes in probability at the ends of the probability interval), and reflective (i.e. it assigns identical weights to given loss- and gain-probabilities).

[108] Numerous further papers investigate the weighting functions. Thus, Tversky and Wakker (1995) confront risk with uncertainty and introduce a method for comparing weighting functions of the same individual for different sources of uncertainty based on the property of subadditivity. Wu and Gonzales (1996) propose and test preference conditions (stronger than subadditivity) that are necessary and sufficient for explaining the concavity-convexity of decision weights. Prelec (1998) suggests a family of so-called "compound invariant functions" that fulfill the empirical requirements of weighting functions. They also show how to generate sub-proportional weighting functions and estimate the inflexion point.

[109] Further estimates are assessed in Camerer and Ho (1994) (i.e. $\gamma = 0.56$), obtained for a power value

Further qualitatively similar formulations are proposed in Prelec (1998)[110] and yield:

$$w^+(p) = w^-(p) = \exp[-(-\ln p)^\epsilon] \tag{3.9}$$

or:

$$w^+(p) = p^\gamma, \quad w^-(p) = p^\delta. \tag{3.10}$$

The respective courses are depicted in Figure 3.3.[111]

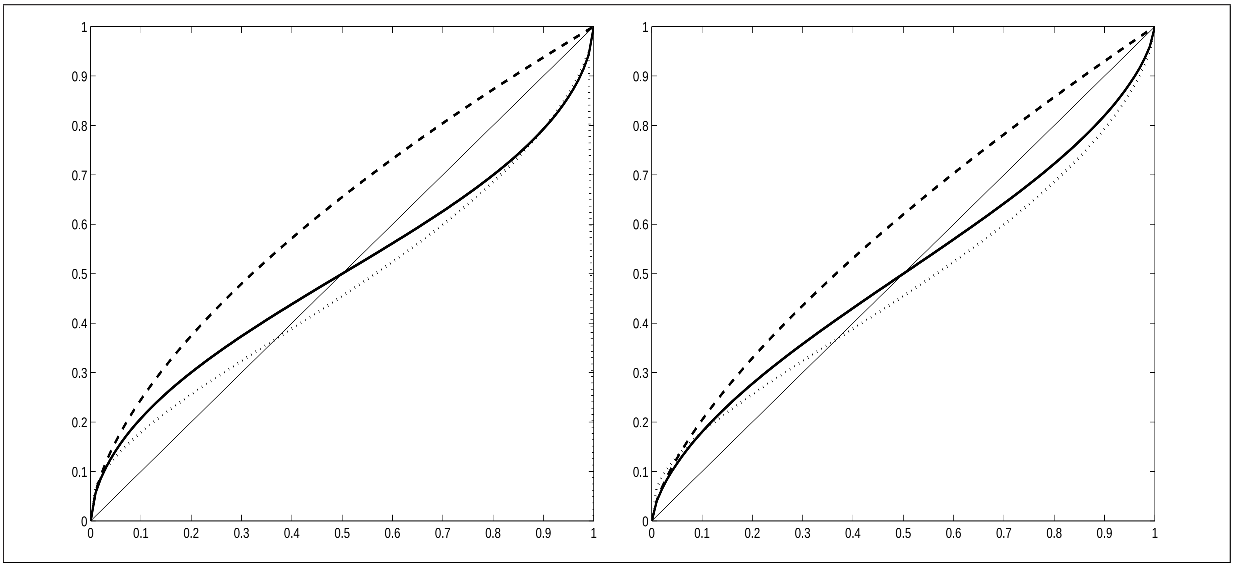


Figure 3.3: Possible PT-weighting for the gain domain w^+ (left panel) and loss domain w^- (right panel) from Equations (3.8) (continuous line), (3.10) (dashed line), and (3.9) (dotted line), for $\gamma = 0.61$, $\delta = 0.69$, $\epsilon = 0.65$

Therefore, in line with the expressions for the value function and the decision weights in Equations (3.18) and (3.7), the prospective value results in:

$$V_i = \pi_i^+ x_i^\alpha - \pi_i^- \lambda(-x_i)^\beta. \tag{3.11}$$

The concept of prospective value is central to the models in Sections 3.2 and 3.3. We rely yet on linear value functions $\alpha = \beta = 1$ and simple probability weights $\pi(p) = p$.

function (x_i^α, where $\alpha = 0.225$).

[110] Prelec (1998) also suggests more general functionals that belong to the same family of compound invariant functions, namely $w^{+/-}(p) = \psi^{+/-} \exp[-\xi^{+/-}(-\ln p)^{\epsilon^{+/-}}]$, but considers the form in Equation (3.9) as more stable with respect to the inflection point over different levels of non-linearity.

[111] Further functionals are estimated in Wu and Gonzales (1996), in particular $p^\gamma/[p^\gamma + (1-p)^\gamma]^\delta$, where $\gamma = 0.721$ and $\delta = 1.565$, and $\delta p^\gamma/[\delta p^\gamma + (1-p)^\gamma]$, where $\gamma = 0.68$ and $\delta = 0.84$.

3.1.3 Loss aversion and Myopia

As stressed above, one of the main innovations of PT is to introduce the notion of **loss aversion** as a common feature of decision making under uncertainty. This concept stands for the empirically observed asymmetric impact of losses and gains on behavior.[112]

In general, loss aversion is modeled in parallel with risk aversion. According to Rabin (2000a,b), this is necessary as EUT cannot properly describe the risk aversion over modest stakes.[113] Two main reasons speak in favor of this assertion and became thus the main points in which PT disagrees from EUT: First, the investor risk attitude cannot be fully described by means of a sole parameter, which is the risk aversion.[114] This idea underlies in PT to the formulation of the value function. Second, preferences are not necessarily linearly dependent on the outcome probabilities, an idea that has fostered the introduction of the probability weighting functions.

Köbberling and Wakker (2005) consider loss aversion as a natural component of the risk attitude. It can be clearly delimited from two further components, namely the utility (expressed by the value function) and the probability weighting (reflected in the decision weights). Being related to the kink of the value function at the origin, the loss aversion can be measured by the ratio of the left and right derivatives of the value function at the reference point. This ratio corresponds to the coefficient λ of the PT-value function in Equation (3.18) and is referred to as the *index of loss aversion*.[115] For linear specifications of the value function, the index of loss aversion is identical to the loss aversion coefficient λ.

Various *estimators* of the coefficient of loss aversion λ are to be found in the literature. The first of them is provided in Tversky and Kahneman (1992). It is inferred on the basis of an experimental sample of untrained students and, under the assumption of a power value function as in Equation (3.18), amounts to $\lambda = 2.25$. Moreover, Benartzi and Thaler (1995) assess the parameter λ of a piecewise linear value function[116] to be 2.77. The estimations rely on Monte Carlo replications of real market data – of stocks, bonds, and treasury bills – between 1926-1990. In the context of an original procedure designed

[112]In other words, the decrease in utility generated due to the marginal loss is higher in absolute value than the increase from a marginal gain of the same size. As underlined in Hastie and Dawes (2001), losses hurt more than gains (specifically twice as much, since the estimated $\lambda > 2$).

[113]In particular, EUT predicts either absurdly high risk aversion over large stakes, or risk-neutrality over small stakes.

[114]The risk aversion is fully described in the EUT framework by the marginal utility of wealth.

[115]Note that the index of loss aversion cannot be defined for CRRA utility functions, but is well compatible to CARA specifications.

[116]That is equivalent to the expression in Equation (3.18), where $\alpha = \beta = 1$.

to test for loss aversion, Schmidt and Traub (2002) obtain $\lambda = 1.43$ from experimental data. The same estimate amounts to 2.87 when merely loss averse choices are considered. The subjects (students) are assumed to perceive choice alternatives according to CPT, i.e. with the specific value function and non-linear decision weights.[117] Furthermore, the loss aversion coefficient obtained by Berkelaar, Kouwenberg, and Post (2004) amounts to 2.50 for a piecewise concave power utility function with a kink at the reference point.[118] For the same utility function but as a result of simultaneous GMM estimation of both the loss aversion parameter λ and the risk aversion parameter γ, they derive $\lambda = 2.711$. Contrary to the previously mentioned studies that draw back on numerical estimation, Berkelaar, Kouwenberg, and Post (2004) derive closed-form solutions for the optimal portfolio of the representative loss averse agent in a discrete one-period equilibrium setting.[119] Their estimations are based on historical US stock market data between 1927-2002.

The concept of loss aversion has been further extended in the literature in order to (better) accommodate with empirically observed anomalies such as the equity premium puzzle.[120] One popular extension introduced in Benartzi and Thaler (1995), and referred to as **myopic loss aversion** (abbr. mLA), relies on the joint effect of loss aversion and narrow framing. Once more, loss aversion refers to the increased discomfort generated by losses compared to the pleasure of making gains of the same size. The *narrow framing* represents an aspect of *mental accounting*,[121] that accounts for the tendency to evaluate performance more frequently. This tendency results from the excessive attention paid to financial prospects.[122] Thaler, Tversky, Kahneman, and Schwartz (1997) define myopic investors as traders who narrowly frame both decisions and outcomes. The former type of narrow framing manifests with respect to short-term choices and the latter one concerns frequent evaluations of performance. For example, regarding the criticism expressed

[117] A further interesting conclusion in Schmidt and Traub (2002) is that loss aversion may be due to the extent rather than to the occurrence of loss averse choices. In the considered sample, women appear to perform more and higher loss averse choices compared to men.

[118] Specifically, this function is $U = \lambda W^\gamma/\gamma + (1-\lambda)\theta^\gamma/\gamma$, for $W \leq \theta$ and W^γ/γ, for $W > \theta$, where $\gamma = 1$.

[119] More details on this model are given at the end of this section.

[120] This anomaly is first revealed in Mehra and Prescott (1985) and refers to the reluctance to invest in stocks in spite of their higher premium relative to bonds. In EUT terms, this corresponds to a tremendously high risk aversion and hence contradicts the estimates obtained from historical market data.

[121] Introduced in Thaler (1980), mental accounting denotes the perception (i.e. the encoding and evaluation) of current and future financial outcomes in mentally distinct partitions to which different levels of utility are assessed. Due to the presence of loss aversion, the aggregation of these different accounts over outcomes (cross-sectionally) and over time is dynamic.

[122] According to Kahneman and Lovallo (1993), the isolated evaluation of single risky prospects prevents the reduction of risk obtained by pooling together different risk sources.

above that EUT cannot give a clear account of the risk-averse behavior with respect to small risks, Rabin and Thaler (2001) stress that this behavior precisely originates in the excessive focus on small risks due to their isolated consideration.

There has been numerous attempts to study the (myopic) loss aversion and its implications. Thus, Benartzi and Thaler (1995) show how this phenomenon can explain the equity premium puzzle[123] and becomes manifest not only for individual investors, but also for institutions, such as pension funds, or foundations and endowment funds. In their work, one year is considered to be the optimal frequency with which investors check their risky portfolios.[124] For a piecewise linear value function and linear probability functions, they show that this frequency explains the empirical equity premium of 6.5% per year. Simulating returns from real market data, Benartzi and Thaler (1995) also provide evidence for how the prospective utility of pure stock portfolios overpasses the one of bonds only for revision frequencies higher than eight months. Finally, they derive the composition of an optimal portfolio of risky and risk-free assets that, for an evaluation horizon of one year, amounts to approximatively $30 - 55\%$ stocks.

Furthermore, Barberis, Huang, and Santos (2001) investigate the price evolution in an aggregated equilibrium setting where investors derive utility from two sources: consumption and fluctuations in financial wealth. They account therefore for a narrow framing that gives rise to the fact that financial investments are considered as an equally important determinant of individual utility besides consumption. The loss aversion in the original PT terms – described by the coefficient λ – is extended in order to allow for the influence of past performance, i.e. for past series of gains and losses resulting from past market movements. This extension provides for explaining the empirically found equity premium.[125] The results are extended in Barberis and Huang (2004a,b) for the case when non-expected – instead of expected – utility is maximized.[126]

[123] Benartzi and Thaler (1995) stress that loss aversion determines investors to shift their attention from utility of consumption to utility of returns. Thus, higher returns are demanded in exchange for the higher volatility perceived in consequence of more frequent portfolio evaluations.

[124] Intuitively, yearly evaluations are motivated by the fact that individual investors fill taxes annually and most reports (of brokers, mutual funds, retirement accounts, etc.) are issued with this frequency.

[125] In essence, they derive aggregate equilibrium prices first for the case when consumption and dividends follow identical processes, then for the more realistic situation with distinct but correlated processes. The model generates high returns as well as excessive return volatility, long-term predictability in returns, high equity premiums, and weak correlation of returns and consumption growth, as observed in practice. The results are tested for a wide range of parameter values and starting from US stock and bond prices, as well as consumption data between 1926-1995. This setting underlies our theoretical framework and will be detailed in Section 3.2.

[126] As shown in Barberis, Huang, and Thaler (2006), the maximization of non-expected utility complies better with the common empirical finding that people who reject small-stakes gambles accept riskier

McQueen and Vorkink (2004) adapt the preference-based equilibrium model in Barberis, Huang, and Santos (2001) in order to capture the investor sensitivity to news.[127] Thus, they explain the volatility clustering of low-frequency returns,[128] as well as other stylized facts on returns, such as the time-varying excess returns, the high risk premia, and the skewness.[129]

Several direct experiments confirm mLA. Thus, Thaler, Tversky, Kahneman, and Schwartz (1997) design an experimental setting that allows to perform separate tests on loss aversion and myopia. The results of their tests confirm these two phenomena, as well as the role of loss aversion vs. risk aversion.[130] At the same time, Gneezy and Potters (1997) directly test mLA on a group of students whose evaluation horizon (i.e. degree of myopia) is manipulated through the supplied feedback information. Their results are extended to a competitive (experimental) setting in Gneezy, Kapteyn, and Potters (2003), where agents face each other by trading units of one risky asset during several successive

large-stakes gambles. The latter from the following three possible specifications is found to perform well: recursive utility with expected-utility certainty equivalent; non-recursive utility with non-expected utility, second-order risk averse certainty equivalent; and non-recursive utility with non-EU, first-order risk averse certainty equivalent. Yet on average, if narrow framing is not taken into account, all of these specifications appear to have difficulties in explaining the attitude to large- vs. small-scale risks. Considering both narrow framing and non-expected recursive utility with first-order risk aversion proves to be sufficient for elucidating the stock market participation and the equity premium puzzles. Barberis and Huang (2004a,b) formalize these ideas in an equilibrium setting and apply them for deriving optimal consumption and portfolio choices. To this end, they rely on the data set in Barberis, Huang, and Santos (2001) and perform robustness checks for various values of the model parameters.

[127] In particular, investors derive utility from consumption and changes in financial wealth, as in Barberis, Huang, and Santos (2001). In addition, the financial wealth and its utility depend not only on loss aversion, but also on a so-called mental scorecard that captures the attentiveness payed by investors to news that affect their portfolios. Positive (negative) news surprises increase (decrease) the scorecard, and there is gradual adaptation to new scorecard levels. Investor responses to news depend further on the past portfolio performance, so that cumulated shocks of the same sign make them more attentive to subsequent shocks.

[128] The tests are performed for monthly and quarterly data.

[129] The equilibrium equations are derived and the model numerically solved for values of the observable parameters that are in line with previous work, and for model-specific parameters that best match historical data. The model predicts that negative news pose a triple threat to prices: by themselves and by the resulting increases in risk aversion and in sensitivity to news. In contrast, positive shocks represent just a double threat, since the resulting price increase is counterbalanced by the higher expected volatility. This explains the asymmetric impact of news on volatility. The conditional expected returns decrease subject to increasing scorecards, while the expected standard deviation increases more for more negative (than for more positive) scorecards and is minimal for zero scorecards. The skewness also increases with news surprises, namely more for negative than for positive shocks. It also varies with the loss aversion coefficient, the memory of the scorecard, and the sensitivity to past performance. For simulated (exponential GARCH) returns, the model is proved to perform better than other approaches in explaining volatility clustering. Finally, volatility predictions and the performance of the chosen scorecard equation in explaining conditional excess returns, clustered volatility, and skewness, yield good performance according to several different tests.

[130] Myopia is controlled by compelling subjects to hold their investments for certain periods and by providing information on outcomes with different frequencies, such as one month, one year, and five years. The experience of losses that can induce risk aversion is eliminated by translating all outcomes into the gain domain.

auctions. The results reinforce the existence of mLA and point out a positive impact on prices.[131] Haigh and List (2005) adapt the setting in Gneezy and Potters (1997) in order to accommodate for the behavior of real players in financial markets. Their combined lab-field experiment compares the behavior of professional futures and option traders at the Chicago Board Exchange with the one of students. Apparently, professional traders suffer from mLA to an even greater extent than students.[132]

However, these empirical tests investigate mLA by isolating it from other decisional elements of potential importance. New research replicating some of the empirical studies mentioned above reach different conclusions in this regard. Thus, Langer and Weber (2005) demonstrate theoretically[133] and provide empirical evidence for the fact that the interaction of different PT-factors such as loss aversion (formalized in Equation (3.18) as $\lambda > 1$), diminishing sensitivity (expressed by the condition $\alpha < 1$), and non-linear probability weights (π) can entail situations when myopia causes an *increase* of the willingness to invest.[134] In addition, Blavatskyy and Pogrebna (2005) draw attention to the fact that the non-linear perception of probabilities manifests simultaneously but contrary to loss aversion and can even reverse the effect of the latter when probability distortions are sufficiently pronounced.[135] Also, Fellner and Sutter (2005) conclude that mLA depends

[131] In particular, prices are found to be higher when the evaluation is aggregated over longer time intervals.

[132] The difference in behavior between more and less frequent feedback is found to be statistically significant for all traders and only for a part of the students. This result is supported by further panel regressions.

[133] They draw back on the theoretical results in Langer and Weber (2001) who prove that lotteries with low probabilities of high losses (e.g. bank loans) can be perceived as less attractive in aggregated than in segregated presentation. This contradicts the general finding that segregation entails unfavorable valuations, which intuitively complies with mLA. In particular, Langer and Weber (2001) use the original CPT value function that exhibits both loss aversion and diminishing sensitivity, i.e. $\alpha = \beta = 0.88$ and $\lambda = 2.25$ in Equation (3.18). They study two-outcome lotteries with a fixed difference (of 2500) between outcomes. The theoretical results receive support from simulations and two empirical studies and are extended for larger lottery portfolios.

[134] Accordingly, myopia increases the attractiveness of investments with low probabilities of high losses, such as bonds. The empirical support of this finding consists of feasible combinations of parameters (α, λ) generated for lotteries considered in previous empirical studies such as Samuelson (1963), Gneezy and Potters (1997), as well as by the direct reinterpretation of the results of these studies. However, a student experiment directly conducted in Langer and Weber (2005) suggests that lotteries with high gain probabilities but small gain sizes are preferred when the feedback is more frequent, which comes in line with the results in Gneezy and Potters (1997).

[135] In particular, their critic addresses the experimental studies (conducted in the laboratory) of Gneezy and Potters (1997) and those (implemented in the field with professional traders) of Haigh and List (2005). These studies find evidence of mLA considering piecewise linear value functions and simple probability weighting. These appear to be at odds with the random sampling of subjects, but can be fully explained if non-linear probability weighting functions, as in CPT, are taken into account. In essence, mLA and non-linear probability weighting exhibit opposed effects, where the latter prevails for high probability distortions (i.e. for high curvature parameters of the weighting functions). In general, the combined effect of loss aversion and non-linear probability weights can become non-linear.

not only on the feedback frequency, but also on the length of the investment horizon (i.e. of the investment flexibility).[136] However, an experimental test meant to disentangle between feedback frequency and investment flexibility conducted in Bellemare, Krause, Kröger, and Zhang (2005) confronts with these findings indicating that mLA can be rather attributed to the feedback frequency as assumed in earlier empirical tests.

Further works address the formation of an *optimal portfolio for loss averse investors* in specific equilibrium settings. In this vein, Gomes (2005) develops a general equilibrium model where traditional risk averse investors (i.e. with CRRA power utility) meet loss averse investors (with PT-utility).[137] He shows that loss-averse investors switch between a low-wealth strategy (with possibly nil risky investments) and a high-wealth strategy (that resembles the portfolio insurance commonly used in practice). Thus, they formulate discontinuous demands on risky assets. Moreover, the presence of loss-averse investors yields an increased trading volume that attains its maximum at the strategy switching point. The trading volume is shown to be positively (negatively) correlated with return volatility when loss averse investors follow the portfolio insurance strategy (the low-wealth strategy).[138] This model provides possible explanations for several puzzling phenomena observed in practice, such as the low rate of stock market investments, the use of portfolio insurance strategies, or the disposition effect.[139]

Moreover, Berkelaar, Kouwenberg, and Post (2004) provide an exact solution on portfolio optimization in complete markets with continuous prices and loss-averse investors.

[136] Their experimental setup accounts for two treatments: an exogenous one, similar to Gneezy and Potters (1997), and an endogenous one, where the subjects can choose between long and short revision horizons. When the choice can be made once at the beginning of the trade, subjects prove to be indifferent between the two alternatives on average. When subjects are first faced with a default horizon and then given the possibility of switching at a fixed cost, they mostly prefer to switch (specifically, they switch sooner to the shorter horizon when the longer horizon is given by default). This reduces – but does not fully eliminate – the effects of mLA. Another finding is that subjects with endogenous investment horizons (i.e. with the possibility of switching) react to previous performance, that is more intensely compared to the case when the investment horizon is predetermined. Specifically, subjects in the endogenous treatment respond positively (negatively) to a higher total number of previous gains (number of gains in the previous three rounds).

[137] In particular, an extended definition of the value function is used here. It adds a domain of extremely high losses to the original gains and loss domains. For extreme losses, the decreasing marginal utility of consumption dominates the psychological effect of losses. Also, Gomes (2005) considers the implication of a dynamic reference that is designed as the linear combination of the last-period reference point and the current wealth.

[138] Specifically, the optimal portfolios of both investor categories are first derived in a static one-period setting with binomial risky returns for four different cases: CRRA utility, and an extended value function for zero, negative, and positive surplus wealth. Subsequently, equilibrium returns and volumes are computed in a more general setting (first with two, then with T periods) and compared between the cases when investors have low and high surplus wealth.

[139] Shefrin and Statman (1985) define this effect as the disposition to sell winning stocks too early and to hold losers too long.

The loss aversion is is formalized by the kink of the utility function at the origin. Hence, it can be accounted for in twofold manner: first through a concave kinked power utility, and second by the two-piece power function of PT.[140] The results are further extended for dynamic (i.e. stochastic) reference points.[141] The wealth percentage invested in risky assets is shown to decrease, for sufficiently high initial wealth, as investors approach the end of the trading horizon. Subsequently, the same authors estimate the aggregate level of loss aversion from historical US market data by first fixing the level of risk-aversion and then using a joint estimation GMM procedure.[142] Empirically, loss aversion cannot be distinguished from risk-aversion, i.e. the two concepts can be considered to be empirical substitutes.

Finally, Berkelaar and Kouwenberg (2006) extend the setting in Berkelaar, Kouwenberg, and Post (2004) and prove that the presence of myopic loss averse investors entails both high levels and high changes in volatility of equilibrium prices. However, the loss aversion heterogeneity – given by different initial wealth levels – appears to smooth out these extreme movements.[143]

[140] The derivation relies on the martingale methodology that allows to reexpress the dynamic optimization problem in static terms. The static problem is solved following the technique for non-concave and non-differentiable problems implemented in Basak and Shapiro (2001). In good states of the world, the kinked-power utility investors behave similar to the classic CRRA investors, while the wealth of loss averse investors with PT-utility drops discontinuously to zero. In intermediate and bad states, both types of loss averse investors act like portfolio insurers. Also, for both investors, the wealth fraction invested in risky assets follows a U-pattern, being higher in extreme (bad or good) states of the world.

[141] They restate the optimization problem with dynamic reference into an equivalent one with static reference.

[142] To this end, they apply a discrete one-period representative agent setting. The loss aversion coefficient can be derived from the ratio of the upside and downside expectations of excess returns. The estimate derived for a fixed risk aversion level amounts to $\lambda = 2.50$. However, as loss aversion appears to decrease subject to higher risk aversion, they also apply a simultaneous estimation procedure for both parameters and obtain non-significant simultaneous estimates, but a significant $\lambda = 2.711$ for $\gamma = 1$ and a significant $\gamma = -0.916$ for $\lambda = 1$.

[143] Their setting represents a pure-exchange economy with continuous prices. First, Berkelaar and Kouwenberg (2006) consider the case when loss averse agents exhibit homogenous reference points, i.e. have identical levels of initial wealth. The price evolution follows a threefold pattern ranging from extremely high to extremely low levels in good and bad economic states, respectively, while the volatility takes the opposite course. In the sequel, the model is extended in order to allow for heterogeneity, in that it considers two distinct groups of loss-averse agents. Accordingly, heterogenous reference points cannot be acceptably substituted by one representative loss averse investor. The price and volatility changes are now lower, yet the volatility is higher on average.

3.2 One-dimensional utility: risky vs. risk-free financial assets

3.2.1 Introduction

The main concern of investors in financial markets is how to optimally allocate money among different types of assets. Portfolio theory teaches us that the optimal allocation results from the maximization of expected portfolio returns subject to given levels of market risk. In spite of the appealing intuition, such an optimization is not an easy task, especially for laymen. The reason is that it involves the selection from a huge variety and quantity of financial instruments existent in practice, and it often requires advanced mathematical skills. The natural response of real financial environments to this difficulty has been the specialization of the investment activity between professional and non-professional investors. Non-professional investors – in other words people whose main occupation does not concern financial investing and/or who lack the necessary knowledge, expertise, time, or any combination of them for making more sophisticated investment decisions – rely often on the help of professional portfolio managers in devising an optimal mix of risky assets. In other words, they often delegate the security and asset allocation to professional managers.[144]

In particular, one can think of the decision process of non-professional investors as unfolding in two main steps: First, they determine the total sum of money to be invested in financial markets (in technical terms, the budget constraint). Second, in order to optimally split this money among different financial instruments, they ask for professional advice. In so doing, non-professional investors commit the technical details of the optimization of their asset portfolio to professional managers, who dispose of sufficient resources to this end. Moreover, non-professional investors provide managers with information about the level of risk they are ready to bear (the risk constraint). Acting on this information,

[144] In essence, this practical tendency of work division between non-professional investors and professionals conforms with portfolio theory. According to the top-down strategy, portfolio optimization can be described by means of a threefold decision procedure: A first step, referred to as the *capital allocation decision*, deals with the choice between risky and risk-free assets. A second so-called *asset allocation decision* focuses on the further selection of different classes of risky assets. The third *security allocation decision* is concerned with the specific securities to be held within each particular risky asset class chosen before. In practice, the last two decisions are usually made by professional portfolio managers with no intervention of their (non-professional) clients. In contrast, as far as the first decision is concerned, the participation of these non-professional investors becomes necessary, since it allows to managers to determine the capital allocation that best fits the individual risk-profiles of their clients.

managers finally derive the optimal capital allocation for their particular clients. What is important for non-professional investors in this context is simply how their wealth can be (optimally) split between risky and risk-free assets.

It is the *decision process of the non-professional investors* that our work focuses on. This process – although of indisputable practical importance – has been somewhat neglected in the literature so far. The extensive research on portfolio optimization deals with more sophisticated details, such as of choosing among different categories of risky assets, that we consider to usually be the responsibility of the portfolio managers.

In particular, we are interested in this section in how non-professional investors split their money between risky and risk-free assets. Since this decision depends on the individual risk profile, we also study the investor attitude towards financial losses. Note that our work does not contribute to the understanding of professional investor decisions, but gives insight into how non-professional ones "operate" on financial markets. In our setting, non-professional investors start from questioning what is their acceptable monetary loss from risky investments. This information depends on individual risk profiles that affect the quantity of money that they are going to invest in risky assets. Also, the frequency of evaluating risky portfolio changes the risk profile and hence their overall performance.

Our work extends the portfolio optimization setting in Campbell, Huisman, and Koedijk (2001), where risk is quantified in form of Value-at-Risk, by explicitly accounting for the formation of what we denote as the *individual VaR* (abbr. VaR*). Our VaR* relies on the subjective perceptions of non-professional investors and expresses the maximum loss that is acceptable for each individual. It is formulated in line with the extended prospect theory in Barberis, Huang, and Santos (2001).

We first analyze how non-professional investors set their subjective VaR*, specifically contingent upon their loss aversion, the past performance of the risky portfolio, the current value of the risky investment, and the expected risk premium. We show how VaR* flows into the portfolio optimization undertaken by the professional manager in form of a risk constraint. We derive the optimal wealth percentages to be invested in the risky portfolio and in risk-free assets and study how they vary in time and subject to different portfolio evaluation frequencies. Furthermore, we introduce an extended measure, termed as the global first-order risk aversion (gRA), that attempts to provide additional information concerning the loss attitude of non-professional investors. We comment on how the frequency of evaluating risky performance can directly and indirectly impact the investor attitude towards risky investments and on how this twofold influence can be estimated.

Our theoretical results are supported and amended by findings relying on the S&P 500 index and the US three-month treasury bills between 1982-2006. The past performance appears to drive the current perception of the risky portfolio. Investors allocate on average between 1-35% of their wealth to risky assets, where the main source of this substantial fluctuation is the portfolio evaluation frequency. The proportion of risky investments decreases fast when portfolio performance is checked more often than once a year, which complies with the notion of myopic loss aversion introduced in Benartzi and Thaler (1995).

Furthermore, we conduct an extended analysis of the perceived utility of the risky portfolio and of the loss attitude in what we denote as the evaluation-frequency domain. Specifically, we propose a twofold segmentation in dependence on the portfolio evaluation frequency of both the prospective value and the global first-order risk aversion. Only evaluation frequencies higher than once a year are of practical relevance. In this context, both variables suggest annual evaluations as being optimal for generating positive attitudes towards risky investments.

Finally, variables aimed at providing an equivalence between the traditional VaR-approach and the estimates in our VaR*-framework – such as equivalent significance levels, loss aversion coefficients, and investments in risky assets – point out that the actual reluctance towards financial losses of non-professional investors might be underestimated.

The remainder of this section is organized as follows: Section 3.2.2 presents the main theoretical considerations. We start with a brief review of the optimal portfolio selection model with exogenous VaR* by Campbell, Huisman, and Koedijk (2001), then take on the value function formulation in Barberis, Huang, and Santos (2001). The notion of VaR* is subsequently introduced. Finally, concentrating on how individual investors perceive the value of the risky portfolio, we derive the prospective value and introduce our extended measure of loss aversion. Section 3.3.3 illustrates the implementation of our theoretical model. We discuss the impact of the evaluation frequency and of the past performance on the wealth percentages invested in the risky portfolio. Also, we extensively investigate the evolution of the prospective value and of the extended loss-attitude measure subject to various evaluation frequencies. Our model is further restated in terms of established models with VaR risk constraints. Section 3.2.4 summarizes the results and concludes. Mathematical proofs and further findings are included in Appendix A.3.2.

3.2.2 Theoretical model

This section contains the main theoretical considerations of our work. We start by reviewing the portfolio selection model in Campbell, Huisman, and Koedijk (2001). This model uses VaR as its measure of risk. Our own setting, subsequently formulated, incorporates the individual perception of risky projects as captured in the extended prospect theory framework of Barberis, Huang, and Santos (2001). We detail the construction of our measure of individual loss aversion VaR* and its implications for the wealth allocation decisions of non-professional investors. We also add to the formal representation of investor attitudes towards financial losses by introducing the notion of global first-order risk aversion (gRA). Moreover, we briefly address how the prospective value and this extended loss-attitude measure may vary subject to different portfolio evaluation frequencies.

Optimal portfolio selection with "exogenous" VaR

The model in Campbell, Huisman, and Koedijk (2001) follows the common procedure of portfolio optimization, where market risk is assessed by means of Value-at-Risk (abbr. VaR). In particular, financial assets are chosen in order to maximize expected returns, subject to a twofold restriction: the budget and risk constraints. Investors can borrow or lend extra money at the fixed market interest rate – which is equivalent with an investment in risk-free assets. The maximum expected loss from holding the risky portfolio should not exceed what we call an *exogenous* VaR (abbr. VaR^{ex}). This VaR^{ex} stands for the risk level that the non-professional client is disposed to bear. It is indicated to the portfolio manager in form of a single, fixed number.[145] In this model, managers do *not* account for how VaR^{ex} forms in the client perception. They consider it as constraint, exogenous to the optimization problem.[146]

The objective of the optimization problem in Campbell, Huisman, and Koedijk (2001) is maximizing the next-period wealth W_{t+1}. This wealth results from what the components of the risky portfolio and the risk-free assets are expected to return. The risky portfolio consists of $i = 1, \ldots, n$ financial assets with single time-t prices $p_{i,t}$ and portfolio weights $w_{i,t}$, such that $\sum_{i=1}^{n} w_{i,t} = 1$. Moreover, $a_{i,t}$ is the number of shares of the asset i

[145] The VaR^{ex} further enters the portfolio optimization problem in form of a threshold level, being thus *exogenous* to it.

[146] Specifically, managers interpret the client indication (a single number) in terms of the theoretical concept of VaR, i.e. of two elements: a confidence level and an investment horizon.

contained in the portfolio at time t.[147] Formally, we can state the portfolio optimization problem as follows:

$$W_{t+1}(w_t) = (W_t + B_t)E_t[R_{t+1}(w_t)] - B_t R_f \underset{w_t}{\longrightarrow} \max. \tag{3.12}$$

$$\text{s.t.} \quad \begin{aligned} & W_t + B_t = \sum_{i=1}^{n} a_{i,t} p_{i,t} = a_t' p_t \quad \text{(budget constraint)} \\ & P_t[W_{t+1}(w_t) \leq W_t - \mathrm{VaR}^{\mathrm{ex}}] \leq 1 - \alpha \quad \text{(risk constraint)}, \end{aligned} \tag{3.13}$$

where $R_{t+1}(w_t)$ stands for the portfolio gross returns at the next trade and $E_t[R_{t+1}(w_t)]$ for the corresponding expected returns. Henceforth, we refer to the gross returns of the risky portfolio by "returns" or "portfolio returns" .

In the above Equations (3.12) and (3.13), B_t denotes the risk-free investment, i.e. the sum of money that can be borrowed ($B_t > 0$) or lent ($B_t < 0$) at the fixed risk-free gross return rate R_f. Note that the maximization in Equation (3.12) is carried over the weights of the risky portfolio w_t but *not* over B_t. The risk-free investment results as a by-product of the optimization procedure.[148] Finally, P_t denotes the conditional probability given the information at time t, and $1 - \alpha$ the chosen confidence level.

After some manipulations, Campbell, Huisman, and Koedijk (2001) obtain the optimal weights of the risky portfolio as:

$$w_t^{\mathrm{opt}} \equiv \arg\max_{w_t} \frac{E_t[R_{t+1}(w_t)] - R_f}{W_t R_f - W_t q_t(w_t, \alpha)}, \tag{3.14}$$

where $q_t(w_t, \alpha)$ represents the quantile of the distribution of portfolio gross returns $R_{t+1}(w_t)$ for the confidence level $1 - \alpha$ (or significance α), i.e. $P_t[R_{t+1}(w_t) \leq q_t(w_t, \alpha)] \leq 1 - \alpha$. Thus, the optimal mix of risky assets depends merely on the distribution of the portfolio gross returns and on the significance level α.

Equation (3.14) shows that, similarly to the traditional mean-variance framework, the *two-fund separation theorem* applies: Neither the (non-professional) investors' initial wealth nor the desired risk level $\mathrm{VaR}^{\mathrm{ex}}$ affects the maximization procedure. In other words, investors first determine the optimal risky portfolio (i.e. the optimal allocation among different risky assets) and second, they decide upon the extra amount of money

[147] Clearly, $a_{i,t} = w_{i,t}(W_t + B_t)/p_{i,t}$.
[148] See the comments concerning the two-fund separation below.

to be borrowed or lent (i.e. invested in risk-free assets). The latter reflects by how much the portfolio VaR, that is defined as:

$$\mathrm{VaR}_t = W_t\Big(q_t(w_t^{\mathrm{opt}}, \alpha) - 1\Big), \tag{3.15}$$

varies according to the investor degree of loss aversion measured by the selected (desired) $\mathrm{VaR}^{\mathrm{ex}}$-level.[149]

The optimal investment in risk-free assets can be then written as:

$$B_t = \frac{\mathrm{VaR}^{\mathrm{ex}} + \mathrm{VaR}_t}{R_f - q_t(w_t^{\mathrm{opt}}, \alpha)}, \tag{3.16}$$

and hence the value of the risky investment at time $t + 1$ yields:

$$S_{t+1} = (W_t + B_t)R_{t+1}. \tag{3.17}$$

Since we consider that non-professional investors are mainly concerned with how to split their money between risky and risk-free assets, the optimal investments in risk-free and risky assets in Equations (3.16) and (3.17) represent fundamental variables in our model. Note that we do not further elaborate on the optimal weights of the risky assets in Equation (3.14), as the details of wealth allocation among the different risky portfolio components are assumed to be left in charge of portfolio managers.

The individual loss level VaR*

Coming from the main ideas of the setting in Campbell, Huisman, and Koedijk (2001), our model goes a step further by asking how non-professional investors actually arrive at their desired level of loss aversion. We elaborate on the construction of an *individual loss level*, that we denote as VaR*, and on its implications for the wealth allocation between risky and risk-free assets. As far as the optimization procedure presented above is concerned, we can think of VaR* formally replacing $\mathrm{VaR}^{\mathrm{ex}}$ in the above equations, but remaining an exogenous input (or constraint). However, the value of this risk constraint forms in our approach on the basis of individual behavioral parameters and affects the final

[149] Note that $\mathrm{VaR}^{\mathrm{ex}}$ is imposed by the client *prior* to the portfolio formation and enters the portfolio optimization problem in form of a constraint. In contrast, the portfolio VaR is an *output* of this optimization and measures the actual maximum loss that can be incurred at time t at the confidence level $1 - \alpha$ for the obtained optimal portfolio w_t^{opt}.

wealth allocation between risky and risk-free assets, as apparent from Equation (3.16). This extension of the allocation problem motivates us to denote VaR* as the *endogenous* individual loss level.[150]

The value function Investors' desires and attitudes – hence their subjective loss level VaR* – depend on their perception of the value of financial investments. The prospect theory (abbr. PT) in Kahneman and Tversky (1979) and Tversky and Kahneman (1992) suggests how individual perceptions of financial performance can be formalized by means of the so-called *value function* v.[151] Accordingly, human minds take for actual carriers of value not the absolute outcomes of a project, but their changes defined as departures from an individual reference point. The deviations above (below) this reference are labeled as gains (losses). Thus, the value function is kinked at the reference point and exhibits distinct profiles in the domains of gains and losses, being steeper for losses (a property known as *loss aversion*). It also shows diminishing sensitivity in both domains, i.e. it is concave for gains but convex for losses. More details on the PT-value function can be found in Section 3.1.2.

As noted in Barberis, Huang, and Santos (2001), individual perceptions can be additionally influenced by the past performance of risky investments. This past performance is captured by the *cushion* concept. Formally, the cushion corresponds to the difference between the current value of the risky investment S_t and a historical benchmark level of the risky value Z_t (that can e.g. be the price at which the assets were purchased, a more recent value of the risky holdings, or a combination of them).[152] When this difference is positive, investors made money from investing in risky assets in the past, otherwise they made losses.

Our approach relies on the extended formulation of the value function proposed in Equations (15) and (16) by Barberis, Huang, and Santos (2001). In the following, we refer to $x_t = R_{t+1} - R_{ft}$ as the *risk premium*, to $S_t - Z_t$ as the (absolute) *cushion*, and to $z_t = Z_t/S_t$ as the *relative cushion*. The positive (negative) past performance corresponds

[150] The allocation problem is extended to incorporate not only the portfolio optimization in the strict sense, as performed by managers, but also the earlier decision of non-professional investors with respect to the desired risk level.

[151] Note that the concepts on which we base our setting are not entirely elaborated until the CPT of Tversky and Kahneman (1992). Since we are not particularly interested in the formal details and most of these concepts are already present in the original PT in Kahneman and Tversky (1979), we henceforth refer to both theories by the global denomination of PT.

[152] For more details with respect to the interpretation of Z_t see Barberis, Huang, and Santos (2001), p. 9.

to a positive (negative) cushion that can be termed as $Z_t \leq S_t$ ($Z_t > S_t$) or equivalently as $z_t \leq 1$ ($z_t > 1$). The value function takes different courses in dependence on the past performance and can be expressed as follows:[153]

$$v_{t+1} = \begin{cases} v_{t+1}^{\text{prior gains}} & \text{, for } z_t \leq 1 \\ v_{t+1}^{\text{prior losses}} & \text{, for } z_t > 1, \end{cases} \tag{3.18}$$

where:

$$v_{t+1}^{\text{prior gains}} = \begin{cases} S_t x_{t+1} & \text{, for } x_{t+1} + (1 - z_t) R_{ft} \geq 0 \\ \lambda S_t x_{t+1} + (\lambda - 1)(S_t - Z_t) R_{ft} & \text{, for } x_{t+1} + (1 - z_t) R_{ft} < 0, \end{cases} \tag{3.19}$$

and

$$v_{t+1}^{\text{prior losses}} = \begin{cases} S_t x_{t+1} & \text{, for } x_{t+1} \geq 0 \\ \lambda S_t x_{t+1} + k(Z_t - S_t) x_{t+1} & \text{, for } x_{t+1} < 0. \end{cases} \tag{3.20}$$

The parameter λ in Equations (3.19) and (3.20) is termed the *coefficient of loss aversion*. According to PT, investors are loss averse when $\lambda > 1$, while $\lambda = 1$ points to loss neutrality. The parameter $k \geq 0$ captures the influence of previous losses on the perception of current ones: The larger the previous losses are, the more painful the next losses become. We denote it as the *sensitivity to past losses*.

Note that the gain branches of both value functions in Equations (3.19) and (3.20) are invariable to the past performance z_t. The loss branches are yet distinct. However, they both contain a first term $\lambda S_t (R_{t+1} - R_{ft})$ that resembles the original PT, but also a second one revealing the impact of the cushion $S_t - Z_t$. Also, the reference point shifts in dependence on the past performance.[154]

Henceforth, we use the following probability notations:

$$\begin{aligned} \pi_t &= P_t(z_t \leq 1) \\ \omega_t &= P_t(x_{t+1} \geq 0 | z_t > 1) \\ \psi_t &= P_t(x_{t+1} + (1 - z_t) R_{ft} \geq 0 | z_t \leq 1), \end{aligned} \tag{3.21}$$

---

[153] Where we restate the term in the condition of Equation (15) by Barberis, Huang, and Santos (2001) as $R_{t+1} - z_t R_{ft} = x_{t+1} + (1 - z_t) R_{ft}$.

[154] The reference point can be observed in the conditions of the two value functions in Equations (3.19) and (3.20).

where π_t stands for the probability of past gains, and ω_t for the probability of a positive premium given past losses. Finally, we can term ψ_t as the probability of obtaining a risk premium $x_{t+1} + (1 - z_t)R_{ft}$, higher than the risk premium x_{t+1}, that expresses raised expectations resulting from recurrent gains.

The derivation of VaR* In Equation (3.16), the risk-free investment depends, among others, on the risk level VaR^{ex} indicated by the non-professional client to the portfolio manager. The traditional approach does not account for the way in which non-professional investors ascertain this level. This ascertainment should take place according to individual perceptions of financial losses which can, in line with PT, substantially differ from the actual losses. In this section, we define a new measure of the *individual loss level* (more specifically, the individually accepted or desired loss level) that we denote as VaR*.

In so doing, we start from the literal definition of VaR*: the maximum loss that can be a-priori expected by someone investing in risky assets. We concentrate on the terms "loss", "individual", and "maximum" encompassed by this definition. First, VaR* quantifies *losses*. According to PT, what actually counts for individual (non-professional) investors is not the absolute magnitude of a loss, but rather the subjectively perceived one, as captured by the value function described above. Hence, VaR* should rely on the *subjective value* of losses expressed in the loss branches of the value functions in Equations (3.19) and (3.20). It thus depends on individual features, originating in the subjective view over gains and losses, and can vary over time, for instance with the past performance of risky investments. Moreover, we are looking for a *maximal* value. This is obtained in that, in calculating VaR*, investors ascribe a maximal occurrence probability (of 1) to the losses in the value function, so that $\pi_t(1-\psi_t) + (1-\pi_t)(1-\omega_t) \stackrel{!}{=} 1$.[155] Finally, VaR* should correspond to the concept of Value-at-Risk and hence represent a *quantile*, namely, according to the above considerations, a quantile of the subjective loss distribution.

[155] This condition requires that the *absolute* probability of making a loss, i.e. *irrespective of the prior performance*, is one.

Therefore, we suggest the following formal definition for the individual loss level:[156]

$$
\begin{aligned}
\mathrm{VaR}^*_{t+1} &= E_t[\text{loss-value}_{t+1}] - \varphi\sqrt{Var_t[\text{loss-value}_{t+1}]} \\
&= \lambda S_t E_t[x_{t+1}] - kE_t[x_{t+1}](S_t - Z_t) \\
&\quad + \sqrt{\pi_t(1-\psi_t)}\Big(\sqrt{\pi_t(1-\psi_t)} - \varphi\sqrt{1-\pi_t(1-\psi_t)}\Big)\Big((\lambda-1)R_{ft} + kE_t[x_{t+1}]\Big)(S_t - Z_t) \\
&= \lambda S_t E_t[x_{t+1}] + \Big(\zeta_t(\lambda-1)R_{ft} + (\zeta_t - 1)kE_t[x_{t+1}]\Big)(S_t - Z_t).
\end{aligned}
\tag{3.22}
$$

where "loss-value" stands for the subjective value ascribed to financial losses according to the loss branch of the value functions in Equations (3.19) and (3.20), and the subjectively perceived losses are assumed to follow a distribution (e.g. normal or Student-t) with the lower quantile φ.[157] Moreover, $E_t[x_{t+1}] = E_t[R_{t+1}] - R_{ft}$ denotes the expected risk premium. The last expression in Equation (3.22) is obtained using the simplifying notation $\zeta_t = \sqrt{\pi_t(1-\psi_t)}\Big(\sqrt{\pi_t(1-\psi_t)} - \varphi\sqrt{1-\pi_t(1-\psi_t)}\Big)$.

We distinguish two terms of the VaR*-expression in Equation (3.22): The first one accounts for the expected risky return (relative to the risk-free rate) $S_t E_t[x_{t+1}]$, weighted by the loss aversion coefficient λ. As it consequently resembles the prospective value according to the original PT, we denote this term as the *PT-term*. The last term is responsible for the influence of the previous performance captured by the cushion $S_t - Z_t$ in Barberis, Huang, and Santos (2001). For this reason, we denote it as the *cushion term*. The corresponding weight is a linear combination of the expected risky and the risk-free returns.

Once non-professional investors set their minds about the desired VaR*, they communicate it to the portfolio manager. In the view of the latter, this client indication represents an exogenous risk level that corresponds to $\mathrm{VaR}^{\mathrm{ex}}$ in Equation (3.16) and is applied in order to determine the optimal level of borrowing or lending B_t. When VaR* is lower in absolute value than the portfolio VaR, B_t is negative, which formalizes the profile of more risk-averse investors who prefer to increase the proportion of wealth invested in risk-free assets. In contrast, for a VaR* higher than VaR in absolute value, investors

[156] The derivation of the expectation and the variance of the loss utility is deferred to Appendix A.3.2. Note that we also worked with an alternative specification that fully corresponds to the literal of VaR*, i.e. the maximal *expectation of sustainable losses* $E_t[\text{loss-utility}_{t+1}]$ and hence does not adjust for variance. The simulation results are similar.

[157] Recall that, as already discussed in Section 3.1.1, although VaR is a very popular risk measure, it does not satisfy one of the four properties of coherent risk measures, namely subadditivity. However, VaR becomes subadditive and hence coherent for elliptic joint distributions, such as normal and Student-t with finite variance.

augment their risky investments by borrowing extra money, i.e. they are less risk averse. Thus, analyzing the evolution of B_t (or equivalently of S_t/W_t, as in the subsequent Section 3.3.3) can shed some light on the behavior of non-professional investors confronted with financial losses.

A further interesting topic to investigate lies in estimating the equivalent loss aversion parameter λ_t^* that can be obtained for a fixed $\overline{\mathrm{VaR}^*}$ under the traditional approach.[158] Common assumptions of this approach are significance levels of 1%, 5%, or 10% and no dependency on past performance $k = 0$. The formula of λ^* is then immediate from the definition in Equation (3.22) for $k = 0$.[159] This yields:

$$\lambda_{t+1}^* = \frac{\overline{\mathrm{VaR}^*} + \zeta_t R_{ft}(S_t - Z_t)}{S_t E_t[x_{t+1}] + \zeta_t R_{ft}(S_t - Z_t)}. \tag{3.23}$$

The prospective value of the risky investment

The estimation of the individually maximum acceptable loss level VaR* represents only the first step in our analysis. As discussed above, it directly enters the optimal risk-free investment derived (by the professional manager) as a byproduct of the portfolio optimization procedure. Thus, VaR* dictates the optimal choice of the non-professional investors in terms of wealth percentages allocated between risky and risk-free assets.

However, we are also interested in the attitude of non-professional investors towards financial losses in general, as this attitude influences the level of the individual VaR*. The loss attitude results from the perception of the utility generated by financial investments.[160] The corresponding PT-concept of (subjectively) expected utility is the so-called *prospective value* V and has been discussed in Section 3.1.2.

[158] In other words, the loss aversion that equivalently results under the manager assumption of a fixed, exogenous risk level.

[159] This holds since λ_{t+1}^* depends on the *fixed* $\overline{\mathrm{VaR}^*}$.

[160] In contrast to Barberis, Huang, and Santos (2001), our investors are *not* concerned with consumption and derive utility merely from financial wealth fluctuations.

In our framework, the prospective value of the risky portfolio can be formulated as:[161]

$$\begin{aligned} V_{t+1} &= \pi_t E_t[v_{t+1}^{\text{prior gains}}] + (1-\pi_t)E_t[v_{t+1}^{\text{prior losses}}] \\ &= \pi_t[\psi_t S_t E_t[x_{t+1}] + (1-\psi_t)(\lambda S_t E_t[x_{t+1}] + (\lambda-1)(S_t - Z_t)R_{ft})] \\ &\quad + (1-\pi_t)[\omega_t S_t E_t[x_{t+1}] + (1-\omega_t)(\lambda S_t E_t[x_{t+1}] + k(Z_t - S_t)E_t[x_{t+1}])] \\ &= \Big(\pi_t\psi_t + (1-\pi_t)\omega_t + (\pi_t(1-\psi_t) + (1-\pi_t)(1-\omega_t))\lambda\Big) S_t E_t[x_{t+1}] \\ &\quad + \Big(\pi_t(1-\psi_t)(\lambda-1)R_{ft} - (1-\pi_t)(1-\omega_t)kE_t[x_{t+1}]\Big)(S_t - Z_t). \end{aligned} \tag{3.24}$$

Note the existence of a twofold effect in the prospective-value formula, that is similar to the one discussed for VaR*: The first term of the last expression in Equation (3.24), that we subsequently denote as the *PT-effect*, captures the expected risky-investment value relative to the safe bank investment $S_t E_t[x_{t+1}]$. The corresponding probability weight is the sum of perceived gain and loss probabilities, laxly put $P_t(\text{gain}) + \lambda P_t(\text{loss})$. It points out that, as in PT, losses loom larger than gains, being additionally penalized by the loss aversion coefficient λ.

The last term of the prospective value in Equation (3.24) covers the cushion influence and we refer to it as the *cushion effect*. The weight of the cushion $S_t - Z_t$ is in this term a combination of expected losses obtained under the consideration of the performance history. Specifically, when current losses follow past gains – which occurs with the joint probability $\pi_t(1-\psi_t)$ – the past performance (given by the cushion) is valued at the risk-free rate R_{ft} and is amended by how much the loss aversion coefficient λ exceeds the loss-neutral value of 1. Indeed, if risky investments were successful in the past, a current loss has value only compared to the alternative of having put the entire money in risk-free assets. When losses extend from past to present – where $(1-\pi_t)(1-\omega_t)$ is the joint probability of current and past losses – the valuation implies a comparison of the risk-free rate to the risky performance $-E_t[x_{t+1}]$ in view of the sensitivity to past losses k.

We are interested in the evolution of the prospective value not only in time but also for different portfolio evaluation frequencies. The rationale for this is that revising portfolio performance at different time intervals implies, first, drawing back on distinct return values

[161] This definition follows Equation (3.1.2) in Section 3.1.2, where the weights are taken to be identical to the outcome probabilities. We have also applied a slightly different definition of the prospective value. Accordingly, gains continue to be considered as possible events and hence are weighted by the respective occurrence probability. Losses are instead assessed in what we can call a "worst case scenario", i.e. with maximum probability. This is equivalent to saying that losses are accounted for in the form of VaR*. The obtained results are qualitatively similar to applying Equation (3.24).

(and hence on different risk premia). Second, these return changes implicitly impact, at later times, on further model parameters, such as the cushion and the probabilities of past and current gains and losses. Therefore, the prospective value in Equation (3.24) is affected in multiple ways. We analyze this topic theoretically at the end of this theoretical section and in an applied context in Section 3.3.3.

In so doing, we apply a further notion referring to the investor attitudes towards financial risks that attempts to capture more complex dependencies than the simple coefficient of loss aversion λ. According to PT, loss aversion corresponds to risk aversion of first order in the loss domain. In the same spirit, we term the first derivative of the prospective value with respect to the expected risk premium as the *global first-order risk aversion* (abbr. gRA). Formally, gRA yields:

$$\begin{aligned} \mathrm{gRA}_t = \frac{\partial V_{t+1}}{\partial E_t[x_{t+1}]} = &\Big(\pi_t\psi_t + (1-\pi_t)\omega_t + (\pi_t(1-\psi_t) + (1-\pi_t)(1-\omega_t))\lambda\Big)S_t \\ &- (1-\pi_t)(1-\omega_t)k(S_t - Z_t). \end{aligned} \tag{3.25}$$

Thus, gRA reflects the sensitivity – in terms of first-order changes – of the prospective value to the variation of expected returns (or equivalently to the expected risk premium).[162] Due to the linearity of our prospective value in the expected risk premium $E_t[x_{t+1}]$, gRA is independent of this premium.

Moreover, since gRA directly reflects changes in the prospective value – which is proportional to the attractiveness of financial investments – higher gRA-values point to a more relaxed loss attitude. This can be formally recognized in Equation (3.25): The first term increases with the sum invested in risky assets S_t; The second is inversely proportional to the cushion $S_t - Z_t$. Note yet that this second term accounts for the situation when past losses are followed by current losses, which occurs with the probability $(1-\pi_t)(1-\omega_t)$. In such a case, cushions are most probably negative $S_t - Z_t \leq 0$. Smaller (negative) cushions will then render this second term higher. In sum, gRA grows both when investors put more money in risky assets and when they manage to reduce recurrent losses.

[162] As the prospective value is the PT-counterpart of the classic concept of investment utility, gRA is the pendant of a marginal utility with respect to the expected premium.

The impact of the portfolio evaluation frequency

We assume that the frequency at which the risky-portfolio performance is evaluated affects the loss attitude and leads to different investment decisions. Intuitively, the higher is the frequency of performance checks, the higher will be the volatility of the risky portfolio. This makes risky returns less likely to be significantly different from the risk-free rate. In consequence, the investor disappointment concerning the risky portfolio performance becomes more pronounced. Since, according to PT, registered losses are perceived as more painful than gains of similar size, risky investments become even less attractive.

The tendency of performing such frequent checks is termed as *myopia* or *narrow framing* and has been addressed in more detail in Section 3.1.3.[163] The idea that the joint effect of the myopia over financial decisions and the reluctance to make losses can dramatically affect the risk perception and hence the subjective desirability of risky investments comes in line with the concept of *myopic loss aversion* (mLA) by Benartzi and Thaler (1995).

We are interested in testing for mLA in our framework and, more generally, in observing how decisions on wealth allocation and loss attitudes vary at different portfolio evaluation frequencies. To this end, the applicative section examines how the wealth allocation to risky and risk-free assets given by S_t and B_t, the prospective value V and, the extended measure of the loss attitude gRA change at various *evaluation horizons* τ or, equivalently, at various *evaluation frequencies* $1/\tau$. In particular, we will work with τ-values ranging from one day to eight years, where the focus lies on short time lengths (up to one year), which we consider to be more plausible in practice.

The evaluation frequency affects our variables – and hence investors' decisions and attitudes – in two ways: First, through expected returns, which are themselves directly influenced by the evaluation frequency (the *direct transmission mechanism*). Second, through *past* returns which impacts on several model variables (such as the cushions, the past and current gain probabilities, etc.) and turn them to be indirectly depended on the evaluation frequency (the *indirect transmission mechanisms*).

Theoretically, the direct dependence (i.e., on returns) could be studied by holding

[163] According to Barberis and Huang (2006), myopia refers strictly to annual evaluations of gains and losses, hence the term of *narrow framing* would be better suited to describing the underlying phenomenon. In a financial context, narrow framing illustrates the isolated evaluation of stock market risk (i.e. unrelated to the overall wealth risk). As underlined in Barberis and Huang (2004), this isolated evaluation entails an underestimation of the stock desirability, even though, viewed in a wide utility-risk frame, stocks represent a good diversification modality.

all model parameters, besides current return expectations, invariable to the evaluation frequency. This is yet technically impossible, as multiple other parameters are indirectly affected by the evaluation frequency. Nevertheless, the direct effect can be discarded by eliminating the current returns. This is rendered possible by gRA, that represents by definition a derivative with respect to expected returns, where the direct impact is no longer contained. Consequently, studying how the prospective value and gRA vary with respect to the evaluation frequency amounts to examining the *total* and the *indirect* mechanism, respectively and we present this in the applicative section.

The same section will analyze a further related issue: Given that the portfolio evaluation frequency appears to affect investor perceptions of financial losses (and thus the level of risky investments), could the reverse causality hold as well? In other words, for a certain loss aversion value (at time t), is there an evaluation frequency that is *optimal* in the sense that it leads to the most relaxed attitude towards risky investments? If this is the case, financial advisors – whose interest is to attract clients, thus to raise capital – could for instance recommend to their clients to undertake performance checks with this "optimal" frequency that maximizes their risky investments and hence the budget at the manager's disposal.[164] We will search for the "optimal" evaluation frequency τ^* in terms of the maximization, first, of the perceived risky value $V(\tau^*)$ and, second, of the loss acceptance gRA(τ^*).

3.2.3 Application

This section presents findings complying with the theoretical results derived in Section 3.2.2 and based on market data. In particular, we consider daily values of the S&P 500 index, corrected for dividends and stock splits, and of the US three-month treasury-bill nominal returns. These two financial instruments – the stock index and the T-bill – serve as proxies for the risky and the risk-free investment, respectively. Both data series range from 01/02/1962 to 03/09/2006 (11,005 observations).[165]

As a consequence of the financial reform in 1979, which significantly changed the trading conditions, the early 80s mark the beginning of a new era of financial markets. We therefore reckon that only the second part of the data is relevant for inferring current market evolutions and divide our sample into two parts: The "active" data set (from

[164] In the same context, Gneezy and Potters (1997) suggest that managers could manipulate the evaluation period of prospective clients.

[165] Descriptive statistics can be found in Tables **??** and **??** of the Appendix A.3.1.

03/01/1982 to 03/09/2006, 6,010 observations)[166] and the "inactive" data (consisting of the first part of the sample from 01/02/1962 to 03/01/1982). The subsequent investigations are based on the active set, while the previous observations provide a basis for estimating the empirical mean and the standard deviation of the portfolio returns at the "date zero" of trade (03/01/1982). The data contains an outlier, corresponding to the October 1987 market crash, which may distort the results. Since market data serves in our work merely as support for simulating trading behaviors – that we view as more general – this outlier is smoothened out by replacing it with the mean of the ten before and after data points.[167]

We consider that non-professional investors perceive risky investments according to the value functions in Equations (3.19) and (3.20) and calculate their maximum expected loss level according to Equation (3.22). The active data set allows us to run the above presented model and to derive the desired VaR*, as well as the wealth proportion invested in the risky portfolio (i.e., in the S&P 500 index). The remaining money is assumed to be automatically put in the risk-free 3-months T-bill. Moreover, we assume that at date zero the investors' initial wealth is evenly allocated between the risky portfolio and the risk-free asset.[168] We also take the number of investors to be constant, i.e., no investors can enter or exit the market during the trading interval.[169]

We construct daily, weekly, monthly, and up to eleven months (increasing one month at a time), then yearly and further lower frequency returns ranging from one to eight years (with a one-year increment). The case commented throughout the application section of this section relies on values considered in Barberis, Huang, and Santos (2001) for the loss aversion coefficient and the sensitivity to past losses, namely $\lambda = 2.25$ and $k = 3$.[170] The expected portfolio gross returns are taken to be the unconditional mean returns until the last date before the decision time.[171] Further details with respect to the parameter choice

[166]We start from 1982 and not from 1979, since it took several years until the financial reform became operative.

[167]We consider this method to be appropriate for preserving some of the particularities of less probable market events such as crashes. At the same time, it allows for the circumvention of the excessive impacts of extreme outliers.

[168]A similar assumption is made in Thaler, Tversky, Kahneman, and Schwartz (1997).

[169]This assumption implies that the evaluation period is shorter than the lifetime of our loss averse agents or, equivalently, that investors are long-lived beyond the VaR horizon. Identical assumptions are made in Basak and Shapiro (2001), Berkelaar, Kouwenberg, and Post (2004), and Berkelaar and Kouwenberg (2006).

[170]We performed simulations for each $\lambda \in \{0.5; 1; 2.25; 3\}$ and $k \in \{0; 3; 10; 20\}$. The results are qualitatively similar to those commented below.

[171]We also performed simulations for the cases when expected portfolio gross returns were computed as a zero mean process, or as an AR(1) process. The results are qualitatively similar. As unsophisticated

for the presented results will be given in the text.

The evolution of the risky investment

In this section we address how the risky investment develops subject to different portfolio evaluation frequencies and to distinct ways of assessing the cushion.

According to Benartzi and Thaler (1995), loss-averse investors – who evaluate the performance of their portfolios once a year and employ linear value functions with conventional PT parameter values – give rise to a market evolution that can explain the equity return premium observed in practice. In the same spirit, we analyze how wealth allocation decisions of our non-professional investors change due to variations in the *portfolio evaluation frequency*. As in our framework these decisions are intrinsically linked to the past performance of the risky portfolio, we study at the same time the *cushion impact*.

In particular, we are interested in how different ways of assessing cushions contribute to determining the amount of wealth to be split between risky and risk-free assets at different evaluation frequencies. To this end, we apply two cushion definitions: *myopic* and *dynamic cushions*.[172] In calculating myopic cushions, we fix the benchmark level of past performance to be identical to the last-period risky holdings $Z_t = S_{t-1}$, so that the myopic cushion expression yields $S_t - S_{t-1}$. The *dynamic cushions* are based on Equation (18) in Barberis, Huang, and Santos (2001), which assumes a more complicated benchmark formula, in particular $Z_t = \eta Z_{t-1}\bar{R} + (1-\eta)S_t$. Hence, the dynamic cushion results in $\eta(S_t - Z_{t-1}\bar{R})$, where the parameter η measures how far in the past the investor memory stretches.[173] In line with the same authors, we subsequently concentrate on the case where $\eta = 0.9$.[174] We moreover take the variable $\bar{R}$ in the definition of the dynamic cushion as the mean gross return.[175]

investors (such as our non-professional traders) are more likely to rely on simple descriptive statistics from past data, we concentrate here on the case when expected returns are derived from average past returns.

[172]We also consider other cushion definitions. For instance, *cumulative cushions* amass from the date zero of the trade, so that $Z_t = Z_1 = S_0$ (e.g. the purchase price). Moreover, we also define *new myopic cushions* assuming $Z_t = Z_{t-1}R_t$. They attain several less plausible results and hence we do not further report on them.

[173]See Barberis, Huang, and Santos (2001). This parameter allows for adjustments of the benchmark, wherefrom the denomination of "dynamic". Specifically, lower η-values ascribe an increased weight to the current risky value S_t relative to past evolutions captured by $Z_{t-1}\bar{R}$, which corresponds to a more myopic view. In contrast, higher η-values denote a more pronounced conservativeness in assessing the past performance benchmark, as the current term S_t losses in importance relative to the past-oriented $Z_{t-1}\bar{R}$.

[174]In fact, we have considered three values $\eta \in \{0.1; 0.5; 0.9\}$. Results are qualitatively similar to each other.

[175]As no dividend data is available to our analysis, we could not apply the simultaneous estimation

Following Campbell, Huisman, and Koedijk (2001), we start by computing the portfolio VaR in Equation (3.15) for gross returns of the risky portfolio that are either (standard) normally or Student-t (with five degrees of freedom) distributed, and for a significance level of 5%. We take π_t, ψ_t, and ω_t to be the empirical frequencies of the cases where $z_t \leq 1$ (i.e. past gains), $x_{t+1} + (1 - z_t)R_{ft} \geq 0 | z_t \leq 1$ (a premium that is acceptable under a history of gains), and $x_{t+1} \geq 0 | z_t > 1$ (a positive premium, conditional on the cases with past losses), respectively. We derive VaR* according to Equation (3.22) using either myopic or dynamic cushions. This value is then plugged into Equation (3.16) in order to determine the optimal level B_t of borrowing ($B_t > 0$) or lending ($B_t < 0$), that depends on the degree of loss aversion of non-professional investors.

Table 3.1 presents averages of the wealth percentages S_t/W_t invested in the risky portfolio, for both myopic and dynamic cushions, normally distributed and Student-t distributed portfolio gross returns R_t, and at different portfolio evaluation horizons τ up to one year. The current value of the risky investment S_t is derived from Equation (3.17).

	Myopic cushions		**Dynamic cushions**	
Evaluation	**Portfolio returns**		**Portfolio returns**	
frequency	Normal	Student-t	Normal	Student-t
1 year	34.51	25.79	30.50	24.48
6 months	20.23	15.67	19.92	16.08
4 months	16.96	13.23	16.30	13.16
3 months	13.42	10.55	13.00	10.52
1 month	7.70	6.21	7.69	6.29
1 week	3.85	3.13	3.85	3.15
1 day	1.90	1.55	1.90	1.56

Table 3.1: Average wealth percentages invested in risky assets.

Accordingly, our non-professional investors allocate from almost no money to over 30% of their wealth to risky assets. The substantial fluctuation of these sums is mainly caused by the evaluation frequency of risky performance. Specifically, more frequent checks entail lower investments in the risky portfolio, irrespective of the way in which our investors account for past performance (i.e., of the type of cushion). This result is consistent with previous findings on mLA, such as Benartzi and Thaler (1995) and Barberis, Huang, and Santos (2001). It suggests that loss-averse investors who narrowly frame financial projects

procedure of Barberis, Huang, and Santos (2001). Note also that due to the fact that the mean and median of our return sample lie very close to each other, the results with $\bar{R} = \text{mean}[R_t]$ and $\bar{R} = \text{median}[R_t]$ are almost identical.

– by overly focusing on long series of past performances – become extremely loss averse when performing performance evaluations at a high frequency.

At an annual evaluation frequency, non-professional investors who dynamically assess cushions appear to be more loss averse than their myopic peers, and allocate less money to the risky portfolio. This difference becomes however negligible at higher evaluation frequencies. Moreover, irrespective of the type of cushion, the investor reluctance towards risky investments is higher for normally distributed than for Student-t distributed portfolio gross returns.

Since our VaR* is a VaR-type measure and VaR has been proven to be an adequate market-risk quantifier for normal distributions,[176] we henceforth focus on the case with normally distributed returns.

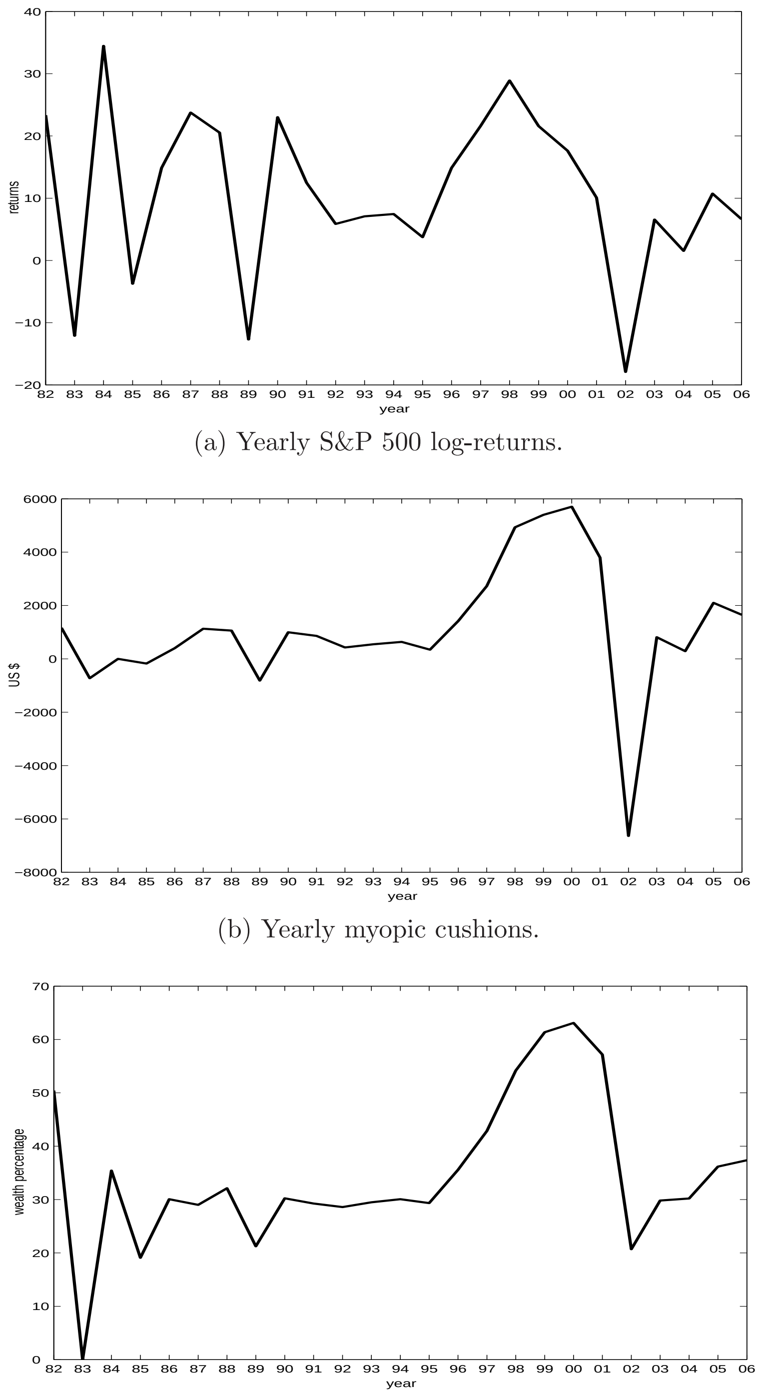
In the sequel, we study the importance of the past risky performance – a concept added to the initial PT representation by Barberis, Huang, and Santos (2001) and formalized by the notion of cushion – with respect to the wealth allocation.[177] In contrast to the above findings in this subsection, we are now interested in the *magnitude* of the cushion effect and its evolution over time.

In order to analyze this issue, we fix the evaluation horizon at one year and plot in Figure 3.4 (Figure A.67 in Appendix A.3.2) the annual returns of the index S&P 500, the evolution of the myopic (dynamic) cushion generated by a series of past gains or losses, and the resulting yearly wealth percentages invested in the risky portfolio. These figures point to a positive correlation of the three variables (returns, cushions, and risky investments).

In line with the idea that loss aversion is sensitive to past performance, we observe in panels c of Figures 3.4 and A.67 that the lower the cushions are, the more loss-averse investors become, since they dispose of less back-up for later contingent losses.

[176] Please refer to Section 3.1.1.

[177] Gneezy and Potters (1997) test for the influence of experienced gains and losses on risk behavior, but find no significant effect. However, as they note on p. 641, their experimental framework deviates from real market settings.



(a) Yearly S&P 500 log-returns.

(b) Yearly myopic cushions.

(c) Yearly wealth percentages invested in S&P 500.

Figure 3.4: Evolution of risky returns, myopic cushions, and wealth percentages invested in the risky portfolio for yearly portfolio evaluations.

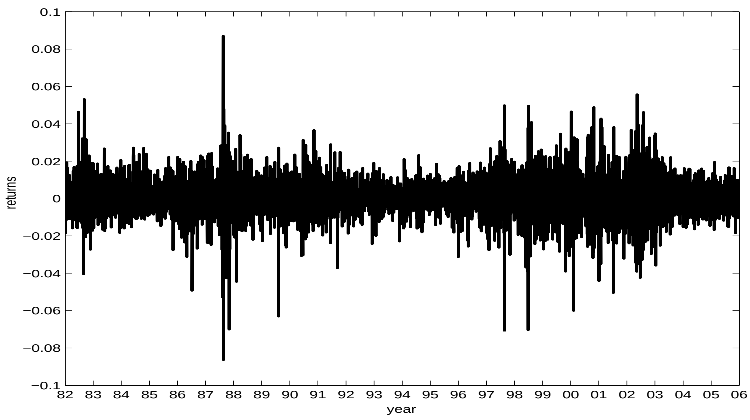
At this point, a further interesting empirical question arises: How long does it take for an investor performing frequent evaluations to quit the risky market? Figure 3.5 (Figure A.68 in Appendix A.3.2) emphasizes the dramatic effect of high evaluation frequencies when investors assess myopic (dynamic) cushions (see panels c). Specifically, non-professional investors who check their portfolio performance every single day put less than 5% of their wealth in risky assets. The reason is that each day can bring substantial changes in the perceived past performance. Therefore, although non-professional investors do not completely quit the risky market, their risky holdings are kept at very low levels during the entire trading interval.

The evolution of the prospective value

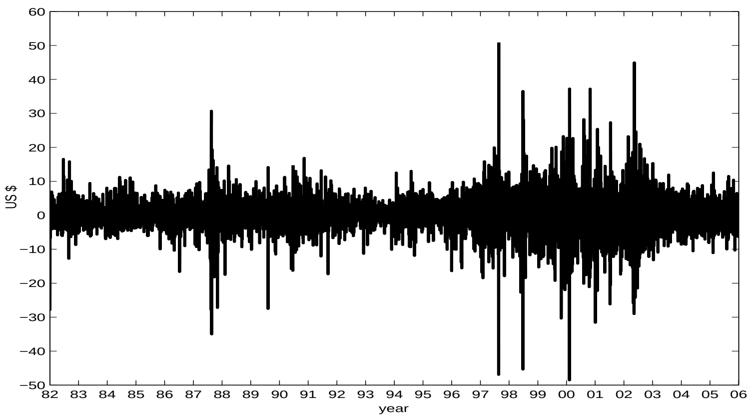
Let us now analyze the prospective value changes in time and with the evaluation frequency. As observed in the theoretical part, the impact of the evaluation horizon τ is twofold: direct, i.e. through expected returns (and thus the expected risk premium), and indirect, i.e. through other model parameters influenced by past returns, such as the cushion or the probabilities of past and current gains and losses. Thus, the prospective value sheds light on the *total* impact of the evaluation frequency on investors' behavior. Henceforth we refer to the descriptions of variables in dependence on the frequency at which the risky performance is checked as representations in the *evaluation-frequency domain*.

We commence our analysis by shortly considering the time evolution of the prospective value in Equation (3.24) and its two components, in order to ascribe the importance of the cushion and PT-effects. Figure 3.6 (Figure A.69 in Appendix A.3.2) illustrates these variables for myopic (dynamic) cushions and evaluation horizons of one year and one day. Note that at both evaluation frequencies, as long as cushions are sufficiently high in absolute value, it is the cushion effect that dictates the shape of the prospective value. This lead role is even more pronounced for daily evaluations, when the expected risk premium is very small and hence the PT-effect weak.[178]

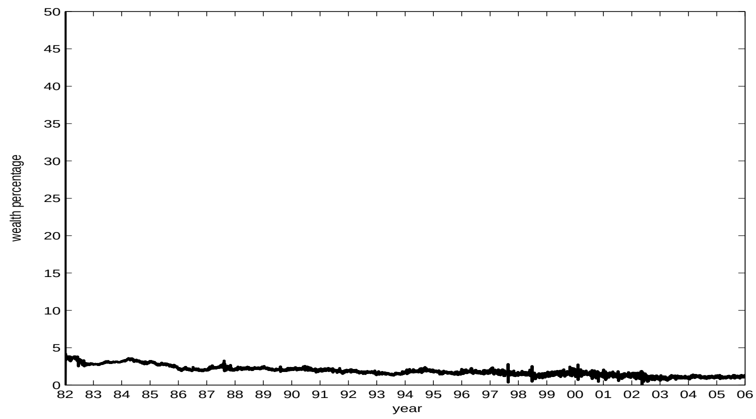
[178]Specifically, in this case the prospective value (black) cannot be visually disentangled from the cushion effect (blue).



(a) Daily S&P 500 returns.

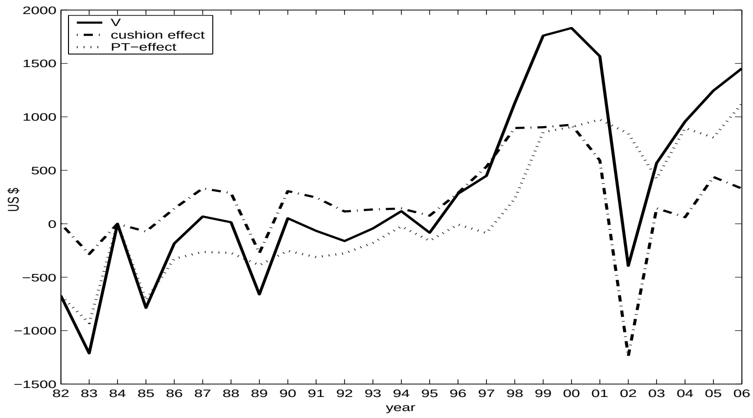


(b) Daily myopic cushions.

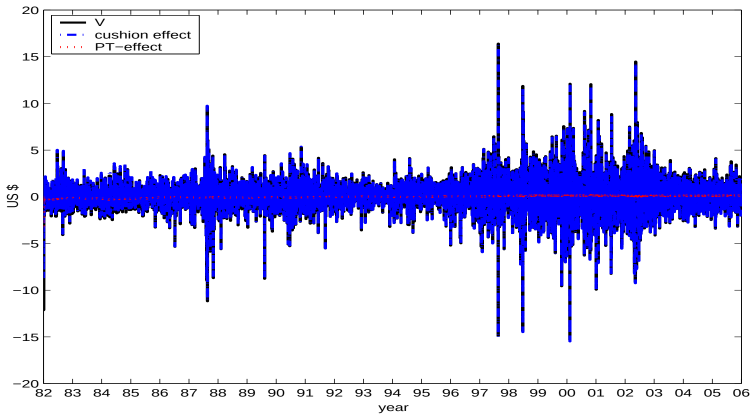


(c) Daily percentage investments in S&P 500.

Figure 3.5: Evolution of risky returns, myopic cushions, and percentages invested in the risky portfolio for daily portfolio evaluations.



(a) Yearly evaluations.



(b) Daily evaluations.

Figure 3.6: Prospective value evolution for yearly and daily evaluations.

In Figure 3.7 (Figure A.70, panel a, in Appendix A.3.2), we plot the prospective value and its two components again, but now as functions of the evaluation horizon τ. This horizon ranges from one month to eight years, where we consider monthly increments of up to one year and yearly increments thereafter.[179]

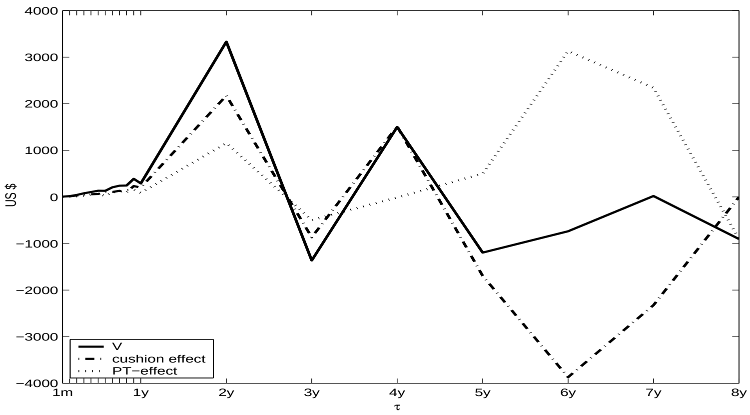


Figure 3.7: Prospective value evolution for myopic cushions and different evaluation frequencies.

It is apparent in Figures 3.7 and A.70 that up to two years the perceived attractiveness of financial investments increases with the evaluation horizons. This tendency is consistent with mLA and characterizes the evolution of both the PT-effect and the cushion effect at higher evaluation frequencies. In fact, the PT-effect is upward sloping over all considered evaluation frequencies, which supports the coherency of mLA within the initial PT-framework.

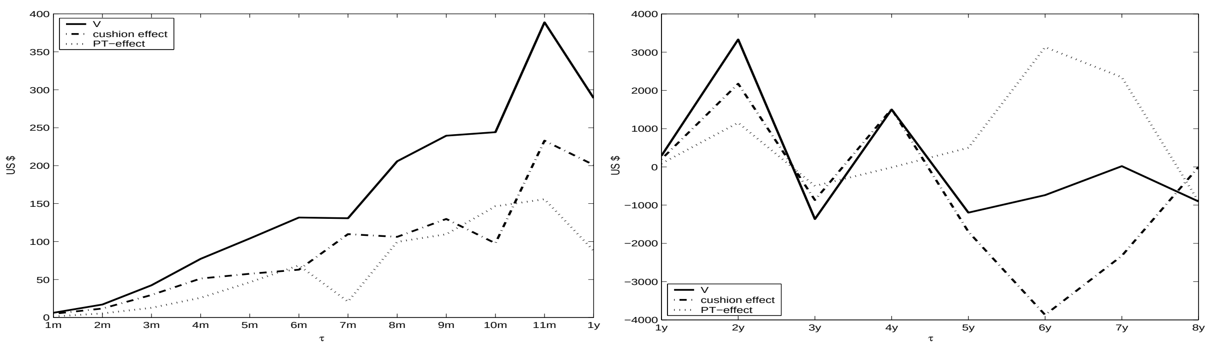
However, for higher evaluation horizons (such as three, five, or six years), the prospective value yields even negative values. This is motivated by the leading role of the cushion effect discussed above, and by the fact that for lower evaluation frequencies cushion values are negative and sufficiently high in order to counterbalance the PT-effect and to dramatically reduce the perceived value of risky investments. Intuitively, when risky performance is checked at longer time intervals, the decision flexibility is lower, since current decisions set the portfolio composition over the entire coming interval of several years. Thus, at lower evaluation frequencies investors would be more wary of registering current losses. As the cushion effect accounts for the perception of possible current losses – a perception which varies depending on the past performance, see the cushion weight in Equation (3.24) – it increases in absolute value for more seldom portfolio checks, but its

[179]In order to obtain a suggestive graphic representation, we consider all evaluation horizons from one to twelve months and discard the observations at one day and one week. An evaluation horizons of eight years implies that investors can only make three portfolio checks during the entire trading interval. Therefore, a further increase of the evaluation horizon would be senseless.

sign is given by the sign of the cushion. Our investors create negative cushions, which gives rise to the observed drop in the cushion effect magnitude and consequently in the prospective value. In sum, checking risky performance less often than once every one or two years appears to deteriorate the perception of the risky investment utility.

Indeed, as documented in Benartzi and Thaler (1995) a decade ago, in practice investors used to perform yearly portfolios checks. Nowadays, due to the high amount of information available at almost no cost and to the enhanced dynamics of market events, financial decisions may be reconsidered more often. However, one year remains as an important anchor in the investors' minds given that, on one hand, various events (such as the release of annual activity reports, taxes, etc.) take place with this frequency and, on the other hand, non-professional investors may not be sufficiently impatient (perhaps because they do not dispose of sufficient time, financial resources, knowledge, experience or the combination of any of them) to perform much more frequent portfolio checks. In our opinion, non-professional investor perceptions reasonably rely on evaluation horizons of one year and less.

Based on these ideas, we delimitate *two distinct segments* of the prospective value in the evaluation horizon domain depicted in Figures 3.7 and A.70. These segments meet at the "critical" horizon of *one year* and are characterized by different evolutions. We denote the segment with evaluation horizons lower than one year as the *left segment* and, as we view it as the (only) one relevant in practice, our subsequent analysis will concentrate on it. The part of the prospective value in the frequency domain encompassing evaluation horizons higher than one year is referred to as the *right segment*. Figure 3.8 (Figure A.70, panels b and c, in Appendix A.3.2) illustrate these two segments separately, for myopic (dynamic) cushions.



(a) $\tau \leq 1$ year. (b) $\tau \geq 1$ year.

Figure 3.8: Prospective value evolution on the two evaluation-frequency segments.

In the left segment, the perceived risky value appears to increase on average with the evaluation horizon. In effect, the curve $V(\tau)$ in panel a of Figure 3.8 (panel b of Figure A.70) is acceptably well described by a polynomial of first order.[180] Accordingly, the subjectively perceived utility of the non-professional investors – captured by the prospective value – should be maximized at the highest frequency of this domain, which is once a year.[181] One year can be hence designated as the *optimal evaluation horizon* with respect to minimizing loss aversion and hence maximizing risky investments $V(\tau^* = 1 \text{ year}) = \max$.

In the same spirit, the lowest evaluation horizon that we consider, namely of one day, entails a minimal expected value of the risky portfolio, pushing investors to step out of the risky market and to allocate (almost) all their money to risk-free assets. In other words, loss-averse investors should check the performance of their risky investments *as seldom as possible* in order to maximize the corresponding prospective value of their investments. Under the practical informational constraints that govern financial markets nowadays, *one year* appears to be the most reasonable evaluation time that would increase the perceived returns of risky investments.

The evolution of the global first-order risk aversion

In this section, we extend the analysis in the frequency domain to our new measure of loss attitudes gRA. In so doing, we study the *indirect* transmission mechanism mentioned at the end of Section 3.2.2. As a derivative of a linear variable, gRA does not contain any direct influence of the evaluation frequency (i.e., through the expected risk premium). The variation of gRA captures thus the collateral impact of τ on other model parameters, such as the cushion $S - Z$, the probability of past gains π, the probability of a positive risk premium given past losses ω, and that of an acceptable premium given past losses ψ.

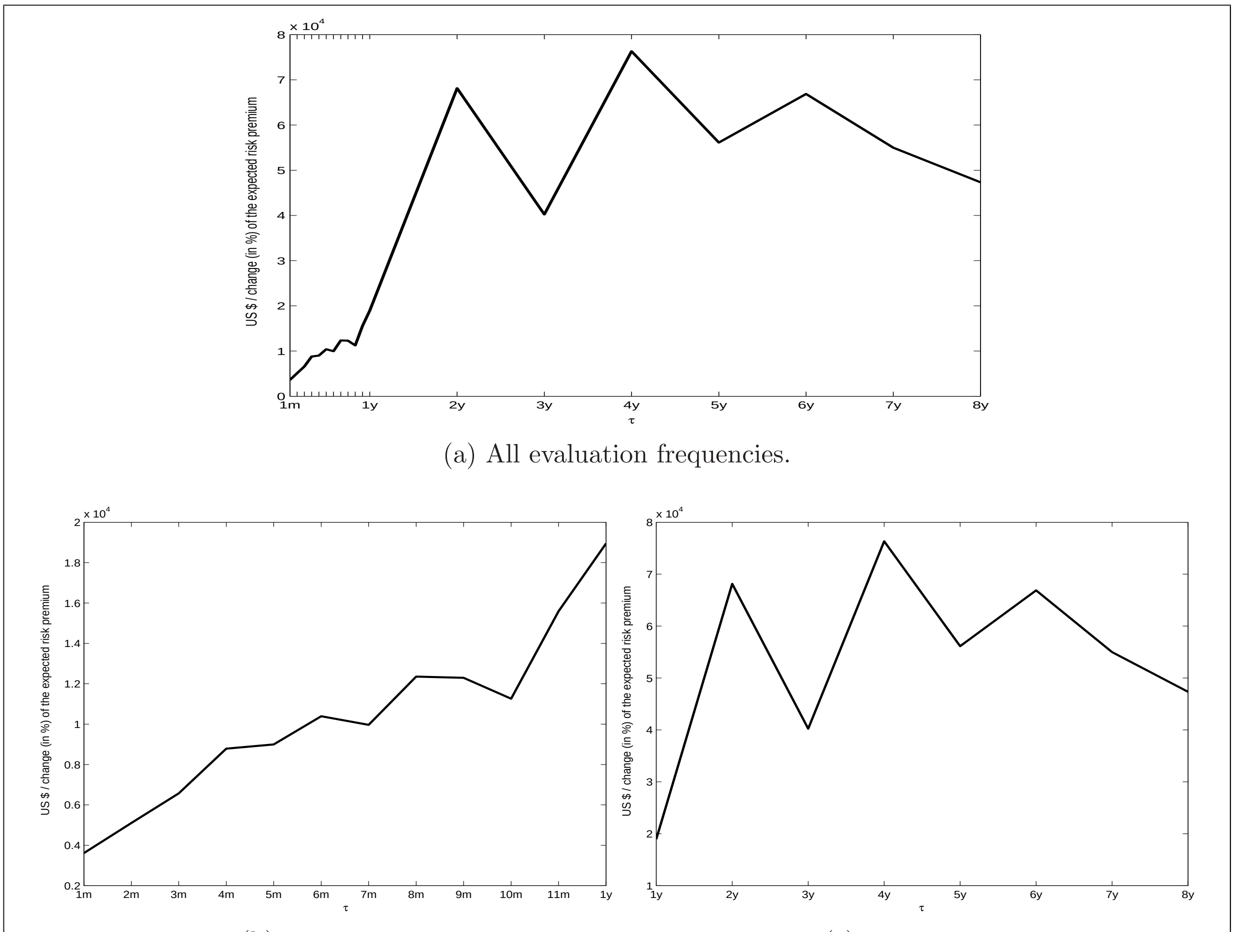
Panel a of Figure 3.9 (Figure A.71 in Appendix A.3.2) illustrates the gRA course for evaluation horizons ranging from one month to eight years and myopic (dynamic) cushions.[182] On average, gRA appears to increase with the evaluation horizon, pointing

[180] Specifically, the adjusted R-squared yields 91.69% (77.44%) for myopic (dynamic) cushions. The estimations are based on polynomial regression fitting performed with the Matlab Curve Fitting Toolbox. All findings in this section are robust across different parameter specifications, such as of the loss aversion coefficient, the sensitivity to past losses, the cushion, returns distribution, expected returns, etc.

[181] In fact, the prospective value in the left segment in Figures 3.8 and A.70 attains its maximum at eleven months. As this value lies closely to the predicted maximum point of one year and as this is a much more noticeable value in investor perception, we consider one year as a sufficiently good approximation of the optimum.

[182] All findings in this section are robust across different parameter specifications.

to a more relaxed attitude towards financial losses as the risky performance is checked less often. Note that this occurs at all frequencies and not only in the left segment, as was the case for the prospective value. Thus, while the impact of the evaluation frequency on the loss perception can be ambiguous in a context where both direct and indirect transmission mechanisms are considered (i.e. for V), the indirect mechanism (captured by gRA) consistently supports the concept of mLA.



(a) All evaluation frequencies.

(b) $\tau \leq 1$ year.

(c) $\tau \geq 1$ year.

Figure 3.9: Evolution of the global first-order risk aversion for different evaluation frequencies and myopic cushions.

The ambiguity of the total transmission mechanism reported for the prospective value appears to be therefore given by its direct component, i.e. manifests through expected returns. The cushion effect, that is highly dependent on returns, distorts the evolution of the prospective value for very seldom portfolio checks and renders it extremely sensitive to the past performance.

Similarly to the prospective value, we consider a segmentation of gRA around one year (see panels b and c in Figure 3.9). In the left segment (panel b), simple lines appear to fit the data acceptably well.[183] Our measure gRA attains its maximum for the lowest frequency of this segment of once a year gRA($\tau^* = 1$ year) = max.[184]

As mentioned in the theoretical part, higher gRA-values represent the result of a more relaxed attitude towards financial losses. Thus, minimizing the loss aversion – as measured by gRA – requires again that portfolio performance should be checked *as seldom as possible*. For the left segment, this is consistent with the recommendation derived with respect to the perception of risky investments captured by the prospective value.

In the right evaluation-frequency segment, the course of gRA is more complex, so that second-order polynomials are necessary for describing the data acceptably well.[185] The maximum of these parabolas is achieved at an evaluation horizon of around five years, which might recommend five-yearly revisions as optimal in this segment.[186] Recall nevertheless that we consider the right segment to be of less practical importance.

In sum, both the total and the indirect mechanisms by which the evaluation frequency impacts on perceptions and decisions suggest that, under practical information constraints, an improvement in the investor attitude towards risky holdings can be achieved for yearly performance evaluations.

A comparison with the "exogenous" portfolio optimization framework

This section proposes a way to "translate" the results obtained in our framework in terms of the "portfolio optimization language" spoken by professional managers. Recall that our investors ascertain *individually* the maximum sustainable level of losses VaR* on the basis of subjective behavioral parameters. In contrast, managers mostly *standardize* the risk definition, (e.g., when risk is measured by means of the VaR concept), to specific confidence levels and time horizons. In order to provide a comparison of these two frameworks – termed as "endogenous" and "exogenous", respectively – we confront the VaR* in our model with the standard VaR used by portfolio managers.

In particular, we perform twofold equivalence computations: First, we start from our VaR*-estimates and derive equivalent significance levels α from the VaR-formula. Second, we apply confidence levels traditionally used in previous research (such as 1% and 10%) to the same VaR-formula and obtain equivalent average coefficients of loss

[183] Specifically, the adjusted R-squared yields 90.5% (91.57%) for myopic (dynamic) cushions.
[184] This statement is now consistent both with the data and the fitted curve.
[185] In particular, the adjusted R-squared yields 49.61% (60.74%) for myopic (dynamic) cushions.
[186] Specifically, the maximum is attained for an evaluation horizon of 4.9859 (5.3178) years for myopic (dynamic) cushions.

aversion and equivalent wealth percentages invested in the risky portfolio, on the basis of the corresponding formulas and estimates in our model.

VaR*-equivalent significance levels Recall that portfolio managers consider the risk level indicated by their clients VaR* in terms of the standard concept of VaR, specifically as VaR$^{\text{ex}}$. In other words, VaR* is equated with the lower quantile of the portfolio gross returns at a given (i.e. fixed) significance level that we denote by α^* (or confidence $1-\alpha^*$). According to Equation (3.16), if the portfolio VaR at time t corresponds to an $\alpha_t > \alpha^*$ (or equivalently, to a confidence level $1-\alpha_t < 1-\alpha^*$), then too much risk would arise by putting the entire wealth in the risky portfolio. The portfolio manager will conclude that a percentage of the investor wealth should be lent (i.e. invested in the risk-free asset) $B_t < 0$. On the contrary, if $\alpha_t < \alpha^*$, the portfolio risk meets the individual risk requirements – being lower than the subjective risk threshold – and investors should borrow extra money $B_t > 0$ and increase their S&P 500-holdings.

In this section, we apply the formulas suggested by portfolio theory in order to determine those significance levels that correspond to the estimates of VaR$^*_{t+1}$ derived from Equation (3.22) on the basis of real market data. The corresponding averages of α^* over time are listed in Table 3.2 for different portfolio evaluation frequencies, normally distributed and Student-t distributed gross returns, myopic and dynamic cushions.

	Myopic cushions		**Dynamic cushions**	
Evaluation	**Portfolio returns**		**Portfolio returns**	
frequency	Normal	Student-t	Normal	Student-t
1 year	0.00	0.00	0.00	0.00
6 months	0.00	0.00	0.00	0.00
4 months	0.00	0.00	0.00	0.00
3 months	0.00	0.00	0.00	0.00
1 month	0.00	0.00	0.00	0.00
1 week	0.00	0.00	0.00	0.00
1 day	0.00	0.00	0.00	0.00

Table 3.2: Portfolio-equivalent significance levels α^* of the estimated VaR$^*_{t+1}$.

The results are striking: The equivalent significance level α^* lies below the commonly acceptable interval (being practically zero). Thus, the assumption of classical portfolio selection models based on the VaR-concept that investors choose significance levels α in the interval $[1, 10]\%$ appears to be at odds with the findings in our VaR*-framework, for any evaluation horizon lower than one year. Even the lowest significance level of 1% used in standard portfolio models is not able to capture the loss aversion of non-professional investors acting according to our setting. In other words, investors may be substantially more risk averse in practice than considered in theory.

Portfolio-equivalent coefficients of loss aversion The same equivalence issue can also be addressed from the opposite viewpoint: We determine the values of λ^*_{t+1} in Equation (3.23) and the average investment in risky assets that result from our VaR*-formula in Equation (3.22). These values correspond to the risk levels used (by managers) according to the conventional VaR-procedure at usual significance levels α of 1% and 10%.

Tables 3.3 and 3.4 (Tables A.2 and A.3 in Appendix A.3.2) present the results of this analysis for normal and Student-t portfolio returns and myopic (dynamic) cushions. Recall that the portfolio VaR in Equation (3.15) is estimated using a 5% significance that is considered the benchmark for the values in these tables (i.e., it corresponds to 100% risky investments).

	Wealth %		λ^*	
Evaluation	**Portfolio returns**		**Portfolio returns**	
frequency	Normal	Student-t	Normal	Student-t
1 year	60.99	36.48	1.02	1.01
6 months	59.72	34.63	0.91	0.90
4 months	59.40	34.17	1.00	0.82
3 months	59.30	34.01	1.43	1.67
1 month	59.04	33.65	0.90	1.62
1 week	58.82	33.34	0.80	0.58
1 day	58.70	33.20	1.00	1.02

Table 3.3: Wealth percentages invested in S&P 500 and the average λ^*, for $\alpha = 1\%$ and myopic cushions.

Accordingly, the equivalent recommendations concerning the money to be invested in risky assets that result from the optimal portfolio allocation under VaR at 1% (10%) significance lie well below (above) the benchmark VaR at 5%. This points to a higher (lower) loss aversion in our endogenous VaR*-framework – after restating it in terms of the exogenous-VaR model – relative to the portfolio risk measured by VaR. Comparing Tables 3.3 and 3.4 (Tables A.2 and A.3 in Appendix A.3.2), we can observe that the lower the significance (or the higher the confidence level) is, the more risk averse the non-professional investors become, as the proportion of wealth to be put in the risky portfolio is smaller than 100%. However, even the lowest percentages in Tables 3.3 and A.2 are

	Wealth %		λ^*	
Evaluation	**Portfolio returns**		**Portfolio returns**	
frequency	Normal	Student-t	Normal	Student-t
1 year	120.80	125.37	1.20	1.02
6 months	121.47	126.11	0.88	1.06
4 months	121.64	126.29	1.00	1.00
3 months	121.70	126.36	1.00	1.00
1 month	121.84	126.50	1.00	1.00
1 week	121.96	126.63	1.00	1.00
1 day	122.00	126.67	1.00	1.00

Table 3.4: Wealth percentages invested in S&P 500 and the average λ^*, for $\alpha = 10\%$ and myopic cushions.

still much higher than those in Table 3.1, where VaR* is treated as endogenous, mainly for high-frequency revisions.

Interestingly, the results for $\alpha = 1\%$ are qualitatively consistent with our previous findings supporting mLA, since the wealth percentages dedicated to risky assets decrease for higher evaluation frequencies, although their variation is much weaker than for the case with $\alpha = 5\%$ considered in the first part of the applicative section. In contrast, when the confidence level increases to $\alpha = 10\%$, this phenomenon is reversed and investors appear to allocate slightly more money to the risky portfolio for more frequent evaluations. As mLA is a widely documented phenomenon, we can conclude that the traditional portfolio optimization framework fails once more to capture the real investor behavior in a consistent way. This problem appears to become more acute for more relaxed assumptions on the risk attitude.

Similar conclusions are reached with respect to the equivalent loss aversion coefficient λ^* derived for conventional significance levels assumed in previous research. Its values in Tables 3.3 and 3.4 (Tables A.2 and A.3 in Appendix A.3.2) for myopic (dynamic) cushions are much lower than the empirical level of 2.25 estimated in the original PT and largely used in previous empirical research.[187] For the majority of the considered combinations of α-values and evaluation frequencies, we obtain $\lambda^* \approx 1$, a level that indicates identical perception over gains and losses according to the value function from Equation (3.24) (and recalling that $k = 0$, i.e., past losses have no influence). Actually, the "neutral" level of 1 is rarely exceeded over the considered evaluation frequencies, for each $\alpha = 1\%$ and 10%. This reinforces our earlier claim that even for low significance levels (e.g. $\alpha = 1\%$,

[187] Such as Barberis, Huang, and Santos (2001) and Benartzi and Thaler (1995).

as is the common case in previous portfolio optimization research) the loss attitude of real investors captured by the specific coefficient λ is *underestimated*.

3.2.4 Summary and conclusions

This section investigates the behavior of non-professional investors facing the problems of fixing a maximal acceptable level for their financial losses and of optimally allocating wealth between a risk-free asset and a risky portfolio. We assume that these investors are loss averse, narrowly frame financial investments, and perceive future portfolio returns as being influenced by past portfolio performance. Their single source of utility consists of financial wealth.

We extend the portfolio allocation model developed in Campbell, Huisman, and Koedijk (2001) in order to incorporate the effect of a desired risk level that is now *subjectively* assessed (VaR*). The first task of our non-professional investors consists of fixing their individual VaR*-level. This is subsequently communicated to professional portfolio managers in charge of finding the optimal composition of the risky portfolio. Based on VaR*, portfolio managers derive the optimal sum of money to be invested in risk-free assets. The non-professional clients are thus provided with a recommendation on how to allocate money between risky and risk-free assets that corresponds to their personal perceptions and preferences.

In modeling investor perceptions over the risky investment that yield the subjective VaR*, we employ the extended subjective valuation of prospective risky investments proposed in Barberis, Huang, and Santos (2001) and rely on the notion of myopic loss aversion introduced in Benartzi and Thaler (1995). The contribution of our work is to integrate these behavioral explanations in the portfolio decision framework mentioned above. Moreover, we enrich the two models with original findings that stem both from theoretical considerations and results obtained on the basis of real market data (specifically, the S&P 500 and the US 3-months T-bill returns).

Considering that investors are merely concerned with financial investments as a source of utility, we build on the theoretical modelling of their perceptions of risky assets (the value function) and define the maximum individually sustainable level of financial losses (VaR*). This VaR*-level serves in deciding upon the optimal amount of money to be invested in the risky portfolio. Also, we propose a way in which non-professional investors can assess the utility of risky prospects (the prospective value). Moreover, we introduce an extended measure, the global first-order risk aversion, that attempts to better capture the actual attitude towards financial losses of real investors. We finally investigate how the portfolio evaluation frequency impacts, through different mechanisms, on the prospective value and on this further measure of the loss attitude. In this context, we also suggest a way to derive the horizon of performance revisions that maximizes risky investments.

The theoretical results are supported and extended by our application. We show that, in sum, our non-professional investors demonstrate myopic loss aversion. They allocate the main part of their wealth to risk-free assets, while a smaller sum (always lower than 35% of wealth) is put into the risky portfolio. This latter sum substantially decreases at higher frequencies at which the risky performance is evaluated.

Furthermore, financial wealth fluctuations determined by the success or failure of previous decisions (the cushion) exert a significant impact on the current portfolio allocation. They make investors without substantial cushion gains firmly refuse holding a large fraction of risky assets.

One year appears to be a critical evaluation horizon under practical market constraints, commonly used in practice, and optimal from the viewpoint of maximizing risky holdings in consequence of a more relaxed attitude towards such holdings. The individual perception of risky investments, captured by the prospective value, and the loss attitude measured by the global first-order risk aversion can be split into two segments with qualitatively distinct evolutions around the annual frequency. Myopic loss aversion holds for the segment of evaluation frequencies of at least one year that can be considered as being the only one of practical relevance. However, the prospective value, which reflects the total impact of the evaluation frequency, reveals a somewhat ambiguous behavior when portfolio performance is checked at time intervals longer than one year. This is apparently due to the direct component of this total impact (i.e. the manifestation through expected returns), since the indirect component (that can be measured by means of the global first-order risk aversion) shows a more consistent evolution over all analyzed evaluation frequencies.

Finally, we carry out estimations aimed at establishing an equivalence between the theoretical portfolio optimization under exogenous VaR-constraints and our extended framework with individual VaR*. The estimated variables (such as significance levels, loss aversion coefficients, and investments in risky assets) suggests an underevaluation of the attitude of non-professional investors towards financial losses.

3.3 Two-dimensional utility: consumption vs. financial assets

3.3.1 Introduction

One of the common decisions in everyday life is the optimal allocation of resources among different activities that generate utility. For all individuals, the first and most important source of utility is consumption. Additionally, people who are active in financial markets derive utility from their investments. This section addresses the behavior of non-professional investors who derive utility from both consumption and financial wealth fluctuations. In other words, we aim at enlarging the perspective provided in Section 3.2 to what we denote as *two-dimensional utility*. Specifically, we account for consumption as additional source of utility besides financial investments.

Similarly to the previous investigations, we are first interested in the attitude of non-professional investors towards financial losses, knowing that they narrowly frame financial investments and change current perceptions subject to the past performance of these investments. Equally, we analyze how non-professional investors split their money between consumption and (risky and risk-free) financial projects as a consequence of their loss attitude.

Our non-professional investors have now to decide upon the optimal wealth allocation between consumption and financial investments in total. The latter category offers a further choice between a risky portfolio and a risk-free asset. We adopt the formal views in Section 3.2 regarding the subjective perception of risky vs. risk-free investments – i.e. the prospective value – and how it enters the wealth-allocation problem of non-professional investors. In this context, loss aversion is quantified by two measures: the loss-aversion coefficient and the global first-order risk aversion gRA. Wealth allocation is expressed by the wealth percentages dedicated to consumption and to (different types of) financial assets. In addition, we rely on the theoretical approach of Barberis, Huang, and colleagues (2001, 2004, 2006), according to which investors decisions rely on the maximization of either expected utility or of recursive non-expected utility with first-order risk aversion. In both cases, the utility function is shaped in order to account for the excessive focus (in technical terms narrow framing or myopia) on financial investments and for the influence of past portfolio performance on the current perceptions of risky investments.

We analyze the loss attitude and wealth allocation in the aggregate equilibrium with a representative investor. The two settings with expected and non-expected utility require specific conditions for attaining this equilibrium: For instance, under non-expected utility the acceptable values of some behavioral parameters are constrained to belong to smaller sets and the actual influence of past performance must be nil. We derive the equilibrium equations in each setting and then infer the variables of interest from these equations. The single variable for which both settings deliver expressions in equilibrium is the prospective value. It further serves for obtaining equilibrium-equivalent measures of the loss attitude, specifically the loss-aversion coefficient and gRA. The expected-utility setting entails, in addition, estimates of the discounting factor. In the same setting, the wealth-allocation variables can only be assessed on average. Under non-expected utility, the percentages of total wealth allocated to consumption and of post-consumption wealth invested in risky assets are, besides the prospective value, direct equilibrium estimates.

The theoretical part is implemented based on the same data set as in Section 3.2. In particular, we consider S&P 500 and 3-months T-bill nominal returns as proxies for a well-diversified risky portfolio and the risk-free investment, respectively. In addition, we employ quarterly data of the aggregate per-capita consumption that provide for consumption values at (only) two different evaluation horizons of the risky-portfolio performance: one year and three months. This allows us to analyze the myopic aversion, i.e. the enhanced reluctance towards financial investments in general and risky investments in particular, manifested at higher evaluation frequencies.

We simulate how non-professional investors behave in an environment where consumption and financial markets are characterized by general parameters, such as the risk-free returns and the dynamics of consumption and of expected returns, derived from the sets of real data at hand. We account for various investor profiles by choosing different combinations of our behavioral parameters, such as the degree of narrow framing, the consumption-related risk aversion, the weight of financial utility, the sensitivity to past losses, the way of accounting for past performance, etc. Moreover, in order to avoid the impossibility of covering current consumption needs from financial revenues over the entire investing interval, we consider that investors periodically dispose of exogenous additional incomes, the level of which can vary as well.

The numerical findings can be directly compared – that is, as values – between settings only for the cases based on identical assumptions. Therefore, our comments on the differences and resemblances of expected and non-expected utility maximization refer first to general aspects that should be understood in a qualitative sense. Then, we briefly examine quantitative aspects.

In essence, the two settings deliver different recommendations based on the two measures of loss aversion – the loss-aversion coefficient and gRA –, These settings agree yet, in the main, with respect to the findings on wealth allocation. In particular, loss aversion can manifest in multiple ways and depends on the measure used to quantify it. Thus, the maximizers of expected utility are less reluctant towards financial losses and allocate higher percentages of their total wealth to financial assets in total. At the same time, they are yet willing to invest less money in risky assets in particular. Myopic loss aversion can be tested only over two evaluation frequencies, but in multiple ways. It is supported only under expected-utility maximization and only when loss attitudes are measured by the loss-aversion coefficient (what we denote to be myopic loss aversion in the strict sense); It also holds with respect to the perception of risky investments captured by the prospective value (that is, in the large sense), but this time exclusively under non-expected utility; None of the two settings provides evidence for myopic loss aversion with respect to the money dedicated to risky assets (in other words, in the monetary sense). Moreover, the non-expected utility maximization appears to be somewhat better suited to describe individual behaviors, based on the robustness of the estimates and the more intuitive economic interpretation.

The remainder of this section is organized as follows: Section 3.3.2 builds upon the theoretical framework. In particular, we commence by a brief review of the general purpose of the model in Section 3.2, where the focus is on the variables of interest for the present section. Then, this model is extended to allow for two-dimensional utility and the approaches with expected and non-expected utility are detailed. Section 3.3.3 presents the implementation of our theoretical model first for expected utility and second for non-expected utility. The main findings in these two settings are subsequently confronted with each other. Finally, Section 3.3.4 summarizes our findings and concludes. Further numerical results are included in Appendix A.3.3.

3.3.2 Theoretical model

This section presents the theoretical framework describing how non-professional investors perceive financial risks and accordingly allocate their wealth between consumption and financial assets in order to maximize perceived utility. In line with Barberis and Huang (2004, 2006), we adopt two distinct formulations of the maximization problem: first around expected utility and second, around recursive non-expected utility with first-order risk aversion. Both settings account for the narrow framing of financial projects and for the influence of past performance on the perceived value of risky investments.

A one-dimensional utility framework

One of the most important human decisions is how to allocate resources, in particular money, among different type of activities. These activities may be either necessary and/or can generate further revenues. In the latter category, investing in financial assets has nowadays become one of the most popular alternatives. This widespread trend of ordinary people turning into "investors" is due, at least in part, to the formidable accessibility of information concerning financial markets, at almost no cost and almost in real time. Under the pressure of the huge amount and intensity of such information, even non-professional investors may have no choice but to become "overly concerned" with their financial investments. This phenomenon of putting excessive emphasis on financial investments is denoted in technical terms as *narrow framing* or *myopia*. Under narrow framing, the central decision refers to how to allocate the "right" amount of money first to financial investments in general, then across different financial assets. Also, financial investments are perceived as distinct and overly important generators of individual utility. This dissociates the decisions upon wealth allocation from the naturally larger context with multiple generators of utility (such as consumption or other factors that do not exclusively apply to financial markets).

Drawing on this idea, Section 3.2 has modelled the attitude towards financial losses and the decision making of *non-professional investors* regarding the optimal wealth allocation among different financial assets. Non-professional investors have been defined as people whose principal concern is not financial investing and who lack of necessary resources for making more sophisticated investment decisions. It has also been assumed that they derive utility merely from financial investments.

The present section extends the model in Section 3.2 for the case when not only financial investments, but also consumption, generate utility. In the sequel, we briefly review the above model structure and variables that serve as support for the present two-dimensional utility framework.

Investor attitudes depend on the subjective perception of financial investments and on the possible losses associated with these investments. In Section 3.2, perceptions have been modeled according to the extended prospect-theory framework by Barberis, Huang, and Santos (2001). Accordingly, risky performance is mentally split – with respect to a subjective reference point – in gains and losses; Moreover, losses loom larger than gains of the same size, and past performance influences current perceptions.

In Section 3.2, we have explained that decisions on wealth allocation often imply the aid of professional managers. In essence, non-professional investors are interested in how to split their money between the two main categories of financial assets: risky and risk-free. The more refined decision upon the composition of the risky portfolio is committed to professional portfolio managers. They find a solution for the optimal wealth allocation among different risky assets and a risk-free one in a portfolio optimization framework based on Campbell, Huisman, and Koedijk (2001), where market risk is measured by the Value-at-Risk (VaR). We have introduced an individualized VaR-level, denoted as VaR*, which is based on the psychological profile of non-professional clients and hence on their subjective perceptions of financial risk, and considered its implications on wealth allocation.

As mentioned above, the assignment of the individual loss level VaR* in investors' minds can be formalized following Barberis, Huang, and Santos (2001). Thus, the subjective perception of one unit of risky investment (relative to the risk-free rate) is captured by the *extended value function* v. As in the original prospect theory of Kahneman and Tversky (abbr. PT),[188] this value function accounts for the distinct perception of gains and losses with respect to a subjective reference point and for the higher reluctance towards losses. In addition, the extended value function is designed to capture the possible influence of past performance on current risk perceptions.[189] We further apply the value function definitions proposed in Equations (3.18-3.20) in the above Section 3.2.

Section 3.2 has extensively analyzed the *cushion* $S_t - Z_t$, defined as the difference between the current value of the risky investment S_t and a benchmark level for the past portfolio performance Z_t. In so doing, we have considered two distinct cushion definitions: the *myopic cushions* for which the benchmark level of past performance was taken to be identical to the last-period risky holdings $Z_t = S_{t-1}$; and the *dynamic cushions* which

[188]Please refer to Section 3.1.2 and specifically to Kahneman and Tversky (1979) and Tversky and Kahneman (1992) for more details on PT.

[189]Which makes, among others, the reference point vary subject to past losses or gains.

assumed the same benchmark to be a combination of past references and current risky investment values $Z_t = \eta Z_{t-1}\bar{R} + (1-\eta)S_t$, where the parameter η measured how far in the past the investor memory stretches. Hence, myopic cushions amount to $S_t - S_{t-1}$ and dynamic ones to $\eta(S_t - Z_{t-1}\bar{R})$.

Based on the perception of financial investments captured by the value function, Equation (3.22) further defined the *individual loss level* VaR* of non-professional investors as the quantile of the subjective loss distribution. Once being formed in investors' minds, VaR* is communicated to the portfolio managers in the form of a fixed number. Managers interpret it as a fixed risk level and incorporate it into the problem of optimal capital allocation among financial investments as a risk constraint. This problem is solved following the portfolio optimization model with VaR as a risk measure by Campbell, Huisman, and Koedijk (2001). The optimization procedure delivers first the optimal weights w_t^* of the risky portfolio components from Equation (3.14), and second the *optimal amount of money B_t to be borrowed or lent* from Equation (3.16). The latter variable plays an important role for non-professional investors, who are above all interested in how to split their money between risky and risk-free assets. Therefore, it represents one of the main variables in the subsequent model and gives account of the wealth-allocation decisions of our investors in the market equilibrium.

Central to the above analysis, therefore also to the following, is the derivation of the so-called *prospective value* V from Equation (3.24). This variable captures the subjectively perceived utility of the risky portfolio (relatively to the risk-free rate) and hence is related to the attitude adopted towards financial losses. One goal of the present work is to determine the prospective value ascribed – in the equilibrium of the aggregate market – to financial investments by investors who derive utility from two main sources (consumption and financial investments).

Drawing on the idea that individual attitudes towards financial losses can be measured by means of the loss-aversion coefficient, it is interesting to compute a loss-aversion coefficient $\bar{\lambda}$ that is equivalent to the prospective value $\bar{V}$ in the market equilibrium. From Equation (3.24), $\bar{\lambda}$ formally yields:

$$\bar{\lambda}_{t+1} = \frac{\bar{V}_{t+1} - \Big(\pi_t\psi_t + (1-\pi_t)\omega_t\Big)S_t E_t[x_{t+1}] + \Big(\pi_t(1-\psi_t)R_{ft} + (1-\pi_t)(1-\omega_t)kE_t[x_{t+1}]\Big)(S_t - Z_t)}{\Big(\pi_t(1-\psi_t) + (1-\pi_t)(1-\omega_t)\Big)S_t E_t[x_{t+1}] + \pi_t(1-\psi_t)R_{ft}(S_t - Z_t)}. \tag{3.26}$$

The coefficient $\bar{\lambda}$ plays a central role in our model, as it stands for an equilibrium-equivalent measure of the attitude towards financial losses. Note that established research (based on PT) works often with values of 2.25 for the loss-aversion coefficient.

However, the simple loss-aversion coefficient fails to capture the influence of past performance that is yet explicitly considered in the extended PT by Barberis, Huang, and Santos (2001). Consequently, Equation (3.25) has introduced a further measure of the loss attitude denoted as the *global first-order risk aversion* (abbr. gRA). It is formally defined as the first derivative of the prospective value with respect to the expected risk premium. In the applied part of the present section, we analyze the evolution of gRA in the two-dimensional utility equilibrium. In essence, higher gRA-values point to more relaxed loss attitudes, as this measure directly reflects changes in the attractiveness of financial investments captured by the prospective value.

A two-dimensional utility framework

As already mentioned, Section 3.2 considers that investors are merely concerned with financial investments and the utility they generate. However, in practice, such considerations – that is, focusing on financial utility alone – appear to be better suited to professional investors than to non-professional ones. The activity of the former demands a strictly investment-oriented perspective, and their main task reduces to making money that is going to be reinvested in financial markets. In contrast, non-professional investors sooner regard financial investments as a source of income dedicated to covering consumption needs.[190] In other words, consumption should be the main generator of individual utility for non-professional investors. However, financial investments might be perceived as an equally important source of utility. The main reason resides in the above mentioned narrow framing, i.e. the excessive focus on financial investments, which appears to be driven by the fear of registering losses when faced with financial risks.

Based on these considerations, we now extend the setting in Section 3.2 by allowing for *two sources* of individual utility: financial wealth fluctuations and consumption. In so doing, we rely on Barberis, Huang, and Santos (2001), Barberis and Huang (2004), and Barberis and Huang (2006). The present section details the theoretical background of our contribution.

[190] Campbell, Huisman, and Koedijk (2001) note that the simple VaR-framework without consumption is sufficiently informative for describing decision making of (non-professional) investors under risk.

In essence, the above wealth allocation problem based on one-dimensional utility is augmented with an additional step: splitting money between consumption and financial investments. Strictly speaking, our non-professional investors decide first on how much money should be dedicated to consumption needs and to financial assets *in total*; Only afterwards they can partition the latter sum between risk-free and risky assets, as shown in Section 3.2. As the performance of risky investments is mostly measured with respect to risk-free assets,[191] we can formally merge these two successive steps into a single decision. The common goal is then the maximization of total utility derived from consumption *and* risky – relative to risk-free – financial investments.

Following Barberis and Huang (2004), we consider an aggregate market which lacks perfect substitution. Thus, we can focus on absolute pricing and avoid possible arbitrage opportunities generated by narrow framing. The total utility is formulated in order to account for the above-mentioned two-dimensional origin and yields the sum of discounted utilities of consumption $U(C)$ and of perceived values of financial investments $\tilde{V}$, that is:[192]

$$U = U(C) + \tilde{V} = \sum_{t=0}^{\infty}\Big(\rho^t U(C_t) + \rho^{t+1} b_t \tilde{V}_{t+1}\Big), \tag{3.27}$$

where ρ is referred to as the *discounting factor* and $0 < \rho < 1$.[193] According to Equation (3.27), at each time t the current consumption is discounted with ρ^t, while the prospective value – that encompasses subjective perception of the *next-period* performance[194] – has to be provided with a corresponding ρ^{t+1}.

In line with Barberis, Huang, and Santos (2001), b_t is an exogenous scaling factor designed to map the perceived value of gains and losses into consumption units. It follows the rule stated in their Equation (11), namely $b_t = b_0 \bar{C}_t^{-\gamma}$, where $\bar{C}_t$ represents the exogenous *aggregate per-capita consumption* at time t,[195] and b_0 measures the *degree of narrow framing*. Finally, we denote γ is as the *consumption-related coefficient of risk aversion*.

[191] Recall that the reference points of the perceived risky value from Equations (3.19) and (3.20) include the risk-free rate R_{ft}.

[192] Strictly speaking, $\tilde{V}$ corresponds to the prospective value V from Equation (3.24), *before* taking expectations. Recall that the prospective value stands for the perceived utility of financial investments. Being obtained from the value functions weighted by the pure occurrence probabilities of possible outcomes, it is equivalent to an expected value.

[193] In the applied part, we consider a finite investment duration T that is yet sufficiently long in order to allow for reaching an equilibrium.

[194] Recall that the prospective value encompasses the future returns R_{t+1}.

[195] The exogeneity is related here to the subjective viewpoint of the individual investor. It points out the fact that b_t is independent of every individual feature related to risk aversion or loss aversion.

In line with Barberis and Huang (2006), it is now possible to develop an equilibrium framework in the aggregate market with a representative investor.[196] We derive the equilibrium conditions in two different settings: first, when investors maximize expected utility, and second, when a recursive non-expected utility function with first-order risk aversion is optimized. Throughout, we formally incorporate the assumptions of narrow framing and dependence of current decisions on past portfolio performance.

We are furthermore interested in the phenomenon of *myopic loss aversion* (mLA). Introduced by Benartzi and Thaler (1995) and supported by numerous experimental tests (as detailed in Section 3.1.3), mLA refers to the fact that narrow framing (or myopia) strengthens the loss aversion, so that investors reduce their risky investments when risky performance is checked on more frequently. In view of the manifold possibilities to quantify the loss attitude, we refine the notion of mLA in the following sense: We denote as *mLA in the strict sense* the enhancement of loss aversion with the evaluation frequency. According to our model, the loss aversion can be quantified either by the loss-aversion coefficient or by the extended measure gRA. Thus, mLA in the strict sense holds if either the loss-aversion coefficient increases or gRA decreases with the evaluation frequency. As both loss-aversion measures are derived from individual perceptions of risky investments, we can also measure *mLA in the large sense* with respect to the prospective value. We can support mLA in the large sense if the prospective value falls at higher evaluation frequencies. Finally, *mLA in the monetary sense* is defined as the decrease of monetary risky holdings – in percentages of the total wealth – in consequence of more frequent portfolio evaluations. In addition, we can speak about *myopic aversion towards financial investments* when the wealth percentages dedicated to consumption increase with the evaluation frequency. All these different aspects of myopic aversion will be analyzed in Section 3.3.3. Note that our data sets constrain us to check on mLA only at two evaluation frequencies.

The expected-utility approach We commence by considering the approach of Barberis, Huang, and Santos (2001), where the representative investor aims at maximizing total *expected utility* generated by both consumption and financial wealth changes.[197]

[196]Henceforth, we use the denominations of "representative investor" and "investors" interchangeably, drawing upon the idea that the latter stands for a group with homogenous preferences. In essence, the actions of all investors in equilibrium can be summarized by the corresponding choices of the representative investor.

[197]As demonstrated in Barberis, Huang, and Santos (2001), this framework can explain the emergence of equity premiums of the magnitude observed in practice.

We refer to the utility of consumption in the traditional CRRA-terms and hence assume $U(C_t) = \dfrac{C_t^{1-\gamma}}{1-\gamma}$. The utility of financial investments is measured by the prospective value in Equation (3.24). Then, the maximization problem of the non-professional investors from Equation (3.27) results in:

$$E_t[U] = E_t\left[\sum_{i=0}^{\infty}\Big(\rho^i\frac{C_i^{1-\gamma}}{1-\gamma} + b_0\rho^{i+1}\bar{C}_i^{-\gamma}v(G_{i+1})\Big)\right] \xrightarrow[C_t,\theta_t]{} \text{max.}, \tag{3.28}$$

where v is the value function from Equation (3.18).

Moreover, G_{t+1} represents the next-period value of the risky investment and yields:

$$G_{t+1} = \theta_t(W_t - C_t)(R_{t+1} - R_{ft}), \tag{3.29}$$

where W_t stands for the *total wealth* and θ_t for the *fraction of post-consumption wealth allocated to the risky portfolio.*

The following equations provide for the formal compatibility of the one-dimensional utility framework in Section 3.2 and the current two-dimensional utility framework. Specifically, the post-consumption wealth proportion put in risky assets θ_t, the current value of the risky investment S_t, as well as the amount of money B_t borrowed ($B_t > 0$) or lent ($B_t < 0$) are reformulated in order to correspond to the total wealth W_t that now encompasses not only financial wealth, but also consumption. They yield:

$$\theta_t = \frac{W_t - C_t + B_t}{W_t - C_t} \tag{3.30a}$$

$$S_t = \theta_t(W_t - C_t) \tag{3.30b}$$

$$B_t = (W_t - C_t)\frac{\text{VaR}^* + \text{VaR}}{(W_t - C_t)R_{ft} - \text{VaR}}. \tag{3.30c}$$

and thus the post-consumption wealth fraction allocated to risk-free assets entails $1 - \theta_t = -B_t/(W_t - C_t)$. In addition, the next-period total wealth results from the current financial investment and can be expressed as:[198]

$$W_{t+1} = (W_t - C_t)\Big(\theta_t R_{t+1} + (1-\theta_t)R_{ft}\Big). \tag{3.31}$$

Note that the maximization in Equation (3.28) is carried out with respect to both the

[198] A part C_{t+1}/W_{t+1} is subsequently allocated to consumption, but consumption only generates utility and no wealth.

consumption C_t and the wealth fraction invested in risky assets θ_t (and hence to the value of the risky investment S_t). Thus, the corresponding Euler equations for optimality at equilibrium yield:[199]

$$\rho R_f E\left[\left(\frac{\bar{C}_{t+1}}{\bar{C}_t}\right)^{-\gamma}\right] = 1 \tag{3.32a}$$

$$\rho E\left[R_{t+1}\left(\frac{\bar{C}_{t+1}}{\bar{C}_t}\right)^{-\gamma}\right] + b_0\rho\bar{C}_t^{-\gamma}E[\bar{v}(G_{t+1})] = 1. \tag{3.32b}$$

According to our approach, $E[\bar{v}(G_{t+1})]$ represents the *prospective value at equilibrium* that we denote by $\bar{V}$. Our goal is to provide an empirical estimate $\hat{V}$ of this prospective value from which we can derive the equilibrium-equivalent loss-aversion coefficient $\hat{\lambda}$ that follows Equation (3.26).

In order to perform the estimation of $\bar{V}$, additional assumptions concerning the consumption and return dynamics are necessary. In line with Equations (68)-(70) in Barberis and Huang (2006), we take:[200]

$$\log\left(\frac{C_{t+1}}{C_t}\right) = c + \sigma_c\epsilon_{t+1} \tag{3.33a}$$

$$\log(R_{t+1}) = r + \sigma_r\eta_{t+1} \tag{3.33b}$$

$$\begin{pmatrix}\epsilon_{t+1}\\ \eta_{t+1}\end{pmatrix} \sim N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & \sigma_{cr}\\ \sigma_{cr} & 1\end{pmatrix}\right), \text{ i.i.d. over time.} \tag{3.33c}$$

Thus, for a constant risk-free rate R_f, the expected-equilibrium Equations (3.32) en-

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[199] See Equations (27) and (28) in Barberis, Huang, and Santos (2001).

[200] Note that Barberis, Huang, and Santos (2001) assume that returns develop following the dividends payed by the risky asset $R_{t+1} = (P_{t+1} + D_{t+1})/P_t$. If this can be considered as a sound assumption on an annual basis, the following problem emerges in terms of higher portfolio evaluation frequencies: While prices vary daily, dividends are released only once every three months or even at longer time intervals. (For instance, according to data from finance.yahoo.com, the mean frequency of dividends releases amounts to approximatively 4.5 months.) This generates a non-smooth dividend evolution that accounts, on the one hand, for the dates of dividend release (when dividends truly change in value) and, on the other hand, for the in-between periods (when no dividends are distributed to investors, meaning that dividends can be considered as constant from one trade to the other). Formally, between two successive dividend releasing times $(u, u+1]$, we have:

$$D_{t+1} = \begin{cases} D_u, & \text{for } t \in (u, u+1) \\ D_{u+1}, & \text{for } t = u+1, \end{cases} \qquad \text{hence} \qquad \frac{D_{t+1}}{D_t} = \begin{cases} 1, & \text{for } t \in (u, u+1) \\ \dfrac{D_{u+1}}{D_u}, & \text{for } t = u+1. \end{cases}$$

tail:[201]

$$\exp\left(-\gamma c + \frac{\gamma^2\sigma_c^2}{2}\right) = \frac{1}{\rho R_f} \tag{3.34a}$$

$$\exp\left(-\gamma c + r + \frac{\gamma^2\sigma_c^2 + \sigma_r^2}{2} - \gamma\sigma_{cr}\right) + b_0 \bar{C}_t^{-\gamma}\bar{V} = \frac{1}{\rho}. \tag{3.34b}$$

The non-expected utility approach Although the expected-utility maximization represents the most widespread theoretical approach so far, Barberis, Huang, and Thaler (2006) claim that another specification captures better the utility of decisions under risk: the *non-expected recursive utility with first-order risk aversion* (abbr. R-FORA). Yet, simple R-FORA specifications account merely for loss aversion and hence have to be extended in order to accommodate with both loss aversion and narrow framing, since these two phenomena appear to be crucial in financial markets. Henceforth, we refer to the R-FORA setting with narrow framing as the non-expected utility approach.

We rely on the approach proposed in Barberis and Huang (2006), according to whom investors maximize a recursive utility-function U_t, that is defined as:

$$U_t = \diamond\Big\langle C_t, \mu(U_{t+1} + b_0 E_t[v(G_{t+1})] | \digamma_t)\Big\rangle, \tag{3.35}$$

where:

$$\diamond\langle C, y\rangle = \Big((1-\beta)C^{1-\gamma} + \beta y^{1-\gamma}\Big)^{\frac{1}{1-\gamma}},\ \text{for } 0<\beta<1,\ 0<\gamma\neq 1 \tag{3.36a}$$

$$\mu(y) = \Big(E[y^{1-\gamma}]\Big)^{\frac{1}{1-\gamma}},\ \text{for } 0<\gamma\neq 1 \tag{3.36b}$$

$$G_{t+1} = \theta_t(W_t - C_t)(R_{t+1} - R_{ft}) \tag{3.36c}$$

$$v(y) = \begin{cases} y, & \text{for } y \geq 0 \\ \lambda x, & \text{for } y < 0, \end{cases} \quad \text{for } \lambda > 1. \tag{3.36d}$$

Here, $\diamond\langle .,.\rangle$ is an aggregator function, $\mu(.)$ the certainty equivalent of the distribution of future utility conditional on the information $\digamma_t$ at time t and G_{t+1} the next-period value of the risky investment. The parameter β is henceforth referred to as the *weight of*

[201] The derivation is easy under the partial result that, for $x \sim N(\mu, \sigma^2)$, we have $E[\exp(x)] = \exp(\mu + \sigma^2/2)$. Also, for $x_i \sim N(\mu_i, \sigma_i^2)$, where $i = 1, 2$, i.i.d. over time and with covariance σ_{12}, we can write $E[\exp(x_1 + x_2)] = \exp(\mu_1 + \mu_2 + (\sigma_1^2 + \sigma_2^2)/2 + \sigma_{12})$.

financial utility.

According to Barberis and Huang (2004), the certainty equivalent $\mu(.)$ is assumed to be homogenous of degree one and, in order to ensure tractability, the individual value function v must also be homogenous. Consequently, for the equilibrium conditions to be necessary and sufficient, v has to take the piecewise-linear form in Equation (3.36d).[202] In other words, a good behaved equilibrium with non-expected utility does *not* allow for the influence of past performance on current perceptions of financial risk as proposed in the extended PT-framework by Barberis, Huang, and Santos (2001), but merely for loss aversion as in the initial PT of Kahneman and Tversky.

Therefore, the non-expected utility equilibrium reduces to imposing the condition of nil cushions $S_t = Z_t$ in all equations of the theoretical model in Section 3.2. This condition can be interpreted as a particular case with dynamic cushions $\eta(S_t - Z_{t-1}\bar{R})$, where $\eta = 0$ or, in other words, when investors have no memory of the past performance. It is this case that we extensively study in the applicative section and which provides a basis for the comparison between the settings with expected and non-expected utility. Obviously, all influence of the sensitivity to past losses k on our model estimates is discarded.

Coming back to the non-expected utility maximization, we again restrict our analysis to the general equilibrium for aggregate markets with a representative investor, in line with Equations (60)-(62) and with the subsequent Example 6.1 for stock markets in Barberis and Huang (2004). Our focus remains on non-professional investor decisions concerning wealth allocation among consumption, the risky portfolio with gross returns R_t, and the risk-free asset with the gross return R_{ft}.

Let α_t be the fraction of total wealth dedicated to consumption, which is formally:

$$\alpha_t = \frac{C_t}{W_t}. \tag{3.37}$$

When a fraction θ_t of post-consumption wealth is to be invested in the risky portfolio and another fraction of total wealth α_t to be consumed, the following (Euler) equations yield

[202]For more details on this specification, see Barberis and Huang (2004). The same authors demonstrate that the two fund separation theorem continues to hold with the above specification of recursive non-expected utility.

necessary and sufficient conditions at equilibrium:

$$\beta R_{ft} E_t \left[\left(\frac{\bar{C}_{t+1}}{\bar{C}_t} \right)^{-\gamma} \right] \left(\beta E_t \left[\left(\frac{\bar{C}_{t+1}}{\bar{C}_t} \right)^{-\gamma} R^{\mathrm{tot}}_{t+1} \right] \right)^{\frac{\gamma}{1-\gamma}} = 1 \tag{3.38a}$$

$$\frac{E_t \left[\left(\frac{\bar{C}_{t+1}}{\bar{C}_t} \right)^{-\gamma} (R_{t+1} - R_{ft}) \right]}{E_t \left[\left(\frac{\bar{C}_{t+1}}{\bar{C}_t} \right)^{-\gamma} \right]} + b_0 R_{ft} \left(\frac{\beta}{1-\beta} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{1-\alpha_t}{\alpha_t} \right)^{-\frac{\gamma}{1-\gamma}} E_t[v(R_{t+1} - R_{ft})] = 0 \tag{3.38b}$$

$$\frac{E_t \left[\left(\frac{\bar{C}_{t+1}}{\bar{C}_t} \right)^{-\gamma} (R^{\mathrm{tot}}_{t+1} - R_{ft}) \right]}{E_t \left[\left(\frac{\bar{C}_{t+1}}{\bar{C}_t} \right)^{-\gamma} \right]} + b_0 R_{ft} \left(\frac{\beta}{1-\beta} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{1-\alpha_t}{\alpha_t} \right)^{-\frac{\gamma}{1-\gamma}} \theta_t E_t[v(R_{t+1} - R_{ft})] = 0, \tag{3.38c}$$

where $R^{\mathrm{tot}}_{t+1} = \theta_t R_{t+1} + (1-\theta_t) R_{ft}$ is the *total gross return of financial assets.*[203] Equation (3.38c) is derived from (3.38b) by multiplying it with θ_t.

The next period financial wealth formulated in Equation (3.31) can be now rewritten as $W_{t+1} = (W_t - C_t) R^{\mathrm{tot}}_{t+1}$ and thus we obtain:

$$R^{\mathrm{tot}}_{t+1} = \frac{\alpha_t}{\alpha_{t+1}(1-\alpha_t)} \frac{C_{t+1}}{C_t}. \tag{3.39}$$

Assuming time constancy for the portfolio wealth fraction θ, the consumption ratio α, and the risk-free return R_f, the total gross return results in:

$$R^{\mathrm{tot}}_{t+1} = \frac{1}{1-\alpha} \frac{C_{t+1}}{C_t} \Rightarrow \log(R^{\mathrm{tot}}_{t+1}) = -\log(1-\alpha) + c + \sigma_c \epsilon_{t+1}. \tag{3.40}$$

[203] Specifically, the total return of the combination between risky and risk-free assets.

Thus, the equilibrium Equations (3.38) yield:

$$\beta^{\frac{1}{1-\gamma}}(1-\alpha)^{-\frac{\gamma}{1-\gamma}} R_f E\left[\left(\frac{\bar{C}_{t+1}}{\bar{C}_t}\right)^{-\gamma}\right]\left(E\left[\left(\frac{\bar{C}_{t+1}}{\bar{C}_t}\right)^{1-\gamma}\right]\right)^{\frac{\gamma}{1-\gamma}} = 1 \tag{3.41a}$$

$$\frac{E\left[\left(\frac{\bar{C}_{t+1}}{\bar{C}_t}\right)^{-\gamma}(R_{t+1}-R_f)\right]}{E\left[\left(\frac{\bar{C}_{t+1}}{\bar{C}_t}\right)^{-\gamma}\right]} + b_0 R_f\left(\frac{\beta}{1-\beta}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{1-\alpha}{\alpha}\right)^{-\frac{\gamma}{1-\gamma}} E[\bar{v}(R_{t+1}-R_f)] = 0 \tag{3.41b}$$

$$\frac{E\left[\left(\frac{\bar{C}_{t+1}}{\bar{C}_t}\right)^{-\gamma}(R^{\text{tot}}_{t+1}-R_f)\right]}{E\left[\left(\frac{\bar{C}_{t+1}}{\bar{C}_t}\right)^{-\gamma}\right]} + b_0 R_f\left(\frac{\beta}{1-\beta}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{1-\alpha}{\alpha}\right)^{-\frac{\gamma}{1-\gamma}} \theta E[\bar{v}(R_{t+1}-R_f)] = 0. \tag{3.41c}$$

We proceed similarly to the expected-utility setting by assuming the same parameter dynamics of consumption and returns as in Equations (3.33). Also, we consider that $\bar{V} = E[\bar{v}(R_{t+1}-R_f)]$, where v corresponds to the value functions in Equation (3.18) under the condition that $S_t = Z_t$, are equivalent in expectation to the prospective value in Equation (3.24). Thus, the equilibrium Equations (3.41) can be further restated as follows:

$$\beta^{\frac{1}{1-\gamma}}(1-\alpha)^{-\frac{\gamma}{1-\gamma}} R_f \exp\left(\frac{\gamma\sigma_c^2}{2}\right) = 1 \tag{3.42a}$$

$$\begin{aligned} &-\exp\left(-\gamma c + r + \frac{\gamma^2\sigma_c^2+\sigma_r^2}{2} - \gamma\sigma_{cr}\right) + R_f \exp\left(-\gamma c + \frac{\gamma^2\sigma_c^2}{2}\right) \\ &\quad = b_0 R_f\left(\frac{\beta}{1-\beta}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{1-\alpha}{\alpha}\right)^{-\frac{\gamma}{1-\gamma}} \exp\left(-\gamma c + \frac{\gamma^2\sigma_c^2}{2}\right)\bar{V} \end{aligned} \tag{3.42b}$$

$$\begin{aligned} &-\frac{1}{1-\alpha}\exp\left((1-\gamma)c + \frac{(1-\gamma)^2\sigma_c^2}{2}\right) + R_f \exp\left(-\gamma c + \frac{\gamma^2\sigma_c^2}{2}\right) \\ &\quad = b_0 R_f\left(\frac{\beta}{1-\beta}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{1-\alpha}{\alpha}\right)^{-\frac{\gamma}{1-\gamma}} \exp\left(-\gamma c + \frac{\gamma^2\sigma_c^2}{2}\right)\theta\bar{V}. \end{aligned} \tag{3.42c}$$

3.3.3 Application

This section presents numerical findings based on the theoretical results from the above Section 3.3.2. We first review the general assumptions made in order to facilitate the estimation procedure and to render comparable the two settings with expected and non-expected utility. The estimation results are subsequently detailed for each of these settings separately. The exposition focuses on two main aspects: On the one hand, we address the evolution of the attitude towards financial losses. This attitude is described first by the loss-aversion coefficient and second by our extended measure gRA. On the other hand, we examine the decisions on wealth allocation among consumption, risky, and risk-free assets. This is quantified by the wealth fractions dedicated to the respective sources of utility. Throughout, we are also interested if mLA (in the strict, large, and monetary sense) continues to manifest in equilibrium.

General assumptions

The first data set that underlies our estimations is that considered in Section 3.2. It includes nominal returns of the stock index S&P 500 and of the three-months treasury bill – as proxies for the risky and the risk-free investment, respectively – from 01/02/1962 to 03/09/2006 (11,005 daily observations). This data is again divided into two parts: The observations before 03/01/1982 serve to estimate the empirical mean and the standard deviation of the portfolio returns at the date considered to be the beginning of the trade, namely on 03/01/1982; The second part, from 03/01/1982 to 03/09/2006 (6,010 observations), is the actual data used for performing simulations.

Additionally, aggregate per-capita consumption data between 01/02/1962 and 12/31/2005 sampled at quarterly intervals, provide a basis for the calculation of the log-consumption's mean and variance.[204] Note that the consumption data set only allows to assess consumption values corresponding to portfolio evaluations horizons of one year and three months. Consequently, we cannot replicate the more detailed analysis in Section 3.2 regarding the impact of the evaluation frequency on investor decisions. We will, however, analyze how the recommendation of our model changes for these two evaluation frequencies.

After smoothing out the outlier corresponding to the October 1987 market crash,[205] quarterly and yearly returns are constructed from the actual data set and used to derive

[204] This data was provided by the Department of Commerce, Bureau of Economic Analysis and Bureau of the Census. Descriptive statistics can be found in Table A.1 in Appendix A.3.1.

[205] Specifically, the outlier is replaced with the mean of the ten before and after data points.

the main variables that describe the loss attitude and the optimal wealth allocation of our non-professional investors. In so doing, we assume that investors start by spreading their wealth equally between consumption and financial assets. The latter fraction is further allocated in equal parts to the risky index and the risk-free T-bill. Our investors are long-lived beyond the VaR-horizon and are not allowed to quit the market during the trading period. Portfolio gross returns are assumed to be normally distributed, and future portfolio returns to be estimated as the unconditional mean of past returns. In addition, the risk-free returns, the mean log-consumption, and the mean risky returns are set identical to the means of the corresponding variable, computed over the actual trade period from 03/01/1982 to 03/09/2006, specifically as $\hat{R}_f = \text{mean}[R_{ft}]$, $\hat{c} = \text{mean}[\log(C_{t+1}) - \log(C_t)]$, and $\hat{r} = \text{mean}[\log(R_t)]$, respectively.

A final and more specific assumption, common to both approaches with expected and non-expected utility, tackles an issue emerging from our considerations that investors are long-lived and view financial investments as single source of wealth:[206] It is possible that financial investments do not generate sufficient revenues in order to cover consumption needs over the entire trade interval. We circumvent this potential problem by considering that, at each time t, investors dispose of additional incomes I_t.[207] Such incomes represent, for instance, the wages earned by non-professional investors from their main employment.[208] They are *exogenous*, that is, they stem from outside of those investments that constitute the decision making object at hand. Under this assumption, the total wealth in Equation (3.31) results from both financial investments and additional incomes and yields at $t+1$:

$$W_{t+1} - I_{t+1} = (W_t - C_t)\Big(\theta_t R_{t+1} + (1-\theta_t)R_{ft}\Big). \tag{3.43}$$

The additional income I_t may cover a part of the consumption needs of the current

[206] Recall that both consumption and financial investments generate *utility*, yet only the latter is "productive" and can effectively augment *wealth*.

[207] Note that Barberis and Huang (2004, 2006) avoid this problem by fixing the percentage of post-consumption wealth invested in risky assets θ. We consider this as rather inappropriate in our framework for two reasons: First, Barberis and Huang (2004, 2006) exclusively work with non-expected utility, while our aim is to render comparable two settings, namely those with expected and non-expected utility. Second, our model relies on the idea that θ depends on the subjective VaR* (see Equations (3.30)). Hence, imposing constancy on this parameter would eliminate the whole analyzed mechanism of how individual perceptions of financial investments are reflected in the wealth allocation.

[208] As their name indicates, *non-professional* investors mainly earn their living from other activities (developed for example as employees of a company) than from financial investments. The latter merely represent a secondary source of revenues.

period and hence we define it as follows:[209]

$$I_t = \frac{C_t}{\alpha\delta}, \tag{3.44}$$

where α represents the percentages of total wealth dedicated to consumption and δ is an arbitrary constant. Of course, both $\alpha, \delta > 0$.

Apparently, for $\delta \leq 1/\alpha$ ($\delta > 1/\alpha$) the additional income exceeds (does not entirely meet) the consumption needs of the period $I_t \geq C_t$ ($I_t < C_t$). We distinguish two particular cases which, due to their lack of practical meaning, are of *no* interest in the present framework: First, for $\delta = 1$, the current additional income yields a fraction of the consumption needs $I_t = C_t/\alpha$ and investors should assign no money to financial assets in total $R_{t+1}^{\text{tot}} = 0$.[210] Second, for $\delta = 1/\alpha$ the additional income covers exactly the current consumption $I_t = C_t$. Then, the total financial investment would be $R_{t+1}^{\text{tot}} = C_{t+1}/C_t$ and hence independent of α, which eliminates any connection between the investment decision and the subjective perception of financial investments in the non-expected utility equilibrium.[211] Consequently, we should be looking for values of $\delta \in \mathbb{R}^+ \setminus \{0, 1, 1/\alpha\}$.[212]

We choose α-values that conform with the equilibrium estimates of the total-wealth percentages allocated to consumption in the non-expected-utility setting.[213] In particular, we vary the level of the additional income I_t by changing the parameters β and δ. The rationale is that α depends on the financial weight parameter β, according to Equations (3.41). We observe that I_t increases (decreases) subject to higher values of β (δ), as higher additional incomes are equivalent to more relaxed requirements for financial

[209] This assumption permits an easy formal manipulation and ensures the comparability of the two settings. At the same time, it allows for sufficient flexibility with respect to the choice of the income magnitude.

[210] Moreover, in this case the percentage of post-consumption wealth assigned to risky assets in equilibrium from Equation (3.48b) yields $\theta = R_f/(R_f - \exp(r + \sigma_r^2/2 - \gamma\sigma_{cr})) \geq 1$. This induces investors to throw caution to the wind, borrowing even more money, which is then channeled into the risky portfolio.

[211] Recall first that V relies on subjective perceptions and is derived on the basis of behavioral parameters according to Equation (3.24). Then this connection is ensured by the interdependency of α and $\bar{V}$ from Equations (3.48). This interdependency is central to our work since we assume that all decisions rely on individual perceptions and attempt to analyze how they change subject to different perception parameters. For $\delta = 1/\alpha$, the percentage of post-consumption wealth allocated to risky assets in equilibrium from Equation (3.48b) becomes independent of α, specifically $\theta = (R_f - \exp(c + (1-2\gamma)\sigma_c^2/2))/(R_f - \exp(r + \sigma_r^2/2 - \gamma\sigma_{cr}))$.

[212] Note that for the purpose of comparability, we consider identical additional incomes in both settings with expected and non-expected utility.

[213] We are forced to do so, as this setting is the only one that delivers direct values of this variable in the market equilibrium. Under expected utility, the percentages of total wealth to be consumed can be only assessed on average from the equilibrium estimates.

investments.[214] We take $\beta \in \{0.1; 0.5; 0.9; 0.98\}$, where the last value is the one estimated in Barberis and Huang (2004), and $\delta \in \{0.1; 0.5; 0.9; 2; 10; 100\}$. Note that, in the expected-utility setting, β has no direct intuitive meaning and thus it is easier to interpret the changes in the additional income resulting from different (β, δ)-combinations.

Further assumptions concern the remaining model parameters, that are in the main of behavioral nature and that we vary in order to study their influence on our main equilibrium variables. In particular, we work with different values for the initial loss-aversion coefficient $\lambda \in \{1; 2.25; 3\}$, where only the latter two corroborate with the non-expected equilibrium according to Equation (3.36d). We rely on Barberis, Huang, and Santos (2001) and Barberis and Huang (2006) in choosing the risk-aversion degree $\gamma \in \{0.5; 1; 1.5\}$ for the expected-utility setting, where higher values point to increased aversion. In line with the condition $\gamma \neq 1$ from Equation (3.36a), we use $\gamma \in \{0.5; 1.5\}$ in the non-expected-utility setting. Furthermore, we consider narrow-framing degrees $b_0 \in \{0.001; 0.1; 0.5; 1; 5; 10; 100; 1000\}$, where the first value stands for the situation with (almost) no narrow framing since $b_0 \neq 0$ according to Equation (3.45).[215] Following the same authors, we also account for no, moderate, and high influence of past losses on the perception of risky investments $k \in \{0; 3; 10\}$. Recall yet that k plays no role in the non-expected utility equilibrium. Finally, the maximizers of expected utility are assumed to assess cushions either myopically as $S_t - S_{t-1}$ or dynamically as $\eta(S_t - Z_{t-1}\bar{R})$ with different memory-length parameters $\eta \in \{0; 0.1; 0.5; 0.9; 1\}$ and with $\bar{R} = \text{mean}[R_t]$. The non-expected utility equilibrium allows only for memoryless dynamic cushions $\eta = 0$.

Since the lack of space does not allow for an extensive presentation of all obtained results, we subsequently focus on few cases that appear to be the most realistic and that entail plausible estimates in both settings, thus providing a support for comparison. Further interesting situations are explicitly indicated.In particular, we refer to risk-averse investors $\gamma = 0.5$ who narrowly frame financial investments $b_0 \geq 1$. We furthermore

[214]Due to our assumption that $I_t = C_t/(\alpha\delta)$ and according to $\alpha = 1 - \beta^{\frac{1}{\gamma}} R_f^{\frac{1-\gamma}{\gamma}} \exp((1-\gamma)\sigma_c^2/2)$ in the subsequent Equation (3.48a).

[215]In so doing, we are in line with Barberis, Huang, and Santos (2001) and less with Barberis and Huang (2004, 2006) who choose $b_0 < 1$. The reason is the following: Recall that the weight b_t of the utility derived from risky investments in Equation (3.27) is defined as the product of the narrow-framing coefficient b_0 with the perceived aggregate consumption $\bar{C}_t^{-\gamma}$. Thus, this weight falls with the consumption-related risk aversion γ and hence higher b_0-values are required in order to generate reasonable values. In our sample mean$[C_t] \approx 46000$, so that yet the highest considered $b_0 = 1000$ yields for $\gamma = 1$ a contribution of around 0.02174 of the perceived risky utility to the total utility, while for $\gamma = 1.5$ this contribution drops to 0.0001. Similarly for $\gamma = 0.5$, $b_0 = 1$ is the lowest value that yields reasonable mean$[b_t] = 0.00466$. Indeed, the estimates obtained for $b_0 < 1$ are in the main part implausible.

consider three qualitatively different levels of the average additional income I, i.e. low, middle, and high, that correspond to the combinations $(\beta = 0.1, \delta = 0.9)$, $(\beta = 0.5, \delta = 0.5)$, and $(\beta = 0.9, \delta = 0.1)$, respectively. Also, we briefly address the middle-range income combination $(\beta = 0.9, \delta = 2)$ at the end of the applicative sections on the expected and non-expected utility equilibriums.[216] In addition, we merely refer to the value of the initial loss-aversion coefficient the most used in literature $\lambda = 2.25$.[217] In line with Section 3.2, our expected-utility maximizers use long-memory dynamic cushions with $\eta = 0.9$.

The expected-utility approach

We start by estimating the main variables that quantify the loss attitude and the wealth allocation in the expected-utility equilibrium. Our interest lies in how they change subject to different psychological profiles of the representative investor. In the sequel, we report on *average* changes of these variables subject to the *ceteris paribus* variation of chosen parameters.

The main equilibrium variables in the expected-utility setting are the discounting factor ρ and the prospective value $\bar{V}$.[218] They are obtained by plugging in the parameter values assumed at the beginning of this applicative part into the following reformulations of Equations (3.34):

$$\rho = \frac{1}{R_f} \exp\left(\gamma c - \frac{\gamma^2 \sigma_c^2}{2}\right) \tag{3.45a}$$

$$\bar{V} = \frac{\bar{C}_t^{\gamma}}{b_0} \left(\frac{1}{\rho} - \exp\left(-\gamma c + r + \frac{\gamma^2 \sigma_c^2 + \sigma_r^2}{2} - \gamma \sigma_{cr}\right)\right). \tag{3.45b}$$

From the expression of the prospective value in equilibrium in Equation (3.45b), we can further assess the corresponding loss-aversion coefficient $\bar{\lambda}$ according to Equation (3.26). All estimates are henceforth denoted by a $\hat{.}$ symbol.

Before detailing the estimation results, let us make an important remark regarding the interpretation of the prospective value in equilibrium: Combining the above Equations

[216] The main part of the model estimates obtained for $\delta > 1$, result in implausible values of our variables in at least one of the settings. The single exception is the combination $(\beta = 0.9, \delta = 2)$. It corresponds to additional incomes of middle level, specifically somewhat higher than those give by $(\beta = 0.5, \delta = 0.5)$.

[217] This is not to be mistaken for the equilibrium estimate $\hat{\lambda}$ from Equation (3.26). Recall that the initial coefficient λ is employed in order to derive VaR* from Equation (3.22), that affects the optimal wealth allocation in Equation (3.16) and hence all further model variables.

[218] Of course, we could also fix ρ and estimate γ. However, this procedure proves to be more delicate, given that Equation (3.34a) is a second-order equation in γ, such that the existence and number of real roots depends on the sign of its determinant $c^2 - 2\sigma_c^2 \log(\rho R_f)$. In the case with either none or two distinct real solutions, we cannot provide an economic interpretation of the equilibrium.

(3.45), we can rewrite $\bar{V}$ as follows:

$$\bar{V} = \frac{\bar{C}_t^{\gamma}}{b_0} \exp\left(-\gamma c + \frac{\gamma^2 \sigma_c^2}{2}\right)\left(R_f - \exp\left(r + \frac{\sigma_r^2}{2} - \gamma \sigma_{cr}\right)\right).$$

This expression is proportional to a factor that indicates the trade-off between the revenues from risk-free vs. risky investments, which is the last right-hand term in parentheses $R_f - \exp(r + \sigma_r^2/2 - \gamma\sigma_{cr})$.[219] This term *decreases* as risky investments become *more* profitable and so also does $\bar{V}$. In other words, the prospective value is directly proportional to the profitability of risky with respect to risk-free assets taken *with negative sign*. Recall now that the prospective value stands for the perceptions of the benefits of risky investments – being, as its name says, a "value" – and hence it is expected to *grow* as risky assets become *more* profitable. Therefore, only the absolute values of the prospective value in equilibrium $|\bar{V}|$ can be meaningfully interpreted and our subsequent comments will exclusively refer to such absolute values.[220]

The main equilibrium estimates under maximization of expected utility for yearly (quarterly) portfolio evaluations and our usual parameter values, that is $\lambda = 2.25$, $\gamma = 0.5$, $b_0 \geq 1$, myopic and long-memory dynamic cushions, and low, middle, and high additional incomes I, are illustrated in Table 3.5 (Table 3.6).

First, as implied by Equation (3.45a), the estimates $\hat{\rho}$ of the discounting factor do not vary with any of our behavioral parameters b_0, β, δ, k, or the type of cushion, and hence we discard them from Tables 3.5 and 3.6. In particular, $\hat{\rho} = 0.99059$ (0.95685) for yearly (quarterly) evaluations. The decline of $\hat{\rho}$ with the frequency of the risky-performance evaluation can be intuitively explained by the fact that the notion of immediacy might change its meaning when investors check more often on their portfolios: Perceived time intervals are shorter and thus the preference for immediacy should drop.

Thus, our estimates $\hat{\rho}$ of the discounting factor lie close to the value of 0.98 assumed by Barberis, Huang, and Santos (2001) and Barberis and Huang (2006), speaking for the validity of our approach. Unreported results show also that the discounting is weaker for higher consumption-related risk-aversion γ. However, $\gamma = 1.5$ entails ρ-values that lie

[219] We can approximate this factor by $R_f - \exp(r + \sigma_r^2/2) \approx R_f - E_t[R_{t+1}]$, which is the expected risk premium taken with a negative sign. This approximation holds in our data set, since the correlation between risky returns and consumption takes a very low value $\sigma_{cr} = 0.0168$ (0.0056) for yearly (quarterly) evaluations.

[220] This effect is specific to the present equilibrium framework (independently of whether expected or non-expected utility is maximized), since in the one-sided utility framework from Section 3.2 we also obtain negative estimates $\hat{V}$, but there, these values are meaningful as such.

		Myopic cushions			Dynamic cushions with $\eta = 0.9$		
		$k = 0$	$k = 3$	$k = 10$	$k = 0$	$k = 3$	$k = 10$
Low I ($\beta = 0.1, \delta = 0.9$)		$b_0 = 1$					
	$\lvert\hat{V}\rvert$	229.8	229.8	229.8	229.8	229.8	229.8
	$\hat{\lambda}$	0.88272	0.69103	0.52197	0.040833	-0.013175	-0.079496
		$b_0 = 5$					
	$\lvert\hat{V}\rvert$	45.961	45.961	45.961	45.961	45.961	45.961
	$\hat{\lambda}$	0.55173	0.51857	0.49687	0.098019	0.08707	0.073462
		$b_0 = 10$					
	$\lvert\hat{V}\rvert$	22.98	22.98	22.98	22.98	22.98	22.98
	$\hat{\lambda}$	0.51035	0.49702	0.49373	0.10517	0.099601	0.092582
		$b_0 = 100$					
	$\lvert\hat{V}\rvert$	2.298	2.298	2.298	2.298	2.298	2.298
	$\hat{\lambda}$	0.47311	0.47762	0.49091	0.1116	0.11088	0.10979
		$b_0 = 1000$					
	$\lvert\hat{V}\rvert$	0.2298	0.2298	0.2298	0.2298	0.2298	0.2298
	$\hat{\lambda}$	0.46939	0.47568	0.49062	0.11224	0.11201	0.11151
Middle I ($\beta = 0.5, \delta = 0.5$)		$b_0 = 1$					
	$\lvert\hat{V}\rvert$	229.8	229.8	229.8	229.8	229.8	229.8
	$\hat{\lambda}$	0.61664	0.59439	0.57693	0.15208	0.14411	0.13325
		$b_0 = 5$					
	$\lvert\hat{V}\rvert$	45.961	45.961	45.961	45.961	45.961	45.961
	$\hat{\lambda}$	0.49851	0.49925	0.50786	0.12027	0.11853	0.11601
		$b_0 = 10$					
	$\lvert\hat{V}\rvert$	22.98	22.98	22.98	22.98	22.98	22.98
	$\hat{\lambda}$	0.48374	0.48735	0.49923	0.11629	0.11533	0.11386
		$b_0 = 100$					
	$\lvert\hat{V}\rvert$	2.298	2.298	2.298	2.298	2.298	2.298
	$\hat{\lambda}$	0.47045	0.47665	0.49146	0.11271	0.11245	0.11192
		$b_0 = 1000$					
	$\lvert\hat{V}\rvert$	0.2298	0.2298	0.2298	0.2298	0.2298	0.2298
	$\hat{\lambda}$	0.46912	0.47558	0.49068	0.11236	0.11216	0.11172
High I ($\beta = 0.9, \delta = 0.1$)		$b_0 = 1$					
	$\lvert\hat{V}\rvert$	229.8	229.8	229.8	229.8	229.8	229.8
	$\hat{\lambda}$	0.47727	0.48285	0.4968	0.11589	0.11545	0.11465
		$b_0 = 5$					
	$\lvert\hat{V}\rvert$	45.961	45.961	45.961	45.961	45.961	45.961
	$\hat{\lambda}$	0.47064	0.47694	0.49183	0.11303	0.1128	0.11229
		$b_0 = 10$					
	$\lvert\hat{V}\rvert$	22.98	22.98	22.98	22.98	22.98	22.98
	$\hat{\lambda}$	0.46981	0.4762	0.49121	0.11267	0.11246	0.112
		$b_0 = 100$					
	$\lvert\hat{V}\rvert$	2.298	2.298	2.298	2.298	2.298	2.298
	$\hat{\lambda}$	0.46906	0.47553	0.49065	0.11235	0.11216	0.11173
		$b_0 = 1000$					
	$\lvert\hat{V}\rvert$	0.2298	0.2298	0.2298	0.2298	0.2298	0.2298
	$\hat{\lambda}$	0.46898	0.47547	0.4906	0.11232	0.11213	0.1117

Table 3.5: The main variable estimates in the expected-utility equilibrium for yearly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, and using various cushion-assessment methods, degrees of past-loss sensitivity k, additional-income levels I, and narrow-framing degrees b_0.

		Myopic cushions			Dynamic cushions with $\eta = 0.9$		
		$k = 0$	$k = 3$	$k = 10$	$k = 0$	$k = 3$	$k = 10$
Low I ($\beta = 0.1, \delta = 0.9$)		$b_0 = 1$					
	$\lvert\hat{V}\rvert$	460.89	460.89	460.89	460.89	460.89	460.89
	$\hat{\lambda}$	2.8189	2.499	1.9725	-0.77164	-0.84913	-0.98526
		$b_0 = 5$					
	$\lvert\hat{V}\rvert$	92.179	92.179	92.179	92.179	92.179	92.179
	$\hat{\lambda}$	3.4252	3.1768	2.6648	-1.3932	-1.4394	-1.7254
		$b_0 = 10$					
	$\lvert\hat{V}\rvert$	46.089	46.089	46.089	46.089	46.089	46.089
	$\hat{\lambda}$	3.5009	3.2616	2.7514	-1.4709	-1.5132	-1.8179
		$b_0 = 100$					
	$\lvert\hat{V}\rvert$	4.6089	4.6089	4.6089	4.6089	4.6089	4.6089
	$\hat{\lambda}$	3.5691	3.3378	2.8293	-1.5409	-1.5796	-1.9011
		$b_0 = 1000$					
	$\lvert\hat{V}\rvert$	0.46089	0.46089	0.46089	0.46089	0.46089	0.46089
	$\hat{\lambda}$	3.576	3.3454	2.8371	-1.5478	-1.5863	-1.9095
Middle I ($\beta = 0.5, \delta = 0.5$)		$b_0 = 1$					
	$\lvert\hat{V}\rvert$	460.89	460.89	460.89	460.89	460.89	460.89
	$\hat{\lambda}$	3.4378	3.1978	2.6327	-1.4292	-1.4706	-1.6006
		$b_0 = 5$					
	$\lvert\hat{V}\rvert$	92.179	92.179	92.179	92.179	92.179	92.179
	$\hat{\lambda}$	3.5489	3.3166	2.7734	-1.5247	-1.5637	-1.6614
		$b_0 = 10$					
	$\lvert\hat{V}\rvert$	46.089	46.089	46.089	46.089	46.089	46.089
	$\hat{\lambda}$	3.5628	3.3314	2.791	-1.5367	-1.5754	-1.669
		$b_0 = 100$					
	$\lvert\hat{V}\rvert$	4.6089	4.6089	4.6089	4.6089	4.6089	4.6089
	$\hat{\lambda}$	3.5753	3.3448	2.8068	-1.5474	-1.5858	-1.6758
		$b_0 = 1000$					
	$\lvert\hat{V}\rvert$	0.46089	0.46089	0.46089	0.46089	0.46089	0.46089
	$\hat{\lambda}$	3.5766	3.3461	2.8084	-1.5485	-1.5869	-1.6765
High I ($\beta = 0.9, \delta = 0.1$)		$b_0 = 1$					
	$\lvert\hat{V}\rvert$	460.89	460.89	460.89	460.89	460.89	460.89
	$\hat{\lambda}$	3.5729	3.3423	2.8038	-1.545	-1.5834	-1.6729
		$b_0 = 5$					
	$\lvert\hat{V}\rvert$	92.179	92.179	92.179	92.179	92.179	92.179
	$\hat{\lambda}$	3.576	3.3455	2.8077	-1.5479	-1.5863	-1.6758
		$b_0 = 10$					
	$\lvert\hat{V}\rvert$	6.089	46.089	46.089	46.089	46.089	46.089
	$\hat{\lambda}$	3.5763	3.3459	2.8081	-1.5483	-1.5866	-1.6762
		$b_0 = 100$					
	$\lvert\hat{V}\rvert$	4.6089	4.6089	4.6089	4.6089	4.6089	4.6089
	$\hat{\lambda}$	3.5767	3.3463	2.8086	-1.5486	-1.587	-1.6765
		$b_0 = 1000$					
	$\lvert\hat{V}\rvert$	0.46089	0.46089	0.46089	0.46089	0.46089	0.46089
	$\hat{\lambda}$	3.5767	3.3463	2.8086	-1.5486	-1.587	-1.6766

Table 3.6: The main variable estimates in the expected-utility equilibrium for quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, and using various cushion-assessment methods, degrees of past-loss sensitivity k, additional-income levels I, and narrow-framing degrees b_0.

slightly above 1, suggesting that a too high consumption-related risk-aversion might be incompatible with our expected-utility framework.[221] Henceforth, we concentrate on the case with $\gamma = 0.5$ that appears to be the only one plausible and common to both expected and non-expected utility approaches.

Let us secondly analyze the subjective utility of financial investments quantified by the prospective value. As expected, the corresponding estimates in the aggregate equilibrium $\hat{V}$ are indeed negative across all considered configurations of parameters. This is exemplified in Tables 3.5 and 3.6 for our usual cases. Note also that the evolution of $|\hat{V}|$ is independent of each the sensitivity to past losses k, the additional-income levels given by the (β, δ)-combinations, and even the way in which cushions are assessed. However, $|\hat{V}|$ decreases considerably subject to higher narrow framing b_0, so that a more intense focus on financial investments appears to worsen the perception of the utility they are generating. Moreover, $|\hat{V}|$ is smaller for quarterly data. Thus, the perceived utility of financial investments ameliorates when the risky performance is evaluated more often, a result that is at odds with the concept of mLA considered in the large sense.

We subsequently turn our attention to the investor attitude towards financial losses. One variable that captures this attitude is the loss-aversion coefficient and we denote its estimates by $\hat{\lambda}$. Table 3.5 reveals that, when risk averse investors revise risky performance yearly, $\hat{\lambda}$ lies substantially below the value of 2.25 advanced in the original prospect theory and in various subsequent theoretical and experimental studies. This situation changes yet radically for quarterly evaluations, as can be observed in Table 3.6. Now, investors who assess cushions myopically are clearly loss averse. This holds across all considered combinations of parameters and underpins mLA in the strict sense: (Even) in the aggregate equilibrium, investors become more reluctant to financial losses when they check on risky performance more often.[222]

Note also that the values of $\hat{\lambda}$ for yearly evaluations and myopic cushions are smaller than 1, thus speaking for loss-loving attitudes. Earlier findings show that such attitudes are not very likely to occur in practice. We can interpret this somewhat puzzling result in

[221] Specifically, for yearly (quarterly) evaluations $\hat{\rho}$ amounts to 0.99842 (0.98208) for $\gamma = 1$ and to 1.0063 (1.0079) for $\gamma = 1.5$. Further implausible results are obtained for $\gamma = 1.5$ with respect to the wealth percentages dedicated to consumption $\bar{C}/\hat{W}$ (please refer to the comments below).

[222] Unreported results show that, for yearly (quarterly) data and other things being equal, $\hat{\lambda}$ increases (decreases) with γ and even becomes extremely high (and negative) for $\gamma = 1.5$, at least for lower values of the narrow-framing coefficient b_0. This underlies our assumption that higher γ-values are meaningful only when coupled with sufficiently high b_0-values. Merely the variation observed for quarterly data appears to be intuitively meaningful: Investors who are less risk averse should be also less loss averse.

manifold ways: for instance, as a necessary condition for reaching the market equilibrium under maximization of two-dimensional expected utility. Markets with investors who are reluctant to financial risks (and behave according to our model) might then not attain a steady state. It is nevertheless possible that $\hat{\lambda}$ does not accurately measure the actual loss attitude. This reinforces the potential relevance of our second loss-attitude measure, gRA, on the estimates of which we comment below.[223]

As anticipated, having more money – from exogenous sources – at their disposal, renders investors less loss averse and $\hat{\lambda}$ diminishes, on average, with the additional income I. However, this holds only for yearly and not for quarterly data. The frequency of performance checks thus appears to overcome the role played by additional incomes and hence to drive loss aversion.

The influence of the narrow focus on loss aversion appears to be also dependent on the portfolio evaluation frequency: $\hat{\lambda}$ decreases (increases) with b_0 for yearly (quarterly) evaluations, other things being equal. In other words, it is not the importance ascribed by investors to financial investments as a source of utility, the one that changes their attitude towards possible losses, but how often they check on the risky performance. Frequent checks and a pronounced narrow framing result in a high loss reluctance, while seldom checks, even under an increasing narrow framing, entail a lower reluctance. This corroborates with mLA in the strict sense.[224]

In addition, $\hat{\lambda}$ clearly diminishes subject to higher sensitivities towards past losses k for quarterly portfolio evaluations. This is somewhat counterintuitive, since investors who behave aversely towards past losses (i.e. have a high k) should remain averse towards future ones as well (high λ).[225] The evaluation frequency plays yet a very important role also in this respect: For yearly evaluations, the same type of variation can be observed only for lower narrow framing b_0 and/or lower additional incomes I.[226]

[223]Another possible explanation would be that an equilibrium framework with CRRA expected utility might not be able to capture the actual behavior of investors confronted with financial losses, possibly due to the incompatibility of such an aggregate perspective and the individually oriented framework with VaR*-based decisions. However, we discard such an alternative, due to the straightforward intuition of the results obtained for quarterly data.

[224]Moreover, $\hat{\lambda}$ is extremely high (positive or negative) for $b_0 = 0.001$ and hence implausible. This supports our assumption that the lack of narrow framing is incompatible with the present framework.

[225]Also, the other loss-aversion measure gRA constantly falls with k across all considered combinations of parameters (see the comments on gRA below), pointing to a relaxation of the loss attitude.

[226]A possible explanation for the opposite change of the loss-aversion coefficient $\hat{\lambda}$ with the sensitivity to past losses k could be that investors who are less sure of being able to cover current consumption needs might be guided by the principle "all or nothing": They take the chance of investing in financial assets, as this chance – in other words gambling – hides not only the danger of losses, but also the promise of gains. Indeed, for yearly data, the average percentages of total wealth dedicated to risky assets $(1 - \bar{C}/\hat{W})\hat{\theta}$

Assessing cushions dynamically with $\eta = 0.9$ appears to worsen the situation and to render investors even more loss reluctant. The same Tables 3.5 and 3.6 reveal that the corresponding loss-aversion coefficient $\hat{\lambda}$ is substantially lower than with myopic cushions when the risky performance is evaluated once a year. The evolution patterns observed for myopic cushions are reversed.[227] In addition, quarterly portfolio checks entail negative and hence implausible estimates of the loss-aversion coefficient. In sum, this type of cushion appears to be less compatible with our expected-utility equilibrium setting.[228]

As mentioned above, we attempt to refine the analysis of loss attitudes by means of our extended measure gRA. Table 3.7 (Table 3.8) presents the estimates of gRA for yearly (quarterly) evaluations, and our usual case with different cushion-types and levels of additional income. Recall that these estimates are obtained by plugging the prospective value in equilibrium $\hat{V}$ into Equation (3.25).

For the majority of cases with myopic cushions, the evolution of the estimated gRA is opposite to that of $\hat{\lambda}$: Other things being equal, gRA augments with the degree of narrow framing b_0, as well as with the additional income given by the (β, δ)-combinations, but declines when higher penalties k are imposed on past losses.[229] This radical difference between the courses of gRA and $\hat{\lambda}$ is yet consistent with the design of these two measures: Recall that gRA reflects first-order changes in the perceived utility of financial investments and hence higher values of gRA point to a *more* relaxed loss attitude; In contrast, higher $\hat{\lambda}$ reveals *less* acceptance of possible losses. However, gRA grows – and hence reveals a higher loss acceptance – for more frequent portfolio evaluations, which contradicts mLA in the strict sense.

Similarly to $\hat{\lambda}$, gRA mostly takes negative values for quarterly data and high sensitivity to past losses $k > 0$, when cushions are dynamically assessed with $\eta = 0.9$.[230] Interestingly, relative to myopic cushions, the long-memory dynamic cushions yield lower gRA-values,

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from Table 3.9 increase with k. However, the changes are extremely small. Moreover, the opposite occurs for quarterly data, as apparent in Table 3.10.

[227] Specifically, $\hat{\lambda}$ under long-memory dynamic cushions is lower for higher degrees of narrow framing b_0, higher additional incomes given by the considered (β, δ)-combinations, and higher sensitivity to past losses k.

[228] From all considered types of dynamic cushion, only $\eta \geq 0.5$ entails plausible estimates for $\gamma = 0.5$, when portfolio evaluations are performed once every three months, and across all considered combinations of the remaining parameters. For other γ-values, the variations of $\hat{\lambda}$ under dynamic cushions are mostly irregular.

[229] Moreover, extremely risk-averse investors with $\gamma = 1.5$ exhibit negative gRA-values.

[230] Note however that negative values of gRA are not necessarily implausible, they simply reflect decreases in the perceived utility V with the expected risk premium, which might occur under less common market conditions.

		Myopic cushions			Dynamic cushions with $\eta = 0.9$		
		$k = 0$	$k = 3$	$k = 10$	$k = 0$	$k = 3$	$k = 10$
	$b_0 = 1$	14663	14444	13954	10078	10210	10531
Low I	$b_0 = 5$	19317	19098	18608	14246	14378	14699
($\beta = 0.1$,	$b_0 = 10$	19898	19680	19189	14767	14899	15220
$\delta = 0.9$)	$b_0 = 100$	20422	20203	19713	15236	15368	15689
	$b_0 = 1000$	20474	20255	19765	15283	15415	15736
	$b_0 = 1$	216770	214900	210630	162420	164300	168780
Middle I	$b_0 = 5$	221420	219560	215280	166590	168470	172950
($\beta = 0.5$,	$b_0 = 10$	222000	220140	215860	167110	168990	173470
$\delta = 0.5$)	$b_0 = 100$	222520	220660	216380	167580	169460	173940
	$b_0 = 1000$	222580	220720	21644	167620	169500	173980
	$b_0 = 1$	7099800	7043500	6913400	5355200	5418000	5567000
High I	$b_0 = 5$	7104500	7048200	6918000	5359400	5422200	5571200
($\beta = 0.9$,	$b_0 = 10$	7105100	7048800	6918600	5359900	5422700	5571700
$\delta = 0.1$)	$b_0 = 100$	7105600	7049300	6919100	5360400	5423200	5572200
	$b_0 = 1000$	7105600	7049400	6919200	5360400	5423200	5572200

Table 3.7: The estimated global first-order risk aversion (gRA) in the expected-utility equilibrium for yearly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, and using various cushion-assessment methods, degrees of past-loss sensitivity k, additional-income levels I, and narrow-framing degrees b_0.

		Myopic cushions			Dynamic cushions with $\eta = 0.9$		
		$k = 0$	$k = 3$	$k = 10$	$k = 0$	$k = 3$	$k = 10$
	$b_0 = 1$	177320	164150	146660	27621	23117	15123
Low I	$b_0 = 5$	190580	177410	160360	11010	6506.2	-2495.4
($\beta = 0.1$,	$b_0 = 10$	192240	179070	162070	8933.2	4429.8	-4697.8
$\delta = 0.9$)	$b_0 = 100$	193730	180560	163610	7064.4	2561	-6679.9
	$b_0 = 1000$	193880	180710	163770	6877.5	2374.2	-6878.1
	$b_0 = 1$	1286300	1216000	1065900	33238	3879.1	-60680
Middle I	$b_0 = 5$	1299600	1229300	1079200	16627	-12732	-77291
($\beta = 0.5$,	$b_0 = 10$	1301300	1230900	1080800	14551	-14808	-79368
$\delta = 0.5$)	$b_0 = 100$	1302800	1232400	1082300	12682	-16677	-81236
	$b_0 = 1000$	1302900	1232600	1082500	12495	-16864	-81423
	$b_0 = 1$	47725000	45387000	40072000	54100	-1011100	-3384700
High I	$b_0 = 5$	47738000	45400000	40085000	37488	-1027700	-3.401300
($\beta = 0.9$,	$b_0 = 10$	47739000	45402000	40087000	35412	-1029800	-3403400
$\delta = 0.1$)	$b_0 = 100$	47741000	45403000	40088000	33543	-1031700	-3405300
	$b_0 = 1000$	47741000	45404000	40088000	33356	-1031800	-3405500

Table 3.8: The estimated global first-order risk aversion (gRA) in the expected-utility equilibrium for quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, and using various cushion-assessment methods, degrees of past-loss sensitivity k, additional-income levels I, and narrow-framing degrees b_0.

which suggests that the acceptance of financial losses depreciates when investors pay more attention to the past performance.

In general, gRA supports the above findings for the case when loss attitudes are captured by the simple loss-aversion coefficient $\hat{\lambda}$, with the exception of mLA which is now contradicted. It also behaves somewhat more consistently to the variation of the behavioral investor profile.

In the sequel, we turn to the problem of wealth allocation in equilibrium. The expected-utility setting does not allow for the *direct* derivation, i.e. from the Euler equations, of any relevant wealth-allocation variable. Consequently, we compute the average values of the wealth proportions dedicated to consumption and to different types of financial assets.

The mean fractions of total wealth to be consumed $\bar{C}/\hat{W}$ and of the post-consumption wealth assigned to risky assets $\hat{\theta}$ are exemplified in Table 3.9 (Table 3.10) for yearly (quarterly) portfolio evaluations, in our usual cases with myopic and long-memory dynamic cushions, and for the three additional income levels. Note that these tables do not anymore contain the narrow-framing coefficient b_0, as all variables that refer to the wealth allocation are independent of it. Moreover, the wealth allocation can be considered to be robust, on average, with respect to the type of cushion, as the corresponding changes are of at most 0.1 percentage points.

		Myopic cushions			Dynamic cushions with $\eta = 0.9$		
		$k = 0$	$k = 3$	$k = 10$	$k = 0$	$k = 3$	$k = 10$
Low I	$\bar{C}/\hat{W}$	0.14555	0.14576	0.1462	0.14574	0.14594	0.14637
($\beta = 0.1$,	$\hat{\theta}$	-0.034714	-0.034729	-0.034818	-0.039377	-0.039509	-0.039873
$\delta = 0.9$)	$(1 - \bar{C}/\hat{W})\hat{\theta}$	-0.029661	-0.029667	-0.029727	-0.033638	-0.033743	-0.034037
Middle I	$\bar{C}/\hat{W}$	0.020167	0.02018	0.020209	0.020182	0.020195	0.00011144
($\beta = 0.5$,	$\hat{\theta}$	0.11555	0.11573	0.11612	0.11151	0.11165	0.11193
$\delta = 0.5$)	$(1 - \bar{C}/\hat{W})\hat{\theta}$	0.11322	0.1134	0.11377	0.10926	0.10939	0.10966
High I	$\bar{C}/\hat{W}$	0.00080024	0.00080057	0.0008013	0.00080065	0.00080099	0.00080177
($\beta = 0.9$,	$\hat{\theta}$	0.1338	0.13397	0.13433	0.12978	0.12993	0.13023
$\delta = 0.1$)	$(1 - \bar{C}/\hat{W})\hat{\theta}$	0.13369	0.13386	0.13422	0.12968	0.12982	0.13013

Table 3.9: The estimated wealth allocation in the expected-utility equilibrium for yearly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, and using various cushion-assessment methods, degrees of past-loss sensitivity k, and additional-income levels I.

For additional-income levels corresponding to $\beta \leq 0.9$, investors who evaluate the

		Myopic cushions			Dynamic cushions with $\eta = 0.9$		
		$k = 0$	$k = 3$	$k = 10$	$k = 0$	$k = 3$	$k = 10$
Low I	$\bar{C}/\hat{W}$	0.22693	0.22883	0.23078	0.22418	0.22671	0.22965
($\beta = 0.1$,	$\hat{\theta}$	0.11972	0.11396	0.10677	0.078252	0.07101	0.060977
$\delta = 0.9$)	$(1 - \bar{C}/\hat{W})\hat{\theta}$	0.092553	0.087881	0.082134	0.060709	0.054911	0.046974
Middle I	$\bar{C}/\hat{W}$	0.050256	0.050422	0.050783	0.050175	0.050396	0.050873
($\beta = 0.5$,	$\hat{\theta}$	0.31304	0.31143	0.30617	0.27412	0.27242	0.26694
$\delta = 0.5$)	$(1 - \bar{C}/\hat{W})\hat{\theta}$	0.29731	0.29573	0.29062	0.26037	0.25869	0.25336
High I	$\bar{C}/\hat{W}$	0.0032122	0.0032145	0.0032205	0.0032133	0.0032159	0.0032221
($\beta = 0.9$,	$\hat{\theta}$	0.35583	0.35572	0.35458	0.31859	0.31878	0.3184
$\delta = 0.1$)	$(1 - \bar{C}/\hat{W})\hat{\theta}$	0.35469	0.35457	0.35344	0.31757	0.31776	0.31738

Table 3.10: The estimated wealth allocation in the expected-utility equilibrium for quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, and using various cushion-assessment methods, degrees of past-loss sensitivity k, additional-income levels I, and narrow-framing degrees b_0.

financial performance once every year (three months), are risk averse with $\gamma = 0.5$, and loss averse with $\lambda = 2.25$, dedicate to consumption average percentages of their total wealth $\bar{C}/\hat{W}$ of up to 14.7% (23.1%). The same investors appear to behave myopically averse towards financial investments in total: They allocate less money to financial assets – and implicitly more to consumption – for more frequent portfolio evaluations.

The computed mean percentages $\bar{C}/\hat{W}$ fall for higher additional incomes, since covering current consumption needs from financial revenues becomes less stringent. Note that this decline of $\bar{C}/\hat{W}$ with I is extremely pronounced. It should be recalled that the total wealth includes not only financial revenues, but also the additional income. As the latter is sufficient for fulfilling consumption needs, the remaining part of total wealth becomes available for being invested in financial assets. This makes the total wealth increase faster than consumption, which is even more pronounced for higher β-values, resulting in small ratios $\bar{C}/\hat{W}$.

When investors are more reluctant to past financial losses, they might also allocate less money to financial ventures and proportionally more to consumption, and indeed we observe that $\bar{C}/\hat{W}$ rises with k. The changes are yet very small, especially for higher additional incomes that allow investors to be more "generous" with financial investments in total. As expected, the mean wealth percentages dedicated to consumption decrease for $\gamma = 1$, other things being equal, that is when investors are more reluctant towards the risks related to consumption.[231]

Moreover, when cushions are myopically accumulated and yearly (quarterly) portfolio checks performed, between 7.1-13.4% (10.6-35.7%) of the wealth remaining after consumption $\hat{\theta}$ is allocated, on average, to risky assets. The corresponding values for long-memory dynamic cushions are only slightly lower, specifically 6.4-13.1% (6.1-32.1%). The values of $\hat{\theta}$ grow with the magnitude of the additional income. Again, this holds across all considered configurations of parameters for $\gamma \leq 1$ and $\beta \leq 0.9$, with a single exception: For $\gamma = 0.5$, the combination $(\beta = 0.1, \delta = 0.9)$ entails negative $\hat{\theta} < 0$. According to Equation (3.30a), this shows the preference for risk-free vs. risky assets: After putting aside the sum necessary for consumption, risk-averse investors who dispose of very scarce additional revenues prefer to lend (save at the risk free rate) money $(B_t \leq C_t - W_t < 0)$ instead of investing it in risky assets. This situation changes as soon as the additional incomes are sufficiently high. Furthermore, the mean post-consumption wealth percentages to be invested in risky assets do not change very much with the consumption-related risk aversion γ, other things being equal.[232]

Finally, recall that in the present two-dimensional utility setting $\hat{\theta}$ stands for the portions of *post-consumption* wealth invested in risky assets. The corresponding percentages of *total* wealth assigned to risky investments can be obtained by multiplying $(1 - \bar{C}/\hat{W})$ by $\hat{\theta}$. Across all considered additional incomes, the average values $(1 - \bar{C}/\hat{W})\hat{\theta}$ amount to 6.7-13.5% (8.2-35.5%) for yearly (quarterly) portfolio evaluations and myopic cushions, while the corresponding percentages for long-memory dynamic cushions are again slightly lower, namely 6-13% (4.7-32%), showing that investors split their money between consumption and risk-free assets. The sole exception is again the case with lowest additional incomes $(\beta = 0.1, \delta = 0.9)$, where for yearly data $(1 - \bar{C}/\hat{W})\hat{\theta} \approx -3\%$. Having more money from investment-exogenous sources at their disposal augments the investor openness towards risky assets and hence $(1 - \bar{C}/\hat{W})\hat{\theta}$ grows with I. The changes with k are very small, so that we can consider the average percentages of total wealth dedicated to risky assets to be invariant to the sensitivity to past losses. Moreover, when investors are more wary towards consumption-related risks, i.e. for higher γ, the mean total-wealth

[231]Specifically, $\bar{C}/\hat{W}$ rises up to 10% (18.7%) for yearly (quarterly) evaluations and $\gamma = 1$. Note that for $\gamma = 1.5$ these percentages rise again, which reinforces our earlier claim that such high values of the consumption-related risk aversion are incompatible with the present framework.

[232]For instance, for $\gamma = 1$ and across both types of cushion we obtain $\hat{\theta}$ between 1.7-13.5% (11.4-35.7%) for yearly (quarterly) evaluations.

portions invested in risky assets decrease, other things being equal.[233]

We can hence conclude that mLA in the monetary sense does *not* hold under the maximization of two-dimensional expected utility: Average deposits in risky assets, as portions of total wealth, increase for more frequent evaluations of the risky portfolio.

In addition, for yearly (quarterly) portfolio revisions the maximizers of two-dimensional expected utility allocate smaller (higher) sums to risky assets than their peers from Section 3.2 who are merely concerned with financial utility.[234] Although consumption clearly plays an important role in changing attitudes towards risky investments and hence decisions on wealth allocation, the evaluation frequency of the risky performance is again the driving factor in this respect.

We close this section by some brief comments on the robustness checks performed first for further values of the initial loss-aversion coefficient λ, and second for middle additional incomes resulting from the combination $(\beta = 0.9, \delta = 2)$.[235]

As expected, the estimates describing loss aversion and wealth allocation in the expected-utility equilibrium vary proportional with λ.

The above results are replicated for $(\beta = 0.9, \delta = 2)$ in Tables A.4-A.9 in Appendix A.3.3. In the main, the variation patterns observed for our usual cases with $\delta < 1$ are preserved. Regarding the variables' values, they mostly come close to those for middle-range additional incomes given by $(\beta = 0.5, \delta = 0.5)$ for myopic cushions, but, under long-memory dynamic cushions, they sooner resemble those for the high additional income with $(\beta = 0.9, \delta = 0.1)$. This underlines the importance of β in the latter case and, correspondingly, the relative insensitivity to δ of the estimates obtained with dynamic cushions.[236] Interestingly, the equilibrium-equivalent loss-aversion coefficient $\hat{\lambda}$ in the case with myopic cushions is noticeably higher for lower degrees of narrow framing $b_0 = 1$

[233]For instance, $\gamma = 1$ yields across both types of cushion values of $(1 - \bar{C}/\hat{W})\hat{\theta}$ between 1.5-13.5% (9.32-35.6%) for yearly (quarterly) evaluations, where the values are, on average, lower for dynamic cushions.

[234]Recall that, according to Table 3.1 in Section 3.2, 34.51% (13.42%) of wealth is invested in risky assets when utility is one-dimensional, portfolios are evaluated yearly (quarterly), cushions are myopic, and expected returns normally distributed. Note that such a comparison is possible only for similar parameter values, such as a sensitivity to past losses $k = 3$, and myopic cushions and long-memory dynamic cushions $\eta = 0.9$, and hence only in the expected utility framework.

[235]Recall that the initial loss-aversion coefficient is employed in order to derive VaR* from Equation (3.22) and the prospective value from Equation (3.24) and that we consider further values $\lambda \in \{1; 3\}$. Regarding the combination $(\beta = 0.9, \delta = 2)$, recall also that this is the single additional income with $\delta > 1$ that yields plausible estimates in both settings, as mentioned in Footnote 216.

[236]Of course, the values for myopic cushions are slightly higher for $(\beta = 0.9, \delta = 0.1)$, as the corresponding additional incomes are also somewhat higher than those for $(\beta = 0.5, \delta = 0.5)$. For dynamic cushions, the variables attain values that are almost identical to those for $(\beta = 0.9, \delta = 0.1)$.

than what we observed before; Specifically, it lies above 1 even for yearly data and hence shows loss aversion. With dynamic cushions, $\hat{\lambda}$ is almost stable with respect to b_0 for yearly evaluations. Moreover, the percentages of total wealth to be put in risky assets $(1 - \bar{C}/\hat{W})\hat{\theta}$ are lower (higher) for yearly (quarterly) evaluations and myopic cushions than those corresponding to $(\beta = 0.5, \delta = 0.5)$.

The non-expected utility approach

In this section, we proceed analogously to the expected-utility approach by estimating the main variables in the equilibrium with non-expected utility and analyzing how they change *on average* subject to the *ceteris paribus* variation of chosen parameters.

Recall that in the non-expected equilibrium, cushions are the result of a memoryless dynamic assessment, i.e. with $\eta = 0$, so that the sensitivity to past losses k exerts no influence on the equilibrium variables. Moreover, from the test values of the consumption-related risk aversion and of the initial loss-aversion coefficient, the non-expected equilibrium allows only for $\gamma \in \{0.5; 1.5\}$ and $\lambda \in \{2.25; 3\}$, respectively. Finally, the parameter β can be directly interpreted in the context of Equation (3.36a) as the weight put on that utility piece which stems from financial investments.

The main equilibrium variables are now the percentage of total wealth dedicated to consumption α, the post-consumption wealth invested in risky assets θ, and the prospective value $\bar{V}$.[237] They are derived under the assumption of periodical additional incomes of $I_t = C_t/(\alpha\delta)$. Accordingly, the total gross returns from financial investments in Equation (3.40) results in:

$$\begin{aligned} R_{t+1}^{\mathrm{tot}} &= \frac{1}{1-\alpha}\frac{C_{t+1} - \alpha I_{t+1}}{C_t} = \frac{1}{1-\alpha}\frac{\delta - 1}{\delta}\frac{C_{t+1}}{C_t} \\ &\Rightarrow \log(R_{t+1}^{\mathrm{tot}}) = \log(\delta - 1) - \log(\delta) - \log(1-\alpha) + c + \sigma_c \epsilon_{t+1}. \end{aligned} \tag{3.46}$$

and hence the equilibrium Equation (3.42c) changes to:

$$\begin{aligned} &-\frac{\delta - 1}{\delta(1-\alpha)} \exp\left((1-\gamma)c + \frac{(1-\gamma)^2\sigma_c^2}{2}\right) + R_f \exp\left(-\gamma c + \frac{\gamma^2\sigma_c^2}{2}\right) \\ &\quad = b_0 R_f \left(\frac{\beta}{1-\beta}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{1-\alpha}{\alpha}\right)^{-\frac{\gamma}{1-\gamma}} \exp\left(-\gamma c + \frac{\gamma^2\sigma_c^2}{2}\right)\theta\bar{V}. \end{aligned} \tag{3.47}$$

[237] Again, it is possible to proceed in the opposite way: fixing α and θ and deriving γ from Equation (3.42a). However, the double or possibly non-real roots of the resulting quadratic equation in γ are difficult to interpret from an economical point of view. Equation (3.42a) has real roots if and only if $\beta R_f \geq \exp(-(2\log(1-\alpha) + 2r_f - \sigma_c^2)^2/(8\sigma_c^2))$.

For a fixed weight of financial utility β in Equation (3.36a), we derive α and θ by dividing Equations (3.42b) and (3.47) and plugging the result into Equation (3.42a). Equation (3.42b) is reformulated in order to obtain an expression for $\bar{V}$. We obtain the following expressions:

$$\alpha = 1 - \beta^{\frac{1}{\gamma}} R_f^{\frac{1-\gamma}{\gamma}} \exp\left(\frac{(1-\gamma)\sigma_c^2}{2}\right) \tag{3.48a}$$

$$\begin{aligned}\theta &= \frac{R_f - \dfrac{\delta - 1}{\delta(1-\alpha)} \exp\left(c + \dfrac{(1-2\gamma)\sigma_c^2}{2}\right)}{R_f - \exp\left(r + \dfrac{\sigma_r^2}{2} - \gamma\sigma_{cr}\right)} \\ &= \frac{\delta \beta^{\frac{1}{\gamma}} R_f^{\frac{1}{\gamma}} - (\delta - 1)\exp\left(c - \dfrac{\gamma\sigma_c^2}{2}\right)}{\delta \beta^{\frac{1}{\gamma}} R_f^{\frac{1-\gamma}{\gamma}} \left(R_f - \exp\left(r + \dfrac{\sigma_r^2}{2} - \gamma\sigma_{cr}\right)\right)}\end{aligned} \tag{3.48b}$$

$$\begin{aligned}\bar{V} &= \frac{1}{b_0 R_f}\left(\frac{\alpha\beta}{(1-\alpha)(1-\beta)}\right)^{-\frac{\gamma}{1-\gamma}} \left(R_f - \exp\left(r + \frac{\sigma_r^2}{2} - \gamma\sigma_{cr}\right)\right) \\ &= \frac{1}{b_0}\beta(1-\beta)^{\frac{\gamma}{1-\gamma}} \exp\left(\frac{\gamma\sigma_c^2}{2}\right) \frac{R_f - \exp\left(r + \dfrac{\sigma_r^2}{2} - \gamma\sigma_{cr}\right)}{\left(1 - \beta^{\frac{1}{\gamma}} R_f^{\frac{1-\gamma}{\gamma}} \exp\left(\dfrac{(1-\gamma)\sigma_c^2}{2}\right)\right)^{\frac{\gamma}{1-\gamma}}}.\end{aligned} \tag{3.48c}$$

Unlike with expected utility, the non-expected utility equilibrium provides direct estimates of the main wealth-allocation variables, namely the proportion of total wealth dedicated to consumption α and the post-consumption wealth fraction to be put in risky assets θ.

Note that the percentages θ from Equation (3.48b) are inversely proportional to each the additional-income parameter δ and the total-wealth percentage dedicated to consumption in equilibrium α. Thus, according to Equation (3.48a) and to our assumption that $I_t = C_t/(\alpha\delta)$, θ increases for each higher weight β and higher δ-value.

Some further observations regarding the interpretation of $\bar{V}$ and θ have to be made: In both Equations (3.48b) and (3.48c) we observe the presence of the same term that stands for the profitability, taken *with inverse sign*, of risky investments, that is $R_f - \exp(r + \sigma_r^2/2 - \gamma\sigma_{cr})$. The prospective value is directly proportional to this term, according to the first-line expression in Equation (3.48c), while the post-consumption wealth percentages invested in risky assets evolve inversely proportional to it, as implied by Equation (3.48b). In our data set, this term is always negative. Highly negative values in fact arise in situations of higher profitability, when, in consequence, both $\bar{V}$ and θ should be highly positive. Thus, a meaningful interpretation can be given only to the absolute values $|\bar{V}|$ and $|\theta|$. Note also that values of $|\theta| > 1$ point to the fact that investors borrow extra-money at the risk-free rate and invest it in the risky portfolio.

One more note on how to interpret the results in the non-expected-utility setting regards the following: We expect that the estimated loss-aversion coefficients in equilibrium take negative values $\hat{\lambda} < 0$. This negative sign is imposed, so to speak "by theoretical construction", namely by the sign of $\bar{V}$ and the condition that $\eta = 0$. In particular, the nil cushions $S_t - Z_t = 0$ implied by the memoryless process with $\eta = 0$, transform Equation (3.26) as follows:

$$\bar{\lambda}_{t+1} = \frac{\bar{V} - \Big(\pi_t\psi_t + (1-\pi_t)\omega_t\Big)S_tE_t[x_{t+1}]}{\Big(\pi_t(1-\psi_t) + (1-\pi_t)(1-\omega_t)\Big)S_tE_t[x_{t+1}]}.$$

The prospective value $\bar{V}$ always dominates the second term in the numerator of this expression.[238] Therefore, our negative – and hence meaningful – equilibrium estimates $\hat{V} < 0$ necessarily imply negative values of $\hat{\lambda}$ from the above expression, although the negative sign is only artificial. Again, an economic interpretation can be given only to the absolute values $|\hat{\lambda}|$. This is always true with $\eta = 0$.[239]

Let us now analyze the obtained estimates and their variation with the chosen behavioral parameters. In so doing, we concentrate again on the case with investors who are both risk- and loss-averse, i.e. with $\gamma = 0.5$ and an initial $\lambda = 2.25$, narrowly frame financial investments $b_0 \geq 1$, and dispose of investment-exogenous incomes I of low, middle, or high magnitudes given by the combinations $(\beta = 0.1, \delta = 0.9)$, $(\beta = 0.5, \delta = 0.5)$, and $(\beta = 0.9, \delta = 0.1)$, respectively.[240]

Table 3.11 (Table 3.12) presents the equilibrium estimates of the variables in Equations (3.48) for yearly (quarterly) evaluations of risky performance.

Similarly to the expected-utility setting, the prospective value $|\hat{V}|$ decreases subject to the intensity of narrow framing b_0, so that the perception of financial investments depreciates when more attention is payed to these investments. However, financial investments

[238] This second term is always smaller than 1 due to the combination of probabilities $\pi_t\psi_t + (1-\pi_t)\omega_t$.
[239] And will be used in the comparative section at the end of the applicative part.
[240] Unreported results show that the highest value of the weight of financial utility $\beta = 0.98$ yields implausible estimates of the wealth fraction dedicated to consumption in equilibrium $\hat{\alpha} < 0$. Naturally, in this case the consumption is almost irrelevant as source of utility in investors' perception relative to financial investments.

		$b_0 = 1$	$b_0 = 5$	$b_0 = 10$	$b_0 = 100$	$b_0 = 1000$
	$\lvert\hat{V}\rvert$	0.16819	0.033638	0.016819	0.0016819	0.00016819
Low I	$\lvert\hat{\lambda}\rvert$	2.0141	2.0145	2.0146	2.0146	2.0146
$(\beta = 0.1, \delta = 0.9)$	$\hat{\alpha}$	0.98927	0.98927	0.98927	0.98927	0.98927
	$\lvert\hat{\theta}\rvert$	6.4843	6.4843	6.4843	6.4843	6.4843
	$\lvert\hat{V}\rvert$	0.63158	0.12632	0.063158	0.0063158	0.00063158
Middle I	$\lvert\hat{\lambda}\rvert$	2.0143	2.0145	2.0146	2.0146	2.0146
$(\beta = 0.5, \delta = 0.5)$	$\hat{\alpha}$	0.73178	0.73178	0.73178	0.73178	0.73178
	$\lvert\hat{\theta}\rvert$	2.7058	2.7058	2.7058	2.7058	2.7058
	$\lvert\hat{V}\rvert$	1.2705	0.25411	0.12705	0.012705	0.0012705
High I	$\lvert\hat{\lambda}\rvert$	2.0146	2.0146	2.0146	2.0146	2.0146
$(\beta = 0.9, \delta = 0.1)$	$\hat{\alpha}$	0.13095	0.13095	0.13095	0.13095	0.13095
	$\lvert\hat{\theta}\rvert$	6.4843	6.4843	6.4843	6.4843	6.4843

Table 3.11: The main variable estimates in the non-expected utility equilibrium for yearly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memoryless dynamic cushions with $\eta = 0$, various additional incomes I and narrow-framing degrees b_0.

		$b_0 = 1$	$b_0 = 5$	$b_0 = 10$	$b_0 = 100$	$b_0 = 1000$
	$\lvert\hat{V}\rvert$	0.15926	0.031851	0.015926	0.0015926	0.00015926
Low I	$\lvert\hat{\lambda}\rvert$	1.2486	1.2547	1.2555	1.2561	1.2562
$(\beta = 0.1, \delta = 0.9)$	$\hat{\alpha}$	0.98982	0.98982	0.98982	0.98982	0.98982
	$\lvert\hat{\theta}\rvert$	6.9149	6.9149	6.9149	6.9149	6.9149
	$\lvert\hat{V}\rvert$	0.58727	0.11745	0.058727	0.0058727	0.00058727
Middle I	$\lvert\hat{\lambda}\rvert$	1.2532	1.2556	1.2559	1.2562	1.2562
$(\beta = 0.5, \delta = 0.5)$	$\hat{\alpha}$	0.74562	0.74562	0.74562	0.74562	0.74562
	$\lvert\hat{\theta}\rvert$	2.8611	2.8611	2.8611	2.8611	2.8611
	$\lvert\hat{V}\rvert$	0.89663	0.17933	0.089663	0.0089663	0.00089663
High I	$\lvert\hat{\lambda}\rvert$	1.2561	1.2562	1.2562	1.2562	1.2562
$(\beta = 0.9, \delta = 0.1)$	$\hat{\alpha}$	0.17581	0.17581	0.17581	0.17581	0.17581
	$\lvert\hat{\theta}\rvert$	6.9149	6.9149	6.9149	6.9149	6.9149

Table 3.12: The main variable estimates in the non-expected utility equilibrium for quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memory-less dynamic cushions with $\eta = 0$, various additional incomes I and narrow-framing degrees b_0.

are now subjectively considered to be less attractive when the risky performance is evaluated more often, i.e. $|\hat{V}|$ is lower for quarterly portfolio evaluations. This comes in line with mLA in the large sense.

We again study the attitude towards financial losses resorting to our two specific measures: the loss-aversion coefficient and gRA. Tables 3.11 and 3.12 suggest that the equilibrium-equivalent loss-aversion coefficient $|\hat{\lambda}|$ is almost invariant with respect to the degree of narrow framing b_0, but only as long as there is indeed narrow framing $b_0 > 0.001$. The same invariance holds approximatively, with respect to the average additional income I, too.[241] Finally, $|\hat{\lambda}|$ grows just slightly for higher $\gamma = 1.5$ and thus can be considered to be robust to the consumption-related risk aversion as well.

In essence, $|\hat{\lambda}|$ lies above the value of 1 across all considered configurations of parameters, as long as there is narrow framing $b_0 > 0.001$.[242] For the cases considered in Tables 3.11 and 3.12, $|\hat{\lambda}| \approx 2$ (1.25) for yearly (quarterly) evaluations, which speaks for loss aversion. The smaller values obtained for quarterly portfolio revisions contradict mLA in the strict sense.

We refine our analysis concerning the loss attitude by focusing on the extended measure gRA. It can be derived from the main equilibrium estimate $\hat{V}$ in Equation (3.48c), according to Equation (3.25). The corresponding gRA-values for yearly (quarterly) data and our usual cases are included in Table 3.13 (Table 3.14).

Relative to the simple loss-aversion coefficient, our extended measure gRA appears again to capture more consistently the loss attitude subject to individual behavioral profiles: Its values in the non-expected equilibrium setting are always positive and change with the behavioral parameters for both evaluation frequencies, as it is to be intuitively expected. In particular, gRA falls with the degree of narrow framing b_0, other things being equal, showing that a narrower focus on financial investments is coupled with a higher reluctance towards potential losses from these investments. Moreover, gRA grows with the magnitude of the average additional income I. In particular, it is insensitive to the free-choice parameter δ, but grows with the weight β of the financial utility. This comes in line with the idea that more relaxed attitudes towards risky investments are to be expected when investors perceive the importance of these investments to be higher. Unreported results suggest that extremely risk-averse investors with $\gamma = 1.5$ fear possible

[241] Strictly speaking, $|\hat{\lambda}|$ grows in each b_0 and I, but the changes are of the order 10^{-3} and smaller.

[242] Specifically, $|\hat{\lambda}| \in [1.0561, 2.0146]$ ($[1.1114, 1.2562]$) for yearly (quarterly) data and $b_0 > 0.001$. In contrast, $b_0 = 0.001$ yields very high values of $\hat{\lambda}$.

	$b_0 = 1$	18.654
Low I	$b_0 = 5$	3.7308
$(\beta = 0.1, \delta = 0.9)$	$b_0 = 10$	1.8654
	$b_0 = 100$	0.18654
	$b_0 = 1000$	0.018654
	$b_0 = 1$	70.05
Middle I	$b_0 = 5$	14.01
$(\beta = 0.5, \delta = 0.5)$	$b_0 = 10$	7.005
	$b_0 = 100$	0.7005
	$b_0 = 1000$	0.07005
	$b_0 = 1$	140.92
High I	$b_0 = 5$	28.184
$(\beta = 0.9, \delta = 0.1)$	$b_0 = 10$	14.092
	$b_0 = 100$	1.4092
	$b_0 = 1000$	0.14092

Table 3.13: The estimated global first-order risk aversion (gRA) in the non-expected utility equilibrium for yearly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memoryless dynamic cushions with $\eta = 0$, various additional incomes I and narrow-framing degrees b_0.

	$b_0 = 1$	54.682
Low I	$b_0 = 5$	10.936
$(\beta = 0.1, \delta = 0.9)$	$b_0 = 10$	5.4682
	$b_0 = 100$	0.54682
	$b_0 = 1000$	0.054682
	$b_0 = 1$	201.64
Middle I	$b_0 = 5$	40.329
$(\beta = 0.5, \delta = 0.5)$	$b_0 = 10$	20.164
	$b_0 = 100$	2.0164
	$b_0 = 1000$	0.20164
	$b_0 = 1$	307.87
High I	$b_0 = 5$	61.573
$(\beta = 0.9, \delta = 0.1)$	$b_0 = 10$	30.787
	$b_0 = 100$	3.0787
	$b_0 = 1000$	0.30787

Table 3.14: The estimated global first-order risk aversion (gRA) in the non-expected utility equilibrium for quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memory-less dynamic cushions with $\eta = 0$, various additional incomes I and narrow-framing degrees b_0.

financial losses to a greater extent, as the corresponding gRA-estimates are lower. Finally, gRA is higher for quarterly performance evaluations, which is again at odds with mLA in the strict sense.

Having analyzed the investor attitude towards financial losses in the non-expected utility equilibrium, we turn now to the problem of wealth allocation among consumption, risky, and risk-free financial assets. Our equilibrium fractions of total wealth to be consumed $\hat{\alpha}$ and of post-consumption wealth to be invested in risky assets $\hat{\theta}$ are particularized in Tables 3.11 and 3.12 for our usual cases. They are both independent of the narrow-framing coefficient b_0.

Moreover, Equation (3.48a) indicates that α does not vary with δ. Therefore, the weight of financial utility β is the only parameter that dictates the changes of α with the magnitude of the additional income I. In particular, the estimated wealth fractions dedicated to consumption $\hat{\alpha}$ are considerably lower when investors ascribe a higher importance β to financial investments as a source of utility. Specifically, when β grows from 0.1 to 0.9, these wealth fractions drop from over 98.9% to around 13.1% (17.6%) for yearly (quarterly) checks on the risky performance. Note also that $\hat{\alpha}$ is slightly higher for increased evaluation frequencies, which underpins the idea of myopic aversion with respect to financial investments in general. Finally, extremely risk-averse investors with $\gamma = 1.5$ allocate less money to consumption (and proportionally more to financial assets).[243] This counterintuitive result leads to the conclusion that, similarly to the expected utility framework, too high a consumption-related risk aversion might be incompatible with the present framework.

Concerning the post-consumption wealth percentages to be put in risky assets, Tables 3.11 and 3.12 suggest that investors who have more money at their disposal, i.e. higher additional incomes I, allocate smaller fractions of their wealth after consumption $|\hat{\theta}|$ to risky assets. Moreover, recall that $|\hat{\theta}| > 1$ stands for the case when investors enhance their investments in risky assets by borrowing additional sums of money at the risk-free rate. This appears to be the case for all combinations $(\beta \geq 0.5, \delta = 0.9)$ and both evaluation frequencies. In particular, $|\hat{\theta}|$ lies above 65.3% (65.6%) for yearly (quarterly) portfolio evaluations. Thus, $|\hat{\theta}|$ is somewhat higher, on average, for more frequent portfolio evaluations, which are the only relevant ones as far as the aversion to financial risks is concerned. The differences are yet very small and do not reflect how the *total* wealth is

[243]Specifically, $\hat{\alpha} \in [0.78957, 0.08951]$ ($[0.7858, 0.073222]$) for yearly (quarterly) evaluations and $\gamma = 1.5$.

split between risky and risk-free assets.

Note also that $|\hat{\theta}|$ takes extremely high – and hence implausible – values for the combinations ($\beta = 0.1, \delta <= 0.5$).[244] The reason is that $\beta = 0.1$ stands for the case when investors consider consumption as the main source of utility and consequently allocate the main part of their wealth α to it. Then, these tremendously high percentages of remaining wealth entail reasonable values for the percentages of *total* wealth dedicated to risky assets $(1-\hat{\alpha})|\hat{\theta}|$ (see also the subsequent comments referring to this latter variable). The same should be kept in mind when we observe that $|\hat{\theta}|$ falls when investors are extremely risk-averse $\gamma = 1.5$.

In order to analyze mLA in the monetary sense, we estimate the fractions $(1 - \hat{\alpha})|\hat{\theta}|$ of total wealth dedicated to risky assets in equilibrium. The results for our usual case with $\lambda = 2.25$, $\gamma = 0.5$ and for yearly (quarterly) portfolio evaluations are to be found in Table 3.15 (Table 3.16).

Low I ($\beta = 0.1, \delta = 0.9$)	0.069569
Middle I ($\beta = 0.5, \delta = 0.5$)	0.72575
High I ($\beta = 0.9, \delta = 0.1$)	5.6351

Table 3.15: The estimated total-wealth fractions dedicated to risky assets $(1 - \hat{\alpha})|\hat{\theta}|$ in the non-expected utility equilibrium for yearly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memoryless dynamic cushions with $\eta = 0$, and various additional incomes I.

Low I ($\beta = 0.1, \delta = 0.9$)	0.07036
Middle I ($\beta = 0.5, \delta = 0.5$)	0.72782
High I ($\beta = 0.9, \delta = 0.1$)	5.6991

Table 3.16: The estimated total-wealth fractions dedicated to risky assets $(1 - \hat{\alpha})|\hat{\theta}|$ in the non-expected utility equilibrium for quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memoryless dynamic cushions with $\eta = 0$, and various additional incomes I.

Across all considered configurations of parameters, multiplying $1 - \hat{\alpha}$ by $|\hat{\theta}|$ amounts to around 6.96-563.5% (7-569.9%) of total wealth dedicated to risky assets when their performance is evaluated yearly (quarterly). Thus, mLA in the monetary sense does not hold in the non-expected-utility setting either: When investors revise risky performance

[244]Specifically, $|\hat{\theta}| \in [53.716, 478.8]$ ($[57.586, 513.63]$) for yearly (quarterly) evaluations.

more often, they dedicate similar to slightly higher portions of their total wealth to risky assets.

The estimated percentages $(1-\hat{\alpha})|\hat{\theta}|$ grow dramatically for higher additional incomes I, which shows that having more money at their disposal renders investors much more open to risky investments. This variation can be split, ceteris paribus, into an increase in β and a decrease in δ. The intuition for the change with β is straightforward: The chances that risky assets are perceived as more attractive should be higher when investors manifest a more pronounced inclination to financial investments in general – that is when these investments are considered as a sufficiently important source of utility relative to consumption. Moreover, when investors dispose of more money in consequence of lower δ-values, other things being equal, their attitude towards financial assets in total does not change, as α is independent of δ. However, when investors are confronted with the more refined choice between risky and risk-free assets, they decide to put more money in the former category. Thus, their reluctance towards financial risk appears to fall.[245]

Surprisingly, $(1-\hat{\alpha})|\hat{\theta}|$ are higher for $\gamma = 1.5$, although extremely risk-averse investors should be disposed to allocate less money to risky investments.[246] This supports our earlier claim that $\gamma = 1.5$ does not fit in our equilibrium framework.

Furthermore, note that the estimates of all variables are almost identical to those presented above for higher initial $\lambda = 3$.

The results for $(\beta = 0.9, \delta = 2)$ are included in Tables A.10-A.14 in Appendix A.3.3. Although the additional income corresponding to this (β, δ)-combination is of middle level, most estimates resemble rather those for high additional incomes given by $(\beta = 0.9, \delta = 0.1)$. This supports the findings in the expected-utility setting for dynamic cushions, where the free-choice parameter δ appears to play a secondary role compared to the weight of financial utility β with respect to investors' decisions. Noticeable discrepancies emerge only for the wealth variables related to the investment in risky assets: For $\delta = 2$, $|\hat{\theta}|$ lies substantially below the corresponding values for $\delta \geq 0.5$ and even below 1. This attests the fact that investors borrow money in order to put it in risky assets. Moreover, $(1-\hat{\alpha})|\hat{\theta}|$ are also much lower than for our usual $\delta \geq 0.5$ and also decline for quarterly evaluations of the risky performance, which underpins mLA in the monetary sense. Thus, investors who ascribe higher weights $\beta = 0.9$ to the financial utility and dispose of scarce additional

[245] Note that the ceteris paribus influence of δ is more pronounced that the one of β.

[246] Specifically, $(1-\hat{\alpha})|\hat{\theta}|$ varies between 19-581% and the differences between the evaluation frequencies are very small.

incomes $\delta = 2$ appear to be identically reluctant towards financial investments in total (as $\hat{\alpha}$ remains at the same level), but less open to risky investments (since $(1-\hat{\alpha})|\hat{\theta}|$ is lower) with respect to their peers who have more money from exogenous sources $\delta = 0.1$.

Expected vs. non-expected utility

A last question emerging in the present context is which of the two settings with expected and non-expected utility describes better the behavior of non-professional investors who derive utility from both consumption and financial investments, narrowly frame the latter, and reluctantly perceive financial losses. On the one hand, the expected-utility setting offers the advantage of being formally less complex and more intuitive. On the other hand, the non-expected utility approach provides immediate estimates of more variables of interest, especially of those related to the optimal wealth allocation.

A rigorous comparison of these two settings is yet not straightforward. In spite of the "preventive measures" adopted in order to ensure such a comparison (namely taking similar parameter values), they rely on different equilibrium conditions, employ distinct estimation procedures, and hence deliver different results.

This section attempts to put together the pieces of evidence gathered so far and to enrich them with further comparative results. We commence by a brief confrontation, *in a qualitative sense*, of the common and specific results under expected and non-expected utility. In particular, we rely on the general conclusions and recommendations of the two settings underlined in the above applicative sections. A comparison *in a quantitative sense* can only be based on the case that is common to both frameworks, namely that with memoryless dynamic cushions $\eta = 0$. We close this section with some remarks referring to the findings for this case under expected-utility maximization.

A qualitative comparison First, our qualitative findings with respect to the subjective value of financial investments for individual investors are, in main, consistent between the settings: A more intense narrow framing of financial assets b_0 yields higher prospective values in equilibrium $|\hat{V}|$. However, mLA in the large sense holds only under non-expected utility, in that the perception of financial investments depreciates at higher performance-evaluation frequencies, which is yet not the case under expected utility.

Second, recall that we measure the loss attitude by means of two variables: the loss-aversion coefficient and the global first-order risk aversion. Within each setting, these two measures are consistent with each other: The estimated loss-aversion coefficient – that is $\hat{\lambda}$ with expected utility and $|\hat{\lambda}|$ with non-expected utility – and gRA vary in opposite directions subject to behavioral parameters (such as the narrow-framing degree b_0, and the sensitivity to past losses k) and to the additional income I. This is to be expected, since the former coefficient is proportional to the loss reluctance and the latter to the loss acceptance. However, this does not necessarily hold with respect to the evaluation frequency: When the loss attitude is measured by the loss-aversion coefficient, mLA in the strict sense holds with expected utility (and hence the loss aversion increases with the portfolio evaluation frequency), but not with non-expected utility. When, in contrast, gRA quantifies the loss attitude, both settings reject mLA in the strict sense. We also note some problems encountered with respect to the loss-aversion coefficient, for instance its inconsistent variation with k under the maximization of expected utility; Also, this coefficient is very low for yearly evaluations in the expected-utility setting and indicates a loss-loving attitude. In light of its more clear and intuitive variation patterns, gRA appears to be somewhat better suited as a measure of loss attitudes in both settings.

Third, the wealth allocation is quantified by means of the wealth fractions dedicated to consumption and to risky financial assets. Under expected utility, we can merely approximate these variables, while under non-expected utility they result as equilibrium values. In spite of this fundamental discrepancy in the methodology, the estimates provided by the two settings behave similarly: The wealth allocation is invariant with respect to the degree of narrow framing b_0 in both settings. As shown by the wealth fractions dedicated to consumption, $\bar{C}/\hat{V}$ with expected utility and $\hat{\alpha}$ with non-expected utility, investors are myopically averse towards financial investments in general, since they allocate more money to consumption – and proportionally less to financial assets in total – when the risky performance is evaluated more often. This aversion decreases when higher additional incomes I are available. This behavior is more pronounced for expected-utility maximization. Moreover, mLA in the monetary sense does not hold in any of the two settings, since constant to higher total-wealth fractions – that is $(1-\bar{C}/\hat{V})\hat{\theta}$ with expected utility and $(1-\hat{\alpha})|\hat{\theta}|$ with non-expected utility – are dedicated to risky assets. The only exception is observed for $\delta = 2$ under non-expected utility. The part of total wealth to be put in risky assets grows with the additional income I. However, both maximizers of expected and of non-expected utility appear to behave myopically averse towards financial investments in general, since they dedicate larger fractions of their total wealth $\bar{C}/\hat{V}$ and $\hat{\alpha}$ respectively to consumption and hence proportionally less to financial investments.

Finally, both settings speak rather against the compatibility of too high consumption-related risk aversion coefficients γ with the equilibrium framework. In particular, $\gamma = 1.5$ mostly delivers implausible estimates of certain variables. The dynamic cushion appears to be less well suited to the estimations under expected utility. Also, from the two parameters that determine the change in the additional income I, β appears to be more important than δ in eliciting investor reactions in both settings. Recall yet that β can be directly interpreted as the weight of financial utility only with non-expected utility. Note also that too high values of each β, such as $\beta = 0.98$, and δ, such as $\delta > 2$, deliver implausible results in both settings.

Note that, in general, the estimates under non-expected utility maximization are robust to changes in the behavioral profile. This is indeed the expected result, as we analyze here the aggregate market with a single *representative* investor (and hence pool consider behavioral profiles "on average"). Therefore, we incline to sustain the claim of Barberis, Huang, and Thaler (2006) that non-expected utility better describes decision making under risk.

A quantitative comparison The direct comparison of the two theoretical approaches is possible only under identical parametric assumptions. The most important of them regards the memoryless dynamic cushions with $\eta = 0$, which is the sole case when the non-expected utility equilibrium has a solution. Appendix A.3.3 presents average estimation results under the maximization of expected utility, when investors assess cushions dynamically with $\eta = 0$, and for the usual values $\lambda = 2.25$, $\gamma = 0.5$, and the various degrees of narrow framing b_0 and of the additional income I. Obviously, all variables are now independent of the sensitivity to past losses k and in consequence of the nil cushions induced by $\eta = 0$.

Note first that the prospective value in equilibrium entails much higher estimates $|\hat{V}|$ under expected than under non-expected utility, as revealed by Tables 3.11 and A.15 (Tables 3.12 and A.16) for yearly (quarterly) evaluations. The similarity among the values of $|\hat{V}|$ in the expected-utility setting for different types of cushion confirms the dominant role of the theoretical setting in which the estimations are performed. It appears thus that, compared to expected-utility maximization, the non-expected utility substantially depreciates the perception of financial utility. As in general with expected utility, mLA in the large sense does not hold for $\eta = 0$, since $|\hat{V}|$ is lower at higher frequencies of the risky-performance evaluation.

Secondly note that, mostly, the two measures of the loss attitude – the loss-aversion coefficient $|\hat{\lambda}|$ and gRA – continue to evolve consistently with each other, but contrary to the corresponding non-expected utility estimates. Yet, gRA exhibits again clearer variation patterns.

Recall that, as discussed in the applicative section on the non-expected utility equilibrium, only the absolute values $|\hat{\lambda}|$ have economical meaning for $\eta = 0$, irrespective of which type of utility is maximized. Table A.15 shows that the combination of yearly portfolio evaluations, low degrees of narrow framing $b_0 \leq 5$, and low additional-income levels ($\beta = 0.1, \delta = 0.9$) yield implausible estimates $|\hat{\lambda}|$. Since these appear to be isolated cases, we consider that the investor interest in financial assets in general might be simply too low for applying our equilibrium framework. All other values of $|\hat{\lambda}|$ in Tables A.15 and A.16 are positive. Moreover, for quarterly evaluations, these values vary much more with each of the additional income I and the degree of narrow framing b_0 compared to the non-expected utility estimates from Table 3.12. Specifically, $|\hat{\lambda}|$ under expected utility starts lower but exceeds then the non-expected utility estimates when either I or b_0 increase, other things being equal. For yearly performance checks, $|\hat{\lambda}|$ is mostly smaller under expected than under non-expected utility. On average, in the expected-utility setting for memoryless dynamic cushions, $|\hat{\lambda}|$ is mostly higher than 1, showing thus loss aversion. In addition, $|\hat{\lambda}|$ augments with the evaluation frequency, suggesting mLA in the strict sense.

The extended loss-attitude measure gRA in the expected-utility equilibrium with memoryless dynamic cushions can be observed in Tables A.17 and A.18 for our usual cases. Surprisingly, gRA does not vary with the additional income I. Its values are substantially higher than with non-expected utility for both evaluation frequencies, which outlines the picture of less loss-reluctant investors. As in the non-expected-utility setting, gRA decreases here with the degree of narrow framing b_0. It is also lower for yearly evaluations which comes once again at odds with mLA in the strict sense.

Third, the wealth-allocation variables with expected utility and memoryless dynamic cushions, although they are derived indirectly in the form of mean values, change similarly to the corresponding non-expected utility estimates subject to the behavioral parameters. The average total-wealth percentages dedicated to consumption $\bar{C}/\hat{W}$ are much lower than

$\hat{\alpha}$.[247] The investors in the expected-utility setting with memoryless dynamic cushions are thus more open to financial investments in general.

Also, the percentages of post-consumption wealth to be invested in risky assets $\hat{\theta}$ with expected utility are considerably lower than the direct estimates $|\hat{\theta}|$ with non-expected utility.[248] They never exceed the limit of 1 showing that investors put only a part of the money for financial projects in risky ones and the rest with the bank.

Finally, the average fractions of total wealth dedicated to risky assets $(1 - \bar{C}/\hat{W})\hat{\theta}$ in the expected-utility setting with memoryless cushions lie substantially below the corresponding direct estimates $(1 - \hat{\alpha})|\hat{\theta}|$ under non-expected utility. This attests a higher reluctance towards risky assets. It holds for both yearly and quarterly risky performance evaluations, and the differences become considerable for increased additional incomes I. The estimates $(1 - \bar{C}/\hat{W})\hat{\theta}$ for memoryless dynamic cushions remain yet lower for less frequent portfolio revisions. This speaks once against mLA in the monetary sense.[249]

In sum, when cushions are memoryless dynamic, both expected and non-expected-utility settings point to similar evolutions of the main variables subject to behavioral profiles. Investors who maximize expected utility appear to be substantially less loss-reluctant, as measured by gRA, and substantially more risk-reluctant as measured by the total-wealth percentages dedicated to risky assets.

3.3.4 Summary and conclusions

This section extends the model in Section 3.2 to a two-dimensional utility framework. We are interested in how non-professional investors, who now derive utility from both consumption and narrowly-framed financial investments, behave when faced with financial risk. We also study how these investors change their perception of losses and how they consequently split their money between consumption, and (risky vs. risk-free) financial assets.

Following Barberis, Huang, and Thaler (2006) and Barberis and Huang (2004, 2006), we consider an aggregate market with a representative investor who maximizes subjective utility. The equilibrium is derived in two distinct settings, in particular considering the

[247] Specifically, the maximum $\bar{C}/\hat{W}$ lies around 14.6% (21.8%) of total wealth for yearly (quarterly) evaluations, $\gamma = 0.5$, and $\beta \leq 0.9$.

[248] Mostly, $\hat{\theta}$ yields 6.4-12.7% (6.9-30.5%) for yearly (quarterly) evaluations. The only exception is found for the lowest additional incomes ($\beta = 0.1, \delta = 0.9$) for which $\hat{\theta} \approx -4.3\%$.

[249] In particular, $(1 - \hat{\alpha})|\hat{\theta}|$ yields 6-12.7% (5.4-30.4%) for yearly (quarterly) evaluations. Only for the lowest additional incomes ($\beta = 0.1, \delta = 0.9$), we obtain $\hat{\theta} \approx -3.7\%$.

maximization of either expected or non-expected utility. In both situations, we explicitly account for the narrow framing of financial investments, as well as for the impact of past performance on current perceptions. Note that each of the two settings requires specific conditions in order to attain the aggregate equilibrium. For instance, the non-expected utility equilibrium does not allow for the influence of past performance. It also restricts the set of feasible values of several behavioral parameters, such as the risk- and loss-aversion coefficients.

Both settings deliver direct equilibrium estimates of the prospective value, i.e. of the subjective utility of financial investments. From this variable, we obtain equilibrium-equivalent measures of the loss attitude, such as the loss-aversion coefficient and the global first-order risk aversion. In addition, the expected-utility setting provides estimates of the coefficient by which utility is discounted in time. Byproducts of the estimation procedure in the non-expected-utility setting are wealth-allocation variables, such as the percentages of total wealth dedicated to consumption and of post-consumption wealth to be invested in risky assets.

The theoretical results are subsequently tested and extended in an applied context. We use the S&P 500 and the 3-months T-bill nominal returns, as proxies for a well diversified risky portfolio and the risk-free investment, respectively, as well as quarterly aggregate per-capita consumption data between 1982-2006. In order to avoid the impossibility of covering current consumption needs from financial revenues throughout the entire investing period, we also consider that investors dispose of exogenous additional incomes at each decision time. These incomes are shaped in order to ensure the equivalency of the two settings with expected and non-expected utility. General market parameters (such as the risk-free returns and the dynamics of consumption and of expected returns) are estimated on the basis of the above real data. As such an estimation is not possible for the behavioral parameters (such as the degree of narrow framing, the risk aversion to consumption, the weight of financial utility, the sensitivity to past losses, the way of accounting for past performance, etc.), we work with wide value-sets of these behavioral parameters in order to detect plausible (combinations of) values. We investigate how the main variables that express loss attitudes and wealth allocation, change subject to different behavioral profiles of non-professional investors, at different levels of the additional income, and for two distinct horizons of risky performance evaluation, specifically of one year and three months.

The two settings with expected and non-expected utility can be straightforwardly compared only in a qualitative sense, that is with respect to their general recommendations and to the variation patterns of the main variables. A quantitative comparison of the variable values is possible only for the cases based on common identical assumptions, from which the most important regards the memoryless cushions.

The common and specific numerical findings of the two settings can be summarized as follows: First, the prospective value mostly grows with the narrow-framing degree of financial investments under the maximization of both expected and non-expected utility, showing an improvement in the perceived benefits of such investments. Moreover, myopic loss aversion in the large sense, i.e. the depreciations of the perceived financial utility in consequence of the more frequent evaluation of risky performance, manifests only under non-expected utility maximization.

Second, the two measures of the loss attitude are consistent with each other within each setting, but exhibit contrary variation patterns between the settings. In particular, myopic loss aversion in the strict sense, i.e. the increase in the loss reluctance with the evaluation frequency, holds only under the maximization of expected utility and only for the loss-aversion coefficient. Relative to this coefficient, gRA appears to be better suited as a measure of loss aversion. The direct comparison that is merely possible across common cases suggests that maximizers of expected utility are substantially less loss reluctant than their non-expected utility peers.

Third, while the non-expected-utility setting allows for the direct derivation – i.e. in the form of equilibrium values – of wealth-allocation variables, in the expected-utility setting these variables can be assessed only on average. However, both the mean estimates under expected utility and the direct estimates under non-expected utility vary in similar directions between the two settings. Thus, wealth allocation is invariant with respect to the narrow-framing degree but strongly influenced by the additional income. Investors who check on risky performance more often allocate less money to financial assets in total, so that they behave myopically averse towards financial investments in total. Myopic loss aversion in the monetary sense, i.e. the decline of the total-wealth percentages to be put in risky assets for higher evaluation frequencies, does not hold in any of the two settings. The direct comparison points at an increased monetary openness towards financial investments in total under expected utility, coupled with a substantially lower acceptance of risky investments in particular.

Note also that too high degrees of consumption-related risk aversion appear to be incompatible with the present equilibrium framework. Moreover, accounting for consumption as an additional source of utility appears to play an important role with respect to wealth allocation. Relative to the case discussed in Section 3.2, when investors are merely concerned with financial utility, the maximizers of two-dimensional expected utility allocate substantially less (more) money to risky assets when their performance is revised yearly (quarterly).

Since the estimates under non-expected utility are more informative, more robust, and change more intuitively with the behavioral investor profile, we consider this setting to be better suited to describe behavior and decision making of non-professional investors.

APPENDIX A

Appendix

A.1 Imperfect information, practical trading rules, and asset prices

A.1.1 Proofs

We make the following notations for the denominators of X_t^B and X_t^S:

$$\begin{aligned}\Pi_B &= 1 + (2\alpha_t p_t - 1)n_{\mathbf{a}} + (2\beta_t q_t^H p_t + 2\beta_t q_t^L (1 - p_t) - 1)n_{\mathbf{b}} \\ \Pi_S &= 1 + (2\alpha_t (1 - p_t) - 1)n_{\mathbf{a}} + (2\beta_t (1 - q_t^H) p_t + 2\beta_t (1 - q_t^L)(1 - p_t) - 1)n_{\mathbf{b}}.\end{aligned} \tag{A.1}$$

We observe that, due to the fact that we work with probabilities (i.e. variables that lie in $[0, 1]$), the variables defined in Equations (A.1) are positive.

Proposition 1

The proof is trivial by law of iterated expectations:

$$E_t[X_{t+1}] = E[E[V|h_{t+1}]|h_t] = E_t[V] = X_t.$$

Proposition 2

The proof is trivial by the fact that the ask and bid prices in Equations (1.12) depend on the probabilities q_t^H and q_t^L. If these probabilities are history dependent, so will be transaction prices as well.

Proposition 3

The proof of the first part of Proposition 3 is trivial from Equations (1.9):

$$X_t^B \overset{(1)}{=} V^L + (V^H - V^L)P_t(V = V^H | x_{it} = B) \leq V^L + (V^H - V^L)1 = V^H$$
$$X_t^S \overset{(1)}{=} V^L + (V^H - V^L)P_t(V = V^H | x_{it} = S) \geq V^L + (V^H - V^L)0 = V^L.$$

For the second part, observe that according to Equations (1.7), (1.8), and (1.9), the spread is proportional to:

$$P_t(x_{it} = B | V = V^H) - P_t(x_{it} = B | V = V^L) = \alpha_t n_{\mathbf{a}} + \beta_t (q_t^H - q_t^L) n_{\mathbf{b}} \geq 0.$$

Corollary 3.1

When there are no informed traders trading in the market, i.e. $n_{\mathbf{c}} = 1$ (or equivalently $n_{\mathbf{a}} = n_{\mathbf{b}} = 0$), the prices refer only to past information and are thus equal $X_t^B = X_t^S = V^H p_t + V^L(1 - p_t)$ and the spread is nil $S_t = 0$.

If there are merely perfectly informed traders trading in the market, i.e. $n_{\mathbf{a}} = 1$, the bid and ask prices reach their maximum:

$$X_t^B = V^H$$
$$X_t^S = V^L,$$

as well as the spread $S_t = V^H - V^L$.

In contrast, with exclusively imperfectly informed traders active in the market, i.e. $n_{\mathbf{b}} = 1$, the value of the spread depends on the accuracy of the trader information and is lower than for $n_{\mathbf{a}} = 1$:

$$X_t^B = V^L + (V^H - V^L) \frac{q_t^H p_t}{q_t^H p_t + q_t^L (1 - p_t)} \leq V^H$$
$$X_t^S = V^L + (V^H - V^L) \frac{(1 - q_t^H) p_t}{1 - q_t^H p_t - q_t^L (1 - p_t)} \geq V^L.$$

Finally, with both fully and imperfectly informed traders but no liquidity traders active in the market, i.e. $n_{\mathbf{a}} + n_{\mathbf{b}} = 1$, the spread depends again on the accuracy of the trader information:

$$X_t^B \overset{(1)}{=} V^L + (V^H - V^L)p_t \frac{\alpha_t n_{\mathbf{a}} + \beta_t q_t^H n_{\mathbf{b}}}{\alpha_t p_t n_{\mathbf{a}} + \beta_t (q_t^H p_t + q_t^L (1-p_t)) n_{\mathbf{b}}} \leq V^H$$
$$X_t^S \overset{(1)}{=} V^L + (V^H - V^L)p_t \frac{\beta_t (1 - q_t^H) n_{\mathbf{b}}}{\alpha_t (1-p_t) + \beta_t (1 - q_t^H p_t - q_t^L (1-p_t)) n_{\mathbf{b}}} \geq V^L.$$

Note that the spread is lower than for $n_{\mathbf{a}} = 1$ but higher than for $n_{\mathbf{b}} = 1$.

For the latter cases with $n_{\mathbf{c}} = 0$,[1] the spread is positive $S_t \geq 0$. This results easily from Equations (1.9), where:

$$S_t = X_t^B - X_t^S \overset{(1)}{=} (V^H - V^L)(P_t(V = V^H | x_{it} = B) - P_t(V = V^H | x_{it} = S)),$$

which, by reexpressing $P_t(V = V^H | x_{it} = B)$ and $P_t(V = V^H | x_{it} = S)$ as posterior probabilities in line with Equations (1.10) and going back on Formulas (1.11), reduces to comparing:

$$\begin{aligned} &P_t(x_{it} = B | V = V^H) P_t(x_{it} = S | V = V^L) - P_t(x_{it} = B | V = V^L) P_t(x_{it} = S | V = V^H) \\ &\quad = (1 - (2\alpha_t - 1)n_{\mathbf{a}} + (2\beta_t q_t^H - 1)n_{\mathbf{b}})(1 - (2\alpha_t - 1)n_{\mathbf{a}} + (2\beta_t - 1)n_{\mathbf{b}}) \\ &\quad - (1 - n_{\mathbf{a}} + (2\beta_t q_t^L - 1)n_{\mathbf{b}})(1 - n_{\mathbf{a}} + (2\beta_t - 1)n_{\mathbf{b}}) \end{aligned}$$

to zero.

In sum, when the market maker fears higher informational asymmetries relative to traders, the spread is enhanced.

Proposition 4

Equations (1.14) and (1.12) yield for perfectly informed traders:

$$\begin{aligned} &E_t[G_{\mathbf{a}t} | x_{\mathbf{a}t} = B] = V^H - X_t^B \\ &\quad = (V^H - V^L)(1-p_t) \frac{1 - n_{\mathbf{a}} + (2\beta_t q_t^L - 1)n_{\mathbf{b}}}{1 + (2\alpha_t p_t - 1)n_{\mathbf{a}} + (2\beta_t q_t^H p_t + 2\beta_t q_t^L (1-p_t) - 1)n_{\mathbf{b}}} \geq 0 \\ &E_t[G_{\mathbf{a}t} | x_{\mathbf{a}t} = S] = X_t^S - V^L \\ &\quad = (V^H - V^L)p_t \frac{1 - n_{\mathbf{a}} + (2\beta_t (1 - q_t^H) - 1)n_{\mathbf{b}}}{1 + (2\alpha_t (1-p_t) - 1)n_{\mathbf{a}} + (2\beta_t (1 - q_t^H)p_t + 2\beta_t (1 - q_t^L)(1-p_t) - 1)n_{\mathbf{b}}} \geq 0. \end{aligned}$$

[1]However, our assumption $n_{\mathbf{c}} > 0$ excludes the realization of one of the latter three situations. We analyze them here only with the purpose of revealing the importance of adverse selection costs.

In contrast, imperfectly informed traders obtain:

$$
\begin{aligned}
&E_t[G_{\mathbf{b}t}|x_{\mathbf{b}t}=B]\\
&\quad=(V^H-V^L)p_t\left(\frac{q_t^H}{q_t^Hp_t+q_t^L(1-p_t)}-\frac{1+(2\alpha_t-1)n_{\mathbf{a}}+(2\beta_tq_t^H-1)n_{\mathbf{b}}}{1+(2\alpha_tp_t-1)n_{\mathbf{a}}+(2\beta_tq_t^Hp_t+2\beta_tq_t^L(1-p_t)-1)n_{\mathbf{b}}}\right)\\
&\quad=\frac{(V^H-V^L)p_t(1-p_t)((q_t^H-q_t^L)(1-n_{\mathbf{a}}-n_{\mathbf{b}})-2\alpha_tq_t^Ln_{\mathbf{a}})}{\Big(q_t^Hp_t+q_t^L(1-p_t)\Big)\Big(1+(2\alpha_tp_t-1)n_{\mathbf{a}}+(2\beta_tq_t^Hp_t+2\beta_tq_t^L(1-p_t)-1)n_{\mathbf{b}}\Big)}\\
&E_t[G_{\mathbf{b}t}|x_{\mathbf{b}t}=S]\\
&\quad=(V^H-V^L)p_t\frac{1-n_{\mathbf{a}}+(2\beta_t(1-q_t^H)-1)n_{\mathbf{b}}}{1+(2\alpha_t(1-p_t)-1)n_{\mathbf{a}}+(2\beta_t(1-q_t^H)p_t+2\beta_t(1-q_t^L)(1-p_t)-1)n_{\mathbf{b}}}\\
&\quad-(V^H-V^L)p_t\frac{1-q_t^H}{(1-q_t^H)p_t+(1-q_t^L)(1-p_t)}\\
&=\frac{(V^H-V^L)p_t(1-p_t)((q_t^H-q_t^L)(1-n_{\mathbf{a}}-n_{\mathbf{b}})-2\alpha_t(1-q_t^H)n_{\mathbf{a}})}{\Big((1-q_t^H)p_t+(1-q_t^L)(1-p_t)\Big)\Big(1+(2\alpha_t(1-p_t)-1)n_{\mathbf{a}}+(2\beta_t(1-q_t^H)p_t+2\beta_t(1-q_t^L)(1-p_t)-1)n_{\mathbf{b}}\Big)},
\end{aligned}
$$

where the conditions stated in Remark 4.1 entail positivity.

The liquidity traders make losses by each trade:

$$
\begin{aligned}
&E_t[G_{\mathbf{c}t}|x_{\mathbf{c}t}=B]=V^Hp_t+V^L(1-p_t)-X_t^B\\
&\quad=-2(V^H-V^L)p_t(1-p_t)\frac{\alpha_tn_{\mathbf{a}}+\beta_t(q_t^H-q^L)n_{\mathbf{b}}}{1+(2\alpha_tp_t-1)n_{\mathbf{a}}+(2\beta_tq_t^Hp_t+2\beta_tq_t^L(1-p_t)-1)n_{\mathbf{b}}}\leq 0\\
&E_t[G_{\mathbf{c}t}|x_{\mathbf{c}t}=S]=X_t^S-V^Hp_t-V^L(1-p_t)\\
&\quad=2(V^L-V^H)p_t(1-p_t)\frac{\alpha_tn_{\mathbf{a}}+\beta_t(q_t^H-q^L)n_{\mathbf{b}}}{1+(2\alpha_tp_t-1)n_{\mathbf{a}}+(2\beta_tq_t^Hp_t+2\beta_tq_t^L(1-p_t)-1)n_{\mathbf{b}}}\leq 0.
\end{aligned}
$$

Proposition 5

From Equations (1.12), we derive:

$$
\begin{aligned}
\frac{\partial X_t^B}{\partial p_t}&\overset{(1)}{=}\frac{(V^H-V^L)(1-n_{\mathbf{a}}-n_{\mathbf{b}}+2\alpha_tn_{\mathbf{a}}+2\beta_tq_t^Hn_{\mathbf{b}})}{\Pi_B^2}(1-n_{\mathbf{a}}-n_{\mathbf{b}}+2\beta_tq_t^Ln_{\mathbf{b}})\geq 0\\
\frac{\partial X_t^S}{\partial p_t}&\overset{(1)}{=}\frac{(V^H-V^L)(1-n_{\mathbf{a}}-n_{\mathbf{b}}+2\beta_t(1-q_t^H)n_{\mathbf{b}})}{\Pi_S^2}(1-n_{\mathbf{a}}-n_{\mathbf{b}}+2\alpha_tn_{\mathbf{a}}+2\beta_t(1-q_t^L)n_{\mathbf{b}})\geq 0
\end{aligned}
$$

Thus:

$$
\left.\begin{aligned}
\frac{\partial^2 X_t^B}{\partial p_t^2}&=-4\frac{\partial X_t^B}{\partial p_t}\frac{\alpha_tn_{\mathbf{a}}+\beta_t(q_t^H-q_t^L)n_{\mathbf{b}}}{\Pi_B}\leq 0\\
\frac{\partial^2 X_t^S}{\partial p_t^2}&=4\frac{\partial X_t^S}{\partial p_t}\frac{\alpha_tn_{\mathbf{a}}+\beta_t(q_t^H-q_t^L)n_{\mathbf{b}}}{\Pi_S}\geq 0
\end{aligned}\right.\quad\Rightarrow\quad\frac{\partial^2 S_t}{\partial p_t^2}=\frac{\partial^2 X_t^B}{\partial p_t^2}-\frac{\partial^2 X_t^S}{\partial p_t^2}\leq 0,
$$

With the notations:

$$\Upsilon_1 = 1 - n_{\mathbf{a}} - n_{\mathbf{b}} + 2\beta_t q_t^L n_{\mathbf{b}}$$
$$\Upsilon_2 = 1 - n_{\mathbf{a}} - n_{\mathbf{b}} + 2\alpha_t n_{\mathbf{a}} + 2\beta_t(1 - q_t^L) n_{\mathbf{b}}$$
$$\Upsilon_3 = 2\Big(\alpha_t n_{\mathbf{a}} + \beta_t(q_t^H - q_t^L) n_{\mathbf{b}}\Big),$$

$p_t^{\max}$ solves the following equation:

$$(\Upsilon_1 + \Upsilon_2)\Upsilon_3\Big(\Upsilon_3(\Upsilon_1 - \Upsilon_2 + \Upsilon_3)p^2 - 2\Upsilon_1\Upsilon_2 p + \Upsilon_1\Upsilon_2 = 0\Big).$$

Under the conditions that:[2]

$$\Upsilon_1 + \Upsilon_2 > 0$$
$$\Upsilon_3 > 0$$
$$\Upsilon_1\Upsilon_2(\Upsilon_1 + \Upsilon_3)(\Upsilon_2 - \Upsilon_3) \geq 0,$$

the equation has real solutions and the spread is maximized for:

$$p_t^{\max} = \begin{cases} \dfrac{\sqrt{\Upsilon_1\Upsilon_2}\Big(\sqrt{\Upsilon_1\Upsilon_2} \pm \sqrt{(\Upsilon_1 + \Upsilon_3)(\Upsilon_2 - \Upsilon_3)}\Big)}{\Upsilon_3(\Upsilon_1 - \Upsilon_2 + \Upsilon_3)}, \text{ for } \Upsilon_1 - \Upsilon_2 + \Upsilon_3 \neq 0 \\ 0.5, \text{ for } \Upsilon_1 - \Upsilon_2 + \Upsilon_3 = 0, \end{cases}$$

$$\text{s.t. } p_t^{\max} \in [0, 1].$$

Obviously, from Equations (1.12) the spread $S_t = 0$ for both extreme values of the prior probability $p_t \in \{0; 1\}$.

In the particular case with $\alpha_t = 0.20$, $\beta_t q_t^H = 0.67$, and $\beta_t q_t^L = 0.33$, we have $\beta_t(q_t^H + q_t^L) = 1$ and hence $\Upsilon_2 - \Upsilon_2 + \Upsilon_3 = 0$, so that $p_t^{\max} = 0.5$.

Proposition 6

Writing:

$$p_{t+1} = P_{t+1}(V = V^H) = P_t(V = V^H | X_{t+1}^B, X_{t+1}^S) = \frac{P_t(X_{t+1}^B, X_{t+1}^S | V = V^H) P_t(V = V^H)}{P_t(X_{t+1}^B, X_{t+1}^S)},$$

and taking:

$$R_t = \frac{P_t(X_{t+1}^B, X_{t+1}^S | V^H)}{P_t(X_{t+1}^B, X_{t+1}^S | V^L)},$$

[2]Obviously, $\Upsilon_1, \Upsilon_2, \Upsilon_3 > 0$.

we obtain:

$$\frac{p_{t+1}}{1-p_{t+1}} = \frac{p_t}{1-p_t}R_t = \frac{p_1}{1-p_1}\prod_{s=1}^{t} R_s.$$

For instance, when the true risky value is high $V = V^H$ (low $V = V^L$), the ratios R_t of the probabilities that the current spread emerges subject to a high with respect to a low risky value eventually converge to infinity (zero). Thus, $p_{t+1}/(1-p_{t+1})$ becomes the product of higher and higher (lower and lower) values and converges to infinity (zero). In essence, the market maker infers valuable information from trades (which are public information) and all market participants should be able to recognize the true value after a sufficient number of transactions.

Proposition 7.1

From Equations (1.12) and given the positivity of the variables defined in Equations (A.1), we obtain:

$$\begin{aligned}
\frac{\partial X_t^B}{\partial \beta_t q_t^H} &\overset{(2)}{=} -\frac{(V^L - V^H)(1-p_t)}{\Pi_B^2}\Big(1 - n_{\mathbf{a}} + (2\beta_t q_t^L - 1)n_{\mathbf{b}}\Big)2p_t n_{\mathbf{b}} \geq 0\\
\frac{\partial X_t^S}{\partial \beta_t q_t^H} &\overset{(2)}{=} \frac{(V^L - V^H)(1-p_t)}{\Pi_S^2}\Big(1 + (2\alpha_t - 1)n_{\mathbf{a}} + (2\beta_t(1 - q_t^L) - 1)n_{\mathbf{b}}\Big)2p_t n_{\mathbf{b}} \leq 0\\
\frac{\partial X_t^B}{\partial \beta_t q_t^L} &\overset{(1)}{=} -\frac{(V^H - V^L)p_t}{\Pi_B^2}\Big(1 + (2\alpha_t - 1)n_{\mathbf{a}} + (2\beta_t q_t^H - 1)n_{\mathbf{b}}\Big)2(1-p_t)n_{\mathbf{b}} \leq 0\\
\frac{\partial X_t^S}{\partial \beta_t q_t^L} &\overset{(1)}{=} \frac{(V^H - V^L)p_t}{\Pi_S^2}\Big(1 - n_{\mathbf{a}} + (2\beta_t(1 - q_t^H) - 1)n_{\mathbf{b}}\Big)2(1-p_t)n_{\mathbf{b}} \geq 0,
\end{aligned}$$

so that:

$$\begin{aligned}
\frac{\partial S_t}{\partial \beta_t q_t^H} &= \frac{\partial X_t^B}{\partial \beta_t q_t^H} - \frac{\partial X_t^S}{\partial \beta_t q_t^H} \geq 0\\
\frac{\partial S_t}{\partial \beta_t q_t^L} &= \frac{\partial X_t^B}{\partial \beta_t q_t^L} - \frac{\partial X_t^S}{\partial \beta_t q_t^L} \leq 0.
\end{aligned}$$

Hence:

$$\begin{aligned}
\frac{\partial^2 X_t^B}{\partial(\beta_t q_t^H)^2} &= -2\frac{\partial X_t^B}{\partial \beta_t q_t^H}\frac{2p_t n_{\mathbf{b}}}{\Pi_B} \leq 0\\
\frac{\partial^2 X_t^S}{\partial(\beta_t q_t^H)^2} &= -2\frac{\partial X_t^S}{\partial \beta_t q_t^H}\frac{-2p_t n_{\mathbf{b}}}{\Pi_S} \leq 0\\
\frac{\partial^2 X_t^B}{\partial(\beta_t q_t^L)^2} &= -2\frac{\partial X_t^B}{\partial \beta_t q_t^L}\frac{2(1-p_t)n_{\mathbf{b}}}{\Pi_B} \geq 0\\
\frac{\partial^2 X_t^S}{\partial(\beta_t q_t^L)^2} &= -2\frac{\partial X_t^S}{\partial \beta_t q_t^L}\frac{-2(1-p_t)n_{\mathbf{b}}}{\Pi_S} \geq 0
\end{aligned}$$

and no clear-cut conclusion can be formulated about the curvature of the spread.

Proposition 7.2

From Equations (1.12), we derive:

$$\frac{\partial X_t^B}{\partial n_{\mathbf{b}}} \overset{(1)}{=} \frac{2p_t(1-p_t)(V^H - V^L)}{\Pi_B^2}\Big(\beta_t(q_t^H - q_t^L)(1-n_{\mathbf{a}}) + \alpha_t(1-2\beta_t q_t^L)n_{\mathbf{a}}\Big)$$
$$\frac{\partial X_t^S}{\partial n_{\mathbf{b}}} \overset{(2)}{=} \frac{2p_t(1-p_t)(V^L - V^H)}{\Pi_S^2}\Big(\beta_t(q_t^H - q_t^L)(1-n_{\mathbf{a}}) + \alpha_t(1-2\beta_t(1-q_t^H))n_{\mathbf{a}}\Big),$$

where the assumptions in Proposition 7.2 and Remark 7.2.1 result in positivity (negativity) of the ask (bid). Note that more general positivity (negativity) conditions are $\beta_t\,(q_t^H - q_t^L)\,(1 - n_{\mathbf{a}}) + \alpha_t(1-2\beta_t q_t^L)n_{\mathbf{a}}$ $(\beta_t(q_t^H - q_t^L)(1-n_{\mathbf{a}}) + \alpha_t(1-2\beta_t(1-q_t^H))n_{\mathbf{a}})$ for ask (bid). Moreover, for $\alpha_t = 0$, $\dfrac{\partial X_t^B}{\partial n_{\mathbf{b}}} \geq 0$ and $\dfrac{\partial X_t^S}{\partial n_{\mathbf{b}}} \leq 0$, as stressed in Remark 7.2.3. Hence:

$$\frac{\partial^2 X_t^B}{\partial n_{\mathbf{b}}^2} = 2\frac{\partial X_t^B}{\partial n_{\mathbf{b}}}\frac{1-2\beta_t q_t^H p_t - 2\beta_t q_t^L(1-p_t)}{\Pi_B}$$
$$\frac{\partial^2 X_t^S}{\partial n_{\mathbf{b}}^2} = -2\frac{\partial X_t^S}{\partial n_{\mathbf{b}}}\frac{1-2\beta_t + 2\beta_t q_t^H p_t + 2\beta_t q_t^L(1-p_t)}{\Pi_S},$$

which, given that the denominators of both ask and bid prices are strictly positive, yields the results stressed in Remarks 7.2.1 and 7.2.2. Moreover, as $\beta_t \leq 1$, $2\beta_t\Big(q_t^H p_t + 2\beta_t q_t^L(1-p_t)\Big) \geq 1$ implies that $2\beta_t\Big(1 - q_t^H p_t - 2\beta_t q_t^L(1-p_t)\Big) \leq 1$ and similarly $2\beta_t\Big(1 - q_t^H p_t - 2\beta_t q_t^L(1-p_t)\Big) > 1$ results in $2\beta_t\Big(q_t^H p_t + 2\beta_t q_t^L(1-p_t)\Big) < 1$ which yields the last statement in Remark 7.2.2.

Proposition 8.1

From Equations (1.12), we have:

$$\left.\begin{aligned}\frac{\partial X_t^B}{\partial \alpha} &\overset{(2)}{=} -\frac{V^L - V^H}{\Pi_B^2}2p_t n_{\mathbf{a}} \geq 0\\ \frac{\partial X_t^S}{\partial \alpha} &\overset{(1)}{=} -\frac{V^H - V^L}{\Pi_S^2}2(1-p_t)n_{\mathbf{a}} \leq 0.\end{aligned}\right. \quad \Rightarrow \quad \frac{\partial S_t}{\partial \alpha} = \frac{\partial X_t^B}{\partial \alpha} - \frac{\partial X_t^S}{\partial \alpha} \geq 0.$$

Thus:

$$\frac{\partial^2 X_t^B}{\partial \alpha^2} = -2\frac{\partial X_t^B}{\partial \alpha}\frac{2p_t n_{\mathbf{a}}}{\Pi_B} \leq 0$$
$$\frac{\partial^2 X_t^S}{\partial \alpha^2} = -2\frac{\partial X_t^S}{\partial \alpha}\frac{2(1-p_t)n_{\mathbf{a}}}{\Pi_S} \geq 0$$

so that:

$$\frac{\partial^2 S_t}{\partial \alpha^2} = \frac{\partial^2 X_t^B}{\partial \alpha^2} - \frac{\partial^2 X_t^S}{\partial \alpha^2} \leq 0.$$

Proposition 8.2

From Equations (1.12), we derive:

$$\frac{\partial X_t^B}{\partial n_\mathbf{a}} \overset{(1)}{=} \frac{2p_t(1-p_t)(V^H-V^L)}{\Pi_B^2}\Big(\beta_t(q_t^H-q_t^L)n_\mathbf{b}+2\alpha_t\beta_t q_t^L n_\mathbf{b}+\alpha_t(1-n_\mathbf{b})\Big)\geq 0$$
$$\frac{\partial X_t^S}{\partial n_\mathbf{a}} \overset{(2)}{=} \frac{2p_t(1-p_t)(V^L-V^H)}{\Pi_B^2}\Big(\beta_t(q_t^H-q_t^L)n_\mathbf{b}+2\alpha_t\beta_t(1-q_t^H)n_\mathbf{b}+\alpha_t(1-n_\mathbf{b})\Big)\leq 0.$$

so that:

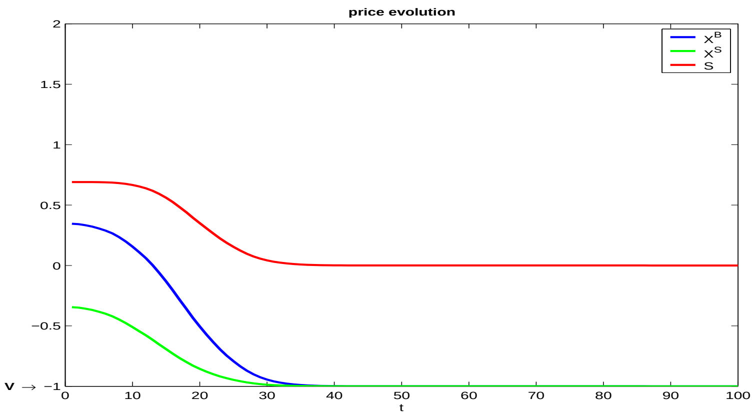
$$\frac{\partial S_t}{\partial n_\mathbf{a}}=\frac{\partial X_t^B}{\partial n_\mathbf{a}}-\frac{\partial X_t^S}{\partial n_\mathbf{a}}\geq 0.$$

The second order derivatives result then trivially in:

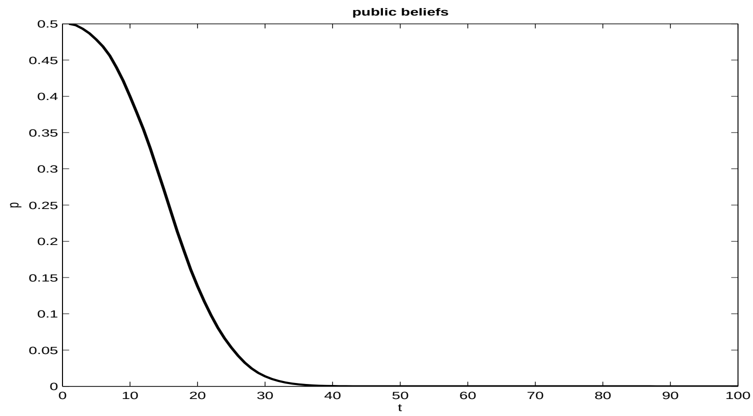
$$\frac{\partial^2 X_t^B}{\partial n_\mathbf{a}^2}=2\frac{\partial X_t^B}{\partial n_\mathbf{a}}\frac{1-2\alpha_t p_t}{\Pi_B}$$
$$\frac{\partial^2 X_t^S}{\partial n_\mathbf{a}^2}=-2\frac{\partial X_t^S}{\partial n_\mathbf{a}}\frac{2\alpha_t(1-p_t)-1}{\Pi_S}.$$

Obviously, as stated in Remark 8.2.1, as long as the probability that perfectly informed traders receive information $\alpha_t \leq 0.5p_t$, the ask evolves convexly. In addition, for $\alpha_t \leq 0.5(1-p_t)$, the bid is concave. As $\alpha \leq 1$, the second affirmation in Remark 8.2.1 holds trivially, due to the fact that $2\alpha_t p_t \geq 1$ implies that $2\alpha_t(1-p_t) \leq 1$, and similarly, $2\alpha_t(1-p_t) \geq 1$ results in $2\alpha_t p_t \leq 1$.

A.1.2 Graphics

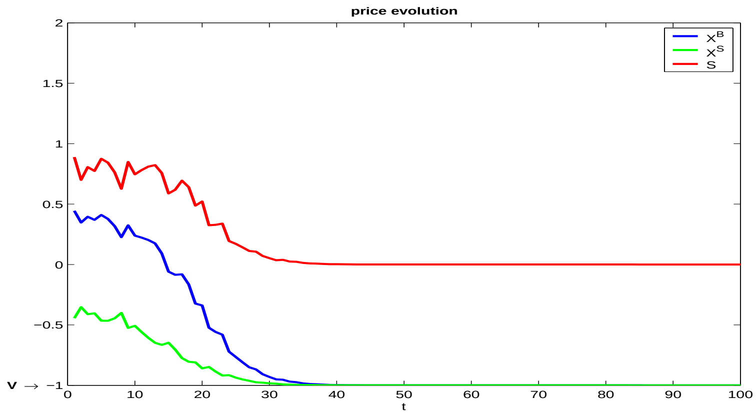


(a) Prices.

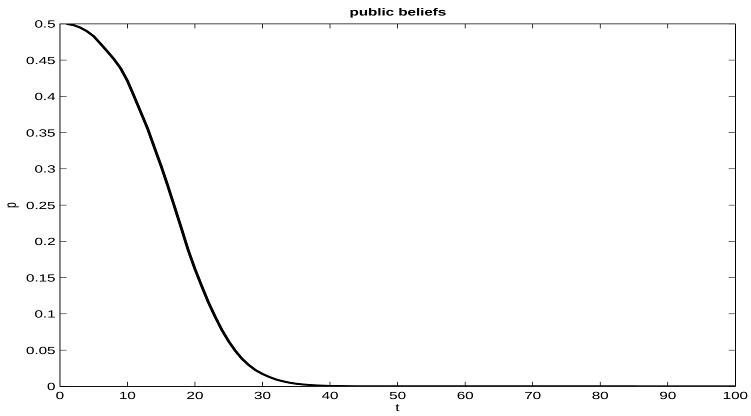


(b) Public beliefs.

Figure A.1: The evolution of prices and public beliefs in time, in a bad economy $V = V^L$, for proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, a probability of perfect information $\alpha_t = 0.20$, and accuracies of imperfect information $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$.

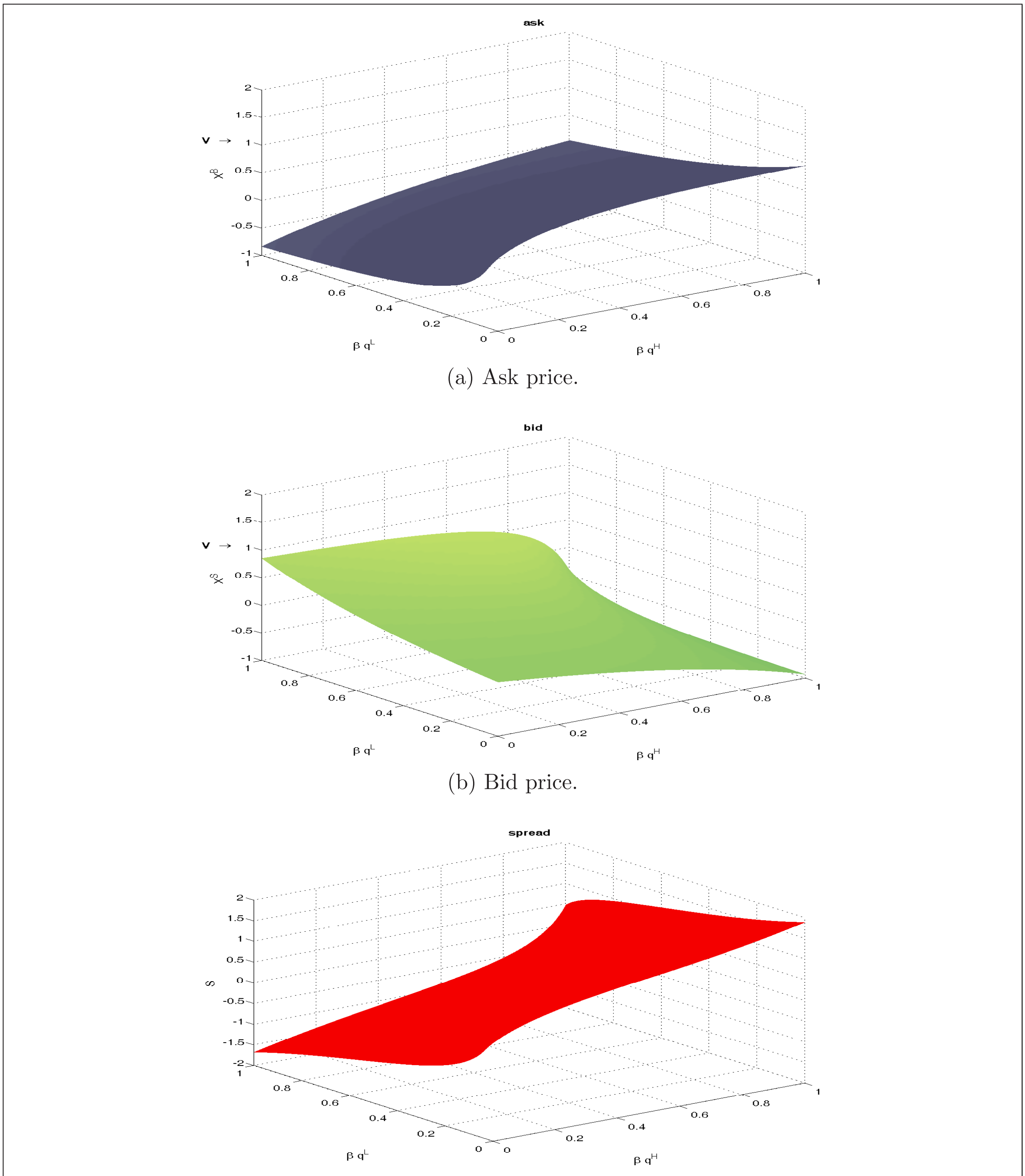


(a) Prices.



(b) Public beliefs.

Figure A.2: The evolution of prices and public beliefs in time, in a bad economy $V = V^L$, for proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, a random probability of perfect information α_t, and random accuracies of imperfect information βq^H and βq^L of middle accuracy 0.50.

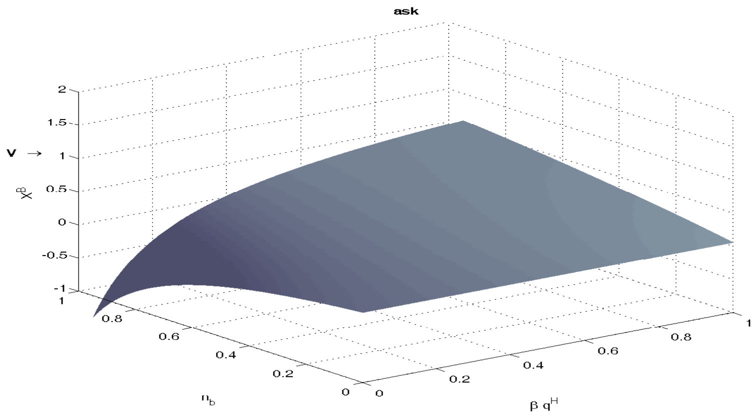


(a) Ask price.

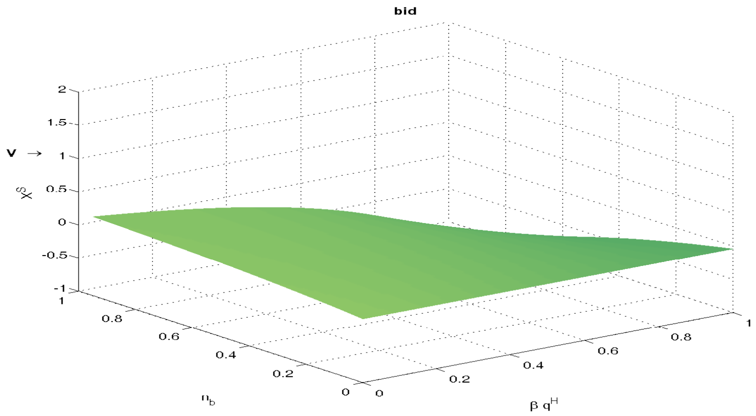
(b) Bid price.

(c) Spread.

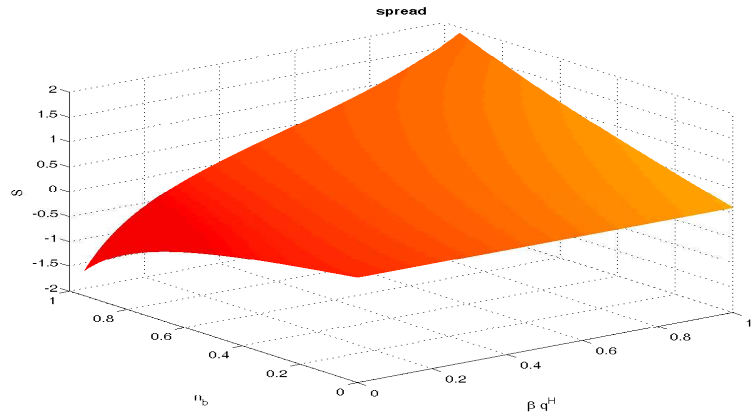
Figure A.3: Price evolution at time t subject to the accuracies of imperfect information βq^H and βq^L, in a good economy $V = V^H$, for proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, public beliefs $p_t = 0.50$, and a probability of perfect information $\alpha = 0.20$.



(a) Ask price.

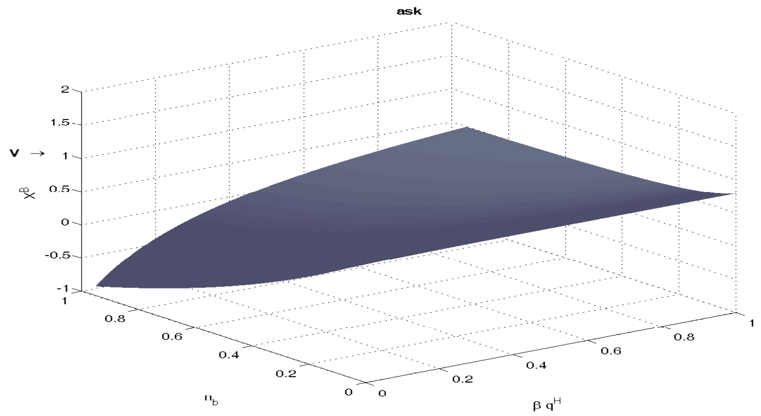


(b) Bid price.

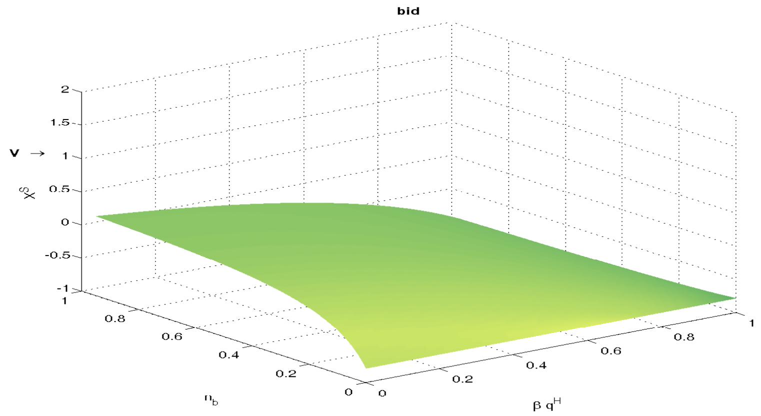


(c) Spread.

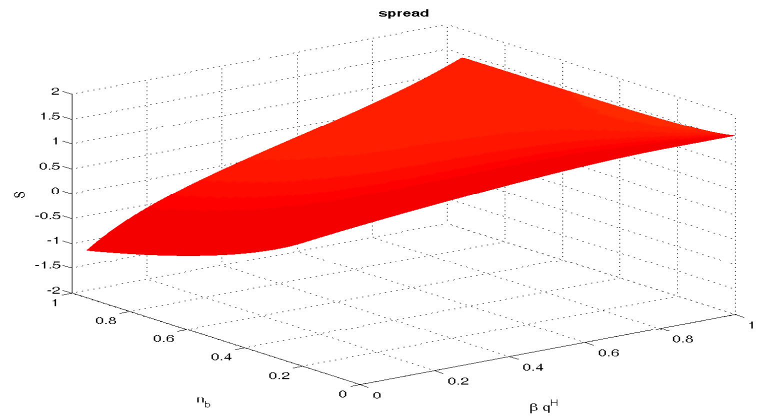
Figure A.4: Price evolution at time t subject to the accuracy of imperfect information in a good economy βq^H and to the proportion of imperfectly informed traders $n_{\mathbf{b}}$, in good economy $V = V^H$, for a fixed proportion of perfectly informed traders $n_{\mathbf{a}} = 20\%$, public beliefs $p_t = 0.50$, a probability of perfect information $\alpha = 0.20$, and an accuracy of imperfect information in a bad economy $\beta_t q_t^L = 0.33$.



(a) Ask price.

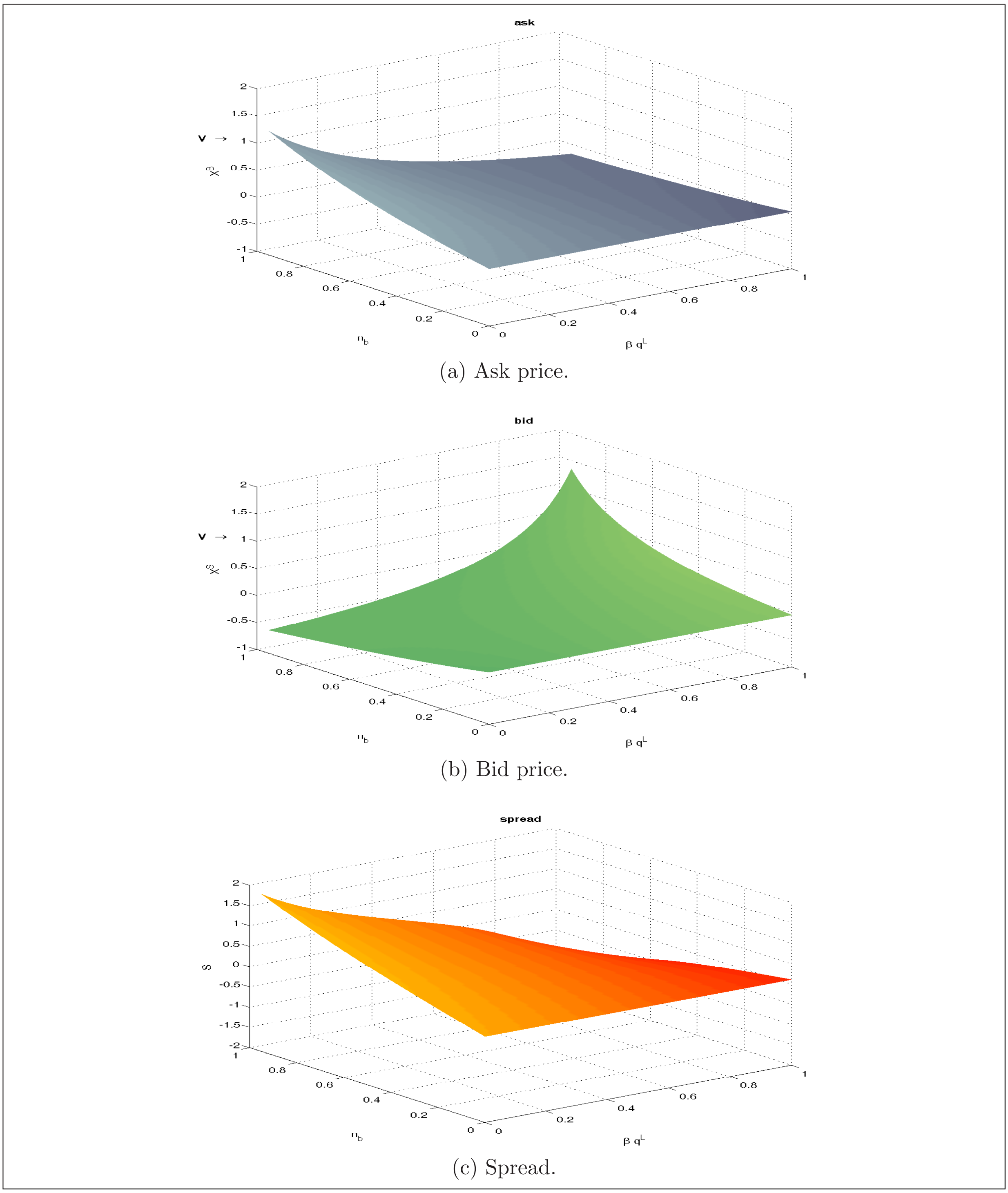


(b) Bid price.



(c) Spread.

Figure A.5: Price evolution at time t subject to the accuracy of imperfect information in a good economy βq^H and to the proportion of imperfectly informed traders $n_{\mathbf{b}}$, in a good economy $V = V^H$, for a fixed proportion of liquidity traders $n_{\mathbf{c}} = 5\%$, public beliefs $p_t = 0.50$, probability of perfect information $\alpha = 0.20$, and an accuracy of imperfect information in a bad economy $\beta_t q_t^L = 0.33$.

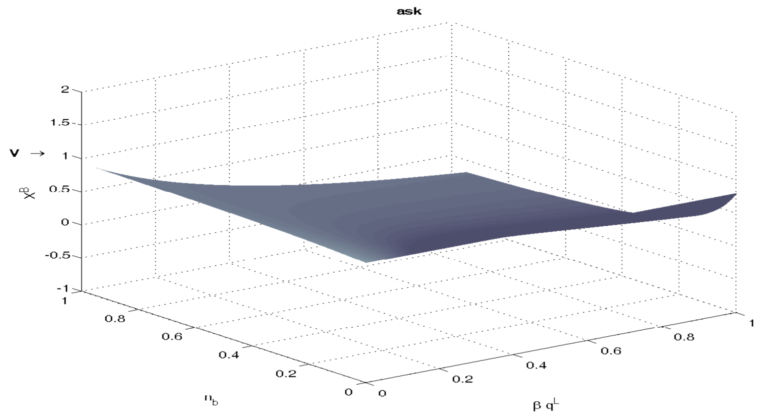


(a) Ask price.

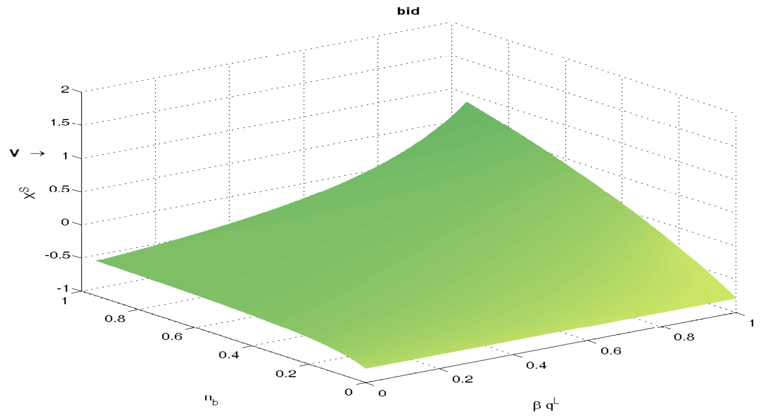
(b) Bid price.

(c) Spread.

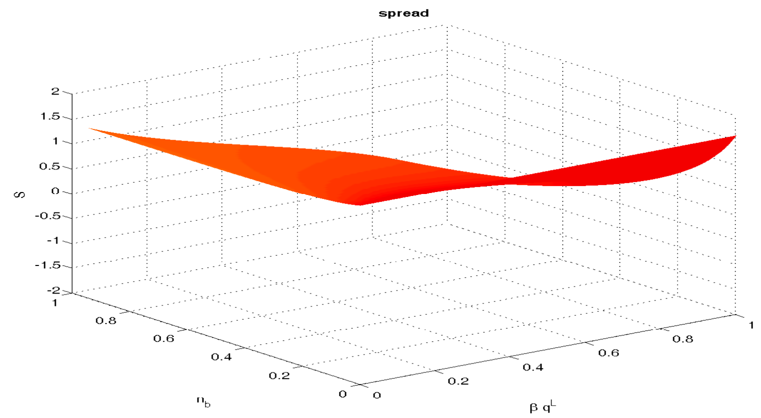
Figure A.6: Price evolution at time t subject to the accuracy of imperfect information in a bad economy $\beta_t q_t^L$ and to the proportion of imperfectly informed traders $n_{\mathbf{b}}$, in a good economy $V = V^H$, for a fixed proportion of perfectly informed traders $n_{\mathbf{a}} = 20\%$, public beliefs $p_t = 0.50$, a probability of perfect information $\alpha = 0.20$, andan accuracy of imperfect information in a good economy $\beta_t q_t^H = 0.67$.



(a) Ask price.

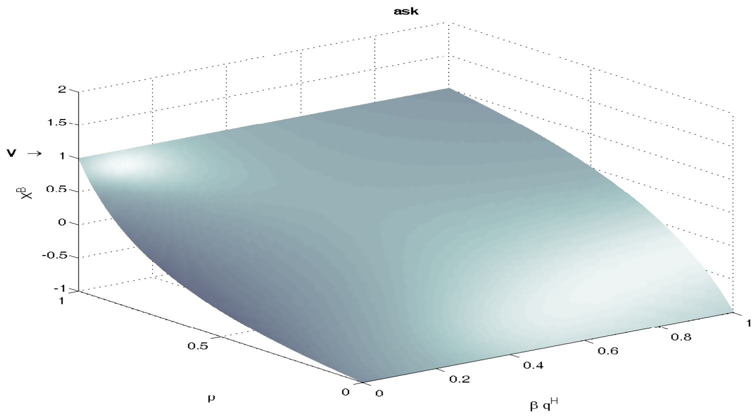


(b) Bid price.

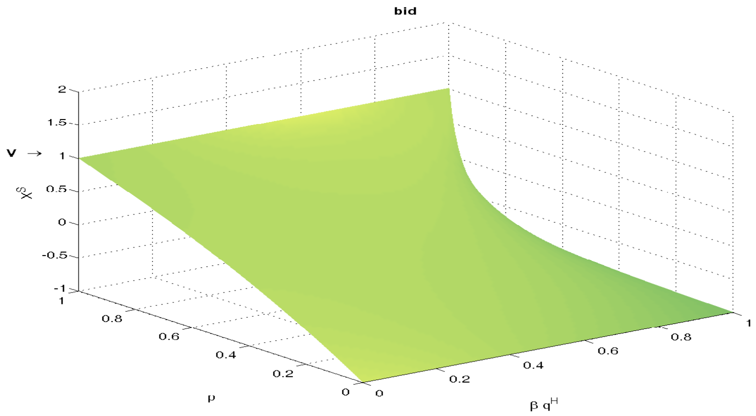


(c) Spread.

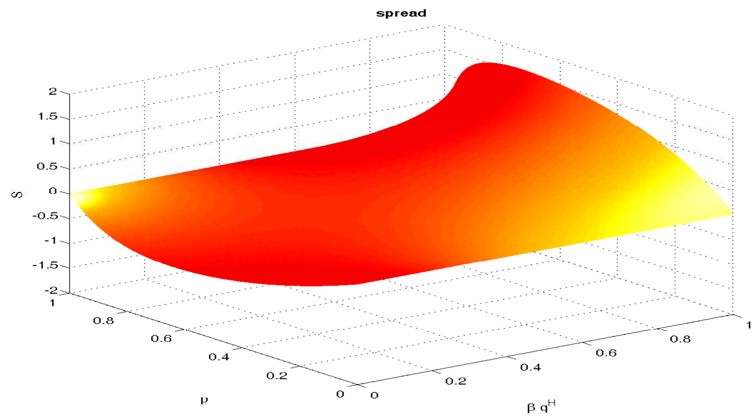
Figure A.7: Price evolution at time t subject to the accuracy of imperfect information in a bad economy $\beta_t q_t^L$ and to the proportion of imperfectly informed traders $n_{\mathbf{b}}$, in good economy $V = V^H$, for a fixed proportion of liquidity traders $n_{\mathbf{c}} = 5\%$, public beliefs $p_t = 0.50$, a probability of perfect information $\alpha = 0.20$, and an accuracy of imperfect information in a good economy $\beta_t q_t^H = 0.67$.



(a) Ask price.

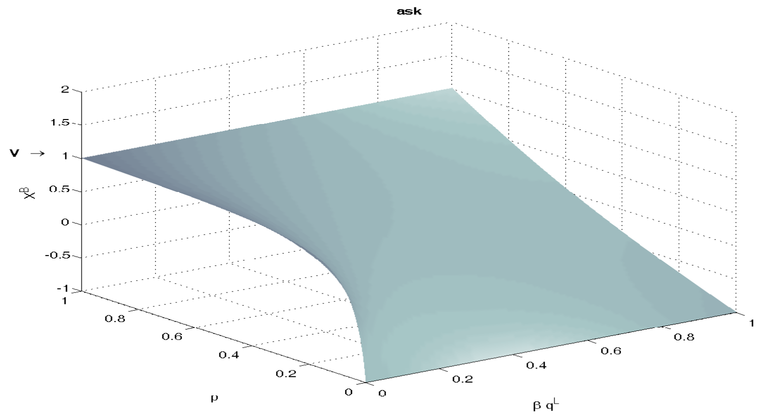


(b) Bid price.

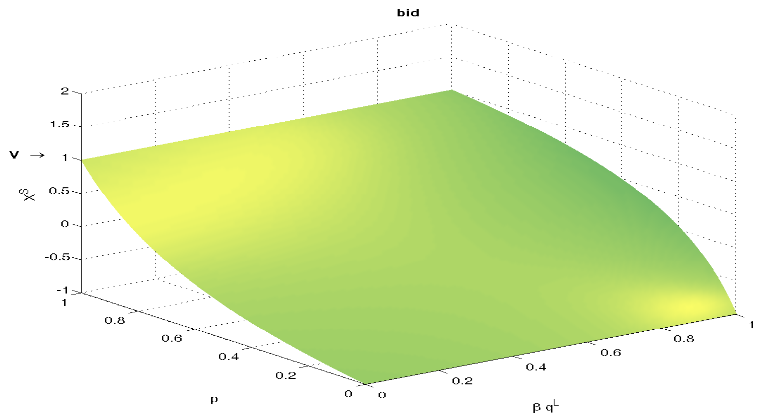


(c) Spread.

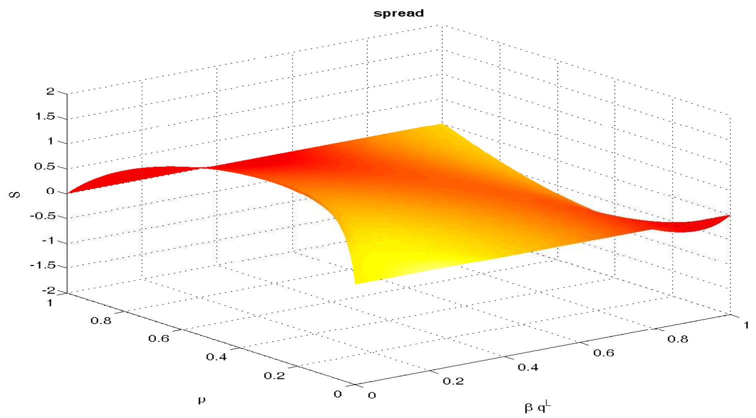
Figure A.8: Price evolution at time t subject to the accuracy of imperfect information in a good economy $\beta_t q_t^H$ and to the public beliefs p_t, in a good economy $V = V^H$, for proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, a probability of perfect information $\alpha = 0.20$, and an accuracy of imperfect information in a bad economy $\beta_t q_t^L = 0.33$.



(a) Ask price.

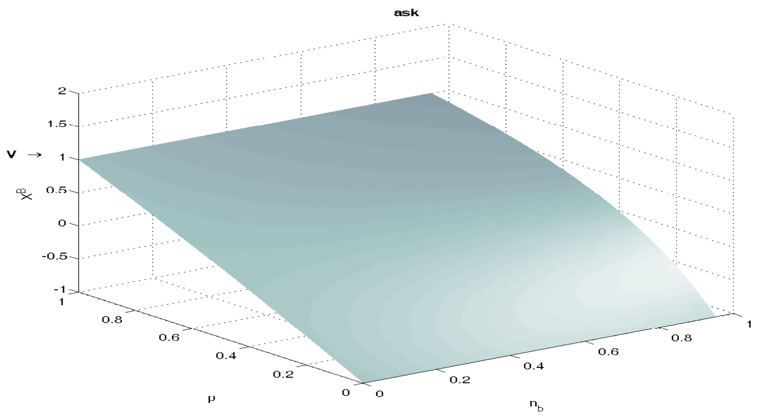


(b) Bid price.

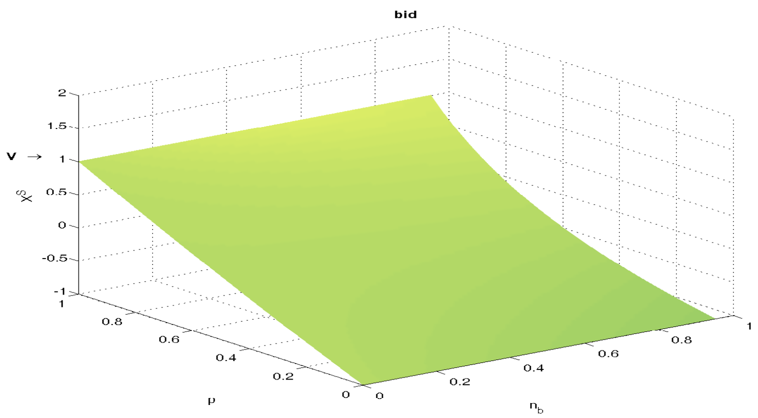


(c) Spread.

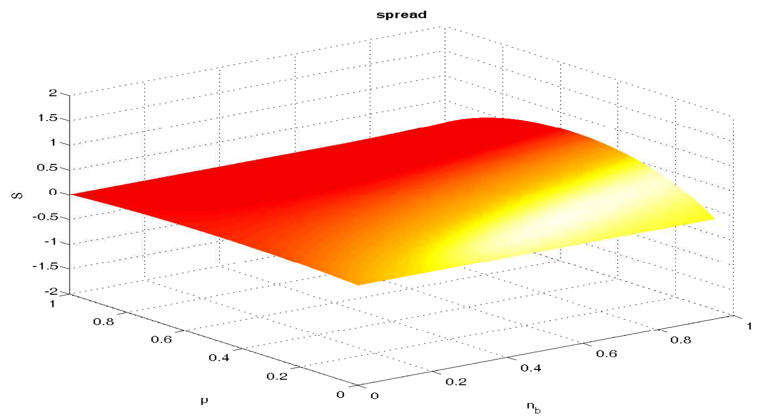
Figure A.9: Price evolution at time t subject to the accuracy of imperfect information in bad economy $\beta_t q_t^L$ and to the public beliefs p_t, in a good economy $V = V^H$, for proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, a probability of perfect information $\alpha = 0.20$, and an accuracy of imperfect information in a good economy $\beta_t q_t^H = 0.67$.



(a) Ask price.

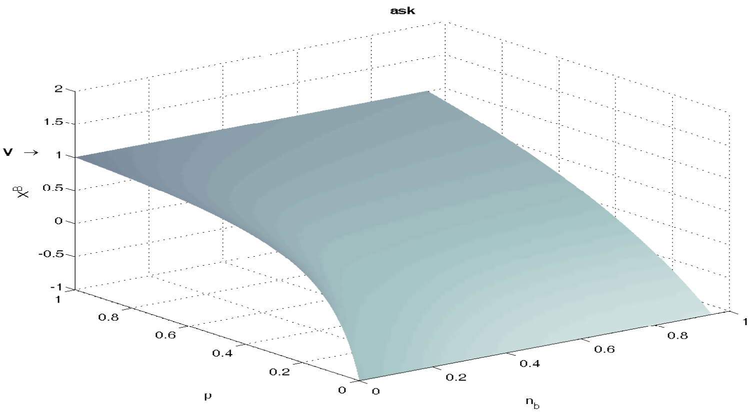


(b) Bid price.

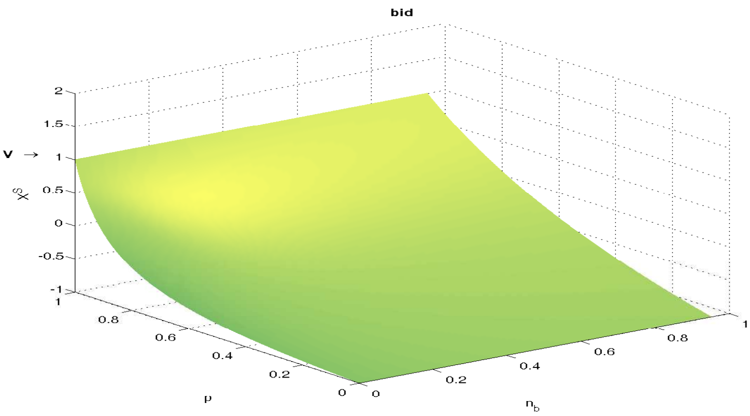


(c) Spread.

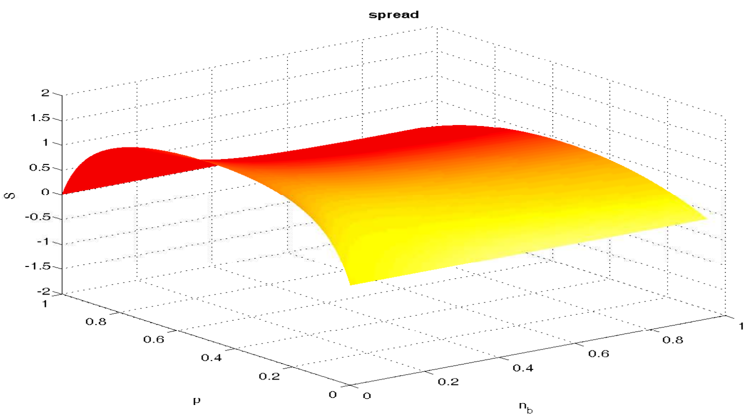
Figure A.10: Price evolution at time t subject to the public beliefs p_t and to the proportion of imperfectly informed traders $n_{\mathbf{b}}$, in a good economy $V = V^H$, for a fixed proportion of perfectly informed traders $n_{\mathbf{a}} = 20\%$, a probability of perfect information $\alpha = 0.20$, and accuracies of imperfect information $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$.



(a) Ask price.

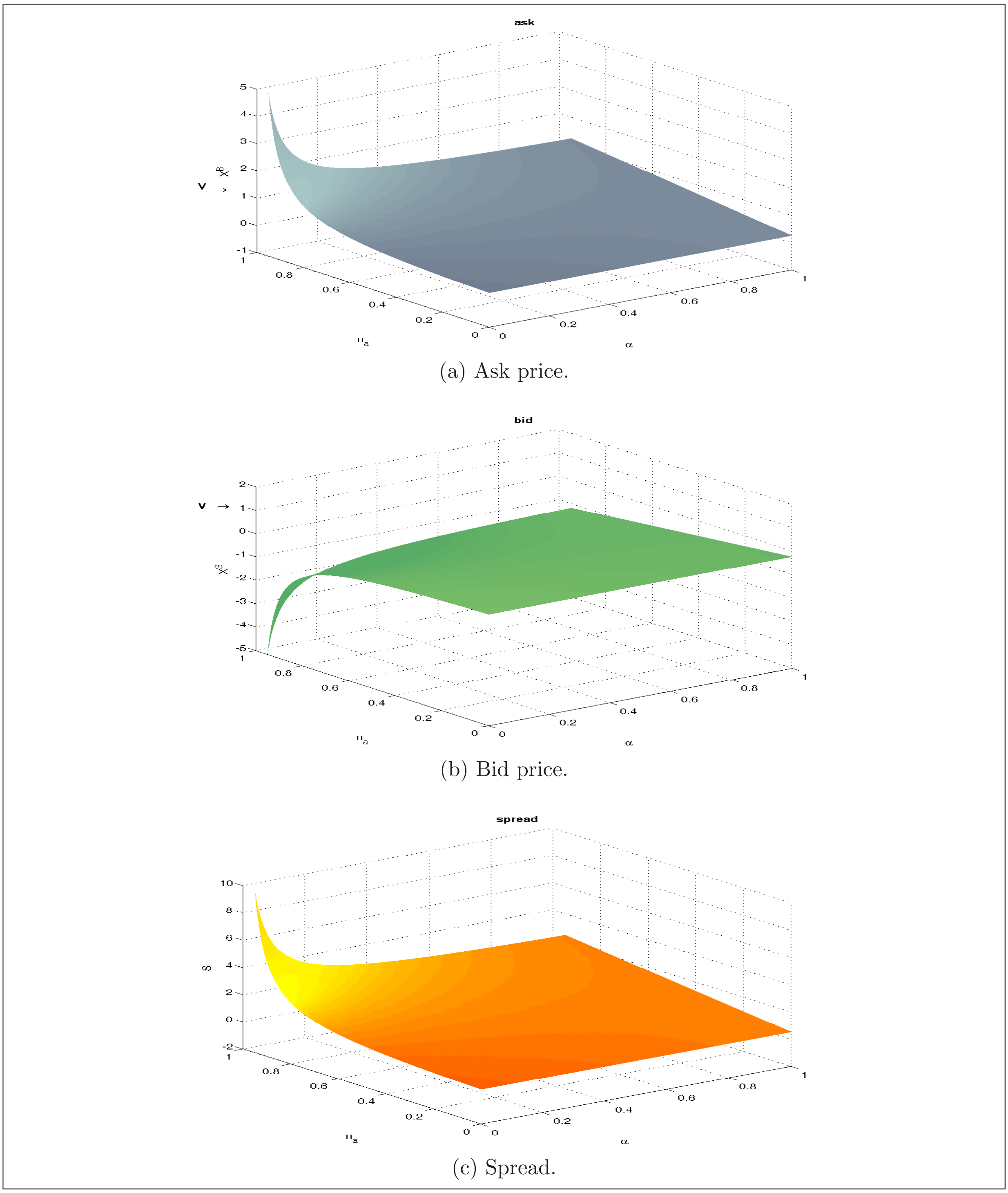


(b) Bid price.



(c) Spread.

Figure A.11: Price evolution at time t subject to the public beliefs p_t and to the proportion of imperfectly informed traders $n_{\mathbf{b}}$, in a good economy $V = V^H$, for a fixed proportion of liquidity traders $n_{\mathbf{c}} = 5\%$, a probability of perfect information $\alpha = 0.20$, and accuracies of imperfect information $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$.

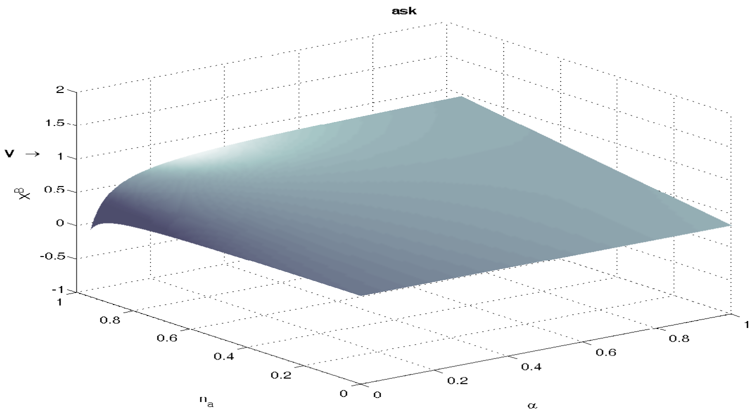


(a) Ask price.

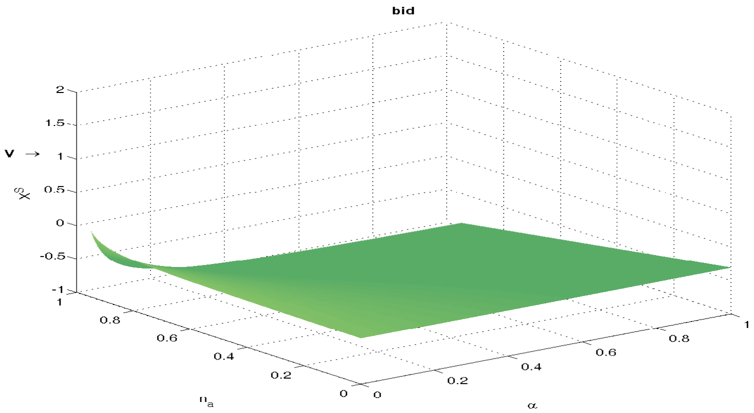
(b) Bid price.

(c) Spread.

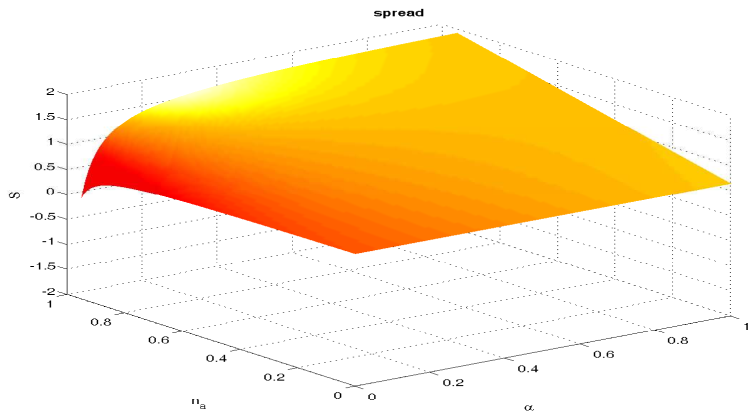
Figure A.12: Price evolution at time t subject to the probability of perfect information α_t and to the proportion of perfectly informed traders $n_{\mathbf{a}}$, in a good economy $V = V^H$, for public beliefs $p_t = 0.5$, a fixed proportion of imperfectly informed traders $n_{\mathbf{b}} = 75\%$, a probability of perfect information $\alpha_t = 0.20$, and accuracies of imperfect information $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$.



(a) Ask price.

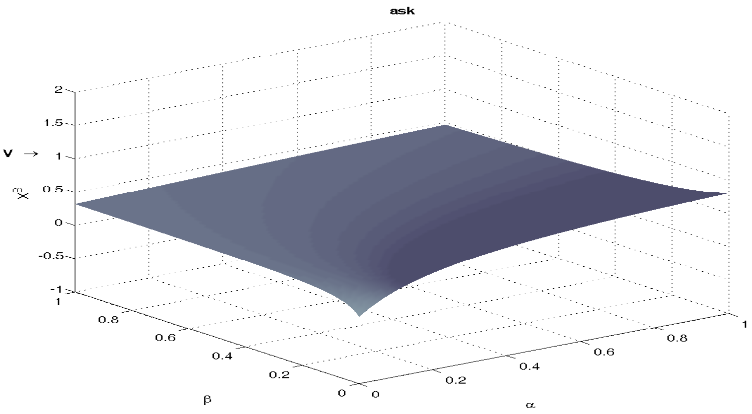


(b) Bid price.

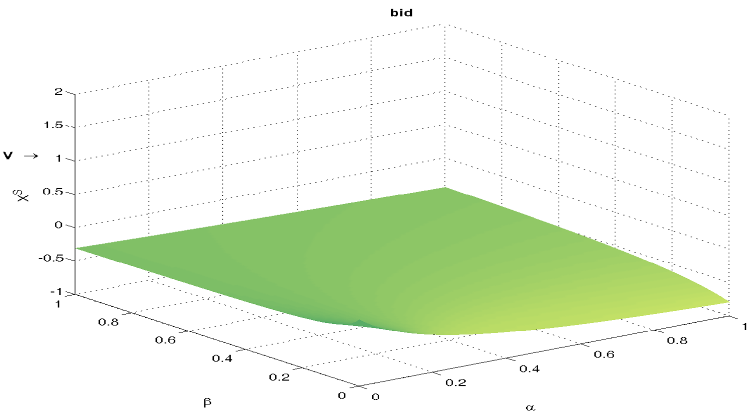


(c) Spread.

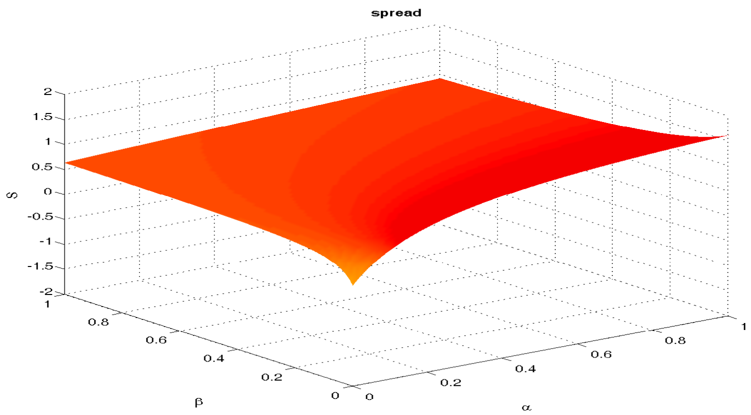
Figure A.13: Price evolution at time t subject to the probability of perfect information α_t and to the proportion of perfectly informed traders $n_{\mathbf{a}}$, in a good economy $V = V^H$, for public beliefs $p_t = 0.5$, a fixed proportion of liquidity traders $n_{\mathbf{c}} = 5\%$, and accuracies of imperfect information $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$.



(a) Ask price.

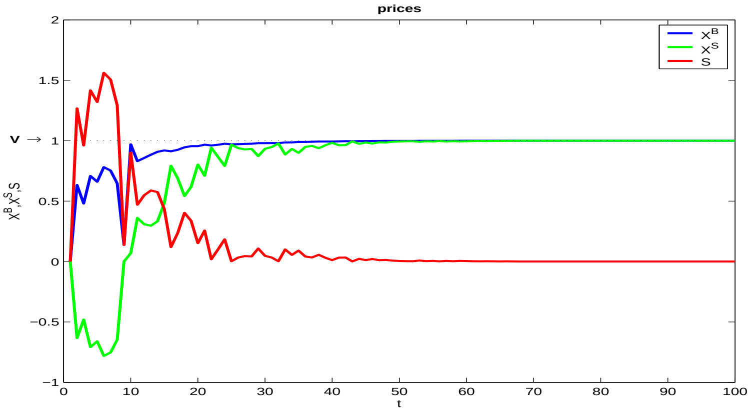


(b) Bid price.

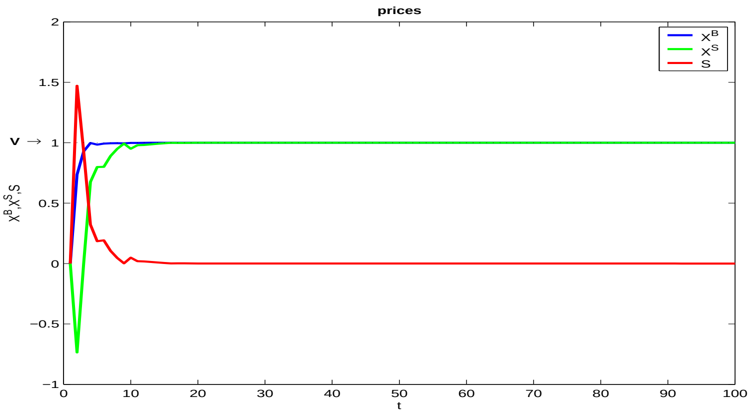


(c) Spread.

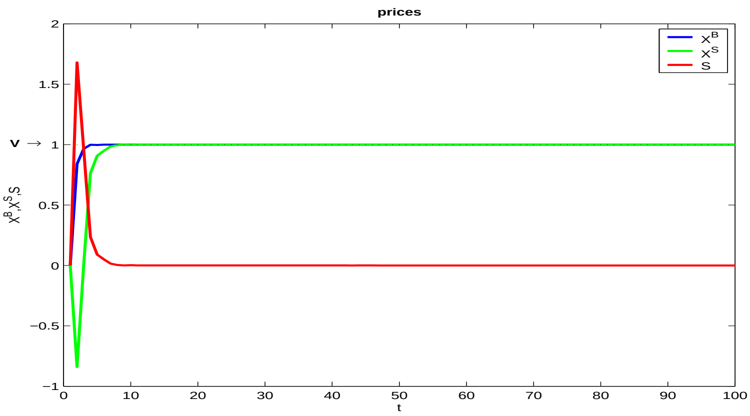
Figure A.14: Price evolution at time t subject to the probabilities of perfect and imperfect information α_t and β_t, in a good economy $V = V^H$, for proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 75\%$, public beliefs $p_t = 0.50$, and accuracies of imperfect information $\beta_t q_t^H = 0.67$ and $\beta_t q_t^L = 0.33$.



(a) High proportion of imperfectly informed traders $n_{\mathbf{b}} = 75\%$.

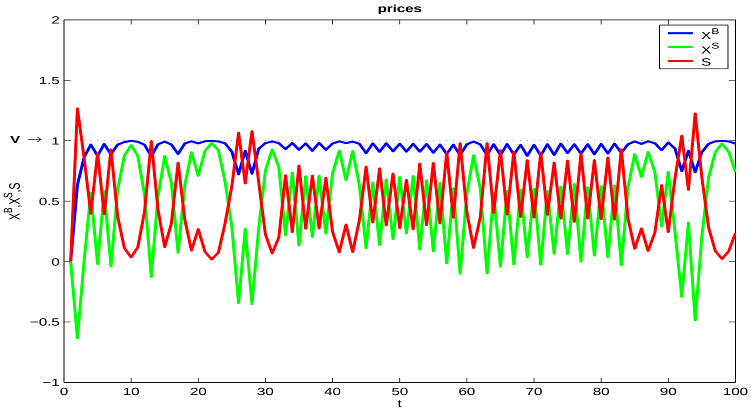


(b) Middle proportion of imperfectly informed traders $n_{\mathbf{b}} = 50\%$.

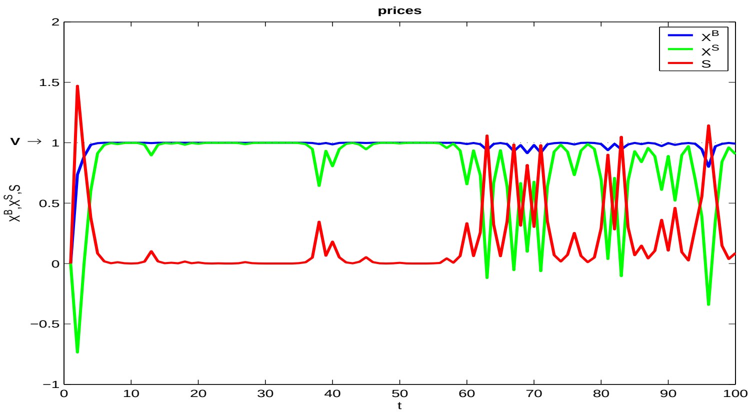


(c) Low proportion of imperfectly informed traders $n_{\mathbf{b}} = 25\%$.

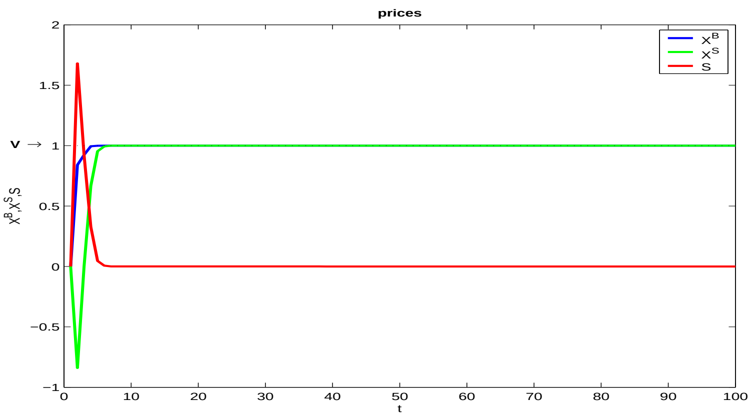
Figure A.15: Price evolution for the momentum strategy (TA-1) for different proportions of imperfectly informed traders $n_{\mathbf{b}}$, in a good economy $V = V^H$, for a random probability of perfect information $\alpha_t \sim U[0,1]$, a proportion of liquidity traders $n_{\mathbf{c}} = 5\%$, and random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.



(a) High proportion of imperfectly informed traders $n_{\mathbf{b}} = 75\%$.

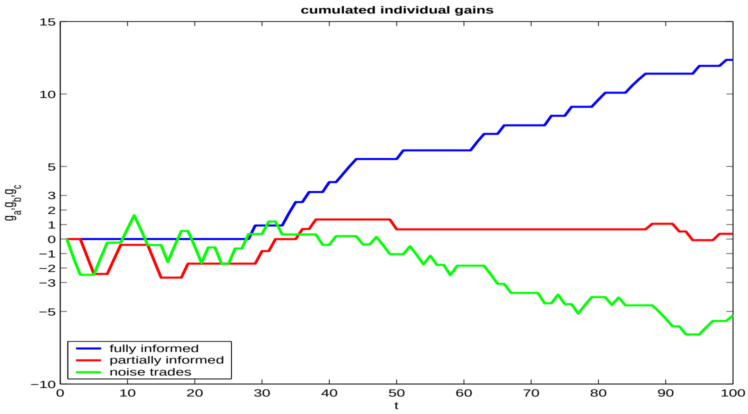


(b) Middle proportion of imperfectly informed traders $n_{\mathbf{b}} = 50\%$.

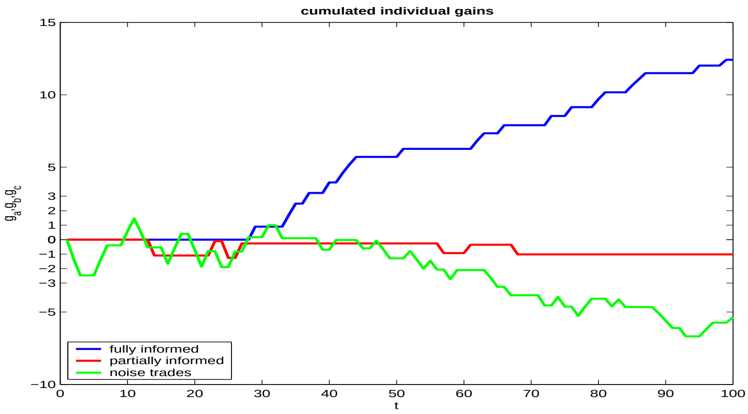


(c) Low proportion of imperfectly informed traders $n_{\mathbf{b}} = 25\%$.

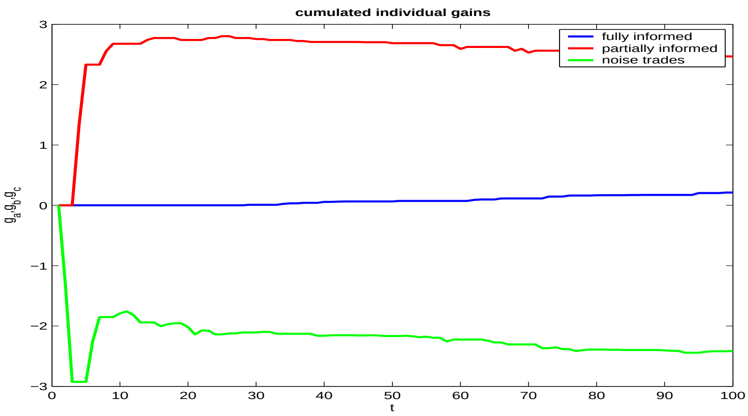
Figure A.16: Price evolution for the fundamentalist strategy (FA) for different proportions of imperfectly informed traders $n_{\mathbf{b}}$, in good economy $V = V^H$, for a random probability of perfect information $\alpha_t \sim U[0,1]$, a proportion of liquidity traders $n_{\mathbf{c}} = 5\%$, and random probabilities of imperfect information βq^H and βq^L of middle accuracy 0.50.



(a) Momentum strategy (TA-1), for random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.



(b) Moving-average strategy (TA-2), for random probabilities of imperfect information $\beta_t q_t^H$ and $\beta_t q_t^L$ of middle accuracy 0.50.



(c) Fundamentalist strategy (FA), for random probabilities of imperfect information βq^H and βq^L of middle accuracy 0.50.

Figure A.17: The evolution of individual cumulated gains for all three imperfectly informed strategies, in a good economy $V = V^H$, for a probability of perfect information $\alpha = 0.20$, and proportions of informed traders $n_{\mathbf{a}} = 20\%$ and $n_{\mathbf{b}} = 25\%$.

A.2 Emotions and financial decision making

The Equations (2.9) are an immediate consequence of the following lemma:

Lemma 1. *Let $a, \mu \in \mathbb{R}^n$, $\beta \in \mathbb{R}$, and Σ be a positive definite $n \times n$ matrix. Then we have:*

$$(x-\mu)^T \Sigma^{-1}(x-\mu) + (a^T x - \beta)^2 = (x-\mu_+)^T \Sigma_+^{-1}(x-\mu_+) + \Delta,$$

where:

$$\Sigma_+ = \Sigma - \frac{(\Sigma a)(\Sigma a)^T}{a^T \Sigma a + 1}, \quad \mu_+ = \mu - \frac{\mu^T a - \beta}{a^T \Sigma a + 1}\Sigma a, \quad \Delta = \frac{(\mu^T a - \beta)^2}{a^T \Sigma a + 1}.$$

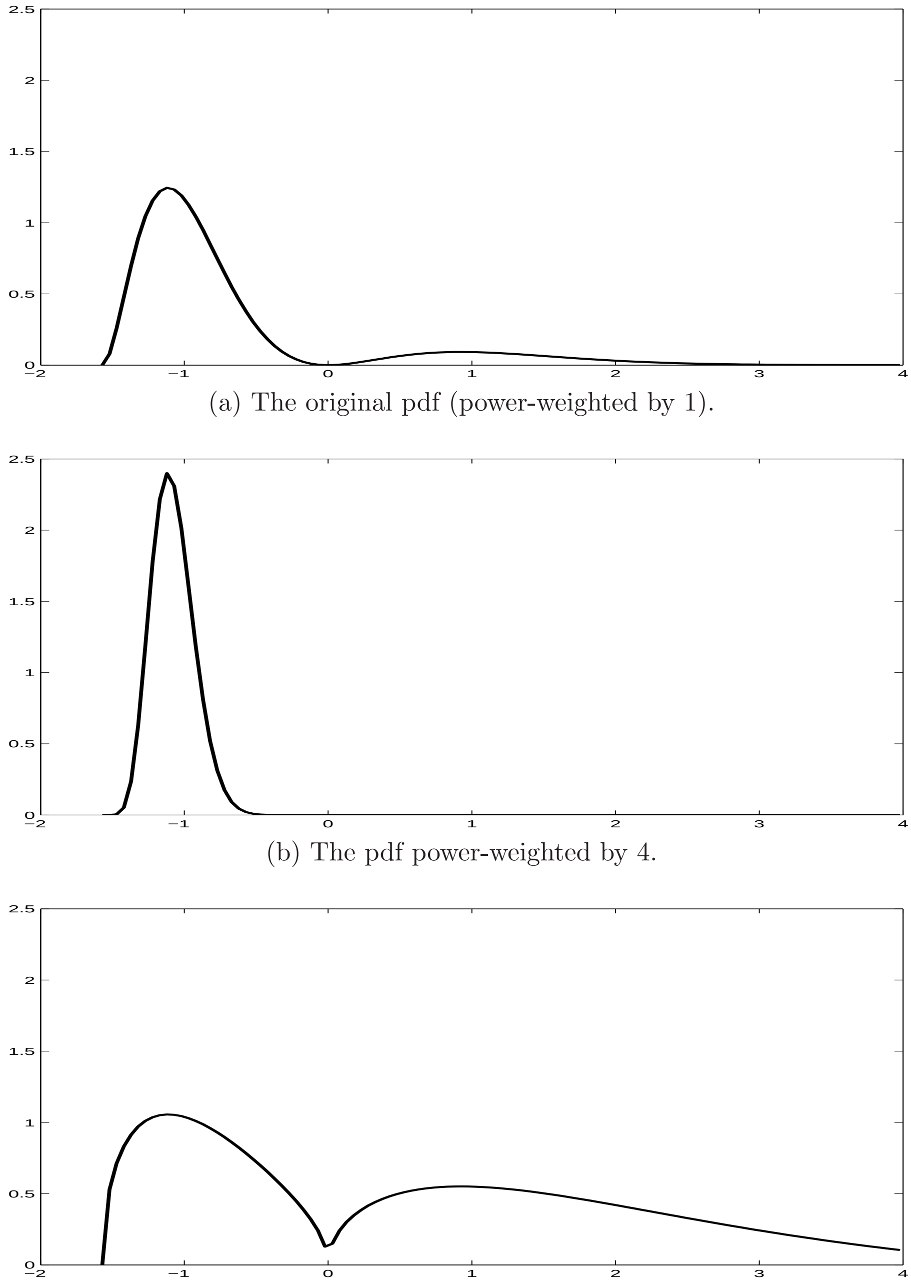
Moreover, we have:

$$\det(\Sigma_+) = \frac{\det(\Sigma)}{a^T \Sigma a + 1}.$$

It is easy to check that $\Sigma_+^{-1} = \Sigma^{-1} + aa^T$. Then the first statement of the above Lemma 1 can be proven by simply replacing on the right-hand side Σ_+, μ_+, and Δ by their values, and checking that the equality holds. Finally, by the multiplicativity of the determinant, we have:

$$\det(\Sigma_+) = \frac{\det(\Sigma)\det\Big(I - (\Sigma^{1/2}a)(\Sigma^{1/2}a)^T\Big)}{a^T \Sigma a + 1}.$$

All the eigenvalues of this last matrix equal 1, except one of them, which equals $1 \,/\, (\, a^T\, \Sigma\, a + 1\,)$.

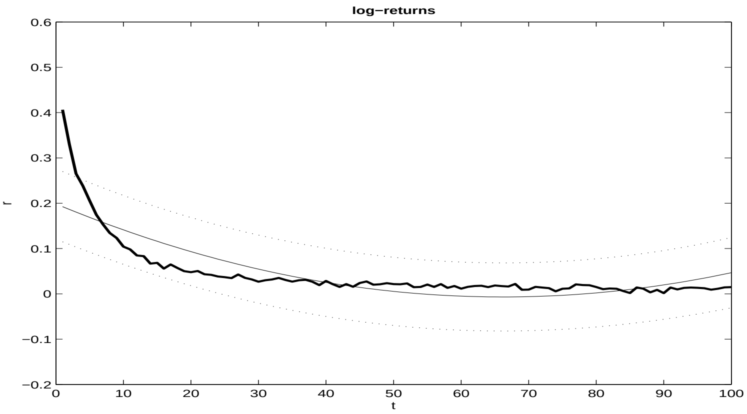


(a) The original pdf (power-weighted by 1).

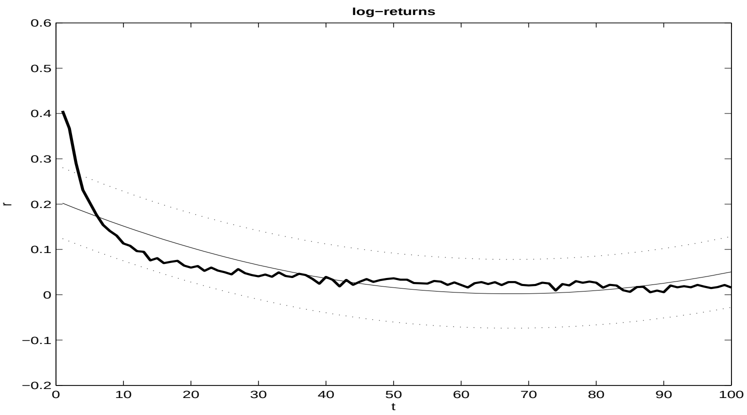
(b) The pdf power-weighted by 4.

(c) The pdf power-weighted by 1/16.

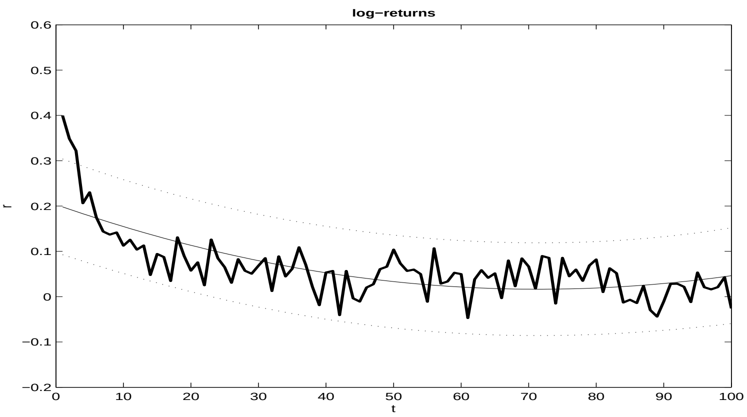
Figure A.18: Changes of a pdf under different power weights.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

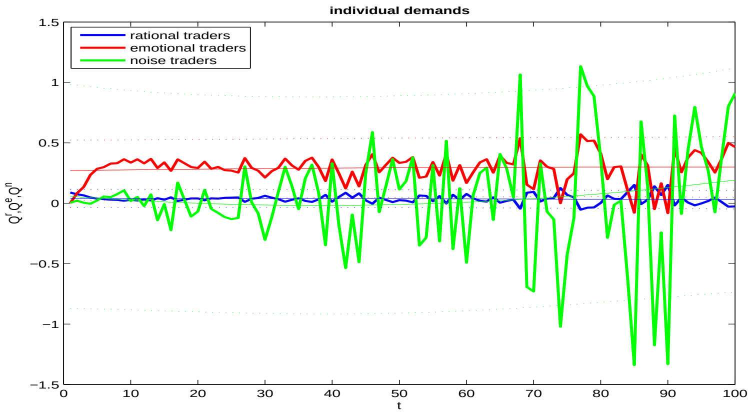


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

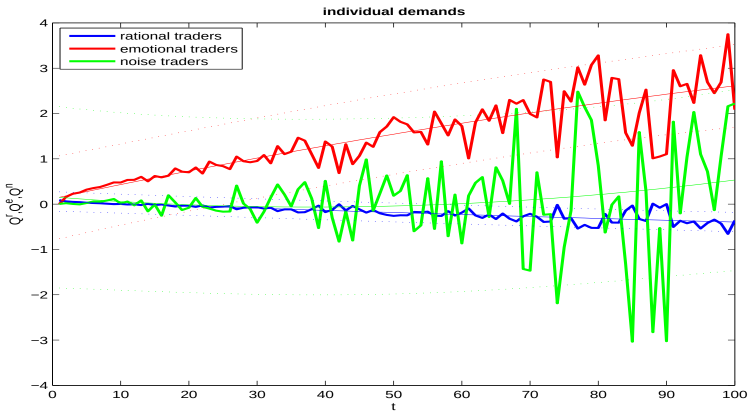


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

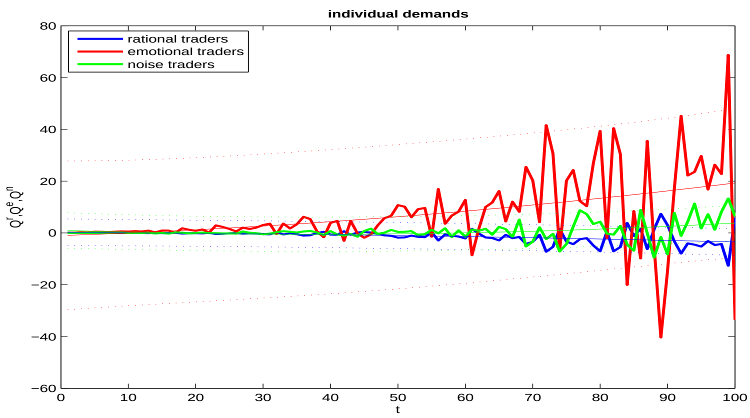
Figure A.19: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

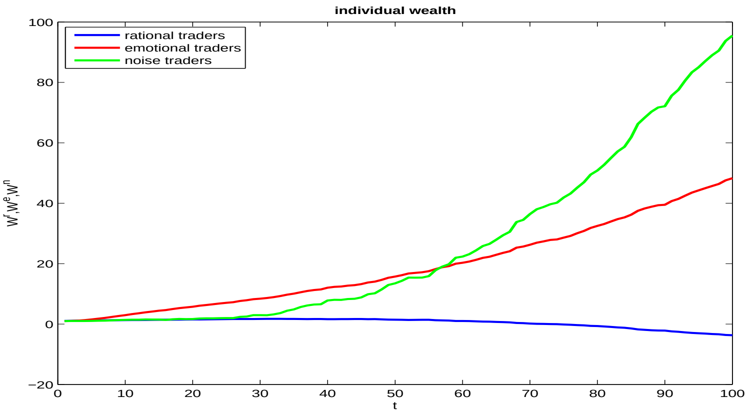


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

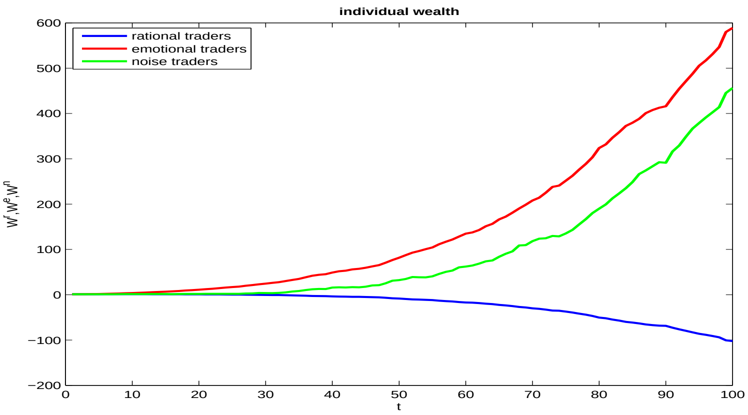


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

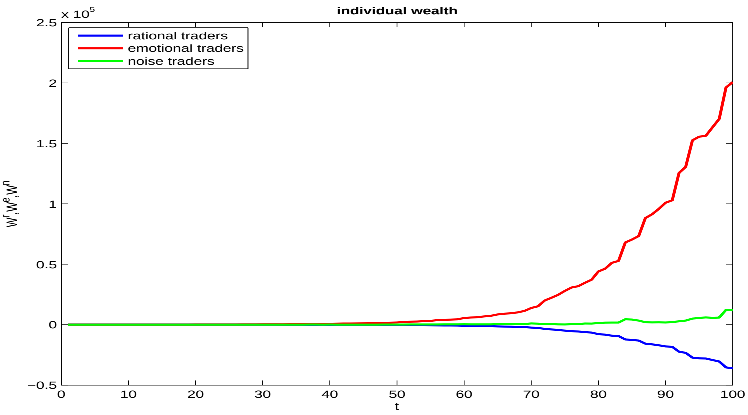
Figure A.20: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.



(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

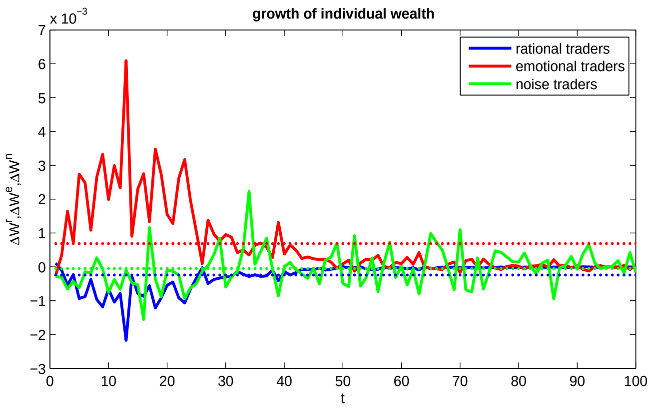


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure A.21: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.

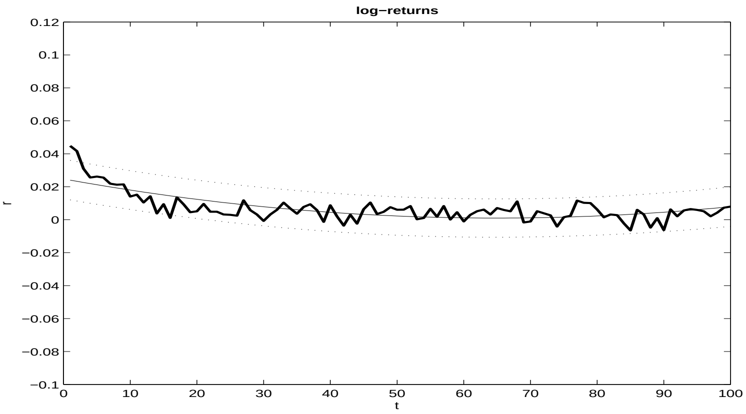
(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

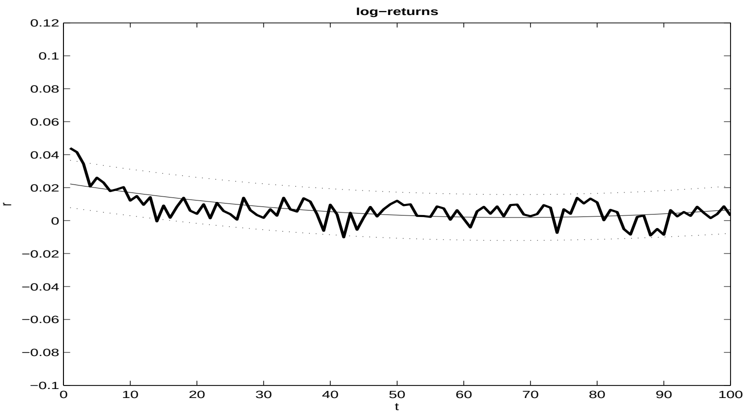


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

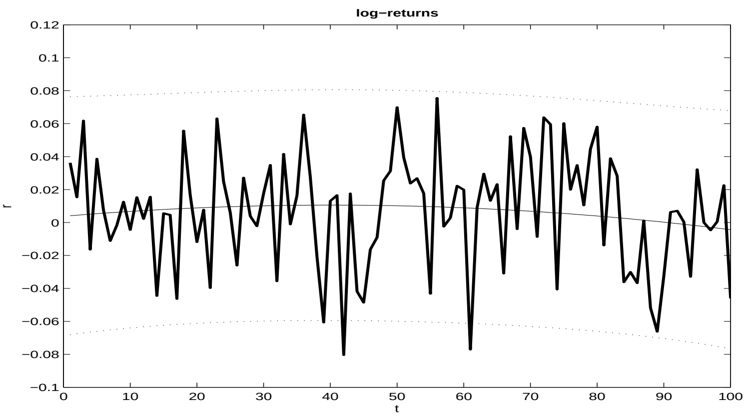
Figure A.22: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

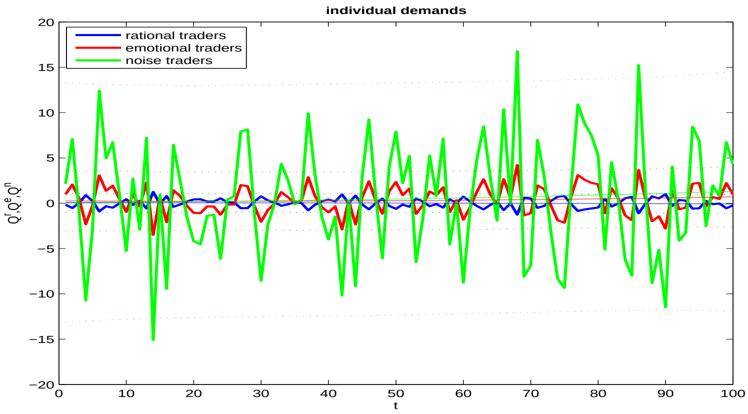


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

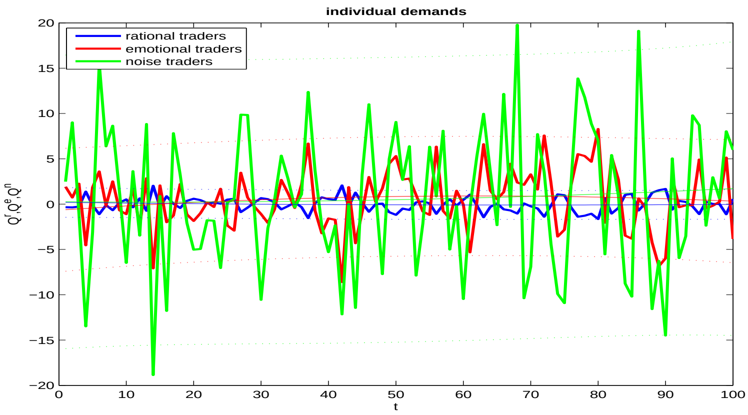


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

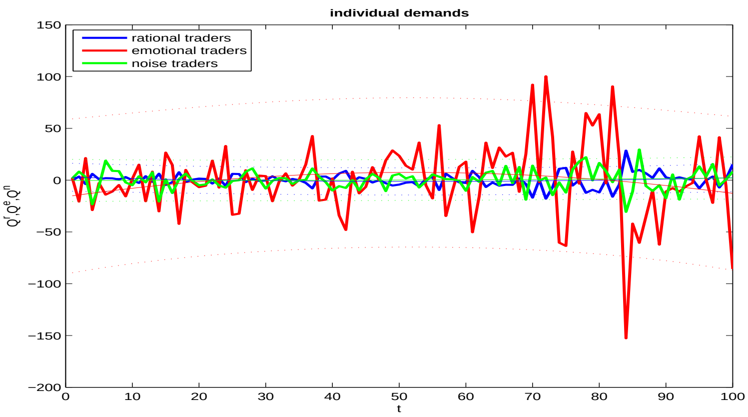
Figure A.23: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ continuing parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

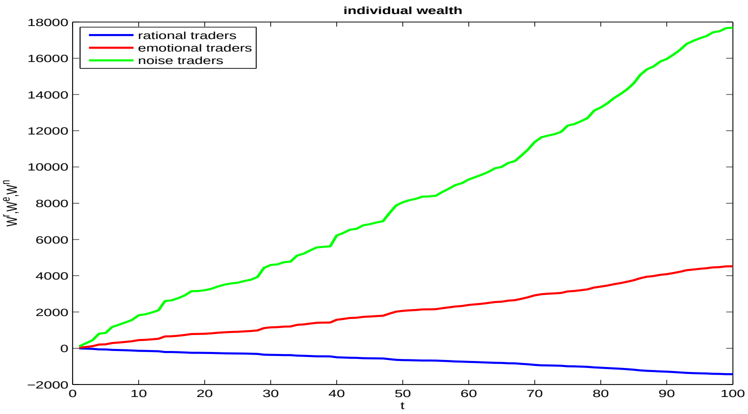


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

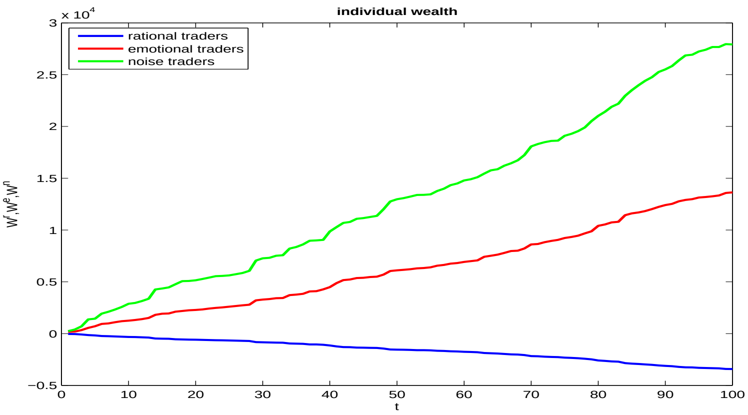


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

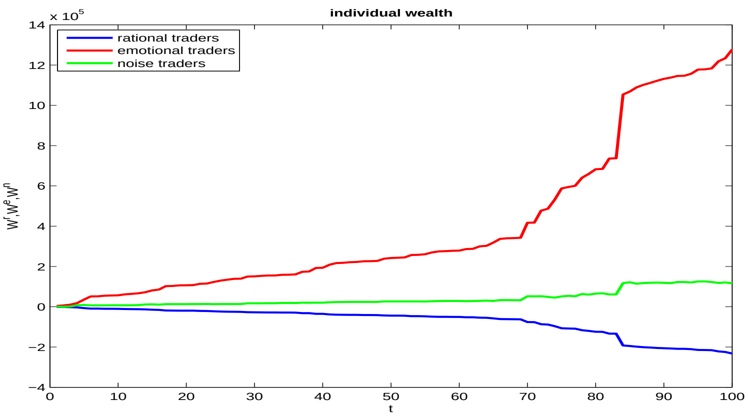
Figure A.24: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ continuing parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

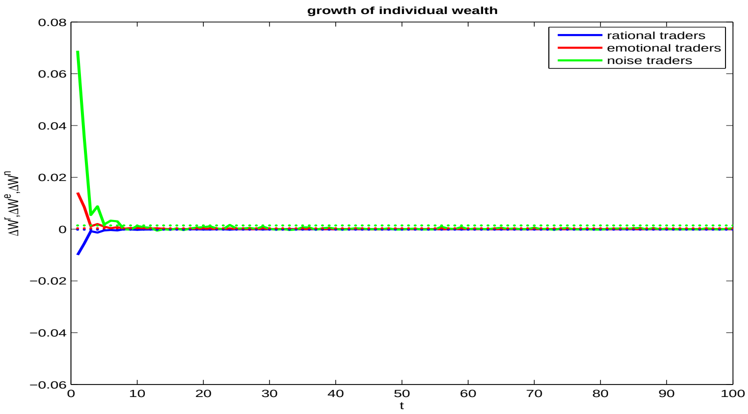


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

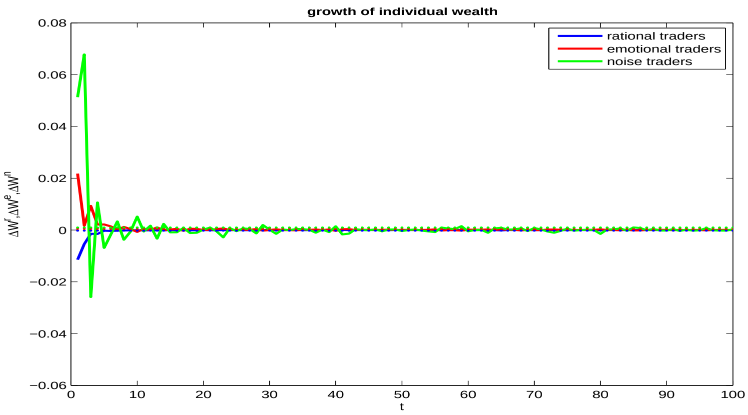


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

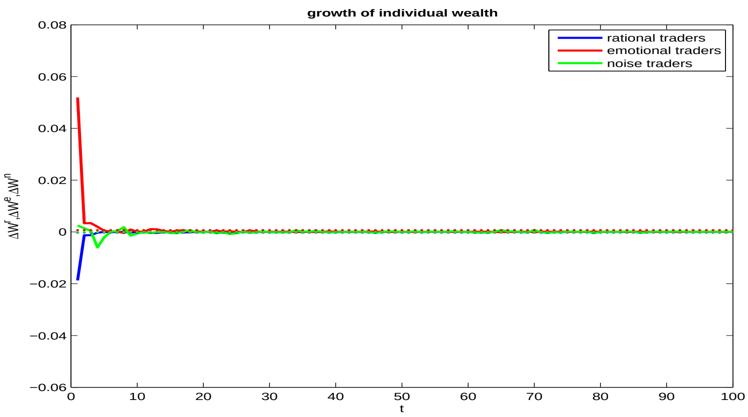
Figure A.25: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ continuing parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

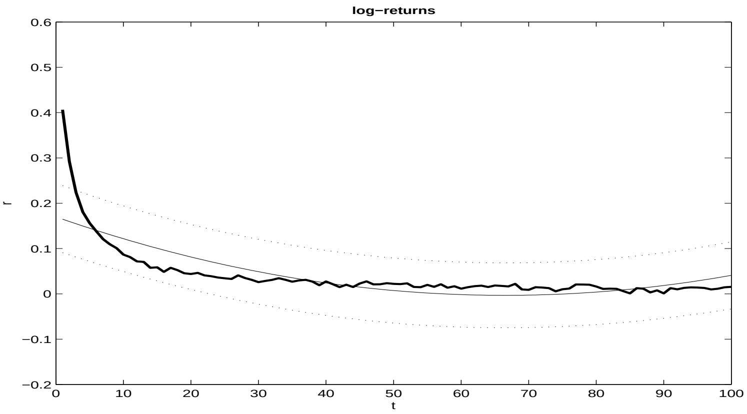


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

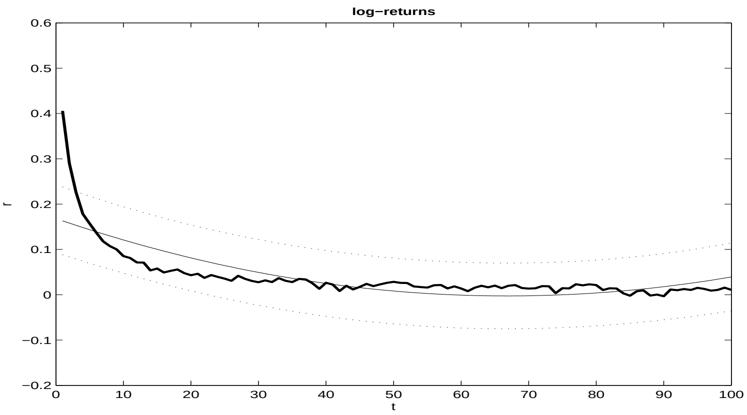


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure A.26: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ continuing parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.



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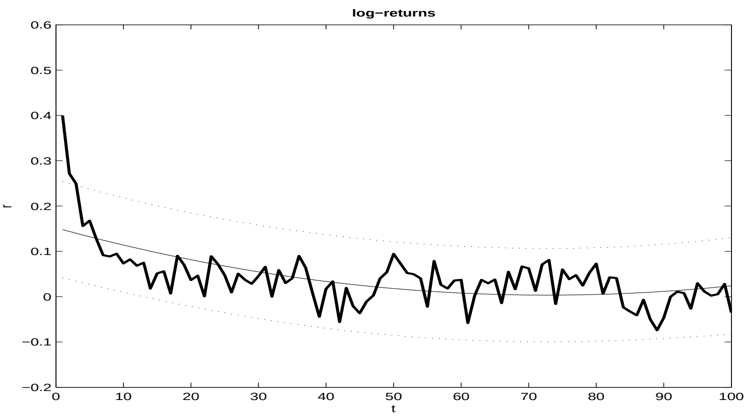
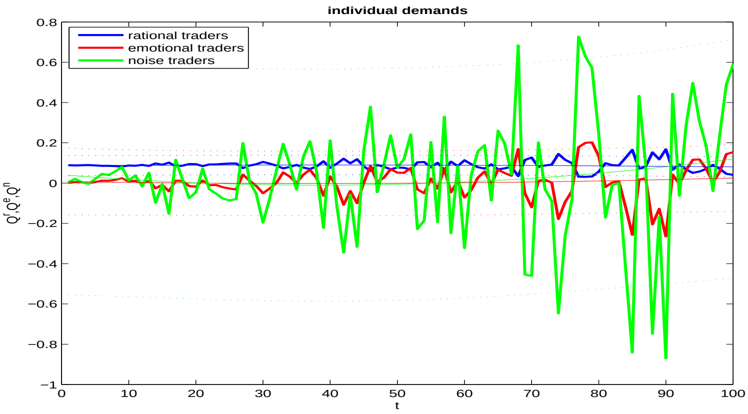
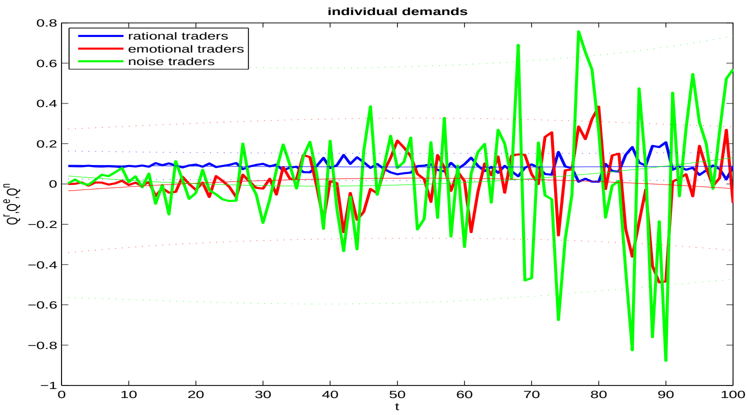


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure A.27: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from current demands (Rule qd-2), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.



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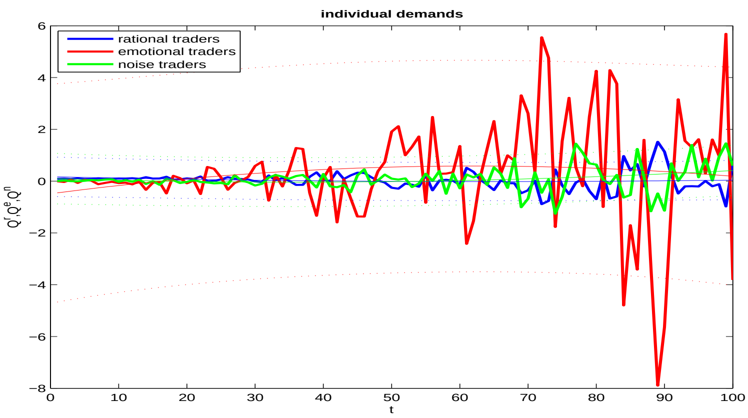
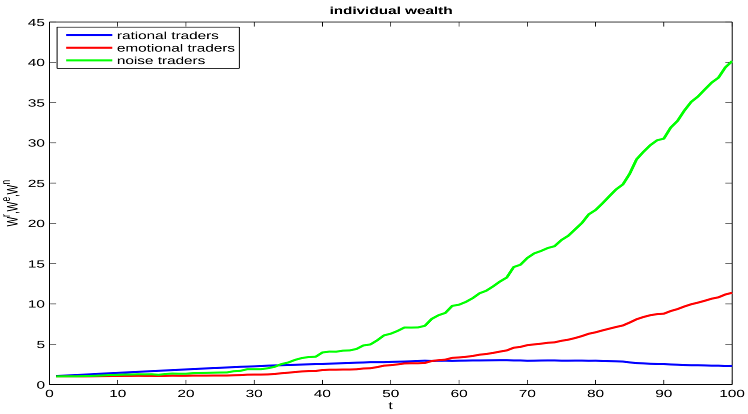
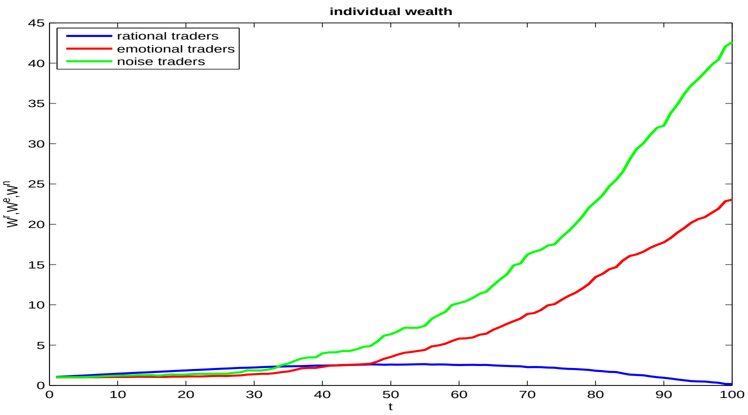


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

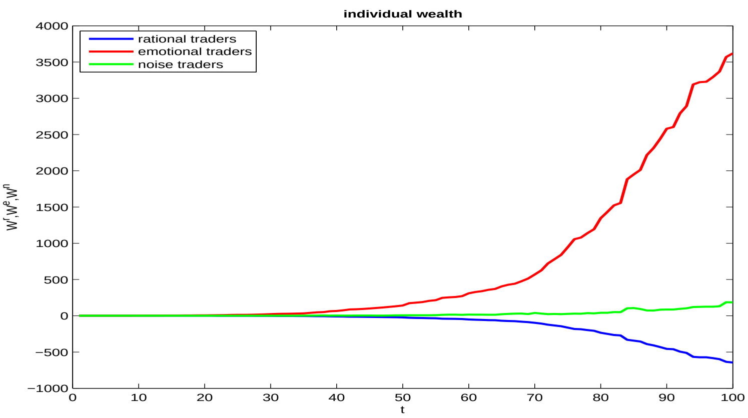
Figure A.28: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from current demands (Rule qd-2), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.



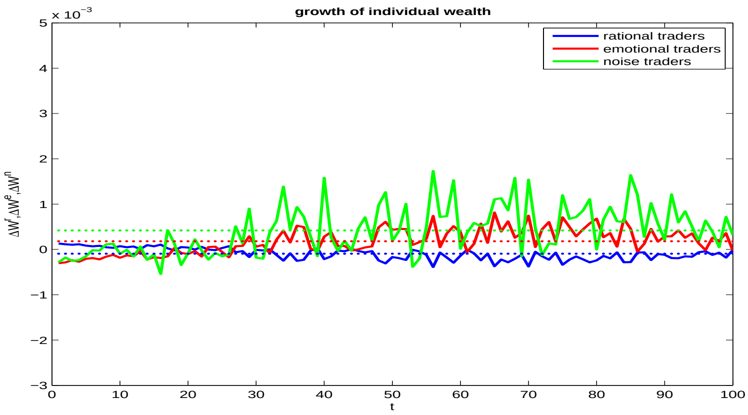
(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.



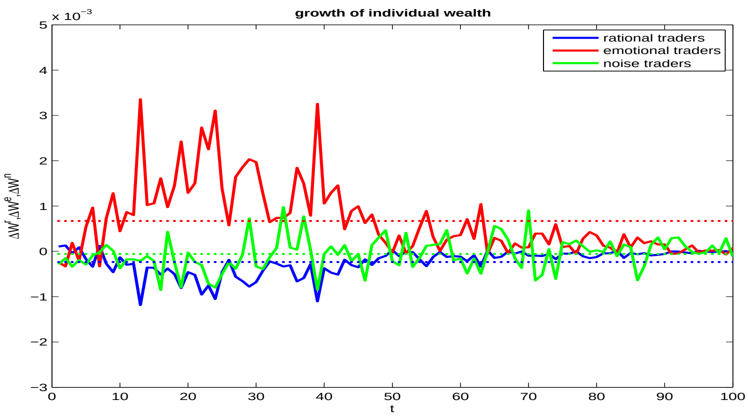
(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure A.29: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from current demands (Rule qd-2), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.

(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

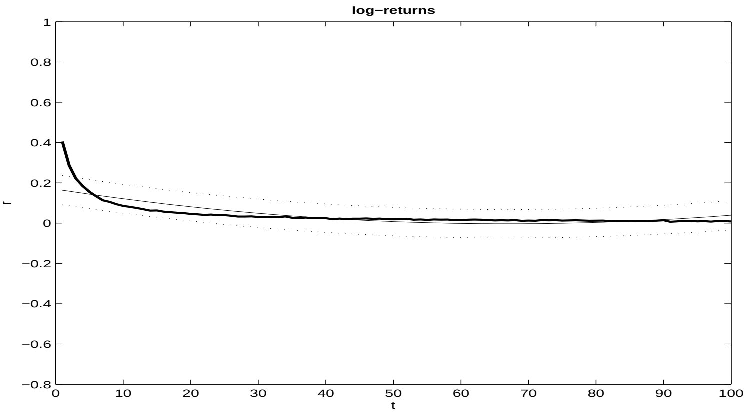


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

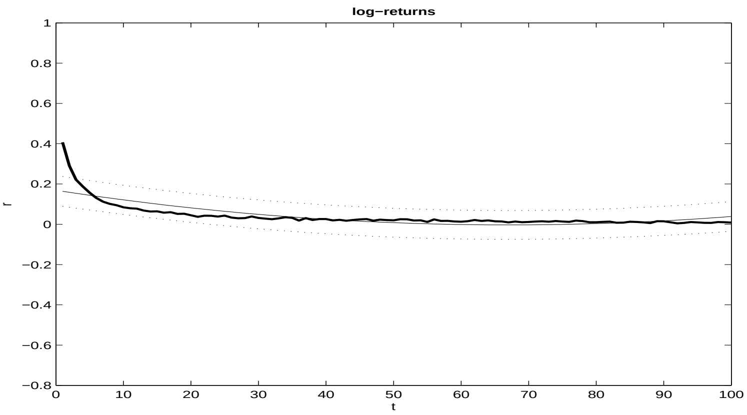


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

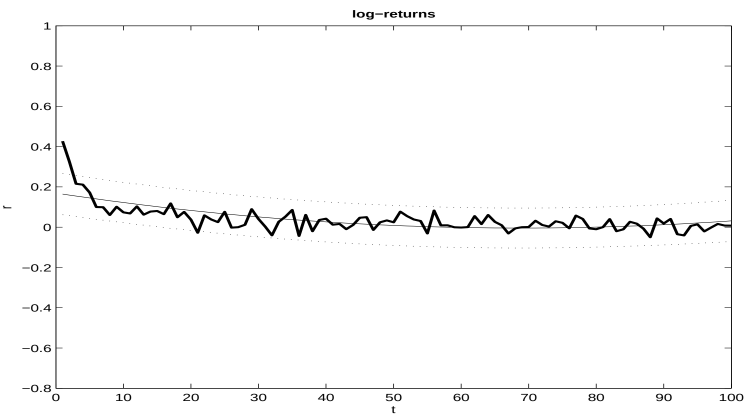
Figure A.30: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from current demands (Rule qd-2), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

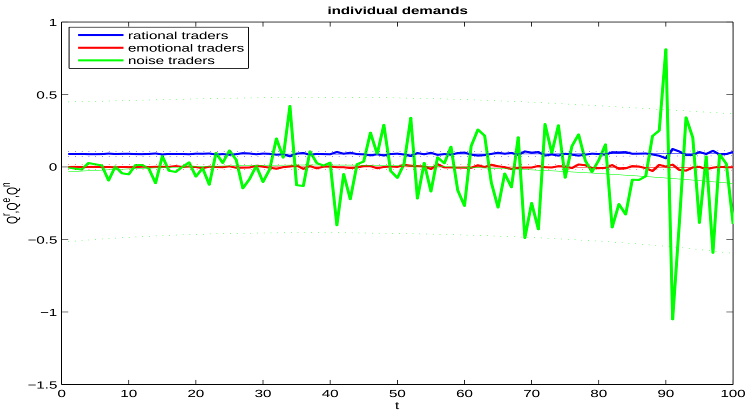


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

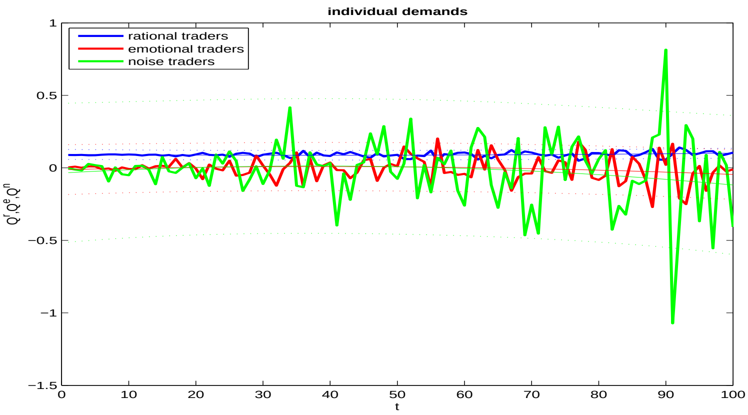


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

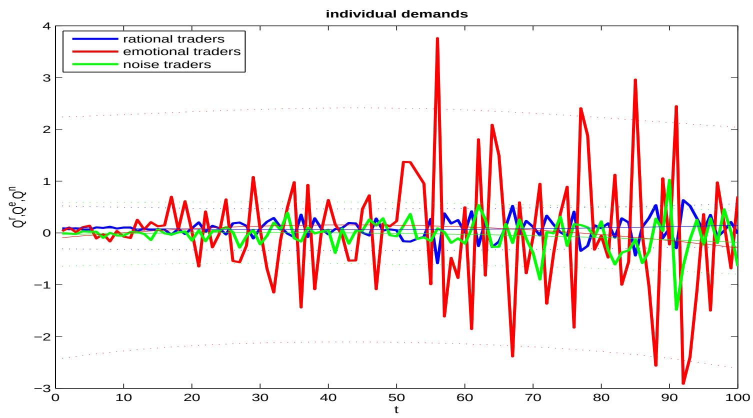
Figure A.31: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = 0, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

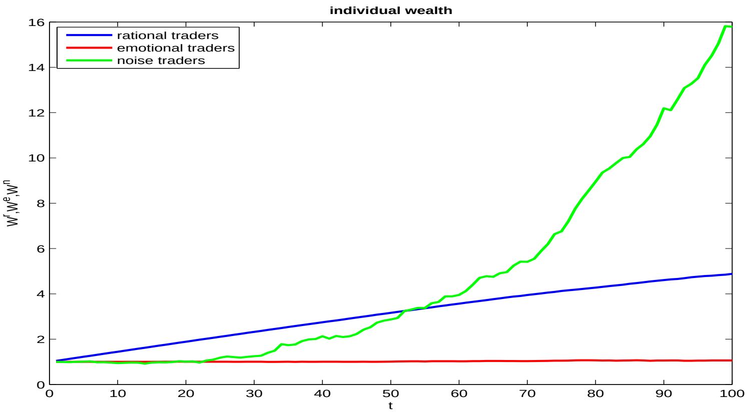


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

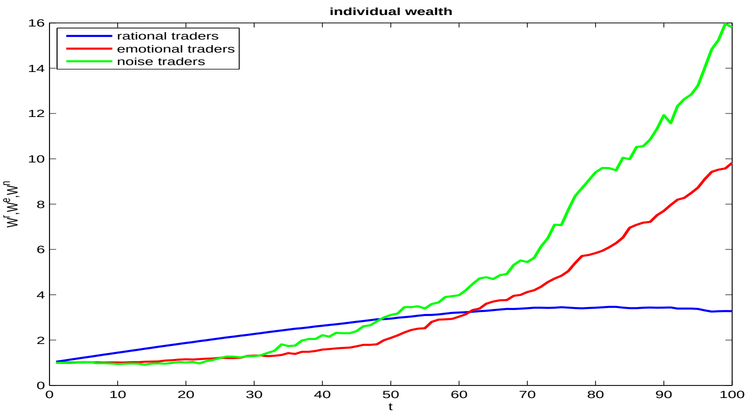


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

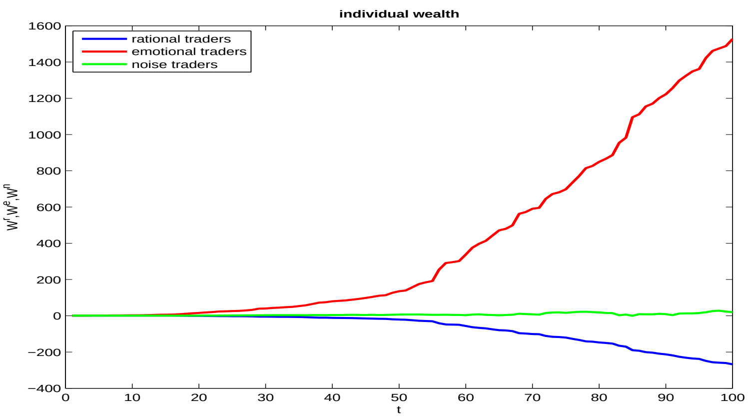
Figure A.32: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = 0, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

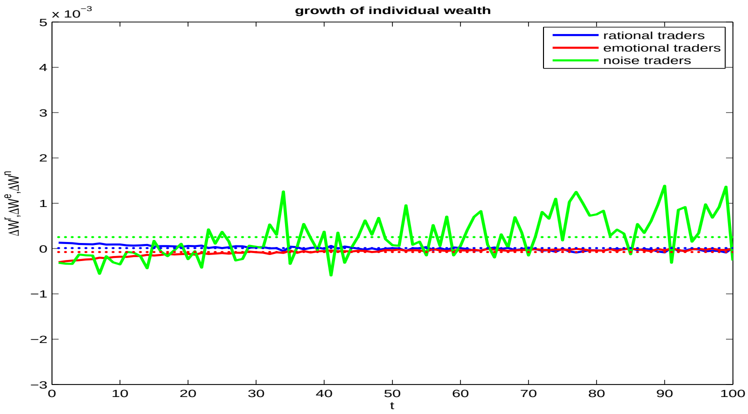


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

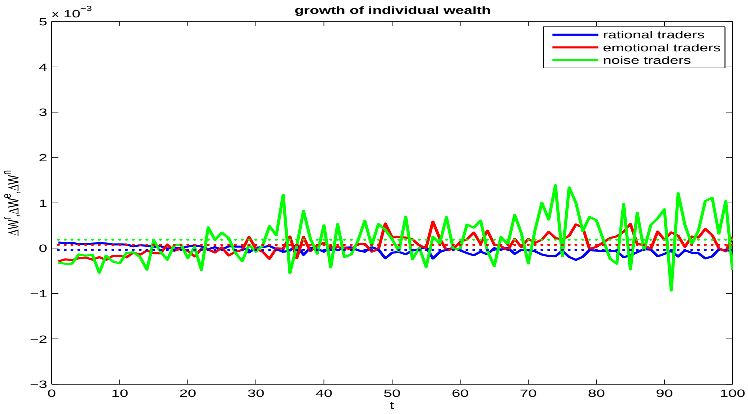


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure A.33: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = 0, \beta^e = 1$.



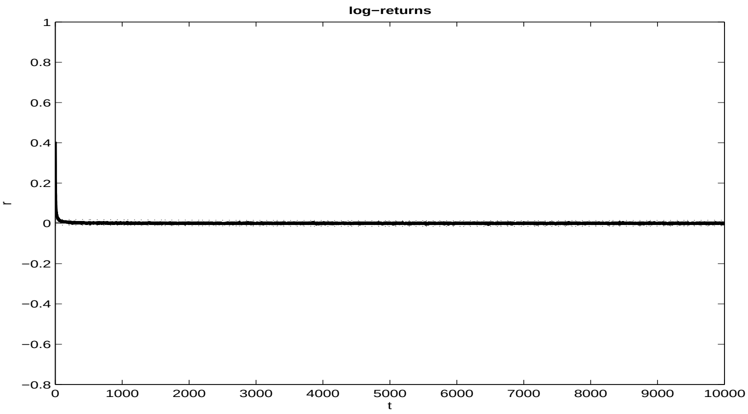
(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.



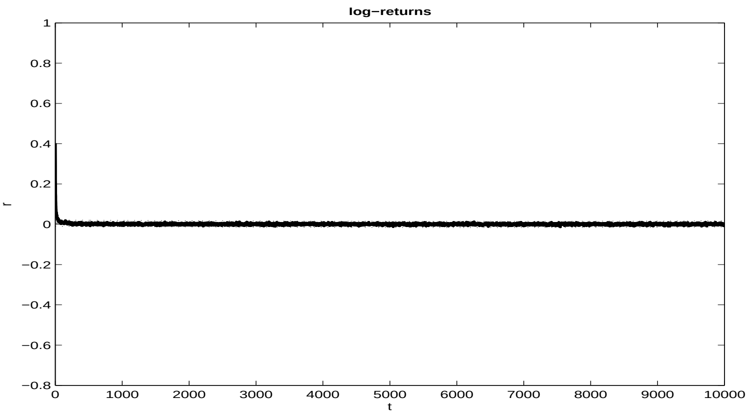
(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

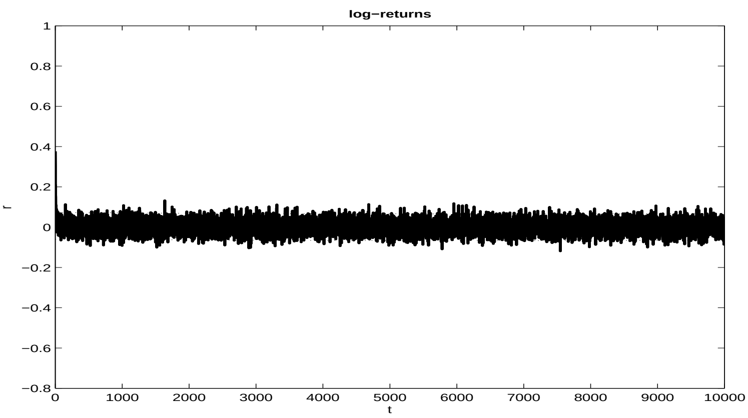
Figure A.34: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = 0, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

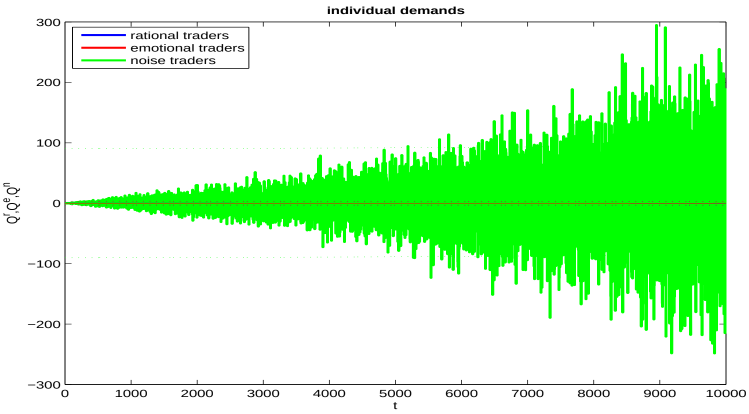


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

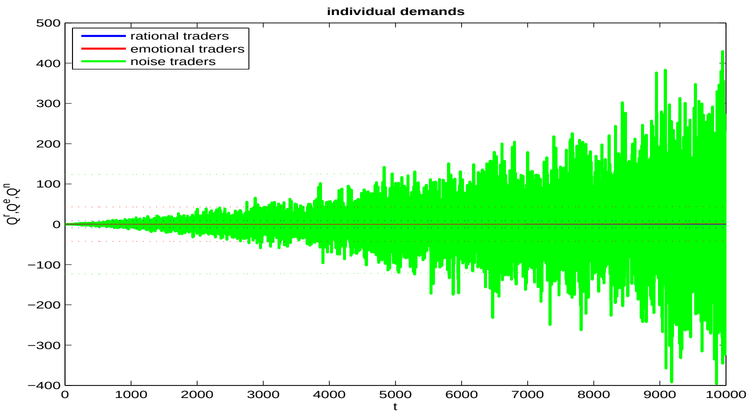


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

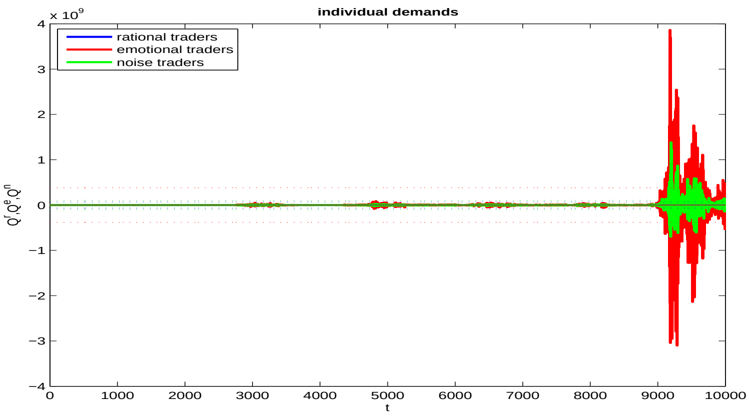
Figure A.35: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), dynamic belief updating at limit (Rule d-2), over $n = 10$ independent and long rounds of $T = 10000$ trades, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

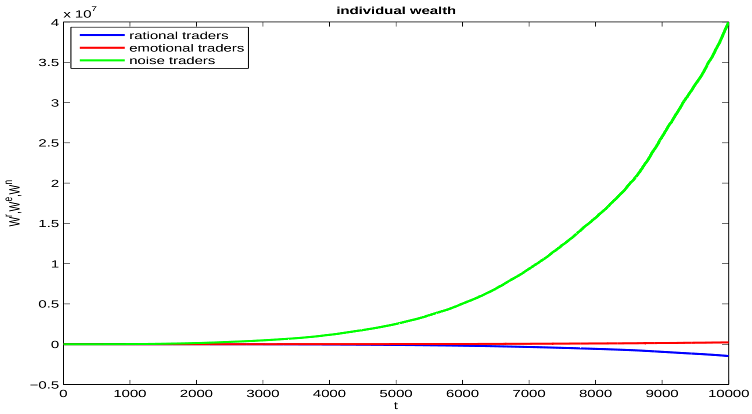


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

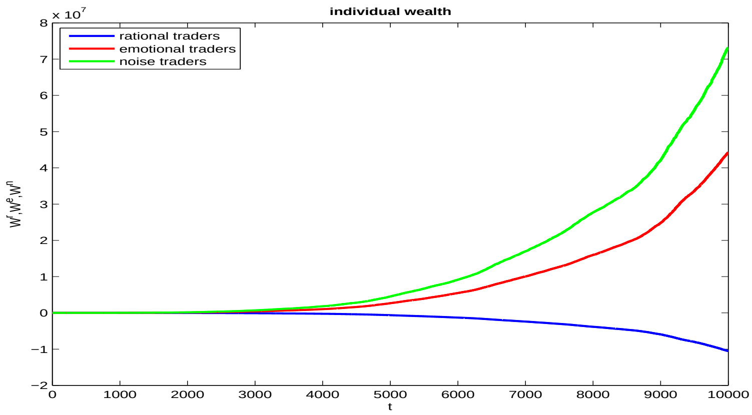


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

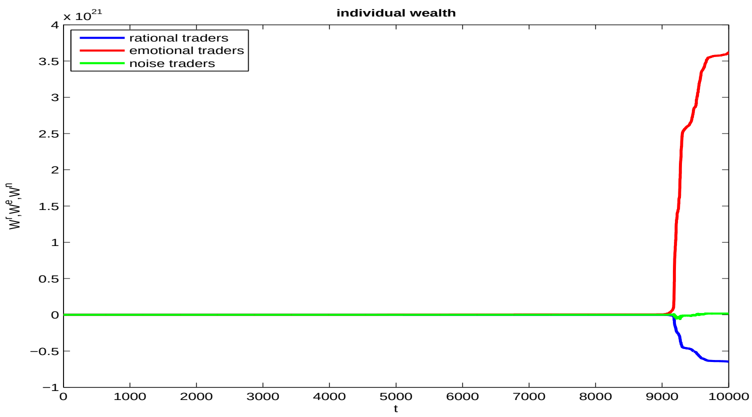
Figure A.36: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), dynamic belief updating at limit (Rule d-2), over $n = 10$ independent and long rounds of $T = 10000$ trades, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

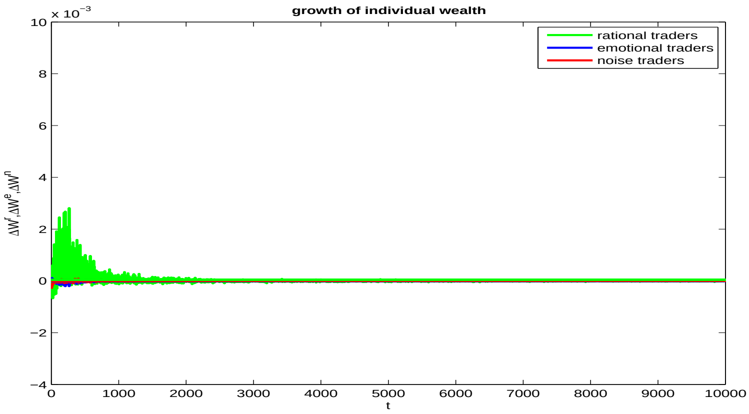


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

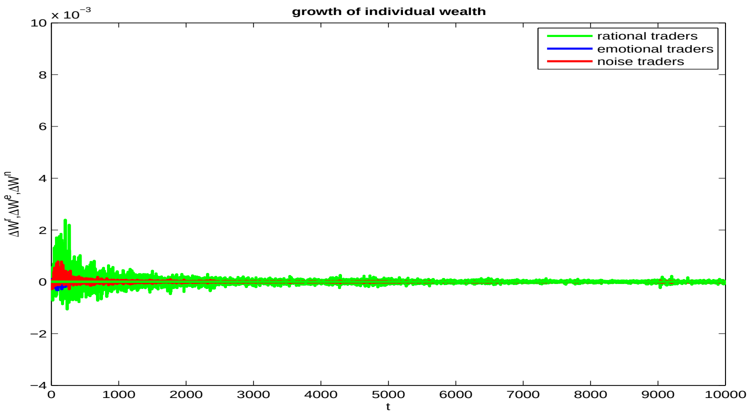


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure A.37: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), dynamic belief updating at limit (Rule d-2), over $n = 10$ independent and long rounds of $T = 10000$ trades, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.



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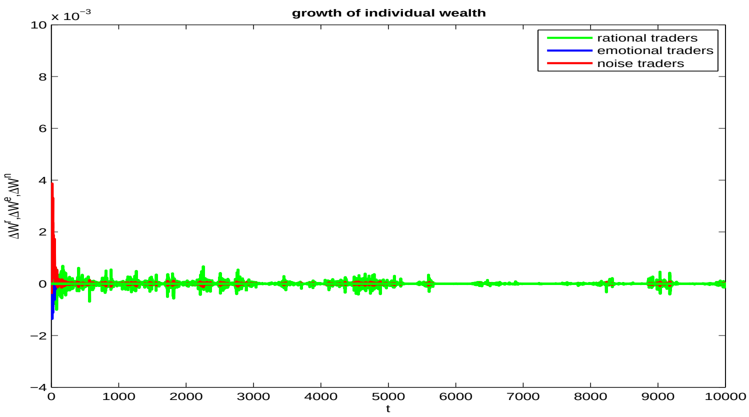
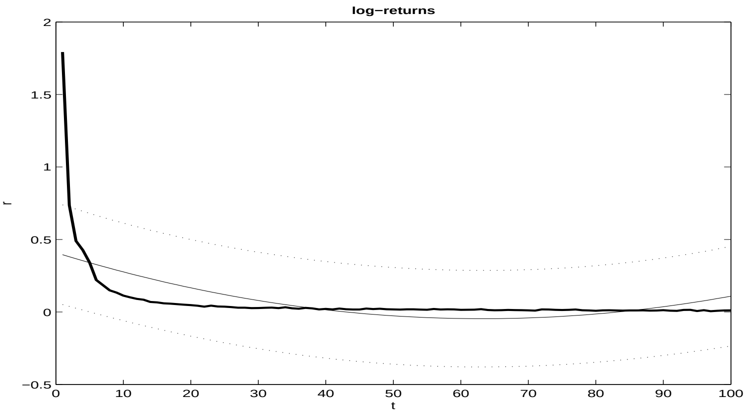
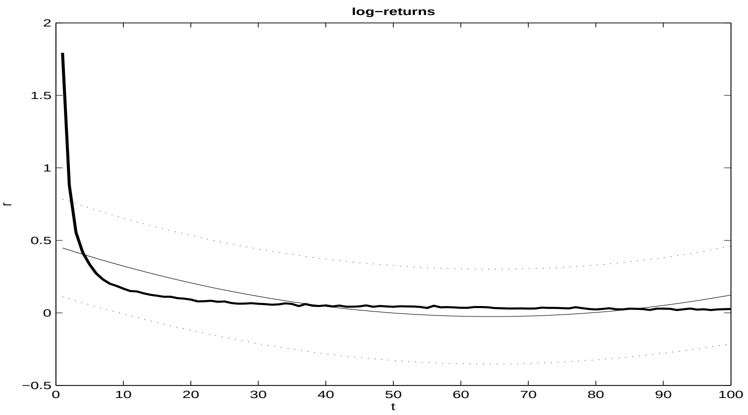


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

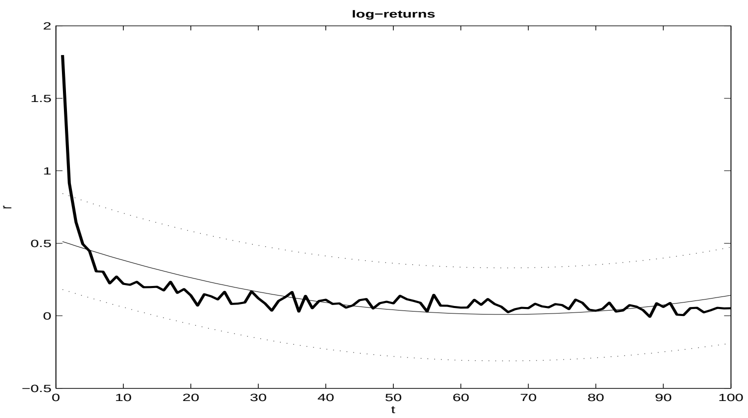
Figure A.38: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), dynamic belief updating at limit (Rule d-2), over $n = 10$ independent and long rounds of $T = 10000$ trades, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

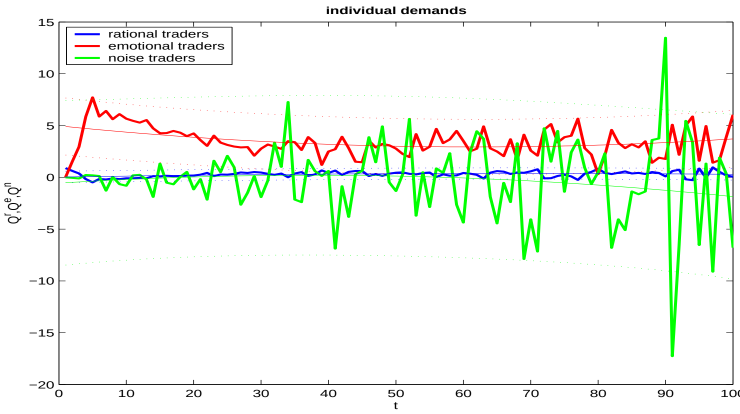


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

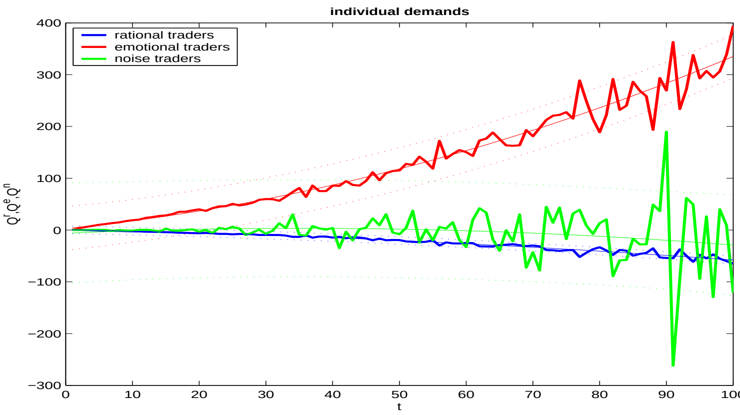


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure A.39: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 10$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.



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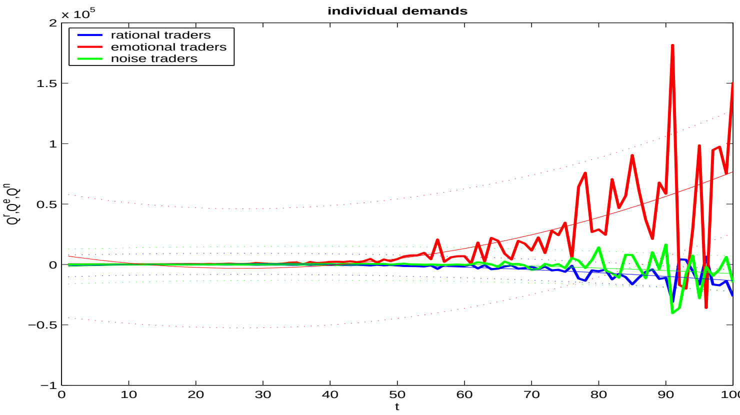
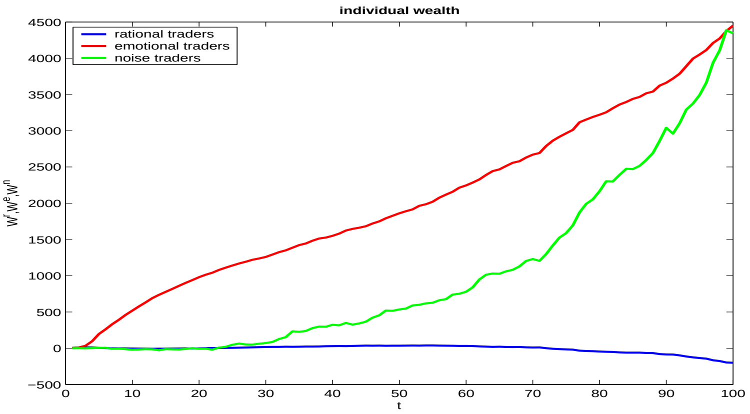
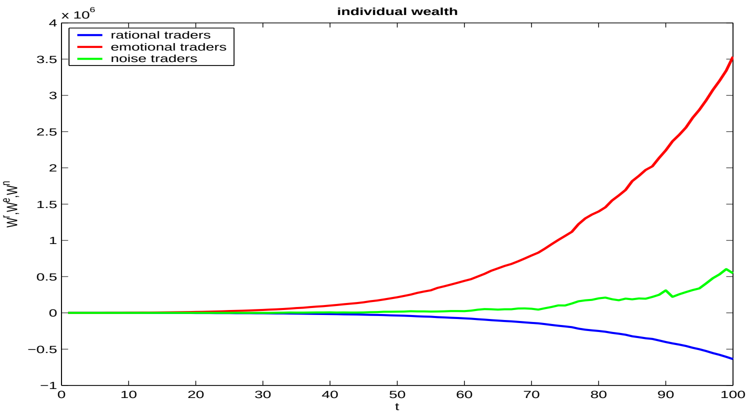


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

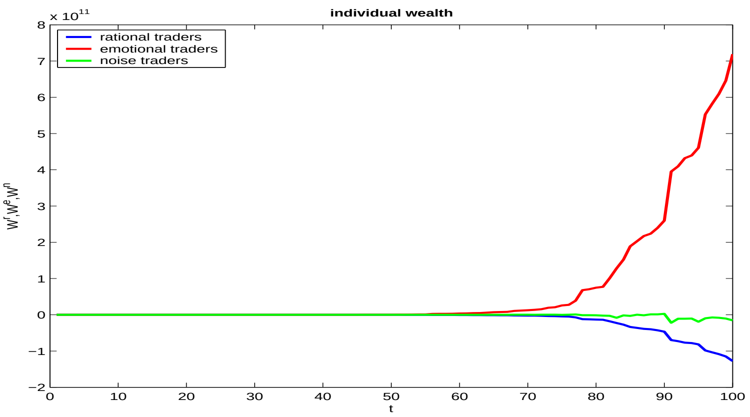
Figure A.40: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 10$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

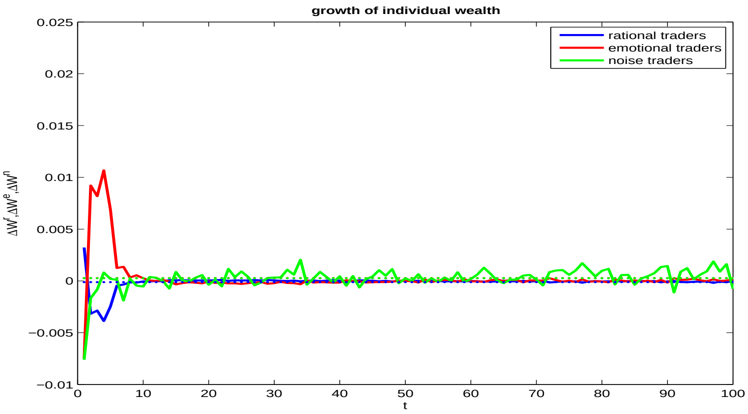


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

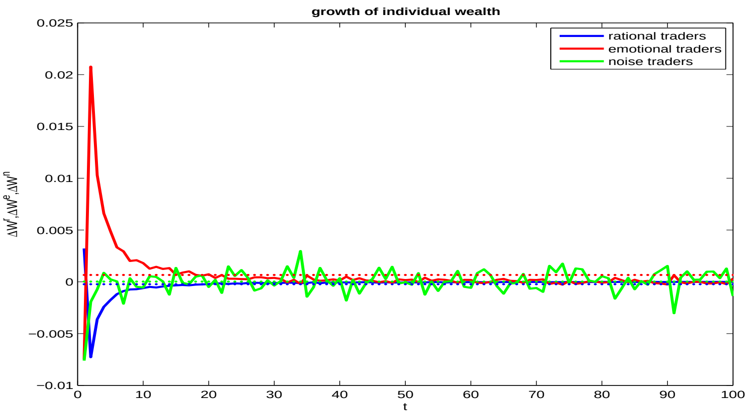


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

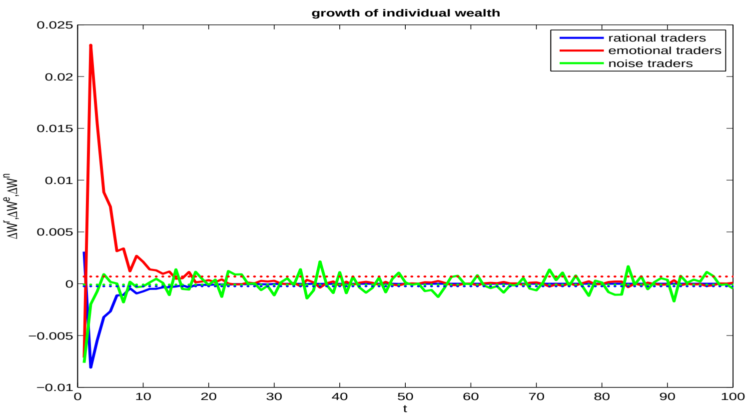
Figure A.41: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 10$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

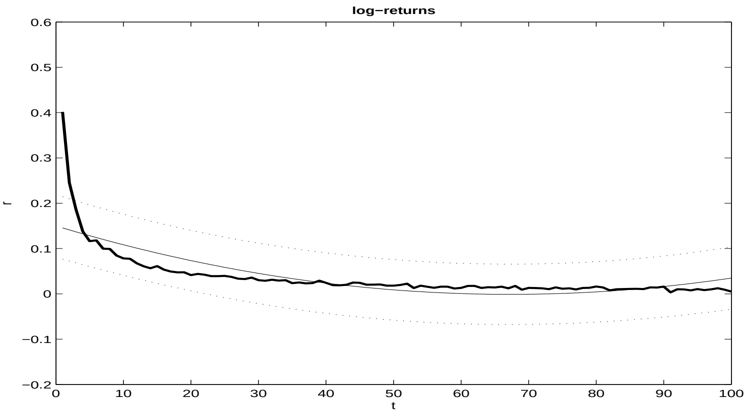


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

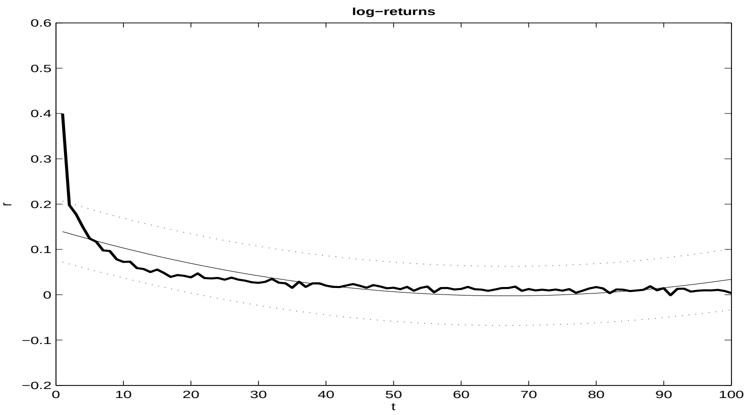


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

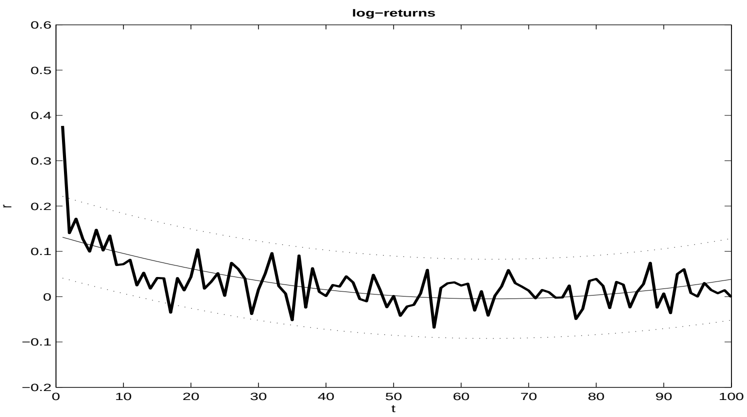
Figure A.42: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 10$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

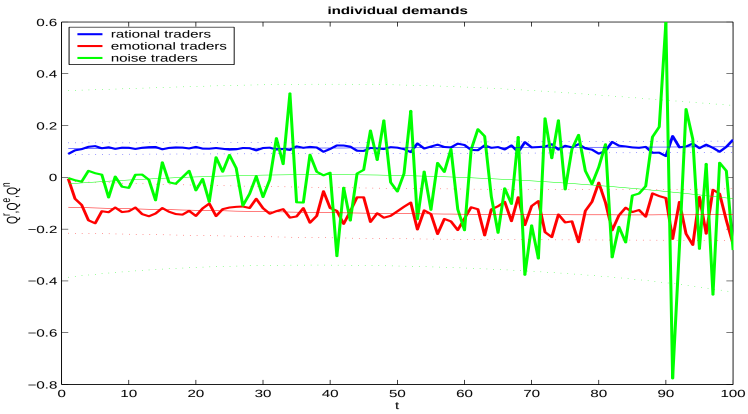


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

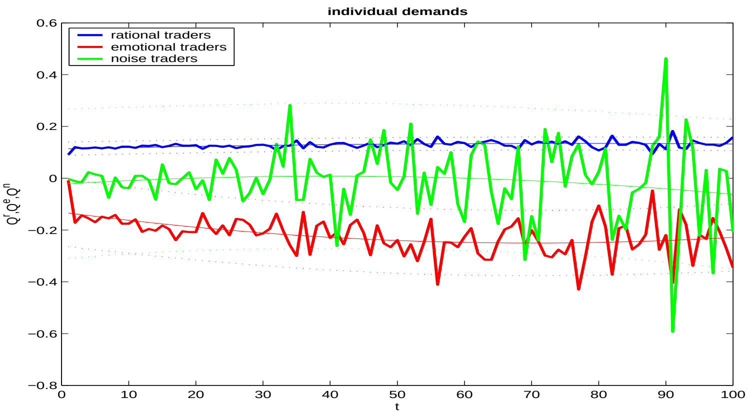


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

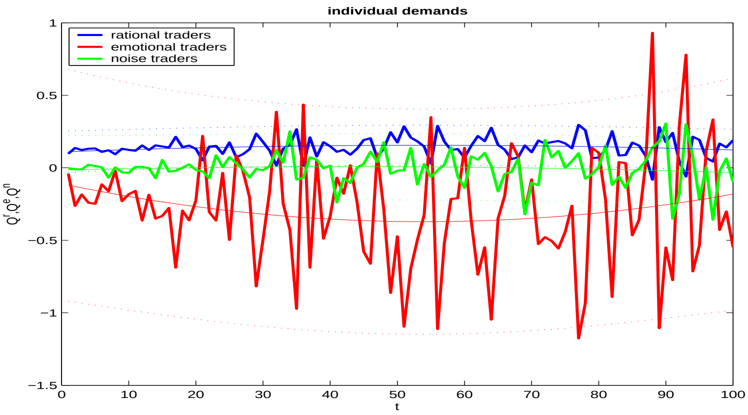
Figure A.43: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = -1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

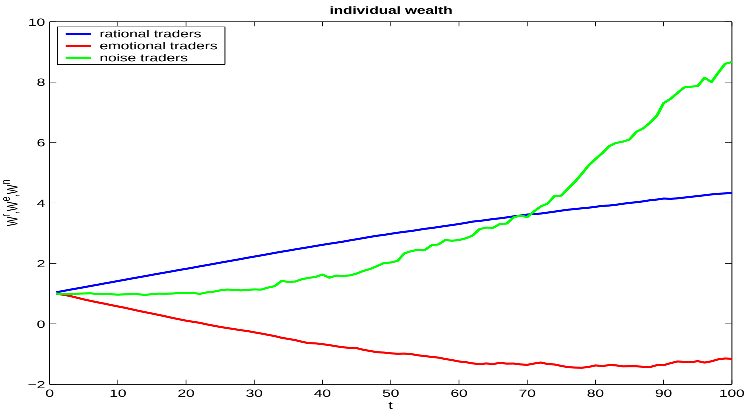


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

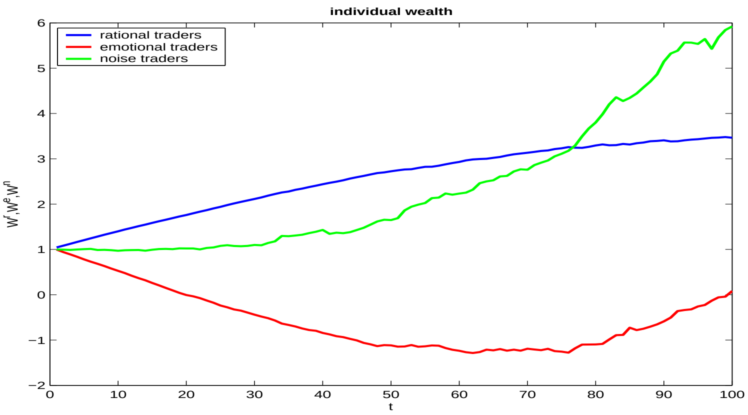


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

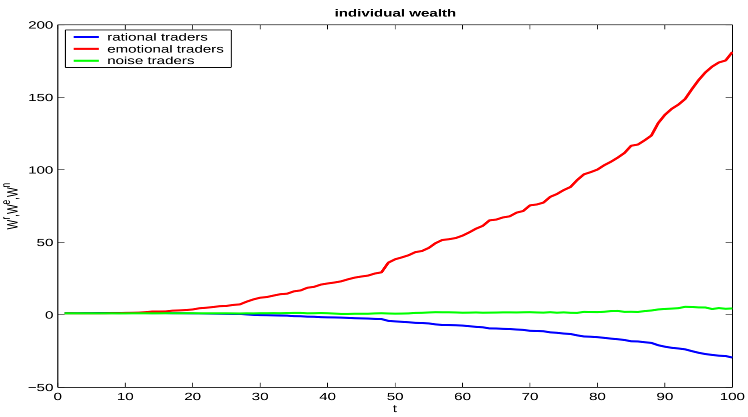
Figure A.44: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = -1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

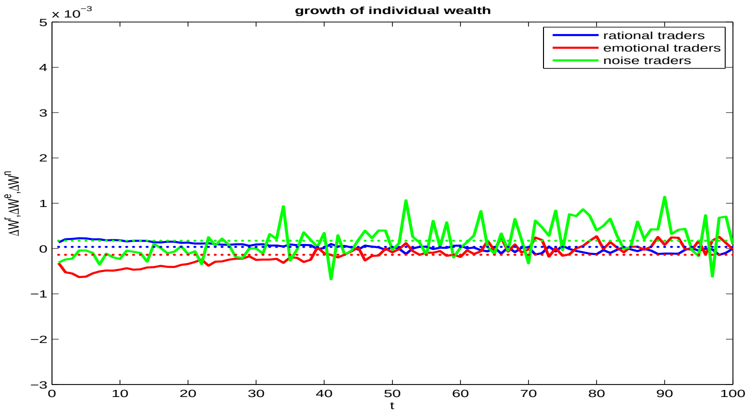


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

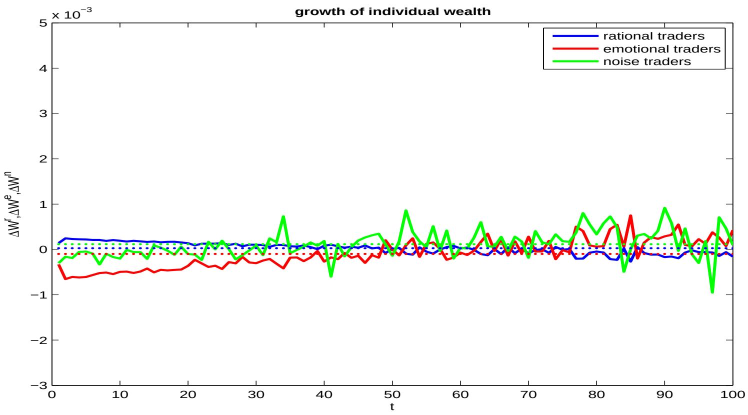


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

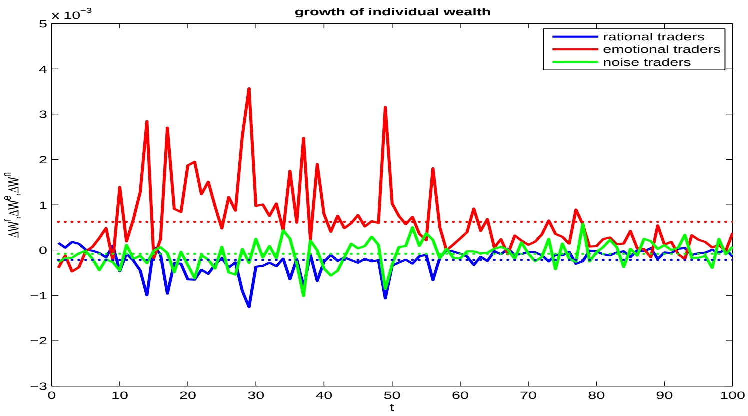
Figure A.45: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = -1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

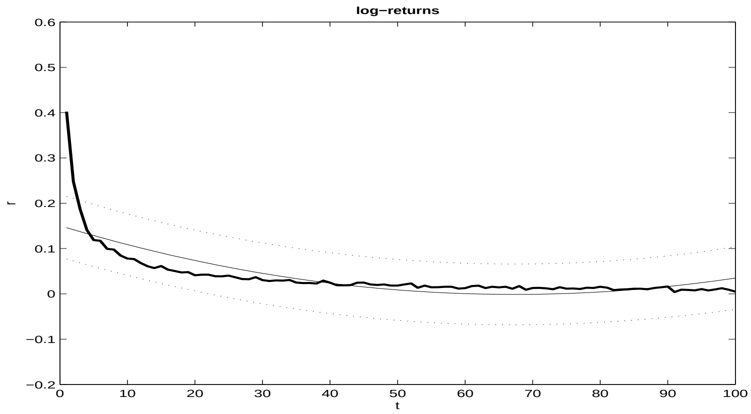


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

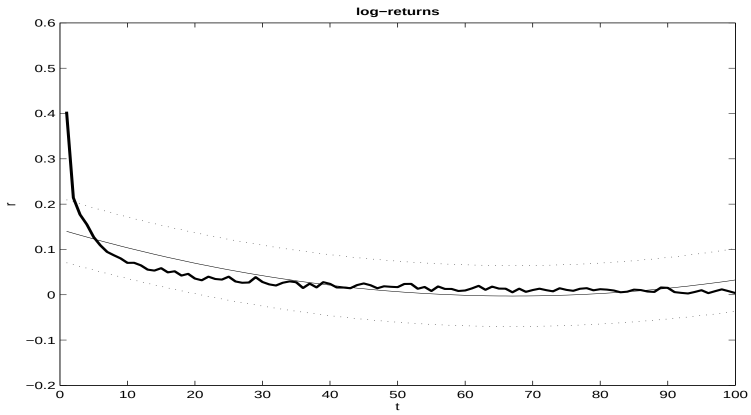


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

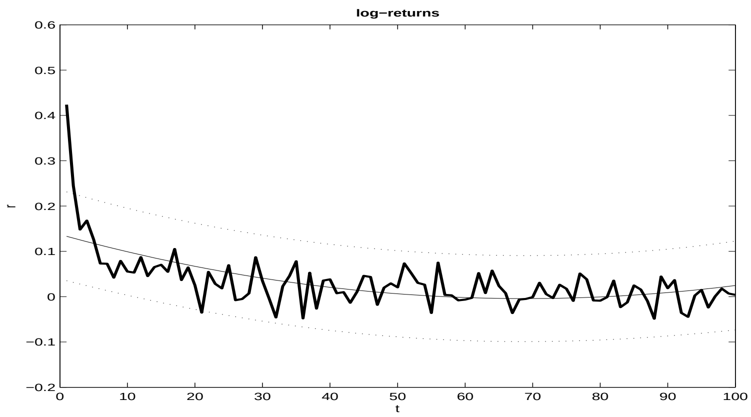
Figure A.46: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = -1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

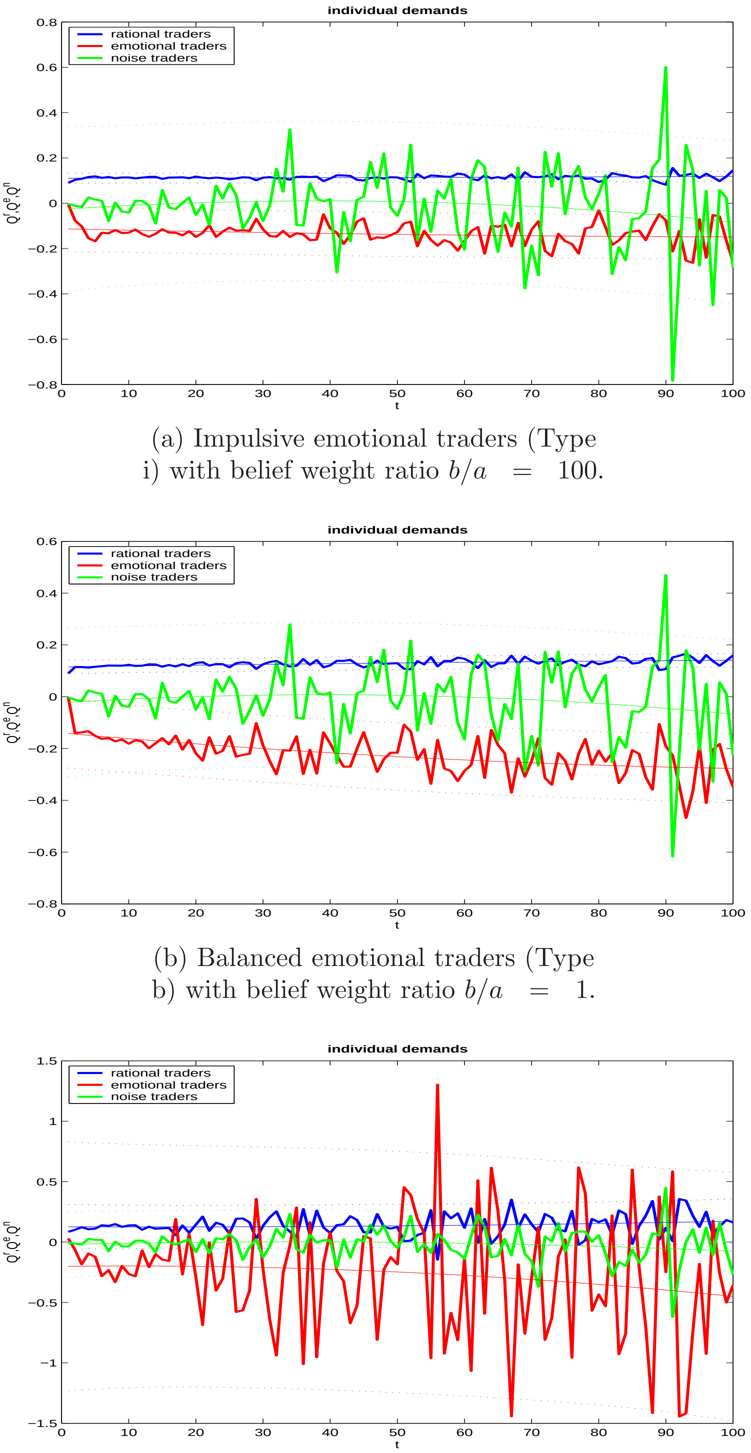


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.



(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure A.47: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = -N^e/N, \beta^e = 1$.

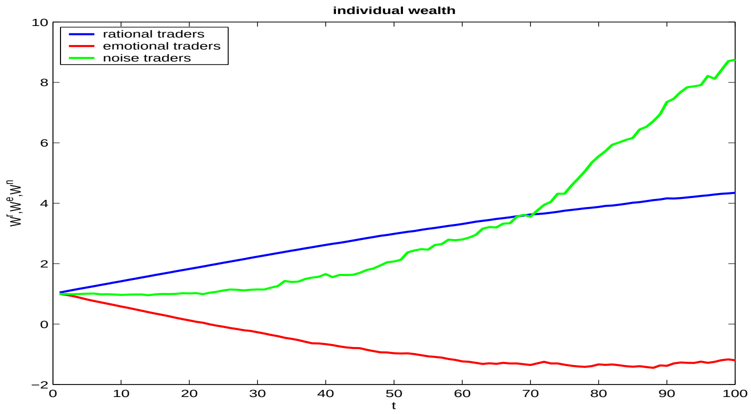


(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

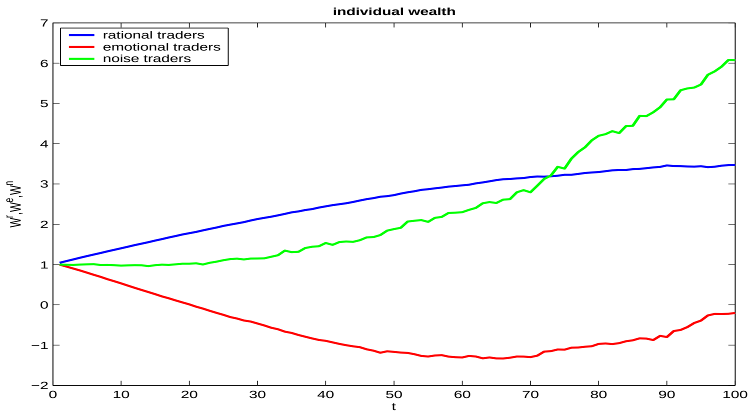
(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

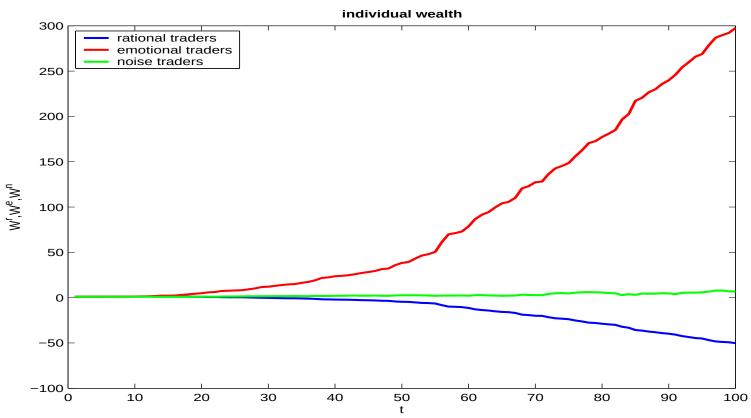
Figure A.48: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = -N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.



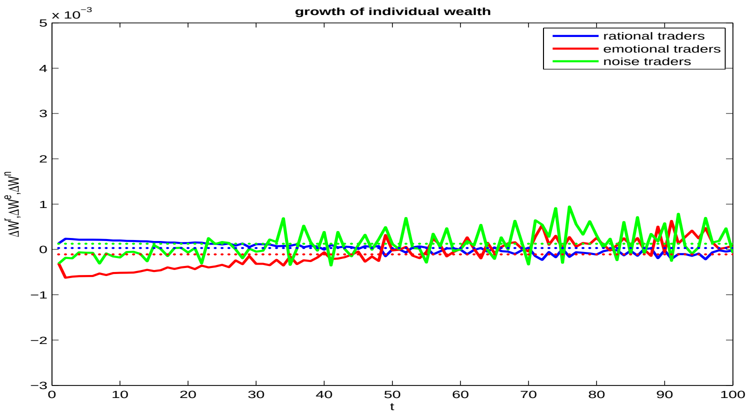
(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.



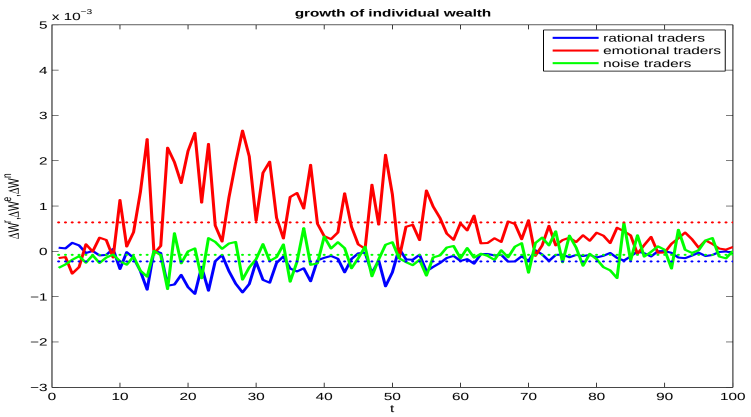
(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure A.49: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = -N^e/N, \beta^e = 1$.

(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

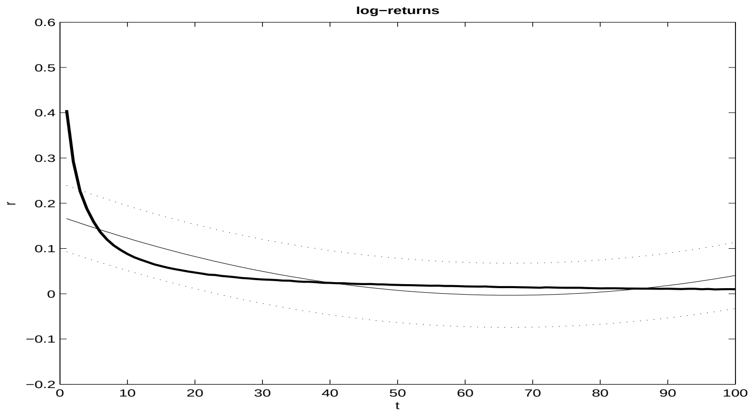


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

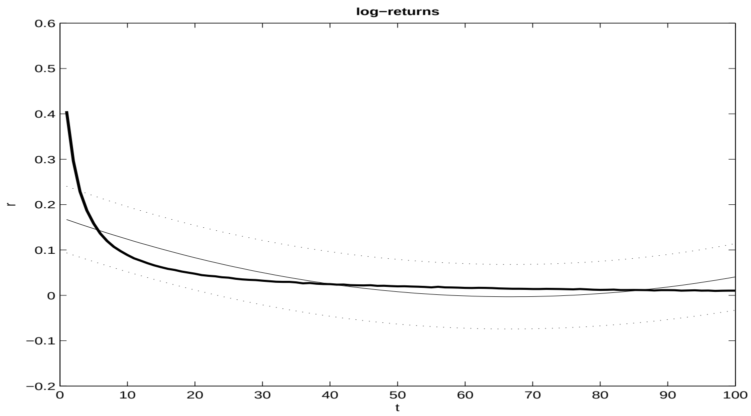


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

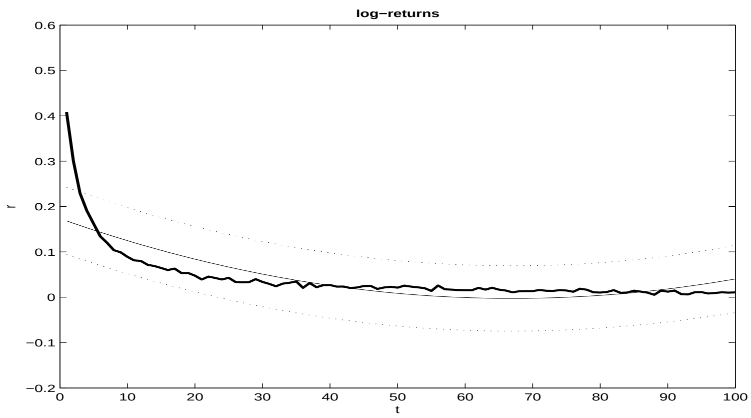
Figure A.50: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = -N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

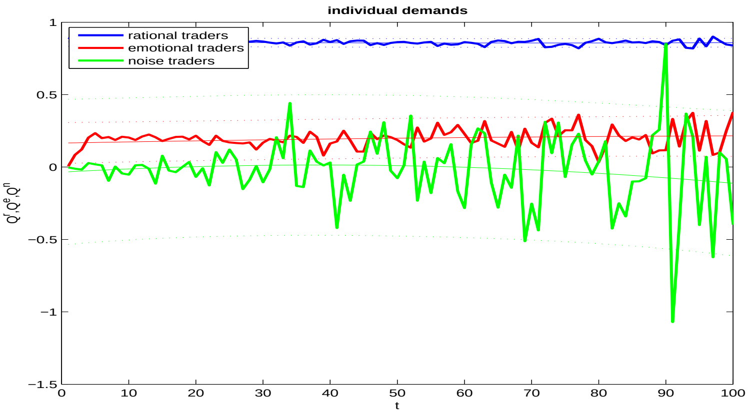


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

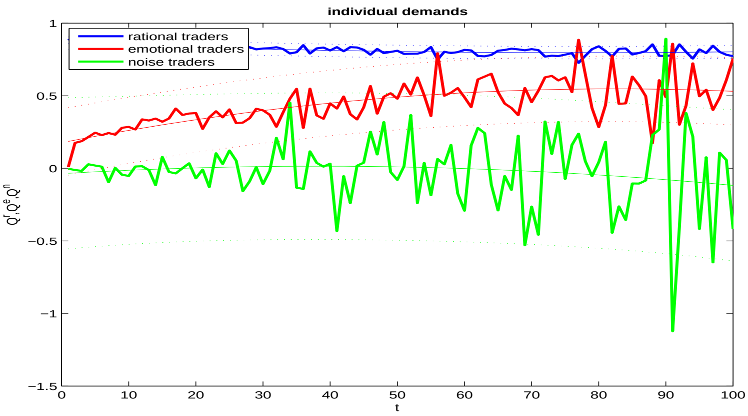


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

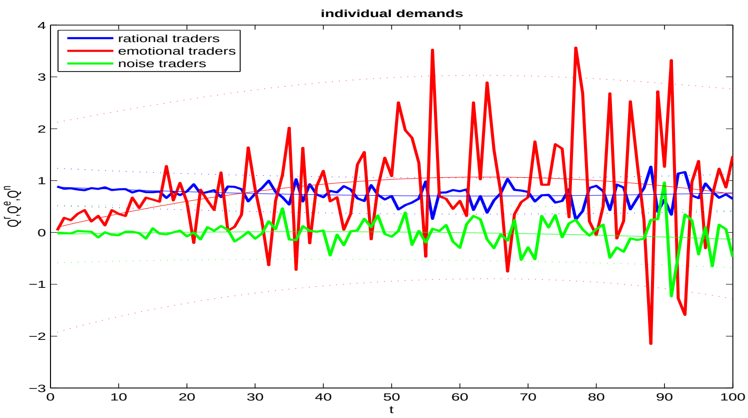
Figure A.51: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

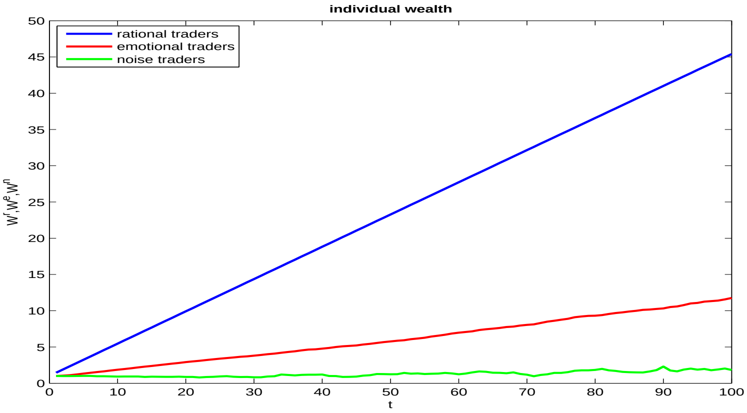


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

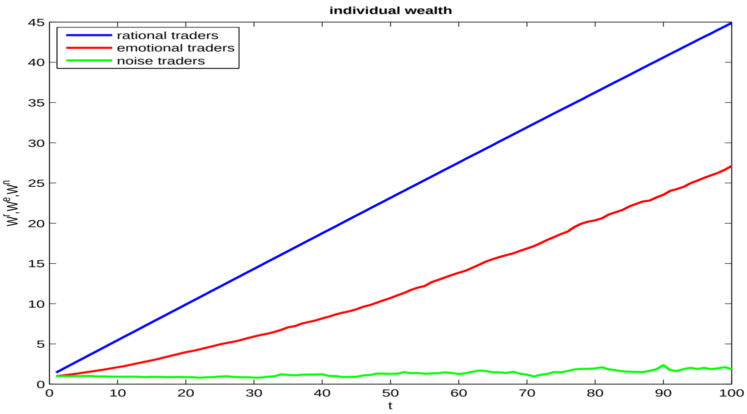


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

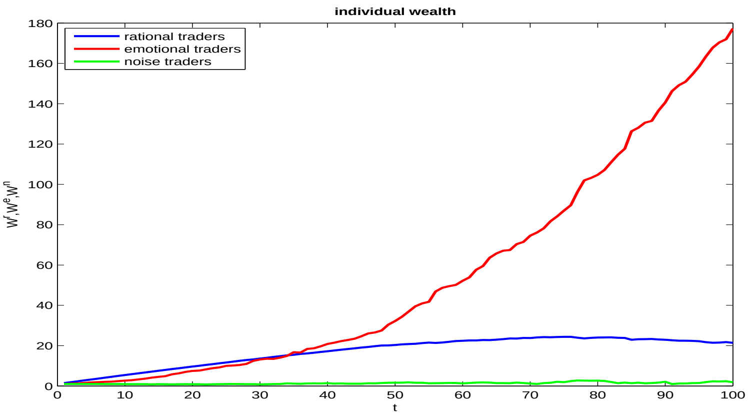
Figure A.52: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

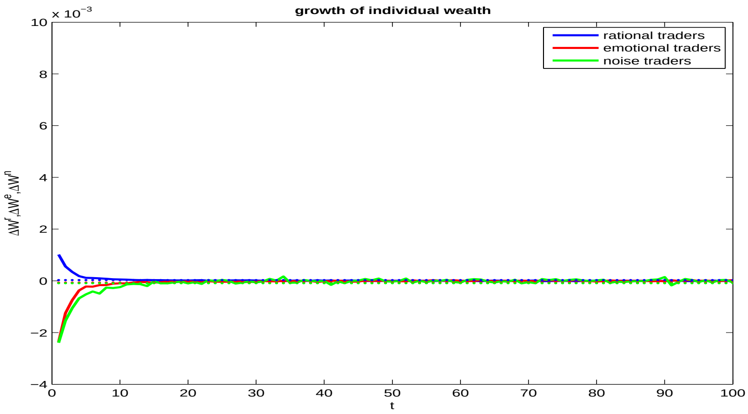


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

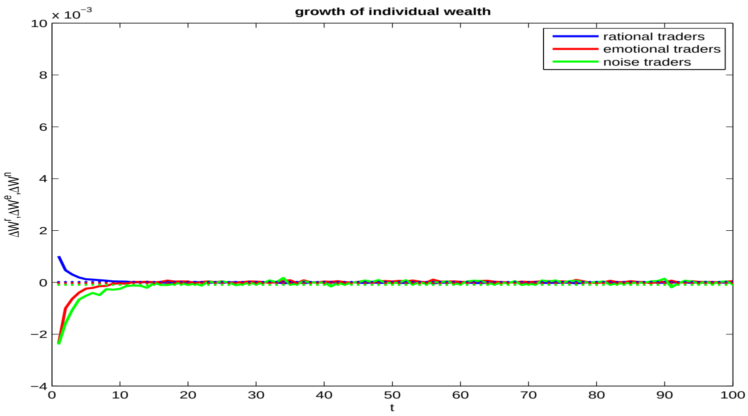


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

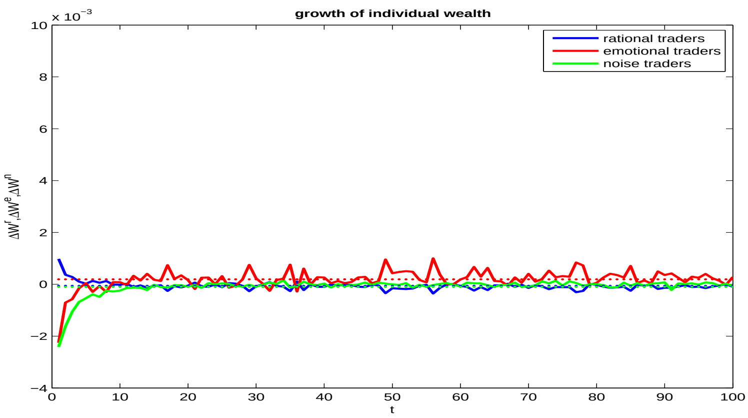
Figure A.53: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

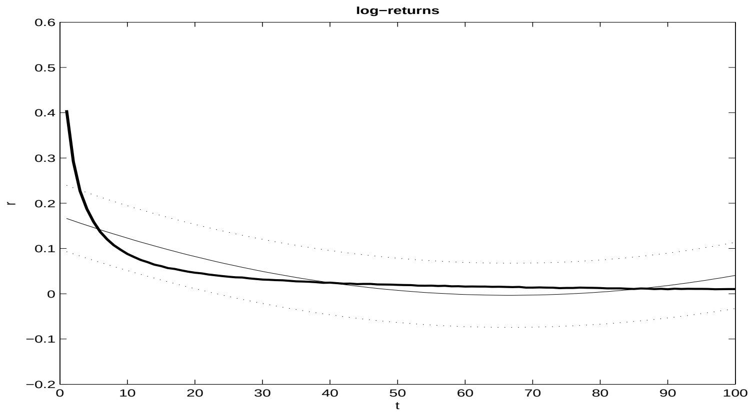


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

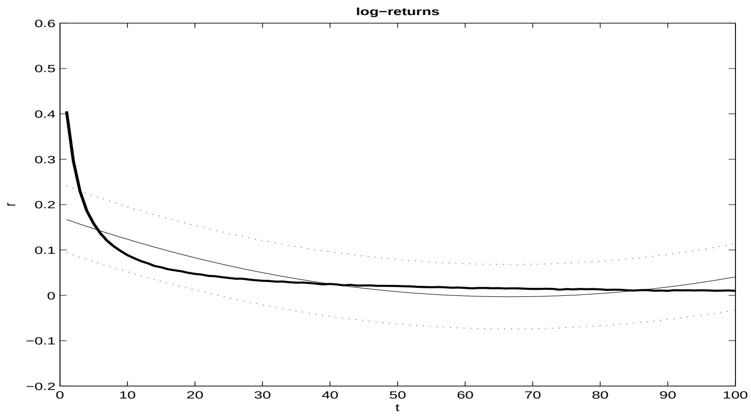


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

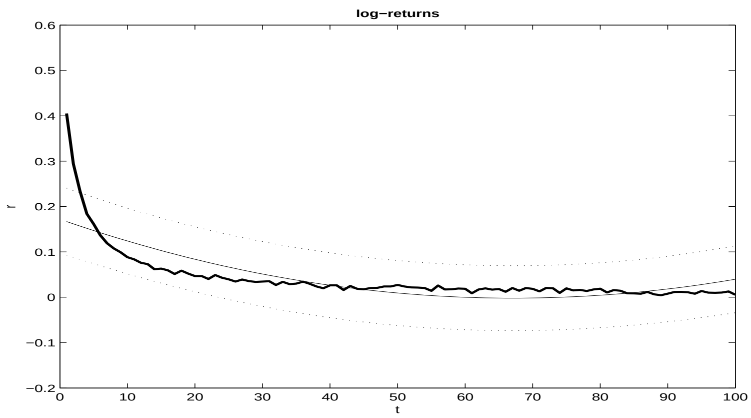
Figure A.54: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), independent emotional and noise trader noise (Scenario 1), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

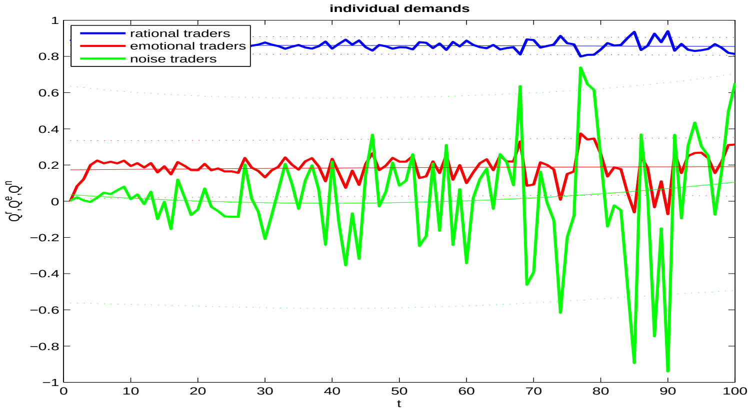


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

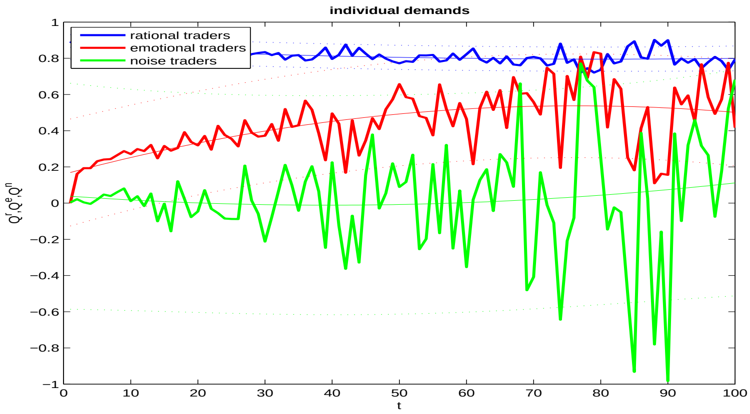


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

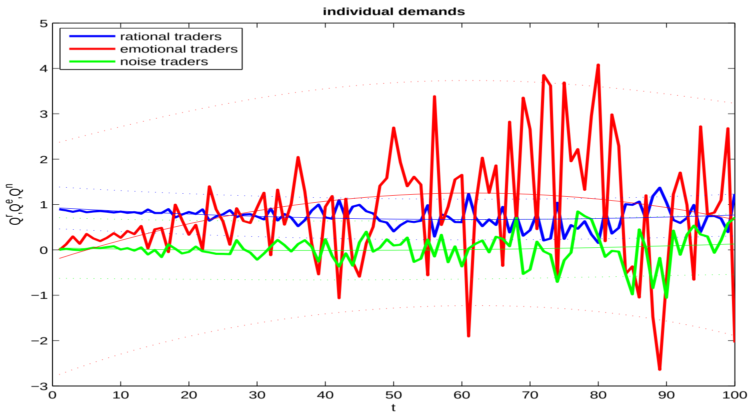
Figure A.55: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

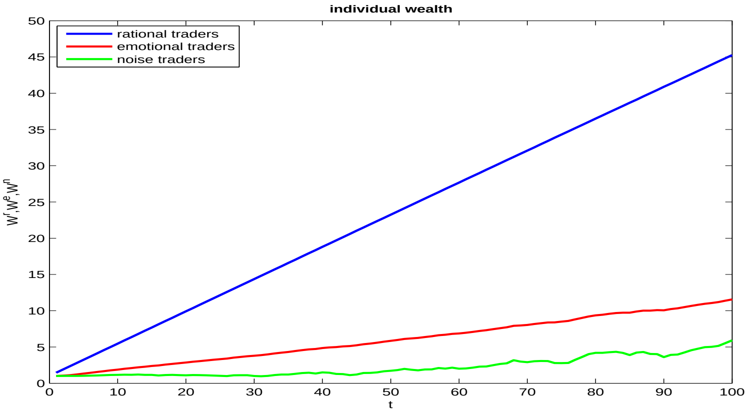


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

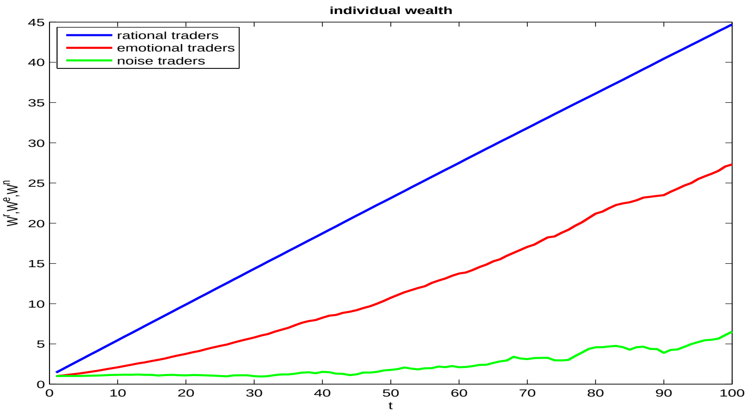


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

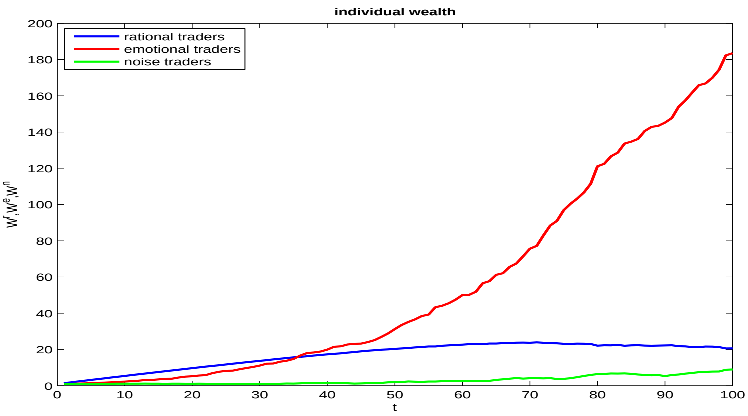
Figure A.56: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

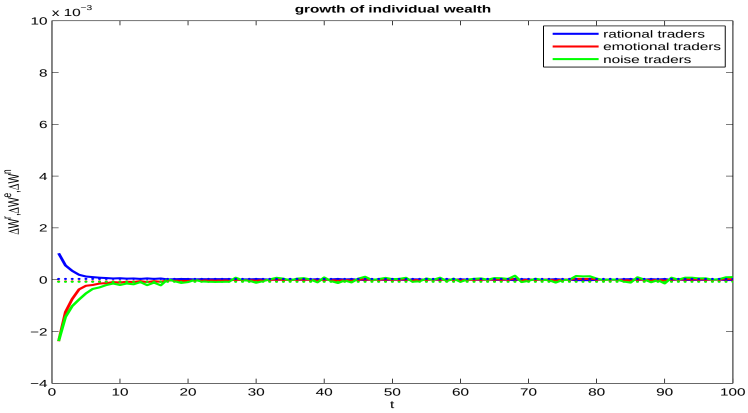


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

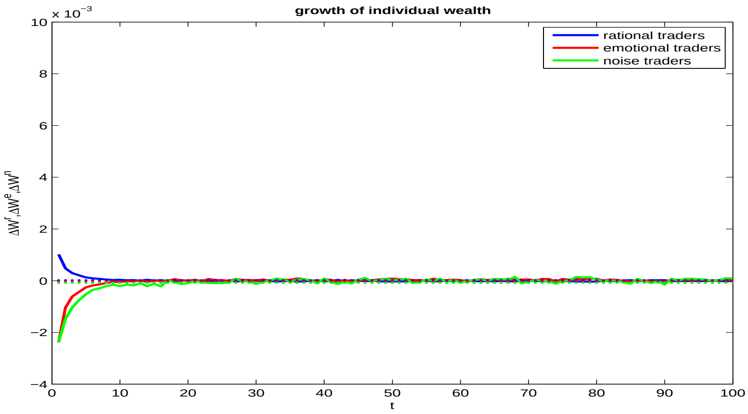


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

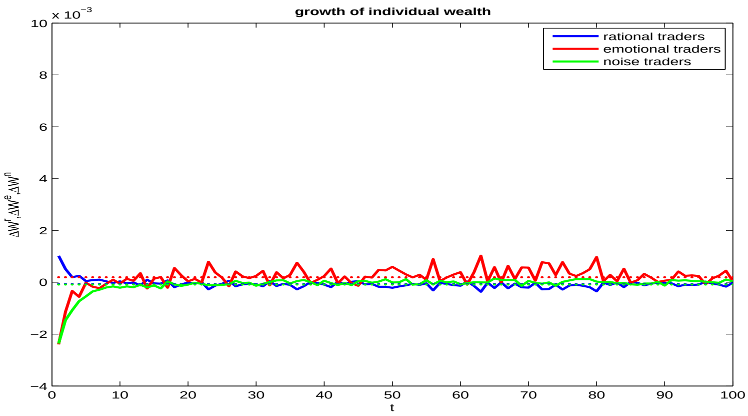
Figure A.57: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

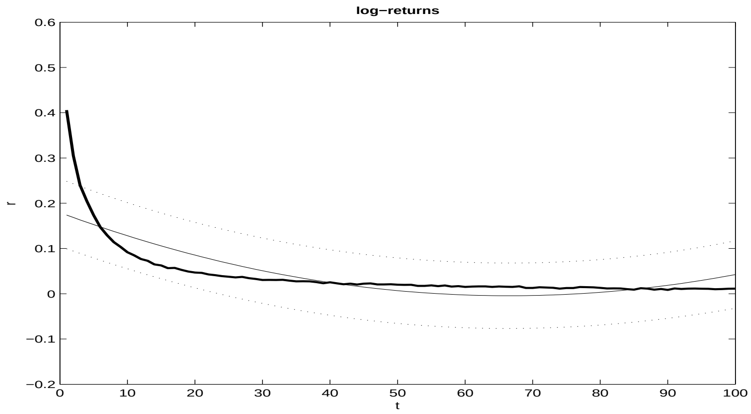


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

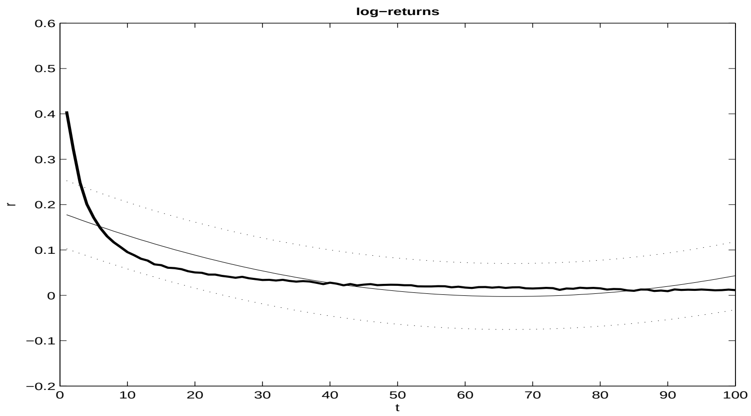


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

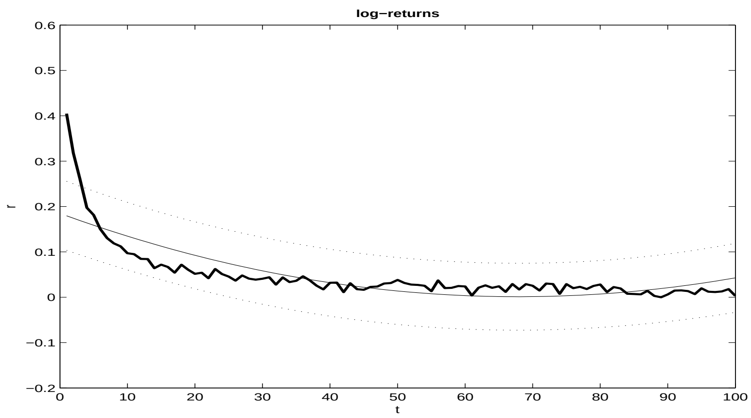
Figure A.58: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 70\%, N^e/N = 25\%, N^n/N = 5\%$ (Case A), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

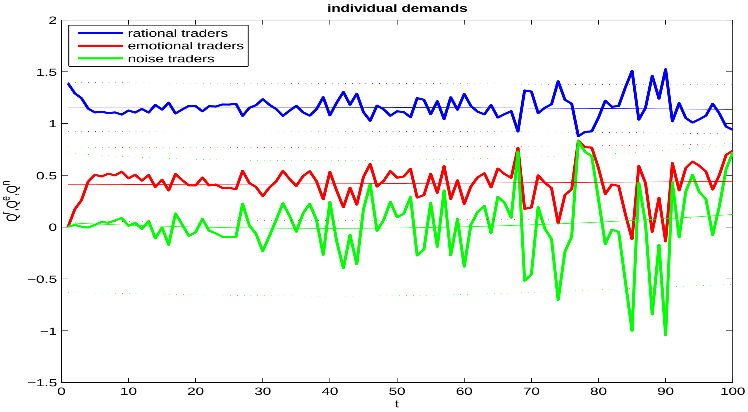


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

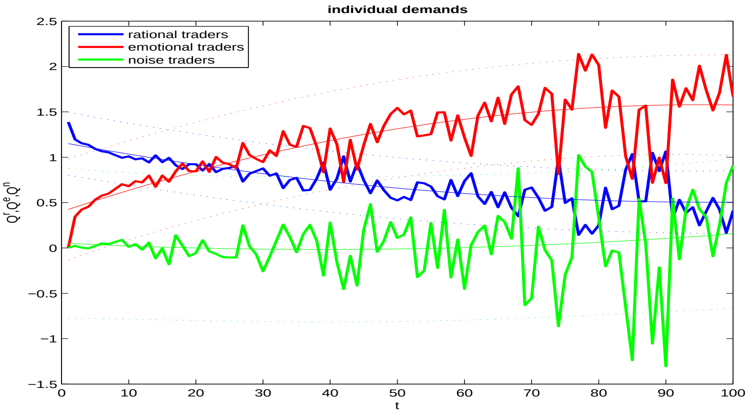


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

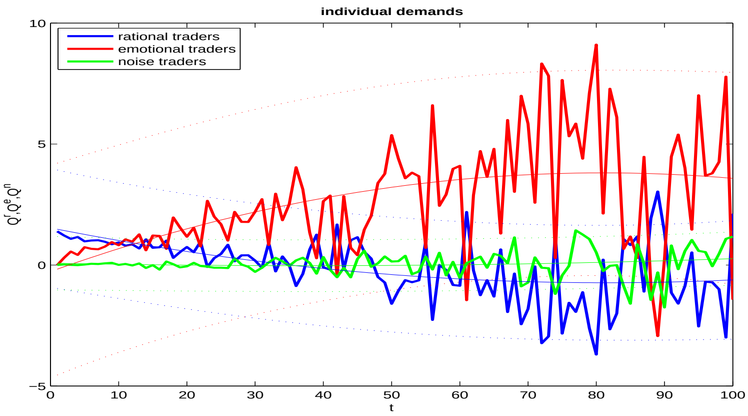
Figure A.59: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 45\%, N^e/N = 50\%, N^n/N = 5\%$ (Case B), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

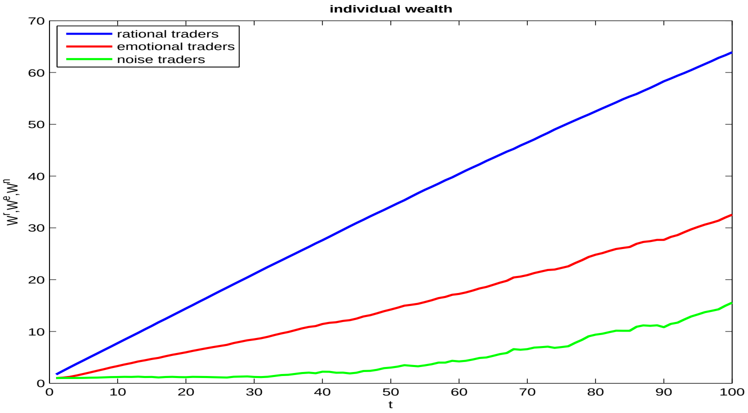


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

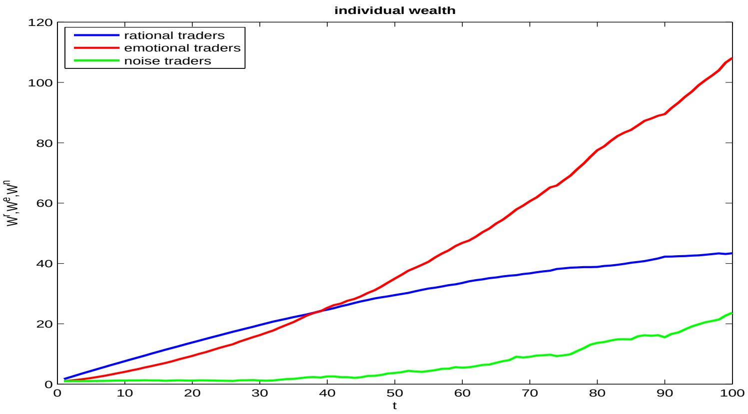


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

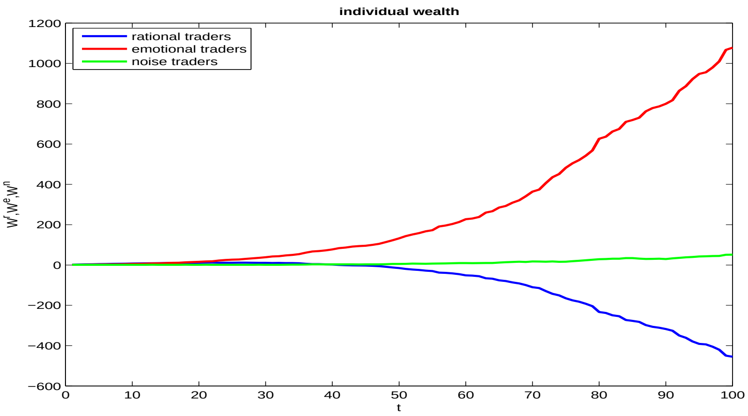
Figure A.60: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 45\%, N^e/N = 50\%, N^n/N = 5\%$ (Case B), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

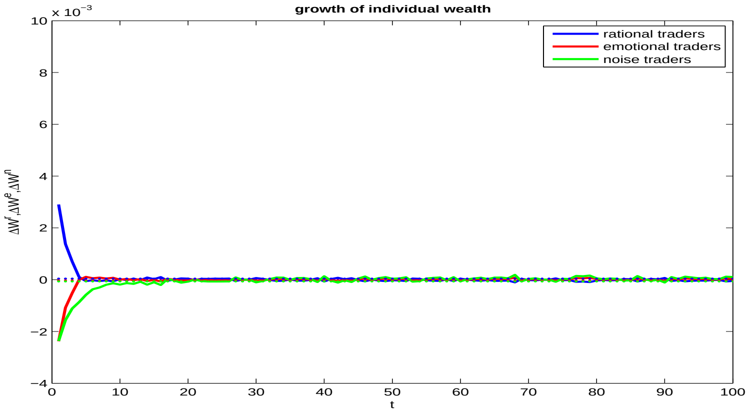


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

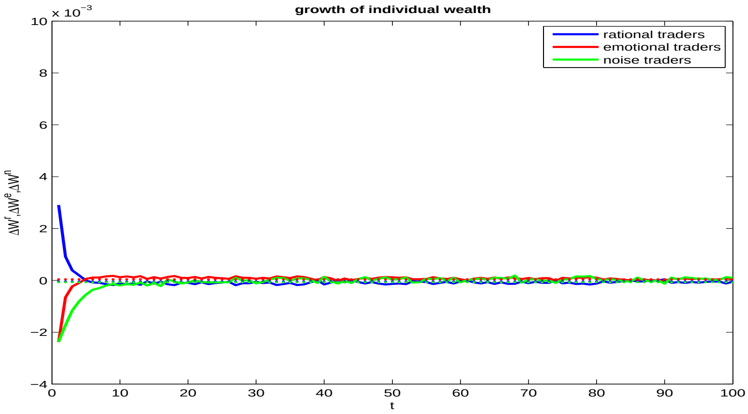


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

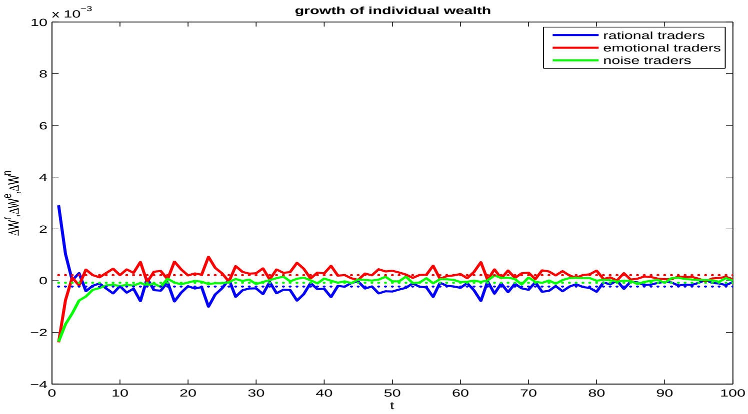
Figure A.61: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 45\%, N^e/N = 50\%, N^n/N = 5\%$ (Case B), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

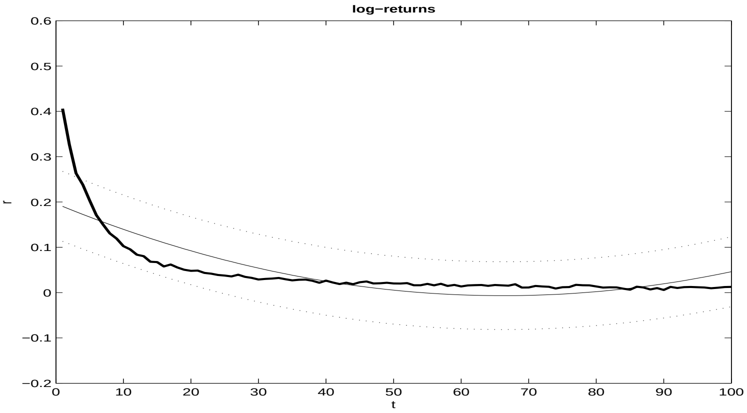


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

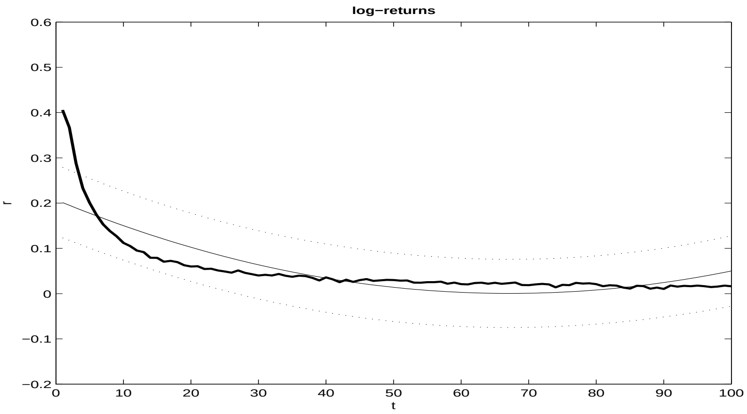


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

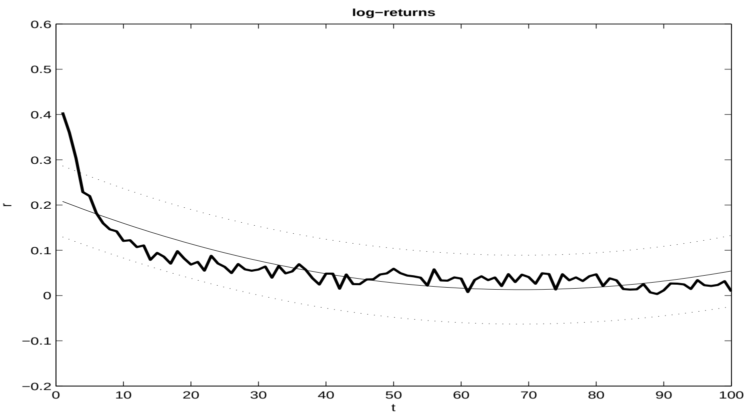
Figure A.62: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 45\%, N^e/N = 50\%, N^n/N = 5\%$ (Case B), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

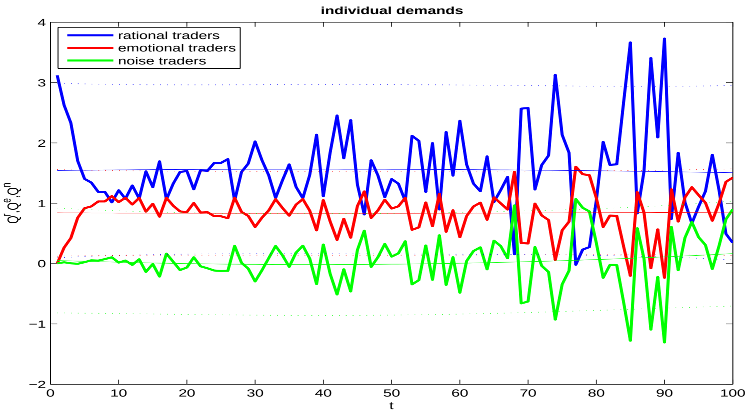


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

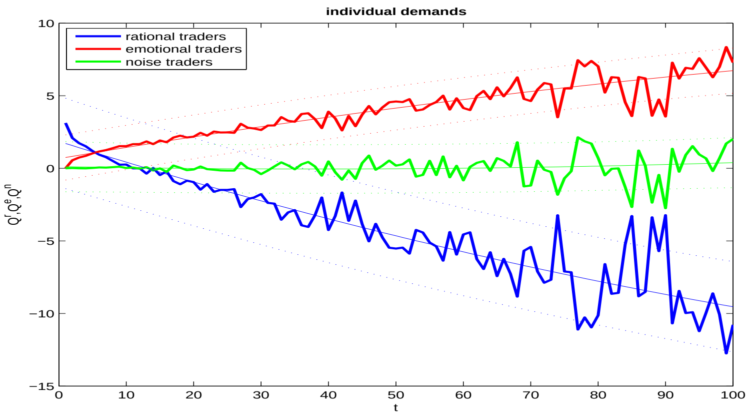


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

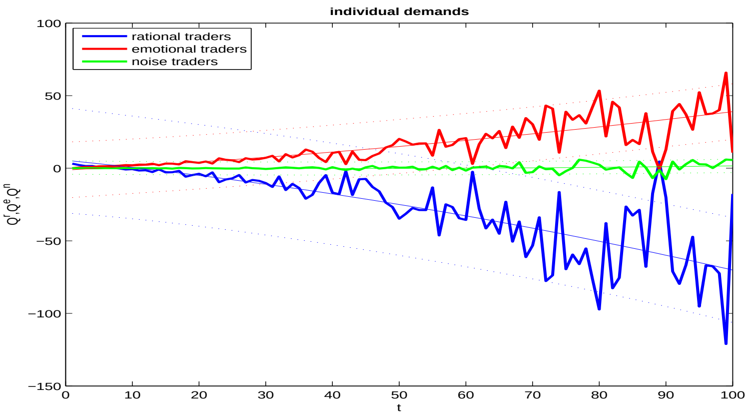
Figure A.63: The evolution of log-returns for different emotional types, with trader proportions $N^r/N = 20\%, N^e/N = 75\%, N^n/N = 5\%$ (Case C), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

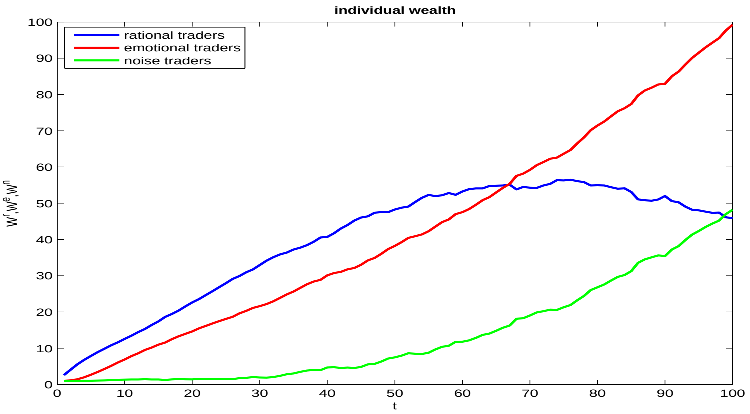


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

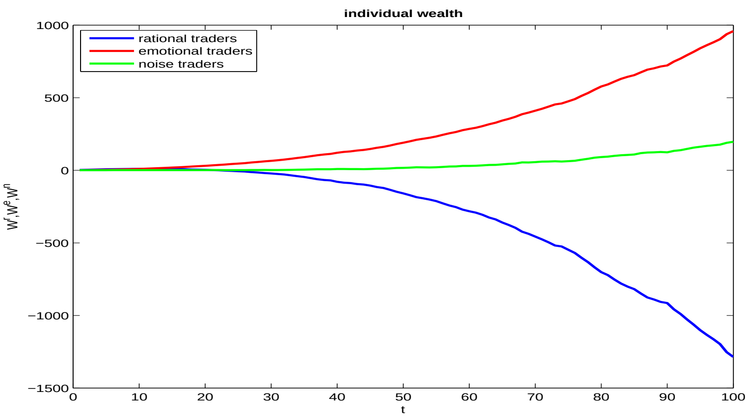


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

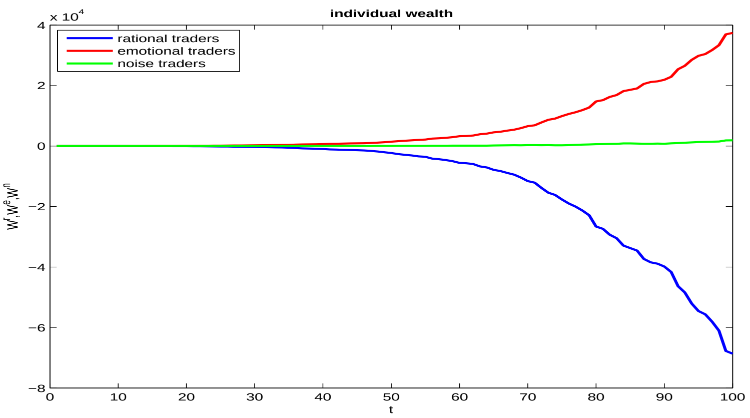
Figure A.64: The evolution of individual demands for different emotional types, with trader proportions $N^r/N = 20\%, N^e/N = 75\%, N^n/N = 5\%$ (Case C), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.

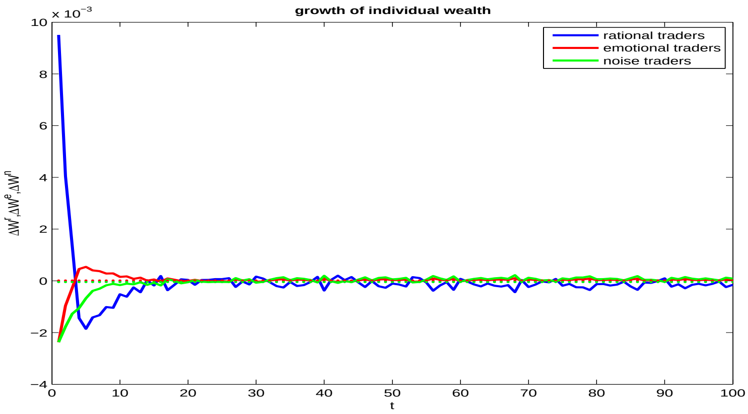


(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.

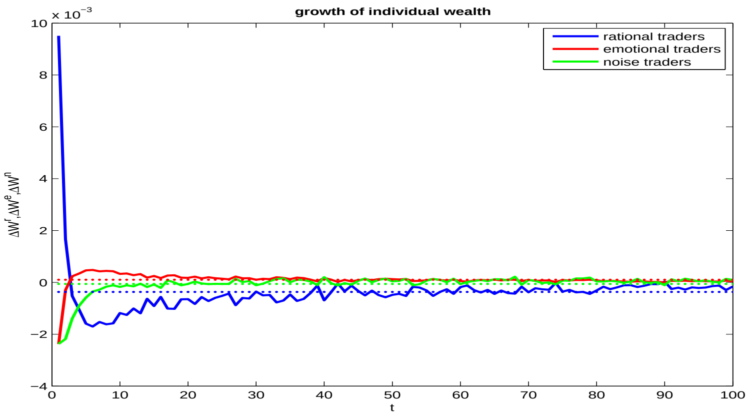


(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

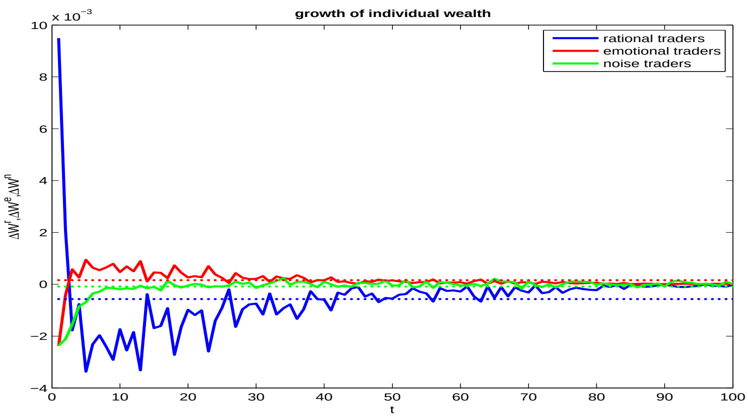
Figure A.65: The evolution of individual wealth for different emotional types, with trader proportions $N^r/N = 20\%, N^e/N = 75\%, N^n/N = 5\%$ (Case C), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.



(a) Impulsive emotional traders (Type i) with belief weight ratio $b/a = 100$.



(b) Balanced emotional traders (Type b) with belief weight ratio $b/a = 1$.



(c) Conservative emotional traders (Type c) with belief weight ratio $b/a = 0.01$.

Figure A.66: The evolution of the growth of individual wealth for different emotional types, with trader proportions $N^r/N = 20\%, N^e/N = 75\%, N^n/N = 5\%$ (Case C), identical emotional and noise trader noise (Scenario 2), updating of mean prior beliefs from past returns (Rule qd-1), over $n = 10$ independent parallel trade rounds, in a more liquid market $\lambda = 0.008$, for a true risky value $V = 1$, and emotional belief parameters $k_{t-1} = 0, k^e = N^e/N, \beta^e = 1$.

A.3 How non-professional investors face financial risk: loss aversion and wealth allocation

A.3.1 Descriptive statistics

	S&P 500		3-month T-bill		Consumption	
	Evaluation frequency		Evaluation frequency		Evaluation frequency	
	Quarterly	Yearly	Quarterly	Yearly	Quarterly	Yearly
Mean	0.017	0.066	0.017	0.073	0.016	0.052
Median	0.018	0.071	0.017	0.070	0.001	0.049
Std.Dev.	0.079	0.136	0.006	0.026	0.008	0.022
Kurtosis	2.661	-0.9659	0.623	0.974	0.673	-1.084
Skewness	-0.671	-0.205	0.951	1.042	-0.018	0.165
Max.	0.290	0.345	0.036	0.142	0.042	0.090
Min.	-0.302	-0.207	0.009	0.037	-0.010	0.011
Obs.	175	43	175	43	175	43

Table A.1: Log-differences of the S&P 500 index and of the 3-month T-bill returns for quarterly and yearly portfolio evaluations.

A.3.2 One-dimensional utility: risky vs. risk-free financial assets

According to Equations (3.19) and (3.20), we have:

$$\begin{aligned}
E_t[\text{loss-value}_{t+1}] &= \pi_t(1-\psi_t)(\lambda S_t E_t[x_{t+1}] + (\lambda-1)(S_t-Z_t)R_{ft}) \\
&\quad + (1-\pi_t)(1-\omega_t)(\lambda S_t E_t[x_{t+1}] - k(S_t-Z_t)E_t[x_{t+1}]) \\
&\underset{\text{cond.}}{=} \lambda S_t E_t[x_{t+1}] + \Big(\pi_t(1-\psi_t)((\lambda-1)R_{ft} + kE_t[x_{t+1}]) - kE_t[x_{t+1}]\Big)(S_t-Z_t).
\end{aligned}$$

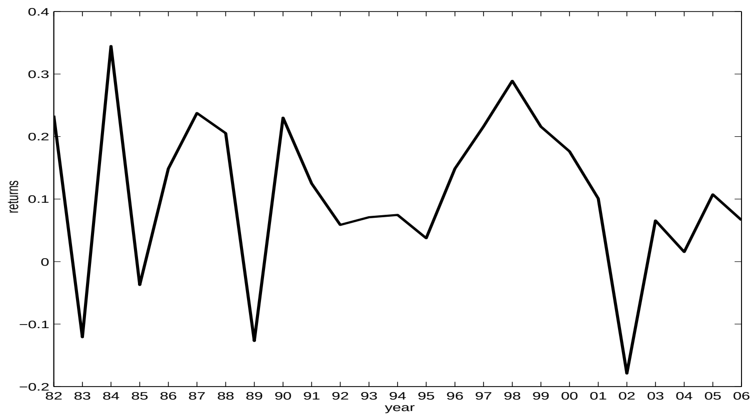
Also, the expectation of the squared loss value and consequently the loss variance result in:

$$\begin{aligned}
E_t[\text{loss-value}^2_{t+1}] &= \pi_t(1-\psi_t)(\lambda S_t E_t[x_{t+1}] + (\lambda-1)(S_t-Z_t)R_{ft})^2 \\
&\quad + (1-\pi_t)(1-\omega_t)(\lambda S_t E_t[x_{t+1}] - k(S_t-Z_t)E_t[x_{t+1}])^2 \\
&= (\lambda S_t E_t[x_{t+1}])^2 + \Big(\pi_t(1-\psi_t)((\lambda-1)^2 R^2_{ft} - k^2 x^2_{t+1}) + k^2 x^2_{t+1}\Big)(S_t-Z_t)^2 \\
&\quad + 2\Big(\pi_t(1-\psi_t)((\lambda-1)R_{ft} + kE_t[x_{t+1}]) - kE_t[x_{t+1}]\Big)\lambda S_t E_t[x_{t+1}](S_t-Z_t),
\end{aligned}$$

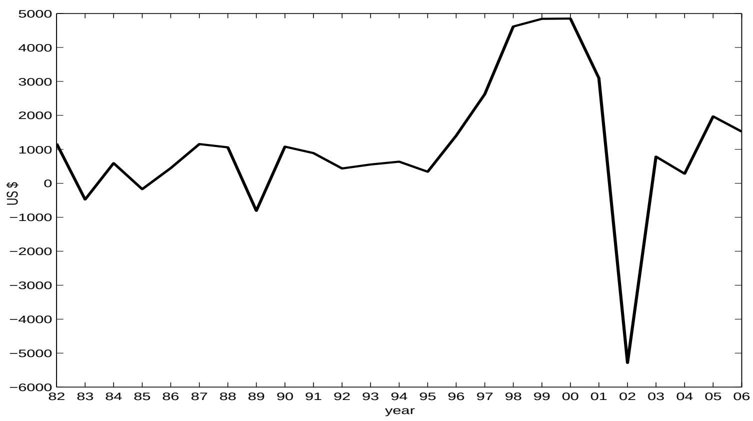
and:

$$
\begin{aligned}
Var_t[\text{loss-value}_{t+1}] &= E_t[\text{loss-value}^2_{t+1}] - E^2_t[\text{loss-value}_{t+1}] \\
&= \pi_t(1-\psi_t)\Big(1-\pi_t(1-\psi_t)\Big)\Big((\lambda-1)R_{ft} + kE_t[x_{t+1}]\Big)^2 (S_t - Z_t)^2.
\end{aligned}
$$

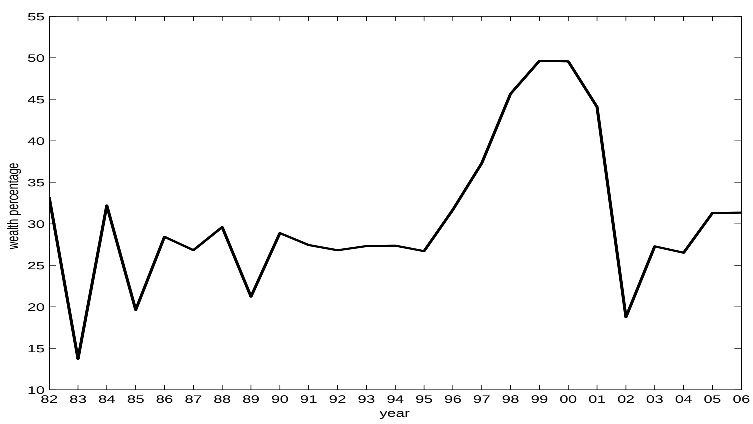
The variance of the loss value is exclusively based on past performance, being generated only by the cushion $S_t - Z_t$.



(a) Yearly S&P 500 log-returns.

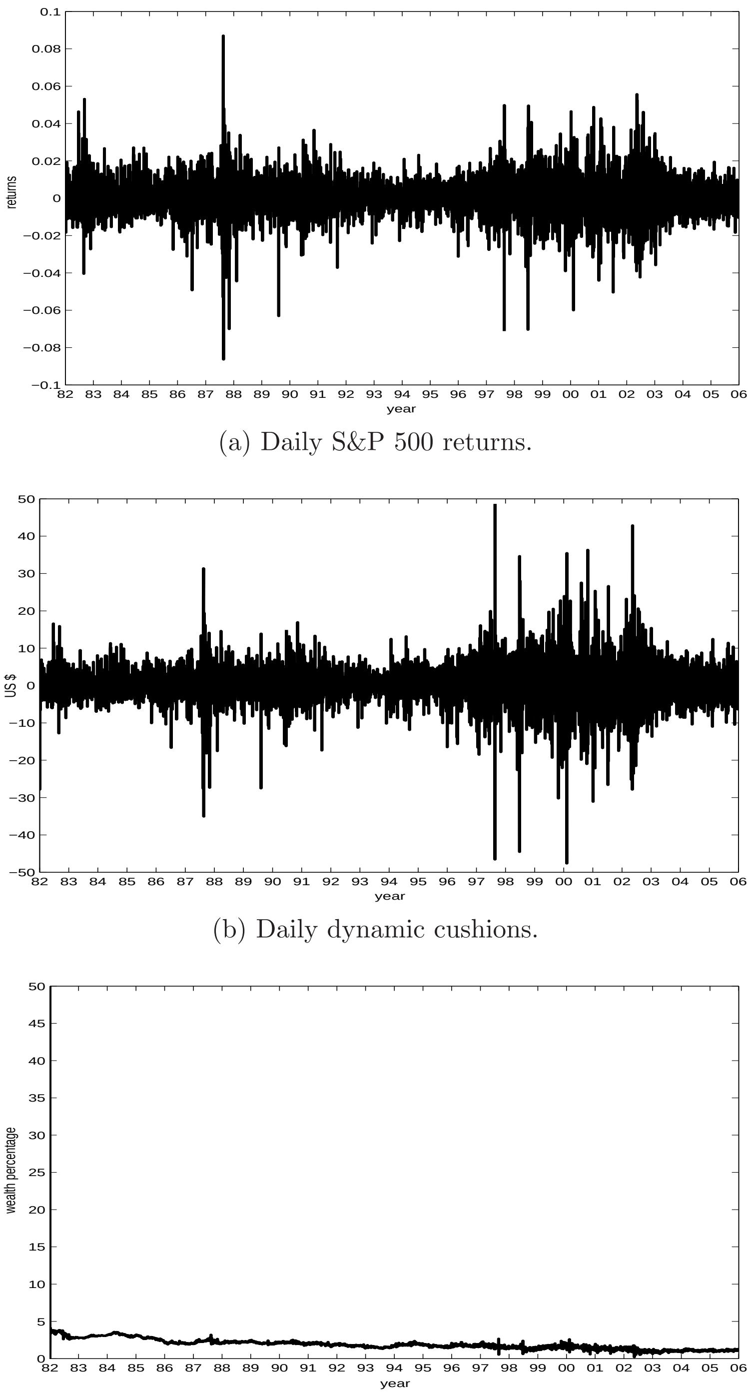


(b) Yearly dynamic cushions.



(c) Yearly wealth percentage invested in S&P 500.

Figure A.67: Evolution of risky returns, dynamic cushions, and wealth percentages invested in the risky portfolio for yearly portfolio evaluations.

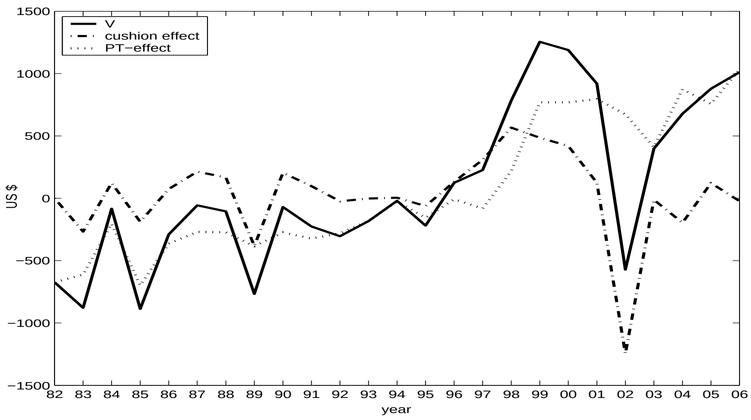


(a) Daily S&P 500 returns.

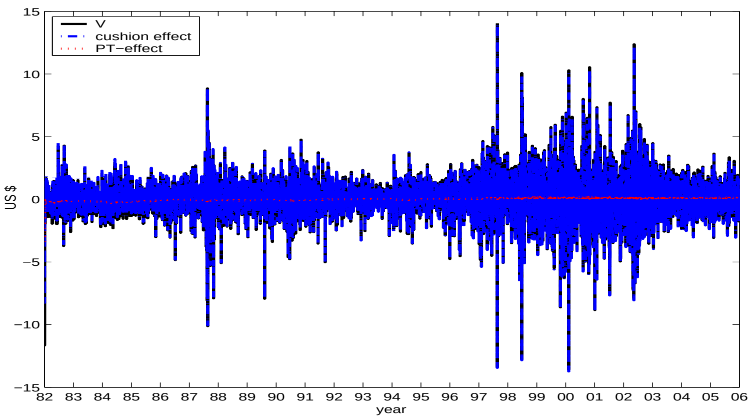
(b) Daily dynamic cushions.

(c) Daily percentage investments in S&P 500.

Figure A.68: Evolution of risky returns, dynamic cushions, and percentages invested in the risky portfolio for daily portfolio evaluations.

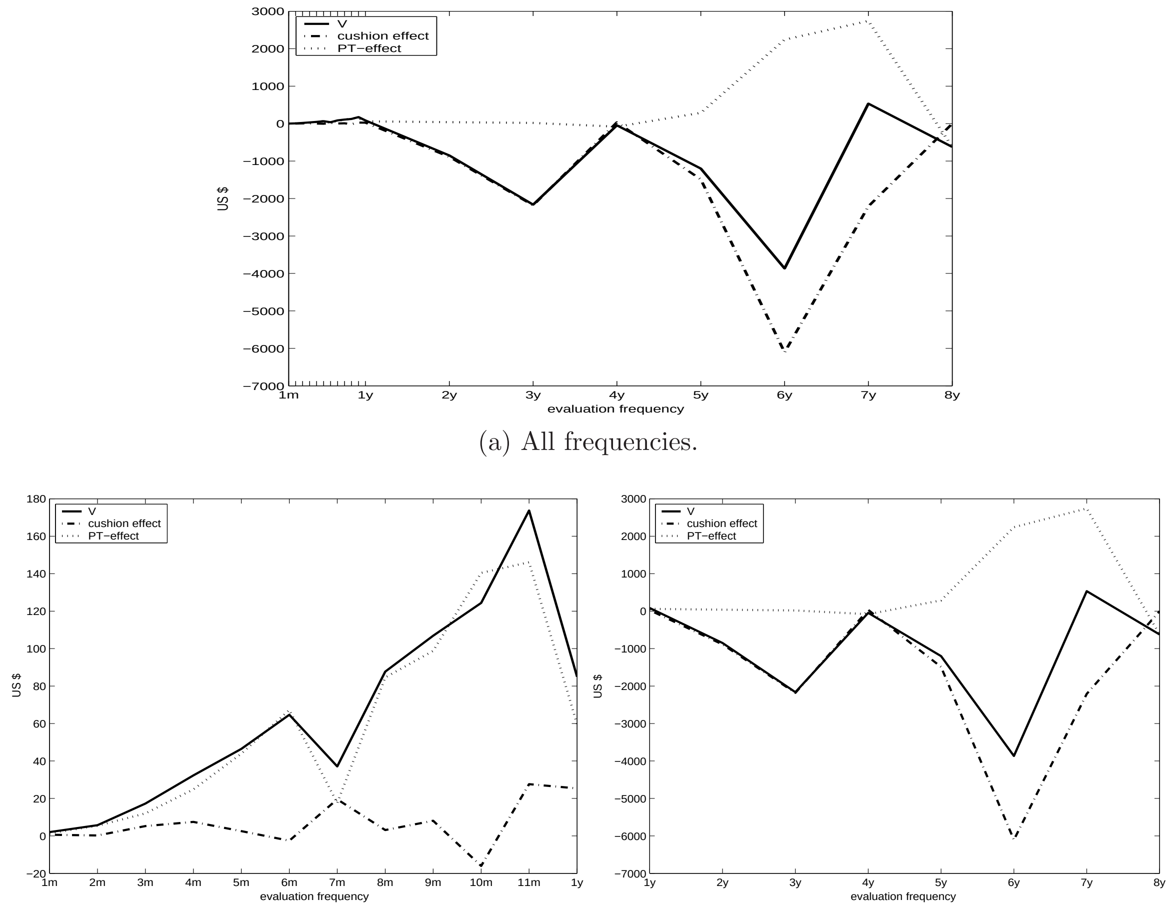


(a) Yearly evaluations.



(b) Daily evaluations.

Figure A.69: Prospective value evolution for dynamic cushions and yearly and daily evaluations.

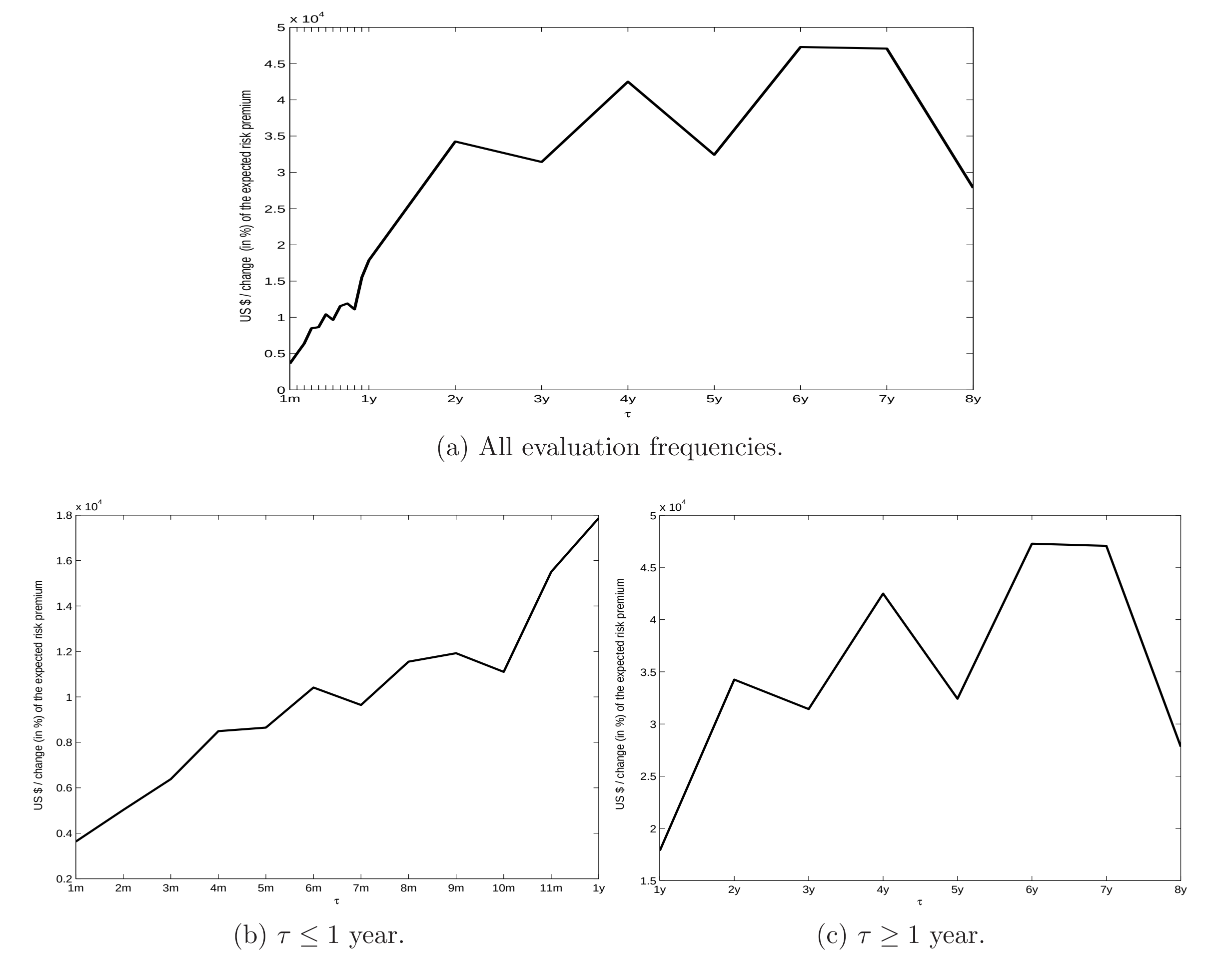


(a) All frequencies.

(b) $\tau \leq 1$ year.

(c) $\tau \geq 1$ year.

Figure A.70: Prospective value evolution for dynamic cushions and different evaluation frequencies.



(a) All evaluation frequencies.

(b) $\tau \leq 1$ year.

(c) $\tau \geq 1$ year.

Figure A.71: Evolution of the global first-order risk aversion for different evaluation frequencies and dynamic cushions.

	Wealth %		λ^*	
Evaluation frequency	**Portfolio returns** Normal	Student-t	**Portfolio returns** Normal	Student-t
1 year	61.00	36.48	1.10	1.04
6 months	59.73	34.63	0.96	0.94
4 months	59.40	34.17	0.81	0.94
3 months	59.30	34.01	0.82	1.39
1 month	59.04	33.65	0.97	1.05
1 week	58.82	33.34	1.03	0.99
1 day	58.70	33.20	1.01	0.98

Table A.2: Wealth percentages invested in risky assets and the average λ^*, for $\alpha = 1\%$ and dynamic cushions.

	Wealth %		λ^*	
Evaluation frequency	**Portfolio returns** Normal	Student-t	**Portfolio returns** Normal	Student-t
1 year	120.80	125.37	1.09	1.03
6 months	121.47	126.11	1.01	0.99
4 months	121.64	126.29	1.00	0.99
3 months	121.70	126.36	1.00	1.00
1 month	121.84	126.50	1.00	1.00
1 week	121.96	126.63	1.00	1.00
1 day	122.00	126.67	1.00	1.00

Table A.3: Wealth percentages invested in risky assets and the average λ^*, for $\alpha = 10\%$ and dynamic cushions.

A.3.3 Two-dimensional utility: consumption vs. financial assets

The expected and non-expected utility approaches with $\delta = 2$

	Myopic cushions			Dynamic cushions with $\eta = 0.9$		
	$k = 0$	$k = 3$	$k = 10$	$k = 0$	$k = 3$	$k = 10$
	$b_0 = 1$					
$\lvert\hat{V}\rvert$	229.8	229.8	229.8	229.8	229.8	229.8
$\hat{\lambda}$	1.7282	1.8298	2.2411	0.11589	0.11545	0.11465
	$b_0 = 5$					
$\lvert\hat{V}\rvert$	45.961	45.961	45.961	45.961	45.961	45.961
$\hat{\lambda}$	0.72081	0.74632	0.8407	0.11303	0.1128	0.11229
	$b_0 = 10$					
$\lvert\hat{V}\rvert$	22.98	22.98	22.98	22.98	22.98	22.98
$\hat{\lambda}$	0.59489	0.61089	0.66565	0.11267	0.11246	0.112
	$b_0 = 100$					
$\lvert\hat{V}\rvert$	2.298	2.298	2.298	2.298	2.298	2.298
$\hat{\lambda}$	0.48157	0.489	0.5081	0.11235	0.11216	0.11173
	$b_0 = 1000$					
$\lvert\hat{V}\rvert$	0.2298	0.2298	0.2298	0.2298	0.2298	0.2298
$\hat{\lambda}$	0.47024	0.47682	0.49234	0.11232	0.11213	0.1117

Table A.4: The main variable estimates in the expected-utility equilibrium for yearly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, and using various cushion-assessment methods, degrees of past-loss sensitivity k, additional-income levels I given by ($\beta = 0.9, \delta = 2$), and various narrow-framing degrees b_0.

	Myopic cushions			Dynamic cushions with $\eta = 0.9$		
	$k = 0$	$k = 3$	$k = 10$	$k = 0$	$k = 3$	$k = 10$
	$b_0 = 1$					
$\lvert\hat{V}\rvert$	460.89	460.89	460.89	460.89	460.89	460.89
$\hat{\lambda}$	3.4758	3.2393	2.6812	-1.545	-1.5834	-1.6729
	$b_0 = 5$					
$\lvert\hat{V}\rvert$	92.179	92.179	92.179	92.179	92.179	92.179
$\hat{\lambda}$	3.5565	3.3249	2.7831	-1.5479	-1.5863	-1.6758
	$b_0 = 10$					
$\lvert\hat{V}\rvert$	46.089	46.089	46.089	46.089	46.089	46.089
$\hat{\lambda}$	3.5666	3.3356	2.7959	-1.5483	-1.5866	-1.6762
	$b_0 = 100$					
$\lvert\hat{V}\rvert$	4.6089	4.6089	4.6089	4.6089	4.6089	4.6089
$\hat{\lambda}$	3.5757	3.3452	2.8073	-1.5486	-1.587	-1.6765
	$b_0 = 1000$					
$\lvert\hat{V}\rvert$	0.46089	0.46089	0.46089	0.46089	0.46089	0.46089
$\hat{\lambda}$	3.5766	3.3462	2.8085	-1.5486	-1.587	-1.6766

Table A.5: The main variable estimates in the expected-utility equilibrium for quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, and using various cushion-assessment methods, degrees of past-loss sensitivity k, additional-income levels I given by $(\beta = 0.9, \delta = 2)$, and various narrow-framing degrees b_0.

	Myopic cushions			Dynamic cushions with $\eta = 0.9$		
	$k = 0$	$k = 3$	$k = 10$	$k = 0$	$k = 3$	$k = 10$
$b_0 = 1$	228020	226180	221900	5355200	5418000	5567000
$b_0 = 5$	232680	230830	226560	5359400	5422200	5571200
$b_0 = 10$	233260	231410	227140	5359900	5422700	5571700
$b_0 = 100$	233780	231930	227660	5360400	5423200	5572200
$b_0 = 1000$	233830	231990	227710	5360400	5423200	5572200

Table A.6: The estimated global first-order risk aversion (gRA) in the expected-utility equilibrium for yearly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, and using various cushion-assessment methods, degrees of past-loss sensitivity k, additional-income levels I given by $(\beta = 0.9, \delta = 2)$, and various narrow-framing degrees b_0.

	Myopic cushions			Dynamic cushions with $\eta = 0.9$		
	$k = 0$	$k = 3$	$k = 10$	$k = 0$	$k = 3$	$k = 10$
$b_0 = 1$	1758500	1671900	1474900	54100	-1011100	-3384700
$b_0 = 5$	1771700	1685200	1488100	37488	-1027700	-3401300
$b_0 = 10$	1773400	1686900	1489800	35412	-1029800	-3.403400
$b_0 = 100$	1774900	1688300	1491300	33543	-1031700	-3405300
$b_0 = 1000$	1775000	1688500	1491400	33356	-1031800	-3405500

Table A.7: The estimated global first-order risk aversion (gRA) in the expected-utility equilibrium for quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, and using various cushion-assessment methods, degrees of past-loss sensitivity k, additional-income levels I given by $(\beta = 0.9, \delta = 2)$, and various narrow-framing degrees b_0.

	Myopic cushions			Dynamic cushions with $\eta = 0.9$		
	$k = 0$	$k = 3$	$k = 10$	$k = 0$	$k = 3$	$k = 10$
$\bar{C}/\hat{W}$	0.024975	0.024985	0.025009	0.00080065	0.00080099	0.00080177
$\hat{\theta}$	0.10957	0.10973	0.11008	0.12978	0.12993	0.13023
$(1 - \bar{C}/\hat{W})\hat{\theta}$	0.10683	0.10699	0.10733	0.12968	0.12982	0.13013

Table A.8: The estimated wealth allocation in the expected-utility equilibrium for yearly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, and using various cushion-assessment methods, degrees of past-loss sensitivity k, additional-income levels I given by $(\beta = 0.9, \delta = 2)$, and various narrow-framing degrees b_0.

	Myopic cushions			Dynamic cushions with $\eta = 0.9$		
	$k = 0$	$k = 3$	$k = 10$	$k = 0$	$k = 3$	$k = 10$
$\bar{C}/\hat{W}$	0.098052	0.098111	0.098267	0.0032133	0.0032159	0.0032221
$\hat{\theta}$	0.3911	0.39106	0.39003	0.31859	0.31878	0.3184
$(1 - \bar{C}/\hat{W})\hat{\theta}$	0.35276	0.35269	0.3517	0.31757	0.31776	0.31738

Table A.9: The estimated wealth allocation in the expected-utility equilibrium for quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, and using various cushion-assessment methods, degrees of past-loss sensitivity k, additional-income levels I given by $(\beta = 0.9, \delta = 2)$, and various narrow-framing degrees b_0.

	$b_0 = 1$	$b_0 = 5$	$b_0 = 10$	$b_0 = 100$	$b_0 = 1000$
$\lvert\hat{V}\rvert$	1.2705	0.25411	0.12705	0.012705	0.0012705
$\lvert\hat{\lambda}\rvert$	2.0139	2.0143	2.0145	2.0146	2.0146
$\hat{\alpha}$	0.13095	0.13095	0.13095	0.13095	0.13095
$\lvert\hat{\theta}\rvert$	0.25236	0.25236	0.25236	0.25236	0.25236

Table A.10: The main variable estimates in the non-expected utility equilibrium for yearly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memory-less dynamic cushions with $\eta = 0$, additional-income levels I given by $(\beta = 0.9, \delta = 2)$, and various narrow-framing degrees b_0.

	$b_0 = 1$	$b_0 = 5$	$b_0 = 10$	$b_0 = 100$	$b_0 = 1000$
$\lvert\hat{V}\rvert$	0.89663	0.17933	0.089663	0.0089663	0.00089663
$\lvert\hat{\lambda}\rvert$	1.2451	1.254	1.2551	1.2561	1.2562
$\hat{\alpha}$	0.17581	0.17581	0.17581	0.17581	0.17581
$\lvert\hat{\theta}\rvert$	0.22905	0.22905	0.22905	0.22905	0.22905

Table A.11: The main variable estimates in the non-expected utility equilibrium for quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memory-less dynamic cushions with $\eta = 0$, additional-income levels I given by $(\beta = 0.9, \delta = 2)$, and various narrow-framing degrees b_0.

$b_0 = 1$	140.92
$b_0 = 5$	28.184
$b_0 = 10$	14.092
$b_0 = 100$	1.4092
$b_0 = 1000$	0.14092

Table A.12: The estimated global first-order risk aversion (gRA) in the non-expected utility equilibrium for yearly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memory-less dynamic cushions with $\eta = 0$, additional-income levels I given by $(\beta = 0.9, \delta = 2)$, and various narrow-framing degrees b_0.

$b_0 = 1$	307.87
$b_0 = 5$	61.573
$b_0 = 10$	30.787
$b_0 = 100$	3.0787
$b_0 = 1000$	0.30787

Table A.13: The estimated global first-order risk aversion (gRA) in the non-expected utility equilibrium for quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memory-less dynamic cushions with $\eta = 0$, additional-income levels I given by $(\beta = 0.9, \delta = 2)$, and various narrow-framing degrees b_0.

Evaluation frequency	
Yearly	Quarterly
0.21931	0.18878

Table A.14: The estimated total-wealth fractions dedicated to risky assets $(1-\hat{\alpha})|\hat{\theta}|$ in the non-expected utility equilibrium for yearly and quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memory-less dynamic cushions with $\eta = 0$, and additional-income levels I given by $(\beta = 0.9, \delta = 2)$.

The expected-utility approach with memory-less dynamic cushions $\eta = 0$

		$b_0 = 1$	$b_0 = 5$	$b_0 = 10$	$b_0 = 100$	$b_0 = 1000$
Low I	$\lvert\hat{V}\rvert$	229.8	45.961	22.98	2.298	0.2298
$(\beta = 0.1, \delta = 0.9)$	$\lvert\hat{\lambda}\rvert$	-9.0563	-0.80628	0.22497	1.1531	1.2459
Middle I	$\lvert\hat{V}\rvert$	-229.8	-45.961	-22.98	-2.298	-0.2298
$(\beta = 0.5, \delta = 0.5)$	$\lvert\hat{\lambda}\rvert$	0.17237	1.0394	1.1478	1.2454	1.2551
High I	$\lvert\hat{V}\rvert$	-229.8	-45.961	-22.98	-2.298	-0.2298
$(\beta = 0.9, \delta = 0.1)$	$\lvert\hat{\lambda}\rvert$	1.2183	1.2486	1.2524	1.2558	1.2562

Table A.15: The main variable estimates in the expected utility equilibrium for yearly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memory-less dynamic cushions with $\eta = 0$, various additional incomes I and narrow-framing degrees b_0.

Low I	$\lvert\hat{V}\rvert$	460.89	92.179	46.089	4.6089	0.46089
$(\beta = 0.1, \delta = 0.9)$	$\lvert\hat{\lambda}\rvert$	0.38945	1.6896	1.8521	1.9984	2.013
Middle I	$\lvert\hat{V}\rvert$	460.89	92.179	46.089	4.6089	0.46089
$(\beta = 0.5, \delta = 0.5)$	$\lvert\hat{\lambda}\rvert$	1.7513	1.962	1.9883	2.012	2.0143
High I	$\lvert\hat{V}\rvert$	460.89	92.179	46.089	4.6089	0.46089
$(\beta = 0.9, \delta = 0.1)$	$\lvert\hat{\lambda}\rvert$	2.0064	2.013	2.0138	2.0145	2.0146

Table A.16: The main variable estimates in the expected utility equilibrium for quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memory-less dynamic cushions with $\eta = 0$, various additional incomes I and narrow-framing degrees b_0.

$b_0 = 1$	75116
$b_0 = 5$	15023
$b_0 = 10$	7511.6
$b_0 = 100$	751.16
$b_0 = 1000$	75.116

Table A.17: The estimated global first-order risk aversion (gRA) in the expected-utility equilibrium for yearly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, and memory-less dynamic cushions with $\eta = 0$.

$b_0 = 1$	51111
$b_0 = 5$	10222
$b_0 = 10$	5111.1
$b_0 = 100$	511.11
$b_0 = 1000$	51.111

Table A.18: The estimated global first-order risk aversion (gRA) in the expected-utility equilibrium for quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, and memory-less dynamic cushions with $\eta = 0$.

Low I	$\bar{C}/\hat{W}$	0.14555
$(\beta = 0.1, \delta = 0.9)$	$\hat{\theta}$	-0.034714
	$(1 - \bar{C}/\hat{W})\hat{\theta}$	-0.029661
Middle I	$\bar{C}/\hat{W}$	0.020167
$(\beta = 0.5, \delta = 0.5)$	$\hat{\theta}$	0.11555
	$(1 - \bar{C}/\hat{W})\hat{\theta}$	0.11322
High I	$\bar{C}/\hat{W}$	0.00080024
$(\beta = 0.9, \delta = 0.1)$	$\hat{\theta}$	0.1338
	$(1 - \bar{C}/\hat{W})\hat{\theta}$	0.13369

Table A.19: The estimated wealth allocation in the expected-utility equilibrium for yearly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memory-less dynamic cushions with $\eta = 0$, and various additional incomes I.

Low I	$\bar{C}/\hat{W}$	0.22693
$(\beta = 0.1, \delta = 0.9)$	$\hat{\theta}$	0.11972
	$(1 - \bar{C}/\hat{W})\hat{\theta}$	0.092553
Middle I	$\bar{C}/\hat{W}$	0.050256
$(\beta = 0.5, \delta = 0.5)$	$\hat{\theta}$	0.25945
	$(1 - \bar{C}/\hat{W})\hat{\theta}$	0.29731
High I	$\bar{C}/\hat{W}$	0.0032122
$(\beta = 0.9, \delta = 0.1)$	$\hat{\theta}$	0.35583
	$(1 - \bar{C}/\hat{W})\hat{\theta}$	0.35469

Table A.20: The estimated wealth allocation in the expected-utility equilibrium for quarterly portfolio evaluations, initial loss aversion $\lambda = 2.25$, risk-aversion $\gamma = 0.5$, memory-less dynamic cushions with $\eta = 0$, and various additional incomes I.

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Eidesstattliche Erklärung

Hiermit versichere ich an Eides Statt, dass ich die vorliegende Diplomarbeit selbständig, ohne fremde Hilfe und ohne Benutzung anderer als der von mir angegebenen Quellen angefertigt habe. Alle aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche gekennzeichnet. Diese Arbeit hat in gleicher oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegen und wurde bisher nicht veröffentlicht.

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